MODULAR STRUCTURE OF CHIRAL FERMI FIELDS IN CONFORMAL QUANTUM FIELD THEORY

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Tag der mündlichen Prüfung:

If people do not believe that mathematics is simple
it is only because they do not realise how complicated life is.

— John von Neumann

ABSTRACT

The following thesis deals with the modular theory of Fermi fields in low dimensions; in particular, making use of the algebraic approach to quantum field theory, we have investigated the behaviour of two-dimensional theories which split into two separate copies of chiral fields, each one of them depending on one lightray variable at a time only.

The remarkable result we have found is the existence of a vacuum preserving isomorphism β connecting the vacuum states between the algebra of N Fermi fields localised in one single interval I and the algebra of one Fermi field localised in N disjoint intervals $E_N = I_1 \cup \ldots \cup I_N$. Since this map preserves the vacuum states, it therefore intertwines the respective modular groups; as a result, the modular automorphism flow for a Fermi field localised in several intervals turns out to mix the field among different points, with the mixing itself being described through suitable differential equations. Moreover, using the fact that Wick products are as well preserved, one can even embed via β the sub-theories of local observables, as currents and the stress-energy tensor. Consequently, since the isomorphism β is multi-local, a new class of multi-local gauge transformations and diffeomorphisms arise.

Interestingly enough, such characterisation of the modular group for multi-local algebras was already presented by [Casini and Huerta, 2009] using different techniques, and so far it is a special feature of free Fermi fields only (although outlooks of generality are fascinating to investigate).

The isomorphism that we have found is deeply related to the split property and the way fields transform under diffeomorphism covariance. In particular, it only differs from the action of diffeomorphisms by a gauge transformation, whose features we have characterised in the cases at hand, namely for the local algebras of Fermi fields, currents and stress-energy tensor.

ZUSAMMENFASSUNG

Die folgende Doktorarbeit befasst sich mit der Modulartheorie von Fermifeldern in niedrigen Dimensionen; insbesondere untersuchen wir das Verhalten der chiralen Felder, nachdem Felder in zwei Dimensionen in zwei ein-dimensionale Lichtstrahlkomponenten zerlegt worden sind. Wir wenden den algebraischen Zugang zur Quantenfeldtheorie an, in dem man sich mit lokalen Algebren befasst.

Wir finden einen Isomorphismus β zwischen der Algebra von N Fermifeldern, die in einem einzelnen Interval I lokalisiert sind, und der Algebra eines Fermifelds, das in mehreren verschieden Intervallen

$E_N = I_1 \cup \ldots \cup I_N$ lokalisiert ist, der den Grundzustand erhält. Daher verknüpft dieser die korrespondierenden Grundzustandmodulargruppen. Weil dieser Isomorphismus nicht-lokal ist, ergibt sich eine Mischung für die Modulargruppe der Multi-Interval-Algebra, die das Feld in verschiedenen Punkten in den unterschiedlichen Intervallen mischt.

Diese Characterisierung der Modulargruppen für die Multi-Interval-Algebra ist nur für freie chirale Fermifelder bekannt. Da dieser Isomorphismus auch Wick Produkte erhält, können auch lokale Observablen, wie die Ströme und der Energie-Impuls-Tensor, damit eingebettet werden. Wegen dieses Merkmals kann man multi-lokale Eichsymmetrien und Diffeomorphismen generieren, deren Verhalten wir auch untersucht haben.

Der Isomorphismus, den wir gefunden haben, setzt sich interessanterweise zusammen aus dem Split-Isomorphismus einer geeigneten Wirkung der Diffeomorphismen und einer Eichtransformation. Das gleiche Verhalten kann man auch auf die Untertheorien der Strömen und des Energie-Impuls-Tensors einschränken, was wir uns im letzten Kapitel angesehen haben.

CONTENTS

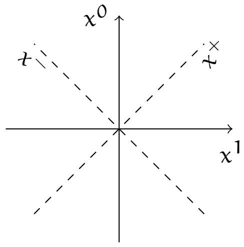
INTRODUCTION

The main theoretical ingredient of algebraic quantum field theory is the concept of field, which is supposed to implement the principle of locality. Observables, identified with the quantities that can be experimentally measured in a laboratory, must satisfy Einstein causality and additional physical requirements that are seen to be realised in nature. Fields therefore appear as the building blocks in order to construct such observables and, though they may themselves be observables, they need not to. The idea lying at the basis of quantum field theory is the assignment of fields to each space-time region, where events are supposed to take place. This reflects into the assignment of a net of algebras onto the Minkowski space; physical measurements correspond, roughly speaking, to states on the algebras and all the most important physical quantities experimentalists are interested in can usually be traced back to the evaluations of scalar product or particular combinations thereof, as for example correlations functions and scattering amplitudes: notable in this sense is the Lehmann-Symanzik-Zimmermann formula reducing scattering amplitudes to time-ordered correlations functions and their poles.

The algebraic approach to quantum field theory deals with the mathematical properties of all these ingredients from the point of view of operator algebras. A marvelous walkthrough these aspects is provided by [Haag, 1992] and [Roberts, 2004] who give complete explanations of why this is a necessary issue. The developement of such a formalism is the key tool to the understanding of quantum field theory itself and encodes almost all the features that we find as realised in nature. Many results have been achieved thanks to the possibility to handle these mathematical tools, especially after very important insights by Takesaki and Tomita, [Borchers, 1999], [Takesaki, 1970], [Takesaki, 2002], who reduced the origin of space-time symmetries to abstract properties of von Neumann algebras, opening a brand new research field consequently.

A very important role in physics is played by systems which exhibit special symmetries, because this characteristic helps a lot to reduce their complexity. In particular we have been concerned with models being symmetric under conformal transformations, that is the set of transformations preserving the angles in the Minkowski space-time. In low dimensions, namely two, this symmetry happens to reduce to very strict requirements with a well-known mathematical structure described by the Virasoro algebra. Investigation of the properties of such algebras leads to amazing results and progresses in the area. The Virasoro generators are moreover the modes of the stress-energy tensor, which generates space-time diffeomorphisms of the theory. As a consequence, a two-dimensional conformal field theory is basically a quantum field theory endowed with a stress-energy tensor whose generators must satisfy specific algebraic properties and commutation relations. Also, the theory contains a special class of fields, the "primary fields", whose transformations properties are very much related to how these fields commute with the stress-energy tensor itself.

Interestingly enough, conformal symmetries can be found very often in actual physical systems. Most of the times this goes along with scaling invariance and, although the two properties do not coincide, they are nevertheless very often interchanged. Models with no proper scale dimensions, as for example massless models, are usually conformally invariant and form the prototypes we can look at, not to mention the huge amount of results, models and features carried by string theory, which is the straightforward application of conformal field theory. However, within the already mentioned two-dimensional models, a special class is given by the so called chiral theories, a group of models where the fields only depend on the "light-cone" variables $x^{\pm} \coloneqq x^0 \pm x^1$. Those theories decouples into two copies of singular theories, either of them being concerned with the one variable x^+ or x^-, respectively. This means that the whole business reduces to a one-dimensional theory, and the original model can be reconstructed eventually taking the tensor product of the two one-dimensional copies. The term chiral becomes then synonym of one-dimensional world living on a light-ray:



Each real line supports both the time-like property (positivity of the energy) and the space-like commutativity (causality). Moreover the real line can be taken onto the unit circle (minus a point) via the Cayley transformations and thus we shall basically be concerned with fields living on a circle, where the conformal transformations acquire the form of general diffeomorphisms.

Going back to the mathematical questions, we have already stated that a revolutionising result was found by Tomita and Takesaki and undergoes the name of modular theory. Starting with a von Neumann algebra and a cyclic and separating state one can automatically construct an inner group of automorphisms σ_t whose explicit form depends on the algebra itself and on the state provided. In some special case, where the algebras are generated by local fields localised in particular space-time regions, this group of automorphisms happens to coincide with some symmetry group occurring in physics (Lorentz boosts, dilations). This result opens a brand new horizon of questions, because it seems that the space-time symmetries lie behind the physical content, back in the algebraic properties of the quantities at hand. It is tempting to generalise such results and further investigate them. The main content of this thesis is exactly modular theory for Fermi fields in one dimension: in particular, we have been looking at fields localised in disjoint intervals, trying to derive and explain the features of their modular theory. It turns out that whenever we choose the fields to be localised in many disjoint intervals, the action of the modular group introduces a mixing among those different intervals on top of a geometric action moving the points, ([Casini and Huerta, 2009], [Longo, Martinetti, and Rehren, 2009]). This result can be traced back to the existence of a vacuum preserving isomorphism moving the fermions all around the circle [Rehren and Tedesco, 2013]. We have widely exploited this feature considering different representations of the algebras and different situations at hand, varying the geometric positions of the intervals and comparing the new results to previous statements. Besides modular theory itself, this work gave us a deeper understanding of how Fermi fields behave on the circle.

Also, since products of Fermi fields generate observables as currents and the stress-energy tensor, these subtheories can be embedded via the mentioned isomorphism and new characteristics emerge. Currents generate gauge transformations which are therefore delocalised all around the circle, as well as new multi-local diffeomorphisms given by the embedded stress-energy tensor. As a consequence, all the standard constructions we have for fermions and related models can be rephrased in terms of this new aspect, giving rise to a new class of perspectives.

As for the organisation of the material, this thesis is divided into different parts. In the beginning we provide the standard description of the mathematical framework lying behind algebraic quantum field theory, following the lines of [Haag, 1992]. We introduce the technical aspects of von Neumann algebras and the world of conformal field theory in the field theoretical setting.

We then move to the analysis of the modular theory for fermions localised in different intervals, showing the new aspects together with new insights on the standard constructions. We ought to mention that part of the ideas were triggered by the original work of Casini and Huerta, [Casini and Huerta, 2009], where the authors calculated the modular group for fermions in disjoint intervals using methods coming from density matrices and statistical mechanics. We took their starting point to rephrase everything in the language of algebraic quantum field theory and operator algebras. Other ideas came from different works on boson-fermion correspondences, [Anguelova, 2011] as well as others, and we tried to contribute attacking the problems from the angle of local quantum physics.

A third part describes the class of models which can be obtained out of Fermi fields, mainly concerning currents and their features, in the light of the new background provided. The multi-local features restrict to these subalgebras with the help of suitable gauge transformations, which can be related to the diffeomorphisms covariance in a limpid way.

Part I

PRELIMINARIES

> The German term “Nahwirkungsprinzip” is more impressive than the somewhat colourless word “locality.”
>
> R. Haag, *Local quantum physics*.

1 INTRODUCTION TO QUANTUM FIELD THEORY

Contents

A rigorous inspection of the behaviour of quantum field theory showed some common general features which were seen to be always realised, no matter the physical system at hand. Such features have been thus taken as defining properties (axioms) of the quantum field theory itself and the study of their mathematical properties leads to the characterisation of algebraic quantum field theory. We shall introduce such postulates following the example given by the standard textbook in this area, namely [Haag, 1992].

The main objects any physical theory deals with, no matter whether classical or quantised, are fields $x \mapsto \phi(x)$ (whose mathematical properties have to fulfill the requirements of the model at hand). Their role is to implement the principle of locality; *observables* are the quantities that can be directly reproduced in a laboratory and they can in general be read off and reconstructed once the field content is assigned. Fields themselves may also be observables, though they need not to.

1.1 GENERAL POSTULATES: WIGHTMAN AXIOMS

A. *Fields*: Fields are operator valued distributions on Minkowski space. This means that the linear assignment $f \mapsto \phi(f)$ gives back an (usually unbounded) operator on some Hilbert space $\mathcal{H}$ with dense domain $\mathfrak{D}(\phi(f)) \subseteq \mathcal{H}$. The assignment has to be thought as a smearing

$$\phi(f) = \int_{\mathcal{M}} d^4x\, \phi(x) f(x)$$

with f belonging to some suitable functional space $\mathcal{F}$. The further assumption $\phi(f)\mathfrak{D} \subset \mathfrak{D}$ ensures that we may operate arbitrarily many times with fields upon vectors $\in \mathfrak{D}$.

B. *Poincaré group and transformation properties*: The Hilbert space $\mathcal{H}$ carries a unitary representation $U(g)$ of the covering of the Poincaré group $\overline{\mathfrak{P}}$. The spectrum of the energy-momentum operators P^μ is contained in the forward light cone and this ensures consistency with special relativity, $p^2 := m^2 \geqslant 0, p^0 \geqslant 0$. Moreover, let $\overline{\mathfrak{L}} \subset \overline{\mathfrak{P}} = \mathbb{R}^4 \rtimes \overline{\mathfrak{L}}$ be the Lorentz subgroup of the

Poincaré group and let $U(\Lambda, a)$ be a representation of $\overline{\mathfrak{P}}$ with $\Lambda \in \overline{\mathfrak{L}}$, $a \in \mathbb{R}^4$. Fields transform under $\overline{\mathfrak{P}}$ as

$$U(\Lambda, a)\,(\phi(x))\,U^*(\Lambda, a) = S(\Lambda^{-1})\,\phi(\Lambda x + a), \quad S(\Lambda^{-1}) \in \overline{\mathfrak{L}}.$$

In a nutshell the choice of $S(\Lambda)$ characterises the "spin" of the field.

C. *Hermiticity*: Given a field $\phi(f)$, the theory contains also the hermitian conjugate field $\phi(f)^*$ defined so that

$$(\Phi, \phi(f)\Psi) = (\phi^\dagger(\bar{f})\Phi, \Psi).$$

Fields may be self-adjoint, $\phi(x) = \phi^\dagger(x)$ and thus $(\Phi, \phi(f)\Psi) = (\phi(\bar{f})\Phi, \Psi)$ given Φ, Ψ.

D. *Locality*: If the supports of the test functions f and g are space-like to each other, then fields must satisfy either of the following commutation relations

$$[\phi(f), \phi(g)]\Psi = 0 \quad \text{or} \quad \{\phi(f), \phi(g)\}\Psi = 0, \quad \Psi \in \mathfrak{D}.$$

Fields of the former type are called "bosonic", whereas fields of the latter type are called "fermionic". Due to Einstein causality observables must commute at space-like distances, therefore fermionic fields by themselves cannot be observables, whilst bosonic fields may.

E. *Vacuum state and completeness*: There exists a unique state $\Omega \in \mathcal{H}$ invariant under $U(g)$, $g \in \overline{\mathfrak{P}}$. Such a state is referred to as the "vacuum state". Also, by acting upon the vacuum with an arbitrary polynomial in the fields $\phi(f)$ one can approximate any operator acting on $\mathcal{H}$.

It turns out that these properties are easily realised by free fields satisfying linear equations, while constructions in terms of interacting fields are very difficult to achieve.

Definition: Let Ω be the vacuum vector. The vacuum expectation values

$$w^{(n)}(x_1, \ldots, x_n) \coloneqq (\Omega, \phi(x_1) \ldots \phi(x_n)\Omega)$$

are called (Wightman) n-points correlation functions, though they are, more precisely, tempered distributions on $\mathbb{R}^{4n}$.

A fundamental result in this respect is the "reconstruction theorem" [Haag, 1992], namely, under some suitable assumptions that we do not discuss in here, the whole fields content can be derived out of the knowledge of all correlation functions.

1.2 FERMI FIELDS VERSUS BOSE FIELDS

As previously stated, fields appearing in nature must satisfy particular restrictions on the way they commute between each other, this being express by either commutation or anti-commutation relations. Fields of the former kind are referred to as "Bose fields" whereas fields of the latter kind are usually referred to as "Fermi fields". In particular, those fields that belong to integer "spin representations" of the Lorentz group (in the sense of $S(\Lambda)$, as we have seen before) are Bose fields, while those ones that belong to half-odd integer representations are Fermi fields. Such particular feature characterises the spin-statistic theorem ([Haag, 1992]). As a first remark notice that Bose fields might in principle be already observables, because they automatically fulfill Einstein causality; on the other hand Fermi fields do not, and observables must be constructed as particular combinations of them (currents and stress-energy tensor, as we will show later on). However, we shall show the explicit construction of operator algebras based on the above commutation relations in the very special case when the space-time is one-dimensional, where this has to be understood as previously mentioned, namely as decomposition in terms of light-ray variables.

Let us construct fermionic fields first. Take $\mathcal{H}$ as any Hilbert space of functions with an involution $\Gamma \mid (\Gamma f)(x) = \overline{f(x)}$. Through the following linear assignment $f \mapsto \psi(f)$, which can be thought as an integral smearing, we can construct the set

$$\mathrm{CAR}(\mathcal{H},\Gamma) \coloneqq \overline{\{\psi(f) \mid f \in \mathcal{H},\ (\Gamma f)(x) = \overline{f(x)}\}}_{\|(\,\cdot\,)\|}.$$

The norm of an operator in such a set is uniquely fixed by the relation $\psi(f)^* = \psi(\Gamma f)$ and by the anti-commutation relations

$$\{\psi(f)^*, \psi(g)\} = (\Gamma f, g)_{\mathcal{H}}\,\mathbf{1}; \quad f, g \in \mathcal{H} \tag{1.2.1}$$

According to the choice of the Hilbert space one can realise either real fields or complex fields. The standard choice is to take $\mathcal{H} = L^2(\mathbb{R}, dx)$ to have real fields and two such copies $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) = L^2(\mathbb{R}) \otimes \mathbb{C}$ to have complex fields. The norm is then seen to satisfy the inequality $\|\psi(f)\| \leqslant \|f\|_{\mathcal{H}}$ and therefore the operators are bounded by the norm of the functions in $\mathcal{H}$. By choosing a projection $P \mid \Gamma P \Gamma = \mathbf{1} - P$ one can decompose fields into creation and annihilation modes $\psi(f) = \psi\left(Pf + \Gamma P \Gamma f\right) = \psi(Pf) + \psi(\Gamma P \Gamma f) = \psi^+(f) + \psi^-(f)$ and also define two-point function as

$$\omega_P\left(\psi(f)\psi(g)\right) \coloneqq (\Gamma f, Pg)_{\mathcal{H}}.$$

Positivity is ensured by positivity in the Hilbert space and higher order correlation functions can be defined [Boeckenhauer, 1996] as

$$\omega_P\left(\psi(f_1)\ldots\psi(f_{2n})\right) \coloneqq$$
$$(-1)^{1/2\,n(n-1)} \sum_{\sigma \in P_n} \operatorname{sign}\sigma \prod_{j=1}^{n} \omega_P\left(\psi(f_{\sigma(j)})\psi(f_{\sigma(n+j)})\right) \tag{1.2.2}$$

with all the odd correlation functions vanishing. Such a state is usually called quasi-free. The corresponding irreducible GNS representation gives the state in terms of scalar product as expressed in the previous paragraph.

Example (Real Fermi field)*:* The real Fermi field on the real line can be decomposed into Fourier modes as

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dk\, a(k)\, e^{-ikx}$$

with the reality condition $a^*(k) = a(-k)$ and anti-commutation relations $\{a(k), a(k')^*\} = \delta(k-k')$. At the level of distributions, commutation relations for the fields themselves are

$$\{\psi(x), \psi(y)\} = \delta(x-y), \quad x, y \in \mathbb{R}.$$

Taking into account that $a(k)$ annihilates the vacuum for each k, the one point function is easily seen to vanish, $\omega_0(\psi(x)) = 0$, whereas the vacuum two-point function is

$$\omega_0(\psi(x)\psi(y)) = \int_{\mathbb{R}} dk\, e^{-ikx} \int_{\mathbb{R}} dk'\, e^{-ik'y}\, \omega_0(a(k)a(k'))$$

which becomes, after using the anti-commutation relations for the Fourier modes

$$\omega_0(\psi(x)\psi(y)) = \int_0^\infty dk\ e^{-ik(x-y)} = \lim_{\varepsilon \to 0^+} \int_0^\infty dk\ e^{-ik(x-y)-k\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{-i}{x-y-i\varepsilon} \tag{1.2.3}$$

and we shall encounter this formula many times later on (e.g 4.1). The projection defining the two-point function is the projection onto the positive modes, $P = \chi(]0,\infty[)$ such that

$$(Pf)(x) = \int_0^\infty dk\, \tilde{f}(k)\ e^{-ikx}\,.$$

The construction of Bose field works similarly, with the exception that commutation relations pose some obstructions for the norm of the operators to be bounded. However one starts from the assignment $f \mapsto \phi(f)$ and defines the Weyl operators as the exponential $W(f) = e^{i\phi(f)}$. Commutation relations are then implemented by means of a skew-symmetric two form $\sigma\colon (f,g) \mapsto \sigma(f,g)$ as

$$W(f)W(g) = e^{i/2\, \sigma(f,g)}\, W(f+g).$$

The set of all $W(f)$ is a *-algebra and imposing the condition $\|W(f)\| = 1$ ensures that it has a unique C^* norm. The set

$$\overline{\{W(f) \mid f \in \mathcal{H}\}}_{\|(\ .\ )\|} =: \mathrm{CCR}(\mathcal{H}, \sigma)$$

is then turned into a C^*-algebra. Notice in turn that unitarity and the Weyl commutation relations imply $W(0) = 1$ and $W(f)^* = W(-f)$. Along the same lines as before, representations may emerge assigning the state $\omega(W(f)) = e^{-1/2\, \|f\|^2}$.

2 CONFORMAL FIELD THEORY

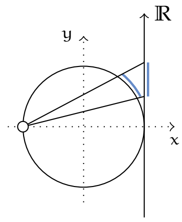
Contents

Conformal field theories may in general be regarded as quantum field theories whose symmetry group is the conformal group, namely the group of angles preserving transformations of the space-time (see definition below in 2.1). Motivations to investigate such a mathematical structure lie in many different models appearing in nature: physical realisations can be found, for example, in the free Maxwell theory, the massless Dirac field in 4-dim, not to mention the whole construction of string theory and all the related areas, as well as applied models in material sciences and engineering. A very interesting class of models, and in particular the actual models we shall be looking at, occurs in two-dimensional theories which are chirally invariant: in this case the observables depend on the so-called "light-cone" variables $x_\pm := x^0 \pm x^1$ only as

$$\phi(x^0, x^1) = \phi^+(x^+) \otimes \mathbf{1}^- \pm \mathbf{1}^+ \otimes \phi^-(x^-)$$

and the set of observables $\mathcal{A}(O) = \mathcal{A}(I) \otimes \mathcal{A}(J) \subset \mathcal{B}(O)$ can be decomposed into their respective chiral parts, $\mathcal{A}(I)$ and $\mathcal{A}(J)$, with O given by $O = I \times J$. Two independent one-dimensional copies living on the light rays $x_\pm \in \mathbb{R}$ are therefore obtained and the entire theory can be reconstructed by taking back the tensor product. The real line where each of the variables $x_\pm$ lives can be compactified on the circle S^1 via the Cayley transform

$$C\colon \mathbb{R} \to S^1 \setminus \{-1\}$$
$$x \mapsto z = \frac{1 + ix}{1 - ix}$$



after identifying $C^{-1}(S^1) \equiv \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. This allows us to look at theories defined on the circle, from which two-dimensional chirally invariant field theories can be reconstructed by following back the above procedure. So to speak, in case of chiral conformal field theories, the space-time is (two copies of) the unit circle whose open intervals form the set of space-time regions under investigation. In particular, we shall see that the mutual position of such intervals will carry causality and all the rest of properties that physics requires to be fulfilled through mathematical axioms.

The conformal group of the two-dimensional theory can be decomposed as $\mathrm{Conf}_2 = \mathrm{Conf}_1 \times \mathrm{Conf}_1$, where $\mathrm{Conf}_1 = \mathrm{Diff}(S^1)$ is identified with the group of the orientation preserving diffeomorphisms on the compactified real line (see below).

Example (Massless Dirac field in two dimensions)*:* The massless Dirac equation in two dimensions reads

$$i\partial\!\!\!/ \Psi(x^0, x^1) = 0$$

which can be turned into

$$(\partial_0 + \partial_1 \gamma^5)\Psi = 0$$

where $\gamma^5 = \gamma^0\gamma^1$. By using the chiral projection $P_\pm = 1/2(\mathbf{1} \pm \gamma^5)$ the Dirac spinor decouples into $\Psi = P_+\Psi_+ + P_-\Psi_-$, with γ^+, γ^- eigenvectors of γ^5 with eigenvalues ± 1. Introducing the light cone coordinates $x_\pm = x^0 \pm x^1$ leads to

$$\partial_\pm \Psi_\pm(x_+, x_-) = 0,$$

thus $\Psi_\pm \equiv \Psi_\pm(x_\mp)$, only depending on one variable at a time. Therefore the argument introduced above directly applies.

2.1 CONFORMAL TRANSFORMATIONS

Conformal transformations are maps $f\colon \mathcal{M}^d \to \mathcal{M}^d$ preserving angles in the d-dimensional Minkowski space-time $\mathcal{M}^d$: this means that the only possible way the metric may transform is up to a scaling (positive) factor $g'_{\mu\nu}(x') = e^{\omega(x)} g_{\mu\nu}(x)$. Working out the definition we are led to the following set of transformations ([Evans and Kawahigashi, 1998]):

Table 1: Conformal transformations

Generator	transformation	
P_μ	translations	$x'^\mu = x^\mu + a^\mu$
$M_{\mu\nu}$	Lorentz	$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \Lambda \in SO(p,q)$
D	dilations	$x'^\mu = \lambda x^\mu, \quad \lambda \in \mathbb{R}$
K_μ	special conformal	$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}$.

The first two classes generate the Poincaré group $SO(p,q) \ltimes \mathbb{R}^d$ and together with the dilations they generate the Weyl group. In the case $d \neq 2$ the whole conformal group is $(d+1)(d+2)/2$-dimensional. The generators obey the following commutation relations, which in turn define the conformal algebra [di Francesco, Mathieu, and Sénéchal, 1997]

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i\,(g_{\mu\nu}D - M_{\mu\nu}) \\
[K_\rho, M_{\mu\nu}] &= i\,(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) \\
[P_\rho, M_{\mu\nu}] &= i\,(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\,(g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}).
\end{aligned} \tag{2.1.1}$$

For our purposes we restrict to the two-dimensional case, where, interestingly enough, the conditions for a map to be conformal reduce to the Cauchy-Riemann equations. In terms of complex variables they are holomorphic and anti-holomorphic maps $z \mapsto f(z)$, $\bar{z} \mapsto \bar{f}(\bar{z})$ such that $\partial_{\bar{z}} f(z) = \partial_z \bar{f}(\bar{z}) = 0$. When the complex variable corresponds to the Cayley transform of a lightray coordinate $x_\pm$, namely on a compactified Minkowski space-time $\mathcal{M}^2 = S^1 \times S^1$, then the conformal group is identified with $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$, two commuting copies of diffeomorphisms of the circle which we are going to look once at a time.

Example: Here we show some examples of what simple conformal transformations on the plane look like:

$f(z) = z^3$ $\quad$ $f(z) = 1/z^2$

$f(z) = 1/z$ $\quad$ $f(z) = \sqrt{z}$

2.2 THE VIRASORO ALGEBRA

Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving diffeomorphisms on S^1, which in turn coincides with the conformal transformations leaving the circle invariant. Its Lie algebra corresponds to the algebra of smooth vector fields on the circle whose complexification gives rise to the Witt algebra with basis elements $l_n := -z^{n+1}\frac{d}{dz}$, such that

$$[l_n, l_m] = (n-m)l_{m+n}.$$

Since we are looking for projective unitary representations of positive energy we shall be concerned with its unique non-trivial central extension (see [Evans and Kawahigashi, 1998]), the Virasoro algebra, given in terms of generators L_n

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}m(m^2-1)\,\delta_{n+m}\mathbf{1}. \tag{2.2.1}$$

L_0 is referred to as the conformal Hamiltonian and we are interested in irreducible unitary representations π of the above algebra with positive energy, namely the spectrum of L_0 is required to be positive. Those representations have been fully classified (see [Friedan, Qiu, and Shenker, 1984a]) and are given in terms of pairs (c, h) where c is the central term appearing in (2.2.1) and h is the lowest weight

$$\pi_{(c,h)}(L_0)\,|h\rangle = h\,|h\rangle \qquad \pi_{(c,h)}(L_m)\,|h\rangle = 0,\ m > 0.$$

Positivity of the energy implies $h \geqslant 0$ and from unitarity it follows $L_n{}^* = L_{-n}$. These conditions give restrictions on the possible admissible pairs (c, h) and we have that ([Friedan, Qiu, and Shenker, 1984b]) either $c \geqslant 1$ and $h \geqslant 0$ or

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, \ldots$$

and

$$h = h_{p,q}(c) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$$

with $p = 1, 2, \ldots, m-1$ and $q = 1, \ldots, p$. Once the lowest weight $|h\rangle$ is given the whole representation space (Verma module $V(h, c)$) can be obtained as a span of

$$|v\rangle = L_{-n_1}\ldots L_{-n_m}\,|h\rangle\,, \qquad n_1 \geqslant \ldots \geqslant n_m > 0.$$

The set of vectors obtained with fixed m forms a subspace $\mathcal{H}^m$ of energy $h + (n_1 + \ldots + n_m)$. The Hilbert space is then obtained as completion of the quotient of $\oplus_{m=0}^{\infty}\mathcal{H}^m$ with respect to the null vectors [Evans and Kawahigashi, 1998].

2.2.1 *The Möbius group*

Let us now look at the action of $SL(2, \mathbb{R})$ on the compactified real line $\overline{\mathbb{R}} = C^{-1}(S^1)$ by

$$x \mapsto gx = \frac{ax+b}{cx+d} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det g = 1.$$

$SL(2,\mathbb{R})$ does not act faithfully on $\overline{\mathbb{R}}$ whereas so does its quotient with respect to the kernel $PSL(2,\mathbb{R}) := SL(2,\mathbb{R}) / \{\pm\mathbf{1}\}$. We call $PSL(2,\mathbb{R})$ the Möbius group and we see it can be identified, after Cayley transform, with $PSU(1,1)$ acting on the circle as

$$z \mapsto \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}} \qquad C(g) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \det C(g) = 1.$$

Notable one-parameters subgroups are given by rotations, translations and dilations, whose action

$$\begin{aligned} R(\theta)z &= e^{i\theta} z, & z &\in S^1 \\ \delta_s x &= e^s x, & x &\in \mathbb{R} \\ \tau_t x &= x + t, & x &\in \mathbb{R} \end{aligned}$$

is displayed as matrices in $PSL(2,\mathbb{R})$ as

$$R(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \quad \delta_s = \begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{-\frac{s}{2}} \end{pmatrix}, \quad \tau_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

A convenient basis for the Lie algebra complexification can be given in terms of elements $\{L_0, L_{\pm 1}\}$ (see [Longo, 2008]) satisfying

$$[L_1, L_{-1}] = -2L_0 \qquad [L_0, L_1] = -L_1 \qquad [L_0, L_{-1}] = L_{-1}$$

namely they generate the closed subalgebra of (2.2.1) with $m, n = 0, \pm 1$. The generators of translations P, rotations K and dilations D can be obtained from

$$\begin{aligned} L_0 &= \frac{1}{2}(P+K) \\ L_{\pm 1} &= \frac{1}{2}(P-K) \pm iD. \end{aligned}$$

We assume that there exist a unique vector $|0\rangle$ such that $L_0 |0\rangle = L_{\pm 1} |0\rangle = 0$ and ergo $U(g)|0\rangle = |0\rangle, \forall g \in PSL(2,\mathbb{R})$. We call such a vector the vacuum state and refer to this feature saying that the Möbius group is the only subgroup of the conformal transformations on the circle preserving the vacuum state. Of course this straightforwardly emerges also by looking at the explicit realisation of l_n as $-z^{n+1}\frac{d}{dz}$.

The Virasoro algebra generated by the L_n contains many copies of the Lie algebra of the cover of the Möbius group. In particular one may define for each $n > 0$

$$L^{(\pm n)} = \frac{1}{n} L_{\pm n}, \quad L^{(0)} = \frac{1}{n} L_0 + \frac{c}{24}\frac{n^2-1}{n}$$

with commutation relations

$$[L^{(n)}, L^{(-n)}] = 2L^{(0)}, \quad [L^{(\pm n)}, L^{(0)}] = \pm L^{(\pm n)}.$$

The subgroup generated by this sub-Lie algebra is isomorphic ([Longo and Xu, 2004]) to the n^{th} covering of the Möbius group $PSU(1,1)^{(n)}$ acting on $z \in S^1$ as

$$g^{(n)}(z) := \sqrt[n]{\frac{\alpha z^n + \beta}{\bar{\beta} z^n + \bar{\alpha}}}.$$

Equivalently, this group can be defined as the set of all elements $g \in \mathrm{Diff}(S^1)$ for which there exists a Möbius transformation ϕ such that $g(z)^n = \phi(z^n)$. Clearly, this is nothing but the definition we just gave above.

As a remark, we shall very often use in the following the concept on n-dilations in the context of modular theory, where such transformations will be exactly defined as

$$\delta_t^{(n)}(z) = \sqrt[n]{\delta_t(z^n)}$$

and thus they appear as standard dilations in $PSU(1,1)^{(n)}$. Here δ_t are the single-interval dilations defined as the subgroup of the Möbius group preserving the intervals, having the boundaries as fixed points (for the precise definition see 4.4).

2.3 THE QUARKS CONSTRUCTION

Let us assume the theory contains many complex fields

$$\psi^i(z) = \sum_s \psi^i_s \, z^{-s-1/2}$$

satisfying fermionic anti-commutation relations, with (s,r) running either in $\mathbb{Z}+1/2$ (vacuum representation) or in $\mathbb{Z}$ (Ramond representation). Define now the a^{th} current as ("quark construction" [Evans and Kawahigashi, 1998]):

$$J^a(z) := \frac{1}{2}\sum_{i,j} :\psi^{*i}\, \tau^a_{ij} \psi^j :(z) \qquad (2.3.1)$$

where $\tau^a \in \mathfrak{g} \subset \mathfrak{u}(n)$ is a basis of some matrix Lie algebra. The normal ordering :AB: between two operators is defined by subtraction of the vacuum expectation value $:AB: := AB - \omega_0(AB)\mathbf{1}$. This is a standard definition for observables in the field theoretical setting in order to avoid divergences that might otherwise occur when calculating scattering amplitudes and correlation functions. As a straightforward consequence of such definition the vacuum expectation value of any normal ordered product vanishes as it is

$$\omega_0(:AB:) = \omega_0(AB - \omega_0(AB)\mathbf{1}) = \omega_0(AB) - \omega_0(AB) = 0.$$

By using the fermionic anticommutation relations one finds that, expanding in Fourier modes on the circle, $z \in S^1$,

$$J^a(z) = \sum_{n\in\mathbb{Z}} j^a_n \, z^{-n-1}, \quad J^a(x) = -\frac{dz}{dx} J^a(z(x))$$

and thereby

$$[j^a_n, j^b_m] = f^{ab}{}_c\, j^c_{n+m} + n\,\delta_{n+m,0}\,\kappa^{ab}\,k \tag{2.3.2}$$

Here $f^{ab}{}_c$ are the structure constants and k is a positive integer, called the "level", that depends on the Lie algebra $\mathfrak{g}$ and its matrix representation in $\mathfrak{u}(n)$ chosen for the construction. It characterises the model. Furthermore κ^{ab} is the Killing form of $\mathfrak{g}$. The latter is the trace of the adjoint action in the Lie algebra $\mathrm{Ad}\colon \mathfrak{g} \to GL(\mathfrak{g}), x \mapsto [x, (\,\cdot\,)]$

$$\kappa^{ab} = \mathrm{tr}(\mathrm{Ad}\,X^a \circ \mathrm{Ad}\,X^b)$$

(see [Rehren, 2013], [Fuchs, 1992]). Equation (2.3.2) defines the non-abelian current algebra for $\mathfrak{g} \subset \mathfrak{u}(n)$ at level k.

In the abelian case the commutation relations for the current look $2\pi i\,[j(x), j(y)] = \delta'(x-y)$; the central operator

$$Q = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, j(x) \tag{2.3.3}$$

is referred to as the "charge". In terms of Fourier modes $j(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1}$ the charge Q emerges as the mode j_0.

The two-point vacuum correlation function for the current can be easily calculated in terms of the fermionic one by implementing the quark construction. In fact we have

$$\omega_0(j(x)j(y)) = \omega_0(:\psi^*\psi:(x)\,:\psi^*\psi:(y)).$$

Standard tools in quantum field theories allow to work out product of normally ordered operators and we remand the reader to any textbook for explicit proofs. In particular these are given in terms of pairing between operators at different points and in the case at hand the only contractions that matter are

$$\begin{aligned}
\omega_0(j(x)j(y)) &= \omega_0(:\psi^*\psi:(x)\,:\psi^*\psi:(y)) \\
&= \omega_0(\psi(x)^*\psi(x)\,\psi(y)^*\psi(y)) + \omega_0(\psi(x)^*\psi(x)\,\psi(y)^*\psi(y)) \\
&= \omega_0(\psi(x)^*\psi(y))\,\omega_0(\psi(x)\psi(y)^*) + 0 \\
&= \omega_0(\psi(x)^*\psi(y))^2
\end{aligned}$$

hence once the fermionic two-point function is given, its square determines $\omega_0(j(x)j(y))$. However, the current algebra possesses a continuum of representations given by the charged states $\omega_q = \omega_0 \circ \rho_q, q \in \mathbb{R}$, where ρ_q are automorphisms acting on the currents as $\rho_q(j(x)) = j(x) + 2q/(1+x^2)$. The one and two-point functions are given by

$$\begin{aligned}
\omega_q(j(x)) &= \frac{2q}{1+x^2} \\
\omega_q(j(x)j(y)) &= \frac{4q^2}{(1+x^2)(1+y^2)} + \frac{-1}{(x-y)^2}
\end{aligned}$$

which read, in terms of the z variable on the circle

$$\omega_q(j(z)) = \frac{q}{z}$$
$$\omega_q(j(z)j(w)) = \frac{q^2}{wz} + \frac{1}{(w-z)^2}$$

Provided the currents, one can construct the "Sugawara" stress-energy tensor as

$$T_S(z) := \xi\, \kappa_{ab} :J^a J^b:(z) \tag{2.3.4}$$

with ξ being a normalisation constant. The Fourier expansion on the circle reads, in terms of modes,

$$T_S(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}$$

and thereby the below commutations relations follow

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} m(m^2-1)\,\delta_{n+m}\mathbf{1}$$
$$i\,[T(x), T(y)] = -\,(T(x)+T(y))\,\delta'(x-y) + \frac{c}{24}\delta'''(x-y)\mathbf{1} \tag{2.3.5}$$

where the central charge c can be expressed as ([di Francesco et al., 1997], [Fuchs, 1992])

$$c = \frac{k}{k+g}\dim\mathfrak{g} \tag{2.3.6}$$

where g is a group factor determined by group theory (dual Coxeter number). We have purposely introduced again the notation L_n as in (2.2.1) for the Fourier modes of the stress-energy tensor to explicitely remark that its modes exactly satisfy the commutation relations defining the Virasoro algebra (2.2.1). If the theory admits unitary implementations for $z \mapsto g(z)$ then we can write

$$\alpha_g(\phi(z)) = \phi'(g(z)) = U(g)\phi(z)U(g)^*; \quad U(g) = e^{iT(f)}$$

$g(z)$ being $g(z) = \exp(f)(z)$. The zero mode L_0 is the conformal Hamiltonian and it generates the time evolution automorphism of the current algebra according to

$$\alpha_t(a) = e^{itL_0}\, a\, e^{-itL_0}\,.$$

The Fermi fields possess by themselves their own full stress-energy tensor given by

$$T_F(z) = \frac{1}{2}\sum_i^N :\psi^{*i}(z)\,\partial_z\psi^i(z): + \frac{\varepsilon}{16}\frac{N}{z^2} \tag{2.3.7}$$

with $\varepsilon = 0, 1$ for vacuum representation and Ramond representation, respectively. Again, this stress-energy satisfies commutation relations of the type (2.3.5). Nevertheless, in general T_F differs from T_S; the difference can be computed (see 5.4) as a new stress-energy tensor given by $T_F = T_S + T_{coset}$ with central charge given by the difference of the two initial central charges: $c_{coset} = c_F - c_S$. The class of models where the difference $T_F - T_S$ happens to be zero are referred to as conformal embeddings.

2.4 PRIMARY FIELDS

In the field theoretical setting each vector $|v\rangle \in V(c,h)$ of finite energy can be thought as $|v\rangle = \lim_{z\to 0} \phi(z)\,|0\rangle$, $|0\rangle$ being a conformally invariant *vacuum state* (state-field correspondence). By the spectrum condition (positivity of L_0), vector-valued distribution on S^1 as $\phi(z)\,|0\rangle$ can be analytically continued to functions in the interior of the circle, so that the limit is well defined.

Definition: Fields corresponding to lowest weight vectors $|h\rangle$ are said to be primary and of scaling dimension h.

By exploiting the properties of the operators L_n one finds, for primary fields, the following commutation relations

$$[L_n, \phi(z)] = h(n+1)z^n\phi(z) + z^{n+1}\partial_z\phi(z) \tag{2.4.1}$$

which can be exponentiated to

$$\phi(z) = \left(\frac{dg(z)}{dz}\right)^h \phi'(g(z)) \tag{2.4.2}$$

$z \mapsto g(z)$ being any general diffeomorphism of the circle. In particular the behaviour of conformally invariant fields under infinitesimal conformal transformations acquires the forms

$$\begin{aligned} i[P,\phi(x)] &= \partial_x\phi(x) \\ i[D,\phi(x)] &= (x\partial_x + h)\phi(x) \\ i[K,\phi(x)] &= (x^2\partial_x + 2hx)\phi(x) \end{aligned}$$

which can be derived from (2.4.1) in case $n = 0, \pm 1$.

Quite often the theory may also contain further fields, which do not transform as above because they are obtained out of non-lowest weight vectors. Such fields, called *secondary* or *descendant*, have additional contributions in the transformation laws due to further contributions in the commutation relations (2.4.1).

Example: The stress-energy tensor defined in (2.3.4) transforms as

$$T(z) = \left(\frac{dg(z)}{dz}\right)^2 T'(g(z)) + \frac{c}{12}s(g(z),z)$$

where

$$s(g(z),z) = \frac{d^3g}{dz^3}\Big/\frac{dg}{dz} - \frac{3}{2}\left(\frac{d^2g}{dz^2}\Big/\frac{dg}{dz}\right)^2$$

is the Schwarzian derivative. The additional term cancels out if $g \in$ Möb (T is *quasi-primary*).

Example: Fermi fields $\psi(z)$ and currents $J(z)$ are primary fields of dimensions $1/2$ and 1, respectively. They therefore transform as

$$\psi(z) = \sqrt{\frac{dg(z)}{dz}}\,\psi'(g(z)) \quad \text{and} \quad J(z) = \frac{dg(z)}{dz}\,J'(g(z)).$$

2.5 CONFORMAL NETS

The section at hand deals with some definitions about conformal nets and their representations in the algebraic setting. For this purpose let $\mathcal{I}$ be the set of non-empty, non-dense open intervals on the circle S^1.

Definition: A *conformal net* on S^1 is an assignment of von Neumann algebras $I \in \mathcal{I} \to \mathcal{A}(I) \subset \mathcal{B}(\mathcal{H})$ such that the following properties hold ([Carpi, 2004]):

(i) *Isotony*:
$$\mathcal{A}(I_1) \subset \mathcal{A}(I_1) \qquad \text{if } I_1 \subset I_2.$$

(ii) *Locality*:
$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \qquad \text{if } I_1 \cap I_2 = \emptyset.$$

(iii) *Möbius covariance*, namely a strongly continuos unitary representation $U(g) \in \mathcal{H}$ of the Möbius group exists such that
$$U(g)\mathcal{A}(I)\,U(g)^* = \mathcal{A}(g\,I) \qquad g \in \text{Möb}.$$

(iv) *Positivity of the energy*: spect $(U(L_0)) \geqslant 0$, L_0 being the generator of the one-parameter subgroup of rotations $R(\theta)z = e^{i\theta} z$.

(v) *Existence and uniqueness of the vacuum*:
$$\exists!\ \Omega \in \mathcal{H} \mid \text{Ker}\,(U(L_n)) = \mathbb{C}\Omega.$$

Also, Ω is assumed to be cyclic, i. e. $a\Omega$ is dense in $\mathcal{H}$, and separating, i. e. $a_1\Omega = a_2\Omega \Rightarrow a_1 = a_2$, for the whole algebra $\mathcal{A}\left(S^1\right) = \bigvee_{I\in\mathcal{I}}\mathcal{A}(I)$.

From the above properties further consequences can be proven, as well as:

(vi) *Factoriality*: The algebras $\mathcal{A}(I)$ are type III_1 factors.

(vii) *Reeh-Schlieder property*:
$$\Omega \text{ is cyclic and separating for } \mathcal{A}(I),\ \forall I \in \mathcal{I}.$$

(viii) *Irreducibility*: The von Neumann algebra generated by all the intervals exhausts all $\mathcal{B}(\mathcal{H})$, i. e.
$$\bigvee_{I\in\mathcal{I}}\mathcal{A}(I) = \mathcal{B}(\mathcal{H})$$

(ix) *Haag duality*:
$$\mathcal{A}(I)' = \mathcal{A}\left(I'\right)\ \forall I \in \mathcal{I}.$$

(x) *Bisognano-Wichmann property*: from Modular Theory it follows that the modular operator associated to the pair $(\mathcal{A}(I), \Omega)$ is
$$\Delta^{it}_{(I,\Omega)} = U\left(\Lambda_I^{-2\pi t}\right)$$

where Λ_I is the one parameter subgroup of Möb preserving the interval I (corresponding to the dilations if $C(I) = \mathbb{R}_+$).

Along the same lines a conformal net is said to be diffeomorphisms covariant if it admits a strongly continuos projective unitary representation V of $\mathrm{Diff}(S^1)$ such that

$$V(h)\mathcal{A}(I)\,V(h)^* = \mathcal{A}(hI) \qquad h \in \mathrm{Diff}(S^1).$$

Definition (Strong additivity)*:* The net is said to be *strongly additive* if, for every pair of intervals I_1, I_2 obtained by removing a single point from I, i. e. $I = I_1 \cup I_2 \cup \{P\}$ we have

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)\,.$$

Definition (Split property)*:* The net $I \to \mathcal{A}(I)$ is said to be *split* if, for any two intervals I, J with disjoint closure, a von Neumann algebras isomorphism

$$\chi\colon \mathcal{A}(I) \vee \mathcal{A}(J) \to \mathcal{A}(I) \otimes \mathcal{A}(J)$$

exists such that $\chi(xy) = x \otimes y$, $x \in \mathcal{A}(I)$, $y \in \mathcal{A}(J)$. Whenever one of the two intervals is contained (along with its closure) into the other, say, $\bar{I} \subset J$, this is equivalent ([Longo, 2008]) to the existence of an intermediate type I factor $\mathcal{M}$, $\mathcal{A}(I) \subset \mathcal{M} \subset \mathcal{A}(J)$. It is essential that the two interval neither touch nor overlap.

The split map is given in terms of a canonical unitary between the representing Hilbert spaces $V\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that

$$V\left(\mathcal{A}(I) \vee \mathcal{A}(J)\right) V^* = \mathcal{A}(I) \otimes \mathcal{A}(J)$$

As a consequence for any given normal states φ_i on $\mathcal{A}(I_i)$ there exists a normal state φ on the total algebra $\vee_I \mathcal{A}(I)$ such that

$$\varphi(a_1 a_2) = \varphi_1(a_1) \cdot \varphi_2(a_2).$$

In the language of field theories this property is never fulfilled by the vacuum state, because splitting the correlation functions into products would eliminate all correlations between fields in different points, $\omega_0(a(x)b(y)) \neq \omega_0(a(x)) \cdot \omega_0(b(y))$. This means that the state given by $\omega_0 \circ V^*$ onto $\mathcal{A}(I) \otimes \mathcal{A}(J)$ is an "excited state".

A complete and full characterisation of the split property can be found in the literature and we refer the reader to the references. In particular it can be shown ([Longo, 2008] or [D'Antoni, Longo, and Radulescu, 2001]) that if the conformal Hamiltonian L_0 satisfies the trace-class condition, namely

$$\mathrm{tr}(e^{-\beta L_0}) < \infty \quad \forall \beta > 0$$

then the conformal net is split.

A *representation* of a conformal net is a family $\{\pi_I\}$ where π_I is a representation of $\mathcal{A}(I)$ on some Hilbert space $\mathcal{H}_{\pi_I}$ such that

$$\pi_J|_{\mathcal{A}(I)} = \pi_I, \quad I \subset J.$$

A unitary equivalence class $[\pi]$ of representations on a separable Hilbert space is called a *sector*. Since the von Neumann algebras are by definition subsets of $\mathcal{B}(\mathcal{H})$ they are already realised on some Hilbert space: we refer to their defining representation as to the *vacuum sector* of the theory. Furthermore a representation is said to be Möbius (diffeomorphisms) covariant ([Carpi, 2004]) if there is a strongly continuous unitary representation U_π of the Möbius (diffeomorphisms) group such that

$$U_\pi(g)\pi_I\left(\mathcal{A}\left(I\right)\right)U_\pi(g)^* = \pi_{g(I)}\left(\mathcal{A}\left(g(I)\right)\right)$$

namely

$$\operatorname{Ad} U_\pi(g) \circ \pi_I = \pi_{g(I)} \circ \operatorname{Ad} U(g(I)).$$

3 BASICS OF VON NEUMANN ALGEBRAS

Contents

The following sections provide an elementary introduction to the basic ingredients we shall be dealing with, namely operators on Hilbert spaces and von Neumann algebras ([Jones, 2009]). This is because the main features of a physical theory are encoded into the fields content, which in turn happen to emerge as operator valued distributions assigned to each point of the space-time, $x \mapsto \phi(x)$ ([Haag, 1992]). For this reason a systematic analysis of their mathematical properties is needed, and tools ought to be developed in order to better understand their algebraic underlying structure.

3.1 BASIC DEFINITIONS AND OPERATOR TOPOLOGIES

Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{D}(A) \subset \mathcal{H}$. An operator on $\mathcal{H}$ whose domain is $\mathfrak{D}(A)$ is a linear map $A\colon \mathfrak{D}(A) \to \mathcal{H}$.

Definition (Operator norm)*:* Let $x \in \mathfrak{D}(A) \mid x \neq 0$ and $A\colon x \mapsto A(x) \in \mathcal{H}$. The operator norm of A is defined as

$$\|A\| = \sup_{x \in \mathfrak{D}(A) \neq 0} \frac{\|Ax\|}{\|x\|}. \qquad (3.1.1)$$

If $\|A\| < \infty$ then the operator A is said to be bounded.

Property: An operator A is bounded if and only if it is continuous.

Property: A bounded (and therefore continuous) operator A defined on a dense subset $\mathfrak{D}(A) \subset \mathcal{H}$ can be uniquely extended to the whole $\mathcal{H}$ by continuity.

As a remark we notice that, by continuity, the convergence of the sequence $x_n \to x$ in $\mathfrak{D}(A)$ implies the convergence of the sequence $Ax_n \to Ax$.

Definition (Closed operator)*:* Let $x_n \in \mathfrak{D}(A)$ such that $x_n \to x$. Let us also assume that $Ax_n \to y$. The operator A is called *closed* if the previous assumptions imply $x \in \mathfrak{D}(A)$ and $y = Ax$. Equivalently, an operator is closed if its graph is closed in the direct sum $\mathcal{H} \oplus \mathcal{H}$.

Take now an operator A, not necessarily closed, and assume that the closure of its graph in $\mathcal{H} \oplus \mathcal{H}$ happens to be the graph of some operator $\overline{A}$, i.e. if $\overline{G(A)} = G(\overline{A})$. Then A is said to be "closable" and $\overline{A}$ its closure.

Definition (Adjoint operator)*:* Let $F_x \colon y \in \mathfrak{D}(A) \to (x, Ay) \in \mathbb{C}$ for any operator A. The set of all points $\{ x \in \mathcal{H} \mid F_x \text{ is continuous} \}$ is defined as $\mathfrak{D}(A^*)$. On this domain, by means of Riesz representation theorem, $\exists!\, z \in \mathcal{H} \mid F_x(y) = (z, y)$. The operator A^* adjoint of A is defined as $A^* x = z$ on $\mathfrak{D}(A^*)$.

Given any two operators A and B we say that $A \subset B$ if $\mathfrak{D}(A) \subset \mathfrak{D}(B)$ and $Ax = Bx$ on their common domain, i. e. $x \in \mathfrak{D}(A)$. A densely defined operator is called *symmetric* if $A \subset A^*$ and *self-adjoint* if $A = A^*$.

Henceforth let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded operators on a Hilbert space $\mathcal{H}$, whose domains then coincide with the whole space, that is $\mathfrak{D} = \mathcal{H}$. We assign the following topologies on $\mathcal{B}(\mathcal{H})$ ([Jones, 2009]):

Definition (Topologies on $\mathcal{B}(\mathcal{H})$)*:* Let T_n be a sequence of operators and T a "limit point" in $\mathcal{B}(\mathcal{H})$:

(i) *Norm topology*: $T_n \to T$ in norm if $\|T_n - T\| \to 0$ in the norm topology defined above in (3.1.1).

(ii) *Strong topology*: $T_n \to T$ strongly if $\forall\, x \in \mathcal{H}$ then $\|T_n x - Tx\| \to 0$ in the vector norm of $\mathcal{H}$.

(iii) *Weak topology*: $T_n \to T$ weakly if $\forall\, x, y \in \mathcal{H}$ then $(T_n x, y) \to (Tx, y)$ as complex functionals, i.e. $F_{x,y}(T_n) \to F_{x,y}(T)$.

It is easy to verify that a natural order among these topologies exists, namely

$$\text{norm topology} \;\triangleright\; \text{strong topology} \;\triangleright\; \text{weak topology}$$

meaning that if a sequence of operators T_n converges to T in a topology on the left then it converges to T in a topology on the right. Stronger topologies have more open sets than weaker ones, and therefore if a set is closed in a weak topology it is also closed in all the stronger ones.

Definition (von Neumann algebra)*:* A von Neumann algebra is a subset $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ which is closed under the weak operator topology and contains the identity. Its commutant $\mathcal{M}'$ is defined as the set $\mathcal{M}' := \{ m' \in \mathcal{B}(\mathcal{H}) \mid [m', m] = 0,\ m \in \mathcal{M} \}$ (similarly for $(\mathcal{M}')' = \mathcal{M}''$ and so forth).

Property (von Neumann bicommutant theorem)*:* Let $\mathcal{M} = \mathcal{M}^*$ be a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. The following assumptions are equivalent:

(i) $\mathcal{M} = \mathcal{M}''$.

(ii) $\mathcal{M}$ is weakly closed.

(iii) $\mathcal{M}$ is strongly closed.

3.2 CLASSIFICATION OF FACTORS

Definition (Factor)*:* The centre of an algebra is the set of all elements within that algebra which commute with all the rest, that is $Z(\mathcal{M}) = \mathcal{M}' \cap \mathcal{M}$. A von Neumann algebra $\mathcal{M}$ whose centre is trivial is called a factor, i.e. $Z(\mathcal{M}) = \mathbb{C}\mathbf{1}$.

Definition (Projections)*:* $p \in \mathcal{B}(\mathcal{H})$ is called a *projection* if and only if $p^2 = p = p^*$. Likewise $v \in \mathcal{B}(\mathcal{H})$ is a *partial isometry* if $v^*v = p$ is a projection. Given two projections p and q, we say they are equivalent ($p \approx q$) if there is a partial isometry v such that $vv^* = p$ and $v^*v = q$.

Given two projections p, q we say that $p \leqslant q$ if and only if their ranges are $p\mathcal{H} \subseteq q\mathcal{H}$. In addition, a projection p is said to be minimal if, $\forall\, q \leqslant p$, either $q = 0$ or $q = p$. Consider now any $q \neq p, q < p$; if there is a partial isometry $v \in \mathcal{M}$ such that $vv^* = p$ and $v^*v = q$ then the projection p is said to be *infinite* (otherwise p is called *finite*). In a nutshell, then, a finite projection has no equivalent subprojections, whereas infinite projections do. Consequently a von Neumann algebra is called infinite if its identity is infinite, otherwise it is finite.

Definition (Murray-von Neumann classification of factors)*:* Projections allow us to classify factors according to the following:

(i) A factor $\mathcal{M}$ with a minimal projection is called a Type I factor.

(ii) A factor $\mathcal{M}$ with no minimal projections but non-zero finite projections is called a Type II factor.

(iii) An infinite factor $\mathcal{M}$ admitting a non-zero linear functional (trace) $\mathrm{tr}\colon \mathcal{M} \to \mathbb{C}$ such that

  a) $\mathrm{tr}(xy) = \mathrm{tr}(yx) \quad x, y \in \mathcal{M}$,

  b) $\mathrm{tr}(x^*x) \geqslant 0$,

  c) tr is ultraweakly continuous,

  is called a Type II_1 factor. The trace is said to be normalised if $\mathrm{tr}(\mathbf{1}) = 1$.

(iv) A factor of the form $\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$, with $\mathcal{M}$ Type II_1 and $\dim \mathcal{H} = \infty$ is called a Type II_∞ factor.

Table 2: Type of factors

Type	"working" definition
I	with minimal projection; also $\exists \mathcal{H} \mid \mathcal{M} \cong \mathcal{B}(\mathcal{H})$.
II	no minimal projection but non-zero finite projections.
II_1	infinite factor with no minimal projections and a trace $\mathrm{tr}\colon \mathcal{M} \to \mathbb{C}$.
II_∞	$\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$, $\mathcal{M}$ Type II_1 and $\dim \mathcal{H} = \infty$.
III	the rest, i.e. no minimal projections, no non-zero finite projections.

(v) A factor $\mathcal{M}$ with no minimal projections, no non-zero finite projections is called a Type III factor.

3.3 INTRODUCTION TO MODULAR THEORY

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$ a cyclic and separating vector for $\mathcal{M}$. The anti-linear operator

$$S_0\colon a\Omega \mapsto a^*\Omega, \quad a \in \mathcal{M} \tag{3.3.1}$$

is closable ([Bratteli and Robinson, 1979a]) and in general unbounded. However, let $S = \overline{S_0}$ be its closure and $S = J\Delta^{1/2}$ the respective polar decomposition. We call Δ the modular operator and J the modular conjugation associated with the pair $(\mathcal{M}, \Omega)$. Via functional calculus the strongly continuous unitary group

$$\Delta^{it} = e^{it\ln\Delta}, \quad t \in \mathbb{R}$$

may be defined and its adjoint action

$$\sigma^t_{(\mathcal{M},\Omega)}(m) := \Delta^{it}\, m\, \Delta^{-it}, \quad m \in \mathcal{M},\ t \in \mathbb{R} \tag{3.3.2}$$

induces a one-parameter automorphisms group of $\mathcal{M}$ called the *modular automorphisms group*. A fundamental result in this respect is the

Theorem 3.3.1 (Tomita-Takesaki ([Bratteli and Robinson, 1979a]))*:* Under the previous assumptions the following statements hold: the operator $J = J^*$ is anti-unitary and

$$J\mathcal{M}J = \mathcal{M}' \tag{3.3.3}$$

$$\sigma^t_{(\mathcal{M},\Omega)}(\mathcal{M}) = \mathcal{M}, \quad \forall\, t \in \mathbb{R}. \tag{3.3.4}$$

The algebra is sent into its commutant by the adjoint action of the modular conjugation J and the modular group acting of $\mathcal{M}$ exhausts all the algebra itself. Since all these quantities explicitly depend on the pair $(\mathcal{M}, \Omega)$ we have different realisations of the modular automorphism group according to this choice. The characterisation of its shape, according to the choices of $(\mathcal{M}, \Omega)$ in some particular cases, is the main topic of the work at hand.

The trivial case, i.e. when the algebra is commutative, is very easy to handle; take $a, b \in \mathcal{M}$ and look at $(S\, a\Omega, S\, b\Omega)$, with S defined as (3.3.1):

$$\begin{aligned}(S\, a\Omega, S\, b\Omega) &= (a^*\Omega, b^*\Omega)\\ &= (\Omega, ab^*\Omega)\end{aligned}$$

by commutativity it follows then

$$\begin{aligned}&= (\Omega, b^* a\Omega) = (b\Omega, a\Omega)\\ &= \overline{(a\Omega, b\Omega)}\end{aligned}$$

therefore S is antiunitary and hence $\Delta = |S| = \mathbb{1}$. This implies that the action of the modular group is trivial on each element of the algebra.

3.3.1 *Kubo-Martin-Schwinger (KMS) condition*

Let $\mathcal{M}$ be a von Neumann algebra and φ a faithful normal state on $\mathcal{M}$. Let furthermore σ_t be a weakly continuous one-parameter group of automorphisms of $\mathcal{M}$. Fixed $a, b \in \mathcal{M}$ consider $F_{a,b}(t) := \varphi(a\, \sigma_t(b))$ as a function in the variable $t \in \mathbb{R}$. The state φ is said to satisfy the Kubo-Martin-Schwinger (KMS) condition at inverse temperature $T = \beta^{-1}$, $0 < \beta < \infty$ if

(i) $F_{a,b}(t)$ can be analytically continued in the strip $0 < \Im\mathfrak{m}(t) < \beta$. It is continuous at the boundaries $\Im\mathfrak{m}(t) = 0, \beta$ and

(ii) $F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$.

Notice that if the algebra were commutative then every state would satisfy the KMS condition at $\beta = 0$, by commutativity. If this only happens as a special feature of the state at hand then φ is said to be *tracial*, otherwise $\beta \neq 0$ measures the deviation of φ from being a trace. Next note that a state is KMS with respect to σ_t at $T^{-1} = \beta \neq 0$ if and only if it is KMS with respect to $\sigma_{-\beta t}$ at $T = -1$; therefore by rescaling the group parameter one can always refer to state of temperature -1. Albeit we shall not discuss this issue any further, KMS states characterise equilibrium states in quantum thermodynamics where the one-parameter group σ_t plays the role of a given time evolution; however, to whom it may concern, a full characterisation and a systematic study of KMS states is provided in [Bratteli and Robinson, 1979b].

The remarkable connection with modular theory is that a normal state happens to be a KMS state with respect to its own modular group ([Haag, 1992]); the converse is also true, namely the modular group is the only one-parameter group of inner automorphisms satisfying the KMS condition on the state where it comes from. This feature will be fully used in the following to characterise and investigate properties of the modular group related to different states and algebras.

3.3.2 *Bisognano-Wichmann property*

As a matter of example let us consider a very important result obtained by Bisognano and Wichmann ([Bisognano and Wichmann, 1975]), and so far the only (up to geometric transformations) explicit characterisation of modular operators for space-time regions.

Let $\mathcal{W}_{R,L} := \{x \in \mathcal{M}^4 \mid x^1 \gtrless |x^0|\}$ be the right (left) wedge region as a subset of the space-time $\mathcal{M}^4$. There is exaclty a one-parameter subgroup of the Lorentz group preserving the wedge, namely mapping the wedge into itself:

$$\Lambda(t) = \begin{pmatrix} \cosh(t) & -\sinh(t) & 0 & 0 \\ -\sinh(t) & \cosh(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.5}$$

Let the local von Neumann algebra $\mathcal{A}(\mathcal{W})$ be generated by Wightmann fields $W(f)$ and let us choose as a cyclic and separating vector the vacuum Ω as Gelfand-Naimark-Segal (GNS) of $\omega_0(W(f)) = e^{-1/2\|f\|^2}$. The modular group associated to the pair $(\mathcal{A}(\mathcal{W}), \Omega)$ coincides with the unitary representation of the subgroup preserving the wedge and the modular conjugation acts as reflection through the edge of the wedge, changing sign to both x^0 and x^1

$$\Delta^{it} = U(\Lambda_{\mathcal{W}}(-2\pi t)) \qquad J = U(r_{\mathcal{W}}). \tag{3.3.6}$$

However, since the vacuum is invariant under Poincaré transformations g, this result can be generalised to any region of the form $\mathcal{W}' = g\mathcal{W}$. Setting ([Longo, 2008])

$$\Lambda_{\mathcal{W}'}(t) = g\,\Lambda_{\mathcal{W}}\,g^{-1} \qquad r_{\mathcal{W}'} = g\,r_{\mathcal{W}}\,g^{-1}$$

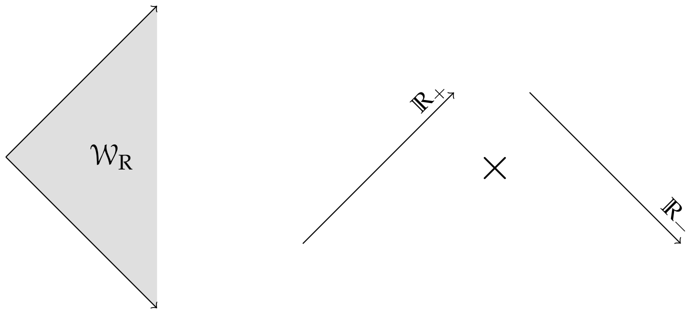
we obtain ([Guido, 2011] and [Guido, Longo, and Wiesbrock, 1998])

$$\Delta^{it} = U(\Lambda_{\mathcal{W}'}(-2\pi t)) \qquad J = U(r_{\mathcal{W}'}).$$

It can be shown (again [Guido, 2011]) that the Bisognano-Wichmann property also holds for conformally covariant theories that split into tensor products of two nets on a line, as described in 2.5. In fact, in two dimensions a wedge reduces to the cartesian product of two half-lines $\mathbb{R}_+ \times \mathbb{R}_-$. In the particular case where $I = \mathbb{R}^+$ the modular group explicitly acts as dilations with a scaling factor of $e^{-2\pi t}$

$$\sigma_t(\,\cdot\,) = \delta_{\lambda_t}(\,\cdot\,), \qquad \lambda_t = e^{-2\pi t}$$

The two-dimensional modular flow is then $\delta_{\lambda_t} \otimes \delta_{\lambda_t^{-1}}$. Intervals can be obtained as conformal transformations of the real line and hence in this case the modular group, corresponding to the subgroup of the Lorentz transformations preserving the wedge, is replaced by the subgroup of the Möbius transformations fixing each interval, either on the circle picture S^1 or on the real line.



As already mentioned above, the modular group σ^t_Ω uniquely satisfies the KMS condition on Ω and hence it characterises thermal equilibrium states for an observer whose dynamics is given by σ^t_Ω. According to this interpretation, the Bisognano-Wichmann property for wedge regions states that the vacuum state is a thermal equilibrium state with temperature $T = -1$ for an observer accelerated with Lorentz boosts. This is the case for an observer moving around the event horizon of a black hole and the example provides an explanation of the Unruh effect, by which the vacuum state behaves like a thermal states for observers moving in a gravitational field ([Connes and Rovelli, 2008; Martinetti and Rovelli, 2003]). In fact, the trajectory of an observer moving in such an event horizon (corresponding in turn to a wedge region) with constant acceleration a is given by the orbits of the Lorentz boosts of the form (3.3.5) with the proper "physical" time τ being t/a. Therefore the trajectory in the wedge can be parametrised as

$$x^\mu(\tau) = 1/a\,(\sinh(a\tau), \cosh(a\tau), 0, 0)$$

and the evolution at later times is $x(\tau_0 + \tau) = \Lambda(a\tau)x(\tau_0)$. On the other hand, as we have seen in equation (3.3.6), the modular group with respect to the vacuum state act as a Lorentz boost of parameter $-2\pi s$ and satisfies the KMS property. Therefore, by uniqueness, the relation between the physical time τ and the modular parameter s has to be $-2\pi s = a\tau$, in order to give back the Unruh inverse temperature $\beta = -d\tau/ds = 2\pi/a$.

3.3.3 *Reconstruction of the translations*

As just stated, the modular group associated with the pair $(\mathcal{A}(\mathcal{W}), \Omega)$, $\mathcal{W}$ being a wedge region, acts like the associated group of Lorentz boosts and this preserves the wedge itself. Remarkable results by Borchers [Borchers, 1992] and Wiesbrock [Wiesbrock, 1993a,b, 1992] showed that it is possible, under suitable conditions, to recover the translations group out of modular data.

Definition: Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. Let furthermore $U(a) := e^{iaH}$, $a \in \mathbb{R}$, be a continuous one-parameter group of unitaries with positive generator H leaving Ω invariant, i. e. $U(a)\Omega = \Omega$. The triple $(\mathcal{M}, H \geqslant 0, \Omega)$ is called a "Borchers triple". Also, the triple is said to satisfy the *half-sided translations* condition if $\mathrm{Ad}\,U(a)\mathcal{M} \subset \mathcal{M}$, $a > 0$.

Theorem 3.3.2 ([Borchers, 1992])*:* Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω and denote as Δ, J the modular data of the pair $(\mathcal{M}, \Omega)$. Then, given a half-sided translated Borchers triple as above, the following holds

$$\Delta^{it} U(a) \Delta^{-it} = U(e^{-2\pi t} a) \tag{3.3.7}$$

$$J U(a) J = U(-a), \tag{3.3.8}$$

namely $U(a)$ is seen to satisfy translations-dilations commutation relations with Δ^{it}.

A stronger result provided by Wiesbrock holds true: given two algebras in suitable position one can automatically recover the unitaries $U(a)$ out of their modular operators only; the construction is showed below.

Definition: Let $\mathcal{N} \subset \mathcal{M}$ be two von Neumann algebras with common cyclic and separating vector Ω and denote with $\Delta_{\mathcal{M}}, \Delta_{\mathcal{N}}$ the respective modular operators. We call the inclusion $\mathcal{N} \subset \mathcal{M}$ half-sided modular ([Wiesbrock, 1993a]) if $\Delta_{\mathcal{M}}^{-it} \mathcal{N} \Delta_{\mathcal{M}}^{it} \subset \mathcal{N}$ for $t \geqslant 0$.

Theorem 3.3.3 ([Wiesbrock, 1993a])*:* Under the previous assumptions of half-sided modular inclusion $\mathcal{N} \subset \mathcal{M}$ let $H := 1/2\pi \, (\ln \Delta_{\mathcal{N}} - \ln \Delta_{\mathcal{M}})$; the triple $(\mathcal{M}, H \geqslant 0, \Omega)$ is a half-sided translated Borchers triple fulfilling theorem 3.3.2.

This result suggests that the information about the translations is contained into the mutual positions of the two algebras and their common cyclic and separating vector. As an important application of such result we mention that wedge regions and their translated indeed satisfy the half-side modular inclusions and therefore the above results directly apply. Representations of the Lorentz boosts emerge as modular operator $\Delta_{\mathcal{M}}^{it}$ (as a consequence of the Bisognano-Wichmann property) and translations may be recovered by means of the Wiesbrock procedure (in two dimensions these exhaust the whole Poincaré group).

Part II

MULTI-GEOMETRIC MODULAR ACTION IN QUANTUM FIELD THEORY

In AQFT it is a long outstanding question, what physical meaning the Tomita-Takesaki modular objects have.

H.-W. Wiesbrock, *Commun. Math. Phys.* **157**, *83 (1993)*.

4

MULTI-GEOMETRIC MODULAR THEORY

Contents

4.1 REPRESENTATIONS OF FERMI FIELDS ON THE CIRCLE

We are now interested in the positive energy representation of real Fermi fields in one dimension, namely fields localised on the real line (or similarly on the circle via Cayley transform) such that $\psi(x)^* = \psi(x)$ which satisfy anti-commutation relations as distribution in the form

$$\{\psi(x), \psi(y)\} = \delta(x-y), \quad x, y \in \mathbb{R}.$$

In terms of the compact picture the equation takes the form

$$\{\psi(z), \psi(w)\} = 2\pi i z\, \delta(z-w) \quad z, w \in S^1$$

and consequently fields are meant to be smeared with functions $f \in L^2(S^1)$. In one dimension, as easily seen, locality is ensured by disjointness. We recall that by means of the anti-commutators (1.2.1) Fermi fields are bounded operators because $\psi(f)\psi(f)^* + \psi(f)^*\psi(f) = \|f\|^2_{\mathcal{H}} \cdot \mathbf{1}$.

By using the GNS construction, representations can arise after choosing appropriate states on the algebra of fields. In particular, we shall look at quasi-free states, namely states whose high order correlations functions can be calculated by using Wick theorem ([Bratteli and Robinson, 1979a] and (1.2.2)) as combinations of two-point functions. Therefore the only ingredient we need is the assignment of $\varphi(\psi(x)\psi(y))$.

The vacuum representation of real Fermi fields in one dimension emerges out of the vacuum two-point function that we already calculated in the example (1.2.3)

$$\omega_0(\psi(x)\psi(y)) = \lim_{\varepsilon\searrow 0} \frac{-i}{x-y-i\varepsilon}.$$

This gives back, via GNS construction, the vacuum representation $\pi_0(\psi(x))$. Using the Cayley transform and the standard transformation laws for fields

$$x \mapsto z = \frac{1+ix}{1-ix}, \quad \psi(z) = \sqrt{-i\,\frac{dz}{dx}}\,\psi(x) = \frac{1-ix}{\sqrt{2}}\,\psi(x)$$

we obtain the periodic representation (Neveu-Schwarz) of fields on the circle given in terms of Fourier modes as

$$\pi_0\left(\psi(z)\right) = \sum_{r\in\mathbb{Z}+1/2} \psi_r\, z^{-r-1/2}$$

with two-point function $\omega_0(\psi(z)\psi(w)) = \lim_{\lambda\nearrow 1} \frac{1}{z-\lambda w}$.

Taking two copies of real Fermi field we obtain a representation for the complex Fermi field $\phi(x) = (\psi_1(x) + i\psi_2(x))/\sqrt{2}$ with anti-commutation relations given by

$$\{\phi(x), \phi(y)^*\} = \{\phi(x)^*, \phi(y)\} = \delta(x-y)$$

and vacuum two-point function

$$\omega_0(\phi(x)\phi(y)^*) = \omega_0(\phi(x)^*\phi(y)) = \omega_0(\psi(x)\psi(y)).$$

On the cirle instead the adjoint relation reads $\phi(z)^* = z\,\phi^\dagger(z)$ and the two-point function becomes again $\omega_0(\phi(z)\phi(w)^*) = \omega_0(\phi(z)^*\phi(w)) = \omega_0(\psi(z)\psi(w))$.

Notice that the vacuum state is invariant under the action of Möbius transformations of the form described in 2.2.1, $\omega_0 \circ \alpha_g = \omega_0$, where $g \in PSL(2,\mathbb{R}) = SL(2,\mathbb{R})\,/\,\{\pm\mathbf{1}\}$ and α_g is its implementation on the algebras as

$$\alpha_g(\psi(x)) = \sqrt{\frac{dg}{dx}}\,\psi(g(x)).$$

Consequently, the vacuum representation is covariant.

Another positive energy representation can be constructed out of the Ramond two-point function

$$\omega_R(\psi(x)\psi(y)) = \lim_{\varepsilon\to 0} \frac{1+xy}{\sqrt{1+x^2}\sqrt{1+y^2}} \cdot \frac{-i}{x-y-i\varepsilon} \tag{4.1.1}$$

whose expression in the compact picture is

$$\omega_R(\psi(z)\psi(w)) = \lim_{\lambda\nearrow 1} \frac{z+w}{2\sqrt{zw}} \cdot \frac{1}{z-\lambda w}\,.$$

This gives rise to the Ramond representation in terms of Fourier modes as

$$\pi_R(\psi(z)) = \sum_{n\in\mathbb{Z}} \psi_n\, z^{-n-1/2}$$

which extends anti-periodically on the circle. This reflects the fact that only local observables, such as currents as bilinear forms in the fields, need to be well defined after rotations of 2π on the circle, but Fermi fields themselves need not to.

4.2 OPERATOR PRODUCT EXPANSIONS

Let us consider the positive energy representations for Fermi fields on the circle as described in the previous paragraph

$$\pi_0\left(\psi(z)\right) = \sum_{r\in\mathbb{Z}+1/2} \psi_r\, z^{-r-1/2}$$

$$\pi_R(\psi(z)) = \sum_{n\in\mathbb{Z}} \psi_n\, z^{-n-1/2}$$

the former being periodic after rotations of $2\pi i$, the latter being anti-periodic

$$\pi_0\left(\psi(e^{2\pi i}\, z)\right) = \pi_0\left(\psi(z)\right)$$
$$\pi_R(\psi(e^{2\pi i}\, z)) = -\pi_R(\psi(z)).$$

Either of these boundary conditions ensure the correct commutation relations between currents once one performs the quarks construction. So to speak, only local observables, as the currents are, must be well defined, but Fermi fields themselves are allowed to carry an additional minus sign without affecting the algebraic relations. For the sake of notations we may write either representations as [Fuchs, 1992]

$$\psi(z) = \sum_s \psi_s\, z^{-s-1/2}$$

and intend $s \in \mathbb{Z}+1/2$ and $s \in \mathbb{Z}$ for the vacuum and Ramond representation, respectively. Anti-commutation relations between Fourier modes can be written as

$$\{\psi_s, \psi_t\} = \delta_{s+t,0}$$

and the modes themselves can be expressed as

$$\psi_s = \frac{1}{2\pi i}\oint dz\, z^{-s-1/2}\, \psi(z).$$

With the help of the above relation we can write the anti-commutator between fields in terms of the analogue between modes through

$$\{\psi(z), \psi(w)\} = \frac{1}{\sqrt{zw}} \sum_{s,r} z^{-s}\, w^{-r} \{\psi_s, \psi_r\}.$$

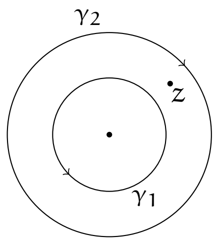
Such an expression may, in principle, be worked out making use of the delta function to help the summation: $\{\psi_s, \psi_t\} = \delta_{s+t,0}$ gives

$$\{\psi(z), \psi(w)\} = \frac{1}{\sqrt{zw}} \sum_r \left(\frac{w}{z}\right)^r = \frac{1}{\sqrt{zw}} \left(\sum_{r \geqslant 0} \left(\frac{w}{z}\right)^r + \sum_{r<0} \left(\frac{w}{z}\right)^r \right);$$

at a first glance problems occur because, although we can split the summation into two contributions, each of them represents a geometric series whose domain of convergency depends on the particular choice of the variables z, w. In particular the former term converges for $|z| > |w|$, the latter otherwise, namely $|z| < |w|$. This means that, in order to make sense, we must introduce in the above equation particular prescriptions on the choice of the allowed variable. To do so we shall proceed as follows: we invert back such formula to have

$$\{\psi_s, \psi_t\} = \frac{1}{2\pi i} \oint dz \, \frac{1}{2\pi i} \oint dw \, z^{s-1/2} \, w^{t-1/2} \{\psi(z), \psi(w)\}$$

and consider, on the right hand side, only those contours of integrations where each single term in the anti-commutator is radially ordered: $\{\psi(z), \psi(w)\} = \psi(z)\psi(w)|_{|z|>|w|} + \psi(w)\psi(z)|_{|z|<|w|}$.



Once we fix the variable $z \in \mathbb{C}$ the integration contours for w must be chosen as γ_1 to ensure $|z| > |w|$ and as γ_2 to ensure the converse. The anti-commutator between the modes becomes then

$$\{\psi_s, \psi_t\} = \frac{1}{2\pi i} \oint_0 dz \left(\frac{1}{2\pi i} \oint_{\gamma_1} dw \, z^{s-1/2} \, w^{t-1/2} \, \psi(z)\psi(w) \right.$$
$$\left. + \oint_{\gamma_2} dw \, z^{s-1/2} \, w^{t-1/2} \, \psi(w)\psi(z) \right).$$

We define the operator product expansion of two Fermi fields as

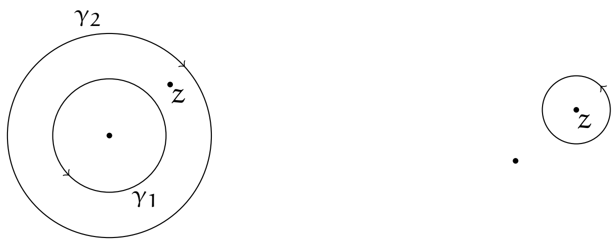
$$\mathrm{OPE}\,(\psi(z)\psi(w)) := \begin{cases} \psi(z)\psi(w) & \text{if } |z| > |w| \\ -\psi(w)\psi(z) & \text{if } |z| < |w| \end{cases}$$

and with the help of this definition the integral can be rewritten as

$$\{\psi_s, \psi_t\} = \frac{1}{2\pi i} \oint_0 dz \, \frac{1}{2\pi i} \oint_{\gamma_1 \cup \gamma_2} dw \, z^{s-1/2} \, w^{t-1/2} \, \mathrm{OPE}\,(\psi(z)\psi(w)).$$

It is even easier if we consider that $\gamma_1 \cup \gamma_2$ can be shrinked down to a loop around the point z so that the integral becomes

$$\{\psi_s, \psi_t\} = \frac{1}{2\pi i} \oint_0 dz \, \frac{1}{2\pi i} \oint_z dw \, z^{s-1/2} \, w^{t-1/2} \, \mathrm{OPE}\,(\psi(z)\psi(w)).$$



It is now pretty clear that if no divergences occur in the OPE then, by means of the Cauchy theorem, the integral on the right hand side vanishes. Therefore poles must occur in order to have reasonable anti-commutators. In general we can assume that fields in the operator product expansions are everywhere analytic except at coinciding points $z = w$ and hence the prototype expansion would have the form of a Laurent series ([Fuchs, 1992])

$$\text{OPE}\,(A(z)B(w)) = \sum_{n=-n_0}^{\infty} c_n(w)(z-w)^n \tag{4.2.1}$$

where the divergences occur for negative n and all the other regular terms, althoug present in the expansion, give no actual contribution to the integrals, nor do they to correlation functions. The singular terms in the expansion are referred to as the "contractions" of the operators and the coefficient c_0 is the "normal ordered product"

$$A\underbracket{(z)B(}w) := \sum_{n=-n_0}^{-1} c_n(w)(z-w)^n, \qquad :A(z)B(z): := c_0(z)$$

so that the OPE is decomposed into

$$\text{OPE}\,(A(z)B(w)) = A\underbracket{(z)B(}w) + :A(z)B(z): + \text{regular terms}.$$

No regular terms occur for free fields, therefore we shall very often omit them, since we are only dealing with such models. The number n_0 and the actual form of the expansion depend upon the fields and the commutation relations we want to realise. In the following two explicit examples of OPE, for Fermi fields and for currents, will reproduce the standard algebras we are used to.

Example (OPE for Fermi fields)*:* The operator product expansion for fermions acquires the form

$$\text{OPE}\,(\psi(z)\psi(w)) = \begin{cases} -\dfrac{1}{z-w} & \text{vacuum rep'n} \\ -\dfrac{1}{z-w}\cdot\dfrac{1}{2}\left(\sqrt{\dfrac{z}{w}}+\sqrt{\dfrac{w}{z}}\right) & \text{Ramond rep'n} \end{cases} \tag{4.2.2}$$

which can be proven right by reproducing the correct relations for the anti-commutators. In fact, substituting back into

$$\{\psi_s, \psi_t\} = \frac{1}{2\pi i}\oint_0 dz\, \frac{1}{2\pi i}\oint_z dw\, z^{s-1/2}\, w^{t-1/2}\ \text{OPE}\,(\psi(z)\psi(w))\,.$$

gives

$$\begin{aligned}
\{\psi_s, \psi_t\} &= \frac{1}{2\pi i}\oint_0 dz\, \frac{1}{2\pi i}\oint_z dw\, z^{s-1/2}\, w^{t-1/2}\, \frac{-1}{z-w} \\
&= \frac{1}{2\pi i}\oint_0 dz\, \frac{-1}{2\pi i}\, z^{s-1/2} \oint_z dw\, w^{t-1/2} \frac{1}{z-w} \\
&= \frac{1}{2\pi i}\oint_0 dz\, \frac{-1}{2\pi i}\, z^{s-1/2}\, 2\pi i \lim_{w\to z}(w-z)\cdot \frac{1}{z-w} w^{t-1/2} \\
&= \frac{1}{2\pi i}\oint_0 dz\, z^{s-1/2}\, z^{t-1/2} = \delta_{s+t,0}.
\end{aligned}$$

Of course the Ramond case works similarly. The OPE helps to easily derive the two-point function: in fact we know by previous arguments that such a function must have poles whenever the two variables approach each other, i. e. in the limit $z \to w$; the only possible contributions come then from the singular terms in the OPE that, in the case at hand, for Fermi fields, are given by (4.2.2). Therefore

$$\begin{aligned}
\omega_0(\psi(z)\psi(w)) &= \frac{1}{z-w} \\
\omega_R(\psi(z)\psi(w)) &= \frac{1}{z-w}\cdot\frac{1}{2}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right) = \frac{1}{2}\cdot\frac{1}{z-w}\cdot\frac{z+w}{\sqrt{zw}}
\end{aligned}$$

reproducing the formulae already shown.

Example (OPE for currents)*:* The operator product expansion for currents takes the form

$$\text{OPE}\left(J^a(z)J^b(w)\right) = \frac{1}{(z-w)^2}\kappa^{ab}\,\kappa - \frac{1}{z-w} f^{ab}{}_c J^c(w)$$

because this correctly reproduces the current algebra

$$[j^a_n, j^b_m] = f^{ab}{}_c\, J^c_{n+m} + n\,\delta_{n+m,0}\,\kappa^{ab}\,k.$$

The two-point function, again, must only contain the singular terms, then it can only be of the form

$$\begin{aligned}
\omega_0(j^a(z)j^b(w)) = \omega_0&\left(\frac{1}{(z-w)^2}\;\kappa^{ab}\,\kappa - \frac{1}{z-w} f^{ab}{}_c J^c(w)\right) \\
&= \frac{1}{(z-w)^2}\kappa^{ab}\,\kappa - \frac{1}{z-w} f^{ab}{}_c\, \omega_0(j^c(w)) \\
&= \frac{1}{(z-w)^2}\kappa^{ab}\,\kappa
\end{aligned}$$

because the one-point function $\omega_0(j(z))$ vanishes.

Example (OPE for stress-energy tensor)*:* Using the quarks construction and the previously calculated OPE for Fermi fields and currents, the operator product expansion for the stress-energy tensor may only take the form

$$\text{OPE}\left(T(z)T(w)\right) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)};$$

and thus the central charge c appears as coefficient for the $1/(z-w)^4$ term in the series.

We want to remark that the normal ordering just defined as the coefficient c_0 in the Laurent expansion for the OPE does in fact correspond to the Wick product of operators used in the standard setting of quantum field theory in the case of free fields, defined in turn subtracting the vacuum expectation value. We have then the identification

$$:AB: = \mathrm{OPE}(AB) - \underbrace{AB} - \text{regular terms} = AB - \omega_0(AB)\mathbf{1}$$

and since now on the two operations will be identified as the same.

4.3 FERMIONISATION IN ONE DIMENSION

As useful for the next purposes we are now going to show a simple example of the feature referred to as "fermionisation" in one dimension. Starting from fields fulfilling fermionic anti-commutation relations one can construct fields satisfying bosonic type commutation relations simply taking particular combinations of the former ones.

In particular we start from fermionic operators of the type $\{a(k), a(k')^*\} = \delta(k-k')$ and define the following bosonic-type operators

$$b(q) := \int_{\mathbb{R}} dk\, a(k+q)^*\, a(k) \qquad b(q)^* := \int_{\mathbb{R}} dk\, a(k-q)^*\, a(k)$$

which we are going to show fulfill bosonic commutation relations as $[b(q), b(q')^*] = q\delta(q-q')$. The commutator is

$$\begin{aligned}
[b(q), b(q')^*] &= b(q)b(q')^* - b(q')^* b(q) \\
&= \int_{\mathbb{R}} dk\, a(k+q)^*\, a(k) \int_{\mathbb{R}} dk'\, a(k'-q')^*\, a(k') \\
&\quad - \int_{\mathbb{R}} dk'\, a(k'-q')^*\, a(k') \int_{\mathbb{R}} dk\, a(k+q)^*\, a(k) \\
&= \int_{\mathbb{R}} dk\, dk' \, \big(a(k+q)^* a(k) a(k'-q')^* a(k') \\
&\quad - a(k'-q')^* a(k') a(k+q)^* a(k)\big)
\end{aligned}$$

at this point we make use of the anti-commutator $a(k)a(k'-q')^* = -a(k'-q')^* a(k) + \delta(k-(k'-q'))$ to switch the operators in the products; the last line becomes

$$\int_{\mathbb{R}} dk' \, \big(a(k'-q'+q)^* a(k') \; - a(k'-q')^* a(k'-q)\big)$$

because any $a(k)^{*2} = a(k)^2 = 0$. Also, we have integrated out the delta functions. At a first sight the above equation may run into problems because after intregrating on the whole real line infinities may arise and it is not clear how to subtract them from each other. In order to solve this issue we make use of the definition of the normal ordered product as $:AB: = AB - \omega_0(AB)\mathbf{1}$. With the help of this substitution we can rewrite the integral as

$$\int_{\mathbb{R}} dk\,\left(:a(k-q'+q)^* a(k): - :a(k-q')^* a(k-q):\right)$$
$$- \int_{\mathbb{R}} dk\,\omega_0(a(k-q+q')^* a(k)) + \int_{\mathbb{R}} dk\,\omega_0(a(k+q')^* a(k+q)).$$

The first contribution containing the normal orderings vanishes after relabelling the variables as $k' - q \to k'$: both terms are exactly the same. The other terms in the vacuum expectation value can be worked out as follows: the domains of integrations are restricted due to the fact that fermionic operators annihilate the vacuum for positive k, hence we can cut out the corresponding factors and end up only with

$$\begin{aligned}
[b(q), b(q')^*] &= \int_{-\infty}^{0} dk\,\omega_0(a(k-q+q')^* a(k)) \\
&\quad - \int_{-\infty}^{-q} dk\,\omega_0(a(k+q')^* a(k+q)) \\
&= \delta(q-q')\left(\int_0^{\infty} dk\,\omega_0(a(k)a(k)^*)\right. \\
&\quad \left. - \int_{-\infty}^{-q} dk\,\omega_0(a(k+q)a(k+q)^*)\right) \\
&= \delta(q-q')\int_0^q dk = \delta(q-q')\cdot q;
\end{aligned}$$

this finally shows that $[b(q), b(q')^*] = q\delta(q-q')$, as to be proven. Along similar lines the authors in [Bischoff and Tanimoto, 2013] show how to recover the $U(1)$-current subalgebra in the algebraic setting, as a subnet of the fermionic Fock space after introducing the correspondence

$$J_n = \sum_{r=-\infty}^{+\infty} :\psi_r{}^* \psi_{n-r}:$$

where the ψ_r are the modes of the free complex Fermi field $\{\psi_r^*, \psi_m\} = \delta_{n+m,0}$.

4.4 BISOGNANO-WICHMANN MODULAR FLOW

Following the standard construction of nets of von Neumann algebras we may assume to be equipped with an assignment of algebras $I \to \mathcal{A}(I)$ of Fermi fields. A standard result by Bisognano and Wichmann ([Bisognano and Wichmann, 1975]) provides the computation of the modular automorphisms group with respect to the vacuum state for chiral conformal field theories. In case $I = \mathbb{R}_+$ the adjoint action corresponds to the dilations of $\delta_t = e^{-2\pi t}$

$$\begin{aligned}
\sigma_t^{\mathbb{R}}(\psi(x)) &= \Delta^{it}\,\psi(x)\,\Delta^{-it} = \\
&\qquad U(D(\delta_t))\,\psi(x)\,U(D(\delta_t))^* = e^{-\pi t}\,\psi(e^{-2\pi t}\,x)
\end{aligned} \tag{4.4.1}$$

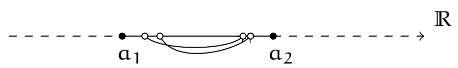
and for every other interval the modular automorphisms are obtained by conjugation with a Möbius transformation $\mu\colon I \to \mathbb{R}_+$ that maps I onto $\mathbb{R}_+$. Since Möbius transformations preserve the vacuum states $\omega_0^{\mathbb{R}_+} \circ \mu = \omega_0^I$ they intertwine the respective modular groups and therefore

$$\sigma_t^I = \mu^{-1} \circ \sigma_t^{\mathbb{R}_+} \circ \mu$$

which allows to calculate the action of the modular automorphisms group on $\psi(x) \in \mathcal{A}\,(I)$ as

$$\sigma_t^I(\psi(x)) = \sqrt{\frac{\delta_t\, \mu'(x)}{\mu'(\mu^{-1}(\delta_t\, \mu(x)))}}\, \psi(\mu^{-1}(\delta_t\, \mu(x)))$$

namely the subgroup of Möbius transformations preserving the interval I, fixing its boundaries.



Since the same is true for the Cayley transform, namely $\omega_0^{\mathbb{R}_+} = \omega_0^{S^1_+} \circ C$, then the modular flow on the circle is nothing but

$$\sigma_t^{S^1_+} = C \circ \sigma_t^{\mathbb{R}_+} \circ C^{-1}.$$

This property ensures that the action of the modular group on Fermi fields localised in one interval is geometric. Now the question that naturally arises and that we want to address is what the modular group is whenever fields are localised in different intervals instead. As we shall see, the action can no more be geometric because of conflicts with algebraic properties otherwise. As first, we introduce the following characterisation of von Neumann algebras due to Takesaki [Takesaki, 1970] and we rephrase it in the language of nets with cyclic and separating vectors.

Definition (conditional expectation)*:* Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of von Neumann algebras and let φ be a cyclic and separating state on $\mathcal{M}$. A linear map $\mathcal{E}\colon \mathcal{M} \to \mathcal{N}$ is called the "conditional expectation" of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ if

(i) $\varphi \circ \mathcal{E} = \varphi|_{\mathcal{N}}$

(ii) $\mathcal{E}(x) = x, \quad x \in \mathcal{N}$

(iii) $\mathcal{E}(x)\Omega = P_I\, x\, \Omega \quad x \in \mathcal{M}$

where Ω emerges out of the GNS representation of φ and P_I projects onto $\mathcal{H}_I = \{\, x\,\Omega \mid x \in \mathcal{A}\,(I)\,\}$.

Theorem 4.4.1 (Takesaki, [Takesaki, 1970])*:* Let $\mathcal{N} \subset \mathcal{M}$ and φ as above. The existence of a conditional expectation $\mathcal{E}\colon \mathcal{M} \to \mathcal{N}$ is equivalent to the global invariance $\sigma_t^{\varphi}(\mathcal{N}) = \mathcal{N}$ under the modular automorphism group.

Let us now assume we take two (for the sake of simplicity) any disjoint intervals $I_1, I_2 \mid I_1 \cup I_2 =: E_2$ and consider the action of the modular group of the algebra $\mathcal{A}(E_2)$ with respect to the vacuum state. The action being still geometric within each of the disjoint intervals would imply that $\sigma_t(\mathcal{A}(I_k)) \subset \mathcal{A}(I_k)$ and as a consequence conditions for the Takesaki's theorem would apply. The Reeh-Schlieder property ensures that the vacuum is cyclic and separating for each of the two subalgebras, hence the subset $\mathcal{H}_I$ is dense in $\mathcal{H}$. Therefore $P_I = \mathbb{1}$ and $\mathcal{E}\Omega = x\Omega$, $x \in \mathcal{A}(E_2)$. Since Ω is also separating then $\mathcal{E}x = x$ however you choose $x \in \mathcal{A}(E_2)$ and thus $\mathcal{A}(E_2)$ must coincide with $\mathcal{A}(I_k)$, which is not the case at hand. From this we realise that the action of the modular automorphism group (with respect to the vacuum state) on Fermi fields localised in disjoint intervals cannot be *purely* geometric in order to avoid conflicts with Takesaki's theorem. We shall see that a mixing with pointwise coefficients is realised with free Fermi fields.

4.4.1 *Geometric flow for product states*

Of course, the situation is quite different if we choose states that decompose as the product of many vacuum states. This choice would destroy all the correlations among different intervals and no restrictions given by the Takesaki theorem would therefore apply. This issue has been investigated by the authors of [Longo et al., 2009] and we shall summarise here the important results.

Equipped with a net of Fermi algebras $I \to \mathcal{A}(I)$ the modular group with respect to the vacuum state acts geometrically within each interval on localised fields (Bisognano-Wichmann property), the geometric flow being given by the subgroup δ_t of the Möbius group preserving the interval (dilations on $I = \mathbb{R}_+$).

Let I be an interval on the circle and $E_N = \sqrt[N]{I}$ the symmetric N-interval generated, i. e. the set of all points z such that $z^N \in I$. The N^{th} covering of the Möbius group, as introduced in 2.2.1, acts on the net as

$$U^{(N)}\left(\Lambda_I^{-2\pi t}\right) \mathcal{A}(E_N)\, U^{(N)}\left(\Lambda_I^{-2\pi t}\right)^* = \mathcal{A}\left(\delta_t^{(N)}(E_N)\right)$$

where the geometric flow corresponds to the n-dilations $\delta_t^{(n)}(z) = \sqrt[n]{\delta_t(z^n)}$. Therefore each sub-interval is separately preserved by this action. If we assume the net to be split, then on $E_N = I_1 \cup \ldots \cup I_n$ the split isomoprhism provides a correspondence $\chi_{E_N} \colon \vee_{k=1}^N \mathcal{A}(I_k) \to \otimes_{k=1}^N \mathcal{A}(I_k)$. With the help of such a map we can construct a product state as follows: let $U(g_k)$ implement a family of diffeomorphisms acting like $z_k \mapsto z^N$ on each of the I_k and let ω_0 be the respective vacuum state. Define now ([Longo et al., 2009]) the Kawahigashi-Longo state on the algebra of the multi-intervals $\mathcal{A}(E_N)$

$$\varphi_{E_N} := \left(\bigotimes_{k=1}^{n} \omega_0 \circ \mathrm{Ad}(U(g_k^{-1}))\right) \circ \chi_{E_N}. \tag{4.4.2}$$

The modular automorphisms flow for such a product state is, by construction, geometric within each sub-interval of E_N, the geometric flow being given by the N-dilations:

$$\sigma_t^{\varphi_{E_N}}(\mathcal{A}(I_k)) = \mathcal{A}\left(\delta_t^{(N)}(I_k)\right).$$

The same construction can be generalised to non-symmetric intervals accordingly. We choose a general family of diffeomorphisms $g_k \colon I \to I_k$ with the property that, given $z \in I$, then $z_k = g_k(z) \in I_k$. The factorisation is then

$$\varphi_{E_N} \coloneqq \left(\bigotimes_{k=1}^{n} \omega_0 \circ \mathrm{Ad}(U(g_k))\right) \circ \chi_{E_N}$$

and the modular group still acts geometrically, the geometric flow being now given by $\delta_t^{E_N}(z) = (g_k^{-1} \circ \delta_t^{(1)} \circ g_k)(z)$ instead; obviously, this reduces to the square root map once you go back to $g_k(z) =$ brances of $\sqrt[N]{z}$. However, we are going to examine the issue of non-symmetric intervals deeper, later on.

4.5 THE RESULT OF CASINI AND HUERTA

We shall focus henceforth on the modular theory for Fermi fields localised in disjoint intervals and the prototype notation will be $\psi(x) \in \mathcal{A}(E_N)$, with $E_N = I_1 \cup \ldots \cup I_N$. We shall also switch very often from the real line picture to the compact picture on the unit circle via Cayley transform. Natural notation is $\psi(x_k)$, $x_k \in I_k$ (and similarly for z_k on the circle) to refer to a field evaluated in I_k.

As we have seen, whenever we consider fields localised in different intervals the action of the modular automorphism group with respect to the vacuum state cannot be geometric because of conflicts between Takesaki's theorem and the Reeh-Schlieder property. A priori, for general space-time regions and massive fields, the action of the modular group may be of any fuzzy sort. In fact, when conformal invariance no more holds, one cannot transfer the geometric result of Bisognano and Wichmann via conformal mappings and thereby the modular action has to be non-local. In particular the authors in [Saffary, 2006; Figliolini and Guido, 1989] tried to describe the Tomita operators whence the modular automorphisms group comes in terms of pseudo-differential operators, from which the non-local action of the modular group arises. As for the modular group of a multi-interval algebra with respect to the Fock vacuum in free conformal theories, it will still act linearly on the free fields, because the modular operator S preserves the N-particle subspaces and hence so does the Tomita operator Δ. On the other hand, as we have just seen, it cannot preserve the single-interval subalgebras. If we knew that the action is pointwise, the most general modular flow would be of the form

$$\begin{array}{rlll} \sigma_t(\psi(x_1)) = & c_{11}(x_1,t)\psi(\zeta_t(x_1)) + & \dots & + c_{1N}(x_N,t)\psi(\zeta_t(x_N)) \\ \vdots & \vdots & \vdots & \\ \sigma_t(\psi(x_N)) = & c_{N1}(x_1,t)\psi(\zeta_t(x_1)) + & \dots & + c_{NN}(x_N,t)\psi(\zeta_t(x_N)) \end{array} \tag{4.5.1}$$

or, in a more compact notation,

$$\sigma_t \begin{pmatrix} \psi(x_1) \\ \vdots \\ \psi(x_N) \end{pmatrix} = \begin{pmatrix} c_{11}(x_1,t) & \dots & c_{1N}(x_N,t) \\ \vdots & \ddots & \vdots \\ c_{N1}(x_1,t) & \dots & c_{NN}(x_N,t) \end{pmatrix} \begin{pmatrix} \psi(\zeta_t(x_1)) \\ \vdots \\ \psi(\zeta_t(x_1)) \end{pmatrix}. \tag{4.5.2}$$

where $\zeta_t(x)$ is some flow to be determined. Of course, once one switches to the circle picture, the entries of the matrix are different. Indeed, the unexpected finding by Casini and Huerta states that for free Fermi fields, the modular flow is of this form.

The investigation of the modular automorphism group is then related to the evaluation of the coefficient appearing in (4.5.1). The original paper by [Casini and Huerta, 2009] provides the calculation of such coefficients using methods coming from density matrix and hamiltonian flows. It is known that the time evolution generated by the modular Hamiltonian K of a system, with $e^{iKt} = \Delta^{it}$ satisfies the KMS property with respect to the vacuum state and therefore coincides with its modular group (for details we refer the reader again to [Casini and Huerta, 2009] and references therein). Thermal states are characterised by density matrices of the form $\rho \sim e^{-K}$ and correlators can be expressed as $(\Omega, (\,\cdot\,)\Omega) = \mathrm{tr}(\rho \cdot (\,\cdot\,))$. This brings up a relation between the Hamiltonian and the n-point functions (especially the two-point function) that can be used to compute the modular dynamics $[K, \psi]$. Exponentiating the commutator one gets the entire time evolution $e^{itK}\,\psi(x)\,e^{-itK}$ corresponding to the modular action $\sigma^t_{\omega_0}(\psi(x)) = \Delta^{it}\psi(x)\Delta^{-it}$. As a consequence the argument is that knowledge of the Hamiltonian evolution flow and density matrix allows, at least formally, the computation of the modular automorphisms group in terms of kernel of distributions.

Let us consider the algebra of a Fermi field localised in N disjoint open intervals $I_1 \cup \dots I_N =: E_N$, with $I_k =]a_k, b_k[\subset \mathbb{R}$; we call the interval "symmetric" if I_k are the N^{th} roots of an interval $I \subset \mathbb{R}$ (in this case we write $E_N = \sqrt[N]{I}$). Let us introduce the following Casini-Huerta function $X \colon E_N \to \mathbb{R}_+$ mapping each of the intervals monotonously onto $\mathbb{R}_+$ as

$$X(x) = -\prod_{k=1}^{N} \frac{x - a_k}{x - b_k} = \prod_{k=1}^{N} \frac{1 + v_k}{1 + u_k} \cdot \prod_{k=1}^{N} \frac{z - u_k}{z - v_k} \tag{4.5.3}$$

where z, u_k, v_k are the Cayley images of the points x, a_k, b_k. Each $X \in \mathbb{R}_+$ has exactly N pre-images, one in each interval, and we refer to them as to $X_1^{-1}(X), \dots, X_N^{-1}(X)$, $X_j^{-1}(X) \in I_j$. Similarly, we denote by $z_k^{-1}(X)$ the k^{th} pre-image on the circle. Moreover, this function has the remarkable property that in case of symmetric intervals, namely when $z^N \in I$ and thus $z_k = \omega_k z \in I_k$ with $\omega_k^N = 1$, then $X(z^N) = \mu \circ C^{-1}(z^N)$, where μ is a suitable Möbius function.

The original result by Casini and Huerta ([Casini and Huerta, 2009] and [Longo et al., 2009]) states that the modular automorphism group with respect to the vacuum state acts on Fermi field as

$$\sqrt{X_k^{-1}(X)'}\,\sigma_t\left(\psi(X_k^{-1})\right) = \sum_{j=1}^{N} O(X,t)_{kj}\sqrt{X_k^{-1}(\delta_t(X))'}\,\psi\left((X_j^{-1}(\delta_t X))\right) \quad (4.5.4)$$

where the flow $\zeta_t(X_k^{-1}(X))$ corresponds exactly to the geometric flow appearing in the one interval case

$$\zeta_t(X_k^{-1}(X)) = \delta_t(X_k^{-1}(X)) = X_k^{-1}(\delta_t(X))$$

with δ_t being the one parameter subgroup of Möbius transformations preserving the interval $\mathbb{R}_+$. The geometric part moving the points happens to be the same as in the one interval case, plus a mixing among different intervals occurs on top of it. Of course, if one reads the modular flow in terms of hamiltonian evolution, it is easy to understand that such an evolution usually delocalises fields in the pure sense of quantum mechanics, and thus we can no more expect a geometric action within each subinterval.

The matrix $O(X,t)$ appearing in (4.5.4) is an orthogonal cocycle $\in SO(N)$ given in terms of a differential equation ([Longo et al., 2009]) as

$$\partial_t O(X,t) = O(X,t)K(\delta_t X) \quad (4.5.5)$$

where the matrix $K(X)$ on the right hand side is

$$K(X)_{kj} = \begin{cases} 2\pi \dfrac{\sqrt{X_k^{-1}(X)'\,X_j^{-1}(X)'}}{X_k^{-1}(X) - X_j^{-1}(X)} & \text{if } k \neq j \\ 0 \quad \text{otherwise} \end{cases} . \quad (4.5.6)$$

The solution is a coboundary

$$O(X,t) = O(X)^T \cdot O(\delta_t X)$$

where $O(X)$ is the anti-path-ordered exponential

$$O(X) = \overline{P}\left(e^{-\frac{1}{2\pi}\int_{X_0}^{X} dX'\,K(X')} \right) . \quad (4.5.7)$$

As a matter of example we can carry out the explicit form of this matrix in the case of symmetric intervals. In this case, as we have seen, the related points $z_k \in I_k$ are obtained by taking one of the N^{th} roots of z^N, hence $z_k = \omega^k z$, where $\omega = e^{\frac{2\pi i}{N}}$ is the N^{th} root of the unity, $\omega^N = 1$ and $\omega^k = e^{\frac{2\pi i}{N}k}$. Such points can also be obtained out of the Casini-Huerta uniformisation function, calculated for symmetric interval, after a suitable match with a Möbius transformation (this is necessary whenever the Casini-Huerta function has range $\mathbb{R}_+$, to match the Möbius transformation taking the upper semi-circle to the interval I).

However, the symmetric form drastically simplifies the entries of the matrik $K(X)$, as they become

$$K(z)_{kj} = 2\pi \frac{\omega^{\frac{k+j}{2}}}{(\omega^k - \omega^j)z} = -\overline{K}_{kj}\,\frac{1}{z} \tag{4.5.8}$$

with $\overline{K}_{kj}$ a matrix with constant entries

$$\overline{K}_{kj} = -\frac{\omega^{\frac{k+j}{2}}}{(\omega^k - \omega^j)};$$

of course $K(z)_{kj}$ is still zero if $k = j$. Since the matrix $\overline{K}$ is a constant matrix, it commutes with itself at different points and as a consequence the anti-path-ordered exponential reduces to an ordinary exponential

$$O(X) = e^{\frac{1}{2\pi} 2\pi \overline{K} \int_1^z dw \frac{1}{w}} = e^{\overline{K}\ln z} = z^{\overline{K}}; \tag{4.5.9}$$

clearly enough, $z = e^{i\varphi}$ is any point on the circle. The expression we have obtained is rather simple, in the case of symmetric intervals; later on we will introduce a lemma stating the particular form allowed for the spectrum of such a matrix, also calculating the particular diagonal form it acquires after an orthonormal transformation $K = B^{-1}\,D\,B$, with a unitary matrix B and a diagonal matrix D. For the moment we can just plug this expression in the above formula (no matter what these matrices actually are) to obtain

$$z^{\overline{K}} = z^{B^{-1}\,D\,B} = e^{i\varphi B^{-1}\,D\,B} = B^{-1}\,z^{D}\,B$$

and the matrix D can be decomposed over its eigenvalues λ_k as $D = \sum_{k=1}^{n} \lambda_k\, P_k$, where $P_k := |e_k\rangle\langle e_k|$ is the projection over the k^{th} eigenspace. Running once around the circle the change in the variable z is $z \mapsto e^{2\pi i}\,z$ and so the matrix $O(z)$ changes consequently as

$$O(e^{2\pi i}\,z) = B^{-1}\,e^{2\pi i \sum_{k=1}^{n} \lambda_k P_k}\,z^{\overline{K}}\,B.$$

We shall see that the spectrum λ_k of D is basically made of natural numbers so that eventually one obtains $e^{(N+1)i\pi}$ and thus we conclude that the only change in the matrix $O(z)$ is up to a minus sign: $O(e^{2\pi i}\,z) = (-1)^{N+1}\,O(z)$.

As we mentioned, the spectrum of the matrix $K(X)$ is given, in the symmetric case, by the following:

Lemma ([Rehren and Tedesco, 2013]): The matrix $K(X)$ has integer spaced spectrum $\frac{1-n}{2}, \ldots, \frac{n-1}{2}$ in the symmetric case. It is diagonalised by the unitary matrix $\frac{1}{\sqrt{n}}B$ whose entries are $B_{kj} = \omega^{(1/2-k)j}$, ω being given by the root of the unity as above. This implies $B\,K = D\,B$, where D is the diagonal matrix with entries $D_{kk} = \frac{n+1}{2} - k$.

Proof. By direct computation of the left hand side

$$\sum_{j\neq l}^{n} B_{kj}K_{jl} = -\sum_{j=l+1}^{n+l-1} \frac{\omega^{(1/2-k)j}\omega^{\frac{j+l}{2}}}{\omega^j - \omega^l}$$
$$= -\omega^{(1/2-k)l}\sum_{j=1}^{n-1} \frac{\omega^{(1/2-k)j}\omega^{\frac{j+2l}{2}}}{(\omega^j - 1)\omega^l}$$

where we made use of the invariance of the sum under the shift $j \to j+n$. Now we symmetrise again the sum under $j \leftrightarrow n-j$ to have

$$\sum_{j\neq l}^{n} B_{kj}K_{jl} = -B_{kl}\sum_{j=1}^{n-1} \frac{\omega^{(1-k)j}}{\omega^j - 1}$$
$$= -B_{kl}\cdot\frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{\omega^{(1-k)j}}{\omega^j - 1} + \frac{\omega^{(1-k)(n-j)}}{\omega^{n-j} - 1}\right)$$

Notice now that whenever $z \in S^1$ is a phase, contributions of the form

$$\frac{z^m - z^{-m}}{z - z^{-1}} = z^{m-1} + \ldots + z^{1-m}$$

can be simplified cancelling the denominators. This applies as well to ω and the right hand side becomes then

$$-B_{kl}\cdot\frac{1}{2}\sum_{j=1}^{n-1}\frac{\omega^{(1-k)j} - \omega^{kj}}{\omega^j - 1} = B_{kl}\cdot\frac{1}{2}\sum_{j=1}^{n-1}\sum_{\nu=1-k}^{k-1}\omega^{j\nu}.$$

Since $\omega^n = 1$ we derive that $\sum_{j=0}^{n}\omega^{j\nu} = n\,\delta_{\nu,0}$ and thus

$$\sum_{j\neq l}^{n} B_{kj}K_{jl} = B_{kl}\sum_{\nu=1-k}^{k-1}(n\,\delta_{\nu,0} - 1) = B_{kl}\cdot\frac{1}{2}(n - 2k + 1) = D_{kk}\cdot B_{kl}$$

completing the proof. □

4.6 DIFFEOMORPHISMS COVARIANCE

We shall now turn to a very important feature of conformal nets on the circle which follows from the way fields are assumed to transform under diffeomorphisms. We shall follow the guide lines provided by [Longo and Xu, 2004]: we assume the net to be diffeomorphisms covariant and split, i. e. for each couple of intervals I_1, I_2 with disjoint closure, there exist an isomoprhism (the "split map") $\chi\colon \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \to \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ such that $\chi(a_1 a_2) = a_1 \otimes a_2$, however you choose $a_1 \in \mathcal{A}(I_1)$, $a_2 \in \mathcal{A}(I_2)$.

Let now I be an interval on the circle and denote as $\mathcal{A}^N(I) = \mathcal{A}(I) \otimes \ldots \otimes \mathcal{A}(I)$ many copies of the related algebra. We introduce a family of diffeomorphisms $\gamma_j : I \to I_j$ such that, with natural understanding of notations, given $z \in I$ then $\gamma_j(z) = z_j \in I_j$. Diffeomorphisms covariance implies the existence of a continuous projective unitary representation of $\mathrm{Diff}(S^1)$. Once we choose the γ_j their action is implemented on the net by means of unitaries $U(\gamma_j)$:

$$\phi_I^j := \mathrm{Ad}\, U(\gamma)|\mathcal{A}(I) = \mathcal{A}\left(\gamma_j(I)\right) = \mathcal{A}\left(I_j\right)$$

in this respect ϕ_I^j is an isomoprhism $\phi_I^j : \mathcal{A}(I) \to \mathcal{A}\left(I_j\right)$. Therefore taking the tensor product N times we obtain a map

$$\bigotimes_{k=1}^{N} \phi_I^k = \phi_I^1 \otimes \ldots \otimes \phi_I^n : \mathcal{A}^N(I) \to \mathcal{A}(I_1) \otimes \ldots \otimes \mathcal{A}(I_N)$$

acting on the elements as $\phi_I^1(a_1) \otimes \ldots \otimes \phi_I^N(a_N)$. We can now compose everything with the inverse split map χ^{-1} at our disposal in order to bring $\mathcal{A}(I_1) \otimes \ldots \otimes \mathcal{A}(I_N)$ into $\mathcal{A}(I_1 \cup \ldots \cup I_N)$. The assignment we eventually obtain is called the Longo-Xu map

$$\mathrm{LX} : \mathcal{A}^N(I) \to \mathcal{A}(I_1 \cup \ldots \cup I_N) \tag{4.6.1}$$

and it is explicitly realised on the elements as

$$\begin{aligned} \mathrm{LX}(a_1 \otimes \ldots \otimes a_N) &:= \left(\chi^{-1} \circ \bigotimes_{k=1}^{N} \phi_I^k\right)(a_1 \otimes \ldots \otimes a_N) \\ &= \phi_I^1(a_1) \cdot \ldots \cdot \phi_I^N(a_N) \end{aligned}$$

for each $a_1, \ldots, a_N \in \mathcal{A}(I)$. As a matter of example, and very useful for the forthcoming purposes, we shall give the explicit formulae for the Longo-Xu map in case the diffeomorphisms $\gamma_j(z)$ coincide with the inverse root map.

Example (Square root map)*:* For the sake of simplicity let us restrict to a symmetric two-interval $\sqrt{I} = I_1 \cup I_2$, that is the set of points $z \in S^1$ such that $z^2 \in I$. The diffeomorphism at hand is the square root map $z^2 \mapsto \sqrt{z^2} = \pm z$ and we identify the two solutions as the two branches $\mu_1(z^2) = z$, $\mu_2(z^2) = -z$ and thus the two intervals are related to each other as $I_2 = \mathrm{rot}(\pi)(I_1)$. Also, in order to avoid troubles with the discontinuities, we require such interval not to contain the point where the cut in the square root is chosen.

In particular, if we take two copies $\psi^{(1)}, \psi^{(2)}$ of a Fermi field localised in I we have, applying (4.6.1)

$$\begin{aligned} \mathrm{LX}\left(\psi^{(1)}(z^2) \otimes \mathbf{1}\right) &= \mathrm{Ad}\, U\left(\sqrt{(\,\cdot\,)}\right)(\psi^{(1)}(z^2)) \cdot \mathbf{1} \\ \mathrm{LX}\left(\mathbf{1} \otimes \psi^{(2)}(z^2)\right) &= \mathbf{1} \cdot \mathrm{Ad}\, U\left(-\sqrt{(\,\cdot\,)}\right)(\psi^{(2)}(z^2)). \end{aligned}$$

By using the conformal transformation law of the Fermi fields, the adjoint action reduces to

$$\mathrm{Ad}\,U(\gamma)\psi(z) = \sqrt{\frac{\partial\gamma}{\partial z}}\,\psi(\gamma(z))$$

and thus

$$\begin{aligned} LX\left(\psi^{(1)}(z^2)\otimes\mathbf{1}\right) &= \frac{1}{\sqrt{2z}}\,\psi(z) \\ LX\left(\mathbf{1}\otimes\psi^{(2)}(z^2)\right) &= \frac{i}{\sqrt{2z}}\,\psi(-z). \end{aligned}$$

Even more useful for the next issues is the form this map acquires on the complex fermion

$$LX(\phi(z^2)) = LX\left(\psi^{(1)}(z^2)+i\psi^{(2)}(z^2)\right) = \frac{1}{2\sqrt{z}}\left(\psi(z)+\psi(-z)\right) \tag{4.6.2}$$

$$LX(\phi(z^2)^*) = LX\left(\psi^{(1)}(z^2)-i\psi^{(2)}(z^2)\right) = \frac{1}{2\sqrt{z}}\left(\psi(z)-\psi(-z)\right). \tag{4.6.3}$$

The Longo-Xu map allows us to simplify the expression of the Kawahigashi-Longo state that we have introduced before, (4.4.2). In fact, again in case of a symmetric two-interval $\sqrt{I}$, the state (4.4.2) appears to be exactly

$$\begin{aligned} \varphi_{\sqrt{I}} &= (\omega_0\otimes\omega_0)\circ\left(\mathrm{Ad}\,U(z\mapsto z^2)\otimes\mathrm{Ad}\,U(-z\mapsto z^2)\right)\circ\chi^{-1}_{\sqrt{I}} \\ \varphi_{\sqrt{I}} &= (\omega_0\otimes\omega_0)\circ LX^{-1} \end{aligned}$$

This relation is going to be very useful to compare such product state with the vacuum state defined on the algebra of the multi-interval.

4.7 A MULTI-LOCAL ISOMORPHISM

In this section we present a simple isomorphism between the algebra of one real chiral Fermi field and the algebras of n real chiral Fermi fields in the context of nets of von Neumann algebras. Unlike the Longo-Xu map, this isomorphism preservers the vacuum state due to a suitable change of localisation; we first prove the result for symmetric intervals and then extend it to the general case of non-symmetric intervals, using insights and results from [Casini and Huerta, 2009].

As a start-up we recall that in general, due to the split property, we can make use of the split map between any two algebras $\chi\colon \mathcal{A}(I)\vee\mathcal{A}(J)\to\mathcal{A}(I)\otimes\mathcal{A}(J)$ taking $ab\to a\otimes b$. To be more precise, since we shall be dealing with Fermi fields, the tensor product $\otimes^t$ is understood to be "graded", namely for any two operators $A\otimes^t B$ is the true tensor product $A\otimes B$ if either of them is a Bose field, and a twisted tensor product $A\otimes(-1)B$ if both of them are Fermi fields. This is to ensure the correct commutation or anti-commutation relations, respectively. Of course, whenever we are dealing with sums of Bose and Fermi fields, the correct formulae for the tensor product follow by linearity.

However, as shown in the previous paraghraph, this is implemented on the algebras as the Longo-Xu map, especially when the fields at hand satisfy diffeomorphisms covariance. Roughly speaking, this allows us to move the fields around the circle and in particular we can bring any number N of "copies" of one algebra $\mathcal{A}^N(I)$ onto one "delocalised" copy of the same algebra $\mathcal{A}(I_1 \cup \ldots \cup I_N)$ via $LX(a_1 \otimes \ldots \otimes a_N) = \phi_I^1(a_1) \cdot \ldots \cdot \phi_I^N(a_N)$. It is nevertheless clear though, that such a map does not preserve the vacuum state, because product states do destroy correlations between fields at different points, $\omega_0(a(x)b(y)) \neq \omega_0(a(x)) \cdot \omega_0(b(y))$.

The new idea is that, nonetheless, a vacuum preserving isomorphism does exist, it just has to be prepared ad hoc, and, interestingly enough, we shall see eventually that it is strongly connected to the Longo-Xu isomorphism through a general gauge transformation. Also, this new isomorphism will be globally defined thanks to its extension to the entire circle. Anyway, before we start we recall once more the standard notations to be used.

Let $\mathcal{A}^N(I)$ denote N copies of an algebra of fields localised in the interval I on the circle (likewise on the real line, we shall switch the two pictures very often), i. e. $\mathcal{A}(I) \otimes \ldots \otimes \mathcal{A}(I)$. Whenever no representation is explicitly stated we assume the fields to be in their defining vacuum representation, namely $\pi_0(\mathcal{A}(I))$. However, we will try to be clear enough throughout. A symmetric interval is essentially an N^{th} root $\sqrt[N]{I} = I_1 \cup \ldots \cup I_N$ and, with clear understanding of symbols, we refer to $z^N \in I$ and $z_1, \ldots, z_N$ as the roots in each sub-interval I_k, each of them satisfying $z_k^N = z^N \in I$. Even clearer is the following notation: the related points z_k can be written as $z_k = \omega^k z$ if $\omega = e^{\frac{2\pi i}{N}}$ and $z = e^{i\varphi}$ is any fixed point $\in E_N$. This will ensure that each of those roots "squares" to $z^N \in I$. In the special case of $N = 2$ this reduces to $z^2 \in I$ and $I_1 \cup I_2$ is the set of points of the form $z, -z$ as the two solutions to $\sqrt{z^2}$.

A non-symmetric interval E_N, instead, is simply the union of any N intervals with disjoint closure, wherein the related points $z_k \in I_k$ need not be roots of z^N, rather they are defined as roots of a particular N-folded map taking the interval I onto $I_1 \cup \ldots \cup I_N$, where each of the z_k appears as one of the solutions of this equation. In particular, this assignment will be achieved by means of the Casini-Huerta function (4.7.9) whose properties have already been stated.

Fields will be considered both on the real line $\mathbb{R}$ and on the circle S^1, the passage from either picture to the other being achieved by means of the Cayley transform. Conformal fields will consequently change as (the extra factor i is conventional)

$$\phi(z) = \sqrt{-i\frac{dz}{dx}}\,\phi(x) = \frac{1-ix}{\sqrt{2}}\,\phi(x).$$

Fields in the compact picture are just a reparametrisation of fields on the real line, and the extension to the entire circle depends on the representation. As already introduced in the previous paraghraph 4.1, Fermi fields on the real line posses two faithful representations: the vacuum (Neveu- Schwarz) and the Ramond representation, the former extending periodically on the circle, the latter anti-periodically. The starting point will be a real chiral Fermi field versus two copies thereof, also seen as a complex fermion again in the sense of 4.1, where all the notations, two-point functions and commutation rules have already been stated. Obviously fields must be smeared with suitable functions in order to obtain operators on a Hilbert space; nevertheless most of the computations will be clear in the sense of distribution, if not stated otherwise.

Moreover, due to the fact that the fields are assumed to be free (and hence their anti-commutators are multiples of the identity operator) the standard anti-commutation relations can be recovered out of the two-point function as

$$\omega(\psi(z)\psi(w)) + \omega(\psi(w)\psi(z)) = \omega(\omega(\psi(z)\psi(w))+ \\ \omega(\psi(w)\psi(z))) = \omega(\{\psi(z),\psi(w)\}) = \{\psi(z),\psi(w)\}.$$

In the case at hand we recover

$$\{\psi(z),\psi(w)\} = \lim_{\lambda\to 1}\left(\frac{1}{z-\lambda w}+\frac{1}{w-\lambda z}\right) = \frac{2\pi}{z}\,\delta(\varphi-\theta)$$

if $z = e^{i\varphi}$ and $w = e^{i\theta}$, for $\varphi,\theta \in\,]-\pi,\pi[$.

4.7.1 *The symmetric case*

We start with the symmetric case for $N = 2$, namely $z^2 \in I$ and $z, -z \in I_1, I_2$ respectively. A complex Fermi field $\phi(z^2), \phi^*(z^2)$ is localised in I and a real Fermi field $\psi(z)$ in $\sqrt{I} = I_1 \cup I_2$.

Proposition ([Rehren and Tedesco, 2013])*:* Let ϕ and ψ stand for the complex and real fermion in their vacuum representation, as stated above. The linear map

$$\beta\colon \mathcal{A}^2(I) \to \mathcal{A}\left(\sqrt{I}\right) \tag{4.7.1}$$

given by

$$\phi(z^2) \mapsto \frac{1}{2}\left(\psi(z)+\psi(-z)\right) \tag{4.7.2}$$

$$\phi^*(z^2) \mapsto \frac{1}{2z}\left(\psi(z)-\psi(-z)\right) \tag{4.7.3}$$

for $z \in S^1$, induces an isomorphism of CAR algebras preserving the vacuum state on the different algebras: $\omega_0^{(1)} \circ \beta = \omega_0^{(2)}$. Note that the map is well defined because the right hand sides are invariant under $z \mapsto -z$.

Proof. We start showing the inverse of such a map: clearly, summing up the two sides of the equations we obtain

$$2\,\psi(z) = 2\,\beta(\phi(z^2)) + 2z\,\beta(\phi^*(z^2))$$
$$2\,\psi(-z) = 2\,\beta(\phi(z^2)) - 2z\,\beta(\phi^*(z^2))$$

therefore the inverse relation reads

$$\beta^{-1}\,(\psi(\pm z)) = \phi(z^2) \pm z\phi^*(z^2).$$

The adjoint relation is immediate:

$$\beta(\phi(z^2))^* = z^2\,\beta(\phi^*(z^2)),$$

hence β preserves the adjoints too. In terms of the two copies $\phi(x) = (\psi_1(x) + i\psi_2(x))/\sqrt{2}$ the map β can be written as

$$\psi_1(z^2) \mapsto \frac{1}{2\sqrt{2}}\,\psi(z)\left(1+\frac{1}{z}\right) + \frac{1}{2\sqrt{2}}\,\psi(-z)\left(1-\frac{1}{z}\right)$$
$$\psi_2(z^2) \mapsto \frac{1}{2\sqrt{2}}\,\psi(z)\left(1-\frac{1}{z}\right) + \frac{1}{2\sqrt{2}}\,\psi(-z)\left(1+\frac{1}{z}\right)$$

with inverse given by

$$\beta^{-1}\,(\psi(\pm z)) = \ \psi_1(z^2)(1 \pm z) + i\psi_2(z^2)(1 \mp z).$$

Now we turn to the vacuum preserving features. The simplest proof proceeds by brute force plugging the right hand side of (4.7.1) into the two-point function and evaluating the result:

$$\begin{aligned}\omega_0 \circ \beta\left(\phi^*(z^2)\phi(w^2)\right) &= \omega_0\left(\beta(\phi^*(z^2))\beta(\phi(w^2))\right)\\ &= \omega_0\left(\frac{1}{2z}(\psi(z)-\psi(-z)) \cdot \frac{1}{2}(\psi(w)+\psi(-w))\right)\end{aligned}$$

this brings four contributions:

$$\begin{aligned}&\frac{1}{4z}\,(\omega_0(\psi(z)\psi(w)) - \omega_0(\psi(-z)\psi(w))\\ &\qquad + \omega_0(\psi(z)\psi(-w)) - \omega_0(\psi(-z)\psi(-w)))\end{aligned}$$

which can be summed up using the formula for the vacuum two-point function for the real chiral Fermi field

$$\begin{aligned}\omega_0 \circ \beta\left(\phi^*(z^2)\phi(w^2)\right) &= \frac{1}{4z}\left(\frac{1}{z-w} - \frac{1}{-z-w} + \frac{1}{z+w} - \frac{1}{-z+w}\right)\\ &= \frac{1}{4z}\,2\left(\frac{1}{z-w} + \frac{1}{z+w}\right)\\ &= \frac{1}{2z} \cdot \frac{2z}{z^2-w^2} = \frac{1}{z^2-w^2}\\ &= \omega_0\left(\phi^*(z^2)\phi(w^2)\right)\end{aligned}$$

as to be proven. By exploiting Wick theorem, this equality extends to all n-points functions, since these are just sums of products of two-point functions, see equation (1.2.2). As already pointed out, the standard anti-commutation relations follow from this correlation function, therefore they remain preserved as well.

Another proof proceeds by simply looking at Fourier modes. In the vacuum representation, where we assume the fields are evaluated, we have

$$\phi(z^2) = \sum_{r\in\mathbb{Z}+1/2} \phi_r \,(z^2)^{-r-1/2}, \quad \psi(z) = \sum_{r\in\mathbb{Z}+1/2} \psi_r \, z^{-r-1/2}.$$

A simple look at the right hand sides of (4.7.1) displays that, for example,

$$\begin{aligned} \beta\left(\phi(z^2)\right) &= \sum_{r\in\mathbb{Z}+1/2} \beta(\phi_r)\,(z^2)^{-r-1/2} = \frac{1}{2}\left(\psi(z)+\psi(-z)\right) \\ &= \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2} \phi_r \, z^{-r-1/2}\left(1+(-1)^{-r-1/2}\right) \\ &= \sum_{p\in\mathbb{Z}} \psi_{2p-1/2}(z^2)^{-p} = \sum_{r\in\mathbb{Z}+1/2} \psi_{2r+1/2}(z^2)^{-r-1/2} \end{aligned}$$

therefore the isomorphism appears as a relabelling of the Fourier modes $\phi_r \mapsto \psi_{2r+1/2}$. Similarly for the adjoint field we have $\phi_r^* \mapsto \psi_{2r-1/2}$; the variable r runs into $\mathbb{Z}+1/2$ for the vacuum representation. In terms of these Fourier modes the anti-commutation relations read

$$\{\phi_r, \phi_s^*\} = \{\psi_r, \psi_s\} = \delta_{r+s,0}$$

and a simple look shows that

$$\{\beta(\phi_r), \beta(\phi_s^*)\} = \{\psi_{2r+1/2}, \psi_{2s-1/2}\} = \delta_{2r+1/2+2s-1/2,0} = \delta_{r+s,0};$$

the vacuum state is the only state which is annihilated by all $\psi_r, r > 0$ and by all $\phi_r, \phi_r^*, r > 0$, respectively. Of course the relabelling does not change these conditions and ergo the vacuum is still sent into itself. The adjoint relations in terms of modes are $\phi_r^* = (\phi_{-r})^*$ and we can easily see that, by making use of $\beta(\phi(z^2))^* = z^2\,\beta(\phi^*(z^2))$ multiplication by z^2 becomes a shift by -1 in terms of modes, and this is exactly $2r+1/2-1 = 2r-1/2$, indeed what happens to ϕ_r^* under the action of β.

We have thus shown that β is an isomorphism that preserves the vacuum state, both in the "local" setting (by looking at the two-point function) and from the algebraic perspective of anti-commutators. This is due to change of localisation from the point z^2 to the points $z, -z$ with suitable coefficients that must adjust the form of two-point function eventually. We shall see later on that such coefficients will acquire a more general form described by the Casini-Huerta function (4.7.9) in the context of modular theory and this will come as a very special feature; with any other function $I_k \to I$ the statement is no more true. The reduction to special symmetric case brings back the form we have just analysed. However, it is interesting to show what the map β becomes on the real line, instead. Since $\mathbb{R} = C^{-1}(S^1 \setminus \{-1\})$ then

$$\beta^{\mathbb{R}} = C^{-1} \circ \beta^{S^1} \circ C.$$

The square root assignment $z^2 \mapsto \sqrt{z^2} = \pm z$ becomes, after Cayley transform, $C^{-1}(z^2) = q(x) = 2x/(1-x^2)$, whose two "square roots" are the points $x = C^{-1}(z)$ and $-1/x = C^{-1}(-z)$. We have that, on the real line, $\beta^{\mathbb{R}}$ is

$$\begin{aligned} \phi(q(x)) &\mapsto \frac{1}{q(x)} \cdot \frac{1}{1-ix}\Big(x\psi(x) + i\psi(-1/x)\Big) \\ \phi^*(q(x)) &\mapsto \frac{1}{q(x)} \cdot \frac{1}{1+ix}\Big(x\psi(x) - i\psi(-1/x)\Big) \end{aligned}$$

and the inverse relation is simply

$$\beta^{-1}(\psi(x)) = \frac{1-ix}{1-x^2}\,\phi(q(x)) + \frac{1+ix}{1-x^2}\,\phi^*(q(x)).$$

One might still ask to show why the vacuum correlation function is preserved on the real line too; this can be easily proven by brute force, though we prefer to show a more elegant solution as follows: the vacuum states on the real line and on the circle are related via $\omega_0^{\mathbb{R}} = \omega_0^{S^1} \circ C$:

$$\begin{aligned} \omega_0^{\mathbb{R}} \circ \beta^{\mathbb{R}} &= \omega_0^{S^1} \circ C \circ \beta^{\mathbb{R}} \\ &= \omega_0^{S^1} \circ C \circ C^{-1} \circ \beta^{S^1} \circ C \\ &= \omega_0^{S^1} \circ \beta^{S^1} \circ C \\ &= \omega_0^{S^1} \circ C = \omega_0^{\mathbb{R}} \end{aligned}$$

as to be proven. □

With a little care the result we have just presented can be straightforwardly generalised to the case of symmetric N-intervals. Before we do so, we notice that a closer look to β brings the following inspection to the coefficients appearing on the right hand sides:

$$\begin{aligned} \phi(z^2) &\mapsto \frac{z^0}{2}\left(\psi(\omega^0 z) + \psi(\omega^1 z)\right) \\ \phi^*(z^2) &\mapsto \frac{z^{-1}}{2}\left(\psi(\omega^0 z) - \psi(\omega^1 z)\right) \end{aligned}$$

where we have introduced the roots of the unity $\omega^{0,1} = \pm 1$ as in the fashion previously described. Even more compact is the form:

$$\phi^{(k)}(z^2) \mapsto \frac{z^{1-k}}{N} \sum_{j=0}^{N-1} \omega^{(1-k)j}\,\psi(\omega^j z) \qquad k = 1, 2.$$

Let us now take any symmetric N-interval with arbitrary N, exploiting the form of $\sqrt[N]{z^N}$ with N solutions $z_1, \ldots, z_N$ such that $z_k =$

$\omega^k z \in I_k$. Similarly we choose N real Fermi fields lying in $\mathcal{A}(I)$ which we can pairwise combine into N/2 complex Fermi fields: in formulae we assign $\psi^{(1)}(z^N), \ldots, \psi^{(N)}(z^N)$ such that $\phi^{(k)^*}(z^N) = \phi^{(N+1-k)}(z^N)$. Then we just have to propose the same ansatz for a symmetric N-interval $\beta\colon \mathcal{A}^N(I) \to \mathcal{A}\left(\sqrt[N]{I}\right)$

$$\phi^{(k)}(z^2) \mapsto \frac{z^{1-k}}{N} \sum_{j=0}^{N-1} \omega^{(1-k)j}\, \psi(\omega^j z) \qquad k = 1, \ldots, N. \tag{4.7.4}$$

In terms of the initial fields and in terms of Fourier modes this becomes

$$\psi^{(k)}(z^N) \mapsto \sum_{r \in \mathbb{Z}+1/2} \psi_{1/2-k+(r+1/2)N}\, (z^N)^{-r-1/2}$$

which corresponds in turn to a relabelling of generators $\psi_r^{(k)} \mapsto \psi_{1/2-k+(r+1/2)N}$. That this is still a vacuum preserving isomorphism can be again verified by noting that the renumbering of generators does not affect the vacuum, or by direct computation of the two-point functions, along the same lines as before.

4.7.2 *The Ramond sector*

The real free Fermi field possesses another faithful representation of positive energy, the Ramond sector, as we have seen in 4.1, induced by the GNS construction from the two-point function (4.1.1). Fields evaluated in the Ramond sector will be expressed as $\pi_R(\psi(z))$. Obviously, as previously stated, their representation on the circle in terms of Fourier modes has a cut at $z = -1$ and extends anti-periodically on the whole S^1:

$$\pi_R(\psi(z)) = \sum_{n \in \mathbb{Z}} \psi_{R,n}\, z^{-n-1/2}.$$

In principle one could just introduce a new field obtained by multiplying the actual Ramond field by $\sqrt{z}$, in order to "cancel" the cut: $\pi_R(\psi(z)) \mapsto \sqrt{z} \cdot \pi_R(\psi(z))$ and so

$$\pi_R(\psi(z)) = \sum_{n \in \mathbb{Z}} \psi_{R,n}\, z^{-n}$$

with two-point function

$$\omega_R(\pi_R(\psi(z))\pi_R(\psi(w))) = \frac{1}{2} \cdot \frac{z+w}{z-w}.$$

Indeed, if we do so, the transformation law for the new defined conformal field under diffeomorphisms $z \mapsto \gamma(z)$ changes by an extra factor of $\sqrt{z/\gamma(z)}$. Commutation relations between modes have the form $\{\psi_{R,n}, \psi_{R,m}\} = \delta_{n+m,0}$. In particular, the zero mode squares to one: $2\psi_{R,0}^2 = \mathbf{1}$.

Proposition: A similar isomorphism like in (4.7.1) can be introduced as

$$\beta_R \colon \pi_0(\mathcal{A}(I)) \otimes^t \pi_R(\mathcal{A}(I)) \to \pi_R\left(\mathcal{A}\left(\sqrt{I}\right)\right) \tag{4.7.5}$$

taking the tensor product of fields in the vacuum and in the Ramond sector and defining

$$\pi_R(\psi(z^2)) \otimes^t \mathbf{1}_0 \mapsto \frac{1}{2}\Big(\pi_R(\psi(z)) + \pi_R(\psi(-z))\Big) \tag{4.7.6}$$

$$\mathbf{1}_R \otimes^t \pi_0(\psi(z^2)) \mapsto \frac{1}{2z}\Big(\pi_R(\psi(z)) - \pi_R(\psi(-z))\Big). \tag{4.7.7}$$

This isomorphism still preserves the vacuum state in the form $\omega_R \circ \beta_R = \omega_R \otimes \omega_0$.

Proof. We look again at the Fourier modes and the right hand sides present a relabelling $\psi_{R,n} \mapsto \psi_{R,2n}$ and $\psi_{R,n} \mapsto \psi_{R,2n+1}$, respectively; this ensures that the correct commutation relations and two-point function directly follow. □

4.7.3 *The non-symmetric case*

We turn now the attention to the vacuum representation in the non-symmetric case and try to follow the same arguments as before, in order to figure out whether an analogous map, playing the role of β, can be derived for non-symmetric intervals. As we said, we take E_N to be any union of N disjoint intervals on the real line (likewise on the circle via Cayley map); thus we are looking for

$$\beta' \colon \mathcal{A}^N(I) \to \mathcal{A}(E_N) \tag{4.7.8}$$

that still preserves the vacuum state $\omega_0^{(N)} \circ \beta = \omega_0 \otimes \cdots \otimes \omega_0$.

For $N = 2$ every non-symmetric two-interval can be obtained by applying a Möbius transformation μ on a symmetric one. Therefore the generic β_2' is nothing but $\mu \circ \beta_2$ and the vacuum preservation is ensured by the fact that any Möbius transformation preserves itself the vacuum state. This result in no more true for $N \neq 2$ and in the general case we need arguments coming from modular theory to help our constructions.

In order to introduce such a map we briefly recall a general result: let the intervals be given by $I_k =]a_k, b_k[\subset \mathbb{R}$ and define the function $X(x)$, $x \in \mathbb{R}$ by

$$X(x) = -\prod_{k=1}^{N} \frac{x - a_k}{x - b_k}. \tag{4.7.9}$$

This function maps each interval monotonously onto $\mathbb{R}_+$, so that every $X \in \mathbb{R}_+$ happens to have exactly n pre-images (which we refer to as $X_1^{-1}(X), \dots, X_n^{-1}(X)$), one in each interval, $X_j^{-1}(X) \in I_j$. We borrow the formulae appearing in [Casini and Huerta, 2009] for reasons that will become clear later. In particular, we look at the form of the modular automorphisms group for Fermi fields localised in disjoint intervals, equation (4.5.4)

$$\sqrt{X_k^{-1}(X)'}\,\sigma_t\left(\psi(X_k^{-1})\right) = \sum_{j=1}^{N} O(X,t)_{kj}\sqrt{X_k^{-1}(\delta_t(X))'}\,\psi\left((X_j^{-1}(\delta_t X))\right).$$

Here fields are evaluated on the real line as functions of the variable $X \in \mathbb{R}_+$. We thus have a collection of N fields: $\psi_1(X), \dots, \psi_N(X)$. The map β' is explicitly given by

$$\beta'(\psi_i(X)) = \sum_{r=1}^{N} O(X)_{ir}\sqrt{X_r^{-1}(X)'}\,\psi(X_r^{-1}(X)) \tag{4.7.10}$$

where the mixing matrix $O(X)$ is the solution of (4.5.5) given in terms of the anti-path ordered exponential

$$O(X) = \overline{P}\exp\left(-\frac{1}{2\pi}\int_{X_0}^{X} dX'\,K(X')\right) \tag{4.7.11}$$

the matrix $K(X)$ being

$$K(X)_{kj} = \begin{cases} 2\pi\,\dfrac{\sqrt{X_k^{-1}(X)'\,X_j^{-1}(X)'}}{X_k^{-1}(X) - X_j^{-1}(X)} & \text{if } k \neq j \\ 0 \quad \text{otherwise} \end{cases}.$$

The idea is that all the information is encoded into the special form of the function (4.7.9) and the dependence of $O(X)$ on $K(X)$ as in (4.7.11). In order to prove that the above β' preserves the vacuum state we show the equality of the two-point functions proving that they satisfy the same differential equation with common initial conditions. Equality of the two-point function means

$$(\omega_0 \circ \beta')(\psi_i(X)\,\psi_j(Y)) = \omega_0^{(2)}(\psi_i(X)\,\psi_j(Y)) = \frac{-i}{X-Y}\,\delta_{ij}$$

Expanding the left hand side we are led to

$$\begin{aligned}(\omega_0 \circ \beta')(\psi_i(X)\,\psi_j(Y)) &= \omega_0\left(\sum_{r=1}^{N} O(X)_{ir}\sqrt{X_r^{-1}(X)'}\psi(X_r^{-1}(X))\right.\\ &\qquad\left.\sum_{s=1}^{N} O(Y)_{js}\sqrt{Y_s^{-1}(Y)'}\psi(Y_s^{-1}(Y))\right)\\ &= \sum_{r,s=1}^{N} O(X)_{ir}O(Y)_{js}\sqrt{X_r^{-1}(X)'}\sqrt{Y_s^{-1}(Y)'}\\ &\qquad \omega_0(\psi(X_r^{-1}(X))\psi(Y_s^{-1}(Y)))\\ &\stackrel{!}{=} \omega_0^{(2)}(\psi_i(X)\,\psi_j(Y)) = \frac{-i}{X-Y}\,\delta_{ij}\,.\end{aligned}$$

Multiplying both sides by $X-Y$ and taking the derivative with respect to X gives

$$\sum_{r,s=1}^{N} O(X)_{ir}O(Y)_{js}\sqrt{Y_s^{-1}(Y)'}\left(\sum_{p\neq r}(X-Y)K(X)_{rp}\sqrt{X_r^{-1}(X)'}\omega_0(p,s)\right.$$
$$\left.+\frac{\partial}{\partial X}\left((X-Y)\sqrt{X_r^{-1}(X)'}\omega_0(r,s)\right)\right)=0$$

as a consequence, for each r,s it must be proven that [1]

$$\sum_{p\neq r}(X-Y)K(X)_{rp}\sqrt{X_r^{-1}(X)'}\omega_0(p,s)$$
$$+\frac{\partial}{\partial X}\left((X-Y)\sqrt{X_r^{-1}(X)'}\omega_0(r,s)\right)=0.$$

The explicit form of $K(X)$ and the derivatives help us to obtain the easier to handle equation

$$\frac{1}{2}\frac{(X_r^{-1})''}{(X_r^{-1})'}-\sum_{p\neq r}\frac{(X_p^{-1})'}{X_r^{-1}-X_p^{-1}}=\sum_{p=1}^{n}\frac{(X_p^{-1})'}{X_p^{-1}-Y_s^{-1}}-\frac{1}{X-Y}\tag{4.7.12}$$

It is now fundamental that the dependence $X_p^{-1}\equiv X_p^{-1}(X)$ is given by the function (4.7.9); this is because the inverse roots X_p^{-1} appear as roots of the polynomial $\mathcal{P}_X(x)=X\prod_{k=1}^{N}(x-b_k)+\prod_{k=1}^{N}(x-a_k)=0$. After the decomposition in terms of its solutions we obtain the useful identity

$$(X+1)\prod_{k=1}^{N}(x-X_k^{-1}(X))=\mathcal{P}_X(x)=X\prod_{k=1}^{N}(x-b_k)+\prod_{k=1}^{N}(x-a_k)$$

which in turn becomes, after factorising out $\prod_{k=1}^{N}(x-b_k)$ on the RHS and evaluating it in $x=Y$

$$\prod_{k=1}^{N}(Y^{-1}-X_k^{-1}(X))=\prod_{k=1}^{N}(Y^{-1}-b_k)\left(\frac{X-Y}{X+1}\right)\tag{4.7.13}$$

where Y^{-1} is any of the inverse roots of $\mathcal{P}(Y)$. This equation is the starting point to obtain both sides of (4.7.12) as follows: taking the derivative with respect to X of the logarithm of (4.7.13) gives back

$$\sum_{k=1}^{N}\frac{(X_k^{-1})'}{X_k^{-1}-Y^{-1}}=\frac{1}{X-Y}-\frac{1}{X+1}$$

therefore the right hand side of (4.7.12) turns out to be nothing but $-1/(X+1)$. Obtaining the right hand side is slightly trickier and we proceed as follows: we start again from (4.7.13) taking the logarithm

1 In the following line $\omega_0(p,s)$ stands for $\omega_0(\psi(X_p^{-1}(X))\psi(Y_s^{-1}(Y)))$.

and derivative with respect to X and then we multiply both sides by $(X-Y)(X_j^{-1}-Y^{-1})$. This gives

$$(X-Y)(X_r^{-1})' + \sum_{p\neq r} \frac{(X_p^{-1})'(X_r^{-1}-Y^{-1})(X-Y)}{X_p^{-1}-Y^{-1}} = \\ X_r^{-1}-Y^{-1} - \frac{(X_r^{-1}-Y^{-1})(X-Y)}{X+1}$$

from which we take again the derivative with respect to the variable X twice and eventually evaluate it in the point $Y^{-1} = X_r^{-1}$ (and therefore $Y = X$). In the summation term we make use of $(ND^{-1})'' = N''D^{-1}$ whenever both N and N' vanish in the limit (which is the case at hand). Carrying the algebra out gives

$$2(X_r^{-1})'' - 2\sum_{p\neq j} \frac{(X_p^{-1})'(X_r^{-1})'}{X_r^{-1}-X_p^{-1}} = (X_r^{-1})'' - \frac{(X_r^{-1})'}{X+1}$$

namely nothing but (4.7.12). Now we are left with the common initial conditions to be proven (we choose $X = Y$): once again, equality of the two-point function $\omega_0 \circ \beta' = \omega_0^{(2)}$ reads

$$\sum_{r,s=1}^{N} O(X)_{ir}O(Y)_{js}\sqrt{X_r^{-1}(X)'}\sqrt{Y_s^{-1}(Y)'}\,\frac{-i}{X_r^{-1}-Y_s^{-1}} = \frac{-i}{X-Y}\,\delta_{ij}.$$

Multiplying both sides by $X-Y$ and splitting the sum into $r = s$ and $r \neq s$ leads us to

$$\lim_{X\to Y}\sum_{r=1}^{N}\left(\sum_{r\neq s} O(X)_{ir}O(Y)_{js}\sqrt{X_r^{-1}(X)'}\sqrt{Y_s^{-1}(Y)'}\frac{X-Y}{X_r^{-1}-Y_s^{-1}} + \right. \\ \left. O(X)_{ir}O(Y)_{jr}\sqrt{X_r^{-1}(X)'}\sqrt{Y_r^{-1}(Y)'}\frac{X-Y}{X_r^{-1}-Y_r^{-1}}\right) = \delta_{ij}.$$

The first term vanishes in the limit, whereas the divergence in the second term gives back $1/(Y_r^{-1})'$ which cancels the square roots. Orthogonality of the matrix $O(Y)$ ensures then the result.

The symmetric case on the circle can be recovered as a special case of the general one. In fact the evaluation of $X(z)$ turns out to be the composition of the inverse Cayley transform with a Möbius transformation onto $\mathbb{R}_+$. Using trigonometric identities like $\prod_{k=1}^{N}(z-\omega_k) = z^N - \omega^N$ we find

$$X(z) = -\frac{(-1)^N - \nu^N}{(-1)^N - u^N}\cdot\frac{z^N-u^N}{z^N-\nu^N}$$

which is indeed a Möbius transform of $\frac{z^N-1}{i(z^N+1)} = C^{-1}(z^N)$. Therefore

$$X(x) = (C^{-1}\circ\mu\circ(z\mapsto z^N)\circ C)(x)$$

where $\mu\colon I \to S^1_+$ is the Möbius transform of the said form. In the general non-symmetric case the map $z \mapsto z^N$ is replaced by any general N-folded map g

$$X(x) = (C^{-1}\circ\mu\circ g\circ C)(x)$$

and μ here is arbitrary. The passage from the circle to the real line, in the explicit formulae of β', is then achieved with the help of suitable Möbius transformations, which do not affect the vacuum state. Consequently the invariance of the vacuum two-point function on the real line ensures the same statement on the circle, and viceversa.

The existence of an isomorphism $\mathcal{A}^N(I) \to \mathcal{A}(E_N)$ preserving the vacuum state implies that the corresponding GNS vacuum representations are isomorphic as well. This, in turn, produces a homomorphism of many copies of the local algebra of a single theory into the algebra of a single fermion localised in many intervals, represented on the Fock space.

4.7.4 *Multi-local fermionisation and gauge transformations*

The multi-local isomorphism β provides a correspondence between fermions localised essentially at our will. As we have seen, Fermi theories sort of automatically contain (non)-abelian current algebras, due to the fact that these can be embedded using the standard quarks construction from sufficiently many free fermions.

This feature, in its general behaviour, is referred to in the literature as "fermionisation", since it allows to express bosons (the currents) in terms of products of Fermi fields. In particular currents are expressed as Wick products (equation (2.3.1)) with subtraction of the vacuum expectation value; since β preserves the vacuum state it extends to Wick products and therefore embeds the currents giving rise to a new feature which is the delocalisation of the components of the current itself. We shall be more precise showing the construction of such objects on the circle, because most of the formulae drastically simplify.

Let us start from the vacuum representation and take the case $N = 2$; we look in particular at symmetric intervals, for the sake of simplicity, though the same construction can be easily generalised with different coefficients eventually. In terms of the complex fermion the current is expressed as

$$j(z) \coloneqq :\phi^*\phi:(z) = i:\psi_1\psi_2:(z); \tag{4.7.14}$$

we are going to embed such formula with the help of (4.7.1). As so, we have (we evaluate on z^2 due to β)

$$\begin{aligned}
\beta(j(z^2)) &= \beta\left(:\phi^*\phi:(z^2)\right) \\
&= \beta\left(\phi^*(z^2)\phi(z^2) - \omega_0(\phi^*(z^2)\phi(z^2))\right) \\
&= \beta(\phi^*(z^2))\beta(\phi(z^2)) - \omega_0 \circ \beta(\phi^*(z^2)\phi(z^2)) \\
&= :\beta(\phi^*)\beta(\phi):(z^2).
\end{aligned}$$

The feature of β to preserve the vacuum state means that it can be taken into the Wick product due to commutativity $\beta \circ {:}(\,\cdot\,){:} = {:}(\,\cdot\,){:} \circ \beta$. Consequently we have

$$\begin{aligned}\beta(j(z^2)) &= {:}\frac{1}{2z}\left(\psi(z) - \psi(-z)\right)\frac{1}{2}\left(\psi(z) + \psi(-z)\right){:}\\ &= \frac{1}{4z}{:}\psi(z)^2 - \psi(-z)\psi(z) + \psi(z)\psi(-z) - \psi(-z)^2{:}\end{aligned}$$

Fermions anti-commute and so $\psi(z)^2 = \psi(-z)^2 = 0$; moreover we have $-\psi(-z)\psi(z) = \psi(z)\psi(-z)$. Making use of such relations we come to the *multi-local fermionisation* formula on the circle:

$$\beta(j(z^2)) = \frac{1}{2z}{:}\psi(z)\psi(-z){:} \tag{4.7.15}$$

The same expression, evaluated on the real line, looks like:

$$\beta(j(q(x)) = \frac{-i}{2x}\cdot\frac{(1-x^2)^2}{(1+x^2)}{:}\psi(x)\psi(-1/x){:}\,.$$

The current embedded with the isomorphism β happens to be delocalised in two anti-podal points on the circle, z and $-z$. This feature justifies the term "multi-local" fermionisation (also in [Rehren and Tedesco, 2013]) here and henceforth, because the fermionisation is shared between pairs of different points. We notice that this new representation of the current is periodic on the circle under the change $z \to -z$; in fact, both numerator (due to the anti-commutators) and denominator acquire a minus sign, cancelling each other altogether. Furthermore, since the expectation value of a Wick product is always zero, we still have $\omega_0 \circ \beta(j(z^2)) = 0$. On the other hand the two-point function is

$$\begin{aligned}\omega_0 \circ \beta(j(z^2)j(w^2)) &= \frac{1}{4zw}\,\omega_0({:}\psi(z)\psi(-z){:}\,{:}\psi(w)\psi(-w){:})\\ &= \frac{1}{4zw}\Big(\omega_0(\psi(z)\psi(-z)\,\psi(w)\psi(-w))\\ &\quad - \omega_0(\psi(z)\psi(-z)\,\psi(w)\psi(-w))\Big)\\ &= \frac{1}{4zw}\Big(- \omega_0(\psi(z)\psi(w))\cdot\omega_0(\psi(-z)\psi(-w))\\ &\quad + \omega_0(\psi(z)\psi(-w))\cdot\omega_0(\psi(-z)\psi(w))\Big)\\ &= \frac{1}{4zw}\cdot\frac{4zw}{(z^2-w^2)^2} = \omega_0(j(z^2)j(w^2))\end{aligned}$$

that is, the two-point function is preserved too.

The embedded current can be decomposed into Fourier modes on the circle

$$\beta(j(z^2)) = \frac{1}{2z}\sum_{m,k\in\mathbb{Z}+1/2}^{N}{:}\psi_m\psi_k{:}(-1)^{-k-1/2}z^{-m-k-1}$$

if $m+k=n$ is odd then $(-1)^{k-n-1/2} = (-1)^{-k-1/2}$ and the sum vanishes because of the anti-commutation relations: each contribution has its own opposite. Therefore the only allowed powers of $m+k$ are even powers of the form $m+k=2p$, which lead us to

$$\beta(j_n) = \sum_{m=0}^{\infty} \psi_{n-m-1/2}\,\psi_{n+m+1/2}(-1)^{n+m+1}.$$

The complex fermion is invariant under gauge transformations generated by its own embedded currents: if $j(z) = {:}\varphi^*\varphi{:}(z)$ then Weyl-type operators are implemented by smearing with suitable test functions $f\colon S^1 \to \mathbb{R}$ in order to obtain $W(f) = e^{ij(f)}$. Gauge transformations are then given by

$$\begin{aligned}\varphi'(z) &= \alpha_f(\varphi(z)) = W(f)\varphi(z)W(f)^* = e^{-if(z)}\,\varphi(z)\\ \varphi^{*\prime}(z) &= \alpha_f(\varphi(z)^*) = W(f)\varphi(z)^*W(f)^* = e^{if(z)}\,\varphi(z).\end{aligned}$$

It is now very interesting to embed the gauge transformations themselves with the help of β, in order to bring new delocalised gauge symmetries, a brand new feature that we are going to present and fully exploit. If β embedds the current then it generates embedded gauge transformations on the free fermion of the form

$$\begin{aligned}\beta(W(f))\psi(z)\beta(W(f))^* &= (\beta\circ\alpha_f\circ\beta^{-1})\psi(z)\\ &= (\beta\circ\alpha_f)(\varphi(z^2) + z\,\varphi(z^2)^*)\\ &= \beta(\alpha_f(\varphi(z^2)) + z\,\alpha_f(\varphi(z^2)^*))\\ &= e^{-if(z^2)}\,\beta(\varphi(z^2)) + z\,e^{if(z^2)}\,\beta(\varphi(z^2)^*)\end{aligned}$$

eventually we end up with

$$\beta(W(f))\psi(z)\beta(W(f))^* = \cos f(z^2)\psi(z) - i\sin f(z^2)\psi(-z). \quad (4.7.16)$$

The new remarkable feature is the bilocal mixing of $\psi(z)$ and $\psi(-z)$, reflecting the non-locality of the isomorphism β. Of course the same calculations can be performed in the situation $N > 2$: in this case non-abelian currents of the form $j_{rs}(z) := {:}\varphi_r^*\varphi_s{:}(z)$ can be embedded and we obtain representations of all these theories in the Fock space of a single real free fermion. In the many interval case β delocalises the fields onto $2N$ points (equations (4.7.4) and, in general, (4.7.10)) and therefore expressions like ${:}\varphi_r^*\varphi_s{:}(z)$ present sums of fields in pairwise coupled points $\psi(z_j)\psi(z_l)$ with position dependent coefficients. Gauge transformations change accordingly, having multi-local contributions from different points.

The same argument can be run in the Ramond sector: the fact that the current $j(z) := {:}\varphi^*\varphi{:}(z) = i{:}\psi_1\psi_2{:}(z)$ satisfies commutation relations in purely algebraic and independent of the representation. As a consequence one can then take the two fields ψ_1 and ψ_2 in two different representations and *twist* the product. In fact, by taking ψ_1 in the vacuum sector and ψ_2 in the Ramond one, the isomorphism (4.7.5) embeds the resulting current into the Ramond algebra $\pi_R\left(\mathcal{A}\left(\sqrt{I}\right)\right)$.

The Wick product here is defined as the subtraction with respect to the corresponding Ramond two-point function; the result is, in the compact picture:

$$\beta_R(j(z^2)) = \frac{1}{2iz^2} \cdot {:}\pi_R(\psi(z))\pi_R(\psi(-z)){:}_R.$$

The new current, as expected, is delocalised at two anti-podal points in the Ramond representation; the Wick product is essentially the standard operator product, because the Ramond expectation value vanishes at the case at hand in the points $z, -z$. Also, the formula changes sign under $z \to -z$, i. e. it is anti-periodic in the variable z^2, expressing the fact that the new current is twisted. Of course, the one-point function is still zero, whereas the two-point function becomes now:

$$\begin{aligned}\omega_R \circ \beta_R(j(z^2)j(w^2)) &= -\frac{1}{4z^2w^2}\,\omega_R({:}\pi_R(\psi(z))\pi_R(\psi(-z)){:}\\ &\qquad {:}\pi_R(\psi(w))\pi_R(\psi(-w)){:})\\ &= \frac{1}{2zw} \cdot \frac{z^2+w^2}{(z^2-w^2)^2}\end{aligned}$$

after making use of the usual Wick contractions within the expectation value. This formula has been previously mentioned by [Anguelova, 2011] as the “twisted” representation of the current, in the context of vertex operator algebras (we refer the reader to the references therein).

4.7.5 *Multi-local diffeomorphisms*

In the previous paragraph we showed the construction of the multi-local current and the resulting multi-local fermionisation and gauge transformations. It is natural to extend the investigation to the stress-energy tensor of such a theory and look for the corresponding multi-local diffeomorphisms that are generated. Again, the special case $N = 2$ for symmetric intervals is a guideline, because formulae simplify and this allows to better understand the features and the behaviours without getting lost in the nasty coefficient for the general case. Fundamental is again the characteristic of β to preserve the vacuum state in order to be extended to Wick products: $\beta \circ {:}(\,\cdot\,){:} = {:}(\,\cdot\,){:} \circ \beta$.

We start in the vacuum representation, as usual. The real free fermion contains the stress-energy tensor of central charge $c = 1/2$

$$T^{1/2}(z) := \frac{-1}{4\pi}\,{:}\psi\partial_z\psi{:}(z)$$

whereas the complex fermion is, roughly speaking, two copies thereof, with $c = 1$

$$T^{c=1}(z) := \frac{-1}{4\pi}\,{:}\psi_1\partial_z\psi_1{:}(z) + \frac{-1}{4\pi}\,{:}\psi_2\partial_z\psi_2{:}(z) = \frac{-1}{4\pi}\,{:}\phi^*\overleftrightarrow{\partial}_z\phi{:}(z).$$

In terms of the currents, the stress-energy tensor can be expressed as

$$T(z) = \frac{1}{4\pi}{:}j^2{:}(z),$$

that is nothing but the abelian version of (2.3.4).

The action of β brings to the embedded stress-energy tensor which we compute to be

$$\begin{aligned}
\beta\big(T^1(z^2)\big) &= \frac{-1}{4\pi}\,\beta\big(:\phi^*\overleftrightarrow{\partial}_z\phi:(z)\big) \\
&= \frac{-1}{4\pi}:\beta(\phi(z^2)^*)\partial_{z^2}\beta(\phi(z^2)) - \partial_{z^2}(\beta(\phi(z^2)^*))\beta(\phi(z^2)): \\
&= \underbrace{\frac{-1}{4\pi}\,\frac{1}{2z^3}:(\psi(z)-\psi(-z))(\psi(z)-\psi(-z)):}_{\sim\beta(j(z^2))} \\
&+ \frac{-1}{4\pi}\cdot\frac{1}{4}\cdot\frac{1}{2z^2}:\Big((\psi(z)-\psi(-z))\partial_z(\psi(z)+\psi(-z)) - \\
&(\partial_z(\psi(z)-\psi(-z)))(\psi(z)+\psi(-z))\Big): \\
&= -\frac{1}{8\pi z^2}\beta(j(z^2)) + \frac{1}{4z^2}\big(T^{1/2}(z)+T^{1/2}(-z)\big),
\end{aligned}$$

expressed as embedding of two real fermions of central charge $c = 1/2$. The remarkable feature is the presence of two delocalised stress-energy tensors in $z, -z$ plus an additional contribution proportional to the embedded current. We shall see later that this further term can be cancelled out by composition with a particular automorphism of the current algebra, though.

The real and complex fermions are invariant under diffeomorphisms generated by its own stress-energy tensor. If $f\colon S^1 \to S^1$ is a general diffeomorphism, then the unitary operators $V(\gamma_t) = e^{itT(f)}$ implement its action as

$$\psi'(\gamma(z)) = \delta_\gamma(\psi(z)) = V(\gamma)\psi(z)V(\gamma)^* = \sqrt{\gamma'(z)}\,\psi(\gamma(z))$$

where $if(z)/z \in \mathbb{R}$ integrates to diffeomorphisms given by the one-parameter group $\partial_t\gamma_t(z) = -(f\circ\gamma_t)(z)$. Also, $\gamma(z)$ is meant as $\gamma_t|_{t=1}(z)$. Simpler to write down is its infinitesimal action δ_f^0 expressed by the commutator

$$\delta_f^0(\psi(z)) = i\,[T(f),\psi(z)] = \Big(-f(z)\partial_z - \frac{1}{2}f'(z)\Big)\psi(z).$$

We can now make use of the equation for the embedded stress-energy tensor

$$\beta\big(T^1(z^2)\big) = -\frac{1}{8\pi z^2}\beta(j(z^2)) + \frac{1}{4z^2}\big(T^{1/2}(z)+T^{1/2}(-z)\big) \qquad (4.7.17)$$

in order to derive and calculate the corresponding multi-local diffeomorphisms. Similarly to the case of currents we have the action $i\left[\beta\big(T^1(f)\big),\psi(z)\right] = (\beta\circ\delta_f^0\circ\beta^{-1})\,\psi(z)$. The contribution due to the two anti-podal parts $T^{1/2}(z)$ and $T^{1/2}(-z)$ gives rise to a term proportional to $\delta_f^0(\psi(z^2))$, while the contribution proportional to the embedded current gives back two anti-podal terms in $\psi(z), \psi(-z)$; in details we obtain

$$\begin{aligned} i\left[\beta(T^1(f)), \psi(z)\right] &= (\beta \circ \delta_f^0 \circ \beta^{-1})\, \psi(z) \\ &= \Big(-\frac{1}{2z} f(z^2)\partial_z - \frac{1}{2} f'(z^2) \Big) \psi(z) \\ &+ \frac{1}{4z^2} f(z^2) \left(\psi(z) - \psi(-z) \right). \end{aligned}$$

Again, we have a mixing of $\psi(z)$ and $\psi(-z)$ (due to the current), on top of a geometric action due to the stress-energy tensor itself. Of course, in equation (4.7.17) everything is expressed in terms of the two real copies ψ_1, ψ_2 of the free fermion; nevertheless one can work it back in terms of the complex fermion $\phi(z), \phi(z)^*$: in this case, when acting with the inverse action β^{-1}, terms proportional to $:\phi^* \partial_z \phi:(z)$ will appear and therefore we will have eventually mixed pairings ϕ, ϕ^* expressing some sort of multi-local "charged" conjugation.

As previously mentioned (see chapter 2.3), the current algebra possesses automorphisms of the form

$$\rho^q(j(z)) = j(z) + \frac{q}{z}, \qquad q \in \mathbb{R}$$

which give rise to charged states $\omega_q := \omega_0 \circ \rho^q$. On the actual Weyl operators those automorphisms are realised as

$$\gamma_q(W(f)) = e^{i\rho^q(j(z))(f)} = e^{iq\int_{S^1} dz \frac{f(z)}{2\pi i z}}\, W(f),$$

giving rise to different inequivalent representations whenever one chooses $\gamma_{q_1}, \gamma_{q_2}$ with $q_1 \neq q_2$. Moreover, these automorphisms happen to be even innerly implemented by unitaries if a real function φ exists such that $q/z = -i\varphi'(z)$

$$\gamma_q(\,\cdot\,) = \mathrm{Ad}(W(-\phi))(\,\cdot\,)$$

and $\gamma_q(\mathcal{A}(I)) = \mathcal{A}(I)$. In fact the above equation can be taken as definition for each representation $\gamma_q(W(f))$ ([Carpi, 2004]). However, since the stress-energy tensor is contained as embedded into the theory of currents, it turns out that composition of β with such a ρ^q exactly "undoes" the additional contribution due to the current in (4.7.17) if we shift back $j(z) \mapsto j(z) + q/z$ for $q = 1/4$. The price to pay is the appearance of a constant shift $\sim z^{-4}$:

$$\beta \circ \rho^{1/4}(T^1(z^2)) = \frac{1}{4z^2}\big(T^{1/2}(z) + T^{1/2}(-z)\big) + \frac{1}{64\pi z^4}.$$

Although we have not introduced the issue yet, we want to emphasise that the constant term popping up is nothing but the Schwarz derivative of the square root automatically generated because the stress-energy tensor is a quasi-primary field. In fact the above formula will coincide with the Longo-Xu map applied to a doubled theory of stress-energy tensors, as we shall see later on in paraghraph 5.5.

Anyway, it is always very useful to rephrase the picture in terms of Fourier modes: as known, $T^1(z) = 1/2\pi \sum_{n\in\mathbb{Z}} L^1_n z^{-n-2}$; then the embedding looks like, in terms of Virasoro generators:

$$\beta(L^1_n) = -\frac{1}{4}\beta(j_n) + \frac{1}{2}L^{1/2}_{2n}$$

for the general case, and

$$\beta \circ \rho^{1/4}(L^1_n) = -\frac{1}{4}\beta(j_n) + \frac{1}{2}L^{1/2}_{2n} + \left(\frac{1}{4}\beta(j_n) + \frac{1}{32}\delta_{n,0}\right) = \frac{1}{2}L^{1/2}_{2n} + \frac{1}{32}\delta_{n,0}$$

for the subtracted $q = 1/4$ current. Here the subtraction of the current modes due to the composition with a charged automorphism is even more evident. In contrast, the first (general) formula involves also the modes of the current, still, which may in turn be expressed in terms of the real Fermi fields. This emphasises that the embedded diffeomorphisms come along with embedded gauge transformations, i. e. a mixing of $\psi(z)$ and $\psi(-z)$, as described before.

In the Ramond sector the stress-energy tensor presents an additional term by definition, as shown in equation (2.3.7). Consequently, the action of β_R produces different subtractions that cancel the gauge term $\beta(j(z^2))$ which we had to deal with in the vacuum sector. As such, the formula in the Ramond representation becomes easier even without composition with a charged automorphism of the currents:

$$\beta_R(T^1(z^2)) = \frac{1}{4z^2}\Big(\pi_R(T^{1/2}(z)) + \pi_R(T^{1/2}(-z))\Big) + \frac{1}{64\pi z^4}.$$

Finally, a last remark to conclude this section: the expression of the multi-local transformations in terms of Virasoro generator is very useful to understand a brand new picture, which will be pretty convenient when we shall turn to modular theory. The subgroup $L_0, L_{\pm 1}$ of the Virasoro algebra generates the Möbius group, which in turn contains rotations, dilations and translations. Its multi-local version, given by $\beta(L_0), \beta(L_{\pm 1})$ produces the corresponding multi-local rotations, dilations and translations. The passage between a single theory in one interval to a delocalised theory in many intervals can then by achieved making use of the same formulae, just taking care of replacing $L_n \mapsto \beta(L_n)$. The multi-local behaviour will then be taken into account by the presence of β, automatically, producing mixing among different components all the time.

4.8 MULTI-LOCAL MODULAR THEORY FOR FERMI FIELDS

The most remarkable feature of β is that it preservers the vacuum state, and therefore its action extends to Wick products and so forth, giving rise to multi-local transformations among different intervals. So far we have not introduced any correspondence with modular theory yet, but as we are going to see, very important applications can be derived implementing the Bisognano-Wichmann property under the action of β. In fact, the modular theory for one-dimensional fermions is well known and has been widely investigated, producing the famous result that the action of the modular group with respect to the vacuum state on the algebra localised in one interval is geometric within the interval and can be expressed in terms of dilations preserving the interval. The interesting idea is that, since β *moves* the theory from one interval to many, preserving the vacuum state, we can expect a rationale to describe the modular theory for Fermi fields localised in many disjoint intervals on the circle (respectively on the real line). In order to characterise the topic in detail we are going to introduce a very strong result, which is going to play a pretty fundamental role: whenever two states of two algebras $\mathcal{A}_1, \mathcal{A}_2$ are intertwined by an isomorphism, such isomorphism also intertwines the modular groups of the two algebras.

Theorem 4.8.1: If $\beta\colon \mathcal{A}^N(I) \to \mathcal{A}(E_N)$ is a vacuum preserving isomorphism, then it intertwines the respective modular groups: $\sigma_N^t = \beta \circ \sigma_1^t \circ \beta^{-1}$.

Proof. The proof proceeds by verifying the KMS property for σ_N^t with respect to the composed state $\omega_0 \circ \beta$. Therefore, let

$$F_{a,b}^N(t) \coloneqq F_{a,b}^{\omega_0\circ\beta}(t) = \big(\Omega, \beta^{-1}(a)\beta^{-1}(\sigma_N^t(b))\Omega\big)$$

be the KMS functional we have to look at, for $a, b \in \mathcal{A}(E_N)$. We have to prove that $F_{a,b}^N(t)$ admits analytic continuation in $t \mapsto t - i$ and in addition $F_{a,b}^N(t-i) = \big(\Omega, \beta^{-1}(\sigma_N^t(b))\beta^{-1}(a)\Omega\big)$.

$$\begin{aligned} F_{a,b}^N(t) &= \big(\Omega, \beta^{-1}(a)\beta^{-1}(\sigma_N^t(b))\Omega\big) \\ &= \big(\Omega, \beta^{-1}(a)\beta^{-1} \circ \beta \circ \sigma_1^t \circ \beta^{-1}(b)\Omega\big) \\ &= \big(\Omega, \beta^{-1}(a)\sigma_1^t(\beta^{-1}(b))\Omega\big) \end{aligned}$$

the analytic continuation in $t - i$ is ensured by the analytic properties of σ_1^t, which we know by hypothesis to be a true modular group. Therefore

$$\begin{aligned} F_{a,b}^N(t-i) &= \big(\Omega, \beta^{-1}(a)\sigma_1^{t-i}(\beta^{-1}(b))\Omega\big) \\ &= \big(\Omega, \sigma_1^t(\beta^{-1}(b))\beta^{-1}(a)\Omega\big) \\ &= \big(\Omega, \beta^{-1} \circ \sigma_N^t \circ \beta(\beta^{-1}(b))\beta^{-1}(a)\Omega\big) \\ &= \big(\Omega, \beta^{-1}(\sigma_N^t(b))\beta^{-1}(a)\Omega\big) \end{aligned}$$

as to be proven. □

Let us apply this result right away to the modular theory of fermions localised in disjoint intervals. We assume to have a generic N-interval on the real line, say E_N, and one Fermi field in any of those points, which we refer to as $\psi(X_k^{-1})$. If such points X_k^{-1} are the inverse roots of the Casini-Huerta special function (4.7.9) then the action of the modular group of the delocalised fermion is simply

$$\sigma_N^t(\psi(X_k^{-1})) = (\beta \circ \sigma_1^t \circ \beta^{-1})(\psi(X_k^{-1})). \tag{4.8.1}$$

The explicit expression for β is given by (4.7.10)

$$\beta(\psi_i(X)) = \sum_{r=1}^{n} O(X)_{ir} \sqrt{X_r^{-1}(X)'}\, \psi(X_r^{-1}(X)).$$

Since $O(X)$ is an orthogonal matrix, its inverse is simply given by its transpose $O(X)^T$ and thus

$$\sqrt{X_k^{-1}(X)'}\,\sigma_t\left(\psi(X_k^{-1}(X))\right) = \\ \sum_{j=1}^{N} O(X,t)_{kj} \sqrt{X_k^{-1}(\delta_t(X))}'\psi\left((X_j^{-1}(\delta_t X))\right)$$

reproducing exactly the well-known result found by Casini and Huerta. Of course, this gives an explanation for the fact that the geometric part of the modular flow is the same as in the one interval case

$$\zeta_t(X_k^{-1}(X)) = \delta_t(X_k^{-1}(X)) = X_k^{-1}(\delta_t(X)) \tag{4.8.2}$$

with δ_t being the one parameter subgroup of Möbius transformations preserving the interval. This is because the modular action is taken into account by σ_1^t anyway, and β just introduces a mixing among different intervals, with the only role to mix the coefficients appearing in the formulae; therefore the geometric action is similar to the initial one (up to an action of X_K^{-1}, as showed in (4.8.2)), on top of a mixing traced back to the existence of a multi-local isomorphism.

Example: In the special case of symmetric intervals on the circle, the geometric action of the modular flow acquires the form $\sqrt[N]{\delta_t(z^N)} = \delta_t^{(N)}(z)$, which in turn coincides with the N-dilations as elements of $PSU(1,1)^{(n)}$, as introduced in 2.2.1 (see also [Longo et al., 2009]).

Property: Just by manifestly looking at (4.5.4) one can factor out suitable linear combinations of fields which diagonalise the modular mixing, also due to the special form of the matrix $O(X)$ as coboundary. In particular, by construction, the following combination

$$D_k(X) := \sum_{j=1}^{N} O(X)_{kj} \sqrt{X_j^{-1}(X)'}\, \psi(X_j^{-1}(X))$$

diagonalises the modular mixing in disjoint intervals

$$\sigma_N^t(D_k(X)) = e^{-\pi t}\, D_k(\delta_{-2\pi t}(X)).$$

Proof. Work the left hand side out making use of $O(X,t) = O(X)^T \cdot O(\delta_t X)$. □

Indeed, this is no accident and no surprise. The original ideas came from the investigation of the formula appearing in the paper by Casini and Huerta, [Casini and Huerta, 2009]; although in the beginning it may seem surprising, if, for whatever reason, the modular group for fields in disjoint intervals is assigned in terms of a mixing matrix $O(X)$, then a vacuum preserving isomorphism *must* exist, whose coefficients are exactly given by the matrix $O(X)$. Consequently, an expression like (4.7.10) must be read off and must preserve the vacuum state, justifying its peculiar form in terms of a very special uniformisation function (4.7.9). The further investigations have been proofs a posteriori of a result that must hold true at the algebraic level, as a feature of the algebras itself, as we are going to show. In fact, although in the general case the mixing cannot be computed explicitly but it is only determined as the solution to some differential equations, we can anyway make use of a remarkable result by [Takesaki, 2002] to derive this property only using the fact that β intertwines the modular groups.

Theorem 4.8.2*:* Let ω, ξ be a pair of faithful semi-finite normal weights on a von Neumann algebra $\mathcal{M}$ and let $\mathbf{D}_{1/2}$ be the horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R} - i/2$. Assume the cocycle derivative u_t can be extended to a member of $\mathcal{A}_{\mathcal{M}}(\mathbf{D}_{1/2})$ such that $\exists\, M > 0 \mid \|u_{-i/2}\| \leqslant M$, where $\mathcal{A}_{\mathcal{M}}(\mathbf{D}_{1/2})$ means the set of all bounded functions (with values in the algebra) on $\overline{\mathbf{D}}_{1/2}$ which are holomorphic in $\mathbf{D}_{1/2}$; then:

$$\omega(m) = \xi\left(u^*_{-i/2} m u_{-i/2}\right), \quad m \in \mathcal{M}.$$

Let henceforth $\mathcal{M}$ be a factor: by means of the previous theorem we are now able to prove the following

Corollary 4.8.3*:* Let ω, ξ be two faithful normal states on $\mathcal{M}$ such that $\sigma^t_\omega(m) = \sigma^t_\xi(m)$, $\forall\, m \in \mathcal{M}$. Then $\omega = \xi$.

Proof: The cocycle derivative $u_t \in \mathcal{M}$ commutes with the images of the modular flows, because by hypothesis

$$\sigma^t_\omega(m) = u_t \sigma^t_\xi(m) u^*_t = \sigma^t_\xi(m)$$

and viceversa, therefore $[\sigma^t_{\omega,\xi}(m), u_t] = 0$. By virtue of Tomita-Takesaki theorem $\sigma^t_\omega(a)$ exhausts all the algebra, hence $u_t \in Z(\mathcal{M}) = \mathbb{C}\mathbb{1}$, because $\mathcal{M}$ is a factor. By the cocycle property u_t is a continuous one parameter group, and being unitary implies $u_t = e^{it\alpha} \in S^1$. This allows the assumptions of the previous theorem to hold and consequently we obtain

$$\omega(m) = e^{\alpha}\, \xi(m), \quad m \in \mathcal{M}$$

but since the states are normalised we conclude that they have to coincide because $\alpha = 0$. □

Corollary 4.8.4*:* Let ω be a state on $\mathcal{M}$ and $\alpha \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma^t_\omega = \alpha^{-1} \circ \sigma^t_\omega \circ \alpha$. Then α preserves ω, i. e. $\omega \circ \alpha = \omega$.

Proof: $\alpha^{-1} \circ \sigma_\omega \circ \alpha$ satisfies the KMS condition on $\omega \circ \alpha$, therefore it coincides with its modular group, which in turns happens to be σ_ω by hypothesis. By using the previous corollary we then conclude that $\sigma^t_{\omega\circ\alpha} = \sigma^t_\omega$ implies $\omega \circ \alpha = \omega$. □

Corollary 4.8.5*:* Let $\mu \colon \mathcal{M}_1 \to \mathcal{M}_2$ be an isomorphism of von Neumann algebras. Let furthermore ω_1, ω_2 be states on $\mathcal{M}_1$ and $\mathcal{M}_2$ whose modular groups are σ_1, σ_2 respectively. The condition $\sigma^t_2 = \mu \circ \sigma^t_1 \circ \mu^{-1}$ implies $\omega_2 \circ \mu = \omega_1$ and viceversa.

Proof: By KMS condition the modular group for $\omega_2 \circ \mu$ is

$$\begin{aligned} \sigma^t_{\omega_2\circ\mu} &= \mu^{-1} \circ \sigma^t_2 \circ \mu \\ &= \mu^{-1} \circ \mu \circ \sigma^t_1 \circ \mu^{-1} \circ \mu = \sigma^t_1. \end{aligned}$$

By 4.8.3 the statement directly follows. □

In the context of Fermi conformal field theory $\beta' \colon \mathcal{A}^N(I) \to \mathcal{A}(E_N)$ plays the role of μ; σ^t_1 and σ^t_N are the modular groups with respect to the respective vacuum states ω_0 on $\mathcal{A}(E_N)$ and $\omega_0^{(N)}$ on $\mathcal{A}^N(I)$. By [Casini and Huerta, 2009] $\sigma^t_N = \beta \circ \sigma^t_1 \circ \beta^{-1}$ and therefore, by 4.8.5,

$$\omega_0^{(N)} \circ \beta = \omega_0 \otimes \cdots \otimes \omega_0.$$

We have provided an independent proof of the result given by Casini and Huerta, tracing back the modular properties to the existence of a vacuum preserving isomorphism at the level of the algebras. The passage from the real line to the circle picture can be easily achieved by noting once again that, due to the special form of $X(z)$ we have $X(z) = (\mu \circ C^{-1})(z)$, C^{-1} being the inverse Cayley map and $z \in I$. Since the modular flow on the circle is nothing but

$$\sigma_t^{S^1_+} = C \circ \sigma_t^{\mathbb{R}_+} \circ C^{-1}.$$

everything is pushed back to a Möbius transformation μ, which preserve the vacuum state and therefore intertwines the respective modular group, as already stated. Hence, a similar result holds on the circle. In the general non-symmetric case, though equations apply with no exceptions, the actual computations are most of the times very difficult to be performed.

The same intertwining property holds true for the Tomita conjugation J; let $a \in \mathcal{M}$ be a generic element in the von Neumann algebra and Ω be the corresponding cyclic and separating vector, then, it follows by modular theory that

$$\begin{aligned} \mathrm{Ad}\, J(a)\Omega &= \mathrm{Ad}(\Delta^{1/2} S)(a)\Omega \\ &= \mathrm{Ad}\, \Delta^{1/2}(a^*)\Omega \\ &= \mathrm{Ad}\, \Delta^{it}|_{t=-i/2}(a^*)\Omega \end{aligned}$$

and thus the action of J is reduced to the action of Δ^{it} on a^*. It is straightforward to conclude that if β preserves the vacuum state then

$$\mathrm{Ad}\, J_N = \beta \circ \mathrm{Ad}\, J_1 \circ \beta^{-1}$$

and by the previous computation

$$\mathrm{Ad}\, J_n(a) = \sigma_N^{-i/2}(a^*)$$

where now a can be taken to belong to any local algebra $\mathcal{A}(I)$. It is evident that, due to the presence of the modular group in many intervals σ_N, the action of J_N is still multi-local, as expected by consistency arguments.

4.8.1 *Modular theory in the Ramond sector*

One can as well reproduce the same lines in the Ramond sector, exploiting $\sigma_N^t = \beta \circ \sigma_1^t \circ \beta^{-1}$. The difference in this case is that the modular theory for Fermi fields in the Ramond representation is not known, not even in the one-interval case. We therefore lack the analogue of the Bisognano-Wichmann property and the analogue of equation (4.8.1) will contain unknowns on both sides; nevertheless we can point out a few interesting features.

Let us take the form of the vacuum preserving isomorphism as in equation (4.7.5); this map preserves the vacuum state $\omega_R \circ \beta_R = \omega_R \otimes \omega_0$ and therefore

$$\sigma_{2,R}^t = \beta_R \circ \left(\sigma_{1,R}^t \otimes \sigma_{1,0}^t\right) \circ \beta_R^{-1}. \tag{4.8.3}$$

The idea is to apply the above to the particular combination of fields $\mathbf{1}_R \otimes^t \pi_0(\psi(z^2))$, as appearing in the right hand side of the equations for the Ramond sector. Comparing with (4.7.5) $\beta_R\Big(\mathbf{1}_R \otimes^t \pi_0(\psi(z^2))\Big) = \lambda_R(z)$ is some linear combination of fields in the Ramond sector at the points $z, -z$. Applying (4.8.3) we obtain

$$\begin{aligned}
\sigma_{2,R}^t\left(\lambda_R(z)\right) &= \beta_R \circ \left(\sigma_{1,R}^t \otimes \sigma_{1,0}^t\right) \circ \beta_R^{-1}\left(\lambda_R(z)\right) \\
&= \beta_R \circ \left(\mathbf{1}_R \otimes \sigma_{1,0}^t(\psi(z^2))\right) \\
&= \beta_R \circ \left(\mathbf{1}_R \otimes \sqrt{\delta'_{-2\pi t}(z^2)}\, \pi_0(\psi(\delta_{-2\pi t}(z^2)))\right) \\
&= \sqrt{\delta'_{-2\pi t}(z^2)}\, \lambda_R\left(\sqrt{\delta_t(z^2)}\right) = \sqrt{\delta'_{-2\pi t}(z^2)}\, \lambda_R(\delta_t^{(2)}(z))
\end{aligned}$$

thus $\sigma_{2,R}^t$ acts on the linear combinations λ_R of $\pi_R(\psi(z))$ and $\pi_R(\psi(-z))$ geometrically, like the 2-dilations, because everything is traced back to the vacuum modular flow in one interval, where we can use the Bisognano-Wichmann statement. On the other hand little can be said on the other combination of fields $\pi_R(\psi(z^2)) \otimes^t \mathbf{1}_0$, in fact, posing $\beta_R\Big(\pi_R(\psi(z^2)) \otimes^t \mathbf{1}_0\Big) = \mu_R(z)$ we have

$$\begin{aligned}
\sigma_{2,R}^t\left(\mu_R(z)\right) &= \beta_R \circ \left(\sigma_{1,R}^t \otimes \sigma_{1,0}^t\right) \circ \beta_R^{-1}\left(\mu_R(z)\right) \\
&= \beta_R \circ \Big(\sigma_{1,R}^t(\pi_R(\psi(z^2))) \otimes \mathbf{1}_0\Big)
\end{aligned}$$

and, as we see, we are taken back to the investigation of the Ramond modular flow in one interval $\sigma^t_{1,R}(\pi_R(\psi(z^2)))$, which is still unknown. Yet, the question is whether this modular flow can be geometric inside each interval at all, as in the vacuum case. We are going to show that it cannot, due to an invariance argument.

Whatever the modular group is, it must preserve the state where it comes from, thus we must pose

$$\omega_R \circ \sigma^t_{1,R} = \omega_R.$$

We work in the real line picture and assume, as hypothesis, that the modular flow is linear in the fields and geometric within each interval, that is $\sigma^t_{1,R}(\pi_R(\psi(x)))$ is proportional to $\pi_R(\psi(f(x)))$, where $f\colon \mathbb{R} \to \mathbb{R}$ is any diffeomorphism. Notice that however the linear hypothesis is not totally justified, since, unlike the Neveu-Schwarz case, the Tomita operator does not preserve the particle number, since in a non-Fock representation such as Ramond, a particle number operator does not even exist. Invariance of the Ramond two-point function implies

$$\omega_R \circ \sigma^t_{1,R}(\pi_R(\psi(x))\pi_R(\psi(y))) = \omega_R(\pi_R(\psi(x))\pi_R(\psi(y)));$$

squaring the expression in (4.1.1) we obtain

$$\frac{f'(x)f'(y)}{(1+f(x)^2)(1+f(y)^2)} \cdot \frac{(1+f(x)f(y))^2}{(f(x)-f(y))^2} = \frac{(1+xy)^2}{(1+x^2)(1+y^2)(x-y)^2}.$$

The right hand side vanishes at $xy = -1$, hence by comparison with the left hand side $f(x)f(y) = -1$; evaluating in $y = -1/x$ we find that f must commute with the inversion map on the real line, namely $f(-1/x) = -1/f(x)$; taking derivatives we get $f'(-1/x) = (x^2/f(x)^2) \cdot f'(x)$. Let $f(1) = A$ and thus $f(-1) = -1/A$; also, $f'(1) = B$ and thus $f'(-1) = B/A^2$, using the relations above. Insert now first $y = 1$ and then $y = -1$ into the invariance condition, in order to have two equations for $f'(x)$ which must be equated. We are led to the necessary condition

$$\left(\frac{Af(x)+1}{f(x)-A}\right)^4 = \left(\frac{x+1}{x-1}\right)^4$$

which is solved, omitting the powers, by only

$$f(x) = \frac{(1+A)x-(1-A)}{(1-A)x+(1+A)},$$

the other sign giving an orientation-reversing diffeomorphism. This is nothing but a Möbius rotation, setting $\cos\alpha = (1+A)/\sqrt{2(1+A)}$ and $\sin\alpha = (1-A)/\sqrt{2(1+A)}$. Rotations have no fixed points, hence they cannot be candidates for modular group for interval algebras and therefore we conclude that the interval algebras cannot have purely geometric linear modular action in the Ramond state.

4.9 DIFFEOMORPHISM COVARIANCE VERSUS MULTI-LOCALITY

So far we have seen that there are essentially two distinguished ways to distribute fields around the circle. The first one is by implementing diffeomorphisms covariance, if the net is assumed to fulfill such property; so to speak, under a general change $z \mapsto \mu(z)$ fields change accordingly as $\phi'(\mu(z)) = (\partial\mu/\partial z)^h \phi(\mu(z))$, where h is the field scaling dimension. Unitary operators causing such displacement do exist and thus $\phi'(\mu(z)) = \mathrm{Ad}(U(\mu))(\phi(z))$. This concept has led [Longo and Xu, 2004] to the introduction of the Longo-Xu map as isomorphism of algebras (4.6.1). Because of the split property being involved, the Longo-Xu map does not preserve the vacuum state, because correlations are explicitly broken into product of states. This behaviour does not reflect quantum field theory at all, whose principal feature is that fields and observables must be correlated anyway, affecting each other according to the Einstein causality principle.

On the other hand, after investigation of modular theory for fermions in disjoint intervals, we have found that, however, an isomorphism of algebras preserving the vacuum state does exist in the form given by equation (4.7.10). Albeit in principle the two concepts might seem unrelated, they happen to be intimately connected through suitable gauge transformations. We start showing a simple example thereof and then we move to the general case.

Example: Let us take as diffeomorphisms the square root map $z \mapsto \pm\sqrt{z}$. The action of the corresponding Longo-Xu isomorphism on the complex fermion looks like equation (4.6.2)

$$\mathrm{LX}(\phi(z^2)) = \frac{1}{2\sqrt{z}}\left(\psi(z) + \psi(-z)\right).$$

We can compare this formula with (4.7.2), which shows the same action under the map β, respectively:

$$\beta(\phi(z^2)) = \frac{1}{2}\left(\psi(z) + \psi(-z)\right).$$

Remarkably we see that the two actions are related as

$$\mathrm{LX}(\phi(z^2)) = (z^2)^{-1/4}\,\beta(\phi(z^2)),$$

which can be rewritten as $\beta = \mathrm{LX} \circ \gamma$, where $\gamma \colon \mathcal{A}^2(I) \to \mathcal{A}^2(I)$ acts on the complex fermion as $\gamma(\phi(z)) = z^{1/4}\phi(z)$ and can be interpreted as a gauge transformation, in particular like pointlike rotations $\mathrm{rot}(\varphi/4)$ if $z = e^{i\varphi}$.

This is not accidental, rather it is pretty general. Let us work for the sake of simplicity on the real line, making use of the uniformisation funcion $X(x)$ as variable at hand. Gauge transformations $\gamma \colon \mathcal{A}(I) \to \mathcal{A}(I)$ preserving each subalgebra $\mathcal{A}(I)$ can be defined by their action on $\mathcal{A}^N(I)$ as

$$\gamma(\psi_i) = \sum_{r=1}^{N} O(X)_{ir}\,\psi_r(X) \qquad (4.9.1)$$

with $O(X)$ being the mixing matrix appearing in the Casini-Huerta modular flow for disjoint intervals. As such, $O(X)$ can be read as an $SO(N)$ valued function and γ takes fields at the point X into combination of fields at the same point, with pointlike dependent coefficient (which justifies the name of gauge transformations). Choosing now the funcion $X(x)$ as diffeomorphism to be implemented, we can act with the Longo-Xu map upon (4.9.1) in order to have

$$\begin{aligned} LX \circ \gamma(\psi_i(X)) &= LX\Big(\sum_{r=1}^{N} O(X)_{ir}\psi_r(X)\Big) \\ &= \sum_{r=1}^{N} O(X)_{ir}\sqrt{X_r^{-1}(X)'}\,\psi(X_r^{-1}(X)) \end{aligned}$$

that is $\beta(\psi_i(X))$, however we choose ψ_i as element of the algebras, and thus

$$\beta = LX \circ \gamma. \tag{4.9.2}$$

The vacuum preserving isomorphism β is related to the Longo-Xu map via choice of suitable gauge transformations, expressed by the orthogonal cocycle $O(X)$. Composition with such a map exactly *undoes* the split property and brings back the correlations among fields. In fact, although neither LX nor $O(X)$ preserve the vacuum state by themselves, their joint action does, bringing back the exact form of β as appearing in the modular flow for free fermions in disjoint intervals. We shall see later on, in the following chapters about currents, that this is a general issue which will be present in the embedded models as well. In fact, even though β itself does not completely restrict to the embedded subalgebras, its composition with γ^{-1} does, giving back the Longo-Xu map for currents and stress-energy tensor in the context of doubled theories; however, concerning these topics, we refer the reader to forthcoming chapters 5.2 and 5.5.

4.10 GEOMETRIC VERSUS NON-GEOMETRIC STATES

A very interesting comparison can be done between the modular groups obtained for the vacuum state on the algebra of the multi-intervals and the product state defined in 4.4.1. Before we carry this on we introduce an important characterisation, following the lines of [Summers, 2003]. Given two states ω and ϕ, cyclic and separating on the von Neumann algebra $\mathcal{A}$, there is a relation between the respective modular groups, σ_t^{ω} and σ_t^{ϕ}, in terms of an intertwining operator. In details:

Theorem 4.10.1 (Connes cocycle)*:* Given ω and ϕ as above, then there exists a strongly continuous unitary operator $t \in \mathbb{R} \mapsto U(t)$ belonging to the algebra $\mathcal{A}$ such that:

- $U(t)$ is a cocycle, namely $U(t+s) = U(t)\,\sigma_t^{\omega}(U(s)), \quad s,t \in \mathbb{R}$
- $U(t)$ intertwines the two modular groups:

$$\sigma_t^{\phi}(a) = U(t)\,\sigma_t^{\omega}(a)\,U(t)^* \quad a \in \mathcal{A},\ t \in \mathbb{R}.$$

The cocycle $U(t)$ is usually called the (Connes) derivative of ϕ with respect to ω.

This result can be applied right away to the special case when the two states are given by the Kawahigashi-Longo state ϕ_E and by the vacuum state ω_0. In particular, we can restrict, for the sake of simplicity, to the case of two intervals. We have then two copies $\mathcal{A}^2(I)$ of a fermionic algebra on one interval I and one copy of a fermionic algebra $\mathcal{A}(E_2)$. Let ω_0 be the vacuum state onto the one interval algebra and LX: $\mathcal{A}^2(I) \to \mathcal{A}(E_2)$ be the corresponding Longo-Xu map previously introduced (see 4.6). The two states to be compared are given by:

$$\varphi_{E_2} = (\omega_0 \otimes \omega_0) \circ LX^{-1}$$
$$\omega_0^{(2)} = (\omega_0 \otimes \omega_0) \circ \beta^{-1}$$

whose modular groups are, exploiting the KMS property,

$$\sigma_t^{\varphi_{E_2}} = LX \circ (\sigma_t^{\omega_0} \otimes \sigma_t^{\omega_0}) \circ LX^{-1}$$
$$\sigma_t^{\omega_0^{(2)}} = \beta \circ (\sigma_t^{\omega_0} \otimes \sigma_t^{\omega_0}) \circ \beta^{-1}$$

Now, as we have already pointed out, β is related to the corresponding Longo-Xu map via a gauge transformation inside the initial interval I, $\beta = LX \circ \gamma$. The key point is that these gauge transformations are implemented by currents embedded from the Fermi algebra itself, hence it exists a Weyl operator $W(f) \in \mathcal{A}^2(I)$ such that γ can be obtained as its adjoint action $\mathrm{Ad}(W(f))$:

$$\sigma_t^{\omega_0^{(2)}} = LX \circ \mathrm{Ad}(W(f)) \circ (\sigma_t^{\omega_0} \otimes \sigma_t^{\omega_0}) \circ \mathrm{Ad}(W(f)^*) \circ LX^{-1}. \quad (4.10.1)$$

The modular group $\sigma_t^{\omega_0}$ acts geometrically as the subgroup of the Möbius group preserving the interval, and thus $\sigma_t^{\omega_0} \otimes \sigma_t^{\omega_0}$ is itself a diffeomorphism δ on $\mathcal{A}^2(I)$. We can therefore commute the two actions as

$$\delta \circ \mathrm{Ad}(W(f)^*) = \mathrm{Ad}(\delta(W(f)^*)) \circ \delta$$

where $W(f)^* = W(-f)$ and $W(\delta(f)) = W(f \circ \delta)$. Equation (4.10.1) can be rewritten as

$$\begin{aligned}\sigma_t^{\omega_0^{(2)}} &= LX \circ \mathrm{Ad}(W(f)) \circ \mathrm{Ad}(W(-f \circ \delta)) \circ \delta LX^{-1} \\ &= LX \circ \mathrm{Ad}(W(f)W(-f \circ \delta)) \circ \delta \circ LX^{-1} \\ &= LX \circ \mathrm{Ad}(W(f - (f \circ \delta)) \circ \delta \circ LX^{-1}.\end{aligned}$$

Since δ preserves the boundaries $]a, b[$ of the interval I, then the function $f - (f \circ \delta)$ is zero on such boundaries along with its derivatives; in fact let $h(z) := f(z) - f(\delta_t(z))$

$$h'(z)|_{z=a} = f'(a) - \left.\frac{\partial f}{\partial \delta}\right|_{\delta_t(z=a)} \cdot \left.\frac{\partial \delta_t(z)}{\partial z}\right|_{z=a} =$$
$$f'(a) - f'(\delta_t(a)) \cdot 1 = f'(a) - f'(a) = 0.$$

We can as a consequence decompose $f - (f \circ \delta)$ as the sum of two any functions $g + g'$, where g supported in I and g' supported elsewhere in I′. This brings us to $W(f - (f \circ \delta)) = W(g + g') = W(g)W(g')$ and, since $g'(z) = 0$ if $z \in I$, its adjoint action is the identity within such interval, hence $\mathrm{Ad}(W(g)W(g')) = \mathrm{Ad}(W(g))$ on $\mathcal{A}^2(I)$. Commuting again through the Longo-Xu map gives

$$\begin{aligned}\sigma_t^{\omega_0^{(2)}} &= \mathrm{Ad}(LX(W(g))) \circ LX \circ \delta \circ LX^{-1} \\ &= \mathrm{Ad}\, W(g \circ LX) \circ \sigma_t^{\phi_{E_2}}.\end{aligned}$$

The last term $W(g \circ LX)$ belongs to $\mathcal{A}(E_2)$ and thus, comparing this expression with the definition of Connes derivative, $\mathrm{Ad}(W(g \circ LX))$ is exactly the cocycle intertwining the two modular groups in the vacuum state and in the Kawahigashi-Longo product state. In deriving this formula we remark once more that the property of δ to preserve the boundaries of the interval I has played a fundamental role, because of which the cocycle $\mathrm{Ad}(W(g \circ LX))$ actually belongs to the algebra $\mathcal{A}(E_2)$.

4.11 MULTI-GEOMETRIC TRANSLATIONS

We have seen that the vacuum modular group for delocalised fermions provides a mixing between fields in different intervals and explicit formulae have been carried out. In the case of a symmetric 2-interval $E_2 = I \cup -I$ we can express the action of the modular group in terms of a mixing matrix $O(t, z)$ as

$$\sigma^t_{I \cup -I} \begin{pmatrix} \psi(z) \\ \psi(-z) \end{pmatrix} = O(t, z) \begin{pmatrix} \psi\left(\delta^{(2)}_{-2\pi t}(z)\right) \\ \psi\left(\delta^{(2)}_{-2\pi t}(-z)\right) \end{pmatrix}$$

where $\delta^{(2)}_t(z)$ are the “2-dilations” $z \mapsto \sqrt{\delta^{(1)}_t(z^2)}$. Since $O(z, t)$ is a one-parameter subgroup $\in SO(2)$ it satisfies the cocycle condition

$$O(z, t)\, O\left(\delta^{(2)}_t(z), s\right) = O(z, t + s)$$

which is solved by the coboundary $O(z, t) = O(z)O(\delta^{(2)}_t(z))^{-1}$.

The Bisognano-Wichmann property ensures that these are in fact true 2-dilations satisfying the group composition law. The question is now whether we can find, mutatis mutandis, a similar subgroup representing true translations just by implementing suitable commutation relations (and of course group properties) with the above flows. For this purpose let us start considering the inclusion of intervals $J \subset I$: in case this inclusion is such that $\mathcal{A}(J)$ is half-sided modular included into $\mathcal{A}(I)$, the same holds true for the two-intervals generated by the square roots, namely $\mathcal{A}\left(\sqrt{J}\right) \subset \mathcal{A}\left(\sqrt{I}\right)$. Then, by a notable result of Wiesbrock (see 3.3.3), we can automatically reconstruct the generator of the translations via the difference of the respective modular groups. The adjoint action as in 3.3.2 provides the defining relations for a semidirect product $\mathbb{R} \ltimes \mathbb{R}$ of the form $\delta_s \circ \tau_t = \tau_{e^{-2\pi s}\, t} \circ \delta_s$. The speculation now is that the one-parameter group of "2-translations" can be also written accordingly and acts on the fields as

$$\tau^t_{I\cup -I}\begin{pmatrix}\psi(z)\\ \psi(-z)\end{pmatrix} = P(t,z)\begin{pmatrix}\psi\left(\alpha^{(2)}_{-2\pi t}(z)\right)\\ \psi\left(\alpha^{(2)}_{-2\pi t}(-z)\right)\end{pmatrix}$$

where now $\alpha^{(2)}_t(z)$ are the "2-translations" $z \mapsto \sqrt{\alpha^{(1)}_t(z^2)}$. The matrix $P(z,t) \in SO(2)$ is of course in general not known and has to be derived by using the properties it is subject to. The fact that it is a one-parameter subgroup of SO(2) implies again that it satisfies the cocycle property

$$P(z,t)\, P\left(\alpha^{(2)}_t(z), s\right) = P(z, t+s)$$

and moreover it has to fulfill correct commutation relations with the dilation part, expressed by the matrix $O(z,t)$

$$P(z,t)\, O\left(\alpha^{(2)}_t(z), s\right) = O(z,s)\, P\left(\delta^{(2)}_t(z), e^{-2\pi s}\, t\right).$$

The above relations can be turned easier expressing the matrices in terms of their rotation angles[2] $P(z,t) = e^{i\sigma_2\theta(z,t)}$ and likewise with $O(z,t) = e^{i\sigma_2\phi(z,t)}$: they become in turn

$$\theta(z,t) + \theta(\alpha^{(2)}_t(z), s) = \theta(z, t+s) \qquad \text{(cocycle for } P(z,t)) \qquad (4.11.1)$$

$$\phi(z,t) + \phi(\delta^{(2)}_t(z), s) = \phi(z, t+s) \qquad \text{(cocycle for } O(z,t)) \qquad (4.11.2)$$

$$\theta(z,t) + \phi(\alpha^{(2)}_t(z), s) = \phi(z,s) + \theta(\delta^{(2)}_t(z), e^{-2\pi s}) \quad \text{(CRs)} \qquad (4.11.3)$$

However, the calculations can be simplified a lot once we have understood that the isomoprhism β intertwines the modular group with respect to the vacuum states for the one-interval algebra and the two-intervals algebra. Since the geometric action is then given in terms of 2-dilations, even the 2-translations are intertwined as

$$\tau^t_{I\cup -I} = \beta \circ \tau^t_{I^2} \circ \beta^{-1}. \qquad (4.11.4)$$

In the circle picture the translations are Möbius transformations acting on z variable as

$$z \mapsto \alpha_t(z) = \frac{\gamma z + \beta}{\bar\beta + \bar\gamma}, \qquad \alpha_t = \begin{pmatrix} 1 + it/2 & it/2 \\ -it/2 & 1 - it/2 \end{pmatrix}$$

that allows to calculate explicitly (4.11.4). For example, taking into account the transformation laws for Fermi fields and the actual form of β we have, on $\psi(z)$

$$\tau^t_{I\cup -I}(\psi(z)) = \frac{1}{(2-it) - itz^2}\left(\psi(\alpha_t(z^2))\left(1 + \frac{z}{\alpha_t(z^2)}\right)\right.$$
$$\left. + \psi(-\alpha_t(z^2))\left(1 - \frac{z}{\alpha_t(z^2)}\right)\right) \qquad (4.11.5)$$

and of course likewise on $\psi(-z)$, with the due change of signs carried by β.

2 σ_2 is the Pauli matrix $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

4.12 A REVERSE PICTURE

We have seen that the action of the modular group with respect to the vacuum state is geometric inside one interval (Bisognano-Wichmann property, 3.3.6), whilst it introduces a mixing among different intervals described by [Casini and Huerta, 2009]. Now we aim to construct a state whose modular group exactly switches these behaviours, namely whose action inside one interval is no more geometric, rather it introduces a mixing among the components described by the same matrix as above.

Let us consider, on $\mathcal{A}^N(I)$, $I \subset \mathbb{R}$, the following gauge transformation $\gamma \colon \mathcal{A}^N(I) \to \mathcal{A}^N(I)$

$$\gamma(\psi_i(X)) = \sum_{r=1}^{n} O(X)_{ir} \sqrt{X_r^{-1}(X)'}\, \psi_r(X)$$

with the function $X(x)$ as in (4.7.9) and the matrix $O(X)$ exactly given as in (4.7.11). Define the state φ on $\mathcal{A}^N(I)$ to be $\varphi := \omega_0 \circ \gamma^{-1}$; its modular group therefore reads $\sigma^t_\varphi = \gamma \circ \sigma^t_{\omega_0} \circ \gamma^{-1}$ by verification of the KMS condition. Explicitly, this gives

$$\begin{aligned}\sigma^t_\varphi(\psi_i(X)) &= (\gamma \circ \sigma^t_{\omega_0} \circ \gamma^{-1})(\psi_i(X)) \\ &= (\gamma \circ \sigma^t_{\omega_0}) \left(\sum_{r=1}^{n} O(X)_{ir} \sqrt{X_r^{-1}(X)'} \right)^{-1} \psi_r(X)\end{aligned}$$

therefore

$$\begin{aligned}\sqrt{X_r^{-1}(X)'}\, \sigma^t_\varphi(\psi_i(X)) &= \gamma \left(\sum_{r=1}^{n} O(X)^T_{ir} \sqrt{\delta_t(X)'}\, \psi_r(\delta_t(X)) \right) \\ &= \sum_{r,p=1}^{n} O(X)^T_{ir} \sqrt{\delta_t(X)'}\, O(\delta_t(X))_{rp} \sqrt{X_p^{-1}(\delta_t(X))'}\, \psi_p(\delta_t(X))\end{aligned}$$

Since the matrix $O(X)$ satisfies $O(X)^T O(\delta_t(X)) = O(t, X)$ we end up with

$$\sqrt{X_r^{-1}(X)'}\, \sigma^t_\varphi(\psi_i(X)) = \sum_{p=1}^{n} O(t, X)_{ip} \sqrt{X_p^{-1}(\delta_t(X))'} \sqrt{\delta_t(X)'}\, \psi_p(\delta_t(X))$$

which presents the same mixing appearing in the vacuum modular flow on the union of n disjoint intervals [Longo, Martinetti, and Rehren, 2009, eq (3.1)]. In the same paper the authors also show that a product state of the form $\varphi_E := (\otimes^n_{k=1} \varphi_k) \circ \chi_E$ has modular group with geometric action within n disjoint intervals. Therein χ_E is the isomorphism given by the split property and φ_k are state given by $\varphi_k := \omega_0 \circ \mathrm{Ad}\, U(\gamma_k)$, where $U(\gamma_k)$ implements diffeomorphisms $\gamma_k \colon z \to z^n$ on I_k (to be expanded).

4.13 THE FREE BOSON CASE

It is tempting, after the results obtained in the free Fermi model, to look at the Bose case in order to understand whether a similar isomorphism, playing the role of β, exists and preserves the vacuum state.

The free boson presents some difficulties related to the construction of the Hilbert space in terms of operators and to the fact that it is not a conformal field in the proper sense (though it may be considered as a conformal field of scaling dimension zero). Nevertheless, in the sense of distributions, we may still think to be provided with a net of algebras on the circle whose elements are bosonic fields satisfying suitable commutation relations and two-point functions. Such models are realised as solutions to the massless Klein-Gordon equation, still fulfilling the requirements to be a conformal invariant field theory.

Let $\varphi(x)$ represent such boson field, in the sense of distributions. In order to obtain a well defined Hilbert space such distributions must be smeared with test functions f which integrate to zero, therefore they appear as derivatives of certain test functions; in formulae

$$\varphi(f) = \int_{\mathbb{R}} dx\, f(x)\, \varphi(x) \quad \text{with } g(x) \mid g'(x) = f(x)$$

and $g(x) \to 0$ as $x \to \pm\infty$. If $\varphi(x)$ solves the massless Klein-Gordon equation, then $\varphi'(x) = j(x)$ is a current. Following the basic construction of boson nets of algebras as shown in 1.2 we see that (the minus sign is only "moral")

$$\mathrm{CCR}(\varphi(f)) = -\mathrm{CCR}(j(g)), \quad f(x) = g'(x).$$

To be more precise we start from the massless Klein-Gordon equation in two dimensions, with $\varphi \equiv \varphi(x^0, x^1)$; then define $j_\pm(x_+, x_-) := \partial_\pm \varphi(x^0, x^1)$, where $x_\pm$ are the usual lightcone variables. The equation of motion then implies $\partial_- j_+ = \partial_+ j_- = 0$, in turn the two currents $j_\pm$ depend on only one variable at a time and we can decompose the theory into two one-dimensional copies, that we are going to analyse separately.

Turning back to the search for a vacuum preserving isomorphism, we intend the notations below as similar to the Fermi case, the only difference being the change in the vacuum two-point function, which now becomes [di Francesco et al., 1997]

$$\omega_0(j^*(x)j(y)) = \lim_{\varepsilon \to 0^+} \frac{1}{(x - y - i\varepsilon)^2}.$$

The passage to the circle picture is pretty easy, and the two-point function simply becomes

$$\omega_0(j^*(z)j(w)) = \frac{1}{(z - w)^2}.$$

Along the same lines as in the free Fermi case we suppose to have a complex Bose field (i. e. two copies of a real Bose fiel) $j(z), j^*(z)$ on $\mathcal{A}^2(I)$. For the sake of simplicity we work out the symmetric case, and thus we assume to have a real Bose field $j(z)$ localised in $\sqrt{I}$. The standard idea to be undertaken is to define the analogue of $\beta\colon \mathcal{A}^2(I) \to \mathcal{A}\left(\sqrt{I}\right)$ as

$$\begin{aligned} j^*(z^2) &\mapsto c_1(z)j(z) + c_2(z)j(-z) \\ j(z^2) &\mapsto c_3(z)j(z) + c_4(z)j(-z) \end{aligned}$$

and to analyse whether $\omega_0 \circ \beta = \omega_0$, according to the choice of the coefficients and to the special form of the vacuum state. Once we plug the ansatz in we obtain

$$\begin{aligned} \omega_0(j^*(z^2)j(w^2)) &= \omega_0 \circ \beta(j^*(z^2)j(w^2)) \\ &= \omega_0\left((c_1(z)j(z) + c_2(z)j(-z))\right. \\ &\left.\cdot(c_3(w)j(w) + c_4(w)j(-w)\right). \end{aligned}$$

Using the explicit form of the two-point function the above equation becomes

$$\begin{aligned} \frac{1}{(z^2-w^2)^2} = c_1(z)c_3(w)\frac{1}{(z-w)^2} + c_2(z)c_3(w)\frac{1}{(-z-w)^2} \\ + c_1(z)c_4(w)\frac{1}{(z+w)^2} + c_2(z)c_4(w)\frac{1}{(-z+w)^2} \end{aligned}$$

but as we see, no choice of coefficients can fulfill this equation, in fact

$$\begin{aligned} \frac{1}{(z^2-w^2)^2} = (c_1(z)c_3(w) + c_2(z)c_4(w))\frac{1}{(z-w)^2} \\ + (c_2(z)c_3(w) + c_1(z)c_4(w))\frac{1}{(z+w)^2} \end{aligned}$$

gives rise to the condition

$$\begin{aligned} \big(c_1(z)c_3(w) + c_2(z)c_4(w)\big)(z+w)^2 \\ + \big(c_2(z)c_3(w) + c_1(z)c_4(w)\big)(z-w)^2 = 1 \end{aligned}$$

after multiplying both sides by $(z^2-w^2)^2$. This gives rise in turn to the set of equations

$$\begin{aligned} c_1(z)c_3(w) + c_2(z)c_4(w) &= \frac{1}{2(z+w)^2} \\ c_2(z)c_3(w) + c_1(z)c_4(w) &= \frac{1}{2(z-w)^2} \end{aligned}$$

whose solutions obstruct the ansatz $c_1(z), c_2(z) = f(z)$ only, and likewise for $c_3(w), c_4(w)$. Therefore it is not possible to introduce an analogous multi-local isomorphism preserving the vacuum state for the free boson.

5 CURRENTS MODELS AND EMBEDDINGS

Contents

5.1 LOOP GROUPS

We shall now briefly describe the construction of nets of von Neumann algebras on the circle in the framework of loop groups as shown in [Pressley and Segal, 1986].

Let G be a compact Lie group whose Lie algebra is denoted by $\mathfrak{g}$: the set of all smooth maps $LG := \{ g \mid S^1 \to G \}$ equipped with pointwise multiplication $(g \cdot h)(z) = g(z)h(z)$ is an infinite dimensional Lie group called the loop group. As such, it possesses a Lie algebra which can be shown, as expected, to be the set of all maps $L\mathfrak{g} = \{ g \mid S^1 \to \mathfrak{g} \}$. As set of maps, both of them can be equipped with the standard topologies of uniform convergence and differential structure, and thus a smooth map $\exp\colon L\mathfrak{g} \to LG$ exists and is a local homeomorphism in the connected neighbourhood near the identity.

A localised subgroup $L_I G$ is the set of all such functions taking the trivial value $\mathbb{1}_G$ outside the interval I, essentially

$$LG := \{ g \mid S^1 \to G \} \qquad L_I G := \{ g \mid g(z) = \mathbb{1}_G \in G, \text{ if } z \notin I \}.$$

Since now on we shall focus on projective unitary representations of the compact Lie group G, where "projective" means that products are preserved up to a complex phase. In order to make formal sense of such a concept we introduce the following definition:

Definition (2-cocycle)*:* Let G be a group. A 2-cocycle is a map $\omega\colon G \times G \to S^1$ satisfying, however we choose $f, g, h \in G$

$$\omega(f, g)\omega(fg, h) = \omega(f, gh)\omega(g, h);$$

also, ω must be trivial on the identity element $\omega(\mathbb{1}_G, g) = \omega(g, \mathbb{1}_G) = 1$. By pointwise multiplication the set of all 2-cocycles forms a group. Moreover, if there exists $\beta\colon G \to S^1$ such that

$$\omega(f, g) = \frac{\beta(f)\beta(g)}{\beta(fg)}$$

then the 2-cocycle is said to be a coboundary.

With the help of the above definition we can therefore define projective representations of a loop group as maps from the groups itself into the set of unbounded operators on some Hilbert space preserving products

$$W(g_1)W(g_2) = \omega(g_1, g_2)W(g_1 g_2), \quad g_1, g_2 \in LG$$

ω being a cocycle of LG. The assignment $I \to \mathcal{A}(I)$ defined as

$$\mathcal{A}(I) := \{ W(g) \mid g \in L_I G \}''$$

defines a net of local algebras whenever the cocycle ω is local. Nets of von Neumann algebras defined out of loop groups representations have the property to only have finitely many inequivalent representations.

In the field theoretical setting such construction are realised by taking the analogue of the Weyl operators for current algebras. Let τ_a be a basis of the Lie algebra $\mathfrak{g}$ of G and define $f(z) = \tau_a f^a(z)$. We define the smeared current $j(f) := \oint_{S^1} dz\, j_a(z) f^a(z)$

$$[j(f_1), j(f_2)] = j[f_1, f_2] + k \oint_{S^1} dz\, \omega(f_1, f_2)(z), \quad f_1, f_2 \in L\mathfrak{g}$$

(notice here that, despite the same notation, ω is an additive cocycle playing the infinitesimal version of the previous one introduced for Weyl relations). The corresponding Weyl operators are

$$W(g) := e^{ij(f)} \qquad g(z) = \exp(f)(z)$$

whose collection generates the local net of von Neumann algebras $\mathcal{A}(I)$ as $\operatorname{supp} f \subset I$.

In this context gauge transformations γ are defined as automorphisms $\gamma\colon \mathcal{A}(I) \to \mathcal{A}(I)$ that preserve every local subalgebra and they may be inner implemented by means of the unitaries $W(g)$. For instance, on currents, $\gamma(J^a) = W(g) J^a W(g)^*$ acts as

$$\gamma(J^a \tau_a) = g(z)^{-1} J^a \tau_a g(z) + k\, g(z)^{-1} \partial_z g(z);$$

the matrices τ_a form a basis in the Lie algebra $\mathfrak{g}$ and the transformation law for the actual fields (the currents) follow by multiplying and taking traces with respect to τ_a. For example, in case $G = SU(2)$ a basis for the Lie algebra is given in terms of the Pauli matrices and thus, after using

$$\sigma_i\, \sigma_j = i\, \epsilon_{ij}{}^k \sigma_k + \delta_{ij} \mathbf{1}$$

we obtain, taking into account that Pauli matrices are traceless themselves

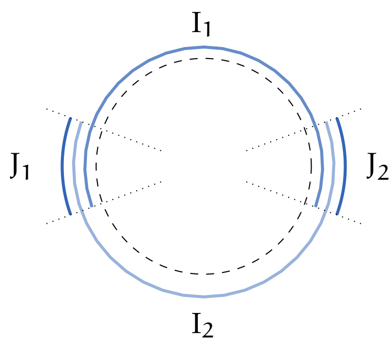
$$\gamma(J^a) = f_{ab}\, J^b + \frac{1}{2}\, i \operatorname{tr}\left(g(z)^{-1} \partial_z g(z) \sigma_a\right)$$

with $f_a{}^b\, \sigma_b = g(z)^{-1} \sigma^a g(z)$. On the other hand, let now $h\colon S^1 \to G$ be a function periodic up to a central element $h_0 \in Z(G)$. Any transformation of the form

$$\gamma(J) = \operatorname{Ad} h^{-1}(J) + h^{-1} dh$$

still fulfils the requirements to be an automorphism, because the central element cancels out; nevertheless it is not implemented by unitaries of the form $W(h)$ because h is no more an element of the loop group. Therefore, by using elements in $Z(G)$ we can construct automorphisms which are no more implemented by unitaries but which are still inner symmetries. This is an interesting feature in the context of representation theory of loop groups, because the composition of states with such automorphisms gives rise to inequivalent representations and all simple sectors are of this form. As a consequence, different sectors arise according to how many central elements the Lie group G has. Thus compact Lie groups with trivial centre only have one simple sector (the vacuum sector).

Example (non-trivial sectors)*:* Let I_1 and I_2 be two intervals on the circle such that their intersection is the union of two disjoint intervals J_1 and J_1: moreover, let $\mathcal{A}(J_1)$ and $\mathcal{A}(J_2)$ come accompanied with



two different localised representations $\pi_1(\mathcal{A}(J_1))$, $\pi_2(\mathcal{A}(J_2))$ different from the defining vacuum representation. If now $U\colon \pi_1 \to \pi_2$ is a map intertwining such representations, namely $U\pi_1(a_1) = \pi_2(a_1)U$, then U belongs, by Haag duality, to both $\pi_0(\mathcal{A}(I_1))$ and $\pi_0(\mathcal{A}(I_2))$; yet, the operator U may differ when evaluated in π_1 and π_2, that is $\pi_1(U) \neq \pi_2(U)$. One can prove, using Doplicher-Haag-Roberts theory of localised endomorphisms, that in case the net $I \to \mathcal{A}(I)$ has only the vacuum sector no such problem occurs and the vacuum representation is always faithful. This also helps to globally define the whole algebra $\mathcal{A}\left(S^1\right)$ as the C^*-algebra generated by all the $\vee_I \pi_0(\mathcal{A}(I)) = \mathcal{B}(\mathcal{H})$.

We want to remark once more that the quarks construction, as showed in the previouos chapters, provides, in the field theoretical setting, the relation between Fermi fields and currents expressed as Wick products thereof.

5.2 currents models

We have previously seen that the isomorphism β provides a map $\beta\colon \mathcal{A}^{(N)}(I) \to \mathcal{A}(\sqrt[N]{I})$ preserving the vacuum state and its representation $\pi_0 \circ \beta = \pi_0$. In the particular case of $N = 2$ a complex Fermi field localised in one interval I is "decomposed" into its symmetric and antisymmetric part

$$\phi(z^2) = \frac{1}{\sqrt{2}}\left(\psi^{(1)}(z^2) + i\psi^{(2)}(z^2)\right) \mapsto \frac{1}{2}\left(\psi(z) + \psi(-z)\right)$$
$$\phi^*(z^2) = \frac{1}{\sqrt{2}}\left(\psi^{(1)}(z^2) - i\psi^{(2)}(z^2)\right) \mapsto \frac{1}{2z}\left(\psi(z) - \psi(-z)\right).$$

The resulting representation is a twisted representation $\pi_0 \otimes^t \pi_0$ because β intrinsically carries a twist on some fields, namely $\beta \circ \mathrm{rot}(2\pi) = \beta \circ \wp$ where $\wp$ is the flip automorphism flipping the tensor product $\wp\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that $\wp(x \otimes y) = y \otimes x$.

The idea is now to extend this map to embedded models, as well as currents and stress-energy tensor, trying to preserve its features. We are then looking for an isomorphism which gives a correspondence $\mathcal{A}(I) \otimes \mathcal{A}(I) \to \mathcal{A}\left(\sqrt{I}\right)$ also at the level of currents and stress-energy tensor, decomposing the fields into their symmetric and antisymmetric parts. Since the restriction of β does not, in general, preserve this embedded subalgebras we should expect an additional gauge transformation to compose β with in order to achieve the result.

Let us denote by $\mathcal{A}^J(I)$ the algebra of currents localised in the interval I. The purpose is to explicitly construct a map $\iota\colon \mathcal{A}^J(I) \otimes \mathcal{A}^J(I) \to \mathcal{A}^J(\sqrt{I})$ making use of β. We start by taking two real Fermi fields $\psi(z)$ and $\psi'(z)$ localised in $\sqrt{I}$ and apply β^{-1} in order to obtain two complex Fermi fields $\phi(z^2), \phi'(z^2)$ localised I. These fields can in turn be decomposed into their respective real and imaginary parts as

$$\phi(z^2) = \frac{1}{\sqrt{2}}\left(\psi^{(1)}(z^2) + i\psi^{(2)}(z^2)\right)$$
$$\phi'(z^2) = \frac{1}{\sqrt{2}}\left(\psi^{(3)}(z^2) + i\psi^{(4)}(z^2)\right)$$

and thus we have generated four real Fermi fields, $\psi^{(1)}, \ldots, \psi^{(4)}$. Combinations of such fields can be used to generate current algebras models with gauge group $O(4)$, since in principle we can combine any two Fermi fields into Wick products $:\psi^i\psi^j:$; in particular the construction runs as follows: let us take the $U(1) = SU(2)$ current constructed out of the combination of the two initial fields $j(z) \coloneqq 2i :\psi\psi':(z)$ and embed by β^{-1}, taking into account the inverse formula in [Rehren and Tedesco, 2013]. We have

$$\beta^{-1}(\psi(z)\psi'(z)) = \left(\phi(z^2) + z\phi^*(z^2)\right)\left(\phi'(z^2) + z\phi'^*(z^2)\right)$$
$$\beta^{-1}(\psi(z)\psi'(z) - \psi(-z)\psi'(-z)) = 2z\,\phi^*(z^2)\phi'(z^2) + 2z\,\phi(z^2)\phi^*(z^2);$$

as we see, only terms coupling hermitian products of $\phi^*\phi$ appear, and thus we may conclude this current is neutral, the total charge being zero. Everything can be expressed in terms of the initial four fermions, and since β preserves the vacuum state and then the Wick products, we find for the even modes of such current

$$\beta^{-1}\left(z\,j(z) - z\,j(-z)\right) = 2z^2\left(J_{13}(z^2) + J_{24}(z^2)\right)$$

with $J_{13}(z) \coloneqq 2i :\psi^{(1)}\psi^{(3)}:(z)$ and similarly for $J_{24}(z)$. The odd modes, instead, present charged combinations $\phi(z^2)\phi'(z^2) + z^2\, \phi^*(z^2)\phi'^*(z^2)$, giving rise to

$$\beta^{-1}\,(z\,j(z) + z\,j(-z)) = J_+(z^2) + z^2\, J_-(z^2)$$

with $J(z) \coloneqq J_{13}(z) - J_{24}(z) + i\,(J_{14}(z) + J_{23}(z))$. In a more compact way the above relations can be written as

$$\beta^{-1}(\text{even}) = 2z^2\, J_0(z^2)$$
$$\beta^{-1}(\text{odd}) = J(z^2) + z^2\, J^*(z^2).$$

Commutation relations between these currents show particular features: the commutator $[J_+, J_-]$ produces the third generator of $\mathfrak{su}(2)$ current algebra with $J_3 \coloneqq J_{12} + J_{34}$, yet both J_+, J_- commute with J_0. The structure is the one of a $\mathfrak{u}(2) \subset \mathfrak{o}(4)$ current algebra $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ where J_0 plays the role of the diagonal part in $\mathfrak{u}(1)$, the rest being the $\mathfrak{su}(2)$ current algebra. However, the action of β^{-1} only gives back the commuting currents J_0 and $J_\pm$. We may as well reverse the picture and look at how the commuting currents, J_0 and $J_\pm$, localised in $\mathcal{A}^J(I)$ are decomposed into symmetric and antisymmetric part of a single current localised in $\mathcal{A}^J(\sqrt{I})$.

With the help of suitable gauge transformations we can reduce the combination $J_+(z) + z\,J_-(z)$ to just a single current. SU(2) gauge transformations γ on Fermi fields transform the embedded currents $J^a(x) = :\psi^*\sigma^a\psi:(x)$ as

$$\gamma(J^a)\sigma_a = g(z)^{-1} J^a \sigma_a g(z) + k\, g(z)^{-1}\partial_z g(z)$$

where $g \in SU(2)$ and transformations for the actual fields J^a follow by multiplying both sides by σ_c and taking traces. Also, use $\sigma_a\sigma_b = i\varepsilon_{ab}{}^{c}\,\sigma_c + \delta_{ab}\mathbb{1}$. We obtain eventually

$$\gamma(J^a) = f^a{}_b\, J^b + \frac{1}{2}\,\kappa\, \mathrm{tr}\left(g(z)^{-1}\partial_z g(z)\sigma^a\right) \tag{5.2.1}$$

with $f^a{}_b(z)\,\sigma^b = g(z)^{-1}\sigma^a g(z)$. Such gauge transformations are automorphisms of the algebras, though they may not preserve the vacuum state. Exploiting the above equation with the group element $g(z) = e^{-i(\varphi/4)\sigma_3}\, e^{-i(\varphi/4)\sigma_2}$ gives back exactly

$$\gamma\left(J_3(z^2)\right) = \sqrt{z}\,\left(J_+(z^2) + z^2\, J_-(z^2)\right).$$

Moreover, since J_0 plays the role of the diagonal part $\mathfrak{u}(1)$ in $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ the map γ can be regarded as acting diagonally on J_0. Therefore we have

$$(\beta \circ \gamma)\left(J_0(z^2)\right) = \frac{1}{2z}\,(j(z) - j(-z)) \tag{5.2.2}$$

$$(\beta \circ \gamma)\left(J_3(z^2)\right) = \frac{1}{2z}\,(j(z) + j(-z))\,. \tag{5.2.3}$$

Defining $\iota = \beta \circ \gamma$ gives us the map we looked for at the level of the currents. Interestingly enough, this map produces an anti-periodic field when acting on J_3. Therefore

$$\iota \colon \mathcal{A}^J(I) \otimes \mathcal{A}^J(I) \to \mathcal{A}^J(\sqrt{I}).$$

The scenario we are dealing with looks now like, according to the scaling dimension:

$$\text{Fermi}^{\ d=\frac{1}{2}} \qquad \begin{matrix} \phi(z^2) \\ \phi^*(z^2) \end{matrix} \ \xrightarrow{\ \beta\ } \ \begin{matrix} \psi(z)+\psi(-z) \\ z^{-1}\left(\psi(z)-\psi(-z)\right) \end{matrix}$$

$$\text{Currents}^{\ d=1} \qquad \begin{matrix} J_0(z^2) \\ J_3(z^2) \end{matrix} \ \xrightarrow{\ \beta\circ\gamma\ } \ \begin{matrix} z^{-1}\left(j(z)-j(-z)\right) \\ z^{-1}\left(j(z)+j(-z)\right) \end{matrix}$$

Of course, the twist emerges once we compose β with $\mathrm{rot}(2\pi)$ and this in turn emerges from the mere commutation relations of any chiral field with the rotations $e^{2\pi i L_0}$,

$$i[L_0, \phi(z)] = i(z\partial_z + h)\phi(z)$$

which integrates to

$$e^{itL_0}\,\phi(z)\,e^{-itL_0} = e^{ith}\,\phi(e^{it}\,z).$$

This means that for scaling dimension $h = 1/2$ we have a minus sign at $t = 2\pi$ if we evaluate fields in the vacuum representation, while no minus sign occurs in the Ramond representation, since the latter presents an additional $\sqrt{z}$ which absorbs the -1. In contrast, no minus sign may appear for local fields with integer scaling dimension.

Now, if we start from non-abelian current algebra with level κ and structure constants ${f_{ab}}^c$, we can easily construct models with twice the level just by taking the symmetric and anti-symmetric parts. In detail we start with [Fuchs, 1992]

$$\left[J^a(z), J^b(w)\right] = {f^{ab}}_c\, J^c(z)\frac{1}{w}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^n + \frac{1}{zw}\,\kappa\, h^{ab}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^n \tag{5.2.4}$$

and construct

$$\begin{aligned} J_a^{2\kappa}(z^2) &\coloneqq J_a^c(z^2)\otimes \mathbf{1} + \mathbf{1}\otimes J_a^c(z^2) \\ \Delta_a^{2\kappa}(z^2) &\coloneqq J_a^c(z^2)\otimes \mathbf{1} - \mathbf{1}\otimes J_a^c(z^2) \end{aligned}$$

clearly both fields belong to $\mathcal{A}^J(I)\otimes\mathcal{A}^J(I)$. These quantities satisfy current algebras commutation relations like (5.2.4) with twice the central charge

$$\begin{aligned} \left[J_a^{2c}(z^2), J_b^{2c}(w^2)\right] = {f_{ab}}^c\, J_c^{2c}(z^2)\frac{1}{w^2}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^{2n} \\ + \frac{1}{z^2w^2}\,2c\,h_{ab}\sum_{n\in\mathbb{Z}} n\left(\frac{z}{w}\right)^{2n} \end{aligned}$$

as for the other combinations

$$\left[J^{2c}_a(z^2), \Delta^{2c}_b(w^2)\right] = f_{ab}{}^c \Delta^{2c}_c(z^2)\frac{1}{w^2}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^{2n+1}$$

$$\left[\Delta^{2c}_a(z^2), \Delta^{2c}_b(w^2)\right] = f_{ab}{}^c J^{2c}_c(z^2)\frac{1}{w^2}\sum_{n\in\mathbb{Z}}\left(\frac{z}{w}\right)^{2n+1} + \frac{1}{z^2w^2}\, 2c\, h_{ab}\sum_{n\in\mathbb{Z}}(n+1/2)\left(\frac{z}{w}\right)^{2n+1}$$

These commutation relations happen to be satisfied by the odd and even modes of a single current localised in $\sqrt{I}$, namely the assignment

$$\alpha\left(J^{2c}_a(z^2)\right) = \frac{1}{2z}\left(j_a(z) - j_a(-z)\right) \tag{5.2.5}$$

$$\alpha\left(\Delta^{2c}_a(z^2)\right) = \frac{1}{2z}\left(j_a(z) + j_a(-z)\right) \tag{5.2.6}$$

preserves the commutation relations, as easily seen by summing up the Fourier modes. In detail let us restrict to the abelian case: the Fourier decomposition of the current $J(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1}$ gives

$$\alpha\left(J^{2c}_n\right) = j^c_{2n}, \quad n\in\mathbb{Z}$$
$$\alpha\left(\Delta^{2c}_\nu\right) = j^c_{2n} \quad \nu\in\mathbb{Z}+1/2$$

which, by using $[j_n, j_m] = n\,\delta_{n+m,0}\, c\,\mathbf{1}$ are seen to satisfy the commutation relations

$$\left[\alpha\left(J^{2c}_n\right), \alpha\left(J^{2c}_m\right)\right] = n\,\delta_{n+m,0}\, 2c\,\mathbf{1} = 2\,\alpha[j_n, j_m]$$
$$\left[\alpha\left(J^{2c}_n\right), \alpha\left(\Delta^{2c}_\nu\right)\right] = n\,\delta_{n+\nu,0}\, 2c\,\mathbf{1} = 2\,\alpha[j_n, j_\nu]$$
$$\left[\alpha\left(\Delta^{2c}_\mu\right), \alpha\left(\Delta^{2c}_\nu\right)\right] = \mu\,\delta_{\mu+\nu,0}\, 2c\,\mathbf{1} = 2\,\alpha[j_\mu, j_\nu].$$

Summing up the Fourier series we obtain

$$\left[\alpha\left(J^{2c}(z^2)\right), \left(J^{2c}(w^2)\right)\right] = \alpha\left[\left(J^{2c}(z^2)\right), \left(J^{2c}(w^2)\right)\right]$$
$$\left[\alpha\left(J^{2c}(z^2)\right), \left(\Delta^{2c}(w^2)\right)\right] = \alpha\left[\left(J^{2c}(z^2)\right), \left(\Delta^{2c}(w^2)\right)\right]$$
$$\left[\alpha\left(\Delta^{2c}(z^2)\right), \left(\Delta^{2c}(w^2)\right)\right] = \alpha\left[\left(\Delta^{2c}(z^2)\right), \left(\Delta^{2c}(w^2)\right)\right].$$

In the abelian case the right hand side coincides with the quantities (5.2.2) we previously calculated as $\beta\circ\gamma$, therefore we may write

$$\alpha\left(J^{2c}(z^2)\right) = (\beta\circ\gamma)\left(J^{2c}_0(z^2)\right)$$
$$\alpha\left(\Delta^{2c}(z^2)\right) = (\beta\circ\gamma)\left(J^{2c}_3(z^2)\right)$$

meaning

$$\left(\alpha^{-1}\circ\beta\circ\gamma\right)\left(J_0(z^2)\right) = J(z^2)\otimes\mathbf{1} + \mathbf{1}\otimes J(z^2) \tag{5.2.7}$$

$$\left(\alpha^{-1}\circ\beta\circ\gamma\right)\left(J_3(z^2)\right) = J(z^2)\otimes\mathbf{1} - \mathbf{1}\otimes J(z^2) \tag{5.2.8}$$

with the currents on the left hand side having twice the central charge of the currents on the right hand side.

5.2.1 *The Kac-Frenkel construction*

We have seen in the previous section that starting from two real Fermi fields the inverse action of β^{-1} gives back two complex Fermi fields which in turn can be decomposed into their real and imaginary parts. This brings us four Fermi fields whose combinations construct a non-abelian current model. In particular we can get a $\mathfrak{u}(2)$ current model constituted by the currents

$$\begin{aligned} J_0 &= J_{13} + J_{24} \\ J &= J_{13} - J_{24} + i\,(J_{14} + J_{23}) \\ J_1 &= J_{13} - J_{24} \\ J_2 &= J_{14} + J_{23} \\ J_3 &= J_{12} + J_{34} \end{aligned}$$

whose commutation relations are

$$[J_i, J_j] = \epsilon_{ij}{}^{k} J_k \quad i,j,k = 1,2,3 \tag{5.2.9}$$

$$[J_i, J_0] = 0 \tag{5.2.10}$$

giving raise to $\mathfrak{u}(2) = \mathfrak{su}(2)$ (the former) $\oplus\mathfrak{u}(1)$ (the latter). This also gives back the commutations relations $[J, J^*]$ with $J = J_1 + i\,J_2$.

The same $\mathfrak{u}(2)$ algebra can be derived by using the Kac-Frenkel construction ([Kac, 1998]) out of two commuting currents J_0 (playing the role of the diagonal $\mathfrak{u}(1)$ current) and J_3 as follows: the unitary Weyl operators on currents $W(f) = e^{ij(f)}$ evaluated on sharp test functions $G_u(x) = q \cdot \theta(x-u)$ become the "vertex operators" ([Longo and Rehren, 2009])

$$V_q(x) = :e^{iq\int_\infty^x du\, j(u)}:;$$

the operators $V_{\pm\sqrt{2}}(x)$ can be decomposed as

$$J^\pm(x) \coloneqq :e^{\pm\sqrt{2}i\int_\infty^x du\, j(u)}: = J^1(x) \pm iJ^2(x)$$

and J_1, J_2, together with $J_3 = j/\sqrt{2}$, generate an $\mathfrak{su}(2)$ current algebra and J_0 plays the role of the $\mathfrak{u}(1)$ contribution. After performing such construction care must be taken to the fact that the vertex operators in general do not act on the same Hilbert space as the constituting currents.

5.3 STRESS-ENERGY TENSOR MODELS

The same game can be played with the stress-energy tensor and its Virasoro generators. Starting from a single Fermi field $\psi(z)$ one can construct the related stress-energy tensor (again following the quarks construction) as

$$T^{c=1/2}(z) = \frac{-1}{4\pi} :\psi\partial_z\psi:(z) = \frac{-1}{8\pi} :\psi\overleftrightarrow{\partial}_z\psi:(z)$$

with central charge $c = 1/2$. Commutation relations follow by implementing its decomposition in terms of Virasoro generators

$$T(z) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

where L_n commute as in (2.2.1). We obtain, on the real line and on the circle picture respectively:

$$[T(x), T(y)] = i\,(T(x) + T(y))\,\delta'(x-y) - i\,\frac{c}{24\pi}\,\delta'''(x-y)$$

$$\begin{aligned}[T(z), T(w)] = &\frac{-1}{2\pi}\left(\frac{T(z)}{w^2} + \frac{T(w^2)}{z^2}\right) \sum_{n \in \mathbb{Z}} n \left(\frac{z}{w}\right)^n \\ &- \frac{c}{48\pi^2}\,\frac{1}{z^2 w^2} \sum_{n \in \mathbb{Z}} (n^3 - n) \left(\frac{z}{w}\right)^n .\end{aligned}$$

In case of a complex Fermi field $\phi(z) = \psi^{(1)}(z) + i\psi^{(2)}(z)$ the related stress-energy tensor is exactly two copies of the individual stress-energy tensors constructed out of each of the two real Fermi fields $\psi^{(1)}(z)$ and $\psi^{(2)}(z)$

$$T^{c=1}(z) = \frac{-1}{4\pi} :\psi^{(1)}\partial_z\psi^{(1)}:(z) + \frac{-1}{4\pi} :\psi^{(2)}\partial_z\psi^{(2)}:(z) = \frac{-1}{8\pi} :\phi^* \overset{\leftrightarrow}{\partial}_z \phi:(z)$$

The central charge is $2 \cdot 1/2 = 1$. Let us now define again

$$\begin{aligned} T^{2c}(z) &\coloneqq T^c(z) \otimes \mathbf{1} + \mathbf{1} \otimes T^c(z) \\ D^{2c}(z) &\coloneqq T^c(z) \otimes \mathbf{1} - \mathbf{1} \otimes T^c(z) \end{aligned}$$

and likewise the assignment

$$\alpha\left(T^{2c}(z^2)\right) \coloneqq \frac{T^c(z) + T^c(-z)}{4z^2} + \frac{c}{32\pi z^4} \tag{5.3.1}$$

$$\alpha\left(D^{2c}(z^2)\right) \coloneqq \frac{T^c(z) - T^c(-z)}{4z^2} \tag{5.3.2}$$

which implies, for the Virasoro modes

$$\alpha\left(L^{2c}_n\right) = \frac{1}{2} L^c_{2n} + \frac{c}{16}\,\delta_{n,0} \qquad n \in \mathbb{Z}$$

$$\alpha\left(D^{2c}_\nu\right) = \frac{1}{2} L^c_{2\nu} \qquad \nu \in \mathbb{Z} + 1/2.$$

Virasoro relations for the generators L^c_n imply

$$\begin{aligned} \left[\alpha\left(L^{2c}_m\right), \alpha\left(L^{2c}_n\right)\right] &= (m-n)\alpha\left(L^{2c}_{m+n}\right) + \frac{2c}{12}(m^3 - m)\,\delta_{m+n,0} \\ \left[\alpha\left(L^{2c}_m\right), \alpha\left(D^{2c}_\nu\right)\right] &= (m-\nu)\alpha\left(D^{2c}_{m+\nu}\right) \\ \left[\alpha\left(D^{2c}_\mu\right), \alpha\left(D^{2c}_\nu\right)\right] &= (\mu-\nu)\alpha\left(D^{2c}_{\mu+\nu}\right) + \frac{2c}{12}(\mu^3 - \mu)\,\delta_{\mu+\nu,0} \end{aligned}$$

summing up the Fourier modes we obtain

$$\begin{aligned} \left[\alpha\left(T^{2c}(z^2)\right), \alpha\left(T^{2c}(w^2)\right)\right] &= \alpha\left[\left(T^{2c}(z^2)\right), \left(T^{2c}(w^2)\right)\right] \\ \left[\alpha\left(T^{2c}(z^2)\right), \alpha\left(D^{2c}(w^2)\right)\right] &= \alpha\left[\left(T^{2c}(z^2)\right), \left(D^{2c}(w^2)\right)\right] \\ \left[\alpha\left(D^{2c}(z^2)\right), \alpha\left(D^{2c}(w^2)\right)\right] &= \alpha\left[\left(D^{2c}(z^2)\right), \left(D^{2c}(w^2)\right)\right] \end{aligned}$$

meaning that α, again, preserves the commutation relations. We have thus embedded two copies of the stress-energy tensor algebra of central charge $c = 1/2$ localised in I into one copy of the same algebra localised in $\sqrt{I}$. Denoting such algebra as $\mathrm{Vir}_{1/2}(I)$ we have

$$\alpha\colon \mathrm{Vir}_{1/2}(I) \otimes \mathrm{Vir}_{1/2}(I) \subset \mathrm{Vir}_1(I) \to \mathrm{Vir}_{1/2}(\sqrt{I}).$$

Notice that the assignment α turns out to be nothing but the composition of β with an automorphism of the current algebra $\rho^{\frac{1}{4}}$ as in [Rehren and Tedesco, 2013] with $\rho^q(j(z)) = j(z) + q/z$. Thus we have the identification $\alpha = \beta \circ \rho^{\frac{1}{4}}$ and ρ plays the role of the gauge transformation γ we have to compose β with in order to obtain suitable homomorphisms. As a consequence, though β itself does not restric to subalgebras, compositions with suitable inner automorphisms do and we can extend the picture to fields of conformal scaling dimension 2

$$\text{Fermi}^{\;d=\frac{1}{2}} \qquad \begin{matrix} \phi(z^2) \\ \phi^*(z^2) \end{matrix} \;\xrightarrow{\beta}\; \begin{matrix} \psi(z) + \psi(-z) \\ z^{-1}\left(\psi(z) - \psi(-z)\right) \end{matrix}$$

$$\text{Currents}^{\;d=1} \qquad \begin{matrix} J_0(z^2) \\ J_3(z^2) \end{matrix} \;\xrightarrow{\beta\circ\gamma}\; \begin{matrix} z^{-1}\left(j(z) - j(-z)\right) \\ z^{-1}\left(j(z) + j(-z)\right) \end{matrix}$$

$$\text{Virasoro}^{\;d=2} \qquad \begin{matrix} T^1(z^2) \\ D^1(z^2) \end{matrix} \;\xrightarrow{\beta\circ\rho^{\frac{1}{4}}}\; \begin{matrix} z^{-2}\left(T^{\frac{1}{2}}(z) + T^{\frac{1}{2}}(-z)\right) \\ z^{-2}\left(T^{\frac{1}{2}}(z) - T^{\frac{1}{2}}(-z)\right) \end{matrix}$$

Again, we have a manifestation of the twist carried by β as flip on one of the two fields

$$\wp\left(T^1(z)\right) = T^1(z),$$
$$\wp\left(\Delta^1(z)\right) = -\Delta^1(z).$$

5.4 COSET MODELS

The quarks construction as described in 2.3 shows that starting from a Lie algebra $\mathfrak{g} \subset \mathfrak{u}(n)$ one can construct a stress-energy tensor with a suitable normalisation ξ as

$$T_S(z) \coloneqq \xi\, \kappa_{ab} : J^a J^b :(z)$$

whose central charge is given by (2.3.6) and whose Fourier modes satisfy the Virasoro algebra commutation relations. Let now take $\mathfrak{h}$ as a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let us apply the Sugawara construction to $\mathfrak{h}$ and $\mathfrak{g}$ respectively. In general, the two stress-energy tensors, which we denote as $T^{\mathfrak{h}}, T^{\mathfrak{g}}$ do not coincide. By taking $T^{\mathfrak{g}/\mathfrak{h}}(z) \coloneqq T^{\mathfrak{g}}(z) - T^{\mathfrak{h}}(z)$ (as shown in [Goddard, Kent, and Olive, 1986], [Goddard, Kent, and Olive, 1985]) we can construct another stress-energy tensor ("coset" SET) commuting with the $T^{\mathfrak{h}}$ whose central charge is the difference of the central charges of the constituent models

$$c^{\mathfrak{g}/\mathfrak{h}} = c^{\mathfrak{g}} - c^{\mathfrak{h}}.$$

An important class of such models is given by the so called "diagonal" embeddings of an algebra into many of its commuting copies, $\mathfrak{g} \subset \mathfrak{g} \oplus \ldots \oplus \mathfrak{g}$. The total current is then just the sum of each single current

$$J^a(x) = (j^a(x) \otimes \mathbf{1} \otimes \ldots \mathbf{1} + \ldots + \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes j^a(x))$$

and since the different copies commute with one other the level of the resulting algebra is just the sum of the level of each diagonal component. As a consequence the total central charge for the stress-energy tensor is

$$c = \left(\frac{k_1}{k_1 + g} + \ldots + \frac{k_n}{k_n + g} + \frac{k_1 + \ldots + k_n}{k_1 + \ldots k_n + g} \right) \dim \mathfrak{g}.$$

For example, taking k copies of (level $k = 1$) $\mathfrak{su}(n)$ current models we can construct an $\mathfrak{su}(n)$ at level k current model. Also, by iteration of this method one can get representations of Virasoro algebras with $c < 1$ just by taking diagonal embeddings of $\mathfrak{su}(n)_{k+1}$ into $\mathfrak{su}(n)_k \oplus \mathfrak{su}(n)_1$, the total central charge being given by

$$T^{\mathfrak{su}(n)_k \oplus \mathfrak{su}(n)_1 / \mathfrak{su}(n)_{k+1}}, \qquad c_k + c_1 - c_{k+1} = 1 - \frac{6}{(k+2)(k+3)}.$$

Similarly, the stress-energy tensor of two complex Fermi fields has central charge $c = 2$, while the stress-energy tensor for the embedded $\mathfrak{su}_1$ has $c = 1$; therefore one can construct a coset stress-energy tensor whose central charge is $c = 2 - 1 = 1$ and this happens to be exactly the abelian contribution $\mathfrak{u}(1)$ into $\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ of the form

$$T(x) = \frac{1}{4\pi} {:}j^2{:}(x)$$

(see, for example, [Fuchs, 1992]).

5.5 EMBEDDING VIA LONGO-XU MAP

We have obtained the results previously mentioned by using the embedding of β and some gauge transformations suitably chosen. We shall now show that the same result can be easily achieved by using the general transformation properties of conformal fields under diffeomorphisms and the Longo-Xu map ([Longo and Xu, 2004] and 4.6). As a recall we assume to be equipped with a conformal net $I \to \mathcal{A}(I)$ fulfilling the split property: therefore, given I_1, I_2 there exists an isomorphism $\chi\colon \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \to \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ such that $\chi(x_1 x_2) = x_1 \otimes x_2$, however you choose $x_1 \in I_1, x_2 \in I_2$. Diffeomorphisms of the net $\mu\colon I \to I_j$ are implemented on the algebras by means of adjoint action of unitaries $U(\mu)$ and this provides isomorphisms $\mathcal{A}(I) \to \mathcal{A}(I_j)$ given by ([Longo and Xu, 2004])

$$\phi_I := \operatorname{Ad} U(\mu) | \mathcal{A}(I).$$

Consequently the Longo-Xu map $LX = \chi \circ \phi_I^N$ gives an isomorphism

$$LX\colon \mathcal{A}^N(I) \to \mathcal{A}(I_1 \cup \ldots \cup I_N)$$

explicitly realised as $LX(x_1 \otimes \ldots \otimes x_N) = \phi_I(x_1) \cdot \ldots \cdot \phi_I(x_N)$.

Let us now restrict ourselves to the particular case of two intervals and diffeomorphisms given in terms of square root maps, namely $\mu_1(z^2) = z$, $\bar{\mu}(z^2) = -z$. They are implemented on fields as

$$\phi_I(\varphi(z)) = \operatorname{Ad} U(\sqrt{(\,\cdot\,)})\varphi(z) = \left(\frac{\partial\mu(z)}{\partial z}\right)^h \varphi(\mu(z))$$

h being the conformal scaling dimension. The corresponding Longo-Xu map is $LX\colon \mathcal{A}^2(I) \to \mathcal{A}\left(\sqrt{I}\right)$. On the doubled currents

$$\begin{aligned} J^{2c}(z^2) &= J^c(z^2) \otimes \mathbf{1} + \mathbf{1} \otimes J^c(z^2) \\ \Delta^{2c}(z^2) &= J^c(z^2) \otimes \mathbf{1} - \mathbf{1} \otimes J^c(z^2) \end{aligned}$$

the Longo-Xu map acts as

$$LX\left(J^{2c}(z^2)\right) = \left(\frac{1}{2z}\right)^1 j^c(z)\mathbf{1} + \mathbf{1}\left(\frac{-1}{2z}\right)^1 j^c(-z) = \frac{j(z) - j(-z)}{2z}$$

$$LX\left(\Delta^{2c}(z^2)\right) = \left(\frac{1}{2z}\right)^1 j^c(z)\mathbf{1} - \mathbf{1}\left(\frac{-1}{2z}\right)^1 j^c(-z) = \frac{j(z) + j(-z)}{2z}$$

which exactly correspond to equations (5.2.5) and (5.2.6). Consequently we derive that $\alpha = \beta \circ \gamma = LX$ and thus β and LX are related to each other through a gauge transformation γ. Of course this must be the case, since the Longo-Xu map does not preserve the vacuum state (diffeomorphisms "destroy" correlations), while β does.

Similarly we can apply the Longo-Xu map to the stress-energy tensor and its doubled copy

$$\begin{aligned} T^{2c}(z) &\coloneqq T^c(z) \otimes \mathbf{1} + \mathbf{1} \otimes T^c(z) \\ D^{2c}(z) &\coloneqq T^c(z) \otimes \mathbf{1} - \mathbf{1} \otimes T^c(z) \end{aligned}$$

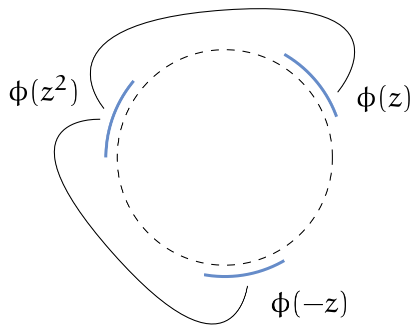
keeping in mind that T does not transform as a primary field under diffeomorphisms, rather it is quasi-primary and an extra contribution due to the Schwarz derivative occurs.

$$\begin{aligned} LX\left(T^{2c}(z^2)\right) &= \left(\left(\frac{1}{2z}\right)^2 T(z) + \frac{c}{12} s(g(z), z)\right)\mathbf{1} \\ &\quad + \mathbf{1}\left(\left(\frac{-1}{2z}\right)^2 T(-z) + \frac{c}{12} s(g(z), z)\right) \\ &= \frac{T(z) + T(-z)}{4z^2} + \frac{c}{32\pi z^4}. \end{aligned}$$

On the other hand the Schwarzian derivative cancels out if we take the difference

$$LX\left(D^{2c}(z^2)\right) = \frac{T(z) - T(-z)}{4z^2}.$$

We have obtained equations (5.3.1) and (5.3.2) just via mere application of the diffeomorphisms invariance and the split property, which we assume to hold for the net at hand. We deduce again $\alpha = \beta \circ \rho^{\frac{1}{4}} = LX$.



The picture that we have now is that, for each scaling dimension, i.e. for Fermi fields, embedded currents and stress-energy tensor, although β does not exactly restrict to the respective subalgebra, its composition with suitable gauge transformations gives back exactly the Longo-Xu map, which in turn is the manifestation of the diffeomorphisms covariance of the net (assumed the split property to hold). Both maps, β and LX, somehow "distribute" fields around the circle $I \to \sqrt{I}$ and they are related to each other via tailor made gauge transformations:

$$\beta = LX \circ \text{gauge}$$

where these acquires the explicit forms

$$\begin{aligned} \text{Fermi fields:} \quad & LX^{d=\frac{1}{2}} = \beta \circ O^{CH} \\ \text{Currents:} \quad & LX^{d=1} = \beta|_J \circ \gamma \\ \text{Stress-energy tensor:} \quad & LX^{d=2} = \beta|_{Vir} \circ \rho^{\frac{1}{4}} \end{aligned}$$

(of course we can read off $\gamma = O^{CH}|_J$ and likewise $\rho^{\frac{1}{4}} = O^{CH}|_{Vir}$).

5.6 MODULAR THEORY FOR CURRENTS

In the previous paragraphs we introduced the doubled theory of currents $\mathcal{A}^J \otimes \mathcal{A}^J$ as embedded from fermions using the quark construction: given a theory $\mathcal{A}^2(I)$ describing two fermions in one interval we can generate the embedded theory of currents $\mathcal{A}^J(I) \hookrightarrow \mathcal{A}^2(I)$. Furthermore, we found out that the restriction of $\beta|_J$ can still be written as $\beta|_J = LX|_F \circ \gamma$, where $LX|_F$ denotes the restriction of the Longo-Xu map to the embedded fermions. Of course, since β preserves the vacuum state for Fermions, so does it restriction to currents. Nonetheless, one may wonder how the very particular form for gauge transformations on currents, equation (5.2.1), may fit so that the correlators are eventually preserved. This is due to the particular action of β on Fermi fields; in fact its peculiarity is to distribute fields onto anti-podal points and this feature reflects on the currents, as in formula (4.7.15). The presence of delocalised factors in, say, $z, -z$ exactly cancels out the additional central term appearing in the gauge transformations so that everything cancels out eventually preserving the form of the vacuum expectation values. As a matter of example we shall present the case of a doubled theory of currents originated from four fermions.

Example: Let $\mathcal{A}^2(I)$ be the theory describing two fermions, say ψ_1, ψ_2 and $\mathcal{A}^2(I) \otimes \mathcal{A}^2(I)$ its double. The four fermions generated thereof can be labelled as

$$\Psi^1 = \psi_1 \otimes \mathbf{1}, \quad \Psi^2 = \psi_2 \otimes \mathbf{1}, \quad \Psi^3 = \mathbf{1} \otimes \psi_1, \quad \Psi^4 = \mathbf{1} \otimes \psi_2.$$

These four fermions can generate $\binom{4}{2} = 6$ different currents $J_{ij}(z) := \Psi^i(z)\Psi^j(z) \in \mathcal{A}^J(I) \otimes \mathcal{A}^J(I)$ (no Wick product occurs because the vacuum expectation values vanish anyway) generating in turn a non-abelian current algebra with gauge group $O(4)$. We shall see that the action of the Longo-Xu map on these currents is local on some pairing, whereas it is non-local on some others. In fact, let us take the action of diffeomorphisms $\mu_j \colon I \to I_j$ as $\mu_1(z) = \sqrt{z}, \mu_2(z) = -\sqrt{z}$ upon, for instance, $J_{12}(z) = \Psi^1(z)\Psi^2(z) = \psi_1(z)\psi_2(z) \otimes \mathbf{1}$; we have

$$\begin{aligned} LX(J_{12}(z)) &= LX(\psi_1(z)\psi_2(z) \otimes \mathbf{1}) \\ &= \sqrt{\mu_1'(z)}\,\psi_1(\mu_1(z))\,\sqrt{\mu_1'(z)}\,\psi_2(\mu_1(z)) \cdot \mathbf{1} \end{aligned}$$

because the diffeomorphisms both act on the first term in the tensor product, distributing the fields in $\mu_1(z)$. Then

$$LX(J_{12}(z)) = \mu_1'(z)\,\psi_1(\mu_1(z))\,\psi_2(\mu_1(z)) = \mu_1'(z)\,{:}\psi_1\psi_2{:}(\mu_1(z))$$

exploiting ${:}\psi_1\psi_2{:} = \psi_1\psi_2 - \omega_0(\psi_1\psi_2) = \psi_1\psi_2 - 0$. We can consequently state that the current J_{12} is distributed locally at the point $\mu_1(z)$; the same happens for those other currents having the initial fermions in the same position in the tensor product, like, for example, $J_{34}(z) = \Psi^3(z)\Psi^4(z) = \mathbf{1} \otimes \psi_1(z)\psi_2(z)$

$$LX(J_{34}(z)) = \mu_2'(z)\,{:}\psi_1\psi_2{:}(\mu_2(z));$$

we conclude then that the two possible local actions are the following ones:

$$LX(J_{12}(z)) = \mu_1'(z)\,j_{12}(\mu_1(z)), \qquad LX(J_{34}(z)) = \mu_2'(z)\,j_{12}(\mu_2(z)).$$

The other possible pairings present non-local contributions, as well as, for example, $J_{13}(z) = \Psi^1(z)\Psi^3(z) = \psi_1(z) \otimes^t \psi_1(z)$

$$LX(J_{13}(z)) = \sqrt{\mu_1'(z)}\,\psi_1(\mu_1(z))\,\sqrt{\mu_2'(z)}\,\psi_1(\mu_2(z));$$

because of the presence of the same fermion field ψ_1, the vacuum expectation value is non-zero and therefore we get

$$\begin{aligned} LX(J_{13}(z)) &= \sqrt{\mu_1'(z)\mu_2'(z)}\,\psi_1(\mu_1(z))\,\psi_1(\mu_2(z)) \\ &= \sqrt{\mu_1'(z)\mu_2'(z)}\,{:}\psi_1(\mu_1(z))\,\psi_1(\mu_2(z)){:} - \frac{\sqrt{\mu_1'(z)\mu_2'(z)}}{\mu_1(z) - \mu_2(z)} \end{aligned}$$

which is delocalised in the two points $\mu_1(z), \mu_2(z)$.

The action of the diffeomorphisms is a "true" action only on some of the currents, taking them into actual currents localised elsewhere. This can be viewed as a true action on the currents of the subgroup $O(2) \times O(2) \subset O(4)$, while the remaining currents are moved in $\mu_1(z), \mu_2(z)$ without summing up again to actual currents.

We can use this argument to reconstruct the form of the one-point function. In fact, since $\omega_0(J(z)) = 0$, we expect $\beta|_J$ to preserve $\omega_0 \circ \beta|_J = \omega_0$. Acting on currents we obtain

$$\begin{aligned}\omega_0 \circ \beta|_J(J(z)) &= \omega_0 \circ LX \circ \gamma(J(z)) \\ &= \omega_0 \circ LX(J(z) + \text{central term})\end{aligned}$$

where $(J(z) + \text{central term})$ has to be intended as in equation (5.2.1) and the central term is of the form $1/2\,\kappa\ \text{tr}\left(g(z)^{-1}\partial_z g(z)\sigma^a\right)$. Using the explicit form of the Lie algebra valued gauge transformations and the structure constants the trace sums up to either zero or the identity, the only numerical prefactors being derivatives of the diffeomorphisms μ in the point z, which cancel the presence of the additional vacuum expectation value in some delocalised currents appearing in the model once we act with the Longo-Xu map. Again, the multilocal behaviour of β helps to prevent obstructions and to preserve the one-point function:

$$\begin{aligned}\omega_0 \circ \beta|_J(J(z)) &= \omega_0 \circ LX(J(z) + \text{central term}) \\ &= \omega_0\left(\underbrace{J(z) - \frac{\sqrt{\mu_1'(z)\mu_2'(z)}}{\mu_1(z)-\mu_2(z)} + \text{central term}}_{\longleftarrow\ \text{cancellations}\ \longrightarrow}\right) \\ &= \omega_0(J(z)) = 0.\end{aligned}$$

On the other hand, if the action were strictly local on all the involved currents, then obstructions for the vacuum one-point function would definitely occur, because cancellations would no more take place and the additional central term arising from the gauge transformations could not be wiped off. As a consequence, if we start from a pure local theory of currents, no vacuum preserving isomorphism in the form $\beta = LX \circ \gamma$ may exist. As distribution $\omega_0(j(x)) = 0$ means that there can be no class of functions f such that $\omega_0(j(f)) \neq 0$. From this it directly follows that a similar argument applies to the non-existence of a vacuum preserving isomorphism for Bose fields; in fact, if this were true then we would have

$$\omega_0 \circ \beta\,(\varphi(f)) = \omega_0\,(\varphi(f))$$

choosing f integrating to zero. Then, since $\varphi(f) = -j(g)$, with $f(x) = g'(x)$ then this would imply

$$\omega_0 \circ \beta\,(j(g)) = \omega_0\,(j(g)) = 0$$

and this cannot hold due to the explicit form of $\beta = LX \circ \text{gauge}$. In fact, as we pointed out, $\beta(j) \sim j + \text{const.} \cdot \mathbf{1}$ and thus

$$\omega_0 \circ \beta\,(j(g)) = \omega_0(j(g)) + \int_{\mathbb{R}} dx\, g(x) = 0 + \int_{\mathbb{R}} dx\, g(x)$$

and this is again if also $g(x)$ integrates itself to zero, but this cannot go along with the integral of $f(x) = g'(x)$ being zero as well.

Example: Let us take again the doubled theory of two fermions, $\mathcal{A}^2(\mathrm{I}) \otimes \mathcal{A}^2(\mathrm{I})$, containing the four fermions as stated in the previous example

$$\Psi^1 = \psi_1 \otimes \mathbf{1}, \quad \Psi^2 = \psi_2 \otimes \mathbf{1}, \quad \Psi^3 = \mathbf{1} \otimes \psi_1, \quad \Psi^4 = \mathbf{1} \otimes \psi_2.$$

The corresponding stress-energy tensor is, by construction,

$$T(z) = -\frac{1}{4\pi} \sum_{i=1}^{4} {:}\Psi^i \partial_z \Psi^i{:}(z)$$

and thus

$$\begin{aligned} T^{2c}(z) = &-\frac{1}{4\pi} {:}\psi_1 \partial_z \psi_1{:}(z) \otimes \mathbf{1} - \frac{1}{4\pi} {:}\psi_2 \partial_z \psi_2{:}(z) \otimes \mathbf{1} \\ &+ \mathbf{1} \otimes -\frac{1}{4\pi} {:}\psi_1 \partial_z \psi_1{:}(z) + \mathbf{1} \otimes -\frac{1}{4\pi} {:}\psi_2 \partial_z \psi_2{:}(z), \end{aligned}$$

which is nothing but $T^{2c}(z) = T^c(z) \otimes \mathbf{1} + \mathbf{1} \otimes T^c(z)$. Due to the fact that there are no mixed terms paired in the tensor products, the action of the LX map is local on each component of the tensor product individually. We have

$$\begin{aligned} \mathrm{LX}\big(T^{2c}(z)\big) &= \mathrm{LX}\big(T^c(z) \otimes \mathbf{1} + \mathbf{1} \otimes T^c(z)\big) \\ &= \Big(\mu_1'(z)^2\, T^c(\mu_1(z)) + \frac{c}{12} s(\mu_1(z), z)\Big) \cdot \mathbf{1} \\ &\quad + \mathbf{1} \cdot \Big(\mu_2'(z)\, T^c(\mu_2(z)) + \frac{c}{12} s(\mu_2(z), z)\Big). \end{aligned}$$

On the other hand if one considers the Sugawara stress-energy tensor $T_s(z) = \xi\, \kappa_{ab} {:}J^a J^b{:}(z)$ with currents given by $J_{ij}(z) = \Psi^i(z)\Psi^j(z)$ then anti-local components may occur, according to the choice of the Lie algebra.

Part III

BACK MATTER

> I still belong to the minority of people who believe that the universe is four dimensional.
>
> — D. Buchholz, Rome, July 2013.

A

APPENDIX

Contents

A.1 ON THE PASSAGE TO TWO-DIMENSIONAL MODELS

As aforementioned, we focused our attention on one-dimensional chiral models where fields depend on the light-cone variables $x_\pm$ only. Taking two such theories, respectively described by the nets of algebras $\mathcal{A}(I_+), \mathcal{A}(I_-)$ (with obvious understanding of notations), the chiral two-dimensional model is given by the tensor product $\mathcal{A}(\mathcal{O}) = \mathcal{A}(I_+) \otimes \mathcal{A}(I_-) \subset \mathcal{B}(\mathcal{O})$, with the space-time region $\mathcal{O}$ given by $I_+ \times I_-$. For the tensor product theory of observables the vacuum state (actually its GNS representation) is the tensor product $\Omega \otimes \Omega$ (acting on $\mathcal{H} \otimes \mathcal{H}$) and therefore the modular theory derived thereof decomposes into tensor products as well. In fact the anti-linear operator (3.3.1) becomes $S_0 \colon a(\Omega \otimes \Omega) \mapsto a^*(\Omega \otimes \Omega)$, $a \in \mathcal{A}(O)$ and the corresponding modular operator is the tensor product $\Delta_{\mathcal{O}}^{it} = \Delta_+^{it} \otimes \Delta_-^{it}$, giving rise to modular automorphisms group as $\sigma^t = \sigma^t_{I_+} \otimes \sigma^t_{I_-}$ by verification of the KMS condition. In particular $\mathcal{O}$ are double cones if I_+, I_- are two intervals, the forward light cone $\mathcal{V}_+$ as $\mathbb{R}_+ \times \mathbb{R}_+$ and the right wedge as $\mathbb{R}_+ \times \mathbb{R}_-$. Replacing everywhere $\mathbb{R}_+ \to \mathbb{R}_-$ gives the backward light cone and the left wedge, respectively.

A.2 MORE ON THE GEOMETRIC ACTION OF MODULAR GROUPS FOR SPECIAL REGIONS

It has been pointed out that the most of the modular theory relies on the result of Bisognano and Wichmann (3.3.6) expressing the modular group and the modular conjugation for Wightmann fields localised in wedge regions. This results allows some sort of generalisations to the cases of space-time regions that can be obtained as geometric transformations of the wedges, provided the vacuum vector to be invariant under such transformations. In particular, we shall recall here a remarkable result found out by [Hislop and Longo, 1982] for double cones and massless scalar fields obeying the Klein-Gordon equation.

The original result by Hislop and Longo refers to the four-dimensional case, but nevertheless it can be transferred to two dimensions. In particular, in order to do so, the scalar field $\phi(f)$ has to be smeared with test functions which are light-cone variables derivatives, namely $f = \partial_\pm g$, g being an appropriate test function. We proceed by noticing that double cones $\mathcal{O}$ can be mapped into wedge regions $\mathcal{W}$ by means of the inversion map

$$\rho\colon (x^0, x^1) \mapsto \rho(x^0, x^1) = \frac{1}{|x|^2}\left(-x^1, -x^0\right), \quad |x|^2 = (x^0)^2 - (x^1)^2.$$

Given $\phi(f)$ as a solution of the Klein-Gordon equation $\phi(\Box f) = 0$, with f a function of the said form, the authors showed that this action can be implemented on the one-particle Hilbert space $\mathcal{H}$ through U_ρ ([Hislop and Longo, 1982])

$$U_\rho \phi(f)\Omega = \phi(f_\rho)\Omega$$

where $f_\rho(x) = -(|x|^2)^{-3} f(\rho(x))$. Due to the conformal symmetry, U_ρ extends to a unitary operator $\Gamma(U_\rho)$ onto the Fock space $\mathcal{F}(\mathcal{H})$ preserving the vacuum state whose action is given by $\Gamma(U_\rho)\phi(f)\Gamma(U_\rho)^* = \phi(f_\rho)$ and gives rise to an isometry between the Weyl algebra of the double cone and the wedge region $\Gamma(U_\rho)\mathcal{A}(\mathcal{O})\Gamma(U_\rho)^* = \mathcal{A}(\rho(\mathcal{O})) = \mathcal{A}(\mathcal{W})$. Since such a unitary preserves the vacuum state, it does also connect the modular objects corresponding to $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\mathcal{W})$ as

$$\begin{aligned} J_{\mathcal{O}} &= \Gamma(U_\rho)\, J_{\mathcal{W}}\, \Gamma(U_\rho)^* \\ \Delta_{\mathcal{O}}^{it} &= \Gamma(U_\rho)\, \Delta_{\mathcal{W}}^{it}\, \Gamma(U_\rho)^* \end{aligned}$$

resulting in a geometric action of the modular group within the double cone given in terms of conformal transformations as

$$x_\pm = \frac{1 + x_\pm - e^{-s}(1 - x_\pm)}{1 + x_\pm + e^{-s}(1 - x_\pm)}$$

with $x_\pm$ the standard light-cone variables and s a real dilation parameter.

A similar geometric transformation can be introduced to map the open double cone $\mathcal{O}'$ into the forward light cone $\mathcal{V}_+$ so that the algebras $\mathcal{A}(\mathcal{O}')$ and $\mathcal{A}(\mathcal{V}_+)$ are equivalent by means of the unitary $\Gamma(T(1/2)U_\rho T(-1))$, where $T(\lambda)$ implements time translations. The relations

$$\begin{aligned} J_{\mathcal{O}'} &= \Gamma\big(T(1/2)U_\rho T(-1)\big)\, J_{\mathcal{V}_+}\, \Gamma\big(T(1/2)U_\rho T(-1)\big)^* \\ \Delta_{\mathcal{O}'}^{it} &= \Gamma\big(T(1/2)U_\rho T(-1)\big)\, \Delta_{\mathcal{V}_+}^{it}\, \Gamma\big(T(1/2)U_\rho T(-1)\big)^* \end{aligned}$$

reproducing a well known result of Buchholz [Buchholz, 1977] stating that the modular operator and conjugation for the forward light cone are respectively given by dilations and CPT inversion mapping $\mathcal{V}_+$ onto the backward light cone $\mathcal{V}_-$, that is $\Delta_{\mathcal{V}_+}^{it} = \Gamma(\delta(2\pi\lambda))$ and $J_{\mathcal{V}_+} = \Gamma(-CPT)$.

In case of massive theories the action of the modular group is known only for wedge regions, again due to the Bisognano-Wichmann property. Since massive theories are in general not conformally invariant this result cannot be transferred to double cones and similar regions, unlike the massless cases. It can be shown that the general action has to be non-local and presumably given in terms of pseudo-differential operators; in particular if $\delta = \partial_t \Delta^{it}|_{t=0}$ is the infinitesimal generator of the modular group, then $\delta = \delta_0 + \delta_m$, where δ_0 is the standard massless generator and δ_m is expressed in terms of the action of a pseudo-differential operator depending on the mass. We refer the reader to [Saffary, 2006; Yngvason, 1994] for progresses in these directions.

A.3 CORRELATIONS FUNCTIONS IN CONFORMAL FIELD THEORY

In this section we are going to have a closer look at the explicit form of correlations functions in conformal field theory. In particular we shall see that conformal invariance, especially in low dimensions, poses strong restrictions to the form of such correlations functions and almost fixes them all, up to some constants, in the case of two and three points functions.

In two dimensions the conformal group reduces to the set of holomorphic and anti-holomorphic functions and in the special case of chiral theories we are allowed to look at each copy singularly. This means that, as already pointed out, the conformal group decomposes into two copies, each one of them is generated by the Virasoro algebra (2.2.1)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}m(m^2-1)\,\delta_{n+m}\mathbf{1}.$$

A special class of fields is given by primary fields, as described in 2.4, which have the special property to transform as in (2.4.2)

$$\phi(z) = \left(\frac{dg}{dz}\right)^h \phi'(g(z))$$

under conformal transformations, h being the conformal dimension of the fields. Here $z \mapsto g(z)$ is the conformal mapping for either of the holomorphic or anti-holomorphic variables, one at a time. This corresponds to the finite exponential form of the infinitesimal commutation relations between such fields and the generators of the Virasoro algebra

$$[L_n, \phi(z)] = h(n+1)z^n\phi(z) + z^{n+1}\partial_z\phi(z).$$

Correlations functions are defined as vacuum expectation values of products of fields (in the sense of tempered distribution)

$$w^n(x_1, \dots, x_n) \coloneqq (\Omega, \phi_1(x_1)\dots\phi_n(x_n)\Omega)$$

and the idea is now that, if we require the above quantities to be invariant under conformal transformations, we may fix the form of such functions up to some degrees of freedom. In particular we have to impose that $w^n(x_1, \dots, x_n) = w'^n(x'_1, \dots, x'_n)$, namely

$$(\Omega, \phi_1(x_1)\dots\phi_n(x_n)\Omega) = (\Omega, \phi'_1(x'_1)\dots\phi'_n(x'_n)\Omega)$$

where, with obvious understanding of notations, $\phi'_k(x'_k)$ is the new field after a change under conformal transformations given by $\phi'(x') = U\,\phi(x)\,U^*$. Exploiting such formula we find interesting results.

A.3.1 *The two-point function*

Let us concentrate first on the two-point function for primary fields $w^{(2)}(x_1, x_2) = (\Omega, \phi_1(x_1)\phi_2(x_2)\Omega)$ and let us impose invariance under translations, dilations and special conformal transformations keeping in mind that primary fields change as

$$
\begin{aligned}
i[P, \phi(x)] &= \partial_x \phi(x) \\
i[D, \phi(x)] &= (x\partial_x + h)\phi(x) \\
i[K, \phi(x)] &= (x^2\partial_x + 2hx)\phi(x)
\end{aligned}
$$

where we have seen in chapter 2.2.1 that P, D, K can be expressed in terms of $L_0, L_{\pm 1}$. Invariance under translations requires

$$
(\Omega, [P, \phi_1(x_1)\phi_2(x_2)]\Omega) = 0;
$$

using $[A, BC] = B[A, C] + [A, B]C$ we are led to

$$
\begin{aligned}
0 &= (\Omega, \phi_1(x_1)\,[P, \phi_2(x_2)]\Omega) + (\Omega, [P, \phi_1(x_1)]\,\phi_2(x_2)\Omega) \\
&= (\partial_{x_1} + \partial_{x_2})w^{(2)}(x_1, x_2)
\end{aligned}
$$

and thus $w^{(2)}(x_1, x_2)$ depends only on the difference of the two variables $w^{(2)}(x_1, x_2) = w^{(2)}(x_1 - x_2)$, which is the standard form required by translations invariance. Along the same lines, for dilations invariance we have

$$
\begin{aligned}
0 &= (\Omega, [D, \phi_1(x_1)\phi_2(x_2)]\Omega) \\
&= (x_1\partial_{x_1} + h_1 + x_2\partial_{x_2} + h_2)w^{(2)}(x_1, x_2)
\end{aligned}
$$

introducing the variable $x = x_1 - x_2$, as we have seen above, we obtain $(x\partial_x + h_1 + h_2)w^{(2)}(x) = 0$ and thus

$$
\frac{1}{w^{(2)}(x)}\,dw^{(2)}(x) = -(h_1 + h_2)\,\frac{1}{x}\,dx
$$

which integrates to $w^{(2)}(x) = c_{12}\,x^{-(h_1+h_2)}$; keep in mind that we want the correlations functions to diverge whenever the two points coincide, therefore for $x \to 0$. This implies that $h_1 + h_2$ must be positive. Last, but not the least, we impose invariance under special conformal transformations

$$
\begin{aligned}
0 &= (\Omega, [K, \phi_1(x_1)\phi_2(x_2)]\Omega) \\
&= (x_1^2\partial_{x_1} + 2h_1x_1 + x_2^2\partial_{x_2} + 2h_2x_2)w^{(2)}(x_1, x_2).
\end{aligned}
$$

To help the computation we can plug in the form $w^{(2)}(x) = c_{12}\, x^{-(h_1+h_2)}$ and work it out:

$$\begin{aligned}
0 &= \left(x_1^2(-1)(h_1+h_2)(x)^{-1} + 2h_1x_1 + x_2^2(h_1+h_2)(x)^{-1} + 2h_2x_2\right) w^{(2)}(x) \\
&= \left((x)^{-1}(h_1+h_2)(x_2^2 - x_1^2) + 2h_1x_1 + 2h_2x_2\right) w^{(2)}(x) \\
&= \left(-(x_1+x_2)(h_1+h_2) + 2h_1x_1 + 2h_2x_2\right) w^{(2)}(x) \\
&= (h_1-h_2)(x_1-x_2) w^{(2)}(x)
\end{aligned}$$

interestingly enough then, the two fields are correlated only if the two scaling dimensions coincide, $h_1 = h_2$. Of course, all the calculations must be intended in the sense of distribution, therefore for primary fields the two-point function takes the form

$$w^{(2)}(x_1 - x_2) = \lim_{\varepsilon \to 0^+} \left(\frac{c_{12}}{x_1 - x_2 - i\varepsilon} \right)^{2h}$$

where the normalisation constant c_{12} is the only parameter left free and can be calculated by imposing further requirements, as well as positivity of the scalar product in the Hilbert space (in the sense of operators) and spectrum conditions. It is straightforward now to derive back the expression of the two-point function for fermions and currents: substitution of $h = 1/2$ and $h = 1$ gives the results we have already stated by performing explicit calculations on the fields themselves.

A.3.2 *The three-point function*

Similar arguments can be undertaken for the three-point function too. Again, translations invariance states

$$(\Omega, [P, \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)]\Omega) = (\partial_{x_1} + \partial_{x_2} + \partial_{x_3}) w^{(3)}(x_1, x_2, x_3) = 0$$

meaning that $w^{(3)}(x_1, x_2, x_3)$ must depend on the pairwise difference of the variables $w^{(3)}(x_1 - x_2, x_1 - x_3, x_2 - x_3)$. Dilations invariance brings homogeneity

$$w^{(3)}(x_1, x_2, x_3) = \frac{c_{123}}{(x_1 - x_2)^a \cdot (x_1 - x_3)^b \cdot (x_2 - x_3)^c}$$

and special conformal invariance fixes the exponents a, b, c to be $a = h_1 + h_2 - h_3$, $b = h_2 + h_3 - h_1$, $c = h_3 + h_1 - h_2$; thus fields of different scaling dimension may still have non-vanishing three-point function.

Higher correlations functions might in principle be similarly derived, with the only difference that in this case Möbius invariance does not give enough restrictions as in the case of two and three points functions. Nevertheless the idea is always to start with the n-point function $w^{(n)}(x_1, \ldots, x_n)$ and impose invariance under a general change after conformal transformations; for each of the Möbius generators we have, in principle:

$$(\Omega, [L_n, \phi_1(x_1) \ldots \phi_n(x_n)]\Omega) = 0$$

and by multiple application of the Leibniz rule for commutators the above equation can be turned into a differential equation for the correlator as $Dw^{(n)}(x_1, \ldots, x_n) = 0$, where D is a differential operator, depending on case by case. Such differential equations are usually referred to as "Ward identities" and can be used to test concrete models, although, as we said, they do not restrict enough the form of the Wightman n-points functions.

CONCLUSIONS AND OUTLOOKS

The ideas that we have shown allow many more future perspectives, both from the point of view of modular theory itself and for what concerns the investigation of embedded theory of observables as currents models and so forth. The first bunch of questions arising are related to possible generalisations of the result of Casini and Huerta to the free Bose fields, trying to look at the corresponding relation between density matrix (containing the modular Hamiltonian) and correlators, that in principle should give back the modular "time evolution" for bosons localised in disjoint intervals, as similar to the case of Fermi fields in two dimensions.

Then one could try to extend such results to the massive case, again both for the Bose fields and for the Fermi ones, taking advantage of some already existing results (mainly by [Figliolini and Guido, 1989; Saffary, 2006]) who showed that in the massive cases the action of the modular group for the free Bose field in particular space-time regions is given in terms of a pseudo-differential operator depending on the mass. Here the techniques mostly go in the direction of functional analysis and differential equations, although the insights from the algebraic approach can still be instrumental.

The understanding of the free Bose fields automatically leads to the characterisation of the currents models and their modular theory, also moving the interest to the study of loop group models and their representations. In particular, we have seen that suitably chosen gauge transformations help us to trace modular theory back to an underlying isomorphism between algebras and perhaps a complete understanding of such gauge maps in a more general setting (maybe of non-standard form) can help very much to have further developments in that area. Interestingly enough, the conjecture that the existence of a vacuum preserving isomorphism is connected to the absence of sectors is still an open problem.

Very challenging is also the characterisation of modular theory in higher dimensions. The two-dimensional case can be pretty much derived taking the tensor product of two one-dimensional theories and therefore all the modular objects can be easily derived. On the other hand, models in three and four dimensions are a wide open area of investigation and the first attempt could be trying to understand which features remain the same and which other features present totally different behaviours instead.

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BIBLIOGRAPHY

I. Anguelova. Boson-fermion correspondence of type B and twisted vertex algebras. *arXiv:1112.3913*, 2011.

I. Anguelova. Twisted vertex algebras, bicharacter construction and boson-fermion correspondences. *arXiv:1206.4026*, 2012.

A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys.*, B241:333–380, 1984.

M. Bischoff. Construction of models in low-dimensional Quantum Field Theory using operator algebraic methods, 2012. URL http://www.theorie.physik.uni-goettingen.de/~bischoff/publications.php.

M. Bischoff and Y. Tanimoto. Construction of wedge-local nets of observables through Longo-Witten endomorphisms II. *Comm. Math. Phys.*, 317:667–695, 2013.

J. J. Bisognano and E. H. Wichmann. On the duality condition for a hermitian scalar field. *J. Math. Phys.*, 16, 1975.

J. Boeckenhauer. Localized endomorphisms of the chiral Ising model. *Comm. Math. Phys.*, 177:265–304, 1996.

H.-J. Borchers. The CPT theorem in two-dimensional theories of local observables. *Comm. Math. Phys.*, 143(2):315–332, 1992.

H.-J. Borchers. On Revolutionizing of Quantum Field Theory with Tomita's Modular Theory. *J. Math. Phys.*, 41, 1999.

H. J. Borchers and J. Yngvason. Modular groups of quantum fields in thermal states. *J. Math. Phys.*, 40:601–625, 1999.

O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics I*. Texts and monographs in physics. Springer Verlag, 1979a. ISBN 0-387-09187-4.

O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics II*. Texts and monographs in physics. Springer Verlag, 1979b. ISBN 0-387-10381-3.

R. Brunetti, D. Guido, and R. Longo. Modular structure and duality in conformal quantum field theory. *Comm. Math. Phys.*, 156:201–219, 1993.

D. Buchholz. On the structure of local quantum fields with non-trivial interaction. pages 146–153. Proc. of the Int. Conf. on Operator Algebras, Ideals and their applications in Theoretical Physics, Teubner Verlagsgesellschaft, 1977.

D. Buchholz, G. Mack, and I. Todorov. The current algebra on the circle as a germ of local field theories. *Nucl. Phys., B, Proc. Suppl.*, 5 (2):20–56, 1988.

S. Carpi. On the representation theory of Virasoro nets. *Comm. Math. Phys.*, 244(2):261–284, 2004.

H. Casini and M. Huerta. Reduced density matrix and internal dynamics for multicomponent regions. *Class. Quant. Grav.*, 26:185005, 2009. doi: 10.1088/0264-9381/26/18/185005.

A. Connes and C. Rovelli. Von Neumann algebra automorphisms and time thermodynamics relations in general covariant field theories. *Class. Quant. Grav*, 11:2899–2918, 2008.

C. D'Antoni, R. Longo, and F. Radulescu. Conformal nets, maximal temperature and models from free probability. *J. Oper. Theory*, 45 (1):195–208, 2001.

P. di Francesco, P Mathieu, and D. Sénéchal. *Conformal field theory*. Graduate texts in contemporary physics. Springer, 1997. ISBN 0-387-94785-X.

D. A. Evans and Y. Kawahigashi. *Quantum symmetries on operator algebras*. Clarendon press Oxford, 1998. ISBN 0 19 851175 2.

F. Figliolini and D. Guido. The tomita operator for the free scalar field. *Ann. Inst. Henri Poincaré*, 51:419–435, 1989.

D. Friedan, Z. Qiu, and S. Shenker. Conformal invariance, unitarity, and critical exponents in two dimensions. *Phys. Rev. Lett.*, 52(18): 1575–1578, 1984a.

D. Friedan, Z. Qiu, and S. Shenker. Conformal invariance, unitarity, and critical exponents in two dimensions. *Phys. Rev. Lett.*, 52:1575–1578, Apr 1984b.

J. Fröhlich and F. Gabbiani. Operator algebras and conformal field theory. *Comm. Math. Phys.*, 155(3):569–640, 1993.

J. Fuchs. *Affine Lie Algebras and Quantum Groups*. Cambridge University Press, 1992. ISBN 0521415934.

T. Furlan, G. M. Sotkov, and I. T. Todorov. Two-dimensional conformal quantum field theory. *Riv. N. Cim.*, 12(6):1–62, 1989.

P. Goddard, A. Kent, and D. Olive. Virasoro algebras and coset space models. *Phys. Lett.*, 152:88–92, 1985.

P. Goddard, A. Kent, and D. Olive. Unitary representations of Virasoro and Super-Virasoro algebras. *Comm. Math. Phys.*, 103:105–119, 1986.

D. Guido. Modular theory for the von Neumann algebras of local quantum physics. *Cont. Math.*, 534:97–120, 2011.

D. Guido, R. Longo, and H.-W. Wiesbrock. Extensions of conformal nets and superselection sectors. *Comm. Math. Phys.*, 192:217–244, 1998.

R. Haag. *Local quantum physics*. Springer Verlag, 1992. ISBN 3-540-53610-8.

P. D. Hislop and R. Longo. Modular structure of the local algebras associated with the free massless scalar field theory. *Comm. Math. Phys.*, 84:271–85, 1982.

V. F. R. Jones. Von Neumann algebras, 2009. URL http://math.berkeley.edu/~vfr/math20909.html.

V. Kac. *Vertex algebras for beginners*, volume 10 of *University lecture series*. American mathematical society, 1998. ISBN 0-8218-1396-X.

F. Lledo. Modular theory by example. *Cont. Math.*, 534:73–96, 2011.

R. Longo. Lecture Notes on Conformal Nets, 2008. URL http://www.mat.uniroma2.it/longo/Lecture_Notes.html.

R. Longo and K.-H. Rehren. How to remove the boundary in CFT. *Comm. Math. Phys.*, 285:1165–1182, 2009.

R. Longo and F. Xu. Topological sectors and dichotomy in conformal field theory. *Comm. Math. Phys.*, 251:321–364, 2004.

R. Longo, P. Martinetti, and K. H. Rehren. Geometric modular action for disjoint intervals and boundary conformal field theory. *Rev. Math. Phys.*, 22:331–354, 2009.

P. Martinetti and C. Rovelli. Diamond's temperature: Unruh effect for bounded trajectories and thermal time hypothesis. *Class. Quant. Grav.*, 20:4919–4931, 2003.

R. R. Paunov and I. Todorov. Local quantum field theories involving the $U(1)$ current algebra on the circle. *Lett. Math. Phys.*, 17:215–230, 1989.

A. Pressley and G. Segal. *Loop groups*. Clarendon press Oxford, 1986. ISBN 019853535X.

M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. ISBN 0-12-585050-6. Functional analysis.

K.-H. Rehren. Konforme Quantenfeldtheorie. Lecture notes, 1997.

K.-H. Rehren. Symmetries in quantum field theory. GK1493 lectures series on mathematical physics, 2012.

K.-H. Rehren. Lecture notes on conformal QFT. Erlangen fall school in geometry and quantum gravity, 2013.

K.-H. Rehren and G. Tedesco. Multilocal fermionization. *Lett. Math. Phys.*, 103:19–36, 2013. doi: 10.1007/s11005-012-0582-5.

J. E. Roberts. More lectures on Algebraic Quantum Field Theory. *Lectures notes in mathematics*, 1831:263–342, 2004.

T. Saffary. On the generator of massive modular groups. *Lett. Math. Phys.*, 77:235–248, 2006.

B. Schroer and Wiesbrock H.-W. Modular theory and geometry. *Rev. Math. Phys.*, 12(01):139–158, 2000.

S. J. Summers. Tomita-Takesaki modular theory. *Encyc. Math. Phys.*, 2003.

M. Takesaki. *Tomita's theory of modular Hilbert algebras and its applications*. Lecture notes in mathematics. Springer Verlag, 1970. ISBN 978-3-540-04917-3.

M. Takesaki. *Theory of operator algebras II*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, 2002. ISBN 3-540-42914-X.

Y. Tanimoto. Ground state representations of loop algebras. *Ann. Inst. Henri Poincaré*, 12:805–827, 2011.

H.-W. Wiesbrock. A comment on a recent work of Borchers. *Lett. Math. Phys.*, 25:157–159, 1992.

H.-W. Wiesbrock. Half-sided modular inclusions of von Neumann algebras. *Comm. Math. Phys.*, 157(1):83–92, 1993a.

H.-W. Wiesbrock. Conformal quantum field theory and half-sided modular inclusions of von Neumann algebras. *Comm. Math. Phys.*, 158(1):537–543, 1993b.

J. Yngvason. A note on essential duality. *Lett. Math. Phys.*, 1994.