Ramanujan $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraphs based on Bruhat-Tits Buildings of type $\tilde{A}_{d-1}$

Dissertation
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenscaftlichen Fakultäten
der Georg-August-Universität zu Göttingen

vorgelegt von
Alireza Sarveniazi
aus Teheran-Iran

Göttingen, 2003

D7

Referent: Prof. Dr. U. Stuhler

Korefferent: Prof. Dr. L. Smith

Tag der mündlichen Prüfung : 20.01.2004

Contents

0 Preface 5

1 Some basic material 9

1.1 Introduction: . . . 9
1.2 Local Fields and Basics . . . 9
1.3 Haar measures . . . 12
1.4 A review of finite dimensional algebras . . . 12

2 Buildings of type $\tilde{A}_{d-1}$ 17

2.1 Introduction . . . 17
2.2 Affine Buildings . . . 18
2.3 Some More Facts About Affine Buildings . . . 23
2.4 Combinatorics and definition of Ramanujan Hypergraph . . . 26

3 Representation Theory and Hypergraphs 31

3.1 Introduction . . . 31
3.2 Quotient hypergraphs, the general situation . . . 32
3.3 Adjacency operators and Hecke operators . . . 35
3.4 The Ramanujan Property . . . 39

4 Explicit Constructions 45

4.1 Introduction . . . 45
4.2 The Skew polynomial ring $\mathbb{F}_{q^d}\{\tau\}$ . . . 45
4.3 Arithmetic groups associated to the division algebra $\mathbb{F}_{q^d}(\tau)$ . . . 54
4.4 Explicit description of some arithmetic quotients . . . 59
4.5 Explicit construction . . . 63
4.6 Bibliography . . . 73

Chapter 0

Preface

We introduce in this thesis Ramanujan regular hypergraphs, generalizing the definition of Ramanujan graphs introduced by Lubotzky, Philips, Sarnak [33]. Ramanujan graphs are regular graphs, whose adjacency matrices, (or equivalently their Laplacians), have eigenvalues satisfying some natural upper bounds. Graphs satisfying such bounds have many interesting properties. For example they are very powerful expander graphs, which make them interesting objects for many applications from communication networks to computer science. The interesting situation here, is that on the one hand it is known and easy to see that almost all regular graphs in a precise sense are Ramanujan graphs. On the other hand it is difficult to show for an explicitly given regular graph that it is a Ramanujan graph. It is therefore quite surprising and has lot of interest, that some very explicit graphs coming from number theory can be shown to be Ramanujan graphs. These graphs are constructed using unit groups of quaternion algebra and their action on the associated symmetric spaces, which in these examples just are the trees in the sense of [55] or [61]. The quotients of these trees by the action of the unit groups mentioned above give already the Ramanujan graphs we are discussing. To show the Ramanujan property requires deep tools from the theory of automorphic representations. In fact it turns out that the Ramanujan property of these graphs is equivalent to the fact, that the associated representations satisfy the Ramanujan-Petersson conjecture. This is not very difficult to prove because one sees rather directly that the adjacency operators of the quotient graph above is more or less nothing else than the Hecke operator of the corresponding prime. The first examples given where working with quaternion algebras over the rational numbers and were using Deligne's proof of the Ramanujan-Petersson conjecture plus the Jaquet-Langlands correspondence between automorphic representations of quaternion algebras and cuspidal automorphic representations of $GL(2, \mathbb{Q})$ of an appropriate type. A

later variant of this by M.Morgenstern [40] was using instead quaternion algebras over the rational function field $\mathbb{F}_q(t)$ and corresponding results of Drinfeld [21] for the associated automorphic representations. It is this class of examples of Morgenstern, which we will generalize to higher dimensions. This is made possible by the observation, that the quaternion algebras in Morgenstern's examples are nothing else than the quotient (skew) fields of skew polynomial rings which are well known in the theory of Drinfeld modules. So we consider the skew polynomial ring $\mathbb{F}_{q^d}\{\tau\}$ over the field $\mathbb{F}_{q^d}$ of q^d elements, where the indeterminate τ satisfies the rule $\tau \cdot \lambda = \lambda^q \cdot \tau$ for $\lambda \in \mathbb{F}_{q^d}$.
The center of this ring is the polynomial ring $\mathbb{F}_q[t]$, the (skew) quotient field $\mathbb{F}_q(\tau)$ is a division algebra of dimension d^2 over the center $\mathbb{F}_q(t)$, the field of rational functions over $\mathbb{F}_q$.
This division algebra is ramified exactly at the primes $t = 0$ and $t = \infty$. We consider the localization $\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}]$ at a prime $p = p(t)$ of $\mathbb{F}_q[t]$ different from t. The unit group associated with this maximal order over the center $\mathbb{F}_q[t][\frac{1}{p}]$ is acting on the Bruhat-Tits building at the prime p, which is a building of type $\tilde{A}_{d-1}$. The quotients by the unit groups we define, will be simplicial complexes of dimension $(d-1)$. It is these simplicial complexes and their adjacency matrices, we are studying in this thesis. In this thesis, we partly work in the category of simplicial complexes and more specialized buildings and partly in the category of hypergraphs. The procedure to be followed now, is quite similar to the case of graphs. It consists in identifying the adjacency operator of these quotient complexes with some Hecke operator acting on certain corresponding spaces of automorphic forms and then making use of the relevant results about automorphic representations in this context. The main result we are relying on here, is Lafforgue's recent proof of the Langlands conjecture for $GL(n)$ over a function field and in particular the Ramanujan-Petersson conjecture for these cuspidal automorphic forms. What causes difficulties, is that here the Jacquet-Langlands correspondence between automorphic representations for $GL(n)$ and automorphic representations for the division algebra is not effective enough. Using this correspondence, it is possible that noncuspidal automorphic representations come up. Lafforgue's result however requires the automorphic representation to be cuspidal.
Nevertheless, using additionally a result of Arthur and Clozel [1, Weak Lifting theorem, 4.2] it is possible to concude the Ramanujan property at least for the d a prime from Lafforgue's result.
It should be metioned also, that after this thesis was (more or less) finished, preprints of W.-C. Winnie Li [65] and Lubotzky, Samuels and Vishne [35],

[36] become avilable to us. In particular W. Lie using a trick of L.Clozel, reduce the the Ramanujan property of the automorphic representation to the Ramanujan property of the moduli scheme of $\mathcal{D}$ elliptic sheaves [32].
Let us add at this point, that there is another treatment of higher dimensional situations in the literature, namely [24] B.W.Jordan and R.Livne. These results work again with unit groups of quaternion algebras, but allow denominators at more than one prime. Correspondingly, their unit groups are acting on a product of trees. As this is again the GL(2)-case, the Ramanujan-Petersson property holds. Also C.Ballantine [2] has given a hypergraph for $d = 3$ related to the Bruhat-Tits building associted to $\mathrm{PGL}(3; \mathbb{Q}_p)$.
We end this introduction with a description of the content of the different chapters of this thesis.
Chapter 1 gives basic material on locally compact fields, Haar measures and division algebras.
Chapter 2 describes buildings of type $\tilde{A}_n$ and various concepts related to this.
In chapter 3 we describe our examples in general. We indicate some of of the concepts from the theory of automorphic representations. In particular we describe the relationship between adjacency operators and Hecke operators. Finally we discuss in this chapter, what would be needed additionally from the theory of automorphic representations, in particular regarding the work of L.Lafforgue. In the final chapter 4 we specialize our situation to the case of skew polynomial rings and the related quotient (skew) fields. Here we are able to compute explicity the quotient of the building by the action of the discrete groups of units we have chosen.

Aknowledgement
I am indebted to Prof. Dr. Ulrich Stuhler, who supervised my doctoral work, for helpful discussion, continued encouragement and suggestions for presentation in the form my thesis. I am also much thankful to PD. Dr. Thomas Lehmkuhl for useful discussion, specially on Theorem 72. I am also very thankful to Prof. Dr. Larry Smith for his careful reading of this thesis.

Chapter 1

Some basic material

1.1 Introduction:

After a short review about local fields and Haar measure we give a survey of skew polynomial rings. In the category of rings, the ”good” objects are the rings in which we have not only additive inverse elements, but also there exist multiplicative inverses of nonzero elements. These are called division rings, or skew fields. Fields are special examples of division rings. All division rings may be broadly classified into two types, according to whether they are finite dimensional (as vector spaces) over their center or not. The objects we study here are usually finite dimensional central division algebras (rings).

1.2 Local Fields and Basics

A general reference for local fields is chapter I of Andre Weil's book [64].

Definition 1. *A local field is a locally compact topological field, which is not discrete.*

Theorem 2. *A local field is isomorphic as a topological field either to*
i) $\mathbb{R}$ or $\mathbb{C}$, the field of real resp. complex numbers, or to
ii) a finite algebraic extension of the fields $\mathbb{Q}_p$ of p-adic numbers, p a prime or to
iii) a field of Laurent series

$$\mathbb{F}_q((t)) = \{\sum_{i=N}^{\infty} a_i t^i \quad | \quad a_i \in \mathbb{F}_q, \quad N \in \mathbb{Z}\}$$

where $\mathbb{F}_q$ is a finite field of q elements.

Remark. *i*) As usual the field of p-adic numbers $\mathbb{Q}_p$ is obtained from the field $\mathbb{Q}$ of rational numbers by completing with respect to the p-adic valuation $|.|_p$ of $\mathbb{Q}$, which on $\mathbb{Z}$ satisfies

$$|a|_p := p^{-v_p(a)} \quad ,$$

where $a = p^{v_p(a)}a'$ such that p and a' are prime to each other.
ii) A basis for the topology of $\mathbb{F}_q((t))$ is given by the complete open subgroups $t^i\mathbb{F}_q[[t]]$ in the ring of formal power series in t over the finite field of q elements

$\mathbb{F}_q, \quad i \in \mathbb{Z}$.

We consider now the field of rational functions $\mathbb{F}_q(t)$ over the finite field $\mathbb{F}_q$ of q elements, By a theorem of Ostrowski we know all discrete valuations of $\mathbb{F}_q(t)$ up to equivalence. [**?**, page 80], [64, III, Theorem 2].

Theorem 3. *Any discrete nontrivial valuation of* $\mathbb{F}_q((t))$ *is given up to equivalence as either*
1) *The valuation corresponding to an irreducible polynomial* $p = p(t) \in \mathbb{F}_q[t]$, *such that*

$$|a|_p := {c_p}^{v_p(a)} \quad ,$$

for $a = a(t) \in \mathbb{F}_q[t]$, $0 \leq c_p < 1$ *is a real number and* $a = p^{v_p(a)}a'$ *such that* p *and* a' *are prime to each other in* $\mathbb{F}_q[t]$. *Given* $|a|_p$ *in the way above on* $\mathbb{F}_q[t]$, *it extends uniquely to discrete valuation of* $\mathbb{F}_q(t)$.
2) *There is additionally the degree valuation at* ∞, *such that*

$$|a(t)|_\infty := c^{\deg(a)} \qquad , \textit{with} \qquad 0 \leq c \leq 1 \quad \textit{fixed.}$$

for $a = a(t) \in \mathbb{F}_q[t]$. *Again,*

$$|\frac{a(t)}{b(t)}|_\infty := c^{\deg(a)-\deg(b)}$$

Remark. To obtain the product formula below, it is useful to normalize the valuations given above as follows:
Choose $c \in \mathbb{R}$, $0 \leq c < 1$ once and for all, then choose $c_p = c^{\deg(p)}$ for the valuation corresponding to an irreducible prime $p = p(t) \in \mathbb{F}_q[t]$. One obtains immediately the
Productformula: For $a \in \mathbb{F}_q(t)(a \neq 0)$ has

$$\prod_{\nu \in X} |a|_\nu = 1$$

Here X denotes the set of all valuations of $\mathbb{F}_q(t)$ normalized as above.

Definition 4. *For a valuation* $v \in X$, F_v *denotes the completion of* $F = \mathbb{F}_q(t)$ *at* v. $\mathcal{O}_v$ *denotes the corresponding ring of integer, that is:*

$$\mathcal{O}_v := \{x \in F_v \quad | \quad |X|_v \leqslant 1\}$$

Definition 5. *Associated with the field* $F = \mathbb{F}_q(t)$ *of rational functions over the finite field* $\mathbb{F}_q$ *of* q *elements is the locally compact topological ring of adeles over* F,

$$\mathbb{A}(F) = \prod_{\nu \in X} (F_v, \mathcal{O}_\nu),$$

where

$$\prod_{\nu \in X} (F_v, \mathcal{O}_\nu) = \{(x_\nu | \nu \in X) | x_\nu \in F_\nu, \textit{for almost all} \quad \nu \in X \quad x_\nu \in \mathcal{O}_\nu\}.$$

Remark. Denote for $S \subset X$, a finite subset;

$$\mathbb{A}^S(F) := \prod_{\nu \in S} F_\nu \times \prod_{\nu \notin S} \mathcal{O}_\nu,$$

equipped with the product topology. By Tychonoff's theorem $\mathbb{A}^S(F)$ is a locally compact topological ring. Furthermore

$$\mathbb{A}(F) = \varinjlim_{(S)} \mathbb{A}^S(F)$$

obtains the induced topology. $\mathbb{A}(F)$ is a locally compact topological ring also. (see [64, chapter IV], for details). Besides the locally compact topological ring of adeles we use its group of units, the so called group of ideles. This is given as

$$I(F) = \varinjlim_{(S)} I^S(F),$$

where

$$I^S(F) := \prod_{\nu \in S} F_\nu^\times \times \prod_{\nu \notin S} \mathcal{O}_\nu^\times,$$

with the product topology. $I(F)$ is a locally compact topological group, see [64, IV,3, IV,4], for details.
Finally suppose, F_ν is locally compact topological field, ν the related valuation, D is a finite dimensional central skew field over F_ν .

See [**?**] for the following theorem:

Theorem 6. *The valuation* $|\ |_\nu$ *of* F_ν *can be prolongated to a valuation of* D *by the formula:*

$$|a|_\nu := \frac{1}{(D : F_\nu)} |N^D_{F_\nu}(a)|_\nu$$

where $N^D_{F_\nu} : D \longrightarrow F_\nu$ *is the norm of* D *over* F_ν.

1.3 Haar measures

This subsection will be short. We remind the reader of the basic facts about Haar measures which are well known (see [29]). Any locally compact topological group has a left invariant positive measure, that is a measure μ satisfying the following proposition:

Proposition 7. *i)* $\mu(A) \geq 0$ *for any measurable subset* $A \subset G$.
ii) $\mu(gA) = \mu(A)$ *for any measurable subset* A *and* $g \in G$.

In particular gA is measurable again. One has associated with the measure μ the corresponding Haar integral

$$\int_G f(g) d\mu(g)$$

for any $f : G \longrightarrow \mathbb{R}$, f continuous with compact support. For A measurable as above, one has

$$\mu(A) = \int_G \chi_A(g) d\mu(g)$$

where $\chi_A : G \longrightarrow \mathbb{R}$,

$$\chi_A(x) = \begin{cases} 0 & \text{for} \quad x \notin A \\ 1 & \text{for} \quad x \in A \end{cases}$$

is the characteristic function. This measure resp. its associated integral is unique up to a positive scalar. Similarly one has right invariant Haar measures with their associated integrals.
In general, right and left invariant Haar measure are not the same. However for compact topological groups or more generally for topological groups generated by compact open topological subgroups it is true, that one has a bi-invariant Haar measure.

1.4 A review of finite dimensional algebras

In this section we remind the reader of some fundamental facts about central simple algebras A over a field F. In the first part of this section the field F will be arbitrary. Later on we consider the cases of F a finite field resp. a local field in the sense of number theory. Finally we describe the classification for the case that F is a global field. Quite generally we have

Theorem 8. (*Wedderburn*) *Any central simple algebra* A *over a field* F *is isomorphic to a matrix ring* $\mathbb{M}(r, D)$, *where* D *is a central division algebra over* F. r *and* D *are uniquely determined by* A.

For a proof see [27].
The classification of central simple algebras is therefore reduced to the classification of central division algebras over F.

Definition 9. *A splitting field $F \hookrightarrow F_1$ for a central simple algebra A over F has the property, that the tensor product of A with F_1 satisfies*

$$A \otimes_F F_1 \quad \cong \quad \mathbb{M}(n, F_1).$$

Remark. :
i) A splitting field $F \hookrightarrow F_1$ of A is also a splitting field for the unique division algebra D over F, $A \cong \mathbb{M}(r, D)$. Conversely any splitting field $F \hookrightarrow F_1$ of D is also a splitting field of A.
ii) We have :

$$(A : F) = (A \otimes_F F_1 : F_1) = n^2.$$

Therefore it follows, that

$$r^2(D : F) = (A : F) = n^2$$

and in particular

$$(D : F) = (\frac{n}{r})^2.$$

$\frac{n}{r}$ is a natural number.

iii) Splitting fields for A always exist. For example, any maximal commutative subfield $F \subset F_1 \subset D$, where $(F_1 : F) = \frac{n}{r}$, is a splitting field. Such fields F_1 always exist, F_1 can be even chosen to be a separable field extension of F.
Splitting fields F_1, $F \hookrightarrow F_1$, can be used to define the reduced norm of A over F, which is a multiplicative map

$$rn \quad : \quad A \longrightarrow F.$$

This can be defined as follows:
We have the sequence of ring homomorphisms

$$\rho : A \hookrightarrow A \otimes_F F_1 \xrightarrow{\sim} \mathbb{M}(n, F_1)$$

Combining this with the determinant map

$$\det \quad : \quad \mathbb{M}(n, F_1) \longrightarrow F_1,$$

$$\det \cdot \rho \quad : \quad A \longrightarrow F_1$$

is a multiplicative map and it turns out that in fact $\mathrm{Im}(\det \cdot \rho) \subset F$.

Therefore we have the multiplicative map

$$rn := \det \cdot \rho \quad : \quad A \longrightarrow F \tag{1.1}$$

It turns out, that rn is independent of the choice of the splitting field $F \hookrightarrow F_1$, see for all of this [27]. This map is called the reduced norm of A over F. There is a slightly different more direct construction of the reduced norm, which is as follows:
Suppose $F \hookrightarrow F_1$ is a maximal commutative subfield of D, that is:

$$(F_1 : F)^2 = (D : F).$$

We consider D as a F_1-vector space of dimension $(D : F_1) = (D : F)^{1/2}$, where F_1 is acting on D from right. Then for $a \in D$,

$$\begin{array}{rccc} \mathrm{L}_a : & D & \longrightarrow & D \\ & x & \mapsto & ax \end{array}$$

is a F_1-linear map of the F_1-vector space D/F_1. One has

$$rn(a) = \det(\mathrm{L}_a)$$

where L_a is considered as F_1-linear map. The proof of this fact is easy by using the splitting $D \otimes_F F_1 \xrightarrow{\sim} \mathbb{M}(n, F_1)$ and using the first description of the reduced norm.

A special class of central simple algebras are cyclic algebras over F, which can be described as follows:
Suppose, $F_1 \supset F$ is a cyclic Galios extension of F, $Gal(F_1/F) =< \sigma >$, where σ is a specified generator of the cyclic Galios group $Gal(F_1/F)$ of order n. Furthermore α is an additional element, such that

$$A = \bigoplus_{i=0}^{n-1} \alpha^i \cdot F_1, \qquad \text{with the rule}$$

$$\alpha x \alpha^{-1} = \sigma(x)$$

for $x \in F_1$. Additionally, the relation

$$\alpha^n = a \in F^\times \qquad \text{holds.}$$

It is well known, that such an algebra A is a central algebra of dimension n^2 over F.
It is called a cyclic algebra over F and is denoted by $(F_1/F; \sigma; a)$ where

$F_1/F, \sigma, a$ are as above. Not all central simple algebras A over an arbitrary field F are cyclic. This, however is true, if F is a local or global field from number theory. For a general commutative field F, the classification of the central simple algebras resp. skew fields over F is not known. However, for the restricted class of fields related F to number theory, a classification is available.

A) Suppose first, F is finite field.

Theorem 10. (*Wedderburn*) *Any finite dimensional central division algebra D over F is trivial, that is, it is isomorphic to F.*

B) Next, we assume that F is a local field in the sense of number theory (defined in the section 1.2). Then, any division algebra D of F, of dimension $(D : F) = n^2$ is a cyclic algebra, which can be described as follows:
i) If $F \cong \mathbb{C}$, the field of complex numbers, then again $D \cong \mathbb{C}$ is the only division algebra over $\mathbb{C}$ up to isomorphism.
ii) If $F \cong \mathbb{R}$, the field of real numbers, the only possibilities, by a theorem of Frobenius, are :

$$D \cong \mathbb{R} \qquad \text{or} \qquad D \cong \mathbb{H},$$

the Hamiltonian quaternions. $D \cong \mathbb{H}$ is a cyclic algebra over $\mathbb{R} \cong F$ with invariant (see *iii*) below) $\mathrm{inv}(D/F) = \frac{1}{2}$.
iii) Suppose now, F is a local nonarchimedean field, so we have a discrete valuation on F. As already mentioned, F is either a finite extension of $\mathbb{Q}_p$, the field of p-adic numbers, or $F \cong \mathbb{F}_q((t))$, the field of Laurent series over the finite field $\mathbb{F}_q$. The valuation of F extends uniquely to D. One has $e = f = n$ for ramification index e and residue field extension degree f. D is a cyclic algebra, which can be described as follows:
D contains an extension F_1 of F, $(F_1 : F) = n$, which is unramified. F_1 is unique up to conjugation. It is a cyclic Galois extension of F with canonical generator the Frobenius automorphism $Frob : F_1 \longrightarrow F_1$, which is a lift of the Frobenius automorphism of the corresponding cyclic residue field extension f_1 of f.
By Skolem-Noether one has $d \in D^\times$, such that:

$$dxd^{-1} \quad = \mathrm{Frob}(x) \qquad \text{for} \quad x \in F_1.$$

It is immediate to see, that $d^n \in F^\times$.
We consider the valuations v_F, v_{F_1}, and v_D related to F, F_1 D. These are taken as additive valuations and we have normalized v_F, such that $v_F(\pi) = 1$,

for π a uniformizing element of F. Then :

$$\begin{aligned} v_D(d) &= \frac{1}{n} v_D(d^n) = \frac{1}{n} v_F(d^n) \in \mathbb{Q}/\mathbb{Z} \\ &= i/n \quad \text{in} \quad \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

We denote

$$i/n =: \mathrm{inv}(D/F)$$

the invariant of D over F. We have the classical :

Theorem 11. *Two division algebras D_1,D_2 over F are isomorphic iff*

$$inv(D_1/F) = inv(D_2/F).$$

All invariants $i/n \in \mathbb{Q}/\mathbb{Z}$ occur, the corresponding division algebra D/F is of dimension n^2 (where $(i,n) = 1$).

C) Finally, we assume, that F is a global field, X denotes the set of valuations of F up to equivalence. Then one has the

Theorem 12. (*Hasse - Brauer - Noether*)
i) Two division algebras D_1,D_2 over F are isomorphic iff for all $v \in X$,

$$inv(D_1 \otimes_F F_v/F_v) = inv(D_2 \otimes_F F_v) \qquad holds.$$

ii) Any subset of invariants $\{\lambda_v \in \mathbb{Q}/\mathbb{Z} \quad | \quad v \in X\}$ can be realized by a division algebras D/F iff $1, 2), 3)$ below hold :

$$\begin{aligned} &1) almost\ all \quad \lambda_v = 0. \\ &2) \lambda_v \in \{0, \frac{1}{2}\} \quad for\ the\ archimedean\ valuations \quad v \in X, \\ &\quad \lambda_v = 0 \quad for \quad v \in X, \quad F_v \cong \mathbb{C}. \\ &3) \sum_{v \in X} \lambda_v = 0 \quad in\ the\ abelian\ group \quad \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

For references concerning all this, see [64], in particular chapters IX,X, XI.

Chapter 2

Buildings of type $\tilde{A}_{d-1}$

2.1 Introduction

A building is a geometrica-combinatorical objectB. uildings were introduced by Jacques Tits in the $1960's$. They have been studied intensively from that time to present. In general there are three types of buildings. Spherical Buildings, characterized by their finite apartments, have much in common with compact symmetric spaces. Affine buildings, characterized by apartments shaped like real affine space, can be compared to non-compact symmetric spaces. Hyperbolic buildings have no such classical analogue. This note deals almost exclusively with affine buildings. One example of non-compact symmetric spaces are the symmetric spaces $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, upon which space the group $\mathrm{SL}(n,\mathbb{R})$ acts. What happens if we wish to replace $\mathbb{R}$ by a local field F ? Nothing prevents us from forming and studing $\mathrm{SL}(n,F)/\mathrm{SO}(n,F)$, but the global shape of the resulting object differs greatly from that of symmetric space. One reason for this difference is that $\mathrm{SO}(n,F)$ is not a maximal compact subgroup of $\mathrm{SL}(n,F)$, nor indeed, is it compact at all. For many purposes the correct analogue of the symmetric space is something completely different, namely the affine building of $\mathrm{SL}(n,F)$. This building is acted upon by $\mathrm{GL}(n,F)$, and if K is any maximal compact subgroup of $\mathrm{GL}(n,F)$, then the building contains a copy of $\mathrm{GL}(n,F)/K$. It is stretching things only a little to say that affine buildings were invented in order to play the role of symmetric spaces for semi-simple matrix groups over nonarchimedean local fields . The definition of affine building is strictly combinatorial, and there do exist affine buildings which do not correspond to any semi-simple matrix group. Nonetheless, understanding the relation to matrix groups is one of the keys to understanding affine buildings. In particular, unless an affine building is of dimension 1 or 2, or unless it is a Cartesian product and has a

factor of dimension 1 or 2, it will be the affine building of some semi-simple matrix group. In this chapter we discuss the affine buildings associated to the matrix groups PGL(d,F).
The aim of this chapter is to give a very brief introduction, largely self-contained and as elementary as possible to the subject of buildings of type $\tilde{A}_{d-1}$ related to non-archimeadian local fields. Let us mention also that there are three main books devoted to buildings: "Buildings" by K.S.Brown [9], and "Lectures on buildings", by Mark Ronan [46] and the book of J. Garret. There are also extensive survey articles by Mark Ronan [45]. These contain long lists of references to earlier work, especially that of J. Tits.

2.2 Affine Buildings

In this section F denotes a local non archimedean field with valuation ring $\mathcal{O}$, (π) is the maximal ideal with specified uniformizing element $\pi \in \mathcal{O}$.
F^d denotes the d-dimensional vector space over F.

Definition 13. *i) A lattice in F^d is a $\mathcal{O}$-submodule of F^d, for which there exists a basis $\{b_1, \ldots, b_d\}$ of F^d such that $L = \sum_{i=1}^d \mathcal{O} b_i$ holds.*
(ii) Two lattices $L, L' \subset F^d$ are similar iff there exists $\lambda \in F^\times$, such that $L' = \lambda L$. In this situation we denote $L \sim L'$.

Remark. (i) If L is a finitely generated torsion free $\mathcal{O}$-submodule of F^d satisfying $F \cdot L = F^d$, then L is a lattice in the sense of the definition above.
(ii) Of course a similar definition as above can be made for arbitrary abstract vector spaces V over F of finite dimension.

The group $\mathrm{GL}(d, F)$ of F-linear isomorphisms of the vector space F^d acts on the set of lattices by

$$gL = \{g(l) | l \in L\}$$

for $g \in \mathrm{GL}(d, F) = Aut_F(F^d), L \subset F^d$ an $\mathcal{O}$-lattice.

Lemma 14. *The action of $GL(d, F)$ is transitive on $\mathcal{L}$ where $\mathcal{L}$ is the set of all lattices.*

Proof. Denote $L_0 := \bigoplus_{i=1}^d \mathcal{O} e_i$ the standard lattice $\mathcal{O}^d \subset F^d, \{e_1, \ldots, e_d\}$ the standard basis of F^d.
It is sufficient to show, that for an arbitrary lattice $L \in \mathcal{L}$ there exists $g \in \mathrm{GL}(d, F), g.L_0 = L$.
By definition, there exists a basis $\{b_1, \ldots, b_d\}$ of F^d, such that $L = \sum_{i=1}^d \mathcal{O} b_i$.
Define $g \in GL(d, F)$ by $g(e_i) = b_i$ for $i = 1, \ldots, d$. Obviously $g.L_0 = L$ holds. □

Lemma 15. *The action of $GL(d,F)$ on $\mathcal{L}$ induces a corresponding action on $\mathcal{L}/\sim$, which is again transitive.*

Proof. It is enough to remark , that $L \sim L'$ implies $g.L \sim g.L'$ for any element $g \in GL(d,F)$. □

Definition 16. *A simplicial complex X_* is given as*
(i) a set X_0 (the set of vertices of the simplicial complex X_*)
(ii) For any natural number $d \geq 0$, the set of d-simplexes X_d where $X_d \subset \mathbb{P}(X_0)$, the power set of X_0 and for $Y \in X_d$ one has $|Y|= d+1$ for the cardinality $|Y|$ of Y.
(iii) If $Y \in X_d$ and $Y' \subset Y$, $Y' \neq \varnothing$, then $Y' \in X_{d'}, d' \leq d$, where $d'+1 = |Y'|$.

Remark. (a) (iii) says the following:
If Y is a d-simplex, any nonempty subset Y' of $(d'+1)$-elements is a d'-simplex of $X_{\cdot}$.
(b) There are various concepts related to the concept of a simplicial complex as ordered simplicial complex and simplicial set. To all of these, one can associate functorially a topological space $|X_*|$ (the so called realization of X_*) and X_* gives in a certain sense a combinatorial description of the topological space $|X_*|$. For all of these materials see [46, chapter 8].
We define now the so called affine building associated with the group $\mathrm{GL}(d,F)$ (or if one prefers, the projective linear group $\mathrm{PGL}(d,F)$). It will be a simplicial complex of dimension $(d-1)$.

Definition 17. *The affine building $X_* = X_*(F^d)$ (associated to the group $PGL(d,F)$) is a simplicial complex given as follows:*
(i) the set of vertices X_0 is $\mathcal{L}/\sim$, the set of lattices up to similarity in F^d.
(ii) If

$$L_0 \supset L_1 \supset \ldots L_r \supset \pi L_0$$

is a flag of $(r+1)$ different lattices in F^d, then $< L_0, L_1, \ldots, L_r >$ is a r-simplex in X_*, that is $< L_0, L_1, \ldots, L_r > \in X_r$. Any simplex of X_* is obtained in this way.

Remark. (i) As $\dim_k (L_0/\pi L_0) = d$, where $k = \mathcal{O}/\pi\mathcal{O}$ is the residue field, it follows:
$\dim (X_*) = d-1$, that is, the maximum dimension of a simplex in $X_{\cdot}$ is $d-1$.
(ii) There is an obvious action of $GL(d,F)$ on X_*, given by:

$$g < L_0, L_1, \ldots, L_r > = < gL_0, gL_1, \ldots, gL_r >$$

for $g \in \mathrm{GL}(d,F)$, $< L_0, L_1, \ldots, L_r > \in X_*$. This action induces an action of $\mathrm{PGL}(d,F)$, because the center of $\mathrm{GL}(d,F)$ acts trivially on X_*.

(iii) In the language of buildings the $(d-1)$-dimensional simplices are called chambers of X_*. They are given as complete flags:

$$L_0 \supset L_1 \supset \dots L_{d-1} \supset \pi L_0$$

of length $(d-1)$. As is shown in [9], X_* is a building in the sense of [9]. In particular, the apartments of $X_*(F^d)$ are given as follows: if $F^d = \oplus_{i=1}^d W_i$ is a direct decomposition of F^d into one-dimensional linear subspaces,

$$< L > \in \underline{A}(W_1, \dots, W_d) \quad \text{iff} \quad L = \oplus_{i=1}^d (W_i \cap L)$$

for a lattice L. $\underline{A}(W_1, \dots, W_d)$ is the full sub complex of $X_*(F^d)$ generated by these vertices. In more concrete terms one can give the following description of the set of $\underline{A}(W_1, \dots, W_d)$. Suppose $W_i = F\omega_i$, $(i = 1, \dots, d)$, such that $\{\omega_1, \dots, \omega_d\}$ is a basis of F^d. Then $< L > \in \underline{A}(W_1, \dots, W_d)$ iff $L = \oplus_{i=1}^d \mathcal{O}\pi_i^n \omega_i$ for a system of appropriate $n_1, \dots, n_d \in \mathbb{Z}$.
simplicial complex $X_*(F^d)$ is simplicially contractible.
(ii) The associated topological space $|X_*(F^d)|$ is contractible.

Proof. (i) Fix the vertex $< L_0 >=< \mathcal{O}^d >$. We will describe a contraction of $X_.(F^d)$ towards the vertex $< \mathcal{O}^d >$. Suppose $< L >$ is an arbitrary vertex of $X_.(F^d)$. We can assume (up to change by a scalar factor) $L_0 \subseteq L \subseteq \pi^m L_0$ and m is maximal with this property. Then we define the map

$$L \longmapsto T(L) := L + \pi^{m-1} L_0 \quad \text{if} \quad 1 \leq m.$$

If $m = 0$, we define $T(L) = L_0$, as $L = L_0$ exactly holds. One can check the following points easily:
1) T induces a well defined map $\overline{T_0} : X_0(F^d) \longrightarrow X_0(F^d)$ on the vertices of $X_.(F^d)$.
2) $\overline{T_0}$ induces a simplicial morphism

$$\overline{T} : X_.(F^d) \longrightarrow X_.(F^d).$$

3) $\overline{T}$ is homotopic to the identity morphism.
For a proof of these facts see [46]. This shows *i*) in the proposition, *ii*) is an immediate consequence. □

Fixing a vertex $< L >$ in the building $X_*(F^d)$, we consider the link $lk_{X_*}(L)$ of the vertex $< L >$ in $X_*(F^d)$. This is the simplicial complex, given by all simplices $\Delta \in X_*(F^d)$, such that Δ does not contain $< L >$ as a vertex, but $\Delta \cup L$ is a simplex in $X_*(F^d)$.

Proposition 18. *The simplicial complex $lk_{X_*}(L)$ is isomorphic to the Tits building $X_*(V)$ associated to the vector space $V := L/\pi L \cong k^d$*

Proof. The vertices of the Tits building associated to the vector space $V \cong k^d$ are given as $< W >$, where $0 \subsetneqq W \subsetneqq V$ are the proper linear subspaces of V. A r-simplex is given as $< W_0, W_1, \ldots, W_r >$, where $W_0 \supsetneqq W_1 \ldots \supsetneqq W_r$ and the $< W_j >$ are vertices. It is then immediate to see that the morphism

$$lk_{X_*}(L) \longrightarrow X_*(V)$$

$$< L_0, \ldots, L_r > \longmapsto < L_0/\pi L, L_1/\pi L, \ldots, L_r/\pi L >$$

(where $L \supset L_0 \supset \ldots \supset L_r \supset \pi L$ holds) is in fact an isomorphisms of simplicial complexes. □

Corollary 19. *i) The Tits building of V resp. $lk_{X_*}(L)$ is a labeled simplicial complex or building in the sense of [9] by the map $< W > \longmapsto dim(W)$ for $< W > \in X_*(k^d)$. The set of labels is $\{1, \ldots, dim(V) - 1\}$.*
ii) $lk_{X_*}(L)$ is a labeled simplicial complex or even a building.

Proof. i) is clear, ii) follows from i) and the above proposition. □

Remark. Of course the labeling in ii) depends on the choice of the lattice L or the vertex $< L >$. It does not correspond to a global labeling of the whole building $X_*(F^d)$.
On the other hand there is the possibility to give a labeling to $X_*(F^d)$.This is only SL(d, F)-invariant, not GL(d, F)-invariant.It is obtained by fixing a Haar measure μ on the commutative locally compact abelian group F^d such that $\mu(\mathcal{O}^d) = 1$. For any lattice $L \in \mathcal{L}$ there is a lattice $L' \sim L$, such that

$$1 \geq \mu(L') \geq \frac{1}{|k|^d}$$

holds. This follows immediately from the formula

$$\mu(\pi^m L) = \frac{1}{|k|^{md}} \mu(L)$$

Furthermore , one has $|\mu(L')| = |k|^{-d}, \quad 0 \leq d' \leq d - 1$. To a neighboring vertex L' to L one can define a labeling $\lambda : X_*(F^d) \longrightarrow \{0, 1, \ldots, d-1\}$, $\lambda(L') := d'$.

Remark. This global labeling of $X_.(F^d)$ is different from the local labeling considered before.

Suppose now, $\Delta_1, \Delta_2 \in X_.(F^d)$ are two simplices of the building.

Definition 20. *A gallery in the building* $X_\cdot(F^d)$ *is a sequence of chambers* $[C_0,\ldots,C_r]$*, such that* C_i, C_{i+1} *are neighboring chambers in* $X_\cdot(F^d)$ *for* $i = 0,\ldots,r-1$.

Definition 21. *The combinatorial distance between* $\Delta_1,\Delta_2 \in X_\cdot(F^d)$ *is given as:*

$$\min\{(r-1)\in \mathbb{N}\cup\{0\} \quad | \quad \exists \quad \text{a gallery} \quad [C_0,\ldots,C_r], \quad \Delta_1 \subset \overline{C_0}, \Delta_2 \subset \overline{C_r}\}$$

Proposition 22. $\Delta_1,\Delta_2 \in X_\cdot(F^d)$ *are given as above,* $\underline{A}$ *is any apartment, such that* $\Delta_1,\Delta_2 \in \underline{A}$*. Then any shortest gallery* $[C_0,\ldots,C_r]$ *as in the definition above satisfying* $C_0, C_r \in \underline{A}$*, is contained completely in* $\underline{A}$.

Proof. see [9, IV.4.Prop., p.88] □

2.3 Some More Facts About Affine Buildings

(a). Volumes of Lattices :
Discrete lattices in $\mathbb{R}^d$ have Lebesgue measure zero in $\mathbb{R}^d$, but lattices over local fields have positive Haar measure (or volume). The standard lattice has measure

$$\mu_d(L_0) = \mu_d(\mathcal{O}^d) = \mu_d(\mathcal{O} \times \mathcal{O} \times \ldots \mathcal{O}) = 1$$

for the d-fold product measure $\mu_d = \mu_{F^d}$ on F^d. It follows from elementary matrix theory arguments that

$$\mu_d(g(L)) = |\det(g)|_F \, \mu_d(L) \tag{2.1}$$

for $L \in \mathcal{L}$ and $g \in GL(d, \boldsymbol{F})$. In particular

$$\mu_d(g(\mathcal{O}^d)) = |det(g)|_F \tag{2.2}$$

$$\mu_d(aL) = |det(aI)|_F \, \mu_d(L) = |a|_F^d \, \mu_d(L), \qquad a \in F. \tag{2.3}$$

(b). Simultaneous diagonalization of lattices :
Let $\boldsymbol{F}, \mathcal{O}, G = GL(d, \boldsymbol{F})$, and $K = GL(d, \mathcal{O})$ be as before and π a uniformizer. The purpose of this subsection is to prove the following theorem:

Theorem 23. *Let L_1, L_2 be any pair of lattices in $\boldsymbol{F}^d$. Then, there exists a unique d-tuple of integers*
$-\infty < n_1 \leq n_2 \leq \ldots \leq n_d < \infty$ and a(non-unique) basis $\{v_1, v_2, \ldots, v_d\}$ in $\boldsymbol{F}^d$ such that

$$L_1 = \mathcal{O}v_1 + \mathcal{O}v_2 + \ldots + \mathcal{O}v_d \qquad \textit{and} \tag{2.4}$$

$$L_2 = \mathcal{O}\pi^{n_1}v_1 + \mathcal{O}\pi^{n_2}v_2 + \ldots + \mathcal{O}\pi^{n_d}v_d \tag{2.5}$$

Proof. Let for any $m \in \mathbb{Z}$ define m^- as follows:

$$m^- = \begin{cases} 0 & \text{if } m \geq 0 \\ -m & \text{if } m < 0. \end{cases}$$

Consider now the quotient groups

$$S(j) := (L_1 + \pi^j L_2)/L_1 \qquad \text{for} \qquad -\infty < j < \infty.$$

Now the number of elements in $S(j)$ is $n(j) = \prod_{i=1}^d q^{(n_i+j)^-}$ where $q = \text{card}(\mathcal{O}/\pi\mathcal{O})$ as before. The function $n(j)$ determines the integers n_i as claimed in the theorem, up to their order. This implies the uniqueness of the

expressions 2.4 and2.5. It remains to prove that, for every $g \in G$, there exists a diagonal $d \times d$ matrix $D = diag(\pi^{n_1}, \pi^{d_2}, \ldots, \pi^{n_d})$ such that $KgK = KDK$. First note that, if $g \in G$, then $KgK = Kg_1K$ where g_1 is derived from g by one of the following operations (by elementary divisor theory):
(i) transposing two rows or two columns,
(ii) multiplying a row or column by a constant in $\mathcal{O}^\times$
(iii) subtracting an $\mathcal{O}$-multiple of row from another row, or an $\mathcal{O}-$multiple of a column from another column.
Since each of these operation is equivalent to either pre- or post-multiplying g by a matrix in K, the proof now proceeds by induction. The matrix entries g_{ij} of g of maximal norm can be moved to the $(1, 1)$ position by operations of type (i). The matrix entries can be changed into p^{n_1} for some integer n_1 by (ii). Then the remainder of the first row of the matrix, and also the remainder of the first column, can be set equal zero by (iii). We now proceed inductively on the lower block by first moving an element of maximal norm to $(2, 2)$ position, etc. This completes the proof of Theorem 23. □

(c). The underlying graph of a building $X_*(F^d)$:
Associated with the building $X_*(F^d)$ is its underlying graph, consisting only the set of vertices $X_0(F^d)$ and the set of edges (1-simplices) $X_1(F^d)$. One has the usual distance function d of a graph. In particular, for neighboring vertices $x, y \in X_0(F^d)$ one has $d(x, y) = 1$. One has the :

Proposition 24. *: A set $\{x_0, \ldots, x_s\} \subset X_0(F^d)$ is an s-simplex in $X_*(F^d)$ iff $d(x_i, x_j) = 1$ holds for all $i, j \in \{0, \ldots, s\}, \quad i \neq j$.*

Proof. If $\{x_0, \ldots, x_s\}$ is an s-simplex in $X_*(F^d)$, $d(x_i, x_j = 1$ for the x_i, x_j above obviously holds. We show the converse: We can choose lattices $L_j \supset F^d$, such that $< L_j >= x_j$ for $j \in \{0, \ldots, s\}$ and additionally

$$L_0 \quad \supset \quad L_j \quad \supset \quad \pi L_0$$

holds for $j \in \{0, \ldots, s\}$. Upon reordering indices, we can additionally assume, that

$$\mu(L_0) \geqslant \mu(L_1) \ldots \geqslant \mu(L_s) \geqslant \mu(\pi L_0)$$

holds. Considering arbitrary L_i, L_j for $i \leq j$, we can find $a \in F^\times$, such that

$$L_i \quad \supsetneq \quad aL_j \quad \supsetneq \quad \pi L_i$$

holds. We obtain for the volumes

$$1 \quad \leqslant \quad \frac{\mu(L_i)}{\mu(L_j)} \quad \leqslant \quad |\pi|^{-d}$$

and

$$1 \quad < \quad \frac{\mu(L_i)}{\mu(L_j)} \quad = \quad |a|^{-d}\frac{\mu(L_i)}{\mu(aL_j)} \quad < \quad |\pi|^{-d}$$

that is:

$$|a|^d \quad < \quad \frac{\mu(L_i)}{\mu(L_j)} \quad < \quad |\frac{a}{\pi}|^d$$

and alltogether

$$|\pi|^d \quad < \quad |a|^d \quad < \quad |\pi|^{-d}.$$

But then it follows, that $a \in \mathcal{O}^\times$, that is:

$$L_i \quad \supsetneqq \quad L_j \quad \supsetneqq \pi L_i$$

and alltogether in particular

$$L_0 \supset L_1 \supset \ldots \supset L_s \supset \pi L_0,$$

that is, $< L_0, L_1, \ldots, L_s >$ is a simplex in $X_*(F^d)$. □

(d). Description of a standard appartment:
We want to give an explicit numerical description of an appartment $\underline{A}(W_1, \ldots, W_d)$, where the $W_i \subset F^d$ are one-dimensional linear spaces and $\oplus_{i=1}^d W_i = F^d$ is a direct sum decomposition. It suffices to assume for this purpose, that $W_i = Fe_i$, $\{e_1, \ldots, e_d\}$ the standard basis of F^d. A vertex $< L > \in \underline{A}(W_1, \ldots, W_d)$ in this case iff

$$L = \oplus_{i=1}^d (L \cap W_i) = (\mathcal{O}\pi^{n_1} e_1 \oplus \ldots \oplus \mathcal{O}\pi^{n_d} e_d).$$

Therfore L is given by the vector $v(L) = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, a vertex $< L >$ is uniquely given by the residue class of $(n_1, \ldots, n_d) \in \mathbb{Z}^d/\mathbb{Z}(1, \ldots, 1)$.
we are going to give a description of the chambers $< L_0, \ldots, L_d >$ in the appartment $\underline{A}(W_1, \ldots, W_d)$. Denote

$$v(L_j) := (n_1^{(j)}, \ldots, n_d^{(j)})$$
$$v(< L_j >) := (n_1^{(j)}, \ldots, n_d^{(j)}) \qquad \bmod \mathbb{Z}(1, \ldots, 1)$$

Fixing the lexiographical order on the abelian group $\mathbb{Z}^d$, we can denote representatives $L_j \in < L_j >$, such that, upon reordering indices $j \in \{0, \ldots, d\}$, we can assume that

$$v(L_0) < v(L_1) < \ldots < v(L_d) < v(L_0) + (1, \ldots, 1)$$

holds. This is equivalent to

$$L_0 \supset L_1 \supset \ldots < L_d \supset \pi L_0.$$

Finally, if we went to fix types, such that $\mu(L_0) = \mu(\mathcal{O}^d)$, the standard lattice, we have a unique choice of $j \in \{0, \ldots, d\}$, such that

$$\log_q \mu(L_j) \equiv 0 \qquad \mod (d)$$

Upon replacing L_j by an appropriate representative $\pi^\alpha L_j$, we can assume, that

$$\mu(\pi^\alpha L_j) = \mu(\mathcal{O}^d) = 1.$$

We obtain the simplices

$$< \pi^\alpha L_j, \pi^\alpha L_{j+1}, \ldots, \pi^\alpha L_d, \pi^{\alpha+1} L_0, \ldots, \pi^{\alpha+1} L_{j-1}, \ldots)$$

satisfying

$$\begin{array}{l} i) \quad \pi^\alpha L_j \supset \pi^\alpha L_{j+1} \subset \ldots \supset \pi^{\alpha+1} L_{j-1} \supset \pi^{\alpha+1} L_j \\ ii) \mu(\pi^\alpha L_j) = 1 \end{array}$$

and correspondingly for the other volumes.

2.4 Combinatorics and definition of Ramanujan Hypergraph

Introduction

For a finite regular graph, the eigenvalue λ of the adjacency matrix which has the second largest absolute value is of particular importance in estimating different invariants of the graph such as girth, independence number and expansion coefficient. A large expansion coefficient is determined by a small λ as shown in [33]. Lubotzky, Philips and Sarnak, in [33], have constructed a family of expander graphs called Ramanujan graphs. Asymptotically, their graphs have the smallest possible λ. They have called these expanders Ramanujan graphs, because all eigenvalues, except the largest (of course in absolute value), satisfy Ramanujan's conjecture (or more precisely Ramanujan-Petersson conjecture).

We shall give at first the definition of regular Ramanujan hypergraphs. These hypergraphs are a natural generalization of Ramanujan graphs. The eigenvalues of the adjacency operators satisfy inequalities associated of a higher dimensional version of Ramanujan-Perterson conjecture. In order to obtain a natural and simple definition of Ramanujan hypergraphs, first we need some combinatorial definitions and concepts. Our main references here are [4] and [5].

Definition 25. *A graph is a pair of sets* (V_X, E_X) *such that:*
$E_X \subset \{\{x,y\}|\{\{x,y\} \subset V_X\}$ *and* $V_X \neq \varnothing$.
The set V_X *is the set of vertices of* X *and* E_X *is the set of edges of* X*. The vertices* x *and* y *are said to be adjacent if* $\{x,y\}$ *is an edge. The number of vertices adjacent to* x *is denoted by* $d(x)$ *and is said to be the degree of* x*. If every vertex of* X *has degree* s*, then* x *is said to be* s*-regular.*

Definition 26. *iIf* X *is a graph with a finite number of vertices* $\{x_1, x_2, \ldots, x_n\}$*, The adjacency matrix* $A = [a_{ij}]$ *of* X *is the* $n \times n$ *matrix with entries* a_{ij} *equal* 1 *if* x_i *is adjacent to* x_j *and* 0 *otherwise.*
ii) Denoting $L^2(V_X)$ *(or* $L^2(X)$*) the space of functions* $f : V_X \longrightarrow \mathbb{C}$ *with the usual* L^2*-norm and with standard basis the set of delta functions* δ_v*,* $v \in V_X$*, the adjacency matrix induces an operator*

$$A : L^2(V_X) \longrightarrow L^2(V_X)$$

with respect to the basis $\{\delta_v | v \in V_X\}$*.*

Definition 27. *A hypergraph* $\boldsymbol{X}$ *is a set* $\boldsymbol{V}$ *together with a family* $\boldsymbol{\Sigma}$ *of subsets of* $\boldsymbol{V}$*. The elements of* $\boldsymbol{V}$ *and* $\boldsymbol{\Sigma}$ *are called vertices and faces of the hypergraph. If* $\mathbf{S} \in \boldsymbol{\Sigma}$*, the rank of* $\mathbf{S}$ *is the cardinality* $|\mathbf{S}|$ *of* $\mathbf{S}$ *and the dimension of* $\mathbf{S}$ *is given by* $|\mathbf{S}| - \mathbf{1}$*.*

We have seen the definition of labeled simplicial complex, and chamber complex in chapter one. So we are ready to give the definition of labelable hypergraph.

Definition 28. *A chamber complex is a set* Δ*, whose elements are called chambers, together with a set* $\{\sim_i : i \in I\}$ *of equivalence relations on* Δ*. Call* $c, c' \in \Delta$ i*-adjacent if* $c \sim_i c'$*. In this situation we refer to* Δ *as a chamber complex over* I

Definition 29. *Suppose that* c *and* c' *are in* Δ*, and that* $c_0 = c, c_1, \ldots, c_s = c'$ *is a finite sequence of chambers such that* c_{k-1} *is adjacent to and distinct from* c_k *for each* $k \in \{1, 2, \ldots, s\}$*. Then* $(c_0, c_1, \ldots, c_s)$ *is called a* ***gallery*** *from* c *to* c'*. If* $c_{k-1} \sim_{i_k} c_k$ *for each* $k \in \{1, 2, \ldots, s\}$*, then we say that the* ***gallery*** *is of* ***type*** $(i_1, i_2, \ldots, i_s)$

Very occasionally we need to consider slightly more general galleries, where the requirement that $c_{k-1} \neq c_k$ for each k is dropped. Such things will be called ***stutter-galleries***.

Definition 30. *A* ***simplicial complex*** *is a set* X *together with a set* $\mathcal{S}$ *of subsets of* X *such that*

1. *each singleton subset* $\{x\}$ *of* X *belongs to* $\mathcal{S}$.
2. *if* $S \in \mathcal{S}$ *and if* $A \subset S, then A \in \mathcal{S}$;

The sets $S \in \mathcal{S}$ *are called* ***simplexes*** *or* ***simplces***. *If* $S \in \mathcal{S}$ *and* $A \subset S$, *the simplex* A *is called a* ***face*** *of* S. *If* s *is a simplex and* S *is not a proper subset of any simplex* S', *we call* S *a maximal simplex, or a* ***chamber***.

Definition 31. *A simplicial complex* X *is called* ***labeled***, *if there is a set* I *of "labels" or "types", so that each vertex* v *has a type* $\mathbb{t}(v) \in I$, *and such that each chamber has exactly one vertex of each type. In other words,* $\mathbb{t} : X \longrightarrow I$ *is map such that for each chamber* C, *the restriction* $\mathbb{t}_{|_C}$ *of* $\mathbb{t}$ *to* C *is a bijection* $C \longrightarrow I$. *This implies that each chamber has exactly* $Card(I)$ *vertices.*

Under suitable conditions one can obtain labeled simplicial complex from a chamber complex and vice versa. (see [46] and [45]) for more detail.

Remark. Specially our building $X_*(F^d)$ has satisfies all conditions in view of above correspondence between chamber and labeled simplicial complex.

Definition 32. *A hypergraph* X *is labelable if it is a chamber complex and there exist a set* I *and a function which assigns to each vertex of* X *an element of* I *in such a way that the vertices of every chamber are mapped bijectively onto* I.

Definition 33. *Let* $\boldsymbol{X}$ *be a labelable hypergraph* $\boldsymbol{X}$ *with the label set* $\boldsymbol{I} = \{0, 1, \ldots, s\}$, *if every vertex* $x \in \boldsymbol{X}$ *has exactly* $\boldsymbol{n_k}$ *number of neighbor of type* k, *then* $\boldsymbol{X}$ *is called a* $(\boldsymbol{n_1, n_2, \ldots, n_s})$*-regular hypergraph. In this situation we can associate to any vertex* x *of* $\boldsymbol{X}$ *a function* $\mathbb{t}_x$ *defined by* $\mathbb{t}_x(y) := \mathbb{t}(x) - \mathbb{t}(y) \bmod d$. *where* $\mathbb{t}$ *is the label map defined as the definition* (31).

denoted by $\underline{X}$.

Definition 34. *Let* $\boldsymbol{X}$ *be a* $(\boldsymbol{n_1, n_2, \ldots, n_s})$*-regular hypergraph. We define for every* $k \in \{0, 1, \ldots, s\}$ *the* k^{th} *adjacency matrix* $\boldsymbol{A^{(k)}}$ *as follows : denote for every* $k \in \{0, 1, \ldots, s\}$ *and for any two vertices x,y*

$$\boldsymbol{\varepsilon^{(k)}}(x,y) = \begin{cases} 1 & \text{if } \mathbb{t}_x(y) = k \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

now $\boldsymbol{A^{(k)}}(i,j) := \boldsymbol{\varepsilon^{(k)}}(x_i, x_j)\boldsymbol{A}(i,j)$ *where* $\boldsymbol{A}(i,j)$ *is the* ij^{th} *entries of the adjacency matrix of underlying graph* $\underline{\boldsymbol{X}}$.

We are ready now to give our main definition in this thesis, namely the definition of a Ramanujan hypergraph.

Definition 35. *A $(\boldsymbol{n_1, n_2, \ldots, n_{d-1}})$-regular hypergraph $\boldsymbol{X}$ is called a Ramanujan hypergraph with the bound $(\boldsymbol{c_1, c_2, \ldots, c_{d-1}})$ where for all $k \in \{1, 2, \ldots, d-1\}$, $\boldsymbol{c_k}$ are positive real number, if every eigenvalue $\boldsymbol{\lambda^{(k)}}$ of the k^{th} adjacency matrix $\boldsymbol{A^{(k)}}$ of $\boldsymbol{X}$ is either $\boldsymbol{\lambda^{(k)} = n_k}$ or $|\boldsymbol{\lambda^{(k)}}| \leqslant \boldsymbol{c_k}$.*

As we will see in the next chapter the structure $\Gamma \backslash X_0(F^d)$ will be a finite regular hypergraph. These quotients would be our main object of study in the next chapters.

Chapter 3

Representation Theory and Hypergraphs

3.1 Introduction

In this section we describe our candidates for Ramanujan hypergraphs in general. The main point to be explained is the relation between the different adjacency operators in a hypergraph and the corresponding Hecke operators for certain automorphic representations, which are related. To really conclude the Ramanujan property, one has to use the relevant results for automorphic representations. Here the case $d = 2$ is easier than the case of general d, because only in this case there is a fully worked out Jacquet-Langlands correspondence between automorphic representations of unit groups of division algebras and general linear groups over, which should allow to relate the situation to automorphic representations of the general linear group and thereby to the recent results of Lafforgue, Instead we use a result of W.Li resp. L. Clozel, which work with the theory of $\mathcal{D}$-elliptic sheaves and does not make use of the Jacquet-Langlands correspondence.
In section 2.1. we explain our situation, 2.2. gives a short review of some fundamental facts concerning automorphic representations. In particular the Hecke algebra and the standard generators of the local Hecke algebras will be explained. 2.3. identifies the adjacency operators of the quotient complex resp. quotient hypergraph with the corresponding Hecke operators of the automorphic situation. Finally the Ramanujan property is discussed in this section.

3.2 Quotient hypergraphs, the general situation

$F = \mathbb{F}_q(t)$ is again a rational function field over $\mathbb{F}_q$. $\{\underline{p}_1, \ldots, \underline{p}_m\}$ is a set of places of F, such that $p_m = \infty$, $\underline{p}$ is a prime of degree one of F, such that

$$\underline{p} \notin \{\underline{p}_1, \ldots, \underline{p}_m\}$$

$p = p(t)$ is an irreducible polynomial, such that $\underline{p} = (p(t))$. We have the orders $\mathbb{F}_q[t][\frac{1}{p}]$ as well as

$$\mathcal{O}_F^{(p)} := \{x \in \mathbb{F}_q[t][\frac{1}{p}] \quad | \quad v_\infty(x) \geqslant 0\}$$

D is a central division algebra over F of dimension d^2, which is ramified exactly at $\{\underline{p}_1, \ldots, \underline{p}_m\}$. For simplicity we assume, that D is totally ramified in ∞, such that there exists a unique valuation v_∞ of D extending the corresponding valuation of F. $\mathcal{O}_D$ is a maximal order of D over $\mathbb{F}_q[t]$. Then

$$\mathcal{O}_D[\frac{1}{p}] := \mathbb{F}_q[t][\frac{1}{p}] \otimes_{\mathbb{F}_q[t]} \mathcal{O}_D$$

is a maximal order in D over $\mathbb{F}_q[t][\frac{1}{p}]$, similarly

$$\mathcal{O}_D^{(p)} := \{x \in \mathcal{O}_D[\frac{1}{p}] \quad | \quad \tilde{v}_\infty(x) \geqslant 0\}$$

(where $\tilde{v}$ is standard extension of v into D) is a maximal order in D over $\mathcal{O}_F^{(p)}$.

Definition 36.

$$\Gamma(1) := \left(\mathcal{O}_D[\frac{1}{p}]\right)^\times / Z,$$

where Z is the center of the group $\mathcal{O}_D[\frac{1}{p}]^\times$, the unit group of the maximal order $\mathcal{O}_D[\frac{1}{p}]$

We are considering in the following congruence subgroups Γ of $\Gamma(1)$. We have the embedding

$$\Gamma(1) \hookrightarrow D_{\underline{p}}^\times / Z \cong GL(d, \mathbb{F}_q(t)_p)/Z$$

as obtained earlier and the corresponding action of the different groups on the Bruhat-Tits building $X_*({\mathbb{F}_q(t)_p}^d)$.

We consider now only congruence subgroups $\Gamma \subset \Gamma(1)$, which are torsion free. In particular their action on $X_*(\mathbb{F}_q(t)_p{}^d)$ is fixpoint free. Again the quotient complex (or quotient hypergraph, if preferred) is a locally labeled complex resp. locally labeled hypergraph. We have quite generally :

Theorem 37. *The quotient complex*

$$\Gamma\backslash X_*(\mathbb{F}_q(t)_p{}^d)$$

is a finite complex.

Proof. This follows from Godement's criterion. □

Remark. Of course here one can argue also directly, because the class number of a maximal order $\mathcal{O}_D$ (and all the variants above) is finite. We consider the $\mathbb{C}$-vector space of $\mathbb{C}$-valued cochains:

$$C^0(\Gamma\backslash X_*(\mathbb{F}_q(t)_p{}^d)) =: C^0(\Gamma\backslash X_*) := \mathrm{Map}(\Gamma\backslash X_0; \mathbb{C}).$$

Because X_* is a locally labeled simplicial complex, and the action of Γ is compatible with the local labeling, also $\Gamma\backslash X_*(\mathbb{F}_q(t)_p{}^d)$ is a locally labeled simplicial complex.

Definition 38. *For each $i, 1 \leqslant i \leqslant d-1$, we have the i-th adjacency operator*

$$\begin{aligned} A^{(i)} : C^0(\Gamma\backslash X_*) &\longrightarrow C^0(\Gamma\backslash X_*) \\ f &\longmapsto A^{(i)}(f) \end{aligned}$$

where

$$A^{(i)}(f)(x) = \sum_{t(y;x)=i} f(y)$$

Remark. We remind, that $t(y;x) = i$ means:

$$y \in \Gamma\backslash X_0, \quad \text{such that} \quad < x, y > \in \Gamma\backslash X_1,$$

that is, x and y are neighbors resp. there is a one-simplex joining them and furthermore the type of the vertex y with respect to x is i.
Next, we give an adelic description of the set $\Gamma\backslash X_0$ above and similarly of the associated space of $\mathbb{C}$-valued functions. Denote $\mathbb{A}_F$ the adeles over F, similarly

$$D^\times(\mathbb{A}_F) := \prod_{v\in|F|} (D^\times(F_v); D^\times(\mathcal{O}_v))$$

the adeles of the multiplicative group of D, $D^\times$. Z denotes in this context again the center of $D^\times(\mathbb{A})$, we have (in the sense of algebraic groups)

$$(D^\times/Z)(\mathbb{A}_F) = D^\times(\mathbb{A}_F)/Z$$

The order $\mathcal{O}_D$ defines local orders $\mathcal{O}_{D,\underline{r}} \subset D_{\underline{r}}$ for all primes $r \neq \infty$. We have identified earlier

$$D_{\underline{p}} \quad \xrightarrow{\sim} \quad \mathbb{M}(d, F_{\underline{p}})$$

and thereby $\mathcal{O}^\times_{D,\underline{p}}/Z$ with the stabilizer of the standard lattice

$$L_0 = \mathcal{O}^d_{\underline{p}} = (\mathbb{F}_q[t]_p)^d \subset F^d_{\underline{p}}.$$

Definition 39.
i) We have

$$\mathfrak{K}^{(1)} := \prod_{\underline{r}\neq\infty,\underline{p}} (\mathcal{O}^\times_{D,\underline{r}}/Z) \times D^\times_\infty/Z,$$

an open subgroup of $(D^\times(\mathbb{A}_F)/Z$.
ii) Any congruence subgroup $\Gamma \subset \Gamma(1)$ *of finite index defines in a natural way a congruence subgroup* $\mathfrak{K} \subset \mathfrak{K}^{(1)}$ *of finite index.*

Proposition 40. *There is a natural bijection*

$$\Gamma\backslash X_0(\mathbb{F}_q(t)^d_p) \quad \xrightarrow{\sim} \quad D^\times\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})$$

Proof. We can identify

$$X_0(\mathbb{F}_q(t)^d_p) \quad \xrightarrow{\sim} \quad D^\times(F_{\underline{p}})/D^\times(\mathcal{O}_{\underline{p}}).Z.$$

Therefore we obtain a mapping

$$X_0(\mathbb{F}_q(t)^d_p) \quad \longrightarrow \quad D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})$$

This mapping induces a map

$$\Gamma\backslash X_0(\mathbb{F}_q(t)^d_p) \quad \xrightarrow{\sim} \quad D^\times\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}}).$$

We construct now an inverse map. Suppose, given an adele $(x_v) \in D^\times(\mathbb{A}_F)$, and thereby the doubled class

$$D^\times(x_v)(Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})).$$

Using strong approximation theorem and also,that the class number of $\mathbb{F}_q[t]$ is one, we can find $\gamma \in D^\times$, such that

$$D^\times(\gamma x_v)_{v\in|F|}(Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})),$$

is of the form

$$D^\times(y_v)(Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})),$$

where

$$y_v = \begin{cases} 1 & \text{for all} \quad v \neq \underline{p} \\ y_{\underline{p}} \in D^\times(F_{\underline{p}}) & \text{at} \quad \underline{p} \end{cases}$$

We construct the inverse map then as

$$D^\times(\gamma x_v)_v(Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})) \longmapsto \Gamma.(y_{\underline{p}}).Z.D^\times(\mathcal{O}_{\underline{p}})) \in \Gamma\backslash D^\times(F_{\underline{p}})/D^\times(\mathcal{O}_{\underline{p}}).Z.$$

It is immediate to see, that this gives a well defined map, where one has to take into regard additionally, that we have, considering $D^\times(F_{\underline{p}})$ as a subgroup of $D^\times(\mathbb{A}_F)$

$$D^\times(F_{\underline{p}}) \cap \mathfrak{K} = \Gamma \tag{3.1}$$

and

$$X_0(\mathbb{F}_q(t)^d_p) \quad \xrightarrow{\sim} \quad D^\times(F_{\underline{p}})/D^\times(\mathcal{O}_{\underline{p}})Z.$$

□

We have therefore immediately

Corollary 41. *There is a canonical identification*

$$C(\Gamma\backslash X_0(\mathbb{F}_q(t)^d_p);\mathbb{C}) \quad \xrightarrow{\sim} \quad C(D^\times\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}).$$

In particular the Hecke-algebra of biinvariant functions $\mathcal{H}(D^\times(\mathbb{A}_F)//\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}}))$ *is acting on* $C(\Gamma\backslash X_0(\mathbb{F}_q(t)^d_p);\mathbb{C})$*, by convolution. As a subalgebra the spherical Hecke algebra* $\mathcal{H}(D^\times(F_{\underline{p}})//D^\times(\mathcal{O}_{\underline{p}}))$ *is acting on* $C(\Gamma\backslash X_0(\mathbb{F}_q(t)^d_p);\mathbb{C})$.

Proof. The identification of the space of functions follows immediately from Proposition 40. The assertions concerning the action of the Hecke algebra are also obvious, but there will be some discussion in 3.3. □

3.3 Adjacency operators and Hecke operators

In the last section we had identified the set of vertices

$$\Gamma\backslash X_0(F^d_{\underline{p}})$$

of the quotient hypergraph associated to the discrete group $\Gamma \subset D^\times(F_{\underline{p}})/Z$ with the set of adelic double classes

$$D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}}).$$

Correspondingly we have an identification of the spaces of functions

$$C(\Gamma\backslash X_0(\mathbb{F}_q(t)_p^d);\mathbb{C}) \quad \xrightarrow{\sim} \quad C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}).$$

The Hecke algebra associated to $\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})$ in $D^\times(\mathbb{A}_F)$ is acting and in particular the local spherical Hecke algebra

$$\mathcal{H}(D^\times(F_{\underline{p}}); Z.D^\times(\mathcal{O}_{\underline{p}})).$$

For the convenience of the reader we remind at this point about some basic concepts of the theory of automorphic representations in connection with the situation above. So,

$$C(D^\times(F)\backslash(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C})$$

is a finite dimensional subspace of the space of functions

$$C_c(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z;\mathbb{C})$$

On this infinite dimensional $\mathbb{C}$-vector space there is an obvious smooth action R of the locally compact topological group $D^\times(\mathbb{A}_F)$ by

$$R(x)(f)(g) = f(gx),$$

where $g \in D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z$, $x \in D^\times(\mathbb{A}_F)$ resp. $(D^\times/Z)(\mathbb{A}_F)$ (as wanted) and $f \in C_c^\infty(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z;\mathbb{C})$ the space of locally constant functions on $D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z$, with compact support. We have the identification of the subspace of $\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})$-invariants:

$$C_c(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z;\mathbb{C})^{\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})} \quad \xrightarrow{\sim} \quad C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}).$$

There remains the action of the Hecke algebra of biinvariant functions

$$\mathcal{H}(D^\times(\mathbb{A}_F)//\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}})) \\ \cong (\otimes_{\underline{r}\neq\underline{p},\infty}\mathcal{H}(D^\times(F_{\underline{r}})//Z(\mathfrak{K}_{\underline{r}})) \otimes \mathcal{H}(D^\times(F_{\underline{p}})//D^\times(\mathcal{O}_{\underline{p}}))$$

where

$$\mathfrak{K} = (\prod_{\underline{r}\neq\underline{p},\infty} \mathfrak{K}_{\underline{r}}) \times D^\times(F_\infty) \times D^\times(\mathcal{O}_{\underline{p}}).$$

The (local) spherical Hecke algebra

$$\mathcal{H}(D^\times(F_{\underline{p}})//Z.D^\times(\mathcal{O}_{\underline{p}})) \cong \mathcal{H}(\mathrm{GL}(d;F_{\underline{p}})//Z.\mathrm{GL}(d;\mathcal{O}_{\underline{p}}))$$

has as $\mathbb{C}$-vector space the generators

$$\mathrm{GL}(d;\mathcal{O}_{\underline{p}})\ \mathrm{diag}(p^{n_1},\dots,p^{n_d})\ \mathrm{GL}(d;\mathcal{O}_{\underline{p}}).Z/Z$$

where we have

$$(n_1,\dots,n_d)\in\mathbb{Z}^d/Z\cdot(1,\dots,1),\quad n_1\leqslant n_2\leqslant\dots\leqslant n_d.$$

We denote the set

$$A_+ := \{(n_1,\dots,n_d)\in\mathbb{Z}^d/Z\cdot(1,\dots,1)\quad|\quad n_1\leqslant n_2\leqslant\dots\leqslant n_d\}.$$

Proposition 42. *As a $\mathbb{C}$-vector space*

$$\mathcal{H}(D^\times(F_{\underline{p}})/Z.D^\times(\mathcal{O}_{\underline{p}})) \cong \bigoplus_{(n_1,\dots,n_d)\in A_+}\left(\mathcal{H}(D^\times(\mathcal{O}_{\underline{p}}))\ diag(p^{n_1},\dots,p^{n_d})\ \mathcal{H}(D^\times(\mathcal{O}_{\underline{p}}))*Z/Z\right)$$

The different Hecke algebras form, as the word indicates, algebras under the convolution of functions, that is

$$(f_1*f_2)(x) := \int_{D^\times(F_{\underline{p}})/Z} f_1(xy^{-1}d)f_2(y)\mu(y)$$

where $d\mu$ denotes the Haar measure on $D^\times(F_{\underline{p}})/Z$, In particular, we have the double classes

$$D^\times(\mathcal{O}_{\underline{p}}))\ \mathrm{diag}(1,\dots,1,p,\dots,p)\ D^\times(\mathcal{O}_{\underline{p}}))*Z/Z$$

(i-times 1 above, at least one p occurring) and the associated characteristic functions $\chi_{\underline{p},i}$.

Definition 43. *The Hecke operator $T_{\underline{p},i}$ is given as the convolution operator*

$$\begin{aligned}&C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}) \quad\longrightarrow\\ &\quad C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C})\\ &f \quad\longmapsto f*\chi_{\underline{p},i} =: T_{\underline{p},i}(f).\end{aligned}$$

We will now identify the Hecke operators $T_{\underline{p},i}$ with the corresponding adjacency operators. The Hecke operators $T_{\underline{p},i}$ can be seen best using the isomorphism

$$C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}) \quad \longrightarrow \quad C(\Gamma\backslash D^\times(F_{\underline{p}})/Z.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C})$$

As the Hecke operators $T_{\underline{p},i}$ are acting by convolution from the right side, they will be commute with the action of Γ from the left side. It is therefore enough to compute the action of the Hecke operators $T_{\underline{p},i}$ on the space of functions

$$C(D^\times(F_{\underline{p}})/Z.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}) \quad \text{resp.} \quad C^\infty(D^\times(F_{\underline{p}})/Z.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}).$$

As it will turn out, the Hecke operators will be defined locally and act therefore on both spaces. We have the identification

$$\begin{aligned} D^\times(F_{\underline{p}})/Z.D^\times(\mathcal{O}_{\underline{p}}) &\xrightarrow{\sim} \mathrm{GL}(d;F_{\underline{p}})/\mathrm{GL}(d;\mathcal{O}_{\underline{p}}) * Z \\ &\xrightarrow{\sim} X_0(F_{\underline{p}}^d). \end{aligned}$$

Therefore, the action of the Hecke algebra $\mathcal{H}(D^\times(F_{\underline{p}})//Z * D^\times(\mathcal{O}_{\underline{p}}))$ can be seen also on the space of functions $C(X_0(F_{\underline{p}}^d);\mathbb{C})$ as well as $C_c(X_0(F_{\underline{p}}^d);\mathbb{C})$.

Theorem 44. *i) The Hecke operator $T_{\underline{p},i}$ equals the adjacency operator $A^{(i)}$ for each $i = 1,\dots,d-1$.*
ii) The Hecke algebra at $\underline{p}$ and the algebra generated by the adjacency operators $A^{(i)}$ $(i = 1,\dots,d-1)$ act equally.
iii) The action of both algebras commutes with the action of

$$GL(d;F_{\underline{p}}) \cong D^\times(F_{\underline{p}}).$$

Proof. *iii)* is obvious. For the case of adjacency operators it follows directly from the formulas for the $A^{(i)}$, acting on the space of functions $C(X_0(F_{\underline{p}}^d);\mathbb{C})$. For the case of the Hecke algebra it follows, because $\mathrm{GL}(d;F_{\underline{p}}) \cong D^\times(F_{\underline{p}})$ is acting from the left side, where as the Hecke operators are given by convolution with biinvariant functions from the right.
i) As a $\mathrm{GL}(d;F_{\underline{p}}) \cong D^\times(F_{\underline{p}})$-module, $C_c(X_0(F_{\underline{p}}^d);\mathbb{C})$ is generated by the characteristic function

$$\chi_{D^\times(\mathcal{O}_{\underline{p}})*Z/Z} = \chi_{L_0},$$

under the identification above. But

$$
\begin{aligned}
& \chi_{D^\times(\mathcal{O}_{\underline{p}}).Z/Z} * T_{\underline{p},i} \\
= \; & \chi_{D^\times(\mathcal{O}_{\underline{p}}).Z/Z} * \chi_{D^\times(\mathcal{O}_{\underline{p}})} \mathrm{diag}(1,\ldots,1,p,\ldots,p) \chi_{D^\times(\mathcal{O}_{\underline{p}}).Z/Z} \\
= \; & \chi_{D^\times(\mathcal{O}_{\underline{p}})} \mathrm{diag}(1,\ldots,1,p,\ldots,p) \; \chi_{D^\times(\mathcal{O}_{\underline{p}}).Z/Z} \\
= \; & \sum_{\mathrm{type}(L;L')=i} \chi_{L'} \\
= \; & A^{(i)}(\chi_L)
\end{aligned}
$$

Here we have used, that $\chi_{D^\times(\mathcal{O}_{\underline{p}}).Z/Z}$ is the unit element of the Hecke algebra $\mathcal{H}(D^\times(F_{\underline{p}})//Z.D^\times(\mathcal{O}_{\underline{p}}))$. This altogether shows i).
ii) is an immediate consequence of i), as both algebras of operators are generated by the Hecke operators $T_{\underline{p},i}$ resp. the $A^{(i)}$ $(i = 1, \ldots, d-1)$. □

3.4 The Ramanujan Property

As is well known, one can decompose the representation space

$$C_c(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z;\mathbb{C}) \quad = \quad \bigoplus_\pi V_\pi$$

into a direct sum of irreducible automorphic representations, (V_π, π) of $D^\times(\mathbb{A}_F)$. The subspace

$$C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}) \subset C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z;\mathbb{C})$$

can be written then in the form

$$
\begin{aligned}
& C(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}) \\
&= \bigoplus_\pi (V_\pi)^{\mathcal{H}(D^\times(\mathbb{A}_F)//Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}}))} \\
&= \bigoplus_\pi (V_\pi)^{\mathcal{H}(D^\times(\mathbb{A}_F^{(p)})//Z.\mathfrak{K})} \otimes (V_{\underline{p}}^{D^\times(\mathcal{O}_{\underline{p}})})
\end{aligned}
$$

decomposing the representation

$$V_\pi = (V_{\pi^{\underline{p}}} \otimes V_{\underline{p}})$$

and correspondingly

$$(V_\pi)^{\mathcal{H}(D^\times(\mathbb{A}_F)//Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}}))}$$

$$\cong \quad (V_{\pi^{(\underline{p})}}^{\mathcal{H}(D^\times(\mathbb{A}_F^{(p)})//Z.\mathfrak{K})}) \otimes (V_{\underline{p}}^{D^\times(\mathcal{O}_{\underline{p}})}).$$

Therefore, only those representations π occur in the decomposition, for which

$$V_{\underline{p}}^{D^\times(\mathcal{O}_{\underline{p}})} \neq 0$$

If this is the case, $V_{\underline{p}}$ is a spherical representation. As the local Hecke algebra $\mathcal{H}(D^\times(F_{\underline{p}})//D^\times(\mathcal{O}_{\underline{p}}))$ is commutative, it follows that $V_{\underline{p}}^{D^\times(\mathcal{O}_{\underline{p}})}$ has to be one-dimensional. The action of

$$\mathcal{H}(D^\times(F_{\underline{p}})//D^\times(\mathcal{O}_{\underline{p}})),$$

is then given by a character (ring homomorphism)

$$\chi_{\pi_{\underline{p}}} : \mathcal{H}(D^\times(F_{\underline{p}})//D^\times(\mathcal{O}_{\underline{p}})) \quad \longrightarrow \quad \mathbb{C}$$

into the field of complex numbers. In particular,

$$\chi_{\pi_{\underline{p}}}(T_{\underline{p},i}) \in \mathbb{C}$$

and the Ramanujan property of the automorphic representation at $\underline{p}$ deals with these. We have to make one further remark:
Obviously in

$$C^\infty(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z;\mathbb{C}),$$

we have the subrepresentations given by characters

$$\chi : D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z \longrightarrow \mathbb{C}^\times.$$

In particular the trivial character, corresponding to the subspace of constant functions, occurs as $D^\times(\mathbb{A}_F)$-invariant subspace.

Definition 45. *i) An irreducible automorphic representation π, such that $\pi_{\underline{p}}$ is a spherical representation, is said to satisfy the Ramanujan property at $\underline{p}$, iff the eigenvalues $\chi_{\pi_{\underline{p}}}(T_{\underline{p},i})$ satisfy*

$$|\chi_{\pi_{\underline{p}}}(T_{\underline{p},i})| \leqslant \binom{d}{i} q^{\frac{i(d-i)}{2}} |\sigma_i(z_1,\ldots,z_d)|,$$

where $z_1,\ldots,z_d \in \mathbb{C}$ are complex numbers with absolute value $|z_j| = 1$ for $j = 1,\ldots,d$ and $\sigma_i(z_1,\ldots,z_d)$ i-th elementary symmetric polynomial.
ii) The Ramanujan property at the prime $\underline{p}$ holds for the representation

$$C^\infty(D^\times(F)\backslash D^\times(\mathbb{A}_F)/Z;\mathbb{C}),$$

if it holds in the sense above for all automorphic representations π complementary to the invariant subspaces, generated by the multiplicative characters.

Theorem 46. *(W.Li.) The Ramanujan property at $\underline{p}$ holds in the sense of Definition 45 (i).*

Corollary 47. *The quotient hypergraphs $\Gamma\backslash X_0(F_p^d)$ are Ramanujan hypergraphs (simplicial complexes) in the sense of Definition 45 in Chapter 2.*

Remark. In the case $d = 2$, the Ramanujan property can be shown by using the so called Jacquet- Langlands correspondence between automorphic representations of $D^\times(\mathbb{A}_F)$ (then D is a quaternion algebra) and automorphic representations of $\mathrm{GL}(2;\mathbb{A}_F)$, not given by multiplicative characters. For these, the Ramanujan property is a consequence of results of Drinfeld [21]. Though in our situation one has now the recent results of L.Lafforgue, showing the Langlands correspondence for $\mathrm{GL}(d;\mathbb{A}_F)$ for arbitrary d and in particular the Ramanujan property for cuspidal automorphic representations, one can not conclude immediately here the Ramanujan property. What is missing, is a completely worked out Jacquet- Langland correspondence as above for $d = 2$. Nevertheless, due to a trick from L.Clozel, one can conclude the Ramanujan property by working with the moduli scheme of $\mathcal{D}$-elliptic modules [32], instead of working with the moduli scheme of shtukas. Concerning this, then reader has to consult [65]. We close this chapter indicating another method to show the Ramanujan property. We use a theorem of Arthur and Clozel [1, Theorem 4.2.], which gives the Jacquet-Langlands correspondence for the case d a prime number. After that we can use the recent result of Lafforgue to conclude the Ramanujan property for this case. It might be mentioned that we found this approch at a time, when the preprints [65] and [36], [35] avilable to us.

Assume d is a prime number, f is an irreducible element of $\mathbb{F}_q[t]$ with $(\underline{f}) \neq (\underline{p})$ and different from all other primes in which $D = \mathbb{F}_{q^d}(\tau)$ is ramified . The canonical homomorphism

$$\mathbb{F}_q[t]\frac{1}{p}] \longrightarrow \mathcal{O}_D[\frac{1}{p}]/f\mathbb{F}_q[t][\frac{1}{p}],$$

induces following homomorphism :

$$\mathcal{O}_D[\frac{1}{p}] \longrightarrow \mathcal{O}_D[\frac{1}{p}]/f\mathcal{O}_D[\frac{1}{p}],$$

where as before $\mathcal{O}_D = \mathbb{F}_{q^d}\{\tau\}$. So we obtain group homomorphism

$$\alpha_f^{(p)} : \Gamma(1) \longrightarrow (\mathcal{O}_D[\frac{1}{p}]/f\mathcal{O}_D[\frac{1}{p}])^\times/Z.$$

Recall that $\Gamma(1) = (\mathcal{O}_D[\frac{1}{p}])^\times/Z$. We define $\Gamma_f^{(p)} := \ker\alpha_f^{(p)}$. So $\Gamma_f^{(p)}$ is a normal subgroup of $\Gamma(1)$ of finite index and we have:

Theorem 48. (*Main Abstract Theorem*)
(1) $\Gamma_f^{(p)}\backslash X_*(\mathbb{F}_q(t)_p^d)$ *is a finite* $(n_1, \ldots, n_{d-1})$*-regular graph with*

$$n_i = \text{number of } i\text{-dimensional subspaces of } \mathbb{F}_{\widetilde{q}} \text{ where } \widetilde{q} = q^{\deg p},$$

for $i = 1, \ldots, d-1$, *more precisely*

$$n_i = \binom{d}{i}_{\widetilde{q}} := \frac{\prod_{m=d-i+1}^{d}(\widetilde{q}^m - 1)}{\prod_{m=1}^{i}(\widetilde{q}^m - 1)} \ .$$

(2) $\Gamma_f^{(p)}\backslash X_*(\mathbb{F}_q(t)_p^d)$ *is a Ramanujan hypergraph in the sense of Definition 45 from Chapter 2,i.e. It is Ramanujan with the bound* $(c_1, \ldots, c_{d-1})$ *where* $c_i = \binom{d}{i} q^{\frac{i(d-i)}{2}\deg p}$ *for* $i = 1, \ldots, d-1$.

Proof. By Theorem 37 the quotient complex (hypergraph) $\Gamma_f^{(p)}\backslash X_*(\mathbb{F}_q(t)_p^d)$ is a finite. the expression about regularity is inherited from the structure of the Bruhat-Tits Building $X_*(\mathbb{F}_q(t)_p^d)$. So (1) is done. Following isomorphisms are known from Chapter 3:

$$C(D^\times\backslash D^\times(\mathbb{A}_F)/Z.\mathfrak{K}.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C}) \xrightarrow{\sim} C(\Gamma\backslash D^\times(F_{\underline{p}})/Z.D^\times(\mathcal{O}_{\underline{p}});\mathbb{C})$$

$$C(\mathrm{GL}(d;F_{\underline{p}})/\mathrm{GL}(d;\mathbb{F}_q[t]_p).Z;\mathbb{C}) \xrightarrow{\sim} C(\mathrm{PGL}(d;F_{\underline{p}})/\mathrm{PGL}(d;\mathbb{F}_q[t]_p);\mathbb{C})$$
$$\xrightarrow{\sim} \quad C(X_0(F_{\underline{p}}^d);\mathbb{C}).$$

which hold for all congruence subgroups of $\Gamma(1)$. in particular for $\Gamma_f^{(p)}$. Thus we define first associated to f the congruence subgroup

$$J_f := \ker\left(\mathcal{O}_{D,f}^\times/Z \longrightarrow (\mathcal{O}_{D,f}/f\mathcal{O}_{D,f})^\times/Z\right),$$

and we define

$$\mathfrak{K} := \prod_{r \neq p,\infty,f} (\mathcal{O}_{D,\underline{r}}^\times/Z).D^\times/Z(F_{\underline{p}}).D^\times/Z(F_\infty).J_f,$$

and let $\mathfrak{M} := D^\times/Z.\mathfrak{K}$

Remark. For any group G here, we use notation $G^{(1)}$ for the subgroup of G of elements with reduced norm 1.

Applying Strong approximation Theorem two times sequentially, we see that $D^\times/Z(\mathbb{A}).F_\mathbb{A}$ ($F_\mathbb{A}$, diagonal embedding of F in $\mathbb{A}$) is a finite index subgroup of $\mathfrak{M}$. This plus

$$|D^\times/Z(\mathbb{A})/D^\times/Z(\mathbb{A}).F_\mathbb{A}| < \infty \ ,$$

shows $|D^\times/Z(\mathbb{A}).F_{\mathbb{A}}/\mathfrak{M}| < \infty$. Also

$$[D^\times/Z\backslash D^\times/Z(\mathbb{A})/\mathfrak{K} : D^\times/Z\backslash\mathfrak{M}/\mathfrak{K}] < \infty \ .$$

But as we have seen by 3.1

$$\Gamma_f^{(p)} = D^\times(F_{\underline{p}}) \cap \mathfrak{K}.$$

So $D^\times/Z\backslash\mathfrak{M}/\mathfrak{K} \cong \Gamma_f^{(p)}\backslash D_\infty^\times/ZD^\times(F_{\underline{p}})/Z$ and finally

$$[D^\times/Z\backslash D^\times/Z(\mathbb{A})/\mathfrak{K} : \Gamma_f^{(p)}\backslash D_\infty^\times/Z.D^\times(F_{\underline{p}})/Z] < \infty \ .$$

Let $\pi_\infty \otimes \pi_{\underline{p}}$ be an irreducible representation of the right regular representation of $D^\times/Z(F_{\underline{p}})/Z.D^\times/Z(F_{\underline{p}})$ in $C(\Gamma_f^{(p)}\backslash D^\times/Z(F_{\underline{p}})/Z.D^\times/Z(F_{\underline{p}})$. There is an irreducible subrepresentation of $D^\times/Z(\mathbb{A})_F)$ in $\tilde{\pi} = \otimes_r \tilde{\pi}_{\underline{p}}$ such that $\tilde{\pi}_{\underline{p}} = \pi_{\underline{p}}$ and $\tilde{\pi}_\infty = \pi_\infty$.
Suppose $\pi_{\underline{p}}$ does not occur in $V_{\underline{p}}^{D^\times/Z(\mathcal{O}_{\underline{p}}}$ (,i.e. It is not one-dimensional). So $\tilde{\pi}$ occurs in $C(D^\times/Z\backslash D^\times/Z(\mathbb{A}_F);\mathbb{C})$. Since d is assumed a prime number, we can apply the Weak Lifting Theorem [1, Theorem 4.2.], and obtain a cuspidal subrepresentation $\rho_\infty \otimes \rho_{\underline{p}}$ of $C(\mathrm{PGL}(d;F_{\underline{p}})/\mathrm{PGL}(d;\mathbb{F}_q[t]_p);\mathbb{C})$, which is cuspidal and $\rho\underline{p} = \pi\underline{p}$. By recent reslt of L. Lafforgue [26] we have

$$|T_{\underline{p},i}| < \binom{d}{i} q^{\frac{i(d-i)}{2}\deg p}\sigma_i(z'_1,\ldots,z'_d) \ ,$$

where $z'_1,\ldots,z'_d \in \mathbb{C}$ are complex numbers with absolute value $|z'_j| = |\frac{z_j}{q^{(d-1)/2}}| = 1$ for $j = 1,\ldots,d$ and $\sigma_i(z'_1,\ldots,z'_d)$ i-th elementary symmetric polynomial. Choose suitable z_i as in Lafforgue's expression of Ramanujan-Peterson conjecture ,i.e. the proof of 2 is complete. □

Chapter 4

Explicit Constructions

4.1 Introduction

To give explicit examples of Ramanujan regular hypergraphs, we specialize the situation of section 3 even further. So our examples of division algebras, we consider, Ore skew polynomial rings over finite fields or more precisely their quotient fields. These are unramified outside the primes zero and infinity. The arithmetic groups, we consider here are obtained again by allowing denominators at a prime $p = p(t)$ different from zero and infinity. The properties of those arithmetic groups can be described to a large extent by divisibility properties of the skew polynomial ring $\mathbb{F}_{q^d}\{\tau\}$.
In section 4.2 we consider the main properties of skew polynomial rings $\mathbb{F}_{q^d}\{\tau\}$ and some related rings. Section 4.3 describes the arithmetic groups we want to study. Sections 4.4 gives the explicit construction of the Ramanujan hypergraphs, our simplicial complexes we are introduced in. Again these will be described in terms of Cayley graphs of various groups.

4.2 The Skew polynomial ring $\mathbb{F}_{q^d}\{\tau\}$

In this section we collect various well known fundamental facts concerning skew polynomial rings. These rings are well known in the theory of non commutative rings and many of the facts we note below hold in greater generality. However we have written up the relevant properties in the form we will need later on. For more details see [13],[14],[15],[11],[10],[12] and [42], [41].
We consider the finite field $\mathbb{F}_q$ of $q = l^n$ elements of characteristic l. $\mathbb{F}_{q^d}$ is a finite extension of $\mathbb{F}_q$ of degree d.
We will construct now the skew polynomial ring $\mathbb{F}_{q^d}\{\tau\}$. As a set this is

given by

$$\{\sum_{i=0}^{n} a_i\tau^i \quad |n \geq 0, \quad a_i \in \mathbb{F}_{q^d}\}$$

Addition is defined by

$$\sum_{i=0}^{n} a_i\tau^i \quad + \quad \sum_{i=0}^{n} b_i\tau^i := \sum_{i=0}^{n}(a_i + b_i)\tau^i.$$

For $\mathbb{F}_{q^d}\{\tau\}$ one obtains the structure of an infinite dimensional vector space over $\mathbb{F}_q$.
Regarding the multiplication, the fundamental rule is

$$\tau\lambda = \lambda^q\tau$$

for $\lambda \in \mathbb{F}_{q^d}$. There is a unique multiplication on the ring $\mathbb{F}_{q^d}\{\tau\}$ satisfying this rule.

Proposition 49. *The center of the ring $\mathbb{F}_{q^d}\{\tau\}$ is given as $\mathbb{F}_q[\tau^d]$.*

Proof. It is obvious, that $\mathbb{F}_q[\tau^d]$ is contained in the center $Z(\mathbb{F}_{q^d}\{\tau\})$. On the other hand, if $c = \sum_{i=0}^{n} a_i\tau^i$ is a central element in $\mathbb{F}_{q^d}\{\tau\}$, it has to commute with all elements $\lambda \in \mathbb{F}_{q^d}$. This forces the a_i for $i \not\equiv 0(\mod d)$ to be zero. Therefore c is of the form

$$c = \sum_{\substack{i \equiv 0 \mod (d) \\ 0 \leq i \leq n}} a_i(\tau^d))^{d/i}$$

Because $\tau c = c\tau$, it follows additionally, that the a_i occurring are elements in $\mathbb{F}_q$. □

Remark. It is immediate, that

$$\{\alpha^i\tau^j \quad | \quad 0 \leq i, j \leq d-1\}$$

where

$$\mathbb{F}_{q^d} = \mathbb{F}_q(\alpha)$$

is a basis of the left (-right) modules $\mathbb{F}_{q^d}\{\tau\}$ over $\mathbb{F}_q[\tau^d]$.

We denote $\tau^d =: t$. The center $\mathbb{F}_q[t]$ is of course the polynomial ring in the indeterminate t over the finite field $\mathbb{F}_q$. The following proposition is due to Ore, see [42], [41].

Proposition 50. *The skew polynomial ring $\mathbb{F}_q^d\{\tau\}$ is a left resp. right Euclidean ring.*

Proof. One has to show the following property: given polynomials $f(\tau), g(\tau) \in \mathbb{F}_{q^d}\{\tau\}$, there are polynomials $s(\tau)$ and $r(\tau)$ such that

$$f(\tau) = s(\tau)g(\tau) + r(\tau) \quad \text{such that} \deg_\tau(r(\tau)) < \deg_\tau(g(\tau))$$

for the τ-degrees of the polynomials above in the obvious sense. This would show that $\mathbb{F}_{q^d}\{\tau\}$ is a left Euclidean ring. But of course this can be seen by the usual division procedure of polynomials taking only into consideration, that one can write $\lambda\tau = \tau\lambda^{1/q}$ because $\mathbb{F}_{q^d} \longrightarrow \mathbb{F}_{q^d}$, $u \longmapsto u^q$ is a bijection. The property, that the ring $\mathbb{F}_{q^d}\{\tau\}$ is right Euclidean means, one can find in the situation above elements $s'(\tau)$ and $r'(\tau)$ such that

$$f(\tau) = g(\tau)s'(\tau) + r'(\tau) \quad \text{where} \deg_\tau(r'(\tau)) < deg_\tau(g(\tau)) \quad \text{holds.}$$

This is shown in the same way. □

Remark. Of course, all of this can be found in the literature (as given above) even for more general skew polynomial rings $k\{\tau\}$, where k is commutative field with automorphism $\sigma : k \longrightarrow k$, such that the rule

$$\tau\lambda = \sigma(\lambda)\tau$$

holds for $\lambda \in k$.

Corollary 51. *i) Any left ideal I in $\mathbb{F}_{q^d}\{\tau\}$ is a principal ideal of the form $I = \mathbb{F}_{q^d}\{\tau\}.a$, with $a \in I$ appropriate.*
ii) Similarly any right ideal J in $\mathbb{F}_{q^d}\{\tau\}$ is a principal ideal of the form $J = b.\mathbb{F}_{q^d}\{\tau\}$ with $b \in J$ appropriate.

Proof. Given I, choose $g(\tau) \in I$ of minimal degree, if $I \neq (0)$. Otherwise we are ready. Suppose, $f(\tau)$ is an arbitrary element of I. We can find $s(\tau)$ and $r(\tau)$ such that

$$f(\tau) = s(\tau)g(\tau) + r(\tau) \quad \text{where} \deg_\tau(r(\tau)) < deg_\tau(g(\tau)).$$

Because $f(\tau), g(\tau) \in I$, it follows that $r(\tau) \in I$, because I is a left ideal. But then $r(\tau) = 0$, because otherwise we would have a contradiction to the choice of $g(\tau)$ as nonzero element of I with $deg_\tau(g(\tau))$ minimal. Therefore we have $f(\tau) = s(\tau)g(\tau)$.
This implies that $I = \mathbb{F}_{q^d}\{\tau\}g(\tau)$ and therefore I is a principal left ideal.
$ii)$ is shown in the same way. □

Remark. The generating elements as $g(\tau)$ above in i) are uniquely determined up to multiplication by an element of $\mathbb{F}_{q^d}^\times$ from the left side.

We have the following well known structure theorem for finitely generated $\mathbb{F}_{q^d}\{\tau\}$-modules (left or right modules). It corresponds to similar results for finitely generated modules over commutative principal ideal domains:

Theorem 52. *i) Any finitely generated left $\mathbb{F}_{q^d}\{\tau\}$-module M is the direct sum of cyclic left $\mathbb{F}_{q^d}\{\tau\}$-modules:*

$$M = \bigoplus_{i=1}^{r} \mathbb{F}_{q^d}\{\tau\}/\mathbb{F}_{q^d}\{\tau\}f_i(\tau)$$

where we can assume additionally

$$f_1(\tau)|_l f_2(\tau)|_l \ldots |_l f_r(\tau).$$

Here $a|_l b$ means left-divisibility, i.e. there is $r \in \mathbb{F}_{q^d}\{\tau\}$ satisfying $r.a = b$.
ii) If M is a finitely generated torsion free left $\mathbb{F}_{q^d}\{\tau\}$-module, then M is a free $\mathbb{F}_{q^d}\{\tau\}$-module, that is, the $f_i(\tau)$ above are all zero.

Proof. see [13] or take any proof for the corresponding statement in the commutative situation. □

Proposition 53. *i) The ring $\mathbb{F}_{q^d}(\tau) := \mathbb{F}_{q^d}\{\tau\} \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t)$, obtained by extension of the center $\mathbb{F}_q[t]$ of $\mathbb{F}_{q^d}\{\tau\}$, is as a left(right) module free of rank d^2 over the rational field $\mathbb{F}_q(t)$.*
ii) The center of $\mathbb{F}_{q^d}(\tau)$ is $\mathbb{F}_q(t)$ under the canonical embedding of $\mathbb{F}_q(t)$ into $\mathbb{F}_{q^d}(\tau)$.
iii) $\mathbb{F}_{q^d}(\tau)$ is a division algebra. In particular $\mathbb{F}_{q^d}(\tau)$ is a central simple algebra of dimension d^2 over the rational function field $\mathbb{F}_q(t)$.

Proof. *i)* is clear, because $\mathbb{F}_{q^d}\{\tau\}$ is a free module of rank d^2 over $\mathbb{F}_q[t]$.
ii) If $u \in Z(\mathbb{F}_{q^d}(\tau))$, then there exists a nonzero polynomial $f(t) \in \mathbb{F}_q[t]$ such that $f(t).u \in Z(\mathbb{F}_{q^d}\{\tau\}) = \mathbb{F}_q[t]$.
Conversely, $\mathbb{F}_q(t)$, canonically embedded, is in the center of $\mathbb{F}_{q^d}(\tau)$,
iii) To show, that $\mathbb{F}_{q^d}(\tau)$ is a division algebra, consider the canonical homomorphism of algebras

$$\mathbb{F}_{q^d}(\tau) \bigotimes_{\mathbb{F}_q(t)} \mathbb{F}_q((t)) \longrightarrow \mathbb{F}_{q^d}((\tau))$$
$$\text{with} \quad f \otimes g \longmapsto \quad f.g$$

Here we are using following notations:

$$\mathbb{F}_q((t)) = \{\sum_{i=N}^{\infty} a_i t^i \quad | \quad N \in \mathbb{Z} \quad a_i \in \mathbb{F}_q\}$$

is the field of Laurent series (at the place $t=0$ of the field $\mathbb{F}_q(t)$), $\mathbb{F}_{q^d}((\tau))$ is given as the skew field of Laurent series

$$\mathbb{F}_q((\tau)) = \{\sum_{i=N}^{\infty} a_i\tau^i \quad | \quad N \in \mathbb{Z} \quad a_i \in \mathbb{F}_{q^d}\}$$

where we have again the communication rule

$$\tau\lambda = \lambda^q\tau$$

for $\lambda \in \mathbb{F}_{q^d}$. It is immediate to see that $\mathbb{F}_{q^d}((\tau))$ is a skew field. It is a d-dimensional vector space over the field $\mathbb{F}_q((t))$ of Laurent series over $\mathbb{F}_q$. Now, as vector space over $\mathbb{F}_q((t))$ both $\mathbb{F}_{q^d}(\tau) \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q((t))$ and $\mathbb{F}_{q^d}((\tau))$ are d^2-dimensional. Furthermore, the canonical homomorphism

$$\phi : \mathbb{F}_{q^d}(\tau) \bigotimes_{\mathbb{F}_q(t)} \mathbb{F}_q((t)) \longrightarrow \mathbb{F}_{q^d}((\tau))$$

is surjective, as the image contains the elements $\{\alpha^i\tau^j \quad | \quad 0 \leq i, j \leq d-1\}$, where $\mathbb{F}_{q^d} = \mathbb{F}_q(\alpha)$ as above, which form a basis of the vector space $\mathbb{F}_q^d((\tau))$ over $\mathbb{F}_{q^d}((t))$. Then, as a surjective homomorphism between vector spaces of equal dimension, ϕ is an isomorphism. Furthermore, ϕ is compatible with the multiplicative structure, therefore ϕ is even an isomorphism of algebras. Because $\mathbb{F}_{q^d}((\tau))$ is a skew field, the algebra $\mathbb{F}_{q^d}(\tau) \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q((t))$ is a skew field as well. So the algebra $\mathbb{F}_{q^d}(\tau)$ (as a sub algebra) has no zero divisors. Furthermore it is a finite dimensional algebra over its center $\mathbb{F}_q(t)$, which is a field. Then $\mathbb{F}_{q^d}(\tau)$ is a skew field itself. This shows *iii*) □

Remark. *i*)It follows from the considerations above, that the division algebra $\mathbb{F}_{q^d}(\tau)$ over its center $\mathbb{F}_q(t)$, where $t=\tau^d$, is ramified at the place $t=0$ with completion the skew field of Laurent series $\mathbb{F}_{q^d}((\tau))$,
ii) Similarly at the place $t=\infty$, the place corresponding to the degree valuation of $\mathbb{F}_q(t)$, $\mathbb{F}_{q^d}(\tau)$ is totally ramified.

We will show now, that these places are the only places of $\mathbb{F}_{q^d}(\tau)$ over $\mathbb{F}_q(t)$, which are ramified.
The skew field $\mathbb{F}_{q^d}(\tau)$ can be described as a cyclic algebra over its center $\mathbb{F}_q(t)$ in the following way. First we have the cyclic Galois extension $\mathbb{F}_{q^d}(t)$ of $\mathbb{F}_q(t)$ with canonical embedding $\mathbb{F}_{q^d}(t) \hookrightarrow \mathbb{F}_{q^d}(\tau)$ by mapping $t \mapsto t = \tau^d$. Of course

$$Gal(\mathbb{F}_{q^d}(t)/\mathbb{F}_q(t)) \cong Gal(\mathbb{F}_{q^d}/\mathbb{F}_q) \cong <\overline{\tau}>$$

where $\overline{\tau} : \mathbb{F}_{q^d} \longrightarrow \mathbb{F}_{q^d}, \quad u \mapsto u^q$ is given by the Frobenius automorphism. $\mathbb{F}_{q^d}(t)$ is an unramified field extension of $\mathbb{F}_q(t)$ at all places of $\mathbb{F}_q(t)$. As it can be compute the invariants of the skew field $\mathbb{F}_{q^d}(\tau)$ over $\mathbb{F}_q(t)$ at all places of $\mathbb{F}_q(t)$ in the sense of the classical theory. The element $\tau \in \mathbb{F}_q^d(\tau)$ satisfies the rule $\tau\lambda = \lambda^q\tau$ and $\tau^d = t$. But for all places $p = p(t) \neq 0, \infty$ we have $v_p(t) = 0$, which implies that the central cyclic algebra $\mathbb{F}_q^d(\tau)$ is unramified at all places $p \neq 0, \infty$. On the other hand, computing the invariants at the places $t = 0, \infty$ we obtain for the invariants

$$inv_0(\mathbb{F}_q^d(\tau)/\mathbb{F}_q(t) = \frac{1}{d}$$
$$inv_\infty(\mathbb{F}_q^d(\tau)/\mathbb{F}_q(t) = -\frac{1}{d}$$

We have obtained

Theorem 54. :
$\mathbb{F}_{q^d}(\tau)$ is up to isomorphism the unique central algebra over $\mathbb{F}_q(t)$ with the following properties:
i) $\mathbb{F}_{q^d}(\tau)$ is unramified at all places $p = p(t) \neq 0, \infty$ of $\mathbb{F}_q(t)$.
ii) It has invariants $\frac{1}{d}$, $-\frac{1}{d}$ at $t = 0$ resp. $t = \infty$.

Proof. All the properties have been shown, the uniqueness statement is a part of the theorem of Hasse - Brauer - Noether see [64, chapter XIII,3.Theorem 2 and 6. theorem 4] □

Lemma 55. *$\mathbb{F}_{q^d}\{\tau\}$ is a maximal $\mathbb{F}_q[t]$-order in $\mathbb{F}_{q^d}(\tau)$.*

Proof. We have to check only the maximality. Assume that R is an order with $\mathbb{F}_{q^d}\{\tau\} \subseteq R$. By definition of the concept of order, R is finite over $\mathbb{F}_q[t]$, so there exist a nonzero element $f \in \mathbb{F}_q[t]$ such that $Rf \subseteq \mathbb{F}_{q^d}\{\tau\}$. As $\mathbb{F}_{q^d}\{\tau\}$ is a PID (left and right), there exists an element μ in $\mathbb{F}_{q^d}\{\tau\}$ with $R = \mathbb{F}_{q^d}\{\tau\}\mu$. Thus $Rf = \mathbb{F}_{q^d}\{\tau\}\mu f^{-1}$. Now since R is a domain the proof is complete. □

Finally we give another description of the skew polynomial ring $\mathbb{F}_{q^d}\{\tau\}$.
Consider an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. $\overline{\mathbb{F}}_q$ is a vector space over the field $\mathbb{F}_q$ and we have the ring of vector space endomorphisms $\mathrm{End}_{\mathbb{F}_q}(\overline{\mathbb{F}}_q)$. As mentioned above, there is also the skew polynomial ring $\overline{\mathbb{F}}_q\{\tau\}$, satisfying in particular again the rule $\tau\lambda = \lambda^q\tau$ for $\lambda \in \overline{\mathbb{F}}_q$. We choose an embedding

$$\mathbb{F}_{q^d}\{\tau\} \hookrightarrow \overline{\mathbb{F}}_q\{\tau\}$$

by choosing a homomorphism $\mathbb{F}_{q^d} \to \overline{\mathbb{F}}_q$.
With any polynomial $f(\tau) = \sum_{i=0}^n a_i\tau^i$, $f(\tau) \in \overline{\mathbb{F}}_q\{\tau\}$ we associate the polynomial endomorphism

$$\varphi_f : \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q, \quad x \mapsto \sum_{i=0}^n a_i x^{q^i} \tag{4.1}$$

Proposition 56. *(i)*

$$\varphi : \bar{\mathbb{F}}_q\{\tau\} \longrightarrow \quad End_{\mathbb{F}_q}(\bar{\mathbb{F}}_q)$$

$$f(\tau) = \sum_{i=0}^{n} a_i \tau^i \mapsto \varphi_f$$

as above, is an injective homomorphism.
ii) $\ker \varphi_f = \{x \in \bar{\mathbb{F}}_q : \varphi_f(x) = \sum_{i=0}^n a_i x^{q^i} = 0\}$
is an $\mathbb{F}_q$*-vector subspace of* $\bar{\mathbb{F}}_q$*. One has:*

$$\dim_{\mathbb{F}_q} \ker(\varphi_f) \leq n$$

iii) φ_f *is injective, iff* f *is purely inseparable as a polynomial, that is,* $f(\tau)$ *is of the form* $c\tau^n$ *with* $c \neq 0$*.*
iv) If $f(\tau) \neq 0$*,* φ_f *is surjective.*

Proof. *i)* *ii)* and *iii)* are evident, *iv)* follows immediately, because the polynomial equations

$$\sum_{i=0}^{n} a_i x^{q^i} = c$$

have solutions for all $c \in \bar{\mathbb{F}}_q$, iff $f(\tau) \neq 0$. □

Proposition 57. *Given a finite dimensional* $\mathbb{F}_q$*-vector space* $V \subset \bar{\mathbb{F}}_q$*, there is a polynomial* $f(\tau) \in \bar{\mathbb{F}}_q\{\tau\}$*, unique up to a scalar from* $\bar{\mathbb{F}}_q^\times$*, such that*
i) $f(\tau)$ *is not divisible by* τ *(left or right would be equivalent for this).*
ii) $\ker(\varphi_f) = V \subset \bar{\mathbb{F}}_q$*,*
iii) $f(\tau)$ *moreover can be chosen to be in* $\mathbb{F}_{q^d}\{\tau\}$ *iff* $V \subset \bar{\mathbb{F}}_q$ *satisfies* $\varphi_{\tau^d}(V) = V$*, that is, the map*

$$\bar{\mathbb{F}}_q \longrightarrow \quad \bar{\mathbb{F}}_q$$

$$x \mapsto x^{q^d} = \varphi_\tau(x)$$

maps V *bijectively onto itself.*

Proof. Given $V \subset \bar{\mathbb{F}}_q$ a finite dimensional vector space over $\mathbb{F}_q$, we define the polynomial

$$p(x) := \prod_{v \in V} (x - v)$$

By induction with respect to the dimension $\dim_{\mathbb{F}_q}(V)$, it is easy to show that $p(x)$ is a $\mathbb{F}_q$-linear (in particular additive) polynomial function, which therefore is of the form $p(x) = \varphi_f(x)$ for a skew polynomial $f(\tau) \in \bar{\mathbb{F}}_q\{\tau\}$. If there would be another such element $g(\tau) \in \bar{\mathbb{F}}_q\{\tau\}$ satisfying $\ker \varphi_g = V$, then the additive polynomial functions φ_f, $\varphi_g : \bar{\mathbb{F}}_q \longrightarrow \bar{\mathbb{F}}_q$ would have the

same zeros in $\overline{\mathbb{F}}_q$, furthermore they have the same degree. Therefore there exists a $c \in \overline{\mathbb{F}}_q^{\times}$ satisfying $c\varphi_f = \varphi_g$.
iii) If $V \subset \overline{\mathbb{F}}_q$ satisfies $\varphi_{\tau^d}(V) = V$, that is, the map $\overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q$, $x \mapsto x^{q^d} = \varphi_\tau(x)$ maps V bijectively onto itself, then φ_f, as constructed above, satisfies

$$\begin{aligned}\varphi_{\tau^d}\varphi_f(x) &= \Big(\prod_{v\in V}(x-v)\Big)^{q^d}\\ &= \prod_{v\in V}(x^{q^d}-v^{q^d})\\ &= \prod_{v\in V}(x^{q^d}-v) \quad (\text{as} \qquad \tau^d(V)=V)\\ &= \varphi_f(\varphi_{\tau^d}(x))\end{aligned}$$

for all $x \in \bar{\mathbb{F}}_q$. This implies immediately

$$\tau^d f(\tau) = f(\tau)\tau^d$$

which implies, that $f(\tau)$ is of the form

$$f(\tau) = \sum_{i=0}^{n} a_i\tau^i \qquad \text{where} \qquad a_i \in \mathbb{F}_{q^d}.$$

□

Proposition 58. *If* $f(\tau), \quad g(\tau) \in \overline{\mathbb{F}}_q\{\tau\}$ *satisfy* $\ker(\varphi_f) = \ker(\varphi_g)$ *and if* $f(\tau)$ *is not divisible by* τ*, then there is* $m \in \mathbb{N}$*, such that*

$$g(\tau) = \tau^m f(\tau)$$

Proof. Given $g(\tau)$, one can find m maximal, such that $g(\tau) = \tau^m \widetilde{f}(\tau)$ but then $\varphi_{\widetilde{f}}$ is a separable polynomial function satisfying $\ker(\varphi_{\widetilde{f}}) = \ker(\varphi_g)$ and therefore also $\ker(\varphi_{\widetilde{f}}) = \ker(\varphi_f)$, furthermore $\varphi_{\widetilde{f}}$, φ_f are separable polynomial functions, which we can assume to have highest coefficient 1. Therefore we obtain $\widetilde{f} = f$ and therefore also

$$g(\tau) = \tau^m f(\tau)$$

□

Proposition 59. *i) Suppose* $f_1(\tau), f(\tau) \in \overline{\mathbb{F}}_q\{\tau\}$*, are separable polynomials. Then there is a separable polynomial* $f_2(\tau)$ *satisfying*

$$f_2(\tau)f_1(\tau) = f(\tau) \quad \textit{iff} \quad \ker(\varphi_{f_1}) \subseteq \ker(\varphi_f) \qquad \textit{as } \mathbb{F}_q\textit{-subspaces.}$$

ii) If $f_1(\tau), f(\tau) \in \mathbb{F}_{q^d}\{\tau\}$*, are separable polynomials, then there is*

$$f_2(\tau) \in \mathbb{F}_{q^d}\{\tau\}$$

satisfying

$$f_2(\tau)f_1(\tau) = f(\tau)$$

Proof. ($\Longrightarrow$) This direction is trivial.
Conversely, suppose that, $\ker(\varphi_{f_1}) \subseteq \ker(\varphi_f)$. Consider the separable polynomial

$$\varphi_{f_1} : \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q$$

and denote for

$$V := \ker(\varphi_f), \qquad \varphi_{f_1}(V) =: \overline{V} \subset \overline{\mathbb{F}}_q$$

Obviously $\bar{V}$ is given as $\mathbb{F}_q$-subspace in $\overline{\mathbb{F}}_q$ and we can find accordingly $f_2(\tau) \in \overline{\mathbb{F}}_q\{\tau\}$, such that φ_{f_2} is separable and $\ker \varphi_{f_2} = \bar{V}$. Then

$$\varphi_{f_2}\varphi_{f_1} : \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q$$

has kernel V and is again separable with highest coefficient 1, Therefore we obtain $\varphi_f = \varphi_{f_2}\varphi_{f_1}$, which implies immediately $f = f_2 f_1$. This shows i).
ii) is an immediate consequence of i). □

Corollary 60. *Suppose,* $f(\tau) \in \mathbb{F}_{q^d}\{\tau\}$ *is not divisible by* τ*, that is, the corresponding polynomial function* φ_f *is separable.*
Decompositions of the form

$$f(\tau) = f_1(\tau) \dots f_r(\tau)$$

in $\mathbb{F}_{q^d}\{\tau\}$ *are in bijective correspondence with* $t = \tau^d$*-invariant flags of* $\mathbb{F}_q$*-linear subspaces,*

$$0 \subset W_1 \subset \ldots \subset W_r = V$$

where $\qquad W_{r-j} = \ker(\varphi_{f_j} \dots \varphi_{f_r}) \qquad$ *for* $j = 1, \dots, r$*.*

Proof. We can assume in the corollary, that $f(\tau)$ and all of the f_j for $j = 1, \dots, r$ have highest coefficient 1. We then have the map above associating with a decomposition

$$f(\tau) = f_1(\tau) \dots f_r(\tau)$$

the corresponding flag of subspaces

$$0 \subset W_1 \subset \ldots \subset W_r = V$$

where $V = \ker \varphi_f$ and $W_j = \ker (f_{r-j+1}(\tau)) \dots f_r(\tau))$.
Conversely, suppose that, the flag of t-invariant subspaces

$$0 \subset W_1 \subset \dots \subset W_r = V$$

is given. We find $f(\tau) \in \overline{\mathbb{F}}_q\{\tau\}$, separable, with highest coefficient 1, such that $\ker \varphi_f = V$ holds. $f(\tau) \in \overline{\mathbb{F}}_q\{\tau\}$ is an element in $\mathbb{F}_{q^d}\{\tau\}$, because $V = \ker \varphi_f$ is invariant under $t = \tau^d$ (not elementwise however).
Similary we find $\widetilde{f}_j(\tau) \in \mathbb{F}_{q^d}\{\tau\}$, such that $\ker (\varphi_{\widetilde{f}_j}) = W_j$, By repeated application of Proposition.59, we can conclude:

$$\begin{aligned} \widetilde{f}_r(\tau) &= f_1(\tau) \dots f_r(\tau) \\ \widetilde{f}_{r-1}(\tau) &= f_1(\tau) \dots f_{r-1}(\tau) \\ &\vdots \\ \widetilde{f}_1(\tau) &= f_r(\tau) \end{aligned}$$

where the $f_j(\tau) \in \mathbb{F}_{q^d}\{\tau\}$. This shows the Corollary. □

4.3 Arithmetic groups associated to the division algebra $\mathbb{F}_{q^d}(\tau)$

We consider again the division algebra of skew polynomials $D = \mathbb{F}_{q^d}(\tau)$ with center $\mathbb{F}_q(t)$ as in the last section.
Associated with this algebra are the algebraic groups $D^\times$, $D^{(1)}$ and $D^\times/Z$ given as group functors on the category of $\mathbb{F}_q(t)$-algebras R (that is, there is a homomorphism of commutative algebras with unit elements
$\mathbb{F}_q(t) \longrightarrow R$) given by :

$$D^\times(R) := (\mathbb{F}_{q^d}(\tau) \otimes_{\mathbb{F}_q(t)} R)^\times \tag{4.2}$$

the group of units, and similarly

$$D^{(1)}(R) := (\mathbb{F}_{q^d}(\tau) \otimes_{\mathbb{F}_q(t)} R)^{(1)} := \{x \in (\mathbb{F}_{q^d}(\tau) \otimes_{\mathbb{F}_q(t)} R)^{(1)} | nr(x) = 1\} \tag{4.3}$$

where $nr : D^\times(R) \longrightarrow R^\times$ is the reduced norm of the central simple algebra $\mathbb{F}_{q^d}(\tau)$ over $\mathbb{F}_q(t)$, seen as a polynomial map and extended by R.
Finally we have the group functor:

$$R \mapsto (D^\times/Z)(R) = D^\times(R)/Z(R^\times)$$

Remark. Other notation for these groups are $GL(1, D)$, $SL(1, D)$ and $PGL(1, D)$. For a proof, that these group functors are in fact representable by algebraic groups see [6].

Besides these algebraic groups, we have the corresponding group schemes $\underline{D}^\times$, $\underline{D}^{(1)}$ and $\underline{D}^\times/Z$ over the ring $\mathbb{F}_q[t]$ respectively. Over its spectrum space $\mathbb{F}_q[t]$, given similarly by the group valued functors :

$$\mathbb{F}_q[t] - \text{Alg} \qquad \longrightarrow \qquad \text{groups}$$

From the category of commutative $\mathbb{F}_q[t]$-algebras to the category of groups, given by

$$R \mapsto D^\times(R) = (D \otimes_{\mathbb{F}_q[t]} R)^\times$$

The arithmetical groups we are considering here can be described as follows. Suppose $p(t) \in \mathbb{F}_q[t]$ is an irreducible polynomial. Denote by $\mathcal{O} := \mathbb{F}_q[t][\frac{1}{p(t)}]$ the localization of the polynomial ring $\mathbb{F}_q[t]$ with respect to the multiplicative system $\mathcal{S} := \{p(t)^n | n = 0, \ldots\}$, that is, one considers the rational functions in $\mathbb{F}_q(t)$, whose denominators are powers of $p(t)$.
The basic arithmetic groups in our situation are then $\underline{D}^\times(\mathcal{O})$,$\underline{D}^{(1)}(\mathcal{O})$,$(\underline{D}^\times/Z)(\mathcal{O})$, which are given explicitly as

$$\begin{aligned} \underline{D}^\times(\mathcal{O}) =& (\mathbb{F}_q^d\{\tau\} \otimes_{\mathbb{F}_q[t]} \mathcal{O})^\times \\ =& (\mathbb{F}_q^d\{\tau\} \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q[t][\frac{1}{p(t)}])^\times. \end{aligned}$$

Furthermore

$$\underline{D}^{(1)}(\mathcal{O}) = \{x \in \underline{D}^\times(\mathcal{O}) \quad | \quad nr(x) = 1\} \tag{4.4}$$

and $\underline{D}^\times/Z)(\mathcal{O}) = \underline{D}^\times(\mathcal{O})/\mathcal{O}^\times$.
We study these groups in the usual way by their operation on the product of the Bruhat-Tits building of the algebraic group $D^\times$(respectively, $D^{(1)}$, $D^\times/Z$) at the primes missing, which in this case are $(p(t))$ and ∞ of $\mathbb{F}_q(t)$.
As the division algebra $\mathbb{F}_{q^d}(\tau)$ is totally ramified at ∞, the corresponding Bruhat-Tits building is just a point. It is therefore sufficient to consider the Bruhat-Tits building at the prime $(p(t))$. This is the building associated to the algebraic group $D^\times \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q(t)_{(p(t))}$, where $\mathbb{F}_q(t)_{(p(t))}$ is the completion of $\mathbb{F}_q(t)$ at the prime $p(t) := p$. (and similarly for the groups $D^{(1)}$, $D^\times/Z$).
Because D is unramified at $(p(t)) = (p)$, it follows, that, we have an isomorphism

$$D \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q(t)_p \quad \xrightarrow{\sim} \quad \mathbb{M}(d; \mathbb{F}_q(t)_p).$$

Therefore, we have an induced isomorphism

$$(D \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q(t)_p)^\times \quad \xrightarrow{\sim} \quad GL(d; \mathbb{F}_q(t)_p).$$

We also have an induced embedding

$$\begin{aligned}\Gamma &= (\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p(t)}])^\times \\ &= (\mathbb{F}_{q^d}\{\tau\} \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q[t][\frac{1}{p(t)}])^\times \\ &\hookrightarrow \quad (\mathbb{F}_{q^d}(\tau) \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q(t)_p)^\times \quad \cong \quad GL(d; \mathbb{F}_q(t)_p)\end{aligned}$$

and using this, an action of Γ and its subgroups on the Bruhat-Tits building corresponding to $p = p(t)$ respectively also for For $GL(d; \mathbb{F}_q(t)_p)$ and the related subgroups $SL(d; \mathbb{F}_q(t)_p)$ for $D^{(1)}$ and $PGL(d; \mathbb{F}_q(t)_p)$ for $D^\times/Z$. As described in section 2.2, that is the building $X_.(p) := X_.(\mathbb{F}_q(t)_p{}^d)$, associated to the vector space $\mathbb{F}_q(t)_p{}^d$ over the locally compact topological field $\mathbb{F}_q(t)_p$. The problem we have is to understand the quotient $\Gamma \backslash X_.(p)$ for the group Γ under consideration. To be able to do this we add some further considerations.
First, because $\mathbb{F}_{q^d}\{\tau\}$ is a maximal order over $\mathbb{F}_q[t]$ in $\mathbb{F}_{q^d}(\tau) = D$, we can choose the isomorphism above in such away that it induces an isomorphism of the corresponding local orders

$$\mathbb{F}_{q^d}\{\tau\} \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q[t]_p \quad \xrightarrow{\sim} \quad \mathbb{M}(d; \mathbb{F}_q[t]_p),$$

where as before $\mathbb{F}_q[t]_p$ is the valuation ring of $\mathbb{F}_q(t)_p$.
We denote $L_0 := \mathbb{F}_q[t]_p{}^d$, the standard lattice. For any lattice $L \subset \mathbb{F}_q(t)_p{}^d$ over $\mathbb{F}_q[t]_p$, we consider $\mathrm{Hom}_{\mathbb{F}_q[t]_p}(L, L_0)$.
This is in an obvious way a left module over the ring

$$\mathrm{End}_{\mathbb{F}_q[t]_p}(L_0) \cong \mathbb{F}_{q^d}\{\tau\} \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q[t]_p$$

Proposition 61. *There is a bijective correspondence between lattices $L \subset \mathbb{F}_q(t)_p{}^d$ over $\mathbb{F}_q[t]_p$ and $\mathbb{F}_{q^d}\{\tau\}$-left modules M equipped additionally with an isomorphism*

$$\phi : \mathbb{F}_q[t][\frac{1}{p(t)}] \otimes_{\mathbb{F}_q[t]} M \quad \xrightarrow{\sim} \quad \mathbb{F}_{q^d}\{\tau\}[\frac{1}{p(t)}]$$

which are free of rank 1 over $\mathbb{F}_{q^d}\{\tau\}_p = \mathbb{F}_q[t]_p \otimes_{\mathbb{F}_q[t]} \mathbb{F}_{q^d}\{\tau\}$

Proof. Given a pair (M, ϕ), we have to reconstruct the lattice $L \subset {\mathbb{F}_q(t)_p}^d$ over $\mathbb{F}_q[t]_p$. Denoting ϕ_p the obvious extension of ϕ to

$$\phi_p : \mathbb{F}_q(t)_p \otimes_{\mathbb{F}_q(t)} M \quad \xrightarrow{\sim} \quad \mathbb{F}_{q^d}(\tau)_p$$

and the $\tilde{\phi_p}$ as the following composition of maps:

$$\mathbb{F}_q(t)_p \otimes_{\mathbb{F}_q(t)} M \quad \xrightarrow{\sim} \quad \mathbb{F}_{q^d}(\tau)_p \quad \xrightarrow{\sim} \quad \mathbb{M}(d; \mathbb{F}_q(t)_p)$$

where the first map is ϕ_p, and the second is the isomorphism fixed above. We consider the restriction map

$$\tilde{\phi}_{p|} : M_p = \mathbb{F}_q[t]_p \otimes_{\mathbb{F}_q[t]} M \longrightarrow \mathbb{M}(d; \mathbb{F}_q(t)_p)$$

The image is a free module of rank one over $\mathrm{End}(\mathbb{F}_q[t]_p^d)$. It is immediate to see that there exists a unique local lattice $L \subset {\mathbb{F}_q(t)_p}^d$, such that

$$\tilde{\phi}_{p|}(M_p) = \mathrm{Hom}_{\mathbb{F}_q[t]_p}(L, L_0)$$

holds, where $L_0 = {\mathbb{F}_q[t]_p}^d$. This is a version of the Morita- equivalence. Conversely, given the lattice L, we obtain an $\mathbb{F}_{q^d}\{\tau\}$-module M in obvious way from the

$$\mathbb{F}_q[t][\frac{1}{p(t)}] \otimes_{\mathbb{F}_q[t]} M \quad \xrightarrow{\sim} \quad \mathbb{F}_{q^d}\{\tau\}[\frac{1}{p(t)}] \qquad \text{and}$$

$$\mathbb{F}_q[t]_p \otimes_{\mathbb{F}_q[t]} M \xrightarrow{\sim} \quad \mathrm{Hom}_{\mathbb{F}_q[t]_p}(L, L_0)$$

with the canonical identification. It is immediate to see, that these two constructions are inverse to each other. □

Proposition 62. *The group* $\Gamma = (\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p(t)}])^\times$ *acts transitively on the set of lattices* $L \subset {\mathbb{F}_q(t)_p}^d$ *over* $\mathbb{F}_q[t]_p$.

Proof. Consider the $\mathbb{F}_{q^d}\{\tau\}$-module M given by the pair

$$M^{(p)} := (\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}], \mathrm{Hom}(L, L_0))$$

(in the sense of the discussion above). Now any such module (as a left $\mathbb{F}_{q^d}\{\tau\}$-module) is isomorphic to $\mathbb{F}_{q^d}\{\tau\}$. Such an isomorphism α induces an isomorphism $\alpha^{(p)}$ of $\mathbb{F}_{q^d}\{\tau\}_p$-modules

$$M^{(p)} := \mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}] \longrightarrow \mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}]$$

Any such isomorphism is given as right multiplication by a unit

$$g \in \mathbb{F}_{q^d}\{\tau\}[\frac{1}{p(t)}]^{\times}$$

As α induces also an isomorphisms of $\mathbb{F}_{q^d}\{\tau\}_p$-modules

$$M_p = \mathrm{Hom}_{\mathbb{F}_q[t]_p}(L, L_0) \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}_q[t]_p}(L_0),$$

it is immediate to see that this is equivalent to the fact, that

$$g(L) = L_0$$

But this shows the transitivity of the action of Γ on the set of lattices. □

Corollary 63. *The group* $\underline{D}^{\times}(\mathcal{O})^{\times}/Z = (\mathbb{F}_q\{\tau\}[\frac{1}{p}])^{\times}/Z$ *acts transitively on the set of vertices* $X_0(\mathbb{F}_q(t)_p{}^d)$ *of the building* $X_.(\mathbb{F}_q(t)_p{}^d)$.

Proof. We have $Z = (\mathbb{F}_q[t][\frac{1}{p}])^{\times}$, which acts by scalar multiplication on the set of lattices. Therefore the group $(\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}])^{\times}/Z$ induces an action on $X_.(\mathbb{F}_q(t)_p{}^d)$, which is transitive on the set of the vertices $X_0(\mathbb{F}_q(t)_p{}^d)$ by proposition 62. □

Corollary 64. *For any subgroup* Γ *of finite index in* $\Gamma(1) = (\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}])^{\times}/Z$ *the quotient* $\Gamma\backslash X_.(\mathbb{F}_q(t)_p{}^d)$ *is a finite simplicial complex.*

Proof. This is an immediate consequence of Corollary 63. □

Remark. In fact this is a very special case of Godement's compactness theorem mentioned earlier, but in our situation it can be shown in a direct way as above.
We consider in the group $\Gamma(1) = (\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}])^{\times}/Z$ the following subgroup $\Gamma(\tau)$. We consider the composition of group homomorphisms

$$\Gamma(\tau) := \ker(\Gamma(1) \longrightarrow \mathbb{F}_{q^d}\{\{\tau\}\}^{\times}/Z) \longrightarrow \mathbb{F}_{q^d}^{\times}/\mathbb{F}_q^{\times} \quad (4.5)$$

where the first homomorphisms corresponds to the embedding to the place $t = 0$ considered easier and the second homomorphism is the evaluation homomorphism for $\tau = 0$. $\Gamma(\tau)$ is the kernel of the composition of these homomorphisms.

Proposition 65. *Let* $\Gamma(\tau)$ *be as 4.5, then*
i) $\Gamma(\tau)$ *is a torsion free group,*
ii) $\Gamma(\tau)$ *acts fixed point free on the simplicial complex* $X_.(\mathbb{F}_q(t)_p{}^d)$ *and also on its realization* $|X_.(\mathbb{F}_q(t)_p{}^d)|$. *In particular no simplex is mapped to itself in a nontrivial way under the action of* $\Gamma(\tau)$.

Proof. *i*) We consider the embedding

$$(\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}])^\times/Z \hookrightarrow \mathbb{F}_{q^d}\{\{\tau\}\}^\times/Z$$

Any element $g \in \Gamma(\tau)$ $g \neq 1$ is mapped to a power series in τ of the form $(1 + a_i\tau^i + \text{higher terms in} \quad \tau)$, where $a_i \in \mathbb{F}_{q^d}$, $a_i \neq 0$. It is immediate to see that $(1+a_i\tau^i+\text{higher terms in}\tau)^n \neq 1$ if $(n, Char(\mathbb{F}_q) = 1$. On the other hand for any $j \geq 1$

$$\begin{aligned}(1 + a_i\tau^i + \text{higher terms in} \quad \tau)^{p^j} &= (1 + a_i\tau^i(1 + b_1\tau + \ldots))^{p^j} \\ &= 1 + (a_i)^{p^j} + \ldots\end{aligned}$$

Thus the lowest term in τ is $(a_i)^{p^j}$, which obviously is not zero. This shows *i*).
ii) If $g \in \Gamma(\tau)$ $g \neq 1$, stabilizes a simplex, it would have a fixed point in the realization $X_.(\mathbb{F}_q(t)_p{}^d)$. As the stabilizer is discrete (as a subgroup of $\Gamma(\tau)$) and compact (being a closed subset in the compact stabilizer of a point of the building $X_.(\mathbb{F}_q(t)_p{}^d)$), it follows, that such a stabilizer group is finite. This implies immediately that g has a finite order, which is a contradiction. □

Proposition 66. *$\Gamma(\tau)$ is transitive on the set of the vertices of the building $X_.(\mathbb{F}_q(t)_p{}^d)$.*

Proof. We have seen above, that the group $\Gamma = (\mathbb{F}_q\{\tau\}[\frac{1}{p}])^\times$ is transitive on the set of vertices $X_0(\mathbb{F}_q(t)_p{}^d))$. Therfore, given a vertex $< L > \in X_0(\mathbb{F}_q(t)_p{}^d))$, we find $g \in \Gamma(\tau)$, $gL = L_0$, where $L_0 = \mathbb{F}_q[t]_p{}^d$. If there is another $g' \in \Gamma(\tau)$ with $g'L = L_0$, then we have $g^{-1}g'L_0 = L_0$. It follows immediately from Proposition 65 (*i*) that $g^{-1}g' = 1$ and so $g = g'$. This completes the proof of the corollary. □

Corollary 67. *$\Gamma(\tau)$ acts simply transitive on the building $X_.(\mathbb{F}_q(t)_p{}^d)$.*

Proof. Immediately obtained from the above propositions. □

4.4 Explicit description of some arithmetic quotients

In 4.3 we have introduced the arithmetic group $\Gamma(\tau)$

$$\Gamma(\tau) \subset \Gamma(1) = \left(\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}]\right)^\times/Z$$

which acts simply transitive on the building $X_\cdot(\mathbb{F}_q(t)_p^d)$ and also on its topological realization $|X_\cdot(\mathbb{F}_q(t)_p^d)|$.
We consider now arbitrary linear polynomials $p(t) = (t-\lambda)$ for $\lambda \in \mathbb{F}_q$, $\lambda \neq 0$.
Associated to the polynomial $p(t) = t - \lambda$ is the $\mathbb{F}_q$-linear homomorphism

$$\varphi_p : \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q, \quad x \mapsto (x^{q^d} - \lambda x)$$

Denote $V := \ker(\varphi_p) = \{x \in \overline{\mathbb{F}}_q \quad | \quad x^{q^d} - \lambda x = 0\}$
Of course, $V = V(\lambda)$ is explicitly given as $\left(\mathbb{F}_{q^d} . \sqrt[q^d-1]{\lambda}\right)$, a one-dimensional $\mathbb{F}_{q^d}$-sub-vector space in $\overline{\mathbb{F}}_q$.
In particular for the case $\lambda = 1$ we have $V(\lambda = 1) = \mathbb{F}_{q^d} \subset \overline{\mathbb{F}}_q$. We apply Corollary 60 from 4.2. to this situation :

Proposition 68. *There is a bijective correspondence between decompositions*

$$p(t) = f_1(\tau)\dots f_d(\tau)$$

into τ-linear factors $f_j(\tau) \in \mathbb{F}_{q^d}\{\tau\}$, $j = 1, \dots, d$, and arbitrary full flags

$$0 \subset W_1 \subset \dots \subset W_d = V$$

of $\mathbb{F}_q$-linear subspaces $W_j \subset V$ (such that $\dim(W_j) = j$*. This correspondence is given as in corollary 60, section* 4.3.

Proof. We only have to check that an arbitrary full flag

$$0 \subset W_1 \subset \dots \subset W_d = V$$

is $t = \tau^d$-invariant. But on V the relation $\tau^d = \lambda$ resp. $x^{q^d} = \lambda x$ holds. Therefore the action $x \mapsto x^{q^d}$ on V is exactly the homothety $V \longrightarrow V, \quad x \mapsto \lambda x$ with $\lambda \in \mathbb{F}_q$. Because $\lambda . W_j = W_j$, as the W_j are $\mathbb{F}_q$-vector-spaces in V, the result follows. □

We fix now again the standard lattice $L_0 := (\mathbb{F}_{q^d}\{\tau\})^d \subset \mathbb{F}_{q^d}(\tau)_p^d$ as well as the associated vertex $< L_0 >$. By Proposition 18 from Chapter 2 we have the isomorphism of simplicial complexes

$$lk\Big(< L_0 >; X_\cdot(\mathbb{F}_q(t)_p^d)\Big)$$

and the Tits building associated to the $\mathbb{F}_q$-vector-space $(L_0/\pi L_0)$. We remind the reader again of the following situation. We have fixed an isomorphism

$$\mathbb{F}_{q^d}\{\tau\}_{(p)} \xrightarrow{\sim} \mathbb{M}(d; \mathbb{F}_q[t]_p)$$

of the corresponding completions at $p = p(t)$. Upon doing this we get a corresponding standard lattice $\mathbb{F}_q[t]_p^d = L_0$, which is acted upon by $\mathbb{F}_{q^d}\{\tau\}_{(p)}$ resp. $\mathbb{M}(d; \mathbb{F}_q[t]_p)$ by the standard action. We have the quotient $(L_0/\pi L_0)$, where $\pi = p(t)$, isomorphic to $(\mathbb{F}_q[t]/(p(t))^d$ as $\mathbb{F}_q[t]/(p(t) \cong \mathbb{F}_q$-vector space. Furthermore we have $V = V(p(t)) = \ker(\varphi_p) \subset \overline{\mathbb{F}}_q$, which is also an $\mathbb{F}_q$-vector space of dimension d. Both $(L_0/\pi L_0)$ and V are naturally simple modules over

$$\mathbb{F}_{q^d}\{\tau\}/p(t)\mathbb{F}_{q^d}\{\tau\} \cong \mathbb{M}(d; \mathbb{F}_q[t]/(p(t)).$$

An isomorphism between these two modules, so compatible with the action $\mathbb{F}_{q^d}\{\tau\}/(p(t))$ on both sides, by Schur's lemma will be unique up to a central nonzero element of $\mathbb{F}_q[t]/(p(t) \cong \mathbb{F}_q$, that is, up to a scalar, $\neq 0$. This implies in particular a unique isomorphism between the simplicial complexes

$$lk\left(< L_0 >; X.(\mathbb{F}_q(t)_p^d)\right)$$

and the Tits building of the $\mathbb{F}_q$-vector space $V = V(p(t)) = \ker(\varphi_p) \subset \overline{\mathbb{F}}_q$, compatible with the action of $\left(\mathbb{F}_{q^d}\{\tau\}/(p(t))\right)^\times$.

In particular, to any neighboring vertex $< L > \in X_0(\mathbb{F}_q[t]_p^d)$, such that $< L_0, L_1 > \in X_1(\mathbb{F}_q[t]_p^d)$ (1-simplices), there is, unique element $\gamma_L \in \Gamma(\tau)$, because $\Gamma(\tau)$ acts simply transitive on the set of vertices.

We can also obtain explicit descriptions of the elements γ_L using the canonical isomorphisms above.

So, suppose we have a neighboring vertex $< L >$ of $< L_0 >$, so we assume the situation

$$L_0 \supsetneq L \supsetneq p(t)L_0.$$

We have $\gamma_L < L_0 > = < L >$ upon multiplying by an appropriate power of $p(t)$, we can even assume, that $\gamma_L(L_0) = L$.

In particular, we obtain then $\gamma_L(L_0) \subset L_0$ and therefore, because γ_L is free of all other primes, we obtain $\gamma_L \in \mathbb{F}_{q^d}\{\tau\}$. But then we obtain the induced equality

$$\gamma_L.\left(L_0/p(t)L_0\right) = \left(L/p(t)L_0\right),$$

That is, in term of the vector space $V = V(p(t)) \subset \overline{\mathbb{F}}_q : \gamma_L.V = W(\overline{L}) \subset V$. The divisor of $p(t)$ corresponding to γ_L will be therefore given by the equality above. We have proved

Theorem 69. *There is canonical bijection between the sets*

$$\left\{\gamma_L \quad | \quad \gamma_L \in \mathbb{F}_{q^d}\{\tau\}, \gamma_L(L_0) = L, \quad \textit{such that} < L_0, L > \in X_1(\mathbb{F}_q[t]_p^d)\right\}$$

and

$$\left\{ f(\tau) \in \mathbb{F}_{q^d}\{\tau\} \quad | \quad f(\tau) \quad \textit{is a nontrivial divisor of} \quad p(t) \right\},$$

$$\gamma_L \mapsto f(\tau),$$

given by the equation

$$\gamma_L(L_0)/p(t)L_0 \cong f(\tau).V$$

with respect to the canonical identification

$$L_0/p(t)L_0 \quad \stackrel{\sim}{\longrightarrow} \quad V = \ker(\varphi_p).$$

We consider now the Cayley graph of $\Gamma(\tau)$ with respect to the set of generators

$$\left\{ \gamma_L \quad | \quad \gamma_L \in \mathbb{F}_{q^d}\{\tau\}, \gamma_L(L_0) = L, \quad \text{such that} < L_0, L > \in X_1(\mathbb{F}_q[t]_p^d) \right\},$$

We denote $\mathrm{Cayley}(\Gamma(\tau); \{\gamma_L\})$ the associated graph.

Theorem 70. *There is a canonical isomorphism of graphs*

$$\textit{Cayley}(\Gamma(\tau); \{\gamma_L\}) \quad \stackrel{\sim}{\longrightarrow} \quad \tau_{\leqslant 1}\left(X_{.}(\mathbb{F}_q(t)_p^d) \right),$$

where $\tau_{\leqslant 1}\left(X_{.}(\mathbb{F}_q(t)_p^d) \right)$ *is the graph underlying the simplicial complex* $X_{.}(\mathbb{F}_q(t)_p^d)$.

This isomorphism is given by the map

$$\mathrm{Cayley}(\Gamma(\tau); \{\gamma_L\})_{(0)} = \Gamma(\tau) \quad \longrightarrow \quad X_0(\mathbb{F}_q(t)_p^d),$$

$$\gamma \mapsto \gamma < L_0 > .$$

Proof. The map above on the 0-level is a bijection, because $\Gamma(\tau)$ is simply transitive on $X_0(\mathbb{F}_q(t)_p^d)$. Two vertices $< \gamma >$ and $< \gamma' >$, where $\gamma, \gamma' \in \Gamma(\tau)$ define a 1-simplex $< \gamma, \gamma' >$ in the Cayley graph $\mathrm{Cayley}(\Gamma(\tau); \{\gamma_L\})$, iff there is $\gamma_L \in \Gamma(\tau)$, satisfying $\gamma\gamma_L = \gamma'$. But then

$$< \gamma\gamma_L(L_0), \gamma L_0 >= \gamma < \gamma_L(L_0), L_0 >,$$

which obviously is a one-simplex in $X_{.}(\mathbb{F}_q(t)_p^d)$, because $< \gamma_L(L_0), L_0 >$ is a one-simplex in $X_{.}(\mathbb{F}_q(t)_p^d)$ by definition.Furthermore, this isomorphism is label-preserving and induces an isomorphism of the corresponding hypergraphs resp. simplicial complexes. The converse is equally clear, furthermore it is clear, that our isomorphism will preserve labels, which can be introduced via the set of generating elements $\{\gamma_L\}$. By proposition 24 from Chapter 2 the graph structure in our situation induces in a unique way a simplicial structure resp. the structure of a hypergraph. Obviously, our isomorphism induces an isomorphism of this structure. □

Theorem 71. *Suppose, $\Gamma \subset \Gamma(\tau)$ is a normal subgroup of finite index, $\overline{\gamma_L} := proj(\gamma_L)$, where*

$$proj : \Gamma(\tau) \longrightarrow \Gamma(\tau)/\Gamma$$

is the canonical projection. Then the isomorphism of Theorem 70 induces an isomorphism of the graph

$$Cayley(\Gamma(\tau)/\Gamma; \{\overline{\gamma_L}\}) = \longrightarrow \tau_{\leqslant 1}\left(\Gamma \backslash X_{\cdot}(\mathbb{F}_q(t)_p^d)\right)$$

This isomorphism of graphs is again label-preserving and induces an isomorphism of the corresponding hypergraphs resp. simplicial complexes.

Proof. This follows immediately from Theorem 70. □

4.5 Explicit construction

In this section we fix $p(t) = 1 - t$ where as before $t = \tau^d$. Our goal is a suitable linear factorization of $p(t)$ in $\mathbb{F}_{q^d}\{\tau\}$. More precisely we hvae:

Theorem 72. *There are $x_1, \ldots, x_{d-1}, x_d \in \mathbb{F}_{q^d}$ (provided $x_d = 1$) such that*
(a) For any $i \in \{0, \ldots, d-1\}$ there exist the linearly indepndent set $\{y_{d-i}^{(j)}\}$ (provided $y_d^{(0)} = 1$) with

$$\ker \varphi_{\prod_{j=0}^{i}(1-x_{d-j}^{1-q}\tau)} = \bigoplus_{j=0}^{i} \mathbb{F}_q \cdot y_{d-i}^{(j)}.$$

(b) The following linear factorization of $p(t)$ in $\mathbb{F}_{q^d}\{\tau\}$ holds:

$$p(t) = 1 - t = (1-\tau)(1 - x_{d-1}^{1-q}\tau)\ldots(1 - x_1^{1-q}\tau). \tag{4.6}$$

Recall that the $\mathbb{F}_q$-linear map φ is defined by 4.1.

Proof. Let $i = 0$ then $\ker \varphi_{(1-\tau)} = \mathbb{F}_q \cdot 1$ so $y_d^{(0)} = 1$. Assume (Inductive assumption) now, $\{y_{d-i}^{(j)}\}$ is defined for $1 \leq j \leq i < k$ such that (1) holds. We must determine x_{d-k} and $\ker \varphi_{\prod_{i=0}^{k}(1-x_{d-j}^{1-q}\tau)}$.

$$z \in \ker \varphi_{\prod_{i=0}^{k}(1-x_{d-j}^{1-q}\tau)} \Longleftrightarrow \varphi_{(1-x_{d-k}^{1-q}\tau)} \in \ker \varphi_{\prod_{i=0}^{k-1}(1-x_{d-j}^{1-q}\tau)}$$

by inductive assumption

$$\Longleftrightarrow z - x_{d-k}^{1-q} z^q \in \bigoplus_{j=0}^{i} \mathbb{F}_q \cdot y_{d-i}^{(j)} \ .$$

Let

$$V_k^{\perp} := \{u \in \mathbb{F}_q \mid \mathrm{Tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\frac{y_{d-i}^{(j)}}{u}) = 0,\ 0 \leq j \leq k-1\}$$

and choose a nonzero element $x_{d-k} \in V_k^{\perp}$. We have for any $i \leq k$

$$\varphi_{1-x_{d-k}^{1-q}\tau}(z) = z - x_{d-k}^{1-q} z^q = y_{d-i}^{(j)} \Longleftrightarrow (\frac{z}{x_{d-k}}) - (\frac{z}{x_{d-k}})^q = (\frac{y_{d-i}^{(j)}}{x_{d-k}})^q \ .$$

By Theorem Hilbert 90, (see [28, page 215]), we can find for $0 \leq j \leq k$ $\theta_{d-i}^{(j)}$, (provided $\theta_{d-k}^{(0)} = 1$) such that

$$\theta_{d-i}^{(j)} - (\theta_{d-i}^{(j)})^q = (\frac{y_{d-i}^{(j)}}{x_{d-k}})^q,$$

let $y_{d-k}^{(j)} := x_{d-k}\theta_{d-k}^{(j)}$. Then $y_{d-k}^{(j)} \in \ker \varphi_{\prod_{i=0}^{k-1}(1-x_{d-j}^{1-q}\tau)}$.

Cliam : The set $\{y_{d-k}^{(0)}, y_{d-k}^{(1)}, \ldots, y_{d-k}^{(k)}\}$ is linearly independent.

Proof. Of cliam:

$$\text{Let} \quad \sum_{j=0}^{k} c_j y_{d-k}^{(j)} = 0$$

$$\Longrightarrow \quad \sum_{j=0}^{k} c_j \frac{y_{d-k}^{(j)}}{x_{d-k}} = 0$$

$$\Longrightarrow \quad \sum_{j=0}^{k} c_j \theta_{d-k}^{(j)} = 0 \Longrightarrow (\sum_{j=0}^{k} c_j \theta_{d-k}^{(j)})^q = 0 \Longrightarrow \sum_{j=0}^{k} c_j (\theta_{d-k}^{(j)})^q = 0$$

$$\Longrightarrow \quad \sum_{j=0}^{k} c_j (\theta_{d-k}^{(j)} - (\theta_{d-k}^{(j)})^q) = 0$$

$$\Longrightarrow \quad \sum_{j=1}^{k} c_j \frac{y_{d-k+1}^{(j-1)}}{x_{d-k}} = 0 \Longrightarrow \sum_{j=1}^{k} c_j y_{d-k+1}^{(j-1)} = 0$$

Now by inductive assumption we know that the set $\{y^{(0)}_{d-k+1}, y^{(1)}_{d-k+1}, \ldots, y^{(k-1)}_{d-k+1}\}$ is a linearly independent set over $\mathbb{F}_q$. So

$$c_1 = \ldots c_k = 0 \Longrightarrow c_0 = 0.$$

□

We have by definition $\dim V_j^{\perp} = d - j$. So we find $x_d = 1, x_{d-1}, \ldots, x_1$ and a linearly indepndence set $\{y_1^{(0)} = x_1, y_1^{(1)}, \ldots, y_1^{(d-1)}\}$ such that :

$$\ker \varphi_{\prod_{j=0}^{d-1}(1-x_{d-j}^{1-q}\tau)} = \bigoplus_{j=0}^{d-1} \mathbb{F}_q \cdot y_1^{(j)}.$$

This proves (1).
In order to prove (2), we see that the map φ on the both side of 4.6 has the same kernel (in this case, the kernel is $\mathbb{F}_{q^d}$).
Clearly $\ker \varphi_{1-t} = \mathbb{F}_{q^d}$, and also by (1) we have :

$$\ker \varphi_{(1-\tau)(1-x_{d-1}^{1-q}\tau)\ldots(1-x_1^{1-q}\tau)} = \mathbb{F}_{q^d}.$$

Thus by Proposition 58 there is a non-negative integer m such that

$$1 - t = \tau^m(1 + \sum_{i=1}^{d-1} x_i^{1-q}\tau)(1 - x_{d-1}^{1-q}\tau)\ldots(1 - x_1^{1-q}\tau). \tag{4.7}$$

From the following lemma, all factors in the left side of (4.6) have reduced norm equal to $1 - t$. Take the reduced norm from both sides of (4.7). Then we must have $t^m = 1$, so $m = 0$. Thus the proof of (2) is complete. □

Lemma 73. *For any $x, \in \mathbb{F}_{q^d}^{\times}$ we have:*

$$rn(1 - x^{1-q}\tau) = 1 - t.$$

where rn is the reduced norm defined by 1.1 in Chapter 1, Section 4.

Proof. We have

$$\begin{aligned} \mathrm{rn}(1 - x^{1-q}\tau) &= \det \begin{pmatrix} 1 & -x^{1-q} & & \ldots & 0 \\ 0 & 1 & -x^{q-q^2} & \ldots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -x^{q^{d-2}-q^{d-1}} \\ -tx^{q^{d-1}-q^d} & 0 & 0 & \ldots & 1 \end{pmatrix} \\ &= 1 + (-1)^{d+1}(-1)^d x^{\sum_{i=0}^{d-1}(q^i - q^{i+1})} t \\ &= 1 - t. \end{aligned}$$

□

Definition 74. *For any $i = 0, \ldots, d-1$ let F_i be the set of all*

$$\prod_{j=0}^{i}(1 - x_{d-j}^{1-q}\tau),$$

where $x_d = 1$ and $x_j \in \mathbb{F}_{q^d}$ $j = d-i, \ldots, d-1$ such that there exists $(x_{d-i+1}, \ldots, x_1)$ with

$$(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{d-1}) \in \mathcal{B}_{q,d}^{1-t}.$$

We define the fundamental domain of $p(t) = 1 - t$ as the disjoint union of F_i's and we denote it by $FUND_{1-t}$.

Corollary 75. *There is a bijective correspondence between F_is in the above definition and $Gr_i(\mathbb{F}_q)$ (the Grassmanian of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$).*

Proof. The discussion in the Proof of Theorem 72 makes this bijective correspondence clear. □

Now we return to $\Gamma(\tau)$ as 4.5.

Definition 76. *If in (4.5) $p(t) = 1-t$, then we denote instead $\Gamma(\tau)$, $\Gamma^{1-t}(\tau)$, that is:*

$$\Gamma^{1-t}(\tau) := \ker\left(\Gamma^{1-t}(1) \quad \longrightarrow \quad \mathbb{F}_q\{\{\tau\}\}^{\times}/Z\right)$$

where

$$\Gamma^{1-t}(1) := (\mathbb{F}_{q^d}\{\tau\}[\frac{1}{1-t}])^{\times}/(Z = \mathbb{F}_{q^d}^{\times}) \tag{4.8}$$

Remark. From now on, we shall be working with groups modulo their centers. However, our calculations are made in the groups themselves (by taking arbitrary liftings).

Proposition 77. *Let $\Gamma^{1-t}(\tau)$ as definition (76). Then we have:*
i) $\Gamma^{1-t}(\tau)$ is a torsion free group.
ii) $\Gamma^{1-t}(\tau)$ is a finitely generated group which is generated by the set$FUND_{1-t}$.
iii) The Cayley graph of $\Gamma^{1-t}(\tau)$ with respect to the generator set $FUND_{1-t}$ is isomorphic with the vertex set of Building $X_{\cdot}(\mathbb{F}_q(t)_{1-t}^d)$, i.e.$X_0(\mathbb{F}_q(t)_{1-t}^d \cong PGL(d, \mathbb{F}_q(t)_{1-t}/PGL(d, \mathbb{F}_q[t]_{1-t})$.

Proof. We can see directly that any element of $\Gamma^{1-t}(\tau)$ can be written as $1+\tau^s\mu$ in which μ is not divisible by τ. Thus the group $\Gamma^{1-t}(\tau)$ has torsion elements iff there are positive integers N and m respectively such that:

$$(1+\tau^s\mu)^N = (1-t)^m. \tag{4.9}$$

Thus we have :

$$1 + N\tau^s\mu + \tau^s\mu\tau^s\mu + \ldots = 1 - mt + m(m-1)/2t^2 - \ldots.$$

Now if $s \neq d$ we obtain a contradiction, since μ is not divisible by τ and neither is the coefficient of t in the right-hand side of above equation.
So the only possibility is that $s = d$. That is $(1+\tau^s\mu)$ must be in

$$\Gamma^{(1-t)}(t) := \{\alpha \in \Gamma^{1-t}(\tau) \quad | \quad \alpha \equiv 1 \mod t\}$$

which is by [20, Chapter 5], an analytic torsion free group. This proves i).
We have $\mu \in \Gamma^{1-t}(\tau) \iff rn(\mu) = (1-t)^m$ for some positive integer m. Suppose that s is the maximal power of $(1-t)$ dividing μ; then $\gamma = \mu/(1-t)^s$ is not divisible by $1-t$. But $nr(\gamma) = (1-t)^{m-ds}$, and on the other hand, since $\mathbb{F}_{q^d}\{\tau\}$ is a (left) PID, there exists an element $\theta \in \mathbb{F}_{q^d}\{\tau\}$ which generates the ideal $(\gamma, 1-t)$. So $\gamma = \alpha\theta$, and using associativity (multiplication by a unit on the left) θ can be chosen uniquely. Also since θ divides $1-t$, $nr(\theta)$ divides $(1-t)^m$, which shows that $rn(\theta) = (1-t)^i$ for some $i = 1, \ldots, m$, and φ_θ must have the same kernel as some $\varphi_{\prod_{j=0}^{i}(1-x_{d-j}^{1-q}\tau)}$ (again since θ divide $1-t$ and using Proposition 57 iii)). This shows that by Proposition 58 there exists a non-negative integer n such that $\theta = \tau^n \prod_{j=0}^{i}(1-x_{d-j}^{1-q}\tau)$. Take reduced norm of both sides to see n must be 0. Now we can repeat this discussion with $\gamma/\theta = \alpha$. This completes the proof of ii).
iii) is obtained immediately from Corollary 67. □

Definition 78. *Let $f \in \mathbb{F}_q[t]$ be an irreducible element, different from t and $1-t$. Then we define:*

$$\Gamma_f^{1-t}(\tau) := \{\mu \in \Gamma^{1-t}(\tau) \quad | \quad \mu \equiv 1 \mod f\}$$

We are now ready to present our main results of this Chapter.

Theorem 79. (*Main Explicit Theorem*)
Let $f \in \mathbb{F}_q[t]$ be an irreducible element, different from t and $1-t$. Then the Cayley graph of $\Gamma^{1-t}(\tau)/\Gamma_f^{1-t}(\tau)$ with respect to the $\overline{FUND}_{1-t}$ (the image of $FUND_{1-t}$ in the quotient group $\Gamma^{1-t}(\tau)/\Gamma_f^{1-t}(\tau)$), is a Ramanujan $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraph, where n_i is the number of all i-dimensional sub-vector spaces of $\mathbb{F}_{q^d}$ with the bound $(c_1, \ldots, c_{d-1})$ where $c_i = \binom{d}{i} q^{(d-1)/2}$ for $i = 1, 2, \ldots, d-1$.

Proof. We can consider the quotient group

$$\Gamma^{1-t}(\tau)/\Gamma_f^{1-t}(\tau)$$

as $\Gamma_f^{1-t}(\tau)\backslash\Gamma^{1-t}(\tau)$. By above Proposition 77 *iii)* $\Gamma^{1-t}(\tau)$ can be identified with $X_0(\mathbb{F}_q(t)^d_{1-t})$, so the hyper graph properties come from the building structure on $X_.(\mathbb{F}_q(t)^d_{1-t})$, and finiteness from Corollary 64. The Ramanujan property is an immediate consequence of Theorem 48 if d is a prime number, and Corollary 47 in the Chapter 3. □

Notation :
We denote the hypergraph in Theorem 79 with $Hyp_f(1-t)$. Our goal is, to give a very simple form of $Hyp_f(1-t)$.
In order to the simplify calculations, we assume that $\deg f = dn$ for some positive integer n. We define now the isomorphism :

$$\psi : \Gamma_{1-t}(\tau) \longrightarrow PGL(d, \mathbb{F}_{q^{\deg(f)}}) \tag{4.10}$$

by

$$\psi([\sum_{i=0}^{d-1}\alpha_i\tau^i]) = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1} & \overline{\alpha_2} & \dots & \overline{\alpha_{d-1}} \\ \overline{t\sigma(\alpha_{d-1})} & \overline{\sigma(\alpha_0)} & \overline{\sigma(\alpha_1)} & \dots & \overline{\sigma(\alpha_{d-2})} \\ \overline{t\sigma^2(\alpha_{d-1})} & \overline{t\sigma^2(\alpha_0)} & \overline{\sigma^2(\alpha_1)} & \dots & \overline{\sigma^2(\alpha_{d-2})} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{t\sigma^{d-1}(\alpha_1)} & \overline{t\sigma^{d-1}(\alpha_2)} & \overline{t\sigma^{d-1}(\alpha_3)} & \dots & \overline{\sigma^{d-1}(\alpha_0)} \end{pmatrix}$$

where $[\sum_{i=0}^{d-1}\alpha_i\tau^{i-1}]$ is the image $\sum_{i=0}^{d-1}\alpha_i\tau^{i-1}$ in $\Gamma_{1-t}(\tau)$, and
$\overline{} : \mathbb{F}_q[t] \longrightarrow \mathbb{F}_q[t]/(f) \cong \mathbb{F}_{q^{\deg(f)}}$ is the natural map (which extended simply over constant field). We have

$$\ker\psi = \Gamma_f^{1-t}(\tau)$$

Thus the Cayley graph $Hyp_f(1-t)$ is exactly isomorphic to the Cayley graph of $\psi(\Gamma_{1-t}(\tau))$ with respect to the following generators:

$$\psi(\prod_{j=0}^{i}(1-x_{d-j}^{1-q}\tau)) = \prod_{j=1}^{i}\psi(1-x_{d-j}^{1-q}\tau) \tag{4.11}$$

where for any $\quad i = 1, \dots, d-1$

$$\psi(\varphi_j) = \begin{pmatrix} 1 & -x_{d-i}^{1-q} & & \dots & 0 \\ 0 & 1 & -x_{d-i}^{q-q^2} & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -x_{d-i}^{q^{d-2}-q^{d-1}} \\ -tx_{d-i}^{q^{d-1}-q^d} & 0 & 0 & \dots & 1 \end{pmatrix},$$

for all elements $(x_1, \dots, x_{d-1}) \in \mathcal{B}_{q,d}^{1-t}$.

Definition 80. *Motivated by the classical Legendre symbol, we define here an extended Legendre symbols for arbitrary d and $A, B \in \mathbb{F}_q[t]$ (or in $\mathbb{Z}$) by:*

$$\left(\frac{A}{B}\right)_d := \begin{cases} 1 & \textit{if} \quad X^d \equiv A \mod B \quad \textit{has a solution} \\ -1 & \textit{otherwise} \end{cases} \tag{4.12}$$

Theorem 81. *Let q be a an odd prime power and $f(t)$ be irreducible of degree equal to dn for some positive integer n, and let ψ be as (4.10). Then:*

$$Image(\psi) = \begin{cases} PSL(d, \mathbb{F}_{q^{\deg f}}) & \textit{if} \quad \left(\frac{1-t}{f(t)}\right)_d = 1 \\ PGL(d, \mathbb{F}_{q^d}) & \textit{otherwise} \end{cases}$$

Proof. Set

$$\mathcal{U} := D^{(1)}(\mathbb{F}_q[t]_\infty) \prod_{h \neq t, 1-t} J_h \Gamma'^{(1)}(\tau)$$

where $D^{(1)}$ given by 4.3 and

$$J_f := \ker\left(D^{(1)}(\mathbb{F}_q[t]_h) \quad \longrightarrow \quad D^{(1)}(\mathbb{F}_q[t]_h / f\mathbb{F}_q[t]_f)\right)$$

and

$$J_h := D^{(1)}(\mathbb{F}_q[t]_h) \qquad \text{if} \quad h \neq f$$

and finally

$$\Gamma'^{(1)}(\tau) := \ker\left(\mathbb{F}_{q^d}\{\tau\}[\frac{1}{1-t}]^\times \quad \longrightarrow \quad \mathbb{F}_q\{\{\tau\}\}^\times\right).$$

Define

$$\widetilde{\psi} : \mathcal{U} \quad \longrightarrow \quad \mathrm{PGL}(d, \mathbb{F}_{q^{\deg f}})$$

as composition $\mathrm{Proj}_f \mathrm{mod}_f$ where

$$(\zeta_\infty, \ldots, \zeta_f, \ldots) \quad \mapsto \quad \zeta_f,$$

By Theorem 54 we know that $D = \mathbb{F}_{q^d}(\tau)$ is unramified at f. Thus we have an isomorphism $D_f = D \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q(t)_f \cong \mathrm{M}(d, \mathbb{F}_q(t)_f)$ which takes the maximal order $\mathbb{F}_{q^d}\{\tau\} \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q[t]_f$ to the maximal order $\mathrm{M}(d, \mathbb{F}_q[t]_f)$. So

$$\mathbb{F}_{q^d}\{\tau\} \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q[t]_f / f\mathbb{F}_q[t]_f \cong \mathrm{M}(d, \mathbb{F}_q[t]_f / f\mathbb{F}_q[t]_f)$$

and this induces an isomorphism :

$$\begin{aligned}(\mathbb{F}_{q^d}\{\tau\} \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q[t]_f/f\mathbb{F}_q[t]_f)^\times/Z \quad &\cong \quad \mathrm{PGL}(d, \mathbb{F}_q[t]_f/f\mathbb{F}_q[t]_f)\\ &\cong \quad \mathrm{PGL}(d, \mathbb{F}_{q^{\deg f}})\end{aligned}$$

Thus $\widetilde{\psi}$ acts as projection on the f^{th} component and after reduction modulo f sends this component, under the above isomorphism (which can be exactly our ψ) to the element defined by (4.11). Again applying Theorem 54 we see that

$$D^{(1)}(\mathbb{F}_q[t]_{1-t}) \cong \mathrm{SL}(d, \mathbb{F}_q[t]_{1-t}),$$

and so from Strong approximation Theorem, it follows immediately that

$$D^{(1)}(\mathbb{F}_q[t])D^{(1)}(\mathbb{F}_q[t]_{1-t})$$

is dense in $D^{(1)}(\mathbb{A})$, i.e, for the open set $\mathcal{U}$ (and for all open sets) of $D^{(1)}(\mathbb{A})/D^{(1)}(\mathbb{F}_q[t]_{1-t})$, $D^{(1)}(\mathbb{F}_q[t]) \cap \mathcal{U}$ must be dense in $\mathcal{U}$.
Now since $\widetilde{\psi}$ is a continuous function over $\mathcal{U}$ with a finite range, we must have :

$$\widetilde{\psi}(D^{(1)}(\mathbb{F}_q[t]) \cap \mathcal{U}) = \widetilde{\psi}(\mathcal{U}).$$

We can see directly that

$$D^{(1)}(\mathbb{F}_q[t]) \cap \mathcal{U} = \{\mu \in \Gamma'^{(1)}(\tau) \quad | \quad \mu \equiv 1 \mod f\}.$$

So this shows that

$$\begin{aligned}\widetilde{\psi}(D^{(1)}(\Gamma^{1-t}(\tau))) = \psi(D^{(1)}(\Gamma^{1-t}(\tau))) &\supseteq \widetilde{\psi}(\{\mu \in \Gamma'^{(1)}(\tau) \quad | \quad \mu \equiv 1 \mod f\}\\ &= \mathrm{PSL}(d, \mathbb{F}_{q^{\deg f}})\widetilde{\psi}(\mathcal{U}).\end{aligned}$$

since

$$X \in \mathrm{PSL}(d, \mathbb{F}_{q^{\deg f}}) \quad \text{iff} \quad \left(\frac{\det X}{f(t)}\right)_d = 1.$$

It is an immediate result of the following diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & U_d & \longrightarrow & k^\times & \xrightarrow{x \mapsto x^d} & k^{\times d} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{SL}(d;k) & \longrightarrow & \mathrm{GL}(d;k) & \xrightarrow{\det} & k^\times & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{PSL}(d;k) & \longrightarrow & \mathrm{PGL}(d;k) & \longrightarrow & k^\times/k^{\times d} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $k = \mathbb{F}_{q^{\deg f}}$ and $U_d := \{u \in k \mid u^d = 1\}$. Exactness of columns and rows and commutativity of the diagram (for arbitrary k) is well known. We see by assumption that any generator of $\Gamma^{1-t}(\tau)$ is in $\mathrm{PSL}(d, \mathbb{F}_{q^{\deg f}})$ iff $\left(\frac{1-t}{f(t)}\right)_d = 1$. The other case will be handled exactly as the known case for $d = 2$, see [40, Theorem 4.13]. □

Corollary 82. *Associated to any element* $(x_1, \ldots, x_{d-1}) \in \mathcal{B}_{q,d}^{1-t}$ *are the matrices:*

$$M_{i,\ldots,1} := \prod_{j=1}^{d-1} Q_j \tag{4.13}$$

where for any $j = 1, \ldots, d-1$

$$Q_j = \begin{pmatrix} 1 & -x_{d-j}^{1-q} & & \ldots & 0 \\ 0 & 1 & -x_{d-j}^{q-q^2} & \ldots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -x_{d-j}^{q^{d-2}-q^{d-1}} \\ -tx_{d-j}^{q^{d-1}-q^d} & 0 & 0 & \ldots & 1 \end{pmatrix}$$

$$i) \quad \text{If} \quad \left(\frac{1-t}{f(t)}\right)_d = 1$$

Then $Hyp_f(1-t)$ *is exactly isomorphic to the Cayley graph of* $PSL(d, \mathbb{F}_{q^{\deg f}})$ *with respect to the generators 4.13.*

$$ii) \quad \textit{If} \quad \left(\frac{1-t}{f(t)}\right)_d = -1$$

Then $Hyp_f(1-t)$ *is exactly isomorphic to the Cayley graph of* $PGL(d, \mathbb{F}_{q^{\deg f}})$ *with respect to the generators 4.13.*

Proof. This follows immediately from the above Theorem. □

4.6 Bibliography

[1] J. Arthur and L. Clozel. *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula.* Princton University Press, 1989.

[2] C. Ballantine. Ramanujan type buildings. *Canadian J. Math.* **52***, 1121-1148*, 2000.

[3] C. Ballantine, J.A. Rhods, and T.R. Shemanske. Hecke operators for GL_n and buildings. *http://www.math.dartmouth.edu/t$\tilde{r}$s/*, 2002.

[4] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, 1974.

[5] B. Bollobas. Extermal graph theory. *Academic press,London*, 1978.

[6] A. Harish-Chandra Borel. *Arithmetic subgroups of algebraic groups.* Ann of Math. **75**, 485 – 535., 1962.

[7] Z.I. Borevich and I.R. Shafarevich. *Number Theory.* Academic Press, New York, 1966.

[8] R. Brooks. *Combinatorial problems in spectral geometry.* Springer Lect. Notes in Math. 1201, 14 – 32, 1986.

[9] K. Brown. *Buildings.* Springer-Verlag, New York, 1988.

[10] L. Carlitz. The arithmetic of polynomials in galois field. *Amer. J. Acad. Math.* **54***,* 39 – 50.

[11] L. Carlitz. The arithmetic of polynomials in Galois field. *Proc. Nat. Acad. Sci.* (*USA*) **17** ,120 – 122, 1932.

[12] L. Carlitz. On polynomials in galois field. *Amer. J. Acad. Math.* **38***,* 736 – 744, 1932.

[13] L. Carlitz. On certain functions connected with polynomials in Galois-field. *Duke Math. J.*1*,* 137 – 168, 1935.

[14] L. Carlitz. A set of polynomials. *Duke Math. J.* **6***,* 486 – 504, 1940.

[15] L. Carlitz. Some special functions over GF(q, x). *Duke Math. J.* **27***,* 139 – 158, 1960.

[16] P. Cartier. *Representations of p-adic groups: A survey in : Automorphic forms, representations, and L- functions,.* Proc. Sympos. Pure Math. 33, Part I (eds. A. Borel and W. Casselman), 1979, 11 – 155.

[17] D.I. Cartwright. Harmonic functions on buildings of type $\tilde{A}_n$. *Cambridge: University Press*, 1999.

[18] J.W.S Cassels and A. Fröhlich. *Algebraic number theory.* Academic press, 1965.

[19] L. Clozel and J. Milne. *Automorphic forms, Shimura varieties, and L-functions, vols I and II,* 1990.

[20] J.D. Dixon, M.P.F. Sautoy, A. Mann, and D. Segal. *Analytic Pro-p Groups.* London Math. Soc. LNS **157**. Cambridge University Press, 1991.

[21] V.G. Drinfeld. The proof of Petersson's conjecture for GL(2) over a global field of characteristic p. *Functional Analysis and Appl.* 22*,* 28−43, 1988.

[22] B. Farb and R. K. Dennis. *Noncommutative Algebra.* GTM 144 Springer, 1993.

[23] N. Jacobson. *Finite-dimensional Division algebras over Fields.* Springer, New York, 1996.

[24] B.W. Jordan and R. Livne. The Ramanujan property for regular cubical complexes. *Duke Mathematical Journal* **105***,* 85 – 103, 2000.

[25] P. Katzuko. The langlands conjecture for GL(2), of a local field. *Ann. of Math.* 112*,* 381 – 412, 1980.

[26] L. Lafforgue. Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson. *Asterisque,* (243)*, SMF*, 1997.

[27] T.Y. Lam. *Introduction to noncommutative algebra.* Springer-Verlag, New York, 1986.

[28] S. Lang. *Algebra.* Addison Wessly, 1965.

[29] S. Lang. $SL_2(R)$. Springer Verlag, 1985.

[30] R.P. Langlands. Problems in the theory of automorphic forms. *Lectures in modern analysis and applications III, LNM*(170)*, Springer*, 1970.

[31] R.P. Langlands. *Eisenstein series, SLN* 544. Springer-Verlag, 1976.

[32] G. Laumon, M. Rapoport, and U. Stuhler. $\mathcal{D}$-elliptic sheaves and Langlands correspondence. *Invent. Math* **113***,* 217 – 338, 1993.

[33] A. Lubotzky, R. Philips, and P. Sarnak. Ramanujan conjectures and explicit construction of expanders. *Proc. Symp. on Theo. of Camp. Sci (STOC), 86,* 240 – 246, 1988.

[34] A. Lubotzky, R. Philips, and P. Sarnak. Ramanujan graphs. *Combinatorica* **8**, 261 – 277, 1988.

[35] A. Lubotzky, B. Samuels, and U. Vishne. Explicit construction of ramanujan complexes of type $\tilde{A}_n$. *preprint*, 2003.

[36] A. Lubotzky, B. Samuels, and U. Vishne. Ramanujan complex of type $\tilde{A}_n$. *preprint*, 2003.

[37] A. Lubotzky and B. Weiss. Groups and expanders. *Expanding grapgs DIMACS seris vol.* 10. *Amer. Math. Soc,* 95 – 109, 1999.

[38] I.J. Macdonald. *Spherical functions on a group of p-adic-type.* Ramanujan Inst. Publications, 1971.

[39] C. MacDuffee. *An Introduction to Abstract Algebra. J.Wiley.* J.Wiley, 1940.

[40] M. Morgenstern. Existence and explicit construction of $q + 1$-regular ramanujan graphs for every prime power q. *J.o.Combinatorial Theory, series B* **62**, *44-62*, 1994.

[41] O. L. Ore. On a special class of polynomials. *Trans. Amer. Math. Soc.* **35**, 559 – 584, 1933.

[42] O. L. Ore. Theory of non-commutative polynomials. *Ann. of Math.* **34**, 80 – 508, 1933.

[43] D. Peleg and E. Upfal. Constructing disjoint paths on expander graphs. *Combinatorica*, 9(3):289–313, 1989.

[44] J.D. Rogawski. Automorphic representations of the unitary group in tree variables. *Ann. of Math*, 1990.

[45] M. Ronan. Buildings: Main ideas and applications, i,ii. *Bulletin of the London Mathematical Society* **24**, 1992,$1 - 57, 97 - 126$.

[46] M. Ronan. *Introduction to Buildings.* Addison-Wesley, 1989.

[47] W. Rudin. *Fourier Analysis on Groups.* Interscience Publishers, a division of John Wiley & Sons, New York,.

[48] P. Sarnak. The arithmetic and geometry of some hyperbolic three manifolds. *Acta Math.* 253 – 295, 1983.

[49] P. Sarnak. *Some Applications of Modular Forms.* Cambridge Tracts in Mathematics **99**, Cambridge University Press, 1990.

[50] Sarnak.P, Davidoff, and A Vallete. *Elementary Number Theory, Group Theory, and Ramanujan Graph.* Cambridge University Press, 2003.

[51] I. Satake. *Spherical functions and Ramanujan conjecture.* Algebraic groups and discontinuous subgroups, Proc. Symp. in pure Math. IX, AMS. 258 – 264, 1966.

[52] W. Scharlau. *Quadratic and Hermitian Forms.* Die Grundlehren der Mathematischen Wissenschaften 270, Springer, 1985.

[53] J.P. Serre. *A course on arithmetic.* Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[54] J.P. Serre. *Linear Representations of Finite Groups.* Springer-Verlag, Berlin-Heidelberg-New York,.

[55] J.P. Serre. *Trees.* Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[56] G. Shimura. *Introduction to the arithmetic theory of automorphic functions.* pub of Math Soc of Japan, No. 11, 1971.

[57] U. Stuhler. Homological properties of certain arithmetic group in the function field case. *Invent. math.* 57 *p.* 263 – 281, 1980.

[58] T. Sunada. *Linear Fundamental groups and Laplacians.* Lecture Notes in Math. 1933 Springer-Verlag, Berlin-Heidelberg-New York, 214 – 277.

[59] A. Terras. *Fourier Analysis on Finite Groups and Applications, UCDS Notes,* 1994.

[60] A. Terras. *Are finite upper half plane graphs Ramanujan?* DIMAS 125 – 142, 1994.

[61] J. Tits. *A local approach to building.* In The Geometric Vein (Coxeter Festschrift) Springer-Verlag, New York, 317 – 322, 1981.

[62] M.F. Vigneras. *Arithmétique des Algébres de Quaternions.* Lect. Not. in Math. 800, Springer-Verlag, New York, 1980.

[63] A. Weil. Number of solutions of equations over finite fields. *Bull. Amer.Math, Soc.*55, 497 − 508, 1949.

[64] A. Weil. *Basic Number Theory.* Die Grundlehren der Mathematischen Wissenschaften 144, Springer, 1967.

[65] W.-C W.Li. Ramanujan hypergraph. *Preprint,* 2003.