Characteristics for Dependence in Time Series of Extreme Values

Dissertation
zur Erlangung des mathematisch-naturwissenschaftlichen
Doktorgrades
„Doctor rerum naturalium“
der Georg-August-Universität Göttingen

vorgelegt von

Andree Ehlert
aus Buxtehude

Göttingen 2010

Referent: Prof. Dr. Martin Schlather
Korreferent: PD Dr. Ulf-Rainer Fiebig
Tag der mündlichen Prüfung: 31.08.2010

Acknowledgement

It is a pleasure to thank those who made this thesis possible. Foremost, I would like to show my gratitude to my advisor Prof. Dr. Martin Schlather for his never-ending support and guidance from the initial to the final level. I also would like to thank my co-advisor PD Dr. Ulf-Rainer Fiebig for his unhesitating help. Furthermore, I am deeply grateful to Prof. Dr. Emilio Porcu for his encouragement and support. I would like to gratefully acknowledge financial support from the Centre for Statistics in terms of a Lichtenberg scholarship. Last, but not least, I am indebted to my colleagues at the Institute for Mathematical Stochastics, above all Anja and Michael, for inspiring discussions, endless support and providing a friendly working atmosphere as well as relaxing leisure-time distractions.

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Notation

$[a]$	$\{b \in \mathcal{B}_n : b \sim_h a\}$, page 36
$[\![x]\!]$	$\max\{n \in \mathbb{Z} : n < x\}$, page 49
$\lfloor x \rfloor$	$\max\{n \in \mathbb{Z} : n \leq x\}$, page 31
$\mathbf{1}(\cdot)$	indicator function, page 31
$\prec$	total order, page 58
$\prec_p$	partial order, page 55
$\sim_c$	equivalence relation defining congruence, page 34
$\sim_h$	equivalence relation defining homometry, page 36
$\mathcal{B}_n$	$\{0,1\}^n$, page 32
$\mathcal{C}_n$	$\mathcal{C}_n \subseteq \mathcal{B}_n/{\sim_h}$ representing the convex hull of $\mathcal{F}^*_{n,\mathbb{Z}}$, page 34
$\mathcal{D}_{\iota,n}$	set of functions $d(h \mid M)$, $h \in \mathbb{Z}$, for all $M \in \mathcal{M}_{\iota,n}$, $\iota, n \in \mathbb{N} \cup \{\infty\}$, page 41
$d(h)$	summary measure for extremal dependence, page 38
$\boldsymbol{e}_A$	vector in $\mathbb{R}^D$ with the d-th component equal to one if $d \in A \subseteq \{1, \ldots, D\}$ and zero otherwise, page 9
$(\eta_t)_{t\in\mathbb{Z}}$	sequence of i.i.d. standard normal random variables, page 87
$\mathcal{F}^*_{n,Q}$	$\{f^*_S \in \mathbb{R}^Q : S \in \sigma_n\}$, $n \in \mathbb{N} \cup \{\infty\}$, $Q \subseteq \mathbb{R}$, page 31
f^*_S, f_S	set correlation (covariance) function of S, page 31
$f^*_{[b]}$, $f^*_{I_b}$	restriction of $f^*_{U_b}$, $U_b = \bigcup_{j\in I_b}[j-1, j)$, $b \in \mathcal{B}_n$, to $\mathbb{Z}$, page 32
$\hat{g}_t$, g	spectral functions representing a stationary dissipative max-stable process on $\mathbb{Z}$, page 39

$G(\cdot)$, $\tilde{G}(\cdot)$	multivariate extreme value distributions, page 3
Γ	piston (cf. [14]), page 10
$\tilde{g}_t$	spectral functions representing a max-stable process, page 9
$\mathcal{H}^*_{n,\mathbb{Z}}$	$\left\{f^*_{I_b} \in \mathbb{R}^{\mathbb{Z}}, b \in \mathcal{B}_n\right\}$, page 32
I_b	set of indices corresponding to ones in $b \in \mathcal{B}_n$ (e.g. $I_b = \{1,3,4\}$ for $b = (1,0,1,1)$), page 32
$l(\cdot)$, $\tilde{l}(\cdot)$	stable tail dependence functions, page 8
$\mathcal{M}_\iota$, $\mathcal{M}_{\iota,n}$	set of M_3 processes with $J \leq \iota \in \mathbb{N} \cup \{\infty\}$ (up to range $n \in \mathbb{N} \cup \{\infty\}$), page 41
$\boldsymbol{M}_n$	$(\max_{t=1}^n Y_{t,1}, \ldots, \max_{t=1}^n Y_{t,D})$, page 8
M_S	$\max_{t \in S} X_t$, page 81
M_4 process	multivariate maxima of moving maxima process, page 10
$\mu(\cdot)$, $\tilde{\mu}(\cdot)$	exponent measures, page 8
ϕ	adjusted extremal coefficient, page 9
$\phi(h)$	extremal coefficient function for $h \in \mathbb{Z}$, page 38
$\tilde{\phi}$	extremal coefficient, page 9
$\mathbb{R}^D_+$	$[0,\infty)^D$, page 8
$R(\zeta)$	special class of M_3 processes depending on a weight vector ζ, page 41
r_Y	range of the stationary dissipative max-stable process Y, page 39
$\mathbb{S}_D$	$(D-1)$-dimensional unit simplex, page 9
σ_n	ensemble of all Borel sets $S \subseteq [q, n+q)$ for some $q \in \mathbb{R}$, $n \in \mathbb{N} \cup \{\infty\}$, page 31
S_m, $S_{m,h}$	$\{-m, \ldots, -1\}$ $(\cup\{h\})$, page 81
Θ	set of multivariate extremal index functions, page 14
θ, θ_d	univariate extremal index (of the d-th margin), page 8
$\theta(\boldsymbol{v})$	multivariate extremal index, page 8
$V(X)$	set of vertices representing the convex hull of a set X, page 32

$(X_t)_{t\in\mathbb{Z}}$	stationary stochastic process in the domain of attraction of a max-stable process, page 81
$(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$	multivariate max-stable process on $\mathbb{Z}$, page 8
$(Z_{jt})_{t\in\mathbb{Z},j\in I}$	i.i.d. standard Fréchet sequence, page 10

Chapter 1

Introduction: Extremal Analysis of Stationary Time Series

The key questions in classic extreme value theory concern the behavior of the maximum of n independent and identically distributed random variables $\tilde{X}_t$, i.e.

$$\tilde{M}_n = \max(\tilde{X}_1, \ldots, \tilde{X}_n)$$

for large n. It is well-known that for a wide class of distributions there are suitable normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$, such that, in a nontrivial way,

$$P(\tilde{M}_n \leq a_n x + b_n) \to \tilde{G}(x)$$

giving rise to the class of so-called max-stable distributions. The popular Fisher-Tippett theorem states further that every nondegenerate max-stable distribution belongs to either of only three parametric families, namely the Fréchet, Gumbel or Weibull class. It is a natural question, however, whether a similar result also holds for the more general concept of (strictly) stationary stochastic sequences $(X_t)_{t\in\mathbb{Z}}$ with the same marginal distribution as $\tilde{X}_1$. The latter appears to be the adequate framework for most applications. To name a few examples we may refer to sequences of returns for financial data [35], the fluctuation of daily rainfall amounts [8, 9], or the concentration of ground-level ozone [59]. In either case the extremes are usually linked to a specific underlying event such as a financial crisis or a certain persistent atmospheric condition that causes dependence in the observations as it dominates their behavior for some time. The data plotted in Figure 1, for example, correspond to the daily absolute log-returns of the S&P 500 index from 01.07.97 to 29.06.01 and encompass several such underlying financial shocks, namely the 1997 Asian and 1998 Russian crises as well as the dot-com bubble burst in 2000. Correspondingly, we find from Figure 1 that the respective extreme returns tend to appear in clusters of size two or three. At the same time, however, we may reasonably conclude from economic theory that there is still independence in the long run, i.e. between any two clusters that occur sufficiently far apart (e.g. one year). With respect to the latter finding it turns out that a

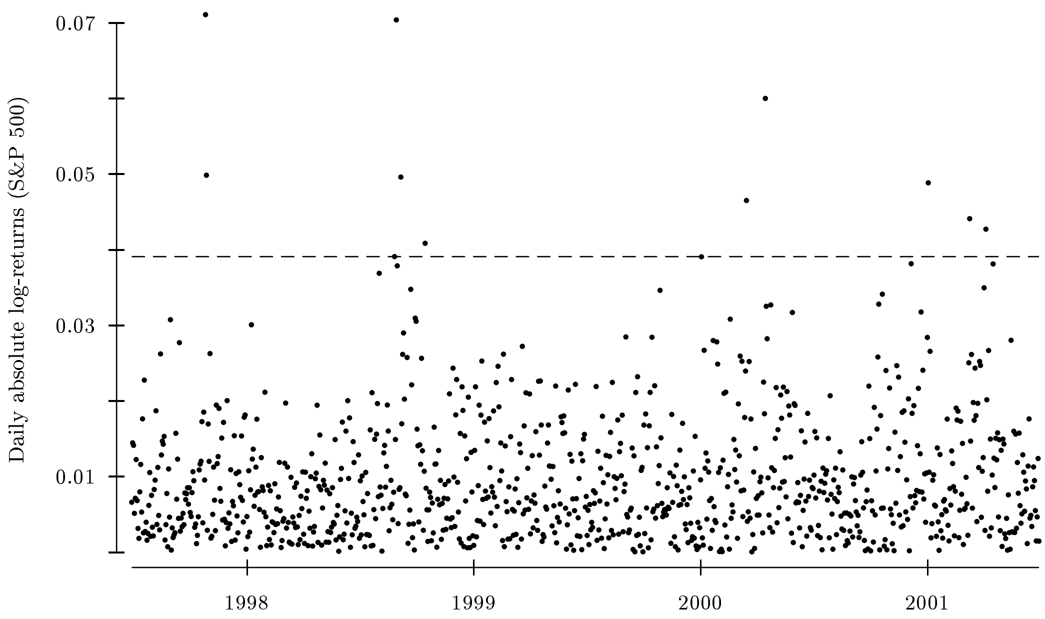

Figure 1: Daily absolute log-returns of the S&P 500 index from 01.07.97 to 29.06.01 ($N = 1009$). The period encompasses the 1997 Asian financial crisis, the 1998 Russian financial crisis as well as the dot-com bubble burst in 2000. The dashed line represents the 99% marginal quantile. The data reflect the stylized fact that extreme financial returns tend to occur in clusters. The year marks indicate the beginning of the respective trading year.

weak long range condition discussed in [31] is sufficient for the possible limit laws of $M_n = \max(X_1, \ldots, X_n)$ to be also necessarily of the nontrivial max-stable form. More precisely, we have that if

$$P(M_n \leq a_n x + b_n) \to G(x)$$

then G is also a max-stable distribution. Most importantly, it is in addition frequently the case that

$$G(x) = \tilde{G}^{\theta}(x)$$

for some $\theta \in [0, 1]$. That is, the results for i.i.d. sequences largely extend to the stationary case, but the parameters of the respective limit distributions will be affected by a single number θ, the so-called extremal index [31]. The latter has become widely accepted as the standard measure for extremal dependence. It allows for several useful interpretations that all roughly characterize the dependence structure in the extremes of the data. In particular, the extremal index reflects the reciprocal of the mean cluster size of extreme events [25]. Due to its influence on the limit distribution for maxima of stationary sequences the extremal index plays a key role in the evaluation of extremal quantiles for dependent data. Moreover, estimates of the average extremal cluster size are a direct matter of interest in numerous applications. For example, with respect to the above S&P 500 data set we get that $\theta \approx 1/3$, see Section 5.3 for details. Note that the resulting limiting mean cluster size of three is roughly in line with a visual inspection of the data. Here, the relevance of the extremal index is straightforward. For example, a financial shock that encompasses three days of successive extremal returns is likely to imply higher economic ruin probabilities than a single day of such large losses. That is, the extremal index is essential to prevent the financial risk from being underestimated. To consider a different context it is well-known that a long-term exposure to high ground-level ozone concentrations can be seriously harmful (and even lethal) in contrast to a short but intensive exposure only. The same is also true for extreme rainfalls on several successive days in contrast to a single day with heavy rain. Given the practical importance of the extremal index it is desirable to have a broad characterization of its behavior. Our first question will therefore concern properties of the extremal index. To this end we shall, however, generalize the above setup to the study of a multivariate D-dimensional stationary sequence where maxima will always be considered componentwise. As before, under a suitable long range independence condition the well-known i.i.d. approach for multivariate extremes [50] generalizes to the stationary case where a so-called multivariate extremal index describes the necessary adjustments to the D-dimensional distribution of the maxima. In particular, we will consider the D-dimensional limits

$$P(M_{n,1} \leq a_{n,1}x_1 + b_{n,1}, \ldots, M_{n,D} \leq a_{n,D}x_D + b_{n,D}) \to G(\boldsymbol{x}),$$
$$P(\tilde{M}_{n,1} \leq a_{n,1}x_1 + b_{n,1}, \ldots, \tilde{M}_{n,D} \leq a_{n,D}x_D + b_{n,D}) \to \tilde{G}(\boldsymbol{x}),$$

where $\boldsymbol{x} \in \mathbb{R}^D$, and as before the independent case carries a tilde. Now, similar to the univariate case we may concentrate on the multivariate extremal index [43]

$$\theta(\boldsymbol{v}) = \frac{\ln G(\boldsymbol{x})}{\ln \tilde{G}(\boldsymbol{x})}$$

for $\boldsymbol{v} \in [\boldsymbol{0}, \boldsymbol{\infty}) \setminus \{\boldsymbol{0}\} \subseteq \mathbb{R}^D$ such that $v_d = -\ln \tilde{G}_d(x_d)$, $d = 1, \ldots, D$. Here, by $\tilde{G}_d$ we denote the d-th margin of the D-variate max-stable distribution $\tilde{G}$. A more detailed setup will be given in Section 2.1. Apparently, the extremal index in the multivariate case turns out to be a function rather than a single number. With respect to the statistical estimation of such a function a detailed description of its behavior is particularly welcome in order to coerce estimates to correspond to the multivariate extremal index of a certain stochastic process. To this end, in Chapter 2 we will discuss an extension of the set of common properties of the multivariate extremal index function. In particular, we will derive sharp bounds for the entire function that appear to be opposed to a former conjecture in [61]. Thereafter, motivated by the above definition of the multivariate extremal index, we will study separately the functions $G(\boldsymbol{x})$ and $\tilde{G}(\boldsymbol{x})$. Regarding the latter, the extremal coefficient [60]

$$\tilde{\phi} = -\ln \tilde{G}\left\{\tilde{G}_1^{\leftarrow}(\exp(-1)), \ldots, \tilde{G}_D^{\leftarrow}(\exp(-1))\right\}$$

has been proposed as a summary measure for the dependence structure of the D-variate marginal distribution $\tilde{G}$. A similar concept will be considered for G. Here, $\tilde{G}_d^{\leftarrow}$, $d =$

$1,\ldots,D$, denote the univariate quantile functions of $\tilde{G}$. Obviously, the characteristics $\theta(\boldsymbol{v})$ and $\tilde{\phi}$ will be strongly related. We will make use of this interrelationship and shall extend a discussion of the mutual properties of $\theta(\boldsymbol{v})$ and $\tilde{\phi}$ started in [36] and [37]. Further, for $D=2$ we will study relatively narrow bounds for valid combinations of $\theta(\boldsymbol{v})$ and the extremal coefficient that are useful e.g. with respect to a consistent simultaneous estimation of such pairs. Most importantly, however, in the sequel when we will leave the D-variate context the extremal coefficients will still play a crucial role. More precisely, we will discuss in Chapters 3 to 5 how the extremal coefficient, an actually multivariate concept, may also be applied to the extremal analysis of univariate time series where we will focus on a so-called extremal coefficient function [56].
The extremal coefficient function is a summary measure that in comparison with the univariate extremal index gives a more detailed description of the extremal clusters as for many problems it is not sufficient to solely address the implications of short range dependence on the distribution of maxima. Under some weak regularity conditions [22] we have

$$\phi(h) = 2 - \lim_{u\to\infty} P(X_h > u \mid X_0 > u), \quad h \in \mathbb{Z}, \tag{1.1}$$

i.e. the extremal coefficient function focusses on the probability of extremes to occur jointly at a certain lag $h \in \mathbb{N}$. It turns out to have an interpretation similar to the usual autocovariance function but for extreme values [56]. In fact, in many applications a number of questions concern all such large (above a certain high threshold) values in a sequence of observations in order to understand the qualitative evolution of a cluster of extremes. For example, assume that in a financial context the average cluster size is two such that, equivalently, $\theta = 0.5$. Now, the grouped extremes may appear e.g. on two subsequent days, or they may as well reflect a different scheme such as a moderate observation on the second day of the cluster in combination with a total cluster duration of three days. The implications for the inherent financial risk may differ substantially between the two scenarios. Here, in contrast to the possible applications for the extremal index discussed above the average number of extreme events that cluster together is not at all a sufficient summary measure. We will discuss such questions in Chapters 3 to 5. In general, when devising adequate cluster characteristics other than the extremal index in order to answer the above questions we will face a tradeoff between the amount of information reflected by the characteristic and its interpretability. A general setup comprising more general cluster functionals is studied in [67]. We shall, however, not follow this approach here and will mainly focus on the abovementioned extremal coefficient function instead. It turns out that with respect to the behavior of valid extremal coefficient functions little is known apart from their positive definite type. In particular, the reconstruction of stochastic example processes from given extremal coefficient functions has not been considered before. We will discuss the latter problem in Chapter 3. First, for the one-dimensional case we will show the equivalence of so-called set correlation functions and the extremal coefficient functions with finite range of dependence on a grid. Note that the rather technical proof of the assertion will be deferred to Chapter 4. The above equivalence will then be useful in order to

determine the set of vertices for the convex set of extremal coefficient functions. This will allow for the construction of simple max-stable processes complying with a given extremal coefficient function and, in addition, will highlight further properties of the latter. We will include an application of this approach and discuss several examples. Further, as to processes with infinite range we will consider a natural extension of the term "long memory" that is well-known in the Gaussian framework to max-stable processes. We will also address the implications of a fixed extremal coefficient function at a certain lag $h \in \mathbb{N}$ on the allowable range of dependence for the underlying process.
As mentioned above Chapter 4 will then be devoted to a constructive proof of equivalence for set correlation and extremal coefficient functions. Apart from the mere theoretical result we will in particular be able to assign the well-known properties of set correlations to the extremal coefficient functions. For example, with respect to the abovementioned desired characterization of the cluster structure it will be easy to show from the set correlations that the extremal coefficient function is unable to distinguish between a certain class of simple cluster types. The problem of such homometric patterns is well-known e.g. in the field of crystallography [46, 47], and will be studied in the extreme value context here.
We will further discuss the implications of the above shortcomings on possible applications of the extremal coefficient function in Chapter 5. This leads us to propose an alternative characteristic that we shall at this point define only tentatively for all $h \in \mathbb{N}$ by

$$\gamma(h) = \lim_{u\to\infty} P(X_h > u \mid X_0 > u, \text{ and } X_0 \text{ first event in the extremal cluster}). \tag{1.2}$$

We will show that in many applications it has a more suitable interpretation and that its properties are often easier to handle in comparison with the extremal coefficient function. In particular, it characterizes the dependence structure of two extremes given that the first observation corresponds to the onset of an extremal cluster. Our focus on the first event in such a cluster is motivated by the fact that e.g. in financial applications the outset of a stress period is in general the point to take adequate measures based on such conditional predictions whereas the extremal coefficient function is not explicitly linked to the beginning of a cluster. We will illustrate the different interpretation of the above characteristics and study some of their general properties. To conclude, an evaluation of our new cluster characteristic, the extremal coefficient function and the extremal index for max-stable as well as the important class of GARCH(1,1) processes will be discussed. To this end, we shall modify a tail chain approach proposed by [57]. Interestingly, the evaluation of $\gamma(h)$ will require the entire framework of [57], i.e. a forward and backward tail chain. This is in contrast to the related analysis of the extremal coefficient function where in principal the forward chain is sufficient [22]. At the same time, with respect to [30] our more general approach yields a simplified algorithm for the evaluation of the extremal index in the GARCH(1,1) case. We will include an example for a GARCH(1,1) model fitted to the S&P 500 data set as well as a small simulation study comprising different GARCH(1,1) parameters.

In order to make the text easier to read we will in general introduce any specific notation chapterwise, i.e. where it first appears. In addition, the most important notational conventions that shall be used throughout are separately summarized above. We will only exceptionally deviate from this setup where it is necessary, e.g. in the closed context of longer proofs. Finally, note that in place of a more detailed introduction at this point we decided to commence each chapter with an outline of the context and a more formal setup.

Chapter 2

The Multivariate Extremal Index

2.1 Multivariate Extremes

The study of componentwise maxima for independent copies of stationary processes on $\mathbb{R}^D$ is a natural question arising in extreme value theory. Its relevance to practice is indicated by numerous applications to extremal phenomena in the environmental or financial context, see e.g. [56, 10, 22]. In theory, the family of limiting processes that emerges from the above setup is fully characterized by the so-called class of max-stable processes that can be seen as infinite dimensional extensions of the multivariate extreme value distributions discussed in Chapter 1. As the latter fail to be of a finite parametric nature particular models for max-stable processes have become a major matter of interest. In this regard we may mention the seminal paper by [60], the extensive class of M_4 processes discussed by [61], and [54] for the spatial case. In Chapter 1 we discussed informally the extremal index as the key parameter to capture the effect of temporal dependence on the limiting distribution of maxima. Recall that an intuitive interpretation of the extremal index emphasizing its relevance to practice is based on its reciprocal value which corresponds to the mean cluster size of extremes of the sequence [25]. In the following we will be concerned with a multivariate generalization of this concept. Then, the corresponding interpretation for the multivariate extremal index is the reciprocal mean cluster size of a univariate sequence that, for each point in time, is given as the maximum of the weighted marginal sequences [61]. That is, the multivariate extremal index is a function of weights comprising each of the respective univariate extremal indices as a special case. This concept will be made precise below. However, the average cluster size for arbitrary weights can, in general, not be determined by knowledge of the univariate extremal indices alone. Given only the latter, the behavior of valid multivariate extremal index functions is therefore an important matter of interest. To be specific, we will consider D-variate, stationary max-stable processes $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}} = \{\boldsymbol{Y}_t = (Y_{t,1}, \ldots, Y_{t,D}),\ t \in \mathbb{Z}\}$, i.e.

$$\begin{aligned} &P^n(Y_{td} \leq ny_{td},\ t = 1, \ldots, k,\ d = 1, \ldots, D) \\ &\quad = P(Y_{td} \leq y_{td},\ t = 1, \ldots, k,\ d = 1, \ldots, D) \end{aligned} \tag{2.1}$$

for all $k, n \in \mathbb{N}$ and $y_{td} \geq 0$. Here, we may assume without loss of generality that the univariate marginal distribution functions F_d are standard Fréchet, i.e. $F_d(y_{0,d}) = \exp\{-y_{0,d}^{-1}\}$ for $y_{0,d} > 0$, and $F_d(y_{0,d}) = 0$, else, $d = 1, \ldots, D$. Let $(\tilde{\boldsymbol{Y}}_t)_{t\in\mathbb{Z}} = \{\tilde{\boldsymbol{Y}}_t = (\tilde{Y}_{t,1}, \ldots, \tilde{Y}_{t,D}),\ t \in \mathbb{Z}\}$ be the associated D-variate sequence of i.i.d. random vectors with the same marginal distribution and let $\boldsymbol{M}_n = (\max_{t=1}^n Y_{t,1}, \ldots, \max_{t=1}^n Y_{t,D})$, and $\tilde{\boldsymbol{M}}_n$ similarly, denote the sequences of componentwise maxima. Then, for any $\boldsymbol{y} = (y_1, \ldots, y_D) \in \mathbb{R}_+^D$ and $[\boldsymbol{0}, \boldsymbol{y}]^c = [\boldsymbol{0}, \boldsymbol{\infty}] \setminus [\boldsymbol{0}, \boldsymbol{y}]$ we have by Theorem 3.1 in [52], Proposition 2.1 of [61] and a tightness argument that

$$\begin{aligned} &\lim_{n\to\infty} P(n^{-1}\boldsymbol{M}_n \leq \boldsymbol{y}) = \exp\{-\mu([\boldsymbol{0}, \boldsymbol{y}]^c)\} = G(\boldsymbol{y}), \\ &\lim_{n\to\infty} P(n^{-1}\tilde{\boldsymbol{M}}_{\boldsymbol{n}} \leq \boldsymbol{y}) = \exp\{-\tilde{\mu}([\boldsymbol{0}, \boldsymbol{y}]^c)\} = \tilde{G}(\boldsymbol{y}) = P(\boldsymbol{Y}_1 \leq \boldsymbol{y}), \end{aligned} \tag{2.2}$$

where $\mu(\cdot)$ and $\tilde{\mu}(\cdot)$ denote the exponent measures as in [50]. Then, for $\boldsymbol{v} \in [0, \boldsymbol{\infty}) \setminus \{\boldsymbol{0}\} \subseteq \mathbb{R}_+^D$, the function

$$\theta(\boldsymbol{v}) = \frac{\mu([\boldsymbol{0}, \boldsymbol{v}^{-1}]^c)}{\tilde{\mu}([\boldsymbol{0}, \boldsymbol{v}^{-1}]^c)} \tag{2.3}$$

introduced by [43], is called the multivariate extremal index. Here, the expression $\boldsymbol{v}^{-1}$ is to be understood componentwise. For $D = 1$ the quotient of the exponent measures reduces to the well-known univariate extremal index $\theta \in (0, 1]$. Note that throughout we will in general exclude the special case $\theta = 0$ that is of limited practical interest, see [32] for a discussion. We will accordingly denote by θ_d the univariate extremal index of the d-th sequence $\{Y_{td}, t \in \mathbb{Z}\}$. As mentioned above, $\theta(\boldsymbol{v})$ is the univariate extremal index of the series $\{\max_d v_d Y_{td}, t \in \mathbb{Z}\}$ [61, Proposition 2.1].
In the following let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_D)$, and let $\boldsymbol{\theta v} = (\theta_1 v_1, \ldots, \theta_D v_D)$ involve the componentwise multiplication. Then,

$$\theta(\boldsymbol{v}) = \frac{l(\boldsymbol{\theta v})}{\tilde{l}(\boldsymbol{v})}, \tag{2.4}$$

where l and $\tilde{l}$ are the two stable tail dependence functions [26],

$$\begin{aligned} &l(\boldsymbol{z}^{-1}) = \mu([\boldsymbol{0}, \boldsymbol{y}]^c), \quad z_d = -(\ln G_d(y_d))^{-1}, \\ &\tilde{l}(\tilde{\boldsymbol{z}}^{-1}) = \tilde{\mu}([\boldsymbol{0}, \boldsymbol{y}]^c), \quad \tilde{z}_d = -(\ln \tilde{G}_d(y_d))^{-1} = y_d, \quad d = 1, \ldots, D. \end{aligned} \tag{2.5}$$

Up to now we are aware of five known properties characterizing $\theta(\boldsymbol{v})$, cf. [2, 43, 48, 61]:

(T1) $\theta(\boldsymbol{v})$ is continuous in $\boldsymbol{v}$,

(T2) $\theta(c\boldsymbol{v}) = \theta(\boldsymbol{v})$, for any constant $c > 0$,

(T3) $\theta(\boldsymbol{e}_d) = \theta_d$, where $\boldsymbol{e}_d$ is the dth unit vector,

(T4) $0 \leq \theta(\boldsymbol{v}) \leq 1$, i.e. $l(\boldsymbol{\theta v}) \leq \tilde{l}(\boldsymbol{v})$,

(T5) $\theta_d > 0$ for all $d = 1, \ldots, D$ iff $\theta(\boldsymbol{v}) > 0$ for all $\boldsymbol{v} \in [\boldsymbol{0}, \boldsymbol{\infty}) \setminus \{\boldsymbol{0}\}$.

By property (T2) we may in particular confine our analysis to the $(D-1)$-dimensional unit simplex $\mathbb{S}_D = \{\boldsymbol{v} \in [\boldsymbol{0}, \boldsymbol{1}] : \|\boldsymbol{v}\|_1 = v_1 + \ldots + v_D = 1\}$, and we shall refer to the restriction of l and $\tilde{l}$ to $\mathbb{S}_D$ as (Pickands) dependence functions, cf. [49] and [2]. We will frequently make use of the following properties [2]:

(L1) $l_{\min}(\boldsymbol{v}) = \max\{v_1, \ldots, v_D\} \leq l(\boldsymbol{v}) \leq l_{\max}(\boldsymbol{v}) = \sum_{d=1}^{D} v_d$,

(L2) $l(\boldsymbol{v})$ is convex,

(L3) $l(c\boldsymbol{v}) = cl(\boldsymbol{v})$, for any constant $c > 0$,

where $\boldsymbol{v} \in [\boldsymbol{0}, \boldsymbol{\infty}) \setminus \{\boldsymbol{0}\}$, and $l_{\min}$ and $l_{\max}$ are also valid dependence functions. For later reference, let A be a subset of $\{1, \ldots, D\}$ and let $\boldsymbol{e}_A$ be a vector in $\mathbb{R}^D$ with the d-th component equal to one if $d \in A$ and zero otherwise. Let $\boldsymbol{1} = \boldsymbol{e}_{\{1,\ldots,D\}}$. Note that the properties (T1) to (T5) above are not sufficient to characterize the function $\theta(\boldsymbol{v})$ completely. As a step towards a better understanding of the multivariate extremal index it will be one of our main results to refine property (T4). In addition to the conjecture in [61] of $l(\boldsymbol{\theta v}) \leq \tilde{l}(\boldsymbol{v})$ to be the only restriction on the two dependence functions we will show further constraints in Section 2.3 which, equivalently, correspond to improved bounds for the function $\theta(\boldsymbol{v})$ given only marginal dependence in terms of θ_d, $d = 1, \ldots, D$. In Section 2.4 the extremal coefficient, $\tilde{\phi} = \tilde{l}(\boldsymbol{1})$, a well-known summary measure for $\tilde{\mu}([\boldsymbol{0}, \boldsymbol{x}]^c)$, will be related to the multivariate extremal index, cf. [60] and [56]. We will first discuss an obvious connection between the univariate extremal indices and the extremal coefficient and give an improved upper bound for the dependence adjusted extremal coefficient, $\phi = l(\boldsymbol{\theta 1})$, a counterpart of $\tilde{\phi}$ that applies to stationary sequences, see [36]. In the main, however, we will concentrate on the fact that $\theta(\boldsymbol{1}) = \phi/\tilde{\phi}$, and show that knowledge of $\tilde{\phi}$ or ϕ, respectively, allows for a significant improvement of the unrestricted bounds for $\theta(\boldsymbol{v})$ considered in Section 2.3. Throughout the chapter we will discuss various example processes.

2.2 Properties of the Multivariate Extremal Index

By [14] a D-dimensional process $(\boldsymbol{Y}_t)_{t\in\mathbb{Z}} = \{\boldsymbol{Y}_t = (Y_{t,1}, \ldots, Y_{t,D}),\ t \in \mathbb{Z}\}$ is max-stable with standard Fréchet margins if and only if

$$Y_{td} = \max_{i\in\mathbb{N}} \tilde{g}_{td}(S_i)U_i, \quad t \in \mathbb{Z},\ d = 1, \ldots, D, \tag{2.6}$$

where $\{U_i, S_i\}_{i=1}^{\infty}$ is a Poisson point process on $\mathbb{R}_+ \times [0,1]$ with intensity $\mathrm{d}u/u^2 \times \mathrm{d}s$, and $\{\tilde{g}_{td}\}_{t\in\mathbb{Z}}$, $d = 1, \ldots, D$, are sequences of nonnegative deterministic spectral functions with $\int_0^1 \tilde{g}_{td}(s)ds = 1$ for all t. Replacing $\{\tilde{g}_t\} \sim \{\tilde{g}_{t+1}\}$ by

$$\{(h_{D(t-1)+1}, \ldots, h_{(Dt)})\} \sim \{(h_{D(t-1+k)+1}, \ldots, h_{D(t+k)})\}$$

in Theorem 5.1 of [14] with $h_{D(t-1)+d} := \tilde{g}_{td}$ gives the spectral representation for stationary D-variate max-stable processes.

Theorem 2.2.1 ([14, Theorem 5.1]). *The elements of a stationary max-stable D-variate process* $(\boldsymbol{Y}_t)$ *are representable through (2.6) with the proper sequence* $\{\tilde{g}_t = (\tilde{g}_{t,1}, \ldots, \tilde{g}_{t,D})\}_{t\in\mathbb{Z}}$. *There exists a piston* Γ *such that* $\tilde{g}_{t+1} \equiv \Gamma(\tilde{g}_t)$.

In order to state a corresponding expression for the multivariate extremal index it is convenient to refer to the following concepts from the literature. Let $[0,1] = S_1 \cup S_2$ be the Hopf decomposition for (2.6) into the dissipative and the conservative part [29], where S_1 is isomorphic to $S_0 \times \mathbb{Z}$ for some measurable set $S_0 \subset S_1$. Theorem 3.1 in [52] states that the extremal index θ_d of the dth component is given by

$$\theta_d = \int_{S_0} \max_{t\in\mathbb{Z}} \tilde{g}_{td}(s)ds.$$

Similarly, by means of Proposition 2.1 in [61] we may conclude that the multivariate extremal index equals

$$\theta(\boldsymbol{v}) = \frac{\int_{S_0} \max_{t\in\mathbb{Z}} \max_{d=1}^{D} v_d \tilde{g}_{td}(s)ds}{\int_{S_0} \sum_{t\in\mathbb{Z}} \max_{d=1}^{D} v_d \tilde{g}_{td}(s)ds + \int_{S_2} \max_{d=1}^{D} v_d \tilde{g}_{0,d}(s)ds}, \quad \boldsymbol{v} \in \mathbb{S}_D.$$

Theorem 2.2.2 ([17, Theorem 2]). *The set of extremal index functions is closed under uniform convergence.*

In the remainder we shall follow the ideas of [16] and [61] who consider discrete and stationary versions of (2.6) given by the decomposition

$$Y_{td} = \max\{M_{td}, S_{td}\}, \quad t \in \mathbb{Z},\, d = 1, \ldots, D, \tag{2.7}$$

where

$$M_{td} = \max_{j\in I} \max_{k\in\mathbb{Z}} a_{jkd} Z_{j,t-k},$$
$$S_{td} = \max_{j\in F} \max_{0\le k\le N_j} \alpha_{jkd} Z^*_{j,t-k}.$$

Here, $I, F \subseteq \mathbb{N} \cup \{0\}$, $\{Z_{jt},\, t \in \mathbb{Z},\, j \in I\}$ and $\{Z^*_{jt},\, 0 \le t \le N_j < \infty,\, j \in F\}$ are independent sequences of i.i.d. standard Fréchet variables where $Z^*_{jt} = Z^*_{j,t+m(N_j+1)}$, $m \in \mathbb{Z}$. The constants a_{jkd}, $j \in I$, $k \in \mathbb{Z}$, and α_{jkd}, $j \in F$, $0 \le k \le N_j$ are non-negative with $\sum_j \sum_k a_{jkd} + \sum_j \sum_k \alpha_{jkd} = 1$ for $d = 1, \ldots, D$. Note that the S_{td} part of the process (2.7) consists of periodic elements and leads to non-ergodic processes [62] whereas the mixing component M_{td} corresponds to the M_4 class of multivariate mixed moving maxima discussed by [61]. Later, beginning with Chapter 3, we will restrict our analysis to the case $D = 1$ and a corresponding class of M_3 processes, cf. (3.28). Note that the latter correspond to the class of so-called dissipative stationary max-stable processes [28]. It will be essential in the following that for the dependence functions of the process (2.7) we get that

$$l(\boldsymbol{\theta v}) = \sum_{j\in I} \max_{k\in\mathbb{Z}} \max_{d=1,\ldots,D} a_{jkd} v_d,$$
$$\tilde{l}(\boldsymbol{v}) = \sum_{j\in I} \sum_{k\in\mathbb{Z}} \max_{d=1,\ldots,D} a_{jkd} v_d + \sum_{j\in F} \sum_{0\le k\le N_j} \max_{d=1,\ldots,D} \alpha_{jkd} v_d.$$

The following proposition states that the results upon the multivariate extremal index for M_4 processes may be generalized to hold for stationary max-stable processes. Moreover, under the conditions of [61, Theorem 2.3] the results obtained in Sections 2.3 and 2.4 hold true also for general stationary processes in the maximum domain of attraction of a max-stable process.

Proposition 2.2.1 ([17, Proposition 1])**.** *The multivariate extremal index of a D-variate stationary max-stable process* $(\boldsymbol{Y}_t)$ *may be approximated uniformly by the multivariate extremal index of an* M_4 *process.*

Whenever we will define in the following a process as in (2.7) or an M_4 process by its coefficients we will tacitly assume all coefficients not explicitly defined to be zero.

2.3 Bounds for the Multivariate Extremal Index

We will now turn to the question of the interdependencies between the two dependence functions l and $\tilde{l}$, and their implications for valid functions $\theta(\boldsymbol{v})$. From the definition of the multivariate extremal index it merely follows that $l(\boldsymbol{\theta v}) \leq \tilde{l}(\boldsymbol{v})$, cf. the conjecture in [61]. We state the following counterexample in order to demonstrate that $l(\boldsymbol{\theta v}) \leq \tilde{l}(\boldsymbol{v})$ is not a sufficient condition for $l(\boldsymbol{\theta v})/\tilde{l}(\boldsymbol{v})$ to serve as a valid multivariate extremal index.

Example 2.3.1. Consider an M_4 process with $D = 2$, $I = 2$ and $\theta_1 = \theta_2 = 0.5$. The condition $l(\boldsymbol{\theta v})/\tilde{l}(\boldsymbol{v}) \leq 1$ allows to fix $\tilde{\phi} = \tilde{l}(\mathbf{1}) = \sum_j \sum_k \max_d a_{jkd} = 1$, say, which is equivalent to $a_{jk1} = a_{jk2}$ for all j, k, see also Corollary 2.4.1 below. Then, it necessarily follows that

$$l(\boldsymbol{\theta v}) = \sum_j \max_k \max_d a_{jkd} v_d = 0.5 \max_d v_d$$

obviously further restricting the requirement $l(\boldsymbol{\theta v})/\tilde{l}(\boldsymbol{v}) \leq 1$. Accordingly, $\theta_1 = \theta_2 = 0.5$ is incompatible with $\theta(\mathbf{1}) = 1$, for example. See Figure 2.3.1 for a sketch of the valid bounds for $\theta(\mathbf{1})$ that are easily derived from the results discussed below.

For a more detailed understanding of the reasoning in Example 2.3.1 we shall introduce a decomposition for the dependence function $\tilde{l}$ in the following theorem and state simple but important properties that we shall use repeatedly in the rest of the chapter.

Lemma 2.3.1. *Let* $\widehat{\sum_k} a_k = \sum_k a_k - \max_k a_k$ *for any sequence of nonnegative constants* a_k *and assume that* $\max_k a_k$ *exists. Then, for any* $a_{kd} \in \mathbb{R}$

$$\max_d \widehat{\sum_k} a_{kd} \leq \widehat{\sum_k} \max_d a_{kd} \,.$$

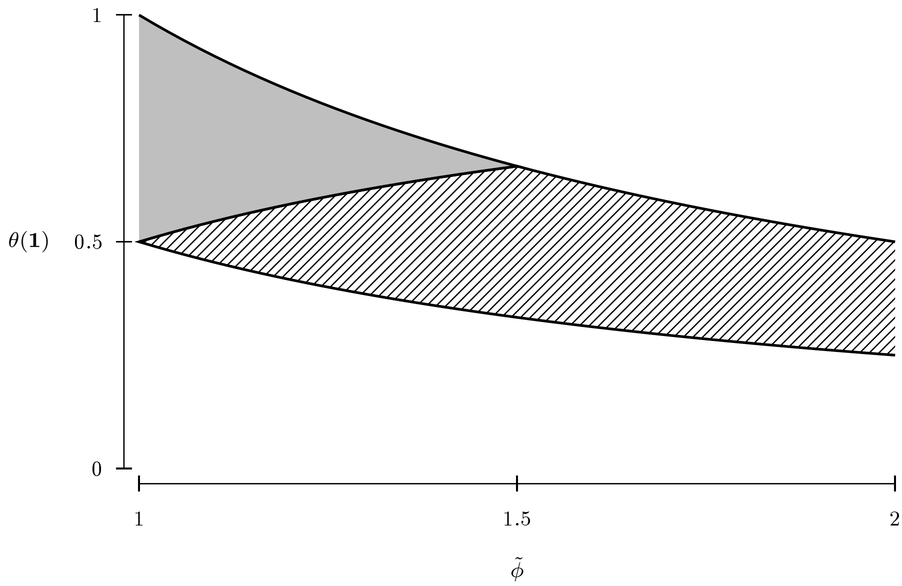

Figure 2.3.1: Bounds for $\theta(\mathbf{1})$ with $\theta_1 = \theta_2 = 0.5$ as functions of the extremal coefficient, see Example 1. Lined: Valid combinations. Gray: Invalid combinations, but consistent with the bound $l(\boldsymbol{\theta v}) \leq \tilde{l}(\boldsymbol{v})$ given in [61].

Proof. For any fixed $d^* \in \{1, \dots, D\}$ let $\varepsilon = \sum_k \{\max_d a_{kd} - a_{kd^*}\}$. Now, $\varepsilon \geq \max_k \max_d a_{kd} - \max_k a_{kd^*}$, and

$$\sum_k a_{kd^*} + \max_k \max_d a_{kd} - \max_k a_{kd^*} \leq \sum_k \max_d a_{kd}, \quad \text{for all } d^* \in \{1, \dots, D\}.$$

□

Theorem 2.3.1. *Let an arbitrary process as in (2.7) with extremal indices* $\theta_1, \dots, \theta_D$ *be given by the coefficients* a_{jkd}, $j \in I$, $k \in \mathbb{Z}$, $d = 1, \dots, D$, *and* α_{jkd}, $j \in F$, $0 \leq k \leq N_j$, $d = 1, \dots, D$. *For all* j, d *let* $a^*_{jkd} = 0$ *for all* k *except one* $k = k(j, d) \in \arg\max_k a_{jkd}$ *where* $a^*_{jkd} = \max_k a_{jkd}$, *and* $\hat{a}_{jkd} = 0$ *for* $k = k(j, d)$ *and* $\hat{a}_{jkd} = a_{jkd}$ *otherwise, i.e.* $\hat{a}_{jkd} = a_{jkd} - a^*_{jkd}$. *Define* $\tilde{l}_\theta(\boldsymbol{v}) = \sum_j \sum_k \max_d a^*_{jkd} v_d$ *and* $\tilde{l}_{1-\theta}(\boldsymbol{v}) = \sum_j \sum_k \max_d \hat{a}_{jkd} v_d + \sum_j \sum_k \max_d \alpha_{jkd} v_d$.

(i) *The functions* $\tilde{l}_\theta$ *and* $\tilde{l}_{1-\theta}$ *are valid dependence functions with sharp upper and lower bounds given by* $\tilde{l}_{\theta,\min}(\boldsymbol{v}) = l_{\min}(\boldsymbol{\theta v}) = \max_d \theta_d v_d$, $\tilde{l}_{\theta,\max}(\boldsymbol{v}) = l_{\max}(\boldsymbol{\theta v}) = \sum_d \theta_d v_d$, $\tilde{l}_{1-\theta,\min}(\boldsymbol{v}) = \max_d (1 - \theta_d) v_d$, *and* $\tilde{l}_{1-\theta,\max}(\boldsymbol{v}) = \sum_d (1 - \theta_d) v_d$.

(ii) *It holds that*

$$\tilde{l}_{\theta,\min}(\boldsymbol{v}) \leq l(\boldsymbol{\theta v}) \leq \tilde{l}_\theta(\boldsymbol{v}), \tag{2.8}$$

$$l(\boldsymbol{\theta v}) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v}) \leq \tilde{l}(\boldsymbol{v}) \leq \tilde{l}_\theta(\boldsymbol{v}) + \tilde{l}_{1-\theta}(\boldsymbol{v}), \tag{2.9}$$

where equality applies for the last inequality if and only if for all $j \in I$ *and* $d = 1, \dots, D$ *we have* $\hat{a}_{jkd} = 0$ *for all* $k \in \{k(j, 1), \dots, k(j, D)\}$.

(iii) For any function $l(\boldsymbol{\theta v})$ a corresponding M_4 process exists such that $\tilde{l}(\boldsymbol{v}) = 1$, $\boldsymbol{v} \in \mathbb{S}_D$.

Proof. (i) From $\tilde{l}_\theta(\boldsymbol{\theta}^{-1}\boldsymbol{v}) = \sum_j \sum_k \max_d a^*_{jkd} v_d / \theta_d$ it follows that the function $\tilde{l}_\theta$ is a dependence function. Analogously for $\tilde{l}_{1-\theta}$. Now, the assertion follows from (L1).
To proof (ii) note that (2.8) follows directly from (i) and the respective definitions. Concerning the left hand side of (2.9), we get by (i) and Lemma 2.3.1 that for all $\boldsymbol{v} \in \mathbb{S}_D$

$$
\begin{aligned}
& l(\boldsymbol{\theta v}) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v}) \\
&= l(\boldsymbol{\theta v}) + \max_d \left\{ \sum_j \sum_k \alpha_{jkd} v_d + \sum_j \sum_k a_{jkd} v_d - \sum_j \max_k a_{jkd} v_d \right\} \\
&\le l(\boldsymbol{\theta v}) + \max_d \left\{ \sum_j \sum_k \alpha_{jkd} v_d \right\} + \max_d \left\{ \sum_j \widehat{\sum}_k a_{jkd} v_d \right\} \\
&\le l(\boldsymbol{\theta v}) + \sum_j \sum_k \max_d \alpha_{jkd} v_d + \sum_j \sum_k \max_d a_{jkd} v_d - \sum_j \max_k \max_d a_{jkd} v_d \\
&= \sum_j \sum_k \max_d a_{jkd} v_d + \sum_j \sum_k \max_d \alpha_{jkd} v_d = \tilde{l}(\boldsymbol{v}).
\end{aligned}
$$

Finally, the right hand side of (2.9) follows from the fact that $\max_d a^*_{jkd} v_d + \max_d \hat{a}_{jkd} v_d \ge \max_d (a^*_{jkd} + \hat{a}_{jkd}) v_d$. Equality holds for all $\boldsymbol{v} \in \mathbb{S}_D$ if and only if $\hat{a}_{jkd} = 0$ for all $k \in \{k(j,1), \ldots, k(j,D)\}$.
With respect to (iii) we have that the swapping of the values of $a_{j,k_1,d}$ and $a_{j,k_2,d}$ does not change l for any j, k_1, k_2, d, so that we may assume for all j and k that $a^*_{jkd} \neq 0$ for at most one value of d. Then,

$$
\tilde{l}_\theta(\boldsymbol{v}) = \sum_j \sum_k \max_d a^*_{jkd} v_d = \sum_j \sum_d a^*_{j,k(j,d),d} v_d = \sum_d \theta_d v_d.
$$

Further, for all j and k, let the $\hat{a}_{jkd}$ be such that $\hat{a}_{jkd} \neq 0$ for at most one value of d, and $\hat{a}_{jkd} = 0$ for all j, d and $k \in \{k(j,1), \ldots, k(j,D)\}$. Then, $\tilde{l}_{1-\theta}(\boldsymbol{v}) = \sum_d (1-\theta_d) v_d$ by the above argumentation. Finally, by (ii) we have $\tilde{l}(\boldsymbol{v}) = \tilde{l}_\theta(\boldsymbol{v}) + \tilde{l}_{1-\theta}(\boldsymbol{v}) = 1$, $\boldsymbol{v} \in \mathbb{S}_D$. ☐

Now, the incompatibility of $\theta_1 = \theta_2 = 0.5$ and $\theta(\mathbf{1}) = 1$, i.e. $l(\boldsymbol{\theta}\mathbf{1}) = \tilde{l}(\mathbf{1})$, in Example 2.3.1 follows also immediately from (2.9). There, we find that $\theta_d = 1$, $d = 1, \ldots, D$, or equivalently, $\tilde{l}_{1-\theta,\min}(\mathbf{1}) = 0$ is a necessary condition for $\theta(\mathbf{1}) = 1$. In addition to Theorem 2.3.1 (i) and (ii) see also [61] where a special case for which $\tilde{l}(\boldsymbol{v}) - l(\boldsymbol{\theta v})$ is convex is discussed. There, using the notation of Theorem 2.3.1, for $D = 2$ a process with $k(j,1) = k(j,2) = 0$, $j \in I$, is considered. Then, by Theorem 2.3.1 (ii) we have that $\tilde{l}(\boldsymbol{v}) = \tilde{l}_\theta(\boldsymbol{v}) + \tilde{l}_{1-\theta}(\boldsymbol{v}) = l(\boldsymbol{\theta v}) + \tilde{l}_{1-\theta}(\boldsymbol{v})$. Now, $\tilde{l}(\boldsymbol{v}) - l(\boldsymbol{\theta v})$ is a valid dependence function by Theorem 2.3.1 (i) and hence also a convex function. The fact that, in general, $\tilde{l}(\boldsymbol{v}) - l(\boldsymbol{\theta v})$ may be neither a dependence function nor a convex function at

all does not, however, allow for the conclusion of arbitrariness of l and $\tilde{l}$ as will become clear in the remainder of this chapter.
Theorem 2.3.2 below gives sharp upper and lower bounds for $\theta(\boldsymbol{v}) = l(\boldsymbol{\theta v})/\tilde{l}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{S}_D$ given θ_d, $d = 1, \ldots, D$, i.e. bounds for the entire multivariate extremal index function given only marginal dependence in terms of $\theta(\boldsymbol{e}_d)$, $d = 1, \ldots, D$.

Theorem 2.3.2. *Let $\Theta(\theta_1, \ldots, \theta_D)$ be the closed set of multivariate extremal index functions of all stationary max-stable processes with univariate extremal indices $\theta_d \in (0, 1]$, $d = 1, \ldots, D$. Define*

$$\theta_{\inf} : \mathbb{S}_D \to (0, 1],\ \boldsymbol{v} \mapsto \inf_{\theta \in \Theta(\theta_1, \ldots, \theta_D)} \theta(\boldsymbol{v}),$$
$$\theta_{\sup} : \mathbb{S}_D \to (0, 1],\ \boldsymbol{v} \mapsto \sup_{\theta \in \Theta(\theta_1, \ldots, \theta_D)} \theta(\boldsymbol{v}).$$

Then,

$$\theta_{\inf}(\boldsymbol{v}) = \max_d \theta_d v_d\,,$$
$$\theta_{\sup}(\boldsymbol{v}) = \frac{\sum_d \theta_d v_d}{\sum_d \theta_d v_d + \max_d (1 - \theta_d) v_d}\,.$$

In particular, $\theta_{\inf}, \theta_{\sup} \in \Theta(\theta_1, \ldots, \theta_D)$.

Proof. Let $\mathcal{A} = \mathcal{A}(\theta_1, \ldots, \theta_D)$ be the class of processes A as in (2.7) with coefficients a_{jkd}, $j \in I$, $k \in \mathbb{Z}$, α_{jkd}, $j \in F$, $0 \le k \le N_j$, such that $l(\boldsymbol{\theta e}_d) = \theta_d$, $d = 1, \ldots, D$. Now, using the same notation as in Theorem 2.3.1,

$$l_{\min}(\boldsymbol{\theta v}) \le \theta(\boldsymbol{v} \mid A) \le \frac{l(\boldsymbol{\theta v} \mid A)}{l(\boldsymbol{\theta v} \mid A) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v})} \le \frac{l_{\max}(\boldsymbol{\theta v})}{l_{\max}(\boldsymbol{\theta v}) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v})},$$

where the lower bound is sharp by property (L1) and Theorem 2.3.1 (iii), the second inequality holds with Theorem 2.3.1 (ii) and the right hand side follows from the discussion of the mapping $x \mapsto \frac{x}{x+a}$, $x, a \ge 0$. To show that the upper bound is sharp consider $A^* \in \mathcal{A}$ with $I = \{1, \ldots, D\}$, $F = \{1\}$, $a^*_{d1d} = \theta_d$ and $\alpha^*_{11d} = 1 - \theta_d$. Proposition 2.2.1 finalizes the proof. ☐

Figure 2.3.2 gives a bivariate example of the above bounds. Theorem 2.3.2 may equivalently be rewritten in terms of an improved lower bound for $\tilde{l}(\boldsymbol{v})$ making use of the additional information obtained by $l(\boldsymbol{\theta v})$, and an improved upper bound for $l(\boldsymbol{\theta v})$ given θ_d and $\tilde{l}(\boldsymbol{v})$.

Corollary 2.3.1. *For any stationary max-stable process with univariate extremal indices $\theta_d \in (0, 1]$, $d = 1, \ldots, D$, it holds that for all $\boldsymbol{v} \in \mathbb{S}_D$*

$$l(\boldsymbol{\theta v})\, \frac{\sum_d \theta_d v_d + \max_d (1 - \theta_d) v_d}{\sum_d \theta_d v_d} \le \tilde{l}(\boldsymbol{v}) \le 1,$$

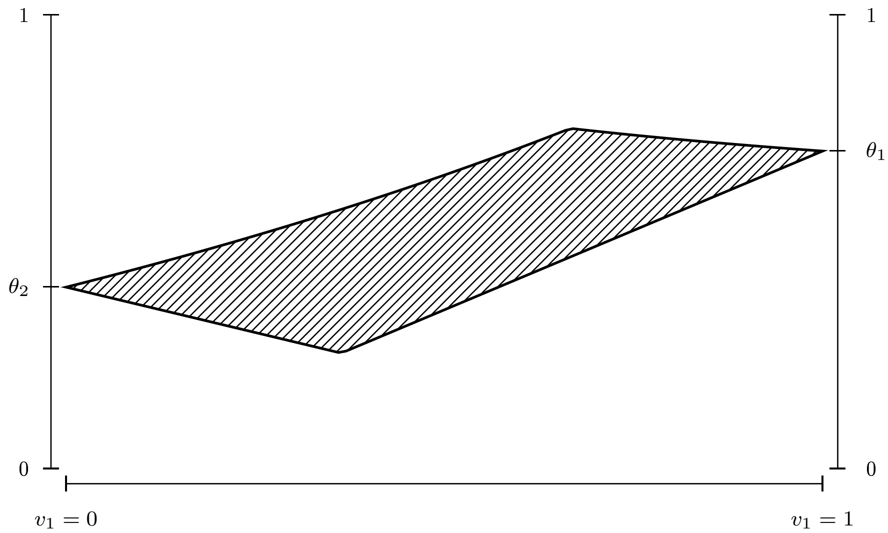

Figure 2.3.2: Bounds for $\theta(\boldsymbol{v})$ in the bivariate case: All admissible multivariate extremal index functions for $\theta_1 = 0.7$, $\theta_2 = 0.4$ are located in the shaded area (including the boundaries).

and

$$\max_d \theta_d v_d \leq l(\boldsymbol{\theta v}) \leq \min\left\{\frac{\tilde{l}(\boldsymbol{v})\sum_d \theta_d v_d}{\sum_d \theta_d v_d + \max_d(1-\theta_d)v_d}, \sum_d \theta_d v_d\right\}.$$

Note that the bounds discussed in Theorem 2.3.2 are applicable e.g. in order to improve on the validity of estimation schemes for the multivariate extremal index function. An example for an estimate of the latter that does not conform to Theorem 2.3.2 may be found in [40].

2.4 Exploring the Extremal Coefficients

The extremal coefficient $\tilde{\phi}$ has been proposed as a summary measure for the in general complex dependence structure of $\tilde{G}(\boldsymbol{x})$ given by $\tilde{\mu}([\boldsymbol{0}, \boldsymbol{x}]^c)$, see [60]. In effect, it is nothing but a single point of the respective dependence function, namely

$$\tilde{\phi}_A = \tilde{l}(\boldsymbol{e}_A) \tag{2.10}$$

where A is a nonempty subset of $\{1, \ldots, D\}$. Nevertheless, the extremal coefficient substantially restricts the possible shape of the entire dependence function $\tilde{l}$, see also properties (L1) and (L2). Note that the extremal coefficient $\tilde{\phi}$ may be interpreted more intuitively in terms of the number of independent variables in a multivariate setting. For a discussion of its further properties see e.g. [56]. So far, however, the bounds for $\theta(\boldsymbol{v})$ derived in Section 2.3 do not incorporate any information in terms of the extremal coefficient. Being a quotient measure of two dependence functions it is therefore natural with respect to $\theta(\boldsymbol{v})$ to consider the effect of a fixed extremal coefficient on the above bounds. Depending on the value of the extremal coefficient it turns out that the bounds may be improved significantly, compare Figure 2.4.1 with Figure 2.3.2. Following [36] we may also look at

$$\phi_A = l(\boldsymbol{\theta} \boldsymbol{e}_A) \tag{2.11}$$

as an adjusted extremal coefficient accounting for temporal dependence. Here, similarly, a single point of the dependence function corresponding to $G(\boldsymbol{x})$ is fixed. It roughly characterizes the entire function l in the above sense. We will therefore also discuss the corresponding improvement of the bounds for $\theta(\boldsymbol{v})$ given ϕ, cf. Figure 2.4.2. Note that the structure of the improvement of the bounds is completely distinct for fixed ϕ and $\tilde{\phi}$, respectively. This will in particular be reflected by the differing complexity of Theorems 2.4.3, 2.4.4 and 2.4.5 below. Before we turn to the interrelationship between $\theta(\boldsymbol{v})$ and the two extremal coefficients, however, we will discuss how the latter themselves are influenced by marginal dependence.

Theorem 2.4.1. *Let A be a non-empty subset of $\{1,\ldots,D\}$. Then, for any stationary max-stable process with univariate extremal indices $\theta_d \in (0,1]$, $d \in A$, the extremal coefficient $\tilde{\phi}_A$ is limited by the sharp bounds*

$$\max_{d\in A} \theta_d + \max_{d \in A} (1-\theta_d) \leq \tilde{\phi}_A \leq |A| \,.$$

Proof. The left inequality follows immediately from Theorem 2.3.1 (ii). The right inequality is well-known, and sharp by Theorem 2.3.1 (iii). An M_4 process with $I = \{1\}$ that reaches the lower bound is given by $a_{11d} = \theta_d$, $a_{1kd} = (1-\theta_d)/K$ for $k = 2,\ldots,K+1$ and K large enough such that $\theta_d \geq (1-\theta_d)/K$ for all d. □

As a consequence of Theorem 2.4.1 the case $\tilde{\phi} = 1$ is restricted to identical marginal dependence of all D series such that $\theta_1 = \ldots = \theta_D$, cf. Example 2.3.1. It also follows from Theorem 2.4.1 that Proposition 2.1(ii) in [37] where $\tilde{G}$ is assumed to have totally dependent margins (i.e. $\tilde{\phi} = 1$) loses generality and must be restricted to the special situation where $\theta_1 = \ldots = \theta_D$. The case is addressed by the following corollary with A defined as above.

Corollary 2.4.1. *If $\tilde{\phi}_A = 1$ then $\theta_m = \theta_n$ for all $m, n \in A$, and $\theta(\boldsymbol{v}) = \theta_m$, $\boldsymbol{v} \in \mathbb{S}_{|A|}$.*

Let now $\tilde{\phi}$ be given. Theorem 2.4.2 below shows that the full set of possible dependence functions l compatible with $\theta_1,\ldots,\theta_D$ is not necessarily admissible for all possible values of the extremal coefficient $\tilde{\phi}$ and vice versa. Equivalently, Theorem 2.4.2 extends the set of properties of $\phi = l(\boldsymbol{\theta}\boldsymbol{1})$ given in [36] by an improved upper bound related to $\tilde{\phi}$.

Theorem 2.4.2. *Let A be a subset of $\{1,\ldots,D\}$ with at least two elements. Then, for any stationary max-stable process with univariate extremal indices $\theta_d \in (0,1]$ for all $d \in A$ and extremal coefficient $\tilde{\phi}_A$ the adjusted extremal coefficient ϕ_A is limited by the sharp bounds*

$$\max_{d\in A} \theta_d \leq l(\boldsymbol{\theta}\boldsymbol{e}_A) = \phi_A \leq \min\left\{ \sum_{d\in A} \theta_d,\ \tilde{\phi}_A - \max_{d\in A}(1-\theta_d) \right\}.$$

Proof. Let $\phi = \phi_A$ and $\tilde{\phi} = \tilde{\phi}_A$ and let us restrict to an $|A|$-variate M_4 process where $|A| > 1$ by assumption. It is a well-known property of any dependence function that $\max_d \theta_d \leq l(\boldsymbol{\theta}\mathbf{1}) \leq \sum_d \theta_d$, see (L1). Further, $l(\boldsymbol{\theta}\mathbf{1}) \leq \tilde{l}(\mathbf{1}) - \max_d(1-\theta_d)$ is a direct consequence of Theorem 2.3.1 (i) and (ii), and hence a closer bound for ϕ is given if $\tilde{\phi} - \max_d(1-\theta_d) < \sum_d \theta_d$.
We first give example processes reaching the bounds for the case $\tilde{\phi} < \sum_d \theta_d + \max_d(1-\theta_d)$. Consider the M_4 process A where $I = \{1,2\}$, $a_{1dd} = c\theta_d$, $a_{21d} = (1-c)\theta_d$, $a_{2kd} = (1-\theta_d)/K$ for $k = 2,\dots,K+1$ and $c \in [0,1)$. Here, $a_{21d} > 0$, and K is chosen such that $a_{21d} \geq a_{2kd}$ for all k and d. Further, let B be the M_4 process where $I = \{1,\dots,D+1\}$, $b_{d1d} = c\theta_d$, $b_{D+1,1,d} = a_{21d}$, $b_{D+1,k,d} = a_{2kd}$, $k = 2,\dots,K+1$, $c \in [0,1)$. Now, for

$$c = \frac{\tilde{\phi} - \max_d \theta_d - \max_d(1-\theta_d)}{\sum_d \theta_d - \max_d \theta_d} \in [0,1)$$

$\tilde{\phi}$ is attained for both processes. Also, A reaches the lower bound and B reaches the upper bound for ϕ.
We now consider the case $\tilde{\phi} \geq \sum_d \theta_d + \max_d(1-\theta_d)$. Let A be a process as in (2.7) where $I = \{1\}$, $F = \{1,2\}$, $a_{1dd} = \theta_d$, $\alpha_{1dd} = c(1-\theta_d)$, $\alpha_{21d} = (1-c)(1-\theta_d)$, $c \in [0,1]$. Further, consider the process B that is also of the form (2.7) where $I = \{1,\dots,D\}$, $F = \{1,2\}$, $b_{d1d} = \theta_d$, $\beta_{1dd} = \alpha_{1dd}$, $\beta_{21d} = \alpha_{21d}$. Now, for

$$c = \frac{\tilde{\phi} - \sum_d \theta_d - \max_d(1-\theta_d)}{\sum_d(1-\theta_d) - \max_d(1-\theta_d)} \in [0,1]$$

$\tilde{\phi}$ is attained for both processes where A reaches the lower bound and B reaches the upper bound for ϕ. □

With respect to the behavior of the multivariate extremal index we have the following corollary.

Corollary 2.4.2. *For any stationary max-stable process with univariate extremal indices $\theta_d \in (0,1]$, $d = 1,\dots,D$, and extremal coefficient $\tilde{\phi}$ the multivariate extremal index is bounded at $\mathbf{1}$ by*

$$\frac{\max_d \theta_d}{\tilde{\phi}} \leq \theta(\mathbf{1}) = \theta(D^{-1}\mathbf{1}) \leq \frac{\min\left\{\tilde{\phi} - \max_d(1-\theta_d),\ \sum_d \theta_d\right\}}{\tilde{\phi}}.$$

In the following three theorems we will generalize the above corollary for the case $D = 2$, i.e. new bounds for the entire multivariate extremal index function will be given for fixed $\tilde{\phi}$ and ϕ, respectively. Due to the complex interdependencies of higher order dependence functions, see e.g. [55], corresponding bounds for $D \geq 3$ are not known yet. From the following theorems note that, in particular, for $\tilde{\phi} = \sum_d \theta_d + \max_d(1-\theta_d)$ in Theorem 2.4.3 the upper bound may not be improved in comparison with Theorem 2.3.2, and for $\tilde{\phi} = D$ in Theorem 2.4.4 the lower bound is unchanged. First, we will state the following example for reference in Theorems 2.4.3 and 2.4.5.

Example 2.4.1. Let $\mathcal{X} = \mathcal{X}(\theta_1, \theta_2, \phi_{\mathcal{X}})$ be the class of processes X as in (2.7) with coefficients x_{jkd}, $j \in I$, $k \in \mathbb{Z}$, $d = 1, 2$, and χ_{jkd}, $j \in F$, $0 \le k \le N_j$, $d = 1, 2$, such that $l(\boldsymbol{\theta e}_d \mid X) = \theta_d$ and $l(\boldsymbol{\theta 1} \mid X) = \phi_X \le \phi_{\mathcal{X}}$. Consider $X^*(\theta_1, \theta_2, \phi_{\mathcal{X}}) \in \mathcal{X}$ with $I = \{1, 2, 3\}$, $F = \{1\}$, $x^*_{d1d} = \phi_{\mathcal{X}} - \theta_{3-d}$, $x^*_{31d} = \sum_d \theta_d - \phi_{\mathcal{X}}$ and $\chi^*_{11d} = 1 - \theta_d$, where $x_{jkd} \ge 0$. Here, $\phi_{X^*} = \phi_{\mathcal{X}}$. Now, using the results of Theorem 2.3.1,

$$\begin{aligned} l(\boldsymbol{\theta v} \mid X^*) = \tilde{l}_\theta(\boldsymbol{v} \mid X^*) &= \sum_d (\phi_{X^*} - \theta_{3-d}) v_d + \Big(\sum_d \theta_d - \phi_{X^*}\Big) \max_d v_d, \\ \tilde{l}_{1-\theta}(\boldsymbol{v} \mid X^*) &= \tilde{l}_{1-\theta,\min}(\boldsymbol{v}), \\ \tilde{l}(\boldsymbol{v} \mid X^*) &= \tilde{l}_\theta(\boldsymbol{v} \mid X^*) + \tilde{l}_{1-\theta}(\boldsymbol{v} \mid X^*). \end{aligned}$$

Then, $l(\boldsymbol{\theta 1} \mid X^*) \ge l(\boldsymbol{\theta 1} \mid X)$, and from the convexity and piecewise linearity of l we may conclude that

$$l(\boldsymbol{\theta v} \mid X^*) \ge l(\boldsymbol{\theta v} \mid X) \quad \text{for all } X \in \mathcal{X}.$$

Now, by Theorem 2.3.1 (ii)

$$\frac{l(\boldsymbol{\theta v} \mid X)}{\tilde{l}(\boldsymbol{v} \mid X)} \le \frac{l(\boldsymbol{\theta v} \mid X)}{l(\boldsymbol{\theta v} \mid X) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v})} \le \frac{l(\boldsymbol{\theta v} \mid X^*)}{l(\boldsymbol{\theta v} \mid X^*) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v})} = \theta(\boldsymbol{v} \mid X^*)$$

for all $X \in \mathcal{X}$ using the same argumentation as in the proof of Theorem 2.3.2 for the second inequality.

Theorem 2.4.3. *Let $D = 2$ and $\Theta(\theta_1, \theta_2, \tilde{\phi})$ be the closed set of multivariate extremal index functions θ of all stationary max-stable processes with univariate extremal indices θ_1, $\theta_2 \in (0, 1]$ and extremal coefficient $\tilde{\phi}$. Define*

$$\theta_{\sup} : \mathbb{S}_2 \to (0, 1],\ \boldsymbol{v} \mapsto \sup_{\theta \in \Theta(\theta_1, \theta_2, \tilde{\phi})} \theta(\boldsymbol{v}).$$

(i) If $\tilde{\phi} \le \sum_d \theta_d + \max_d(1 - \theta_d)$, then $\theta_{\sup} \in \Theta$ and

$$\theta_{\sup}(\boldsymbol{v}) = \left(1 + \frac{\max_d(1 - \theta_d) v_d}{\sum_d (\phi^* - \theta_{3-d}) v_d + (\theta_1 + \theta_2 - \phi^*) \max_d v_d}\right)^{-1},$$

where $\phi^* = \tilde{\phi} - \max_d(1 - \theta_d)$.

(ii) If $\tilde{\phi} > \sum_d \theta_d + \max_d(1 - \theta_d)$, then $\theta_{\sup} \in \Theta$ iff $\tilde{\phi} = D$, and

$$\theta_{\sup}(\boldsymbol{v}) = \left(1 + \frac{\min_d \big\{ \max\{(1 - \theta_d) v_d, (2 - \tilde{\phi}) v_{3-d}\} + (\tilde{\phi} - 1 - \theta_{3-d}) v_{3-d} \big\}}{\sum_d \theta_d v_d}\right)^{-1}.$$

Proof. From Theorem 2.2.2 we have that $\Theta(\theta_1, \theta_2, \tilde{\phi})$ is closed. Let $\mathcal{B} = \mathcal{B}(\theta_1, \theta_2, \tilde{\phi})$ be the class of processes B of the form (2.7) where $l(\boldsymbol{\theta e}_d \mid B) = \theta_d$ and $\tilde{l}(\boldsymbol{1} \mid B) = \tilde{\phi}$.

(i) For $\mathcal{X}$ as in Example 2.4.1, $\mathcal{B}(\theta_1, \theta_2, \tilde{\phi}) \subset \mathcal{X}(\theta_1, \theta_2, \tilde{\phi} - \max_d(1 - \theta_d))$ by Theorem 2.4.2. Now, it is easily verified that $X^*(\theta_1, \theta_2, \tilde{\phi} - \max_d(1 - \theta_d)) \in \mathcal{B}$, and the assertion follows from Example 2.4.1.
(ii) Using the same notation as in Theorems 2.3.1 and 2.3.2 we have that $\mathcal{B}(\theta_1, \theta_2, \tilde{\phi}) \subset \mathcal{A}(\theta_1, \theta_2)$, and hence

$$\theta(\boldsymbol{v} \mid B) \leq \frac{l_{\max}(\boldsymbol{\theta v})}{l_{\max}(\boldsymbol{\theta v}) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v})} =: \theta_{U,1}(\boldsymbol{v}), \quad B \in \mathcal{B}.$$

Further, from $\tilde{l}(\mathbf{1} \mid B) = \tilde{\phi}$ it follows by convexity and piecewise linearity that $\tilde{l}(\boldsymbol{v} \mid B) \geq \sum_d v_d - \max_d\{(2\sum_d v_d - \tilde{\phi})v_d\} = 1 - (2 - \tilde{\phi}) \max_d v_d$, and hence, a second upper bound is given by

$$\theta_{U,2}(\boldsymbol{v}) = \frac{l_{\max}(\boldsymbol{\theta v})}{1 - (2 - \tilde{\phi}) \max_d v_d}.$$

Next, we will show that

$$\begin{aligned}
&\min\{\theta_{U,1}(\boldsymbol{v}),\, \theta_{U,2}(\boldsymbol{v})\} \\
&= \frac{l_{\max}(\boldsymbol{\theta v})}{\max\{l_{\max}(\boldsymbol{\theta v}) + \tilde{l}_{1-\theta,\min}(\boldsymbol{v}),\, 1 - (2 - \tilde{\phi}) \max_d v_d\}} \\
&= \frac{l_{\max}(\boldsymbol{\theta v})}{\min_d \left\{\max\{(1-\theta_d)v_d, (2-\tilde{\phi})v_{3-d}\} + (\tilde{\phi} - 1 - \theta_{3-d})v_{3-d}\right\} + \sum_d \theta_d v_d} \\
&= \frac{l_{\max}(\boldsymbol{\theta v})}{\min\left\{\tilde{l}_1(\boldsymbol{v}),\, \tilde{l}_2(\boldsymbol{v})\right\}} = \theta_{\sup}(\boldsymbol{v})
\end{aligned}$$

is a sharp upper bound for valid dependence functions $\tilde{l}_1$ and $\tilde{l}_2$ consistent with $l(\boldsymbol{\theta v}) = l_{\max}(\boldsymbol{\theta v})$. Here, the second equation follows after some lengthy but elementary calculations. Note that $\theta_{\sup}$ is reached piecewise by the example processes $B_m \in \mathcal{B}$, $m = 1, 2$, as in (2.7) where $b_{m,d1d} = \theta_d$, $\beta_{m,11m} = 1 - \theta_m$, $\beta_{m,(1,1,3-m)} = 2 - \tilde{\phi}$, $\beta_{m,(1,2,3-m)} = \tilde{\phi} - 1 - \theta_{3-m}$, and $\beta_{m,(1,2,3-m)} \geq 0$, $m = 1, 2$, by the assumption on $\tilde{\phi}$. Finally, for $\tilde{\phi} < D$ by lack of convexity of $\min\{\tilde{l}_1, \tilde{l}_2\}$ we have that $\theta_{\sup} \notin \Theta$ whereas for $\tilde{\phi} = D$ it holds that $\tilde{l}_1 = \tilde{l}_2$, and $\theta_{\sup} \in \Theta$. □

Except for specific parameter values the lower bound for $\theta(\boldsymbol{v})$ given θ_1, θ_2 and $\tilde{\phi}$ to be discussed next and represented in Figure 2.4.1 appears to be of a more complex form than the upper bound in the last theorem. For a motivation of the structure of the processes involved we first give the following example process. Namely, for the complex case when $\tilde{\phi} < \sum_d \theta_d + \max_d(1 - \theta_d)$, see the theorem below, it will turn out to be a simple member of two classes of processes reaching the lower bound pointwise for certain values of $\boldsymbol{v} \in \mathbb{S}_2$. Further, for the remaining values of $\boldsymbol{v} \in \mathbb{S}_2$ the process reaches the lower bound piecewise. In the following example and in Theorem 2.4.4 we will make use of a certain partition of $\boldsymbol{v} \in \mathbb{S}_2$. To this end let for $\theta_1 \geq \theta_2$

$$V_1 = \left[0,\, \frac{\tilde{\phi} - 1 + \theta_2 - \theta_1}{2\tilde{\phi} - 2 + \theta_2 - \theta_1}\right], \quad V_3 = \left[\frac{1}{2},\, 1\right], \quad V_2 = [0, 1] \setminus (V_1 \cup V_3),$$

$$V_{2,2} = \left[\frac{\theta_2}{\theta_1+\theta_2}, \frac{1}{2}\right], \quad V_{2,1} = V_2 \setminus V_{2,2}.$$

Example 2.4.2. For $D = 2$, $\theta_1 > \theta_2$, $\tilde{\phi} < 1 + \theta_1$ and $\boldsymbol{v}^* \in \mathbb{S}_2$ let $C = C(\theta_1, \theta_2, \tilde{\phi}, v_1^*)$ be the M_4 process with coefficients c_{jkd}, $j \in I = \{1, 2\}$, $k \in \mathbb{N}$, $d = 1, 2$, where

$$\begin{aligned}
c_{11d} &= (1 - q_d)\theta_d, \\
c_{2dd} &= q_d\theta_d, \\
c_{1kd} &= (1 - \theta_1)/K \text{ for } k = 2, \ldots, K + 1, \\
c_{1k2} &= (\theta_1 - \theta_2)/K \text{ for } k = 2 + K, \ldots, 1 + 2K
\end{aligned}$$

for some $q_d \in [0, 1)$ specified below and K the smallest positive integer such that $c_{11d} \geq c_{1kd}$ for $k > 2$, $d = 1, 2$. Further,

$$\begin{aligned}
q_2 \quad &= \quad \frac{\tilde{\phi} - 1 + \theta_2 - \theta_1}{\theta_2} \in [0, 1)\,, \\
q_1(\boldsymbol{v}^*) \quad &= \quad \begin{cases} \dfrac{\tilde{\phi} - 1}{\theta_1} \in (0, 1)\,, & v_1^* \in V_1, \\ \dfrac{\theta_2}{\theta_1} q_2(1/v_1^* - 1) \in [0, 1)\,, & v_1^* \in V_{2,1}, \\ q_2\,, & v_1^* \in V_{2,2} \cup V_3 \end{cases}
\end{aligned}$$

such that $c_{11d} > 0$. Now, for $Z_1^C = \{(j, k) \in I \times \mathbb{Z} : c_{jkd} > 0,\ d = 1, 2\} = \{(1, k) : k = 1, \ldots, K + 1\}$ and $Z_2^C = \{(j, k) \in Z_1^C : c_{jk1} = \max_k c_{jk1}\} = \{(1, 1)\}$ the meaning of which will become clear in the proof of the next theorem it obviously holds that

1. $\tilde{l}(\mathbf{1} \mid C) = \tilde{\phi}$,
2. $c_{jk1} = c_{jk2}, \quad (j, k) \in Z_1^C \setminus Z_2^C$,
3. $\displaystyle\sum_{(j,k) \in Z_1^C \setminus Z_2^C} c_{jk1} = 1 - \theta_1$,
4. $c_{112} = c_{111}\dfrac{v_1^*}{v_2^*} + \theta_2 - \theta_1\dfrac{v_1^*}{v_2^*} \geq c_{111}\dfrac{v_1^*}{v_2^*},\ c_{222} = c_{211}\dfrac{v_1^*}{v_2^*},\ v_1^* \in V_{2,1},\ v_2^* = 1 - v_1^*$,
5. $c_{112} = c_{111}\theta_2/\theta_1,\ c_{222} = c_{211}\theta_2/\theta_1, \quad v_1^* \in V_{2,2} \cup V_3$.

Theorem 2.4.4. *Let $D = 2$ and $\Theta(\theta_1, \theta_2, \tilde{\phi})$ be the closed set of multivariate extremal index functions θ for all stationary max-stable processes with univariate extremal indices θ_1, $\theta_2 \in (0, 1]$ and extremal coefficient $\tilde{\phi}$. Define*

$$\theta_{\inf} : \mathbb{S}_2 \to (0, 1],\ \boldsymbol{v} \mapsto \inf_{\theta \in \Theta(\theta_1, \theta_2, \tilde{\phi})} \theta(\boldsymbol{v})\,.$$

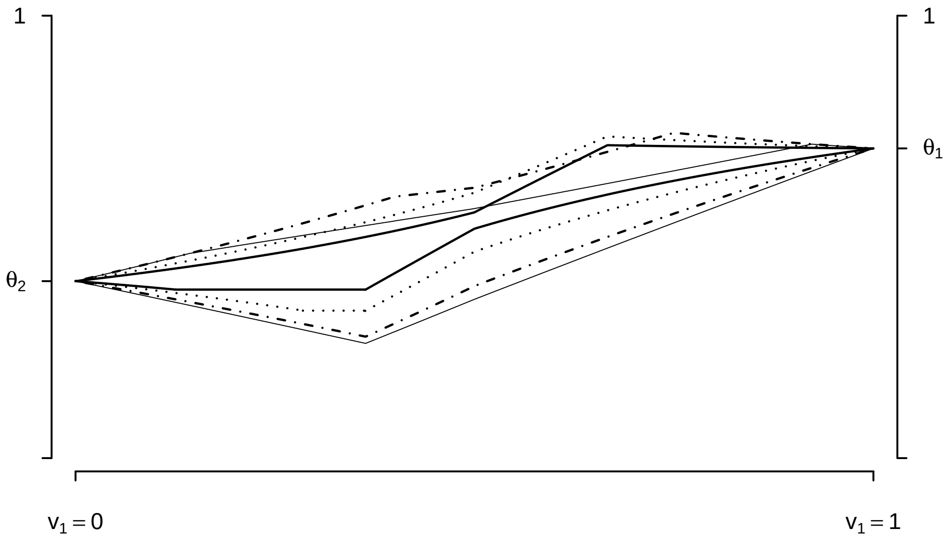

Figure 2.4.1: Upper and lower bounds for $\theta(\boldsymbol{v})$ as in Theorems 2.4.3 and 2.4.4 given $\tilde{\phi} = 1.35$ (thick line), $\tilde{\phi} = 1.5$ $(\cdots)$, $\tilde{\phi} = 1.8$ $(- \cdot -)$ and $\tilde{\phi} = 1.95$ (thin line) for $\theta_1 = 0.7, \theta_2 = 0.4$.

1. *If* $\tilde{\phi} \geq \sum_d \theta_d + \max_d(1 - \theta_d)$ *or* $\theta_1 = \theta_2$, *then* $\theta_{\inf} \in \Theta$ *and*

$$\theta_{\inf}(\boldsymbol{v}) = \frac{\max_d \theta_d v_d}{\tilde{\phi} - 1 + (2 - \tilde{\phi}) \max_d v_d}, \ \boldsymbol{v} \in \mathbb{S}_2.$$

2. *If* $\theta_1 > \theta_2$ *and* $\tilde{\phi} < \sum_d \theta_d + \max_d(1 - \theta_d) = 1 + \theta_1$, *then*

$$\theta_{\inf}(\boldsymbol{v}) = \begin{cases} \dfrac{\theta_2 v_2}{1 - (2 - \tilde{\phi}) v_1}, & v_1 \in V_1, \\ \dfrac{\max_d \theta_d v_d}{\max_d \theta_d v_d + (\tilde{\phi} - \theta_1) v_2}, & v_1 \in V_2, \\ \dfrac{\theta_1 v_1}{1 - (2 - \tilde{\phi}) v_2}, & v_1 \in V_3, \end{cases}$$

where $v_2 = 1 - v_1$. *In particular,* $\theta_{\inf} \notin \Theta$. *The assertion for* $\theta_1 < \theta_2$ *is given by symmetry.*

Proof. Again, by Proposition 2.2.1 it suffices to restrict to the respective bounds of M_4 processes.
1. For $\tilde{\phi} \geq \sum_d \theta_d + \max_d(1 - \theta_d)$ and $\theta_1, \theta_2 \in (0, 1]$ consider a sequence of M_4 processes A with $I = \{1\}$ given by $a_{1dd} = \theta_d$, $a_{1kd} = \frac{2-\tilde{\phi}}{K}$ for $k = D+1, \ldots, D+K$, $a_{1kd} = \frac{\tilde{\phi}-1-\theta_d}{K} \geq 0$ for $k = D + dK + 1, \ldots, D + (d + 1)K$ and K such that $\theta_d \geq a_{1kd}$ for all k, d. Now, $l(\boldsymbol{\theta v} \mid A) = l_{\min}(\boldsymbol{\theta v})$ and $\tilde{l}(\boldsymbol{v} \mid A) = \tilde{l}_{\max}(\boldsymbol{v} \mid \tilde{\phi})$ by convexity and piecewise linearity where $\tilde{l}_{\max}(\boldsymbol{v} \mid \tilde{\phi})$ is the overall maximum of $\tilde{l}(\boldsymbol{v})$ given $\tilde{\phi}$.

For $\theta_1 = \theta_2$ and $\tilde{\phi} < \sum_d \theta_d + \max_d(1-\theta_d) = 1+\theta_1$ consider the process $C(\theta_1, \theta_1, \tilde{\phi}, 0)$ in Example 2.4.2. We have that $l(\boldsymbol{\theta v} \mid C) = l_{\min}(\boldsymbol{\theta v})$ and $\tilde{l}(\boldsymbol{v} \mid C) = \tilde{l}_{\max}(\boldsymbol{v} \mid \tilde{\phi})$ by convexity and piecewise linearity.

2. We consider separately the four subsets V_1, $V_{2,1}$, $V_{2,2}$ and V_3 for v_1 with $\boldsymbol{v} \in \mathbb{S}_2$.

   (i) For $v_1 \in V_1$ consider the process $C(\theta_1, \theta_2, \tilde{\phi}, v_1)$ in Example 2.4.2. Now, $l(\boldsymbol{\theta v} \mid C) = \theta_2 v_2 = l_{\min}(\boldsymbol{\theta v})$ and $\tilde{l}(\boldsymbol{v} \mid C) = 1-(2-\tilde{\phi})v_1 = \tilde{l}_{\max}(\boldsymbol{v} \mid \tilde{\phi})$, $v_1 \in V_1$.

   (ii) Throughout this part we fix $\boldsymbol{v} \in \mathbb{S}_2$ with $v_1 \in V_{2,1}$. Let $\mathcal{B} = \mathcal{B}(\theta_1, \theta_2, \tilde{\phi})$ be the class of M_4 processes B with coefficients b_{jkd}, $j \in I$, $k \in \mathbb{Z}$, $d = 1, 2$, where $l(\boldsymbol{\theta e}_d \mid B) = \theta_d$ and $\tilde{l}(\mathbf{1} \mid B) = \tilde{\phi}$. We will show that a process C with $l(\boldsymbol{\theta v} \mid C) = l_{\min}(\boldsymbol{\theta v})$ exists such that for all $B \in \mathcal{B}$ the inequality $\tilde{l}(\boldsymbol{v} \mid B) - \tilde{l}(\boldsymbol{v} \mid C) \leq l(\boldsymbol{\theta v} \mid B) - l(\boldsymbol{\theta v} \mid C)$ holds. Then, $l(\boldsymbol{\theta v} \mid C)/\tilde{l}(\boldsymbol{v} \mid C) = \theta_{\inf}(\boldsymbol{v})$ by discussion of the mapping $x \mapsto \frac{x+a}{x+b}$, $0 \leq a \leq b$, $x \geq 0$. For the calculation of $\tilde{l}$ it will be advantageous to replace the double index $(j,k) \in I \times \mathbb{Z}$ by a single one, $m \in \mathbb{Z}$. More precisely, let $f : I \times \mathbb{Z} \to \mathbb{Z}$, $(j,k) \mapsto f(j,k)$ be an arbitrary bijective mapping and define $b_{md} = b_{f^{-1}(m),d}$. Then,

$$\tilde{l}(\boldsymbol{v} \mid B) = \sum_{m \in \mathbb{Z}} \max_d b_{md} v_d = \sum_d v_d - m(\boldsymbol{v} \mid B),$$

with $m(\boldsymbol{v} \mid B) = \sum_{m \in \mathbb{Z}} \min_d b_{md} v_d \geq 0$.

Let π be the projection $\pi : I \times \mathbb{Z} \to I$, $(j,k) \mapsto j$, and define $g := \pi \circ f^{-1} : \mathbb{Z} \to I$, $m \mapsto g(m) = j$. Let $b^*_{jd} = b_{j,k^*_{jd},d}$, for $k^*_{jd} \in \arg\max_k b_{jkd}$, $j \in I$, $d = 1, 2$. Since the set of M_4 processes B for which k^*_{jd} is unique is a dense subset of $\mathcal{B}$ we may assume uniqueness of k^*_{jd}. Let $Z_1^B = \{m \in \mathbb{Z} : b_{md} > 0,\ d = 1, 2\}$ such that

$$\sum_{m \in \mathbb{Z}} \min_d b_{md} = \sum_{m \in Z_1^B} \min_d b_{md} = 2 - \tilde{\phi} = m(\mathbf{1} \mid B). \tag{2.12}$$

Let $Z_2^B = \{m \in Z_1^B : b_{m1} = b^*_{g(m),1}\}$. Now,

$$\sum_{m \in Z_1^B \setminus Z_2^B} b_{m1} \leq 1 - \theta_1 \tag{2.13}$$

such that

$$\sum_{m \in Z_1^B \setminus Z_2^B} \min_d b_{md} = 1 - \theta_1 - \mu \tag{2.14}$$

for $0 \leq \mu \leq \min\{1-\theta_1,\ \tilde{\phi} - 1 + \theta_2 - \theta_1\}$, where the latter follows from the fact that

$$1 - \theta_1 - \mu = 2 - \tilde{\phi} - \sum_{m \in Z_2^B} \min_d b_{md} \tag{2.15}$$

by (2.14) and (2.12), and $\sum_{m\in Z_2^B}\min_d b_{md} \leq \min_d \sum_{m\in Z_2^B} b_{md} \leq \min_d \theta_d = \theta_2$. Note that from $\tilde{\phi} < 1+\theta_1$, Eq. (2.12) and Ineq. (2.13) it follows that $Z_2^B \neq \emptyset$. Now, with

$$\sum_{m\in Z_1^B\setminus Z_2^B} \min_d b_{md}v_d \geq \sum_{m\in Z_1^B\setminus Z_2^B} \min_d b_{md} \min_d v_d = (1-\theta_1-\mu)v_1 \tag{2.16}$$

we get

$$m(\boldsymbol{v}\mid B) \geq \sum_{m\in Z_2^B} \min_d b_{md}v_d + (1-\theta_1-\mu)v_1. \tag{2.17}$$

We consider the (disjoint) decomposition $Z_2^B = Z_{2,1}^B \cup Z_{2,2}^B \cup Z_{2,3}^B$ where $m\in Z_{2,1}^B$ if $b_{m1}<b_{m2}$, $m\in Z_{2,2}^B$ if $b_{m2}\leq b_{m1}\leq b_{m2}v_2/v_1$ and $m\in Z_{2,3}^B$ else. Now,

$$\min_d b_{md} = \begin{cases} b_{m1} = b_{m2} - \xi_m,\ m\in Z_{2,1}^B \\ b_{m2},\ m\in Z_{2,2}^B\cup Z_{2,3}^B, \end{cases}$$

$$\min_d\{b_{md}v_d\} = \begin{cases} b_{m1}v_1,\ m\in Z_{2,1}^B\cup Z_{2,2}^B \\ b_{m2}v_2 = (b_{m1}-\eta_m)v_1,\ m\in Z_{2,3}^B, \end{cases}$$

$$\max_d\left\{b^*_{g(m),d}v_d\right\} = \begin{cases} (b_{m2}+\kappa_m)v_2,\ m\in Z_{2,1}^B\cup Z_{2,2}^B \\ (b_{m2}+\kappa_m)v_2 + \max\{0,\ \eta_m v_1 - \kappa_m v_2\},\ m\in Z_{2,3}^B, \end{cases}$$

where $\xi_m = b_{m2}-b_{m1} > 0$, $m\in Z_{2,1}^B$, $\kappa_m = b^*_{g(m),2} - b_{m2} \geq 0$, $m\in Z_2^B$, and $0<\eta_m = b^*_{g(m),1} - b_{m2}v_2/v_1 = b_{m1} - b_{m2}v_2/v_1 \leq 1 - b_{m2}v_2/v_1$, $m\in Z_{2,3}^B$. Let

$$\begin{aligned} s_{m1} &= \sum_{j\in I\setminus\{g(m):m\in Z_2^B\}} b^*_{j1} = \theta_1 - \sum_{m\in Z_2^B} b_{m1} \\ &= \theta_1 - \left\{\sum_{m\in Z_2^B}\min_d b_{md}v_d/v_1 + \sum_{m\in Z_{2,3}^B}\eta_m\right\}, \\ s_{m2} &= \sum_{j\in I\setminus\{g(m):m\in Z_2^B\}} b^*_{j2} = \theta_2 - \sum_{m\in Z_2^B} b^*_{g(m),2} \\ &= \theta_2 - \left\{\sum_{m\in Z_2^B}\min_d b_{md} + \sum_{m\in Z_{2,1}^B}\xi_m + \sum_{m\in Z_2^B}\kappa_m\right\}. \end{aligned} \tag{2.18}$$

Then,

$$\begin{aligned} l(\boldsymbol{\theta v}\mid B) &= \sum_{m\in Z_2^B}\max_d\{b^*_{g(m),d}v_d\} + \sum_{j\in I\setminus\{g(m):m\in Z_2^B\}}\max_d\{b^*_{jd}v_d\} \\ &\geq \sum_{m\in Z_2^B} b_{m2}v_2 + \sum_{m\in Z_2^B}\kappa_m v_2 + \sum_{m\in Z_{2,3}^B}\max\{0,\ \eta_m v_1 - \kappa_m v_2\} \\ &\quad + s_{m1}v_1. \end{aligned} \tag{2.19}$$

Let $\mathcal{C} \subset \mathcal{B}$ be the class of M_4 processes C with coefficients c_{jkd} where

$$c^*_{j2} = \frac{v_1}{v_2} c^*_{j1} + \gamma_j, \quad j \in I,\, \gamma_j \geq 0, \tag{2.20}$$

$$c_{m1} = c_{m2}, \quad m \in Z_1^C \setminus Z_2^C, \tag{2.21}$$

$$\sum_{m \in Z_1^C \setminus Z_2^C} c_{m1} = 1 - \theta_1, \tag{2.22}$$

$$c_{m2} = c^*_{g(m),2} = \frac{v_1}{v_2} c_{m1} + \gamma_{g(m)} \leq c_{m1},\, m \in Z_2^C, \quad \text{and} \tag{2.23}$$

$$\sum_{m \in Z_2^C} \gamma_{g(m)} = \sum_{j \in I} \gamma_j = \theta_2 - \frac{v_1}{v_2}\theta_1. \tag{2.24}$$

Here, Eq. (2.22) replaces the corresponding Ineq. (2.13) above. In particular, $\mathcal{C}$ is not empty by Example 2.4.2. Since $\sum_{m \in Z_1^C \setminus Z_2^C} \min_d c_{md} = 1 - \theta_1$ by (2.21) and (2.22) it holds with (2.23) and (2.12) that

$$\sum_{m \in Z_2^C} \min_d c_{md} = \sum_{m \in Z_2^C} c_{m2} = 1 - \tilde{\phi} + \theta_1. \tag{2.25}$$

By the left hand side of (2.23) we now get with (2.25) that

$$\begin{aligned}\sum_{m \in Z_2^C} c_{m1} v_1 &= \Big(1 - \tilde{\phi} + \theta_1 - \sum_{m \in Z_2^C} \gamma_m\Big) v_2 \\ &= (1 - \tilde{\phi} - \theta_2) v_2 + \theta_1, \quad \boldsymbol{v} \in \mathbb{S}_2, \end{aligned} \tag{2.26}$$

where (2.26) follows from (2.24). Further, by (2.23) and (2.21),

$$\begin{aligned} m(\boldsymbol{v} \mid C) = \sum_{m \in Z_1^C} \min_d c_{md} v_d &= \sum_{m \in Z_2^C} c_{m1} v_1 + \sum_{m \in Z_1^C \setminus Z_2^C} c_{m1} v_1 && (2.27) \\ &= 1 - (\tilde{\phi} - \theta_1 + \theta_2) v_2, && (2.28) \end{aligned}$$

where (2.28) follows with (2.26) and (2.22). To conclude the proof we will make use of the following four results.

First, by (2.20) and the definition of s_{m2} we have

$$l(\boldsymbol{\theta v} \mid C) = l_{\min}(\boldsymbol{\theta v}) = \theta_2 v_2 = s_{m2} v_2 + \sum_{m \in Z_2^B} b^*_{g(m),2} v_2, \quad v_1 \in V_{2,1}. \tag{2.29}$$

Eqs. (2.25) and (2.15) imply

$$s_{m2} = \theta_2 - \Bigg\{ \sum_{m \in Z_2^C} \min_d c_{md} + \mu + \sum_{m \in Z_{2,1}^B} \xi_m + \sum_{m \in Z_2^B} \kappa_m \Bigg\}. \tag{2.30}$$

Further, applying (2.24) and (2.23) twice yields

$$\sum_{m\in Z_2^C} \min_d c_{md} v_2 = \sum_{m\in Z_2^C} c_{m1} v_1 + \theta_2 v_2 - \theta_1 v_1. \tag{2.31}$$

Finally, by (2.27), (2.22) and (2.17)

$$m(\boldsymbol{v} \mid C) - m(\boldsymbol{v} \mid B) \leq \sum_{m\in Z_2^C} c_{m1} v_1 - \sum_{m\in Z_2^B} \min_d b_{md} v_d + \mu v_1. \tag{2.32}$$

Now,

$$\begin{aligned} l(\boldsymbol{\theta v} \mid B) - l(\boldsymbol{\theta v} \mid C) &= l(\boldsymbol{\theta v} \mid B) - \theta_2 v_2 \\ &\geq s_{m1} v_1 - s_{m2} v_2 + \sum_{m\in Z_{2,3}^B} (\eta_m v_1 - \kappa_m v_2) \qquad (2.33) \\ &= \sum_{m\in Z_2^C} c_{m1} v_1 - \sum_{m\in Z_2^B} \min_d b_{md} v_d + \mu v_2 + \sum_{m\in Z_{2,1}^B} \xi_m v_2 + \\ &\sum_{m\in Z_2^B} \kappa_m v_2 - \sum_{m\in Z_{2,3}^B} \kappa_m v_2 \qquad (2.34) \\ &\geq m(\boldsymbol{v} \mid C) - m(\boldsymbol{v} \mid B), \qquad (2.35) \end{aligned}$$

where (2.33) holds with (2.19) and (2.29), (2.34) holds with (2.30), (2.31) and the definition of s_{m1}, and (2.35) finally follows from (2.32).

(iii) Let $\boldsymbol{v} \in \mathbb{S}_2$ with $v_1 \in V_{2,2}$ be fixed. The proof is similar to that of part (ii) and uses the same notation where possible. Let now $\mathcal{C}$ be the class of M_4 processes C with coefficients c_{jkd} where

$$c_{j2}^* = \frac{v_1}{v_2} c_{j1}^* - \gamma_j, \quad j \in I,\ \gamma_j \geq 0, \sum_{j\in I} \gamma_j = \frac{v_1}{v_2}\theta_1 - \theta_2, \tag{2.36}$$

and (2.21) and (2.22) hold. From (2.36) we get for $m \in Z_2^C$ that

$$c_{m2} = \frac{v_1}{v_2} c_{m1} - \gamma_{g(m)} - \varepsilon_m \leq c_{m1}, \tag{2.37}$$

where $0 \leq \varepsilon_m \leq \frac{v_1}{v_2} c_{m1} - \gamma_{g(m)}$ accounts for the fact that $c_{m2} \leq c_{g(m),2}^*$. Again, $\mathcal{C}$ is not empty by Example 2.4.2, where $\gamma_j = c_{j1}^*(v_1/v_2 - \theta_2/\theta_1)$ and

$$l(\boldsymbol{\theta v} \mid C) = l_{\min}(\boldsymbol{\theta v}) = \theta_1 v_1 \tag{2.38}$$

by Eq. (2.36). From (2.37) it follows that $c_{m1} v_1 \geq c_{m2} v_2$, $v_1 \in V_{2,2}$, and hence

$$m(\boldsymbol{v} \mid C) = \sum_{m\in Z_2^C} c_{m2} v_2 + \sum_{m\in Z_1^C \setminus Z_2^C} c_{m1} v_1. \tag{2.39}$$

Further, following the argumentation there, Eq. (2.25) holds with (2.37) instead of (2.23). Now, with (2.39) and (2.22) it follows that

$$m(\boldsymbol{v} \mid C) = (1 - \tilde{\phi} + \theta_1)v_2 + (1 - \theta_1)v_1. \tag{2.40}$$

Consider again the class $\mathcal{B}$ as in part (ii). Using the above decomposition of Z_2^B we may write

$$m(\boldsymbol{v} \mid B) = \sum_{m \in Z_2^B} b_{m2}v_2 - \sum_{m \in Z_{2,1}^B \cup Z_{2,2}^B} (b_{m2}v_2 - b_{m1}v_1) + \sum_{m \in Z_1^B \setminus Z_2^B} \min_d b_{md}v_d.$$

Eq. (2.15) states that $1 - \tilde{\phi} + \theta_1 + \mu = \sum_{m \in Z_2^B} \min_d b_{md} \leq \sum_{m \in Z_2^B} b_{m2}$. Now, using (2.16) and (2.40) it follows that

$$\begin{aligned}\sum_{m \in Z_2^B} b_{m2}v_2 + \sum_{m \in Z_1^B \setminus Z_2^B} \min_d b_{md}v_d &\geq (1 - \tilde{\phi} + \theta_1 + \mu)v_2 + (1 - \theta_1 - \mu)v_1 \\ &\geq (1 - \tilde{\phi} + \theta_1)v_2 + (1 - \theta_1)v_1 \\ &= m(\boldsymbol{v} \mid C),\end{aligned}$$

and hence

$$m(\boldsymbol{v} \mid C) - m(\boldsymbol{v} \mid B) \leq \sum_{m \in Z_{2,1}^B \cup Z_{2,2}^B} (b_{m2}v_2 - b_{m1}v_1). \tag{2.41}$$

Further, with

$$\max_d\{b^*_{g(m),d}v_d\} = \begin{cases} b_{m1}v_1 + (b_{m2}v_2 - b_{m1}v_1 + \kappa_m v_2), & m \in Z_{2,1}^B \cup Z_{2,2}^B \\ b_{m1}v_1 + \max\{0,\, \kappa_m v_2 - \eta_m v_1\}, & m \in Z_{2,3}^B, \end{cases}$$

we get that

$$\begin{aligned}\sum_{m \in Z_2^B} \max_d\{b^*_{g(m),d}v_d\} = &\sum_{m \in Z_2^B} b_{m1}v_1 + \sum_{m \in Z_{2,1}^B \cup Z_{2,2}^B} (b_{m2}v_2 - b_{m1}v_1 + \kappa_m v_2) \\ &+ \sum_{m \in Z_{2,3}^B} \max\{0,\, \kappa_m v_2 - \eta_m v_1\}.\end{aligned} \tag{2.42}$$

By definition,

$$l(\boldsymbol{\theta v} \mid B) = \sum_{m \in Z_2^B} \max_d\{b^*_{g(m),d}v_d\} + \sum_{j \in I \setminus \{g(m): m \in Z_2^B\}} \max_d\{b^*_{jd}v_d\}. \tag{2.43}$$

Now,

$$l(\boldsymbol{\theta v} \mid B) - l(\boldsymbol{\theta v} \mid C) = l(\boldsymbol{\theta v} \mid B) - \theta_1 v_1 \tag{2.44}$$

$$= l(\boldsymbol{\theta v} \mid B) - \sum_{m \in Z_2^B} b_{m1}v_1 - \sum_{j \in I \setminus \{g(m): m \in Z_2^B\}} b^*_{j1}v_1 \tag{2.45}$$

$$\geq \sum_{m \in Z_{2,1}^B \cup Z_{2,2}^B} (b_{m2}v_2 - b_{m1}v_1) \tag{2.46}$$

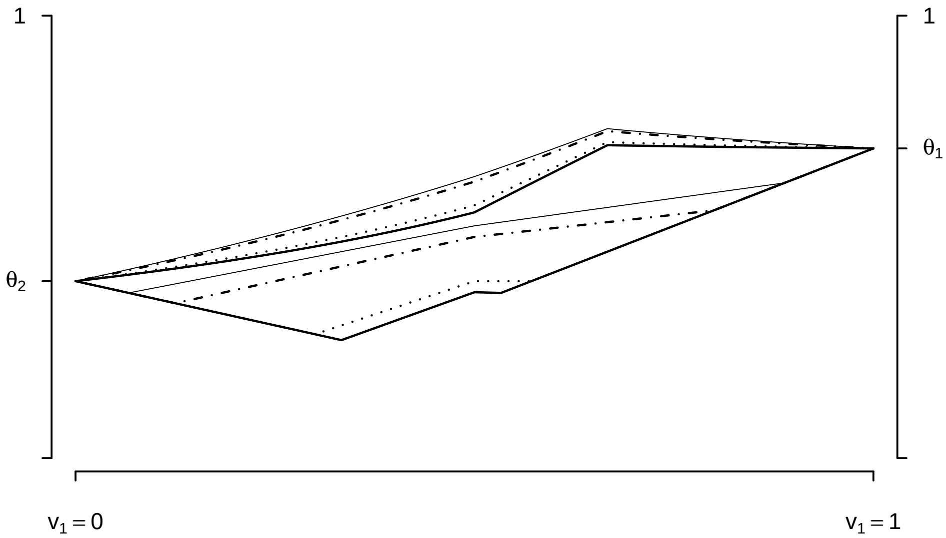

Figure 2.4.2: Upper and lower bounds for $\theta(\boldsymbol{v})$ as in Theorem 2.4.5 given $\phi = 0.75$ (thick line), $\phi = 0.8$ ($\cdots$), $\phi = 1$ ($- \cdot -$) and $\phi = 1.05$ (thin line) for $\theta_1 = 0.7, \theta_2 = 0.4$.

where (2.44) holds with (2.38), (2.45) holds with (2.18), and (2.46) follows from (2.43) and (2.42). Finally, by (2.41) and the argumentation at the beginning of part (ii) we have that $\theta_{\inf}(\boldsymbol{v}) = \theta(\boldsymbol{v} \mid C)$.

(iv) For $v_1 \in V_3$ consider the process $C(\theta_1, \theta_2, \tilde{\phi}, v_1)$ in Example 2.4.2 and apply the same argumentation as in part (i).

□

Finally, we will consider the case where ϕ instead of $\tilde{\phi}$ is given, see Figure 2.4.2.

Theorem 2.4.5. *Let $D = 2$ and $\Theta(\theta_1, \theta_2, \phi)$ be the closed set of multivariate extremal index functions θ for all stationary max-stable processes with univariate extremal indices θ_1, $\theta_2 \in (0, 1]$ and adjusted extremal coefficient $\phi = l(\boldsymbol{\theta 1}) = \tilde{\phi}\,\theta(\mathbf{1})$. Let $\theta_{\inf}$ and $\theta_{\sup}$ be defined in the same way as in Theorems 2.4.3 and 2.4.4. Then,*

$$\theta_{\sup}(\boldsymbol{v}) = \left(1 + \frac{\max_d (1-\theta_d) v_d}{\sum_d (\phi - \theta_{3-d}) v_d + (\theta_1 + \theta_2 - \phi) \max_d v_d}\right)^{-1},$$

$$\theta_{\inf}(\boldsymbol{v}) = \min_d \left\{ \max\{\theta_d - 2(\theta_d - \phi/2) v_{3-d},\, \theta_{3-d} v_{3-d}\} \right\}.$$

Further, $\theta_{\sup} \in \Theta$ for all ϕ, and $\theta_{\inf} \notin \Theta$ if and only if $\max_d \theta_d < \phi < \sum_d \theta_d$.

Proof. Let $\mathcal{B} = \mathcal{B}(\theta_1, \theta_2, \phi)$ be the class of processes B as in (2.7) with coefficients b_{jkd}, $j \in I$, $k \in \mathbb{Z}$, $d = 1, 2$, and β_{jkd}, $j \in F$, $0 \le k \le N_j$, $d = 1, 2$, such that $l(\boldsymbol{\theta e}_d \mid B) = \theta_d$ and $l(\boldsymbol{\theta 1} \mid B) = \phi$. Now, the equality for $\theta_{\sup}$ follows by Example 2.4.1 and the fact that $\mathcal{B} \subseteq \mathcal{X}(\theta_1, \theta_2, \phi)$, where $X^*(\theta_1, \theta_2, \phi) \in \mathcal{B}$.

To show the equality for $\theta_{\inf}$ note that from $l(\boldsymbol{\theta 1} \mid B) = \phi$ it holds by (L1) and (L2) that

$$\begin{aligned} l(\boldsymbol{\theta v} \mid B) &\geq \min_d \{\max\{\theta_d - 2(\theta_d - \phi/2)v_{3-d},\, \theta_{3-d}v_{3-d}\}\} \\ &= \min\{l_1(\boldsymbol{v}),\, l_2(\boldsymbol{v})\} = \theta_{\inf}(\boldsymbol{v}), \end{aligned}$$

where l_1 and l_2 are valid dependence functions, see below. Then, the last equation follows from Theorem 2.3.1 (iii). Further, $\theta_{\inf}$ is not a valid dependence function for $\max_d \theta_d < \phi < \sum_d \theta_d$ by lack of convexity, and $\theta_{\inf}(\boldsymbol{v}) = \sum_d \theta_d v_d$ for $\phi = \sum_d \theta_d$ and $\theta_{\inf}(\boldsymbol{v}) = \max_d \theta_d v_d$ for $\phi = \max_d \theta_d$.
Finally, note that $\theta_{\inf}$ is reached piecewise by the example processes $B_m \in \mathcal{B}$, $m = 1, 2$, of the form (2.7) where $b_{m,11m} = \phi - \theta_{3-m}$, $b_{m,21m} = \sum_d \theta_d - \phi$, $b_{m,(2,2,3-m)} = \theta_{3-m}$, $\beta_{m,1dd} = 1 - \theta_d$. $\square$

Chapter 3

Reconstruction of Max-Stable Processes for Given Extremal Coefficient Functions

3.1 Motivation

With respect to the dependence structure of D-variate stationary max-stable processes we discussed in Chapter 2 the multivariate extremal index as a rough summary measure of the clustering behavior. In particular, we gave an illustrative interpretation of the extremal index in terms of the mean limiting cluster size. Consequently, a given extremal index may in general be realized by a rich class of processes comprising a large variety of different dependence structures. Furthermore, unlike the Gaussian family where the dependence structure is entirely determined by the corresponding autocovariance function the class of max-stable processes cannot be completely characterized by a similar concept. Still, a suitable summary measure for the dependence structure of such processes that goes beyond the meaning of the extremal index is given by the extremal coefficient function, cf. (1.1). It is a conditionally negative definite function and was proposed in [56]. At the same time it is a special case of the extremogram [12]. It will become clear in Section 3.3 how this function is related to the more general notion of extremal coefficients already discussed in Chapter 2. Although in most applications the extremal coefficient function gives a more detailed idea of the dependence structure than the extremal index we remark that the former may not be understood rigorously as a refinement of the extremal index. In particular, the latter can in general not be reconstructed uniquely for a given extremal coefficient function. Recall also (1.2) for an alternative approach to the dependence structure within extremal clusters that is not affected by this shortcoming. It will be studied in detail in Chapter 5.

The matter of the extremal coefficient function may best be understood alluding to the usual autocovariance. Similar to the latter the extremal coefficient function is a dependence measure for pairwise (temporal or spatial) separations of a process at a given lag $h \in \mathbb{R}^D$. This concept will be covered more formally in Section 3.3. Although it remains a summary measure of the dependence structure, i.e. it neither characterizes the multivariate marginals of the process nor the bivariate dependence structure over space or time completely, it has a convenient interpretation that is appropriate to most applications. Moreover, as in the Gaussian case a summability condition on the extremal coefficient function will allow for a corresponding characterization of max-stable processes as having short or long memory. Further, while it does not determine the dependence structure of a max-stable process any given extremal coefficient function still imposes significant restrictions on the admissible set of underlying processes. Here, we will exploit in more detail the structure of extremal coefficient functions in order to recover corresponding max-stable processes, i.e. such processes that are able to generate the respective extremal coefficient function. Although the latter classes are extensive it has been an open question to state such valid member processes explicitly. Based on the well-known fact that the set of extremal coefficient functions is convex [56] we will, in particular, focus on convex decompositions of those functions, i.e. a representation of the latter in terms only of the vertices of their hull. It will be instructive at this point to have an early look at Figure 3.4.1 below where as an example for a range of $n = 5$ we display the abovementioned vertex extremal coefficient functions. Put differently, all valid extremal coefficient functions on $\mathbb{Z}$ up to range five are given by some convex combination of the functions included in the figure. As a crucial point we will discuss in detail the determination of the set of such vertices. The latter will then give rise to a reconstruction scheme for max-stable processes associated to given extremal coefficient functions. In particular, we shall introduce a sparse reference class of max-stable processes that is intimately related to the above set of vertices. The class of processes depends on a weight vector that may readily be chosen such as to reproduce any valid extremal coefficient function. The reconstruction of max-stable example processes from given extremal coefficient functions is then essentially reduced to the determination of suitable weights. Note that throughout we will confine our analysis to the one-dimensional case in discrete time.

The chapter will be organized as follows. In Section 3.2 we shall first introduce the concept of set correlation functions. Following a brief discussion of their properties we will restrict to their evaluation on a grid, and determine the vertices of their convex set. We point out that the analysis in Section 3.2 is self contained and independent of the concepts commonly used in extreme value theory. We will, however, show in Section 3.3 that the ensembles of set correlation functions and extremal coefficient functions actually coincide on a grid. The reason to work with set correlation functions first is that in order to analyze their structure and determine the vertices of their set we may refer directly to well-known concepts from the literature, in particular the problem of homometry [46, 47]. In Section 3.3 we will then formally refer to the theory of extremes and recall two essential concepts already discussed in Chapter 2, namely max-stable processes and extremal coefficients. Section 3.4 will be primarily devoted to the setup of the abovementioned reconstruction scheme. An example of the latter in addition to some related applications will be discussed in Section 3.5. Finally, the

usefulness of partial knowledge of the extremal coefficient function for assertions on the range of the underlying process will be considered in Section 3.6.
Throughout, following standard conventions we will write $S + q = \{x + q : x \in S\}$, and accordingly $aS = \{ax : x \in S\}$, for a set $S \subseteq \mathbb{R}$, and $q, a \in \mathbb{R}$. We will denote the indicator function of a set $S \subseteq \mathbb{R}$ by $\mathbf{1}(x \in S)$. Further, we will assume all operations that involve vectors to apply componentwise, and denote by "$\subset$" a proper inclusion whereas "$\subseteq$" does not preclude equality. For $x \in \mathbb{R}$ let $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

3.2 Set Correlation Functions and Basic Notions

To begin with in this section we will concentrate exclusively on set correlation functions, a concept shown in Section 3.3 to be equivalent to the extremal coefficient functions on a grid. In our approach we will show first that the ensemble $\mathcal{F}^*_{n,\mathbb{Z}}$ of set correlation functions with finite range $n \in \mathbb{N}$ that are evaluated on $\mathbb{Z}$ is a convex set. We will further determine its vertices, see Lemma 3.2.1 and Theorem 3.2.1 below. To begin with, it will be instructive to incorporate the relevant concepts successively into the well-known framework of general covariograms. To this end, for an integrable and square integrable function $w(x)$ in $\mathbb{R}$ we will define the covariogram by the convolution product

$$f(h) = \int w(x)w(x+h)dx, \quad h \in \mathbb{R}. \tag{3.1}$$

Note that two fundamental properties of the covariogram, namely symmetry and positive definiteness, are immediate from (3.1), and will be essential in the following. As an important special case of (3.1) we will next consider the length of the intersection of a set with its translation. More precisely, for $w(x) = \mathbf{1}(x \in S)$, $S \in \sigma_\infty$, let

$$f_S(h) = \int \mathbf{1}(x \in S)\mathbf{1}(x \in (S-h))dx = |S \cap (S-h)|, \quad h \in \mathbb{R}, \tag{3.2}$$

denote the set covariance function of S, also termed the geometric covariogram [38], where σ_∞ stands for the ensemble of all Borel sets $S \subseteq \mathbb{R}$ with $0 < |S| < \infty$. Note that later on we shall use the notation $|\cdot|$ also to indicate the cardinality of a set where no confusion may arise. For later reference we introduce $\sigma_n \subseteq \sigma_\infty$ in order to represent accordingly all Borel sets $S \subseteq [q, n+q)$ for some $q \in \mathbb{R}$. The number $n \in \mathbb{N}$ will later be referred to as finite range. For convenience, in the following we shall without loss of generality consider the set correlation functions $f^*_S(h) = f_S(h)/f_S(0)$ for all $h \in \mathbb{R}$, $S \in \sigma_\infty$. To provide some preliminary insight into the behavior of f^*_S note that by (3.2) we have in particular that $f^*_S(0) = 1$, $\int f^*_S(h)dh = |S|$, and that $f^*_S(h)$ is not differentiable at the origin [7]. As a further restriction of (3.1) and (3.2) we shall henceforth confine our analysis to the evaluation of f^*_S on a subset $Q \subseteq \mathbb{R}$, i.e. we consider $f^*_S(h)$, $h \in Q$, and put $\mathcal{F}^*_{n,Q} = \{f^*_S \in \mathbb{R}^Q : S \in \sigma_n\}$ for any $n \in \mathbb{N} \cup \{\infty\}$. Note that $f^*_S \in \mathcal{F}^*_{\infty,Q}$ might be in $\mathcal{F}^*_{n,Q}$ for some $n \in \mathbb{N}$ although S is unbounded. The following elementary lemma provides a fundamental background for the rest of our analysis.

Lemma 3.2.1. *For all* $n \in \mathbb{N} \cup \{\infty\}$ *and all* $p \in \mathbb{N}$ *the set* $\mathcal{F}^*_{n,p^{-1}\mathbb{Z}}$ *is convex.*

Proof. Let $f^*_{S_1}, f^*_{S_2} \in \mathcal{F}^*_{n,p^{-1}\mathbb{Z}}$, $n \in \mathbb{N} \cup \{\infty\}$. Consider first the case $n \in \mathbb{N}$. Without loss of generality we may assume that $S_i \subseteq [0, n)$, $i = 1, 2$. For $\lambda \in [0, 1]$ put

$$S_3 = \bigcup_{i \in \mathbb{Z}} \left[\left(\left[0, \frac{\lambda}{p}\right) \cap \lambda \left(S_1 - \frac{i-1}{p} \right) \right) \right. \\ \left. \cup \left(\left(\left[0, 1 - \frac{\lambda}{p}\right) \cap (1-\lambda) \left(S_2 - \frac{i-1}{p} \right) \right) + \frac{\lambda}{p} \right) + \frac{i-1}{p} \right].$$

Now, we have that $S_3 \in \sigma_n$, and $f^*_{S_3}(h) = \lambda f^*_{S_1}(h) + (1-\lambda) f^*_{S_2}(h)$, $h \in \mathbb{Z}/p$, holds by (3.2). If $n = \infty$, the assertion follows for $S_i \subseteq \mathbb{R}$, $i = 1, 2$. ☐

In the following, by $V(\mathcal{F}^*_{n,\mathbb{Z}})$ we will denote the (unknown) set of vertices representing the convex hull of $\mathcal{F}^*_{n,\mathbb{Z}}$. It will be a consequence of Proposition 3.2.1 below that $V(\mathcal{F}^*_{n,\mathbb{Z}})$ is contained in a natural superset with finite cardinality for any $n \in \mathbb{N}$, i.e. $|V(\mathcal{F}^*_{n,\mathbb{Z}})| \leq 2^n$. The superset will be determined by the set of all 2^n binary vectors that, however, entails substantial redundancies to be discussed below. We will introduce simple set correlation functions $f^*_{U_b}$ for $U_b = \bigcup_{j \in I_b} [j-1, j)$, where I_b is the set of indices corresponding to ones in $b = (b_1, \ldots, b_n) \in \mathcal{B}_n = \{0, 1\}^n$ (e.g. $I_b = \{1, 3, 4\}$ for $b = (1, 0, 1, 1)$). For the restriction of $f^*_{U_b}$, $b \in \mathcal{B}_n$, to $\mathbb{Z}$ we shall for simplicity introduce the notation $f^*_{I_b}$, and put $\mathcal{H}^*_{n,\mathbb{Z}} = \left\{ f^*_{I_b} \in \mathbb{R}^{\mathbb{Z}}, b \in \mathcal{B}_n \right\}$. For later reference, note that by (3.2), in particular,

$$f^*_{I_b}(h) = \sum_{k \in \mathbb{Z}} \min\{b_k, b_{k+h}\} |I_b|^{-1} = \sum_{k \in \mathbb{Z}} b_k b_{k+h} |I_b|^{-1}, \quad h \in \mathbb{Z},\, b \in \mathcal{B}_n, \tag{3.3}$$

where $b_k = 0$ for $k \in \mathbb{Z} \setminus \{1, \ldots, n\}$.

Proposition 3.2.1. *For all* $n \in \mathbb{N}$ *we have that* $V(\mathcal{F}^*_{n,\mathbb{Z}}) \subseteq \mathcal{H}^*_{n,\mathbb{Z}}$*. Further,* $V(\mathcal{F}^*_{\infty,\mathbb{Z}}) \subseteq \bigcup_{n=1}^{\infty} \mathcal{H}^*_{n,\mathbb{Z}}$.

Proof. In order to show the first assertion let $n \in \mathbb{N}$ and $S \in \sigma_n$. Without loss of generality we may assume that $S \subseteq [0, n)$. We will show that

$$f^*_S(h) = \sum_{b \in \mathcal{B}_n} f^*_{I_b}(h) \mu_b, \quad h \in \mathbb{Z},$$

where $0 \leq \mu_b \leq 1$, $b \in \mathcal{B}_n$, $\sum_{b \in \mathcal{B}_n} \mu_b = 1$. To this end, for all $b \in \mathcal{B}_n$ let

$$\delta_b = [0, 1) \cap \bigcap_{i \in I_b} (S + 1 - i) \tag{3.4}$$

and put

$$\Delta_b = \delta_b \cap \bigcap_{a \in \mathcal{B}_n : I_b \subset I_a} \delta_a^c \subseteq [0,1). \tag{3.5}$$

Now, we find that

$$\Delta_a \cap \Delta_b = \delta_a \cap \delta_b \cap \Bigg(\bigcup_{\substack{\omega \in \mathcal{B}_n : I_a \subset I_\omega \\ \text{or } I_b \subset I_\omega}} \delta_\omega \Bigg)^c = \emptyset \quad \text{for all } a, b, \in \mathcal{B}_n \text{ with } a \neq b. \tag{3.6}$$

Here, the last equality follows from the fact that by (3.4) we have

$$\delta_a \cap \delta_b \subseteq \bigcup_{\substack{\omega \in \mathcal{B}_n : I_a \subset I_\omega \\ \text{or } I_b \subset I_\omega}} \delta_\omega, \quad a, b, \in \mathcal{B}_n,\ a \neq b.$$

Let $S_b = \dot\bigcup_{i \in I_b} (\Delta_b + i - 1)$, $b \in \mathcal{B}_n$, where the union is disjoint by (3.5). By (3.5) and (3.6) we get in particular that

$$S_a \cap (S_b + h) = \emptyset \quad \text{for all } h \in \mathbb{Z}, \text{ and all } a, b, \in \mathcal{B}_n \text{ with } a \neq b. \tag{3.7}$$

Further, (3.4) and (3.5) yield for all $b \in \mathcal{B}_n$ that $S_b \subseteq S$, and hence

$$\dot\bigcup_{b \in \mathcal{B}_n} S_b \subseteq S. \tag{3.8}$$

Next, note that $x \in S$ by (3.4) implies that $x \in \delta_b + i - 1$ for some $i \in \{1, \dots, n\}$ and $b \in \mathcal{B}_n$ with $I_b = \{i\}$. By (3.5) we get further that $x \in \Delta_a + i - 1$ for some $a \in \mathcal{B}_n$ with $I_b \subseteq I_a$. Altogether we now find that $x \in S$ implies

$$x \in \dot\bigcup_{b \in \mathcal{B}_n} (\Delta_b + i - 1) \subseteq \dot\bigcup_{b \in \mathcal{B}_n} \dot\bigcup_{i \in I_b} (\Delta_b + i - 1) = \dot\bigcup_{b \in \mathcal{B}_n} S_b,$$

and hence $S = \dot\bigcup_{b \in \mathcal{B}_n} S_b$ by (3.8). Then, from (3.7) and (3.2) we get for $\mu_b = |S_b|/|S| = |\Delta_b||I_b|/|S|$, $b \in \mathcal{B}_n$, that

$$f_S^*(h) = \sum_{b \in \mathcal{B}_n} f_{S_b}^*(h)\mu_b = \sum_{b \in \mathcal{B}_n} f_{I_b}^*(h)\mu_b$$

where the second equality holds by definition of S_b and $f_{I_b}^*$. We finally consider the second assertion. For any $S \in \sigma_\infty$ let

$$L_n = \bigcup_{z \in \mathbb{Z}} \Bigg(z + \bigcup_{i \in \mathbb{Z} \setminus \{-n, \dots, n\}} ((S - i) \cap [0,1)) \Bigg), \quad n \in \mathbb{N},$$

and $S = (S \cap L_n) \cup (S \cap L_n^c)$. Then,

$$f_S^* = |S \cap L_n^c||S|^{-1} f_{S \cap L_n^c}^* + |S \cap L_n||S|^{-1} f_{S \cap L_n}^* \in \mathcal{F}_{\infty,\mathbb{Z}}^*$$

and $f_{S \cap L_n^c}^* \in \mathcal{F}_{2n+1,\mathbb{Z}}^*$. Now, for $n \to \infty$ we have that $|S \cap L_n^c||S|^{-1} \to 1$, and the second summand tends to 0. □

Next, via the introduction of suitable equivalence relations we will successively discard certain redundancies within $\mathcal{B}_n$ and finally determine a set $\mathcal{C}_n \subseteq \mathcal{B}_n$ with $V(\mathcal{H}^*_{n,\mathbb{Z}}) = \{f^*_{I_b} \in \mathbb{R}^{\mathbb{Z}}, b \in \mathcal{C}_n\}$. In particular, we will demonstrate that the immediate idea of congruence for any two sets I_a and I_b, $a, b \in \mathcal{B}_n$, is a sufficient condition for $f^*_{I_a} = f^*_{I_b}$ only whereas the concept of homometry that we shall discuss below is necessary and sufficient. Still, we will also study the former equivalence relation in more detail as the number of noncongruent and homometric vectors $a, b \in \mathcal{B}_n$ will turn out to be relatively small, cf. Proposition 3.2.2 and Table 3.2.1. To formalize the notion of congruence first define reflections $r_u : \{0,1\}^n \to \{0,1\}^n$, $u \in \{0,1\}$, $r_1((x_1,\dots,x_n)) = (x_n,\dots,x_1)$, $r_0 = id$, and translations $s_t : \{0,1\}^n \to \{0,1\}^n$, $t \in \mathbb{Z}$,

$$s_t((x_1,\dots,x_n)) = \begin{cases} (0,\dots,0,x_1,\dots,x_{n-t}) & \text{if } x_{n-t+1},\dots,x_n = 0 \text{ and } t \geq 0, \\ (x_{-t+1},\dots,x_n,0,\dots,0) & \text{if } x_1,\dots,x_{-t} = 0 \text{ and } t \leq -1, \\ (x_1,\dots,x_n) & \text{else.} \end{cases}$$

Now, for all $a, b \in \mathcal{B}_n$ we will define congruence by the equivalance relation $a \sim_c b$, $a = s_t \circ r_z(b)$ for some $(t,z) \in \{-n+1,\dots,n-1\} \times \{0,1\}$. We will denote the quotient set of $\mathcal{B}_n$ with respect to $\sim_c$ by $\mathcal{B}_n/{\sim_c}$ and state the following result for $|\mathcal{B}_n/{\sim_c}|$, i.e. the number of non-congruent patterns in $\mathcal{B}_n$.

Proposition 3.2.2. *We have that*

$$|\mathcal{B}_n/{\sim_c}| = 2^{n-2} + 2^{\lfloor (n-2)/2 \rfloor} + 2^{\lfloor (n-1)/2 \rfloor}, \quad n \in \mathbb{N}. \tag{3.9}$$

In particular, we have $|\mathcal{B}_n/{\sim_c}| \sim 2^{n-2}$.

Proof. Let $\mathcal{B}_{n,1} = \{b \in \mathcal{B}_n : b_1 = 1\} \subseteq \mathcal{B}_n$ where applying the translation defined above we have that $b = s_t(a)$ for all $b \in \mathcal{B}_n \setminus \{0\}$ and some $(t,a) \in \{0,\dots,n-1\} \times \mathcal{B}_{n,1}$. Hence, by definition of the equivalence relation $\sim_c$ we find that

$$|\mathcal{B}_n/{\sim_c}| = |\mathcal{B}_{n,1}/{\sim_c}| + 1. \tag{3.10}$$

Next, consider the partition $\mathcal{B}_{n,1,N} \cup \mathcal{B}_{n,1,E}$ of $\mathcal{B}_{n,1}$ where $\mathcal{B}_{n,1,N} = \{b \in \mathcal{B}_{n,1} : b_n = 0\}$ and $\mathcal{B}_{n,1,E} = \mathcal{B}_{n,1} \setminus \mathcal{B}_{n,1,N}$. We obviously get that $a \not\sim_c b$ for any $a \in \mathcal{B}_{n,1,N}$ and any $b \in \mathcal{B}_{n,1,E}$ such that

$$|\mathcal{B}_{n,1}/{\sim_c}| = |\mathcal{B}_{n,1,N}/{\sim_c}| + |\mathcal{B}_{n,1,E}/{\sim_c}|. \tag{3.11}$$

Note that by definition of $\mathcal{B}_{n,1}$ and $\mathcal{B}_{n,1,N}$ we have that $b \in \mathcal{B}_{n-1,1}$ if and only if $(b,0) \in \mathcal{B}_{n,1,N}$ such that, in particular, $|\mathcal{B}_{n-1,1}/{\sim_c}| = |\mathcal{B}_{n,1,N}/{\sim_c}|$. Applying the latter equality successively to (3.11) we find with (3.10) that

$$|\mathcal{B}_n/{\sim_c}| = \sum_{j=1}^{n} |\mathcal{B}_{j,1,E}/{\sim_c}| + 1. \tag{3.12}$$

For $S_n = \{b \in \mathcal{B}_{n,1,E} : b_k = b_{n-k+1}, k = 1, \ldots, n\}$, i.e. the set of all symmetric vectors $b \in \mathcal{B}_{n,1,E}$, we now consider the partition

$$\mathcal{B}_{n,1,E} = A_n \cup S_n \tag{3.13}$$

where $A_n = \mathcal{B}_{n,1,E} \setminus S_n$. It is immediate that S_n can be identified with its quotient set with respect to $\sim_c$, i.e. $S_n/\sim_c = S_n$. Moreover, with respect to the set $A_n \subseteq \mathcal{B}_{n,1,E}$ of asymmetric vectors for all $a \in A_n$ we have that $r_1(a) = b$ for some $b \in A_n$, $b \neq a$. Note that $s_t(a) = a$ for all $(t, a) \in \mathbb{Z} \times \mathcal{B}_{n,1,E}$, and hence we get that $|A_n/\sim_c| = \frac{1}{2}|A_n|$. Further, the definition of S_n yields that $a \not\sim_c b$ for any $a \in S_n$ and any $b \in A_n$ such that

$$|\mathcal{B}_{n,1,E}/\sim_c| = |A_n/\sim_c| + |S_n/\sim_c| = \frac{1}{2}|A_n| + |S_n| = \frac{1}{2}|\mathcal{B}_{n,1,E}| + \frac{1}{2}|S_n| \tag{3.14}$$

where the second equality follows from the above remarks and the third equality holds by (3.13). Note that $\mathcal{B}_{n,1,E}$ is the ensemble of all $b \in \mathcal{B}_n$ with $b_1 = b_n = 1$ and cardinality

$$|\mathcal{B}_{n,1,E}| = \sum_{m=0}^{n-2} \binom{n-2}{m}. \tag{3.15}$$

For the number of symmetric sequences $|S_n|$ we find by case differentiation that, for $n \geq 3$,

$$|S_n| = \begin{cases} \sum_{m=0}^{n-2} \binom{(n-2)/2}{m/2} & \text{if } m, n \text{ even,} \\ \sum_{m=0}^{n-2} \binom{(n-3)/2}{m/2} & \text{if } n \text{ odd and } m \text{ even,} \\ \sum_{m=0}^{n-2} \binom{(n-3)/2}{(m-1)/2} & \text{if } m, n \text{ odd,} \\ 0 & \text{else.} \end{cases} \tag{3.16}$$

Finally, using (3.12), and by case differentiation upon (3.16) we get that, for $n \geq 3$,

$$\begin{aligned} |\mathcal{B}_n/\sim_c| &= 5 + \frac{1}{2}\sum_{j=2}^{n-2}\sum_{m=0}^{j}\binom{j}{m} + \frac{1}{2}\sum_{j=1}^{\lfloor (n-2)/2 \rfloor}\sum_{m=0}^{j}\binom{j}{m} + \sum_{j=1}^{\lfloor (n-3)/2 \rfloor}\sum_{m=0}^{j}\binom{j}{m} \\ &= 2^{n-2} + 2^{\lfloor (n-2)/2 \rfloor} + 2^{\lfloor (n-1)/2 \rfloor}. \end{aligned}$$

It is readily seen that the r.h.s. also holds for $n = 1$ and $n = 2$. □

Note from Proposition 3.2.2 that a correction for congruence in $\mathcal{B}_n$ will asymptotically reduce the number of relevant binary vectors by three quarters. Next, in order to motivate the abovementioned notion of homometry we shall consider an alternative interpretation of the set correlation that focusses on the mutual differences between the elements in I_b, i.e.

$$f^*_{I_b}(h) = |\{(x,y) \in I_b^2 : x+h=y\}||I_b|^{-1}, \quad h \in \mathbb{Z}. \tag{3.17}$$

The concept of homometry, also known as the turnpike or partial digest problem, is typically specified by equations similar to (3.17). In particular, given all distances between points on the line is it possible to retrieve the corresponding sets I_b, $b \in \mathcal{B}_n$, up to congruence? Put differently, if any is there a unique class $[b] \in \mathcal{B}_n/{\sim_c}$, $b \in \mathcal{B}_n$, identified by a given set correlation function $f^* \in \mathcal{H}^*_{n,\mathbb{Z}}$? The answer goes back at least as far as [46, 47] in the context of the analysis of diffraction patterns in crystallography where the set covariance is related to so-called multisets and where it is also well-known that $|\mathcal{B}_n/{\sim_c}| > |\mathcal{H}^*_{n,\mathbb{Z}}|$, $n \geq 12$, cf. Table 3.2.1. In line with the above discussion two patterns $a, b \in \mathcal{B}_n$ are called homometric if, $a \sim_h b$, $f^*_{I_a} = f^*_{I_b}$, cf. [47]. Denote by $[a] = \{b \in \mathcal{B}_n : b \sim_h a\}$ the equivalence class of a, i.e. we identify the equivalence class $[a]$ with the corresponding function $f^*_{I_a}$ and put $f^*_{[a]} = f^*_{I_a}$, $a \in \mathcal{B}_n$. Let

$$\mathcal{B}_n/{\sim_h} = \{[b] : b \in \mathcal{B}_n\}.$$

Note that emphasizing its computational complexity the problem has been discussed more recently in [33]. For later reference we may define $|[b]| := |I_a|$ for some $a \in [b]$, $b \in \mathcal{B}_n$, by the following lemma.

Lemma 3.2.2. *From $a \in [b]$ for some $b \in \mathcal{B}_n$ it follows that $|I_a| = |I_b|$.*

Proof. By definition we have that $a \in [b]$, $b \in \mathcal{B}_n$, is equivalent to $f^*_{I_a} = f^*_{I_b}$ such that, in particular, $\max I_a - \min I_a = \max I_b - \min I_b =: r$. The latter yields by (3.3) that $f_{I_a}(r) = f_{I_b}(r) = 1$. Now, by definition we get $|I_a| = f_{I_a}(r)/f^*_{I_a}(r) = f_{I_b}(r)/f^*_{I_b}(r) = |I_b|$. □

By the above discussion we may now restrict to any representative of $\mathcal{B}_n/{\sim_h}$ as candidate vectors generating $V(\mathcal{H}^*_{n,\mathbb{Z}})$. For

$$\mathcal{C}_n = \left\{ [a] \in \mathcal{B}_n/{\sim_h} : f^*_{[a]} \neq \sum_{[b] \in \mathcal{B}_n/{\sim_h} \setminus \{[a]\}} f^*_{[b]} \mu_{[b]}, \text{ for all } \mu_{[b]} \in [0,1] \right\} \subseteq \mathcal{B}_n/{\sim_h} \tag{3.18}$$

we get that

$$\left\{ f^*_{[b]} \in \mathbb{R}^{\mathbb{Z}} : [b] \in \mathcal{C}_n \right\} = V(\mathcal{H}^*_{n,\mathbb{Z}}) = V(\mathcal{F}^*_{n,\mathbb{Z}}) \tag{3.19}$$

where the second equality follows from Proposition 3.2.1 and the fact that $\mathcal{H}^*_{n,\mathbb{Z}} \subseteq \mathcal{F}^*_{n,\mathbb{Z}}$. Note that beyond the idea of homometry we are not aware of a suitable concept that yields the set $\mathcal{C}_n$ directly from $\mathcal{B}_n$.

Theorem 3.2.1. *For all $S \in \sigma_n$, $n \in \mathbb{N} \cup \{\infty\}$, there is $\mathcal{X} \subseteq \mathcal{C}_n$ such that*

$$f^*_S(h) = \sum_{[b] \in \mathcal{X}} f^*_{[b]}(h) \mu_{[b]}, \quad h \in \mathbb{Z}, \tag{3.20}$$

where $0 < \mu_{[b]} \leq 1$, $[b] \in \mathcal{X}$, $\sum_{[b] \in \mathcal{X}} \mu_{[b]} = 1$. Reversely, given the r.h.s. of (3.20) a set $S \in \sigma_n$ exists such that (3.20) holds. In particular, $|\mathcal{X}| \leq n$, $n \in \mathbb{N}$.

n	$\lvert\mathcal{B}_n\rvert = 2^n$	$\lvert\mathcal{B}_n/{\sim_c}\rvert$	$\lvert\mathcal{B}_n/{\sim_c}\rvert - \lvert\mathcal{B}_n/{\sim_h}\rvert$	$\lvert\mathcal{B}_n/{\sim_h}\rvert - \lvert\mathcal{C}_n\rvert$
4	16	8	0	0
5	32	14	0	1
6	64	24	0	2
7	128	44	0	2
8	256	80	0	4
9	512	152	0	7
10	1024	288	0	19
11	2048	560	0	36
12	4096	1088	2	73
13	8192	2144	8	131
14	16384	4224	20	259
15	32768	8384	36	523
16	65536	16640	73	958
17	131072	33152	128	1762
18	262144	66048	234	3379
19	524288	131840	394	–
20	1048576	263168	682	–

Table 3.2.1: Number of equivalence classes with respect to congruence, cf. Proposition 3.2.2, and homometry where $|\mathcal{H}^*_{n,\mathbb{Z}}| = |\mathcal{B}_n/{\sim_h}|$. We also state the number of homometric equivalence classes in the interior of $\mathcal{C}_n$, i.e. a number $|\mathcal{B}_n/{\sim_h}| - |\mathcal{C}_n|$ of set correlation functions $f^*_{[b]}$, $b \in \mathcal{B}_n/{\sim_h}$, are convex combinations of some $f^*_{[a]}$, $[a] \in \mathcal{C}_n$, $a \neq b$, cf. (3.18). The latter result has been obtained by a search algorithm and gives a lower bound. Because of computational limitations we do not report the results for $n \geq 19$.

Proof. The proof of Proposition 3.2.1 yields that for all $S \in \sigma_n$, and $\bar{\mu}_b = |S_b|/|S|$ we have

$$f^*_S(h) = \sum_{b\in\mathcal{B}_n} f^*_{S_b}(h)\bar{\mu}_b = \sum_{b\in\mathcal{B}_n} f^*_{I_b}(h)\bar{\mu}_b = \sum_{[b]\in\mathcal{C}_n} f^*_{[b]}(h)\hat{\mu}_{[b]} = \sum_{[b]\in\mathcal{X}\subseteq\mathcal{C}_n} f^*_{[b]}(h)\mu_{[b]}, \quad h \in \mathbb{Z},$$

where the third equality follows by (3.19). The existence of $\mathcal{X} \subseteq \mathcal{C}_n$ with $|\mathcal{X}| \leq n$, is a consequence of Carathéodory's theorem [6] and (3.19). Finally, note that

$$\sum_{[b]\in\mathcal{C}_n} \hat{\mu}_{[b]} = \sum_{[b]\in\mathcal{X}} \mu_{[b]} = \sum_{b\in\mathcal{B}_n} \bar{\mu}_b = 1$$

where the weights $\hat{\mu}_{[b]}$, $[b] \in \mathcal{C}_n$, and $\mu_{[b]}$, $[b] \in \mathcal{X}$, are not unique in general. □

The fact that in general $\mathcal{B}_n/{\sim_h}$ may include interior points of $\mathcal{C}_n$ is referred to in Table 3.2.1 where it is shown that $|\mathcal{B}_n/{\sim_h}| > |\mathcal{C}_n|$ for $n \geq 5$, cf. also Section 3.5.3. Note that the results for $|\mathcal{B}_n/{\sim_h}|$ and $|\mathcal{C}_n|$ in Table 3.2.1 have been obtained by simulation, cf. [4]. Related questions have also been studied by [23] and [51].

3.3 Relations Between Extremal Coefficient and Set Correlation

Recall Definition (2.6) and Theorem 2.2.1 for a stationary max-stable process $(Y_t)_{t\in\mathbb{Z}}$ for $D = 1$ with standard Fréchet margins. For its finite dimensional distributions we then have

$$P(Y_1 \le y_1, \ldots, Y_k \le y_k) = \exp\left(-\int_0^1 \bigvee_{t=1}^{k} \frac{\Gamma^t(\tilde{g}_0(s))}{y_t} ds\right) \tag{3.21}$$

for all $k \ge 1$ and $y_t \ge 0$, $t = 1, \ldots, k$. Here, the so-called spectral functions (see Section 2.2) $\tilde{g}_t : [0,1] \to \mathbb{R}_+$ are such that $\int_0^1 \tilde{g}_t(s)ds = 1$ for all t where as before $\tilde{g}_{t+1} = \Gamma(\tilde{g}_t)$ for a piston Γ, see [14]. Note that we put $\Gamma^t = \Gamma \circ \ldots \circ \Gamma$ for the t-fold composition of Γ. As a summary measure reflecting the temporal (spatial) dependence structure of Y the metric

$$d(h) = \int |\tilde{g}_0(s) - \Gamma^h(\tilde{g}_0(s))| ds, \quad h \in \mathbb{Z}, \tag{3.22}$$

has been proposed, see e.g. [13]. Following its standard usage in the literature we shall, however, not directly refer to $d(h)$ but define the equivalent extremal coefficient function [56] as a transformation of (3.22) that is given by

$$\phi(h) = \frac{d(h)+2}{2} = \int \max\{\tilde{g}_0(s), \Gamma^h(\tilde{g}_0(s))\} ds, \quad h \in \mathbb{Z}. \tag{3.23}$$

Note that (3.21) yields a more intuitive interpretation of $\phi(h)$, i.e.

$$P(Y_0 \le y, Y_h \le y) = P(Y_0 \le y)^{\phi(h)}, \quad y > 0, h \in \mathbb{Z}, \tag{3.24}$$

and also

$$\phi(h) = 2 - \lim_{y\to\infty} P(Y_h > y \mid Y_0 > y), \quad h \in \mathbb{Z}, \tag{3.25}$$

as introduced in Chapter 1, cf. (1.1). Both representations particularly emphasize the relevance to practice of the extremal coefficient function, see also [22]. Especially (3.25) provides a convenient interpretation in terms of the conditional probability of an extreme event to follow a preceding extreme event at lag h. Note that $\phi(h) = 2$, $h \in \mathbb{Z}$, by (3.24) is equivalent to independence of Y_t and Y_{t+h} for all $t \in \mathbb{Z}$. We remark that (3.24) for a fixed lag $h \in \mathbb{N}$ closely corresponds to the definition of $\tilde{\phi}$ in (2.10). More precisely, let the vector $\boldsymbol{e}_{\{1,h+1\}} \in \mathbb{R}^{h+1}$ be defined as in Section 2.1. By (2.10) we now have

$$\begin{aligned}\tilde{\phi}_{\{1,h+1\}} &= y\tilde{l}(y^{-1}\boldsymbol{e}_{\{1,h+1\}}) = y\tilde{\mu}\left(\left[\mathbf{0}, y\boldsymbol{e}^{-1}_{\{1,h+1\}}\right]^c\right) \\ &= -y\ln\tilde{G}\left(y\boldsymbol{e}^{-1}_{\{1,h+1\}}\right) = -y\ln P(Y_1 \le y, Y_{h+1} \le y) \\ &= -y\ln P(Y_1 \le y)\phi(h \mid Y) = \phi(h \mid Y), \quad y > 0,\end{aligned}$$

where (2.5) yields the second equality and the third equality is a consequence of (2.2). The last two equalities hold by the assumption of standard Fréchet margins for Y, and (3.24), respectively. Here, in contrast to Section 2.1 the function $\tilde{G}$, and $\tilde{\mu}$ or $\tilde{l}$ equivalently, reflect the $(h+1)$-variate marginal distribution of the stationary max-stable process Y. We shall next restrict the above framework to the class of dissipative stationary max-stable processes, see e.g. [28]. By [65] such a process $(Y_t)_{t\in\mathbb{Z}}$ has the representation

$$Y_t = \max_{i\in\mathbb{N}} U_i \hat{g}_{t-z_i}(S_i), \quad t \in \mathbb{Z},$$

where as above $\hat{g}_t : [0,1] \to \mathbb{R}_+$ with $\int_0^1 \hat{g}_t(s)ds = 1$ for all t. Here, $\{(U_i, S_i, z_i)\}_{i=1}^{\infty}$ is a Poisson point process on $[0,\infty)\times\mathcal{S}\times\mathbb{Z}$ with intensity measure $u^{-2}\mathbf{1}(u>0)du\times dS\times 1$. Without loss of generality we may again assume that $\mathcal{S} = [0,1]$ and $dS = \mathbf{1}(s \in [0,1))ds$. For $g(t) = \hat{g}_{\lfloor t\rfloor}(t-\lfloor t\rfloor)$, $t\in\mathbb{R}$, we have that

$$Y_t = \max_{i\in\mathbb{N}} U_i g(t-z_i), \quad t \in \mathbb{Z}, \tag{3.26}$$

where $\{(U_i, z_i)\}_{i=1}^{\infty}$ is a Poisson point process on $[0,\infty)\times\mathbb{R}$ with intensity measure $u^{-2}\mathbf{1}(u>0)du \times dz$. Note that in (3.26) the single spectral function g completely charaterizes the dependence structure of the dissipative max-stable process Y on $\mathbb{Z}$. In the following we will write Y_g where it is advantageous to indicate that the process Y is generated by g. We have in particular that (3.23) simplifies to

$$\phi(h \mid g) = \int \max\{g(s), g(s+h)\}ds, \quad h \in \mathbb{Z}, \tag{3.27}$$

and denoting by $\mathrm{supp}(g)$ the support of g the range of Y_g is given by

$$r_{Y_g} = \inf\{m \in \mathbb{N} : |\mathrm{supp}(g) \cap (\mathrm{supp}(g)+t)| = 0 \text{ for all } |t| \geq m,\, t\in\mathbb{Z}\},$$

i.e. $(Y_1,\ldots,Y_k)$ and $(Y_{k+q},\ldots,Y_{k+q+l})$ are independent for all $q \geq r_Y$, $k,l\in\mathbb{N}$. For the ensemble of extremal coefficient functions we shall next discuss a summability condition, and put $\Phi_{\infty,\mathbb{Z}} = \left\{\phi\in[1,2]^{\mathbb{Z}} : \sum_{h\in\mathbb{Z}}(2-\phi(h)) < \infty\right\}$. We will denote by $\Phi_{n,\mathbb{Z}}$ the restriction of $\Phi_{\infty,\mathbb{Z}}$ to those underlying processes with finite range $r_Y \leq n$. The above classification of extremal coefficient functions motivates the following analogy to the term “long memory” [3] that usually refers to the non-summabilty of the autocovariance function. We will propose an analogous notion for max-stable processes.

Definition 3.3.1. A second order weakly stationary random process on $\mathbb{Z}$ with covariance function ρ has a long memory [3] if $\sum_{h\in\mathbb{Z}}|\rho(h)| = \infty$. A stationary random process Y on $\mathbb{Z}$ with existing extremal coefficient function ϕ has a long memory if $\phi(\cdot \mid Y) \notin \Phi_{\infty,\mathbb{Z}}$, i.e. the correlation function of the random process $\mathbf{1}(Y>n)$ is not absolutely summable in the limit as $n\to\infty$.

Proposition 3.3.1 ([18, Proposition 3])**.** *Any stationary max-stable process Y on $\mathbb{Z}$ with standard Fréchet margins and summable function $2-\phi(\cdot \mid Y)$ is dissipative.*

The following theorem is essential to the integration of the results discussed in Section 3.2 into the extreme value context. It characterizes every summable function $2-\phi$ for max-stable processes on $\mathbb{Z}$ as a special set correlation function. Note that its proof will be based on the rather lengthy arguments preceding Corollary 4.5.2 in Chapter 4.

Theorem 3.3.1. *For all* $n \in \mathbb{N} \cup \{\infty\}$ *we have* $\mathcal{F}^*_{n,\mathbb{Z}} = \{2 - \phi : \phi \in \Phi_{n,\mathbb{Z}}\}$.

Proof. Let $\xi \in \mathcal{F}^*_{n,\mathbb{Z}}$, $n \in \mathbb{N} \cup \{\infty\}$. Then, there is $S \in \sigma_n$ such that $f^*_S(h) = \xi(h)$, $h \in \mathbb{Z}$. Further, for $g(x) = \mathbf{1}(x \in S)|S|^{-1}$ we have $\phi(h \mid g) = 2 - \xi(h) \in \Phi_{n,\mathbb{Z}}$ by (3.27) and (3.2). The reverse direction is a direct consequence of Corollary 4.5.2 in Chapter 4, and Proposition 3.3.1. □

Now, Theorem 3.3.1 yields in particular that a discrete-time max-stable random process has a long memory if and only if its extremal coefficient function cannot be represented by a set correlation function. Note that Definition 3.3.1 also characterizes certain dissipative processes as having a long memory. Our point of view therefore differs from the interpretation in [52] where the definition for short memory phenomena coincides with a process being purely dissipative. Consider e.g. a dissipative process as in (3.26) with spectral function $g(s) = s^{-2}\mathbf{1}(s \geq 1)$ that has a long memory according to Definition 3.3.1. With respect to Theorem 3.3.1 note also that we have discussed three equivalent concepts representing the extremal coefficient function on a grid, namely ϕ, d and f, cf. (3.27), (3.22) and (3.2). We will henceforth mainly be concerned with two questions related to the above setup. Namely, in what way is the class of extremal coefficient functions restricted by the right-hand side of (3.27), and how can processes of the form given in (3.26) be reconstructed for given extremal coefficient functions? To this end, from now on we will focus on so-called M_3 processes, also termed mixed moving maxima. For $D = 1$ the processes correspond to the M_4 class introduced in Section 2.2. More precisely, the processes are discrete versions of (3.26) where

$$M_t = \max_{j=1}^{J} \max_{k \in \mathbb{Z}} a_{jk} Z_{j,t-k}, \quad t \in \mathbb{Z}, \tag{3.28}$$

for some $J \in \mathbb{N}$ and a sequence $\{Z_{jt}, j \in \{1, \ldots, J\}, t \in \mathbb{Z}\}$ of i.i.d. standard Fréchet variables, i.e. $P(Z_{jt} \leq u) = \exp(-u^{-1})$, $u > 0$. Further, $a_{jk} \geq 0$, $j \in \{1, \ldots, J\}$, $k \in \mathbb{Z}$, and $\sum_{j=1}^{J} \sum_{k \in \mathbb{Z}} a_{jk} = 1$ such that the marginal distributions of the M_3 processes are also standard Fréchet. Note that we obtain (3.28) from (3.26) by choosing

$$g(x) = J \sum_{j=1}^{J} \sum_{k \in \mathbb{Z}} a_{jk} \mathbf{1}(x \in k + J^{-1}[j-1, j)), \quad x \in \mathbb{R}. \tag{3.29}$$

We will consider the following useful classification of M_3 processes. To this end, by $\mathcal{M}_\iota$ we will denote the set of all M_3 processes with $J \leq \iota \in \mathbb{N} \cup \{\infty\}$. Note that for their special structure the elements of $\mathcal{M}_1$ are canonically referred to as moving maxima or

M_2 processes. Further, we will put $\mathcal{M}_{\iota,n}$ for the restriction of $\mathcal{M}_\iota$ to processes up to range $n \in \mathbb{N}$. The extremal coefficient function $\phi(h \mid M)$ using (3.27) and (3.29) equals

$$\phi(h \mid M) = (d(h \mid M) + 2)/2 \tag{3.30}$$

where

$$d(h \mid M) = \sum_{j=1}^{J} \sum_{k \in \mathbb{Z}} |a_{jk} - a_{j,k+h}|, \quad h \in \mathbb{Z}, M \in \mathcal{M}_\infty. \tag{3.31}$$

For later reference, by $\mathcal{D}_{\iota,n}$ we will accordingly denote the set of functions $d(h \mid M)$, $h \in \mathbb{Z}$, for all $M \in \mathcal{M}_{\iota,n}$, $\iota, n \in \mathbb{N} \cup \{\infty\}$.

3.4 A Class of Simple Processes for Given Extremal Coefficients

In the following we will turn the results for set correlation functions obtained in Section 3.2 into the construction of actual max-stable processes corresponding to given extremal coefficient functions. In particular, we will assign to each vertex of $\mathcal{F}^*_{n,\mathbb{Z}}$ a simple class of M_2 processes that represents the respective vertex extremal coefficient functions, cf. Corollary 3.4.1 below. We will then focus on weighted maxima of those classes in order to incorporate the convexity of $\Phi_{n,\mathbb{Z}}$. To this end, consider the following sparse class $R(\zeta) \subseteq \mathcal{M}_{|\mathcal{C}_n|,n}$ of M_3 processes. Let $\mathcal{G} = \{\zeta = (\zeta_{[b]})_{[b] \in \mathcal{C}_n} \in [0,1]^{|\mathcal{C}_n|} : \sum_{[b] \in \mathcal{C}_n} \zeta_{[b]} |[b]| = 1\}$, and for all $\zeta \in \mathcal{G}$ define

$$\begin{aligned} R(\zeta) = \Big\{ (R_t)_{t \in \mathbb{Z}} : R_t = \max_{[b] \in \mathcal{C}_n} \zeta_{[b]} \max_{k=1}^{n} r_{[b],k} Z_{[b],t-k}, t \in \mathbb{Z}, \\ \text{and } r_{[b]} = (r_{[b],1}, \ldots, r_{[b],n}) \in [b] \Big\} \end{aligned} \tag{3.32}$$

where as before by $\{Z_{[b],i}, [b] \in \mathcal{C}_n, i \in \mathbb{Z}\}$ we denote a sequence of i.i.d. standard Fréchet variables. Note from (3.32) that any complete vector of representatives $r = (r_{[b]})_{[b] \in \mathcal{C}_n}$ determines a particular process $R \in R(\zeta)$ for any given $\zeta \in \mathcal{G}$. In the following proposition we will state an essential property of the class $R(\zeta)$.

Proposition 3.4.1. *We have that $\phi(h \mid A) = \phi(h \mid B)$, $h \in \mathbb{Z}$, for all $A, B \in R(\zeta)$, $\zeta \in \mathcal{G}$.*

Proof. By (3.32) for any fixed $R \in R(\zeta)$ there is a unique vector of representatives $r \in \mathcal{B}_n^{|\mathcal{C}_n|}$. Consequently, we find by (3.30) that

$$\begin{aligned} \phi(h \mid R) &= \sum_{[b] \in \mathcal{C}_n} \zeta_{[b]} \sum_{k \in \mathbb{Z}} \max\left\{r_{[b],k}, r_{[b],k+h}\right\} = 2 - \sum_{[b] \in \mathcal{C}_n} \zeta_{[b]} \sum_{k \in \mathbb{Z}} \min\left\{r_{[b],k}, r_{[b],k+h}\right\} \\ &= 2 - \sum_{[b] \in \mathcal{C}_n} \zeta_{[b]} |[b]| f^*_{[b]}(h), \quad h \in \mathbb{Z}, \end{aligned}$$

where the last equality holds by (3.3), and where we tacitly assume that $r_{[b],k} = 0$ for all $k \in \mathbb{Z} \setminus \{1, \dots, n\}$, $[b] \in \mathcal{C}_n$. To conclude the proof note that the r.h.s. is independent of r. □

The next corollary follows immediately from the proof of Proposition 3.4.1. It identifies the abovementioned classes of M_3 processes $R(\zeta)$ that generate the vertex extremal coefficient functions.

Corollary 3.4.1. *Let $\zeta_{[b]} = |[b]|^{-1}$ for any fixed $[b] \in \mathcal{C}_n$, and let $\zeta_{[a]} = 0$ for all $[a] \in \mathcal{C}_n$, $[a] \neq [b]$. Then, $2 - \phi(h \mid R) = f^*_{[b]}(h) \in V(\mathcal{H}^*_{n,\mathbb{Z}})$ for all $R \in R(\zeta)$.*

Referring to Corollary 3.4.1 we shall in the following denote the vertex extremal coefficient functions by $\phi(h \mid [b]) = \phi(h \mid R) = 2 - f^*_{[b]}(h)$ for any $R \in R(\zeta)$ with $\zeta_{[b]} = |[b]|^{-1}$, $[b] \in \mathcal{C}_n$. The functions are displayed in Figure 3.4.1 for the case $n = 5$. We will show in Corollary 3.4.2 below that the restriction to the class $R(\zeta)$, $\zeta \in \mathcal{G}$, is admissible in order to represent any extremal coefficient function $\phi \in \Phi_{n,\mathbb{Z}}$. An actual example for the reconstruction of processes based on the classes $R(\zeta)$ will be discussed in more detail in Section 3.5.3.

Corollary 3.4.2. *For any extremal coefficient function $\phi \in \Phi_{n,\mathbb{Z}}$ there is a $\zeta \in \mathcal{G}$ with $|\{\zeta_{[b]} : \zeta_{[b]} > 0, [b] \in \mathcal{C}_n\}| \leq n$ such that for all $R \in R(\zeta)$ we have $\phi(h \mid R) = \phi(h)$, $h \in \mathbb{Z}$.*

Proof. By Theorem 3.3.1 there is $S \in \sigma_n$ such that $\phi(h) = 2 - f^*_S(h)$, $h \in \mathbb{Z}$. Further, Theorem 3.2.1 with $\zeta_{[b]} = \mu_{[b]}/|[b]|$, $[b] \in \mathcal{X}$, yields that

$$f^*_S(h) = \sum_{[b] \in \mathcal{X}} f^*_{[b]}(h)\zeta_{[b]}|[b]| = 2 - \phi(h \mid R) = 2 - \sum_{[b] \in \mathcal{X}} \phi(h \mid [b])\zeta_{[b]}|[b]|, \quad h \in \mathbb{Z},$$

for any process $R \in R(\zeta)$. Here, the second equality follows from the proof of Proposition 3.4.1 and the third equality is immediate from the definition of $\phi(\cdot \mid [b])$. □

Finally, it will be instructive to recall that any vertex extremal coefficient function $\phi(\cdot \mid [b])$ reflects a class $[b] \in \mathcal{C}_n$ of homometric vectors rather than a unique vector $b \in \mathcal{B}_n$. In particular, the signature pattern [68] of a process is in general not determined by the extremal coefficient function, see the discussion in Section 5.1 below. Even for a given function $\phi(\cdot \mid R) \in \Phi_{n,\mathbb{Z}}$ where $R \in R(\zeta)$, $\zeta \in \mathcal{G}$, the signature pattern corresponding to R is at best determined up to homometry, cf. Section 3.2.

3.5 Examples

3.5.1 Simplification of Arbitrary M_3 Processes with Given Coefficients

Let $A \in \mathcal{M}_{J,n}$, $J, n \in \mathbb{N}$, be given by the coefficients $a_{jk} \geq 0$, $j \in \{1, \dots, J\}$, $k \in \mathbb{Z}$. Due to the bounded range n of A we may assume without loss of generality that $a_{jk} = 0$,

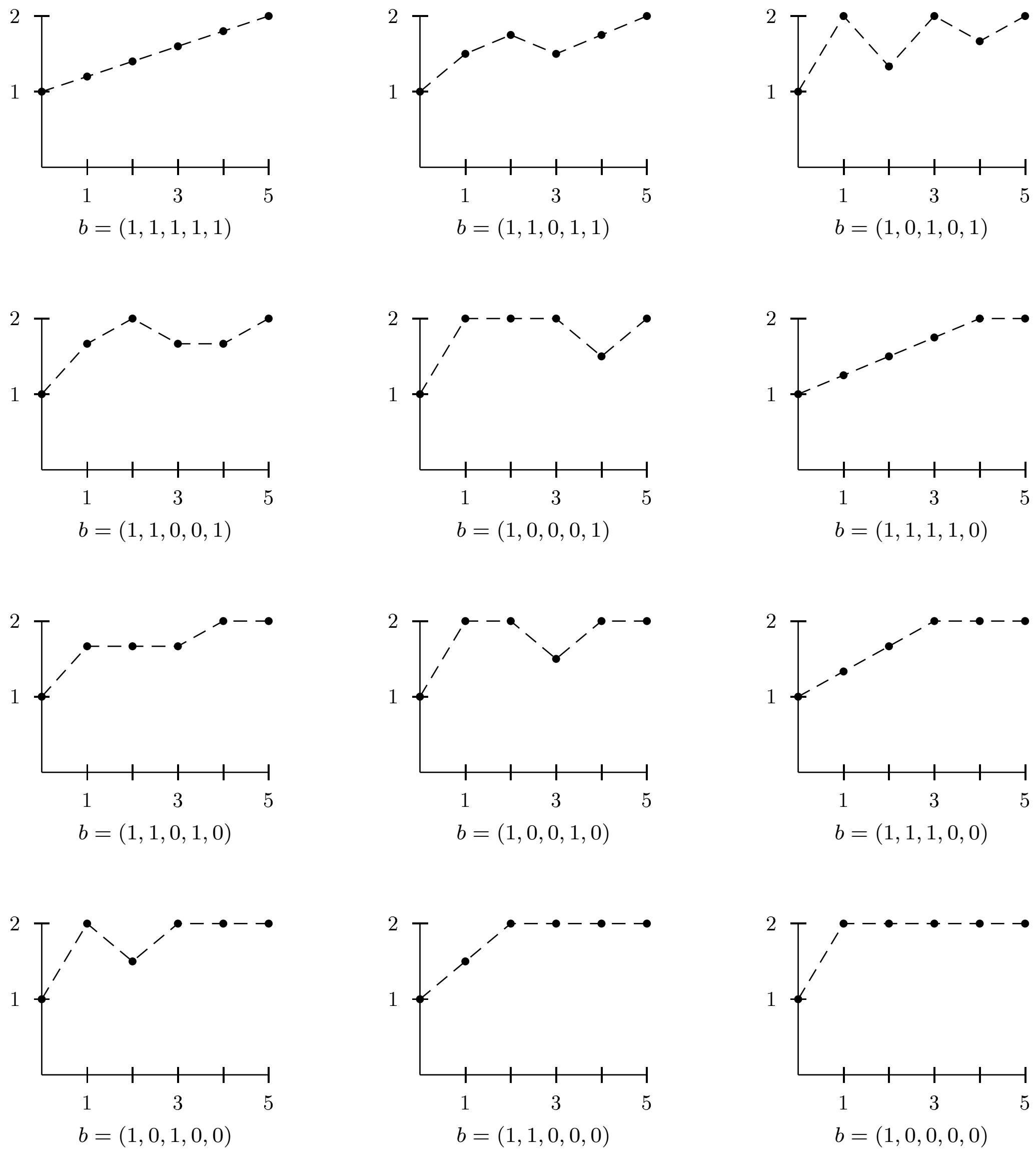

Figure 3.4.1: Vertex extremal coefficient functions $\phi(h \mid [b])$, $[b] \in \mathcal{C}_n$, for $n = 5$ and $h = 0, \ldots, 5$. The respective equivalence classes are identified by the corresponding representatives $b \in \mathcal{B}_n$. Note that we only tentatively include the lines joining the points as we confine our analysis to $\mathbb{Z}$.

$j \in \{1, \ldots, J\}$, $k \in \mathbb{Z} \setminus \{1, \ldots, n\}$. Define

$$\psi : [0,1]^n \to [0,1]^n$$
$$b \mapsto \max\left(b - \min\{b_i : b_i > 0\}, 0\right).$$

Let the M_3 process $C \in \mathcal{M}_{(Jn),n}$ carry a third index l in addition to j, k, and let C be defined by the coefficients $c_{jlk} \geq 0$, $j \in \{1, \ldots, J\}$, $l, k \in \{1, \ldots, n\}$, that is

$$C_t = \max_{j=1}^{J} \max_{l=1}^{n} \max_{k \in \mathbb{Z}} c_{jlk} Z_{jl,t-k}, \quad t \in \mathbb{Z}, \tag{3.33}$$

where the sequence $\{Z_{jli}, j \in \{1,\ldots,J\}, l \in \{1,\ldots,n\}, i \in \mathbb{Z}\}$ again represents i.i.d. standard Fréchet variables. Further, let $c_{jl} = (c_{jl,1},\ldots,c_{jl,n}) = \psi^{l-1}(a_j) - \psi^l(a_j)$ where $a_j = (a_{j,1},\ldots,a_{j,n})$ are the coefficients of A, and $\psi^l = \psi \circ \ldots \circ \psi$ gives the l-fold composition of ψ. We will make use of the following simple fact.

Lemma 3.5.1. *For all $a_1, a_2, m \in \mathbb{R}$ let $b_1 = \max(a_1 - m, 0)$, $b_2 = \max(a_2 - m, 0)$, $c_1 = \min(a_1, m)$ and $c_2 = \min(a_2, m)$. Then $|a_1 - a_2| = |b_1 - b_2| + |c_1 - c_2|$.*

Now, by a repeated application of Lemma 3.5.1 it follows that

$$d(h \mid A) = d(h \mid C) = \sum_{j=1}^{J}\sum_{l=1}^{n}\sum_{k=1}^{n} |c_{jlk} - c_{jl,k+h}|, \quad h \in \mathbb{Z}.$$

We will finally emphasize that the vertex extremal coefficient functions may be identified naturally from C. To this end, for $c_{jl} \neq 0$ let $m_{jl} = \max_k c_{jlk}$ and $\hat{c}_{jl} = c_{jl}/m_{jl}$ such that by definition of c_{jl} we have $\hat{c}_{jl} \in \mathcal{B}_n$ for all $j \in \{1,\ldots,J\}$ and all $l \in \{1,\ldots,n\}$. Next, put

$$C_{jl,t} = |[\hat{c}_{jl}]|^{-1} \max_{k\in\mathbb{Z}} \hat{c}_{jlk} Z_{jl,t-k}, \quad t \in \mathbb{Z},$$

such that (3.30) yields $C_{jl} \in \mathcal{M}_{1,n}$, and $\phi(h \mid C_{jl}) = |[\hat{c}_{jl}]|^{-1} \sum_{k\in\mathbb{Z}} \max\{\hat{c}_{jlk}, \hat{c}_{jl,k+h}\}$. Further, using (3.33) we get that $C_t = \max_{j=1}^{J} \max_{l=1}^{n} |[\hat{c}_{jl}]| m_{jl} C_{jl,t}$, $t \in \mathbb{Z}$, and, accordingly, by (3.30) we now have

$$\phi(h \mid C) = \sum_{j=1}^{J}\sum_{l=1}^{n} m_{jl} |[\hat{c}_{jl}]| \phi(h \mid C_{jl}). \tag{3.34}$$

The fact that $\hat{c}_{jl} \in \mathcal{B}_n$ by (3.32) yields that $C_{jl} \in R(\zeta)$ for $\zeta_{[\hat{c}_{jl}]} = |[\hat{c}_{jl}]|^{-1}$ such that for all processes C_{jl} with $\hat{c}_{jl} \in [b]$, $[b] \in \mathcal{C}_n$, we find by Corollary 3.4.1 that $\phi(\cdot \mid C_{jl}) = \phi(\cdot \mid [b])$. Finally, (3.34) gives

$$\phi(h \mid A) = \phi(h \mid C) = \sum_{[b]\in\mathcal{C}_n} \beta_{[b]} \phi(h \mid [b]), \quad h \in \mathbb{Z},$$

where $\beta_{[b]} = \sum_{j=1}^{J}\sum_{l=1}^{n} m_{jl} |[\hat{c}_{jl}]| \mathbf{1}(\hat{c}_{jl} \in [b])$ for all $[b] \in \mathcal{C}_n$. To conclude the example note that applying the arguments discussed in Sections 3.2 and 3.4 we may further reduce the appropriate index set to $\mathcal{X} \subseteq \mathcal{C}_n$.

3.5.2 Blind Reconstruction of M_2 Processes

We shall now turn to the blind retrieval of a real example process for an extremal coefficient function of a stationary max-stable process in discrete time with finite range n. Here, we will first restrict to the class of M_2 processes, that is we put $|I| = 1$, in order to show that given a priori knowledge about the index set I there are alternative approaches for the reconstruction of processes that do not necessarily resort to Corollary 3.4.2. Below we shall discuss such an approach. To this end, let $d \in \mathcal{D}_{1,n}$ be given. Then, there is an unknown (not necessarily unique) M_2 process X that is determined by its coefficients $x_1, \ldots, x_n$ such that by (3.31) we have

$$d(h \mid X) = \sum_{k=1}^{2n} |x_k - x_{k-h}|, \quad h = 1, \ldots, n. \tag{3.35}$$

In order to turn (3.35) into more tractable systems of linear equations we will make use of the following lemma that can be easily seen.

Lemma 3.5.2. *Let $x_i \geq 0$, $i = 1, \ldots, n$, and $x_i = 0$, else. There is a permutation π on $\{1, \ldots, n\}$ such that $x_{\pi^{-1}(1)} \geq \ldots \geq x_{\pi^{-1}(n)}$ and*

$$\sum_{i \in \mathbb{Z}} |x_i - x_{i-h}| = \sum_{i=1}^{n} \alpha_{\pi,h,i} x_i, \quad h = 1, \ldots, n, \tag{3.36}$$

where

$$\alpha_{\pi,h,i} = 2\left[\mathbf{1}(\pi(i) < \pi(i-h)) + \mathbf{1}(\pi(i) < \pi(i+h)) - 1\right] \in \{-2, 0, 2\} \tag{3.37}$$

for all $h, i \in \{1, \ldots, n\}$, and $\pi(i) = \infty$ for all $i \in \mathbb{Z} \setminus \{1, \ldots, n\}$. Further, $\sum_{i=1}^{n} \alpha_{\pi,h,i} = 2h$, $h = 1, \ldots, n$. The sequence of coefficients $\alpha_{\pi,h,i}$, $h, i = 1, \ldots, n$, is unique for a given permutation π, and vice versa.

Now, for the unknown M_2 process X according to Lemma 3.5.2 there is a (not necessarily unique) permutation π such that $x_{\pi^{-1}(1)} \geq \ldots \geq x_{\pi^{-1}(n)}$ and such that by (3.35) and (3.36) we have

$$d(h \mid X) = \sum_{i=1}^{n} \alpha_{\pi,h,i} x_i, \quad h = 1, \ldots, n. \tag{3.38}$$

Note that as π is unknown so is the sequence $\alpha_{\pi,h,i}$, $h, i = 1, \ldots, n$, and hence running through all possible permutations we will have that (3.38) represents $n!$ systems of linear equations. Here, in each case the coefficients $\alpha_{\pi,h,i}$ are given by (3.37). However, by the assumption that $d \in \mathcal{D}_{1,n}$ an appropriate permutation π will be associated to at least one of the linear systems, and a corresponding solution $x_1, \ldots, x_n$ representing such a process X exists. The latter can be found for instance via a linear program [4]. Note also that for any $d \in \mathcal{D}_{\infty,n}$ the above approach will reveal whether any solution to (3.38) exists at all, i.e. whether $d \in \mathcal{D}_{1,n} \subseteq D_{\infty,n}$.

3.5.3 Blind Reconstruction of M_3 Processes

As indicated by the above discussion we find that even with respect to the function $d(h \mid A)$ for an arbitrary process $A \in \mathcal{M}_{2,n}$ it appears to be nontrivial to state whether also $d(h \mid A) \in \mathcal{D}_{1,n}$. Put differently, given $A \in \mathcal{M}_{2,n}$ we ask whether there is a

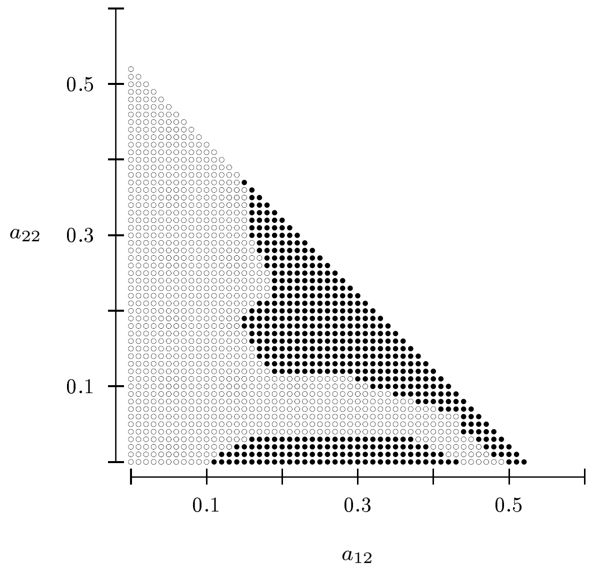

Figure 3.5.1: Admissible combinations of a_{12} and a_{22} for the process A discussed in Section 3.5.3 where $d(h \mid A) \in \mathcal{D}_{1,n}$ ($\circ$) and $d(h \mid A) \notin \mathcal{D}_{1,n}$ ($\bullet$).

process $B \in \mathcal{M}_{1,n}$ such that $d(h \mid A) = d(h \mid B)$, $h \in \mathbb{Z}$. Except for some pathological examples we are not aware of a suitable analytic criterion that focusses directly on the coefficients of A. Thus, using (3.38) and the method outlined above we will check by a trial and error procedure whether for simulated processes $A \in \mathcal{M}_{2,n}$ with arbitrary coefficients a_{jk}, $j \in \{1, 2\}$, $k \in \{1, \ldots, 5\}$, we have that $d(h \mid A) \in \mathcal{D}_{1,n}$. We give particular such processes $A(a_{12}, a_{22})$ where $d(h \mid A(a_{12}, a_{22})) \notin \mathcal{D}_{1,n}$ for at least some a_{12}, a_{22} in Table 3.5.1. In order to get more insight into the sensitivity of our results to changes of the coefficients we run through all admissible values of a_{12} and a_{22} with all other coefficients fixed and state whether $d(h \mid A(a_{12}, a_{22})) \in \mathcal{D}_{1,n}$. The result is given in Figure 3.5.1. Apart from a certain tradeoff between a_{12} and a_{22} along the upper right boundary the figure appears to reveal no specific structure.

a_{jk}	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$j=1$	0.01	a_{12}	0.02	0.05	0.21
$j=2$	$0.52 - a_{12} - a_{22}$	a_{22}	0.12	0.06	0.01

Table 3.5.1: Coefficients of the process $A(a_{12}, a_{22})$ discussed in Section 3.5.3.

Next, we will discuss an example for the reconstruction of max-stable processes that makes use of Corollary 3.4.2, that is we do not consider the above instances where $J = 1$. We will put $n = 5$ in order to cover at the same time the case $|\mathcal{B}_n/{\sim_h}| > |\mathcal{C}_n|$ discussed in Section 3.2. To this end, from the class of processes A discussed above we

b	$[b]$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$(1,1,1,1,1)$	1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1
$(1,1,0,1,1)$	2	ζ_2	ζ_2		ζ_2	ζ_2
$(1,0,1,0,1)$	3	ζ_3		ζ_3		ζ_3
$(1,1,0,0,1)$	4	ζ_4	ζ_4			ζ_4
$(1,0,0,0,1)$	5	ζ_5				ζ_5
$(1,1,1,1,0)$	6	ζ_6	ζ_6	ζ_6	ζ_6	
$(1,1,0,1,0)$	7	ζ_7	ζ_7		ζ_7	
$(1,0,0,1,0)$	8	ζ_8			ζ_8	
$(1,1,1,0,0)$	9	ζ_9	ζ_9	ζ_9		
$(1,0,1,0,0)$	10	ζ_{10}		ζ_{10}		
$(1,1,0,0,0)$	11	ζ_{11}	ζ_{11}			
$(1,0,0,0,0)$	12	ζ_{12}				

Table 3.5.2: Example coefficients $\zeta_{[b]}r_{[b],k}$, $k=1,\dots,5$, for a specific process $R \in R(\zeta) \subseteq \mathcal{M}_{12,5}$, cf. (3.32). Here, $(\zeta_1,\dots,\zeta_{12}) = \zeta$ where we use the notational convention explained after (3.39). See Figure 3.4.1 for an illustration of the vertex extremal coefficient functions $\phi(\cdot \mid [b])$, $[b] \in \mathcal{C}_n$, that are retrievable from any $R \in R(\zeta)$ if $\zeta_{[b]} = |[b]|^{-1}$. Note also that the case $b = (1,1,1,0,1)$ is not included in the table as $[b] \in \mathcal{B}_n/\sim_h$ but $[b] \notin \mathcal{C}_n$.

arbitrarily choose $A(0.15, 0.18)$ with $d(h \mid A(0.15, 0.18)) \notin \mathcal{D}_{1,n}$ where, in particular,

h	1	2	3	4	5
$d(h \mid A(0.15, 0.18))$	1.06	1.46	1.54	1.96	2.00

From now on, we will assume $d(h) = d(h \mid A)$ to be given and consider the process $A \in \mathcal{M}_{2,n} \setminus \mathcal{M}_{1,n}$ to be unknown. Let $\mathcal{G}_{d(h)} = \{\zeta \in \mathcal{G} : d(h \mid R) = d(h), h \in \mathbb{Z}, R \in R(\zeta)\}$ be the set of all vectors $\zeta \in \mathcal{G}$ that determine sets $R(\zeta)$ of suitable candidate processes. Note that $\mathcal{G}_{d(h)}$ is nonempty by Corollary 3.4.2. We will focus on the following system of linear equations

$$d(h) = d(h \mid R), \quad R \in R(\zeta), h \in \mathbb{Z}, \tag{3.39}$$

where by Proposition 3.4.1 we may choose $R \in R(\zeta)$ arbitrarily. A particular process $R \in R(\zeta)$ is given in Table 3.5.2. To simplify notation we shall replace the indices $[b]$, $[b] \in \mathcal{C}_n$, by $1,\dots,12$ according to the second column in Table 3.5.2. We now get from (3.39) and (3.31) for R as in Table 3.5.2 that

$$\begin{aligned}
d(1 \mid R) =&\ 2\zeta_1 + 4\zeta_2 + 6\zeta_3 + 4\zeta_4 + 4\zeta_5 + 2\zeta_6 + 4\zeta_7 + 4\zeta_8 + 2\zeta_9 + 4\zeta_{10} + 2\zeta_{11} + 2\zeta_{12}\\
d(2 \mid R) =&\ 4\zeta_1 + 6\zeta_2 + 2\zeta_3 + 6\zeta_4 + 4\zeta_5 + 4\zeta_6 + 4\zeta_7 + 4\zeta_8 + 4\zeta_9 + 2\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}\\
d(3 \mid R) =&\ 6\zeta_1 + 4\zeta_2 + 6\zeta_3 + 4\zeta_4 + 4\zeta_5 + 6\zeta_6 + 4\zeta_7 + 2\zeta_8 + 6\zeta_9 + 4\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}\\
d(4 \mid R) =&\ 8\zeta_1 + 6\zeta_2 + 4\zeta_3 + 4\zeta_4 + 2\zeta_5 + 8\zeta_6 + 6\zeta_7 + 4\zeta_8 + 6\zeta_9 + 4\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}\\
d(5 \mid R) =&10\zeta_1 + 8\zeta_2 + 6\zeta_3 + 6\zeta_4 + 4\zeta_5 + 8\zeta_6 + 6\zeta_7 + 4\zeta_8 + 6\zeta_9 + 4\zeta_{10} + 4\zeta_{11} + 2\zeta_{12}.
\end{aligned}$$

Numerically, if $\phi(h)$ is a valid extremal coefficient function, i.e. $\mathcal{G}_{d(h)}$ is nonempty, a particular element $\zeta \in \mathcal{G}_{d(h)}$ may be determined by expanding (3.39) to a linear program. Here, using [4] we find e.g.

$$\zeta = (\zeta_1, \ldots, \zeta_{12}) = (0.020, 0, 0, 0, 0, 0.085, 0, 0.105, 0, 0.040, 0.135, 0)$$

as a valid (not necessarily unique) solution. We point out that according to Corollary 3.4.2 there are $n = 5$ nonzero elements in ζ.

Remark 3.5.1. For all processes $R \in R(\zeta)$, $\zeta \in \mathcal{G}$, we have that $\sum_{[b]\in\mathcal{C}_n} \zeta_{[b]} = \theta_R$ where θ_R denotes the extremal index or, equivalently, the expected inverse cluster size, see Chapter 2. Note also that $1/n \leq \theta_R \leq 1$, i.e. the range n of R imposes a lower bound on the extremal index.

3.5.4 Necessary Conditions for Valid Extremal Coefficient Functions

Apart from the reconstruction of max-stable processes for given extremal coefficient functions the technique applied in Section 3.5.3 is applicable also to evaluate whether a supposed extremal coefficient function of order n is valid for max-stable processes on $\mathbb{Z}$. To our knowledge, in the literature so far only necessary conditions for extremal coefficient functions to be admissible have been discussed [10, 56]. Linking the results for first order variograms (madograms) discussed by [39] to extremal coefficient functions it is shown in [10] that every valid extremal coefficient function $\phi(h)$ for all $h, k \in \mathbb{R}$ satisfies

$$\phi(h+k) \leq \phi(h)\phi(k), \tag{3.40}$$

$$\phi(h+k)^\tau \leq \phi(h)^\tau + \phi(k)^\tau - 1, \quad 0 \leq \tau \leq 1, \tag{3.41}$$

$$\phi(h+k)^\tau \geq \phi(h)^\tau + \phi(k)^\tau - 1, \quad \tau \leq 0. \tag{3.42}$$

In addition, it is well-known that $\phi(h)$ is positive semi-definite, cf. [56]. We give an example showing that conditions (3.40) to (3.42) are indeed not sufficient. The construction of such an example is not evident but substantially facilitated by knowledge of the vertex extremal coefficient functions displayed in Figure 3.4.1. Consider e.g. the following function $p : \mathbb{Z} \to [1, 2]$, $p(-h) = p(h)$, with

h	0	1	2	3	4
$p(h)$	1	5/3	5/3	3/2	2

and $p(h) = 2$, $h \geq 5$. Note that

$$p(x) = \phi(x \mid [b]) \quad \text{for } x \in \{0, 3, 4, 5\} \tag{3.43}$$

and that

$$p(x) \neq \phi(x \mid [b]) \quad \text{for } x \in \{1, 2\} \tag{3.44}$$

for $b = (1, 0, 0, 1, 0) \in \mathcal{B}_n$. Further, by Figure 3.4.1 we easily find that

$$(\phi(3 \mid [b]), \phi(4 \mid [b]), \phi(5 \mid [b])) \neq \sum_{[a] \in \mathcal{C}_n \setminus \{[b]\}} (\phi(3 \mid [a]), \phi(4 \mid [a]), \phi(5 \mid [a]))\, \mu_{[a]} \quad (3.45)$$

for any $\mu_{[a]} \in [0, 1]$, $[a] \in \mathcal{C}_n \setminus \{[b]\}$. Now, using (3.43) to (3.45) we get from the convexity of $\Phi_{n,\mathbb{Z}}$ that p is not a valid extremal coefficient function. However, it is readily verified that p still satisfies (3.40) to (3.42).

3.6 Restrictions on the Range of Extremal Coefficient Functions

In the following we will study a lower bound on the range of a max-stable process if the corresponding extremal coefficient is known for a fixed $h \in \mathbb{N}$ only. More precisely, if for any fixed $h \in \mathbb{N}$ the extremal coefficient $\phi(h)$ is given we will specify the smallest lag $\bar{h} \geq h$ for which $\phi(\tilde{h}) = 2$ for all $\tilde{h} \geq \bar{h}$ is possible. In practice, the approach will be applicable to the study of the actual (bounded) memory spread of short memory processes, cf. Definition 3.3.1. Consider for instance the question of a lower bound on the memory of financial markets after shocks when information is limited to estimates of a single extremal coefficient.

Theorem 3.6.1. *Let $\phi(h \mid Y) \in [1, 2)$ be given for some fixed $h \in \mathbb{N}$ and some max-stable process $Y \in \mathcal{M}_\infty$. We have that $Y \notin \mathcal{M}_{\infty, r_\phi}$ for any*

$$r_\phi \in \begin{cases} \mathbb{N} & \text{if } \phi(h) = 1, \\ \{1, \ldots, [\![(\phi(h) - 1)^{-1}]\!] h\} & \text{else,} \end{cases}$$

where $[\![x]\!] = \max\{n \in \mathbb{Z} : n < x\}$ for any $x \in \mathbb{R}$. On the other hand, if $\phi(h) \in (1, 2)$, for some $h \in \mathbb{N}$, then a process $Y \in \mathcal{M}_{\infty, [\![(\phi(h)-1)^{-1}]\!]h+1}$ with $\phi(h \mid Y) = \phi(h)$ exists.

Proof. The assertion for $\phi(h) = 1$, $h > 0$, follows directly from Theorem 1.4.1(2) in [53]. The proof for $\phi(h) \in (1, 2)$ will be based on the M_3 representation for dissipative max-stable processes discussed in Section 3.3, and comprises three steps. First, within the classes $\mathcal{M}_{\infty, K+h-1}$ of all M_3 processes with maximum range $K + h - 1$, $K \in \mathbb{N}h + 1 = \{h + 1, 2h + 1, \ldots\}$, we will define a simple M_3 process $A_{K,h} \in \mathcal{M}_{1,K}$ of range K. Then, we will show that $A_{K,h} \in \mathcal{M}_{\infty, K+h-1}$ minimizes $\phi(h \mid B)$ for all $B \in \mathcal{M}_{\infty, K+h-1}$. Based on this finding we may conclude in step three that all processes $Z \in \mathcal{M}_\infty$ with $\phi(h \mid Z) = \phi(h)$ are at least of range $[\![(\phi(h) - 1)^{-1}]\!] h + 1$. We will give an example in order to show that the bounds are sharp.

1. For any $K \in \mathbb{N}h + 1$ let the process $A_{K,h} \in \mathcal{M}_{1,K}$ be given by the coefficients $a_{K,k}$, $k \in \mathbb{Z}$, where

$$a_{K,ih+1} = \left(\frac{K - 1}{h} + 1 \right)^{-1}, \quad i \in \{0, 1, \ldots, (K - 1)/h\}, \quad (3.46)$$

and all other coefficients zero. In particular, by (3.31) we have

$$d(h \mid A_{K,h}) = 2a_{K,1}. \tag{3.47}$$

Without loss of generality we let $B \in \mathcal{M}_{\infty,K+h-1}$ be given by

$$0 \leq b_{1k} = a_{K,k} + \varepsilon_{1k} \leq 1, \quad k \in \{1,\dots,K+h-1\}, \tag{3.48}$$

where the $a_{K,k}$ are chosen according to (3.46). Further, for $j \in \{2,3,\dots\}$ and $k \in \{1,\dots,K+h-1\}$ we let

$$0 \leq b_{jk} = \varepsilon_{jk} \leq 1 \tag{3.49}$$

and tacitly assume all other coefficients to be zero. Now, from the fact that $\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} b_{jk} = 1$ we get by (3.48) and (3.49) that

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} \varepsilon_{jk} = 0 \tag{3.50}$$

and, in particular,

$$\sum_{i=0}^{\frac{K-1}{h}} \varepsilon_{1,ih+1} \leq 0. \tag{3.51}$$

2. We show that for all processes $B \in \mathcal{M}_{\infty,K+h-1}$ it holds that

$$d(h \mid B) \geq d(h \mid A_{K,h}), \quad K \in \mathbb{N}h+1. \tag{3.52}$$

To this end, note that by (3.31) we find that (3.52) is equivalent to

$$-\varepsilon_{11} - \varepsilon_{1,K} \leq \sum_{j=1}^{\infty}\sum_{l=1}^{h}\sum_{i=0}^{\frac{K-1}{h}-1} |\varepsilon_{j,l+(i+1)h} - \varepsilon_{j,l+ih}| + \sum_{j=1}^{\infty}\sum_{\substack{l=1\\ j+l>2}}^{h} (\varepsilon_{jl} + \varepsilon_{j,K+l-1}). \tag{3.53}$$

Now, (3.48) and (3.49) yield that $\varepsilon_{j,l}, \varepsilon_{j,K+l-1} \geq 0$ for all $j \in \mathbb{N}$ and $l \in \{1,\dots,h\}$ with $j+l > 2$, such that (3.53) holds if $\min\{\varepsilon_{11}, \varepsilon_{1,K}\} \geq 0$. In order to show (3.53) for the case $\min\{\varepsilon_{11}, \varepsilon_{1,K}\} < 0$ put $N = \{ih+1,\ i = 0,1,\dots,(K-1)/h\}$ and for $j \in \mathbb{N}$, $l \in \{1,2,\dots,h\}$ let $S_{jl} = \sum_{i\in N+l-1} \varepsilon_{ji}$, $\bar{\mu}_{jl} = S_{jl}|N|^{-1}$ and $\mu_{jl,\max} = \max_{i\in N+l-1} \varepsilon_{ji}$. Further, let

$$\mu_{1,\min} = -\min\{\mu_{1,1,\max}, 0\}. \tag{3.54}$$

Now, we find that

$$\begin{aligned} -\varepsilon_{11} - \varepsilon_{1,K} &\leq |\varepsilon_{11}| + |\varepsilon_{1,K}| \leq |\varepsilon_{11} - \mu_{1,\min}| + |\varepsilon_{1,K} - \mu_{1,\min}| + 2\mu_{1,\min} \\ &\leq \sum_{i=0}^{\frac{K-1}{h}-1} |\varepsilon_{1,ih+1} - \varepsilon_{1,(i+1)h+1}| + 2\mu_{1,\min}. \end{aligned} \tag{3.55}$$

Also, by (3.54) we get for $\min\{\varepsilon_{11}, \varepsilon_{1,K}\} < 0$ that $\min_{i\in N} \varepsilon_{1,i} \leq \mu_{1,\min} \leq \max_{i\in N} \varepsilon_{1,i}$ which yields the second inequality in (3.55). Next, we have

$$\mu_{1,\min} \leq \frac{-S_{1,1}}{|N|} = \frac{1}{|N|} \sum_{j=1}^{\infty} \sum_{\substack{l=1\\ j+l>2}}^{h} S_{j,l} = \sum_{j=1}^{\infty} \sum_{\substack{l=1\\ l+j>2}}^{h} \bar{\mu}_{jl} \leq \sum_{j=1}^{\infty} \sum_{\substack{l=1\\ j+l>2}}^{h} \mu_{j,l,\max}. \tag{3.56}$$

Here, if $\max_{i\in N}\{\varepsilon_{1,i}\} \geq 0$ note that $\mu_{1,\min} = 0$, such that the first inequality is obvious from the fact that $S_{1,1} \leq 0$ by (3.51). Else, if $\max_{i\in N}\{\varepsilon_{1,i}\} < 0$ we get that $\mu_{1,\min} = \min_{i\in N}\{-\varepsilon_{1,i}\} \leq |N|^{-1} \sum_{i\in N} -\varepsilon_{1,i} = -S_{1,1}|N|^{-1}$ which in that case yields the first inequality. Further, the first equality in (3.56) holds by (3.50) and the second equality as well as the second inequality are immediate. Finally, for all $j \in \mathbb{N}$ and all $l \in \{1, \ldots, h\}$ with $l + j > 2$ we have

$$\begin{aligned} 2\mu_{j,l,\max} &= |\mu_{j,l,\max} - \varepsilon_{jl}| + |\mu_{j,l,\max} - \varepsilon_{j,l+K-1}| + \varepsilon_{jl} + \varepsilon_{j,l+K-1} \\ &\leq \sum_{i=0}^{\frac{K-1}{h}-1} |\varepsilon_{j,l+(i+1)h} - \varepsilon_{j,l+ih}| + \varepsilon_{jl} + \varepsilon_{j,l+K-1}. \end{aligned} \tag{3.57}$$

where the equality follows from the fact that $0 \leq \varepsilon_{jl} \leq \mu_{jl,\max}$ for $l + j > 2$. Now, (3.53) holds by (3.55) to (3.57).

3. Let $\mathcal{Z}(d(h)) \subseteq \mathcal{M}_\infty$ be the class of all M_3 processes Z with $d(h \mid Z) = d(h)$. By (3.52) it follows that $\mathcal{Z}(d(h)) \cap \mathcal{M}_{\infty,\kappa+h-1} = \emptyset$ for all

$$\kappa \in \{K \in \mathbb{N}h + 1 : d(h \mid A_{K,h}) > d(h)\} = \{K \in \mathbb{N}h + 1 : K < [\![2/d(h)]\!]h + 1\}$$

where (3.47) yields the equality. Let $K^* = [\![2/d(h)]\!]h + 1$ where $K^* < \infty$ from the fact that by assumption $d(h) > 0$. In particular, we now have that

$$d(h \mid A_{K^*,h}) \leq d(h) < d(h \mid A_{K^*-1,h}). \tag{3.58}$$

It remains to show that a process $Z^* \in \mathcal{M}_{\infty,K^*} \cap \mathcal{Z}(d(h))$ exists. To this end, let Z^* be given by $z_k^* = a_{K^*,k} - \varepsilon_k + \delta_k$, $k \in \{1, 2, \ldots, K^*\}$ where $a_{K^*,k}$ are the coefficients of $A_{K^*,h}$, cf. (3.46). Further, we put $\varepsilon_{ih+1} = \frac{1}{2}a_{K^*,1}(d(h) - 2a_{K^*,1})$, $i \in \{0, 1, \ldots, (K^* - 1)/h\}$, $\delta_2 = \frac{1}{2}(d(h) - 2z_1^*)$ and all other coefficients zero such that Z^* is of range K^*. Note that (3.58) yields

$$0 \leq \frac{1}{2}d(h) - a_{K^*,1} < a_{K^*-1,1} - a_{K^*,1} = \frac{h}{K^* - 1} - \frac{h}{K^* - 1 + h} < 1$$

such that $0 \leq \varepsilon_{1+ih} < a_{K^*,1}$, $i \in \{0, 1, \ldots, (K^* - 1)/h\}$. Further, using (3.58) we have

$$2z_1^* = 2(a_{K^*,1} - \varepsilon_1) < 2a_{K^*,1} = d(h \mid A_{K^*,h}) \leq d(h)$$

which yields that $\delta_2 > 0$. Finally, $d(h \mid Z^*) = d(h)$ is a consequence of (3.31).

□

Chapter 4

A Constructive Proof for the Extremal Coefficient of a Dissipative Max-Stable Process on $\mathbb{Z}$ being a Set Correlation

4.1 Formal Setup

In this chapter we will develop successively a certain sequence of sets that form the basis of the assertion in Corollaries 4.5.1 and 4.5.2. The latter was already referred to in the proof of Theorem 3.3.1 above. There, we stated that the sets of extremal coefficient functions for dissipative max-stable processes and those of set correlations actually coincide on $\mathbb{Z}$. To begin with we shall briefly restate the definition of the extremal coefficient function in terms of a spectral function g as the latter will be the starting point of our analysis. Recall that a stationary dissipative max-stable process Y on $\mathbb{Z}$ with standard Fréchet margins has the representation $Y_t = \max_{i\in\mathbb{N}} U_i g(t - z_i)$, $t \in \mathbb{Z}$, cf. Section 3.3. Here, $g : \mathbb{R} \to \mathbb{R}_+$ with $\int g(s)ds = 1$, and $\{(U_i, z_i)\}_{i=1}^{\infty}$ is a Poisson point process on $[0, \infty) \times \mathbb{R}$ with intensity measure $u^{-2}\mathbf{1}(u > 0)du \times dz$. In particular, the spectral function g completely characterizes the dependence structure of Y. As a suitable summary measure with properties similar to the usual autocovariance function in Section 3.3 we discussed the extremal coefficient function

$$\phi(h \mid g) = \int \max\{g(s), g(s + h)\}\, ds, \quad h \in \mathbb{Z}, \tag{4.1}$$

that has been proposed by [56]. In the following we will consider a sequence $(g_n)_{n\in\mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, of non-negative step functions such that $g_n \uparrow \xi$ for a suitable function $\xi : \mathbb{R} \to \mathbb{R}_+$ with $\phi(\cdot \mid \xi) = \phi(\cdot \mid g)$, and hence $\phi(\cdot \mid g_n) \to \phi(\cdot \mid g)$ as $n \to \infty$. Our main result will be the construction of a bounded monotonic sequence of sets, i.e.

$(S_n)_{n\in\mathbb{N}_0} \uparrow S$, $|S| < \infty$, associated to (g_n) such that

$$2\int g_n(s)\,ds - \phi(h \mid g_n) = |S_n \cap (S_n - h)|, \quad n \in \mathbb{N}_0, h \in \mathbb{Z}. \tag{4.2}$$

Hence, our analysis will imply that for any extremal coefficient function (4.1) on $\mathbb{Z}$ an equivalent representation as a set covariance function $|S \cap (S-h)|$, $h \in \mathbb{Z}$, given by a certain set $S \subset \mathbb{R}$, $|S| < \infty$, exists. The reverse is straightforward, cf. Section 3.3. Consequently, the ensembles for set covariance and extremal coefficient functions for dissipative processes can be shown to actually coincide on a grid. For reasons of content this result was already stated in Section 3.3 without proof, see Theorem 3.3.1.
To be specific, let $(g_n)_{n\in\mathbb{N}_0} \uparrow \xi$ be a monotonically increasing sequence of step functions with nonnegative coefficients a_{nki}, $n \in \mathbb{N}_0$, $k \in K_n = \{-n, \dots, n\}$, $i = (i_1, \dots, i_n) \in \{0,1\}^n$, and all other coefficients zero. Here, $i_s \in \{0,1\}$, $s = 1, \dots, n$, and $i = i_0 = \emptyset$ if $n = 0$. Throughout this chapter, we will put $[i]_2 = \sum_{j=1}^{n} i_j 2^{n-j}$ and $i|_\tau = (i_1, \dots, i_\tau)$, $\tau = 1, \dots, n$, where $i|_0 = \emptyset$. Note that the use of a binary number for the index i will be advantageous later on. As before, for all $x \in \mathbb{R}$ we put $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$, and we will assume from now on that

$$g_n(x) = \sum_{s=0}^{n} a_{s,k,i|_s}, \text{ where } k = \lfloor x \rfloor \text{ and } i \in \{0,1\}^n \text{ with } [i]_2 = \lfloor 2^n(x-k) \rfloor. \tag{4.3}$$

According to (4.1) let $\phi(h \mid g_n)$, $h \in \mathbb{Z}$, $n \in \mathbb{N}_0$, denote the extremal coefficient function of the stationary dissipative process Y_{g_n} generated by the spectral function g_n where by (4.3) we find that

$$\begin{aligned}\phi(h \mid g_n) = 2^{-n+1} &\sum_{k\in K_n} \sum_{i\in\{0,1\}^n} \sum_{s=0}^{n} a_{s,k,i|_s} - \\ &2^{-n} \sum_{k\in K_n} \sum_{i\in\{0,1\}^n} \min\left\{ \sum_{s=0}^{n} a_{s,k,i|_s}, \sum_{s=0}^{n} a_{s,k+h,i|_s} \right\}, \quad h \in \mathbb{Z}.\end{aligned} \tag{4.4}$$

Example 4.1.1. For the continuous spectral function $g = \xi$ given in Figure 4.1.1 we sketch the first three elements of a monotonic sequence of step functions $(g_n)_{n\in\mathbb{N}_0} \uparrow \xi$ given by (4.3) with

$$\begin{array}{lllll} a_{0,0,\emptyset} = \frac{1}{10} & & a_{1,0,0} = \frac{1}{15} & & a_{2,-2,(0,0)} = \frac{1}{5} \\ a_{1,-1,0} = \frac{1}{30} & & a_{1,1,0} = \frac{1}{15} & & a_{2,0,(1,0)} = \frac{1}{30} \\ a_{1,-1,1} = \frac{2}{15} & & a_{1,1,1} = \frac{1}{30} & & a_{2,2,(1,0)} = \frac{1}{15}. \end{array}$$

To give a preliminary idea of our construction principle for a suitable sequence (S_n) consider the sets A_{ski}, $s = 0, 1, 2$, $k \in K_s$, $i \in \{0,1\}^s$, given in Figure 4.3.1. Their formal structure will be studied below. We put

$$S_n = \dot{\bigcup_{s=0}^{n}} \dot{\bigcup_{k\in K_s}} \dot{\bigcup_{i\in\{0,1\}^s}} (A_{ski} + k), \quad n \in \mathbb{N}_0, \tag{4.5}$$

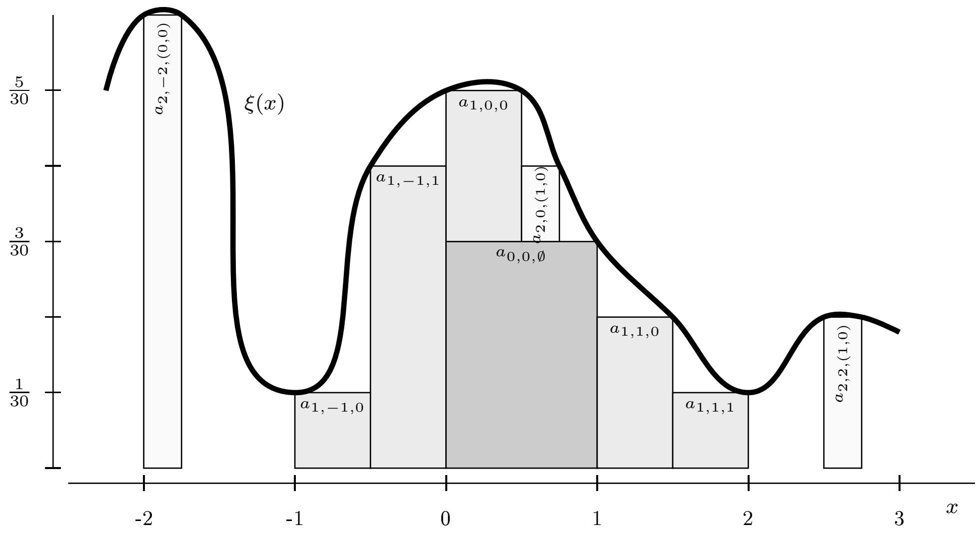

Figure 4.1.1: Approximation of a continuous spectral function $g = \xi$ by the step function g_2 defined in (4.3). The corresponding coefficients are given in Example 4.1.1. Note that the bars a_{nki} do not necessarily touch the graph of ξ.

such that (S_n) is clearly monotonic, and we will show in Theorem 4.5.1, cf. Section 4.5, that (4.2) holds. For the above small set of example coefficients the latter may be readily verified graphically using Figures 4.1.1 and 4.3.1. We point out that in (4.2) the requirement of monotonicity for (S_n) appears to be a fundamental restriction. More precisely, the determination of an arbitrary sequence (S_n) such that (4.2) holds is straightforward. Throughout the rest of our analysis we will mainly be concerned with the construction of suitable sets A_{ski} as well as the discussion of their properties. However, note that in Figure 4.3.1 we also include certain intervals B_{sbi}, $s = 0, 1, 2$, $i \in \{0, 1\}^s$, for some tedious index b that will be discussed below. At this point the intervals B_{sbi} may best be thought of as placeholders. In particular, they indicate allowable locations for the sets A_{ski}. Further, the intervals B_{sbi} will necessarily have to be constructed jointly with A_{ski}. As an only preliminary remark in this direction note that the construction of $B_{s+1,a,j}$ for $j \in \{(i, 0), (i, 1)\}$ and some suitable index a will be shown in (4.16) below to depend on certain intersections of A_{ski} on the index k ranging over particular orderable subsets of K_s. The latter will specifically be reflected by the index a. To conclude the example, note that in Theorem 4.5.1 we will essentially make use of the fact that

$$2^{-n} g_n(k + [i]_2 2^{-n}) = \left| \dot{\bigcup_{s=0}^{n}} A_{s,k,i|s} \cap \dot{\bigcup_{b}} B_{nbi} \right| \tag{4.6}$$

for all $k \in K_n$ and all $i \in \{0, 1\}^n$. It will be helpful later on to check at this point that (4.6) holds for the above example using Figures 4.1.1 and 4.3.1.

Throughout the chapter we will as before denote a proper inclusion by "$\subset$" whereas we shall use "$\subseteq$" for an inclusion that does not preclude equality. Further, we will understand $[x,y)=\emptyset$ if $y<x$, and $A^0=\emptyset$ for any set A. For $n\in\mathbb{N}_0$ let $\mathbb{B}_n=\{0,1\}^{n^2}$, $b=(b_1,\dots,b_n)\in\mathbb{B}_n$, where $b_s\in\{0,1\}^{2s-1}$, $s=1,\dots,n$, $b=b_0=\emptyset$ if $n=0$, and N_{b_s} is the set of indices corresponding to zeros in b_s, e.g. $N_{b_s}=\{2,5\}$ for $b_s=(1,0,1,1,0)$. Let $E_{n,0}=\mathbb{B}_n\times\{0,1\}^n$ and $E_n=E_{n,0}\setminus(\{0\}\times\{0,1\}^n)$. We will put $b|_\tau=(b_0,\dots,b_\tau)$, and accordingly $(b,i)|_\tau=(b|_\tau,i|_\tau)$ for $\tau=0,\dots,n$. Our approach will be organized as follows. In Section 4.2 we will define suitable intervals B_{nbi} and discuss their relevant properties. To this end, we shall study an order on the joint index $(b,i)\in E_{n,0}$ that will later refer to the allocation of the intervals B_{nbi} on the line. We will formally introduce the order in (4.7) below. The nature of the order will then be largely revealed by part 2 of Lemma 4.2.1. It will be shown in Lemma 4.2.2 that the order is total on a suitable subset of $E_{n,0}$. In particular, the definition of B_{nbi} in (4.16) will be restricted to this subset in a natural way. As we will be able to draw some important conclusions on the intervals B_{nbi} even for arbitrary sets A_{nki} we will defer the actual joint definition of B_{nbi} and A_{nki} to Section 4.3. There, in Corollaries 4.3.1 and 4.3.2 we will show that the assertions of two auxiliary assumptions made for step n, cf. (A1) and (A2) in Section 4.2, hold true by induction in step $n+1$. In Section 4.3 we will further discuss two important properties of A_{nki} in Lemmata 4.3.1 and 4.3.2. Thereafter we will study a decomposition of A_{nki} in Section 4.4 that will eventually be useful in the proof of Theorem 4.5.1 in Section 4.5 where we will show that (4.2) holds. Our main result will be stated in Corollaries 4.5.1 and 4.5.2 where for the latter we will make use of the fact that for any given spectral function g there is a suitable function ξ as a limit of step functions with $\phi(\cdot\mid g)=\phi(\cdot\mid\xi)$, cf. [19].

4.2 A Sequence of Auxiliary Sets

To begin with, we will equip the sets $E_{n,0}$, $n\in\mathbb{N}$, with the following partial order "$\prec_p$". For $(b,i)\in E_{n,0}$ let

$$\begin{aligned}\{(a,j)\in E_{n,0}:(a,j)\prec_p(b,i)\}=\Big\{&(a,j)\in E_{n,0}:\exists\tau\le n\text{ such that }a|_\tau=b|_\tau\\&\text{and }a|_\kappa\ne b|_\kappa, a|_\kappa, b|_\kappa\ne 0\text{ for all }\tau<\kappa\le n.\text{ Further, }[j|_\tau]_2<[i|_\tau]_2,\\&\text{or }j|_\tau=i|_\tau\text{ and }N_{a_{\tau+1}}\subset N_{b_{\tau+1}}\Big\}\cup\Big\{(a,j)\in E_{n,0}:\exists\delta\le n\text{ such}\\&\text{that }b|_\delta=0, a|_\delta\ne 0\text{ and }b|_\lambda\ne 0\text{ for all }\delta<\lambda\le n\Big\}.\end{aligned}\tag{4.7}$$

For later reference note that by (4.7), in particular,

$$(b,i)\prec_p(0,j),\quad\text{for all }j\in\{0,1\}^n\text{ and }(b,i)\in E_n.\tag{4.8}$$

Further, we have that

$$(b,i)\in E_n\text{ if }(b,i)\prec_p(a,j)\text{ for any }(a,j)\in E_n.\tag{4.9}$$

As indicated above, we will show in Lemma 4.2.2 below that for all $n\in\mathbb{N}_0$ the functions $g_0,\dots,g_n$ generate a suitable subset $\tilde{E}_{n,0}=\tilde{E}_{n,0,g}\subseteq E_{n,0}$ for which the above order is total. An essential step to the construction of $\tilde{E}_{n,0}$ will be provided by the following lemma whose proof is obvious.

Lemma 4.2.1. *For $k = 1, \dots, K$, let $q_k \in [0,1]$ and put $q_\infty = 1$. Define* $\max_{k\in\emptyset} q_k = 0$ *and* $\min_{k\in\emptyset} q_k = 1$ *and let $x_b = \max_{k\in N_b} q_k$ and $y_b = \min_{k\notin N_b} q_k$ for all $b \in \{0,1\}^K$. Put $M_u = \{l : q_l \le u\}$, $u \in \mathbb{R}$, and $U_k = \{b \in \{0,1\}^K : N_b = M_u$ for some $u \in \{q_l : q_l < q_k\} \cup \{0\}\}$. For all $k \in \{1,\dots,K\} \cup \{\infty\}$ we have that*

1. *A partition of $[0, q_k)$ is given by $\{[x_b, y_b),\, b \in U_k\}$*
2. *$\{N_b,\, b \in U_k\}$ is strictly totally ordered under inclusion*
3. *$y_a \le x_b$ for $N_a \subset N_b$, $a, b \in U_k$, or if $a = b \notin U_\infty$*
4. *$U_k = \{a \in U_\infty : N_a \subseteq N_b\}$, $b \in \arg\max_{a\in U_k} |N_a|$.*

Example 4.2.1. For $K = 6$ and $q_1 = 0.2$, $q_2 = 0.3$, $q_3 = 0$, $q_4 = 0.4$, $q_5 = 0.1$ and $q_6 = 0.2$ we consider the partition of $[0, q_2)$ given by part 1 of Lemma 4.2.1. We have $\{q_l : q_l < q_2\} \cup \{0\} = \{0, 0.1, 0.2\}$ and $M_0 = \{3\}$, $M_{0.1} = \{3,5\}$ and $M_{0.2} = \{1,3,5,6\}$ such that $U_2 = \{(1,1,0,1,1,1), (1,1,0,1,0,1), (0,1,0,1,0,0)\}$. In particular, part 1 of Lemma 4.2.1 yields $[0, 0.3) = [0, 0.1) \cup [0.1, 0.2) \cup [0.2, 0.3)$, and parts 2 and 3 are obvious. Concerning part 4 of the lemma we have $U_\infty = \{(1,1,0,1,1,1), (1,1,0,1,0,1), (0,1,0,1,0,0), (0,0,0,1,0,0), (0,0,0,0,0,0)\}$. To verify the assertion note finally that $\arg\max_{a\in U_2} |N_a| = \{(0,1,0,1,0,0)\}$.

For later reference by parts 1 to 3 of Lemma 4.2.1 for all $b \in U_\infty$ we readily have

$$[x_b, y_b) = \sum_{\substack{a\in U_\infty:\\ N_a\subset N_b}} |[x_a, y_a)| + [0, y_b - x_b). \tag{4.10}$$

For all $n \in \mathbb{N}_0$ we will next define successively the sets $\tilde{E}_{n,0}$ and B_{nbi}. For that purpose we shall frequently apply the notation introduced in Lemma 4.2.1. Note carefully, however, that we will necessarily have to extend the subscripts by the indices $n \in \mathbb{N}_0$ and $(b,i) \in \tilde{E}_{n,0}$. Consequently, for all $k \in K_n \cup \{\infty\}$ and $(b,i) \in E_{n,0}$ let now

$$U_{nkbi} = \left\{a \in \{0,1\}^{2n+1} : N_a = M_{nubi} \text{ for some } u \in \{q_{nlbi} : q_{nlbi} < q_{nkbi}\} \cup \{0\}\right\} \tag{4.11}$$

where, as above, $M_{nubi} = \{k : q_{nkbi} \le u\}$, $u \in \mathbb{R}$. Further, we put

$$q_{nkbi} = |A_{nki} \cap B_{nbi}| \tag{4.12}$$

and

$$q_{n,\infty,b,i} = |B_{nbi}| \tag{4.13}$$

for arbitrary sets A_{nki} that will be chosen to depend on $g_0, \dots, g_{n-1}$ in (4.35) below, i.e. q_{nkbi}, $k \in K_n \cup \{\infty\}$, are arbitrary numbers up to $q_{nkbi} \le |B_{nbi}|$. Note that by (4.11) we have

$$0 \notin U_{nkbi} \quad \text{if } k \in K_n,\ (\text{i.e. } k \ne \infty). \tag{4.14}$$

Let now

$$\tilde{E}_{n,0} = \Big\{((b,u),(i,j)),(b,i) \in \tilde{E}_{n-1,0}, u \in U_{n-1,\infty,b,i}, j \in \{0,1\}\Big\} \\ \cup (\{0\} \times \{0,1\}^n) \tag{4.15}$$

and put $\tilde{E}_n = \tilde{E}_{n,0} \backslash (\{0\} \times \{0,1\}^n)$. We will discuss below that the union in (4.15) can in fact be disjoint. In particular, for any $n \in \mathbb{N}_0$ we will have that $0 \notin U_{\infty,0,i}$, $i \in \{0,1\}^n$, cf. the proof of Corollary 4.3.1. Note that in the following we shall occasionally truncate the above indexation where no confusion may arise.

Lemma 4.2.2. *For all $n \in \mathbb{N}$ the order "$\prec_p$" given in (4.7) is total on $\tilde{E}_{n,0}$.*

Proof. Let the order "$\prec_p$" be total on $\tilde{E}_{n-1,0}$ and let $(a,j),(b,i) \in \tilde{E}_{n-1,0}$. By (4.15) it is sufficient to show that either $((a,\alpha),(j,\iota)) = ((b,\beta),(i,\epsilon))$ or $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ or $((a,\alpha),(j,\iota)) \succ_p ((b,\beta),(i,\epsilon))$ for all $\alpha \in U_{\infty,a,j} \cup \{0\}$, $\beta \in U_{\infty,b,i} \cup \{0\}$ and all $\iota, \epsilon \in \{0,1\}$. By symmetry we may assume that $(a,j) \preceq (b,i)$. Let first $(a,j) = (b,i)$. Then, $U_{\infty,a,j} = U_{\infty,b,i}$, and the following threefold distinction is a partition of all $((a,\alpha),(j,\iota))$ and $((b,\beta),(i,\epsilon))$ with $(a,j) = (b,i)$. In either case we will show that an ordering by "$\prec_p$" exists where we will omit the trivial relation of equality.

1. Let $\alpha = \beta \in U_{\infty,a,j}$ and $\iota, \epsilon \in \{0,1\}$ such that $[(j,\iota)]_2 < [(i,\epsilon)]_2$. Then, $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (4.7) for $\tau = n$. (δ does not exist.)
2. Let $(a,\alpha),(b,\beta) \neq 0$, $\alpha, \beta \in U_{\infty,a,j}$, $\alpha \neq \beta$, and $\iota, \epsilon \in \{0,1\}$. Then, $(a,\alpha)|_\tau = (b,\beta)|_\tau$ and $(a,\alpha)|_\kappa \neq (b,\beta)|_\kappa$ for all $\tau < \kappa \leq n$, only if $\tau = n-1$. Further, $(j,\iota)|_{n-1} = j = i = (i,\epsilon)|_{n-1}$, and $N_\alpha \subset N_\beta$ (or $N_\alpha \supset N_\beta$) by part 2 of Lemma 4.2.1 and the fact that $\alpha, \beta \in U_{\infty,a,j}$. ($\delta$ does not exist.)
3. Let $(b,\beta) = 0$, $\alpha \in U_{\infty,a,i}$, $\alpha \neq 0$, and $\iota, \epsilon \in \{0,1\}$. Then, $\delta = n$, and the fact that $(a,\alpha) \neq 0$ yields $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (4.7). (τ does not exist.)

Next, let $(a,j) \prec_p (b,i)$. Then, by (4.7) there is $\tau \leq n-1$ such that $a|_\tau = b|_\tau$ and $a|_\kappa \neq b|_\kappa$, $a|_\kappa, b|_\kappa \neq 0$ for all $\tau < \kappa \leq n-1$, or there is $\delta \leq n-1$ such that $b|_\delta = 0$, $a|_\delta \neq 0$ and $b_\lambda \neq 0$ for all $\delta < \lambda \leq n-1$. According to (4.7) we may distinguish three cases that yield the ordering $(a,j) \prec_p (b,i)$. We will consider them separately and show that in either case also $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ for all $\alpha, \beta \in \{0,1\}^{2n-1}$ and all $\iota, \epsilon \in \{0,1\}$.

1. Let $a|_\tau = b|_\tau$, $a|_\kappa \neq b|_\kappa$, $a|_\kappa, b|_\kappa \neq 0$ for all $\tau < \kappa \leq n-1$, and $[j|_\tau]_2 < [i|_\tau]_2$. Now, $(a,\alpha)|_{\tau+1} = (b,\beta)|_{\tau+1}$ only if $\tau = n-1$. (δ does not exist.) Then, the fact that $[(j,\iota)|_{\tau+1}]_2 < [(i,\epsilon)|_{\tau+1}]_2$ yields $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (4.7). If $(a,\alpha)|_{\tau+1} \neq (b,\beta)|_{\tau+1}$ also $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ by (4.7) using that $(a,\alpha)|_\tau = (b,\beta)|_\tau$ and $[(j,\iota)|_\tau]_2 < [(i,\epsilon)|_\tau]_2$.

2. Let $a|_\tau = b|_\tau$, $a|_\kappa \neq b|_\kappa$, $a|_\kappa, b|_\kappa \neq 0$ for all $\tau < \kappa \leq n$, and $j|_\tau = i|_\tau$, $N_{a_{\tau+1}} \subset N_{b_{\tau+1}}$. ($\delta$ does not exist.) Then, $\tau < n$, and for all $\alpha, \beta \in \{0,1\}^{2n-1}$ and all $\iota, \epsilon \in \{0,1\}$ we trivially also have $(a,\alpha)|_\tau = (b,\beta)|_\tau$, $[(j,\iota)|_\tau]_2 = [(i,\epsilon)|_\tau]_2$ and $N_{(a,\alpha)_{\tau+1}} \subset N_{(b,\beta)_{\tau+1}}$. The above yields further that $(a,\alpha)|_{\tau+1} \neq (b,\beta)|_{\tau+1}$ and $(a,\alpha)|_{\tau+1}, (b,\beta)|_{\tau+1} \neq 0$. Comparing with (4.7) we find that $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$.
3. Finally, let $b_\delta = 0$, $a|_\delta \neq 0$ and $b|_\lambda \neq 0$ for all $\delta < \lambda \leq n$. (τ does not exist.) Now, also $(b,\beta)|_\delta = 0$ and $(a,\alpha)|_\delta \neq 0$. If $(b,\beta)|_{\delta+1} \neq 0$ then $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$ is immediate by (4.7). If $(b,\beta)|_{\delta+1} = 0$ then $\delta = n-1$, and $(a,\alpha)|_{\delta+1} \neq 0$ yields by (4.7) that $((a,\alpha),(j,\iota)) \prec_p ((b,\beta),(i,\epsilon))$.

□

We will denote the respective total order by "$\prec$". For $(b,i) \in \tilde{E}_{n,0}$ let

$$B_{nbi} = \begin{cases} \left[x_b, \dfrac{x_b+y_b}{2}\right) + \displaystyle\sum_{(a,j)\prec(b,i)|_{n-1}} |B_{n-1,a,j}|, & b \neq 0,\ i_n = 0, \\ \left[\dfrac{x_b+y_b}{2}, y_b\right) + \displaystyle\sum_{(a,j)\prec(b,i)|_{n-1}} |B_{n-1,a,j}|, & b \neq 0,\ i_n = 1, \\ 2^{-n}\left[0, \max\limits_{k\in\mathbb{Z}} g_n\left(k+[i]_2 2^{-n}\right) - \max\limits_{k\in\mathbb{Z}} g_{n-1}\left(k+[i]_2 2^{-n}\right)\right) \\ \quad + \displaystyle\sum_{(b,j)\prec(0,i)} |B_{nbj}|, & b = 0, \end{cases} \tag{4.16}$$

where retaining the notation of Lemma 4.2.1 we put

$$x_b = x_{n,b,i|_{n-1}} = \max_{k\in N_{b_n}-n} q_{n-1,k,(b,i)|_{n-1}} \tag{4.17}$$

and

$$y_b = y_{n,b,i|_{n-1}} = \min_{k\notin N_{b_n}-n} q_{n-1,k,(b,i)|_{n-1}}. \tag{4.18}$$

Note that part 3 of Lemma 4.2.1 and (4.15) yield in particular that by applying (4.16) to any $(b,i) \notin \tilde{E}_{n,0}$ we get $B_{nbi} = \emptyset$. In Figure 4.3.1 we give a successive construction of B_{nbi} up to $n = 2$. There, we use the coefficients discussed in Example 4.1.1 and we anticipate (4.35) in order to fix A_{nki}. Next, note that (4.16) for all $n \in \mathbb{N}_0$ yields

$$|B_{naj}| = |B_{nai}| \quad \text{for all } (a,j),(a,i) \in \tilde{E}_n \text{ with } j|_{n-1} = i|_{n-1} \tag{4.19}$$

where we point out that (4.19) does not hold for $a = 0$, cf. the intervals $B_{1,0,0}$ and $B_{1,0,1}$ in Figure 4.3.1. As indicated above we shall in the following work under the assumption that for a fixed $n \in \mathbb{N}_0$, we have

$$B_{m,0,i} = \dot{\bigcup_{\substack{(a,j)\in\tilde{E}_{m+1}:\\(a,j)|_m=(0,i)}}} B_{m+1,a,j} \quad \text{for all } m < n \text{ and } i \in \{0,1\}^m. \tag{A1}$$

The assumption will be relaxed in Section 4.3.

Lemma 4.2.3. *Assume (A1). Then, for all $m \le n$ the following holds.*

1. *For $(b,i) \in \tilde{E}_{m,0}$ we have*

$$B_{mbi} = [0, |B_{mbi}|) + \sum_{(a,j) \prec (b,i)} |B_{maj}|. \tag{4.20}$$

2. *For $(b,i) \in \tilde{E}_m$ we have*

$$B_{mbi} = \dot{\bigcup_{\substack{(a,j)\in \tilde{E}_{m+1}: \\ (a,j)|_m=(b,i)}}} B_{m+1,a,j}. \tag{4.21}$$

Proof. For the proof of (4.20) we may restrict to the case $b \neq 0$ as (4.20) is immediate from (4.16) for $b = 0$. In the following, let $(a,j) \in \tilde{E}_{m-1,0}$ for any $m \le n$. To begin with, in (4.22) to (4.24) we will discuss simple but important preliminaries that follow easily from the above setup. Let first $\gamma \in U_{\infty,a,j} = U_{m-1,\infty,a,j}$ be arbitrary. Using (4.17) and (4.18) we then have by (4.10) that

$$\left[x_{(a,\gamma)}, \frac{x_{(a,\gamma)} + y_{(a,\gamma)}}{2}\right) = \sum_{\substack{\beta\in U_{\infty,a,j}: \\ N_\beta \subset N_\gamma}} \left|[x_{(a,\beta)}, y_{(a,\beta)})\right| + \left[0, \frac{y_{(a,\gamma)} - x_{(a,\gamma)}}{2}\right) \tag{4.22}$$

and, accordingly,

$$\begin{aligned}\left[\frac{x_{(a,\gamma)} + y_{(a,\gamma)}}{2}, y_{(a,\gamma)}\right) = &\sum_{\substack{\beta\in U_{\infty,a,j}: \\ N_\beta \subset N_\gamma}} \left|[x_{(a,\beta)}, y_{(a,\beta)})\right| + \frac{y_{(a,\gamma)} - x_{(a,\gamma)}}{2} \\ &+ \left[0, \frac{y_{(a,\gamma)} - x_{(a,\gamma)}}{2}\right).\end{aligned} \tag{4.23}$$

Further, we get by (4.15) that

$$\begin{aligned}\sum_{\substack{\beta\in U_{\infty,a,j}, \\ N_\beta \subset N_\gamma}} \sum_{l\in\{0,1\}} |B_{m,(a,\beta),(j,l)}| &= \sum_{\substack{(c,l)\in \tilde{E}_{m,0}: N_{c_m} \subset N_\gamma, \\ (c,l)|_{m-1}=(a,j)}} |B_{mcl}| \\ &= \sum_{\substack{(c,l)\in \tilde{E}_{m,0}: (c,l)\prec((a,\gamma),(j,0)), \\ (c,l)|_{m-1}=(a,j)}} |B_{mcl}|\end{aligned} \tag{4.24}$$

where the latter equality holds by (4.7). Next, let $\gamma \in U_{\infty,a,j}$ such that $(a,\gamma) \neq 0$. Note that depending on the above choice of $(a,j) \in \tilde{E}_{m-1,0}$ this constitutes an additional restriction on γ only if $a = 0$. Now, using (4.16) in (4.22) and (4.23), we find that for any $i \in \{(j,0),(j,1)\}$

$$B_{m,(a,\gamma),i} = \sum_{\substack{\beta \in U_{\infty,a,j}: \\ N_\beta \subset N_\gamma}} \sum_{l \in \{0,1\}} |B_{m,(a,\beta),(j,l)}| + \mathbf{1}(i_m = 1)|B_{m,(a,\gamma),(j,0)}|$$
$$+ [0, |B_{m,(a,\gamma),i}|) + \sum_{\substack{(c,l) \in \tilde{E}_{m-1,0}: \\ (c,l) \prec (a,j)}} |B_{m-1,c,l}|$$
$$= \sum_{\substack{(c,l) \in \tilde{E}_m : (c,l) \prec ((a,\gamma),i), \\ (c,l)|_{m-1} = (a,j)}} |B_{mcl}| + [0, |B_{m,(a,\gamma),i}|) + \sum_{\substack{(c,l) \in \tilde{E}_{m-1,0}: \\ (c,l) \prec (a,j)}} |B_{m-1,c,l}| \tag{4.25}$$

where the latter equality holds by (4.24). Note that by (4.9) and the fact that $(a,\gamma) \neq 0$ we may restrict to $\tilde{E}_m$ instead of $\tilde{E}_{m,0}$ for the first sum in (4.25). We next consider the special case $(c,l) \in \tilde{E}_{m-1}$. Then,

$$[0, |B_{m-1,c,l}|) = \dot{\bigcup_{\beta \in U_{\infty,c,l}}} \left\{ \left[x_{(c,\beta)}, \frac{x_{(c,\beta)} + y_{(c,\beta)}}{2} \right) \cup \left[\frac{x_{(c,\beta)} + y_{(c,\beta)}}{2}, y_{(c,\beta)} \right) \right\}$$
$$= \dot{\bigcup_{\beta \in U_{\infty,c,l}}} \dot{\bigcup_{i \in \{0,1\}}} B_{m,(c,\beta),(l,i)} - \sum_{\substack{(a,i) \in \tilde{E}_{m-1}: \\ (a,i) \prec (c,l)}} |B_{m-1,a,i}|$$
$$= \dot{\bigcup_{\substack{(a,i) \in \tilde{E}_m: \\ (a,i)|_{m-1} = (c,l)}}} B_{mai} - \sum_{\substack{(a,i) \in \tilde{E}_{m-1}: \\ (a,i) \prec (c,l)}} |B_{m-1,a,i}| \tag{4.26}$$

where the first equality is immediate by part 1 of Lemma 4.2.1 and (4.13), and the second equality holds by (4.16) as we have $c \neq 0$ by assumption. The third equality is a consequence of (4.15). Note that in the second and third equality we also make use of (4.9) in order to justify the sets $\tilde{E}_{m-1}$ instead of $\tilde{E}_{m-1,0}$. Now, by (4.26) and (A1), we have for all $(c,l) \in \tilde{E}_{m-1,0}$ that

$$|B_{m-1,c,l}| = \sum_{\substack{(a,i) \in \tilde{E}_m: \\ (a,i)|_{m-1} = (c,l)}} |B_{mai}|, \tag{4.27}$$

and (4.20) for $b \neq 0$ holds by (4.25) and (4.27). Note that we use (4.27) in order to resolve the last sum in (4.25), and that, in particular, (A1) is a necessary assumption even though $(a,\gamma) \neq 0$. In order to proof (4.21) consider (4.26) for $m+1$ instead of m, i.e.

$$[0, |B_{mci}|) = \dot{\bigcup_{\substack{(a,j) \in \tilde{E}_{m+1}: \\ (a,j)|_m = (c,i)}}} B_{m+1,a,j} - \sum_{\substack{(a,j) \in \tilde{E}_m: \\ (a,j) \prec (c,i)}} |B_{maj}|, \quad (c,i) \in \tilde{E}_m, \tag{4.28}$$

and the assertion follows by (4.20). □

In particular, by (4.20) we have for $m \leq n$ that

$$B_{mbi} \cap B_{maj} = \emptyset, \quad (b,i),(a,j) \in \tilde{E}_{m,0},\ (b,i) \neq (a,j). \tag{4.29}$$

Using (A1), and (4.21) repeatedly yields for all $(b,i) \in \tilde{E}_{m,0}$, $m < n$, that

$$B_{mbi} = \dot{\bigcup_{\substack{(a,j)\in\tilde{E}_n : a|_{m+1}\neq 0,\\ (a,j)|_m=(b,i)}}} B_{naj}. \tag{4.30}$$

Note that the restriction $a_{m+1} \neq 0$ affects the case $b = 0$ only, cf. (A1) where the index (a,j) does not run over $a = 0$. Further, using (4.19) repeatedly we have by (4.30) for all $(b,i) \in \tilde{E}_{m,0}$, $m < n$, and any $j \in \{0,1\}^n$ with $j|_m = i$ that

$$2^{m-n}|B_{mbi}| = \sum_{\substack{a\in\mathbb{B}_n:(a,j)\in\tilde{E}_n,\\ a|_m=b, a|_{m+1}\neq 0}} |B_{naj}|. \tag{4.31}$$

Note that in (4.31) the summation is over $a \in \mathbb{B}_n$ only, and j is fixed. We shall assume next that for a fixed $n \in \mathbb{N}_0$ and all $m < n$, $k \in K_m$ and $j \in \{0,1\}^m$ there is a unique $(\omega,(j,1)) = (\omega_{mkj},(j,1)) \in \tilde{E}_{m+1}$ such that

$$\sum_{\substack{(a,i)\in\tilde{E}_{m+1}:i|_m=j,\\ (a,i)\preceq(\omega,(j,1))}} |B_{m+1,a,i}| = 2^{-m}\sum_{p=0}^{m} a_{p,k,j|_p}. \tag{A2}$$

The assumption (A2) will also be relaxed in Section 4.3.

Lemma 4.2.4. *Let $n \in \mathbb{N}_0$, $k \in K_n$, $i \in \{0,1\}^n$ and $j = i|_{n-1}$. Assume (A1) and let $(\omega,(j,1)) = (\omega_{n-1,k,j},(j,1)) \in \tilde{E}_n$ according to (A2). We have*

$$\sum_{a:(\omega,(j,1))\prec(a,i)} |B_{nai}| \geq 2^{-n}a_{nki}.$$

Proof. By (4.31) for all $i \in \{0,1\}^n$, $n \in \mathbb{N}_0$, we get that

$$\begin{aligned}\sum_{m=0}^{n} 2^m|B_{m,0,i|_m}| &= 2^n\sum_{m=0}^{n-1}\sum_{\substack{a:(a,i)\in\tilde{E}_n, a|_m=0,\\ a|_{m+1}\neq 0}} |B_{nai}| + 2^n|B_{n,0,i}| \\ &= 2^n\sum_{a:(a,i)\in\tilde{E}_{n,0}} |B_{nai}|. \end{aligned}\tag{4.32}$$

Let $k^* = k^*_{i,n} \in \arg\max_k g_n(k + [i]_2 2^{-n})$. Then, by (4.3) we have

$$\begin{aligned} g_n(k^* + [i]_2 2^{-n}) &= \sum_{s=0}^{n} a_{s,k^*,i|_s} = \sum_{p=0}^{n}\left(\max_k\sum_{s=0}^{p} a_{s,k,i|_s} - \max_k\sum_{s=0}^{p-1} a_{s,k,i|_s}\right) \\ &= \sum_{p=0}^{n} 2^p|B_{p,0,i|_p}| \end{aligned}\tag{4.33}$$

where the third equality holds by (4.16). Combining (4.32) and (4.33) yields for all $k \in K_n$, $i \in \{0,1\}^n$ and $j = i|_{n-1}$ that

$$\sum_{a:(\omega,(j,1))\prec(a,i)} |B_{nai}| = 2^{-n}\sum_{s=0}^{n} a_{s,k^*,i|_s} - \sum_{a:(a,i)\preceq(\omega,(j,1))} |B_{nai}|$$
$$= 2^{-n}\sum_{s=0}^{n} a_{s,k^*,i|_s} - 2^{-n}\sum_{s=0}^{n-1} a_{s,k,i|_s} \geq 2^{-n} a_{nki} \qquad (4.34)$$

where we use (A2) and (4.19) for the second equality. $\square$

4.3 The Sequence S_n: Building Blocks and Properties

The sets S_n, $n \in \mathbb{N}_0$, are given in (4.5) where at this point for all $n \in \mathbb{N}_0$, $k \in K_n$, $i \in \{0,1\}^n$ and for $(\omega,(i|_{n-1},1)) = (\omega_{n-1,k,i|_{n-1}},(i|_{n-1},1)) \in \tilde{E}_n$ according to (A2), we will define essentially

$$A_{nki} = \dot{\bigcup_{\substack{b:(b,i)\\ \in \tilde{E}_{n,0}}}} \left\{ B_{nbi} \cap \left([0, 2^{-n}a_{nki}) + \sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| + \sum_{\substack{a:(a,i)\\ \preceq(\omega,(i|_{n-1},1))}} |B_{nai}| \right) \right\}. \qquad (4.35)$$

Here, the union is disjoint by (4.20). For the final definition of A_{nki} we refer to Corollary 4.3.2 below. In Figure 4.3.1 we depict the sets A_{nki} and B_{nbi} up to $n = 2$ for the coefficients discussed in Example 4.1.1. Note that for all $(b,i) \in \tilde{E}_{n,0}$ and all $(c,j) \in \tilde{E}_{m,0}$, $m \leq n$, we have

$$\sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| + \sum_{a:(a,i)|_m\preceq(c,j)} |B_{nai}|$$
$$= \begin{cases} \displaystyle\sum_{(a,j)\prec(b,i)} |B_{naj}| - \sum_{\substack{a:(c,j)\prec(a,i)|_m,\\ (a,i)\prec(b,i)}} |B_{nai}|, & (c,j)\prec(b,i)|_m \\ \displaystyle\sum_{(a,j)\prec(b,i)} |B_{naj}| + \sum_{\substack{a:(b,i)\preceq(a,i),\\ (a,i)|_m\preceq(c,j)}} |B_{nai}|, & \text{else,} \end{cases} \qquad (4.36)$$

such that, by (4.20) and (4.35), we get

$$A_{nki}\cap B_{nbi} = \begin{cases} \displaystyle\sum_{(a,j)\prec(b,i)} |B_{naj}| + \Bigg[0, \min\Bigg\{2^{-n}a_{nki} \\ \displaystyle\qquad - \sum_{\substack{a:(\omega,(i|_{n-1},1))\prec\\ (a,i)\prec(b,i)}} |B_{nai}|, |B_{nbi}|\Bigg\}\Bigg), & (\omega,(i|_{n-1},1))\prec(b,i), \\ \emptyset, & \text{else.} \end{cases} \qquad (4.37)$$

Next, we find that by Lemma 4.2.4 and (4.37) for all $i \in \{0,1\}^n$, $k \in K_n$, there is a

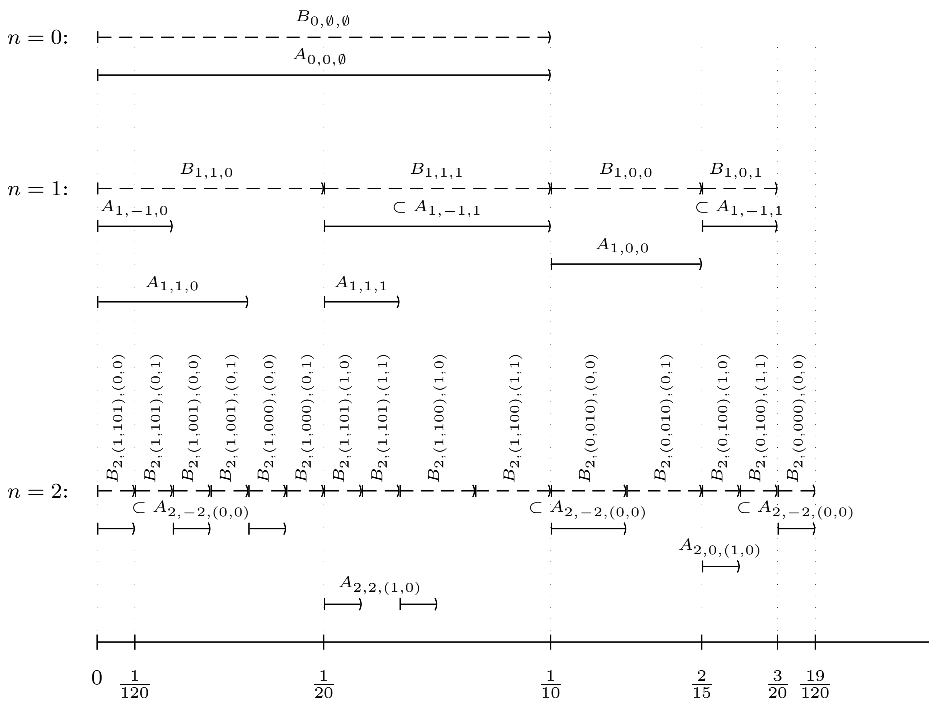

Figure 4.3.1: Jointly successive construction of B_{nbi} and A_{nki} (top to bottom) using (4.16) and (4.35) up to $n = 2$. The corresponding coefficients are given in Example 4.1.1. In the figure we denote by "$\subset X$" a subset of a set X.

unique $(\tilde{\omega}, i) = (\tilde{\omega}_{nki}, i) \succ (\omega, (i|_{n-1}, 1)) = (\omega_{n-1,k,i|_{n-1}}, (i|_{n-1}, 1))$, $(\tilde{\omega}, i) \preceq (0, i)$, such that

$$0 \leq 2^{-n} a_{nki} - \sum_{a:(\omega,(i|_{n-1},1))\prec(a,i)\prec(\tilde{\omega},i)} |B_{nai}| \leq |B_{n,\tilde{\omega},i}|. \tag{4.38}$$

Hence, by (4.37) and (4.38)

$$A_{nki} \cap B_{nbi} = \begin{cases} B_{nbi}, & (\omega, (i|_{n-1}, 1)) \prec \\ & (b, i) \prec (\tilde{\omega}, i), \\ \sum\limits_{(a,j)\prec(\tilde{\omega},i)} |B_{naj}| + [0, |A_{nki} \cap B_{nbi}|], & (b, i) = (\tilde{\omega}, i), \\ \emptyset, & \text{else.} \end{cases} \tag{4.39}$$

Note that (4.39) and (4.35) yield

$$A_{nki} \subseteq \dot{\bigcup_{b:(\omega,(i|_{n-1},1))\prec(b,i)\preceq(\tilde{\omega},i)}} B_{nbi}, \tag{4.40}$$

such that, in particular,

$$A_{nki} = \dot{\bigcup_{b:(\omega,(i|_{n-1},1))\prec(b,i)\preceq(\tilde{\omega},i)}} A_{nki} \cap B_{nbi}. \tag{4.41}$$

For later reference we find that the latter yields

$$|A_{nki}| = 2^{-n} a_{nki} \tag{4.42}$$

where we use (4.37) and (4.38).

Corollary 4.3.1. *Assume (A1) and (A2). For all $i \in \{0,1\}^n$ we have*

$$B_{n,0,i} = \dot{\bigcup_{\substack{(a,j)\in \tilde{E}_{n+1}: \\ (a,j)|_n=(0,i)}}} B_{n+1,a,j}. \tag{4.43}$$

Proof. Let $k^* = k^*_{i,n}$ as in the proof of Lemma 4.2.4, and let

$$(\omega, (i|_{n-1}, 1)) = (\omega_{n-1,k^*,i|_{n-1}}, (i|_{n-1}, 1)) \in \tilde{E}_n$$

as in (A2). We now get by (4.34) for $k = k^*$ that for any $i \in \{0,1\}^n$

$$2^{-n} a_{n,k^*,i} = \sum_{a:(\omega,(i|_{n-1},1))\prec(a,i)} |B_{nai}| = \sum_{\substack{a:(\omega,(i|_{n-1},1))\prec \\ (a,i)\preceq(0,i)}} |B_{nai}| \tag{4.44}$$

where the second equality holds by (4.8). Using further that from (A2) and (4.8) we have $(\omega, (i|_{n-1}, 1)) \prec (0, i)$ we get by (4.37) that

$$\begin{aligned} &A_{n,k^*,i} \cap B_{n,0,i} \\ &\quad = \sum_{(a,j)\prec(0,i)} |B_{naj}| + \Bigg[0, \min\Bigg\{2^{-n} a_{n,k^*,i} - \sum_{\substack{a:(\omega,(i|_{n-1},1)) \\ \prec(a,i)\prec(0,i)}} |B_{nai}|,\ |B_{n,0,i}|\Bigg\}\Bigg) \\ &\quad = \sum_{(a,j)\prec(0,i)} |B_{naj}| + [0, |B_{n,0,i}|) = B_{n,0,i}. \end{aligned} \tag{4.45}$$

Here, the second equality follows by (4.44) and the last equality corresponds to (4.20). By (4.45), using (4.12) and (4.13) we find that $q_{n,k^*,0,i} = q_{n,\infty,0,i}$ such that by (4.11), in particular,

$$U_{k^*,0,i} = U_{\infty,0,i}. \tag{4.46}$$

Further, (4.46) and (4.14) yield

$$0 \notin U_{\infty,0,i}. \tag{4.47}$$

Now,

$$\begin{aligned} [0, |B_{n,0,i}|) = [0, q_{n,k^*,0,i}) &= \dot{\bigcup_{\beta\in U_{k^*,0,i}}} [x_{(0,\beta)}, y_{(0,\beta)}) = \dot{\bigcup_{\beta\in U_{k^*,0,i}}} \dot{\bigcup_{j\in\{0,1\}}} B_{n+1,(0,\beta),(i,j)} \\ &- \sum_{(a,j)\prec(0,i)} B_{naj} = \dot{\bigcup_{\substack{(a,j)\in \tilde{E}_{n+1}: \\ (a,j)|_n=(0,i)}}} B_{n+1,a,j} - \sum_{(a,j)\prec(0,i)} B_{naj}. \end{aligned} \tag{4.48}$$

Here, the first equality follows from (4.45) and (4.13), the second equality holds by part 1 of Lemma 4.2.1, and the third equality is a consequence of (4.16) where we use that by (4.14) $0 \notin U_{k^*,0,i}$. The last equality holds by (4.46) and (4.15) where we again may restrict the union to $\tilde{E}_{n+1}$ instead of $\tilde{E}_{n+1,0}$ by (4.14). Finally, (4.20) and (4.48) yield the assertion. □

Corollary 4.3.2. *Assume (A1) and (A2). For all $k \in K_n$ and $j \in \{0,1\}^n$ there is a unique $(\omega,(j,1)) = (\omega_{nkj},(j,1)) \in \tilde{E}_{n+1}$ such that*

$$\sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j,\\(a,i)\preceq(\omega,(j,1))}} |B_{n+1,a,i}| = 2^{-n}\sum_{p=0}^{n} a_{p,k,j|_p}. \tag{4.49}$$

Proof. Let $k \in K_n$ and $(\tilde{\omega},j) = (\tilde{\omega}_{nkj},j) \in \tilde{E}_{n,0}$ as in (4.38). Further, let $z \in \arg\max_{\beta\in U_{k\tilde{\omega}j}} |N_\beta|$. Then, $z \neq 0$ by (4.14), and z is unique by part 2 of Lemma 4.2.1. By part 4 of Lemma 4.2.1 we have

$$U_{k\tilde{\omega}j} = \{\beta \in U_{\infty,\tilde{\omega},j} : N_\beta \subseteq N_z\}. \tag{4.50}$$

Further,

$$\begin{aligned}[0,q_{n,k,\tilde{\omega},j}) &= \dot{\bigcup_{\beta\in U_{k,\tilde{\omega},j}}} [x_{(\tilde{\omega},\beta)},y_{(\tilde{\omega},\beta)}) = \dot{\bigcup_{\substack{\beta\in U_{\infty,\tilde{\omega},j}:\\N_\beta\subseteq N_z}}} [x_{(\tilde{\omega},\beta)},y_{(\tilde{\omega},\beta)}) \\ &= \dot{\bigcup_{\substack{\beta\in U_{\infty,\tilde{\omega},j}:\\N_\beta\subseteq N_z}}} \dot{\bigcup_{l\in\{0,1\}}} B_{n+1,(\tilde{\omega},\beta),(j,l)} - \sum_{(a,i)\prec(\tilde{\omega},j)} |B_{nai}| \end{aligned} \tag{4.51}$$

where the first equality holds by part 1 of Lemma 4.2.1 and the second equality is a consequence of (4.50). The third equality then follows by (4.16) and (4.14). Next, by (4.39) and (4.51) we get that

$$\begin{aligned} A_{nkj}\cap B_{n\tilde{\omega}j} &= \dot{\bigcup_{\substack{\beta\in U_{\infty,\tilde{\omega},j}:\\N_\beta\subseteq N_z}}} \dot{\bigcup_{l\in\{0,1\}}} B_{n+1,(\tilde{\omega},\beta),(j,l)} = \dot{\bigcup_{\substack{(a,i)\in\tilde{E}_{n+1}:(a,i)|_n=(\tilde{\omega},j),\\N_{a_{n+1}}\subseteq N_z}}} B_{n+1,a,i} \\ &= \dot{\bigcup_{\substack{(a,i)\in\tilde{E}_{n+1}:(a,i)\preceq((\tilde{\omega},z),(j,1)),\\(a,i)|_n=(\tilde{\omega},j)}}} B_{n+1,a,i} \end{aligned} \tag{4.52}$$

where the second equality holds by (4.15). Here, by (4.50) and (4.14) we have that $\beta \neq 0$ such that (4.8) justifies the restriction to $\tilde{E}_{n+1}$ instead of $\tilde{E}_{n+1,0}$. The third equality is a consequence of (4.7) and Lemma 4.2.2. Further, by (4.41) and (4.39)

$$\begin{aligned} A_{nkj} &= \biguplus_{\substack{b:(\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1)) \\ \prec(b,j)\prec(\tilde{\omega},j)}} B_{nbj} \quad \uplus \quad (A_{nkj} \cap B_{n\tilde{\omega}j}) \\ &= \biguplus_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j, \\ (\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1))\prec(a,i)|_n\prec(\tilde{\omega},j)}} B_{n+1,a,i} \uplus \biguplus_{\substack{(a,i)\in\tilde{E}_{n+1}:(a,i)\preceq((\tilde{\omega},z),(j,1)), \\ (a,i)|_n=(\tilde{\omega},j)}} B_{n+1,a,i} \\ &= \biguplus_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j,(a,i)\preceq((\tilde{\omega},z),(j,1)), \\ (\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1))\prec(a,i)|_n}} B_{n+1,a,i} \end{aligned} \tag{4.53}$$

where the second equality follows from (4.21) and (4.52). Thus,

$$\sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j,(a,i)\preceq((\tilde{\omega},z),(j,1)), \\ (\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1))\prec(a,i)|_n}} |B_{n+1,a,i}| = |A_{nkj}| = 2^{-n}a_{nkj} \tag{4.54}$$

where the second equality holds by (4.42). Further, (A2) yields

$$\begin{aligned} 2^{-n+1}\sum_{p=0}^{n-1} a_{p,k,j|_p} &= \sum_{\substack{(b,l)\in\tilde{E}_n:l|_{n-1}=j|_{n-1}, \\ (b,l)\preceq(\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{nbl}| \\ &= \sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_{n-1}=j|_{n-1}, \\ (a,i)|_n\preceq(\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{n+1,a,i}| \end{aligned} \tag{4.55}$$

where the second equality is a consequence of (4.21). Note that by (4.19)

$$\begin{aligned} \frac{1}{2}\sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_{n-1}=j|_{n-1}, \\ (a,i)|_n\preceq(\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{n+1,a,i}| &= \sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j, \\ (a,i)|_n\preceq(\omega_{n-1,k,j|_{n-1}},(j|_{n-1},1))}} |B_{n+1,a,i}| \\ &= 2^{-n}\sum_{p=0}^{n-1} a_{p,k,j|_p}. \end{aligned} \tag{4.56}$$

Here, the second equality holds by (4.55). The assertion follows by (4.56) and (4.54) where we put

$$\omega_{nkj} = (\tilde{\omega}_{nkj}, z). \tag{4.57}$$

□

Now, for (4.43) and (4.49) to hold for all $n \in \mathbb{N}_0$, by induction on Corollaries 4.3.1 and 4.3.2 it is sufficient to note that (A1) and (A2) hold trivially in the case $n = 0$. In particular, for all $n \in \mathbb{N}_0$, we have

$$B_{n0i} = \biguplus_{\substack{(a,j)\in\tilde{E}_{n+1}: \\ (a,j)|_n=(0,i)}} B_{n+1,a,j}, \tag{4.58}$$

and for all $n \in \mathbb{N}_0$, $k \in K_n$ and $j \in \{0,1\}^n$ there is a unique $(\omega,(j,1)) = (\omega_{nkj},(j,1)) \in \tilde{E}_{n+1}$ such that

$$\sum_{\substack{(a,i)\in\tilde{E}_{n+1}:i|_n=j,\\ (a,i)\preceq(\omega,(j,1))}} |B_{n+1,a,i}| = 2^{-n}\sum_{p=0}^{n} a_{p,k,j|_p}. \tag{4.59}$$

Combining (4.58) and (4.21) yields that for all $n \in \mathbb{N}_0$

$$B_{nbi} = \biguplus_{\substack{(a,j)\in\tilde{E}_{n+1}:\\ (a,j)|_n=(b,i)}} B_{n+1,a,j}, \quad (b,i)\in\tilde{E}_{n,0}, \tag{4.60}$$

and by a repeated applicaction of (4.60) we get that for all $m, n \in \mathbb{N}_0$, $m < n$,

$$B_{mbi} = \biguplus_{\substack{(a,j)\in\tilde{E}_n:a|_{m+1}\neq 0,\\ (a,j)|_m=(b,i)}} B_{naj}, \quad (b,i)\in\tilde{E}_{m,0}. \tag{4.61}$$

Lemma 4.3.1. *For all $m, n \in \mathbb{N}_0$, $k \in K_m$ and $j \in \{0,1\}^m$, $i \in \{0,1\}^n$ with $n \neq m$ or $j \neq i$ we have*

$$A_{nki} \cap A_{mkj} = \emptyset.$$

Proof. For $m = n$ the assertion is immediate by (4.40) and (4.29). To prove the case $m < n$ let $(b,i) \in \tilde{E}_{n,0}$ such that $(b,i)|_{m+1} \preceq (\omega_{m,k,i|_m},(i|_m,1))$. Then, by (4.38) we have

$$(\omega_{m,k,i|_m},(i|_m,1)) \prec (\tilde{\omega}_{m+1,k,i|_{m+1}}, i|_{m+1}) = (\omega_{m+1,k,i|_{m+1}},(i|_{m+1},1))|_{m+1}$$

where the equality holds by (4.57). Now, $(b,i)|_{m+1} \prec (\omega_{m+1,k,i|_{m+1}},(i|_{m+1},1))|_{m+1}$, and by the second part in the proof of Lemma 4.2.2 the latter implies that $(b,i)|_{m+2} \prec (\omega_{m+1,k,i|_{m+1}},(i|_{m+1},1))$ if $m+2 \leq n$. Proceeding iteratively, we have that

$$(b,i) \prec (\omega_{n-1,k,i|_{n-1}},(i|_{n-1},1)) \quad \text{if } (b,i)|_{m+1} \preceq (\omega_{m,k,i|_m},(i|_m,1)). \tag{4.62}$$

Further, note that

$$A_{mkj} = \biguplus_{\substack{(b,i)\in\tilde{E}_{m+1}:i|_m=j,(b,i)\preceq(\omega_{mkj},(j,1)),\\ (\omega_{m-1,k,j|_{m-1}},(j|_{m-1},1))\prec(b,i)|_m}} B_{m+1,b,i} = \biguplus_{\substack{(b,i)\in\tilde{E}_n:i|_m=j,(b,i)|_{m+1}\preceq(\omega_{mkj},(j,1)),\\ (\omega_{m-1,k,j|_{m-1}},(j|_{m-1},1))\prec(b,i)|_m}} B_{nbi} \tag{4.63}$$

where the first equality follows from (4.53) and the second equality is a consequence of (4.61). Now, comparing (4.40) and (4.63) for $j = i|_m$ yields the assertion by (4.62). To finalize the proof let $j \neq i|_m$. By (4.35) we find that

$$A_{mkj} \subseteq \biguplus_{b:(b,j)\in\tilde{E}_{m,0}} B_{mbj} = \biguplus_{(c,l)\in\tilde{E}_{n,0}:l|_m=j} B_{ncl} \tag{4.64}$$

where the equality holds by (4.61). Using (4.29) the assertion follows by (4.35) and (4.64). The case $m > n$ follows by symmetry. ☐

Lemma 4.3.2. *For all $n \in \mathbb{N}_0$ and $k \in K_n$ we have*

$$\dot{\bigcup_{s=0}^{n}} \dot{\bigcup_{i\in\{0,1\}^s}} A_{ski} \subseteq [0,1). \tag{4.65}$$

Proof. For $m < n \in \mathbb{N}$ we have by (4.61) that

$$\dot{\bigcup_{\substack{(b,i)\in\tilde{E}_n:b|_{m+1}\neq 0,\\ (b,i)|_m=(0,j)}}} B_{nbi} = B_{m,0,j}, \quad j \in \{0,1\}^m, \tag{4.66}$$

such that (4.29) with (4.66) yields

$$B_{m,0,j} \cap B_{p,0,l} = \emptyset \quad \text{for } m \neq p \text{ or } j \neq l. \tag{4.67}$$

Consequently, we get that

$$\dot{\bigcup_{m=0}^{n-1}} \dot{\bigcup_{j\in\{0,1\}^m}} B_{m,0,j} = \dot{\bigcup_{m=0}^{n-1}} \dot{\bigcup_{j\in\{0,1\}^m}} \dot{\bigcup_{\substack{(b,i)\in\tilde{E}_n:b|_{m+1}\neq 0,\\ (b,i)|_m=(0,j)}}} B_{nbi} = \dot{\bigcup_{(b,i)\in\tilde{E}_n}} B_{nbi}, \tag{4.68}$$

and by (4.20) and (4.7) for all $i \in \{0,1\}^n$ it holds that

$$\dot{\bigcup_{(b,j)\in\tilde{E}_n}} B_{nbj} \,\dot{\cup}\, \dot{\bigcup_{[j]_2<[i]_2}} B_{n,0,j} = \dot{\bigcup_{(b,j)\prec(0,i)}} B_{nbj}. \tag{4.69}$$

Now, from (4.68) and (4.69) we find that for all $i \in \{0,1\}^n$

$$\sum_{(b,j)\prec(0,i)} |B_{nbj}| = \sum_{m=0}^{n-1} \sum_{j\in\{0,1\}^m} |B_{m,0,j}| + \sum_{[j]_2<[i]_2} |B_{n,0,j}| \tag{4.70}$$

such that by (4.16) and (4.70)

$$B_{n,0,i} = [0, |B_{n,0,i}|) + \sum_{m=0}^{n-1} \sum_{j\in\{0,1\}^m} |B_{m,0,j}| + \sum_{[j]_2<[i]_2} |B_{n,0,j}|. \tag{4.71}$$

Next, using (4.66) we have for all $i \in \{0,1\}^n$

$$\dot{\bigcup_{\substack{b:(b,i)\in\tilde{E}_n,b|_m=0,\\ b|_{m+1}\neq 0}}} B_{nbi} \subseteq B_{m,0,i|_m}, \tag{4.72}$$

and (4.67) and (4.72) yield

$$\dot{\bigcup_{m=0}^{n}} B_{m,0,i|_m} \supseteq \dot{\bigcup_{b:(b,i)\in\tilde{E}_{n,0}}} B_{nbi}, \quad \text{for all } i \in \{0,1\}^n. \tag{4.73}$$

We get from (4.35) that $A_{nki} \subseteq \dot{\bigcup}_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi}$ for all $k \in K_n$, $i \in \{0,1\}^n$, such that using Lemma 4.3.1

$$\begin{aligned}\dot{\bigcup_{s=0}^{n}} \dot{\bigcup_{i\in\{0,1\}^s}} A_{ski} &\subseteq \bigcup_{s=0}^{n} \bigcup_{i\in\{0,1\}^s} \bigcup_{b:(b,i)\in\tilde{E}_{s,0}} B_{sbi} \subseteq \bigcup_{s=0}^{n}\bigcup_{m=0}^{s}\bigcup_{i\in\{0,1\}^s} B_{m,0,i|_m} \\ &= \bigcup_{s=0}^{n}\bigcup_{m=0}^{s}\bigcup_{j\in\{0,1\}^m} B_{m,0,j} = \dot{\bigcup_{s=0}^{n}}\dot{\bigcup_{i\in\{0,1\}^s}} B_{s,0,i} \\ &= \left[0, \sum_{s=0}^{n}\sum_{i\in\{0,1\}^s} |B_{s,0,i}|\right), \qquad k\in K_n, \end{aligned} \tag{4.74}$$

where the second inclusion follows by (4.73), the second equality holds by (4.67), and the last equality is a consequence of (4.71). Next, note that

$$\begin{aligned}\sum_{s=0}^{n}\sum_{i\in\{0,1\}^s} |B_{s,0,i}| &= \sum_{s=0}^{n}\sum_{i\in\{0,1\}^s} 2^{-s}\left(\max_{k\in\mathbb{Z}} g_s\left(k+[i]_2 2^{-s}\right) - \max_{k\in\mathbb{Z}} g_{s-1}\left(k+[i]_2 2^{-s}\right)\right) \\ &\le \sum_{s=0}^{n}\sum_{i\in\{0,1\}^s}\sum_{k\in\mathbb{Z}} 2^{-s}\left(g_s\left(k+[i]_2 2^{-s}\right) - g_{s-1}\left(k+[i]_2 2^{-s}\right)\right) \\ &= \sum_{s=0}^{n}\sum_{i\in\{0,1\}^n}\sum_{k\in\mathbb{Z}} 2^{-n}\left(g_s\left(k+[i]_2 2^{-n}\right) - g_{s-1}\left(k+[i]_2 2^{-n}\right)\right) \\ &= \sum_{i\in\{0,1\}^n}\sum_{k\in\mathbb{Z}} 2^{-n} g_n\left(k+[i]_2 2^{-n}\right) = \int g_n(x)\,dx \le 1. \end{aligned} \tag{4.75}$$

Here, the first equality follows directly from (4.16), and the first inequality is a consequence of the fact that for $a_k \le b_k \in \mathbb{R}$ we have $0 \le \max_k b_k - \max_k a_k \le \sum_k (b_k - a_k)$. As to the second equality we use that for any $j \in \{0,1\}^s$ we have $g_s\left(k+[j]_2 2^{-s}\right) = g_s\left(k+[i]_2 2^{-n}\right)$, $i \in \{0,1\}^n$, $i|_s = j$, and $|\{i \in \{0,1\}^n : i|_s = j\}| = 2^{n-s}$. The last inequality reflects the assumption of unit Fréchet margins of the max-stable process generated by g. Finally, (4.74) and (4.75) yield the assertion. □

4.4 A Useful Decomposition of the Sets A_{nki}

Recall from (4.16) and (4.35) that the sets B_{mbi} and A_{mki} are defined in a joint successive way. The following notion of $D_{n,mki}$ will generalize the sets A_{mki}. In contrast to (4.35), however, for $m < n$ they will require the corresponding sets B_{nbi} to be already defined. More precisely, we put

$$D_{n,mki} = \dot{\bigcup_{b:(b,i)\in\tilde{E}_{n,0}}} \Bigg\{ B_{nbi} \cap \Bigg([0, 2^{-n} a_{m,k,i|_m}) + \sum_{\substack{a:(a,i)\in\tilde{E}_{n,0}:\\ (a,i)|_m \preceq (\omega,(i|_{m-1},1))}} |B_{nai}| + \sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| \Bigg) \Bigg\} \tag{4.76}$$

for all $n \in \mathbb{N}_0$, $m \leq n$, $k \in K_m$ and $i \in \{0,1\}^n$ where

$$(\omega, (i|_{m-1}, 1)) = (\omega_{m-1,k,i|_{m-1}}, (i|_{m-1}, 1))$$

as in (4.59). In particular, we readily find by (4.76) that

$$D_{n,mki} \subseteq \dot{\bigcup_{b:(b,i)\in\tilde{E}_{n,0}}} B_{nbi}, \tag{4.77}$$

and (4.76) and (4.35) yield that $D_{n,nki} = A_{nki}$. Further, for $i, j \in \{0,1\}^n$, $i \neq j$, we get by (4.29) and (4.76) that

$$D_{n,mki} \cap D_{n,p,k+h,j} = \emptyset, \quad \text{for all } m, p \leq n \in \mathbb{N}_0,\ h \in \mathbb{N}_0. \tag{4.78}$$

Next, using (4.77) and (4.61) we have for all $m < n$, $(b,j) \in \tilde{E}_{m,0}$ and all $i \in \{0,1\}^n$ with $i|_m = j$ that

$$D_{n,mki} \cap B_{mbj} = D_{n,mki} \cap \dot{\bigcup_{\substack{a:(a,i)\in\tilde{E}_n, a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}}} B_{nai} = \dot{\bigcup_{\substack{a:(a,i)\in\tilde{E}_n, a|_{m+1}\neq 0,\\ (a,i)|_m=(b,j)}}} (B_{nai} \cap D_{n,mki})$$

such that, for later reference,

$$\dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} (D_{n,mki} \cap B_{mbj}) = \dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} \dot{\bigcup_{\substack{a:(a,i)\in\tilde{E}_{n,0},(a,i)|_m=(b,j),\\ (a,i)|_{m+1}\neq(0,i|_{m+1})}}} (D_{n,mki} \cap B_{nai}) \tag{4.79}$$

where the union on i is disjoint by (4.76) and (4.29).

Lemma 4.4.1. *For all $m < n \in \mathbb{N}_0$, $k \in K_m$, $i \in \{0,1\}^n$ and $j = i|_m$ we have*

$$D_{n,mki} \subseteq \dot{\bigcup_{\substack{a:(a,j)\in\tilde{E}_{m,0},\\ (\omega,(j|_{m-1},1))\prec(a,j)\preceq(\tilde{\omega},j)}}} B_{maj}$$

where $(\omega, (j|_{m-1}, 1)) = (\omega_{m-1,k,j|_{m-1}}, (j_{m-1}, 1))$ and $(\tilde{\omega}, j) = (\tilde{\omega}_{mkj}, j)$.

Proof. Note first that (4.76) and (4.20) give

$$
\begin{aligned}
D_{n,mki} \cap B_{nbi} = \Bigg(\sum_{(a,j)\prec(b,i)} |B_{naj}| + [0, |B_{nbi}|) \Bigg) \cap \Bigg([0, 2^{-n} a_{m,k,i|_m}) \\
+ \sum_{\substack{a:(a,i)\in\tilde{E}_{n,0}, \\ (a,i)|_m \preceq (\omega,(i|_{m-1},1))}} |B_{nai}| + \sum_{\substack{(a,j)\prec(b,i), \\ j\neq i}} |B_{naj}| \Bigg).
\end{aligned} \tag{4.80}
$$

Next, applying (4.36) to (4.80) we find similar as in (4.37) that

$$
D_{n,mki} \cap B_{nbi} = \begin{cases} \displaystyle\sum_{(a,j)\prec(b,i)} |B_{naj}| + \Bigg[0, \min\Bigg\{ 2^{-n} a_{m,k,i|_m} \\ \quad \displaystyle - \sum_{\substack{a:(\omega,(i|_{m-1},1))\prec(a,i)|_m, \\ (a,i)\prec(b,i)}} |B_{nai}|, |B_{nbi}| \Bigg\} \Bigg), & (\omega,(i|_{m-1},1)) \prec (b,i)|_m, \\ \emptyset, & \text{else.} \end{cases} \tag{4.81}
$$

Further, using (4.19) repeatedly we get

$$
\sum_{(\omega,(j|_{m-1},1))\prec(a,j)\preceq(\tilde{\omega},j)} |B_{maj}| = 2^{n-m} \sum_{c:(\omega,(j|_{m-1},1))\prec(c,i)|_m\preceq(\tilde{\omega},j)} |B_{nci}| \tag{4.82}
$$

such that by (4.82) and (4.38) for all $j \in \{0,1\}^m$ and all $i \in \{0,1\}^n$ with $i|_m = j$

$$
2^{-n} a_{mkj} - \sum_{\substack{c:(c,i)\in\tilde{E}_{n,0}, \\ (\omega,(j|_{m-1},1))\prec(c,i)|_m\preceq(\tilde{\omega},j)}} |B_{nci}| \leq 0. \tag{4.83}
$$

Now, by (4.81) and (4.83) for all $(b,i) \in \tilde{E}_{n,0}$ with $(b,i)|_m \preceq (\omega,(i|_{m-1},1))$ or $(\tilde{\omega}, i|_m) \prec (b,i)|_m$ we have

$$
D_{n,mki} \cap B_{nbi} = \emptyset. \tag{4.84}
$$

Finally, (4.77) and (4.84) yield that for all $i \in \{0,1\}^n$ with $i|_m = j$

$$
D_{n,mki} \subseteq \biguplus_{\substack{b:(b,i)\in\tilde{E}_{n,0}, \\ (\omega,(j|_{m-1},1))\prec(b,i)|_m\preceq(\tilde{\omega},j)}} B_{nbi} = \biguplus_{\substack{a:(a,j)\in\tilde{E}_{m,0}, \\ (\omega,(j|_{m-1},1))\prec(a,j)\preceq(\tilde{\omega},j)}} B_{maj} \tag{4.85}
$$

where the equality holds by (4.61). □

By (4.81) and (4.85) we may now state for later reference that

$$
|D_{n,mki}| = 2^{-n} a_{m,k,i|_m}. \tag{4.86}
$$

Lemma 4.4.2. *For all $m < n \in \mathbb{N}_0$, $k \in K_m$ and $j \in \{0,1\}^m$ we have*

$$A_{mkj} = \dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} D_{n,mki}.$$

Proof. Note that by (4.40) and Lemma 4.4.1 it is sufficient to show for all $(b,j) \in \tilde{E}_{m,0}$ with $(\omega,(j|_{m-1},1)) \prec (b,j) \preceq (\tilde{\omega},j)$ that

$$B_{mbj} \cap A_{mkj} = B_{mbj} \cap \dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} D_{n,mki}. \tag{4.87}$$

To this end, we shall consider a twofold case differentiation. First, let $2^{-m}a_{mkj} - \sum_{a:(\omega,(j|_{m-1},1))\prec(a,j)\prec(b,j)} |B_{maj}| \le |B_{mbj}|$. Then, by (4.37)

$$\begin{aligned}B_{mbj} \cap A_{mkj} &= \sum_{(a,i)\prec(b,j)} |B_{mai}| + \Big[0, 2^{-m}a_{mkj} - \sum_{a:(\omega,(j|_{m-1},1))\prec(a,j)\prec(b,j)} |B_{maj}|\Big) \\ &= \dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} \dot{\bigcup_{\substack{a:(a,i)|_m=(b,j),\\ (a,i)|_{m+1}\preceq(\omega,(j,1))}}} B_{nai} \end{aligned}\tag{4.88}$$

where the second equality follows by (4.63) and (4.61). Now, (4.88) and a repeated application of (4.19) yield that for all $i \in \{0,1\}^n$ with $i|_m = j$

$$\begin{aligned}\sum_{\substack{a:(a,i)|_m=(b,j),\\ (a,i)|_{m+1}\preceq(\omega,(j,1))}} |B_{nai}| &= 2^{m-n}\Bigg(2^{-m}a_{mkj} - \sum_{\substack{a:(\omega,(j|_{m-1},1))\prec\\ (a,j)\prec(b,j)}} |B_{maj}|\Bigg) \\ &= 2^{-n}a_{mkj} - \sum_{\substack{c:(\omega,(j|_{m-1},1))\prec\\ (c,i)|_m\prec(b,j)}} |B_{nci}| \end{aligned}\tag{4.89}$$

and by (4.89), in particular,

$$\sum_{\substack{c:(c,i)\preceq(a,i),\\ (c,i)|_m=(b,j)}} |B_{nci}| \le 2^{-n}a_{mkj} - \sum_{c:(\omega,(j|_{m-1},1))\prec(c,i)|_m\prec(b,j)} |B_{nci}| \tag{4.90}$$

for all $a \in \mathbb{B}_n$ with $(a,i)|_m = (b,j)$ and $(a,i)|_{m+1} \preceq (\omega,(j,1))$. Further, by (4.89) we have for $(\omega,(j,1)) \prec (a,i)|_{m+1}$ that

$$2^{-n}a_{mkj} - \sum_{c:(\omega,(j|_{m-1},1))\prec(c,i)|_m\prec(b,j)} |B_{nci}| - \sum_{\substack{c:(c,i)\prec(a,i),\\ (c,i)|_m=(b,j)}} |B_{nci}| \le 0. \tag{4.91}$$

Now, by (4.20) we get

$$\begin{aligned}
\dot{\bigcup_{\substack{a:(a,i)|_m=(b,j),\\(a,i)|_{m+1}\preceq(\omega,(j,1))}}} B_{nai} &= \dot{\bigcup_{\substack{a:(a,i)|_m=(b,j),\\(a,i)|_{m+1}\preceq(\omega,(j,1))}}} \Big([0,|B_{nai}|) + \sum_{(c,l)\prec(a,i)} |B_{ncl}|\Big) \\
&= \dot{\bigcup_{a:(a,i)|_m=(b,j)}} \Bigg(\sum_{(c,l)\prec(a,i)} |B_{ncl}| + \Bigg[0, \min\Bigg\{2^{-n}a_{mkj} \\
&\quad - \sum_{c:(\omega,(j|_{m-1},1))\prec(c,i)|_m\prec(b,j)} |B_{nci}| - \sum_{\substack{c:(c,i)\prec(a,i),\\(c,i)|_m=(b,j)}} |B_{nci}|, |B_{nai}|\Bigg\}\Bigg)\Bigg) \\
&= \dot{\bigcup_{a:(a,i)|_m=(b,j)}} (D_{n,mki}\cap B_{nai}) \qquad (4.92)
\end{aligned}$$

where we use (4.90) to (4.91) for the second equality and (4.81) for the last equality. Next, by (4.88) and (4.92) we find that

$$\begin{aligned}
B_{mbj}\cap A_{mkj} &= \dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} \dot{\bigcup_{a:(a,i)|_m=(b,j)}} (D_{n,mki}\cap B_{nai}) \\
&= B_{mbj}\cap \dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} D_{n,mki} \qquad (4.93)
\end{aligned}$$

where the last equality holds by (4.79). To conclude the proof consider now the case $2^{-m}a_{mkj} - \sum_{a:(\omega,(j|_{m-1},1))\prec(a,j)\prec(b,j)} |B_{maj}| > |B_{mbj}|$. Then, we have by (4.37) that

$$\begin{aligned}
B_{mbj}\cap A_{mkj} &= \sum_{(a,i)\prec(b,j)} |B_{mai}| + [0,|B_{mbj}|) \\
&= B_{mbj} = \dot{\bigcup_{\substack{i\in\{0,1\}^n:\\ i|_m=j}}} \dot{\bigcup_{\substack{a:(a,i)\in\tilde{E}_n, a|_{m+1}\neq 0,\\(a,i)|_m=(b,j)}}} B_{nai} \qquad (4.94)
\end{aligned}$$

where the second equality is a consequence of (4.20) and the last equality holds by (4.61). In particular, using (4.94) and applying (4.19) repeatedly we find that

$$\sum_{\substack{a:(a,i)\in\tilde{E}_{n,0}, a|_{m+1}\neq 0,\\(a,i)|_m=(b,j)}} |B_{nai}| = 2^{m-n}|B_{mbj}| < 2^{-n}a_{mkj} - \sum_{\substack{c:(\omega,(j|_{m-1},1))\prec\\(c,i)|_m\prec(b,j)}} |B_{nci}| \qquad (4.95)$$

where the inequality merely reflects the above assumption for the second case. Now, for any $a\in\mathbb{B}_n$ such that $(a,i)|_m=(b,j)$ we have by (4.95) that

$$2^{-n}a_{mkj} - \sum_{\substack{c:(\omega,(j|_{m-1}))\prec\\(c,i)|_m\prec(b,j)}} |B_{nci}| - \sum_{\substack{c:(c,i)\prec(a,i),\\(c,i)|_m=(b,j)}} |B_{nci}| > |B_{nai}|. \qquad (4.96)$$

Hence, we get that

$$\begin{aligned}
\biguplus_{\substack{a:(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\(a,i)|_m=(b,j)}} B_{nai} &= \biguplus_{\substack{(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\(a,i)|_m=(b,j))}} \left(\sum_{(c,l)\prec(a,i)} |B_{ncl}| + [0,|B_{nai}|)\right) \\
&= \biguplus_{\substack{(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\(a,i)|_m=(b,j)}} \left(\sum_{(c,l)\prec(a,i)} |B_{ncl}| + \left[0, \min\left\{2^{-n}a_{mkj}\right.\right.\right. \\
&\quad \left.\left.\left. - \sum_{\substack{c:(\omega,(j|_{m-1},1))\prec\\(c,i)|_m\prec(b,j)}} |B_{nci}| - \sum_{\substack{c:(c,i)\prec(a,i),\\(c,i)|_m=(b,j)}} |B_{nci}|, |B_{nai}|\right\}\right)\right) \\
&= \biguplus_{\substack{a:(a,i)\in\tilde{E}_{n,0},a|_{m+1}\neq 0,\\(a,i)|_m=(b,j)}} (D_{n,mki}\cap B_{nai}) \qquad (4.97)
\end{aligned}$$

where the first equality holds by (4.20) and the second equality follows from (4.96). The last equality corresponds to (4.81). Now, similar to the above, the result follows by combining (4.94) and (4.97) first, and using (4.79) to conclude. ☐

Note that by (4.77) and (4.29) we get that

$$\biguplus_{j\in\{0,1\}^n} D_{n,mkj} \cap \biguplus_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi} = D_{n,mki}, \quad i\in\{0,1\}^n, \qquad (4.98)$$

such that (4.98) and Lemma 4.4.2 yield an equivalent representation of $D_{n,mki}$, namely

$$D_{n,mki} = A_{m,k,i|_m} \cap \biguplus_{b:(b,i)\in\tilde{E}_{n,0}} B_{nbi} \quad \text{for all } i\in\{0,1\}^n.$$

By Lemma 4.4.2 and Lemma 4.3.1 we have

$$\biguplus_{j\in\{0,1\}^m} A_{mkj} = \biguplus_{i\in\{0,1\}^n} D_{n,mki}, \quad m\le n\in\mathbb{N}_0, k\in K_m. \qquad (4.99)$$

Further, Lemma 4.4.2 yields that $D_{n,mki}\subseteq A_{mkj}$ for all $j\in\{0,1\}^m$, $k\in K_m$ and all $i\in\{0,1\}^n$ with $i|_m = j$. Then, the fact that for all $n\in\mathbb{N}_0$, $k\in K_n$ and $i\in\{0,1\}^n$ we have

$$D_{n,mki}\cap D_{n,pki} = \emptyset, \quad m<p\le n, \qquad (4.100)$$

holds by Lemma 4.3.1.

4.5 Main Result

In the following theorem we shall make use of the sets S_n given in (4.5) where the unions are now seen to be disjoint by Lemma 4.3.2.

Theorem 4.5.1. *The sequence of sets $(S_n)_{n\in\mathbb{N}_0} \uparrow S$ is monotonic, and*

$$2^{-n+1}\sum_{k\in K_n}\sum_{i\in\{0,1\}^n}\sum_{s=0}^{n} a_{s,k,i|_s} - \phi(h \mid g_n) = |S_n \cap (S_n - h)|, \quad n \in \mathbb{N}_0,\, h \in \mathbb{Z}.$$

Proof. By Lemma 4.3.2 we have

$$\begin{aligned}
|S_n \cap (S_n - h)| &= \sum_{k\in\mathbb{Z}} \left| \dot{\bigcup_{s=0}^{n}} \dot{\bigcup_{i\in\{0,1\}^s}} A_{ski} \quad \cap \quad \dot{\bigcup_{s=0}^{n}} \dot{\bigcup_{i\in\{0,1\}^s}} A_{s,k+h,i} \right| \\
&= \sum_{k\in\mathbb{Z}} \left| \dot{\bigcup_{i\in\{0,1\}^n}} \dot{\bigcup_{s=0}^{n}} D_{n,ski} \quad \cap \quad \dot{\bigcup_{i\in\{0,1\}^n}} \dot{\bigcup_{s=0}^{n}} D_{n,s,k+h,i} \right| \\
&= \sum_{k\in\mathbb{Z}} \left| \dot{\bigcup_{i\in\{0,1\}^n}} \left(\dot{\bigcup_{s=0}^{n}} D_{n,ski} \quad \cap \quad \dot{\bigcup_{s=0}^{n}} D_{n,s,k+h,i} \right) \right| \\
&= \sum_{k\in\mathbb{Z}} \sum_{i\in\{0,1\}^n} \left| \dot{\bigcup_{s=0}^{n}} D_{n,ski} \quad \cap \quad \dot{\bigcup_{s=0}^{n}} D_{n,s,k+h,i} \right| \qquad (4.101)
\end{aligned}$$

where the second equality holds by (4.99), and (4.78) gives the third equality. Next, note that (4.76) and (4.100) yield

$$\begin{aligned}
\dot{\bigcup_{s=0}^{n}} D_{n,ski} &= \dot{\bigcup_{b:(b,i)\in\tilde{E}_{n,0}}} \Bigg\{ B_{nbi} \cap \Bigg(\sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| \\
&\qquad + \dot{\bigcup_{s=0}^{n}} \Bigg([0, 2^{-n} a_{s,k,i|_s}) + \sum_{\substack{a:(a,i)\in\tilde{E}_{n,0},\\ (a,i)|_s \preceq (\omega,(i|_{s-1},1))}} |B_{nai}| \Bigg) \Bigg) \Bigg\} \\
&= \dot{\bigcup_{b:(b,i)\in\tilde{E}_{n,0}}} \Bigg\{ B_{nbi} \cap \Bigg(\sum_{\substack{(a,j)\prec(b,i),\\ j\neq i}} |B_{naj}| + \Bigg[0, 2^{-n}\sum_{s=0}^{n} a_{s,k,i|_s} \Bigg) \Bigg) \Bigg\} \qquad (4.102)
\end{aligned}$$

where the second equality holds by (4.59) and (4.19). Using (4.102) we get that

$$\dot{\bigcup_{s=0}^{n}} D_{n,ski} \quad \cap \quad \dot{\bigcup_{s=0}^{n}} D_{n,s,k+h,i} = \dot{\bigcup_{s=0}^{n}} D_{n,ski}$$

if and only if

$$\sum_{s=0}^{n} a_{s,k,i|_s} \leq \sum_{s=0}^{n} a_{s,k+h,i|_s},$$

for any $n \in \mathbb{N}_0$, $h \in \mathbb{Z}$, $k \in K_n$ and $i \in \{0,1\}^n$. Combining the latter result with (4.101) and (4.4) yields the assertion where

$$\left| \dot{\bigcup_{s=0}^{n}} D_{n,ski} \right| = 2^{-n} \sum_{s=0}^{n} a_{s,k,i|_s}$$

follows directly from (4.86). □

Corollary 4.5.1. *For any extremal coefficient function ϕ of a dissipative max-stable process on $\mathbb{Z}$ whose spectral function g may be approximated by (4.3) there exists a measurable set $S \subset \mathbb{R}$ such that*

$$2 - \phi(h \mid g) = |S \cap (S - h)|, \qquad h \in \mathbb{Z}.$$

Proof. Using (4.3) we find that

$$\int g_n(x)dx = 2^{-n} \sum_{k \in K_n} \sum_{i \in \{0,1\}^n} \sum_{s=0}^{n} a_{s,k,i|_s} \to 1 \quad (n \to \infty)$$

by the fact that $g_n \uparrow g$, and by $\int g(x)dx = 1$. Now, the assertion follows directly from Theorem 4.5.1. □

The following corollary extends the above result to general dissipative max-stable processes, i.e. the assumption (4.3) on the spectral function g is abandoned. As indicated in Section 4.1 its proof may be based on the construction of a suitable function ξ such that $g_n \uparrow \xi$ and $\phi(\cdot \mid \xi) = \phi(\cdot \mid g)$, see [19] for details.

Corollary 4.5.2 ([19, Corollary 3])**.** *For any extremal coefficient function ϕ of a dissipative max-stable process on $\mathbb{Z}$ there exists a measurable set $S \subset \mathbb{R}$ such that*

$$2 - \phi(h) = |S \cap (S - h)|, \qquad h \in \mathbb{Z}.$$

Chapter 5

A Novel Characteristic for the Dependence Structure of Clustered Extremes

5.1 Exploring Extremal Clusters

In the preceding chapters we characterized extremal clusters of stationary max-stable processes by basically two well-known summary measures. We discussed the extremal index that reflects the expected size of such clusters and we studied the extremal coefficient function that describes bivariate dependencies at all lags of such processes. In the following we will abandon the restriction to max-stable processes and extend the study of extremal measures to the rich class of processes that lie in the domain of attraction of a max-stable process, cf. [22]. Note that the latter assumption will hold in particular for the important GARCH family of time series models that we shall consider in detail below. In the first place, however, we shall turn to a more general problem. That is, we will critically examine the informational value of the existing measures θ and ϕ with respect to typical questions about the structure of extremes that may come up in many applications. Our reasoning will then give rise to the proposal of a novel summary measure for extremal dependence. Note that the current chapter has been motivated to a large extent by the study of homometric patterns in Section 3.3, i.e. such simple cluster types that are not distinguishable by the extremal coefficient function. As in that section we used a rather technical setup we will clarify below by a brief example the possible implications of such patterns in practice. Although we shall for convenience incorporate our discussion into a stylized financial context whose extremal behavior has been studied in detail [41, 20] we remark that all arguments will likewise cover further fields of application. To begin with, it appears to be reasonable to require an appropriate measure for the within extremal cluster structure to address the following questions:

(Q1) What is the probability for a second, third etc. extreme event occuring two, three

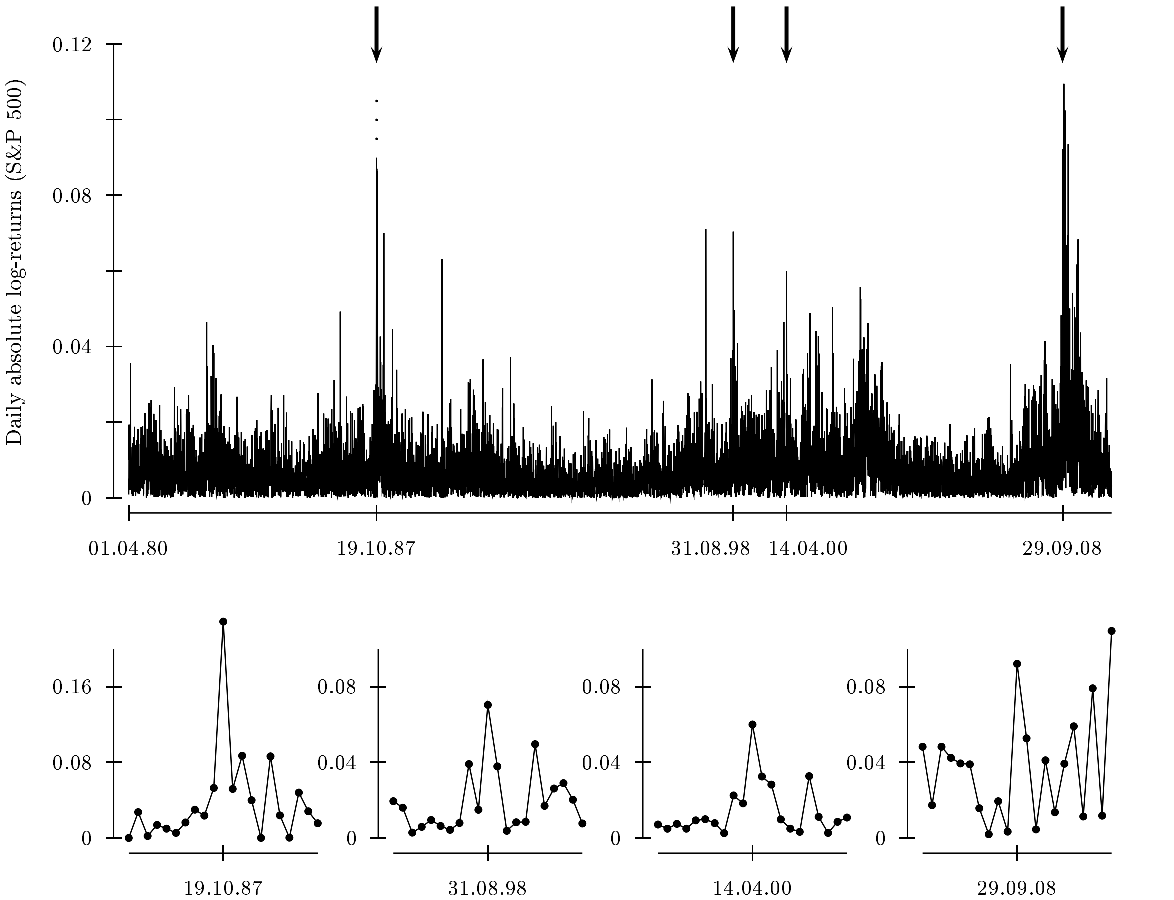

Figure 5.1.1: Volatility clustering for the daily absolute log returns of the S&P 500 index from 01.04.80 to 31.03.10 (7569 records). The marked extreme events are enlarged in the respective bottom pictures and correspond to the "Black Monday" in 1987, the Russian financial crisis in 1998 and the dot-com bubble burst in 2000. The remaining event in 2008 is surrounded by the unusual volatility attributable to the so-called subprime mortgage crisis.

etc. days after the outset of a financial crisis?

(Q2) What is the structure of a cluster of high-level exceedances and how does the course of extreme events (i.e. the evolution of a stress period or crisis over time) typically look like?

(Q3) How may the memory spread of financial markets with respect to shocks be characterized, i.e. how long does a crisis typically last?

Hence, many possible questions focus on expected events in the near future given a *first* extreme event today (i.e. the beginning of a temporary shock or crisis). Note that the above setup may readily be extended to a multivariate approach that we will not consider here. To illustrate the relevance of these matters, in Figure 5.1.1 we display the daily absolute logarithmic returns of the S&P 500 index ranging from 1980 to 2010. We understand that an extremal cluster is formed by several adjacent high-level exceedances where different clusters are generally separated by longer periods

k	1	2	3	4	5	6	7	8	9	10	11	12
$P1$	1/3	1/3		1/3								
$P2$	1/3		1/3	1/3								
$P3$	1/6	1/6	1/6				1/6		1/6			1/6
$P4$	1/6	1/6					1/6	1/6		1/6		1/6

Table 5.1.1: Coefficients a_k of the M_2 processes $P1$ to $P4$ discussed in Example 5.1.1.

of low levels. Here, we focus on the within-cluster dependence structure and assume observations in different clusters to be independent. The courses of four typical clusters indicated by an arrow in the upper plot are depicted in the bottom subfigures. Apart from an obvious volatility clustering at high levels they also suggest the presence of a roughly common pattern. In particular, the outset of each cluster of extremes is fairly well distinguishable in the first three cases which illustrates the relevance of conditioning on the beginning of a crisis when judging probabilities for future extreme events. In contrast, the event in 2008 is accompanied by an unusually long-lasting volatility cluster which complicates the identification of such a definite starting point. Still, in all four subfigures we find that just giving the average cluster size (where using [64] we get that $\theta \approx 1/3$, see also Example 5.3.1) does not reveal the characteristic structure of extreme events that is evident from the above plots. More precisely, the latter show a clearly visible second peak five to six trading days after the start of each cluster. In practice, e.g. for financial institutions the expected location of such events within a cluster of extremes is essential in order to efficiently react to the pattern of inherent risk they describe. We shall therefore require a suitable characteristic that reaches beyond the extremal index to reliably indicate strength and location of subsequent extreme events within a crisis. The shortcomings of the extremal coefficient function in this regard may best be understood by the following example.

Example 5.1.1. Consider the M_2 processes $P1$ to $P4$ that are given by their coefficients a_k according to Table 5.1.1. Recall from (3.28) that the coefficients in Table 5.1.1 in particular determine the structure of the extremal clusters for the respective processes, also called their signature patterns [68]. According to Section 2.2 the corresponding extremal indices are given by $\theta_{P1} = \theta_{P2} = 1/3$ and $\theta_{P3} = \theta_{P4} = 1/6$. Further, we contrast the extremal coefficient function ϕ with a new characteristic γ in Table 5.1.2. We will give a formal definition of the latter in Section 5.2 but at this point it will be sufficient to preliminarily recall the representation given in Chapter 1, i.e.

$$\gamma(h \mid Y) = \lim_{u\to\infty} P(Y_h > u \mid Y_0 > u, \text{ and } Y_0 \text{ first event in the extremal cluster}) \quad (5.1)$$

for all $h \in \mathbb{N}$. For convenience we repeat (3.25) where the extremal coefficient function is given by

$$2 - \phi(h \mid Y) = \lim_{u\to\infty} P(Y_h > u \mid Y_0 > u), \quad h \in \mathbb{Z},$$

which highlights the similar construction of the two characteristics. It is therefore even more remarkable that their interpretation will differ substantially. First and

	h	0	1	2	3	4	5	6	7	8	9	10	11	12
$2-\phi(h)$	$P1$	1	1/3	1/3	1/3	0		$\cdots$						
	$P2$	1	1/3	1/3	1/3	0		$\cdots$						
$\gamma(h)$	$P1$	1	1	0	1	0		$\cdots$						
	$P2$	1	0	1	1	0		$\cdots$						
$2-\phi(h)$	$P3$	1	1/3	1/3	1/6	1/6	1/3	1/3	1/6	1/6	1/6	1/6	1/6	0
	$P4$	1	1/3	1/3	1/6	1/6	1/3	1/3	1/6	1/6	1/6	1/6	1/6	0
$\gamma(h)$	$P3$	1	1	1	0	0	0	1	0	1	0	0	1	0
	$P4$	1	1	0	0	0	0	1	1	0	1	0	1	0

Table 5.1.2: Extremal measures ϕ and γ for processes $P1$ to $P4$ discussed in Example 5.1.1.

obvious, note that the limiting probabilities defining ϕ are not tied to the beginning of an extremal cluster which is in contrast to (Q1) discussed above. Moreover, we will show in Corollary 5.2.2 below that γ traces the evolution of an extremal cluster in the way suggested by (Q1) and (Q2). In particular, Table 5.1.2 shows that for $P1$ and $P2$ the extremal coefficient function is unable to distinguish between a simple reflection of the processes whereas γ exactly mirrors the pattern of extremes. The problem, however, is not restricted to the rather obvious case of congruent patterns as can be seen from the behavior of ϕ for $P3$ and $P4$ which represent an example of the homometric noncongruent patterns discussed in Section 3.2. Here, we find once more that γ conveys the information requested by (Q1) and (Q2). Concerning (Q3), however, Table 5.1.2 suggests that both ϕ and γ are suitable characteristics in order to reflect the duration of extremal clusters.

The above simple example underlines the ability of γ to draw a priori conclusions about the shape of an extremal cluster. Note that from the point of view of a financial institution scenarios of the kind $P1$ and $P2$ will clearly require a different management of risk that is not distinguishable by the extremal coefficient function. More precisely, under $P1$ measures against two subsequent shocks and hence a larger risk at the beginning of the cluster will have to be taken from the outset whereas under $P2$ those events will occur only at the end of the cluster. In particular, they will be signalized by a single extreme event two days in advance. In addition to the favourable properties of γ with respect to questions of the above kind the new characteristic will turn out to have a remarkably easy relation with the extremal index which is not the case for the extremal coefficient function. We will discuss those questions in Section 5.2 where we will also study further properties both of γ and ϕ. In Section 5.3 we will evaluate the above measures in the GARCH(1,1) case using a modified tail chain approach. We shall conclude with an example regarding the above S&P 500 data set.

5.2 Properties of Dependence Measures

In the following we will study the above extremal measures for stationary processes in the domain of attraction of a max-stable process, see [22] for details. To this end, we will assume throughout for the stationary stochastic process $(X_t)_{t\in\mathbb{Z}}$ that a corresponding max-stable limiting process $(Y_t)_{t\in\mathbb{Z}}$ exists, cf. (2.1). More precisely, we will require all finite dimensional distributions of (X_t) to belong to the maximum domain of attraction of a multivariate extreme value distribution. Here, without loss of generality we may restrict the latter to standard Fréchet margins, i.e. $F_{Y_0}(x) = \exp(-x^{-1})$, $x > 0$. It will be advantageous later on to cover the above assumptions equivalently by means of a multivariate regular variation condition, i.e.

$$xP\left(\left(\frac{X_1}{x}, \ldots, \frac{X_m}{x}\right) \in \cdot\right) \xrightarrow{v} \mu(\cdot), \quad (x \to \infty), \tag{C1}$$

for a Radon measure μ on $[-\infty, \infty]^m \backslash \{0\}^m$ and all $m \in \mathbb{N}$. Here, by $\xrightarrow{v}$ we denote vague convergence, see e.g. [50, 41] for details. Throughout, we will put $M_S = \max_{t\in S} X_t$ for an index set S where $M_\emptyset = 0$ and $M_{m,n} = M_{\{m,\ldots,n\}}$, $m \leq n$. Further, we will frequently make use of the sets $S_m = \{-m, \ldots, -1\}$, $S_{m,h} = S_m \cup \{h\}$ and $S_{m,0,h} = S_{m,h} \cup \{0\}$. We shall refer to the familiar extremal index θ of the process (X_t) using a particularly illustrative definition, i.e.

$$\theta(h) = \lim_{n\to\infty} P(X_t \leq n,\, t \in S_{r_n,h} \mid X_0 > n), \quad h \in \mathbb{Z}, \tag{5.2}$$

such that $\theta(-1) = \theta$ corresponds to (2.3) for $D = 1$ under a weak mixing condition [45], see also [61]. Here, $r_n \to \infty$ is a suitable sequence that, in particular, satisfies $r_n/n \to 0$ as $n \to \infty$. The limiting probability in (5.2) also provides an alternative interpretation of the extremal index in terms of the probability of a high-level exceedance being the first (last) in a cluster of extremes. Recall that in general we have $\theta \in [0, 1]$ where the case $\theta = 0$ may reasonably be excluded a priori in most applications. We will therefore assume throughout that for some increasing sequence r_n with $r_n/n \to \infty$ as $n \to \infty$ the well-known condition

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(M_{-r_n,-m-1} > n \mid X_0 > n) = 0 \tag{C2}$$

holds, see e.g. [45], which restricts the influence of an extreme observation over time. In particular, (C2) is sufficient for $\theta > 0$, and it guarantees the existence of the limit in (5.2), cf. Proposition 5.2.1. Note that (C2) may be stronger than necessary but holds for the important class of GARCH(1,1) processes that we will consider below, cf. Corollary 5.3.1. See also [58] for further examples. We will show that the following more general concept

$$\theta_{m,n}(h) = P(X_t \leq n, t \in S_{m,h} \mid X_0 > n), \quad h \in \mathbb{Z}, \tag{5.3}$$

is sufficient under (C2) in order to investigate any of the extremal measures discussed above.

Proposition 5.2.1. *Under conditions (C1) and (C2) we have that*

$$\theta(h) = \lim_{m\to\infty} \lim_{n\to\infty} \theta_{m,n}(h), \quad h \in \mathbb{Z}.$$

In particular, $\theta = \lim_{m\to\infty} \lim_{n\to\infty} \theta_{m,n}(-1) > 0$.

Proof. Note first that the limit $\theta_m(h) = \lim_{n\to\infty} \theta_{m,n}(h)$ exists by (C1). Consequently, for all $m \in \mathbb{N}$, $h \in \mathbb{Z}$, we have

$$\begin{aligned}
0 &\leq \theta_m(h) - \limsup_{n\to\infty} P(X_t \leq n,\, t \in S_{r_n,h} \mid X_0 > n) \\
&\leq \theta_m(h) - \liminf_{n\to\infty} P(X_t \leq n,\, t \in S_{r_n,h} \mid X_0 > n) \\
&= \limsup_{n\to\infty} \big[\theta_{m,n}(h) - P(X_t \leq n,\, t \in S_{r_n,h} \mid X_0 > n)\big] \\
&= \limsup_{n\to\infty} P(M_{-r_n,-m-1} > n, X_t \leq n, t \in S_{m,h} \mid X_0 > n) \\
&\leq \limsup_{n\to\infty} P(M_{-r_n,-m-1} > n \mid X_0 > n).
\end{aligned}$$

Now, by (C2) the r.h.s. tends to 0 as $m \to \infty$, and we get that

$$\theta(h) = \lim_{m\to\infty} \theta_m(h) = \lim_{m\to\infty} \lim_{n\to\infty} \theta_{m,n}(h), \quad h \in \mathbb{Z},$$

as $\theta_m(h)$ decreases in m. Finally, the fact that $\theta > 0$ follows from the discussion after Condition 10.8 in [2]. □

Note that under the conditions of Proposition 5.2.1 we may in general not conclude that $\theta(h) > 0$, $h \in \mathbb{N}$. We will now give a formal definition of the extremal characteristics discussed in Section 5.1 for stationary processes in the domain of attraction of a max-stable process. First, for all $h \in \mathbb{Z}$ the extremal coefficient function following [56] is given by

$$\phi(h) = 1 + \theta_0(h) = 2 - \lim_{n\to\infty} P(X_h > n \mid X_0 > n) = 2 - \lim_{n\to\infty} P(Y_h > n \mid Y_0 > n) \quad (5.4)$$

where the third equality by (C1) holds for the max-stable limiting process (Y_t), cf. (3.25). Concerning the questions raised in Section 5.1 it may be beneficial to replace (5.4) by a similar probability that is tied to the first extreme event in a cluster as in (5.1). We therefore propose to modify the above concept and consider as a closely related characteristic the function $\gamma(h)$ which we shall define by

$$\gamma(h) = 1 - \frac{\theta(h)}{\theta} = \lim_{m\to\infty} \gamma_m(h), \quad h \in \mathbb{N}_0, \quad (5.5)$$

where $\gamma_m(h) = \lim_{n\to\infty} P(X_h > n \mid X_0 > n, X_t \leq n, t \in S_m)$. Although Definitions (5.4) and (5.5) appear to be closely related, in the following we will discuss that their properties may differ substantially.

To begin with, note that θ and $\phi(h)$, $h \in \mathbb{N}$, are obviously invariant under time reversal of the process (X_t), i.e. $\theta(X_t,\, t \in \mathbb{Z}) = \theta(X_{-t},\, t \in \mathbb{Z})$ and $\phi(h) = \phi(-h)$, but that in general neither $\theta(h \mid X_t,\, t \in \mathbb{Z}) = \theta(h \mid X_{-t},\, t \in \mathbb{Z})$ nor $\gamma(h \mid X_t,\, t \in \mathbb{Z}) = \gamma(h \mid X_{-t},\, t \in \mathbb{Z})$, $h \in \mathbb{N}$, cf. Example 5.1.1. From the above definitions we readily have that

$$0 \leq \theta(h) \leq \theta \leq \phi(h) - 1 \leq 1$$

for each $h \in \mathbb{N}$. The following theorem gives an exact relationship between the sum of the function $\gamma(h)$, $h \in \mathbb{N}_0$, and the extremal index.

Theorem 5.2.1. *Under conditions (C1) and (C2) we have that*

$$\sum_{h\in\mathbb{N}_0}\gamma(h)=\frac{1}{\theta}. \tag{5.6}$$

Proof. By (5.5) it suffices to show that $\sum_{h\in\mathbb{N}_0}(\theta-\theta(h))=1$. To this end, note that for all $h\in\mathbb{N}$ we have

$$\begin{aligned}\theta-\theta(h)&=\lim_{m\to\infty}\lim_{n\to\infty}P(X_{-m-h}\le n,\dots,X_{-1-h}\le n,\,X_{-h}>n\mid X_0>n)\\&=\lim_{m\to\infty}\lim_{n\to\infty}P(X_{-m}\le n,\dots,X_{-1-h}\le n,\,X_{-h}>n\mid X_0>n)\\&\quad-\lim_{m\to\infty}\lim_{n\to\infty}P(M_{-m-h,-m-1}>n,\,X_{-m}\le n,\dots,X_{-1-h}\le n,\\&\quad X_{-h}>n\mid X_0>n)\end{aligned}$$

where the latter term is bounded from above by 0 through (C2). Now, for all $p\in\mathbb{N}$ we get that

$$\sum_{h=0}^{p}(\theta-\theta(h))=\lim_{m\to\infty}\lim_{n\to\infty}P(X_{-m}\le n,\dots,X_{-p-1}\le n\mid X_0>n)$$

where

$$\begin{aligned}&\lim_{p\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}P(X_{-m}\le n,\dots,X_{-p-1}\le n\mid X_0>n)\\&\qquad\ge\liminf_{n\to\infty}P(X_{-r_n}\le n,\dots,X_{-q-1}\le n\mid X_0>n)\quad\text{for all } q\in\mathbb{N}.\end{aligned}$$

Finally, condition (C2) yields that the r.h.s. tends to 1 as $q\to\infty$. ☐

In particular, Theorem 5.2.1 highlights the fact that under the above conditions $\theta=1$ is equivalent to $\gamma(h)=0$, $h\in\mathbb{N}$. Note also that, depending on the point of view, (5.6) may provide a refined interpretation of the extremal index, see [20, Section 8.1.2] for a discussion. Further, the proof of Theorem 5.2.1 yields the following limiting relationship between θ and $\theta(h)$.

Corollary 5.2.1. *Under the conditions of Theorem 5.2.1 we have that*

$$\theta=\lim_{n\to\infty}\frac{1}{n}\sum_{h=0}^{n}\theta(h).$$

In the following we shall consider the M_3 representation for dissipative max-stable processes as in (3.28). We will first derive the corresponding expressions for the above extremal characteristics.

Theorem 5.2.2. *For an M_3 process A defined by its coefficients $a_{jp} \geq 0$, $j \in I$, $p \in \mathbb{Z}$, we have*

$$\theta(h \mid A) = \sum_{j\in I}\sum_{p\in\mathbb{Z}}(a_{jp} - \bar{a}_{j,p,h})_+, \quad h \in \mathbb{Z},$$

where $\bar{a}_{j,p,h} := \max\{a_{j,-\infty}, \ldots, a_{j,p-1}, a_{j,p+h}\}$.

Proof. By definition of A we find that

$$\begin{aligned} P(A_t \leq n,\, t \in S) &= P(Z_{j,t-k} \leq n/a_{jk},\, j \in I,\, k \in \mathbb{Z},\, t \in S) \\ &= P(Z_{j,p} \leq n/a_{jk},\, j \in I,\, p \in \mathbb{Z},\, k \in S + p) \\ &= \exp\Big\{ -\frac{1}{n}\sum_{j\in I}\sum_{p\in\mathbb{Z}} \max_{k\in S+p} a_{jk}\Big\} \end{aligned} \tag{5.7}$$

where from the fact that $\sum_j \sum_k a_{jk} = 1$ it follows immediately that

$$\sum_{j\in I}\sum_{p\in\mathbb{Z}} \max_{k\in S+p} a_{jk} \leq |S|. \tag{5.8}$$

Moreover, for all $j \in I$ we have

$$\sum_{p\in\mathbb{Z}} \max_{k\in S_{m,0,h}+p} a_{jk} = \sum_{p\in\mathbb{Z}} \max_{k\in S_{m,h}+p} a_{jk} + \sum_{p\in\mathbb{Z}}\Big(a_{jp} - \max_{k\in S_{m,h}+p} a_{jk}\Big)_+. \tag{5.9}$$

Now, using (5.3), and (5.7) to (5.9) we find that

$$\begin{aligned} \theta_m(h) &= \lim_{n\to\infty} \exp\Big\{ -\frac{1}{n}\sum_{j\in I}\sum_{p\in\mathbb{Z}} \max_{k\in S_{m,h}+p} a_{jk}\Big\} \\ &\quad \cdot \lim_{n\to\infty} \frac{\Big(1 - \exp\Big\{ -\frac{1}{n}\sum_{j\in I}\sum_{p\in\mathbb{Z}}\Big(a_{jp} - \max_{k\in S_{m,h}+p} a_{jk}\Big)_+\Big\}\Big)}{1 - \exp\{-1/n\}} \\ &= \sum_{j\in I}\sum_{p\in\mathbb{Z}}\Big(a_{jp} - \max_{k\in S_{m,h}+p} a_{jk}\Big)_+. \end{aligned}$$

Finally,

$$\theta(h) = \lim_{m\to\infty}\theta_m(h) = \sum_{j\in I}\sum_{p\in\mathbb{Z}}(a_{jp} - \bar{a}_{j,p,h})_+,$$

where $\bar{a}_{j,p,h} = \lim_{m\to\infty}\max_{k\in S_{m,h}+p} a_{jk}$. □

Note in particular that Theorem 5.2.2 generalizes the well-known fact that $\theta_A = \theta(-1 \mid A) = \sum_{j\in I}\max_{p\in\mathbb{Z}} a_{jp}$, cf. [61]. As a useful corollary of Theorems 5.2.1 and 5.2.2 to any dependence function $\gamma(h)$, $h \in \mathbb{N}_0$, we are now able to associate a simple M_2 example process $\tilde{A} = \tilde{A}(\gamma)$ that represents $\gamma(h)$.

Corollary 5.2.2. *Assume (C1) and (C2). For any function $\gamma(h)$, $h \in \mathbb{N}_0$, according to (5.5) let the M_2 process $\tilde{A}$ be given by the coefficients*

$$\tilde{a}_k = \begin{cases} \dfrac{\gamma(k)}{\sum_{h\in\mathbb{N}_0}\gamma(h)} = \theta\gamma(k), & k \in \mathbb{N}_0, \\ 0, & -k \in \mathbb{N}. \end{cases}$$

Then, we have that $\gamma(h) = \gamma(h \mid \tilde{A})$, $h \in \mathbb{N}_0$.

Note that $\tilde{A}$ has the following useful properties that can be easily seen. First, we find that $\tilde{a}_0 = \theta \geq \tilde{a}_h$, $h \in \mathbb{N}_0$. That is, any extremal cluster of $\tilde{A}$ starts with a value driven by $\tilde{a}_0$ that dominates the following realizations in the same cluster. Second, note that $\gamma(h) = \tilde{A}_h/\tilde{A}_0 = \tilde{a}_h/\tilde{a}_0$, $h \in \mathbb{N}_0$, where $\tilde{A}_0$ represents the starting point of the current cluster. In particular, we also have that $\gamma(h) = 0$ is equivalent to $\tilde{a}_h = 0$, for all $h \in \mathbb{N}$. To sum up, in order to illustrate and actually visualize the implications of $\gamma(h)$ for arbitrarily complex processes it may be useful to study the simple process $\tilde{A}$ instead. Note that the possibility to readily state a valid example process for any valid function $\gamma(h)$ is in sharp contrast to the difficulties encountered for the reconstruction of example processes conforming to an extremal coefficient function $\phi(h)$, see Chapter 3. Further, as opposed to the latter characteristic which is necessarily positive definite, we get from Corollary 5.2.2 that any summable function $\gamma : \mathbb{N}_0 \to [0,1]$ with $\gamma(0) = 1$ is a valid such extremal dependence function.
In contrast to Theorem 5.2.1 no definite interrelation between the extremal index θ and the extremal coefficient function $\phi(h)$ exists. In particular, θ may not be reconstructed from ϕ. However, we are able to state a sharp lower bound in terms of θ on the sum of $2 - \phi(h)$, $h \in \mathbb{N}$.

Theorem 5.2.3. *For any max-stable process we have that*

$$\lfloor 1/\theta \rfloor \left(1 - (1 + \lfloor 1/\theta \rfloor)\theta/2\right) \leq \sum_{h\in\mathbb{N}} \left(2 - \phi(h)\right). \tag{5.10}$$

Proof. Note first that any max-stable process with summable function $2-\phi(h)$, $h \in \mathbb{N}$, is dissipative by Proposition 3.3.1. Then, by Theorem 3.1 in [52] we have in particular that $\theta > 0$. The proof may therefore be based on the M_3 representation as in (3.28). Let $\mathcal{A}(\theta)$ be the class of M_3 processes A given by the coefficients a_{jk}, $j \in I, k \in \mathbb{Z}$, with extremal index $\theta > 0$. Put

$$\Xi_A = \sum_{h\in\mathbb{N}} \left(2 - \phi(h \mid A)\right), \quad A \in \mathcal{A}(\theta),$$

where $\phi(h \mid A) = \sum_{j\in I}\sum_{k\in\mathbb{Z}} \max_{l\in\{0,h\}+k} a_{jl}$, see Theorem 5.2.2. Next, note that for all $h \in \mathbb{N}$ and all $j \in I$ we have

$$2\sum_{k\in\mathbb{Z}} a_{jk} - \sum_{k\in\mathbb{Z}} \max_{l\in\{0,h\}+k} a_{jl} = \sum_{k\in\mathbb{Z}} a_{jk} \left[\mathbf{1}(a_{jk} < a_{j,k-h}) + \mathbf{1}(a_{jk} \leq a_{j,k+h})\right]$$

such that by $\sum_{j\in I}\sum_{k\in\mathbb{Z}} a_{jk} = 1$ we get

$$\begin{aligned}\Xi_A &= \sum_{j\in I}\sum_{k\in\mathbb{Z}} a_{jk} \sum_{h\in\mathbb{N}} [\mathbf{1}(a_{jk} < a_{j,k-h}) + \mathbf{1}(a_{jk} \le a_{j,k+h})] \\ &= \sum_{k\in\mathbb{N}} (k-1) \sum_{j\in I} a_{j,(k)}. \end{aligned} \tag{5.11}$$

Here, for all $j \in I$ we denote by $a_{j,(k)}$ the k-th largest coefficient (including ties). Next, for any $A \in \mathcal{A}(\theta)$ let

$$b_k = \sum_{j\in I} a_{j,(k)}, \quad k \in \mathbb{N}, \tag{5.12}$$

and all other coefficients zero, define the M_2 process $B = B(A) \in \mathcal{A}(\theta)$ such that by (5.11) we have $\Xi_A = \Xi_B$. Let further the M_2 process $B^* \in \mathcal{A}(\theta)$ be given by $b_k^* = \theta$ for $k = 1, \ldots, \lfloor 1/\theta \rfloor = q$, $b_{q+1}^* = 1 - \theta q$ and all other coefficients zero. Put $\delta_k = b_k^* - b_{(k)}$ where

$$\sum_{k\in\mathbb{N}} \delta_k = 0 \tag{5.13}$$

and

$$\delta_k \ge 0 \quad \text{for } k = 1, \ldots, q. \tag{5.14}$$

Here, the latter inequality holds by the fact that $\theta = \max_k b_k$, cf. Theorem 5.2.2. Further,

$$\delta_k \le 0 \quad \text{for } k = q+2, \ldots, \tag{5.15}$$

follows from $b_k^* = 0$ for $k = q+2, \ldots$. Now,

$$\Xi_{B^*} - \Xi_B = \sum_{k=2}^{q+1} (k-1) b_k^* - \sum_{k\in\mathbb{N}} (k-1) b_{(k)} = \sum_{k\in\mathbb{N}} (k-1)\delta_k \le 0$$

by (5.13) to (5.15). Finally, as a consequence of (5.11) we have that $\Xi_{B^*} = \lfloor 1/\theta \rfloor \{1 - (1 + \lfloor 1/\theta \rfloor)\theta/2\}$. □

Remark 5.2.1. It can be seen by a simple example that a finite upper bound for the r.h.s. of (5.10) does not exist for a fixed extremal index $\theta < \infty$. To this end, consider an M_2 process with $a_0 = \theta$, $a_1 = \ldots = a_N = (1-\theta)/N$. Then, by (5.11) we have that

$$\begin{aligned}\sum_{h\in\mathbb{N}} (2 - \phi(h)) &= \sum_{n=0}^{N-1} ((1-\theta) - n(1-\theta)/N) \\ &= (1-\theta)(N+1)/2 \to \infty \quad (N \to \infty).\end{aligned}$$

Remark 5.2.2. To conclude this section we remark that processes with $\gamma(h) = 2 - \phi(h)$, $h \in \mathbb{N}$, are easily constructed. Let, for example, an M_2 process with $\theta = 0.5$ be defined by the coefficients $a_k = 0.5^k$, $k \in \mathbb{N}$, and all other coefficients zero. Then, by Theorem 5.2.2 we find that $\gamma(h) = 2 - \phi(h) = 0.5^h$, $h \in \mathbb{N}_0$.

5.3 Application: GARCH(1,1)

Recall from Figure 5.1.1 the stylized fact that financial returns tend to reflect a fluctuating and at the same time clustered volatility over time. The class of GARCH(p, q) processes [21, 5, 63] was originally proposed in order to model this behavior. The finding that the family of GARCH models also parallels real financial data through both heavy tails for its one-dimensional margins as well as a clustering of extreme values has further increased its appeal. In contrast, the latter property does not hold for the class of so-called stochastic volatility processes [11]. In general, for the GARCH family to our knowledge there are no analytic expressions for the extremal measures discussed in Section 5.2. We will therefore consider a suitable simulation technique that is based on a so-called tail chain approach studied by [57]. The tail chain is a certain process that mirrors the behavior of the original sequence when started at a high level [58]. It is therefore of particular interest for the evaluation of extremal characteristics. The tail chain resembles a random walk and, in particular, may be looked at in a forward as well as backward direction. Our setup generalizes a similar method proposed by [15] and [30] whose algorithm focusses on the square of the original process in an intermediate step. It is therefore restricted to GARCH models with symmetric innovations whereas our approach may readily be extended to the asymmetric case that we will, however, not consider here. In addition, our procedure also covers the abovementioned simultaneous evaluation of the forward and backward behavior of the GARCH sequence when started at a high value. Note that this property will be indispensible for the evaluation of the characteristic $\gamma(h)$, $h \in \mathbb{N}$, with the backward direction being in general more tedious. In the following, we will restrict to the GARCH(1,1) model only as a generalization of the tail chain approach to higher order GARCH(p, q) processes has not been considered yet. Moreover, from an applied point of view it appears to be difficult for any volatility model to outperform the GARCH(1,1) approach [24] such that the latter is of special practical importance. Note also that we cover the well-known ARCH(1) model in terms of a GARCH(1,0) setup. The GARCH(1,1) model is defined by

$$X_t = \sigma_t \eta_t, \quad t \in \mathbb{Z}, \tag{5.16}$$

where for the volatility sequence of conditional variances we have that

$$\sigma_t = \sqrt{\alpha_0 + (\alpha_1 \eta_{t-1}^2 + \beta)\sigma_{t-1}^2} = \Phi(\sigma_{t-1}, \eta_{t-1}). \tag{5.17}$$

Here, $(\eta_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. standard normal distributed random variables where η_t and σ_t are independent for fixed t, and

$$\lim_{s\to\infty} s^{-1}\Phi(s, z) = \phi(z) = (\alpha_1 z^2 + \beta)^{1/2}. \tag{5.18}$$

We shall henceforth assume that $\alpha_0 > 0$ in order to preclude the possibility of a degenerate solution to (5.16). Let further $\alpha_1 > 0$, $\beta \geq 0$ and $\alpha_1 + \beta < 1$ such that there exist strictly stationary processes which fulfill the conditions (5.16) and (5.17), cf. [44]. Note that the case $\beta = 0$ corresponds to the ARCH(1) model mentioned above. In order to evaluate the extremal measures $\theta, \phi(h)$, $h \in \mathbb{Z}$, and $\gamma(h)$, $h \in \mathbb{N}$, discussed in Section 5.2 we will concentrate on the joint limiting distribution

$$\lim_{x\to\infty} \mathcal{L}\left(\frac{X_{-l}}{x}, \dots, \frac{X_u}{x} \,\middle|\, X_0 > x\right) \tag{5.19}$$

where $l, u \in \mathbb{N}_0$. Here and in the following we will denote by $\mathcal{L}(X)$ the law of a random vector X. Note that an application of the abovementioned tail chain approach requires a first order Markov structure of the underlying process. We shall therefore make use of the decomposition suggested by (5.16), and will initially focus only on the tail chain of the volatility sequence (5.17) as a separate process which is first order Markov. We will then apply a result obtained in [27] that covers the replacement of the conditioning event $\sigma_0 > x$ that is present in the tail chain of the volatility sequence by the event $|X_0| > x$, i.e. a condition on the related overall process (X_t). Based on this finding we will show in a final step that by the special structure of (X_t) it is straightforward to recover the desired distribution in (5.19). The following well-known lemma (see e.g. [42, Theorem 2.1]) shows that the random variables $|X_0|$ and σ_0 are both regularly varying with a certain index $2\kappa > 0$.

Lemma 5.3.1. *For any stationary solution of (5.16) the equation*

$$E\left(\left[\alpha_1 \eta_0^2 + \beta\right]^\kappa\right) = 1$$

has a unique positive solution. Further, for all $y > 0$ we then have that

$$\lim_{x\to\infty} \frac{P(|X_0| > yx)}{P(|X_0| > x)} = \lim_{x\to\infty} \frac{P(\sigma_0 > yx)}{P(\sigma_0 > x)} = y^{-2\kappa}.$$

The following proposition establishes a preliminary tail chain for the (nonnegative) volatility sequence (5.17) that will form the basis for an appropriate tail chain of the process (X_t) in (5.31) below.

Proposition 5.3.1 (Theorem 5.2, [57]). *Let the stationary process $(\sigma_t)_{t\in\mathbb{Z}}$ be given by (5.17). Then, for all $l, u \in \mathbb{N}_0$, as $x \to \infty$,*

$$\mathcal{L}\left(\frac{\sigma_{-l}}{x}, \dots, \frac{\sigma_u}{x} \,\middle|\, \sigma_0 > x\right) \to \mathcal{L}\left(\hat{\sigma}_{-l}, \dots, \hat{\sigma}_u\right)$$

where

$$\hat{\sigma}_{\pm t} = \prod_{i=0}^{t} A_{\pm i}, \quad t \in \mathbb{N}_0, \tag{5.20}$$

$$P(A_0 > x) = x^{-2\kappa}, \quad x \geq 1, \tag{5.21}$$

$$A_t = \phi(\hat{\eta}_{t-1}) = \left(\alpha_1 \hat{\eta}_{t-1}^2 + \beta\right)^{1/2}, \quad t \in \mathbb{N}, \tag{5.22}$$

for an i.i.d. sequence $(\hat{\eta}_t)_{t\in\mathbb{N}}$ *with the same distribution as* η_0 *in (5.16). Further,* A_0 *is independent of all other variables, and* A_{-t}*,* $t \in \mathbb{N}$*, are i.i.d. symmetric random variables independent of* $(A_t)_{t\in\mathbb{N}}$ *where for* $0 < x \le \beta^{-1/2}$ *we have*

$$P(A_{-1} \le x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{x^{-2}-\beta}{\alpha_1}\right)^{1/2}}^{\infty} (\alpha_1 z^2 + \beta)^{\kappa} \exp\left(-\frac{1}{2}z^2\right) dz. \tag{5.23}$$

Proof. By Lemma 5.3.1 and (5.17) the process (σ_t) satisfies the conditions of Theorem 5.2 in [57]. Following his proof which is substantially simplified by the symmetry of (5.16) it remains to show that $(\hat{\sigma}_t/\hat{\sigma}_0)_{t\in\mathbb{Z}}$ represents a so-called back and forth tail chain $BFTC(2\kappa, \nu)$. The latter corresponds to (5.20) by (4.4) and (4.5) in Segers where $\mathcal{L}(A_t) = \nu$ and $\mathcal{L}(A_{-t}) = \nu^*$, $t \in \mathbb{N}$, for a certain relation between the measures ν and ν^*, and A_0 is independent of $A_{\pm t}$, $t \in \mathbb{N}$. Here, (5.21) is equivalent to (5.3.ii) in Segers. Further, by (2.6) and (4.4) in Segers we get (5.22). Finally, the measures ν and ν^* are related via (3.7) in Segers which yields (5.23) by the fact that

$$E(f(A_{-1})) = E\left[\mathbf{1}\left((\alpha_1\eta_1^2 + \beta)^{-1/2} \le x\right)(\alpha_1\eta_1^2 + \beta)^{\kappa}\right] = P(A_{-1} \le x)$$

for $f(z) = \mathbf{1}(z \le x)$. □

Lemma 5.3.2. *For the stationary processes* (X_t) *and* (σ_t) *given by (5.16) and (5.17) there is a random vector* $(\tilde{\sigma}_0, \tilde{\eta}_0)$ *such that*

$$\lim_{x\to\infty} \mathcal{L}\left(\frac{\sigma_0}{x}, \frac{\sigma_1}{x} \,\Big|\, |X_0| > x\right) = \mathcal{L}(\tilde{\sigma}_0, \tilde{\sigma}_1)$$

where

$$\tilde{\sigma}_1 = \tilde{\sigma}_0 \phi(\tilde{\eta}_0) \tag{5.24}$$

and

$$\begin{aligned} P(\tilde{\sigma}_0 > &y,\, \tilde{\eta}_0 \le x) = \\ &\frac{y^{-2\kappa}}{E(\eta_0^{2\kappa})} \int_{-\infty}^{x} \left[\mathbf{1}(1 \ge y|z|)(|z|y)^{-1} + \mathbf{1}(1 < y|z|)\right]^{-2\kappa} F_{\eta_0}(dz) \end{aligned} \tag{5.25}$$

for all $y > 0$ *and* $x \in \mathbb{R}$.

Proof. By Lemma 5.3.1 we have that

$$\lim_{x\to\infty} P\left(\frac{|X_0|}{x} > 1 \,\Big|\, \eta_0 = z, \frac{\sigma_0}{x} > y\right) = \left[\mathbf{1}(1 \ge y|z|)(|z|y)^{-1} + \mathbf{1}(1 < y|z|)\right]^{-2\kappa}.$$

Now, applying successively Lemmata 3.3.2 and 3.3.1 in [27] yields the assertion. □

As to the symmetric distribution of $\tilde{\eta}_0$ it follows from (5.25) that

$$P(\tilde{\eta}_0 \le x) = \frac{1}{2} + \frac{1}{2} F_\Gamma\left(\frac{1}{2}x^2, 1, \kappa + \frac{1}{2}\right), \quad x \ge 0, \tag{5.26}$$

where by $F_\Gamma(x, k, \omega)$ we denote the Gamma distribution function with shape parameter ω and scale parameter k. Using Lemma 5.3.2 the following proposition now arises as a special case of Theorem 3.5.2 in [27].

Proposition 5.3.2. *Let the stationary processes (X_t) and (σ_t) be given by (5.16) and (5.17). Then, for all $l, u \in \mathbb{N}_0$, as $x \to \infty$,*

$$\mathcal{L}\left(\frac{\sigma_{-l}}{x}, \dots, \frac{\sigma_u}{x} \,\middle|\, |X_0| > x\right) \to \mathcal{L}\left(\tilde{\sigma}_{-l}, \dots, \tilde{\sigma}_u\right) \tag{5.27}$$

with $(\tilde{\sigma}_0, \tilde{\sigma}_1)$ as in Lemma 5.3.2, and

$$\tilde{\sigma}_t = \tilde{\sigma}_{t-1} A_t, \quad t \in \mathbb{N} \setminus \{1\}, \quad \textit{and} \quad \tilde{\sigma}_{-t} = \tilde{\sigma}_{-t+1} A_{-t}, \quad t \in \mathbb{N}, \tag{5.28}$$

for A_t as in (5.22) and (5.23), $t \in \mathbb{Z} \setminus \{0, 1\}$, independent of $(\tilde{\sigma}_0, \tilde{\sigma}_1)$.

Next, we shall turn the r.h.s. in (5.27) into a more suitable form in order to simulate from the limiting measure in (5.19). To this end, note that by (5.17) we have

$$\mathcal{L}(\eta_t) = \mathcal{L}\left(\left(\left(\frac{\sigma_{t+1}}{\sigma_t}\right)^2 - \frac{\alpha_0}{\sigma_t^2} - \beta\right)^{1/2} \alpha_1^{-1/2} B_t\right), \quad t \in \mathbb{Z}, \tag{5.29}$$

for a sequence $(B_t)_{t\in\mathbb{Z}}$ of i.i.d. random variables independent of $(\sigma_t)_{t\in\mathbb{Z}}$ with

$$P(B_0 = 1) = P(B_0 = -1) = 1/2. \tag{5.30}$$

Now, by Proposition 5.3.2 we get

$$\mathcal{L}\left(\frac{X_{-l}}{x}, \dots, \frac{X_u}{x} \,\middle|\, |X_0| > x\right) \to \mathcal{L}\left(\tilde{\sigma}_{-l}\tilde{\eta}_{-l}, \dots, \tilde{\sigma}_u\tilde{\eta}_u\right) = \mathcal{L}\left(\tilde{X}_{-l}, \dots, \tilde{X}_u\right) \tag{5.31}$$

where we apply (5.29) and the continuous mapping theorem. Here, (5.28) yields that

$$\tilde{\eta}_t = \left(\frac{A_{t+1}^2 - \beta}{\alpha_1}\right)^{1/2} \tilde{B}_t, \tag{5.32}$$

$$\tilde{\eta}_{-t} = \left(\frac{A_{-t}^{-2} - \beta}{\alpha_1}\right)^{1/2} \tilde{B}_{-t}, \quad t \in \mathbb{N}, \tag{5.33}$$

for a sequence $(\tilde{B}_t)_{t\in\mathbb{Z}}$ with the same distribution as $(B_t)_{t\in\mathbb{Z}}$, and independent of $(A_t)_{t\in\mathbb{Z}}$. Now, by (5.22), in particular, $\mathcal{L}(\tilde{\eta}_t) = \mathcal{L}(\eta_t)$, $t \in \mathbb{N}$. Further, we have that $\tilde{X}_0$ is symmetric where

$$P(|\tilde{X}_0| > y) = y^{-2\kappa}, \quad y \geq 1, \tag{5.34}$$

by Lemma 5.3.1 and the definition in (5.31). For simulation from the r.h.s. of (5.31) it will at first be advantageous to write

$$\mathcal{L}\left(\tilde{X}_{\pm t}\right) = \mathcal{L}\left(|\tilde{X}_0| \prod_{i=1}^{t} \frac{\tilde{X}_{\pm i}}{\tilde{X}_{\pm(i-1)}}\right) = \mathcal{L}\left(|\tilde{X}_0| \prod_{i=1}^{t} \frac{\tilde{\sigma}_{\pm i}\tilde{\eta}_{\pm i}}{\tilde{\sigma}_{\pm(i-1)}\tilde{\eta}_{\pm(i-1)}}\right), \quad t \in \mathbb{N},$$

such that replacing for (5.24), (5.28), (5.32) and (5.33) yields

$$
\begin{aligned}
&\mathcal{L}\left(\tilde{X}_{-l},\dots,\tilde{X}_{-1},\tilde{X}_0,\tilde{X}_1,\dots,\tilde{X}_u\right)\\
&= \mathcal{L}\Bigg(|\tilde{X}_0|\prod_{i=2}^{l}\left(B_{-i}A_{-i}\left(\frac{A_{-i}^{-2}-\beta}{A_{-i+1}^{-2}-\beta}\right)^{1/2}\right),\dots,|\tilde{X}_0|\frac{B_{-1}A_{-1}}{|\tilde{\eta}_0^{-1}|}\left(\frac{A_{-1}^{-2}-\beta}{\alpha_1}\right)^{1/2}, \qquad (5.35)\\
&\quad |\tilde{X}_0|B_0,|\tilde{X}_0|\phi(|\tilde{\eta}_0|)\frac{\tilde{\eta}_1}{|\tilde{\eta}_0|},\dots,|\tilde{X}_0|\phi(|\tilde{\eta}_0|)\frac{\tilde{\eta}_1}{|\tilde{\eta}_0|}\prod_{i=2}^{u}\left((\alpha_1\tilde{\eta}_{i-1}^2+\beta)^{1/2}\frac{\tilde{\eta}_i}{\tilde{\eta}_{i-1}}\right)\Bigg).
\end{aligned}
$$

Now, the r.h.s. in (5.35) highlights the fact that the drawing of mutually independent i.i.d. random variables A_t, $t \in \{-l,\dots,-1\}$, according to (5.23), B_t, $t \in \{-l,\dots,0\}$, as in (5.30), and the standard normal variables η_t, $t \in \{1,\dots,u\}$, is sufficient in order to simulate from the r.h.s. in (5.31). Further, the random variables $|\tilde{X}_0|$ and $|\tilde{\eta}_0|$ whose distribution is directly related to the condition $|X_0| > x$, cf. (5.34) and (5.26), are also independent both of the above variables and of each other. The latter can be seen e.g. from [27, Lemma 3.1]. Finally, conditioning on $X_0 > x$ in (5.19) instead of $|X_0| > x$ leads to the same limit distribution as in (5.35) but with $B_0 = 1$ almost surely.
In Table 5.3.1 we report the results of a simulation study for θ_m, $\phi(h)$ and $\gamma_m(h)$, $h = 1,\dots,5$, where $m = 500$. It is based on (5.35) according to the respective characteristic. The evaluation of probabilities depends on $N = 10000$ replications. In order to reflect the stylized fact that $\alpha_1+\beta$ is close to one in many applications, we fix $\alpha_1+\beta = 0.99$. Note, in particular, that in accordance with the discussion in Sections 5.1 and 5.2 Table 5.3.1 suggests that for the GARCH(1,1) class there is no simple relationship between the characteristics $\phi(h)$ and $\gamma(h)$. For the last column in Table 5.3.1 we refer to Theorem 5.2.1. Note that the latter applies to the stationary GARCH(1,1) as it is well-known that (C1) holds [1], and (C2) will be considered in the following corollary. In particular, we have that $\lim_{m\to\infty}\theta_m(h) = \theta(h)$, $h \in \mathbb{N}\cup\{1\}$. We also remark that the mixing condition referred to after (5.2) holds for the GARCH(1,1) class, see [34]. First, we will confirm the following result for later reference.

Lemma 5.3.3. *We have that* $E(\tilde{\sigma}_1) < \infty$.

Proof. First, by (5.24) and the independence of $|\tilde{X}_0|$ and $|\tilde{\eta}_0|$ we find that $E(\tilde{\sigma}_1) = E(\phi(|\tilde{\eta}_0|)/|\tilde{\eta}_0|)E(|\tilde{X}_0|)$. Here, $E(|\tilde{X}_0|) < \infty$ by (5.34) and the fact that $\kappa > 1$ if $\alpha_1+\beta < 1$, see e.g. [41]. Further, we have by (5.22) that

$$E(\phi(|\tilde{\eta}_0|)/|\tilde{\eta}_0|) = E\left(\left[\alpha_1+\beta/|\tilde{\eta}_0|^2\right]^{1/2}\right) \le \alpha_1^{1/2}+\beta^{1/2}E(1/|\tilde{\eta}_0|).$$

Finally, using (5.26) we get that

$$E(1/|\tilde{\eta}_0|) = \frac{(1/2)^{\kappa-1/2}}{\Gamma(\kappa+1/2)}\int_0^\infty x^{2\kappa-1}\exp\left(-\frac{1}{2}x^2\right)dx = 2^{-1/2}\frac{\Gamma(\kappa)}{\Gamma(\kappa+1/2)}.$$

□

Corollary 5.3.1. *For the stationary GARCH(1,1) process given by (5.16) a sequence r_n exists such that (C2) holds.*

Proof. Note first that (C2) is implied by the stronger condition

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=m+1}^{r_n} P(X_{-k}>n \mid X_0>n)=0.$$

Further, by (5.22) we have that $E(\prod_{i=2}^k A_i)=\lambda^{k-1}$, $\lambda<1$, such that, in particular,

$$P\left(x\tilde\eta_k\prod_{i=2}^k\phi(\tilde\eta_{i-1})>1\right)\le x\lambda^{k-1}E\left(\tilde\eta_k\mathbf{1}\left(\tilde\eta_k>0\right)\right)=\frac{x\lambda^{k-1}}{\sqrt{2\pi}},\quad x>0.$$

Now, by (5.35) we have that for all $k\in\mathbb{N}$

$$\begin{aligned}\lim_{n\to\infty}P(X_k>n\mid X_0>n)&=P\left(\frac{\tilde X_0\phi(\tilde\eta_0)}{\tilde\eta_0}\tilde\eta_k\prod_{i=2}^k\phi(\tilde\eta_{i-1})>1\right)\\&=\int_0^\infty P\left(x\tilde\eta_k\prod_{i=2}^k\phi(\tilde\eta_{i-1})>1\right)F_{\tilde\sigma_1}(dx)\\&\le\frac{\lambda^{k-1}}{\sqrt{2\pi}}\int_0^\infty xF_{\tilde\sigma_1}(dx)\end{aligned}\tag{5.36}$$

where the second equality holds by (5.24) and the fact that $\tilde\sigma_1$ is independent of the sequence $(\tilde\eta_1,\ldots,\tilde\eta_k)$. By (5.36) there is $n^*=n^*(k)\in\mathbb{N}$ such that $P(X_k>n\mid X_0>n)\le 2E(\tilde\sigma_1)\lambda^{k-1}/\sqrt{2\pi}$ for all $n\ge n^*$. Consequently, for any $r\in\mathbb{N}$ we get that

$$\sum_{k=1}^r P(X_k>n\mid X_0>n)\le\frac{2E(\tilde\sigma_1)}{\sqrt{2\pi}}\sum_{k=1}^r\lambda^{k-1}\tag{5.37}$$

for all $n\ge N(r)=\max\{r^2,\max_{k=1,\ldots,r}n^*(k)\}$. With $r_n=N^{-1}(n):=\sup\{r\in\mathbb{N}_0: N(r)<n\}$ we have $r_n\to\infty$ and $r_n/n\to0$ as $n\to\infty$, and

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=m}^{r_n}P(X_k>n\mid X_0>n)\le\frac{2E(\tilde\sigma_1)}{\sqrt{2\pi}}\lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=m}^{r_n}\lambda^{k-1}=0$$

where we use Lemma 5.3.3 for the last equality. □

Example 5.3.1. We fit the GARCH(1,1) model given by (5.16) to the S&P 500 data set discussed in Section 5.1. The estimated parameters [64] are

$$\hat\alpha_0=0.1\times10^{-5}\ (10^{-7}),\quad\hat\alpha_1=0.072\ (0.002),\quad\hat\beta=0.920\ (0.003)\tag{5.38}$$

where the ML standard errors are given in brackets. Note that $\hat\alpha_1+\hat\beta=0.992$ being close to one is a common result for long financial return series, see e.g. [41] for a discussion. We include an evaluation of the correspoding extremal measures by the above tail chain approach in the last row of Table 5.3.1. Next, in order to assess the accuracy for estimators of the extremal measures when applied to raw data we generate $N = 1000$ independent GARCH(1,1) processes according to (5.38) of the same length as the S&P 500 dataset (7569 records) in Section 5.1. The empirical quantiles resulting from the so-called blocks estimator [66, 2] for a block length of $m = 126$ are given in Table 5.3.2 where we let the thresholds range from the empirical 0.95 quantile up to the 0.995 quantile. As to the choice of the block length note that extremal events occuring in two distinct blocks are assumed to be independent. Here, six trading months corresponding to 126 days appear to be a reasonable order of magnitude. Further, in order to evaluate the quality of the GARCH(1,1) approach in (5.38) with respect to the observed extremal behavior of the S&P 500 series we directly estimate the extremal index of the latter by the above blocks method without making any model assumptions, cf. the last row in Table 5.3.2. Given that our block length is a valid choice the fact that the results fall within the simulated pointwise confidence intervals indicates a satisfactory agreement with the behavior of the estimated extremal index in the GARCH(1,1) case. Note, however, that the block estimator is not directly applicable in order to assess probabilities such as (5.5). To our knowledge, a so-called runs estimation scheme [2] appears to be the only available alternative. Unfortunately, the runs estimator performs poor even in case of the extremal index. We therefore refrain from the statistical estimation of the characteristic $\gamma(h)$, $h \in \mathbb{N}$, for the S&P 500 data. With respect to $\phi(h)$, $h \in \mathbb{Z}$, note that valid estimates should belong to a certain class of positive definite functions, see [56] and [18] for a discussion. Appropriate estimation schemes, however, have not been considered satisfactorily so far and are a matter of current research.

α_1	β	κ	$\hat{\theta}_m$	$2-\hat{\phi}(1)$	$2-\hat{\phi}(2)$	$2-\hat{\phi}(3)$	$2-\hat{\phi}(4)$	$2-\hat{\phi}(5)$	$\hat{\gamma}_m(1)$	$\hat{\gamma}_m(2)$	$\hat{\gamma}_m(3)$	$\hat{\gamma}_m(4)$	$\hat{\gamma}_m(5)$	$\left(\sum_{h=0}^{500}\hat{\gamma}_m(h)\right)^{-1}$
0.99	0	1.014	0.570	0.251	0.167	0.125	0.090	0.063	0.213	0.139	0.104	0.071	0.052	0.573
0.15	0.84	1.478	0.207	0.153	0.144	0.139	0.138	0.140	0.061	0.063	0.065	0.054	0.064	0.199
0.11	0.88	1.838	0.245	0.110	0.104	0.104	0.101	0.093	0.052	0.042	0.038	0.047	0.034	0.247
0.09	0.90	2.203	0.304	0.089	0.085	0.081	0.073	0.080	0.045	0.035	0.034	0.030	0.028	0.302
0.07	0.92	2.885	0.397	0.055	0.050	0.053	0.051	0.052	0.022	0.020	0.020	0.025	0.022	0.419
0.04	0.95	5.991	0.854	0.007	0.007	0.006	0.004	0.006	0.005	0.004	0.003	0.002	0.004	0.858
0.072	0.920	2.476	0.317	0.063	0.064	0.066	0.064	0.065	0.021	0.020	0.027	0.019	0.023	0.305

Table 5.3.1: Extremal measures ($m = 500$) for selected GARCH(1,1) processes with $\alpha_1 + \beta = 0.99$, as well as the process fitted in Example 5.3.1. The results are based on $N = 10000$ runs of the tail chain. The approximate confidence intervals are smaller than ± 0.01 for all entries.

	$q_{0.950}$	$q_{0.955}$	$q_{0.960}$	$q_{0.965}$	$q_{0.970}$	$q_{0.975}$	$q_{0.980}$	$q_{0.985}$	$q_{0.990}$	$q_{0.995}$
$\hat{\theta}^B_{0.025}$	0.210	0.203	0.198	0.198	0.192	0.189	0.179	0.163	0.159	0.137
$\hat{\theta}^B_{0.975}$	0.515	0.505	0.512	0.497	0.506	0.510	0.514	0.525	0.548	0.629
$\hat{\theta}^B_{S\&P\ 500}$	0.305	0.301	0.339	0.347	0.348	0.330	0.325	0.355	0.363	0.403

Table 5.3.2: Blocks estimation ($m = 126$) of the extremal index for different thresholds represented by the respective quantiles q for $N = 1000$ independent GARCH(1,1) processes of length 7569 according to (5.38). The first and second row represent the simulated 95% confidence intervals. In the last row we include the blocks estimation of the extremal index $\theta_{S\&P\ 500}$ for the S&P 500 data set discussed in Section 5.1.

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