L^2-invariants of nonuniform lattices in semisimple Lie groups

Dissertation
zur Erlangung des mathematisch-naturwissenschaftlichen Doktorgrades

"Doctor rerum naturalium"

der Georg-August-Universität Göttingen

im Promotionsprogramm Mathematik
der Georg-August University School of Science (GAUSS)

vorgelegt von

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Göttingen, 2013

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Tag der mündlichen Prüfung: 17.04.2013

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CHAPTER 1

Introduction

L^2-invariants have an *analytic* definition for closed Riemannian manifolds and a *topological* definition for finite CW complexes. A central idea is to lift classical topological notions to the universal covering taking into account the free action of the fundamental group by deck transformations. Let us consider the simplest example, the *L^2-Betti numbers*. Given a connected finite CW complex X with fundamental group Γ, the universal covering $\widetilde{X}$ has a cellular chain complex of $\mathbb{Z}\Gamma$-modules $C_p(\widetilde{X})$. We complete it to the L^2-cellular chain complex $C_p^{(2)}(\widetilde{X}) = \ell^2\Gamma \otimes_{\mathbb{Z}\Gamma} C_p(\widetilde{X})$. A cellular basis of $C_p(X)$ endows each $C_p^{(2)}(\widetilde{X})$ with the structure of a Hilbert space with isometric Γ-action. So the differentials determine the Γ-equivariant L^2-Laplacian $\Delta_p = d_p^* d_p + d_{p+1} d_{p+1}^*$ on $C_p^{(2)}(\widetilde{X})$. We define the L^2-Betti numbers of $\widetilde{X}$ to be the *von Neumann dimensions* of the harmonic L^2-chains, $b_p^{(2)}(\widetilde{X}) = \dim_{\mathcal{N}(\Gamma)} \ker \Delta_p$. Note that L^2-Betti numbers are a priori real valued as the von Neumann dimension is induced by the trace of the group von Neumann algebra $\mathcal{N}(\Gamma)$. It turns out that L^2-Betti numbers provide powerful invariants with many convenient properties. Their alternating sum gives the Euler characteristic and a positive L^2-Betti number obstructs nontrivial self-coverings and nontrivial circle actions. The p-th *Novikov–Shubin invariant* of $\widetilde{X}$, denoted by $\widetilde{\alpha}_p(\widetilde{X})$, captures information on eigenspaces of Δ_p in a neighborhood of zero. It takes values in $[0, \infty] \cup \{\infty^+\}$ that measure with respect to von Neumann dimension how slowly aggregated eigenspaces grow for small positive eigenvalues. Finally the third L^2-invariant we will consider is the *L^2-torsion* of $\widetilde{X}$ denoted by $\rho^{(2)}(\widetilde{X}) \in \mathbb{R}$. It is the L^2-counterpart of classical Reidemeister torsion and it is only defined if $\widetilde{X}$ is det-*L^2-acyclic* which essentially means that $b_p^{(2)}(\widetilde{X}) = 0$ for $p \geq 0$.

We obtain the analytic definition of L^2-Betti numbers, Novikov–Shubin invariants and L^2-torsion when we replace Δ_p by the Laplace–de Rham operator acting on p-forms of the universal covering of a closed Riemannian manifold. The key observation of the theory is that if we choose a triangulation, analytic and topological L^2-invariants agree. This flexibility effects that beside their apparent relevance for geometry and topology, L^2-invariants have additionally shown up in contexts as diverse as algebraic K-theory, ergodic theory, type II_1 factors, simplicial volume, knot theory and quantum groups. The subject of our concern is not yet in the list: group theory. Groups enter the picture when we consider aspherical spaces so that the L^2-invariants, being homotopy invariants, depend on the fundamental group only. Thus if a group Γ has a finite CW model for $B\Gamma$ we set $b_p^{(2)}(\Gamma) = b_p^{(2)}(E\Gamma)$, $\widetilde{\alpha}_p(\Gamma) = \widetilde{\alpha}_p(E\Gamma)$ and $\rho^{(2)}(\Gamma) = \rho^{(2)}(E\Gamma)$ if $E\Gamma$ is det-L^2-acyclic in which case we say that Γ itself is det-L^2-acyclic. Note that L^2-Betti numbers and Novikov–Shubin invariants of arbitrary group actions have been defined in [28, 68] and [69] so that $b_p^{(2)}(\Gamma)$ and $\widetilde{\alpha}_p(\Gamma)$ are in fact defined for any group Γ. An interesting case occurs if a group happens to have a closed manifold model for $B\Gamma$, because then the equality of topological and analytic L^2-invariants permits to calculate invariants of discrete groups by geometric methods.

A class of groups that has extensively been studied in this context is given by torsion-free uniform lattices in semisimple Lie groups. Such a $\Gamma \subset G$ acts properly and thus freely on the symmetric space $X = G/K$ where $K \subset G$ is a maximal compact subgroup. Since X is contractible, the locally symmetric space $\Gamma \backslash X$ is a closed manifold model of $B\Gamma$. M. Olbrich [85] has built on previous work by J. Lott and E. Hess–T. Schick to compute the three L^2-invariants of Γ with the analytic approach. We will recall the precise statement in Theorem 3.19. The computation uses $(\mathfrak{g}, K)$-cohomology as well as the Harish-Chandra–Plancherel Theorem. Uniform lattices in semisimple Lie groups can be seen as the chief examples of CAT(0) groups. Similarly, their geometric counterpart, the closed locally symmetric spaces of noncompact type, form the main examples of nonpositively curved manifolds. Therefore they often serve as a test ground for general assertions on nonpositive curvature. It is however fairly restrictive to require that lattices be uniform as this already rules out the most natural example $\mathrm{SL}(n, \mathbb{Z})$ which is central to number theory and geometry. In fact, a theorem of D. A. Kazhdan and G. A. Margulis [57] characterizes the nonuniform lattices in semisimple linear Lie groups without compact factors as those lattices that contain a unipotent element. Therefore nonuniform lattices possess infinite unipotent subgroups. Group theoretically this expels nonuniform lattices from the CAT(0) region in M. Bridson's universe of finitely presented groups [20]. However, they stay in the nonpositively curved area as they form the key examples of CAT(0) *lattices* for which an interesting structure theory has recently been developed in [23, 24]. Geometrically the locally symmetric spaces $\Gamma \backslash X$ of torsion-free nonuniform lattices Γ provide infinite $B\Gamma$s with cusps or ends and the unipotent subgroups are reflected in certain nilmanifolds that wind around the ends.

The purpose of this thesis is to calculate L^2-invariants of nonuniform lattices in semisimple Lie groups using suitable compactifications of locally symmetric spaces. Of course the compactification has to be homotopy equivalent to the original $\Gamma \backslash X$ to make sure it is a $B\Gamma$. One way to achieve this is to simply chop off the ends. An equivalent construction due to A. Borel and J.-P. Serre suggests to add boundary components at infinity so that $\Gamma \backslash X$ forms the interior of a compact manifold with *corners*. To expand on this, let us first suppose that Γ is *irreducible* and $\mathrm{rank}_{\mathbb{R}} G > 1$. Then G. Margulis' celebrated arithmeticity theorem says we may assume there exists a semisimple linear algebraic $\mathbb{Q}$-group $\mathbf{G}$ such that $G = \mathbf{G}^0(\mathbb{R})$ and such that Γ is *commensurable* with $\mathbf{G}(\mathbb{Z})$. We assemble certain nilmanifolds N_P and so-called *boundary symmetric spaces* $X_{\mathbf{P}} = M_{\mathbf{P}}/K_P$ to *boundary components* $e(\mathbf{P}) = N_P \times X_{\mathbf{P}}$ associated with the rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$. We define a topology on the *bordification* $\overline{X} = X \cup \bigcup_{\mathbf{P}} e(\mathbf{P})$ specifying which sequences in X will converge to points in which boundary components $e(\mathbf{P})$. The Γ-action on X extends freely to $\overline{X}$. The bordification $\overline{X}$ is still contractible but now has a compact quotient $\Gamma \backslash \overline{X}$ called the *Borel–Serre compactification* of the locally symmetric space $\Gamma \backslash X$. For not necessarily arithmetic torsion-free lattices in semisimple Lie groups with $\mathrm{rank}_{\mathbb{R}}(G) = 1$, H. Kang [56] has recently constructed a finite $B\Gamma$ by attaching nilmanifolds associated with real parabolic subgroups.

We will use these two types of compactifications to conclude information on Novikov–Shubin invariants and L^2-torsion of Γ. For the L^2-Betti numbers, however, the problem can more easily be reduced to the uniform case by the work of D. Gaboriau [40]. To state the result let us recall that the *deficiency* of G is given by $\delta(G) = \mathrm{rank}_{\mathbb{C}}(G) - \mathrm{rank}_{\mathbb{C}}(K)$ and that every symmetric space X of noncompact type has a dual symmetric space X^d of compact type. There is moreover a canonical choice of a Haar measure μ_X on G which gives $\mu_X(\Gamma \backslash G) = \mathrm{vol}(\Gamma \backslash X)$ for the induced G-invariant measure in case Γ is torsion-free.

Theorem 1.1. *Let G be a connected semisimple linear Lie group with symmetric space $X = G/K$ fixing the Haar measure μ_X. Then for each $p \geq 0$ there is a constant $B_p^{(2)}(X) \geq 0$ such that for every lattice $\Gamma \leq G$ we have*

$$b_p^{(2)}(\Gamma) = B_p^{(2)}(X)\,\mu_X(\Gamma\backslash G).$$

Moreover $B_p^{(2)}(X) = 0$ unless $\delta(G) = 0$ and $\dim X = 2p$, when $B_p^{(2)}(X) = \frac{\chi(X^d)}{\mathrm{vol}(X^d)}$.

As an example, let us consider the modular group $\mathrm{PSL}(2,\mathbb{Z})$. We obtain $B_1^{(2)}(\mathbb{H}^2) = \frac{1}{2\pi}$ because the dual of the hyperbolic plane is the 2-sphere. Integrating the volume form $\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}$ over the interior of the standard fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$ acting on the upper half-plane, we obtain $\mu_{\mathbb{H}^2}(\mathrm{PSL}(2,\mathbb{Z})\backslash\mathrm{PSL}(2,\mathbb{R})) = \frac{\pi}{3}$. Thus $b_1^{(2)}(\mathrm{PSL}(2,\mathbb{Z})) = \frac{1}{6}$. Note that generally $b_p^{(2)}(\Lambda) = [\Gamma : \Lambda]\, b_p^{(2)}(\Gamma)$ for finite index subgroups. This is interesting because $\mathrm{PSL}(2,\mathbb{Z})$ contains the free group F_2 on two letters. As $BF_2 = S^1 \vee S^1$, it is easy to see that $b_1^{(2)}(F_2) = 1$. So we conclude that every embedding $F_2 \to \mathrm{PSL}(2,\mathbb{Z})$ has either infinite index or index six. If one takes the isomorphism $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/3 * \mathbb{Z}/2$ for granted, this can also be shown with the help of Wall's rational Euler characteristic [107].

It remains to investigate Novikov–Shubin invariants and L^2-torsion. To the author's knowledge, the only results in this direction for nonuniform lattices have been obtained in the hyperbolic case. J. Lott and W. Lück give bounds for $\widetilde{\alpha}_p(\Gamma)$ if $G = \mathrm{SO}^0(3,1)$ [65] in the context of computing L^2-invariants of 3-manifolds. In a follow-up paper W. Lück and T. Schick [72] compute $\rho^{(2)}(\Gamma)$ for $G = \mathrm{SO}^0(2n+1,1)$ as follows.

Theorem 1.2. *There are certain nonzero numbers $T^{(2)}(\mathbb{H}^{2n+1})$ such that for every torsion-free lattice $\Gamma \subset \mathrm{SO}^0(2n+1,1)$ we have $\rho^{(2)}(\Gamma) = T^{(2)}(\mathbb{H}^{2n+1})\mathrm{vol}(\Gamma\backslash\mathbb{H}^{2n+1})$.*

The first constants $T^{(2)}(\mathbb{H}^{2n+1})$ for $n = 1,2,3$ are $-\frac{1}{6\pi}$, $\frac{31}{45\pi^2}$ and $-\frac{221}{70\pi^3}$. In the hyperbolic case the nilpotent Lie groups defining the boundary nilmanifolds are actually abelian so that the structure of Kang's compactification is quite transparent. The boundary is a finite disjoint union of flat manifolds which thus are finitely covered by tori. We check that the calculations of Lott–Lück for Novikov–Shubin invariants in the special case $G = \mathrm{SO}^0(3,1)$ hold more generally to give

Theorem 1.3. *Let Γ be a lattice in $\mathrm{SO}^0(2n+1,1)$. Then $\widetilde{\alpha}_n(\Gamma) \leq 2n$.*

For uniform $\Gamma \subset \mathrm{SO}^0(2n+1,1)$ J. Lott had computed $\widetilde{\alpha}_n(\Gamma) = \frac{1}{2}$ [63, Proposition 46]. It follows from the Cartan classification that the groups $G = \mathrm{SO}^0(2n+1,1)$ are up to finite coverings the only connected semisimple Lie groups without compact factors and with $\mathrm{rank}_{\mathbb{R}}(G) = 1$ that define a symmetric space of nonvanishing fundamental rank. So by Theorem 1.1 the remaining examples $\mathrm{SO}^0(2n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ and $F_{4(-20)}$ have lattices with nonvanishing middle L^2-Betti number. This prevents an easy generalization of Theorem 1.3 to give bounds on middle Novikov–Shubin invariants in these cases. We can however say something about Novikov–Shubin invariants right below the top dimension.

Theorem 1.4. *Let G be a connected semisimple linear Lie group of $\mathrm{rank}_{\mathbb{R}}(G) = 1$ with symmetric space $X = G/K$. Suppose that $n = \dim X \geq 3$. Let $P \subset G$ be a proper real parabolic subgroup. Then for every nonuniform lattice $\Gamma \subset G$*

$$\widetilde{\alpha}_{n-1}(\Gamma) \leq \tfrac{d(N_P)}{2}.$$

Here $d(N_P)$ denotes the degree of polynomial growth of the unipotent radical N_P of P. This theorem contrasts Olbrich's result that all uniform lattices have ∞^+ as Novikov–Shubin invariant in this high dimension. On a second thought this is maybe not so surprising because Novikov–Shubin invariants tend to be finite for infinite amenable groups. While no lattice $\Gamma \subset G$ is amenable, we have already mentioned that a torsion-free nonuniform lattice Γ has infinite unipotent subgroups which are geometrically reflected in the nilmanifolds at infinity of the symmetric space. These take their toll and bound Novikov–Shubin invariants. The L^2-torsion in turn is only defined for lattices acting on det-L^2-acyclic symmetric spaces X which according to Theorem 1.1 is equivalent to $\delta(G) > 0$. So Theorem 1.2 of Lück–Schick answers all the questions on L^2-torsion when $\operatorname{rank}_{\mathbb{R}}(G) = 1$.

Let us now assume that G is a connected semisimple linear Lie group without compact factors and with $\operatorname{rank}_{\mathbb{R}}(G) > 1$. Then one version of Margulis arithmeticity says that for every irreducible lattice $\Gamma \subset G$ there exists a connected semisimple linear algebraic $\mathbb{Q}$-group $\mathbf{G}$ such that Γ and $\mathbf{G}(\mathbb{Z})$ are abstractly commensurable (Corollary 4.4). Therefore [69, Theorem 3.7.1] says that Γ and all arithmetic subgroups of $\mathbf{G}(\mathbb{Q})$ have equal Novikov–Shubin invariants. Moreover G and $\mathbf{G}(\mathbb{R})$ define the same symmetric space X. So it remains to analyze the arithmetic case where the Borel–Serre bordification $\overline{X}$ is available. Let q be the *middle dimension* of X, so either $\dim X = 2q$ or $\dim X = 2q + 1$.

Theorem 1.5. *Let* $\mathbf{G}$ *be a connected semisimple linear algebraic* $\mathbb{Q}$*-group. Suppose that* $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$ *and* $\delta(\mathbf{G}(\mathbb{R})) > 0$*. Let* $\mathbf{P} \subset \mathbf{G}$ *be a proper rational parabolic subgroup. Then for every arithmetic subgroup* $\Gamma \subset \mathbf{G}(\mathbb{Q})$

$$\widetilde{\alpha}_q(\Gamma) \leq \delta(M_{\mathbf{P}}) + d(N_P).$$

The new phenomenon that occurs is that apart from the nilmanifolds N_P, boundary symmetric spaces $X_{\mathbf{P}} = M_{\mathbf{P}}/K_P$ show up in $\partial\overline{X}$ whenever $\operatorname{rank}_{\mathbb{R}}(\mathbf{G}) > \operatorname{rank}_{\mathbb{Q}}(\mathbf{G})$. Certain subgroups of Γ act cocompactly on $X_{\mathbf{P}}$ and N_P so that ultimately the theorem reduces to Olbrich's work in order to control the boundary symmetric space and to a theorem of M. Rumin [97] which gives bounds for the Novikov–Shubin invariants of graded nilpotent Lie groups.

In the most complicated case of arbitrary $\operatorname{rank}_{\mathbb{R}}(\mathbf{G}) \geq \operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) > 1$, the structure of ends is intriguing. In fact the boundary $\partial\overline{X}$ is connected and can be built up by $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) - 1$ consecutive pushouts attaching boundary components of increasing dimensions which result in a smooth manifold with corners. If $\delta(G) > 0$, it is possible to bound the middle Novikov–Shubin invariant of Γ by going over to the boundary, $\widetilde{\alpha}_q(\overline{X}) \leq \widetilde{\alpha}_q(\partial\overline{X})$. But Novikov–Shubin invariants only satisfy a very weak version of additivity with respect to pushouts so that it remains unclear if $\widetilde{\alpha}_q(\partial\overline{X})$ is finite. For the L^2-torsion, however, we are able to cover half of all cases.

Theorem 1.6. *Let* $\mathbf{G}$ *be a connected semisimple linear algebraic* $\mathbb{Q}$*-group. Suppose that* $\mathbf{G}(\mathbb{R})$ *has positive, even deficiency. Then every torsion-free arithmetic lattice* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *is* det-L^2*-acyclic and*

$$\rho^{(2)}(\Gamma) = 0.$$

Unlike Novikov–Shubin invariants, L^2-torsion behaves additively with respect to pushouts in the same way as the ordinary Euler characteristic does. The projection to $\Gamma\backslash\partial\overline{X}$ of the closures $\overline{e(\mathbf{P})}$ of boundary components in $\partial\overline{X}$ are total spaces of fiber bundles of manifolds with corners. We identify the basis with the Borel–Serre compactification of the boundary locally symmetric space $\Gamma_{M_{\mathbf{P}}}\backslash\overline{X}_{\mathbf{P}}$ for a certain induced lattice $\Gamma_{M_{\mathbf{P}}}$. The typical fiber is given by the closed nilmanifold $\Gamma \cap N_P\backslash N_P$. A theorem due to C. Wegner [108] says that the L^2-torsion of finite aspherical CW-complexes with infinite elementary amenable fundamental group vanishes. Using additivity and a product formula for fiber bundles, the nilfibers therefore finally effect that $\rho^{(2)}(\partial\overline{X})$ vanishes. This is sufficient for the conclusion of the theorem because $\dim X$ has the same parity as $\delta(\mathbf{G}(\mathbb{R}))$ and in even dimensions

$\rho^{(2)}(\partial\overline{X}) = 2\rho^{(2)}(\overline{X})$ as a consequence of Poincaré duality. Also note that by this equality Theorem 1.6 is trivial for uniform lattices.

L^2-torsion obeys a simpler product formula than Novikov-Shubin invariants do. Therefore we can get rid of the irreducibility assumption and invoke Margulis arithmeticity for a statement about all lattices in semisimple Lie groups with positive, even deficiency. To do so, let us say a group Γ is *virtually* det-L^2*-acyclic* if a finite index subgroup Γ' has a finite det-L^2-acyclic Γ'-CW model for $E\Gamma'$. In that case its *virtual* L^2*-torsion* is well-defined by setting $\rho^{(2)}_{\mathrm{virt}}(\Gamma) = \frac{\rho^{(2)}(\Gamma')}{[\Gamma:\Gamma']}$.

Theorem 1.7. *Let G be a connected semisimple linear Lie group with positive, even deficiency. Then every lattice $\Gamma \subset G$ is virtually* det-L^2*-acyclic and*

$$\rho^{(2)}_{\mathrm{virt}}(\Gamma) = 0.$$

For example $\rho^{(2)}_{\mathrm{virt}}(\mathrm{SL}(n,\mathbb{Z})) = 0$ if $n > 2$ and $n \equiv 1$ or $2 \mod 4$. In the case of odd deficiency in contrast, our methods break down completely. For one thing, the equation $\rho^{(2)}(\partial\overline{X}) = 2\rho^{(2)}(\overline{X})$ is no longer true. For another, Theorem 1.2, Olbrich's Theorem 3.19 and Conjecture 1.12 below suggest that we should expect nonzero L^2-torsion also for nonuniform lattices if $\delta(G) = 1$. But the corresponding nonzero constants $T^{(2)}(X)$ that occur in Theorem 3.19 seem to hint at an intimate connection of the L^2-torsion of Γ with the representation theory of G. So it seems unlikely to come up with those values by mere topological means.

The computation of L^2-invariants is a worthwhile challenge in itself. Yet we want to convince the reader that the problem is not isolated within the mathematical landscape. The following conjecture goes back to M. Gromov [44, p. 120]. We state it in a version that appears in [67, p. 437].

Conjecture 1.8 (Zero-in-the-spectrum Conjecture)**.** *Let M be a closed aspherical Riemannian manifold. Then there is $p \geq 0$ such that zero is in the spectrum of the minimal closure of the Laplacian*

$$(\Delta_p)_{\min}\colon \mathrm{dom}((\Delta_p)_{\min}) \subset L^2\Omega^p(\widetilde{M}) \to L^2\Omega^p(\widetilde{M})$$

acting on p-forms of the universal covering $\widetilde{M}$ with the induced metric.

The conjecture has gained interest due to its relevance for seemingly unrelated questions, see [64] for an expository article. For one example, the zero-in-the-spectrum conjecture for M with $\Gamma = \pi_1(M)$ is a consequence of the *strong Novikov conjecture* for Γ which in turn is contained in the *Baum–Connes conjecture* for Γ. Following the survey [67, Chapter 12], let us choose a Γ-triangulation X of $\widetilde{M}$. We define the homology $\mathcal{N}(\Gamma)$-module $H^{\Gamma}_p(X;\mathcal{N}(\Gamma)) = H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*(X))$ where we view the group von Neumann algebra $\mathcal{N}(\Gamma)$ as a discrete ring. Then the zero-in-the-spectrum conjecture has the equivalent algebraic version that for some $p \leq \dim M$ the homology $H^{\Gamma}_p(X;\mathcal{N}(\Gamma))$ does not vanish. L^2-invariants enter the picture in that for a general finite Γ-CW complex X we have $H^{\Gamma}_p(X;\mathcal{N}(\Gamma)) = 0$ for $p \geq 0$ if and only if $b^{(2)}_p(X) = 0$ and $\widetilde{\alpha}_p(X) = \infty^+$ for $p \geq 0$.

Therefore Olbrich's theorem implies that closed locally symmetric spaces $\Gamma\backslash X$ coming from uniform lattices satisfy the conjecture. The statement of the conjecture does not immediately include locally symmetric spaces $\Gamma\backslash X$ coming from nonuniform lattices because they are not compact. But since already the strong Novikov conjecture is known for large classes of groups, including Gromov hyperbolic groups, it should pay off to think about generalizing the formulation of the zero-in-the-spectrum conjecture. One such generalization would be to cross out the word "aspherical" in the statement of Conjecture 1.8 above. But then there are counterexample due to M. Farber and S. Weinberger [36]. Compare also [50]. So we should stick with aspherical spaces and try to relax the condition "closed manifold" instead. This gives a question that W. Lück has asked, see [67, p. 440].

Question 1.9. *If a group* Γ *has a finite CW-model for* $B\Gamma$*, is there* $p \geq 0$ *such that* $H_p^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$ *does not vanish?*

Now this question makes sense for nonuniform lattices, and as we said, L^2-Betti numbers and Novikov–Shubin invariants provide a way to answer it. In our case Theorem 1.4 gives number (i) and Theorems 1.1 and 1.5 give number (ii) of the following result.

Theorem 1.10. *The answer to Question* 1.9 *is affirmative for*

(i) torsion-free nonuniform lattices of connected semisimple linear Lie groups G *with* $\operatorname{rank}_{\mathbb{R}}(G) = 1$*,*

(ii) torsion-free arithmetic subgroups of connected semisimple linear algebraic $\mathbb{Q}$*-groups* $\mathbf{G}$ *with* $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$.

In a different direction, recall that two lattices Γ and Λ, uniform or not, in the same noncompact Lie group H give the prototype example of *measure equivalent* groups in the sense of M. Gromov. The group H together with the left and right actions $\Gamma \curvearrowright H \curvearrowleft \Lambda$ provides a *measure coupling*, meaning H endowed with Haar measure μ is an infinite Lebesgue space and the two actions are free, commute and both have finite measure fundamental domains X and Y. The ratio $\frac{\mu(X)}{\mu(Y)}$ is called the *index* of the measure coupling. It is explained in [38, p. 1061] that it follows from the work of R. J. Zimmer [111] that lattices in different higher rank simple Lie groups are not measure equivalent. A remarkable rigidity theorem due to A. Furman [38, Theorem 3.1] therefore says that the measure equivalence class of a lattice Γ in a higher rank simple Lie group G coincides up to finite groups with the set of all lattices in G. On the other hand, Furman explains how it follows from [90] that all countable amenable groups form one single measure equivalence class. Moreover he uses the measure coupling of two measure equivalent groups Γ and Λ to induce unitary Λ-representations to unitary Γ-representations, thereby showing that Kazhdan's Property (T) is a measure equivalence invariant [38, Corollary 1.4]. In this context, Furman proposes the problem of finding other measure equivalence invariants of groups, besides amenability and Property (T) [38, Open question 3, p. 1062]. Since such an invariant cannot distinguish amenable groups, one should probably consider invariants that have turned out to be useful in the "opposite" Property (T) world. In particular, typical quasi-isometry invariants like growth functions, cohomological dimension or Gromov hyperbolicity fail to be measure equivalence invariant.

In a far-reaching paper D. Gaboriau [40] has proven that the property of having a zero p-th L^2-Betti number is indeed a measure equivalence invariant. More precisely, he shows that if Γ and Λ have a measure coupling of index c, then $b_p^{(2)}(\Gamma) = c \cdot b_p^{(2)}(\Lambda)$. On the other hand, Novikov–Shubin invariants are not invariant under measure equivalence. This is immediate for amenable groups, for example $\widetilde{\alpha}_1(\mathbb{Z}^n) = \frac{n}{2}$. Beyond that, for $G = \mathrm{Sp}(n,1)$ and $G = F_{4(-20)}$ Theorem 1.4 gives Property (T) counterexamples, see [59, Remark 10]. These are also counterexamples to the relaxed version that for two measure equivalent groups Γ, Λ we had $\widetilde{\alpha}_p(\Gamma) = \infty^+ \Leftrightarrow \widetilde{\alpha}_p(\Lambda) = \infty^+$. The now obvious question for the L^2-torsion has already been asked by W. Lück and R. Sauer [67, Question 7.35, p. 313].

Question 1.11. *Let* Γ *and* Λ *be measure equivalent,* det-L^2*-acyclic groups. Is it true that* $\rho^{(2)}(\Gamma) = 0 \Leftrightarrow \rho^{(2)}(\Lambda) = 0$*?*

This question of course includes the question whether $\rho^{(2)}(\Gamma) = 0$ whenever Γ is amenable and has a finite $B\Gamma$. As mentioned, C. Wegner has verified this for elementary amenable groups. H. Li and A. Thom have very recently given the complete affirmative answer by identifying the L^2-torsion of Γ with the entropy of a certain algebraic action of Γ [62]. Meanwhile in view of Gaboriau's theorem and the similar behavior of L^2-Betti numbers and L^2-torsion, Question 1.11 has become the following more precise conjecture [71, Conjecture 1.2].

Conjecture 1.12 (Lück–Sauer–Wegner)**.** *Let* Γ *and* Λ *be* det-L^2*-acyclic groups. Assume that* Γ *and* Λ *are measure equivalent of index* c*. Then* $\rho^{(2)}(\Gamma) = c \cdot \rho^{(2)}(\Lambda)$*.*

In fact, Lück–Sauer–Wegner only assume the groups to be L^2-acyclic and make it part of the conclusion of the conjecture that they are of det $\geq$ 1-*class*, see Remark 3.7 (iii). They prove the conjecture if measure equivalence is replaced by the way more rigorous notion of *uniform measure equivalence* of groups. In case of finitely generated amenable groups for example, uniform measure equivalence classes and quasi-isometry classes agree [101, Lemma 2.25; 103, Theorem 2.1.7]. Regarding the original Conjecture 1.12, our Theorem 1.7 and the above discussion of the work of Zimmer and Furman translate as follows.

Theorem 1.13. *Let* $\mathcal{L}^{\text{even}}$ *be the class of* det-L^2*-acyclic groups that are measure equivalent to a lattice in a connected simple linear Lie group with even deficiency. Then Conjecture* 1.12 *holds true and Question* 1.11 *has affirmative answer for* $\mathcal{L}^{\text{even}}$*.*

Of course in fact $\rho^{(2)}(\Gamma) = 0$ for all $\Gamma \in \mathcal{L}^{\text{even}}$, which one might find unfortunate. On the other hand, $\mathcal{L}^{\text{even}}$ contains various complete measure equivalence classes of det-L^2-acyclic groups so that Theorem 1.13 certainly has substance. Gaboriau points out in [39, p. 1810] that apart from amenable groups and lattices in connected simple linear Lie groups of higher rank, no more measure equivalence classes of groups have completely been understood so far. The same reference gives a concise survey on further measure equivalence invariants of groups.

Among the open problems we will list, we find the odd deficiency case of Theorem 1.7 most exigent. A promising strategy seems to be a generalization of the methods in [72] where the asymptotic equality of the analytic L^2-torsion of a finite-volume hyperbolic manifold and the cellular L^2-torsion of a compact exhaustion is proved. Such a generalization will require analytic estimations of heat kernels and thus a detailed understanding of the asymptotic geometry of symmetric spaces. In particular a suitable coordinate system that allows one to make precise what "chopping off the ends" in the higher rank case should mean is desirable. This has led us to considerations about adapting Chevalley bases of complex semisimple Lie algebras to a given real structure. As the main result we construct a basis for every real semisimple Lie algebra such that the structure constants are (half-)integers which can be read off from the root system of the complexification together with the involution determining the real structure. One application gives coordinates for symmetric spaces in a uniform way. They single out maximal flat totally geodesic submanifolds and complementing nilmanifolds given by Iwasawa N-groups. The structure of the Iwasawa N-groups is likewise made explicit. These results are of independent interest and have appeared as a preprint in [55].

The outline of the remaining chapters is as follows. In Chapter 2 we give a detailed exposition on the Borel–Serre compactification widely following the modern approach in [15]. We include a brief survey on the similar Kang compactification designed for nonarithmetic lattices in rank one groups. Chapter 3 details the definitions and facts from [67] about L^2-invariants that are essential for our purposes. Chapter 4 forms the core of the thesis where the theorems as outlined in this introduction are proven. Chapter 5 concludes with the results on integral structures in real semisimple Lie algebras we mentioned lastly.

Acknowledgements

I gratefully acknowledge financial support from the Education Abroad Program of the University of California as well as from the German-American Fulbright Commission that allowed me to learn the necessary background material for this thesis during one year of graduate studies in the inspiring environment of UC Berkeley. The main research period undertaken at Göttingen University was generously funded by the German Research Foundation (DFG) within the project "L^2-invariants and quantum groups". I am indepted to my advisor Thomas Schick for constant assistance and encouragement and to my parents for support during years of difficult research.

CHAPTER 2

Borel–Serre compactification

In this chapter we introduce the Borel–Serre compactification of a locally symmetric space mostly following the modern treatment by A. Borel and L. Ji [15, Chapter III.9, p. 326]. The construction is built on the structure theory of rational parabolic subgroups of a reductive linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. We will present this theory incorporating methods of Harish-Chandra [46] in order to allow for disconnected groups $\mathbf{G}$. This enables us to recover the recursive character of the construction which is pronounced in the original treatment by A. Borel and J.-P. Serre [17].

The outline of sections is as follows. In Section 1 we recall basic notions of linear algebraic groups, their arithmetic subgroups and associated locally symmetric spaces. We recall a criterion to decide whether such a locally symmetric space is compact. Section 2 studies rational parabolic subgroups and their Langlands decompositions. These induce horospherical decompositions of the symmetric space. We classify rational parabolic subgroups up to conjugacy in terms of parabolic roots. The general sources for the background material in Sections 1 and 2 are [10], [11] and [15]. We will however give precise references whenever we feel the stated fact would not exactly be standard. Section 3 introduces and examines the bordification, a contractible manifold with corners which contains the symmetric space as an open dense set. In Section 4 we see that the group action extends cocompactly to the bordification. The compact quotient gives the desired Borel–Serre compactification. We will examine its constituents to some detail. Finally Section 5 gives a brief survey on Kang's compactification of locally symmetric spaces defined by lattices in rank one simple Lie groups. Throughout the presentation, all concepts will be illustrated in the example of the simplest symmetric space: the hyperbolic plane.

1. Algebraic groups and arithmetic subgroups

Let $\mathbf{G} \subset \mathrm{GL}(n,\mathbb{C})$ be a linear algebraic group defined over $\mathbb{Q}$. A Zariski-closed subgroup $\mathbf{T} \subset \mathbf{G}$ is called a *torus* of $\mathbf{G}$ if it is isomorphic to a product of copies of $\mathbb{C}^* = \mathrm{GL}(1,\mathbb{C})$. If $k = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, then $\mathbf{T}$ is called *k-split* if $\mathbf{T}$ and this isomorphism are defined over k. All maximal k-split tori of $\mathbf{G}^0$, the unit component, are conjugate by elements in $\mathbf{G}^0(k)$ and their common dimension is called the *k-rank* of $\mathbf{G}$. Clearly $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) \leq \mathrm{rank}_{\mathbb{R}}(\mathbf{G}) \leq \mathrm{rank}_{\mathbb{C}}(\mathbf{G})$. The group $\mathbf{G}$ is called *k-anisotropic* if $\mathrm{rank}_k(\mathbf{G}) = 0$. A *$k$-character* on $\mathbf{G}$ is a homomorphism $\mathbf{G} \to \mathbb{C}^*$ defined over k. The k-characters of $\mathbf{G}$ form an abelian group under multiplication which we denote by $X_k(\mathbf{G})$. The *radical* $\mathbf{R}(\mathbf{G})$ of $\mathbf{G}$ is the maximal connected normal solvable subgroup of $\mathbf{G}$. Similarly, the *unipotent radical* $\mathbf{R}_u(\mathbf{G})$ of $\mathbf{G}$ is the maximal connected normal unipotent subgroup of $\mathbf{G}$. As $\mathbf{G}$ is defined over $\mathbb{Q}$, so are $\mathbf{R}(\mathbf{G})$ and $\mathbf{R}_u(\mathbf{G})$. The group $\mathbf{G}$ is called *reductive* if $\mathbf{R}_u(\mathbf{G})$ is trivial and *semisimple* if $\mathbf{R}(\mathbf{G})$ is trivial. Any reductive k-subgroup of a general k-group $\mathbf{G}$ is contained in a maximal reductive k-subgroup. The maximal reductive k-subgroups are called *Levi k-subgroups*. They are conjugate under $\mathbf{R}_u(\mathbf{G})(k)$ [17, Section 0.4, p. 440]. The k-group $\mathbf{G}$ is the semidirect product of any Levi k-subgroup $\mathbf{L}$ by the unipotent radical, $\mathbf{G} = \mathbf{R}_u(\mathbf{G}) \rtimes \mathbf{L}$.

From now on we will assume that the linear algebraic $\mathbb{Q}$-group $\mathbf{G}$ is reductive and that it satisfies the following two conditions.

(I) We have $\chi^2 = 1$ for all $\chi \in X_{\mathbb{Q}}(\mathbf{G})$.
(II) The centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{T})$ of each maximal $\mathbb{Q}$-split torus $\mathbf{T} \subset \mathbf{G}$ meets every connected component of $\mathbf{G}$.

This class of groups appears in [46, p.1]. Condition (I) implies that $X_{\mathbb{Q}}(\mathbf{G}^0)$ is trivial. Thus $\mathbf{G}$ has $\mathbb{Q}$-anisotropic center. Note that the structure theory of reductive algebraic groups is usually derived for connected groups, see for example [11, Chapter IV]. But if one tries to enforce condition (I) for a connected reductive $\mathbb{Q}$-group $\mathbf{H}$ by going over to $\bigcap_{\chi \in X_{\mathbb{Q}}(\mathbf{H})} \ker \chi^2$, the resulting group will generally be disconnected. That is why we impose the weaker condition (II) which will turn out to be good enough for our purposes.

The group $\mathbf{G} \subset \mathrm{GL}(n, \mathbb{C})$ is an affine variety in $M(n+1, \mathbb{C}) \cong \mathbb{C}^{(n+1)^2}$ by means of the embedding $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{smallmatrix}\right)$. The *integer points* $\mathbf{G}(\mathbb{Z})$ given by the intersection $\mathbf{G} \cap M(n+1, \mathbb{Z})$ form a subgroup of $\mathbf{G}$. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called *arithmetic* if it is *commensurable* with $\mathbf{G}(\mathbb{Z})$. This means $\Gamma \cap \mathbf{G}(\mathbb{Z})$ has finite index both in Γ and in $\mathbf{G}(\mathbb{Z})$. If $\varphi\colon \mathbf{G} \to \mathbf{G}'$ is a $\mathbb{Q}$-isomorphism, then $\mathbf{G}'(\mathbb{Z})$ is commensurable with $\varphi(\mathbf{G}(\mathbb{Z}))$ [93, Proposition 4.1, p. 171]. It follows that the set of arithmetic subgroups of $\mathbf{G}$ is closed under conjugation with elements in $\mathbf{G}(\mathbb{Q})$.

The real points $G = \mathbf{G}(\mathbb{R})$ form a reductive Lie group with finitely many connected components [11, Section 24.6(c)(i), p. 276]. Due to condition (I), an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a *lattice* in G, which means the quotient space G/Γ has finite G-invariant measure. This is a deep result of A. Borel and Harish-Chandra [13, Theorem 9.4, p. 522] that generalizes classical reduction theories of quadratic forms to the setting of general arithmetic groups. Selberg's Lemma [2] says that Γ has torsion-free subgroups of finite index. We want to assume that Γ is torsion-free to begin with. This ensures that Γ acts freely and properly from the left on the *symmetric space* $X = G/K$ where K is a maximal compact subgroup of G. Corresponding to K there is a *Cartan involution* θ_K on G which extends to an algebraic involution of $\mathbf{G}$ [17, Definition 1.7, p. 444]. If G is semisimple, θ_K is the usual Cartan-involution. The symmetric space X is connected because K meets every connected component of G. In general, it is the product of a symmetric space of noncompact type and a Euclidean factor. The quotient $\Gamma\backslash X = \Gamma\backslash G/K$ is called a *locally symmetric space*. The locally symmetric space $\Gamma\backslash X$ is a connected finite-volume Riemannian manifold and in fact a *classifying space* for Γ because its universal covering X is contractible. The question under which further conditions on $\mathbf{G}$ the quotient $\Gamma\backslash G$ or equivalently the locally symmetric space $\Gamma\backslash X$ is actually compact has also been settled in the work of Borel and Harish-Chandra [13, Theorem 11.8, p. 529]. An alternative proof with different methods has independently been given by G. D. Mostow and T. Tamagawa [80, p. 452]. For the third part see [7, Theorem 5.4(b), p. 43].

Proposition 2.1. *The following are equivalent.*

(i) The locally symmetric space $\Gamma\backslash X$ *is compact.*
(ii) No nontrivial element in $\mathbf{G}(\mathbb{Q})$ *is unipotent.*
(iii) The group $\mathbf{G}$ *is* $\mathbb{Q}$*-anisotropic.*

If $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) > 0$, then the Borel–Serre compactification $\Gamma\backslash\overline{X}$ will be a manifold with "corners" that contains $\Gamma\backslash X$ as an open dense subset. The maximal codimension of the corners is given by $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G})$. In this sense the $\mathbb{Q}$-rank of $\mathbf{G}$ measures how intricate the structure of $\Gamma\backslash X$ at infinity is. A high $\mathbb{Q}$-rank allows for a rich combinatorial structure of *rational parabolic subgroups* of $\mathbf{G}$ which are crucial for understanding the structure of $\Gamma\backslash X$ at infinity as we will see next.

2. Rational parabolic subgroups

If $\mathbf{G}$ is connected, a closed $\mathbb{Q}$-subgroup $\mathbf{P} \subset \mathbf{G}$ is called a *rational parabolic subgroup* if $\mathbf{G}/\mathbf{P}$ is a complete (equivalently projective) variety. If $\mathbf{G}$ is not connected, we say that a closed $\mathbb{Q}$-subgroup $\mathbf{P} \subset \mathbf{G}$ is a *rational parabolic subgroup* if it is the normalizer of a rational parabolic subgroup of $\mathbf{G}^0$. These definitions are compatible because rational parabolic subgroups of connected groups are self-normalizing. It is clear that $\mathbf{P}^0 = \mathbf{P} \cap \mathbf{G}^0$, and condition (II) on $\mathbf{G}$ ensures that $\mathbf{P}$ meets every connected component of $\mathbf{G}$ [46, Lemma 1, p. 2], so $\mathbf{G}/\mathbf{P}$ is complete.

Given a rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ we set $\mathbf{N_P} = \mathbf{R}_u(\mathbf{P})$ and we denote by $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$ the *Levi quotient* of $\mathbf{P}$. Let $\mathbf{S_P} \subset \mathbf{L_P}$ be the maximal central $\mathbb{Q}$-split torus and set $\mathbf{M_P} = \bigcap_{\chi \in X_{\mathbb{Q}}(\mathbf{L_P})} \ker \chi^2$. The $\mathbb{Q}$-group $\mathbf{M_P}$ is reductive and satisfies conditions (I) and (II). It complements $\mathbf{S_P}$ as an *almost direct product* in $\mathbf{L_P}$ [46, p. 3]. This means $\mathbf{L_P} = \mathbf{S_P M_P}$ and $\mathbf{S_P} \cap \mathbf{M_P}$ is finite. For the groups of real points $L_\mathbf{P} = \mathbf{L_P}(\mathbb{R})$, $A_\mathbf{P} = \mathbf{S_P}(\mathbb{R})^0$ and $M_\mathbf{P} = \mathbf{M_P}(\mathbb{R})$ the situation is even better behaved. One can verify that $L_\mathbf{P} = A_\mathbf{P} M_\mathbf{P}$ but now the finite group $A_\mathbf{P} \cap M_\mathbf{P}$ is actually trivial because $A_\mathbf{P}$ is torsion-free. Since both $A_\mathbf{P}$ and $M_\mathbf{P}$ are normal, the product is direct. We would like to lift these decompositions to some Levi k-subgroup of $\mathbf{P}$. The following result due to A. Borel and J.-P. Serre asserts that the maximal compact subgroup $K \subset G$ singles out a canonical choice for doing so [17, Proposition 1.8, p. 444]. The caveat is that $k = \mathbb{Q}$ needs to be relaxed to $k = \mathbb{R}$. We view $x_0 = K$ as a base point in the symmetric space X.

Proposition 2.2. *Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup and let $K \subset G$ be maximal compact. Then $\mathbf{P}$ contains one and only one $\mathbb{R}$-Levi subgroup $\mathbf{L}_{\mathbf{P},x_0}$ which is stable under θ_K.*

We remark that for a given $\mathbf{P}$, the maximal compact subgroup K which is identified with the base point $x_0 = K$ in X can always be chosen such that $\mathbf{L}_{\mathbf{P},x_0}$ is a $\mathbb{Q}$-group. In fact, $\mathbf{L}_{\mathbf{Q},x_0}$ is then a $\mathbb{Q}$-group for all parabolic subgroups $\mathbf{Q} \subset \mathbf{G}$ that contain $\mathbf{P}$. This follows from the proof of [15, Proposition III.1.11, p. 273]. In this case we will say that x_0 is a *rational base point* for $\mathbf{P}$. In general however, there is no universal base point x_0 such that the θ_K-stable Levi subgroups of all rational parabolic subgroups would be defined over $\mathbb{Q}$ [42, Section 3.9, p. 151].

The canonical projection $\pi\colon \mathbf{L}_{\mathbf{P},x_0} \to \mathbf{L_P}$ is an $\mathbb{R}$-isomorphism. The groups $\mathbf{S_P}$ and $\mathbf{M_P}$ lift under π to the $\mathbb{R}$-subgroups $\mathbf{S}_{\mathbf{P},x_0}$ and $\mathbf{M}_{\mathbf{P},x_0}$ of $\mathbf{P}$. The rational parabolic subgroup $\mathbf{P}$ thus has the decomposition

$$\mathbf{P} = \mathbf{N_P} \mathbf{S}_{\mathbf{P},x_o} \mathbf{M}_{\mathbf{P},x_0} \cong \mathbf{N_P} \rtimes (\mathbf{S}_{\mathbf{P},x_0} \mathbf{M}_{\mathbf{P},x_0}) \tag{2.3}$$

where $\mathbf{L}_{\mathbf{P},x_0} = \mathbf{S}_{\mathbf{P},x_0} \mathbf{M}_{\mathbf{P},x_0}$ is an almost direct product. Similarly the Lie groups $L_\mathbf{P}$, $A_\mathbf{P}$ and $M_\mathbf{P}$ lift to the Lie subgroups $L_{\mathbf{P},x_0}$, $A_{\mathbf{P},x_0}$ and $M_{\mathbf{P},x_0}$ of the *cuspidal group* $P = \mathbf{P}(\mathbb{R})$.

Definition 2.4. The point $x_0 \in X$ yields the *rational Langlands decomposition*

$$P = N_P A_{\mathbf{P},x_0} M_{\mathbf{P},x_0} \cong N_P \rtimes (A_{\mathbf{P},x_0} \times M_{\mathbf{P},x_0}).$$

We intentionally used a non-bold face index for $N_P = \mathbf{N_P}(\mathbb{R})$ because N_P coincides with the unipotent radical of the linear Lie group P. The number s-rank$(\mathbf{P}) = \dim_\mathbb{R} A_{\mathbf{P},x_0}$ is called the *split rank* of $\mathbf{P}$ [53, p. 445]. Let $K_P = P \cap K$ and $K'_P = \pi(K_P)$. Inspecting [17, Proposition 1.8, p. 444] we see that $K_P \subset L_{\mathbf{P},x_0}$ so $K'_P \subset L_\mathbf{P}$. Since K'_P is compact, we have $\chi(K'_P) \subset \{\pm 1\}$ for each $\chi \in X_\mathbb{Q}(\mathbf{L_P})$ so that actually $K'_P \subset M_\mathbf{P}$ and thus $K_P \subset M_{\mathbf{P},x_0}$. Moreover $G = PK$ so that P acts transitively on the symmetric space $X = G/K$.

Definition 2.5. The map $(n, a, mK_P) \mapsto namK$ is a real analytic diffeomorphism

$$N_P \times A_{\mathbf{P},x_0} \times X_{\mathbf{P},x_0} \cong X$$

of manifolds called the *rational horospherical decomposition* of X with respect to $\mathbf{P}$ and x_0 and with *boundary symmetric space* $X_{\mathbf{P},x_0} = M_{\mathbf{P},x_0}/K_P$.

Note that $K_P \subset M_{\mathbf{P},x_0}$ is maximal compact as it is even so in P [17, Proposition 1.5, p. 442]. Write an element $p \in P$ according to the rational Langlands decomposition as $p = nam$ and write a point $x_1 \in X$ according to the rational horospherical decomposition as $x_1 = (n_1, a_1, m_1K_P)$. Then we see that the left-action of P on X is given by

$$nam.(n_1, a_1, m_1K_P) = (n\ {}^{am}n_1, aa_1, mm_1K_P),$$

where we adopt the convention to write ${}^hg$ for the conjugation hgh^{-1}.

Example 2.6. Let $\mathbf{G} = \mathbf{SL}(2,\mathbb{C})$. The diagonal subgroup $\mathbf{S} = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) : a \in \mathbb{C}^*\right\}$ is an example of a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$ so that $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$. The group $\mathbf{P} = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right) : a \in \mathbb{C}^*, b \in \mathbb{C}\right\}$ is a both minimal and maximal rational parabolic subgroup. Its unipotent radical is $\mathbf{N_P} = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right) : b \in \mathbb{C}\}$. The subgroup $K = \mathrm{SO}(2,\mathbb{R})$ of $G = \mathrm{SL}(2,\mathbb{R})$ is maximal compact. It provides a rational base point $x_0 = K$ for $\mathbf{P}$ so that we can identify $\mathbf{L_P} = \mathbf{S_P} \cong \mathbf{L}_{\mathbf{P},x_0} = \mathbf{S}_{\mathbf{P},x_0}$ from the start. The $\mathbb{Q}$-character group of $\mathbf{L_P}$ is given by $X_{\mathbb{Q}}(\mathbf{L_P}) = \{\chi^k : k \in \mathbb{Z}\}$, where χ sends $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) \in \mathbf{S}$ to a. Thus $\mathbf{M}_{\mathbf{P},x_0} = \{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)\}$. We obtain the rational Langlands decomposition of P with respect to x_0

$$P \cong N_P \rtimes (A_{\mathbf{P},x_0} \times M_{\mathbf{P},x_0})$$

with $N_P = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1\end{smallmatrix}\right) : b \in \mathbb{R}\}$, $A_{\mathbf{P},x_0} = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) : a \in \mathbb{R}_{>0}\right\}$ and $M_{\mathbf{P},x_0} = \{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)\}$. As $K_P = M_{\mathbf{P},x_0}$, the boundary symmetric space $X_{\mathbf{P},x_0}$ is a point. The rational horospherical decomposition of $X = G/K$ with respect to $\mathbf{P}$ and x_0 reduces to

$$X \cong N_P \times A_{\mathbf{P},x_0}.$$

Since $G = \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SO}^0(2,1)$, the symmetric space X can be identified with the hyperbolic plane. In the upper half-plane model, $\{1\} \times A_{\mathbf{P},x_0}$ is the imaginary coordinate axis whose N_P-translates are geodesic vertical lines that connect points from the boundary line $\mathbb{R}$ to the point at infinity. Accordingly $N_P \times \{1\}$ and its $A_{\mathbf{P},x_0}$-translates are the horizontal lines which are *horocycles* or one-dimensional *horospheres* that join at the point at infinity. This explains the terminology. For the opposite parabolic subgroup $\mathbf{P}^- = \left\{\left(\begin{smallmatrix} a & 0 \\ b & a^{-1}\end{smallmatrix}\right) : a \in \mathbb{C}^*, b \in \mathbb{C}\right\}$ the one-dimensional subspace $\{1\} \times A_{\mathbf{P}^-,x_0}$ of X is again the imaginary axis but now its N_{P^-}-translates are geodesic half-circles with both ends in the boundary line $\mathbb{R}$, one end being the origin. Accordingly $N_{P^-} \times \{1\}$ and its $A_{\mathbf{P}^-,x_0}$-translates are pinched circles tangent to the boundary line $\mathbb{R}$ at zero, see Figure 2.7. A generic rational parabolic subgroup $\mathbf{P}$ has a nonzero boundary point in $\mathbb{R}$ as limit point for the geodesics and tangent point for the horospheres in the corresponding horospherical decomposition of X. In this sense one might want to say that a rational parabolic subgroup singles out a "direction to infinity" in the symmetric space X.

The horospherical decomposition realizes the symmetric space X as the product of a nilmanifold, a flat manifold and yet another symmetric space $X_{\mathbf{P},x_0}$. The isomorphism π identifies the latter one with the symmetric space $X_{\mathbf{P}} = M_{\mathbf{P}}/K'_P$. It is the symmetric space of the reductive $\mathbb{Q}$-group $\mathbf{M_P}$ which meets conditions (I) and (II). The group $\mathbf{M_P}$ inherits the arithmetic lattice $\Gamma'_{M_{\mathbf{P}}}$ which is the image of $\Gamma_P = \Gamma \cap \mathcal{N}_G(P)$ under the projection $P \to P/N_P \cong L_P$. Here we have $\Gamma'_{M_{\mathbf{P}}} \subset M_{\mathbf{P}}$ because $\chi(\Gamma'_{M_{\mathbf{P}}}) \subset \{\pm 1\}$ for all $\chi \in X_{\mathbb{Q}}(\mathbf{L_P})$ as $\chi(\Gamma'_{M_{\mathbf{P}}}) \subset \mathbf{GL}(1,\mathbb{Q})$ is arithmetic. In general $\Gamma'_{M_{\mathbf{P}}}$ might have torsion elements. But there is a condition on Γ that ensures it does not.

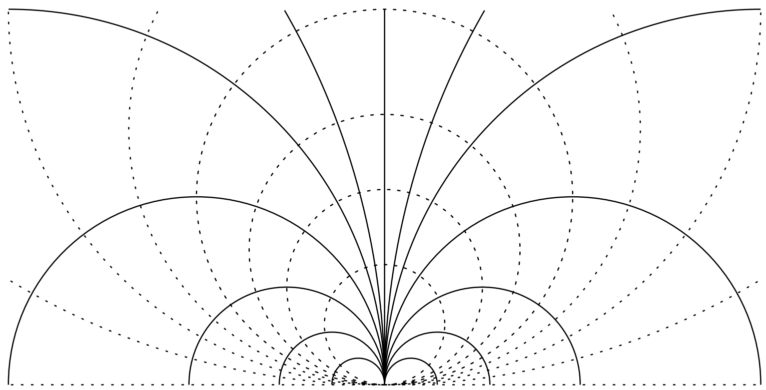


FIGURE 2.7. Horospherical decomposition of the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. Horospheres and geodesics intersect orthogonally.

Definition 2.8. A matrix $g \in \mathrm{GL}(n,\mathbb{Q})$ is called *neat* if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of g is torsion-free. A subgroup of $\mathrm{GL}(n,\mathbb{Q})$ is called neat if all of its elements are neat.

The notion of neatness is due to J.-P. Serre. It appears first in [10, Section 17.1, p. 117]. A neat subgroup is obviously torsion-free. Every arithmetic subgroup of a linear algebraic $\mathbb{Q}$-group has a neat subgroup of finite index [10, Proposition 17.4, p. 118] and neatness is preserved under morphisms of linear algebraic groups [10, Corollaire 17.3, p. 118]. Therefore $\Gamma'_{M_{\mathbf{P}}}$ is neat if Γ is, and in that case $\Gamma'_{M_{\mathbf{P}}}$ acts freely and properly on the boundary symmetric space $X_{\mathbf{P}}$. We observe that $\mathrm{rank}_{\mathbb{Q}}(\mathbf{M_P}) = \mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) - \dim A_{\mathbf{P}}$. In this sense the locally symmetric space $\Gamma'_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$ is closer to being compact than the original $\Gamma\backslash X$. This is a key observation for the construction of the Borel–Serre compactification. If in particular $\mathbf{P}$ is a *minimal* rational parabolic subgroup, then $\mathbf{S}_{\mathbf{P},x_0} \subset \mathbf{P}$ is G-conjugate to a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$ so that $\mathrm{rank}_{\mathbb{Q}}(\mathbf{M_P}) = 0$ and thus $\Gamma'_{M_{\mathbf{P}}}\backslash X_{\mathbf{P}}$ is compact by Proposition 2.1.

Now the group $\mathbf{M_P}$ has itself rational parabolic subgroups $\mathbf{Q}'$ whose cuspidal subgroups Q' have a Langlands decomposition $Q' = N_{Q'} A_{\mathbf{Q}',x_0'} M_{\mathbf{Q}',x_0'}$ with respect to the base point $x_0' = K_P'$. The isomorphism π identifies those groups as subgroups of $M_{\mathbf{P},x_0}$. We set $N_Q^* = N_P N_{Q'} \cong N_P \rtimes N_{Q'}$, $A^*_{\mathbf{Q},x_0} = A_{\mathbf{P},x_0} A_{\mathbf{Q}',x_0'} = A_{\mathbf{P},x_o} \rtimes A_{\mathbf{Q}',x_0'}$ and $M^*_{\mathbf{Q},x_0} = M_{\mathbf{Q}',x_0'}$. Then we define $Q^* = N_Q^* A^*_{\mathbf{Q},x_0} M^*_{\mathbf{Q},x_0}$. The group Q^* is the cuspidal group of a rational parabolic subgroup $\mathbf{Q}^*$ of $\mathbf{G}$ such that $\mathbf{Q}^* \subset \mathbf{P}$. Equivalently, $\mathbf{Q}^*$ is a rational parabolic subgroup of $\mathbf{P}$. The Langlands decomposition of Q^* with respect to x_0 is the decomposition given in its construction.

Lemma 2.9. *The map $\mathbf{Q}' \mapsto \mathbf{Q}^*$ gives a bijection of the set of rational parabolic subgroups of $\mathbf{M_P}$ to the set of rational parabolic subgroups of $\mathbf{G}$ contained in $\mathbf{P}$.*

This is [46, Lemma 2, p. 4]. We use the inverse of this correspondence to conclude that for every rational parabolic subgroup $\mathbf{Q} = \mathbf{Q}^* \subset \mathbf{P}$ we obtain a *rational horospherical decomposition of the boundary symmetric space*

$$X_{\mathbf{P},x_0} \cong X_{\mathbf{P}} \cong N_{Q'} \times A_{\mathbf{Q}',x_0'} \times X_{\mathbf{Q}',x_0'}. \tag{2.10}$$

In the case $\mathbf{P} = \mathbf{G}$ condition (I) gives $\mathbf{M}_{\mathbf{G},x_0} = \mathbf{G}$ so that we get back the original rational horospherical decomposition of Definition 2.5.

In the rest of this section we will recall the classification of rational parabolic subgroups of $\mathbf{G}$ up to conjugation in $\mathbf{G}(\mathbb{Q})$ in terms of parabolic roots. The reference for this material is [46, Chapter 1, pp. 3–4]. Still let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup and let $x_0 = K$ be a base point. Let $\mathfrak{g}^0$, $\mathfrak{p}$, $\mathfrak{n}_P$, $\mathfrak{a}_{\mathbf{P},x_0}$ and $\mathfrak{m}_{\mathbf{P},x_0}$ be the Lie algebras of the Lie groups G, P, N_P, $A_{\mathbf{P},x_0}$ and $M_{\mathbf{P},x_0}$. From the viewpoint of algebraic groups, these Lie algebras are given by $\mathbb{R}$-linear left-invariant derivations of the field of rational functions defined over $\mathbb{R}$ on the unit components of $\mathbf{G}$, $\mathbf{P}$, $\mathbf{N_P}$, $\mathbf{S}_{\mathbf{P},x_0}$ and $\mathbf{M}_{\mathbf{P},x_0}$, respectively. A linear functional α on $\mathfrak{a}_{\mathbf{P},x_0}$ is called a *parabolic root* if the subspace

$$\mathfrak{n}_{P,\alpha} = \{n \in \mathfrak{n}_P : \operatorname{ad}(a)(n) = \alpha(a)n \text{ for all } a \in \mathfrak{a}_{\mathbf{P},x_0}\}$$

of $\mathfrak{n}_P$ is nonzero. We denote the set of all parabolic roots by $\Phi(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$. If $l = \dim \mathfrak{a}_{\mathbf{P},x_0}$, there is a unique subset $\Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0}) \subset \Phi(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$ of l *simple parabolic roots* such that every parabolic root is a unique linear combination of simple ones with nonnegative integer coefficients. The group $A_{\mathbf{P},x_0}$ is exponential so that $\exp\colon \mathfrak{a}_{\mathbf{P},x_0} \to A_{\mathbf{P},x_0}$ is a diffeomorphism with inverse "log". Therefore we can evaluate a parabolic root $\alpha \in \Phi(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$ on elements $a \in A_{\mathbf{P},x_0}$ setting $a^\alpha = \exp(\alpha(\log a))$ where now "exp" is the ordinary real exponential function.

The subsets of $\Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$ classify the rational parabolic subgroups of $\mathbf{G}$ that contain $\mathbf{P}$ as we will now explain. Let $I \subset \Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$ be a subset and let $\Phi_I \subset \Phi(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$ be the set of all parabolic roots that are linear combinations of simple roots in I. Set $\mathfrak{a}_I = \bigcap_{\alpha\in I} \ker \alpha$ and $\mathfrak{n}_I = \bigoplus_{\alpha\in\Sigma} \mathfrak{n}_{P,\alpha}$ where $\Sigma = \Sigma(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$ denotes the set of all parabolic roots which do not lie in Φ_I. Consider the sum $\mathfrak{p}_I = \mathfrak{n}_I \oplus \mathfrak{z}(\mathfrak{a}_I)$ of $\mathfrak{n}_I$ and the centralizer of $\mathfrak{a}_I$ in $\mathfrak{g}^0$. Let $P_I = \mathcal{N}_G(\mathfrak{p}_I)$ be the normalizer of $\mathfrak{p}_I$ in G. If $x_1 \in X$ is a different base point, then $x_1 = p.x_0$ for some $p \in P$ and $\mathfrak{a}_{\mathbf{P},x_1} = {}^p\mathfrak{a}_{\mathbf{P},x_0}$ as well as $\mathfrak{n}_{(I^p)} = {}^p\mathfrak{n}_I$. It follows that the group P_I, thus its Zariski closure $\mathbf{P}_I$, is independent of the choice of base point. Since rational base points exist for $\mathbf{P}$, the Lie algebra of $\mathbf{P}_I$, which as a variety is given by $\mathbb{C}$-linear left-invariant derivations of the field of rational functions on $\mathbf{P}_I^0$, is defined over $\mathbb{Q}$. It follows that $\mathbf{P}_I$ is a $\mathbb{Q}$-group [46, p. 1]. In fact, $\mathbf{P}_I$ is a rational parabolic subgroup of $\mathbf{G}$ with cuspidal group P_I. Let N_I and A_I be the Lie subgroups of P_I with Lie algebras $\mathfrak{n}_I$ and $\mathfrak{a}_I$. Then $N_I \subset P_I$ is the unipotent radical and $A_I = \mathbf{S}_{\mathbf{P}_I,x_0}(\mathbb{R})^0$. The parabolic roots $\Phi(\mathfrak{p}_I, \mathfrak{a}_I)$ are the restrictions of $\Sigma(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$ to $\mathfrak{a}_I$ where simple parabolic roots restrict to the simple ones $\Delta(\mathfrak{p}_I, \mathfrak{a}_I)$ of $\mathfrak{p}_I$.

Every rational parabolic subgroup of $\mathbf{G}$ that contains $\mathbf{P}$ is of the form $\mathbf{P}_I$ for a unique $I \subset \Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P},x_0})$. The two extreme cases are $\mathbf{P}_\emptyset = \mathbf{P}$ and $\mathbf{P}_{\Delta(\mathfrak{p},\mathfrak{a}_{\mathbf{P},x_0})} = \mathbf{G}$. If $\mathbf{P}$ is minimal, the groups $\mathbf{P}_I$ form a choice of so called *standard rational parabolic subgroups.* Every rational parabolic subgroup of $\mathbf{G}$ is $\mathbf{G}(\mathbb{Q})$-conjugate to a unique standard one. Whence there are only finitely many rational parabolic subgroups up to conjugation in $\mathbf{G}(\mathbb{Q})$. There are even only finitely many when we restrict ourselves to conjugating by elements of an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$. This is clear from the following result of A. Borel [46, p. 5].

Proposition 2.11. *Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Then the set $\Gamma\backslash \mathbf{G}(\mathbb{Q}) / \mathbf{P}(\mathbb{Q})$ is finite.*

3. Bordification

From now on we drop x_0 and x_0' from our notation. The resulting notational collisions $A_{\mathbf{P}} = A_{\mathbf{P},x_0}$, $M_{\mathbf{P}} = M_{\mathbf{P},x_o}$ and $X_{\mathbf{P}} = X_{\mathbf{P},x_0}$ regarding Levi quotients and Levi subgroups are justified by Proposition 2.2 and the discussion throughout the preceding section. We will use the symbol "$\dot\bigcup$" for general disjoint unions in topological spaces, whereas the symbol "$\coprod$" is reserved for the true categorical coproduct.

Let $\mathbf{P} \subset \mathbf{G}$ be a rational parabolic subgroup. It determines the rational horospherical decomposition $X = N_P \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ of Definition 2.5. Define the *boundary component* of $\mathbf{P}$ by $e(\mathbf{P}) = N_P \times X_{\mathbf{P}}$. Then as a set, the *Borel–Serre bordification* $\overline{X}$ of the symmetric space X is given by the countable disjoint union

$$\overline{X} = \coprod_{\mathbf{P} \subset \mathbf{G}} e(\mathbf{P})$$

of all boundary components of rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$. This includes the symmetric space $X = e(\mathbf{G})$. In order to topologize the set $\overline{X}$ we introduce different coordinates on $e(\mathbf{P})$ for every parabolic subgroup $\mathbf{Q} \subset \mathbf{P}$. We do so by writing the second factor in $e(\mathbf{P}) = N_P \times X_{\mathbf{P}}$ according to the rational horospherical decomposition of the boundary symmetric space $X_{\mathbf{P}} = N_{Q'} \times A_{\mathbf{Q}'} \times X_{\mathbf{Q}'}$ given in (2.10). From the preparation of Lemma 2.9 we have $N_Q = N_P N_{Q'}$ and $M_Q = M_{Q'}$ so that we are left with

$$e(\mathbf{P}) = N_Q \times A_{\mathbf{Q}'} \times X_{\mathbf{Q}}. \tag{2.12}$$

The closed sets of $\overline{X}$ are now determined by the following *convergence class of sequences* [15, I.8.9–I.8.13, p. 113].

A sequence (x_i) of points in $e(\mathbf{P})$ converges to a point $x \in e(\mathbf{Q})$ if $\mathbf{Q} \subset \mathbf{P}$ and if for the coordinates $x_i = (n_i, a_i, y_i)$ of (2.12) and $x = (n, y)$ of $e(\mathbf{Q}) = N_Q \times X_{\mathbf{Q}}$ the following three conditions hold true.

(i) $a_i^\alpha \to +\infty$ for each $\alpha \in \Phi(\mathfrak{q}', \mathfrak{a}_{\mathbf{Q}'})$,
(ii) $n_i \to n$ within N_Q,
(iii) $y_i \to y$ within $X_{\mathbf{Q}}$.

A general sequence (x_i) of points in $\overline{X}$ converges to a point $x \in e(\mathbf{Q})$ if for each $\mathbf{P} \subset \mathbf{G}$ every infinite subsequence of (x_i) within $e(\mathbf{P})$ converges to x.

Note that in the case $\mathbf{Q} = \mathbf{P}$ the set $\Phi(\mathfrak{q}', \mathfrak{a}_{\mathbf{Q}'})$ is empty so that condition (i) is vacuous. We therefore obtain the convergence of the natural topology of $e(\mathbf{P})$. In particular, the case $\mathbf{Q} = \mathbf{P} = \mathbf{G}$ gives back the natural topology of X. It is clear that we obtain the same set $\overline{X}$ with the same class of sequences if we go over from $\mathbf{G}$ to $\mathbf{G}^0$. We thus may cite [15, Section III.9.2, p. 328] where it is stated that this class of sequences does indeed form a convergence class of sequences. This defines the topology of $\overline{X}$.

Example 2.13. As in Example 2.6, let $\mathbf{G} = \mathbf{SL}(2, \mathbb{C})$. We have identified the symmetric space X with the upper half plane. Within the Riemann sphere $\mathbb{C} \cup \infty$, it thus has the natural boundary $\mathbb{R} \cup \infty$. The boundary symmetric space $X_{\mathbf{P}}$ is a point for every rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Thus $e(\mathbf{P}) = N_P$ is homeomorphic to the real line. The bordification $\overline{X}$ is now constructed from X by adding one real line $e(\mathbf{P})$ for each point in $\mathbb{Q} \cup \infty$. The topology on $\overline{X}$ ensures that for each $n \in N_P = e(\mathbf{P})$ the curve $a \mapsto n \times \exp(a) \times \mathrm{pt} \in N_P \times A_{\mathbf{P}} \times X_{\mathbf{P}} \cong X$ with time parameter $a \in \mathfrak{a}_{\mathbf{P}} \cong \mathbb{R}$ is the unique geodesic in X converging to $n \in e(\mathbf{P})$. Thus in Figure 2.7, the boundary component $e(\mathbf{P})$ can be thought of as an additional horosphere at infinity which parametrizes the geodesics converging to zero.

Since a sequence (x_i) in $e(\mathbf{P})$ can only converge to a point $x \in e(\mathbf{Q})$ if $\mathbf{Q} \subset \mathbf{P}$, it is immediate that the *Borel–Serre boundary* $\partial\overline{X} \subset \overline{X}$ of $\overline{X}$ defined as

$$\partial\overline{X} = \bigcup_{\mathbf{P} \subsetneq \mathbf{G}} e(\mathbf{P}) \tag{2.14}$$

is closed in $\overline{X}$. Whence its complement $e(\mathbf{G}) = X \subset \overline{X}$ is open. The following proposition sharpens [15, Lemma III.16.2, p. 371].

Proposition 2.15. *The closure of the boundary component $e(\mathbf{P})$ in the bordification $\overline{X}$ can be canonically identified with the product*

$$\overline{e(\mathbf{P})} = N_P \times \overline{X}_{\mathbf{P}}$$

where $\overline{X}_{\mathbf{P}}$ is the Borel–Serre bordification of the boundary symmetric space $X_{\mathbf{P}}$.

PROOF. By construction of the convergence class of sequences we have

$$\overline{e(\mathbf{P})} = \dot{\bigcup_{\mathbf{Q}\subset\mathbf{P}}} e(\mathbf{Q}). \tag{2.16}$$

In terms of the rational parabolic subgroup $\mathbf{Q}' \subset \mathbf{M_P}$ of Lemma 2.9 the boundary component $e(\mathbf{Q})$ can be expressed as

$$e(\mathbf{Q}) = N_Q \times X_{\mathbf{Q}} = N_P \times N_{Q'} \times X_{\mathbf{Q}'} = N_P \times e(\mathbf{Q}'). \tag{2.17}$$

In the distributive category of sets we thus obtain

$$\overline{e(\mathbf{P})} = \coprod_{\mathbf{Q}\subset\mathbf{P}} e(\mathbf{Q}) = \coprod_{\mathbf{Q}'\subset\mathbf{M_P}} N_P \times e(\mathbf{Q}') = N_P \times \coprod_{\mathbf{Q}'\subset\mathbf{M_P}} e(\mathbf{Q}') = N_P \times \overline{X}_{\mathbf{P}}.$$

We have to verify that this identifies the spaces $\overline{e(\mathbf{P})}$ and $N_P \times \overline{X}_{\mathbf{P}}$ also topologically if we assign the bordification topology to $\overline{X}_{\mathbf{P}}$. For this purpose we show that the natural convergence classes of sequences on $\overline{e(\mathbf{P})}$ and $N_P \times \overline{X}_{\mathbf{P}}$ coincide. Let us refine our notation and write $\mathbf{Q}' = \mathbf{Q}|\mathbf{P}$ to stress that $\mathbf{Q}' \subset \mathbf{M_P}$. Let $\mathbf{R} \subset \mathbf{Q}$ be a third rational parabolic subgroup. Then the equality $M_{\mathbf{Q}} = M_{\mathbf{Q}|\mathbf{P}}$ implies the cancellation rule $\mathbf{R}|\mathbf{Q} = (\mathbf{R}|\mathbf{P})|(\mathbf{Q}|\mathbf{P})$. Incorporating coordinates for $e(\mathbf{Q})$ with respect to $\mathbf{R}$ as in (2.12), equation (2.17) can now be written as

$$e(\mathbf{Q}) = N_R \times A_{\mathbf{R}|\mathbf{Q}} \times X_{\mathbf{R}} = N_P \times (N_{R|P} \times A_{(\mathbf{R}|\mathbf{P})|(\mathbf{Q}|\mathbf{P})} \times X_{\mathbf{R}|\mathbf{P}}).$$

Here the product $N_{R|P} \times A_{(\mathbf{R}|\mathbf{P})|(\mathbf{Q}|\mathbf{P})} \times X_{\mathbf{R}|\mathbf{P}}$ gives the coordinates (2.12) for $e(\mathbf{Q}|\mathbf{P})$ with respect to $\mathbf{R}|\mathbf{P}$. Let (n_i, a_i, y_i) be a sequence in $e(\mathbf{Q})$ converging to $(n, y) \in e(\mathbf{R})$. We decompose uniquely $n_i = n_i^P n_i^{R|P}$ and $n = n^P n^{R|P}$ according to $N_R = N_P N_{R|P} \cong N_P \rtimes N_{R|P}$. Then firstly $n_i^P \to n^P$ in N_P. Secondly $(n_i^{R|P}, a_i, y_i)$ is a sequence in $e(\mathbf{Q}|\mathbf{P})$ that converges to $(n^{R|P}, y) \in e(\mathbf{R}|\mathbf{P})$ according to the convergence class of the bordification $\overline{X}_{\mathbf{P}}$. Since the convergence class of $N_P \times \overline{X}_{\mathbf{P}}$ consists of the memberwise products of convergent sequences in N_P and the sequences in the convergence class of $\overline{X}_{\mathbf{P}}$, this clearly proves the assertion. □

One special case of this proposition is $\overline{e(\mathbf{G})} = \overline{X}$. The other important special case occurs when $\mathbf{P}$ is a minimal rational parabolic subgroup. Then $\operatorname{rank}_{\mathbb{Q}}(\mathbf{M_P}) = 0$ so that $\overline{X}_{\mathbf{P}} = X_{\mathbf{P}}$ which means that $e(\mathbf{P})$ is closed.

As we have $\overline{e(\mathbf{P})} = \dot\bigcup e(\mathbf{Q})$, the union running over all $\mathbf{Q} \subset \mathbf{P}$, we should also examine the subset

$$\underline{e(\mathbf{P})} = \dot{\bigcup_{\mathbf{Q}\supset\mathbf{P}}} e(\mathbf{Q}) \subset \overline{X}.$$

To this end consider the rational horospherical decomposition $X = N_P \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ of X given $\mathbf{P}$. Let $\Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P}}) = \{\alpha_1, \ldots, \alpha_l\}$ be a numbering of the simple parabolic roots. The map $a \mapsto (a^{-\alpha_1}, \ldots, a^{-\alpha_l})$ defines a coordinate chart $\varphi_{\mathbf{P}} \colon A_{\mathbf{P}} \to (\mathbb{R}_{>0})^l$. The minus signs make sure the "point at infinity" of $A_{\mathbf{P}}$ will correspond to the origin in $\mathbb{R}^l$. Let $\overline{A}_{\mathbf{P}}$ be the closure of $A_{\mathbf{P}}$ in $\mathbb{R}^l$ under the embedding $\varphi_{\mathbf{P}}$. Given $\mathbf{Q} \supset \mathbf{P}$, let $I \subset \Delta = \Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P}})$ be such that $\mathbf{Q} = \mathbf{P}_I$ (see Section 2, p. 14) and set

$$A_{\mathbf{P},\mathbf{Q}} = \exp\Big(\bigcap_{\alpha\in\Delta\setminus I} \ker\alpha\Big)$$

Since the simple roots $\Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P}})$ restrict to the simple roots $\Delta(\mathfrak{p}_I, \mathfrak{a}_I)$, we obtain inclusions $A_{\mathbf{P},\mathbf{Q}} \times \overline{A}_{\mathbf{Q}} \subset \overline{A}_{\mathbf{P}}$. If $o_{\mathbf{Q}} \in \overline{A}_{\mathbf{Q}}$ denotes the origin, these inclusions combine to give a disjoint decomposition

$$\overline{A}_{\mathbf{P}} = \dot{\bigcup_{\mathbf{Q}\supset\mathbf{P}}} A_{\mathbf{P},\mathbf{Q}} \times o_{\mathbf{Q}}$$

of the corner $\overline{A}_{\mathbf{P}}$ into the corner point (for $\mathbf{Q} = \mathbf{P}$), the boundary edges, the boundary faces, . . . , the boundary hyperfaces and the interior (for $\mathbf{Q} = \mathbf{G}$). In the coordinates $e(\mathbf{Q}) = N_P \times A_{\mathbf{P}'} \times X_{\mathbf{P}}$ as in (2.12), the group $A_{\mathbf{P}'}$ can be identified with the group

$A_{\mathbf{P},\mathbf{Q}}$ [15, Lemma III.9.7, p. 330]. It follows that the subset $N_P \times A_{\mathbf{P},\mathbf{Q}} \times o_{\mathbf{Q}} \times X_{\mathbf{P}}$ in $N_P \times \overline{A}_{\mathbf{P}} \times X_{\mathbf{P}}$ can be identified with $e(\mathbf{Q})$ and hence

$$\underline{e(\mathbf{P})} \cong N_P \times \overline{A}_{\mathbf{P}} \times X_{\mathbf{P}} \tag{2.18}$$

has the structure of a real analytic manifold with corners. For a proof that the involved topologies match, we refer to [15, Lemmas III.9.8–10, pp. 330–332]. The manifold $\underline{e(\mathbf{P})}$ is called the *corner* in $\overline{X}$ corresponding to the rational parabolic subgroup $\mathbf{P}$. The corners $\underline{e(\mathbf{P})}$ are open. With their help neighborhood bases of boundary points in $\overline{X}$ can be described [15, Lemma III.9.13, p. 332]. These demonstrate that $\overline{X}$ is a Hausdorff space [15, Proposition III.9.14, p. 333]. The corners $\underline{e(\mathbf{P})}$ form an open cover of the bordification $\overline{X}$. One verifies that their analytic structures are compatible to conclude the following result [15, Proposition III.9.16, p. 335].

Proposition 2.19. *The bordification $\overline{X}$ has a canonical structure of a real analytic manifold with corners.*

Corollary 2.20. *The bordification $\overline{X}$ is contractible.*

PROOF. According to [17, Appendix] the corners of $\overline{X}$ can be smoothened to endow $\overline{X}$ with the structure of a smooth manifold with boundary. The collar neighborhood theorem thus implies that $\overline{X}$ is homotopy equivalent to its interior. The interior X is contractible as we have already remarked in Section 1. □

Another corollary of Proposition 2.19 together with Proposition 2.15 is that the closures of boundary components $\overline{e(\mathbf{P})}$ are real analytic manifolds with corners as well. In fact, the inclusion $\overline{e(\mathbf{P})} \subset \overline{X}$ realizes $\overline{e(\mathbf{P})}$ as a *submanifold with corners* of $\overline{X}$. Note that topologically a manifold with corners is just a manifold with boundary. We conclude this section with the observation that

$$\overline{e(\mathbf{P})} \cap \overline{e(\mathbf{Q})} = \overline{e(\mathbf{P} \cap \mathbf{Q})} \tag{2.21}$$

if $\mathbf{P} \cap \mathbf{Q}$ is rational parabolic. Otherwise the intersection is empty. Dually,

$$\underline{e(\mathbf{P})} \cap \underline{e(\mathbf{Q})} = \underline{e(\mathbf{R})}$$

where now $\mathbf{R}$ denotes the smallest rational parabolic subgroup of $\mathbf{G}$ that contains both $\mathbf{P}$ and $\mathbf{Q}$. If $\mathbf{R} = \mathbf{G}$, the intersection equals X.

4. Quotients

We extend the action of $\mathbf{G}(\mathbb{Q})$ on X to an action on $\overline{X}$. Given $g \in \mathbf{G}(\mathbb{Q})$ and a rational parabolic subgroup $\mathbf{P}$, let $k \in K$, $n \in N_P$, $a \in A_{\mathbf{P}}$ and $m \in M_{\mathbf{P}}$ such that $g = kman$. Note that we have swapped m and n compared to the order in the rational Langlands decomposition in Definition 2.4. This ensures that a and n are unique. In contrast, the elements k and m can be altered from right and left by mutually inverse elements in K_P. Their product km is however well-defined. We therefore obtain a well-defined map $g.\colon e(\mathbf{P}) \to e({}^k\mathbf{P})$ setting

$$g.(n_0, m_0 K_P) = ({}^{kma}(nn_0), {}^{k}(mm_0)K_{{}^kP}). \tag{2.22}$$

Using the convergence class of sequences, one checks easily that this defines a continuous and in fact a real analytic action of $\mathbf{G}(\mathbb{Q})$ on $\overline{X}$ which extends the action on X [15, Propositions III.9.15–16, pp. 333–335]. The restricted action of $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is proper [15, Proposition III.9.17, p. 336] and thus free because Γ is torsion-free. The quotient $\Gamma\backslash\overline{X}$ is therefore Hausdorff and in fact a real analytic manifold with corners. It is called the *Borel–Serre compactification* of the locally symmetric space $\Gamma\backslash X$ in view of the following result.

Theorem 2.23. *The real analytic manifold with corners* $\Gamma\backslash\overline{X}$ *is compact.*

By Corollary 2.20 the Borel–Serre compactification $\Gamma\backslash\overline{X}$ is a classifying space for Γ. It is therefore of key importance that it is compact. So let us briefly comment on the proof [15, Theorem III.9.18, p. 337].

PROOF. For $t > 0$ let $A_{\mathbf{P},t} = \{a \in A_{\mathbf{P}} : a^{\alpha} > t \text{ for each } \alpha \in \Delta(\mathfrak{p}, \mathfrak{a}_{\mathbf{P}})\}$. For any two bounded sets $U \subset N_P$ and $V \subset X_{\mathbf{P}}$, the subset

$$\mathfrak{S}_{\mathbf{P},U,t,V} = U \times A_{\mathbf{P},t} \times V$$

of $N_P \times A_P \times X_{\mathbf{P}} = X$ is called a *Siegel set* of X associated with $\mathbf{P}$. According to Proposition 2.11 there is a finite system $\mathbf{P}_1, \ldots, \mathbf{P}_r$ of Γ-representatives of rational parabolic subgroups. It follows from [10, Théorème 15.5, p. 104] that there are associated Siegel sets $U_1 \times A_{\mathbf{P}_1,t_1} \times W_1, \ldots, U_r \times A_{\mathbf{P}_r,t_r} \times W_r$ which project to a cover of $\Gamma\backslash X$. We can assume that the sets U_i and W_i are compact. The sets $A_{\mathbf{P}_i,t_i}$ have compact closure $\overline{A}_{\mathbf{P}_i,t_i}$ in $\overline{A}_{\mathbf{P}_i}$. In view of (2.18) the closures of the Siegel sets within $\overline{X}$ are given by $U_i \times \overline{A}_{\mathbf{P},t_i} \times W_i$ and thus are compact. The Γ-translates of the compact sets $U_i \times \overline{A}_{\mathbf{P},t_i} \times W_i$ are closed because Γ acts properly discontinuously. Since $X \subset \overline{X}$ is dense, they cover $\overline{X}$. Therefore the projections of the sets $U_i \times \overline{A}_{\mathbf{P},t_i} \times W_i$ form a finite cover of $\Gamma\backslash\overline{X}$ by compact sets. □

The subgroup $\Gamma_P = \Gamma \cap \mathcal{N}_G(P)$ of Γ leaves $e(\mathbf{P})$ invariant. Let us denote the quotient by $e'(\mathbf{P}) = \Gamma_P\backslash e(\mathbf{P})$. Since $g.e(\mathbf{P}) \cap e(\mathbf{P}) = \emptyset$ for every $g \in \Gamma$ that does not lie in Γ_P, we have the following disjoint decomposition of the quotient $\Gamma\backslash\overline{X}$ [15, Proposition III.9.20, p. 337].

Proposition 2.24. *Let* $\mathbf{P}_1, \ldots, \mathbf{P}_r$ *be a system of representatives of* Γ*-conjugacy classes of rational parabolic subgroups in* $\mathbf{G}$*. Then*

$$\Gamma\backslash\overline{X} = \bigsqcup_{i=1}^{r} e'(\mathbf{P}_i).$$

Example 2.25. In the setting of Example 2.13 let $\Gamma \subset \mathbf{SL}(2,\mathbb{Q})$ be any torsion-free arithmetic subgroup. The quotient $e'(\mathbf{P}) = \Gamma_P\backslash e(\mathbf{P})$ is homeomorphic to the circle S^1. The locally symmetric space $\Gamma\backslash X$ is a *hyperbolic surface* and has finitely many ends or *hyperbolic cusps*. From Proposition 2.24 we see that one obtains its Borel–Serre compactification $\Gamma\backslash\overline{X}$ by adding one circle $e'(\mathbf{P})$ at the infinity of each of the hyperbolic cusps.

The closure of $e'(\mathbf{P})$ in $\Gamma\backslash\overline{X}$ is compact and has the decomposition

$$\overline{e'(\mathbf{P})} = \bigsqcup_{\mathbf{Q}\subset\mathbf{P}} e'(\mathbf{Q}). \tag{2.26}$$

This follows from the compatibilities $e'(\mathbf{P}) = \nu(e(\mathbf{P}))$ and $\overline{e'(\mathbf{P})} = \nu(\overline{e(\mathbf{P})})$ and from (2.16) where $\nu\colon \overline{X} \to \Gamma\backslash\overline{X}$ denotes the canonical projection [17, Proposition 9.4, p. 476]. By (2.16) and the remarks preceding Proposition 2.24 we see that $\overline{e'(\mathbf{P})} = \nu(\overline{e(\mathbf{P})})$ also equals $\Gamma_P\backslash\overline{e(\mathbf{P})}$. We will examine this latter quotient.

Let $\Gamma_{N_P} = \Gamma \cap N_P$. The rational Langlands decomposition 2.4 defines a projection $P \to M_{\mathbf{P}}$. Let $\Gamma_{M_{\mathbf{P}}}$ be the image of Γ_P under this projection. Equivalently, $\Gamma_{M_{\mathbf{P}}}$ is the canonical lifting under π of the group $\Gamma'_{M_{\mathbf{P}}}$ defined on p. 12, see [14, Proposition 2.6, p. 272]. We should however not conceal a word of warning. The lift $\Gamma'_{M_{\mathbf{P}}} \to \Gamma_{M_{\mathbf{P}}}$ does not necessarily split the exact sequence

$$1 \longrightarrow \Gamma_{N_P} \longrightarrow \Gamma_P \longrightarrow \Gamma'_{M_{\mathbf{P}}} \longrightarrow 1,$$

not even if the suppressed base point was rational for $\mathbf{P}$. By [14, Propositions 2.6 and 2.8, p. 272] we have $\Gamma_P \subset N_P\Gamma_{M_{\mathbf{P}}} = N_P\Gamma_P$. We analyze how the action of Γ_P on $\overline{e(\mathbf{P})}$ behaves regarding the decomposition $\overline{e(\mathbf{P})} = N_P \times \overline{X}_{\mathbf{P}}$ of Proposition 2.15.

Proposition 2.27. *Let $p \in \Gamma_P$ and let $p = mn$ be its unique decomposition with $m \in \Gamma_{M_{\mathbf{P}}}$ and $n \in N_P$. Let $(n_0, x) \in N_P \times \overline{X}_{\mathbf{P}} = \overline{e(\mathbf{P})}$. Then*

$$p.(n_0, x) = ({}^{m}(nn_0), m.x).$$

Proof. There is a unique rational parabolic subgroup $\mathbf{Q} \subset \mathbf{P}$ and there are unique elements $n_0' \in N_{Q'}$ and $m_0' \in M_{Q'}$ such that

$$x = (n_0', m_0' K_{Q'}) \in N_{Q'} \times X_{\mathbf{Q}'} = e(\mathbf{Q}') \subset \overline{X}_{\mathbf{P}}.$$

We decompose $m \in M_{\mathbf{P}}$ as $m = km'a'n'$ with $k \in K_P$, $m' \in M_{\mathbf{Q}'}$, $a' \in A_{\mathbf{Q}'}$ and $n' \in N_{Q'}$. By (2.17) we have $N_P \times e(\mathbf{Q}') = e(\mathbf{Q}) = N_Q \times X_{\mathbf{Q}}$ and under this identification our element (n_0, x) corresponds to $(n_0 n_0', m_0' K_Q)$. We have $p = km'a'(n'n)$ with $m' \in M_{\mathbf{Q}'} = M_{\mathbf{Q}}$, $a' \in A_{\mathbf{Q}'} \subset A_{\mathbf{Q}}$ and $n'n \in N_Q$. According to (2.22) the element p therefore acts as

$$p.(n_0 n_0', m_0' K_Q) = ({}^{km'a'}(n'nn_0n_0'), {}^{k}(m'm_0')K_{{}^{k}Q}).$$

For the left-hand factor we compute

$$\begin{aligned} {}^{km'a'}(n'nn_0n_0') = {}^{km'a'}({}^{n'}(nn_0)n'n_0') = {}^{km'a'n'}(nn_0)\ {}^{km'a'}(n'n_0') = \\ = {}^{m}(nn_0)\ {}^{km'a'}(n'n_0'). \end{aligned}$$

Transforming back from $N_Q \times X_{\mathbf{Q}}$ to $N_P \times e(\mathbf{Q}')$ we therefore obtain

$$p.(n_0, x) = ({}^{m}(nn_0), ({}^{km'a'}(n'n_0'), {}^{k}(m'm_0')K_{{}^{k}Q})) = ({}^{m}(nn_0), m.x). \qquad \square$$

If Γ is neat, then Proposition 2.27 makes explicit that we have a commutative diagram

$$\begin{array}{ccc} \overline{e(\mathbf{P})} & \longrightarrow & \Gamma_P\backslash\overline{e(\mathbf{P})} \\ \downarrow & & \downarrow \\ \overline{X}_{\mathbf{P}} & \longrightarrow & \Gamma_{M_{\mathbf{P}}}\backslash\overline{X}_{\mathbf{P}} \end{array}$$

of bundle maps of manifolds with corners. The bundle structure of $\Gamma_P\backslash\overline{e(\mathbf{P})}$ will later be of particular interest.

Theorem 2.28. *Suppose that $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a neat arithmetic subgroup. Then the manifold with corners $\overline{e'(\mathbf{P})} = \Gamma_P\backslash\overline{e(\mathbf{P})}$ has the structure of a real analytic fiber bundle over the manifold with corners $\Gamma_{M_{\mathbf{P}}}\backslash\overline{X}_{\mathbf{P}}$ with the compact nilmanifold $\Gamma_{N_P}\backslash N_P$ as typical fiber.*

Also for later purposes we remark that the Borel–Serre compactification $\Gamma\backslash\overline{X}$ clearly has a finite CW-structure such that the closed submanifolds $\overline{e'(\mathbf{P})}$ are subcomplexes. The bordification $\overline{X}$ is a regular covering of this finite CW complex with deck transformation group Γ, in other words a finite free Γ-CW complex in the sense of [105, Section II.1, p. 98]. In the sequel we want to assume that $\overline{X}$ is endowed with this Γ-CW structure as soon as a torsion-free arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is specified. Then Corollary 2.20 and Theorem 2.23 say in more abstract terms that the bordification $\overline{X}$ is a cofinite classifying space $E\Gamma$. In fact, something better is true. The bordification is a model for the classifying space $\underline{E}\Gamma$ for proper group actions for every general, not necessarily torsion-free, arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$. This means every isotropy group is finite and for every finite subgroup $\Lambda \subset \Gamma$ the fix point set $\overline{X}^{\Lambda}$ is contractible (and in particular nonempty). This was pointed out in [1, Remark 5.8, p. 546] and L. Ji thereafter supplied a proof in [52, Theorem 3.2, p. 520].

5. Nonarithmetic lattices

The $\mathbb{Q}$-structure of the group $\mathbf{G}$ is crucial for the construction of the Borel–Serre bordification $\overline{X}$ because requiring parabolic subgroups to be rational specifies a countable subcollection of all $\mathbb{R}$-parabolic subgroups of $\mathbf{G}$ which are supposed to determine boundary components. The price to pay is that only the action of rational points $g \in \mathbf{G}(\mathbb{Q})$ can naturally be extended to the boundary because conjugates ${}^{g}\mathbf{P}$ of rational parabolic subgroups $\mathbf{P}$ have to be rational again. Therefore the Borel–Serre bordification $\overline{X}$ only permits a natural action by arithmetic subgroups $\Gamma \subset \mathbf{G}(\mathbb{Q})$. We will recall in Corollary 4.4 that for the geometer interested in lattices in semisimple Lie groups, this is unproblematic as long as the group has real rank at least two. However, honestly nonarithmetic lattices are known to exist in the rank one simple Lie groups $\mathrm{SO}^0(n,1)$ for all $n \geq 2$ and in $\mathrm{SU}(n,1)$ at least if $n \leq 3$. So in the rank one case one should look for a different type of bordification.

Such a bordification has been suggested by H. Kang [56]. The idea is to imitate the construction of the Borel–Serre compactification in its essential ideas but to work with real Langlands and real horospherical decompositions instead of the rational ones. The central point is to find an additional geometric condition that distinguishes a countable set of real parabolic subgroups whose boundary components still cover all directions to infinity of the locally symmetric space. To make this more precise, let G be a connected semisimple linear Lie group. As usual $K \subset G$ denotes a maximal compact subgroup and $X = G/K$ is the associated symmetric space. By [110, Theorem 3.37, p. 38] there is a connected semisimple linear algebraic $\mathbb{R}$-group $\mathbf{G}$ such that $\mathbf{G}(\mathbb{R})^0 = G$. A *real parabolic subgroup* $\mathbf{P} \subset \mathbf{G}$ is an $\mathbb{R}$-subgroup such that $\mathbf{G}/\mathbf{P}$ is a complete (equivalently projective) variety. In that case we call $P = \mathbf{P}(\mathbb{R})^0$ a parabolic subgroup of G. Proposition 2.2 is in fact stated for real parabolic subgroups $\mathbf{P} \subset \mathbf{G}$ in [17, Proposition 1.8, p. 444]. We thus obtain the *real Langlands decomposition* and associated *real horospherical decomposition*

$$P = N_P \times A_P \times M_P \quad \text{and} \quad X = N_P \times A_P \times X_P$$

working over the field $\mathbb{R}$ instead of $\mathbb{Q}$. If $\mathbf{P}$ happens to be a rational parabolic subgroup, then real and rational decompositions agree if and only if $\mathrm{rank}_{\mathbb{R}}(\mathbf{G}) = \mathrm{rank}_{\mathbb{Q}}(\mathbf{G})$ [15, Remark III.1.12, p. 274]. Now let $\mathrm{rank}_{\mathbb{R}}(G) = 1$. Similar to the Borel–Serre bordification, we define as boundary component $e(P) = N_P \times X_P$. Since every proper parabolic subgroup $P \subset G$ is minimal (and maximal), we have $M_P \subset K$, as can be seen from the description of the Lie algebra of P in terms of restricted roots [15, Section I.1.3, p. 30]. Therefore in fact $e(P) = N_P$ if P is proper, and $e(G) = X$ for the improper parabolic subgroup G. Let $\Gamma \subset G$ be a lattice. Motivated by the "rational boundary components" in [5] we will say that a parabolic subgroup $P \subset G$ is *geometrically rational* with respect to Γ if $\Gamma \cap N_P$ is a lattice in N_P. Note that lattices in nilpotent Lie groups are always uniform. Let Δ_Γ be the set of geometrically rational parabolic subgroups of G. Trivially $G \in \Delta_\Gamma$. As a set, Kang's compactification is given by

$$\overline{X}_\Gamma = \coprod_{P \in \Delta_\Gamma} e(P).$$

The topology on $\overline{X}_\Gamma$ is defined by a convergence class of sequences just like it was done for the Borel–Serre bordification. The action of Γ on $\overline{X}_\Gamma$ can likewise be defined using horospherical coordinates. It is proper and cocompact. The bordification $\overline{X}_\Gamma$ is a smooth manifold with boundary $\partial \overline{X}_\Gamma = \coprod_{P \subsetneq G} e(P)$ and with interior X. Again it follows that it is contractible, thus a cofinite $E\Gamma$ if Γ is torsion-free. As the main result of his thesis Kang proves that in fact $\overline{X}_\Gamma$ has the structure of a finite Γ-CW complex and is a model for the classifying space for proper Γ-actions $\underline{E}\Gamma$.

CHAPTER 3

L^2-invariants

In this chapter we review L^2-Betti numbers, Novikov–Shubin invariants and L^2-torsion of CW complexes and Riemannian manifolds with group actions following [67, Chapters 1–3]. The outline of sections is as follows. We introduce the three invariants abstractly in terms of spectral density functions of morphisms of Hilbert $\mathcal{N}(\Gamma)$-modules in Section 1. Section 2 applies this theory to the Laplacians of the L^2-chain complex of a finite free Γ-CW complex. This gives the cellular or topological versions of L^2-invariants. We list convenient properties that facilitate their computation. In Section 3 we replace the cellular Laplacians by the form Laplacians of a free proper cocompact Riemannian Γ-manifold. This yields the analytic L^2-invariants. In the case of L^2-torsion one has to cope with some complications as we discuss in detail. We cite a theorem which says that the analytic invariants equal their cellular counterparts on a free proper cocompact Riemannian Γ-manifold with equivariant triangulation. If the Riemannian manifold is a symmetric space, analytic L^2-invariants can be defined if Γ only acts with a finite-volume quotient. The resulting values have been computed explicitly as we will recall.

1. Hilbert modules and spectral density functions

Let Γ be a discrete countable group. It acts unitarily from the left on the Hilbert space $\ell^2\Gamma$ of square-integrable functions $\Gamma \to \mathbb{C}$. This Hilbert space has a distinguished vector $e \in \Gamma \subset \ell^2\Gamma$. The Γ-equivariant bounded operators $\mathcal{N}(\Gamma) = \mathcal{B}(\ell^2\Gamma)^\Gamma$ form a weakly closed unital $*$-subalgebra of $\mathcal{B}(\ell^2\Gamma)$ called the *group von Neumann algebra* of Γ. Let V be a Hilbert space with isometric left Γ-action. We call V a *Hilbert $\mathcal{N}(\Gamma)$-module* if there is a Hilbert space H such that V embeds Γ-equivariantly and isometrically into $H \otimes \ell^2\Gamma$. A Hilbert $\mathcal{N}(\Gamma)$-module V is called *finitely generated* if H can be chosen finite-dimensional. Homomorphisms of $\mathcal{N}(\Gamma)$-Hilbert modules are Γ-equivariant bounded operators. An orthonormal basis $\{\xi_i\}$ of H defines the *von Neumann trace* on the set of positive endomorphisms $H \otimes \ell^2\Gamma \to H \otimes \ell^2\Gamma$ setting $\operatorname{tr}_{\mathcal{N}(\Gamma)}(f) = \sum_i \langle f(\xi_i \otimes e), \xi_i \otimes e\rangle \in [0,\infty]$. It is independent of the basis $\{\xi_i\}$. By means of an embedding any Hilbert $\mathcal{N}(\Gamma)$-module V inherits its own unique von Neumann trace from this construction. Define the *von Neumann dimension* of V by $\dim_{\mathcal{N}(\Gamma)}(V) = \operatorname{tr}_{\mathcal{N}(\Gamma)}(\operatorname{id}_V)$.

Now let $f\colon \operatorname{dom}(f) \subset U \to V$ be a possibly unbounded closed densely defined Γ-equivariant operator of Hilbert $\mathcal{N}(\Gamma)$-modules. The selfadjoint operator $f^*f\colon \operatorname{dom}(f^*f) \subset U \to U$ defines a family $\{E_\lambda^{f^*f}\}$ of Γ-equivariant spectral projections.

Definition 3.1. The *spectral density function* of f is the monotone non-decreasing right continuous function $F(f)\colon [0,\infty) \to [0,\infty]$ given by

$$F(f)(\lambda) = \dim_{\mathcal{N}(\Gamma)}(\operatorname{im}(E_{\lambda^2}^{f^*f})).$$

In all what follows let us assume that f is a *Fredholm operator* which means that there is $\lambda > 0$ such that $F(\lambda) < \infty$. This is automatic if U has finite von Neumann dimension.

Definition 3.2. The L^2-*Betti number* of f is given by

$$b^{(2)}(f) = F(f)(0) \ \in [0, \infty).$$

Thus the L^2-Betti number of f is the von Neumann dimension of the Hilbert $\mathcal{N}(\Gamma)$-module $\ker(f)$.

Definition 3.3. The *Novikov–Shubin invariant* of f is given by

$$\alpha(f) = \liminf_{\lambda \to 0^+} \frac{\log(F(f)(\lambda) - F(f)(0))}{\log(\lambda)} \ \in [0, \infty]$$

unless $F(f)(\varepsilon) = b^{(2)}(f)$ for some $\varepsilon > 0$ in which case we set $\alpha(f) = \infty^+$.

The Novikov–Shubin invariant measures how slowly the density function grows in a neighborhood of zero. The fractional expression is so chosen that we obtain $\alpha(f) = k$ if the spectral density function happens to be a polynomial with highest order k. For the case that $F(f)$ is constant in a neighborhood of zero, we have introduced the formal symbol $\alpha(f) = \infty^+$ and we decree $r < \infty < \infty^+$ for all $r \in \mathbb{R}$. The value $\alpha(f) = \infty^+$ thus indicates a spectral gap at zero.

Let us now restrict to the case that $f \colon U \to V$ is a morphism of Hilbert $\mathcal{N}(\Gamma)$-modules whose von Neumann dimensions are finite. Recall that the spectral density function $F = F(f)$ determines a Lebesgue-Stieltjes measure $\mathrm{d}F$ on the Borel measure space $[0, \infty)$.

Definition 3.4. The *Fuglede–Kadison determinant* of f is given by

$$\det_{\mathcal{N}(\Gamma)}(f) = \exp\left(\int_{0^+}^{\infty} \log(\lambda) \mathrm{d}F(\lambda)\right) \in [0, \infty).$$

We agree that this definition shall not exclude the possibility of the diverging integral $\int_{0^+}^{\infty} \log(\lambda)\mathrm{d}F(\lambda) = -\infty$ in which case $\det_{\mathcal{N}(\Gamma)}(f) = 0$. We call f of *determinant class* if $\int_{0^+}^{\infty} \log(\lambda)\mathrm{d}F(\lambda) > -\infty$. The symbol 0^+ means that we exclude zero and integrate over the measure subspace $(0, \infty)$. If for instance Γ is finite, we obtain $\det_{\mathcal{N}(\Gamma)}(f) = (\prod_{i=1}^{r} \lambda_i)^{\frac{1}{2|\Gamma|}}$ with the positive eigenvalues $\lambda_1, \ldots, \lambda_r$ of the positive endomorphism $f^* f$. In case f is invertible this is just the $|\Gamma|$-th root of the ordinary determinant of $|f| = \sqrt{f^* f}$. Now let $\{f_p\} = \{f_p\}_{p=0}^{\infty}$ be a whole family of determinant class morphisms $f_p \colon U \to V$ such that $f_p = 0$ for almost all p.

Definition 3.5. The L^2-*torsion* of $\{f_p\}$ is given by

$$\rho^{(2)}(\{f_p\}) = -\frac{1}{2} \sum_p (-1)^p \, p \, \log(\det_{\mathcal{N}(\Gamma)}(f_p)) \ \in \mathbb{R}.$$

We see that the three abstract L^2-*invariants* of $\{f_p\}$ absorb more and more spectral information. The p-th L^2-Betti number is the value of the spectral density function of f_p at zero. The p-th Novikov–Shubin invariant describes the growth behavior of the spectral density function of f_p in a neighborhood of zero. Finally the abstract L^2-torsion depends on the full spectral density function of each f_p.

2. Cellular L^2-invariants

Let X be a finite free Γ-CW-complex in the sense of [105, Section II.1, p. 98]. Equivalently, X is a Galois covering of a finite CW-complex with deck transformation group Γ. Let $C_*(X)$ be the cellular $\mathbb{Z}\Gamma$-chain complex. The L^2-*chain complex* $C_*^{(2)}(X) = \ell^2\Gamma \otimes_{\mathbb{Z}\Gamma} C_*(X)$ is a finite chain complex of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules whose differentials $c_p \colon C_p^{(2)}(X) \to C_{p-1}^{(2)}(X)$ are Fredholm operators induced from the differentials in $C_*(X)$. These define the p-th *Laplace operator* $\Delta_p \colon C_p^{(2)}(X) \to C_p^{(2)}(X)$ given by $\Delta_p = c_{p+1}c_{p+1}^* + c_p^* c_p$. We say that X is *of determinant class* if Δ_p (equivalently c_p) is of determinant class for all $p \geq 0$.

Definition 3.6 (Cellular L^2-invariants)**.**

(i) The p-th L^2*-Betti number* of X is given by $b_p^{(2)}(X;\mathcal{N}(\Gamma)) = b^{(2)}(\Delta_p)$.

(ii) The p-th *Novikov–Shubin invariant* of X is given by $\widetilde{\alpha}_p(X;\mathcal{N}(\Gamma)) = \alpha(\Delta_p)$.

(iii) Assume that $b_p^{(2)}(X) = 0$ for all $p \geq 0$ and that X is of determinant class. Then the L^2*-torsion* of X is given by $\rho^{(2)}(X;\mathcal{N}(\Gamma)) = \rho^{(2)}(\{\Delta_p\})$.

In what follows we will say that X is det-L^2*-acyclic* if it satisfies the conditions in (iii). Moreover, we will frequently suppress $\mathcal{N}(\Gamma)$ from our notation.

Remark 3.7.

(i) By [67, Lemma 1.18, p. 24] we get alternatively $b_p^{(2)}(X) = \dim_{\mathcal{N}(\Gamma)}(H_p^{(2)}(X))$ where the $\mathcal{N}(\Gamma)$-module $H_p^{(2)}(X) = \ker c_p / \overline{\operatorname{im} c_{p+1}}$ is called the *p-th reduced* L^2*-homology* of X.

(ii) For many purposes it seems to be more convenient to work with a finer version of Novikov–Shubin invariants defined as $\alpha_p(X) = \alpha(c_p|_{\operatorname{im}(c_{p+1})^\perp})$. We gain back the above version by the formula $\widetilde{\alpha}_p(X) = \frac{1}{2}\min\{\alpha_p(X), \alpha_{p+1}(X)\}$.

(iii) The assumption that all L^2-Betti numbers of X vanish, in other words that X is L^2*-acyclic*, will make sure that $\rho^{(2)}$ is a homotopy invariant, at least if Γ lies within a large class of groups $\mathcal{G}$ that notably contains all residually finite groups [102]. In this reference it is also shown that the *determinant conjecture* holds for the class $\mathcal{G}$. This conjecture does not only state that X is of determinant class but makes the even stronger assertion that Γ is of det ≥ 1*-class*. This means that any $A \in M(m, n, \mathbb{Z}\Gamma)$ induces a morphism $r_A^{(2)} \colon (\ell^2\Gamma)^m \to (\ell^2\Gamma)^n$ with $\det_{\mathcal{N}(\Gamma)}(r_A^{(2)}) \geq 1$. For our later purpose, it will be enough to know that lattices in connected semisimple linear Lie groups belong to $\mathcal{G}$. This follows because they are finitely generated [110, Theorem 4.58, p. 62], thus residually finite by an old theorem of A. Malcev [74].

(iv) A finite free Γ-CW-pair (X, A) defines a *relative* L^2*-chain complex* $C_*^{(2)}(X, A)$. Its Laplacians define the *relative* L^2-invariants $b_p^{(2)}(X, A)$, $\alpha_p(X, A)$ and also $\rho^{(2)}(X, A)$ provided (X, A) is det-L^2-acyclic.

We will use the standard terminology that a group *virtually* has a property P if some finite-index subgroup has the property P.

Theorem 3.8 (Selected properties of cellular L^2-invariants)**.**

(i) **Homotopy invariance.** *Let $f \colon X \to Y$ be a weak Γ-homotopy equivalence of finite free Γ-CW-complexes. Then*

$$b_p^{(2)}(X) = b_p^{(2)}(Y) \quad \text{and} \quad \alpha_p(X) = \alpha_p(Y) \text{ for all } p \geq 0.$$

Suppose that X or Y is L^2-acyclic and that $\Gamma \in \mathcal{G}$. Then

$$\rho^{(2)}(X) = \rho^{(2)}(Y).$$

(ii) **Poincaré duality.** *Let the Γ-CW-pair $(X, \partial X)$ be an equivariant triangulation of a free proper cocompact orientable Γ-manifold of dimension n with possibly empty boundary. Then*

$$b_p^{(2)}(X) = b_{n-p}^{(2)}(X, \partial X) \quad \text{and} \quad \alpha_p(X) = \alpha_{n+1-p}(X, \partial X).$$

Suppose X is det-L^2*-acyclic. Then so is $(X, \partial X)$ and*

$$\rho^{(2)}(X) = (-1)^{n+1}\rho^{(2)}(X, \partial X).$$

Thus $\rho^{(2)}(X) = 0$ if the manifold is even-dimensional and has empty boundary.

(iii) **First Novikov–Shubin invariant.** *Let X be a connected free finite Γ-CW complex. Then the group Γ is finitely generated and it determines $\alpha_1(X)$. More precisely*

(a) $\alpha_1(X) < \infty$ *if and only if* Γ *is virtually nilpotent. In that case* $\alpha_1(X)$ *equals the growth rate of* Γ.
(b) $\alpha_1(X) = \infty$ *if and only if* Γ *is amenable but not virtually nilpotent.*
(c) $\alpha_1(X) = \infty^+$ *if and only if* Γ *is finite or is not amenable.*

(iv) **Euler characteristic and fiber bundles.** *Let* X *be a connected finite CW-complex. Then the classical Euler characteristic* $\chi(X)$ *can be computed as*

$$\chi(X) = \sum_{p\geq 0}(-1)^p \, b_p^{(2)}(\widetilde{X}).$$

Let $F \to E \to B$ *be a fiber bundle of connected finite CW-complexes. Assume that the inclusion* $F_b \to E$ *of one (then every) fiber induces an injection of fundamental groups. Suppose that* $\widetilde{F_b}$ *is* det-L^2*-acyclic. Then so is* $\widetilde{E}$ *and*

$$\rho^{(2)}(\widetilde{E}) = \chi(B) \cdot \rho^{(2)}(\widetilde{F}).$$

(v) **Aspherical CW-complexes and elementary amenable groups.** *Let* X *be a finite CW-complex with contractible universal covering. Suppose that* $\Gamma = \pi_1(X)$ *is of* $\det \geq 1$*-class and contains an elementary amenable infinite normal subgroup. Then*

$$b_p^{(2)}(\widetilde{X}) = 0 \text{ for } p \geq 0, \quad \alpha_p(\widetilde{X}) \geq 1 \text{ for } p \geq 1 \quad \text{and} \quad \rho^{(2)}(\widetilde{X}) = 0.$$

The proofs are given in [67, Theorem 1.35, p. 37, Theorem 2.55 p. 97, Theorem 3.93, p. 161, Corollary 3.103, p. 166, Theorem 3.113, p. 172, Lemma 13.6, p. 456]. The assertion $\rho^{(2)}(\widetilde{X}) = 0$ in (v) is due to C. Wegner [109] who has recently given a slight generalization in [108]. For a survey on amenable and elementary amenable groups see [67, Section 6.4.1, p. 256]. What lies behind these properties is that to some extent, homological algebra can be developed for Hilbert $\mathcal{N}(\Gamma)$-modules. J. Cheeger and M. Gromov pioneered this idea to conclude information on L^2-Betti numbers [27, Section 2]. Subsequently, consequences for Novikov–Shubin invariants and L^2-torsion have been examined by J. Lott–W. Lück [65] and by W. Lück–M. Rothenberg [70]. We will give a short account of this theory in the next theorem.

Let C_* be a finite chain complex of finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules. As in Remark 3.7 (i) we define the p-th *reduced* L^2*-homology* of the chain complex C_* as $H_p^{(2)}(C_*) = \ker c_p/\overline{\operatorname{im} c_{p+1}}$. Let $\{\Delta_p\}$ be the Laplacians of C_*. We set $b_p^{(2)}(C_*) = b^{(2)}(\Delta_p)$, $\widetilde{\alpha}_p(C_*) = \alpha(\Delta_p)$ and $\alpha_p(C_*) = \alpha(c_p|_{\operatorname{im}(c_{p+1})^\perp})$. We call C_* L^2*-acyclic* if $b_*^{(2)}(C_*) = 0$ and of *determinant class* if Δ_p is of determinant class for every p. If C_* is of determinant class, we set $\rho^{(2)}(C_*) = \rho^{(2)}(\{\Delta_p\})$. A sequence of Hilbert $\mathcal{N}(\Gamma)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$ is called *exact* if $\ker p = \operatorname{im} i$ and *weakly exact* if $\ker p = \overline{\operatorname{im} i}$. A morphism $U \xrightarrow{f} V$ of Hilbert $\mathcal{N}(\Gamma)$-modules is called a *weak isomorphism* if $0 \to U \xrightarrow{f} V \to 0$ is weakly exact. In that case $\dim_{\mathcal{N}(\Gamma)} U = \dim_{\mathcal{N}(\Gamma)} V$.

Theorem 3.9 (L^2-invariants and short exact sequences)**.** *Consider the short exact sequence* $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{j_*} E_* \to 0$ *of finitely generated Hilbert* $\mathcal{N}(\Gamma)$*-chain complexes.*

(i) We have a long weakly exact homology sequence

$$\cdots \xrightarrow{H_{p+1}^{(2)}(j_*)} H_{p+1}^{(2)}(E_*) \xrightarrow{\partial_{p+1}} H_p^{(2)}(C_*) \xrightarrow{H_p^{(2)}(i_*)} H_p^{(2)}(D_*) \xrightarrow{H_p^{(2)}(j_*)} H_p^{(2)}(E_*) \xrightarrow{\partial_p} \cdots$$

(ii) We have the inequalities

$$\frac{1}{\alpha_p(D_*)} \leq \frac{1}{\alpha_p(C_*)} + \frac{1}{\alpha_p(E_*)} + \frac{1}{\alpha(\partial_p)},$$
$$\frac{1}{\alpha_p(E_*)} \leq \frac{1}{\alpha_{p-1}(C_*)} + \frac{1}{\alpha_p(D_*)} + \frac{1}{\alpha(H_{p-1}^{(2)}(i_*))},$$
$$\frac{1}{\alpha_p(C_*)} \leq \frac{1}{\alpha_p(D_*)} + \frac{1}{\alpha_{p+1}(E_*)} + \frac{1}{\alpha(H_p^{(2)}(j_*))}.$$

(iii) Suppose that C_*, D_* and E_* are L^2-acyclic and that two of them are of determinant class. Then all three are of determinant class and if additionally $\det_{\mathcal{N}(\Gamma)}(i_*) = \det_{\mathcal{N}(\Gamma)}(j_*)$, then

$$\rho^{(2)}(D_*) = \rho^{(2)}(C_*) + \rho^{(2)}(E_*).$$

In (ii) some straightforward rules [67, Notation 2.10, p. 76] are understood to make sense of these inequalities when a Novikov–Shubin invariant takes one of the values 0, ∞ or ∞^+. We briefly discuss three further conclusions which will be of particular importance for our later applications.

Lemma 3.10. *Let the Γ-CW-pair $(X, \partial X)$ be an equivariant triangulation of a free proper cocompact orientable L^2-acyclic Γ-manifold. Then for each $p \geq 1$*

$$\tfrac{1}{2} \min\{\alpha_p(X), \alpha_{n-p}(X)\} \leq \alpha_p(\partial X).$$

Proof. We apply the last inequality of Theorem 3.9 (ii) to the sequence of the pair $(X, \partial X)$. As $b_p^{(2)}(X) = 0$, we have $\alpha_p(H_p^{(2)}(j_*)) = \infty^+$ so that

$$\frac{1}{\alpha_p(\partial X)} \leq \frac{1}{\alpha_p(X)} + \frac{1}{\alpha_{p+1}(X, \partial X)}.$$

The lemma follows because $\alpha_{p+1}(X, \partial X) = \alpha_{n-p}(X)$ by Theorem 3.8 (ii). □

Note that the lemma yields $\widetilde{\alpha}_q(X) \leq \alpha_q(\partial X)$ if $\dim X = 2q + 1$ or $\dim X = 2q$. In the latter case it gives in fact more precisely $\alpha_q(X) \leq 2\alpha_q(\partial X)$.

Lemma 3.11. *Let the Γ-CW-pair $(X, \partial X)$ be an equivariant triangulation of a free proper cocompact orientable Γ-manifold of even dimension. Assume X is* det-*L^2-acyclic. Then so is ∂X and*

$$\rho^{(2)}(X) = \tfrac{1}{2}\rho^{(2)}(\partial X).$$

Proof. See [67, Exercise 3.23, p. 209]. Theorem 3.8 (ii) says the pair $(X, \partial X)$ is det-L^2-acyclic and $\rho^{(2)}(X, \partial X) = (-1)^{n+1}\rho^{(2)}(X)$. By Theorem 3.9 (i) the boundary ∂X is L^2-acyclic. Applying Theorem 3.9 (iii) we conclude that ∂X is of determinant class and $\rho^{(2)}(X) = \rho^{(2)}(\partial X) + \rho^{(2)}(X, \partial X)$. □

Lemma 3.12. *Consider the pushout of finite free Γ-CW complexes*

$$\begin{array}{ccc} X_0 & \xrightarrow{j_2} & X_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow \\ X_1 & \longrightarrow & X \end{array}$$

where j_1 is an inclusion of a Γ-subcomplex, j_2 is cellular and X carries the induced Γ-CW-structure. Assume that X_i is det-*L^2-acyclic for $i = 0, 1, 2$. Then so is X and*

$$\rho^{(2)}(X) = \rho^{(2)}(X_1) + \rho^{(2)}(X_2) - \rho^{(2)}(X_0).$$

PROOF. See [67, Theorem 3.93(2), p. 161]. The pushout is *cellular* in the sense of [66, Definition 3.11, p. 41]. Therefore we obtain a short exact Mayer-Vietoris sequence of finitely generated Hilbert $\mathcal{N}(\Gamma)$-chain complexes

$$0 \to C_*^{(2)}(X_0) \to C_*^{(2)}(X_1) \oplus C_*^{(2)}(X_2) \to C_*^{(2)}(X) \to 0.$$

Theorem 3.9, (i) and (iii) imply the lemma. □

We conclude this section with the remark that L^2-invariants, being homotopy invariants by Theorem 3.8 (i), yield invariants for groups whose classifying spaces have a finite CW-model $B\Gamma$. For this purpose we set $b_p^{(2)}(\Gamma) = b_p^{(2)}(E\Gamma; \mathcal{N}(\Gamma))$ as well as $\alpha_p(\Gamma) = \alpha_p(E\Gamma; \mathcal{N}(\Gamma))$. We say that Γ is *det-L^2-acyclic* if $E\Gamma$ is, and set $\rho^{(2)}(\Gamma) = \rho^{(2)}(E\Gamma; \mathcal{N}(\Gamma))$ in that case. In fact, L^2-Betti numbers have been generalized to arbitrary Γ-spaces and thus to arbitrary groups [28, 68]. Novikov–Shubin invariants can likewise be defined for general groups [69]. So we shall allow ourselves to talk about $b_p^{(2)}(\Gamma)$, $\alpha_p(\Gamma)$ and $\widetilde{\alpha}_p(\Gamma)$ for any countable discrete group Γ. Only for the L^2-torsion such a generalization has not (yet) been given.

3. Analytic L^2-invariants

Let M be a cocompact free proper Riemannian Γ-manifold of dimension n without boundary. Our main example is any Galois covering of a closed connected Riemannian n-manifold with deck transformation group Γ. Consider the pre-Hilbert space $\Omega_c^p(M)$ of compactly supported p-forms associated with the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$. On this space Γ acts isometrically by pulling back forms. Using a fundamental domain of the Γ-action on M one can construct a Γ-equivariant isomorphism of the L^2-completion $L^2\Omega^p(M)$ of $\Omega_c^p(M)$ and $\ell^2\Gamma \otimes L^2\Omega^p(\Gamma\backslash M)$ [4, pp. 57 and 65]. Therefore $L^2\Omega^p(M)$ is endowed with the structure of a Hilbert $\mathcal{N}(\Gamma)$-module. The de Rham differential $d^p \colon \Omega_c^p(M) \to L^2\Omega^{p+1}(M)$ has the adjoint $\delta^p \colon \Omega_c^p(M) \to L^2\Omega^{p-1}(M)$. The Laplacian $\Delta_p \colon \Omega_c^p(M) \to L^2\Omega^p(M)$ given by $\Delta_p = d^{p-1}\delta^p + \delta^{p+1}d^p$ is a densely defined Γ-equivariant unbounded operator. Let Δ_p^a be its *minimal closure* [67, p. 55] which in fact equals the *maximal closure* according to [4, Proposition 3.1, p. 53]. Similarly let $d_{\min}^p$ be the minimal closure of d^p with domain $\operatorname{dom}(d_{\min}^p)$ and let $d_{\min}^{p\perp}$ be the restriction of $d_{\min}^p$ to $\operatorname{dom}(d_{\min}^p) \cap \operatorname{im}(d_{\min}^{p-1})^\perp$. The spectral density functions $F(\Delta_p^a)$ and $F(d_{\min}^{p\perp})$ have only finite values so that Δ_p^a and $d_{\min}^{p\perp}$ are Fredholm [67, Lemma 2.66(1), p. 104].

Definition 3.13 (Analytic L^2-Betti numbers and Novikov–Shubin invariants)**.**

(i) The p-th *analytic L^2-Betti number* of M is given by $b_p^{(2a)}(M) = b^{(2)}(\Delta_p^a)$.

(ii) The p-th *analytic Novikov–Shubin invariant* of M is $\widetilde{\alpha}_p^{(a)}(M) = \alpha(\Delta_p^a)$.

Let us also define the refined analytic Novikov–Shubin invariant $\alpha_p^{(a)}(M) = \alpha(d_{\min}^{p-1\perp})$. We obtain $\widetilde{\alpha}_p^{(a)}(M) = \frac{1}{2}\min\{\alpha_p^{(a)}(M), \alpha_{p+1}^{(a)}(M)\}$ [67, Lemma 2.66(2), p. 104]. Of course one would like to define the analytic L^2-torsion by setting $\rho^{(2a)}(M) = \rho^{(2)}(\{\Delta_p^a\})$. While this is essentially what it will be, we need to find a replacement for the Fuglede–Kadison determinant $\det_{\mathcal{N}(G)}(\Delta_p)$ in Definition 3.5 which we have only defined for morphisms of Hilbert $\mathcal{N}(\Gamma)$-modules with finite von Neumann dimension. A similar problem does already occur when one tries to find the analytic counterpart to the classical *Reidemeister torsion* of M. Following [67, Sections 3.1.3 and 3.5.1, pp. 123, 178], we review how to resolve the issue in that case because this will guide us to the definition of analytic L^2-torsion. The

Reidemeister torsion is given by

$$\rho(M;V) = -\frac{1}{2}\sum_{p\geq 0}(-1)^p\, p\, \log(\det_{\mathbb{R}}(\Delta_p)) \in \mathbb{R}$$

if we require additionally that M is acyclic. Here $\Delta_p\colon V\otimes_{\mathbb{Z}\Gamma} C_p(X) \to V\otimes_{\mathbb{Z}\Gamma} C_p(X)$ is the cellular Laplacian, where X is a smooth equivariant triangulation $X \to M$ and V is a fixed finite-dimensional orthogonal Γ-representation. Now one would like to replace the cellular Laplacian with the form Laplacian $\Delta_p\colon \Omega^p(M;V) \to \Omega^p(M;V)$ but one has to cope with what the determinant of a positive automorphism of infinite-dimensional vector spaces should be. To this end we observe that if $\lambda_1,\dots,\lambda_r$ are the eigenvalues of the cellular Laplacian Δ_p, listed according to their multiplicities, then

$$\log(\det_{\mathbb{R}}(\Delta_p)) = -\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\left(\sum_{i=1}^{r}\lambda_i^{-s}\right).$$

Therefore let us set $\zeta_p(s) = \sum_{\lambda>0}\lambda^{-s}$, summing over all positive eigenvalues of the form Laplacian $\Delta_p\colon \Omega^p(M;V)\to\Omega^p(M;V)$. The eigenvalues grow fast enough to ensure that the series converges to define a holomorphic function for $\mathrm{Re}(s) > \frac{n}{2}$. It has a meromorphic extension to the whole complex plain without pole in zero. We define the *analytic Reidemeister torsion* or *Ray–Singer torsion* of M by

$$\rho^a(M;V) = \frac{1}{2}\sum_{p\geq 0}(-1)^p\, p\, \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\zeta_p(s).$$

J. Cheeger [26] and W. Müller [81] independently proved the conjecture of D. B. Ray and I. M. Singer [94, p. 151] that Ray–Singer torsion equals Reidemeister torsion. In our L^2-setting, the passage from the finite-dimensional orthogonal representation V to the infinite-dimensional unitary representation $\ell^2\Gamma$ effects that the spectrum of the Laplacian can no longer be assumed discrete. Nevertheless, we can use the Γ-*function* $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t$ to rewrite

$$\zeta_p(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum_{\lambda>0}e^{-\lambda t}\mathrm{d}t. \tag{3.14}$$

The widespread use of the Γ-function throughout mathematics should prevent any confusion with our notation "Γ" for the group acting on M. The sum $\sum_{\lambda>0}e^{-\lambda t}$ now has an obvious generalization to our L^2-Laplacian. It is given by

$$\theta_p^{\perp}(t) = \int_0^\infty e^{-t\lambda}\mathrm{d}F(\lambda) - b_p^{(2a)}(M) \tag{3.15}$$

which is the *Laplace transform* $\theta_p(t) = \int_0^\infty e^{-t\lambda}\mathrm{d}F(\lambda)$ of the spectral density function F of $\Delta_p\colon \Omega_c^p(M) \to L^2\Omega^p(M)$ subtracted by the p-th analytic L^2-Betti number of M because the eigenvalue zero was explicitly excluded in the sum. In order to substitute the sum in (3.14) by (3.15) we have to discuss convergence of the integral in (3.14). Fix $\varepsilon > 0$. For $t \to 0$ one verifies again that $\frac{1}{\Gamma(s)}\int_0^\varepsilon t^{s-1}\theta_p^{\perp}\mathrm{d}t$ defines a holomorphic function for $\mathrm{Re}(s) > \frac{n}{2}$ with meromorphic extension to $\mathbb{C}$ and no pole in zero. The convergence for $t\to\infty$ is the problematic part. If $\alpha_p^a(M) = \infty^+$, then $\theta_p^{\perp}$ decays exponentially, the integral converges and we can simplify

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\frac{1}{\Gamma(s)}\int_\varepsilon^\infty t^{s-1}\theta_p^{\perp}(t)\mathrm{d}t = \int_\varepsilon^\infty \frac{\theta_p^{\perp}(t)}{t}\mathrm{d}t.$$

In the general case, however, we do not see any ad hoc reason why the small eigenvalues of Δ_p should ensure that $\theta_p^{\perp}$ decays fast enough to yield a convergent integral. Instead, we introduce a bit of new terminology. We call M of *analytic* *determinant class* if $\int_\varepsilon^\infty \frac{\theta_p^\perp(t)}{t}\mathrm{d}t < \infty$ for $p = 0, \ldots, n$ and one (then all) $\varepsilon > 0$. Finally, we are in the position to give the following definition.

Definition 3.16 (Analytic L^2-torsion)**.** Let M be of analytic determinant class. Then the *analytic L^2-torsion* of M is given by

$$\rho^{(2a)}(M) = \frac{1}{2}\sum_{p\geq 0}(-1)^p\, p\left(\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^\varepsilon t^{s-1}\theta_p^\perp(t)\mathrm{d}t \;+\; \int_\varepsilon^\infty \frac{\theta_p^\perp(t)}{t}\mathrm{d}t\right).$$

Note that in the analytic picture we had no need to require that M were analytically L^2-acyclic. The Laplace transform $\theta_p(t) = \int_0^\infty e^{-t\lambda}\mathrm{d}F(\lambda)$ is precisely the von Neumann trace of the operator $e^{-t\Delta_p^a}$ defined by spectral calculus. According to [4, Proposition 4.16, p. 63] this trace can be calculated as

$$\theta_p(t) = \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{C}}(e^{-t\Delta_p^a}(x,x))\mathrm{dvol}$$

where $\mathcal{F}$ is a fundamental domain for the Γ-action on M and $e^{-t\Delta_p^a}(x,y)$ denotes the heat kernel associated with Δ_p^a. If M happens to be a homogeneous manifold, for example a symmetric space, then $\mathrm{tr}_{\mathbb{C}}(e^{-t\Delta_p^a}(x,x))$ is constant throughout $x \in M$ whence $\rho^{(2)}(M) = C(M)\,\mathrm{vol}(\Gamma\backslash M)$ with a constant $C(M)$ independent of Γ. This is in fact only one special case of a way more general peculiarity of analytic L^2-invariants.

Theorem 3.17 (Proportionality principle)**.** *Given a simply connected Riemannian manifold M, there are constants $B_p^{(2)}(M)$, $A_p(M)$ and $T^{(2)}(M)$ such that for every free proper cocompact isometric action $\Gamma \curvearrowright M$ (of analytic determinant class)*

$$\begin{aligned} b_p^{(2a)}(M;\mathcal{N}(\Gamma)) &= B_p^{(2)}(M)\,\mathrm{vol}(\Gamma\backslash M),\\ \alpha_p^{(a)}(M;\mathcal{N}(\Gamma)) &= A_p(M),\\ \rho^{(2a)}(M;\mathcal{N}(\Gamma)) &= T^{(2)}(M)\,\mathrm{vol}(\Gamma\backslash M). \end{aligned}$$

The theorem is proven in [67, Theorem 3.183, p. 201]. For the relationship between topological and analytic L^2-invariants we obtain the best possible result.

Theorem 3.18 (Topological and analytic L^2-invariants)**.** *Let M come equipped with a finite equivariant Γ-triangulation. Then*

$$b_p^{(2)}(M) = b_p^{(2a)}(M) \quad \textit{and} \quad \alpha_p(M) = \alpha_p^{(a)}(M) \quad \textit{for each } p.$$

The Γ-CW-complex M is of determinant class if and only if the Riemannian manifold M is of analytic determinant class. If so and if M is L^2-acyclic, then

$$\rho^{(2)}(M) = \rho^{(2a)}(M).$$

The result is due to J. Dodziuk for the L^2-Betti numbers [33], to A. V. Efremov for the Novikov–Shubin invariants [35] and lastly to D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald for the L^2-torsion [22]. This bridge theorem between topological and analytical methods makes L^2-invariants powerful because strong properties such as homotopy invariance or proportionality are apparent in one picture while arcane in the other.

One advantage of the analytic picture is that as soon as a simply connected Riemannian manifold M has any cocompact action by isometries, the constants $B_p^{(2)}(M)$, $A_p(M)$ and $T^{(2)}(M)$ are defined. We can then take Theorem 3.17 as the *definition of analytic L^2-invariants* for the Γ-manifold M *if* Γ *only acts with finite volume quotient* and not necessary cocompactly. This applies in particular to the case that M is a *symmetric space of noncompact type*, $M = G/K$ for a semisimple Lie group G with maximal compact subgroup K. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of

G and K. Recall that the *deficiency* of G is given by $\delta(G) = \operatorname{rank}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}) - \operatorname{rank}_{\mathbb{C}}(\mathfrak{k}_{\mathbb{C}})$. We may assume $G \subset \mathrm{GL}(n, \mathbb{R})$ and obtain compact subgroups $K \subset U \subset \mathrm{GL}(n, \mathbb{C})$ corresponding to $\mathfrak{k} \subset \mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{gl}(n, \mathbb{C})$. We call $M^d = U/K$ the *dual symmetric space* of M of *compact type*. It inherits a unique Riemannian metric from M by requiring that multiplication with "i" give an isometry $T_K(M) \to T_K(M^d)$.

Theorem 3.19 (L^2-invariants of symmetric spaces)**.** *Let $M = G/K$ be a symmetric space of noncompact type and let $m = \delta(G)$ and $n = \dim(M)$.*

(i) We have $B_p^{(2)}(M) = 0$ unless $m = 0$ and $n = 2p$ in which case

$$B_p^{(2)}(M) = \frac{\chi(M^d)}{\operatorname{vol}(M^d)}.$$

(ii) We have $A_p(M) = \infty^+$ unless $m > 0$ and $p \in [\frac{n-m}{2} + 1, \frac{n+m}{2}]$ in which case

$$A_p(M) = m.$$

(iii) We have $T^{(2)}(M) = 0$ unless $m = 1$ in which case $M = X_0 \times X_1$ is a product of a symmetric space $X_0 = G_0/K_0$ of noncompact type with $\delta(G_0) = 0$ and $X_1 = X_{p,q} = \mathrm{SO}^0(p,q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ with p, q odd or $X_1 = X_{\mathrm{SL}} = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$. The constant is then given by $T^{(2)}(M) = (-1)^{\frac{\dim(X_0)}{2}} \frac{\chi(X_0^d)}{\operatorname{vol}(X_0^d)} T^{(2)}(X_1)$ with

$$T^{(2)}(X_{p,q}) = (-1)^{\frac{pq-1}{2}} \frac{\chi(X_{p-1,q-1}^d)}{\operatorname{vol}(X_{p,q}^d)} \pi Q_{p+q-1} \quad \textit{or} \quad T^{(2)}(X_{\mathrm{SL}}) = \frac{2\pi}{3 \operatorname{vol}(X_{\mathrm{SL}}^d)}$$

where the Q_k are certain positive rational numbers.

Part (i) can already be found in [12]. Parts (ii) and (iii) are due to M. Olbrich [85] generalizing previous work of J. Lott [63] and E. Hess–T. Schick [49]. We note that $n - m$ (thus $n + m$) is always even and positive. It is of course a consequence of the classical Cartan classification of symmetric spaces that $\delta(G) = 1$ implies the specific form of M described in (iii). To make sure that the formula for $T^{(2)}(X_{p,q})$ includes the case of hyperbolic space, let us moreover agree that $X_{p-1,q-1}$ and its dual is a point if $p = 1$ or $q = 1$. The first few numbers Q_k are $Q_3 = \frac{1}{3}$, $Q_5 = \frac{31}{45}$ and $Q_7 = \frac{221}{210}$. There is an interesting yet unhandy general formula for Q_k involving the Weyl dimension polynomial for finite dimensional representations of compact Lie groups [85, Proposition 5.3, p. 235]. If we assign the invariant metric to X_{SL} which is induced from the standard trace form on $\mathfrak{sl}(3, \mathbb{R})$, we obtain $\operatorname{vol}(X_{\mathrm{SL}}^d) = 4\pi^3$ whence $T^{(2)}(X_{\mathrm{SL}}) = \frac{1}{6\pi^2}$ [85, Proposition 1.4, p. 223]. The Killing form metric would in turn give $T^{(2)}(M) = \frac{1}{6^6\pi^2}$.

CHAPTER 4

L^2-invariants of lattices

This chapter brings together the two preceding ones. We will apply the cellular definitions of L^2-invariants to the Borel-Serre compactification and Kang's compactification for lattices in rank one groups in order to conclude the results on L^2-invariants of lattices in semisimple Lie groups as stated in the introduction. The outline of sections is as follows. In Section 1 we will recall that a theorem of D. Gaboriau reduces the computation of L^2-Betti numbers of nonuniform lattices to the well-known uniform case. Section 2 about Novikov-Shubin invariants begins with a precise explanation how Margulis arithmeticity reduces the case of irreducible lattices in higher rank groups to arithmetic subgroups of $\mathbb{Q}$-groups. We then prove Theorem 1.5 which gives an upper bound for the middle Novikov-Shubin invariant in the case of positive fundamental rank and rational rank one. We illustrate the Theorem in a concrete example. Then we turn to the rank one case where we apply Kang's bordification to prove Theorem 1.3 which gives an upper bound for the middle Novikov-Shubin invariant of lattices in $\mathrm{SO}^0(2n+1,1)$. Lastly we prove Theorem 1.4 which gives an upper bound for the Novikov-Shubin invariant of a nonuniform lattice right below the top dimension. This disproves the idea that cellular and analytic Novikov-Shubin invariants could also be equal for nonuniform lattices. In Section 3 we prove the vanishing of (virtual) L^2-torsion for lattices in even deficiency groups. We make no assumption on the rational rank so that the full structure theory of the Borel-Serre compactification will come into play. Section 4 on related results and problems concludes the chapter.

1. L^2-Betti numbers

Let us recall the following definition due to M. Gromov [43, Section 0.5.E, p. 16]. We give it in the equivalent version that appears in [38, Definition 1.1, p. 1059]. A *Lebesgue measure space* is a standard Borel space with a σ-finite measure.

Definition 4.1. Two countable groups Γ and Λ are called *measure equivalent* if there exists an infinite Lebesgue measure space (Ω, μ) with commuting, free, measure preserving actions of Γ and Λ such that both actions admit fundamental domains of finite measure.

The space (Ω, μ) together with the actions of Γ and Λ is called a *measure coupling* of Γ with Λ. If $X \subset \Omega$ and $Y \subset \Omega$ are fundamental domains of finite measure for the actions of Γ and Λ respectively, then the ratio $c = \frac{\mu(X)}{\mu(Y)} > 0$ is called the *index* of the measure coupling. Scaling the translation action $\mathbb{Z} \curvearrowright \mathbb{R}$ shows that in general a pair of measure equivalent groups can have measure couplings with varying indices. The standard example of measure equivalent groups are two lattices Γ and Λ, uniform or not, in the same locally compact second countable group H. Since H has lattices, it is unimodular so that it provides itself a measure coupling with its Haar measure where Γ and Λ act by left and right multiplication.

Theorem 4.2 (D. Gaboriau). *Let Γ and Λ be two countable measure equivalent groups with a measure coupling of index c. Then for all $p \geq 0$*

$$b_p^{(2)}(\Gamma) = c \cdot b_p^{(2)}(\Lambda).$$

In fact Gaboriau defines L^2-Betti numbers for (countable standard measure preserving) Borel relations building on the theory of L^2-cohomology for group actions on general spaces developed by J. Cheeger and M. Gromov [28]. In case the equivalence relation is induced by a free measure preserving action of Γ on a standard Borel space (without atoms) these L^2-Betti numbers equal the L^2-Betti numbers of Γ defined by Cheeger and Gromov. In this sense Gaboriau's theorem is a successful implementation of a third viewpoint on L^2-invariants: measure theory. Since any infinite amenable group is measure equivalent to $\mathbb{Z}$ [90, Theorem 6, p. 163], the theorem shows that all L^2-Betti numbers of infinite amenable groups vanish.

If G is a connected semisimple Lie group, then an invariant metric on the symmetric space $X = G/K$ fixes a Haar measure μ_X on G by requiring

$$\int_G f(g) \mathrm{d}\mu_X(g) = \int_{G/K} \int_K f(gk) \mathrm{d}\nu(k) \,\mathrm{dvol}(gK)$$

for integrable functions f where the Haar measure ν on K is normalized to have total measure $\nu(K) = 1$. If $\Gamma \subset G$ is a torsion-free lattice, then clearly $\mu_X(\Gamma\backslash G) = \mathrm{vol}(\Gamma\backslash X)$ for the induced invariant measure.

Theorem 1.1. *Let G be a connected semisimple linear Lie group with symmetric space $X = G/K$ fixing the Haar measure μ_X. Then for each $p \geq 0$ there is a constant $B_p^{(2)}(X) \geq 0$ such that for every lattice $\Gamma \leq G$ we have*

$$b_p^{(2)}(\Gamma) = B_p^{(2)}(X)\, \mu_X(\Gamma\backslash G).$$

Moreover $B_p^{(2)}(X) = 0$ unless $\delta(G) = 0$ and $\dim X = 2p$, when $B_p^{(2)}(X) = \frac{\chi(X^d)}{\mathrm{vol}(X^d)}$.

Proof. According to [9, Theorem C, p. 112] G possesses a uniform lattice Λ. By Selberg's Lemma [2] we may assume that Λ is torsion-free. Let $A \subset G$ and $B \subset G$ be fundamental domains for the left and right actions of Γ and Λ respectively. If $B' \subset G$ is a fundamental domain for the left action of Λ, then $\mu_X(B) = \mu_X(B')$ because G is unimodular. Theorem 4.2, Theorem 3.18 and Theorem 3.17 imply

$$b_p^{(2)}(\Gamma) = \tfrac{\mu_X(A)}{\mu_X(B')}\, b_p^{(2)}(\Lambda) = \tfrac{\mu_X(A)}{\mathrm{vol}(\Lambda\backslash X)}\, b_p^{(2a)}(X; \mathcal{N}(\Lambda)) = \mu_X(\Gamma\backslash G)\, B_p^{(2)}(X).$$

The information on the constant $B_p^{(2)}(X)$ was stated in Theorem 3.19 (i). □

2. Novikov–Shubin invariants

It was one of the great 20$^{\text{th}}$ century breakthroughs in mathematics when G. Margulis realized that for higher rank semisimple Lie groups, taking integer points of algebraic $\mathbb{Q}$-groups is essentially the only way to produce lattices. Recall that a lattice Γ in a connected semisimple Lie group G without compact factors is called *reducible* if G admits infinite connected normal subgroups H and H' such that $G = HH'$, such that $H \cap H'$ is discrete and such that $\Gamma/(\Gamma \cap H)(\Gamma \cap H')$ is finite. Otherwise Γ is called *irreducible*.

Theorem 4.3 (Margulis arithmeticity [75, Theorem 1, p. 97]). *Let $\mathbf{G}$ be a connected semisimple linear algebraic $\mathbb{R}$-group with $\mathrm{rank}_{\mathbb{R}}(\mathbf{G}) > 1$ and without direct $\mathbb{R}$-anisotropic factor. Let $\Gamma \subset \mathbf{G}(\mathbb{R})^0$ be an irreducible lattice. Then there is a linear algebraic $\mathbb{Q}$-group $\mathbf{H}$ and an $\mathbb{R}$-epimorphism $\varphi\colon \mathbf{H} \to \mathrm{Ad}\,\mathbf{G}$ such that the Lie group $(\ker\varphi)(\mathbb{R})$ is compact and such that $\varphi(\mathbf{H}(\mathbb{Z}))$ is commensurable with $\mathrm{Ad}\,\Gamma$.*

We reformulate this theorem in a version that is more appropriate for the purposes we have in mind. Two groups are called *abstractly commensurable* if they have isomorphic subgroups of finite index.

Corollary 4.4. *Let G be a connected semisimple linear Lie group of* $\operatorname{rank}_{\mathbb{R}}(G) > 1$ *without compact factors. Let $\Gamma \subset G$ be an irreducible lattice. Then there is a connected semisimple linear algebraic $\mathbb{Q}$-group $\mathbf{H}$ such that Γ and $\mathbf{H}(\mathbb{Z})$ are abstractly commensurable and such that G and $\mathbf{H}(\mathbb{R})$ define isometric symmetric spaces.*

Proof. By [110, Theorem 3.37, p. 38] there is a linear algebraic $\mathbb{R}$-group $\mathbf{G}$ such that $\mathbf{G}(\mathbb{R})^0 = G$. The group $\mathbf{G}$ cannot have $\mathbb{R}$-anisotropic factors because then G would have compact factors. Moreover $\mathbf{G}$ is semisimple, for example because its Lie algebra is the complexification of the Lie algebra of $\mathbf{G}(\mathbb{R})$. Since $(\mathbf{G}^0(\mathbb{R}))^0 = \mathbf{G}(\mathbb{R})^0$, we can assume that $\mathbf{G}$ is connected. By the theorem there is a $\mathbb{Q}$-group $\mathbf{H}$ and an $\mathbb{R}$-epimorphism $\varphi\colon \mathbf{H} \to \operatorname{Ad}\mathbf{G}$ with properties as stated. Since $\ker\varphi$ has compact real points, it cannot contain the additive or multiplicative groups $\mathbf{G}_a$ and $\mathbf{G}_m$ of the field $\mathbb{C}$ as $\mathbb{R}$-subgroups. Therefore $\ker\varphi$ is reductive. The center $Z(\ker\varphi)$ is normal in $\mathbf{H}$ and intersects $\mathbf{H}(\mathbb{Z})$ in a finite group. By replacing $\mathbf{H}$ with $\mathbf{H}/Z(\ker\varphi)$ if necessary, we may therefore assume that $\mathbf{H}$ is semisimple, being an extension of semisimple groups. By Selberg's Lemma [2] the arithmetic subgroup $\mathbf{H}(\mathbb{Z})$ has a torsion-free subgroup of finite index on which φ must restrict to an injection. Similarly $\mathbf{G}$ finitely covers $\operatorname{Ad}\mathbf{G}$ so that a torsion-free finite index subgroup of Γ is mapped injectively to $\operatorname{Ad}\mathbf{G}$. Since $\varphi(\mathbf{H}(\mathbb{Z}))$ is commensurable to $\operatorname{Ad}\Gamma$, we conclude that $\mathbf{H}(\mathbb{Z})$ and Γ have isomorphic subgroups of finite index. Moreover $\mathbf{H}^0$ has finite index in $\mathbf{H}$ so that the connected group $\mathbf{H}^0$ allows the same conclusion. Any maximal compact subgroup $K_{\mathbf{H}} \subset \mathbf{H}^0(\mathbb{R})$ must contain the normal compact subgroup $(\ker\varphi)(\mathbb{R}) \cap \mathbf{H}^0(\mathbb{R})$. Therefore φ induces an isometry $\mathbf{H}^0(\mathbb{R})/K_{\mathbf{H}} \xrightarrow{\sim} G/K$ of symmetric spaces, possibly after rescaling one of the invariant metrics. □

W. Lück, H. Reich and T. Schick have shown in [69, Theorem 3.7.1] that abstractly commensurable groups have equal Novikov–Shubin invariants. Therefore all irreducible lattices in higher rank semisimple Lie groups are covered when we work for the moment with arithmetic subgroups of connected semisimple linear algebraic $\mathbb{Q}$-groups. The rank one case will be treated afterwards. Before we come to the proof of Theorem 1.5, we need to recall the following definition for a compactly generated locally compact group H with compact generating set $V \subset H$ and Haar measure μ (compare [45]).

Definition 4.5. The group H has *polynomial growth of order* $d(H) \geq 0$ if

$$d(H) = \inf\left\{ k > 0 \colon \limsup_{n\to\infty} \tfrac{\mu(V^n)}{n^k} < \infty \right\}.$$

This definition is independent of the choice of V and of rescaling μ [45, p. 336]. If H is discrete and V is a finite symmetric generating set, we get back the familiar definition in terms of metric balls in the Cayley graph defined by word lengths. As in Chapter 2, let $G = \mathbf{G}(\mathbb{R})$ and let $K \subset G$ be a maximal compact subgroup giving rise to the symmetric space $X = G/K$. Let q be the *middle dimension* of X, so either $\dim X = 2q+1$ or $\dim X = 2q$. Let us recall the result we want to prove.

Theorem 1.5. *Let $\mathbf{G}$ be a connected semisimple linear algebraic $\mathbb{Q}$-group. Suppose that $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$ and $\delta(\mathbf{G}(\mathbb{R})) > 0$. Let $\mathbf{P} \subset \mathbf{G}$ be a proper rational parabolic subgroup. Then for every arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$*

$$\widetilde{\alpha}_q(\Gamma) \leq \delta(M_{\mathbf{P}}) + d(N_P).$$

Here the deficiency of a reductive Lie group G' is defined as $\delta(G') = \operatorname{rank}_{\mathbb{C}}(G') - \operatorname{rank}_{\mathbb{C}}(K')$ for a maximal compact subgroup $K' \subset G'$ as in the case of semisimple groups. The deficiency of G' is also known as the *fundamental rank* f-rank(X') of the associated symmetric space $X' = G'/K'$. Note that $\mathbf{G}$ trivially satisfies conditions (I) and (II) of page 10. Since $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$, all proper rational parabolic subgroups are conjugate under $\mathbf{G}(\mathbb{Q})$ so that the constant $\delta(M_{\mathbf{P}}) + d(N_P)$ only depends on $\mathbf{G}$. One example of a group $\mathbf{G}$ as in Theorem 1.5 is of course $\mathbf{G} = \mathrm{SO}(2n+1,1;\mathbb{C})$. But for this special group we will prove the more general Theorem 1.3 anyway. The point of Theorem 1.5 is that no restriction is made on the real rank of $\mathbf{G}$ and we will consider groups $\mathbf{G}$ with higher real rank in Example 4.11 after proving the theorem. The proof will require an estimation of Novikov–Shubin invariants of the boundary components $e(\mathbf{P}) = N_P \times X_{\mathbf{P}}$ of the Borel–Serre bordification $\overline{X}$. Since a product formula is available for Novikov–Shubin invariants, the calculation eventually reduces to Theorem 3.19 (ii) and the following theorem due to M. Rumin [96, Theorem 3.13, p. 144], see also [97, Theorem 4, p. 990].

Theorem 4.6 (M. Rumin)**.** *Let N be a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{n}$ comes with a grading $\mathfrak{n} = \bigoplus_{k=1}^{r} \mathfrak{n}_k$. Fix a left-invariant metric and assume that N possesses a uniform lattice. Then for each $p = 1, \ldots, \dim N$*

$$0 < A_p(N) \leq \sum_{k=1}^{r} k \dim \mathfrak{n}_k.$$

In fact, Rumin gives a finer pinching than the above, which in special cases gives precise values. For example $A_2(N) = \sum_{k=1}^{r} k \dim \mathfrak{n}_k$ if N is *quadratically presented* [96, Section 4.1, p. 146]. We remark that Rumin defines the p-th Novikov–Shubin invariant of N as

$$\alpha_p^R(N) = 2 \liminf_{\lambda \to 0^+} \frac{\log F(d^p_{\min}|_{\ker(d^p)^\perp})(\sqrt{\lambda})}{\log \lambda}$$

[96, equation (1), p. 125]. Since $b_p^{(2a)}(N) = 0$ by Theorem 3.8 (v) and Theorem 3.18, we have $F(d^p_{\min}|_{\ker(d^p)^\perp})(0) = 0$. Moreover $\operatorname{im}(d^{p-1})$ lies dense in $\ker(d^p)$ so that $\ker(d^p)^\perp = \operatorname{im}(d^{p-1})^\perp$ whence $d^p_{\min}|_{\ker(d^p)^\perp} = d^{p\perp}_{\min}$. Finally, substituting $\lambda \mapsto \lambda^2$ cancels out the factor of two so that we have $\alpha_p^R(N) = \alpha_{p+1}^{(a)}(N)$ in our notation. Compare the remark in [95, p. 4] on the confusion in the literature about indexing Novikov–Shubin invariants.

Corollary 4.7. *Let $\mathbf{P} \subset \mathbf{G}$ be a proper rational parabolic subgroup. Then for every torsion-free arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ and each $p = 1, \ldots, \dim N_P$ we have*

$$\alpha_p(N_P; \mathcal{N}(\Gamma_{N_P})) \leq d(N_P).$$

Proof. At the end of Section 2 in Chapter 2 we have seen that the Lie algebra $\mathfrak{n}_P$ of N_P is conjugate to a standard $\mathfrak{n}_I = \bigoplus_{\alpha \in \Sigma} \mathfrak{n}_{P,\alpha}$ and thus graded by the lengths of parabolic roots. Since $[\mathfrak{n}_{P,\alpha}, \mathfrak{n}_{P,\beta}] \subset \mathfrak{n}_{P,\alpha+\beta}$ by Jacobi identity, this graded algebra can be identified with the graded algebra associated with the filtration of $\mathfrak{n}_P$ coming from its lower central series. It thus follows from [45, Théorème II.1, p. 342] that the weighted sum appearing in Theorem 4.6 equals the degree of polynomial growth of N_P. Moreover $\alpha_p(N_P; \mathcal{N}(\Gamma_{N_P})) = A_p(N_P)$ by Theorem 3.17. □

Proposition 4.8. *Suppose* $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$. *Then for every proper rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ and every torsion-free arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ we have*

$$\alpha_q(e(\mathbf{P}); \mathcal{N}(\Gamma_P)) \leq \text{f-rank}(X_{\mathbf{P}}) + d(N_P).$$

Proof. Fix such $\mathbf{P} \subset \mathbf{G}$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$. We mentioned below Definition 2.8 that Γ possesses a neat and thus torsion-free subgroup of finite index. It induces a neat subgroup of finite index of Γ_P. Since Novikov–Shubin invariants remain unchanged for finite index subgroups, we may assume that Γ itself is neat. Thus $\Gamma_{M_{\mathbf{P}}}$ acts freely on $X_{\mathbf{P}}$. As $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$, every proper rational parabolic subgroup is minimal (and maximal). So the boundary component $e(\mathbf{P})$ is closed as we observed below Proposition 2.15. Therefore the Γ_P-action on $e(\mathbf{P})$ is cocompact. Theorems 3.17 and 3.18 imply

$$\alpha_q(e(\mathbf{P}); \mathcal{N}(\Gamma_P)) = \alpha_q(N_P \times X_{\mathbf{P}}; \mathcal{N}(\Gamma_{N_P} \times \Gamma_{M_{\mathbf{P}}})).$$

This observation enables us to apply the *product formula* for Novikov–Shubin invariants [67, Theorem 2.55(3), p. 97]. It says that $\alpha_q(N_P \times X_{\mathbf{P}}; \mathcal{N}(\Gamma_{N_P} \times \Gamma_{M_{\mathbf{P}}}))$ equals the minimum of the union of the four sets

$$\{\alpha_{i+1}(N_P) + \alpha_{q-i}(X_{\mathbf{P}})\colon\ i = 0, \dots, q-1\},$$
$$\{\alpha_i(N_P) + \alpha_{q-i}(X_{\mathbf{P}})\colon\ i = 1, \dots, q-1\},$$
$$\{\alpha_{q-i}(X_{\mathbf{P}})\colon\ i = 0, \dots, q-1,\ b_i^{(2)}(N_P) > 0\},$$
$$\{\alpha_i(N_P)\colon\ i = 1, \dots, q,\ b_{q-i}^{(2)}(X_{\mathbf{P}}) > 0\}.$$

We need to discuss one subtlety here. Applying the product formula requires us to verify that both N_P and $X_{\mathbf{P}}$ have the *limit property*. This means that "lim inf" in Definition 3.3 of the Novikov–Shubin invariants equals "lim sup" of the same expression. But this follows from the explicit calculations in [97] and [85]. Note that the third set above is actually empty because of Theorem 3.8 (v). The group $\mathbf{M_P} = \mathbf{Z_P}\mathbf{M'_P}$ is the almost direct product of its center $\mathbf{Z_P}$ and the derived subgroup $\mathbf{M'_P} = [\mathbf{M_P}, \mathbf{M_P}]$ which is semisimple. Accordingly, the boundary symmetric space $X_{\mathbf{P}} = X_{\mathbf{P}}^{\mathrm{Eucl}} \times X_{\mathbf{P}}^{\mathrm{nc}}$ is the product of a Euclidean symmetric space and a symmetric space of noncompact type. Clearly $\text{f-rank}(X_{\mathbf{P}}^{\mathrm{Eucl}}) = \dim X_{\mathbf{P}}^{\mathrm{Eucl}}$ so that

$$\text{f-rank}(X_{\mathbf{P}}) = \text{f-rank}(X_{\mathbf{P}}^{\mathrm{Eucl}} \times X_{\mathbf{P}}^{\mathrm{nc}}) = \dim X_{\mathbf{P}}^{\mathrm{Eucl}} + \text{f-rank}(X_{\mathbf{P}}^{\mathrm{nc}}).$$

As $\text{s-rank}(\mathbf{P}) = 1$ we get $\dim e(\mathbf{P}) = \dim X - 1$ with $\dim X = 2q$ or $\dim X = 2q+1$. Let us set $n = \dim N_P$, hence $\dim X_{\mathbf{P}} = \dim X - 1 - n$. Now we distinguish two cases. First we assume that $\text{f-rank}(X_{\mathbf{P}}) = 0$. Then $X_{\mathbf{P}} = X_{\mathbf{P}}^{\mathrm{nc}}$ is even-dimensional and we obtain from Theorem 3.19 (i) that $b^{(2)}_{q-\lceil \frac{n}{2} \rceil}(X_{\mathbf{P}}) > 0$. Here for a real number $a \in \mathbb{R}$ we denote by $\lceil a \rceil$ and $\lfloor a \rfloor$ the smallest integer not less than a and the largest integer not more than a, respectively. Therefore the Novikov–Shubin invariant $\alpha_{\lceil \frac{n}{2} \rceil}(N_P)$ appears in the fourth set above and is bounded by $d(N_P)$ according to Corollary 4.7. Now let us assume $\text{f-rank}(X_{\mathbf{P}}) > 0$. We compute $q - \lceil \frac{n}{2} \rceil = \lfloor \frac{\dim X_{\mathbf{P}} + 1}{2} \rfloor$ if $\dim X = 2q$ and $q - \lfloor \frac{n}{2} \rfloor = \lceil \frac{\dim X_{\mathbf{P}}}{2} \rceil$ if $\dim X = 2q+1$. We claim that both values lie in the interval $[\frac{1}{2}(\dim X_{\mathbf{P}} - \text{f-rank}(X_{\mathbf{P}})) + 1, \frac{1}{2}(\dim X_{\mathbf{P}} + \text{f-rank}(X_{\mathbf{P}}))]$. This is clear if $\dim X_{\mathbf{P}}$ is odd because then both values equal $\frac{\dim X_{\mathbf{P}} + 1}{2}$ which is the arithmetic mean of the interval limits. If on the other hand $\dim X_{\mathbf{P}}$ is even, then both values equal $\frac{\dim X_{\mathbf{P}}}{2}$. The fundamental rank $\text{f-rank}(X_{\mathbf{P}})$ is then likewise even and thus $\text{f-rank}(X_{\mathbf{P}}) \geq 2$. Therefore $\frac{1}{2}(\dim X_{\mathbf{P}} - \text{f-rank}(X_{\mathbf{P}})) + 1 \leq \frac{\dim X_{\mathbf{P}}}{2}$ and the claim is verified. It follows from [67, equation (5.14), p. 230] that in the two cases $\alpha_{q-\lceil \frac{n}{2} \rceil}(X_{\mathbf{P}})$ and $\alpha_{q-\lfloor \frac{n}{2} \rfloor}(X_{\mathbf{P}})$ are bounded by $\text{f-rank}(X_{\mathbf{P}}^{\mathrm{nc}}) + \dim X_{\mathbf{P}}^{\mathrm{Eucl}} = \text{f-rank}(X_{\mathbf{P}})$. Moreover $\alpha_{\lceil \frac{n}{2} \rceil}(N_P) \leq d(N_P)$ and $\alpha_{\lfloor \frac{n}{2} \rfloor}(N_P) \leq d(N_P)$ again by Corollary 4.7 so that either the number $\alpha_{\lceil \frac{n}{2} \rceil}(N_P) + \alpha_{q-\lceil \frac{n}{2} \rceil}(X_{\mathbf{P}})$ or the number $\alpha_{\lfloor \frac{n}{2} \rfloor}(N_P) + \alpha_{q-\lfloor \frac{n}{2} \rfloor}(X_{\mathbf{P}})$ appears in the second of the four sets above and both are bounded by $d(N_P) + \text{f-rank}(X_{\mathbf{P}})$. So in any case we conclude $\alpha_q(e(\mathbf{P})) \leq \text{f-rank}(X_{\mathbf{P}}) + d(N_P)$. □

We make one last elementary observation to prepare the proof of Theorem 1.5.

Lemma 4.9. *Let the discrete group Γ act freely and properly on the path-connected space X. Let $Y \subset X$ be a simply connected subspace which is invariant under the action of a subgroup $\Lambda \leq \Gamma$. Then the induced homomorphism $\Lambda = \pi_1(\Lambda \backslash Y) \to \pi_1(\Gamma \backslash X)$ is injective.*

PROOF. From covering theory we obtain a commutative diagram of groups

$$\begin{array}{ccc} \pi_1(\Lambda\backslash Y) & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \\ \pi_1(\Gamma\backslash X) & \longrightarrow & \Gamma. \end{array}$$

The upper map is an isomorphism and the right hand map is injective. So the left hand map must be injective as well. □

PROOF (OF THEOREM 1.5). Again by Selberg's Lemma and stability of Novikov-Shubin invariants for finite index subgroups [69, Theorem 3.7.1], we may assume that Γ is torsion-free. The bordification $\overline{X}$ is L^2-acyclic by Theorem 1.1. According to Lemma 3.10 we thus have $\widetilde{\alpha}_q(\overline{X}) \leq \alpha_q(\partial\overline{X})$. Recall from (2.14) that the Borel–Serre boundary $\partial\overline{X} = \bigcup_{\mathbf{P}\subsetneq\mathbf{G}} e(\mathbf{P})$ is given by the disjoint union of all boundary components of proper rational parabolic subgroups. Since $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$, every proper rational parabolic subgroup is minimal so all the boundary components are closed. As $\overline{X}$ is normal (T_4), the Borel–Serre boundary is in fact the coproduct $\partial\overline{X} = \coprod_{\mathbf{P}\ \min} e(\mathbf{P})$ of all boundary components of minimal rational parabolic subgroups. Proposition 2.24 implies that there is a finite system of representatives $\mathbf{P}_1, \ldots, \mathbf{P}_k$ of Γ-conjugacy classes of minimal rational parabolic subgroups which give the decomposition $\Gamma\backslash\partial\overline{X} = \coprod_{i=1}^k e'(\mathbf{P}_i)$. It thus follows from Lemma 4.9 applied to each $e(\mathbf{P}_i) \subset \overline{X}$ and $\Gamma_{P_i} \leq \Gamma$ that $\partial\overline{X} = \coprod_{i=1}^k e(\mathbf{P}_i) \times_{\Gamma_{P_i}} \Gamma$. According to [67, Lemma 2.17(3), p. 82] we obtain $\alpha_q(\partial\overline{X}) = \min_i \{\alpha_q(e(\mathbf{P}_i) \times_{\Gamma_{P_i}} \Gamma)\}$. Since the minimal rational parabolic subgroups $\mathbf{P}_1, \ldots, \mathbf{P}_k$ are $\mathbf{G}(\mathbb{Q})$-conjugate, we have in fact $\alpha_q(\partial\overline{X}) = \alpha_q(e(\mathbf{P}_1) \times_{\Gamma_{P_1}} \Gamma)$. The induction principle for Novikov–Shubin invariants [67, Theorem 2.55(7), p. 98] in turn says that $\alpha_q(e(\mathbf{P}_1) \times_{\Gamma_{P_1}} \Gamma; \mathcal{N}(\Gamma)) = \alpha_q(e(\mathbf{P}_1); \mathcal{N}(\Gamma_{P_1}))$ which is bounded from above by $\text{f-rank}(X_{\mathbf{P}_1}) + d(N_{P_1})$ according to Proposition 4.8. □

We want to discuss how the upper bound $\delta(M_{\mathbf{P}}) + d(N_P)$ appearing in Theorem 1.5 can actually be computed for a particular choice of $\mathbf{G}$. To this end we shall allow ourselves a brief digression on the classification theory of semisimple algebraic groups over a general field k as outlined in [104]. Let K be the separable closure of k and let $\mathcal{G} = \operatorname{Gal}(K/k)$ be the absolute Galois group of k. Let $\mathbf{G}$ be a semisimple algebraic k-group. Then any maximal torus $\mathbf{T} \subset \mathbf{G}$ is K-split and contains a maximal k-split torus $\mathbf{S}$. Let $\mathbf{Z}$ be the maximal central k-anisotropic torus of the centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})$ of $\mathbf{S}$. Then the derived subgroup $\mathcal{Z}_{\mathbf{G}}(\mathbf{S})'$ is called the *semisimple anisotropic kernel* and the group $\mathbf{Z}\mathcal{Z}_{\mathbf{G}}(\mathbf{S})'$ is called the *reductive anisotropic kernel* of $\mathbf{G}$. Both are well-defined up to k-isomorphism. A Theorem of J. Tits [104, Theorem 2, p. 43] says that the k-isomorphism type of $\mathbf{G}$ is determined by its K-isomorphism type, the semisimple anisotropic kernel and the *Tits index*. The Tits index is given by the triple $(\Delta, \Delta_0, *)$ where Δ denotes a set of simple roots in the root system $\Phi(\mathbf{G}, \mathbf{T}) \subset X_K(\mathbf{T})$ of $\mathbf{G}$, the subset $\Delta_0 \subset \Delta$ is given by the simple roots in Δ which restrict to zero on $\mathbf{S}$ and "$*$" denotes the *star action* of $\mathcal{G}$ on Δ defined as follows. The Galois group $\mathcal{G}$ naturally acts on the characters $X_K(\mathbf{T})$ such that the root system $\Phi(\mathbf{G}, \mathbf{T})$ is an invariant subset. An element $\sigma \in \mathcal{G}$ maps the simple roots Δ to yet another set of simple roots $\sigma(\Delta)$. Since the Weyl group of $\Phi(\mathbf{G}, \mathbf{T})$ acts simply transitively on Weyl chambers, there is a unique Weyl group element w such that $w(\sigma(\Delta)) = \Delta$ and we define $\sigma* = w \circ \sigma$. Tits indices can be visualized by the Dynkin diagram of $\Phi(\mathbf{G}, \mathbf{T})$ representing the simple roots Δ where elements in the same $*$-orbit are drawn close to one another and where the *distinguished orbits*, those that do not lie in Δ_0, are circled. An example is presented in Figure 4.10 where on the right hand side each of the two upper nodes is close to the facing lower node.

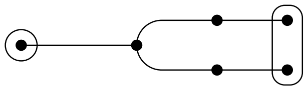


FIGURE 4.10. The Tits index of exceptional type ${}^2E_{6,2}^{16}$ which occurs for $k = \mathbb{Q}$ but does not exist over finite or $\mathfrak{p}$-adic fields.

The notation for Tits indices follows the pattern ${}^gX_{n,r}^t$ where X_n denotes the type of the Dynkin diagram and r gives the number of distinguished orbits, which is equal to the k-rank of $\mathbf{G}$. The index g is the order of the effectively acting quotient of $\mathcal{G}$ and t is a further characteristic number which we agree to be the dimension of the reductive anisotropic kernel in the case of exceptional types; for classical types we put t in parentheses and we let it denote the degree of a certain division k-algebra which can be used to define $\mathbf{G}$. If $g = 1$, we say the group is of *inner* type, otherwise of *outer* type. The Tits index of the semisimple anisotropic kernel is obtained by dropping the distinguished vertices and the edges starting or ending in it. J. Tits lists the possible indices in [104, Table II, pp. 54-61].

Now let us specialize to a group $\mathbf{G}$ over $k = \mathbb{Q}$ as in Theorem 1.5. We first explain how to compute the number $d(N_P)$. The Lie algebra $\mathfrak{n}_P$ of N_P has the decomposition $\mathfrak{n}_P = \bigoplus_{\alpha\in\Sigma} \mathfrak{n}_{P,\alpha}$ as we saw at the end of Section 2 in Chapter 2 so that $\mathfrak{n}_P$ is graded by parabolic root lengths. In view of the formula in Theorem 4.6 it only remains to determine Σ and the *multiplicities* m_α given by the dimensions of the root spaces $\mathfrak{n}_{P,\alpha}$. Note from below Proposition 2.2 that we can choose a base point $x_0 = K$ such that the decomposition $\mathbf{P} = \mathbf{N_P}\mathbf{S}_{\mathbf{P},x_o}\mathbf{M}_{\mathbf{P},x_0} = \mathbf{N_P}\mathbf{S_P}\mathbf{M_P}$ in equation (2.3) consists of $\mathbb{Q}$-groups. Since $\mathbf{P}$ is minimal, the torus $\mathbf{S_P}$ is in fact a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$. Associated with $\mathbf{S_P}$ we have the *restricted roots* $\Phi(\mathbf{G}, \mathbf{S_P}) \subset X_{\mathbb{Q}}(\mathbf{S_P})$ and the minimal parabolic subgroup $\mathbf{P}$ corresponds to a choice of positive restricted roots $\Phi^+(\mathbf{G}, \mathbf{S_P}) \subset \Phi(\mathbf{G}, \mathbf{S_P})$ which can be identified with Σ. Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus that contains $\mathbf{S_P}$. We turn the $\mathbb{R}$-vector space $X_{\overline{\mathbb{Q}}}(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ into a Euclidean space by choosing an inner product $\langle \cdot, \cdot \rangle$ invariant under the (finite) Weyl group $\mathcal{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. We can identify $X_{\mathbb{Q}}(\mathbf{S_P}) \otimes_{\mathbb{Z}} \mathbb{R}$ with the subspace of $X_{\overline{\mathbb{Q}}}(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ orthogonal to the characters vanishing on $\mathbf{S_P}$. Note that characters over k-split tori are automatically defined over k so that we have a restriction map $X_{\overline{\mathbb{Q}}}(\mathbf{T}) \to X_{\mathbb{Q}}(\mathbf{S_P})$ which corresponds to the orthogonal projection $X_{\overline{\mathbb{Q}}}(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R} \to X_{\mathbb{Q}}(\mathbf{S_P}) \otimes_{\mathbb{Z}} \mathbb{R}$. The subset of positive roots in $\Phi(\mathbf{G}, \mathbf{T})$ which do not restrict to zero on $\mathbf{S_P}$ maps surjectively to $\Phi^+(\mathbf{G}, \mathbf{S_P})$, which specifies Σ. Since root spaces over the algebraic closure $\overline{\mathbb{Q}}$ are one-dimensional, the multiplicities m_α for $\alpha \in \Sigma$ are moreover given by the number of roots in $\Phi(\mathbf{G}, \mathbf{T})$ that restrict to α.

Next we turn our attention to the summand $\delta(\mathbf{M_P})$. From comparison with standard parabolic subgroups we see that the Lie algebra of P has the decomposition $\mathfrak{p} = \mathfrak{n}_P \oplus \mathfrak{z}(\mathfrak{a}_P)$. Accordingly the centralizer of $\mathbf{S_P}$ is given by $\mathcal{Z}_{\mathbf{G}}(\mathbf{S_P}) = \mathbf{S_P}\mathbf{M_P}$. In the proof of Proposition 4.8 we had written $\mathbf{M_P} = \mathbf{Z_P}\mathbf{M}'_{\mathbf{P}}$ as the almost direct product of the center and the derived subgroup. The torus $\mathbf{Z_P}$ is $\mathbb{Q}$-anisotropic because $\mathbf{M_P}$ satisfies condition (I), p. 10. By the above, $\mathbf{M}'_{\mathbf{P}}$ is the derived subgroup of $\mathcal{Z}_{\mathbf{G}}(\mathbf{S_P})$ as well. This shows that $\mathbf{M_P}$ is the reductive anisotropic kernel and $\mathbf{M}'_{\mathbf{P}}$ is the semisimple anisotropic kernel of $\mathbf{G}$. It follows from [104, equation (1), p. 40] that $\dim \mathbf{Z_P} = |\Delta| - |\Delta_0| - r$ where r denotes the number of distinguished orbits in the Tits index of $\mathbf{G}$. In particular $\mathbf{Z_P}$ is trivial, and thus $\mathbf{M_P}$ and $\mathbf{M}'_{\mathbf{P}}$ coincide, if $\mathbf{G}$ is of inner type. In general we have $\delta(M_{\mathbf{P}}) = \operatorname{rank}_{\mathbb{R}}(\mathbf{Z_P}) + \delta(M'_{\mathbf{P}})$. As mentioned we obtain the Tits index of $\mathbf{M}'_{\mathbf{P}}$ over $\mathbb{Q}$ by removing the distinguished orbits of the Tits index of $\mathbf{G}$. It is however the Tits index over $\mathbb{R}$ which is relevant for determining $\delta(M_{\mathbf{P}})$. Thus some further inspection in the particular cases is necessary as we want to illustrate in the following example.

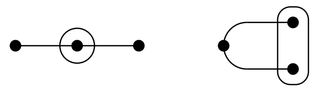
Example 4.11. Upon discussions with F. Veneziano and M. Wiethaup we have come up with the family of senary diagonal quadratic forms

$$Q^p = \langle 1, 1, 1, -1, -p, -p \rangle$$

over $\mathbb{Q}$ where p is a prime congruent to 3 mod 4. Let $\mathbf{G}^p = \mathrm{SO}(Q^p; \mathbb{C})$ be the $\mathbb{Q}$-subgroup of $\mathrm{SL}(6; \mathbb{C})$ of matrices preserving Q^p. By Sylvester's law of inertia, the groups $\mathbf{G}^p$ are $\mathbb{R}$-isomorphic to $\mathrm{SO}(3, 3; \mathbb{C})$, so that $\mathbf{G}(\mathbb{R}) \cong \mathrm{SO}(3, 3)$ which has deficiency one. Over $\mathbb{Q}$ there is an obvious way of splitting off one hyperbolic plane,

$$Q^p = \langle 1, -1 \rangle \perp \langle 1, 1, -p, -p \rangle,$$

but the orthogonal complement $\langle 1, 1, -p, -p \rangle$ is $\mathbb{Q}$-anisotropic. To see this, recall from elementary number theory that if a prime congruent to 3 mod 4 divides a sum of squares, then it must divide each of the squares. It thus follows from infinite descent that the Diophantine equation $x_1^2 + x_2^2 = p(x_3^2 + x_4^2)$ has no integer and thus no rational solution other than zero. Therefore $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}^p) = 1$ and $\mathbf{G}^p$ satisfies the conditions of Theorem 1.5. The group $\mathbf{G}^p$ is $\overline{\mathbb{Q}}$-isomorphic to $\mathrm{SO}(6; \mathbb{C})$ which accidentally has $\mathrm{SL}(4; \mathbb{C})$ as a double cover and thus is of type A_3. Since $\mathbf{G}^p$ has precisely one distinguished orbit, only two indices in Tit's list are possible, ${}^1A^2_{3,1}$ and ${}^2A^1_{3,1}$, as pictured.



To decide which one is correct, note that the hyperbolic plane in the above decomposition of Q^p gives an obvious embedding of a one-dimensional $\mathbb{Q}$-split torus $\mathbf{S}$ into $\mathbf{G}^p$. Let $\mathbf{P}$ be a minimal parabolic subgroup corresponding to a choice of positive restricted roots of $\mathbf{G}^p$ with respect to $\mathbf{S} = \mathbf{S_P}$. The centralizer $\mathcal{Z}_{\mathbf{G}^p}(\mathbf{S_P})$ obviously contains a $\mathbb{Q}$-subgroup that is $\mathbb{R}$-isomorphic to $\mathrm{SO}(2, 2; \mathbb{C})$ so that $\mathrm{SO}(2, 2; \mathbb{C}) \subset \mathbf{M}'_{\mathbf{P}}$ as an $\mathbb{R}$-embedding. Because of the exceptional isomorphism $D_2 = A_1 \times A_1$, the Dynkin diagram of $\mathbf{M}'_{\mathbf{P}}$ must contain two disjoint nodes. Removing the distinguished orbits, we therefore see that only the left hand Tits index ${}^1A^2_{3,1}$ can correspond to $\mathbf{G}^p$. Since it is of inner type, the center $\mathbf{Z_P}$ of $\mathbf{M_P}$ is trivial and in fact $\mathbf{M_P} = \mathbf{M}'_{\mathbf{P}} \cong_{\mathbb{R}} \mathrm{SO}(2, 2; \mathbb{C})$. Thus $\delta(M_{\mathbf{P}}) = \delta(\mathrm{SO}(2, 2)) = \delta(\mathrm{SL}(2; \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})) = 0$.

Let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus containing $\mathbf{S_P}$. The root system $\Phi(\mathbf{G}, \mathbf{T})$ is three-dimensional so that everything needed to compute $d(N_P)$ can be seen visually in Figure 4.12. In the Tits index of ${}^1A^2_{3,1}$, the left hand node corresponds to the arrow pointing up front, the center node corresponds to the arrow pointing down right and the right hand node corresponds to the arrow pointing up rear. Since both the left and right nodes of the Tits index do not lie in distinguished orbits, the subspace $X_{\mathbb{Q}}(\mathbf{S_P}) \otimes_{\mathbb{Z}} \mathbb{R}$ is given by the intersection of the planes orthogonal to their corresponding arrows which is the line going through the centers of the left face and right face of the cube. It follows that the restricted root system $\Phi(\mathbf{G}^p, \mathbf{S_P})$ is of type A_1 and that four roots of $\Phi(\mathbf{G}^p, \mathbf{T})$ restrict to each of the two roots in $\Phi(\mathbf{G}^p, \mathbf{S_P})$. Thus we have only one root of length one and multiplicity four in $\Sigma = \Phi^+(\mathbf{G_P}, \mathbf{S_P})$ which gives $d(N_P) = 4$. The symmetric space of $\mathbf{G}^p(\mathbb{R})$ has dimension nine, so Theorem 1.5 gives

$$\widetilde{\alpha}_4(\mathbf{G}^p(\mathbb{Z})) \leq 4.$$

Note that the bound is uniform in p even though the quadratic forms Q^p and hence the groups $\mathbf{G}^p$ are definitely not mutually $\mathbb{Q}$-isomorphic. Since $\mathrm{SO}(6; \mathbb{C})$ is doubly

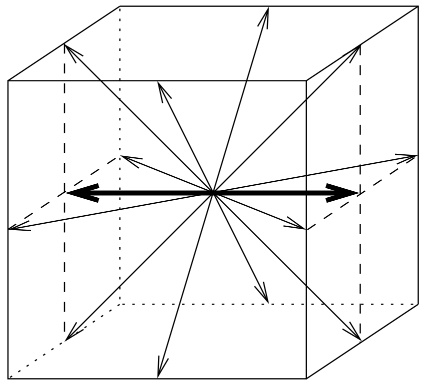


FIGURE 4.12. The root system of type A_3 with the restricted root system of the Tits index ${}^1A^2_{3,1}$ depicted by thick arrows.

covered by $\mathrm{SL}(4;\mathbb{C})$, we can take the preimage of $\mathbf{G}^p(\mathbb{Z})$ to get nonuniform lattices in $\mathrm{SL}(4;\mathbb{R})$ whose fourth Novikov-Shubin invariant is equally bounded by four.

Now we come to the case of real rank one semisimple Lie groups, where nonarithmetic lattices exist. Since the construction of Kang's compactification for lattices in rank one semisimple Lie groups largely parallels the Borel–Serre compactification, we easily obtain the statement for not necessarily arithmetic lattices acting on odd-dimensional hyperbolic space.

Theorem 1.3. *Let Γ be a lattice in $\mathrm{SO}^0(2n+1,1)$. Then $\widetilde{\alpha}_n(\Gamma) \leq 2n$.*

PROOF. We may assume that Γ is torsion-free, so Kang's bordification $\overline{X}_\Gamma$ is a finite Γ-CW model for $E\Gamma$, see Chapter 2, Section 5. Due to Theorem 1.1, the bordification $\overline{X}_\Gamma$ is L^2-acyclic. We conclude from Lemma 3.10 that $\widetilde{\alpha}_n(\Gamma) = \widetilde{\alpha}_n(\overline{X}_\Gamma) \leq \alpha_n(\partial\overline{X}_\Gamma)$. According to [56, p. 122], the boundary components $e(P)$ are closed ("type ***C2***") in $\overline{X}_\Gamma$ if $P \neq G$. Therefore the boundary $\partial\overline{X}_\Gamma$ is the coproduct $\partial\overline{X}_\Gamma = \coprod_{P\in\Delta_\Gamma} e(P)$. Moreover, it follows from the proof of [56, Proposition IV.23, p. 137] that there are only finitely many geometrically rational parabolic subgroups $P_1, \ldots, P_k \in \Delta_\Gamma$ up to Γ-conjugacy. Whence $\partial\overline{X}_\Gamma = \coprod_{i=1}^k N_{P_i} \times_{\Gamma_{P_i}} \Gamma$ and as in the preceding proof we obtain $\alpha_n(\partial\overline{X}_\Gamma) = \alpha_n(N_{P_1}; \mathcal{N}(\Gamma_{N_{P_1}}))$. Since $N_{P_1} \cong \mathbb{R}^{2n}$, as we will recall in Section 6.1 of Chapter 5, the latter term is bounded by $2n$ according to Theorem 4.6. □

We can give some sparse information about the Novikov–Shubin invariants outside middle dimension. The first observation is that only the value ∞^+ occurs in the first and in the top degree $n = \dim X = \dim G - \dim K$. Indeed, it is well-known that lattices Γ in noncompact semisimple Lie groups are not amenable [106, Example 2.7, p. 240 and Proposition 2.5, p. 241]. Thus $\alpha_1(\Gamma) = \infty^+$ according to Theorem 3.8 (iiic). But also $\alpha_n(\Gamma) = \infty^+$. For nonuniform lattices this follows from [65, Lemma 3.5.5, p. 34] because either Kang's compactification or the Borel-Serre compactification provide a topological manifold with nonempty boundary as classifying space of a finite-index subgroup of Γ. For uniform lattices the assertion follows from Theorem 3.8 (ii). For nonuniform lattices in rank one groups, we obtain moreover an upper bound in the degree right below the top dimension.

Theorem 1.4. *Let G be a connected semisimple linear Lie group of $\mathrm{rank}_{\mathbb{R}}(G) = 1$ with symmetric space $X = G/K$. Suppose that $n = \dim X \geq 3$. Let $P \subset G$ be a* *proper real parabolic subgroup. Then for every nonuniform lattice* $\Gamma \subset G$

$$\widetilde{\alpha}_{n-1}(\Gamma) \leq \tfrac{d(N_P)}{2}.$$

PROOF. Again we may assume that Γ is torsion-free. We apply the third inequality of Theorem 3.9 (ii) to the sequence of the pair $(\overline{X}_\Gamma, \partial\overline{X}_\Gamma)$ given by Kang's compactification, see Chapter 2, Section 5. Since $n \geq 3$, we have $b_1^{(2)}(\overline{X}_\Gamma) = 0$ by Theorem 1.1 and therefore $\alpha(H_1^{(2)}(j_*)) = \infty^+$ so that the inequality takes the form

$$\frac{1}{\alpha_1(\partial\overline{X}_\Gamma)} \leq \frac{1}{\alpha_1(\overline{X}_\Gamma)} + \frac{1}{\alpha_2(\overline{X}_\Gamma, \partial\overline{X}_\Gamma)}.$$

We have $\alpha_1(\overline{X}_\Gamma) = \infty^+$ by Theorem 3.8 (iiic). Using Theorem 3.8 (ii) we thus obtain $\alpha_{n-1}(\overline{X}_\Gamma) \leq \alpha_1(\partial\overline{X}_\Gamma)$. As in the above proof of Theorem 1.3 we get $\alpha_1(\partial\overline{X}_\Gamma) = \alpha_1(e(P); \mathcal{N}(\Gamma_{N_P}))$. Since $e(P) = N_P$, Theorem 3.8 (iiia) says $\alpha_1(e(P); \mathcal{N}(\Gamma_{N_P})) = d(N_P)$. By Remark 3.7 (ii) and since $\alpha_n(\Gamma) = \infty^+$ as explained above, we have $\widetilde{\alpha}_{n-1}(\Gamma) = \frac{1}{2}\min\{\alpha_{n-1}(\Gamma), \alpha_n(\Gamma)\} \leq \frac{1}{2}\min\{d(N_P), \infty^+\} = \frac{d(N_P)}{2}$. □

Note that in fact we proved $\alpha_{n-1}(\Gamma) \leq d(N_P)$ for the alternative version of Novikov-Shubin invariants. The Cartan classification divides the connected simple Lie groups G with $\operatorname{rank}_{\mathbb{R}}(G) = 1$ into five different types. We collect the data relevant for Theorem 1.4 in the following table.

Cartan type	G	X	$\dim X$	$d(N_P)$
BII / DII	$\mathrm{SO}^0(n,1)$	$\mathbb{H}^n_{\mathbb{R}}$	n	$n-1$
AIV	$\mathrm{SU}(n,1)$	$\mathbb{H}^n_{\mathbb{C}}$	$2n$	$2n$
CII	$\mathrm{Sp}(n,1)$	$\mathbb{H}^n_{\mathbb{H}}$	$4n$	$4n+2$
FII	$F_{4(-20)}$	$\mathbb{H}^2_{\mathbb{O}}$	16	22

The groups N_P appear as the nilpotent groups in *Iwasawa decompositions* of G. The growth rates can therefore easily be established by root system considerations as we did in Example 4.11. In that case the relevant information is given in a *Satake diagram* which corresponds to the Tits indices over $\mathbb{R}$ if the Lie group is given by the $\mathbb{R}$-points of a semisimple algebraic $\mathbb{R}$-group. We will give the precise structure of the groups N_P in Chapter 5, Section 6. Except for the groups $\mathrm{SO}^0(2n+1,1)$, all of the groups G have vanishing fundamental rank, so that their lattices have middle L^2-Betti numbers by Theorem 1.1. The theorem therefore says that the nonuniform ones give examples of lattices which both have a nonzero L^2-Betti number and a finite Novikov–Shubin invariant. There are no uniform lattices in semisimple Lie groups with this property. The same observation gives counterexamples to the tempting idea that for any torsion-free lattice Γ we had $\alpha_p(\Gamma) = \alpha_p^{(a)}(X; \mathcal{N}(\Gamma))$ with the definition for the right hand side given on p. 28. For example if $\Gamma \subset \mathrm{SO}^0(4,1)$ is torsion-free nonuniform, in other words Γ is the fundamental group of a noncompact finite-volume hyperbolic 4-manifold, then $\alpha_3(\Gamma) \leq 3$ because in that case $N_P \cong \mathbb{R}^3$, but $\alpha_3^{(a)}(\mathbb{H}^4; \mathcal{N}(\Gamma)) = \infty^+$ by Theorem 3.19 (ii) because $\delta(\mathrm{SO}^0(4,1)) = 0$.

3. L^2-torsion

Recall that the L^2-torsion is only defined for groups which are det-L^2-acyclic. According to Theorem 1.1, for a lattice $\Gamma \subset G$ in a semisimple Lie group this is equivalent to $\delta(G) > 0$. Among the rank one simple Lie groups, the only groups with positive deficiency are $G = \mathrm{SO}^0(2n+1,1)$ which have been treated by W. Lück and T. Schick in Theorem 1.2. For higher rank Lie groups, we again have Margulis arithmeticity available so that the following Theorem will be enough to cover general lattices in even deficiency groups as we will see subsequently.

Theorem 1.6. *Let $\mathbf{G}$ be a connected semisimple linear algebraic $\mathbb{Q}$-group. Suppose that $\mathbf{G}(\mathbb{R})$ has positive, even deficiency. Then every torsion-free arithmetic lattice $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is* det-*L^2-acyclic and*

$$\rho^{(2)}(\Gamma) = 0.$$

Note that in the odd deficiency case, Borel and Serre have proved correspondingly that $\chi(\Gamma) = 0$ in [17, Proposition 11.3, p. 482]. The core idea will also prove successful for the proof of Theorem 1.6 though various technical difficulties arise owed to the considerably more complicated definition of L^2-torsion. A combinatorial argument will reduce the calculation of the L^2-torsion of $\overline{X} = \bigcup_{\mathbf{P}\subseteq\mathbf{G}} \overline{e(\mathbf{P})}$ to the calculation of the L^2-torsion of the manifolds with corners $\overline{e(\mathbf{P})}$ for proper rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$ which form the boundary $\partial\overline{X}$ of the bordification. This in turn is settled by the following proposition.

Proposition 4.13. *Let $\mathbf{P} \subset \mathbf{G}$ be a proper rational parabolic subgroup. Then for every torsion-free arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ the finite free Γ_P-CW complex $\overline{e(\mathbf{P})} \subset \overline{X}$ is* det-*L^2-acyclic and $\rho^{(2)}(\overline{e(\mathbf{P})}; \mathcal{N}(\Gamma_P)) = 0$.*

Proof. L^2-torsion is multiplicative under finite coverings [67, Theorem 3.96(5), p. 164] so that similar to the proof of Proposition 4.8, we may assume that Γ is neat. We have already remarked below Theorem 2.23 that $e(\mathbf{P})$, hence its closure $\overline{e(\mathbf{P})}$, is a Γ_P-invariant subspace of the bordification $\overline{X}$. So $\overline{e(\mathbf{P})}$ regularly covers the subcomplex $\overline{e'(\mathbf{P})}$ of $\Gamma\backslash\overline{X}$ with deck transformation group Γ_P. It thus is a finite free Γ_P-CW complex. In fact $\overline{e(\mathbf{P})}$ is simply connected so that it can be identified with the universal covering of $\overline{e'(\mathbf{P})}$. The nilpotent group Γ_{N_P} is elementary amenable and therefore of det ≥ 1-class [102]. It is moreover infinite because it acts cocompactly on the nilpotent Lie group N_P. This Lie group is diffeomorphic to a nonzero Euclidean space because $\mathbf{P} \subset \mathbf{G}$ is proper. By Theorem 3.8 (v) the universal cover N_P of the finite CW-complex $\Gamma_{N_P}\backslash N_P$ is L^2-acyclic and $\rho^{(2)}(N_P; \mathcal{N}(\Gamma_{N_P})) = 0$. The canonical base point $K_P \in \overline{X}_{\mathbf{P}}$ and Proposition 2.15 define an inclusion $N_P \subset \overline{e(\mathbf{P})}$. Applying Lemma 4.9 to $N_P \subset \overline{e(\mathbf{P})}$ and $\Gamma_{N_P} \subset \Gamma_P$ shows that the fiber bundle $\overline{e'(\mathbf{P})}$ of Theorem 2.28 satisfies the conditions of Theorem 3.8 (iv). We conclude that $\overline{e(\mathbf{P})}$ is det-L^2-acyclic and

$$\rho^{(2)}(\overline{e(\mathbf{P})}, \mathcal{N}(\Gamma_P)) = \chi(\Gamma_{M_\mathbf{P}}\backslash\overline{X}_\mathbf{P})\, \rho^{(2)}(N_P; \mathcal{N}(\Gamma_{N_P})) = 0. \qquad \square$$

We remark that as an alternative to C. Wegner's theorem 3.8 (v), we could have concluded $\rho^{(2)}(N_P; \mathcal{N}(\Gamma_{N_P})) = 0$ from the fact that the nilmanifold $\Gamma_{N_P}\backslash N_P$ topologically is an iterated torus bundle over a torus. It therefore admits various S^1-actions so that the inclusion of an orbit induces an injection on fundamental groups. This also implies vanishing L^2-torsion according to a theorem of W. Lück [67, Theorem 3.105, p. 168].

Proof (of Theorem 1.6). Fix a torsion-free arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$. By Remark 3.7 (iii) the bordification $\overline{X}$ is of determinant class. It is L^2-acyclic by Theorem 1.1 because $\delta(G) > 0$. Lemma 3.11 says that the boundary $\partial\overline{X}$ is det-L^2-acyclic and since $\overline{X}$ is even-dimensional we have proven the theorem when we can show $\rho^{(2)}(\partial\overline{X}; \mathcal{N}(\Gamma)) = 0$. To this end consider the space $Y_k = \coprod_{\text{s-rank}(\mathbf{P})=k} e(\mathbf{P})$ for $k = 1, \ldots, \operatorname{rank}_\mathbb{Q}(\mathbf{G})$, the coproduct of all boundary components $e(\mathbf{P})$ of rational parabolic subgroups $\mathbf{P} \subset \mathbf{G}$ with split rank k. The usual action given in (2.22) defines a free proper action of Γ on Y_k because the split rank of a rational parabolic subgroup is invariant under conjugation with elements in $\mathbf{G}(\mathbb{Q})$. This action extends uniquely to a free proper action on the coproduct $\overline{Y_k} = \coprod_{\text{s-rank}(\mathbf{P})=k} \overline{e(\mathbf{P})}$ of closed boundary components because $Y_k \subset \overline{Y_k}$ is dense. The canonical Γ-equivariant map $\overline{Y_k} \to \overline{X}$ lies in the pullback diagram

$$\begin{array}{ccc} \overline{Y_k} & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow \\ \Gamma\backslash\overline{Y_k} & \longrightarrow & \Gamma\backslash\overline{X}. \end{array}$$

By Proposition 2.24, we have a finite system of representatives of Γ-conjugacy classes of rational parabolic subgroups of $\mathbf{G}$. Let $\mathbf{P}_1^k, \ldots, \mathbf{P}_{r_k}^k$ be an ordering of the subsystem of rational parabolic subgroups with split rank k. Then $\Gamma\backslash\overline{Y_k} = \coprod_{i=1}^{r_k} \overline{e'(\mathbf{P}_i^k)}$. We apply Lemma 4.9 to each inclusion $\overline{e(\mathbf{P}_i^k)} \subset \overline{X}$ and $\Gamma_{P_i^k} \leq \Gamma$ to conclude that $\overline{Y_k} = \coprod_{i=1}^{r_k} \overline{e(\mathbf{P}_i^k)} \times_{\Gamma_{\mathbf{P}_i^k}} \Gamma$. Since every space $\overline{e(\mathbf{P}_i^k)} \times_{\Gamma_{\mathbf{P}_i^k}} \Gamma$ is a Γ-invariant subcomplex of $\partial\overline{X}$, this endows $\overline{Y_k}$ with the structure of a finite free Γ-CW complex such that the equivariant map $\overline{Y_k} \to \partial\overline{X}$ is cellular. By the induction principle for L^2-torsion [67, Theorem 3.93(6) p. 162] and Proposition 4.13 $\overline{Y_k}$ is det-L^2-acyclic and

$$\rho^{(2)}(\overline{Y_k}; \mathcal{N}(\Gamma)) = \sum_{i=1}^{r_k} \rho^{(2)}(\overline{e(\mathbf{P}_i^k)} \times_{\Gamma_{P_i^k}} \Gamma; \mathcal{N}(\Gamma)) = \sum_{i=1}^{r_k} \rho^{(2)}(\overline{e(\mathbf{P}_i^k)}; \mathcal{N}(\Gamma_{P_i^k})) = 0.$$

From Lemma 3.11 we conclude that also the boundary $\partial\overline{Y_k} = \overline{Y_k}\backslash Y_k$ is det-L^2-acyclic. The lemma says moreover that $\rho^{(2)}(\partial\overline{Y_k}; \mathcal{N}(\Gamma)) = 0$ if $\overline{Y_k}$ is even-dimensional. But the same is true if $\overline{Y_k}$ is odd-dimensional because of Theorem 3.8 (ii). Consider the Γ-CW subcomplexes $\overline{X_k} = \bigcup_{\text{s-rank}(\mathbf{P}) \geq k} e(\mathbf{P})$ of $\overline{X}$ where $k = 1, \ldots, \operatorname{rank}_{\mathbb{Q}}(\mathbf{G})$. It follows from (2.21) that they can be constructed inductively as pushouts of finite free Γ-CW complexes

$$\begin{array}{ccc} \partial\overline{Y_k} & \longrightarrow & \overline{X}_{k+1} \\ \downarrow & & \downarrow \\ \overline{Y_k} & \longrightarrow & \overline{X}_k. \end{array} \tag{4.14}$$

The beginning of the induction is the disjoint union $\overline{X}_{\operatorname{rank}_{\mathbb{Q}}(\mathbf{G})} = \bigcup_{\mathbf{P} \text{ min.}} e(\mathbf{P})$ within $\overline{X}$. Since $e(\mathbf{P})$ is closed if $\mathbf{P}$ is minimal, we observe as in the proof of Theorem 1.5 that in fact $\overline{X}_{\operatorname{rank}_{\mathbb{Q}}(\mathbf{G})} = \coprod_{\mathbf{P} \text{ min.}} \overline{e(\mathbf{P})} = \overline{Y}_{\operatorname{rank}_{\mathbb{Q}}\mathbf{G}}$. Therefore Lemma 3.12 verifies that each $\overline{X_k}$ is det-L^2-acyclic and $\rho^{(2)}(\overline{X_k}; \mathcal{N}(\Gamma)) = 0$. This proves the theorem because $\overline{X_1} = \partial\overline{X}$. $\square$

A group Λ has *type F*, if it possesses a finite CW model for $B\Lambda$. If Λ is finitely presented, type F can be algebraically characterized as type FL, meaning that the trivial $\mathbb{Z}\Lambda$-module $\mathbb{Z}$ has a finite free resolution [21, Proposition 6.3, p. 200 and Theorem 7.1, p. 205]. The Euler characteristic of a type F group is defined by $\chi(\Lambda) = \chi(B\Lambda)$. A slight generalization of this is due to C. T. C. Wall [107]. If Λ virtually has type F, its *virtual Euler characteristic* is given by $\chi_{\text{virt}}(\Lambda) = \frac{\chi(\Lambda')}{[\Lambda:\Lambda']}$ for a finite index subgroup Λ' with finite CW model for $B\Lambda'$. This is well-defined because the Euler characteristic is multiplicative under finite coverings. Since the L^2-torsion in many respects behaves like an odd-dimensional Euler characteristic, we want to define its virtual version as well. If a group Γ is virtually det-L^2-acyclic, we define $\rho^{(2)}_{\text{virt}}(\Gamma) = \frac{\rho^{(2)}(\Gamma')}{[\Gamma:\Gamma']}$ for a finite index subgroup Γ' with finite det-L^2-acyclic Γ'-CW model for $E\Gamma'$. Again this is well-defined because $\rho^{(2)}$ is multiplicative under finite coverings.

Lemma 4.15. *Let Λ be virtually of type F and let Γ be virtually* det-L^2*-acyclic. Then $\Lambda \times \Gamma$ is virtually* det-L^2*-acyclic and*

$$\rho^{(2)}_{\mathrm{virt}}(\Lambda \times \Gamma) = \chi_{\mathrm{virt}}(\Lambda) \cdot \rho^{(2)}_{\mathrm{virt}}(\Gamma).$$

PROOF. Let $\Lambda' \leq \Lambda$ and $\Gamma' \leq \Gamma$ be finite index subgroups with finite classifying spaces such that $E\Gamma'$ is det-L^2-acyclic. Applying Theorem 3.8 (iv) to the trivial fiber bundle $B\Gamma' \to B(\Lambda' \times \Gamma') = B\Lambda' \times B\Gamma' \to B\Lambda'$, we obtain that $E(\Lambda' \times \Gamma')$ is det-L^2-acyclic and $\rho^{(2)}(\Lambda' \times \Gamma') = \chi(\Lambda')\rho^{(2)}(\Gamma')$. Therefore

$$\rho^{(2)}_{\mathrm{virt}}(\Lambda \times \Gamma) = \frac{\rho^{(2)}(\Lambda' \times \Gamma')}{[\Lambda \times \Gamma : \Lambda' \times \Gamma']} = \frac{\chi(\Lambda')\rho^{(2)}(\Gamma')}{[\Lambda : \Lambda'][\Gamma : \Gamma']} = \chi_{\mathrm{virt}}(\Lambda)\rho^{(2)}_{\mathrm{virt}}(\Gamma). \qquad \square$$

Theorem 1.7. *Let G be a connected semisimple linear Lie group with positive, even deficiency. Then every lattice $\Gamma \subset G$ is virtually* det-L^2*-acyclic and*

$$\rho^{(2)}_{\mathrm{virt}}(\Gamma) = 0.$$

PROOF. By Selberg's Lemma there exists a finite index subgroup $\Gamma' \subset \Gamma$ which is torsion-free. Thus Γ' can neither meet any compact factor nor the center of G which is finite because G is linear. Therefore we may assume that G has trivial center and no compact factors. Suppose Γ' was reducible. By [110, Proposition 4.24, p. 48] we have a direct product decomposition $G = G_1 \times \cdots \times G_r$ with $r \geq 2$ such that Γ' is commensurable with $\Gamma'_1 \times \cdots \times \Gamma'_r$ where $\Gamma'_i = G_i \cap \Gamma'$ is irreducible in G_i for each i. Again by Selberg's Lemma we may assume that $\Gamma'_1 \times \cdots \times \Gamma'_r$ is torsion-free. If $\mathrm{rank}_{\mathbb{R}}(G_i) = 1$, then Γ_i is type F by Kang's compactification, see Chapter 2, Section 5. If $\mathrm{rank}_{\mathbb{R}}(G_i) > 1$, then Γ_i is virtually type F by Margulis arithmeticity, Corollary 4.4, and the Borel-Serre compactification. Therefore, and by Theorem 1.1 and Remark 3.7 (iii), $\Gamma'_1 \times \cdots \times \Gamma'_r$ and thus Γ is virtually det-L^2-acyclic. Thus we may assume that $\Gamma'_1 \times \cdots \times \Gamma'_r$ is honestly det-L^2-acyclic and we have to show that $\rho^{(2)}(\Gamma'_1 \times \cdots \times \Gamma'_r) = 0$.

Since $\delta(G) > 0$, there must be a factor G_{i_0} with $\delta(G_{i_0}) > 0$. Let H be the product of the remaining factors G_i and let Γ_H be the product of the corresponding irreducible lattices Γ_i. If $\delta(H) > 0$, then Γ_H is det-L^2-acyclic by Theorem 1.1 and $\rho^{(2)}(\Gamma'_1 \times \cdots \times \Gamma'_r) = \rho^{(2)}(\Gamma'_{i_0} \times \Gamma_H) = 0$ by Lemma 4.15 because $\chi(\Gamma'_{i_0}) = 0$ by Theorem 3.8 (iv). If $\delta(H) = 0$, then $\delta(G_{i_0})$ is even, and Lemma 4.15 says that $\rho^{(2)}(\Gamma_H \times \Gamma'_{i_0}) = \chi(\Gamma_H)\rho^{(2)}(\Gamma'_{i_0})$. So we may assume that the original Γ' was irreducible. We have $\mathrm{rank}_{\mathbb{R}}(G) \geq \delta(G) \geq 2$ as follows from [18, Section III.4, Formula (3), p. 99]. By Margulis arithmeticity, Corollary 4.4, Γ' is abstractly commensurable to $\mathbf{H}(\mathbb{Z})$ for a connected semisimple linear algebraic $\mathbb{Q}$-group $\mathbf{H}$. Moreover $\delta(\mathbf{H}(\mathbb{R})) = \delta(G)$ because $\mathbf{H}(\mathbb{R})$ and G define isometric symmetric spaces. Theorem 1.6 completes the proof. $\square$

It remains to give some details for our application to the Lück–Sauer–Wegner conjecture.

Theorem 1.13. *Let $\mathcal{L}^{\mathrm{even}}$ be the class of* det-L^2*-acyclic groups that are measure equivalent to a lattice in a connected simple linear Lie group with even deficiency. Then Conjecture* 1.12 *holds true and Question* 1.11 *has affirmative answer for $\mathcal{L}^{\mathrm{even}}$.*

PROOF. Let $\Gamma \in \mathcal{L}^{\mathrm{even}}$ be measure equivalent to $\Lambda \subset G$ with G as stated. Then $\delta(G) > 0$ by Theorem 1.1 because Γ is L^2-acyclic by assumption. Since Γ has a finite $B\Gamma$, it is of necessity torsion-free so that Γ is a lattice in $\mathrm{Ad}\, G$ by [38, Theorem 3.1, p. 1062]. Theorem 1.7 applied to $\Gamma \subset \mathrm{Ad}\, G$ completes the proof. $\square$

4. Related results and problems

We conclude this chapter with a brief survey on related results and some follow-up questions. Theorem 1.1 gives all L^2-Betti numbers of lattices Γ in semisimple Lie groups G. Moreover the formula $B_p^{(2)}(X) = \frac{\chi(X^d)}{\mathrm{vol}(X^d)}$ expresses the proportionality constant in terms of the topology and geometry of the dual symmetric space. It would however be desirable to explain the constant in terms of the surrounding Lie group G itself in order to do justice to the rigidity phenomena of Margulis and Furman which seem to characterize Γ as some kind of "discrete copy" of G. This has been achieved by H. D. Petersen in his Ph. D. thesis [92] in the more general context of lattices in locally compact groups. Petersen defines L^2-Betti numbers $b_p^{(2)}(G, \mu)$ for second countable, unimodular, locally compact groups G with Haar measure μ and establishes the formula

$$b_p^{(2)}(\Gamma) = b_p^{(2)}(G, \mu)\mu(\Gamma\backslash G) \tag{4.16}$$

for all lattices $\Gamma \subset G$ provided G possesses a uniform one. This gives back Theorem 1.1 if G is a semisimple Lie group. The L^2-Betti numbers of locally compact groups are defined by $b_p^{(2)}(G, \mu) = \dim_{(\mathcal{N}(G),\psi)} H^p(G, L^2G)$ where ψ is the canonical weight of the group von Neumann algebra $\mathcal{N}(G)$. The weight ψ is tracial because G is unimodular. This makes it possible to define the dimension function $\dim_{(\mathcal{N}(G),\psi)}$ which measures the size of the continuous cohomology $H^p(G, L^2G)$. One of the difficulties compared to the discrete case is that ψ is in general only semifinite and not finite. An advantage of the more general setting is that there are many interesting examples of second countable, unimodular, locally compact, *totally disconnected* groups for whose lattices Petersen shows equation (4.16) without assuming the existence of uniform lattices. This is important in view of the example $G = \mathrm{Sp}_{2n}(\mathbb{F}_q((t)))$, the symplectic group over the nonarchimedean local field $\mathbb{F}_q((t))$ of formal Laurent series over the finite field $\mathbb{F}_q$, which for $n \geq 2$ possesses lattices though no such is uniform. Petersen shows $b_n^{(2)}(\mathrm{Sp}_{2n}(\mathbb{F}_q((t))), \mu) > 0$ for large enough q and thus $b_n^{(2)}(\Gamma) > 0$ for every lattice $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{F}_q((t)))$. In this context Petersen coined the slogan that whenever one has a result on some class of discrete groups, one should spare a thought whether it doesn't hold more generally for the corresponding class of totally disconnected groups. So the next step would be:

Problem 4.17. *Give a definition for Novikov-Shubin invariants of locally compact groups and compare the resulting values to the Novikov-Shubin invariants of the various lattices.*

As we hinted at, Petersen's theory of L^2-Betti numbers of locally compact groups is built on W. Lück's general dimension theory of modules over a group von Neumann algebra. These modules split into projective and torsion parts. Novikov-Shubin invariants, or rather their inverse *capacities*, measure the size of the torsion parts in much the same way as L^2-Betti numbers measure the size of the projective parts. So the hope is that capacities can also be defined in the more general situation using group von Neumann algebras of unimodular locally compact groups. Note that by the example $\mathrm{SO}^0(4,1)$ we considered on p. 40, there can be no definition $\alpha_p(G, \mu)$ of Novikov-Shubin invariants for a locally compact group G such that $\alpha_p(G, \mu) = \alpha_p(\Gamma)$ for all lattices $\Gamma \subset G$. So it would be interesting to know what values the most natural definition of $\alpha_p(G, \mu)$ will give in the case of a semisimple Lie group G with symmetric space X. One candidate would be the values $A_p(X)$, which coincide with the Novikov-Shubin invariants of the uniform lattices of G; another possibility would be the values $\alpha_p(\Gamma)$ for an arithmetic subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ of a linear algebraic $\mathbb{Q}$-group $\mathbf{G}$ with $G^0 = \mathbf{G}^0(\mathbb{R})$ and $\mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{R}}G$, which would correspond to the Novikov-Shubin invariants of the arithmetic subgroups for the most natural linear embedding of G. A conceptual reason to favor the first possibility is that Novikov-Shubin invariants are likely to be quasi-isometry invariants of discrete groups [67, Question 7.36, p. 313; 100, Theorem 1.6, p. 480]. As for Theorem 1.5 a solution to the following problem would of course be pleasing.

Problem 4.18. *Relax the conditions* $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$ *and* $\delta(G) > 0$ *in Theorem 1.5.*

In the current proof both conditions are essential. We are using the weak version of additivity for Novikov-Shubin invariants given by Theorem 3.9 (ii). The inequalities are only useful if the third summand vanishes so that we need $\delta(G) > 0$. If one tries to apply the first inequality to the short exact sequence coming from the pushout diagram in (4.14), one obtains $\min\{\alpha_p(\partial\overline{Y}_k), \alpha_p(\overline{X}_k)\} \leq 2\alpha_p(\overline{X}_{k+1})$ but we do not see that $\alpha_p(\partial\overline{Y}_k) \geq \alpha_p(\overline{X}_k)$ except of course when $\partial\overline{Y}_1 = \emptyset$ which happens if and only if $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 1$. However, we are more optimistic about the following problem.

Problem 4.19. *Compute the virtual* L^2*-torsion of all lattices in odd deficiency semisimple Lie groups.*

By the same proof as for Theorem 1.1, Conjecture 1.12 would imply that $\rho^{(2)}_{\mathrm{virt}}(\Gamma) = T^{(2)}(X)\,\mu_X(\Gamma\backslash G)$ for any lattice Γ in a connected semisimple linear Lie group G. In particular this would mean $\rho^{(2)}_{\mathrm{virt}}(\Gamma) \neq 0$ if and only if $\delta(G) = 1$. As we have already remarked in the introduction, for odd deficiency there is hardly any hope for a solution similar to the even case that would rely on the topological structure of the Borel-Serre compactification only. A more promising approach should be to try and generalize the method that has proven successful in the hyperbolic case in [72]. It suggests to look for comparison theorems between the analytic L^2-torsion of the finite-volume locally symmetric interior of the Borel-Serre compactification with the topological L^2-torsion of an exhaustion by compact manifolds that are obtained by chopping off the ends. Such an exhaustion has been described explicitly by E. Leuzinger [60, 61] which could be a helpful reference. The heat kernel manipulations performed by Lück and Schick in [72] make intensive use of the constant sectional curvature structure in the hyperbolic case. This and also the corners that arise in the Borel-Serre compactification in the higher $\mathbb{Q}$-rank case definitely prevent a straightforward generalization of the paper. On the other hand the work of Leuzinger, Olbrich and Rumin provide a set of powerful tools so that Problem 4.19 does not seem hopelessly difficult at this point.

More recently a twisted version $\rho^{(2)}_{\tau}(X)$ of L^2-torsion has come into focus. To explain this, let us assume that Γ is an arithmetic, uniform, torsion-free lattice so that Γ is commensurable with $\mathbf{G}(\mathbb{Z})$ for an anisotropic semisimple $\mathbb{Q}$-group $\mathbf{G}$ with $G = \mathbf{G}(\mathbb{R})$. As usual we write $X = G/K$ for the symmetric space. Choose a rational irreducible representation $\tau\colon \mathbf{G} \to \mathrm{GL}(V)$. Associated with the restriction of τ to Γ is the flat bundle E_τ over $\Gamma\backslash X$ which comes equipped with a distinguished hermitian fiber metric called admissible in [77, Lemma 3.1, p. 375]. Let $\Delta_p(\tau)$ be the Laplacian acting on p-forms on $\Gamma\backslash X$ with values in E_τ and let $\widetilde{\Delta}_p(\tau)$ be the lift to the universal covering X. Then we define the twisted L^2-torsion $\rho^{(2)}_{\tau}(X)$ by the same formula as in Definition 3.16 but using $\widetilde{\Delta}_p(\tau)$ instead of the ordinary "Δ_p". We get back the classical analytic L^2-torsion when τ is the trivial representation. The invariant $\rho^{(2)}_{\tau}(X)$ is of interest because it detects an algebraic property of the arithmetic group Γ: the size of the torsion part of the cohomology modules $H^*(\Gamma, M)$ for the local system defined by a Γ-invariant lattice $M \subset V$. Already from the classical equality of topological Reidemeister torsion and analytic Ray–Singer torsion S. Marshall and W. Müller have concluded that the order of $H^2(\Gamma, M_{2k})$, which is completely torsion, grows exponentially in k^2 in the special case of certain arithmetic uniform lattices Γ in $\mathrm{SL}(2,\mathbb{C})$ with representation $V = S^{2k}(\mathbb{C}^2)$ and arbitrary Γ-stable lattice $M_{2k} \subset S^{2k}(\mathbb{C}^2)$ [76]. This type of result has been generalized by W. Müller and J. Pfaff to all closed odd-dimensional hyperbolic manifolds and more general representations in [83] and subsequently to all closed locally symmetric spaces in [84]. In both cases the strategy is an asymptotic comparison between Ray–Singer and twisted L^2-torsion along a ray of highest weight representations and the computation of the twisted L^2-torsion along the lines of Olbrich [85]. Of course we now ask the following.

Problem 4.20. *Compute the twisted L^2-torsion of finite-volume locally symmetric spaces for suitable rays of highest weight representations. Conclude information about torsion in the cohomology of nonuniform arithmetic groups.*

Müller and Pfaff have attacked the simplest case, when X is hyperbolic space, in [82]. Instead of fixing the lattice and varying the local system, N. Bergeron and A. Venkatesh fix a local system and vary the lattice through a tower of congruence subgroups $\{\Gamma_N\}$ with trivial intersection [8]. These lattices are again assumed to be arithmetic subgroups of an anisotropic semisimple $\mathbb{Q}$-group and thus are in particular uniform lattices. Bergeron-Venkatesh conjecture that the limit

$$\lim_N \frac{\log |H_j(\Gamma_N, M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_N]}$$

always exists and that it is positive (a constant times the volume of $\Gamma\backslash X$) if and only if the ordinary L^2-torsion of Γ does not vanish and j is the middle dimension; in other words if and only if $\delta(G) = 1$ and $\dim X = 2j + 1$. So again an exponential growth of torsion is suspected, this time with respect to increasing the covolume of Γ. To support their conjecture they prove that if $\delta(G) = 1$ and if the arithmetic Γ-module M is strongly acyclic, then

$$\liminf_N \sum_j \frac{\log |H_j(\Gamma_N, M)_{\mathrm{tors}}|}{[\Gamma : \Gamma_N]} \geq c_{G,M} \mathrm{vol}(\Gamma\backslash X) > 0, \tag{4.21}$$

summing over all j with the same parity as $\frac{\dim X - 1}{2}$.

Problem 4.22. *Proof inequality* (4.21) *if* Γ *is a nonuniform arithmetic subgroup.*

While Bergeron–Venkatesh suspect their assumptions of $\{\Gamma_N\}$ being congruence subgroups with trivial intersection both being essential, they say explicitly that they expect (4.21) to hold for suitable sequences of subgroups of the nonuniform arithmetic group $\mathrm{SL}(2, \mathbb{Z}[\mathrm{i}])$ as well [8, p. 3].

Let us quit listing further problems and revisit Problem 4.19 instead, which appears most urgent to us at this point. To actually carry out the suggested strategy of comparing the cellular L^2-torsion of a compact exhaustion with the analytic L^2-torsion of the finite-volume locally symmetric space, a precise understanding of the geometry of the symmetric space is indispensable. As a first step we have uniformly constructed bases for all real semisimple Lie algebras such that the structure constants can be read off from the root system of the complexification with the involution determining the real structure; precisely the data given by a Tits–Satake diagram. Since this is work of independent interest, we give a self-contained presentation in the final chapter of this thesis. One consequence is that we obtain explicit coordinates for all symmetric spaces of noncompact type. These coordinates distinguish both a maximal flat totally geodesic submanifold and the complementing nilmanifold given by an Iwasawa N-group which coincides with the group $N_{\mathbf{P}}$ for a suitable minimal rational parabolic subgroup $\mathbf{P}$ in the case of a standard $\mathbb{Q}$-embedding $G = \mathbf{G}^0(\mathbb{R})$ with $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) = \mathrm{rank}_{\mathbb{R}}(\mathbf{G})$.

CHAPTER 5

Integral structures in real semisimple Lie algebras

In this chapter we construct a convenient basis for all real semisimple Lie algebras by means of an adapted Chevalley basis of the complexification. It determines (half-)integer structure constants which we express in terms of the root system and the automorphism defining the real structure only. Provided the real algebra admits one, the basis exhibits an explicit complex structure. Part of the basis spans the nilpotent algebra of an Iwasawa decomposition. This gives an intrinsic proof that Iwasawa N-groups have lattices. We give explicit realizations of all Iwasawa N-groups in the real rank one case and we construct coordinate charts for symmetric spaces of noncompact type in a uniform way. This chapter is available as a preprint in [55].

1. Summary of results

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and Killing form B. Denote its root system by $\Phi(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$. Given a root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be its root space and let $t_\alpha \in \mathfrak{h}$ be the corresponding *root vector* which is defined by $B(t_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. Set $h_\alpha = \frac{2t_\alpha}{B(t_\alpha,t_\alpha)}$ and for a choice of simple roots $\Delta(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi(\mathfrak{g}, \mathfrak{h})$, set $h_i = h_{\alpha_i}$. The following definition appears in [51, p. 147].

Definition 5.1. A *Chevalley basis* of $(\mathfrak{g}, \mathfrak{h})$ is a basis $\mathcal{C} = \{x_\alpha, h_i : \alpha \in \Phi(\mathfrak{g}, \mathfrak{h}), 1 \leq i \leq l\}$ of $\mathfrak{g}$ with the following properties.

(i) $x_\alpha \in \mathfrak{g}_\alpha$ and $[x_\alpha, x_{-\alpha}] = -h_\alpha$ for each $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$.
(ii) For all pairs of roots $\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{h})$ such that $\alpha + \beta \in \Phi(\mathfrak{g}, \mathfrak{h})$, let the constants $c_{\alpha\beta} \in \mathbb{C}$ be determined by $[x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$. Then $c_{\alpha\beta} = c_{-\alpha-\beta}$.

The existence of a Chevalley basis is easily established. C. Chevalley showed that the structure constants for such a basis are integers. More precisely in 1955 he published the following now classical theorem in [29, Théorème 1, p. 24], see also [51, Theorem 25.2, p. 147].

Theorem 5.2. *A Chevalley basis* $\mathcal{C}$ *of* $(\mathfrak{g}, \mathfrak{h})$ *yields the following structure constants.*

(i) $[h_i, h_j] = 0$ *for* $i, j = 1, \ldots, l$.
(ii) $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$ *for* $i = 1, \ldots, l$ *and* $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$.
(iii) $[x_\alpha, x_{-\alpha}] = -h_\alpha$ *and* h_α *is a* $\mathbb{Z}$*-linear combination of the elements* $h_1, \ldots, h_l$.
(iv) $c_{\alpha\beta} = \pm(r + 1)$ *where* r *is the largest integer such that* $\beta - r\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$.

As is customary we have used the notation $\langle \beta, \alpha \rangle = \frac{2B(t_\beta,t_\alpha)}{B(t_\alpha,t_\alpha)} \in \mathbb{Z}$ with $\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{h})$ for the Cartan integers of $\Phi(\mathfrak{g}, \mathfrak{h})$. The $\mathbb{Z}$-span $\mathfrak{g}(\mathbb{Z})$ of such a basis is obviously a Lie algebra over $\mathbb{Z}$ so that tensor products with finite fields can be considered. Certain groups of automorphisms of these algebras turn out to be simple. With this method Chevalley constructed infinite series of finite simple groups in a uniform way. For $\mathfrak{g}$ exceptional he also exhibited some previously unknown ones [25, p. 1].

But Theorem 5.2 states way more than the mere existence of a basis with integer structure constants. Up to sign, it gives the entire multiplication table of $\mathfrak{g}$ only in terms of the root system $\Phi(\mathfrak{g}, \mathfrak{h})$. The main result of this chapter will be an analogue of Theorem 5.2 for any *real* semisimple Lie algebra $\mathfrak{g}^0$. To make this more precise, let $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}^0$ determined by a Cartan involution θ. Let $\mathfrak{h}^0 \subset \mathfrak{g}^0$ be a θ-stable Cartan subalgebra such that $\mathfrak{h}^0 \cap \mathfrak{p}$ is of maximal dimension. Consider the complexification $(\mathfrak{g}, \mathfrak{h})$ of $(\mathfrak{g}^0, \mathfrak{h}^0)$. The complex conjugation σ in $\mathfrak{g}$ with respect to $\mathfrak{g}^0$ induces an involution of the root system $\Phi(\mathfrak{g}, \mathfrak{h})$. We will construct a real basis $\mathcal{B}$ of $\mathfrak{g}^0$ with (half-)integer structure constants. More than that, we compute the entire multiplication table of $\mathfrak{g}^0$ in terms of the root system $\Phi(\mathfrak{g}, \mathfrak{h})$ and its involution induced by σ. For the full statement see Theorem 5.18.

The idea of the construction is as follows. We pick a Chevalley basis $\mathcal{C}$ of $(\mathfrak{g}, \mathfrak{h})$ and for $x_\alpha \in \mathcal{C}$ we consider twice its real and its imaginary part, $X_\alpha = x_\alpha + \sigma(x_\alpha)$ and $Y_\alpha = \mathrm{i}(x_\alpha - \sigma(x_\alpha))$, as typical candidates of elements in $\mathcal{B}$. It is clear that $\sigma(x_\alpha) = d_\alpha x_{\alpha^\sigma}$ for some $d_\alpha \in \mathbb{C}$ where α^σ denotes the image of α under the action of σ on $\Phi(\mathfrak{g}, \mathfrak{h})$. But to hope for simple formulas expanding $[X_\alpha, X_\beta]$ as linear combination of other elements X_γ, we need to adapt the Chevalley basis $\mathcal{C}$ to get some control on the constants d_α. A starting point is the following lemma of D. Morris [78, Lemma 6.4, p. 480]. We state it using the notation we have established so far. Let τ be the complex conjugation in $\mathfrak{g}$ with respect to the compact form $\mathfrak{u} = \mathfrak{k} \oplus \mathrm{i}\mathfrak{p}$.

Lemma 5.3. *There is a Chevalley basis $\mathcal{C}$ of $(\mathfrak{g}, \mathfrak{h})$ such that for all $x_\alpha \in \mathcal{C}$*
(i) $\tau(x_\alpha) = x_{-\alpha}$,
(ii) $\sigma(x_\alpha) \in \{\pm x_{\alpha^\sigma}, \pm \mathrm{i} x_{\alpha^\sigma}\}$.

In fact Morris proves this for any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is the complexification of a general θ-stable Cartan subalgebra $\mathfrak{h}^0 \subset \mathfrak{g}^0$. With our special choice of a so-called *maximally noncompact* θ-stable Cartan subalgebra $\mathfrak{h}^0$, we can sharpen this lemma. We will adapt the Chevalley basis $\mathcal{C}$ to obtain $\sigma(x_\alpha) = \pm x_{\alpha^\sigma}$ (Proposition 5.8) and we will actually determine which sign occurs for each root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ (Proposition 5.13). By means of a Chevalley basis of $(\mathfrak{g}, \mathfrak{h})$ thus adapted to σ and τ we will then obtain a version of Theorem 5.2 over the field of real numbers (Theorem 5.18). We remark that a transparent method of consistently assigning signs to the constants $c_{\alpha\beta}$ has been proposed by Frenkel–Kac [37].

Various applications will be given. To begin with, if $\mathfrak{g}^0$ admits a complex structure, the basis $\mathcal{B}$ uncovers a particular complex structure explicitly. If a complex structure of $\mathfrak{g}^0$ is known, we explain how the freedom of choices in the construction of $\mathcal{B}$ can be used to reproduce the initial complex structure in this fashion. This leads to a nice characterization of the three different special cases of a semisimple algebra $\mathfrak{g}^0$ (split, compact or complex) in terms of the type of roots of the complexification (Theorem 5.19) which in itself was most likely known before. We see that in the split case the basis $\mathcal{B}$ boils down to twice the Chevalley basis $2\mathcal{C}$. In the compact case it becomes the basis in the standard construction of a compact form of a complex semisimple algebra $\mathfrak{g}$.

As another notable feature of the basis $\mathcal{B}$ we observe that part of it spans the nilpotent algebra $\mathfrak{n}$ in an Iwasawa decomposition $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. In fact, a variant of $\mathcal{B}$ is the disjoint union of three sets spanning the Iwasawa decomposition (Theorem 5.20). For all Iwasawa $\mathfrak{n}$-algebras we obtain integer structure constants whose absolute values have upper bound six. Invoking the classification of complex semisimple Lie algebras, we improve this bound to four (Theorem 5.21). Moreover, with our basis and a fixed set of signs for the constants $c_{\alpha\beta}$, the multiplication table of all exponentiated Iwasawa N-groups can be read off from the root system $\Phi(\mathfrak{g}, \mathfrak{h})$ and the involution σ. By a criterion of Malcev [73] the basis verifies that

Iwasawa N-groups contain uniform lattices (Corollary 5.22). This is a property which only countably many isomorphism types of nilpotent Lie groups possess. Yet it is easy to construct uncountable families of nonisomorphic nilpotent Lie groups. We explain a uniform way of constructing coordinate systems for symmetric spaces of noncompact type (Section 5.4). Finally we illustrate our methods by giving explicit realizations of Iwasawa N-groups case by case for real rank one semisimple Lie algebras (Sections 6.1-6.4). For $\mathfrak{g}^0 = \mathfrak{so}(n,1)$ and $\mathfrak{g}^0 = \mathfrak{su}(n,1)$ we obtain the abelian group $\mathbb{R}^{n-1}$ and the Heisenberg group H^{2n-1} (compare [58, Problem 5, p. 426] and [47, Exercise E1, p. 215]). Similarly, for $\mathfrak{g}^0 = \mathfrak{sp}(n,1)$ and $\mathfrak{g}^0 = \mathfrak{f}_{4(-20)}$ the resulting groups are the quaternionic and octonionic Heisenberg groups $\mathbb{H}H^{4n-1}$ and $\mathbb{O}H^{15}$ (compare [91, Section 9.3, p. 33]).

The outline of sections is as follows. Section 2 recalls the concept of restricted roots as well as the Iwasawa decomposition of real semisimple Lie algebras. Section 3 carries out the adaptation procedure for a Chevalley basis as we have indicated. In Section 4 the Chevalley-type theorem for real semisimple Lie algebras is proven. Sections 5 and 6 conclude with the applications described above.

2. Restricted roots and the Iwasawa decomposition

Again let $\mathfrak{g}^0$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{p}$ determined by a Cartan involution θ. There is a maximal abelian θ-stable subalgebra $\mathfrak{h}^0 \subseteq \mathfrak{g}^0$, unique up to conjugation, such that $\mathfrak{a} = \mathfrak{h}^0 \cap \mathfrak{p}$ is maximal abelian in $\mathfrak{p}$ [48, pp. 259 and 419–420]. The dimension of $\mathfrak{a}$ is called the *real rank* of $\mathfrak{g}^0$, $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}^0 = \dim_{\mathbb{R}} \mathfrak{a}$. Given a linear functional α on $\mathfrak{a}$, let

$$\mathfrak{g}^0_\alpha = \left\{ x \in \mathfrak{g}^0 \colon [h,x] = \alpha(h)x \text{ for each } h \in \mathfrak{a} \right\}.$$

If $\mathfrak{g}^0_\alpha$ is not empty, it is called a *restricted root space* of $(\mathfrak{g}^0, \mathfrak{a})$ and α is called a *restricted root* of $(\mathfrak{g}^0, \mathfrak{a})$. Let $\Phi(\mathfrak{g}^0, \mathfrak{a})$ be the set of restricted roots. The Killing form B^0 of $\mathfrak{g}^0$ restricts to a Euclidean inner product on $\mathfrak{a}$ which carries over to the dual $\mathfrak{a}^*$.

Proposition 5.4. *The set $\Phi(\mathfrak{g}^0, \mathfrak{a})$ is a root system in $\mathfrak{a}^*$.*

For a proof see [48, p. 456]. Note two differences to the complex case. On the one hand, the root system $\Phi(\mathfrak{g}^0, \mathfrak{a})$ might not be reduced. This means that given $\alpha \in \Phi(\mathfrak{g}^0, \mathfrak{a})$, it may happen that $2\alpha \in \Phi(\mathfrak{g}^0, \mathfrak{a})$. On the other hand, reduced root spaces will typically not be one-dimensional. Now choose positive roots $\Phi^+(\mathfrak{g}^0, \mathfrak{a})$. Then define a nilpotent subalgebra $\mathfrak{n} = \oplus\, \mathfrak{g}^0_\alpha$ of $\mathfrak{g}^0$ by the direct sum of all restricted root spaces of positive restricted roots. We want to call it an Iwasawa $\mathfrak{n}$-algebra.

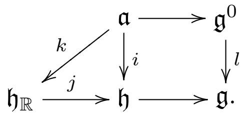
Proposition 5.5 (Iwasawa decomposition)**.** *The real semisimple Lie algebra $\mathfrak{g}^0$ is the direct vector space sum of a compact, an abelian and a nilpotent subalgebra,*

$$\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

A proof is given in [48, p. 250, Theorem 3.4]. The possible choices of positive restricted roots exhaust all possible choices of Iwasawa $\mathfrak{n}$-algebras in the decomposition. Their number is thus given by the order of the Weyl group of $\Phi(\mathfrak{g}^0, \mathfrak{a})$. Let $\mathfrak{g} = \mathfrak{g}^0_{\mathbb{C}}$ be the complexification. Then $\mathfrak{h} = \mathfrak{h}^0_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}$. It determines the set of roots $\Phi(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$. Let $B = B^0_{\mathbb{C}}$ be the complexified Killing form. Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ be the real span of the root vectors t_α for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. It is well-known that the restriction of B turns $\mathfrak{h}_{\mathbb{R}}$ into a Euclidean space.

Proposition 5.6. *We have $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} \oplus \mathrm{i}(\mathfrak{h}^0 \cap \mathfrak{k})$.*

For a proof see [48, p. 259, Lemma 3.2]. In what follows, we will need various inclusions as indicated in the diagram



The compatibility $l^*B = B^0$ is clear. Let $\Sigma = \{\alpha \in \Phi(\mathfrak{g},\mathfrak{h})\colon i^*\alpha \neq 0\}$ be the set of roots which do not vanish everywhere on $\mathfrak{a}$. The following proposition explains the term "restricted roots". It is proven in [48, pp. 263 and 408].

Proposition 5.7.

(i) We have $\Phi(\mathfrak{g}^0,\mathfrak{a}) = i^*\Sigma$.

(ii) For each $\beta \in \Phi(\mathfrak{g}^0,\mathfrak{a})$*, we have* $\mathfrak{g}^0_\beta = (\bigoplus_{\substack{i^*\alpha=\beta\\ \alpha\in\Sigma}} \mathfrak{g}_\alpha) \cap \mathfrak{g}^0$.

Statement (i) says in particular that each $\alpha \in \Sigma$ takes only real values on $\mathfrak{a}$. In fact, $j^*\Phi(\mathfrak{g},\mathfrak{h})$ is a root system in $\mathfrak{h}^*_{\mathbb{R}}$ and the restriction map i^* translates to the orthogonal projection k^* onto $\mathfrak{a}^*$.

3. Adapted Chevalley bases

Recall that σ and τ denote the complex anti-linear automorphisms of $\mathfrak{g}$ given by conjugation with respect to $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{p}$ and the compact form $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$, respectively. Evidently $\theta = l^*(\sigma\tau)$ so that $\sigma\tau$ is the unique complex linear extension of θ from $\mathfrak{g}^0$ to $\mathfrak{g}$ which we want to denote by θ as well. Since σ, τ and θ are involutive, σ and τ commute. Choose positive roots $\Phi^+(\mathfrak{g},\mathfrak{h})$ such that $i^*\Phi^+(\mathfrak{g},\mathfrak{h}) = \Phi^+(\mathfrak{g}^0,\mathfrak{a}) \cup \{0\}$ and let $\Delta(\mathfrak{g},\mathfrak{h}) \subset \Phi^+(\mathfrak{g},\mathfrak{h})$ be the set of simple roots. For $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ let $h_\alpha = \frac{2}{B(t_\alpha,t_\alpha)} t_\alpha$ and set $h_i = h_{\alpha_i}$ for the simple roots $\alpha_i \in \Delta(\mathfrak{g},\mathfrak{h})$ where $1 \leq i \leq l = \operatorname{rank}_{\mathbb{C}}(\mathfrak{g})$. Let $\alpha^\sigma, \alpha^\tau, \alpha^\theta \in \mathfrak{h}^*$ be defined by $\alpha^\sigma(h) = \overline{\alpha(\sigma(h))}$, $\alpha^\tau(h) = \overline{\alpha(\tau(h))}$ and $\alpha^\theta(h) = \alpha(\theta(h))$ where $\alpha \in \mathfrak{h}^*$, $h \in \mathfrak{h}$. If $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ and $x_\alpha \in \mathfrak{g}_\alpha$, then

$$[h,\sigma(x_\alpha)] = \sigma([\sigma(h),x_\alpha]) = \sigma(\alpha(\sigma(h))x_\alpha) = \overline{\alpha(\sigma(h))}\sigma(x_\alpha)$$

so that $\sigma(x_\alpha) \in \mathfrak{g}_{\alpha^\sigma}$ and similarly for τ and θ. Thus in this case α^σ, α^τ and α^θ are roots. From Proposition 5.6 we see directly that $\alpha^\tau = -\alpha$ for each $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$. We adopt a terminology of A. Knapp [58, p. 390] and call a root $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ *real* if it is fixed by σ, *imaginary* if it is fixed by θ and *complex* in all remaining cases. Note that $\alpha^\sigma = -\alpha$ if and only if α is imaginary. A real root vanishes on $\mathfrak{h}^0 \cap \mathfrak{k}$, thus takes only real values on $\mathfrak{h}_0$. An imaginary root vanishes on $\mathfrak{a}$, thus takes purely imaginary values on $\mathfrak{h}^0$. A complex root takes mixed complex values on $\mathfrak{h}^0$. The imaginary roots form a root system $\Phi_{i\mathbb{R}}$ [48, p. 531]. The complex roots $\Phi_{\mathbb{C}}$ and the real roots $\Phi_{\mathbb{R}}$ give a decomposition of the set $\Sigma = \Phi_{\mathbb{C}} \cup \Phi_{\mathbb{R}}$ which restricts to the root system $i^*\Sigma = \Phi(\mathfrak{g}^0,\mathfrak{a})$. Let $\Delta_0 = \Delta(\mathfrak{g},\mathfrak{h}) \cap \Phi_{i\mathbb{R}}$ be the set of simple imaginary roots and let $\Delta_1 = \Delta(\mathfrak{g},\mathfrak{h}) \cap \Sigma$ be the set of simple complex or real roots.

Recall Definition 5.1, Theorem 5.2 and Lemma 5.3 of Section 1. Our goal is to prove the following refinement of Lemma 5.3.

Proposition 5.8. *There is a Chevalley basis* $\mathcal{C} = \{x_\alpha, h_i : \alpha \in \Phi(\mathfrak{g},\mathfrak{h}), 1 \leq i \leq l\}$ *of* $(\mathfrak{g},\mathfrak{h})$ *such that*

(i) $\tau(x_\alpha) = x_{\alpha^\tau} = x_{-\alpha}$ *for each* $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$,

(ii) $\sigma(x_\alpha) = \pm x_{\alpha^\sigma}$ *for each* $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ *and*
$\sigma(x_\alpha) = +x_{\alpha^\sigma}$ *for each* $\alpha \in \Phi_{i\mathbb{R}} \cup \Delta_1$.

Remark 5.9. A. Borel [9, Lemma 3.5, p. 116] has built on early work by F. Gantmacher [41] to prove a lemma which at least assures that $\sigma(x_\alpha) = \pm x_{\alpha^\sigma}$ for all

$\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. But Borel assumes a technical condition, namely that θ leaves invariant an element in $\mathfrak{h}_{\mathbb{R}}$ which is regular in $\mathfrak{g}$. In Proposition 3.7, p. 118 of the same reference he achieves this condition using previous joint work with G. D. Mostow [16, Theorem 4.5]. But with this method he of necessity comes up with a *maximally compact* θ-stable Cartan subalgebra $\mathfrak{h}^0$ of $\mathfrak{g}^0$, which is one that has intersection with $\mathfrak{k}$ of maximal dimension. Since we will be interested in geometric applications such as symmetric spaces and the Iwasawa decomposition, we need to work with a *maximally noncompact* θ-stable Cartan subalgebra $\mathfrak{h}^0$ that has intersection with $\mathfrak{p}$ of maximal dimension. For these types of θ-stable Cartan subalgebras, Borel's technical assumption definitely goes wrong.

We will say that a Chevalley basis $\mathcal{C}$ is *τ-adapted* if it fulfills (i) and *σ-adapted* if it fulfills (ii) of the proposition. We prepare the proof with the following lemma.

Lemma 5.10. *There is a unique involutive permutation $\omega\colon \Delta_1 \to \Delta_1$ and there are unique nonnegative integers $n_{\beta\alpha}$ with $\alpha \in \Delta_1$ and $\beta \in \Delta_0$ such that for each $\alpha \in \Delta_1$*

(i) $\alpha^\theta = -\omega(\alpha) - \sum_{\beta\in\Delta_0} n_{\beta\alpha}\beta$,

(ii) $n_{\beta\,\omega(\alpha)} = n_{\beta\alpha}$ *and*

(iii) ω extends to a Dynkin diagram automorphism $\omega\colon \Delta(\mathfrak{g}, \mathfrak{h}) \to \Delta(\mathfrak{g}, \mathfrak{h})$.

Part (i) is due to I. Satake [99, Lemma 1, p. 80]. As an alternative to Satake's original proof, A. L. Onishchik and E. B. Vinberg suggest a slightly differing argument as a series of two problems in [89, p. 273]. We will present the solutions because they made us observe the additional symmetry (ii) which will play a key role in all that follows. Part (iii) can be found in the appendix of [86, Theorem 1, p. 75], which was written by J. Šilhan.

Proof. Let C be an involutive $(n\times n)$-matrix with nonnegative integer entries. It acts on the first orthant X of $\mathbb{R}^n$, the set of all $v \in \mathbb{R}^n$ with only nonnegative coordinates. We claim that C is a permutation matrix. Since C is invertible, every column and every row has at least one nonzero entry. Thus we observe $|Cv|_1 \geq |v|_1$ for all $v \in X$. Suppose the i-th column of C has an entry $c_{ji} \geq 2$ or a second nonzero entry. Then the standard basis vector $\varepsilon_i \in X$ is mapped to a vector of L^1-norm at least 2. But that contradicts C being involutive.

Let $\alpha \in \Delta_1$. Then α^θ is a negative root, so we can write

$$\alpha^\theta = -\sum_{\gamma\in\Delta_1} n_{\gamma\alpha}\gamma - \sum_{\beta\in\Delta_0} n_{\beta\alpha}\beta$$

with nonnegative integers $n_{\gamma\alpha}$ and $n_{\beta\alpha}$. Consider the transformation matrix of θ acting on $\mathfrak{h}^*$ with respect to the basis $\Delta(\mathfrak{g}, \mathfrak{h})$. In terms of the decomposition $\Delta(\mathfrak{g}, \mathfrak{h}) = \Delta_1 \cup \Delta_0$ it takes the block form

$$\left(\begin{array}{c|c} -n_{\gamma\alpha} & 0 \\ \hline -n_{\beta\alpha} & \mathbf{1} \end{array}\right)$$

with $\mathbf{1}$ representing the $|\Delta_0|$-dimensional unit matrix. The block matrix squares to a unit matrix. For the upper left block we conclude that $(n_{\gamma\alpha})$ is a matrix C as above and thus corresponds to an involutive permutation $\omega\colon \Delta_1 \to \Delta_1$. This proves (i). For the lower left block we conclude that $n_{\beta\alpha} = \sum_{\delta\in\Delta_1} n_{\beta\delta}n_{\delta\alpha} = n_{\beta\omega(\alpha)}$ because $(n_{\delta\alpha})$ is the aforementioned permutation matrix, so $n_{\delta\alpha} = 1$ if $\delta = \omega(\alpha)$ and $n_{\delta\alpha} = 0$ otherwise. This proves (ii).

For (iii) we only mention the construction. Choose canonical generators of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. These yield the decomposition $\operatorname{Aut}\mathfrak{g} = \operatorname{Int}\mathfrak{g} \rtimes \operatorname{Aut}\Delta(\mathfrak{g}, \mathfrak{h})$ of automorphisms of $\mathfrak{g}$ into inner and outer ones, the outer ones being identified with Dynkin diagram automorphisms. The extension of ω is provided by the composition $s\nu$ where s is the outer part of $\theta \in \operatorname{Aut}\mathfrak{g}$ and

ν is the outer part of a *Weyl involution* $w \in \operatorname{Aut}\mathfrak{g}$. A Weyl involution is obtained from the root system automorphism $\alpha \mapsto -\alpha$ for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ using the canonical generators. □

Lastly, we recall a well-known fact on simple roots which is for instance proven in [51, p. 50, Corollary 10.2.A].

Lemma 5.11. *Each positive root $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})$ decomposes as a sum $\alpha_1 + \cdots + \alpha_k$ of simple roots $\alpha_i \in \Delta(\mathfrak{g}, \mathfrak{h})$ such that each partial sum $\alpha_1 + \cdots + \alpha_i$ is a root.*

PROOF (OF PROPOSITION 5.8). Pick a Chevalley basis $\mathcal{C}$ of the pair $(\mathfrak{g}, \mathfrak{h})$. The proofs of Lemma 5.3 (i) by Borel and Morris make reference to the conjugacy theorem of maximal compact subgroups in connected Lie groups. We have found a more hands-on approach that has the virtue of giving a more complete picture of the proposition: The adaptation of $\mathcal{C}$ to τ is gained by adjusting the norms of the x_α. Thereafter the adaptation of $\mathcal{C}$ to σ is gained by adjusting the complex phases of the x_α.

From Definition 5.1 (i) we obtain $-\frac{2t_\alpha}{B(t_\alpha, t_\alpha)} = [x_\alpha, x_{-\alpha}] = B(x_\alpha, x_{-\alpha})t_\alpha$, therefore $B(x_\alpha, x_{-\alpha}) < 0$ because $B(t_\alpha, t_\alpha) > 0$. But also $B(x_\alpha, \tau x_\alpha) < 0$. Indeed, $(x_\alpha + \tau x_\alpha) \in \mathfrak{k}$ where B is negative definite, so $B(x_\alpha + \tau x_\alpha, x_\alpha + \tau x_\alpha) < 0$ and $B(x_\alpha, x_\beta) = 0$ unless $\alpha + \beta = 0$. If constants $b_\alpha \in \mathbb{C}$ are defined by $\tau x_\alpha = b_\alpha x_{-\alpha}$ for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$, we conclude that the b_α are in fact positive real numbers. Moreover, $b_{-\alpha} = b_\alpha^{-1}$ because τ is an involution. We use Definition 5.1 (ii) to deduce $b_{\alpha+\beta} = b_\alpha b_\beta$ from $[\tau x_\alpha, \tau x_\beta] = \tau([x_\alpha, x_\beta])$ whenever $\alpha, \beta, \alpha + \beta \in \Phi(\mathfrak{g}, \mathfrak{h})$. In other words and under identification of α and t_α, the map b defined on the root system $j^*\Phi(\mathfrak{g}, \mathfrak{h})$ extends to a homomorphism from the root lattice $Q = \mathbb{Z}(j^*\Delta(\mathfrak{g}, \mathfrak{h}))$ to the multiplicative group of positive real numbers. We replace each x_α by $\frac{1}{\sqrt{b_\alpha}} x_\alpha$ and easily check that we obtain a Chevalley basis with unchanged structure constants that establishes (i).

Now assume that $\mathcal{C}$ is τ-adapted. It is automatic that $\sigma(x_\beta) = +x_{\beta^\sigma} = x_{-\beta}$ for each $\beta \in \Phi_{i\mathbb{R}}$ because S. Helgason informs us in [48, Lemma 3.3 (ii), p. 260] that for each imaginary root β the root space $\mathfrak{g}_\beta$ lies in $\mathfrak{k} \otimes \mathbb{C}$. But $\mathfrak{k} \otimes \mathbb{C}$ is the fix point algebra of θ, so the assertion follows from (i) and $\sigma = \tau\theta$. We define constants $u_\alpha \in \mathbb{C}$ by $\theta(x_\alpha) = u_\alpha x_{\alpha^\theta}$ for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. As we have just seen, $u_\alpha = 1$ if α is imaginary. In general, the τ-adaptation effects $\sigma(x_\alpha) = \overline{u_\alpha} x_{\alpha^\sigma}$ and $\overline{u_\alpha} = u_{-\alpha}$ because $\sigma = \tau\theta = \theta\tau$. Note that

$$-u_\alpha u_{-\alpha} h_{\alpha^\theta} = [u_\alpha x_{\alpha^\theta}, u_{-\alpha} x_{-\alpha^\theta}] = [\theta(x_\alpha), \theta(x_{-\alpha})] = -\theta(h_\alpha) = -h_{\alpha^\theta},$$

so $u_{-\alpha} = u_\alpha^{-1}$ and $|u_\alpha| = 1$. From $\theta^2(x_\alpha) = x_\alpha$ we get $u_{\alpha^\theta} = u_\alpha^{-1} = u_{-\alpha}$ $(*)$. Next we want to discuss the relation between u_α and $u_{\omega(\alpha)}$ for $\alpha \in \Delta_1$. First assume that for a given two-element orbit $\{\alpha, \omega(\alpha)\}$ the integers $n_{\beta\alpha}$ of Lemma 5.10 vanish for all $\beta \in \Delta_0$. A notable case where this condition is vacuous for all $\alpha \in \Delta_1$, is that of a *quasi-split* algebra $\mathfrak{g}^0$ when $\Delta_0 = \emptyset$. From $n_{\beta\alpha} = 0$ we get $\omega(\alpha)^\theta = -\alpha$. Thus $u_{\omega(\alpha)} = u_{-\omega(\alpha)^\theta} = u_\alpha$ by means of $(*)$. Now assume on the other hand there is $\beta_0 \in \Delta_0$ such that $n_{\beta_0\alpha} > 0$. From Lemma 5.10 (i) and (ii) we get that $-\omega(\alpha)^\theta = \alpha + \sum_{\beta \in \Delta_0} n_{\beta\alpha}\beta$ is the unique decomposition of $-\omega(\alpha)^\theta$ as a sum of simple roots. Lemma 5.11 tells us that that this sum can be ordered as $-\omega(\alpha)^\theta = \alpha_1 + \cdots + \alpha_k$ such that all partial sums $\gamma_i = \alpha_1 + \cdots + \alpha_i$ are roots. Thus

$$x_{-\omega(\alpha)^\theta} = \prod_{i=1}^{k-1} c_{\alpha_{i+1}\gamma_i}^{-1} \operatorname{ad}(x_{\alpha_k}) \cdots \operatorname{ad}(x_{\alpha_2})(x_{\alpha_1}).$$

For one i_0 we have $\alpha_{i_0} = \alpha$ and the remaining α_i are imaginary. Hence by $(*)$

$$\begin{aligned} u_{\omega(\alpha)} x_{-\omega(\alpha)} &= u_{-\omega(\alpha)^\theta} x_{-\omega(\alpha)} = \theta(x_{-\omega(\alpha)^\theta}) = \\ &= \prod_{i=1}^{k-1} c^{-1}_{\alpha_{i+1}\gamma_i}\ u_\alpha \operatorname{ad}(x_{\alpha_k^\theta}) \cdots \operatorname{ad}(x_{\alpha_2^\theta})(x_{\alpha_1^\theta}) = \\ &= \prod_{i=1}^{k-1} \frac{c_{\alpha^\theta_{i+1}\gamma_i^\theta}}{c_{\alpha_{i+1}\gamma_i}}\ u_\alpha\, x_{-\omega(\alpha)} = \pm u_\alpha x_{-\omega(\alpha)}. \end{aligned}$$

Here we used that $c_{\alpha\beta} = \pm c_{\alpha^\theta\beta^\theta}$ by Theorem 5.2 (iv) because θ induces an automorphism of the root system $\Phi(\mathfrak{g},\mathfrak{h})$. It follows that $u_{\omega(\alpha)} = \pm u_\alpha$ and the sign depends on the structure constants of the Chevalley basis only. We want to achieve $u_{\omega(\alpha)} = +u_\alpha$. So for all two-element orbits $\{\alpha, \omega(\alpha)\}$ with $u_\alpha = -u_{\omega(\alpha)}$, replace $x_{\omega(\alpha)}$ and $x_{-\omega(\alpha)}$ by their negatives. This produces a new τ-adapted Chevalley basis $\{x'_\alpha, h_i \colon \alpha \in \Phi(\mathfrak{g},\mathfrak{h})\}$ though some structure constants might have changed sign. Set $\theta(x'_\alpha) = u'_\alpha x'_{\alpha^\theta}$ for all $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$. We claim that $u'_{\omega(\alpha)} = u'_\alpha$ for all $\alpha \in \Delta_1$. The only critical case is that of an $\alpha \in \Delta_1$ with $n_{\beta_0\alpha} > 0$ for some $\beta_0 \in \Delta_0$. But in this case we deduce from Lemma 5.10 that neither $-\alpha^\theta$ nor $-\omega(\alpha)^\theta$ is simple, yet only vectors $x_{\omega(\alpha)}$, $x_{-\omega(\alpha)}$ corresponding to simple roots $\omega(\alpha)$ with $u_{\omega(\alpha)} = -u_\alpha$ have been replaced. So still $u'_{\omega(\alpha)} = u'_\alpha$ if we had $u_{\omega(\alpha)} = u_\alpha$. If $u_{\omega(\alpha)} = -u_\alpha$, the replacement is given by $x'_{\alpha^\theta} = x_{\alpha^\theta}$ and $x'_{\omega(\alpha)^\theta} = x_{\omega(\alpha)^\theta}$ as well as $x'_\alpha = x_\alpha$ whereas $x'_{\omega(\alpha)} = -x_{\omega(\alpha)}$. Thus,

$$\begin{aligned} u'_\alpha x'_{\alpha^\theta} &= \theta(x'_\alpha) = \theta(x_\alpha) = u_\alpha x_{\alpha^\theta} = u_\alpha x'_{\alpha^\theta} \quad \text{and} \\ u'_{\omega(\alpha)} x'_{\omega(\alpha)^\theta} &= \theta(x'_{\omega(\alpha)}) = \theta(-x_{\omega(\alpha)}) = -u_{\omega(\alpha)} x_{\omega(\alpha)^\theta} = -u_{\omega(\alpha)} x'_{\omega(\alpha)^\theta}. \end{aligned}$$

It follows that $u'_{\omega(\alpha)} = -u_{\omega(\alpha)} = u_\alpha = u'_\alpha$. Since $\Delta(\mathfrak{g},\mathfrak{h})$ is a basis of $\mathfrak{h}^*$, there exists $h \in \mathfrak{h}$ such that $e^{\alpha(h)} = u'_\alpha$ and $(-\mathrm{i})\alpha(h) \in (-\pi,\pi]$ for all $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$. From $u'_\beta = 1$ for each $\beta \in \Delta_0$ we get $h \in \bigcap_{\beta\in\Delta_0} \ker(\beta)$ and from $u'_{\omega(\alpha)} = u'_\alpha$ we get $\alpha(h) = \omega(\alpha)(h)$ for each $\alpha \in \Delta_1$. Thus by Lemma 5.10 we have for each $\alpha \in \Delta_1$

$$\alpha^\theta(h) = -\alpha(h).$$

We remark that since $\alpha(\theta h) = \alpha(-h)$ holds true for all $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$, it follows $\theta(h) = -h$, so $h \in \mathrm{i}\mathfrak{a}$. Let $x''_\alpha = e^{-\frac{\alpha(h)}{2}} x'_\alpha$ for each $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$. Then Definition 5.1 (i) and (ii) hold for the new x''_α. But so does Proposition 5.8 (i) because $\alpha(h)$ is purely imaginary for each $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ and because τ is complex antilinear. For $\alpha \in \Delta_1$ we calculate

$$\theta(x''_\alpha) = e^{-\frac{\alpha(h)}{2}} \theta(x'_\alpha) = e^{-\frac{\alpha(h)}{2}} u'_\alpha x'_{\alpha^\theta} = e^{\frac{\alpha(h)}{2}} e^{\frac{\alpha^\theta(h)}{2}} x''_{\alpha^\theta} = x''_{\alpha^\theta}.$$

From now on we will work with the basis $\{x''_\alpha, h_i \colon \alpha \in \Phi(\mathfrak{g},\mathfrak{h})\}$ and drop the double prime. We have $\theta(x_\alpha) = x_{\alpha^\theta}$ for each $\alpha \in \Phi_{\mathrm{i}\mathbb{R}} \cup \Delta_1$. It remains to show $\theta(x_\alpha) = \pm x_{\alpha^\theta}$ for general $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$. First let $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})^+$ be positive and let $\alpha = \alpha_1 + \cdots + \alpha_k$ be a decomposition as in Lemma 5.11. For $1 \le j \le k$ let $\gamma_j = \alpha_1 + \cdots + \alpha_j$. Then we have

$$x_\alpha = \prod_{i=1}^{k-1} c^{-1}_{\alpha_{i+1}\gamma_i}\ \operatorname{ad}(x_{\alpha_k}) \cdots \operatorname{ad}(x_{\alpha_2})(x_{\alpha_1}).$$

Thus

$$\theta(x_\alpha) = \prod_{i=1}^{k-1} c^{-1}_{\alpha_{i+1}\gamma_i}\ \operatorname{ad}(x_{\alpha_k{}^\theta}) \cdots \operatorname{ad}(x_{\alpha_2{}^\theta})(x_{\alpha_1{}^\theta}) = \prod_{i=1}^{k-1} \frac{c_{\alpha_{i+1}{}^\theta\gamma_i^\theta}}{c_{\alpha_{i+1}\gamma_i}}\ x_{\alpha^\theta} = \pm x_{\alpha^\theta}.$$

Finally we compute

$$[\theta(x_{-\alpha}), \pm x_{\alpha^\theta}] = [\theta(x_{-\alpha}), \theta(x_\alpha)] = \theta(h_\alpha) = h_{\alpha^\theta} = [x_{-\alpha^\theta}, x_{\alpha^\theta}],$$

hence $\theta(x_{-\alpha}) = \pm x_{-\alpha^\theta}$. We conclude $\theta(x_\alpha) = \pm x_{\alpha^\theta}$ and $\sigma(x_\alpha) = \pm x_{\alpha^\sigma}$ for all $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$. ☐

The constructive method of proof also settles two obvious questions that remain. Which combinations of signs of the $c_{\alpha\beta}$ can occur for a σ- and τ-adapted Chevalley basis $\{x_\alpha, h_\alpha\}$? And if we set $\sigma(x_\alpha) = \mathrm{sgn}(\alpha)x_{\alpha^\sigma}$ for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$, how can we compute $\mathrm{sgn}(\alpha) \in \{\pm 1\}$? We put down the answers in the following two propositions.

Proposition 5.12. *A set of Chevalley constants* $\{c_{\alpha,\beta} \colon \alpha + \beta \in \Phi(\mathfrak{g}, \mathfrak{h})\}$ *of* $\mathfrak{g}$ *can be realized by a* σ*- and* τ*-adapted Chevalley basis if and only if for each two-element orbit* $\{\alpha, \omega(\alpha)\}$ *of roots in* Δ_1 *with* $n_{\beta_0\alpha} > 0$ *for some* $\beta_0 \in \Delta_0$ *we have*

$$\prod_{i=1}^{k-1} \frac{c_{\alpha_{i+1}{}^\theta \gamma_i^\theta}}{c_{\alpha_{i+1}\gamma_i}} = 1$$

where $-\omega(\alpha)^\theta = \alpha_1 + \cdots + \alpha_k$ *with* $\alpha_i \in \Delta(\mathfrak{g}, \mathfrak{h})$ *and* $\gamma_i = \alpha_1 + \cdots + \alpha_i \in \Phi(\mathfrak{g}, \mathfrak{h})$ *for all* $i = 1, \ldots, k$.

Proof. If the condition on the structure constants holds, take a Chevalley basis of $(\mathfrak{g}, \mathfrak{h})$ which realizes them and start the adaptation procedure of the proof of Proposition 5.8. Thanks to the condition, the replacement $x_\alpha \mapsto x'_\alpha$ in the course of the proof is the identity map. The other two adaptations $x_\alpha \mapsto \frac{1}{\sqrt{b_\alpha}} x_\alpha$ and $x'_\alpha \mapsto x''_\alpha$ leave the structure constants unaffected. Conversely, if $c_{\alpha\beta}$ are the structure constants of a σ- and τ-adapted Chevalley basis, we compute similarly as in the proof of Proposition 5.8 that for each such critical $\alpha \in \Delta_1$ we have

$$x_{-\omega(\alpha)} = \theta(x_{-\omega(\alpha)^\theta}) = \prod_{i=1}^{k-1} \frac{c_{\alpha_{i+1}{}^\theta \gamma_i^\theta}}{c_{\alpha_{i+1}\gamma_i}} x_{-\omega(\alpha)}. \qquad \square$$

In particular, for all quasi-split $\mathfrak{g}^0$ as well as for all $\mathfrak{g}^0$ with $\omega = \mathrm{id}_{\Delta_1}$ all structure constants of any Chevalley basis of $(\mathfrak{g}, \mathfrak{h})$ can be realized by a σ- and τ-adapted one. To compute $\mathrm{sgn}(\alpha)$ first apply σ to the equation $[x_\alpha, x_{-\alpha}] = -h_\alpha$ to get $\mathrm{sgn}(\alpha)\mathrm{sgn}(-\alpha)[x_{\alpha^\sigma}, x_{-\alpha^\sigma}] = -h_{\alpha^\sigma}$, so $\mathrm{sgn}(\alpha) = \mathrm{sgn}(-\alpha)$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. Moreover, we get the recursive formula

$$\mathrm{sgn}(\alpha + \beta) = \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta)\frac{c_{\alpha^\sigma\beta^\sigma}}{c_{\alpha\beta}}$$

for all $\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{h})$ such that $\alpha + \beta \in \Phi(\mathfrak{g}, \mathfrak{h})$. This follows from applying σ to the equation $[x_\alpha, x_\beta] = c_{\alpha\beta}x_{\alpha+\beta}$. Since $\mathrm{sgn}(\alpha) = 1$ for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ the following absolute version of the recursion formula is immediate.

Proposition 5.13. *Let* $\{x_\alpha, h_\alpha\}$ *be a* σ*- and* τ*-adapted Chevalley basis of* $(\mathfrak{g}, \mathfrak{h})$*. If* $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})^+$*, let* $\alpha = \alpha_1 + \cdots + \alpha_k$ *with* $\alpha_i \in \Delta(\mathfrak{g}, \mathfrak{h})$ *and* $\gamma_i = \alpha_1 + \cdots + \alpha_i \in \Phi(\mathfrak{g}, \mathfrak{h})$ *for all* $i = 1, \ldots, k$*. Then*

$$\mathrm{sgn}(\alpha) = \prod_{i=1}^{k-1} \frac{c_{\alpha_{i+1}{}^\sigma \gamma_i^\sigma}}{c_{\alpha_{i+1}\gamma_i}}.$$

It is understood that the empty product equals one. Also note that $c_{\alpha^\sigma\beta^\sigma} = c_{-\alpha^\sigma -\beta^\sigma} = c_{\alpha^\theta\beta^\theta}$. For carrying out explicit computations we still need to comment on how to find a choice of signs for the $c_{\alpha\beta}$ in Theorem 5.2 (ii) as to obtain some set of Chevalley constants to begin with. This problem has created its own industry. One algorithm is given in [98, p. 54]. A similar method is described in [25, p. 58], introducing the notion of *extra special pairs* of roots. A particularly enlightening approach goes back to I. B. Frenkel and V. G. Kac in [37, p. 40]. It starts with the case of *simply-laced* root systems, which are those of one root length only, then tackles the non-simply-laced case. An exposition is given in [54, Chapters 7.8–7.10, p. 105] and also in [32, p. 189]. In this picture the product expression appearing in Propositions 5.12 and 5.13 can be easily computed. So this approach shall be our method of choice. We briefly describe how it works.

Let Φ be a root system of type A_l with $l \geq 1$, D_l with $l \geq 4$ or E_l with $l = 6, 7, 8$ in a Euclidean space V with scalar product $(\cdot, \cdot)$ such that $(\alpha, \alpha) = 2$ for all $\alpha \in \Phi$. Let $Q = \mathbb{Z}\Phi \subset V$ be the root lattice.

Definition 5.14. A map $\varepsilon\colon Q \times Q \to \{1, -1\}$ is called an *asymmetry function* if for all $\alpha, \beta, \gamma, \delta \in Q$ it satisfies the three equations

$$\begin{aligned} \varepsilon(\alpha + \beta, \gamma) &= \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma), \\ \varepsilon(\alpha, \gamma + \delta) &= \varepsilon(\alpha, \gamma)\varepsilon(\alpha, \delta), \\ \varepsilon(\alpha, \alpha) &= (-1)^{\frac{1}{2}(\alpha,\alpha)}. \end{aligned}$$

Immediate consequences of the defining equations are $\varepsilon(\alpha, 0) = \varepsilon(0, \beta) = 1$ and $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$ as well as

$$\varepsilon(\alpha, \beta) = \varepsilon(-\alpha, \beta) = \varepsilon(\alpha, -\beta) = \varepsilon(-\alpha, -\beta)$$

for $\alpha, \beta \in Q$. If $\alpha \in \Phi$ is a root, we have $\varepsilon(\alpha, \alpha) = -1$. We construct an asymmetry function. Choose simple roots $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ and label each edge of the Dynkin diagram of (Φ, Δ) with an arrow pointing to either of the adjacent nodes. The resulting diagram is called an *oriented Dynkin diagram.* Then for $\alpha_i, \alpha_j \in \Delta$ define $\varepsilon(\alpha_i, \alpha_j) = -1$ if either $i = j$ or if α_i and α_j are connected by an edge whose arrow points from α_i to α_j. In all other cases set $\varepsilon(\alpha_i, \alpha_j) = 1$. Then extend ε from $\Delta \times \Delta$ to $Q \times Q$ by the first two equations of Definition 5.14.

Let $\mathfrak{h}(\Phi)$ be a complex vector space with basis $\{\dot{t}_1, \ldots, \dot{t}_l\}$. For each $\alpha \in V$ let $\dot{t}_\alpha = \sum_{i=1}^{l} s_i \dot{t}_i$ if $\alpha = \sum_{i=1}^{l} s_i \alpha_i$ and let $\mathfrak{g}_\alpha(\Phi)$ be a one-dimensional complex vector space with basis $\{x_\alpha\}$. Set

$$\mathfrak{g}(\Phi) = \mathfrak{h}(\Phi) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha(\Phi).$$

A bilinear, antisymmetric map $[\cdot, \cdot]\colon \mathfrak{g}(\Phi) \times \mathfrak{g}(\Phi) \longrightarrow \mathfrak{g}(\Phi)$ is determined by

$$\begin{aligned} [\dot{t}_i, \dot{t}_j] &= 0 && \text{for } 1 \leq i, j \leq l, \\ [\dot{t}_i, x_\alpha] &= (\alpha, \alpha_i)x_\alpha && \text{for } 1 \leq i \leq l \text{ and } \alpha \in \Phi, \\ [x_\alpha, x_{-\alpha}] &= -\dot{t}_\alpha && \text{for } \alpha \in \Phi, \\ [x_\alpha, x_\beta] &= 0 && \text{for } \alpha, \beta \in \Phi, \alpha + \beta \notin \Phi, \beta \neq -\alpha, \\ [x_\alpha, x_\beta] &= \varepsilon(\alpha, \beta)x_{\alpha+\beta} && \text{for } \alpha, \beta, \alpha + \beta \in \Phi. \end{aligned}$$

Proposition 5.15. *This bracket turns* $\mathfrak{g}(\Phi)$ *into a simple Lie algebra of type* A_l, D_l *or* E_l *with Cartan subalgebra* $\mathfrak{h}(\Phi)$ *and root space decomposition as given above.*

The proof is given in [54, Proposition 7.8, p. 106]. Its essential part is the verification of the Jacobi identity. In particular, the proposition identifies Φ with the root system of $\mathfrak{g}(\Phi)$ with respect to the Cartan subalgebra $\mathfrak{h}(\Phi)$.

Proposition 5.16. *The set* $\mathcal{C}(\Phi) = \{x_\alpha, \dot{t}_i \colon \alpha \in \Phi, 1 \leq i \leq l\}$ *is a Chevalley basis of* $(\mathfrak{g}(\Phi), \mathfrak{h}(\Phi))$.

Proof. It only remains to verify that the elements $\dot{t}_\alpha$ coincide with the elements $h_\alpha = \frac{2}{B_\Phi(t_\alpha, t_\alpha)} t_\alpha$. Here h_α and t_α are defined by the Killing form B_Φ of $\mathfrak{g}(\Phi)$ as in the beginning of Section 1. To check this, we define another bilinear form $(\cdot, \cdot)_\Phi$ on $\mathfrak{g}(\Phi)$ by setting $(\dot{t}_i, \dot{t}_j)_\Phi = (\alpha_i, \alpha_j)$ and $(x_\alpha, x_\beta)_\Phi = -\delta_{\alpha,-\beta}$ (Kronecker-δ) for $\alpha, \beta \in \Phi$ as well as $(h, x_\alpha)_\Phi = 0$ for $h \in \mathfrak{h}(\Phi)$ and $\alpha \in \Phi$. It is easily seen that $(\cdot, \cdot)_\Phi$ is invariant. Thus it is proportional to the Killing form B_Φ. So if $(\cdot, \cdot)_\Phi = \lambda^{-1} B_\Phi$ for $\lambda \in \mathbb{C}$, then from the second equation defining the bracket above, we get

$$(\dot{t}_i, \dot{t}_j)_\Phi = (\alpha_i, \alpha_j) = \alpha_i(\dot{t}_j) = B_\Phi(t_{\alpha_i}, \dot{t}_j) = \lambda(t_{\alpha_i}, \dot{t}_j)_\Phi$$

for all $1 \leq i, j \leq l$. It follows that $\dot{t}_\alpha = \lambda t_\alpha$ for all $\alpha \in \Phi$. Hence

$$\dot{t}_\alpha = \tfrac{2}{(\dot{t}_\alpha, \dot{t}_\alpha)_\Phi} \dot{t}_\alpha = \tfrac{2}{B_\Phi(t_\alpha, t_\alpha)} t_\alpha = h_\alpha. \qquad \square$$

Let now Φ be more precisely of type D_{l+1} with $l \geq 3$, A_{2l-1} with $l \geq 2$, E_6 or D_4. Then in the first three cases the Dynkin diagram of (Φ, Δ) has a nontrivial automorphism $\bar{\mu}\colon \Delta \to \Delta$ of order $r = 2$ and in the remaining case an automorphism $\bar{\mu}\colon \Delta \to \Delta$ of order $r = 3$. Choose a $\bar{\mu}$-invariant orientation of the Dynkin diagram inducing the asymmetry function ε. The diagram automorphism $\bar{\mu}$ extends to an outer automorphism $\bar{\mu}\colon \Phi \to \Phi$ of the root system. This induces an outer automorphism μ of the Lie algebra $\mathfrak{g}(\Phi)$ which still has order r. Let

$$\begin{aligned} \Psi_l &= \{\alpha \in \Phi \colon \bar{\mu}(\alpha) = \alpha\}, \\ \Psi_s &= \{r^{-1}\left(\alpha + \bar{\mu}(\alpha) + \cdots + \bar{\mu}^{r-1}(\alpha)\right) : \alpha \in \Phi,\ \bar{\mu}(\alpha) \neq \alpha\}. \end{aligned}$$

Then $\Psi = \Psi_l \dot{\cup} \Psi_s$ is the decomposition into long and short roots of an irreducible root system of type B_l, C_l, F_4 or G_2 respectively. We have a corresponding decomposition of simple roots of Ψ given by $\Pi = \Pi_l \cup \Pi_s$ where

$$\begin{aligned} \Pi_l &= \{\alpha \in \Delta \colon \bar{\mu}(\alpha) = \alpha\}, \\ \Pi_s &= \{r^{-1}\left(\alpha + \cdots + \bar{\mu}^{r-1}(\alpha)\right) : \alpha \in \Delta,\ \bar{\mu}(\alpha) \neq \alpha\}. \end{aligned}$$

If $\alpha \in \Psi_l$, let $\alpha' = \alpha \in \Phi$. If $\alpha \in \Psi_s$, let $\alpha' = \beta$ for some $\beta \in \Phi$ with $\alpha = r^{-1}\left(\beta + \cdots + \bar{\mu}^{r-1}(\beta)\right)$. Define $y_\alpha \in \mathfrak{g}(\Phi)$ by $y_\alpha = x_{\alpha'}$ if $\alpha \in \Psi_l$ and $y_\alpha = x_{\alpha'} + \cdots + x_{\bar{\mu}^{r-1}(\alpha')}$ if $\alpha \in \Psi_s$. Note that we have $\dot{t}_\alpha = \dot{t}_{\alpha'}$ if $\alpha \in \Psi_l$ and $\dot{t}_\alpha = r^{-1}\left(\dot{t}_{\alpha'} + \cdots + \dot{t}_{\bar{\mu}^{r-1}(\alpha')}\right)$ if $\alpha \in \Psi_s$. As usual, let $h_\alpha = \frac{2}{B^\Phi(t_\alpha, t_\alpha)} t_\alpha = \frac{2}{(\dot{t}_\alpha, \dot{t}_\alpha)_\Phi} \dot{t}_\alpha$ for $\alpha \in \Psi$. If $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, let $h_i = h_{\alpha_i}$. For $\alpha \in \Psi$ let $\mathfrak{g}_\alpha(\Psi)$ be the one-dimensional subspace of $\mathfrak{g}(\Phi)$ spanned by y_α. Let $\mathfrak{h}(\Psi)$ be the subspace of $\mathfrak{h}(\Phi)$ spanned by all h_i for $1 \leq i \leq l$. Set

$$\mathfrak{g}(\Psi) = \mathfrak{h}(\Psi) \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha(\Psi).$$

Proposition 5.17. *The fix point algebra of the automorphism μ acting on $\mathfrak{g}(\Phi)$ is given by $\mathfrak{g}(\Psi)$. It is simple of type B_l, C_l, F_4 or G_2 respectively. A Cartan subalgebra is given by $\mathfrak{h}(\Psi)$ which induces the root space decomposition as given above. The set $\mathcal{C}(\Psi) = \{y_\alpha, h_i \colon \alpha \in \Psi, 1 \leq i \leq l\}$ is a Chevalley basis of $(\mathfrak{g}(\Psi), \mathfrak{h}(\Psi))$. If $\alpha, \beta, \alpha + \beta \in \Psi$, we have*

$$[y_\alpha, y_\beta] = \varepsilon(\alpha', \beta')(p + 1) y_{\alpha+\beta}$$

where p is the largest integer such that $\alpha - p\beta \in \Psi$ and where $\alpha', \beta' \in \Phi$ are so chosen that $\alpha' + \beta' \in \Phi$.

The proof is given in [54, Proposition 7.9, p. 108].

4. Integral structures

Let us return to the Lie algebra $\mathfrak{g} = \mathfrak{g}^0 \otimes \mathbb{C}$ with $\mathfrak{g}^0$ real semisimple. Pick a σ- and τ-adapted Chevalley basis $\mathcal{C}$ of $(\mathfrak{g}, \mathfrak{h})$. Set $X_\alpha = x_\alpha + \sigma(x_\alpha)$ and $Y_\alpha = \mathrm{i}(x_\alpha - \sigma(x_\alpha))$ for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. Let $H^1_\alpha = h_\alpha + h_{\alpha^\sigma}$ and $H^0_\alpha = \mathrm{i}(h_\alpha - h_{\alpha^\sigma})$. In other words, X_α, H^1_α are twice the real part and Y_α, H^0_α are twice the imaginary part of x_α, h_α in the complex vector space $\mathfrak{g}$ with real structure σ. Let $Z_\alpha = X_\alpha + Y_\alpha$. Let $\Phi^{+\,*}_{\mathbb{C}}$ be $\Phi^+_{\mathbb{C}}$ with one element from each pair $\{\alpha, \alpha^\sigma\}$ removed and set $\Phi^*_{\mathbb{C}} = \Phi^{+\,*}_{\mathbb{C}} \cup -\Phi^{+\,*}_{\mathbb{C}}$. Here, as always, the plus sign indicates intersection with all positive roots. Pick one element from each two-element orbit $\{\alpha,\omega(\alpha)\}$ in Δ_1 and subsume them in a set Δ_1^*. Consider the sets

$$\mathcal{B}_\mathbb{R}=\{Z_\alpha\colon \alpha\in\Phi_\mathbb{R}\},\quad \mathcal{B}_{\mathrm{i}\mathbb{R}}=\{X_\alpha,Y_\alpha\colon \alpha\in\Phi_{\mathrm{i}\mathbb{R}}^+\},\quad \mathcal{B}_\mathbb{C}=\{X_\alpha,Y_\alpha\colon \alpha\in\Phi_\mathbb{C}^*\},$$
$$\mathcal{H}^1=\{H_\alpha^1\colon \alpha\in\Delta_1\setminus\Delta_1^*\},\quad \mathcal{H}^0=\{H_\alpha^0\colon \alpha\in\Delta_0\cup\Delta_1^*\}$$

and let $\mathcal{B}$ be their union. We agree that $c_{\alpha\beta}=0$ if $\alpha+\beta\notin\Phi(\mathfrak{g},\mathfrak{h})$ and $x_\alpha=0$ thus $X_\alpha=Y_\alpha=Z_\alpha=0$ if $\alpha\notin\Phi(\mathfrak{g},\mathfrak{h})$. Since $\langle\beta,\alpha\rangle$ (see below Theorem 5.2) is linear in β, we may allow this notation for all root lattice elements $\beta\in Q=\mathbb{Z}\Phi(\mathfrak{g},\mathfrak{h})$.

Theorem 5.18. *The set $\mathcal{B}$ is a basis of $\mathfrak{g}^0$ and the subsets $\mathcal{H}^1$ and $\mathcal{H}^0$ are bases of $\mathfrak{a}$ and $\mathfrak{h}^0\cap\mathfrak{k}$. The resulting structure constants lie in $\frac{1}{2}\mathbb{Z}$ and are given as follows.*

(i) *Let $\alpha,\beta\in\Phi(\mathfrak{g},\mathfrak{h})$. Then $[H_\alpha^i,H_\beta^j]=0$ for $i,j\in\{0,1\}$ and*

(ii)
$$[H_\alpha^1,X_\beta]=\langle\beta+\beta^\sigma,\alpha\rangle X_\beta,\quad [H_\alpha^1,Y_\beta]=\langle\beta+\beta^\sigma,\alpha\rangle Y_\beta,$$
$$[H_\alpha^0,X_\beta]=\langle\beta-\beta^\sigma,\alpha\rangle Y_\beta,\quad [H_\alpha^0,Y_\beta]=-\langle\beta-\beta^\sigma,\alpha\rangle X_\beta.$$

(iii) *Let $\alpha\in\Phi_\mathbb{R}$. Then*
$$[Z_\alpha,Z_{-\alpha}]=-\mathrm{sgn}(\alpha)2H_\alpha^1$$
and H_α^1 is a $\mathbb{Z}$-linear combination of elements in $\mathcal{H}^1$.

(iv) *Let $\alpha\in\Phi_{\mathrm{i}\mathbb{R}}^+$. Then*
$$[X_\alpha,Y_\alpha]=H_\alpha^0$$
and H_α^0 is a $\mathbb{Z}$-linear combination of elements H_β^0 for $\beta\in\Delta_0$.

(v) *Let $\alpha\in\Phi_\mathbb{C}^*$. Then*
$$[X_\alpha,X_{-\alpha}]=-H_\alpha^1,\quad [X_\alpha,Y_{-\alpha}]=-H_\alpha^0,\quad [Y_\alpha,Y_{-\alpha}]=H_\alpha^1$$
where H_α^1 and $2H_\alpha^0$ are $\mathbb{Z}$-linear combinations in $\mathcal{H}^1$ and $\mathcal{H}^0$, respectively.

(vi) *Let $\alpha,\beta\in\Phi(\mathfrak{g},\mathfrak{h})$ with $\beta\notin\{-\alpha,-\alpha^\sigma\}$. Then*
$$[X_\alpha,X_\beta]=c_{\alpha\beta}X_{\alpha+\beta}+\mathrm{sgn}(\alpha)c_{\alpha^\sigma\beta}X_{\alpha^\sigma+\beta},$$
$$[X_\alpha,Y_\beta]=c_{\alpha\beta}Y_{\alpha+\beta}+\mathrm{sgn}(\alpha)c_{\alpha^\sigma\beta}Y_{\alpha^\sigma+\beta},$$
$$[Y_\alpha,Y_\beta]=-c_{\alpha\beta}X_{\alpha+\beta}+\mathrm{sgn}(\alpha)c_{\alpha^\sigma\beta}X_{\alpha^\sigma+\beta}.$$

Note that (ii) and (vi) also yield the structure constants involving Z_β because for $\beta\in\Phi_\mathbb{R}$ we have $Z_\beta=X_\beta$ if $\mathrm{sgn}(\beta)=1$ and $Z_\beta=Y_\beta$ if $\mathrm{sgn}(\beta)=-1$. Also, in (vi) there is no reason to prefer α over β and indeed, by anticommutativity we have $\mathrm{sgn}(\alpha)c_{\alpha^\sigma\beta}X_{\alpha^\sigma+\beta}=\mathrm{sgn}(\beta)c_{\alpha\beta^\sigma}X_{\alpha+\beta^\sigma}$ and similarly we obtain $\mathrm{sgn}(\alpha)c_{\alpha^\sigma\beta}Y_{\alpha^\sigma+\beta}=-\mathrm{sgn}(\beta)c_{\alpha\beta^\sigma}Y_{\alpha+\beta^\sigma}$. Of course the basis $2\mathcal{B}$ has integer structure constants.

Proof. By construction the set $\mathcal{B}$ consists of linear independent elements and we have $|\mathcal{B}|=\dim_\mathbb{C}\mathfrak{g}=\dim_\mathbb{R}\mathfrak{g}^0$. So $\mathcal{B}$ is a basis. Moreover, $\theta(H_\alpha^j)=(-1)^jH_\alpha^j$ for all $\alpha\in\Phi(\mathfrak{g},\mathfrak{h})$ so that $\mathcal{H}^1\subset\mathfrak{a}$ and $\mathcal{H}^0\subset\mathfrak{h}^0\cap\mathfrak{k}$. Since $\dim_\mathbb{R}\mathfrak{a}=|\Delta_1|-|\Delta_1^*|$, these subsets generate. We verify the list of relations. Part (i) is clear. For part (ii) we compute

$$[H_\alpha^1,X_\beta]=[h_\alpha+h_{\alpha^\sigma},x_\beta+\mathrm{sgn}(\beta)x_{\beta^\sigma}]=\langle\beta,\alpha\rangle x_\beta+\mathrm{sgn}(\beta)\langle\beta^\sigma,\alpha\rangle x_{\beta^\sigma}+$$
$$+\langle\beta,\alpha^\sigma\rangle x_\beta+\mathrm{sgn}(\beta)\langle\beta^\sigma,\alpha^\sigma\rangle x_{\beta^\sigma}=\langle\beta+\beta^\sigma,\alpha\rangle X_\beta$$

where we used that $\langle\beta^\sigma,\alpha^\sigma\rangle=\langle\beta,\alpha\rangle$. The other three equations follow similarly. Let $\alpha\in\Phi_\mathbb{R}$. Then $Z_\alpha=X_\alpha$ if $\mathrm{sgn}(\alpha)=1$ and $Z_\alpha=Y_\alpha$ if $\mathrm{sgn}(\alpha)=-1$. In the two cases we have $[X_\alpha,X_{-\alpha}]=[2x_\alpha,2x_{-\alpha}]=-4h_\alpha=-2H_\alpha^1$ and $[Y_\alpha,Y_{-\alpha}]=-4[x_\alpha,x_{-\alpha}]=2H_\alpha^1$ so we get the first part of (iii). We verify that H_α^1 is a $\mathbb{Z}$-linear combination within $\mathcal{H}^1$ for general $\alpha\in\Phi(\mathfrak{g},\mathfrak{h})$. Under the Killing form identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ the elements $t_\alpha\in\mathfrak{h}$ correspond to the roots $\alpha\in\mathfrak{h}^*$. The elements $h_\alpha\in\mathfrak{h}$ correspond to the forms $\frac{2\alpha}{B(\alpha,\alpha)}\in\mathfrak{h}^*$ which make up a root system as well, namely the dual root system of $\Phi(\mathfrak{g},\mathfrak{h})$ with simple roots $\{h_\beta\colon \beta\in\Delta(\mathfrak{g},\mathfrak{h})\}$. We thus have

$$h_\alpha=\sum_{\gamma\in\Delta_1}k_\gamma h_\gamma+\sum_{\beta\in\Delta_0}k_\beta h_\beta$$

with certain integers k_γ, k_β which are either all nonnegative or all nonpositive. Since $\beta^\sigma = -\beta$ for $\beta \in \Delta_0$, we have

$$H^1_\alpha = h_\alpha + h_{\alpha^\sigma} = \sum_{\gamma\in\Delta_1} k_\gamma(h_\gamma + h_{\gamma^\sigma}) = \sum_{\gamma\in\Delta_1} k_\gamma H^1_\gamma.$$

From Lemma 5.10 we see $\gamma+\gamma^\sigma = \omega(\gamma)+\omega(\gamma)^\sigma$ and $B(\omega(\gamma),\omega(\gamma)) = B(\gamma,\gamma)$. Thus

$$H^1_\gamma = h_\gamma + h_{\gamma^\sigma} = \tfrac{2t_\gamma}{B(\gamma,\gamma)} + \tfrac{2t_{\gamma^\sigma}}{B(\gamma^\sigma,\gamma^\sigma)} = \tfrac{2t_{\gamma+\gamma^\sigma}}{B(\gamma,\gamma)} = \tfrac{2t_{\omega(\gamma)+\omega(\gamma)^\sigma}}{B(\omega(\gamma),\omega(\gamma))} = h_{\omega(\gamma)} + h_{\omega(\gamma)^\sigma} = H^1_{\omega(\gamma)}$$

and it follows that

$$H^1_\alpha = \sum_{\gamma\in\Delta_1\setminus\Delta_1^*} ((1-\delta_{\gamma,\omega(\gamma)})k_{\omega(\gamma)} + k_\gamma) H^1_\gamma$$

with Kronecker-δ. This proves the second part of (iii). Let $\alpha \in \Phi_{\mathrm{i}\mathbb{R}}$. Then

$$[X_\alpha, Y_\alpha] = [x_\alpha + x_{-\alpha}, \mathrm{i}(x_\alpha - x_{-\alpha})] = 2\mathrm{i}h_\alpha = H^0_\alpha.$$

Since the elements h_α for $\alpha \in \Phi_{\mathrm{i}\mathbb{R}}$ form the dual root system of $\Phi_{\mathrm{i}\mathbb{R}}$, we see that H^0_α is a $\mathbb{Z}$-linear combination of elements $H^0_\beta = 2\mathrm{i}h_\beta$ with $\beta \in \Delta_0$. This proves (iv). To prove (v) note first that for each $\alpha \in \Phi(\mathfrak{g},\mathfrak{h})$ the difference $\alpha - \alpha^\sigma$ is not a root. Indeed, if it were, then from the recursion formula on p. 54 we would get $\mathrm{sgn}(\alpha-\alpha^\sigma) = \mathrm{sgn}(\alpha)\mathrm{sgn}(-\alpha^\sigma)\frac{c_{\alpha^\sigma,-\alpha}}{c_{\alpha,-\alpha^\sigma}} = -1$ contradicting Proposition 5.8 (ii) because $\alpha-\alpha^\sigma = \alpha+\alpha^\theta \in \Phi_{\mathrm{i}\mathbb{R}}$. With this remark the three equations are immediate. It remains to show that H^0_α is a $\frac{1}{2}\mathbb{Z}$-linear combination within $\mathcal{H}^0$. From the above decomposition of h_α as a sum of simple dual roots we get

$$H^0_\alpha = \mathrm{i}(h_\alpha - h_{\alpha^\sigma}) = \sum_{\gamma\in\Delta_1} k_\gamma H^0_\gamma + \sum_{\beta\in\Delta_0} k_\beta H^0_\beta.$$

We still have to take care of H^0_γ for $\gamma \in \Delta_1 \setminus \Delta_1^*$. From Lemma 5.10 we conclude

$$h_{\gamma^\sigma} = \tfrac{2}{B(\gamma,\gamma)} t_{\gamma^\sigma} = h_{\omega(\gamma)} + \sum_{\beta\in\Delta_0} n_{\beta\gamma}\tfrac{B(\beta,\beta)}{B(\gamma,\gamma)} h_\beta$$

and the numbers $m_{\beta\gamma} = n_{\beta\gamma}\frac{B(\beta,\beta)}{B(\gamma,\gamma)}$ are integers. We thus get

$$\begin{aligned} H^0_\gamma &= \mathrm{i}(h_\gamma - h_{\gamma^\sigma}) = \mathrm{i}(h_\gamma - h_{\omega(\gamma)} - \sum_{\beta\in\Delta_0} m_{\beta\gamma}h_\beta) = \\ &= -H^0_{\omega(\gamma)} - 2\mathrm{i}\sum_{\beta\in\Delta_0} m_{\beta\gamma}h_\beta = -H^0_{\omega(\gamma)} - \sum_{\beta\in\Delta_0} m_{\beta\gamma}H^0_\beta. \end{aligned}$$

If $\omega(\gamma) \in \Delta_1^*$, this realizes H^0_γ as a $\mathbb{Z}$-linear combination within $\mathcal{H}^0$. If $\omega(\gamma) = \gamma$, we obtain $H^0_\gamma = -\frac{1}{2}\sum_{\beta\in\Delta_0} m_{\beta\gamma}H^0_\beta$ and this is the only point where half-integers might enter the picture. Finally to prove (vi) use the recursion formula to compute

$$\begin{aligned} [X_\alpha, X_\beta] &= [x_\alpha + \mathrm{sgn}(\alpha)x_{\alpha^\sigma}, x_\beta + \mathrm{sgn}(\beta)x_{\beta^\sigma}] = \\ &= c_{\alpha\beta}x_{\alpha+\beta} + \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta)c_{\alpha^\sigma\beta^\sigma}x_{\alpha^\sigma+\beta^\sigma} + \\ &\quad + \mathrm{sgn}(\alpha)c_{\alpha^\sigma\beta}x_{\alpha^\sigma+\beta} + \mathrm{sgn}(\beta)c_{\alpha\beta^\sigma}x_{\alpha+\beta^\sigma} = \\ &= c_{\alpha\beta}x_{\alpha+\beta} + c_{\alpha\beta}\mathrm{sgn}(\alpha+\beta)x_{\alpha^\sigma+\beta^\sigma} + \\ &\quad + c_{\alpha^\sigma\beta}\mathrm{sgn}(\alpha)x_{\alpha^\sigma+\beta} + c_{\alpha^\sigma\beta}\mathrm{sgn}(\alpha)\mathrm{sgn}(\alpha^\sigma+\beta)x_{\alpha+\beta^\sigma} = \\ &= c_{\alpha\beta}X_{\alpha+\beta} + \mathrm{sgn}(\alpha)c_{\alpha^\sigma\beta}X_{\alpha^\sigma+\beta}. \end{aligned}$$

The other two equations follow similarly. □

5. Consequences and applications

5.1. Special cases of $\mathfrak{g}^0$. We recall that $\mathfrak{g}^0$ is called *split* if $\mathfrak{a} = \mathfrak{h}^0$, *compact* if B^0 is negative definite, that is if $\mathfrak{k} = \mathfrak{g}^0$, and (abstractly) *complex* if $\mathfrak{g}^0$ has an $\mathbb{R}$-vector space automorphism J such that $J^2 = -\mathrm{id}$ and $[JX, Y] = J[X, Y]$ for all $X, Y \in \mathfrak{g}^0$. In the following theorem we will characterize these properties in terms of the three different types of roots in $\Phi(\mathfrak{g}, \mathfrak{h})$. In particular we construct an explicit complex structure J of $\mathfrak{g}^0$ if it admits one. Afterwards we discuss how the three special cases endow the basis $\mathcal{B}$ with additional features.

Theorem 5.19. *The semisimple Lie algebra $\mathfrak{g}^0$ is split, compact or complex if and only if all roots in $\Phi(\mathfrak{g}, \mathfrak{h})$ are real, imaginary or complex, respectively.*

Proof. We have $\Phi_{\mathbb{R}} = \Phi(\mathfrak{g}, \mathfrak{h})$ if and only if $\alpha^\sigma = \alpha$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ if and only if $\Delta_0 = \Delta_1^* = \mathcal{H}^0 = \emptyset$ if and only if $\mathfrak{h}^0 = \mathbb{R}\mathcal{H}^1 = \mathfrak{a}$. Similarly, $\Phi_{\mathrm{i}\mathbb{R}} = \Phi(\mathfrak{g}, \mathfrak{h})$ if and only if $\alpha^\sigma = -\alpha$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ if and only if $\Delta_1 = \mathcal{H}^1 = \emptyset$ if and only if $\mathfrak{h}^0 = \mathbb{R}\mathcal{H}^0 \subset \mathfrak{k}$ and $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}^0 \subset \mathfrak{k}$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ if and only if $\mathfrak{g}^0 = \mathfrak{k}$.

Let $\Phi_{\mathbb{C}} = \Phi(\mathfrak{g}, \mathfrak{h})$. Then $\Delta_0 = \emptyset$ so ω is an order-two permutation of $\Delta(\mathfrak{g}, \mathfrak{h})$. We have $\omega(\alpha) = \alpha^\sigma$ for each $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ by Lemma 5.10. So ω is a Dynkin diagram automorphism. Moreover, ω is fix point free because $\Phi_{\mathbb{R}} = \emptyset$. In particular $|\Delta(\mathfrak{g}, \mathfrak{h})|$ is even and ω does not leave invariant a connected component of the Dynkin diagram. Indeed, if it did, this component would necessarily be of type A_{2n} with ω being the flip because ω is fix point free. But then the sum of the two middle roots would be a real root which is absurd. Thus the Dynkin diagram consists of pairs of isomorphic components swapped by ω. Choose one component from each such pair and let their union be Δ_1^*. Let $\Phi_{\mathbb{C}}^*$ be the root system with simple roots Δ_1^*. We define a complex structure J on $\mathfrak{g}^0$ by means of the basis $\mathcal{B}$ of $\mathfrak{g}^0$, setting

$$X_\alpha \mapsto Y_\alpha, \qquad Y_\alpha \mapsto -X_\alpha \qquad \text{for } \alpha \in \Phi_{\mathbb{C}}^*,$$
$$H_\alpha^1 \mapsto H_{\omega(\alpha)}^0, \qquad H_{\omega(\alpha)}^0 \mapsto -H_\alpha^1 \qquad \text{for } \alpha \in \Delta_1 \setminus \Delta_1^*.$$

It follows that $JH_\alpha^1 = H_{\alpha^\sigma}^0$ and $JH_{\alpha^\sigma}^0 = -H_\alpha^1$ for all $\alpha \in \sigma(\Phi_{\mathbb{C}}^*)$ whereas $JH_\alpha^1 = -H_{\alpha^\sigma}^0$ and $JH_{\alpha^\sigma}^0 = H_\alpha^1$ for all $\alpha \in \Phi_{\mathbb{C}}^*$. By construction $J^2 = -\mathrm{id}$ and inspecting the equations in Theorem 5.18 we easily verify that $[JX, Y] = J[X, Y]$ for all $X, Y \in \mathfrak{g}^0$.

Let $\mathfrak{g}^0$ possess the complex structure J. For this last step compare [89, Example 2, p. 273]. Let $\overline{\mathfrak{g}^0}$ be equal to $\mathfrak{g}^0$ as real Lie algebras but with complex structure $-J$. Then the map $x \otimes y \mapsto y \otimes x$ defines a real form of the complex algebra $\mathfrak{g}^0 \oplus \overline{\mathfrak{g}^0}$ and this real form is clearly isomorphic to $\mathfrak{g}^0$. So the complexifications are isomorphic, that is $\mathfrak{g} \cong \mathfrak{g}^0 \oplus \overline{\mathfrak{g}^0}$. Let $\mathfrak{u}^0$ be a compact form of the complex algebra $\mathfrak{g}^0$ with conjugation τ^0. Then $\mathfrak{g}^0 = \mathfrak{u} \oplus J\mathfrak{u}$ is a Cartan decomposition of the real algebra $\mathfrak{g}^0$ and the Cartan involution θ equals τ^0. Let $\mathfrak{t} \subset \mathfrak{u}$ be maximal abelian. Then as real algebras $\mathfrak{h}^0 = \mathfrak{t} \oplus J\mathfrak{t}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}^0$ and as complex algebras $\mathfrak{h}^0$ is a τ^0-stable Cartan subalgebra of $\mathfrak{g}^0$. The conjugation τ^0 provides an isomorphism $(\overline{\mathfrak{g}^0}, \overline{\mathfrak{h}^0}) \cong (\mathfrak{g}^0, \mathfrak{h}^0)$ of pairs of complex Lie algebras. So the root system $\Phi(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g} \cong \mathfrak{g}^0 \oplus \overline{\mathfrak{g}^0}$ with Cartan subalgebra $\mathfrak{h} = \mathfrak{h}^0 \oplus \overline{\mathfrak{h}^0}$ consists of two orthogonal copies of the root system $\Phi(\mathfrak{g}^0, \mathfrak{h}^0)$ of the complex algebra $\mathfrak{g}^0$. These two copies are swapped by σ. It follows that $\Phi(\mathfrak{g}, \mathfrak{h})$ has neither real nor imaginary roots. □

If $\mathfrak{g}^0$ is split, then $\alpha^\sigma = \alpha$ and $\operatorname{sgn}(\alpha) = 1$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. So in that case we have $\mathcal{B} = 2\mathcal{C}$ and Theorem 5.18 boils down to the list of ordinary Chevalley constants of $\mathfrak{g}$ multiplied by two, compare [51, Theorem 25.2, p. 147]. If $\mathfrak{g}^0$ is compact, then $\alpha^\sigma = -\alpha$ and $\operatorname{sgn}(\alpha) = 1$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. In this case $\mathcal{B}$ gives the basis in the standard construction of a compact real form of a complex semisimple Lie algebra, see [48, equation (2), p. 182]. If $\mathfrak{g}^0$ is complex, we choose Cartan decomposition and θ-stable subalgebra $\mathfrak{h}^0$ as in the proof of Theorem 5.19. We conclude that $\Phi(\mathfrak{g}, \mathfrak{h})$ is the orthogonal sum of two copies of the root system $\Phi(\mathfrak{g}^0, \mathfrak{h}^0)$ of the complex algebra $\mathfrak{g}^0$ swapped by σ. We let $\Phi^*_{\mathbb{C}}$ with simple roots Δ^*_1 be the copy corresponding to $\mathfrak{g}^0$ in the decomposition $\mathfrak{g} \cong \mathfrak{g}^0 \oplus \overline{\mathfrak{g}^0}$. This decomposition is exactly the eigenspace decomposition of the extension of J from $\mathfrak{g}^0$ to $\mathfrak{g}$ with eigenvalues i and $-$i. Then the set

$$\{X_\alpha, H^1_\beta \colon \alpha \in \Phi^*_{\mathbb{C}},\ \beta \in \Delta_1 \setminus \Delta^*_1\} \subset \mathcal{B}$$

is a Chevalley basis of the complex semisimple Lie algebra $\mathfrak{g}^0$ with respect to the complex Cartan subalgebra $\mathfrak{h}^0$. Moreover, $JX_\alpha = J(x_\alpha + \operatorname{sgn}(\alpha)x_{\alpha^\sigma}) = \mathrm{i}x_\alpha + \operatorname{sgn}(\alpha)(-\mathrm{i})x_{\alpha^\sigma} = Y_\alpha$ and similarly for Y_α, H^1_α and $H^0_{\omega(\alpha)}$ so that J coincides with the complex structure constructed from $\mathcal{B}$ above.

5.2. Iwasawa decompositions. We construct a slight modification of the basis $\mathcal{B}$. It is going to be the union of three sets spanning the fixed Iwasawa decomposition $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. We start by discussing the Iwasawa $\mathfrak{n}$-algebra. The observation $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha^\sigma}$ allows us to state Proposition 5.7 (ii) more precisely as

$$\mathfrak{g}^0_\beta = \bigoplus_{\alpha \in \Phi^*_{\mathbb{C}} \colon i^*\alpha = \beta} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha^\sigma}) \cap \mathfrak{g}^0 \quad \bigoplus_{\alpha \in \Phi_{\mathbb{R}} \colon i^*\alpha = \beta} \mathfrak{g}_\alpha \cap \mathfrak{g}^0$$

for each $\beta \in \Phi(\mathfrak{g}^0, \mathfrak{a})$. It follows that the set

$$\mathcal{N} = \{X_\alpha, Y_\alpha, Z_\beta \colon \alpha \in \Phi^{*+}_{\mathbb{C}},\ \beta \in \Phi^+_{\mathbb{R}}\} \subset \mathcal{B}$$

is a basis of $\mathfrak{n}$. The structure constants are given in Theorem 5.18 (vi) so they are still governed by the root system $\Phi(\mathfrak{g}, \mathfrak{h})$. We will compute them explicitly in case $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}^0 = 1$ in Section 6.

Now we consider the maximal compact subalgebra $\mathfrak{k}$. For $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ let $U_\alpha = X_\alpha + \tau X_\alpha = X_\alpha + X_{-\alpha}$ and similarly $V_\alpha = Y_\alpha + \tau Y_\alpha = Y_\alpha - Y_{-\alpha}$ as well as $W_\alpha = Z_\alpha + \tau Z_\alpha = U_\alpha + V_\alpha$. By counting dimensions we verify

$$\mathcal{K} = \mathcal{H}^0 \cup \{U_\alpha, V_\alpha, X_\beta, Y_\beta, W_\gamma \colon \alpha \in \Phi^{*+}_{\mathbb{C}}, \beta \in \Phi^+_{\mathrm{i}\mathbb{R}}, \gamma \in \Phi^+_{\mathbb{R}}\}$$

is a basis of $\mathfrak{k}$. Thus $\mathcal{K} \cup \mathcal{H}^1 \cup \mathcal{N}$ is a basis of $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The elements U_α, V_α, W_γ are by construction $\mathbb{Z}$-linear combinations of elements in $\mathcal{B}$. Conversely, the only elements in $\mathcal{B}$ which do not lie in $\mathcal{K} \cup \mathcal{H}^1 \cup \mathcal{N}$ are $X_{-\alpha}, Y_{-\alpha}$ for $\alpha \in \Phi^{*+}_{\mathbb{C}}$ and $Z_{-\beta}$ for $\beta \in \Phi^+_{\mathbb{R}}$. But for those we have $X_{-\alpha} = U_\alpha - X_\alpha$, $Y_{-\alpha} = -V_\alpha + Y_\alpha$ and $Z_{-\alpha} = \operatorname{sgn}(\alpha)(W_\alpha - Z_\alpha)$. It follows that the change of basis matrices between $\mathcal{B}$ and $\mathcal{K} \cup \mathcal{H}^1 \cup \mathcal{N}$ both have integer entries and determinant ± 1. Theorem 5.18 thus gives the following conclusion.

Theorem 5.20. *The set $\mathcal{K} \cup \mathcal{H}^1 \cup \mathcal{N}$ is a basis of $\mathfrak{g}^0$ spanning the Iwasawa decomposition $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The structure constants lie in $\frac{1}{2}\mathbb{Z}$.*

5.3. Iwasawa N-Groups. Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}^0$. Let K, A, N be the analytic subgroups of G with Lie algebra $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ respectively. Then the map $(k, a, n) \mapsto kan$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G. This is the global Iwasawa decomposition of G, see [48, Theorem 5.1, p. 270]. The groups A and N are simply-connected. Therefore $\mathfrak{g}^0$ determines the groups N and $S = A \ltimes N$ up to Lie group isomorphism. The group N is called the *Iwasawa N-group* of $\mathfrak{g}^0$ and we want to call S the *symmetric space group* of $\mathfrak{g}^0$ with solvable *symmetric space algebra* $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$. A simply-connected nilpotent Lie group is *exponential*, which means that the exponential map $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism, see [88, Example 5, p. 63]. Since the Baker–Campbell–Hausdorff formula terminates for nilpotent Lie algebras, the basis $\mathcal{N}$ also gives a complete algebraic description of Iwasawa N-groups. Indeed, we can identify $N = \mathfrak{n}$ as sets and realize the multiplication as

$$X \cdot Y = \log(\exp X \exp Y) = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}\left([X, [X, Y]] - [Y, [X, Y]]\right) \mp \cdots$$

for $X, Y \in N$. Moreover, we have the following result.

Theorem 5.21. *Every Iwasawa $\mathfrak{n}$-algebra has a basis with integer structure constants of absolute value at most four.*

PROOF. From Theorem 5.18 (vi) we obtain $2|c_{\alpha\beta}|$ as an upper bound of the absolute value of structure constants. Theorem 5.2 (iv) and the well-known fact that root strings are of length at most four, tell us that $c_{\alpha\beta} \in \pm\{1, 2, 3\}$. The Chevalley constants $c_{\alpha\beta} = \pm 3$ can only occur when $\mathfrak{g}$ contains an ideal of type G_2. But G_2 has only two real forms, one compact and one split. A compact form does not contribute to $\mathfrak{n}$. For the split form divide all corresponding basis vectors in $\mathcal{N}$ by two. Let α be the short and β be the long simple root. Then we have just arranged that the equation $[Z_{2\alpha+\beta}, Z_\alpha] = \pm 3 Z_{3\alpha+\beta}$ gives the largest structure constant corresponding to this ideal. If $\mathfrak{g}^0$ happens to have an ideal admitting a complex G_2-structure, then $\mathfrak{g}$ has two G_2-ideals swapped by σ. In that case the corresponding two G_2 root systems are perpendicular. So one of the two summands in every equation of Theorem 5.18 vanishes and the ideal in $\mathfrak{g}^0$ does not yield structure constants larger than three either. □

In [30] it is shown that the Iwasawa $\mathfrak{n}$-algebras of a semisimple Lie algebra $\mathfrak{g}^0$ with $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}^0 = 1$ comprise exactly the "H-type Lie algebras" fulfilling the "J^2-condition". G. Crandall and J. Dodziuk have shown in [31] that every H-type Lie algebra has a basis with integer structure constants which can even be chosen to lie in the set $\{-1, 0, +1\}$. In accordance with this result, we will see in Section 6 that in the rank one case our method also allows for structure constants within this set.

Corollary 5.22. *Every Iwasawa N-group contains a lattice.*

PROOF. According to a criterion of A. I. Malcev [73, Theorem 7, p. 24] the assertion is equivalent to $\mathfrak{n}$ admitting a $\mathbb{Q}$-*structure* which is just a basis with rational structure constants. □

Any lattice in a nilpotent Lie group is uniform. The set of isomorphism classes of nilpotent Lie algebras with $\mathbb{Q}$-structure is clearly countable. A. L. Onishchik and E. B. Vinberg remark in [87, p. 46] that all nilpotent Lie algebras up to dimension six admit $\mathbb{Q}$-structures. On the other hand a continuum of pairwise nonisomorphic seven dimensional six-step nilpotent Lie algebras is constructed in N. Bourbaki [19, Exercise 18, p. 95]. P. Eberlein [34] describes moduli spaces with the homeomorphism type of arbitrary high dimensional manifolds even for two-step nilpotent Lie algebras. So in this somewhat stupid sense most nilpotent Lie groups do not contain lattices.

5.4. Coordinates in symmetric spaces. Recall that a Lie algebra $\mathfrak{t}$ over a field F is called *triangular* if for all $x \in \mathfrak{t}$ the endomorphism $\operatorname{ad}(x)$ has all eigenvalues in F. The symmetric space algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is triangular over $\mathbb{R}$ as is clear inspecting Theorem 5.18 (ii) and (vi). In fact $\mathfrak{a} \oplus \mathfrak{n}$ is maximal triangular in $\mathfrak{g}^0$ as proven by G. D. Mostow in [79, paragraph 2.4, p. 506]. A Lie group H is called triangular if all operators $\operatorname{Ad}(h)$ for $h \in H$ have only real eigenvalues. Clearly, a connected Lie group is triangular if and only if its Lie algebra is. It follows that the symmetric space group $S \subset G$ is simply connected triangular, thus exponential according to [88, Example 6, p. 63]. Let $P = \exp(\mathfrak{p})$ and let $\tilde{\theta}$ be the global geodesic symmetry, that is the automorphism of G with differential θ. Then the assignment $\psi\colon s \mapsto \tilde{\theta}(s)s^{-1}$ defines a diffeomorphism of the closed subgroup S of G onto the closed submanifold P of G, see [48, Proposition 5.3, p. 272]. Moreover, the projection $\pi\colon G \to G/K$ restricts to a diffeomorphism of P onto the globally symmetric space G/K according to Theorem 1.1.(iii), p. 253 of the same reference.

Finally, the basis $\mathcal{H}^1 \cup \mathcal{N}$ of $\mathfrak{s}$ provides a vector space isomorphism $\phi\colon \mathbb{R}^n \to \mathfrak{s}$ with $n = |\Sigma^+| + \operatorname{rank}_{\mathbb{R}} \mathfrak{g}^0$. Hence we get a chain of diffeomorphisms

$$\mathbb{R}^n \xrightarrow{\phi} \mathfrak{s} \xrightarrow{\exp} S \xrightarrow{\psi} P \xrightarrow{\pi} G/K$$

which defines a coordinate system of the globally symmetric space G/K of noncompact type. We have thus constructed coordinate charts for all symmetric spaces of noncompact type in a uniform way. Note moreover that the diffeomorphism ψ restricts to $s \mapsto s^{-2}$ on the closed abelian subgroup $A = S \cap P$ of G. Thus ψ leaves A invariant. It follows that in our coordinates the set $\phi^{-1}(\mathcal{H}^1)$ spans the maximal flat, totally geodesic submanifold $\pi(A)$ of G/K.

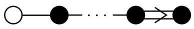
6. Real rank one simple Lie algebras

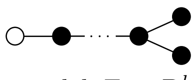
To illustrate our methods we now compute the structure constants of all Iwasawa $\mathfrak{n}$-algebras of simple Lie algebras $\mathfrak{g}^0$ with $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}^0 = 1$. These are precisely the Lie algebras of the isometry groups of rank-1 symmetric spaces of noncompact type. According to the Cartan classification (see [48, table V, p. 518]) the complete list consists of $\mathfrak{so}(n,1)$, $\mathfrak{su}(n,1)$, $\mathfrak{sp}(n,1)$ for $n \geq 2$ and the exceptional $\mathfrak{f}_{4(-20)}$. They correspond to real, complex and quaternionic hyperbolic spaces $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{H}}$ and to the Cayley plane $\mathbb{H}^2_{\mathbb{O}}$. Since $\mathfrak{g}^0$ is of real rank one, $\Phi(\mathfrak{g}^0, \mathfrak{a})$ can only be of type A_1 or BC_1. The corresponding Iwasawa $\mathfrak{n}$-algebra is correspondingly abelian or two-step nilpotent. The Campbell–Baker–Hausdorff formula thus takes a particularly simple form and we have $X \cdot Y = X + Y + \frac{1}{2}[X,Y]$ for X and Y in the Iwasawa N-group of $\mathfrak{g}^0$.

All relevant data identifying the isomorphism type of a real semisimple Lie algebra can be pictured in a convenient diagram which has been introduced in [99].

Definition 5.23. The *Satake diagram* of $\mathfrak{g}^0$ is the Dynkin diagram of $\mathfrak{g}$ with all imaginary roots shaded and each two-element orbit of ω in Δ_1 connected by a curved double-headed arrow.

The Satake diagram is a complete invariant of real semisimple Lie algebras as proven by S. Araki in [3]. It is connected if and only if $\mathfrak{g}^0$ is simple. Satake diagrams of all isomorphism types of real simple Lie algebras are displayed on pp. 32/33 of Araki's article. Note that the Tits indices we have introduced above Example 4.11 give a generalization of this concept in the context of algebraic groups over a general field k.

6.1. Real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. In this case $\mathfrak{g}^0 = \mathfrak{so}(n,1)$ with maximal compact subalgebra $\mathfrak{k} = \mathfrak{so}(n)$. For even n, the Lie algebra $\mathfrak{g}^0$ is of type BII which corresponds to the Satake diagram  with $l = \frac{n}{2}$ nodes. The root system $j^*\Phi(\mathfrak{g},\mathfrak{h}) \subseteq \mathfrak{h}^*_{\mathbb{R}}$ is thus of type B_l for which we use the following common model (see [51, p. 64]). Let E be standard Euclidean l-space and $\varepsilon_i \in E$ the i-th standard vector. Then $\Phi = \{\pm\varepsilon_i\} \cup \{\pm(\varepsilon_i \pm \varepsilon_j)\colon i \neq j\}$ as a union of short and long roots. A natural choice of a set Δ of simple roots is given by the $l-1$ long roots $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{l-1} - \varepsilon_l$ and the short root ε_l. In this order, they correspond to the nodes of the Satake diagram from left to right. The Satake diagram tells us that in this model $\mathfrak{a}^*$ is given by $\mathbb{R}\varepsilon_1$, the orthogonal complement of the subspace spanned by all the shaded roots $\Delta \setminus \{\varepsilon_1 - \varepsilon_2\}$. The orthogonal projection k^* thus becomes $p(v) = \langle v, \varepsilon_1 \rangle \varepsilon_1$ for $v \in E$. Therefore $\Sigma^+ = \{\varepsilon_1, \varepsilon_1 \pm \varepsilon_i\colon i \geq 2\}$ so that $p(\Sigma^+) = \{\varepsilon_1\}$ which says $\Phi(\mathfrak{g}^0, \mathfrak{a})$ is of type A_1. The case $n = 6$ is illustrated in Figure 5.24.

For n odd, $\mathfrak{g}^0$ is of type DII and has the Satake diagram  with $l = \frac{n+1}{2}$ nodes. Thus $\Phi(\mathfrak{g},\mathfrak{h})$ is of type D_l and we use the model $E = \mathbb{R}^l$,

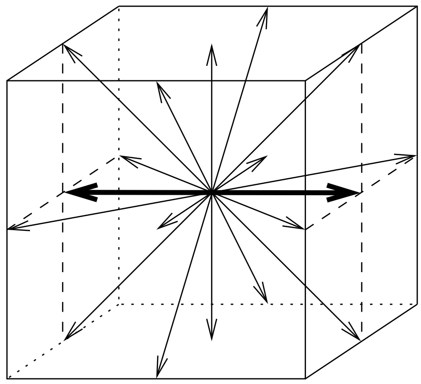
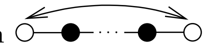


FIGURE 5.24. The root system of $\mathfrak{so}(7;\mathbb{C})$ with restricted root system $\Phi(\mathfrak{so}(6,1),\mathfrak{a})$ depicted as thick arrows. The short root ε_1 is pointing right, the short root ε_2 is pointing upwards and the short root ε_3 is pointing to the front.

$\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) \colon i \neq j\}$ with simple roots $\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_{l-1} + \varepsilon_l\}$. The restriction k^* becomes p as above, so that $\Sigma^+ = \{\varepsilon_1 \pm \varepsilon_i \colon i \geq 2\}$, thus $p(\Sigma^+) = \{\varepsilon_1\}$. Hence in any case $\Phi(\mathfrak{g}^0, \mathfrak{a})$ is of type A_1. But then $\mathfrak{n}$ must be abelian. All structure constants in any basis are zero.

6.2. Complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. In this case $\mathfrak{g}^0 = \mathfrak{su}(n,1)$ with maximal compact subalgebra $\mathfrak{k} = \mathfrak{su}(n)$. For n arbitrary, $\mathfrak{g}^0$ is of type AIV which has the Satake diagram  with $l = n$ nodes. So $\Phi(\mathfrak{g}, \mathfrak{h})$ is of type A_l and as a model let E be the orthogonal complement of $\varepsilon_1 + \cdots + \varepsilon_{l+1}$ in $\mathbb{R}^{l+1}$. Then $\Phi = \{\varepsilon_i - \varepsilon_j \colon i \neq j\}$ and $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \colon 1 \leq i \leq l\}$ is a basis of simple roots. The middle $l-2$ shaded nodes in the Satake diagram tell us that the line $\mathfrak{a}^*$ must lie in the span of ε_1 and ε_{l+1}. The bent arrow in turn says $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_l - \varepsilon_{l+1}$ yield the same restricted root in $\mathfrak{a}^*$ via k^*. But then of necessity $\mathfrak{a}^* = \mathbb{R}(\varepsilon_1 - \varepsilon_{l+1})$. Thus $p(v) = \frac{1}{2}\langle v, \varepsilon_1 - \varepsilon_{l+1}\rangle(\varepsilon_1 - \varepsilon_{l+1})$ for $v \in E$. Now for $i = 1, \ldots, l-1$, set $\alpha_i = \varepsilon_1 - \varepsilon_{i+1}$. Then $\alpha_i = \frac{1}{2}(\varepsilon_1 - \varepsilon_{l+1}) + (\frac{1}{2}\varepsilon_1 - \varepsilon_{i+1} + \frac{1}{2}\varepsilon_{l+1})$ is the decomposition of α_i with respect to $E = \mathfrak{a}^* \oplus \mathfrak{a}^{*\perp}$. It follows that $\alpha_i^\sigma = \frac{1}{2}(\varepsilon_1 - \varepsilon_{l+1}) - (\frac{1}{2}\varepsilon_1 - \varepsilon_{i+1} + \frac{1}{2}\varepsilon_{l+1}) = \varepsilon_{i+1} - \varepsilon_{l+1}$. Let $\beta = \varepsilon_1 - \varepsilon_{l+1}$. Then $\Sigma^+ = \{\alpha_i, \alpha_i^\sigma, \beta\}$ and the projection takes the values $p(\alpha_i) = p(\alpha_i^\sigma) = \frac{1}{2}\beta$ and $p(\beta) = \beta$. Thus $\Phi(\mathfrak{g}^0, \mathfrak{a})$ is of type BC_1. We observe that $\alpha_i + \alpha_i^\sigma = \beta$ while all other sums of two roots in Σ^+ do not lie in Φ. The case $n = 3$ is illustrated in Figure 5.25.
Pick a σ- and τ-adapted Chevalley basis of $(\mathfrak{g}, \mathfrak{h})$. Then $\mathfrak{n}$ has the basis $\mathcal{N}$ consisting of the $2l - 1$ elements

$$Z_\beta,\ X_{\alpha_i},\ Y_{\alpha_i} \quad \text{for } i = 1, \ldots, l-1.$$

The recursion formula on p. 54 gives $\operatorname{sgn}(\beta) = \operatorname{sgn}(\alpha_i)^2 \frac{c_{\alpha_i^\sigma \alpha_i}}{c_{\alpha_i \alpha_i^\sigma}} = -1$. So $Z_\beta = Y_\beta$. Note that the α_i^σ-string through α_i in Φ is of length two, so $c_{\alpha_i^\sigma \alpha_i} = \pm 1$ by Theorem 5.2 (iv). Hence by Theorem 5.18, the only nonzero structure constants are given by

$$[X_{\alpha_i}, Y_{\alpha_i}] = Z_\beta$$

where we have replaced X_{α_i} by $-X_{\alpha_i}$ if $\operatorname{sgn}(\alpha_i) c_{\alpha_i^\sigma \alpha_i} = -1$. In other words, $\mathfrak{n}$ is a 2-step nilpotent Lie algebra isomorphic to the Heisenberg Lie algebra $\mathfrak{h}^{2l-1}$. This Lie algebra is also known as the H-type algebra $\mathfrak{n}_1^{l-1}$ corresponding to the Clifford module C_1^{l-1}, see [30, p. 6]. It has a one-dimensional center with basis $\{Z_\beta\}$.

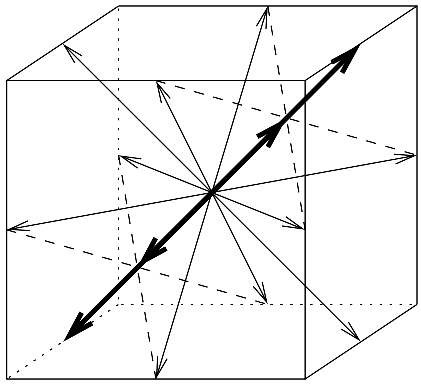


FIGURE 5.25. The root system of $\mathfrak{sl}(4;\mathbb{C})$ with restricted root system $\Phi(\mathfrak{su}(3,1),\mathfrak{a})$ depicted as thick arrows. The root $\varepsilon_1-\varepsilon_2$ is pointing up front, the root $\varepsilon_2-\varepsilon_3$ is pointing down right and the root $\varepsilon_3-\varepsilon_4$ is pointing up back.

6.3. Quaternionic hyperbolic space $\mathbb{H}^n_{\mathbb{H}}$. In this case $\mathfrak{g}^0=\mathfrak{sp}(n,1)$ with maximal compact subalgebra $\mathfrak{k}=\mathfrak{sp}(n)$. For $n\geq 2$ arbitrary, $\mathfrak{g}^0$ is of type CII which has the Satake diagram ●—○—●—⋯—●⇐● with $l=n+1$ nodes. So $\Phi(\mathfrak{g},\mathfrak{h})$ is of type C_l and as a model let $E=\mathbb{R}^l$ and $\Phi=\{\pm 2\varepsilon_i,\pm(\varepsilon_i\pm\varepsilon_j)\colon i\neq j\}$. A basis of the root system is given by $\Delta=\{\varepsilon_1-\varepsilon_2,\dots,\varepsilon_{l-1}-\varepsilon_l,2\varepsilon_l\}$. We have $\mathfrak{a}^*=\mathbb{R}(\varepsilon_1+\varepsilon_2)$, thus $p(v)=\frac{1}{2}\langle v,\varepsilon_1+\varepsilon_2\rangle(\varepsilon_1+\varepsilon_2)$ for $v\in E$. Now for $i=1,\dots,l-2$, set $\alpha_i=\varepsilon_1+\varepsilon_{i+2}$ and $\beta_i=\varepsilon_1-\varepsilon_{i+2}$. It follows $\alpha_i^\sigma=\varepsilon_2-\varepsilon_{i+2}$ and $\beta_i^\sigma=\varepsilon_2+\varepsilon_{i+2}$. Let $\gamma=2\varepsilon_1$ so that $\gamma^\sigma=2\varepsilon_2$ and let $\delta=\varepsilon_1+\varepsilon_2$. Then $\Sigma^+=\{\alpha_i,\alpha_i^\sigma,\beta_i,\beta_i^\sigma,\gamma,\gamma^\sigma,\delta\colon i=1,\dots,l-2\}$ and $p(\gamma)=p(\gamma^\sigma)=\delta$ while $p(\alpha_i^{(\sigma)})=p(\beta_i^{(\sigma)})=\frac{1}{2}\delta$. So again, $\Phi(\mathfrak{g}^0,\mathfrak{a})$ is of type BC_1. We have $\alpha_i+\beta_i=\gamma$, thus $\alpha_i^\sigma+\beta_i^\sigma=\gamma^\sigma$, and $\alpha_i+\alpha_i^\sigma=\beta_i+\beta_i^\sigma=\delta$. All other sums of two roots in Σ^+ do not lie in Φ. The case $n=2$ is featured in Figure 5.26.

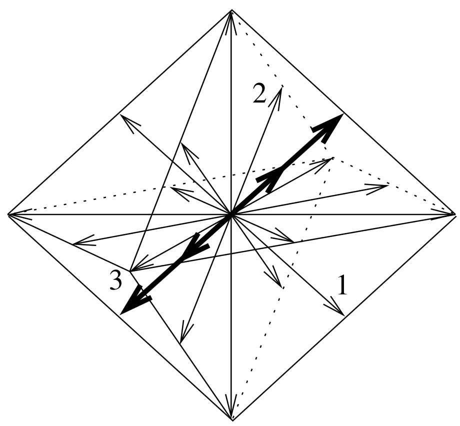


FIGURE 5.26. The root system of $\mathfrak{sp}(3;\mathbb{C})$ with restricted root system $\Phi(\mathfrak{sp}(2,1),\mathfrak{a})$ depicted as thick arrows. The root $\varepsilon_1-\varepsilon_2$ is labeled "1", the root $\varepsilon_2-\varepsilon_3$ is labeled "2" and the root $2\varepsilon_3$ is labeled "3".

We consider ○▶○—⋯—○▶○◀○—⋯—○◀○, a $\bar{\mu}$-invariantly oriented Dynkin diagram of type A_{2l-1}. By Proposition 5.17 it defines the simple complex Lie algebra $\mathfrak{g}(\Psi)$ with Cartan subalgebra $\mathfrak{h}(\Psi)$, Chevalley basis $\mathcal{C}(\Psi)=\{y_\alpha,h_i\}$ and asymmetry function ε. Here Ψ and thus $\mathfrak{g}(\Psi)$ are of type C_l and Ψ has simple roots Π. We have a canonical bijection $\Delta(\mathfrak{g},\mathfrak{h}) \to \Pi$ because the Dynkin diagram of type C_l has no symmetries. This induces an isomorphism $\mathfrak{h} \to \mathfrak{h}(\Psi)$. Pick arbitrary nonzero elements $x_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$. Then the assignment $x_\alpha \mapsto y_\alpha$ and the isomorphism $\mathfrak{h} \to \mathfrak{h}(\Psi)$ extend to a unique isomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{g}(\Psi)$, see [51, Theorem 14.2, p. 75]. Thus $\mathcal{C} = \varphi^{-1}(\mathcal{C}(\Psi))$ is a Chevalley basis of $(\mathfrak{g},\mathfrak{h})$ with the same structure constants as $\mathcal{C}(\Psi)$ in $\mathfrak{g}(\Psi)$. We adapt $\mathcal{C}$ to σ and τ as in the proof of Proposition 5.8. The Satake diagram of $\mathfrak{g}^0$ has no curved arrows, so $\omega = \mathrm{id}_{\Delta_1}$. As remarked below Proposition 5.12, we thus did not change the structure constants when adapting $\mathcal{C}$. The α_i-string through β_i has length three and $\beta_i - \alpha_i$ is a root. The α_i-string through α_i^σ as well as the β_i-string through β_i^σ have length two. Hence $c_{\alpha_i\beta_i} = \pm 2$, $c_{\alpha_i^\sigma\alpha_i} = \pm 1$ and $c_{\beta_i^\sigma\beta_i} = \pm 1$. We compute the sign in these three expressions. For brevity, let us denote the simple roots by $\vartheta_1 = \varepsilon_1 - \varepsilon_2,\ \dots\ ,\ \vartheta_{l-1} = \varepsilon_{l-1} - \varepsilon_l$ and $\vartheta_l = 2\varepsilon_l$. As sums of simple roots, we have

$$\begin{aligned}\alpha_i &= \vartheta_1 + \cdots + \vartheta_{i+1} + 2(\vartheta_{i+2} + \cdots + \vartheta_{l-1}) + \vartheta_l, \quad \alpha_i^\sigma = \vartheta_2 + \cdots + \vartheta_{i+1},\\ \beta_i &= \vartheta_1 + \cdots + \vartheta_{i+1}, \quad \beta_i^\sigma = \vartheta_2 + \cdots + \vartheta_{i+1} + 2(\vartheta_{i+2} + \cdots + \vartheta_{l-1}) + \vartheta_l.\end{aligned}$$

Let $\eta_1, \dots, \eta_{2l-1}$ be the simple roots from left to right in the oriented Dynkin diagram. Then a choice of primed roots written as a sum as in Lemma 5.11 is given by

$$\begin{aligned}{\alpha_i}' &= \eta_1 + \cdots + \eta_{2l-i-2}, & {\alpha_i^\sigma}' &= \eta_{2l-i-1} + \cdots + \eta_{2l-2},\\ {\beta_i}' &= \eta_{2l-i-1} + \cdots + \eta_{2l-1}, & {\beta_i^\sigma}' &= \eta_2 + \cdots + \eta_{2l-i-2}.\end{aligned}$$

From the description of the root system of type A in Section 6.2, we see that ${\alpha_i^\sigma}' + {\alpha_i}'$, ${\beta_i^\sigma}' + {\beta_i}'$ and ${\alpha_i}' + {\beta_i}'$ are roots. We calculate

$$\begin{aligned}\varepsilon({\alpha_i^\sigma}',\ {\alpha_i}') &= \varepsilon(\eta_{2l-i-1},\ \eta_{2l-i-2}) = -1,\\ \varepsilon({\beta_i^\sigma}',\ {\beta_i}') &= \varepsilon(\eta_{2l-i-2},\ \eta_{2l-i-1}) = 1,\\ \varepsilon({\alpha_i}',\ {\beta_i}') &= \varepsilon(\eta_{2l-i-2},\ \eta_{2l-i-1}) = 1.\end{aligned}$$

By Proposition 5.17 we thus get $c_{\alpha_i^\sigma\alpha_i} = -1$, $c_{\beta_i^\sigma\beta_i} = 1$ and $c_{\alpha_i\beta_i} = 2$ for $i = 1, \dots, l-2$. It only remains to compute $\mathrm{sgn}(\alpha_i)$ and $\mathrm{sgn}(\beta_i)$. For $j = 2, \dots, 2l-2$ let us decree ${\vartheta_{j+1}}' = \eta_{2l-j-1}$ and ${\vartheta_{j+1}^\sigma}' = (-\vartheta_{j+1})' = -\eta_{2l-j-1}$. Then ${\vartheta_{j+1}}' + {\alpha_{j-1}^\sigma}'$ and ${\vartheta_{j+1}^\sigma}' + {\alpha_{j-1}}'$ as well as ${\vartheta_{j+1}}' + {\beta_{j-1}}'$ and ${\vartheta_{j+1}^\sigma}' + {\beta_{j-1}^\sigma}'$ are roots. By Propositions 5.13 and 5.17 we have

$$\begin{aligned}\mathrm{sgn}(\alpha_i) = \mathrm{sgn}(\alpha_i^\sigma) &= \prod_{j=2}^{i} \tfrac{\varepsilon({\vartheta_{j+1}^\sigma}',\ {\alpha_{j-1}}')}{\varepsilon({\vartheta_{j+1}}',\ {\alpha_{j-1}^\sigma}')} = \prod_{j=2}^{i} \tfrac{\varepsilon(-\eta_{2l-j-1},\ \eta_1+\cdots+\eta_{2l-j-1})}{\varepsilon(\eta_{2l-j-1},\ \eta_{2l-j}+\cdots+\eta_{2l-2})} =\\ &= \prod_{j=2}^{i} \tfrac{\varepsilon(\eta_{2l-j-1},\ \eta_{2l-j-1})\varepsilon(\eta_{2l-j-1},\ \eta_{2l-j-2})}{\varepsilon(\eta_{2l-j-1},\ \eta_{2l-j})} = \prod_{j=2}^{i} \tfrac{(-1)(-1)}{(+1)} = +1,\\ \mathrm{sgn}(\beta_i) = \tfrac{\varepsilon({\vartheta_2^\sigma}',{\vartheta_1^\sigma}')}{\varepsilon({\vartheta_2}',{\vartheta_1}')} \prod_{j=2}^{i} \tfrac{\varepsilon({\vartheta_{j+1}^\sigma}',{\beta_{j-1}^\sigma}')}{\varepsilon({\vartheta_{j+1}}',{\beta_{j-1}}')} &= -\prod_{j=2}^{i} \tfrac{\varepsilon(-\eta_{2l-j-1},\ \eta_2+\cdots+\eta_{2l-j-1})}{\varepsilon(\eta_{2l-j-1},\ \eta_{2l-j}+\cdots+\eta_{2l-1})} = -1.\end{aligned}$$

Now we have collected all necessary data. The Chevalley basis $\mathcal{C}$ defines the basis $\mathcal{N}$ of the Iwasawa algebra $\mathfrak{n}$ of $\mathfrak{g}^0$ consisting of the $4l-5$ elements

$$X_{\alpha_i},\ Y_{\alpha_i},\ X_{\beta_i},\ Y_{\beta_i},\ X_\gamma,\ Y_\gamma,\ Z_\delta \text{ for } i = 1, \dots, l-2.$$

Since $\operatorname{sgn}(\delta) = -1$ we have $Z_\delta = Y_\delta$. Theorem 5.18 says that the following relations give the nonzero structure constants.

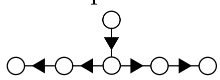
$$[X_{\alpha_i}, X_{\beta_i}] = 2X_\gamma, \qquad [X_{\alpha_i}, Y_{\beta_i}] = 2Y_\gamma, \qquad [X_{\alpha_i}, Y_{\alpha_i}] = -Z_\delta,$$
$$[Y_{\alpha_i}, Y_{\beta_i}] = -2X_\gamma, \qquad [Y_{\alpha_i}, X_{\beta_i}] = 2Y_\gamma, \qquad [X_{\beta_i}, Y_{\beta_i}] = -Z_\delta.$$

The Lie algebra $\mathfrak{n}$ is also isomorphic to the H-type algebra $\mathfrak{n}_3^{l-2,0}$ determined by the Clifford module $C_3^{l-2,0}$. Alternatively, by comparison with the structure constants given in [6, p. 185], we see that $\mathfrak{n}$ is isomorphic to the Lie algebra of the quaternionic Heisenberg group $\mathbb{H}H^{4n-1}$. If we rescale the elements X_{α_i}, Y_{α_i}, X_{β_i}, Y_{β_i}, X_γ, Y_γ by $\frac{1}{2}$ and the element Z_δ by $\frac{1}{4}$, we obtain structure constants within the set $\{-1, 0, +1\}$.

6.4. Octonionic hyperbolic plane $\mathbb{H}^2_{\mathbb{O}}$. This is the exceptional case $\mathfrak{g}^0 = \mathfrak{f}_{4(-20)}$ with maximal compact subalgebra $\mathfrak{so}(9)$. The Cartan label of $\mathfrak{g}^0$ is FII and the Satake diagram is . The root system $\Phi(\mathfrak{g}, \mathfrak{h})$ is of type F_4, so as a model take $E = \mathbb{R}^4$ and $\Phi = \{\pm\varepsilon_i, \pm(\varepsilon_i \pm \varepsilon_j), \pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \colon 1 \le i, j \le 4, i \ne j\}$, a total of 48 roots. Let $\vartheta_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$, $\vartheta_2 = \varepsilon_4$, $\vartheta_3 = \varepsilon_3 - \varepsilon_4$ and $\vartheta_4 = \varepsilon_2 - \varepsilon_3$. Then $\Delta = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ is a set of simple roots. We see $\mathfrak{a}^* = \mathbb{R}\varepsilon_1$ and $p(v) = \langle v, \varepsilon_1\rangle\varepsilon_1$ for $v \in E$. Set $\delta = \varepsilon_1$, $\gamma_i = \varepsilon_1 - \varepsilon_{i+1}$ for $i = 1, 2, 3$ and $\alpha_i = \vartheta_1 + \cdots + \vartheta_i$ for $i = 1, 2, 3, 4$. Then $\Sigma^+ = \{\delta, \gamma_i, \gamma_i^\sigma, \alpha_j, \alpha_j^\sigma \colon i = 1, 2, 3; j = 1, 2, 3, 4\}$ and we compute $p(\delta) = p(\gamma_i^{(\sigma)}) = \delta$ and $p(\alpha_j^{(\sigma)}) = \frac{1}{2}\delta$. So $\Phi(\mathfrak{g}^0, \mathfrak{a})$ is of type BC_1, too. We have $\alpha_i + \alpha_i^\sigma = \delta$ for $i = 1, 2, 3, 4$. Additionally we get

$$\alpha_1 + \alpha_4^\sigma = \gamma_1, \qquad \alpha_1 + \alpha_3^\sigma = \gamma_2, \qquad \alpha_1 + \alpha_2^\sigma = \gamma_3,$$
$$\alpha_2 + \alpha_3 = \gamma_1, \qquad \alpha_2 + \alpha_4 = \gamma_2, \qquad \alpha_3 + \alpha_4 = \gamma_3.$$

Together with six more equations obtained by applying σ, these comprise all sums of two roots in Σ^+ lying in Φ.

We consider the $\bar{\mu}$-invariantly oriented Dynkin diagram  of type E_6. Just like in Section 6.3, we obtain a σ- and τ-adapted Chevalley basis $\mathcal{C}$ of $(\mathfrak{g}, \mathfrak{h})$ whose structure constants arise from the Dynkin diagram as in Proposition 5.17. We see that $\alpha_i - \alpha_i^\sigma$ is not a root, so we have $c_{\alpha_i^\sigma \alpha_i} = \pm 1$ for $i = 1, 2, 3, 4$. On the other hand, if two roots in $\{\alpha_i, \alpha_i^\sigma\}$ add up to some γ_i or γ_i^σ then their difference is a root, too. Since root strings in F_4 are of length at most three, this gives $c_{\alpha_1\alpha_4^\sigma}$, $c_{\alpha_1\alpha_3^\sigma}$, $c_{\alpha_1\alpha_2^\sigma}$, $c_{\alpha_2\alpha_3}$, $c_{\alpha_2\alpha_4}$, $c_{\alpha_3\alpha_4} \in \{\pm 2\}$. We compute the signs. As sums of simple roots we have

$$\alpha_1^\sigma = \vartheta_1 + 3\vartheta_2 + 2\vartheta_3 + \vartheta_4, \qquad \alpha_2^\sigma = \vartheta_1 + 2\vartheta_2 + 2\vartheta_3 + \vartheta_4,$$
$$\alpha_3^\sigma = \vartheta_1 + 2\vartheta_2 + \vartheta_3 + \vartheta_4, \qquad \alpha_4^\sigma = \vartheta_1 + 2\vartheta_2 + \vartheta_3.$$

Denote the upper root of the oriented Dynkin diagram by η_4 and the lower roots from left to right by η_1, η_2, η_3, η_5 and η_6. We make choices of primed roots whose sums are roots by inspecting a standard description of a type E_6 root system as for example in [48, p. 473] or [88, p. 225]. Then we obtain

$$\begin{aligned}
\varepsilon(\alpha_1^{\sigma\prime}, \ \alpha_1{}') &= \varepsilon(\eta_1 + 2\eta_2 + 2\eta_3 + \eta_4 + \eta_5, \ \eta_6) &&= -1,\\
\varepsilon(\alpha_2^{\sigma\prime}, \ \alpha_2{}') &= \varepsilon(\eta_1 + \eta_2 + 2\eta_3 + \eta_4 + \eta_5, \ \eta_5 + \eta_6) &&= 1,\\
\varepsilon(\alpha_3^{\sigma\prime}, \ \alpha_3{}') &= \varepsilon(\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5, \ \eta_3 + \eta_5 + \eta_6) &&= -1,\\
\varepsilon(\alpha_4^{\sigma\prime}, \ \alpha_4{}') &= \varepsilon(\eta_1 + \eta_2 + \eta_3 + \eta_5, \ \eta_3 + \eta_4 + \eta_5 + \eta_6) &&= 1,
\end{aligned}$$

for the roots summing up to δ. For the roots summing up to γ_i we do not get varying values. Indeed, we compute

$$\begin{aligned}
\varepsilon({\alpha_1}',\ {\alpha_4^\sigma}') &= \varepsilon(\eta_6,\ \eta_1+\eta_2+\eta_3+\eta_5) &&= 1,\\
\varepsilon({\alpha_1}',\ {\alpha_3^\sigma}') &= \varepsilon(\eta_6,\ \eta_1+\eta_2+\eta_3+\eta_4+\eta_5) &&= 1,\\
\varepsilon({\alpha_1}',\ {\alpha_2^\sigma}') &= \varepsilon(\eta_6,\ \eta_1+\eta_2+2\eta_3+\eta_4+\eta_5) &&= 1,\\
\varepsilon({\alpha_2}',\ {\alpha_3}') &= \varepsilon(\eta_1+\eta_2,\ \eta_3+\eta_5+\eta_6) &&= 1,\\
\varepsilon({\alpha_2}',\ {\alpha_4}') &= \varepsilon(\eta_1+\eta_2,\ \eta_3+\eta_4+\eta_5+\eta_6) &&= 1,\\
\varepsilon({\alpha_3}',\ {\alpha_4}') &= \varepsilon(\eta_1+\eta_2+\eta_3,\ \eta_3+\eta_4+\eta_5+\eta_6) &&= 1.
\end{aligned}$$

This gives all signs of the above list of constants by Proposition 5.17. Lastly,

$$\begin{aligned}
&\operatorname{sgn}(\alpha_1) && &&= 1,\\
&\operatorname{sgn}(\alpha_2) = \operatorname{sgn}(\alpha_1)\tfrac{\varepsilon({\vartheta_2^\sigma}',\ {\alpha_1^\sigma}')}{\varepsilon({\vartheta_2}',\ {\alpha_1}')} &&= \tfrac{\varepsilon(-\eta_2,\ \eta_1+2\eta_2+2\eta_3+\eta_4+\eta_5)}{\varepsilon(\eta_2,\ \eta_1)} &&= 1,\\
&\operatorname{sgn}(\alpha_3) = \operatorname{sgn}(\alpha_2)\tfrac{\varepsilon({\vartheta_3^\sigma}',\ {\alpha_2^\sigma}')}{\varepsilon({\vartheta_3}',\ {\alpha_2}')} &&= \tfrac{\varepsilon(-\eta_3,\ \eta_1+\eta_2+2\eta_3+\eta_4+\eta_5)}{\varepsilon(\eta_3,\ \eta_1+\eta_2)} &&= -1,\\
&\operatorname{sgn}(\alpha_4) = \operatorname{sgn}(\alpha_3)\tfrac{\varepsilon({\vartheta_4^\sigma}',\ {\alpha_3^\sigma}')}{\varepsilon({\vartheta_4}',\ {\alpha_3}')} &&= -\tfrac{\varepsilon(-\eta_4,\ \eta_2+\eta_3+\eta_4+\eta_5+\eta_6)}{\varepsilon(\eta_4,\ \eta_1+\eta_2+\eta_3)} &&= 1
\end{aligned}$$

by Proposition 5.13. The basis $\mathcal{N}$ of the Iwasawa $\mathfrak{n}$-algebra of $\mathfrak{g}^0$ consists of the 15 elements

$$X_{\alpha_1},\ Y_{\alpha_1},\ X_{\alpha_2},\ Y_{\alpha_2},\ X_{\alpha_3},\ Y_{\alpha_3},\ X_{\alpha_4},\ Y_{\alpha_4},$$
$$X_{\gamma_1},\ Y_{\gamma_1},\ X_{\gamma_2},\ Y_{\gamma_2},\ X_{\gamma_3},\ Y_{\gamma_3},\ Z_\delta.$$

By Theorem 5.18 we have the following nonzero structure constants.

$$\begin{aligned}
&[X_{\alpha_1},Y_{\alpha_1}] = -Z_\delta, && [X_{\alpha_2},Y_{\alpha_2}] = Z_\delta, && [X_{\alpha_3},Y_{\alpha_3}] = Z_\delta, && [X_{\alpha_4},Y_{\alpha_4}] = Z_\delta,\\
&[X_{\alpha_1},X_{\alpha_2}] = 2X_{\gamma_3}, && [X_{\alpha_1},X_{\alpha_3}] = -2X_{\gamma_2}, && [X_{\alpha_1},X_{\alpha_4}] = 2X_{\gamma_1}, && [X_{\alpha_1},Y_{\alpha_2}] = -2Y_{\gamma_3},\\
&[X_{\alpha_1},Y_{\alpha_3}] = 2Y_{\gamma_2}, && [X_{\alpha_1},Y_{\alpha_4}] = -2Y_{\gamma_1}, && [Y_{\alpha_1},X_{\alpha_2}] = 2Y_{\gamma_3}, && [Y_{\alpha_1},X_{\alpha_3}] = -2Y_{\gamma_2},\\
&[Y_{\alpha_1},X_{\alpha_4}] = 2Y_{\gamma_1}, && [Y_{\alpha_1},Y_{\alpha_2}] = 2X_{\gamma_3}, && [Y_{\alpha_1},Y_{\alpha_3}] = -2X_{\gamma_2}, && [Y_{\alpha_1},Y_{\alpha_4}] = 2X_{\gamma_1},\\
&[X_{\alpha_2},X_{\alpha_3}] = 2X_{\gamma_1}, && [X_{\alpha_2},X_{\alpha_4}] = 2X_{\gamma_2}, && [X_{\alpha_2},Y_{\alpha_3}] = 2Y_{\gamma_1}, && [X_{\alpha_2},Y_{\alpha_4}] = 2Y_{\gamma_2},\\
&[Y_{\alpha_2},X_{\alpha_3}] = 2Y_{\gamma_1}, && [Y_{\alpha_2},X_{\alpha_4}] = 2Y_{\gamma_2}, && [Y_{\alpha_2},Y_{\alpha_3}] = -2X_{\gamma_1}, && [Y_{\alpha_2},Y_{\alpha_4}] = -2X_{\gamma_2},\\
&[X_{\alpha_3},X_{\alpha_4}] = 2X_{\gamma_3}, && [X_{\alpha_3},Y_{\alpha_4}] = 2Y_{\gamma_3}, && [Y_{\alpha_3},X_{\alpha_4}] = 2Y_{\gamma_3}, && [Y_{\alpha_3},Y_{\alpha_4}] = -2X_{\gamma_3}.
\end{aligned}$$

The Lie algebra $\mathfrak{n}$ is isomorphic to the H-type algebra $\mathfrak{n}_7^1$ corresponding to the Clifford module C_7^1. Alternatively, it is isomorphic to the Lie algebra of the octonionic Heisenberg group $\mathbb{O}H^{15}$ [91, Section 9.3, p. 33]. A basis of its seven-dimensional center is given by the set $\{X_{\gamma_1}, Y_{\gamma_1}, X_{\gamma_2}, Y_{\gamma_2}, X_{\gamma_3}, Y_{\gamma_3}, Z_\delta\}$. If we rescale $Z_\delta \in \mathcal{N}$ by $\frac{1}{4}$ and the other 14 elements by $\frac{1}{2}$, we obtain structure constants in the set $\{-1, 0, +1\}$.

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