On projective resolutions of simple modules over the Borel subalgebra $S^+(n, r)$ of the Schur algebra $S(n, r)$ for $n \leq 3$

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Introduction

The representation theory of finite groups was introduced by Frobenius between the years 1896 and 1900 (see [8] and [9]). He suggested to his pupil I. Schur that he should examine the representation theory of the infinite group $\mathrm{GL}_n(\mathbb{C})$ of invertible matrices over the field $\mathbb{C}$ of complex numbers. In his doctoral thesis [18] Schur investigated homogeneous representations of $\mathrm{GL}_n(\mathbb{C})$. In particular, he showed that irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ by matrices with r-homogeneous polynomial coefficients are in one-to-one correspondence with the partitions of r into at most n parts. The work was done by studying the space of r-homogeneous polynomial functions in the standard n^2 coordinates of $\mathrm{GL}_n(\mathbb{C})$. In the subsequent work [19] Schur reproved his results by analysing the natural actions of the symmetric group Σ_r and the general linear group $\mathrm{GL}_n(\mathbb{C})$ on $(\mathbb{C}^n)^{\otimes r}$.

For an arbitrary infinite field K the representation theory of the general linear group $\mathrm{GL}_n(K)$ starts with the work of Thrall [21] and the paper of Carter and Lusztig [1]. The main tool is the hyperalgebra $\mathfrak{U}_K$ constructed out of the Kostant $\mathbb{Z}$-form of the universal enveloping algebra of the general linear Lie algebra over $\mathbb{Q}$. In particular, they constructed the 'Weyl modules' as certain subspaces of tensor space, showed they were defined over $\mathbb{Z}$ and specialised to the irreducible modules in characteristic zero. The reduction of these modules modulo p turns out to be neither irreducible nor indecomposable.

In his monograph [11] Green takes another approach, based on the observation that the category of r-homogeneous representations (over the infinite field K) of the general linear group $\mathrm{GL}_n(K)$ is equivalent to the category of modules over a certain finite dimensional algebra, which he calls the Schur algebra and denotes by $S(n, r)$. This algebra can be described as follows. Let V be an n-dimensional vector space over K. Then the permutation group Σ_r acts on the tensor power $V^{\otimes r}$ by the rule

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_r)\sigma = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)},$$

where σ is an element of Σ_r. Then the Schur algebra $S(n, r)$ is the set of all linear operators on the vector space $V^{\otimes r}$ which commute with the above action of the symmetric group Σ_r. We have the natural homomorphism T from the the (infinite dimensional) group algebra $K[\mathrm{GL}_n(K)]$ into the Schur algebra $S(n,r)$ given by the formula

$$T(g)v_1 \otimes v_2 \otimes \cdots \otimes v_n = gv_1 \otimes gv_2 \otimes gv_n,$$

where g is an element of $\mathrm{GL}_n(K)$. It is clear that any finite dimensional module over $S(n,r)$ becomes a $\mathrm{GL}_n(K)$-module through the homomorphism T. It is also not difficult to check that all such modules are r-homogeneous. The main achievement of [11] was showing that every finite dimensional r-homogeneous module over $\mathrm{GL}_n(K)$ can be inflated from a module over the Schur algebra $S(n,r)$ through the homomorphism T.

Further investigation of Schur algebras and their generalisations was undertaken in Donkin's papers [3, 4, 5, 6, 7]. In particular, he has shown in [3] that the category of modules over the Schur algebra $S(n,r)$ is an example of what has become known as a highest weight category.

The notion of highest weight category was introduced in the paper [2] of Cline, Parshall and Scott. The main motivation for this notion were the properties of the category $\mathcal{O}$ of highest weight modules for the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a semi-simple Lie algebra $\mathfrak{g}$ over the field $\mathbb{C}$.

Recall that a poset Λ is called interval-finite if for every $\mu \leq \lambda$ in Λ, the set $[\mu, \lambda] = \{ \tau \in \Lambda \mid \mu \leq \tau \leq \lambda \}$ is finite. The structure of a highest weight category C is controlled by an interval-finite poset Λ, which is called a weight poset. For every $\lambda \in \Lambda$ there are five associated objects in C: the simple object $L(\lambda)$, the standard object $\Delta(\lambda)$, the costandard object $\nabla(\lambda)$, the projective object $P(\lambda)$ and the injective object $I(\lambda)$. The set $\{ L(\lambda) \mid \lambda \in \Lambda \}$ is the full collection of pairwise non-isomorphic simple modules in C. It is required that $L(\lambda)$ is the head of $\Delta(\lambda)$ and the socle of $\nabla(\lambda)$. Moreover, the simple composition factors of $\mathrm{Ker}(\Delta(\lambda) \to L(\lambda))$ and $\nabla(\lambda)/L(\lambda)$ have to be of the form $L(\mu)$ with $\mu < \lambda$. The module $P(\lambda)$ is required to be the projective cover of the standard module $\Delta(\lambda)$ and of the simple module $L(\lambda)$, and $I(\lambda)$ is required to be the injective hull of the costandard module $\nabla(\lambda)$ and the simple module $L(\lambda)$. Moreover, the module $\mathrm{Ker}(P(\lambda) \to \Delta(\lambda))$ has a filtration with composition factors of the form $\Delta(\mu)$ with $\mu > \lambda$, and the quotient module $I(\lambda)/\nabla(\lambda)$ has a filtration with subfactors of the form $\nabla(\mu)$ with $\mu > \lambda$. Recall that the Grothendieck group $\mathrm{K}_0(C)$ is defined as the linear $\mathbb{Z}$-span of (isomorphism classes of) objects of C modulo the relations $F_1 - F_2 + F_3 = 0$ for each short exact sequence

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

in C. From the definition of highest weight category it follows that the modules $\{P_\lambda : \lambda \in \Lambda\}$ and the modules $\{\Delta(\lambda) : \lambda \in \Lambda\}$ are two different bases of the Grothendieck group $\mathrm{K}_0(C)$. In particular, each standard module $\Delta(\lambda)$ can be expressed in $\mathrm{K}_0(C)$ as a linear combination of modules P_μ with $\mu \geq \lambda$. The categorical counterpart of such an expression is a projective resolution of $\Delta(\lambda)$. Thus, it is interesting to have descriptions of explicit projective resolutions for standard modules.

In the case of the category of modules over the Schur algebra $S(n,r)$, the weight poset is the set $\Lambda^+(n,r)$ of all partitions of r into at most n parts. The standard modules in this category are usually called Weyl modules. In [25] Woodcock shows how to get a projective resolution for a Weyl module from a projective resolution of a simple module for the Borel algebra $S^+(n,r)$. The Borel algebra $S^+(n,r)$ was defined in [11] as a subalgebra of the algebra $S(n,r)$ generated by elements of the form $T(g)$, where g is an upper triangular matrix in $\mathrm{GL}_n(K)$. The category of modules over the Borel algebra $S^+(n,r)$ is again a highest weight category, but in this case the weight poset is given by the set $\Lambda(n,r)$ of all decompositions of r into at most n parts. Woodcock proves that for $\lambda \in \Lambda^+(n,r)$ the simple module K_λ over $S^+(n,r)$ is acyclic with respect to the induction functor $\mathrm{Hom}_{S^+(n,r)}(S(n,r), -)$. Thus, if we have an $S^+(n,r)$-projective resolution of K_λ and apply to it the induction functor we get a projective resolution for $\mathrm{Hom}_{S^+(n,r)}(S(n,r), K_\lambda)$, which is known to be isomorphic to the Weyl module V^λ.

Inspired by these results, Santana, in [17], constructs the first two terms of the minimal projective resolution of a simple module over the algebra $S^+(n,r)$, for all $n \in \mathbb{N}$, and the first three terms in the case $n = 2$ over a field of positive characteristic. She also obtains the minimal projective resolutions of simple modules over the algebras $S^+(2,r)$ and $S^+(3,r)$ over a field of zero characteristic. The characteristic zero case was fully examined by Woodcock in [24] using the BGG-resolution.

In this work we consider the case of an infinite field of positive characteristic. Recall that the minimal projective resolution of a module M over a finite dimensional algebra is a projective resolution

$$\cdots \to P_k \xrightarrow{d_k} P_{k-1} \cdots \to P_1 \xrightarrow{d_1} P_0 \to M \to 0$$

of M, such that $\mathrm{Im}(d_k) \subset \mathrm{rad}(P_{k-1})$ for all k. It can be shown that there is a unique projective resolution with this property, and that if M has finite projective dimension then the minimal projective resolution has minimal length among projective resolutions of the module M.

We construct the minimal projective resolution for every simple module over the algebra $S^+(2,r)$ (Theorem 35). In Corollary 40 we show that the global dimension of the algebra $S^+(2,r)$ is given by the formula

$$2\left[\frac{r}{p}\right] + \tau(r),$$

where

$$\tau(t) = \begin{cases} 0, & t \in p\mathbb{Z}, \\ 1, & t \notin p\mathbb{Z}. \end{cases}$$

Further, we derive projective resolutions of minimal length for Weyl modules over the Schur algebra $S(2,r)$, corresponding to the regular weights, by applying the induction functor (Remark 47 and Theorem 51). We also construct (non-minimal) projective resolutions for simple modules over the algebra $S^+(3,r)$ (Theorem 67).

The text is organised as follows. In Chapter 1 we introduce some combinatorial notation, and the definitions of partition, decomposition, tableau and Young diagram.

In Chapter 2 we give the definitions of the Schur algebra and of its upper Borel subalgebra. We also summarise in Theorem 18 the results from [17] concerning projective and simple modules over the algebra $S^+(n,r)$.

In Chapter 3 we introduce the notion of a twisted double complex and show how to use it to construct projective resolutions. The idea goes back to Wall, who used these complexes for the construction of free resolutions of trivial modules over finite groups ([22]).

The main results of the work are proved in Chapter 4 and Chapter 5. The proof is based on two technical tools. The first is the multiplication rule of Green given in Proposition 12, which allows us to derive necessary equalities in the algebras $S^+(2,r)$ and $S^+(3,r)$. The second tool is Theorem 22 which gives us the inductive step in the proofs.

Contents

Chapter 1

Combinatorial notation and definitions

In this work we use the following notation

- The set $\{1, 2, \ldots, n\}$ is denoted by $\mathbf{n}$.
- The set of multi-indices $\{i = (i_1, \ldots, i_r) : i_\rho \in \mathbf{n}\ \forall \rho \in \mathbf{r}\}$ is denoted by $I = I_n = I(n, r)$.
- Let i, $j \in I$. We say that $i \leq j$ if $i_\rho \leq j_\rho$ for all $\rho \in \mathbf{r}$.
- Denote by $G = \Sigma_r$ the group of permutations of $\mathbf{r}$. It acts on I on the right as follows:

  $$i\pi = (i_{\pi(1)}, \ldots, i_{\pi(r)}) \qquad (i \in I,\ \pi \in G).$$

  The group G also acts on $I \times I$ by

  $$(i, j)\pi = (i\pi, j\pi) \qquad (i \in I,\ j \in I,\ \pi \in G).$$

- Let i, $j \in I$. We write $i \sim j$ if i and j belong to the same G-orbit.
- Let (i, j), $(p, q) \in I \times I$. We write $(i, j) \sim (p, q)$ if (i, j) and (p, q) belong to the same G-orbit, that is, $p = i\pi$, $q = j\pi$ for some $\pi \in G$.

We shall use the following combinatorial notions.

Definition 1. A *partition* λ of r is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative weakly decreasing integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that $\sum \lambda_i = r$. The set of all partitions of r is denoted by $\Lambda^+(r)$. The λ_i are the *parts* of the partition. If $\lambda_{n+1} = \lambda_{n+2} = \cdots = 0$, we say λ has *length* at most n. The set of all partitions of length at most n is denoted by $\Lambda^+(n,r)$.

Dropping the condition that the λ_i are decreasing, we say that λ is a *composition* of r. The set of all compositions of r is denoted by $\Lambda(r)$. The set of all compositions of r of length at most n is denoted by $\Lambda(n,r)$.

There are two natural orderings on the set $\Lambda(r)$.

Definition 2. (Dominance order) For $\lambda, \mu \in \Lambda(r)$, we say that λ *dominates* μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all j.

Definition 3. (Lexicographic order) For $\lambda, \mu \in \Lambda(r)$, we write $\lambda \geq \mu$ if $\lambda = \mu$ or the smallest j for which $\lambda_j \neq \mu_j$ satisfies $\lambda_j \geq \mu_j$. This is called the *lexicographic order* on compositions.

There is a connection between compositions of r and multi-indices.

Definition 4. We say that a composition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the *weight* of $i \in I(n,r)$, written $i \in \lambda$ or $\lambda = \mathrm{wt}(i)$, if

$$\lambda_\nu = |\{\rho \in \mathbf{r} : i_\rho = \nu\}|$$

for all $\nu \in \mathbf{n}$.

Definition 5. We write $i \leq j$ for $i,\, j \in I(n,r)$ if $i_\sigma \leq j_\sigma$ for all σ, $1 \leq \sigma \leq r$.

Remark 6. It is clear that $i \leq j$ implies $\mathrm{wt}(i) \trianglerighteq \mathrm{wt}(j)$.

Let us give a definition of tableaux and diagrams.

Definition 7. Let $\lambda \in \Lambda(n,r)$. The *Young diagram* for λ is the subset

$$[\lambda] = \{(i,j) : i,\, j \in \mathbb{N},\, i \geq 1,\, 1 \leq j \leq \lambda_i\}$$

of $\mathbb{Z}^2$. Any map T from $[\lambda]$ to $\mathbb{N}$ is called a λ-*tableau*.

If T is a λ-tableau, we will say that $T(p,q)$ lies in the p-th row and the q-th column. The set $R_p = \{T(p,k) : k \in \mathbb{N}\}$ is called the p-th row of T, and $C_q = \{T(k,q) : k \in \mathbb{N}\}$ is called the q-th column of T.

We shall draw a λ-tableau with row indices increasing from top to bottom and column indices increasing from left to right.

If T maps into $\mathbf{r}$ and is a bijection, then T is called a *basic λ-tableau.* For all $\lambda \in \Lambda(n,r)$, let us fix the λ-tableau of the form

$$T^\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \dots & \dots & \lambda_1 \\ \hline \lambda_1+1 & \lambda_1+2 & \dots & \lambda_1+\lambda_2 & \\ \cline{1-4} \dots & \dots & \dots & & \\ \cline{1-3} r-\lambda_n+1 & \dots & r & & \\ \cline{1-3} \end{array}$$

Let $\lambda \in \Lambda(n,r)$. We have a 1-1 correspondence between $I(n,r)$ and the set of all λ-tableaux given by

$$i \mapsto T_i^\lambda,$$

where T_i^λ has (p,q) entry equal to $i_{T^\lambda(p,q)}$.

Definition 8. T_i^λ is called *row semi-standard* if the entries of each row increase weakly from left to right. T_i^λ is called *column standard* if the entries of each column increase from top to bottom. T_i^λ is called *standard* if it is row semi-standard and column standard. Let us denote $I^\lambda = \{i \in I(n,r) : T_i^\lambda \text{ is standard}\}$.

We denote by $l(\lambda)$ the element of I^λ such that

$$T_{l(\lambda)}^\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & \dots & \dots & 1 \\ \hline 2 & 2 & \dots & 2 & \\ \cline{1-4} \dots & \dots & \dots & & \\ \cline{1-3} n & \dots & n & & \\ \cline{1-3} \end{array} ,$$

that is $l(\lambda) = (1^{\lambda_1}, 2^{\lambda_2}, \dots, n^{\lambda_n})$. Denote by $I(\lambda)$ the set $\{i \in I(n,r) : i \leq l(\lambda),\ T_i^\lambda \text{ is row semi-standard}\}$.

Let $i \in I(n,r)$ be of weight $\lambda \in \Lambda(n,r)$ and $s < t$ be two natural numbers. For a natural number $k < \lambda_t$, denote by $A_{s,t}^k i$ the multi-index i with the first k occurrences of t replaced by s. We denote the weight of $A_{s,t}^k i$ by $\mathrm{R}_{s,t}^k \lambda$. Notice that $\mathrm{R}_{s,t}^k \lambda = (\lambda_1, \dots, \lambda_s + k, \dots, \lambda_t - k, \dots, \lambda_n)$.

Chapter 2

Schur algebras

2.1 Definition of the algebra $S_K(n,r)$

In this section we follow [11] and [14].

Let K be an infinite field (of any characteristic) and V a natural module over $\mathrm{GL}_n(K)$ with basis $\{v_1,\ldots,v_n\}$. Then there is a diagonal action of $\mathrm{GL}_n(K)$ on the r-fold tensor product $V^{\otimes r}$. With respect to the basis $\{v_i = v_{i_1}\otimes\cdots\otimes v_{i_r} : i\in I(n,r)\}$, this action is given by the formula

$$gv_i = gv_{i_1}\otimes\cdots\otimes gv_{i_r}.$$

We denote by $T : \mathrm{GL}_n(K) = \mathrm{GL}(V) \to \mathrm{End}_K(V^{\otimes r})$ the corresponding representation of the group $\mathrm{GL}_n(K) = \mathrm{GL}(V)$.

Definition 9 ([14, Def. 2.1.1]). The Schur algebra $S_K(n,r)$ is the linear closure of the group $\{T(g) : g\in \mathrm{GL}_n(K)\}$.

We denote by $e_{i,j}$ the linear transformation of $V^{\otimes r}$ whose matrix, relative to the basis $\{\, v_i : i\in I(n,r)\,\}$ of $V^{\otimes r}$, has 1 in place (i,j) and zeros elsewhere. The group G acts (on the right) on $\mathrm{End}_K(V^{\otimes r})$ as follows: let $u\in \mathrm{End}_K(V^{\otimes r})$ and $\sigma\in G$, then $u^\sigma(v)\,(u(v\sigma^{-1})\sigma)$, for all $v\in V$. We find that $e^\sigma_{i,j} = e_{i\sigma,j\sigma}$, for all $i,j\in I(n,r)$ and $\sigma\in G$.

Note, that $A = \mathrm{End}_K(V)$ is an G-algebra. We collect some basic results about G-algebras (for an arbitrary group G) in Appendix C.

Theorem 10 ([23, Theorem 4.4E]). *Let K be an infinite field. The natural inclusion of $S_K(n,r)$ into the algebra of G-invariants $A^G = \mathrm{End}_K(V)^G$ is an isomorphism.*

Let X be a transversal of the action of $G = \Sigma_r$ on the set $I(n,r)\times I(n,r)$. We have the following

Proposition 11 ([14, Thm. 2.2.6]). *The set*

$$\left\{ \xi_{i,j} = \sum_{(p,q)\sim(i,j)} e_{p,q} : (i,j) \in X \right\}$$

is a basis for the algebra $S(n,r)$.

Proof. It is clear that the set

$$\left\{ \xi_{i,j} = \sum_{(p,q)\sim(i,j)} e_{p,q} : (i,j) \in X \right\}$$

is a basis of $\mathrm{End}_K(V)^G$. Now, the result follows from Theorem 10. □

Note that $\xi_{i,i} = \xi_{j,j}$ if and only if i and j have the same weight. We will write ξ_λ for $\xi_{i,i}$ if i has weight λ.

In the following we will need to know how to multiply two basis elements $\xi_{i,j}$ and $\xi_{f,h}$ of $S(n,r)$. It is clear that $\xi_{i,j}\xi_{f,h} = 0$ unless $j \sim f$. Therefore, only the formula for $\xi_{i,j}\xi_{j,h}$ is needed. Let G_i denote the stabiliser of i in G and $G_{i,j} = G_i \cap G_j$, $G_{i,j,k} = G_i \cap G_j \cap G_k$. Then, if $[G_{i,h} : G_{i,h,j}]$ denotes the index of $G_{i,h,j}$ in $G_{i,h}$, we have the following

Proposition 12 (Green [14, Thm. 2.2.11]). *Let i, j, l be multi-indices from $I(n,r)$. Then*

$$\xi_{i,j}\xi_{j,l} = \sum_\sigma [G_{i\sigma,l} : G_{i\sigma,j,l}]\xi_{i\sigma,l},$$

where the summation is over a transversal $\{\sigma\}$ of double cosets $G_{i,j}\sigma G_{j,l}$ in G_j.

Proof. Let Y be a transversal of the set of all cosets $G_{i,j}\sigma$ in G, then we can write $\xi_{i,j}$ as

$$\xi_{i,j} = \sum_{\sigma \in Y} e^\sigma_{i,j} = \mathrm{Tr}^P_{P_{i,j}}(e_{i,j})$$

where, for any subgroups H, L of G such that $H \le L$, Tr^L_H denotes the "relative trace" map (see Appendix C). We shall write Tr^G_H as $\mathrm{Tr}(H)$, for any subgroup H of G, to avoid cumbersome suffixes.

We have

$$\xi_{i,j}\xi_{j,l} = \mathrm{Tr}(G_{i,j})(e_{i,j})\,\mathrm{Tr}(G_{j,l})(e_{j,l}).$$

The Mackey formula (see Theorem 86) now gives

$$\xi_{i,j}\xi_{j,l} = \sum_{\tau} \mathrm{Tr}(G_{i,j}^{\tau} \cap G_{j,l})(e_{i,j}^{\tau} e_{j,l}),$$

where the being over a transversal $\{\tau\}$ of the set of all double cosets $G_{i,j}\tau G_{j,l}$ in G. Now, $e_{i,j}^{\tau} e_{j,l}$ is zero unless $j\tau = j$, that is unless $\tau \in G_j$. If $\tau \in G_j$, then $e_{i,j}^{\tau} e_{j,l} = e_{i\tau,j}$. Notice, that $G_{i,j}^{\tau} = \tau^{-1} G_{i,j} \tau = G_{i\tau,j\tau}$ for any i, $j \in I(n,r)$ and $\tau \in G$. Thus

$$G_{i,j}^{\tau} \cap G_{j,l} = G_{i\tau,j} \cap G_{j,l} = G_{i\tau,j,l}$$

and

$$\mathrm{Tr}(G_{i,j}^{\tau} \cap G_{j,l})(e_{i,j}^{\tau} e_{j,l}) = \mathrm{Tr}(G_{i\tau,j,l})(e_{i\tau,l}).$$

Since $G_{i\tau,j,l} \leq G_{i\tau,l}$, the last expression equals

$$[G_{i\tau,l} : G_{i\tau,j,l}]\,\mathrm{Tr}(G_{i\tau}, l)(e_{i\tau,l}) = [G_{i\tau,l} : G_{i\tau,j,l}]\,\xi_{i\tau,l}.$$

□

As a consequence of Proposition 12 and using the definition of $\xi_{i,j}$, we have the

Corollary 13. *For any $i, j \in I(n,r)$,*

$$\xi_{i,i}\xi_{i,j} = \xi_{i,j}\xi_{j,j} = \xi_{i,j}.$$

In particular, each ξ_λ is an idempotent, and

$$1_{S(n,r)} = \sum_{\lambda \in \Lambda(n,r)} \xi_\lambda$$

is an orthogonal decomposition of unity.

Proof. We have $G_j = G_{i,j}G_{j,j}$, so there is only one double coset $G_{i,j}eG_{j,j}$ in G_j. By Proposition 12, $\xi_{i,j}\xi_{j,j} = [G_{i,j} : G_{i,j,j}]\xi_{i,j} = \xi_{i,j}$. Analogously, $\xi_{i,i}\xi_{i,j} = \xi_{i,j}$. The decomposition of unity follows from the definition of the elements ξ_λ. □

Definition 14. Let i, $j \in I(n,r)$ and $\lambda \in \Lambda(n,r)$. The element $C^{\lambda}(i:j) = \xi_{i,l(\lambda)}\xi_{l(\lambda),j}$ is called a *codeterminant.* If i, $j \in I^{\lambda}$, then the corresponding codeterminant is called *standard.*

Denote by Ω the set $\{(i,j,\lambda) : i,\ j \in I^{\lambda},\ \lambda \in \Lambda(n,r)\}$. The following is proved in [14].

Proposition 15 ([14, Thm. 2.4.8]). *The set $\{C^{\lambda}(i:j) : (i,j,\lambda) \in \Omega\}$ is a basis for $S(n,r)$.*

2.2 Definition of the algebra $S^+(n,r)$

The definitions of this section are taken from [17].

Let us denote by $B_n^+(K)$ the subgroup of upper triangular matrices in the general linear group $\mathrm{GL}_n(K)$. Recall that $T\colon \mathrm{GL}_n(K) \to \mathrm{End}(V^{\otimes r})$ is a representation of $\mathrm{GL}_n(K)$.

Definition 16 ([17, Def. 0.1]). The upper Borel subalgebra $S_K^+(n,r)$ of the Schur algebra $S_K(n,r)$ is the linear closure of the group $\{T(g) : g \in B_n^+(K)\}$.

Let $\Omega' = \{(i, l(\lambda)) : \lambda \in \Lambda(n,r),\ i \in I(\lambda)\}$. Note that Ω' is a transversal of the action of $G = \Sigma_r$ on the set $\{(i,j) : i \leq j\}$. The next statement was proved in [12, §§3, 6].

Proposition 17. *1) The algebra $S_K^+(n,r)$ has K-basis $\{\xi_{i,j} : (i,j) \in \Omega'\}$.*

2) The radical ideal $\operatorname{rad} S_K^+(n,r)$ *of* $S_K^+(n,r)$ *has K-basis $\{\xi_{i,j} : (i,j) \in \Omega',\ i \neq j\}$.*

For every $\lambda \in \Lambda(n,r)$, let us define the map $\chi_\lambda : S^+(n,r) \to K$ such that $\chi_\lambda(\xi_\lambda) = 1$ and $\chi_\lambda(\xi_{i,j}) = 0$ otherwise.

The following was proved in [17].

Proposition 18 ([17, Prop. 2.2]). *Let $\lambda \in \Lambda(n,r)$. Then we have the following.*

1) *The map χ_λ is a homomorphism of K-algebras. We denote by K_λ the corresponding one-dimensional module over $S^+(n,r)$.*

2) *The set $\{\, K_\mu \mid \mu \in \Lambda(n,r) \,\}$ is a full collection of pairwise non-isomorphic simple $S^+(n,r)$-modules.*

3) *The set $\{\xi_\mu : \mu \in \Lambda(n,r)\}$ is a full collection of primitive idempotents in $S^+(n,r)$.*

4) *Denote by P_λ the module $S^+(n,r)\xi_{\lambda,\lambda}$. Then the modules P_λ are projective, and the set $\{P_\mu : \mu \in \Lambda(n,r)\}$ is a full collection of pairwise non-isomorphic principal indecomposable $S^+(n,r)$-modules.*

5) *The modules P_λ and* $\operatorname{rad} P_\lambda$ *have K-bases*

$$\{\xi_{i,l(\lambda)} : i \in I(\lambda)\} \quad \textit{and} \quad \{\xi_{i,l(\lambda)} : i \in I(\lambda),\ i \neq l(\lambda)\},$$

respectively.

6) *The simple module K_λ is isomorphic to the quotient module* $P_\lambda/\operatorname{rad} P_\lambda$.

Let us denote $V^\lambda = S(n,r) \otimes_{S^+(n,r)} K_\lambda$. The module V^λ is called the *Weyl module*.

Remark 19. The algebra $S(n,r)$ is quasi-hereditary and $\{V^\lambda : \lambda \in \Lambda^+(n,r)\}$ is a full set of pairwise non-isomorphic standard modules (see Appendix B for more details about quasi-hereditary algebras and highest-weight categories).

Chapter 3

Homological algebra prerequisites

3.1 Twisted double complexes

In this section we introduce the notion of a twisted double complex. Such terminology reflects the fact that twisted double complexes usually arise as double complexes with the differential perturbed by a twisted cochain (cf. [20, §3.3]).

Definition 20. A *twisted double complex* L is a collection of modules

$$\{L_{s,t} : s,\, t \in \mathbb{Z}\}$$

and a collection of maps

$$d_k : L_{s,t} \to L_{s+k-1,t-k}, \quad k \geq 0$$

such that

$$\sum_{k=0}^{n} d_k d_{n-k} = 0$$

for all $n \geq 0$.

Every twisted double complex L defines a total complex $X = \mathrm{Tot}(L)$:

$$X_n = \bigoplus_{s+t=n} L_{s,t}, \quad d = \sum_i d_i : X_n \to X_{n-1}.$$

Let $\mathrm{H}_\bullet(L)$ denote the homology groups of the complex $X = \mathrm{Tot}(L)$. Then we have the following

Theorem 21. *Suppose $L_{s,t} = 0$ if $s < 0$ or $t < 0$, and $\mathrm{H}^{d_0}_{s,t}(L) = 0$ if $s > 0$. Then*

$$\mathrm{H}_t(X) \cong \mathrm{H}^{d_1}_t(\mathrm{H}^{d_0}_{0,\bullet}(L_{\bullet,\bullet})).$$

Proof. Consider the increasing filtration

$$X_k := \bigoplus_{t \le k} L_{s,t}$$

on the complex X. Under the conditions of the theorem we have, for the corresponding spectral sequence,

$$E^2_{s,t} \cong \mathrm{H}^{d_1}_t(\mathrm{H}^{d_0}_{s,\bullet}(L_{\bullet,\bullet})) \cong \begin{cases} 0, & s > 0, \\ \mathrm{H}^{d_1}_t(\mathrm{H}^{d_0}_{0,\bullet}(L_{\bullet,\bullet})), & s = 0. \end{cases}$$

Hence the spectral sequence collapses and

$$\mathrm{H}_t(X) \cong \mathrm{H}^{d_1}_t(\mathrm{H}^{d_0}_{0,\bullet}(L_{\bullet,\bullet})). \quad \square$$

3.2 Projective resolutions

The statement of the next theorem is implicitly contained in [22].

Theorem 22. *Let A be an algebra over a field K and M a module over A. Suppose $N_\bullet$ is a (non-projective) resolution of the module M and $P_{\bullet,t}$ are projective resolutions of the modules N_t for $t \ge 0$. Then the module M has a projective resolution $P_\bullet$ such that*

$$P_n = \bigoplus_{s+t=n} P_{s,t}.$$

Proof. Denote by ϵ_t the augmentation map $P_{0,t} \to N_t$. In the proof of Lemma 2 in [22], it was shown that there exist A-module maps $d_k\colon P_{s,t} \to P_{s+k-1,t-k}$ such that

1) $d_0\colon P_{s,t} \to P_{s-1,t}$ is the differential of the resolution $P_{\bullet,t}$;

2) $d_1\epsilon_{s-1} = \epsilon_s d\colon P_{0,t} \to N_{t-1}$ (where d denotes the differential in N);

3) $\sum_{k=1}^{n} d_k d_{n-k} = 0$, for each $n \in \mathbb{N}$.

Then $P = \{P_{s,t} : s, t \in \mathbb{N}\}$ obtains a structure of a twisted double complex such that

1) $\mathrm{H}^{d_0}_{s,t}(P) = 0$ if $s \geq 1$;

2) $(\mathrm{H}^{d_0}_{\bullet,0}(P), \bar{d}_1)$ and $N_\bullet$ are isomorphic as complexes of A-modules.

We therefore get, by Theorem 21,

$$\mathrm{H}_s(P) \cong \mathrm{H}^{d_1}_s(\mathrm{H}^{d_0}_{0,t}(P)) \cong \mathrm{H}_s(N_\bullet) \cong \begin{cases} M, & s = 0, \\ 0, & s > 0. \end{cases}$$

Thus $\mathrm{Tot}(P)$ is a projective resolution of M. □

Chapter 4

Projective resolutions for $S^+(2,r)$

4.1 The algebra $S^+(2,r)$

Let $\lambda = (\lambda_1, \lambda_2)$ and $i \in I(\lambda)$, that is, $i \leq l(\lambda)$ and T_i^λ is row semi-standard. Then

$$T_i^\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & \dots & 1 & 1 & \dots & \dots & 1 \\ \hline 1 & \dots & 1 & 2 & \dots & 2 & \\ \cline{1-6} \end{array}$$

Therefore $i = l(\mu)$ for some $\mu \geq \lambda$. Let us write $\xi_{\mu,\lambda}$ for $\xi_{l(\mu),l(\lambda)}$. It follows from Proposition 17 that the algebra $S^+(2,r)$ has basis $\{\xi_{\mu,\lambda} : \mu \geq \lambda\}$.

Lemma 23. *Let ν, μ, $\lambda \in \Lambda(2,r)$. If $\nu \geq \mu \geq \lambda$, then*

$$\xi_{\nu,\mu}\xi_{\mu,\lambda} = \binom{\lambda_2 - \nu_2}{\mu_2 - \nu_2}\xi_{\nu,\lambda}.$$

Proof. Let V be a 2-dimensional K-vector space with basis $\{v_1, v_2\}$. Then by definition, $S^+(2,r)$ is a subalgebra of $A = \mathrm{End}_K(V^{\otimes r})$. We will check the above stated equality of linear operators on the basis $\{v_i = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} : i \in I(2,r)\}$ of $V^{\otimes r}$.

If $i \notin \lambda$ then $\xi_{\mu,\lambda}(v_i) = 0$ and $\xi_{\nu,\lambda}(v_i) = 0$ by definition of the maps $\xi_{\mu,\lambda}$ and $\xi_{\nu,\lambda}$.

Now let $i \in \lambda$. Since the action of $S^+(2,r)$ commutes with the action of Σ_r, we can suppose that $i = l(\lambda)$. Then

$$\xi_{\mu,\lambda}(v_{l(\lambda)}) = \sum_{(s,q)\sim(l(\mu),l(\lambda))} e_{s,q}(v_{l(\lambda)}) = \sum_{(s,l(\lambda))\sim(l(\mu),l(\lambda))} v_s = \sum_{s\in\mu:s\leq l(\lambda)} v_s.$$

Multiplying the last equality by $\xi_{\nu,\mu}$ on the left hand side we get

$$\xi_{\nu,\mu}\xi_{\mu,\lambda}(v_i) = \sum_{\substack{s\in\mu \\ s\leq l(\lambda)}} \sum_{\substack{t\in\nu \\ t\leq s}} v_t.$$

Let us compute the coefficient of v_t in the last equation, that is, the number of $s \in \mu$ such that $t \leq s \leq l(\lambda)$.

Since $l(\lambda)(j) = 1$ implies $s(j) = 1$, we have $s(j) = 1$ for all $j \leq \lambda_1$.

Moreover, $t(j) = 2$ implies $s(j) = 2$. Since for the ν_2 values $\nu_1 + 1, \nu_1 + 2, \ldots, r$ of j we have $t(j) = 2$, there are only $\lambda_2 - \nu_2$ places in s with the freedom of choice between 1 and 2. Further, on these $\lambda_2 - \nu_2$ places, 2 appears $\mu_2 - \nu_2$ times. Hence there are exactly $\binom{\lambda_2 - \nu_2}{\mu_2 - \nu_2}$ different s that satisfy the above conditions. Thus

$$\xi_{\nu,\mu}\xi_{\mu,\lambda}(v_{l(\lambda)}) = \binom{\lambda_2 - \nu_2}{\mu_2 - \nu_2} \sum_{t\in\nu: t\leq l(\lambda)} v_t = \binom{\lambda_2 - \nu_2}{\mu_2 - \nu_2} \xi_{\nu,\lambda}(v_{l(\lambda)}). \quad \square$$

We will need the following well-known result.

Proposition 24. *Let $r, s \in \mathbb{N}$ and $r \geq s$. Write*

$$r = \sum_{k=0}^{\infty} r_k p^k, \quad s = \sum_{k=0}^{\infty} s_k p^k,$$

where $0 \leq r_k,\ s_k \leq p - 1$. Then

$$\binom{r}{s} \equiv \binom{r_0}{s_0}\binom{r_1}{s_1}\binom{r_2}{s_2} \cdots \pmod{p}.$$

Here $\binom{r_k}{s_k} = 0$ if $r_k < s_k$.

Proof. We have

$$(x + 1)^r \equiv (x + 1)^{r_0}(x^p + 1)^{r_1}(x^{p^2} + 1)^{r_2} \cdots \pmod{p}.$$

Now compare coefficients of x^s on both sides. $\square$

Proposition 25. *The set*

$$\{\xi_{\lambda,\mu} : \lambda_2 - \mu_2 \text{ is a power of } p\}$$

generates $S^+(2,r)$.

Proof. From Corollary 17 we know that the set $\{\xi_{\rho,\nu} : \nu \geq \rho\}$ is a basis for $S^+(2,r)$. We shall show that each $\xi_{\rho,\nu}$ is a product of elements from $\{\xi_{\lambda,\mu} : \lambda_2 - \mu_2$ is a power of $p\}$. Suppose

$$\rho_2 - \nu_2 = r_0 + r_1 p + r_2 p^2 + \cdots + r_k p^k$$

with $0 \leq r_i \leq p-1$. Let us denote $s_j = \sum_{i=0}^{j} r_i p^i$. Recall, that $R\lambda$ denotes the partition $(\lambda_1+1, \lambda_2-1)$, for $\lambda \in \Lambda(2,r)$. By Lemma 23 and Proposition 24 we have

$$\begin{aligned}\xi_{\mathrm{R}^{s_{j+1}}\nu,\mathrm{R}^{s_j}\nu}\xi_{\mathrm{R}^{s_j}\nu,\nu} &= \binom{\nu_2 - (\mathrm{R}^{s_{j+1}}\nu)_2}{\nu_2 - (\mathrm{R}^{s_j}\nu)_2}\xi_{\mathrm{R}^{s_{j+1}}\nu,\nu} = \binom{s_{j+1}}{s_j}\xi_{\mathrm{R}^{s_{j+1}}\nu,\nu} \\ &= \binom{r_0}{r_0}\cdots\binom{r_j}{r_j}\binom{r_{j+1}}{0}\xi_{\mathrm{R}^{s_{j+1}}\nu,\nu} = \xi_{\mathrm{R}^{s_{j+1}}\nu,\nu}.\end{aligned}$$

By recursion, we get

$$\xi_{\rho,\nu} = \xi_{\rho,\mathrm{R}^{s_k}\nu}\xi_{\mathrm{R}^{s_k}\nu,\mathrm{R}^{s_{k-1}}\nu}\cdots\xi_{\mathrm{R}^{s_0}\nu,\nu}.$$

This reduces the problem to the case $\rho_2 - \nu_2 = rp^k$ with $0 \leq r \leq p-1$. We have for $1 \leq t \leq p-2$ by Lemma 23 and Proposition 24

$$\xi_{\mathrm{R}^{(t+1)p^k}\nu,\mathrm{R}^{tp^k}\nu}\xi_{\mathrm{R}^{tp^k}\nu,\nu} = \binom{(t+1)p^k}{tp^k}\xi_{\mathrm{R}^{(t+1)p^k}\nu,\nu} = (t+1)\xi_{\mathrm{R}^{(t+1)p^k}\nu,\mathrm{R}^{tp^k}\nu}.$$

Therefore, by induction,

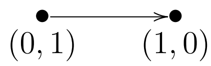
$$r!\xi_{\mathrm{R}^{rp^k}\nu,\nu} = \xi_{\mathrm{R}^{rp^k}\nu,\mathrm{R}^{(r-1)p^k}\nu}\xi_{\mathrm{R}^{(r-1)p^k}\nu,\mathrm{R}^{(r-2)p^k}\nu}\cdots\xi_{\mathrm{R}^{p^k}\nu,\nu}.$$

Since for $0 \leq r \leq p-1$ the number $r!$ is invertible in K, this completes the proof. □

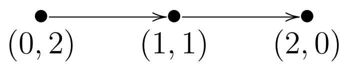
In view of Lemma 23 and Proposition 25 we can consider $S^+(2,r)$ as a path algebra of a quiver with relations[1].

[1] The reader can find a short account about path algebras of quivers (with relations) in Appendix A.

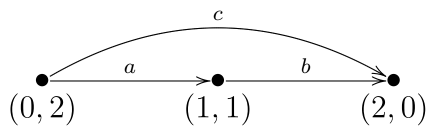
For example, $S^+(2, 1)$ corresponds to the quiver



with no relations. The algebra $S^+(2, 2)$ corresponds to the quiver



with no relations if $\operatorname{char} K \neq 2$ and to the quiver



with the relation $ba = 0$ if $\operatorname{char} K = 2$.

4.2 Some facts about modules over $S^+(2, r)$

Let V be a module over the algebra $S^+(2, r)$. We denote by $V(\lambda)$ the λ-*weight subspace* $\xi_\lambda V$ of V. Since $1 = \sum_{\lambda \in \Lambda(2,r)} \xi_\lambda$, we have $V = \bigoplus_{\lambda \in \Lambda(2,r)} V(\lambda)$. Moreover, morphisms of $S^+(2, r)$-modules preserve weight subspaces. Therefore, a module over the algebra $S^+(2, r)$ can be considered as a collection of spaces $\{V(\lambda) : \lambda \in \Lambda(2, r)\}$ with maps

$$\xi_{\mu,\lambda} \colon V(\lambda) \to V(\mu), \quad \mu \geq \lambda,$$

such that $\xi_{\nu,\mu}\xi_{\mu,\lambda} = \binom{\lambda_2 - \nu_2}{\mu_2 - \nu_2}\xi_{\nu,\lambda}$.

Let us denote by $\operatorname{Supp}(V)$ the set $\{\lambda \in \Lambda(2, r) : V(\lambda) \neq 0\}$.

For the construction of a projective resolution of a simple module K_λ, we will need modules intermediate between simple and projective ones.

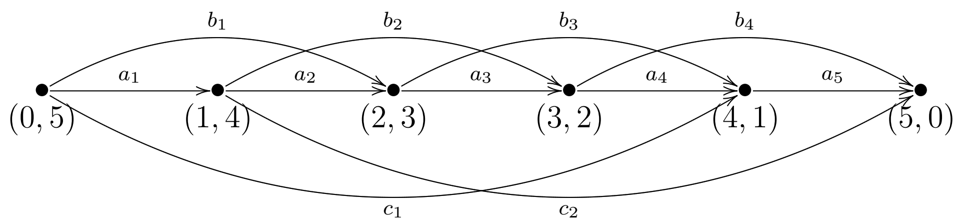
Definition 26. We denote by $P_{\lambda,k}$ the module over the algebra $S^+(2, r)$ with basis $\{v_\mu : \mu \geq \lambda,\ p^k \mid \lambda_2 - \mu_2\}$, where $v_\mu \in P_{\lambda,k}(\mu)$ and the action of $S^+(2, r)$ is given by the formula

$$\xi_{\nu,\mu} v_\mu = \begin{cases} \binom{\lambda_2 - \nu_2}{\mu_2 - \nu_2} v_\nu, & \text{if } p^k \text{ divides } \mu_2 - \nu_2, \\ 0, & \text{otherwise.} \end{cases}$$

We shall prove in Lemma 28 that the modules $P_{\lambda,k}$ are well defined.

Remark 27. To avoid ambiguity, we will sometimes denote v_μ from $P_{\lambda,k}$ by $v_{\mu,\lambda,k}$.

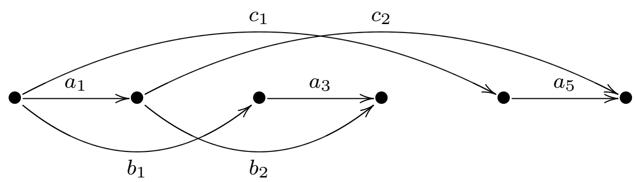
Let us show what the modules $P_{\lambda,k}$ look like in the case $r = 5$ and $p = 2$. Recall that we can consider the algebra $S^+(2, 5)$ as a quiver algebra of the diagram



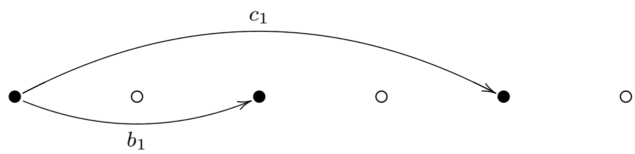
with relations

$$\begin{aligned} a_{i+1}a_i &= 0 && \text{for } 1 \le i \le 4 \\ b_{i+2}b_i &= 0 && \text{for } 1 \le i \le 2 \\ a_{i+2}b_i &= b_{i+1}a_i && \text{for } 1 \le i \le 3 \\ a_5c_1 &= c_2a_1. \end{aligned}$$

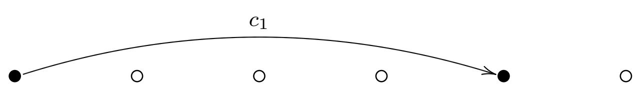
The module $P_{(0,4),0} \cong P_{(0,4)}$ has the form



where bullets (•) denote the non-zero basis elements of $P_{(0,4)}$ and only non-zero maps are shown. The module $P_{(0,4),1}$ has the form



where ∘ means that the corresponding weight space is trivial. The module $P_{(0,4),2}$ is two-dimensional and can be drawn as



Lemma 28. *The modules $P_{\lambda,k}$ are well-defined.*

Proof. We have to check that

$$(\xi_{\rho,\nu}\xi_{\nu,\mu})v_\mu = \xi_{\rho,\nu}(\xi_{\nu,\mu}v_\mu)$$

for all ρ, ν, $\mu \in \Lambda(2,r)$ such that $\rho \geq \nu \geq \mu \geq \lambda$.

If $\mu_2 - \rho_2$ is not divisible by p^k then, by definition of the module structure, we get zero on both sides of the equality.

If p^k divides $\mu_2 - \rho_2$ but $\mu_2 - \nu_2$ is not divisible by p^k then by Lemma 23 and Proposition 24 we have

$$\xi_{\rho,\nu}(\xi_{\nu,\mu}v_\mu) = \xi_{\rho,\nu}0 = 0,$$

and

$$\begin{aligned}(\xi_{\rho,\nu}\xi_{\nu,\mu})v_\mu &= \binom{\mu_2-\rho_2}{\mu_2-\nu_2}\xi_{\rho,\mu}v_\mu \\ &= \binom{0}{(\mu_2-\nu_2)_1}\cdots\binom{0}{(\mu_2-\nu_2)_{k-1}}\binom{(\mu_2-\rho_2)_k}{(\mu_2-\nu_2)_k}\cdots\xi_{\rho,\mu}v_\mu \\ &= 0,\end{aligned}$$

since there exists at least one $i \leq k-1$ such that $(\mu_2 - \nu_2)_i \neq 0$.

If p^k divides $\mu_2 - \nu_2$ and $\mu_2 - \rho_2$, then by Lemma 23

$$(\xi_{\rho,\nu}\xi_{\nu,\mu})v_\mu = \binom{\mu_2-\rho_2}{\mu_2-\nu_2}\binom{\lambda_2-\rho_2}{\mu_2-\rho_2}v_\rho = \frac{(\lambda_2-\rho_2)!}{(\mu_2-\nu_2)!(\nu_2-\rho_2)!(\lambda_2-\mu_2)!}v_\rho,$$

and

$$\xi_{\rho,\nu}(\xi_{\nu,\mu}v_\mu) = \binom{\lambda_2-\nu_2}{\lambda_2-\mu_2}\binom{\lambda_2-\rho_2}{\lambda_2-\nu_2}v_\rho = \frac{(\lambda_2-\rho_2)!}{(\mu_2-\nu_2)!(\lambda_2-\mu_2)!(\nu_2-\rho_2)!}v_\rho. \ \square$$

Lemma 29. *Let $\lambda \in \Lambda(2,r)$. Then $P_{\lambda,k}$ is a cyclic indecomposable module with generator v_λ.*

Proof. Let $\mu \geq \lambda$ and $p^k \mid \lambda_2 - \mu_2$. Then by definition of the $S^+(2,r)$-module structure on $P_{\lambda,k}$

$$\xi_{\mu,\lambda}v_\lambda = \binom{\lambda_2-\mu_2}{\lambda_2-\mu_2}v_\mu = v_\mu.$$

Furthermore, $\operatorname{rad} P_{\lambda,k}$ has basis $\{\, v_\mu : \mu > \lambda, \lambda_2 - \mu_2 \in p^k\mathbb{Z}\,\}$. Therefore $P_{\lambda,k}/\operatorname{rad} P_{\lambda,k}$ is one-dimensional and thus $P_{\lambda,k}$ is indecomposable. $\square$

Remark 30. It follows from the definition that $P_{\lambda,m} \cong K_\lambda$ for $p^m > \lambda_2$ and from Proposition 18 that $P_{\lambda,0} \cong P_\lambda$.

Let us denote by $\mathrm{Ann}(v_{\mu,\lambda,k})$ the annihilator of $v_{\mu,\lambda,k} \in P_{\lambda,k}(\mu)$. Then for any $\nu \neq \mu$ we have $\mathrm{Ann}(v_{\mu,\lambda,k})\xi_\nu = S^+(2,r)\xi_\nu$. Denote $\mathrm{Ann}(v_{\mu,\lambda,k})\xi_\mu$ by $\mathrm{ann}(v_{\mu,\lambda,k})$.

Remark 31. Let $\lambda \in \Lambda(2,r)$ and $l \geq 0$. Since the module $P_{\lambda,l}$ is cyclic with generator $v_{\lambda,l}$, we have a 1-1 correspondence between the set of $S^+(2,r)$-maps from $P_{\lambda,l}$ to an $S^+(2,r)$-module M and the set of elements m in M such that

$$\mathrm{Ann}(v_{\lambda,l}) \subset \mathrm{Ann}(m)$$

or, equivalently, the set of elements m in $M(\lambda)$ such that

$$\mathrm{ann}(v_{\lambda,k}) \subset \mathrm{ann}(m) = \mathrm{Ann}(m)\xi_\lambda.$$

Proposition 32. *Let λ, $\mu \in \Lambda(2,r)$ and $\mu \geq \lambda$. Then*

$$\mathrm{ann}(v_{\mu,\lambda,k}) = \{\xi_{\nu\mu} : \mu_2 - \nu_2 \notin p^k\mathbb{Z}\} \cup \left\{\xi_{\nu\mu} : \binom{\lambda_2 - \nu_2}{\mu_2 - \nu_2} \in p\mathbb{Z}\right\}.$$

In particular,

$$\mathrm{ann}(v_{\mu,\mu,k}) = \{\xi_{\nu\mu} : \mu_2 - \nu_2 \notin p^k\mathbb{Z}\}.$$

Proof. This follows from the definition of the module structure on $P_{\lambda,k}$. □

Proposition 33. *Let λ, $\mu \in \Lambda(2,r)$ and $\mu \geq \lambda$. Suppose $l \geq k$. Then*

$$\mathrm{ann}(v_{\mu,\mu,k}) \subset \mathrm{ann}(v_{\mu,\lambda,l}).$$

Proof. Let $\xi_{\nu\mu} \in \mathrm{ann}(v_{\mu,\mu,k})$. Then $\mu_2 - \nu_2 \notin p^k\mathbb{Z}$. Since $p^l\mathbb{Z} \subset p^k\mathbb{Z}$ we have $\mu_2 - \nu_2 \notin p^l\mathbb{Z}$, that is, $\xi_{\nu\mu} \in \mathrm{ann}(v_{\mu,\lambda,l})$. □

It follows from Proposition 33 and Remark 30 that the map

$$\begin{array}{rccl} \Phi^{\mu,k}_{\lambda,l} & : & P_{\mu,k} & \to P_{\lambda,l} \\ & & v_{\nu,\mu,k} & \mapsto \xi_{\nu,\mu} v_{\mu,\lambda,l} \end{array}$$

for $\mu \geq \lambda$, $l \geq k$, is a well-defined map of $S^+(2,r)$-modules.

Proposition 34. *Let λ, $\mu \in \Lambda(2,r)$ and $\mu \geq \lambda$. Suppose $l \leq k$ and $\lambda_2 - \mu_2 + p^l \in p^k\mathbb{Z}$. Then*

$$\operatorname{ann}(v_{\mu,\mu,k}) = \operatorname{ann}(v_{\mu,\lambda,l}).$$

Proof. The inclusion $\operatorname{ann}(v_{\mu,\lambda,l}) \subset \operatorname{ann}(v_{\mu,\mu,k})$ is proved in the same fashion as Proposition 33. For the reverse inclusion, let $\xi_{\nu\mu} \in \operatorname{ann}(v_{\mu,\mu,k})$. By Proposition 32 we have $\mu_2 - \nu_2 \notin p^k\mathbb{Z}$. If, furthermore, $\mu_2 - \nu_2 \notin p^l\mathbb{Z}$ then $\xi_{\nu\mu} \in \operatorname{ann}(v_{\mu,\lambda,l})$. Thus, we only have to consider the case $\mu_2 - \nu_2 \in p^l\mathbb{Z}\setminus p^k\mathbb{Z}$. We can write $\mu_2 - \nu_2$ in the form $r_0p^l + r_1p^k$ with $1 \leq r_0 \leq p^{k-l} - 1$. Note that $\lambda_2 - \mu_2 = sp^k - p^l$ for some s and hence $\lambda_2 - \nu_2 = (r_0 - 1)p^l + (r_1 + s)p^k$. From Proposition 24 we obtain

$$\binom{\lambda_2 - \nu_2}{\lambda_2 - \mu_2} \equiv \binom{r_0 - 1}{p^{k-l} - 1}\binom{r_1 + s}{s} \equiv 0 \pmod p,$$

since $r_0 - 1 < p^{k-l} - 1$. Therefore $\xi_{\nu,\mu} \in \operatorname{ann}(v_{\mu,\lambda,l})$, as required. □

It follows from Proposition 34 and Remark 30 that the map

$$\begin{array}{cccl} \Psi^{\mu,k}_{\lambda,l} & : & P_{\mu,k} & \to P_{\lambda,l} \\ & & v_{\nu,\mu,k} & \mapsto \xi_{\nu,\mu}v_{\mu,\lambda,l} \end{array}$$

is a well-defined inclusion of $S^+(2,r)$-modules for $l \leq k$ and $\mu \geq \lambda$ such that $\lambda_2 - \mu_2 + p^l \in p^k\mathbb{Z}$.

4.3 Projective resolutions of simple modules over the algebra $S^+(2,r)$

We denote by $\mathbb{N}^\omega$ the set of all sequences of natural numbers with only finitely many non-zero terms. Denote by $e_i \in \mathbb{N}^\omega$ the sequence with 1 in the i-th place and zero elsewhere. We identify $\mathbb{N}^k$ with the subsemigroup of $\mathbb{N}^\omega$ generated by $e_1, e_2, \dots, e_k$. Define the map $|\cdot|\colon \mathbb{N}^\omega \to \mathbb{N}$ by the rule

$$|(n_1, \dots, n_k)| = \sum_{i=1}^k n_i,$$

and the map $f\colon \mathbb{N}^\omega \to \mathbb{N}$ by the rule

$$f(n_1, \dots, n_k) = \sum_{i\geq 1} \left(p\left[\frac{n_i}{2}\right] + \varepsilon(n_i)\right) p^{i-1},$$

Table 4.1: Values of f on $\mathbb{N}^2$

n_2 \ n_1	0	1	2	3	4
0	0	1	p	$p+1$	$2p$
1	p	$p+1$	$2p$	$2p+1$	$3p$
2	p^2	p^2+1	p^2+p	p^2+p+1	p^2+2p
3	p^2+p	p^2+p+1	p^2+2p	p^2+2p+1	p^2+3p
4	$2p^2$	$2p^2+1$	$2p^2+p$	$2p^2+p+1$	$2p^2+2p+1$
5	$2p^2+p$	$2p^2+p+1$	$2p^2+2p$	$2p^2+2p+1$	$2p^2+3p$

where $\varepsilon(n) = 0$ for n even and $\varepsilon(n) = 1$ for n odd. Note, that we denote by $[\,]$ the floor function, that is for $\alpha \in \mathbb{R}$ the number $[\alpha]$ is an integer such that

$$0 \leq \alpha - [\alpha] < 1.$$

We give some values of f on $\mathbb{N}^2$ in Table 4.1. We shall construct a projective resolution of the module $P_{\lambda,k}$ as a total complex of a multiple complex parametrised by $\mathbb{N}^k$, in which the module $P_{\mathrm{R}^{f(n)}\lambda}$ lies at the node $n \in \mathbb{N}^k$. In particular, for $k \geq \log_p(\lambda_2)$ we get a projective resolution of the module K_λ.

Theorem 35. *Let $\lambda \in \Lambda(2,r)$. Then the module $P_{\lambda,k}$ over $S^+(2,r)$ has a minimal projective resolution of the form*

$$\cdots \longrightarrow C_s(\lambda,k) \xrightarrow{d_s} \cdots \xrightarrow{d_2} C_1(\lambda,k) \xrightarrow{d_1} C_0(\lambda,k) \longrightarrow P_{\lambda,k} \longrightarrow 0,$$

where

$$C_s(\lambda,k) = \bigoplus_{n\in\mathbb{N}^k:|n|=s,\, f(n)\leq\lambda_2} P_{\mathrm{R}^{f(n)}\lambda}$$

and

$$d_{s|P_{\mathrm{R}^{f(n)}\lambda}} = \sum_{i=1}^{k} (-1)^{n_1+\cdots+n_{i-1}} \partial_{i,n},$$

where

$$\partial_{i,n} = \Phi^{\mathrm{R}^{f(n)}\lambda,0}_{\mathrm{R}^{f(n-e_i)}\lambda,0} \colon P_{\mathrm{R}^{f(n)}\lambda} \to P_{\mathrm{R}^{f(n-e_i)}\lambda}.$$

Before we prove the theorem, we give some examples for small λ. Let $p = 2$ and $\lambda = (0,8)$. We collect in the following table values of $n \in \mathbb{N}^4$ such that $f(n) \leq 8$:

n				$\|n\|$	$f(n)$
0	0	0	0	0	0
1	0	0	0	1	1
0	1	0	0	1	2
0	0	1	0	1	4
0	0	0	1	1	8
2	0	0	0	2	2
1	1	0	0	2	3
0	2	0	0	2	4
1	0	1	0	2	5
0	1	1	0	2	6
0	0	2	0	2	8
3	0	0	0	3	3

n				$\|n\|$	$f(n)$
2	1	0	0	3	4
1	2	0	0	3	5
0	3	0	0	3	6
2	0	1	0	3	6
1	1	1	0	3	7
0	2	1	0	3	8
4	0	0	0	4	4
3	1	0	0	4	5
2	2	0	0	4	6
1	3	0	0	4	7
0	4	0	0	4	8
3	0	1	0	4	7

n				$\|n\|$	$f(n)$
2	1	1	0	4	8
5	0	0	0	5	5
4	1	0	0	5	6
3	2	0	0	5	7
2	3	0	0	5	8
4	0	1	0	5	8
6	0	0	0	6	6
5	1	0	0	6	7
4	2	0	0	6	8
7	0	0	0	7	7
6	1	0	0	7	8
8	0	0	0	8	8

Thus the resolution from Theorem 35 of the module $P_{(0,8),4} \cong K_{(0,8)}$ looks like

$$\begin{aligned} 0 \longrightarrow P_{(8,0)} &\longrightarrow P_{(8,0)} \oplus P_{(7,1)} \longrightarrow P_{(8,0)} \oplus P_{(7,1)} \oplus P_{(6,2)} \\ &\longrightarrow P_{(8,0)} \oplus P_{(8,0)} \oplus P_{(7,1)} \oplus P_{(6,2)} \oplus P_{(5,3)} \\ &\longrightarrow P_{(8,0)} \oplus P_{(7,1)} \oplus P_{(8,0)} \oplus P_{(7,1)} \oplus P_{(6,2)} \oplus P_{(5,3)} \oplus P_{(4,4)} \\ &\longrightarrow P_{(8,0)} \oplus P_{(7,1)} \oplus P_{(6,2)} \oplus P_{(6,2)} \oplus P_{(5,3)} \oplus P_{(4,4)} \oplus P_{(3,5)} \\ &\longrightarrow P_{(8,0)} \oplus P_{(6,2)} \oplus P_{(5,3)} \oplus P_{(4,4)} \oplus P_{(3,5)} \oplus P_{(2,6)} \\ &\longrightarrow P_{(8,0)} \oplus P_{(4,4)} \oplus P_{(2,6)} \oplus P_{(1,7)} \longrightarrow K_{(0,8)} \longrightarrow 0. \end{aligned}$$

Let $p = 3$ and $\lambda = (0, 10)$. Then we have the following $n \in \mathbb{N}^3$ such that $f(n) \leq 10$:

n			$\|n\|$	$f(n)$
0	0	0	0	0
1	0	0	1	1
0	1	0	1	3
0	0	1	1	9
2	0	0	2	3
1	1	0	2	4

n			$\|n\|$	$f(n)$
0	2	0	2	9
1	0	1	2	10
3	0	0	3	4
2	1	0	3	6
1	2	0	3	10
4	0	0	4	6

n			$\|n\|$	$f(n)$
3	1	0	4	7
5	0	0	5	7
4	1	0	5	9
6	0	0	6	9
5	1	0	6	10
7	0	0	7	10

The corresponding resolution of the module $P_{(0,10),3} \cong K_{(0,10)}$ has the form

$$\begin{aligned}0 \longrightarrow P_{(10,0)} &\longrightarrow P_{(10,0)} \oplus P_{(9,1)} \longrightarrow P_{(9,1)} \oplus P_{(7,3)} \longrightarrow P_{(7,3)} \oplus P_{(6,4)} \\ &\longrightarrow P_{(10,0)} \oplus P_{(6,4)} \oplus P_{(4,6)} \longrightarrow P_{(10,0)} \oplus P_{(9,1)} \oplus P_{(4,6)} \oplus P_{(3,7)} \\ &\longrightarrow P_{(9,1)} \oplus P_{(3,7)} \oplus P_{(1,9)} \longrightarrow P_{(0,10)} \longrightarrow K_{(0,10)} \longrightarrow 0.\end{aligned}$$

We precede the proof of Theorem 35 with a series of lemmata concerning the modules $P_{\lambda,k}$.

Lemma 36. *Let $\lambda \in \Lambda(2,r)$ and $k \geq 1$. Denote $R^{p^k}\lambda$ by μ and $R^{p^{k+1}}\lambda$ by ν. Then there is an exact sequence*

$$0 \longrightarrow P_{\nu,k+1} \xrightarrow{\eta} P_{\mu,k} \xrightarrow{\varphi} P_{\lambda,k} \xrightarrow{\pi} P_{\lambda,k+1} \longrightarrow 0,$$

where $\pi = \Phi^{\lambda,k}_{\lambda,k+1}$, $\varphi = \Phi^{\mu,k}_{\lambda,k}$ and $\eta = \Psi^{\nu,k+1}_{\mu,k}$.

Proof. The map π is surjective since $P_{\lambda,k+1}$ is a cyclic module generated by the vector $v_{\lambda,\lambda,k+1} = \pi(v_{\lambda,\lambda,k})$. Since $\lambda_2 - \mu_2 \notin p^{k+1}\mathbb{Z}$, we have $(P_{\lambda,k+1})_\mu = 0$. Now, $\pi\varphi(v_{\mu,\mu,k})$ is an element of $(P_{\lambda,k+1})_\mu$, and therefore $\pi\varphi(v_{\mu,\mu,k}) = 0$. Thus $\operatorname{Im}\varphi \subset \operatorname{Ker}\pi$. We now show that $\operatorname{Ker}\pi \subset \operatorname{Im}\varphi$.

The kernel of π has basis $\{v_{\rho,\lambda,k} : \lambda_2 - \rho_2 \in p^k\mathbb{Z} \setminus p^{k+1}\mathbb{Z}\}$. Let $v_{\rho,\lambda,k}$ be an element of this basis. We can write $\lambda_2 - \rho_2$ in the form $r_0p^k + r_1p^{k+1}$, where $1 \leq r_0 \leq p-1$. By definition of the map φ we get

$$\begin{aligned}\varphi(v_{\rho,\mu,k}) &= \xi_{\rho,\mu}v_{\mu,\lambda,k} = \binom{\lambda_2-\rho_2}{\lambda_2-\mu_2}v_{\rho,\lambda,k} = \binom{r_0p^k + r_1p^{k+1}}{p^k}v_{\rho,\lambda,k} \\ &= \binom{r_0}{1}\binom{r_1}{0}v_{\rho,\lambda,k} = r_0v_{\rho,\lambda,k}.\end{aligned}$$

Hence $\varphi(r_0^{-1}v_{\rho,\mu,k}) = v_{\rho,\lambda,k}$ and $v_{\rho,\lambda,k} \in \operatorname{Im}\varphi$. We also obtain that

$$\{v_{\rho,\mu,k} : \lambda_2 - \rho_2 \in p^{k+1}\mathbb{Z},\ \rho > \mu\} = \{v_{\rho,\mu,k} : \mu_2 + p^k - \rho_2 \in p^{k+1}\mathbb{Z},\ \rho > \mu\}$$

is a basis for $\operatorname{Ker}\varphi$. Let $v_{\rho,\mu,k}$ be an element of this basis. Then we can write $\mu_2 - \rho_2$ in the form $(p-1)p^k + rp^{k+1}$, where $r \geq 0$. Therefore, by definition of the map η,

$$\begin{aligned}\eta(v_{\rho,\nu,k+1}) &= \xi_{\rho,\nu}v_{\nu,\mu,k} = \binom{\mu_2-\rho_2}{\mu_2-\nu_2}v_{\rho,\mu,k} \\ &= \binom{(p-1)p^k + rp^{k+1}}{(p-1)p^k}v_{\rho,\mu,k} = \binom{p-1}{p-1}\binom{r}{0}v_{\rho,\mu,k} = v_{\rho,\mu,k},\end{aligned}$$

that is, $v_{\rho,\mu,k} \in \operatorname{Im}\eta$ and so $\operatorname{Ker}\varphi \subset \operatorname{Im}\eta$. Now

$$\varphi \circ \eta(v_{\nu,\nu,k+1}) = \xi_{\nu,\mu}\varphi(v_{\mu,\mu,k}) = \xi_{\nu,\mu}\xi_{\mu,\lambda}v_{\lambda,\lambda,k} = \binom{p^{k+1}}{p^k} v_{\lambda,\lambda,k} = 0,$$

so $\operatorname{Im}\eta \subset \operatorname{Ker}\varphi$. The injectivity of the map η follows from Proposition 34. This concludes the proof of the lemma. □

Corollary 37. *Every module $P_{\lambda,k+1}$ has a "k-resolution"*

$$\cdots \longrightarrow P_{\mathrm{R}^{f(m)p^k}\lambda,k} \longrightarrow \cdots \longrightarrow P_{\mathrm{R}^{f(1)p^k}\lambda,k} \longrightarrow P_{\lambda,k} \longrightarrow P_{\lambda,k+1} \longrightarrow 0.$$

Proof. Apply the previous lemma to the modules $P_{\mathrm{R}^{mp^{k+1}}\lambda}$, $m \geq 0$, and glue the resulting exact sequences. □

Corollary 38. *For the $S^+(2, r)$-module $P_{\lambda,1}$, the resolution*

$$\cdots \longrightarrow P_{\mathrm{R}^{f(m)}\lambda} \xrightarrow{d_m} \cdots \xrightarrow{d_2} P_{\mathrm{R}^{f(1)}\lambda} \xrightarrow{d_1} P_\lambda \longrightarrow P_{\lambda,1} \longrightarrow 0$$

is a minimal projective resolution.

Proof. The minimality of the constructed resolution follows from the fact that $\operatorname{Im} d_m$ does not contain $v_{\mathrm{R}^{f(m-1)}}$, since every element of $\operatorname{Supp} P_{\mathrm{R}^{f(m)}\lambda}$ is strictly greater than $\mathrm{R}^{f(m-1)}\lambda$, that is, $\operatorname{Im} d_m \subset \operatorname{rad} P_{\mathrm{R}^{f(m)}\lambda}$. □

Proof. [Proof of Theorem 35] First, we have to check that all sequences

$$\cdots \longrightarrow C_s(\lambda, k) \xrightarrow{d_s} \cdots \xrightarrow{d_2} C_1(\lambda, k) \xrightarrow{d_1} C_0(\lambda, k) \longrightarrow P_{\lambda,k} \longrightarrow 0$$

are well-defined chain complexes, that is, $d_{s-1} \circ d_s = 0$. In view of the definition of d_s, it is enough to check the equalities

$$\partial_{j,n-e_i} \circ \partial_{i,n} = \partial_{i,n-e_j} \circ \partial_{j,n} \colon P_{\mathrm{R}^{f(n)}\lambda} \to P_{\mathrm{R}^{f(n-e_i-e_j)}\lambda}$$

for all $n \in \mathbb{N}^k$ and all i, j such that $1 \leq i, j \leq k$. Since $P_{\mathrm{R}^{f(n)}\lambda}$ is cyclic, we will check the above equality only on the generating vector $v_{\mu,\mu,0}$, where

$\mu = \mathrm{R}^{f(n)}\lambda$. Let $\nu = \mathrm{R}^{f(n-e_i)}\lambda$, $\kappa = \mathrm{R}^{f(n-e_j)}\lambda$ and $\theta = \mathrm{R}^{f(n-e_i-e_j)}\lambda$. Define $\gamma\colon \mathbb{N} \to \mathbb{N}$ by the rule

$$\gamma(n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ p-1, & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\begin{aligned} \nu_2 - \mu_2 &= f(n) - f(n-e_i) \\ &= \left(p\left(\left[\frac{n_i}{2}\right] - \left[\frac{n_i-1}{2}\right] \right) + \epsilon(n_i) - \epsilon(n_i - 1) \right) p^i \\ &= \left\{ \begin{array}{ll} p^i, & \text{if } n_i \text{ is odd,} \\ (p-1)p^i, & \text{if } n_i \text{ is even} \end{array} \right\} = \gamma(n_i)p^i. \end{aligned}$$

Analogously,

$$\kappa_2 - \mu_2 = \theta_2 - \nu_2 = \gamma(n_j)p^j \quad \text{and} \quad \theta_2 - \kappa_2 = \gamma(n_i)p^i.$$

We have

$$\begin{aligned} \partial_{j,n-e_i} \circ \partial_{i,n}(v_{\mu,\mu,0}) &= \partial_{j,n-e_i}(\xi_{\mu,\mu} v_{\mu,\nu,0}) = \xi_{\mu,\nu} v_{\nu,\theta,0} \\ &= \binom{\theta_2 - \mu_2}{\theta_2 - \nu_2} v_{\mu,\theta,0} = \binom{\gamma(n_j)p^j + \gamma(n_i)p^i}{\gamma(n_j)p^j} v_{\mu,\theta,0} \\ &= \binom{\gamma(n_i)}{0}\binom{\gamma(n_j)}{\gamma(n_j)} v_{\mu,\theta,0} = v_{\mu,\theta,0} \end{aligned}$$

and

$$\begin{aligned} \partial_{i,n-e_j} \circ \partial_{j,n}(v_{\mu,\mu}) &= \partial_{i,n-e_j}(\xi_{\mu,\mu} v_{\mu,\theta,0}) = \xi_{\mu,\kappa} v_{\kappa,\theta,0} \\ &= \binom{\theta_2 - \mu_2}{\theta_2 - \kappa_2} v_{\mu,\theta,0} = \binom{\gamma(n_i)p^i + \gamma(n_j)p^j}{\gamma(n_i)p^i} v_{\mu,\theta,0} \\ &= \binom{\gamma(n_i)}{\gamma(n_i)}\binom{\gamma(n_j)}{0} v_{\mu,\theta,0} = v_{\mu,\theta,0}. \end{aligned}$$

Now we prove that the complexes $(C(\lambda,k),d)$ are resolutions of $P_{\lambda,k}$ by induction on k.

Base of induction. The required claim for $k=1$ is proved in Corollary 38.

Inductive step. Suppose we have proved that the complexes $(C(\mu,k),d)$ are resolutions of $P_{\mu,k}$ for all $k \le m$ and all $\mu \in \Lambda(2,r)$. Let us show that the complex $(C(\lambda, m{+}1), d)$ is a resolution of $P_{\lambda,m+1}$. We consider $(C(\lambda, m{+}1), d)$ as a double complex $K_{\bullet,\bullet}$ with

$$K_{s,t} = \bigoplus_{n\in\mathbb{N}^m:|n|=s} P_{\mathrm{R}^{f(n)}\mathrm{R}^{f(t)p^m}\lambda} = C_s(\mathrm{R}^{f(t)p^m}\lambda, m).$$

Then, by the inductive hypothesis, we have

$$\mathrm{H}^{d_0}_s(K_{\bullet,t}) = 0 \text{ for } s > 0 \text{ and } \mathrm{H}^{d_0}_0(K_{\bullet,t}) \cong P_{\mathrm{R}^{f(t)p^m}\lambda,m}.$$

Moreover, the differential $d_1 : P_{\mathrm{R}^{f(t)p^m}\lambda,m} \to P_{\mathrm{R}^{f(t-1)p^m}\lambda,m}$ coincides, up to sign, with the differential from Corollary 37. Applying Corollary 37 and Theorem 21 we get

$$\mathrm{H}_t(K) \cong \mathrm{H}^{d_1}_t\, \mathrm{H}^{d_0}_{0,\bullet}(K_{\bullet,\bullet}) \cong \mathrm{H}^{d_1}_t(P_{\mathrm{R}^{f(t)p^m}\lambda,m}) \cong \begin{cases} 0, & \text{if } t > 0, \\ P_{\lambda,m+1}, & \text{if } t = 0. \end{cases}$$

Therefore $(C(\lambda, m{+}1), d)$ is a projective resolution of $P_{\lambda,m+1}$. Its minimality follows from the fact that $\operatorname{Im} d_j \subset \operatorname{rad} C_{j-1}$, for all $j \geq 1$. □

Corollary 39. *Let $\lambda \in \Lambda(2, r)$. Then the projective dimension of K_λ equals $2\left[\frac{\lambda_2}{p}\right] + \tau(\lambda_2)$, where*

$$\tau(t) = \begin{cases} 0, & t \in p\mathbb{Z}, \\ 1, & t \notin p\mathbb{Z}. \end{cases}$$

Proof. It follows from Theorem 35 that $\operatorname{pdim} K_\lambda = \max\{|n| : f(n) \leq \lambda_2,\, n \in \mathbb{N}^k\}$, where $p^k \geq \lambda_2$. From the definition of the maps f and $|\cdot|$ it follows that if $|n| = |m|$ and $n \trianglerighteq m$, then $f(n) < f(m)$. Thus we can take the maximum over elements of the form $(n_1, 0, \ldots, 0)$. Therefore $\operatorname{pdim} K_\lambda = \max\left\{n_1 : \left[\frac{n_1}{2}\right] p + \varepsilon(n_1) \leq \lambda_2\right\} = 2\left[\frac{\lambda_2}{p}\right] + \tau(\lambda_2)$. □

Corollary 40. *The global dimension of $S^{+}(2, r)$ is $2\left[\frac{r}{p}\right] + \tau(r)$.*

Proof. We have

$$\begin{aligned} \operatorname{gdim}(S^{+}(2, r)) &= \max\{\operatorname{pdim} K_\lambda : \lambda \in \Lambda(2, r)\} \\ &= \operatorname{pdim} K_{0,r} = 2\left[\frac{r}{p}\right] + \tau(r). \quad \square \end{aligned}$$

4.4 Projective resolutions of Weyl modules over $S(2,r)$

In this section we construct a projective resolution, of minimal possible length, of the Weyl module V^λ for each $\lambda \in \Lambda(2,r)^+$, such that $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$. Recall, that Weyl module V^λ is defined as the tensor product $S^(2,r) \otimes_{S^+(2,r)K_\lambda}$. We shall use

Theorem 41 ([25, Corollary 5.2]). *For each $\lambda \in \Lambda^+(n,r)$*

$$\mathrm{Ext}^i_{S^+(n,r)}(S(n,r), K_\lambda) \cong \begin{cases} V^\lambda, & \text{if } i = 0 \\ 0, & \text{if } i > 0. \end{cases}$$

The idea is as follows. We apply the induction functor $S(2,r) \otimes_{S^+(2,r)} (-)$ to the projective resolution of K_λ from Theorem 35. By Theorem 41, this gives a projective resolution of the Weyl module V^λ (for any $\lambda \in \Lambda^+(2,r)$). The problem is that this resolution can have length greater than the projective dimension of the module V^λ. Therefore, we have to modify the resulting resolutions. We are able to do this in the case $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$.

We denote by $L(\mu)$ the simple module with highest weight $\mu \in \Lambda^+(2,r)$ over the Schur algebra $S(2,r)$. Recall, that the projective dimension of V^λ is the maximal integer j such that there is $\mu \in \Lambda^+(2,r)$ such that the extension group $\mathrm{Ext}^j_{S(2,r)}(V^\lambda, L(\mu))$ is non-trivial. It is clear that the group $\mathrm{Ext}^j_{S(2,r)}(V^(\lambda), L(\mu))$ is non-trivial only if λ and μ are in the same block of the algebra $S(2,r)$. Denote by $\delta(\lambda)$ the maximal integer δ such that $\lambda_1 - \lambda_2 + 1 \in p^\delta\mathbb{Z}$.

Theorem 42. *Two weights λ, $\mu \in \Lambda^+(2,r)$ are in the same block of the Schur algebra $S(2,r)$ if and only if*

1) $\delta(\lambda) = \delta(\mu)$;

2) either $\lambda_1 - \mu_1 \in p^{\delta(\lambda)+1}\mathbb{Z}$ or $\lambda_1 - \mu_2 + 1 \in p^{\delta(\lambda)+1}\mathbb{Z}$.

Proof. This result is a direct consequence of [6, Corollary, p.417] and [13, 7.2.(3)]. □

For $\lambda \in \Lambda^+(2,r)$ denote by $d(\lambda)$ the integer $\left[\frac{\lambda_1 - \lambda_2}{p}\right]$.

Theorem 43. *Let $\lambda \in \Lambda^+(2, r)$ be such that $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$, and let $\mu \in \Lambda^+(2, r)$. Then*

1) if $\mu < \lambda$, then for all $j \geq 0$, the group $\operatorname{Ext}^j_{S(2,r)}(V^\lambda, L(\mu))$ is trivial;

2) if $\mu \geq \lambda$ and μ lies in the same block as λ, then $\operatorname{Ext}^{d(\mu)-d(\lambda)}_{S(2,r)}(V^\lambda, L(\mu)) \cong k$ and $\operatorname{Ext}^j_{S(2,r)}(V^\lambda, L(\mu)) = 0$ for all $j > d(\mu) - d(\lambda)$.

Proof. By [3, 2.2d], we have for any two $S(2, r)$ modules M and N and any $j \geq 0$, an isomorphism

$$\operatorname{Ext}^j_{S(2,r)}(M, N) \cong \operatorname{Ext}^j_{\mathrm{GL}_2(K)}(M, N),$$

where $\mathrm{GL}_2(k)$ is the general linear group of rank 2. Now, the first part of the theorem follows from [13, Proposition 6.20]. The second part of the theorem is a reformulation of [16, Lemma 2.1] and [16, Theorem 2.4]. See also [15, Lemma 3.5, Lemma 5.1] for the notation.

Remark 44. Note, that from [3, 2.2d] it also follows that

$$\operatorname{Ext}^j_{S(n,r)}(M, N) \cong \operatorname{Ext}^j_{\mathrm{GL}_n(K)}(M, N)$$

for any two $S(n, r)$-modules M and N, and for all n and r.

Let $\lambda \in \Lambda^+(2, r)$. Denote by r_i the residue of λ_i modulo p. Define the function $T\colon \Lambda^+(2, r) \to \{0, 1\}$ by

$$T(\lambda) = \begin{cases} 0, & r_2 < r_1 + 1 \text{ and } r_1 \neq p - 1, \\ 1, & r_2 > r_1 + 1 \text{ or } r_1 = p - 1. \end{cases}$$

Corollary 45. *Let $\lambda \in \Lambda^+(2, r)$ and $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$. Then the projective dimension of the Weyl module V^λ is $2\left[\frac{\lambda_2}{p}\right] + T(\lambda)$.*

Proof. In order to determine the projective dimension of the Weyl module V^λ we have to determine the maximal integer j such that there is $\mu \in \Lambda^+(2, r)$ such that $\operatorname{Ext}^j(V^\lambda, L(\mu)) \neq 0$. It follows from Theorem 43, that for a given μ from the block of λ, such integer is equal to $d(\mu) - d(\lambda)$. Since d is an increasing function of μ, we have to find the maximal μ in the block of λ. Let $q_i = \left[\frac{\lambda_i}{p}\right]$. It is easy to check that

$$\mu = \begin{cases} ((q_1+q_2+1)p+r_2-1,0), & \text{if } r_1 = p-1 \\ ((q_1+q_2)p+r_2-1, r_1+1), & \text{if } r_2 > r_1+1 \\ ((q_1+q_2)p+r_1, r_2), & \text{if } r_2 < r_1+1 \text{ and } r_1 \neq p-1. \end{cases}$$

Note that $r_2 = r_1+1$ is impossible, since $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$. If $r_1 = p-1$, then $r_2 \neq 0$ and

$$\begin{aligned} \operatorname{pdim} V^\lambda &= d(\mu) - d(\lambda) \\ &= \left[\frac{(q_1+q_2+1)p+r_2-1}{p}\right] - \left[\frac{(q_1-q_2)p+r_1-r_2}{p}\right] \\ &= q_1+q_2+1-(q_1-q_2) = 2q_2+1 = 2\left[\frac{\lambda_2}{p}\right] + T(\lambda). \end{aligned}$$

If $r_2 > r_1+1$, then $r_2 - r_1 - 2 \geq 0$, $r_1 - r_2 \leq -1$ and

$$\begin{aligned} \operatorname{pdim} V^\lambda &= d(\mu) - d(\lambda) \\ &= \left[\frac{(q_1+q_2)p+r_2-r_1-2}{p}\right] - \left[\frac{(q_1-q_2)p+r_1-r_2}{p}\right] \\ &= q_1+q_2-(q_1-q_2-1) = 2q_2+1 = 2\left[\frac{\lambda_2}{p}\right] + T(\lambda). \end{aligned}$$

If $r_2 < r_1+1$ and $r_1 \neq p-1$, then $r_1 - r_2 \geq 0$ and

$$\begin{aligned} \operatorname{pdim} V^\lambda = d(\mu) - d(\lambda) &= \left[\frac{(q_1+q_2)p+r_1-r_2}{p}\right] - \left[\frac{(q_1-q_2)p+r_1-r_2}{p}\right] \\ &= q_1+q_2-(q_1-q_2) = 2q_2 = 2\left[\frac{\lambda_2}{p}\right] + T(\lambda). \quad \square \end{aligned}$$

Corollary 46. *Let $\lambda \in \Lambda^+(2,r)$ and $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$. If $r_2 = 0$ or $r_1 = p-1$ or $r_2 > r_1+1$, then* $\operatorname{pdim} V^\lambda = \operatorname{pdim} K_\lambda$.

Proof. If $r_2 = 0$, then $\tau(\lambda_2) = 0 = T(\lambda)$. If $r_1 = p-1$ then $r_2 \neq 0$, since $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$. Therefore $\tau(\lambda_2) = 1 = T(\lambda)$. If $r_2 > r_1+1$ then again $r_2 \neq 0$ and $\tau(\lambda_2) = 1 = T(\lambda)$. In all these cases it follows from Corollary 40 and Corollary 45 that

$$\operatorname{pdim} V^\lambda = 2\left[\frac{\lambda_2}{p}\right] + T(\lambda) = 2\left[\frac{\lambda_2}{p}\right] + \tau(\lambda_2) = \operatorname{pdim} K_\lambda. \quad \square$$

Remark 47. If $\lambda \in \Lambda^+(2,r)$ satisfies the conditions of Corollary 46, then the projective resolution of the Weyl module V^λ induced from the minimal projective resolution of the $S^+(2,r)$-module K_λ has minimal possible length.

Remark 48. If $p = 2$, then the conditions of Corollary 46 are satisfied for all $\lambda \in \Lambda^+(2,r)$ such that $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$.

Corollary 49. *Let $\lambda \in \Lambda^+(2,r)$. Suppose that $0 \le \lambda_2 \le p-1$. Denote by r_1 the residue of λ_1 modulo p. If $r_1 + 1 > \lambda_2$ and $r_1 \neq p-1$, then the Weyl module V^λ is a projective $S(2,r)$-module.*

Proof. By Corollary 45 we have

$$\operatorname{pdim} V^\lambda = 2\left[\frac{r_2}{p}\right] + T(\lambda) = 2\left[\frac{\lambda_2}{p}\right] + T(\lambda) = 2\cdot 0 + 0 = 0. \quad \square$$

Corollary 50. *Let $\lambda \in \Lambda^+(2,r)$ and $1 \le \lambda_2 \le p-1$. Then there is an exact sequence of $S(2,r)$-modules*

$$0 \longrightarrow S(2,r)\xi_{\mathrm{R}\,\lambda} \longrightarrow S(2,r)\xi_\lambda \longrightarrow V^\lambda \longrightarrow 0.$$

Proof. This is a sequence obtained by applying the functor $S(2,r) \otimes_{S^+(2,r)} (-)$ to the projective resolution of the $S^+(2,r)$-module K_λ:

$$0 \longrightarrow P_{\mathrm{R}\,\lambda} \longrightarrow P_\lambda \longrightarrow K_\lambda \longrightarrow 0.$$

The resulting sequence is exact by Theorem 41. $\square$

Proposition 51. *Let $\lambda \in \Lambda^+(2,r)$. Suppose $\lambda_1 - \lambda_2 + 1 \notin p\mathbb{Z}$, $r_2 \neq 0$, $r_1 \neq p-1$ and $r_1 + 1 > r_2$. Denote by μ the partition $\left(\lambda_1 + \left[\frac{\lambda_2}{p}\right]p, r_2\right)$. Then the Weyl module V^λ has a projective resolution of length $2\left[\frac{\lambda_2}{p}\right]$ of the form*

$$\cdots \longrightarrow \overline{C}_s(\lambda,k) \xrightarrow{d_s} \cdots \xrightarrow{d_2} \overline{C}_1(\lambda,k) \xrightarrow{d_1} \overline{C}_0(\lambda,k) \longrightarrow V^\lambda \longrightarrow 0,$$

where

$$\overline{C}_s(\lambda,k) = \bigoplus_{n\in\mathbb{N}^k:|n|=s,\, f(n)\le\lambda_2} S(2,r)\xi_{\mathrm{R}^{f(n)}\lambda}, \textit{ for } s \le 2\left[\tfrac{\lambda_2}{p}\right] - 1$$

$$\overline{C}_{2\left[\frac{\lambda_2}{p}\right]} = V^\mu \oplus S(2,r)\xi_{\mathrm{R}\,\mu}.$$

Proof. The resolution induced from the resolution of the $S^+(2,r)$-module K_λ constructed in Theorem 35 has the required form, except that

$$S(2,r) \otimes_{S^+(2,r)} C_{2[\frac{\lambda_2}{p}]} = S(2,r)\xi_\mu \oplus S(2,r)\xi_{\mathrm{R}\,\mu}$$

and

$$S(2,r) \otimes_{S^+(2,r)} C_{2[\frac{\lambda_2}{p}]+1} = S(2,r)\xi_{\mathrm{R}\,\mu}.$$

By Corollary 50 the cokernel of the map $S(2,r)\xi_{\mathrm{R}\,\mu} \to S(2,r)\xi_\mu$ is isomorphic to the Weyl module V^μ. Since μ satisfies the conditions of Corollary 49, the module V^μ is projective. □

Chapter 5

Projective resolutions for $S^+(3,r)$

5.1 First reduction

We shall need the following technical lemma.

Lemma 52. *Let* $\lambda \in \Lambda(3,r)$*,* $1 \le s < t \le 3$*, and* $k + l \le \lambda_t$*. Then*

$$\xi_{A_{st}^k A_{st}^l l(\lambda), A_{st}^l l(\lambda)} \xi_{A_{st}^l l(\lambda), l(\lambda)} = \binom{k+l}{k} \xi_{A_{st}^{k+l} l(\lambda), l(\lambda)}.$$

Proof. The proof is analogous to the proof of Lemma 23. □

Recall that by Proposition 17, the algebra $S^+(3,r)$ has basis $\{\xi_{i,l(\lambda)} : \lambda \in \Lambda(3,r),\ i \in I(\lambda)\}$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $i \in I(\lambda)$, that is, $i \le l(\lambda)$ and T_i^λ is row semi-standard. Then T_i^λ has the form

1	...	...	...	...	...	...	1	
1	...	...	1	2	...	2		
1	...	1	2	...	2	3	...	3

Let μ_{12} be the number of occurrences of 1 in the second row of T_i^λ, μ_{13} the number of occurrences of 1 in the third row, and μ_{23} the number of occurrences of 2 in the third row. Recall that for a multi-index j we denote by $A_{st}j$ the multi-index obtained from j by replacing the first occurrence of t by s. Then $i = A_{23}^{\mu_{23}} A_{13}^{\mu_{13}} A_{12}^{\mu_{12}} l(\lambda)$.

Lemma 53. *Let λ and i be as above. Then for $j = A_{12}^{\mu_{12}} l(\lambda)$ we have*

$$\xi_{i,l(\lambda)} = \xi_{i,j}\xi_{j,l(\lambda)}.$$

Proof. We use Proposition 12 for the proof. We have

$$\begin{aligned} l(\lambda) &= (1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}), \\ j &= A_{12}^{\mu_{12}} l(\lambda) = (1^{\lambda_1+\mu_{12}}, 2^{\lambda_2-\mu_{12}}, 3^{\lambda_3}), \\ i &= A_{23}^{\mu_{23}} A_{13}^{\mu_{13}} j = (1^{\lambda_1+\mu_{12}}, 2^{\lambda_2-\mu_{12}}, 1^{\mu_{13}}, 2^{\mu_{23}}, 3^{\lambda_3-\mu_{13}-\mu_{23}}). \end{aligned}$$

Thus

$$\begin{aligned} G_j &\cong \Sigma_{\lambda_1+\mu_{12}} \times \Sigma_{\lambda_2-\mu_{12}} \times \Sigma_{\lambda_3}, \\ G_{i,j} &\cong \Sigma_{\lambda_1+\mu_{12}} \times \Sigma_{\lambda_2-\mu_{12}} \times (\Sigma_{\mu_{12}} \times \Sigma_{\mu_{23}} \times \Sigma_{\lambda_3-\mu_{13}-\mu_{23}}), \\ G_{j,l(\lambda)} &\cong (\Sigma_{\lambda_1} \times \Sigma_{\mu_{12}}) \times \Sigma_{\lambda_2-\mu_{12}} \times \Sigma_{\lambda_3}. \end{aligned}$$

We claim that $G_j = G_{i,j}G_{j,l(\lambda)}$. In fact, suppose $(\sigma_1, \sigma_2, \sigma_3) \in G_j$. Then $(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1, \sigma_2, e)(e, e, \sigma_3)$ and $(\sigma_1, \sigma_2, e) \in G_{i,j}$, $(e, e, \sigma_3) \in G_{j,l(j)}$.

Moreover,

$$\begin{aligned} G_{i,j,l(\lambda)} &\cong (\Sigma_{\lambda_1} \times \Sigma_{\mu_{12}}) \times \Sigma_{\lambda_2-\mu_{12}} \times (\Sigma_{\mu_{12}} \times \Sigma_{\mu_{23}} \times \Sigma_{\lambda_3-\mu_{13}-\mu_{23}}), \\ G_{i,l(\lambda)} &\cong (\Sigma_{\lambda_1} \times \Sigma_{\mu_{12}}) \times \Sigma_{\lambda_2-\mu_{12}} \times (\Sigma_{\mu_{12}} \times \Sigma_{\mu_{23}} \times \Sigma_{\lambda_3-\mu_{13}-\mu_{23}}), \end{aligned}$$

that is, $[G_{i,j,l(\lambda)} : G_{i,l(\lambda)}] = 1$ and by Proposition 12,

$$\xi_{i,l(\lambda)} = \xi_{i,j}\xi_{j,l(\lambda)}. \quad \square$$

Corollary 54. *Let $\lambda \in \Lambda(3,r)$. Denote by v_λ a generator of the projective $S^+(3,r)$-module P_λ. Then P_λ has basis*

$$\{\xi_{A_{23}^{\mu_{23}} A_{13}^{\mu_{13}} A_{12}^{\mu_{12}} l(\lambda), A_{12}^{\mu_{12}} l(\lambda)} \xi_{A_{12}^{\mu_{12}} l(\lambda), l(\lambda)} v_\lambda : \mu_{12} \leq \lambda_2,\ \mu_{13} + \mu_{23} \leq \lambda_3\}.$$

Proof. This follows from Proposition 18 and Lemma 53. $\square$

We denote by $v_{(\mu_{23},\mu_{13},\mu_{12}),\lambda}$ the element

$$\xi_{A_{23}^{\mu_{23}} A_{13}^{\mu_{13}} A_{12}^{\mu_{12}} l(\lambda), A_{12}^{\mu_{12}} l(\lambda)} \xi_{A_{12}^{\mu_{12}} l(\lambda), l(\lambda)} v_\lambda$$

from the module P_λ. For any $S^+(3,r)$-module M, the map from $M(\lambda)$ to $\mathrm{Hom}_{S^+(3,r)}(P_\lambda, M)$ given by the formula

$$m \mapsto \left(\Theta_m^\lambda \colon \xi_{j,l(\lambda)} v_\lambda \mapsto \xi_{j,l(\lambda)} m\right)$$

is an isomorphism since P_λ is generated by v_λ and

$$\mathrm{Ann}(v_\lambda) = \bigoplus_{\mu\neq\lambda} S^+(n,r)\xi_\mu.$$

Definition 55. An $\mathbb{N}$-*sequence* $M_\bullet$ of $S^+(3,r)$-modules is a collection $\{M_i : i \in \mathbb{N}\}$ of $S^+(3,r)$-modules together with $S^+(3,r)$-maps $d_i \colon M_i \to M_{i-1}$.

Proposition 56. *For $\lambda \in \Lambda(3,r)$, consider the $\mathbb{N}$-sequence $D_\bullet$ of $S^+(3,r)$-modules*

$$\cdots \longrightarrow D_s(\lambda) \xrightarrow{d_s} \cdots \xrightarrow{d_2} D_1(\lambda) \xrightarrow{d_1} D_0(\lambda) \longrightarrow 0,$$

where

$$D_s(\lambda) = \bigoplus_{n\in\mathbb{N}^\omega : |n|=s} P_{\mathrm{R}_{1,2}^{f(n)}\lambda}$$

and

$$d_{s|P_{\mathrm{R}_{1,2}^{f(n)}\lambda}} = \sum_{i=1}^{k} (-1)^{n_1+\cdots+n_{i-1}} \partial_{i,n},$$

where

$$\partial_{i,n} = \Theta^{\mathrm{R}_{1,2}^{f(n)}\lambda}_{v_{(\gamma(n_i)p^i,0,0),\mathrm{R}_{1,2}^{f(n-e_i)}\lambda}} : P_{\mathrm{R}_{1,2}^{f(n)}\lambda} \to P_{\mathrm{R}_{1,2}^{f(n-e_i)}\lambda}.$$

Then $D_\bullet$ is a complex. It is exact at all terms except the zero term. The $S^+(3,r)$-module $Q_\lambda := \mathrm{H}_0(D_\bullet)$ has basis $\{\xi_{A_{2,3}^m A_{1,3}^l l(\lambda), l(\lambda)} w_\lambda : m + l \leq \lambda_3\}$, where w_λ is the image of v_λ.

Proof. Notice that all maps in the above sequence are homomorphisms of $S^+(3,r)$-modules.

Let $\nu = \mathrm{R}_{12}^s \lambda$ for some $s \geq 0$. For each pair (μ_{23}, μ_{13}) such that $\mu_{23}+\mu_{13} \leq \lambda_3 = \nu_3$ we denote by $P_\nu(\mu_{23}, \mu_{13})$ the subspace of P_ν with basis

$$\{\xi_{A_{23}^{\mu_{23}} A_{13}^{\mu_{13}} A_{12}^{\mu_{12}}\nu, A_{12}^{\mu_{12}}\nu} \xi_{A_{12}^{\mu_{12}}\nu,\nu} v_\nu \colon \mu_{12} \leq \nu_2\}.$$

Then we have an isomorphism

$$P_\nu \cong \bigoplus_{\mu_{23}+\mu_{13}\leq\lambda_3} P_\nu(\mu_{23}, \mu_{13}).$$

We say that the elements of $P_\nu(\mu_{23}, \mu_{13})$ have degree (μ_{23}, μ_{13}). It follows from Lemma 52 that the maps $\partial_{i,n}$ preserve degree. Therefore the $\mathbb{N}$-sequence $D_\bullet$ decomposes into the direct sum of $\mathbb{N}$-sequences $D_\bullet(\mu_{23}, \mu_{13})$ for $\mu_{23}+\mu_{13} \leq \lambda_3$.

Let $\lambda' = (\lambda_1, \lambda_2) \in \Lambda(2, r - \lambda_3)$. Define $\varphi_\bullet(\mu_{23}, \mu_{13})\colon D_\bullet(\mu_{23}, \mu_{13}) \to C_\bullet(\lambda')$ by the rule

$$\begin{array}{rcrcl} \varphi_s(\mu_{23}, \mu_{13})_{|P_\nu(\mu_{23},\mu_{13})} & : & P_\nu(\mu_{23}, \mu_{13}) & \to & P_{\nu'} \\ & & v_{(\mu_{23},\mu_{13},\mu_{12}),\mathrm{R}_{12}^{f(n)}\lambda} & \mapsto & v_{(\lambda_1+\mu_{12},\lambda_2-\mu_{12}),\mathrm{R}^{f(n)}\lambda'}. \end{array}$$

By Proposition 18 and Corollary 54, all the maps $\varphi_s(\mu_{23}, \mu_{13})$ are isomorphisms of vector spaces. Moreover, from Lemma 52 and Lemma 23 it follows that they commute with the $\partial_{i,n}$. Thus the $\mathbb{N}$-sequence $D_\bullet(\mu_{23}, \mu_{13})$ is isomorphic to a chain complex of vector spaces $C_\bullet(\lambda')$. This shows that $D_\bullet$ is a complex and that

$$\mathrm{H}_s(D_\bullet) \cong \begin{cases} 0, & \text{if } s > 0, \\ \displaystyle\bigoplus_{(\mu_{23},\mu_{13}):\mu_{23}+\mu_{13}\le\lambda_3} \mathrm{H}_0(C_\bullet(\lambda')), & \text{if } s = 0. \end{cases}$$

By Theorem 35 we have $\mathrm{H}_0(C_\bullet(\lambda')) \cong K_{\lambda'}$ and the space $\mathrm{H}_0(C_\bullet(\lambda'))$ is generated by the image of $v_{\lambda',\lambda'}$. This means that $\mathrm{H}_0(D_\bullet)$ has a basis consisting of the images $w_{(\mu_{23},\mu_{13}),\lambda} = \xi_{A_{2,3}^{\mu_{23}} A_{1,3}^{\mu_{13}} l(\lambda), l(\lambda)} w_\lambda$ of the vectors $v_{(\mu_{23},\mu_{13},0),\lambda} = \xi_{A_{2,3}^{\mu_{23}} A_{1,3}^{\mu_{13}} l(\lambda), l(\lambda)} v_\lambda$. □

5.2 Second reduction

For each λ, the module Q_λ is a quotient of P_λ. Let $\pi_\lambda\colon P_\lambda \to Q_\lambda$ be the natural projection. Then the kernel of π_λ has basis

$$\{v_{(\mu_{23},\mu_{13},\mu_{12}),\lambda} : 1 \le \mu_{12} \le \lambda_2,\ \mu_{23} + \mu_{13} \le \lambda_3\}$$

or, in other words,

$$\begin{aligned} \operatorname{ann}(w_\lambda) &= \operatorname{Ann}(w_\lambda) \cap S^+(3,r)\xi_\lambda \\ &= \langle \xi_{A_{23}^{\mu_{23}} A_{13}^{\mu_{13}} A_{12}^{\mu_{12}} l(\lambda), A_{12}^{\mu_{12}} l(\lambda)} \xi_{A_{12}^{\mu_{12}} l(\lambda), l(\lambda)} : 1 \le \mu_{12} \le \lambda_2,\ \mu_{23} + \mu_{13} \le \lambda_3 \rangle. \end{aligned}$$

In particular, $\operatorname{ann}(w_\lambda)$ is generated by the elements $\xi_{A_{12}^t l(\lambda), l(\lambda)}$ for $t \ge 1$ as a left ideal in $S^+(3,r)$.

Proposition 57. *Let* $i = A_{13}^s l(\lambda) = (1^{\lambda_1}, 2^{\lambda_2}, 1^s, 3^{\lambda_3 - s})$ *and* $\nu = \mathrm{R}_{13}^s \lambda = (\lambda_1 + s, \lambda_2, \lambda_3 - s)$. *Define the map* $\Xi_\lambda^\nu\colon Q_\nu \to Q_\lambda$ *by the rule*

$$\xi_{j,l(\nu)} w_\nu \mapsto \xi_{j,l(\nu)} \xi_{i,l(\lambda)} w_\lambda.$$

Then Ξ_λ^ν *is a well-defined map of* $S^+(3,r)$*-modules.*

Before giving the proof, we introduce one more notation. Let $\lambda \in \Lambda(n,r)$ and $\sigma \in \Sigma_n$. We denote by $[\lambda,\sigma]$ the element of Σ_r such that if

$$\sum_{j=1}^{k-1} \lambda_j < i \leq \sum_{j=1}^{k} \lambda_j,$$

then

$$[\lambda,\sigma](i) = \sum_{j=1}^{\sigma(k)-1} \lambda_{\sigma(j)} + i - \sum_{j=1}^{k-1} \lambda_j.$$

For example,

$$[(2,2),(12)] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \text{and} \quad [(1,2,3),(13)] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 2 & 3 & 1 \end{pmatrix}.$$

Proof. We have to check that

$$\operatorname{ann}(w_\nu) \subset \operatorname{ann}(\xi_{i,l(\lambda)} w_\lambda).$$

Since $\operatorname{ann}(w_\nu)$ is generated by the elements $\xi_{A_{12}^t l(\nu), l(\nu)}$, $t \geq 1$, it is enough to show that

$$\xi_{A_{12}^t l(\nu), l(\nu)} \xi_{i,l(\lambda)} w_\lambda = 0$$

for all $t \geq 1$. Let $\sigma = [(\lambda_1, \lambda_2, s, \lambda_3 - s), (23)]$. Then

$$\begin{array}{rclclcl} i\sigma & = & (1^{\lambda_1}, 2^{\lambda_2}, 1^s, 3^{\lambda_3-s})\sigma & = & (1^{\lambda_1}, 1^s, 2^{\lambda_2}, 3^{\lambda_3-s}) & = & l(\nu), \\ (A_{12}^t i)\sigma & = & (1^{\lambda_1+t}, 2^{\lambda_2-t}, 1^s, 3^{\lambda_3-s})\sigma & = & (1^{\lambda_1+s+t}, 2^{\lambda_2-t}, 3^{\lambda_3-s}) & = & A_{12}^t l(\nu), \end{array}$$

and therefore

$$\xi_{A_{12}^t l(\nu), l(\nu)} = \xi_{(A_{12}^t l(\nu))\sigma, l(\nu)\sigma} = \xi_{A_{12}^t i, i}.$$

We shall prove in Lemma 58 that

$$\xi_{A_{12}^t A_{13}^s l(\lambda), A_{13}^s l(\lambda)} \xi_{A_{13}^s l(\lambda), l(\lambda)} = \xi_{A_{13}^s A_{12}^t l(\lambda), l(\lambda)},$$

and by Lemma 53 we have

$$\xi_{A_{13}^s A_{12}^t l(\lambda), l(\lambda)} = \xi_{A_{13}^t A_{12}^s l(\lambda), A_{12}^s l(\lambda)} \xi_{A_{12}^s l(\lambda), l(\lambda)}$$

for $\lambda \in \Lambda(3,r)$ and $t \leq \lambda_2$, $s \leq \lambda_3$. Therefore for $t \geq 1$,

$$\xi_{A_{12}^t l(\nu), l(\nu)} \xi_{i,l(\lambda)} w_\lambda = \xi_{A_{12}^t i, i} \xi_{i,l(\lambda)} w_\lambda = \xi_{A_{13}^s A_{12}^t l(\lambda), A_{12}^t l(\lambda)} \xi_{A_{12}^t l(\lambda), l(\lambda)} w_\lambda = 0,$$

since $\xi_{A_{12}^t l(\lambda), l(\lambda)} \in \operatorname{ann}(w_\lambda)$. $\square$

Lemma 58. *Let $\lambda \in \Lambda(3,r)$, $s \leq \lambda_3$, and $t \leq \lambda_2$. Then*

$$\xi_{A_{12}^t A_{13}^s l(\lambda), A_{13}^s l(\lambda)} \xi_{A_{13}^s l(\lambda), l(\lambda)} = \xi_{A_{13}^s A_{12}^t l(\lambda), l(\lambda)}.$$

Proof. Let

$$\begin{aligned} j &= A_{13}^s l(\lambda) = (1^{\lambda_1}, 2^{\lambda_2}, 1^s, 3^{\lambda_3 - s}), \\ i &= A_{12}^t j = (1^{\lambda_1+t}, 2^{\lambda_2-t}, 1^s, 3^{\lambda_3-s}). \end{aligned}$$

Denote $[(\lambda_1 + \lambda_2, s, \lambda_3 - s), (14)] \in \Sigma_r$ by σ. Then

$$\begin{array}{rcccl} l(\lambda)\sigma &=& (1^{\lambda_1}, 2^{\lambda_2}, 3^s, 3^{\lambda_3-s})\sigma &=& (3^{\lambda_3-s}, 3^s, 1^{\lambda_1}, 2^{\lambda_2}), \\ j\sigma &=& (1^{\lambda_1}, 2^{\lambda_2}, 1^s, 3^{\lambda_3-s})\sigma &=& (3^{\lambda_3-s}, 1^s, 1^{\lambda_1}, 2^{\lambda_2}), \\ i\sigma &=& (1^{\lambda_1+t}, 2^{\lambda_2-t}, 1^s, 3^{\lambda_3-s})\sigma &=& (3^{\lambda_3-s}, 1^s, 1^{\lambda_1+t}, 2^{\lambda_2-t}), \end{array}$$

and

$$\begin{array}{rcccccc} G_{j\sigma} &\cong& \Sigma_{\lambda_3-s} &\times& \Sigma_{\lambda_1+s} &\times& \Sigma_{\lambda_2}, \\ G_{i\sigma,j\sigma} &\cong& \Sigma_{\lambda_3-s} &\times& \Sigma_{\lambda_1+s} &\times& (\Sigma_t \times \Sigma_{\lambda_2}), \\ G_{j\sigma,l(\lambda)\sigma} &\cong& \Sigma_{\lambda_3-s} &\times& (\Sigma_s \times \Sigma_{\lambda_1}) &\times& \Sigma_{\lambda_2}. \end{array}$$

Hence $G_{i\sigma,j\sigma} G_{j\sigma,l(\lambda)\sigma} = G_{j\sigma}$. Moreover,

$$\begin{array}{rcccccc} G_{i\sigma,j\sigma,l(\lambda)\sigma} &\cong& \Sigma_{\lambda_3-s} &\times& (\Sigma_s \times \Sigma_{\lambda_1}) &\times& (\Sigma_t \times \Sigma_{\lambda_2}), \\ G_{i\sigma,l(\lambda)\sigma} &\cong& \Sigma_{\lambda_3-s} &\times& (\Sigma_s \times \Sigma_{\lambda_1}) &\times& (\Sigma_t \times \Sigma_{\lambda_2}), \end{array}$$

that is, $[G_{i\sigma,l(\lambda)\sigma} : G_{i\sigma,j\sigma,l(\lambda)\sigma}] = 1$. Therefore, by Proposition 12,

$$\xi_{i,j}\xi_{j,l(\lambda)} = \xi_{i\sigma,j\sigma}\xi_{j\sigma,l(\lambda)\sigma} = \xi_{i\sigma,l(\lambda)\sigma} = \xi_{i,l(\lambda)}. \quad \square$$

Lemma 59. *Let $\lambda \in \Lambda(3,r)$. Then for all s, t with $s + t \leq \lambda_3$ we have*

$$\xi_{A_{23}^s A_{13}^t l(\lambda), l(\lambda)} = \xi_{A_{23}^s A_{13}^t l(\lambda), A_{13}^t l(\lambda)} \xi_{A_{13}^t l(\lambda), l(\lambda)}.$$

Proof. The proof goes along the same lines as the proof of Lemma 53 . □

Corollary 60. *Let $\lambda \in \Lambda(3,r)$. Then the module Q_λ has basis*

$$\{\xi_{A_{23}^s A_{13}^t l(\lambda), A_{13}^t l(\lambda)} \xi_{A_{13}^t l(\lambda), l(\lambda)} w_\lambda \colon s + t \leq \lambda_3\}.$$

Proof. This is a direct consequence of Proposition 56 and Lemma 59.

Proposition 61. *For $\lambda \in \Lambda(n,r)$, let $E_\bullet$ be an $\mathbb{N}$-sequence of $S^+(3,r)$-modules*

$$\cdots \longrightarrow E_m(\lambda) \xrightarrow{d_m} \cdots \xrightarrow{d_2} E_1(\lambda) \xrightarrow{d_1} E_0(\lambda) \longrightarrow 0,$$

where

$$E_m(\lambda) = \bigoplus_{n\in\mathbb{N}^\omega:|n|=m} Q_{\mathrm{R}_{1,3}^{f(n)}\lambda}$$

and

$$d_{m|Q_{\mathrm{R}_{1,3}^{f(n)}\lambda}} = \sum_{i=1}^{k}(-1)^{n_1+\cdots+n_{i-1}}\partial_{i,n},$$

where

$$\partial_{i,n} = \Xi^{\mathrm{R}_{1,3}^{f(n)}\lambda}_{\mathrm{R}_{1,3}^{f(n-e_i)}\lambda} : Q_{\mathrm{R}_{1,3}^{f(n)}\lambda} \to Q_{\mathrm{R}_{1,3}^{f(n-e_i)}\lambda}.$$

Then $E_\bullet(\lambda)$ is a complex that is exact at all terms except the zero term. The $S^+(3,r)$-module $R_\lambda := \mathrm{H}_0(E_\bullet(\lambda))$ has basis $\{\xi_{A_{2,3}^m l(\lambda),l(\lambda)}u_\lambda\}$, where u_λ is the image of w_λ.

Proof. Let $\nu = \mathrm{R}_{13}^r\lambda = (\lambda_1+r,\lambda_2,\lambda_3-r)$ for some $r \geq 0$. We denote by $Q_\nu(s)$ the subspace of Q_ν generated by

$$\{\xi_{A_{23}^s A_{13}^t l(\nu),A_{13}^t l(\nu)}\xi_{A_{13}^t l(\nu),l(\nu)}w_\nu : 0 \leq t \leq \nu_3 - s\}.$$

We say that the elements of $Q_\nu(s)$ have degree s. Let us show that for $\nu' = \mathrm{R}_{13}^{r'}\lambda$, $r' \leq r$, the maps $\Xi^\nu_{\nu'}$ preserve the degree defined in this way. Denote $r - r'$ by r''. We have

$$\Xi^\nu_{\nu'}(\xi_{A_{23}^s A_{13}^t l(\nu),A_{13}^t l(\nu)}\xi_{A_{13}^t l(\nu),l(\nu)}w_\nu) = \xi_{A_{23}^s A_{13}^t l(\nu),A_{13}^t l(\nu)}\xi_{A_{13}^t l(\nu),l(\nu)}\xi_{A_{13}^{r''}l(\nu'),l(\nu')}w_{\nu'}.$$

Let $\sigma = [(\lambda_1+r', r'', \lambda_2, \lambda_3 - t), (23)] \in \Sigma_r$. Then

$$\begin{aligned}
l(\nu)\sigma &= (1^{\lambda_1+r'}, 1^{r''}, 2^{\lambda_2}, 3^{\lambda_3-r})\sigma \\
&= (1^{\lambda_1+r'}, 2^{\lambda_2}, 1^{r''}, 3^{\lambda_3-r}) = A_{13}^{r''}l(\nu'), \\
(A_{13}^t l(\nu))\sigma &= (1^{\lambda_1+r'}, 1^{r''}, 2^{\lambda_2}, 1^t, 3^{\lambda_3-r})\sigma \\
&= (1^{\lambda_1+r'}, 2^{\lambda_2}, 1^{r''+t}, 3^{\lambda_3-r}) = A_{13}^{r''+t}l(\nu'), \\
(A_{23}^s A_{13}^t l(\nu))\sigma &= (1^{\lambda_1+r'}, 1^{r''}, 2^{\lambda_2}, 1^t, 2^s, 3^{\lambda_3-r-s})\sigma \\
&= (1^{\lambda_1+r'}, 2^{\lambda_2}, 1^{r''+t}, 2^s, 3^{\lambda_3-r-s}) = A_{23}^s A_{13}^{r''+t}l(\nu').
\end{aligned}$$

Therefore, by definition of the elements $\xi_{\bullet,\bullet}$ and by Lemma 52,

$$\begin{aligned}
&\xi_{A_{23}^s A_{13}^t l(\nu), A_{13}^t l(\nu)} \xi_{A_{13}^t l(\nu), l(\nu)} \xi_{A_{13}^{r''} l(\nu'), l(\nu')} w_{\nu'} \\
&\qquad = \xi_{A_{23}^s A_{13}^{r''+t} l(\nu'), A_{13}^{r''+t} l(\nu')} \xi_{A_{13}^{r''+t} l(\nu'), A_{13}^{r''} l(\nu')} \xi_{A_{13}^{r''} l(\nu'), l(\nu')} w_{\nu'} \\
&\qquad = \binom{t+r''}{r''} \xi_{A_{23}^s A_{13}^{t+r''} l(\nu'), A_{13}^{t+r''} l(\nu')} \xi_{A_{13}^{t+r''} l(\nu'), l(\nu')} w_{\nu'} \in Q_{\nu'}(s),
\end{aligned}$$

and so $E_\bullet \cong \bigoplus_{s \le \lambda_3} E_\bullet(s)$. Let $\nu = (\lambda_1, \lambda_2 + r, \lambda_3 - r)$ for some r. Denote $(\lambda_1 + s, \lambda_3 - r - s)$ by ν^s. Define

$$\begin{array}{rrcl}
\psi_s : & E_\bullet(s) & \to & C_\bullet(\lambda^s) \\
& \xi_{A_{23}^s A_{13}^t l(\nu), A_{13}^t l(\nu)} \xi_{A_{13}^t l(\nu), l(\nu)} w_\nu & \mapsto & \xi_{\mathrm{R}^t\, l(\nu^s), l(\nu^s)} v_{\nu^s}.
\end{array}$$

It follows from the definition of the differentials in $E_\bullet(s)$ and $C_\bullet(\lambda^s)$ that ψ_s is a map of $\mathbb{N}$-sequences. Moreover, since ψ_s is a bijection on the basis, it is an isomorphism. Hence $E_\bullet(s)$ is a complex isomorphic to $C_\bullet(\lambda^s)$. By Theorem 56, it is exact at all terms except the zero term and $\mathrm{H}_0(E_\bullet(s)) \cong K$. It is clear that the vector space $\mathrm{H}_0(E_\bullet(s))$ is generated by the image of $\xi_{A_{23}^s l(\nu), l(\nu)} w_\nu$. □

5.3 Third reduction

Let $\lambda \in \Lambda(3, r)$. Then R_λ is a quotient of P_λ. Denote by ρ_λ the natural projection $P_\lambda \to R_\lambda$. Then $\operatorname{Ker} \rho_\lambda$ has basis

$$\xi_{A_{23}^{\mu_{23}} A_{13}^{\mu_{13}} A_{12}^{\mu_{12}} l(\lambda), A_{13}^{\mu_{13}} A_{12}^{\mu_{12}} l(\lambda)} \xi_{A_{13}^{\mu_{13}} A_{12}^{\mu_{12}} l(\lambda), l(\lambda)} v_\lambda,$$

where $\mu_{23} + \mu_{13} \le \lambda_3$, $\mu_{12} \le \lambda_2$, and $\mu_{13} + \mu_{12} \ge 1$.

Lemma 62. *Let $\nu = \mathrm{R}_{23}^s \lambda$. Define the map $\Upsilon_\lambda^\nu \colon R_\nu \to R_\lambda$ by the rule*

$$\xi_{A_{23}^t l(\nu), l(\nu)} u_\nu \mapsto \xi_{A_{23}^t A_{23}^s l(\lambda), A_{23}^s l(\lambda)} \xi_{A_{23}^s l(\lambda), l(\lambda)} u_\lambda.$$

Then Υ_λ^ν is a well-defined map of $S^+(3,r)$-modules.

Proof. The idea of the proof is the same as for Proposition 57 with the only difference being that we need the equalities

$$\begin{aligned}
&\xi_{A_{12}^t A_{23}^s l(\lambda), A_{23}^s l(\lambda)} \xi_{A_{23}^s l(\lambda), l(\lambda)} \\
&\qquad = \sum_{j=0}^{\min(t,s)} \xi_{A_{23}^{s-j} A_{13}^j A_{12}^{t-j} l(\lambda), A_{13}^j A_{12}^{t-j} l(\lambda)} \xi_{A_{13}^j A_{12}^{t-j} l(\lambda), A_{12}^{t-j} l(\lambda)} \xi_{A_{12}^{t-j} l(\lambda), l(\lambda)}
\end{aligned}$$

and

$$\xi_{A_{13}^t A_{23}^s l(\lambda), A_{23}^s l(\lambda)} \xi_{A_{23}^s l(\lambda), l(\lambda)} = \xi_{A_{13}^t A_{23}^s l(\lambda), l(\lambda)} = \xi_{A_{23}^s A_{13}^t l(\lambda), A_{13}^t l(\lambda)} \xi_{A_{13}^t l(\lambda), l(\lambda)}.$$

These are proved in the next two lemmata. □

Lemma 63. *Let $\lambda \in \Lambda(3,r)$. Then, for s and t with $s+t \leq \lambda_3$, we have*

$$\xi_{A_{13}^t A_{23}^s l(\lambda), A_{23}^s l(\lambda)} \xi_{A_{23}^s l(\lambda), l(\lambda)} = \xi_{A_{13}^t A_{23}^s l(\lambda), l(\lambda)},$$

and

$$\xi_{A_{23}^s A_{13}^t l(\lambda), A_{13}^t l(\lambda)} \xi_{A_{13}^t l(\lambda), l(\lambda)} = \xi_{A_{23}^s A_{13}^t l(\lambda), l(\lambda)}.$$

Proof. The proof is the same as for Lemma 58. □

Remark 64. Notice that $\xi_{A_{13}^t A_{23}^s l(\lambda), l(\lambda)} = \xi_{A_{23}^s A_{13}^t l(\lambda), l(\lambda)}$ since for

$$\sigma = [(\lambda_1 + \lambda_2, s, t, \lambda_3 - s - t), (23)] \in \Sigma_r$$

we have

$$\begin{aligned} l(\lambda)\sigma &= (1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3})\sigma = (1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}) = l(\lambda), \\ A_{23}^s A_{13}^t l(\lambda)\sigma &= (1^{\lambda_1}, 2^{\lambda_2}, 1^t, 2^s, 3^{\lambda_3 - s - t})\sigma \\ &= (1^{\lambda_1}, 2^{\lambda_2}, 2^s, 1^t, 3^{\lambda_3 - s - t}) = A_{13}^t A_{23}^s l(\lambda). \end{aligned}$$

Lemma 65. *Let $\lambda \in \Lambda(3,r)$. Then for $s \leq \lambda_3$ and $t \leq \lambda_2 + s$ we have*

$$\begin{aligned} &\xi_{A_{12}^t A_{23}^s l(\lambda), A_{23}^s l(\lambda)} \xi_{A_{23}^s l(\lambda), l(\lambda)} \\ &\quad = \sum_{c=0}^{\min(t,s)} \xi_{A_{23}^{s-c} A_{13}^c A_{12}^{t-c} l(\lambda), A_{13}^c A_{12}^{t-c} l(\lambda)} \xi_{A_{13}^c A_{12}^{t-c} l(\lambda), A_{12}^{t-c} l(\lambda)} \xi_{A_{12}^{t-c} l(\lambda), l(\lambda)}. \end{aligned}$$

Proof. We denote the left hand side of the equality by B and right hand side by D. Then we have to prove that $Bv = Dv$ for each $v \in V^{\otimes r}$, where V is a three-dimensional vector space with basis $\{v_1, v_2, v_3\}$. It is clear that this has to be checked only for the basis elements v_i, where $i \in I(3,r)$. For $i \notin \lambda$, $Bv_i = 0 = Dv_i$.

Suppose now that $i \in \lambda$, and consider v_i. Applying B to this element we get $\sum_j v_j$, where the sum is over j obtained from i in the following way. First we replace 3 by 2 in some s places, then we replace 2 by 1 in t places. In particular, on the second step, some new 2s can be replaced by 1s. We say that j is of type c if there are c such 2s. Now each j of type c can be obtained from i in the following way. First, we replace 2 by 1 in $t-c$ places, then we replace 3 by 1 in c places, and finally we replace 3 by 2 in $s-c$ places. Thus

$$\sum_{j \text{ is of type } c} v_j = \xi_{A_{23}^{s-c} A_{13}^{c} A_{12}^{t-c} i, A_{13}^{c} A_{12}^{t-c} i} \xi_{A_{13}^{c} A_{12}^{t-c} i, A_{12}^{t-c} i} \xi_{A_{12}^{t-c} i, i}.$$

This completes the proof. □

Proposition 66. *For $\lambda \in \Lambda(n, r)$, let $F_\bullet$ be an $\mathbb{N}$-sequence of $S^+(3, r)$-modules*

$$\cdots \longrightarrow F_m \xrightarrow{d_m} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0,$$

where

$$F_m(\lambda, k) = \bigoplus_{n \in \mathbb{N}^\omega : |n| = m} R_{\mathrm{R}_{2,3}^{f(n)} \lambda}$$

and

$$d_{m|R_{\mathrm{R}_{2,3}^{f(n)} \lambda}} = \sum_{i=1}^{k} (-1)^{n_1 + \cdots + n_{i-1}} \partial_{i,m},$$

where

$$\partial_{i,m} = \Upsilon^{A_{2,3}^{f(n)} \lambda}_{A_{2,3}^{f(n-e_i)} \lambda} : R_{\mathrm{R}_{2,3}^{f(n)} \lambda} \to R_{\mathrm{R}_{2,3}^{f(n-e_i)} \lambda}.$$

Then $F_\bullet$ is a complex. It is exact at all terms except the zero term and $\mathrm{H}_0(F_\bullet) \cong K_\lambda$.

Proof. Let $\nu = (\lambda_1, \lambda_2 + r, \lambda_3 - r)$ for some r. Denote $(\lambda_2 + r, \lambda_3 - r)$ by ν''. Define the map $\varphi : F_\bullet \to C_\bullet(\lambda'')$ by the formula

$$\xi_{A_{23}^{s} \nu} u_\nu \mapsto \xi_{\mathrm{R}^s \nu''} v_{\nu''}.$$

It follows from Lemma 23 and Lemma 52 that φ is a map of $\mathbb{N}$-sequences. Since φ is a bijection on bases, it is an isomorphism. Therefore by Theorem 35 we have that $F_\bullet$ is exact at all terms except the zero term, and $\mathrm{H}_0(F_\bullet) \cong K$ as vector spaces. Moreover, $\mathrm{H}_0(F_\bullet)$ is generated by an element of weight λ, and hence $\mathrm{H}_0(F_\bullet) \cong K_\lambda$ as $S^+(3, r)$-modules. □

5.4 Projective resolution for the trivial modules over the algebra $S^+(3,r)$

By Proposition 61, we have a $Q_\bullet(\lambda)$-resolution of modules R_λ and by Proposition 56, a projective resolution $P_\bullet(\nu)$ of Q_ν. Therefore, by Theorem 22, there is a projective resolution of R_λ with n-th term

$$\bigoplus_{k+l=n} \bigoplus_{\substack{n_2\in\mathbb{N}^\omega \\ |n_2|=l}} \bigoplus_{\substack{n_1\in\mathbb{N}^\omega \\ |n_1|=k}} P_{\mathrm{R}_{13}^{f(n_1)}\mathrm{R}_{12}^{f(n_2)}\lambda}.$$

Now, we have an $R_\bullet(\lambda)$-resolution of K_λ and a projective resolution for each R_ν.Therefore, from Theorem 22 we get the following

Theorem 67. *Every simple module K_λ over the algebra $S^+(3,r)$ has a projective resolution*

$$\cdots \longrightarrow C_m \xrightarrow{d_m} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0,$$

where

$$C_m(\lambda,k) = \bigoplus_{n_1,n_2,n_3:|n_1|+|n_2|+|n_3|=m} P_{\mathrm{R}_{2,3}^{f(n_3)}\mathrm{R}_{1,3}^{f(n_2)}\mathrm{R}_{1,2}^{f(n_1)}\lambda}.$$

5.5 Conclusions

The results of the previous section allow us to construct projective resolutions for Weyl modules over the Schur algebra $S(3,r)$. Namely, we apply the induction functor $S(3,r)\otimes_{S^+(3,r)}(-)$ to the resolutions from Theorem 67. By [25, Theorem 5.1], this gives projective resolutions for the Weyl modules V^λ, where $\lambda\in\Lambda^+(3,r)$. Note that this gives neither the minimal projective resolutions nor projective resolutions of minimal length, since the resolutions constructed in Theorem 67 are not of minimal possible length.

The author plans to extend the results of this work to the case $n \geq 3$. It would be also interesting to find a construction for minimal resolutions of one-dimensional modules over $S^+(3,r)$.

Appendix A

Algebras and quivers

In order to work with algebras it is convenient to use the concept of quivers and relations.

A.1 Representations of quivers

Definition 68. A quiver Γ is a directed graph $\Gamma = (V, E, s, t)$ where V is the set of vertices and E is the set of arrows, and s, t are maps $E \to V$. Given an arrow $a \in E$, we say it starts at vertex $s(a)$ and terminates at $t(a)$. The quiver is said to be *finite* provided both V and E are finite sets.

Suppose Γ is a quiver; and K is a fixed field. A *representation* M of a quiver Γ over K is given by (M_v, φ_a) where for any vertex $v \in V$ we have a vector space M_v, and for any arrow $v \xrightarrow{a} w$, there is a linear transformation $\varphi_a \colon M_v \to M_w$. If $M = (M_v, \varphi_a)$ and $N = (N_v, \psi_a)$ are representations of Γ over K then a *map* $\eta \colon M \to N$ is defined to be $\eta = (\eta_v)$ where $\eta_v \colon M_v \to N_v$ is a linear transformation such that for any arrow $v \xrightarrow{a} w$ there is a commutative diagram

$$\begin{array}{ccc} M_v & \xrightarrow{\varphi_a} & M_w \\ {\scriptstyle \eta_v}\downarrow & & \downarrow{\scriptstyle \eta_w} \\ N_v & \xrightarrow{\psi_a} & N_w \end{array}$$

that is, $\eta_w \varphi_a = \psi_a \eta_v$. Denote the category of representation of Γ by $\mathcal{R}(\Gamma)$.

A.2 The path algebra of a quiver

Definition 69. Given $v, w \in V$; then a *path* of length $l \geq 1$ from v to w is of the form $(w|a_l, \ldots, a_1|v)$ with arrow a_i satisfying $t(a_i) = s(a_{i+1})$ for all i,

$1 \leq i \leq l-1$, such that v is the starting point of a_1, and w is the end point of a_l. In addition, we also define for any vertex v of Γ a path of length zero (form v to itself) denoted by v also (or $(v||v)$).

The path algebra $K\Gamma$ of Γ is defined to be the K-vector space with basis the set of all paths in Γ. The product of two paths is taken to be the concatenation if it is againe a path, and zero otherwise. In this way, we obtain an associative K-algebra which has an identity if and only if V is finite (then the identity is given by $\sum_{v \in V} v$). Note that the path algebra is finite-dimensional if and only if V is finite, and there is no cyclic path in Γ.

We denote by $K\Gamma^+$ the ideal of $K\Gamma$ generated by all arrows. Then $(K\Gamma^+)^n$ is the ideal generated by all paths of length $\geq n$.

Proposition 70. *The categories $\mathcal{R}(\Gamma)$ and $K\Gamma$ are equivalent. In particular, $\mathcal{R}(\Gamma)$ is an abelian category.*

Proof. Given $M = (M_v, \varphi_a)$ in $\mathcal{R}(\Gamma)$, define the $K\Gamma$-module T_M with underlying vector space $\bigoplus_{v \in V} M_v$ with action of the algebra as follows:

Let $m \in M_v$, then

$$\begin{aligned} vm &= m, & & \\ wm &= 0 & & \text{for } w \neq v, \\ am &= \varphi_a(m) & & \text{if } a \text{ starts at } v, \\ am &= 0 & & \text{otherwise.} \end{aligned}$$

Suppose T is a $K\Gamma$-module, define $M = (M_v, \varphi_a)$ as follows:

If $v \in V$ then take $M_v = vT$, and if $v \xrightarrow{a} w$ is an arrow, then φ_a is the linear transformation $vT \to wT$ which is given by left multiplication with a.

If $\eta = (\eta_v)$ is a map $M \to N$ and $T = T_M$ and $S = T_N$, then η induces in an obvious way a $K\Gamma$-homomorphism which also denote by η. Any $K\Gamma$-homomorphism arises from a map $M \to N$. □

A.3 Quiver with relations

Definition 71. Let v and w be vertices of a quiver Γ. A *relation* ρ on Γ is an element $\rho = \sum c_\omega \omega \in K\Gamma$ where the ω are paths between two fixed vertices. If $\{\rho_\nu\}_\nu$ is a set of relations for Γ then $(\Gamma, \{\rho_\nu\}_\nu)$ is a *quiver with relations.*

If $\omega = (w|a_n, \ldots, a_1|v)$ is a path in Γ and $M = (M_v, \varphi_a)$ is a representation of Γ, then "ω acts on V" via the linear transformation $\omega(M) = \varphi_{a_n} \cdots \varphi_{a_1}$.

More generally, if ρ is a relation in Γ, say $\rho = \sum c_i\omega_i$, where $c_i \in K$ and each ω_i is a path then $\rho(M) = \sum c_i\omega_i(M)$.

Definition 72. Given a quiver with relations $(\Gamma, \{\rho_\nu\}_\nu)$ and a representation $M = (M_v, \varphi_a)$ of Γ then M is a *representation of* $(\Gamma, \{\rho_\nu\}_\nu)$ if for all ν we have $\rho_\nu(M) = 0$.

Proposition 73. *The category of representations of* $(\Gamma, \{\rho_\nu\}_\nu)$ *is equivalent to the category of modules over* $K\Gamma/I$ *where* I *is the ideal of* $K\Gamma$ *generated by* $\{\rho_\nu\}_\nu$.

Proof. The claim of the proposition is a direct consequence of definitions. □

Appendix B

Quasi-hereditary algebras and highest weight categories

B.1 Hereditary ideals

Let A be a finite dimensional algebra over a field K.

Theorem-Definition 74. *An ideal N of the algebra A is called* the radical *of A and denoted by* $\mathrm{rad}(A)$ *if one of the following equivalent conditions holds:*

1) *The ideal N is the intersection of all maximal left ideals of A.*

2) *The ideal N is the intersection of all maximal right ideals of A.*

3) *The ideal N is the maximal nilpotent ideal in A.*

Definition 75. An ideal I of A is said to be a *hereditary* ideal of A if

1) $J^2 = J$;

2) $J\,\mathrm{rad}(A)J = 0$;

3) J, considered as a left A-module, is projective.

Proposition 76. *If e is an idempotent of A, then $(AeA)^2 = AeA$. Conversely, if J is an ideal of A such that $J^2 = J$, then $J = AeA$ for an idempotent of A.*

Proof. The first assertion is trivial. So assume that $J^2 = J$. The algebra $B = A/\mathrm{rad}(A)$ is semi-simple, therefore any ideal of B is generated by an idempotent. Any idempotent of B is of the form $\bar{e} = e + \mathrm{rad}(A)$ with an idempotent e in A. Thus $J + \mathrm{rad}(A) = AeA + \mathrm{rad}(A)$ for some idempotent

e of A. Now, $J^2 = J$ implies $(J + \mathrm{rad}(A))^i = J + \mathrm{rad}(A)^i$ for all $i \geq 1$; similarly, $(AeA + \mathrm{rad}(A))^i = AeA + \mathrm{rad}(A)^i$ for all $i \geq 1$. But for large i, $\mathrm{rad}(A)^i = 0$, and therefore $J = AeA$. □

Corollary 77. *Let J be a hereditary ideal of a finite-dimensional algebra A. Then there is an idempotent $e \in A$ such that $J = AeA$.*

Proposition 78. *Let e be an idempotent of an algebra A. If the right module $(AeA)_A$ or the left module ${}_A(AeA)$ is projective, then the multiplication map*

$$\mu\colon Ae \otimes_{eAe} eA \to AeA$$

is bijective. Conversely, assume that A is a finite-dimensional and that

$$e\,\mathrm{rad}(A)e = 0.$$

Then, if μ is bijective, both modules $(AeA)_A$ and ${}_A(AeA)$ are projective.

Proof. For any left A-module M, consider the multiplication map

$$\mu_M\colon Ae \otimes_{AeA} eA \otimes_A M \to M.$$

The map μ_M is bijective for $M = Ae$, and therefore for all direct summands of direct sums of the module Ae. Now, there is a surjective A-module homomorphism of the form $\oplus Ae \to AeA$, where the direct sum is indexed by all elements of A. Since ${}_A(AeA)$ is projective, this epimorphism splits, and it follows that μ_{AeA} is bijective. But this means that μ is bijective, since

$$eA \otimes_A AeA \cong eAeA = eA.$$

The same argument applies in the case that ${}_A(AeA)$ is projective.

Now, assume that A is finite-dimensional and $e\,\mathrm{rad}(A)e = 0$. Then $\mathrm{rad}(eAe) = e\,\mathrm{rad}(A)e = 0$ and therefore eAe is semi-simple. In particular, all modules over eAe are projective. Since $(Ae)_{eAe}$ and $(eA)_A$ are projective, the module $(Ae \otimes_{eAe} eA)_A$ is projective also. Thus, the bijectivity of μ implies that $(AeA)_A$ is projective. Similarly, it implies that ${}_A(AeA)$ is projective. □

Corollary 79. *Let $J = AeA$ be a hereditary ideal in a finite-dimensional algebra A. Then the homomorphism*

$$\mu\colon Ae \otimes_{eAe} eA \to AeA = J$$

is bijective. Moreover, J, considered as a right A-module, is projective.

Proposition 80. *Let J be an ideal of a finite dimensional algebra A. Denote by B the algebra A/J. Then $J^2 = J$ if and only if* $\operatorname{Hom}_A(J_A, M_A) = 0$ *for any B-module M. If J is projective, then $J^2 = J$ if and only if* $\operatorname{Hom}_A(J_A, B_A) = 0$.

Proof. First, assume that $J^2 = J$ and let $\varphi\colon J_A \to M_A$ be a homomorphism. Then $\varphi(J) = \varphi(J^2) \subset JM = 0$, and thus $\varphi = 0$. Conversely, let $\operatorname{Hom}_A(J_A, M_A) = 0$ for any B-module M. Write $Y_A = J/J^2$. Since $JY = 0$, Y can be viewed as a B-module. Hence, $\operatorname{Hom}_A(J_A, Y_A) = 0$, and the canonical epimorphism $J_A \to Y_A$ shows that $Y = 0$.

Finally, assume that J_A is projective and that $\operatorname{Hom}_A(J_A, B_A) = 0$. Given a B-module M, let F be a free B-module with an epimorphism $\pi\colon F \to M$. Since J_A is projective, any map $\varphi\colon J_A \to M_A$ lifts to a map $\varphi'\colon J_A \to F_A$ with $\varphi = \pi\varphi'$. But $\operatorname{Hom}_A(J_A, F_A) = 0$, because F is a direct sum of copies of B. □

Definition 81. A finite dimensional associative K-algebra A is called *quasi-hereditary* if there is a chain of (two-sided) ideals in A,

$$0 = J_0 < J_1 < \cdots < J_n = A,$$

such that for any $k \in \{1, 2, \ldots, n\}$, J_k/J_{k-1} is a hereditary ideal of A/J_{k-1}. We call such a chain of idempotent ideals a *hereditary chain* or *defining sequence* for A.

B.2 Highest weight categories

Let $\mathcal{C}$ be a K-finite abelian category. This guarantees that $\operatorname{Hom}(M, N)$ is a finite-dimensional K-vector space for M and N in $\mathcal{C}$, composition is K-bilinear and all objects have composition series. Recall that a composition factor S of an object A in $\mathcal{C}$ is by definition, a composition factor of a subobject of finite length. The multiplicity (possibly infinite) of S in A, denoted $[A : S]$, is defined to be the maximum of the multiplicity of S in all subobjects of A of finite length.

Let Λ be a finite poset.

Definition 82. A category $\mathcal{C}$ over K as above is called a *highest weight category* if there exists an interval-finite poset Λ (the "weights" of $\mathcal{C}$) satisfying the following conditions:

1) There exists a family $\{\Delta(\lambda) : \lambda \in \Lambda\}$ of objects of $\mathcal{C}$ (variously called the *Weyl objects*, the *standard objects* or the *Verma objects*).

2) The head of $\Delta(\lambda)$ is simple; denoting this head by $L(\lambda)$ then $\{L(\lambda)\}$ is complete set of simple objects in $\mathcal{C}$. For each $\lambda \in \Lambda$, the composition factors of $\ker(\Delta(\lambda) \to L(\lambda))$ are all of the form $L(\mu)$, for $\mu < \lambda$.

3) Each $L(\lambda)$ has a projective cover, $P(\lambda)$, in $\mathcal{C}$. There exists an epimorphism $P(\lambda) \to \Delta(\lambda)$ whose kernel is filtered by some $\Delta(\mu)$ with $\mu > \lambda$.

Dual statements exist about the costandard objects $\nabla(\lambda)$ $(\lambda \in \Lambda)$, its simple socle and associated injective hull $I(\lambda)$ of $L(\lambda)$.

Theorem 83 ([2, 3.4]). *Let A be a finite dimensional algebra. The category A-mod of A-modules together with $(\Lambda, \leq)$ is a highest weight category if and only if A is quasi-hereditary.*

We give an informal sketch indicating why this result is true. Somehow one has to construct standard objects for a given quasi-hereditary algebra A with a set of simple modules $\{L(\lambda)\}$.

We take the maximal hereditary chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$$

in A. Then it can be shown that all indecomposable summands of the A-module J_k/J_{k-1} are pairwise isomorphic. We denote by $\Delta(k)$ one of these summands. It is a routine to check that the modules $\Delta(k)$ satisfies the required in the definition of highest weight category.

Appendix C

The Mackey formula for G-Algebras

In this appendix G is an arbitrary finite group.

Definition 84. A G-*algebra* over a field K is a K-algebra, on which G acts as a group of K-algebra homomorphisms.

For each subgroup $H \leq G$ we denote by A^H the subalgebra of G-invariant elements in A. Clearly, if H, L are subgroups of G, then

$$H \leq L \Rightarrow A^L \subset A^H.$$

Definition 85. If H and L are subgroups of G such that $H \leq L$, define the K-linear map $\mathrm{Tr}_H^L \colon A^H \to A^L$, by

$$\mathrm{Tr}_H^L(a) = \sum_{\sigma \in X} a^\sigma,$$

where the sum is over an H-transversal X of L, that is X is a set of representatives of the cosets $H\sigma$ in L.

Because $a \in A^H$, the value of Tr_H^L does not depend on the choice of X. Moreover, $\mathrm{Tr}_H^L(a)^\tau = \mathrm{Tr}_H^L(a)$, since $X\tau$ is an H-transversal of L if X is, for any $\tau \in L$.

Theorem 86 ([10, Lemma 4e]). *If L is a subgroup of G, and D, H are subgroups of L, then for any $a \in A^H$,*

$$\mathrm{Tr}_H^L(a) = \sum_{\sigma \in X} \mathrm{Tr}_{H^\sigma \cap D}^D(a^\sigma),$$

where X is an (H, D) transversal of L, that is X is a set of representatives of the double cosets $H\sigma D$ in L. If $a \in A^H$ and $b \in A^D$, then

$$\mathrm{Tr}_H^L(a)\,\mathrm{Tr}_D^L(b) = \sum_{\sigma\in X} \mathrm{Tr}_{H^\sigma\cap D}^L(a^\sigma b).$$

Proof. For each $\sigma \in X$, let Y_σ be an $H^\sigma \cap D$-transversal of D. Then it is easy to see that

$$Y = \cap_{\sigma\in X}\sigma Y_\sigma$$

is an H-transversal of L and the first equality holds by using this Y as a transversal. Now

$$\begin{aligned}
\mathrm{Tr}_H^L(a)\,\mathrm{Tr}_D^L(b) &= \mathrm{Tr}_D^L(\mathrm{Tr}_H^L(a)b)\\
&= \mathrm{Tr}_D^L\left(\sum_{\sigma\in X}\mathrm{Tr}_{H^\sigma\cap D}^D(a^\sigma)b\right)\\
&= \mathrm{Tr}_D^L\left(\sum_{\sigma\in X}\mathrm{Tr}_{H^\sigma\cap D}^D(a^\sigma b)\right)\\
&= \sum_{\sigma\in X}\mathrm{Tr}_{H^\sigma\cap D}^L(a^\sigma b).
\end{aligned}$$

The last equality follows from the fact that for any subgroups $E \leq D \leq L$ holds

$$\mathrm{Tr}_D^L\left(\mathrm{Tr}_E^D(a)\right) = \mathrm{Tr}_E^L(a).$$

□

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