Geometric Twisted K-homology, T-duality Isomorphism and T-duality for Circle Actions

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Abstract

We discuss topological T-duality and the associated geometric or topological objects in this thesis. Concretely, it consists of three parts. In this first part we prove two versions of geometric twisted K-homology are equivalent and construct the T-duality transformation for geometric twisted K-homology. This gives a dual picture for T-duality transformation of twisted K-groups. In the second part, we show that T-duality isomorphism of twisted K-theory is unique, which gives rise to the conclusion that T-duality isomorphisms through different approach (e.g. algebraic topology, C^*-algebra and groupoid) are the same. We also prove that 2-fold composition of T-duality isomorphism is equal to identity, which is given before in other papers but not proved correctly. In the third part, We discuss T-duality for circle actions. We construct the topological T-duality for countable infinite CW-complexes and use this to describe the T-duality for proper circle actions. Moreover, Mathai and Wu's discussion on the same topic is also equivalent to my construction. We also discuss the relations between this approach and C. Daenzer's groupoid approach.

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Contents

0 Introduction

String theory[54] is a physical theory which is aiming to construct an approach of quantum gravity. It assumes that the basic unit of our universe is not point-like particles but one dimensional strings. Starting from this, people expect that string theory can unify gravity with other fundamental forces naturally. What is interesting is that according to No-Ghost theorem[54], string theory only works for 26 dimensional spacetime manifolds. Combining with super-symmetry, we can get super-string theory, which also has restrictions on the dimension of spacetime manifolds. Although the dimension of super-string theory is largely reduced to 10, it is still far from the classical 4-dimensional spacetime. The way to reducing to classical 4d spacetime is by doing compacification over a special kind of 6d manifolds which are called Calabi-Yao manifolds. Although it is not proved that string theory is the right theory for quantum gravity, it has brought a lot of light to both physics and mathematics.

String theorists believe that there are five kinds of string theories and all of them are mathematical consistent. However, it is not known how to determine which one is the right one for our universe. From 1990s string theorists began to study dualities between these different kinds of super-string theories, such as T-duality, S-duality and U-duality[54]. Briefly speaking, all of these dualities are equivalence between different kinds super-string theories. For example, there exists T-duality between type II A super-string theory and type II B super-string theory , which exchanges momentum and winding number of the equation of D-branes. Also, between type I super-string theory and heterotic $SO(32)$ super-string theory there is another kind of string duality called S-duality, which is also called electric-magnetic duality in [41]. U-duality is a duality combining S-duality and T-duality transformations. T-duality and S-duality both have corresponding constructions in mathematics. In [63], A. Strominger, S.T. Yao and E. Zaslow gave a conjecture which states that mirror symmetry is T-duality. In [41] and [27], relations between S-duality and geometric Langlands program are constructed. In this thesis, we will also study a mathematical construction related to T-duality called topological T-duality.

In string theory, a pair of important objects are D-branes and Ramond-Ramond charge over D-branes, which were first studied by J. Polchinski. D-brane describes the dynamics of strings. In mathematics, Witten ([68]) suggested that Ramond-Ramond charges over D-branes should be represented by elements of (twisted) K-groups of spacetime manifolds instead of de-Rham cohomology classes. Therefore, an equivalence of different super-string theories should induce an isomorphism of twisted K-groups of a spacetime manifold and its T-dual spacetime manifold. With these ideas in mind, we now give a brief description of topological T-duality, which we are going to discuss in this thesis.

Let P and $\hat{P}$ be principal $U(1)$-bundles over a compact topological space B. Let $H \in H^3(P, \mathbb{Z})$ and $\hat{H} \in H^3(\hat{P}, \mathbb{Z})$. And we call H and $\hat{H}$ twists over P and $\hat{P}$ respectively. According to the classification of principal $PU(\mathcal{H})$-bundles there exist $\mathcal{K}$-bundles (Here $\mathcal{K}$ is the C^*-algebra of compact operators over a complex separable Hilbert space) $\mathcal{A}$ and $\hat{\mathcal{A}}$ over P and $\hat{P}$ with Dixmier-Dourady classes H and $\hat{H}$ respectively. Then we can obtain the following diagram (which we call T-dual diagram over B below) from the data above:

$$
\begin{tikzcd}
 & P \times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\
P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\
 & B &
\end{tikzcd}
\tag{0.1}
$$

Here we call $(P, \mathcal{A})$ and $(\hat{P}, \hat{\mathcal{A}})$ are pairs over B. Roughly speaking, $(P, \mathcal{A})$ and $(\hat{P}, \hat{\mathcal{A}})$ are called T-dual to each other if there exists a nice isomorphism u between $j^*(\mathcal{A})$ and

$\hat{j}^*(\hat{\mathcal{A}})$ as follows:

$$\begin{tikzcd}
 & p^*(\mathcal{A}) \arrow[rr, "u"] \arrow[dl] \arrow[dr] & & \hat{p}^*(\hat{\mathcal{A}}) \arrow[dl] \arrow[dr] & \\
\mathcal{A} \arrow[dr] & & P \times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & & \hat{\mathcal{A}} \arrow[dl] \\
 & P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] & \\
 & & B & &
\end{tikzcd} \tag{0.2}$$

in which the restriction of u to each fiber of $P \times_B \hat{P} \to B$ corresponds to the second cohomology class $\sum_{i=1}^{n} x_i \cup \hat{x}_i$ (here x_i and $\hat{x}_i$ are the i-th generators of $H^1(\mathbb{T}^n, \mathbb{Z})$ and $H^1(\hat{\mathbb{T}}^n, \mathbb{Z})$ respectively). We will do more explanation the restriction on u in the end of Section 1.1. In this case, we also call $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}))$ to be a T-duality pair. Moreover, if we denote this isomorphism by u, then we get a triple $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ which we call a T-duality triple. Similarly, if we replace principal S^1-bundle by principal higher dimensional torus bundle, we can still get a similar notion of T-duality triple. The difference is that in higher dimensional case, the required nice isomorphism does not always exist. In [16] and [17], they give the classfication of this T-duality pair and sufficient and necessary conditions for the existence of the nice isomorphism between $\mathcal{K}$-bundles for higher dimensional cases.

With the discussions above in mind, we can expect an isomorphism of the corresponding twisted K-groups of a T-duality pair. Indeed we can construct an isomorphism of twisted K-groups as follows:

$$T := \hat{j}_! \circ u \circ j^* : K^*(P, \mathcal{A}) \to K^{*+n}(\hat{P}, \hat{\mathcal{A}})$$

Here $u : K^*(P \times_B \hat{P}, j^*(\mathcal{A})) \to K^*(P \times_B \hat{P}, \hat{j}^*(\hat{\mathcal{A}}))$ is the isomorphism induced by the nice isomorphism u between $\mathcal{K}$-bundles, j^* and $\hat{j}_!$ are the corresponding pullback and push-forward maps, n is the dimension of the fiber torus.

In the above construction, we get a T-duality isomorphism of twisted K-groups, which represents the T-duality transformation of Ramond-Ramond charges between spacetime manifolds. It is natural to ask if there is a corresponding construction corresponding to the equivalence of D-branes. The answer is yes and D-brane can be represented by the elements of geometric twisted K-homology group $K^g(X, \alpha)$ of the spacetime. We will discuss this in Chapter 2 and get the T-duality transformation of geometric twisted K-homology:

Theorem 0.1. *Let B be a finite CW-complex and $((P, H), (\hat{P}, \hat{H}))$ are T-dual to each other over B as follows.*

$$\begin{tikzcd} & P \times_B \hat{P} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\ & B & \end{tikzcd}$$

Moreover, we assume that $\alpha : P \to K(\mathbb{Z}, 3)$ and $\hat{\alpha} : \hat{P} \to K(\mathbb{Z}, 3)$ satisfy that $\alpha^*([\Theta]) = H$ and $\hat{\alpha}^*([\Theta]) = \hat{H}$ (Here $[\Theta]$ is the positive generator of $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$). Moreover, we assume that α and $\hat{\alpha}$ are both representable (see Definition 2.28). Then the map $T = \hat{p}_* \circ u \circ p^! : K^g_*(X, \alpha) \mapsto K^g_{*+1}(\hat{X}, \hat{\alpha})$ is an isomorphism.

The diagram (0.1) gives the geometric picture of topological T-duality. Besides, it can also be described using C^*-algebra and groupoid languages, which we will discuss more in the next chapter. In any picture of topological T-duality, we always have a T-duality pair and a T-duality isomorphism between twisted K-groups. Another main results of this thesis is about the different models for T-duality isomorphism. Let $\mathbf{T-triple}_1$ be the category with all T-duality triples like $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ as objects and the pullbacks induced by continuous maps between base spaces as morphisms. The exact definition is given in Definition 3.1. In chapter 3 we get the following theorem which states the uniqueness of T-duality isomorphism.

Theorem 0.2. *There exists a unique T-duality isomorphism which satisfies the following axioms for each object in the category $\mathbf{T-triple}_1$ i.e. for any space B and any T-duality triple $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ over B, there is a unique way to assign a T-duality isomorphism between the corresponding twisted K-groups $K^*(P, \mathcal{A})$ and $K^{*+1}(\hat{P}, \hat{\mathcal{A}})$ such that the following axioms are satisfied.*

- ***Axiom 1*** *When the base space is a point , the T-duality isomorphism over a point T_{pt} satisfies the following equalities:*

$$T_{pt}(e_0) = e_1, T_{pt}(e_1) = e_0. \tag{0.3}$$

Here e_0 and e_1 are the positive generators of $K^0(S^1)$ and $K^1(S^1)$ respectively.

- ***Axiom 2*** *If $g : X \to Y$ is a continuous map, then we can pullback a $T-$duality triple over Y to X and get a T-duality pair over Y. The T isomorphisms T_X and T_Y satisfy the following naturality condition:*

$$T_X \circ F^* = \hat{F}^* \circ T_Y, \tag{0.4}$$

in which $F : f^*(P) \to P$, $\hat{F} : f^*(\hat{P}) \to \hat{P}$ are the corresponding maps induced by f and F^*, $\hat{F}^*$ are the maps between twisted K-groups.

- ***Axiom 3*** *Let $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ be a T-duality triple over B. $((P \times S^1, j^*\mathcal{A}), (\hat{P} \times S^1, \hat{j}^*\hat{\mathcal{A}}), u)$ gives a T-duality triple over $B \times S^1$.*

$$\begin{tikzcd}
 & P \times_B \hat{P} \times S^1 \arrow[dl, "j \times id_{S^1}"'] \arrow[dr, "\hat{j} \times id_{S^1}"] & \\
P \times S^1 \arrow[dr, "\pi \times id_{S^1}"'] & & \hat{P} \times S^1 \arrow[dl, "\hat{\pi} \times id_{S^1}"] \\
 & B \times S^1 &
\end{tikzcd}$$

then the following identity holds:

$$T_{S^1 \times B} = Id_{K^*(S^1)} \otimes T_B. \tag{0.5}$$

According to Theorem 0.2, we can define a category $\mathbf{T-TRIPLE}_1$. The objects of $\mathbf{T-TRIPLE}_1$ are pairs $(\mathfrak{D}, T_{\mathfrak{D}})$. Here each $\mathfrak{D}$ is a T-duality triple over base space B and each $T_{\mathfrak{D}}$ is a T-duality isomorphism between the twisted K-groups of T-duality pairs in $\mathfrak{D}$. Moreover, we require $T_{\mathfrak{D}}$ satisfies the axioms in Theorem 0.2. The morphisms of $\mathbf{T-TRIPLE}_1$ are also the pullbacks induced by the continuous maps between base spaces. Then Theorem 0.2 can also be stated as follows

Theorem 0.3. *$\mathbf{T-triple}_1$ and $\mathbf{T-TRIPLE}_1$ are equivalent to each other.*

In string theory, string theorists also study T-duality with Kaluza-Klein monopoles i.e. T-duality for spacetime manifold with some kinds of singularities. This is first discussed by A. Pande in [49] for semi-free S^1-action on smooth manifolds. More generally, we can ask the question if there is a T-duality pair for a space which admits an S^1-action.

This is also called the Missing T-dual problem in [59]. There have been some studies on this problem. For example, Mathai and Wu used equivariant twisted K-theory to give an answer to this problem in [45]. In this thesis, we will also give some other approaches to this problem in Chapter 4.

Now we give the main structure of this thesis.

In Chapter 1, we review different approaches to topological T-duality but don't give the full details, which are useful to our discussion later. In section 1.1, we review the constructions in [12], in which differential forms are used to describe the construction of T-duality pair and T-isomorphism. Their constructions play an important role when we compare two approaches to the singular topological T-duality in Chapter 4. In section 1.2 and 1.3, we introduce Bunke-Schick construction for principal S^1-bundles and higher dimensional torus bundles. In section 1.4, we briefly talk about Mathai and Rosenberg's approach to topological T-duality via noncommutative topology. In addition we review Connes-Thom isomorphism, which provides an analogue list of axioms as we do in chapter 3. In section 1.5, we give A. Schneider's work on the proof of the equivalence between topological approach and C^*-algebra approach to topological T-duality. In section 1.6 and 1.7 of this chapter, we introduce C. Daenzer's approach to topological T-duality using groupoid and discuss the relation between the groupoid approach and other approaches. In the end of this chapter we give some examples of topological T-duality and compute some twisted K-groups using Atiyah-Hirzebruch spectral sequence.

In Chapter 2, we discuss geometric twisted K-homology and T-duality transformation of geometric K-cycles. In section 2.1, we introduce two definitions of geometric twisted K-cycles, which have been discussed in [67] and [7] respectively. In section 2.2 and 2.3, we show that the two definitions are equivalent and use this to prove that the charge map in [7] is an isomorphism. In section 2.4, we give another construction of geometric twisted K-homology using bundle gerbes. In section 2.5, we establish some properties of twisted geometric K-homology. In section 2.6 and 2.7, we construct the T-duality transformation for geometric K-cycles and show that it is an natural isomorphism for representable twists.

In Chapter 3, we discuss the uniqueness of T-isomorphism. In section 3.1 we give some basic notions. In section 3.2, we first prove that the three approaches we discussed in section 1.1, 1.4 and 1.6 satisfies the axioms in (0.2). Then we give theorem (0.2) and complete its proof. In section 3.3, we extend the results in section 3.1 to higher dimensional torus bundles. In section 3.4, we reinterpret the results in section 3.1 using KK-elements. In section 3.5, we use the similar methods to compute the two-folds composition of T-isomorphism and get the following theorem:

Theorem 0.4. *For each object $(P, \mathcal{A})$ over base space B in* ***Pair*** *(see Definition 3.1), there exists a unique isomorphism $\tau_{(P,\mathcal{A})} : K^*(P, \mathcal{A}) \to K^*(P, \mathcal{A})$ which satisfies the axioms below.*

- *(**Axiom 1**) When B is a point, $\tau_{(S^1,0)} = Id$;*
- *(**Axiom 2**) If there is a map $l : X \to B$ then $L^* \circ \tau_{(P,\mathcal{A})} = \tau_{(P_X,\mathcal{A}_X)} \circ L^*$. Here $L : l^*P \to P$ is the map induced by l and $(P_X, \mathcal{A}_X)$ is the pullback pair of $(P, \mathcal{A})$ along $l : X \to B$;*
- *(**Axiom 3**)Consider the pair $(P \times S^1, i^*(\mathcal{A}))$ over $B \times S^1$, here $i : P \times S^1 \to P$ is the projection. Then the isomorphism $\tau_{(P\times S^1,i^*(\mathcal{A}))}$ satisfies*

$$\tau_{(P\times S^1,i^*(\mathcal{A}))} = \tau_{(P,\mathcal{A})} \otimes Id_{K^*(S^1)}. \tag{0.6}$$

Especially, we get the two-fold composition of T-isomorphism of twisted K-group is the identity map.

In section 3.6, we use the results in section 3.5 to discuss axiomatic topological T-duality.

In Chapter 4, we study topological T-duality for manifolds which admit a proper S^1-action . In section 4.1, we first review the construction in [45]. In section 4.2, we construct T-duality pairs for countable infinite CW complexes and prove that the T-duality transformation for twisted K-theory is still an isomorphism. In particular, this implies Mathai and Wu's results. In section 4.3 we use groupoids to give a construction of T-duality pair for a manifold with a twist which also admits a smooth S^1-action. In section 4.4 and 4.5 we compare the construction in section 4.1 with Mathai and Wu's results and give the connections between them. In section 4.6, we discuss topological T-duality for S^1-manifolds using differentiable stacks. We construct the T-duality pair using differentiable stack and give the push-forward map of twisted K-theory for differentiable stacks. In the end of this chapter we give some other possible methods to construct topological T-duality for S^1-manifolds.

In the appendix A we give the classification of principle $PU(\mathbb{H})$-bundles and definition of twisted K-theory. In appendix B we give some basic notions and constructions for differentiable stacks. In appendix C we discuss KK-equivalence and list Universal Coefficient Theorem and K*ü*nneth Theorem for KK-theory.

1 Review of topological T-duality

We review different models of topological T-duality in this chapter, which include algebraic topology approach, C^*-algebra approach and groupoid approach. For simplicity here we only give the basic ideas and some important results and don't go too much into details.

1.1 Bouwknegt, Evslin and Mathai's construction

In [12], P. Bouwknegt, J. Evslin and V. Mathai give a definition of topological T-duality using the de Rham cohomology. They begin with a pair (P, H), where $\pi : P \to B$ is a principal S^1-bundle over base space B and H is a closed 3-form over P. Moreover, they requre that H has integral period. They define the T-dual of (P, H) to be another pair $(\hat{P}, \hat{H})$, where $\hat{\pi} : \hat{P} \to B$ is another principal S^1-bundle over B and $\hat{H}$ is a closed 3-form over $\hat{P}$ with integral period such that

$$c_1(\hat{P}) = \pi_!(H),\ c_1(P) = \hat{\pi}_!(\hat{H}). \tag{1.1}$$

Here $\pi_!$ and $\hat{\pi}$ are the Gysin maps of the two S^1-bundles respectively. We can also see it as integration along the fiber S^1. The details of the fiber integration can be found in [29].

Remark 1.1. Here we need to be careful because the restriction of H makes that the de-Rham cohomology class of $[H]$ lies in the image of $H^3(P, \mathbb{Z})$ into $H^3_{de-Rham}(P)$. Since principal $PU(\mathcal{H})$-bundles are classified by the third integral cohomology group, therefore this condition is necessary when we use H to define twisted K-groups.

Using the data above, they also give constructions of T-isomorphisms between (P, H) and its T-dual $(\hat{P}, \hat{H})$ for twisted de-Rham cohomology and twisted K-theory. Now we state their constructions. Given a pair (P, H) and its T-dual $(\hat{P}, \hat{H})$, we can get a fiber product $P \times_B \hat{P}$. Also we can get a T-dual diagram as follows:

$$\begin{tikzcd} & P \times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\ P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\ & B & \end{tikzcd} \tag{1.2}$$

To get the T-dual space $\hat{P}$ and T-dual twist $\hat{H}$, they choose connections A and $\hat{A}$ over P and $\hat{P}$ respectively. Using (1.1) and the Gysin sequence they get that $j(H)$ and $\hat{j}^*(\hat{H})$ are cohomologous and moreover they get

$$d(\mathcal{B}) = -j^*(H) + \hat{j}^*(\hat{H}), \tag{1.3}$$

where $\mathcal{B} = j^*(A) \wedge \hat{j}^*(\hat{A})$.

Definition 1.2. Let M be a smooth manifold and H be a closed 3-form over M. Let $\Gamma^*(M)$ a $\mathbb{Z}/2\mathbb{Z}$ graded space with $\Gamma^0(M) = \bigoplus_{k=2n} \Omega^k(M)$ and $\Gamma^1 = \bigoplus_{k=2n+1} \Omega^k(M)$. Here $\Omega^i(M)$ is the set of all i-forms over M and n is a non-negative integer. Denote the differential operator of the de-Rham complex by d. Denote $d_H = d + H\wedge$. Then $(\Gamma^*(M), d_H)$ forms a complex. And we call the cohomology group of this complex the twisted de-Rham cohomology group, which is denoted by $H^i(M, H)$ ($i = 0$ or 1).

Given any $\omega \in H^*(P, H)$. They define a T-duality transformation $T : H^*(P, H) \to H^{*-1}(\hat{P}, \hat{H})$ by the following formula:

$$T(\omega) = \hat{j}_! \circ e^{\mathcal{B}} \circ j^*(\omega). \tag{1.4}$$

Here j^* is the pullback map induced by j, $\hat{j}_!$ is the push-forward map induced by $\hat{j}$ (Since $\hat{j} : P \times_B \hat{P} \to \hat{P}$ is S^1-principal bundle, therefore this push-forward map is actually integration on the fiber), $e^{\mathcal{B}}$ is the wedge product with $e^{\mathcal{B}}$.The inverse of T is given by

$$T^{-1} = j_! \circ e^{-\mathcal{B}} \circ \hat{j}^*. \tag{1.5}$$

Therefore T is an isomorphism.

For twisted K-theory (see Appendix A), they also give a T-homomorphism similarly using the correspondence space. The difference is that the changing twist map $u : K^*(P \times_B$

$\hat{P}, j^*(H)) \to K^*(P \times_B \hat{P}, \hat{j}^*(\hat{H}))$ instead of wedge with $e^{\mathcal{B}}$. Here t is defined as follows: We choose a curving f ($\hat{f}$) for the chosen gerbe over P ($\hat{P}$) induced by H ($\hat{H}$),i.e. $df = H, d\hat{f} = \hat{H}$. Then we have that

$$d(\mathcal{B} + f - \hat{f}) = 0.$$

Hence $[\mathcal{B} + f - \hat{f}]$ determines a line bundle over $P \times_B \hat{P}$, which in turn induces a trivial bundle gerbe. The changing twisting map $\Lambda : K^i(P \times_B \hat{P}, j^*H) \to K^i(P \times_B \hat{P}, \hat{j}^*\hat{H})$ is defined by tensoring with this trivial bundle gerbe.The T-isomorphism in [12] is given as follows:

$$T := \hat{j}_! \circ \Lambda \circ p^* : K^i(P, H) \to K^{i+1}(\hat{P}, \hat{H}). \tag{1.6}$$

They prove T is an isomorphism by saying that $j_! \circ \Lambda^{-1} \circ \hat{j}^*$ is the inverse of T. Unfortunately the proof of this in [12] is not strict. We will get the inverse of T-duality isomorphism by proving Theorem 0.4 in section 3.5. In the paper [12] they use closed differential forms (with integer period) as twists. However, as we explained in Remark 1.1, they are essentially using third integral cohomology classes as twists when they are considering twisted K-theory. However, there are different versions of twists which we appear in this thesis. we give a short introduction of them as the end of this section. Given a space B, there are two other different versions of twists in this thesis

- A map from B to $K(\mathbb{Z}, 3)$;
- A $\mathcal{K}$-bundle over B, here $\mathcal{K}$ is the C^*-algebra of compact operators over a complex separable Hilbert space.

These two versions of twists are equivalent in the following sense: According to Theorem A.7, $K(\mathbb{Z}, 3)$ is a model of $BPU(\mathcal{H})$. Therefore there exists a universal $\mathcal{K}$-bundle $\mathfrak{K}$ over $K(\mathbb{Z}, 3)$. For any map $\alpha : B \to K(\mathbb{Z}, 3)$, we can pullback the bundle $\mathfrak{K}$ along α and get a $\mathcal{K}$-bundle over B. Moreover, we can define two categories using these two kinds of twisting. Let $\mathbf{Twist}_1(B)$ be the category of maps from B to $K(\mathbb{Z}, 3)$ and $\mathbf{Twist}_2$ be the category of $\mathcal{K}$-bundles over B. The morphisms are homotopies between maps and isomorphisms between $\mathcal{K}$-bundles respectively. Then the above discussion implies that the equivalence classes of objects in $\mathbf{Twist}_1$ and $\mathbf{Twist}_2$ are both isomorphic to $H^3(B, \mathbb{Z})$. Given two objects α_i ($i = 0,\ 1$) in $\mathbf{Twist}_1$, a homotopy from α_0 to α_1 is a map $B \times [0, 1]$ to $K(\mathbb{Z}, 3)$. Since $[B \times [0, 1], K(\mathbb{Z}, 3)] \cong [B, K(\mathbb{Z}, 2)]$, therefore we can see that a morphism between α_0 and α_1 determines a second integer cohomology class over B. The same conclusion holds for the morphisms in $\mathbf{Twist}_2$ i.e. an isomorphism between two

$\mathcal{K}$-bundles determines a second integer cohomology class. Now we can understand the description of the restriction on the changing twist map u in the last chapter.

1.2 Bunke-Schick Construction

In [16], U. Bunke and T. Schick give another definition of topological T-duality via algebraic topology. A pair over a base space B is a principal bundle and a third cohomology class over the principal bundle. They start with two pairs over the same base space B: (P, H) and $(\hat{P}, \hat{H})$. Here $\pi : P \to B$ and $\hat{\pi} : \hat{P} \to B$ are principal S^1-bundles and $H \in H^3(P, \mathbb{Z}), \hat{H} \in H^3(\hat{P}, \mathbb{Z})$. Denote the associated line bundles of P and $\hat{P}$ by E and $\hat{E}$ respectively. Let $V = E \bigoplus \hat{E}$ and $r : S(V) \to B$ be the unit sphere bundle of V.

Definition 1.3. A class $\mathbf{Th} \in H^3(S(V), \mathbb{Z})$ is called Thom class if $r_!(\mathbf{Th}) = 1 \in H^0(B, \mathbb{Z})$.

Let $i : P \to S(V)$ and $\hat{i} : \hat{P} \to S(V)$ be inclusion of principal S^1-bundle into S^3-bundle.

Definition 1.4. We say that (P, H) and $(\hat{P}, \hat{H})$ are T-dual to each other if there exists a Thom class $\mathbf{Th} \in H^3(S(V), \mathbb{Z})$ such that

$$H = i^*(\mathbf{Th}), \hat{H} = \hat{i}^*(\mathbf{Th}). \tag{1.7}$$

Remark 1.5. In [16] they prove that this definition is equivalent to the one in [12]. One difference between them is that Bunke-Schick construction starts from two pairs which means that they consider a pair and its T-duality pair together. For principal S^1-bundle case, the existence always holds. However, as we will see in the following sections, the existence fails for higher torus bundle cases sometimes. Therefore it is easier to generalize the Bunke-Schick construction to higher cases.

Definition 1.6. Let $q : U \to K(\mathbb{Z}, 2)$ be the universal S^1-bundle and $LK(\mathbb{Z}, 3)$ be the loop space of $K(\mathbb{Z}, 3)$. Since $LK(\mathbb{Z}, 3)$ admits an S^1-action by rotation along the parameter of S^1, we have an associated bundle $U \times_{S^1} LK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 2)$. Let R be the total space of the associated bundle.

The bundle map of the associated bundle $U \times_{S^1} LK(\mathbb{Z}, 3)$ determines a second cohomology class and therefore also determines a principal S^1-bundle $\pi : \mathbf{P} \to R$. Let $\mathbf{h} : \mathbf{P} \to K(\mathbb{Z}, 3)$ be the map $\mathbf{h}(v, u, \gamma) = \gamma(uv^{-1})$, here $v, u \in S^1$ and $\gamma \in LK(\mathbb{Z}, 3)$. $(\mathbf{P}, \mathbf{h})$ is called the universal pair. Denote the isomorphism classes of pairs over B by $P(B)$. Then P is actually a covariant functor. The following proposition in [16] gives a classification space of pairs.

Proposition 1.7. *R is a classifying space of P. i.e., for any pair (P, H) over B, there exists a unique (up to homotopy) continuous map $f : B \to R$ such that $f^*(\mathbf{P}, \mathbf{H}) = (P, H)$.*

Now we explain how they give the T-duality transformations of twisted generalized cohomology theories. The key to doing this is defining the changing twist map u. Let us now show how they do this for the trivial case, i.e. when the base space is a point, which leads to the notion of T-admissibility.

First of all we give a general construction in [16] as a preparation. Assume that $h : I \times Y \to X$ is a homotopy from f_0 to f_1 and $i_k : Y \to I \times Y$ ($k = 0$ or 1) is given by $i_k(y) = (k, y)$. Define $F : I \times Y \to I \times X$ as $F(t, y) = (t, h(t, y))$. For any twist $\mathcal{H}$ over X, the twists $(id_I \times f_0)^* pr_2^*(\mathcal{H})$ and $F^* pr_2^*(\mathcal{H})$ are isomorphic because $h(0, y) = f_0(y)$. We define an isomorphism $u(h) : (id_I \times f_0)^* pr_2^*(\mathcal{H}) \to F^* pr_2^* \mathcal{H}$ to be the unique morphism such that the composition of the following isomorphisms is the identity:

$$f_0^*(\mathcal{H}) \cong i_0^* \circ (id_I \times f_0)^* \circ pr_2^*(\mathcal{H}) \overset{i_0^*(u(h))}{\cong} i_0^* \circ F^* \circ pr_2^*(\mathcal{H}) \cong f_0^*(\mathcal{H}).$$

For the trivial base space case, $P = S^1, \hat{P} = S^1, P \times_B \hat{P} = S^1 \times S^1$. Let $S \subset \mathbb{C}^2$ be the unite sphere, $i : P \to S$ be the inclusion $i(z) = (z, 0)$ and $\hat{i} : \hat{E} \to S$ be the inclusion $\hat{i}(z) = (0, \hat{z})$. Let $p : P \times_B \hat{P} \to P$ and $\hat{p} : P \times_B \hat{P} \to \hat{P}$ be the projections to the first and second factor. We define a homotopy from $i \circ p$ to $\hat{i} \circ \hat{p}$ h as follows:

$$h(t, z, \hat{z}) = 1/\sqrt{2}(\sqrt{1 - t^2} z, t\hat{z}). \tag{1.8}$$

Denote the twist over S determined by the generator of $H^3(S, \mathbb{Z})$ by $\mathcal{K}$. Let $\mathcal{H} = i^*(\mathcal{K})$ and $\hat{\mathcal{H}} = \hat{i}^*(\mathcal{K})$. Define u to be the composition of the following isomorphisms

$$\hat{p}^*(\mathcal{H}) = \hat{p}^* \hat{i}^*(\mathcal{K}) \cong (\hat{i} \circ \hat{p})^*(\mathcal{K}) \overset{u(h)}{\cong} (i \circ p)^*(\mathcal{K}) \cong p^* i^*(\mathcal{K}) = p^*(\mathcal{H}). \tag{1.9}$$

We can see that u induces the changing twist map $u : \mathfrak{H}^*(S^1 \times S^1, i^*(\mathcal{H})) \to \mathfrak{H}^*(S^1 \times S^1, \hat{i}^*(\hat{\mathcal{H}}))$. Here $\mathfrak{H}$ can be any generalized twisted cohomology theory, for example, twisted de-Rham cohomology and twisted K-theory. Then we can define the T-homomorphism as follows:

$$T = \hat{j}_! \circ u \circ j^*, \tag{1.10}$$

$\mathfrak{H}$ is called T-admissible if T is an isomorphism.

Remark 1.8. Since we can choose many different h in the above constructions , therefore there are different changing twist maps. However, they lead to the same T here.

Example 1.9. $\mathbb{Z}_2$-graded twisted de-Rham cohomology theory, twisted K-theory are T-admissible. Twisted $spin^c$ cobordism is not T-admissible because it is not 2-periodic.

The next lemma in [16] shows that there are many T-admissible cohomology theories.

Lemma 1.10 ([16])**.** *Let R be a injective ring, then $\mathbb{Z}_2$-graded twisted cohomology theory with coefficient in R is T-admissible.*

Proof. Use universal coefficient theorem we have the following exact sequence:

$$0 \to Ext(H^*(X,\mathbb{Z}), R) \to H^*(X,R) \to Hom(H^*(X,\mathbb{Z}), R) \to 0.$$

Since R is injective, we have $Ext(H^*(X,\mathbb{Z}), R) = 0$ and we also have the commutative diagram:

$$\begin{tikzcd} H^*(X,R) \arrow[r] \arrow[d, "T_R"] & H^*(X,\mathbb{Z}) \otimes R \arrow[d, "T \otimes Id_R"] \\ H^{*-1}(X,R) \arrow[r] & H^{*-1}(X,\mathbb{Z}) \otimes R \end{tikzcd}$$

Since all of the other morphisms are isomorphisms, so is T_R. □

For more general cases, i.e., when the base space B is not a point, since we still have $j^*(H) = \hat{j}^*(\hat{H})$ we can get the changing twist map u (up to homotopy) similarly. Similarly, we have the T-duality transformation

$$T := \hat{j}_! \circ u \circ j^* \tag{1.11}$$

Remark 1.11. The changing twist map is very essential in topological T-duality. In the trivial case, even all of the twists and principal S^1-bundles are trivial, the changing twist map is actually the only "nontrivial" part. For example, if we don't change twists and do the push-forward to the T-dual part without the changing twist map, then we will always get 0.

1.3 Topological T-duality for Higher Principal Torus Bundles

In the last two sections we discussed topological T-duality for principal S^1-bundles. It is natural to consider if these constructions are applicable for principal higher dimensional torus $\mathbb{T}^n$-bundles. In [13] and [17], this topic is discussed geometrically. Here we first review some of their constructions and give an example to show that in higher dimensional cases the classical T-dual doesn't always exist. This is the missing T-dual problem mentioned in [59]. We will see more about this problem in the remainder of the thesis.

In higher dimensional cases, we can't expect the formula like (1.1) because the pushforward map change the degree of twists more than one in higher dimensional cases. In [13] and [17], they both do the discussion when the T-duality pairs are principal torus bundles with twists. In both papers, they give a sufficient condition on the existence of T-duality pair. In [13], they define a class of H-flux on a principal $\mathbb{T}^2$-bundle called T-dualizable H-fluxes as follows.

Definition 1.12. If H is an H-flux over a principal $\mathbb{T}^2$-bundle $\pi : P \to B$ and there exists a closed $\hat{t}^2$-valued ,2-form on B such that $dH = 0$ and $\iota_X H = \pi^* \hat{F}(X)$ for any $X \in t^2$, then H is called T-dualizable. Here t^2 is the Lie algebra of $\mathbb{T}^2$ and $\hat{t}^2$ is the dual of t^2.

In [17], they start from a notion of T-duality triple. More concretely, they put the original space, its T-dual and changing twist isomorphism together to form a T-duality triple. More exactly, we give the following definition

Definition 1.13. An n-dimensional T-duality triple over B is a triple

$$((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$$

consisting of T^n-bundles $\pi : P \to B, \hat{\pi} : \hat{P} \to B$, where the characteristic classes of $\mathcal{A}$ and $\hat{\mathcal{A}}$ lies in the second filtration step of the Leray-Serre spectral sequence filtration and their leading parts satisfy

$$[\mathcal{A}]^{2,1} = [\sum_{i=1}^{n} y_i \otimes \hat{c}_i] \in^{\pi} E_\infty^{2,1} \tag{1.12}$$

and

$$[\hat{\mathcal{A}}]^{2,1} = [\sum_{i=1}^{n} \hat{y}_i \otimes c_i] \in^{\hat{\pi}} E_\infty^{2,1} \tag{1.13}$$

respectively, and an isomorphism $u : \hat{j}^* \hat{\mathcal{A}} \to j^* \mathcal{A}$ which satisfies the following condition: When we restrict the T-duality diagram to a point b of B, u is an isomorphism corresponding to $[\sum_{i=1}^{n} y_i \cup \hat{y}_i] \in H^2(\mathbb{T}^n)_b \times \hat{\mathbb{T}}^n{}_b, \mathbb{Z}$. Here $y_i \in H^1(\mathbb{T}^n, \mathbb{Z}), \hat{y}_i \in H^1(\hat{\mathbb{T}}^n, \mathbb{Z})$ are respectively the ith generators.

In their picture,a pair $(P, \mathcal{A})$ is T-dualizable if there exists an extension of $(P, \mathcal{A})$ to a T-duality triple $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$. They give a necessary and sufficient condition on when a pair $(P, \mathcal{A})$ admits such an extension.

Theorem 1.14 ([17])**.** *The pair* $(P, \mathcal{A})$ *admits an extension to a* T*-duality triple* $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ *if and only if the Dixmier-Douady class of* $\mathcal{A}$ *lies in* $\mathcal{F}^2H^3(P, \mathbb{Z})$*.*

Here $x \in \mathcal{F}^kH^n(P, \mathbb{Z})$ if for any $(k-1)$-dimensional CW-complex X and a map $\phi : X \to B$ we have $\Phi^*x = 0$, where $\Phi : \phi^*P \to P$ is the induced map.

Remark 1.15. This condition is different from the definition in [13]. It is more general since it also works for higher dimensional torus bundles cases. Even for principal $\mathbb{T}^2$-bundles cases, it is still more general because the Dixmier-Douady class of $\mathcal{A}$ can be torsion elements of $H^3(P, \mathbb{Z})$ and the image of the integer coefficient cohomology group (the inclusion in the real coefficient cohomology group) is isomorphic to the non-torsion part of the integer coefficient cohomology group.

Now let us give a simple example which is not T-dualizable.

Example 1.16. Consider $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ as a principal $\mathbb{T}^2$-bundle over S^1, and we choose the generator H of $H^3(\mathbb{T}^3, \mathbb{Z})$ as the twist. Then we have $H \in \mathcal{F}^3H^3(\mathbb{T}^3, \mathbb{Z})$, therefore the pair $(\mathbb{T}^3, H)$ is not T-dualizable.

Remark 1.17. We can get a little feeling about the T-dual missing problem in the example above. There are different approaches to deal with this problem. We will see in the next section that even if the above example is not T-dualizable in the classic sense, but it still has a noncommutative T-dual space.

1.4 Topological T-duality and Crossed Product

In [43] and [44], Mathai and Rosenberg discuss topological T-duality via C^*-algebra and noncommutative algebraic topology. In this part we review their constructions and also briefly introduce the Thom-Connes isomorphism which is crucial in their approach.

There is an important notion in their approach called continuous trace C^*-algebras. For completeness we list the definition here, more details can be found in [56] and [58]

Definition 1.18. We call a C^*-algebra A a continuous trace C^*-algebra if the spectrum of A, which we denote by $\hat{A}$, is Hausdorff and if the continuous-trace elements $a \in A_+ | Tr\pi(a) < \infty$ for all $\pi \in \hat{A}$ is continuous on $\hat{A}$ are dense in A_+.

The next theorem in [23] gives a classification of continuous-trace C^*-algebras.

Theorem 1.19 ([23]). *Let A be a separable continuous-trace C^*-algebra with spectrum X. Then $A \cong \Gamma_0(\mathfrak{A})$, the algebra of sections vanishing at infinity of a continuous field $\mathfrak{A}$ of elementary C^*-algebras over X. To $\mathfrak{A}$ is associated a characteristic class $\delta(A) \in H^3(X, \mathbb{Z})$(Céch cohomology). If A is stable, that is $A \cong A \otimes \mathcal{K}$, then $\mathfrak{A}$ is locally trivial, with fibers which are isomorphic to $\mathcal{K}$. In this case, A is determined, up to automorphisms fixing X pointwise, by $\delta(A)$. And any class $\delta \in H^3(X, \mathbb{Z})$ arises from a (unique) stable separable continuous-trace C^*-algebra A_δ over X.*

One can see that the data which determines a stable continuous-trace C^*-algebra is the same as a pair we discussed in the last section. Therefore Mathai and Rosenberg start from a principal $\mathbb{T}^n$-bundle $\pi : P \to B$ and a twist $H \in H^3(P, \mathbb{Z})$. They consider the stable continuous-trace C^*-algebra $CT(P, H)$ and use the crossed product by $\mathbb{R}^n$ to define topological T-duality. In [43] they discussed the case for $n = 2$ and in [44] they discussed higher dimensional cases. The discussion of the 1-dimensional case is much earlier than the notion of topological T-duality. It was given by J. Rosenberg ([58]) in 1980s and the main result is as follows.

Theorem 1.20 ([58]). *Let T be any second-countable locally compact space with a homotopy type of a finite CW-complex, and $p : \Omega \to T$ any principal S^1-bundle over T, A a stable continuous-trace algebra with spectrum Ω. Then there is an action α of $\mathbb{R}$ on A, unique up to exterior equivalence, such that every point in $\Omega = \hat{A}$ has stabilizer $\mathbb{Z}$ and the $\mathbb{R}$-action on Ω factors through the $\mathbb{R}/\mathbb{Z} \cong S^1$-action defining p. Furthermore, $(A \rtimes_\alpha \mathbb{R})^\wedge$ together with the dual action of $\mathbb{R}$ defines another principal S^1-bundle $\hat{p} : (A \rtimes_\alpha \mathbb{R})^\wedge \to T$, and the characteristic classes $[p]$ and $[\hat{p}]$ of the bundles p and $\hat{p}$ are related to the Dixmier-Douady classes by the equations*

$$[\hat{p}] = p_!\delta(A), [p] = \hat{p}_!\delta(A \rtimes_\alpha \mathbb{R}),$$

where $p_! : H^3(\Omega, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ and $\hat{p}_! : H^3((A \rtimes_\alpha \mathbb{R})^\wedge, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ are Gysin maps.

We can also do the construction of crossed product for continuous trace C^*-algebras admitting $\mathbb{R}^n$-action, so we can generalize the above construction without too much difficulty. And this is exactly the starting point of Mathai and Rosenberg's paper. Given a principal $\mathbb{T}^n$-bundle $\pi : P \to B$, and a H-flux $H \in H^3(P, \mathbb{Z})$, one can also construct a stable continuous trace C^*-algebra $CT(P, X)$ with spectrum P and Dixmier-Douady class H. Here comes the differences compared with dimension 1 case. Although principal

$\mathbb{T}^n$-bundles always admit an $\mathbb{R}^n$-action which is induced by $\mathbb{T}^n$-action, not all of the corresponding stable continuous trace C^*-algebras admit an $\mathbb{R}^n$-action. This lifting property is determined by the Dixmier-Douady class of $CT(P, H)$. Here we need the notion of Brauer group of a space X: $Br(X)$ and its $\mathbb{R}^n$-equivariant version $Br_{\mathbb{R}^n}(X)$, which one can find in the appendix (Definition A.11). If the Dixmier-Douady class $H \in Br_{\mathbb{R}^n}(X)$, then one can lift the $\mathbb{R}^n$-action to $CT(P, H)$ and therefore one can do the crossed product construction similarly. They call $CT(X, H) \rtimes \mathbb{R}^n$ the T-dual of $CT(X, H)$. The following theorem in [44] gives an alternative description of the above condition:

Theorem 1.21 ([44])**.** *Let $\mathbb{T}$ be a torus, $\mathbb{G}$ its universal covering, and $\pi : P \to B$ be a principal $\mathbb{T}$-bundle. Then the image of the forgetful map $F : Br_{\mathbb{G}}(P) \to H^3(P, \mathbb{Z})$ is precisely the kernel of the map $\iota^* : H^3(P, \mathbb{Z}) \to H^3(\mathbb{T}, \mathbb{Z})$ induced by the inclusion $\iota : \mathbb{T} \hookrightarrow P$ of a torus fiber into P.*

Even if we can do the crossed product construction by $\mathbb{R}^n$, but we still can't expect that $CT(P, H) \rtimes \mathbb{R}^n$ can be realized as a stable continuous trace C^*-algebra over some space in general. This is related to a notion called Mackey obstruction, which we will skip here. Now we give the main conclusion in [44], which gives a sufficient condition for the existence of a classical T-dual:

Theorem 1.22 ([44])**.** *Let $\pi : P \to B$ be a principal $\mathbb{T}^n$-bundle. Let $H \in H^3(P, \mathbb{Z})$ be an H-flux on P that is the kernel of $\iota^* : H^3(P, \mathbb{Z}) \to H^3(\mathbb{T}^n, \mathbb{Z})$, where ι is the inclusion of a fiber. Let $k = \frac{n(n-1)}{2}$. Then:*

1. *If $\pi_!(H) = 0 \in H^1(B, \mathbb{Z}^k)$, then there is a classical T-dual to (P, H) consisting of $\hat{\pi} : \hat{P} \to B$, which is another principal $\mathbb{T}^n$-bundle over B, and $\hat{H} \in H^3(\hat{P}, \mathbb{Z})$. One obtains a picture of the form:*

$$\begin{tikzcd} & P \times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\ P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\ & B & \end{tikzcd}$$

There is a natural isomorphism of twisted K-theory

$$K^*(P, H) \cong K^{*+n}(\hat{X}, \hat{H})$$

2. *If $\pi_!(H) \neq 0 \in H^1(B, \mathbb{Z}^k)$, then a classical T-dual as above does not exist. However, there is a "non-classical" T-dual bundle of noncommutative tori over B. It is not unique, but the non-uniqueness does not affect its K-theory, which is isomorphic to $K^*(P, H)$ with a dimension shift of $n \bmod 2$.*

Here $\pi_!(H) = (\int_{\mathbb{T}^2_1} H, \int_{\mathbb{T}^2_2} H, ..., \int_{\mathbb{T}^2_k} H) \in H^1(B, \mathbb{Z}^k)$, where $\mathbb{T}^2_j$, $j = 1, 2, ..., k$ run through a basis for the possible 2-dimensional subtori in the fibers.

Remark 1.23. At the first glance, the operator $\int_{\mathbb{T}^2_i}$ maps H not to $H^1(B, \mathbb{Z})$ but to $H^1(P/\mathbb{T}^2_i, \mathbb{Z})$. However, the condition that $\pi_!$ is defined on the kernel of $\iota^* : H^3(P, \mathbb{Z}) \to H^3(\mathbb{T}^n, \mathbb{Z})$ makes the statement well defined. This is explained in the Theorem 2.3 of [44] via Machey constructio and Leray-Serre spectral sequence.

Remark 1.24. In the last section we give an example which is not T-dualizable. Here we can see that in this example, we have one of the factors of $\pi_!(H)$ is the generator of $H^1(\mathbb{T}, \mathbb{Z})$, i.e. $\pi_!(H) \neq 0$. So we still get that the example does not have a classical T-dual. According to the part two of the above theorem, we have $(\mathbb{T}^3, H)$ has a non-classical T-dual. Actually, its T-dual can be realized by a bundle of stabilized noncommutative tori fibered over $\mathbb{T}$.

The isomorphism of twisted K-theory is given by the Connes-Thom isomorphism. We will discuss T-duality isomorphism in chapter 3, so we briefly review some properties of the Connes-Thom isomorphism here. In [20], Connes constructed an isomorphism ϕ_A from the ith K group of A which admits an $\mathbb{R}$-action to the $(i + 1)$th K group of $A \rtimes \mathbb{R}$, which satisfies the following axioms:

- **Axiom 1** If $A = \mathbb{C}$, the image by ϕ_A of the positive generator of $K^0(pt)$ is the positively generator of $K^1(\mathbb{R})$;

- **Axiom 2** If B is another C^*-algebra admitting an $\mathbb{R}$-action and $\rho : A \to B$ is an equivariant homomorphism, then
$$(\hat{\rho})_* \circ \phi_A = \phi_B \circ \rho_*;$$

- **Axiom 3** Let SA be the suspension of A, then
$$s_{\hat{A}} \circ \phi_A = \phi_{SA} \circ s_A.$$
Here $s : A \to SA$ is the inclusion map induced by constant loops.

Remark 1.25. Here we want to point out by saying the word "positive generator of $K^1(\mathbb{R})$" we are just making a choice. Also, one can use Chern character to say that the image of the element under Chern character is the positive generator in the associted cohomology group. Besides, there are other ways to represent the positive generator of $K^1(\mathbb{R})$. For example, Fack and Skandalis give an explicit KK-cycle to represent the positive generator of $K^1(\mathbb{R})$.

In [26], they generalize Connes's construction to Kasparov's KK-group and also get similar isomorphisms. Given a C^*-dynamical system $(A, \mathbb{R}, \alpha)$ with A separable and a C^*-algebra B. They construct isomorphisms:

$$\phi^i_\alpha : KK^i(B, A) \to KK^{i+1}(B, A \rtimes \mathbb{R}),$$

and

$$\Phi^i_\alpha : KK^i(A, B) \to KK^{i+1}(A \rtimes \mathbb{R}, B),$$

which satisfy the following three axioms:

- **Axiom 1** If $\alpha^0_{\mathbb{C}}$ is the trivial action of $\mathbb{R}$ on $\mathbb{C}$ and c_1 is the positive generator of $K^0(pt)$, then $\phi^0_{\alpha^0_{\mathbb{C}}}(c_1)$ is the positive generator of $K^1(\mathbb{R})$ and $\Phi^0_{\alpha^0_{\mathbb{C}}}(c_1)$ is the positive generator of $Ext(\mathbb{R})$.

- **Axiom 2** If $\rho : (A, \alpha) \to (B, \beta)$ is an equivariant homomorphism, then

  $$(\hat{\rho})_* \circ \phi^i_\alpha = \phi^i_\beta \circ \rho_*,$$

  and

  $$(\hat{\rho})^* \circ \Phi^i_\alpha = \Phi^i_\beta \circ \rho^*,$$

  where $i \in \mathbb{Z}/2\mathbb{Z}$ and $\hat{\rho} : A \rtimes \mathbb{R} \to B \rtimes \mathbb{R}$ is associated with ρ.

- **Axiom 3** Assume D is separable and E have a countable approximate unit. For $x \in KK^i(B, A)$ and $y \in KK^j(D, E)$ we have

  $$\phi^{i+j}_{id_E \otimes \alpha}(y \otimes_{\mathbb{C}} x) = y \otimes_{\mathbb{C}} \phi^i_\alpha(x).$$

  For $x \in KK^i(A, B)$ and $y \in KK^j(D, E)$, we have

  $$\Phi^{i+j}_{id_E \otimes \alpha}(y \otimes_{\mathbb{C}} x) = \Phi^i_\alpha(x) \otimes_{\mathbb{C}} y.$$

Remark 1.26. The above three axioms are essential to the Connes-Thom isomorphism. Actually, there exists a unique isomorphism which satisfies the three axiom. We will use a similar method to prove that the T-duality isomorphism for twisted K-theory is unique later.

In this section, we see that through C^*-algebra approach the T-dual space of a torus bundle can be noncommutative torus bundle over the base manifold if the twist H does not lie in the kernel of $\pi_!$. Actually, if the twist is more general, the algebra corresponding to the T-dual space can be even non-associative algebra. One can find details about this in [14]. Now we give a simple example to describe different cases of T-duality pair. We can see that the least dimension of the total space is 3 if there is non-associative T-dual space, therefore we just give the different cases for $\mathbb{T}^3$ here to get an impression.

Example 1.27. (1)The T-dual of $\mathbb{T}^3$ as a trivial bundle over a point with trivial H-flux is the dual torus $\hat{\mathbb{T}}^3$ with trivial twist.

(2)The T-dual space of $(\mathbb{T}^3, kdx \wedge dy \wedge dz)$ (Here $\mathbb{T}^3$ is considered as a trivial principal S^1-bundle over $\mathbb{T}^2$ and $k \in \mathbb{Z}$) is $H_{\mathbb{R}}/H_{\mathbb{Z}}$ with trivial twist. Here $H_{\mathbb{R}}$ is the 3-dimensional Heisenberg group and $H_{\mathbb{Z}}$ is the lattice defined by

$$H_{\mathbb{Z}} = \{\begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z}\} \tag{1.14}$$

(3)If we consider $\mathbb{T}^3$ as a trivial $\mathbb{T}^2$-bundle over $\mathbb{T}$, then the T-dual of $(\mathbb{T}^3, kdx\wedge dy\wedge dz)$ is a continuous field of stabilized noncommutative tori, $C^*(H_{\mathbb{Z}}) \otimes \mathcal{K}$.

(4)If we consider $\mathbb{T}^3$ as a trivial $\mathbb{T}^3$-bundle over a point. The T-dual of $(\mathbb{T}^3, kdx \wedge dy \wedge dz)$ is a nonassociative torus, where ϕ is the tricharacher associated to $kdx \wedge dy \wedge dz$.

More details about the above examples can be found in [43], [44] and [14].

In [24] they construct a kind of principal noncommutative torus bundles and they use this notion to give a new and more exact explanation of the T-duality pairs. It will be interesting to investigate their approach and give a more complete version of topological T-duality in a bigger category which includes principal noncommutative torus bundles.

1.5 Schneider's Work

In his thesis [62], A. Schneider introduces a notion of dynamical triple which corresponds to Mathai and Rosenberg's picture of topological T-duality and use this to constructs an equivalence between Bunke-Schick's topological T-duality triples and dynamical triples, which connects the topological approach of topological T-duality and the C^*-algebra one. More concretely, he obtains an explicit formula from the crossed product $CT(P, H) \rtimes \mathbb{R}^n$ to the continuous trace C^*-algebra $CT(\hat{P}, \hat{H})$. We list some of his constructions and results below without complete proofs for reference later.

Denote $\mathbb{R}^n$ by G and $\mathbb{Z}^n$ by N.

Definition 1.28. A dynamical triple (ρ, E, P) over B is a pair (E, P) over B where P is a principal G/N-bundle over B and E is a principal $PU(\mathcal{H})$-bundle over P, together with a continuous action $\rho : E \times G \to E$ which lifts the induced G-action on P such that $\rho(\cdot, g) : E \to E$ is a bundle automorphism for all $g \in G$.

Theorem 1.29 ([62])**.** *Let (ρ, E, P) be a dynamical triple over B and $F = P \times_{PU(\mathcal{H})} \mathcal{K}(\mathcal{H})$. Let $(\hat{\rho}, \hat{E}, \hat{P})$ be the T-dual triple (in the sense of [62]) and $\hat{F}$ be the corresponding C^*-algebra bundle over $\hat{P}$. Then there is an isomorphism of C^*-dynamical systems*

$$(\Gamma(P, H) \rtimes G, \hat{G}, \hat{\alpha^\rho}) \xrightarrow{\cong} (\Gamma(\hat{P}, \hat{H}), \hat{G}, \alpha^{\hat{\rho}}),$$

in which $\hat{G}$ is the dual group of G, $\hat{\rho}$ is the $\hat{G}$ action over $\hat{E}$ and $\hat{\alpha^\rho}$ is the $\hat{G}$ action over $\Gamma(P, H) \rtimes G$ induced by ρ.

We will not give the whole proof of this theorem here. But we will need the explicit formulas for the isomorphisms later, so we explain them a little bit. We first give the isomorphism when the base space is a point. Note that $G \rtimes_{\alpha^\mu} C(G/N, \mathcal{K}(\mathcal{H}))$ is isomorphic to $\overline{C_c(G \times G/N, \mathcal{K}(\mathcal{H}))}$ which is a subalgebra of the algebra of linear operators on $L^2(G \times G/N, \mathcal{H})$. One can find more details about crossed product in the appendix $A.3$. Here $\mu : G \times G/N \to PU(\mathcal{H})$ is a Borel cocyle and $\bar{\mu} : G \times G/N \to U(\mathcal{H})$ is a lifting of μ. Then we introduce a unitary isomorphism between two Hilbert spaces:

$$v : L^2(G \times G/N, \mathcal{H}) \to L^2(\hat{G}, L^2(\widehat{G/N}, \mathcal{H})).$$

For any $F \in L^2(G \times G/N, \mathcal{H})$, $\chi \in \hat{G}$ and $\alpha \in \widehat{G/N}$

$$vF(\chi)(\alpha) = \int_{G \times G/N} < \chi + \alpha^\perp, g >< \alpha, z > \bar{\mu}(-g, z) F(g, z) d(g, z),$$

in which $< \cdot, \cdot >$ is the pair between $\hat{G}$ and G. Under this isomorphism one can transform operators over $L^2(G \times G/N, \mathcal{H})$ to operators over $L^2(\hat{G}, L^2(\widehat{G/N}, \mathcal{H}))$. Schneider defines the isomorphism via u. For any $f \in C_c(G \times G/N, \mathcal{K}(\mathcal{H}))$, he computes $v(f \cdot F)(\chi)(\alpha)$ and define a family of Hilbert-Schmidt operators $f^{\bar{\mu}}$ over $\hat{G}$. Then he define the isomorphism $S_\mu : G \rtimes_{\alpha^\mu} C(G/N, \mathcal{K}(\mathcal{H})) \to C(\hat{G}/N^\perp, \mathcal{K}(L^2(\widehat{G/N}, \mathcal{H})))$ by the following formula:

$$(S_{\bar{\mu}} f)(\chi N^\perp) := Ad(\Lambda(\chi) \otimes Id) f^{\bar{\mu}}(\chi). \tag{1.15}$$

When the base space B is not a point, Schneider chooses a nice open covering $(U_i)_{i\in I}$ of B. Then any section $s \in \Gamma(P, H)$ corresponds to an unique family of functions $f_i \in C(U_i \times G/N, \mathcal{K}(\mathcal{H}))$, which satisfy

$$f_i(u, z) = \zeta_{ji}(u)(z)^{-1}(s_j(u, g_{ji}(u) + z)), u \in U_{ij}, z \in G/N. \tag{1.16}$$

Here g_{ij} and ζ_{ij} are the transition functions of the G/N-bundle $\pi : P \to B$ and principal $PU(\mathcal{H})$-bundle E over P. The strategy is to define the homomorphism piecewise and then prove that they can be glued together. For every open set U_i, Schneider defines the operator:

$$S_i : C(U_i, C_c(G \times G/N, \mathcal{K}(\mathcal{H}))) \to C(U_i \times \hat{G}/N^\perp, \mathcal{K}(L^2(G\hat{/}N, \mathcal{H})))$$

by

$$S_i f_i(u, \chi N^\perp) := (T_{\hat{\mu}_i} f_i(u))(\chi N^\perp) = Ad(\Lambda(\chi)) f_i(u)^{\hat{\mu}_i(u)}(\chi). \tag{1.17}$$

Here $\hat{\mu}_i : U_i \to Z^1_{Bor}(G, L^\infty(G/N, U(\mathcal{H})))$ are a unitary Borel cocycles similar to the $\hat{\mu}$ in the definition of point case. After defining the operator locally, one can glue these S_i together to get the operator S in Theorem 1.29.

1.6 Daenzer's Groupoid Approach

Besides the methods we mentioned in the previous sections, there is another approach to topological T-duality via groupoids given by C.Daenzer. He generalized topological T-duality to noncommutative Lie group action. We will only compare his approach with topological approach of T-duality, so we will only give some of his main constructions here and focus on the case of commutative Lie groups. To do topological T-duality, C. Daenzer first generalizes the notion of principal bundles to the following definition.

Definition 1.30. Let $\mathcal{G}$ be a groupoid, G a locally compact group, and $\rho : \mathcal{G} \to G$ a homomorphism of groupoids. The generalized principal bundle associated to ρ is the groupoid:

$$G \rtimes_\rho \mathcal{G} := (G \times \mathcal{G}_1 \rightrightarrows G \times \mathcal{G}_0);$$

whose source and range maps are respectively given by:

$\bar{s} : (g, \gamma) \mapsto (g\rho(\gamma), s\gamma)$ and $\bar{r} : (g, \gamma) \mapsto (g, r\gamma)$;

and for which the composition is given by

$(g, \gamma_1) \circ (g\rho(\gamma_1), \gamma_2) = (g, \gamma_1\gamma_2)$.

The next lemma shows that Definition 1.30 really gives a generalization of principal bundles.

Lemma 1.31. *When $\mathcal{G}$ is the Cěch groupoid of a good open covering of a topological space X then the gluing functions of a principal G bundle P over X defines a groupoid homomorphism from $\mathcal{G}$ to G.*

Proof. We choose a nice enough open covering U_i such that the restriction of the principal bundle P over every open set is trivial. Assume that the principal bundle P is given by the gluing functions: $f_{ij} : U_i \cap U_j \to G$. We show that these gluing functions induce a groupoid homomorphism $f : \mathfrak{U} \to G$ (Here $\mathfrak{U}$ is the Cech groupoid associated to the open covering U_i). It is obvious that the gluing functions f_{ij} induce a map from $\mathfrak{U}$ to G. The remainder is to prove it is a groupoid homomorphism, for any $\gamma_1 \in U_{ij}$ and $\gamma \in U_{jk}$, $f(\gamma_2 \circ \gamma_1) = f_{ik}(x) = f_{jk}(x) \circ f_{ij}(x) = f(\gamma_2) \circ f(\gamma_1)$. □

Remark 1.32. From the lemma above we can see that a principal G-bundle always corresponds to a generalized principal G-bundle over the Cech groupoid of the base space.

Another important factor of a T-duality pair is the set of twists over principal torus bundles, i.e. some cohomology class over principal torus bundles. C. Daenzer defines a particular kind of groupoid cohomology group in [22]. Before giving his definition, we first review the notion of groupoid cohomology. Let $\mathcal{G}$ be a groupoid and $B \xrightarrow{b} \mathcal{G}_0$ be a left module of $\mathcal{G}$. Let $C^k(\mathcal{G}, B)$:={continuous maps $f : \mathcal{G}_k \to B | b(f(h_1, h_2, ..., h_k)) = rh_1$} and for $f \in C^k(\mathcal{G}, B)$, define an operator δ as follows

$$\begin{aligned}\delta f(g_1, g_2, ..., g_{k+1}) = g_1 \cdot f(g_2, g_3, ..., g_{k+1}) + &\sum_{i=1,...,k} (-1)^i f(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\ &+ (-1)^{k+1} f(g_1, ..., g_k).\end{aligned} \tag{1.18}$$

Then the groupoid cohomology of $\mathcal{G}$ with coefficient B is the cohomology of the complex $(C^\bullet(\mathcal{G}, B), \delta)$. Now we give the constructions of equivariant groupoid cohomology in [22].

Let $\mathcal{G}$ be a G-groupoid, then it induces a G-action on $C^\bullet(\mathcal{G}, B)$ as follows:

$$g \cdot f(g_1, g_2, ..., g_k) = f(g^{-1} \cdot g_1, g^{-1} \cdot g_2, ..., g^{-1} \cdot g_k). \tag{1.19}$$

Then we can construct a double complex:

$$K^{p,q} = (C^p(G, C^q(\mathcal{G}, B)), d, \delta).$$

Here d is the groupoid cohomology differential operator for the groupoid $G \rightrightarrows *$.

Definition 1.33. The G-equivariant cohomology of $\mathcal{G}$ with coefficient B, which we denote by $H^*_G(\mathcal{G}, B)$, is the cohomology of the total complex:

$$tot(K)^n := (\otimes_{p+q=n} K^{p,q}, D = d + (-1)^p \delta). \tag{1.20}$$

Remark 1.34. The cohomology groups here are not Morita invariant. One can form a Morita invariant one by doing injective resolutions. But it is more convenient to use cocycles of this cohomology to represent geometric objects.

A twist in this picture is a second cocycle in the total complex $K^{p,q}$. In general, it can be represented as a triple (σ, λ, β), where $\sigma \in C^0(G, C^2(\mathcal{G}, B)), \lambda \in C^1(G, C^1(\mathcal{G}, B))$ and $\beta \in C^2(G, C^0(\mathcal{G}, B))$ and they satisfy the following condition:

$$\delta\sigma = 1, d\sigma = \delta\lambda, d\lambda = \delta\beta \tag{1.21}$$

Given the above constructions, we now give the construction of classical T-duality in [22]. For simplicity, we first give the dimension 1 case. Let $G = \mathbb{R}, N = \mathbb{Z}$. C. Daenzer started with a pair

$$(G/N \rtimes_{\bar{\rho}} \mathcal{G}, (\sigma, \lambda, 1) \in Z^2_G(G/N \rtimes_{\bar{\rho}} \mathcal{G}; U(1))). \tag{1.22}$$

By Pontryagin duality and generalized Mackey-Rieffel imprimitivity(**Theorem 11.1** in [22]) he obtains a T-duality pair of (1.22) as follows:

$$(\hat{N} \rtimes_{\hat{\lambda}} \mathcal{G}, (\sigma^\vee, \rho, 1) \in Z^2_{\hat{G}}(\hat{N} \rtimes_{\bar{\lambda}} \mathcal{G}; U(1))). \tag{1.23}$$

Here

$$\sigma^\vee(\phi, \gamma_1, \gamma_2) := \sigma(e, \gamma_1, \gamma_2)\lambda(\rho(\gamma_1), \gamma_2) < (\phi\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\rho_1, \rho_2)) > . \tag{1.24}$$

in which $< \cdot, \cdot >$ is the pair between $\hat{N}$ and N. The $\hat{G}$-action is given by

$$\phi' \cdot a(\phi, \gamma) :=< \phi, \rho(\gamma) > a(\phi'^{-1}, \gamma), \tag{1.25}$$

for $\phi' \in \hat{G}$ and $a \in C_c(N \rtimes_{\bar{\lambda}} \mathcal{G})$. We also need to give the definition of $\bar{\lambda} : \mathcal{G} \to \hat{N}$. By definition, $\lambda \in C^1(G, C^1(G/N \rtimes \mathcal{G}, U(1)))$ and $d\lambda = 1$. Therefore we can get the following identities:

$$d\lambda(g_1, g_2)(t, \gamma) = g_1 \cdot \lambda(g_2, t, \gamma) \cdot \lambda(g_1 g_2, t, \gamma)^{-1} \cdot \lambda(g_2, t, \gamma) = 1. \tag{1.26}$$

Choose $g_1 = n \in N, g_2 = g \in G$ or $g_1 = g \in G, g_2 = n \in N$ we have

$$\lambda(n, t, \gamma) = \lambda(gn, t, \gamma)^{-1} \cdot \lambda(g, t, \gamma), \tag{1.27}$$

or

$$\lambda(ng, t, \gamma)^{-1} \cdot \lambda(g, t, \gamma) = g \cdot \lambda(n, t, \gamma) = \lambda(n, g^{-1}t, \gamma). \tag{1.28}$$

Since G is abelian, so we obtain

$$\lambda(n, t, \gamma) = \lambda(n, g^{-1}t, \gamma), \tag{1.29}$$

i.e. when we restrict $\lambda : G \times G/N \times \mathcal{G}_1 \to U(1)$ to $N \times G/N \times \mathcal{G}_1$, it does not depend on t any more. If we choose $g_1, g_2 \in N$ in (1.26), we have

$$\lambda(g_1 g_2, t, \gamma) = \lambda(g_1, t, \gamma) \cdot \lambda(g_2, t, \gamma). \tag{1.30}$$

By $\delta\lambda = d\sigma$ we have

$$\begin{aligned} \delta\lambda(n, t, \gamma_1, \gamma_2) &= \lambda(n, t, \gamma_2) \cdot \lambda(n, t, \gamma_1\gamma_2)^{-1} \cdot \lambda(n, t, \gamma_1); \\ &= d\sigma(n, t, \gamma_1, \gamma_2) = n \cdot \sigma(1, \gamma_1, \gamma_2) \cdot \sigma(1, \gamma_1, \gamma_2)^{-1} = 1. \end{aligned}$$

By the above analysis we get that $\lambda_{N \times G/N \times \mathcal{G}_1}$ is homomorphic in both N and $\mathcal{G}$. It induces a homomorphism from $\mathcal{G}$ to $\hat{N}$, which we call $\bar{\lambda}$. If $\mathcal{G}$ is a Čech groupoid of B and $\bar{\rho} : \mathcal{G} \to S^1$ is the groupoid homomorphism induced by the transition function of principal S^1-bundle $\pi : P \to B$, then the generalized principal bundle $G/N \rtimes_{\bar{\rho}} \mathcal{G}$ is exactly the principal S^1-bundle $\pi : P \to B$. And $\sigma \in Z^2(G/N \rtimes \mathcal{G}, U(1))$ is induced by $H \in H^3(P, \mathbb{Z})$. In this thesis we will construct relations between the above T-duality pair and the T-duality pair in the sense of [12].

1.7 Connections between Daenzer's Construction and Other Approaches

In the appendix of [22], C. Daenzer proved that his approaches to topological T-duality is equivalent to Mathai-Rosenberg's approach.

Theorem 1.35 ([22]). *Let $Q \to X$ be a principal torus bundle trivialized over a good cover of X, let $\mathcal{G}$ denote the Čech groupoid for this cover and let $\rho : \mathcal{G} \to V/\Lambda$ be transition functions presenting Q. Then*

1. *For any $H \in H^3(Q;\mathbb{Z})$ such that $A := A(Q;H)$ admits a V-action, there is a Morita equivalence*
$$A \overset{Morita}{\sim} C^*(V \rtimes \Lambda \rtimes^{\delta\rho} \mathcal{G};\sigma)$$
for some $\sigma \in Z^2(V \rtimes \Lambda \rtimes^{\delta\rho} \mathcal{G}; U(1))$ that is constant in V. If V acts by translation on $C^*(V \rtimes \Lambda \rtimes^{\delta\rho} \mathcal{G};\sigma)$ then the equivalence is V-equivalent.

2. *$[\sigma]$ is the image of H under the composite map*
$$H^3(Q;\mathbb{Z}) \xrightarrow{\sim} H^2(Q;U(1)) \to H^2(V \rtimes \Lambda \rtimes^{\delta\rho} \mathcal{G}; U(1)).$$
Here the second map is induced by the groupoid map $(\theta, x) \to (\theta, 1, x)$ from Q to $V \rtimes \Lambda \rtimes^{\delta\rho} \mathcal{G}$.

3. *Let $\sigma^\vee := \sigma|_{\Lambda\rtimes^{\delta\rho}\mathcal{G}} \in Z^2(\Lambda \rtimes^{\delta\rho} \mathcal{G}; U(1))$. Then for the chosen action of V, there is a $\hat{V}$-equivariant Morita equivalence:*
$$V \ltimes A \overset{Morita}{\sim} V \ltimes C^*(V \rtimes \Lambda \rtimes^{\delta\rho} \mathcal{G};\sigma) \overset{Morita}{\sim} C^*(\Lambda \rtimes^{\delta\rho} \mathcal{G};\sigma^\vee),$$
where $\hat{V}$ acts by the canonical dual action on the left two algebras and on the right most algebra by $\phi \cdot a(\lambda,\gamma) =< \phi, \lambda\rho(\gamma) > a(\lambda,\gamma)$ for $\phi \in \hat{V}$ and $(\lambda,\gamma) \in \Lambda \rtimes^{\delta\rho} \mathcal{G}$.

We can also see some relations between C. Daenzer's picture of topological T-duality and Bouwknegt, Evslin and Mathai's. We will use the notation in the last section and (1.1). In order to see the connections, we need to construct the four relations below:

- $$\mathbb{R}/\mathbb{Z} \rtimes_{\bar{\rho}} \mathcal{G} \dashrightarrow \pi : P \to B; \tag{1.31}$$

- $$\sigma \dashrightarrow H; \tag{1.32}$$

- $$\hat{\mathbb{Z}} \rtimes_{\bar{\lambda}} \mathcal{G} \dashrightarrow \hat{\pi} : \hat{P} \to B; \tag{1.33}$$

- $$\sigma^\vee \dashrightarrow \hat{H}. \tag{1.34}$$

Now we construct the connections one by one.
Starting from a principal S^1-bundle $P \to B$, we choose $\mathcal{G}$ to be a Čech groupoid, then according to Lemma 1.31 we obtain that $S^1 \rtimes_{\bar{\rho}} \mathcal{G}$ is a principal S^1-bundle over B with transition functions induced by $\bar{\rho}$.
From (1.33) that $\delta\sigma = 1$ i.e. σ is a Čech cocycle of $\mathcal{G}$ with coefficient $U(1)$. If we can always choose $\mathcal{G}$ good enough such that $H^k(B,\mathbb{Z}) \cong H^k(\mathcal{G},\mathbb{Z})$. By the exact sequence of *Cěch* cohomology group induced by the exact sequence of sheaves

$$1 \to U(1) \to \mathbb{R} \to \mathbb{Z} \to 1. \tag{1.35}$$

we can get $H^k(\mathcal{G}, U(1)) \cong H^{k+1}(\mathcal{G},\mathbb{Z})$ and moreover $H^k(\mathcal{G}, U(1)) \cong H^{k+1}(B,\mathbb{Z})$. Therefore for any $H \in H^3(B,\mathbb{Z})$, we can always find a closed Čech 2-cocycle σ representing the associated element in Čech cohomology group $H^2(\mathcal{G}, U(1))$.

Remark 1.36. Until now we show that the starting point of Bouwknegt, Evslin and Mathai's approach is equivalent to some special cases of Daenzer's. The difficult point is to construct the relations between the T-dual part.

If $\mathcal{G}$ is a Čech groupoid of B, then we have that $\hat{\mathbb{Z}} \rtimes_{\bar{\lambda}} \mathcal{G}$ is a principal $U(1)$-bundle over B which we denoted by $\hat{P}$. And therefore $C^*(\hat{\mathbb{Z}} \rtimes_{\bar{\lambda}} \mathcal{G}; \sigma^\vee)$ is a continuous trace C^*-algebra over B and its Dixmier-Douady class is the third cohomology class corresponding to $\sigma^\vee \in Z^2(\mathcal{G}, U(1))$ under the following homomorphisms:

$$H^3(\hat{P},\mathbb{Z}) \cong H^2(\hat{P}, U(1)) \to H^2(\hat{\mathbb{Z}} \rtimes_{\bar{\lambda}} \mathcal{G}, U(1)). \tag{1.36}$$

By the construction in Section 12 of [22], we have that

$$C^*(\mathbb{R}/\mathbb{Z} \rtimes_{\bar{\rho}} \mathcal{G}; \sigma) \rtimes_{\lambda} G \overset{Morita}{\backsim} C^*(\hat{\mathbb{Z}} \rtimes_{\bar{\lambda}} \mathcal{G}; \sigma^\vee).$$

According to theorem 1.20, we know that $C^*(\mathbb{R}/\mathbb{Z} \rtimes_{\bar{\rho}} \mathcal{G}; \sigma) \rtimes_{\lambda} G$ is a continuous trace C^*-algebra with spectrum P' and Dixmier-Douady class H' which satisfy the T-duality condition in [12]. Since $C^*(\hat{\mathbb{Z}} \rtimes_{\bar{\lambda}} \mathcal{G}; \sigma^\vee)$ is Morita equivalent to $C^*(\mathbb{R}/\mathbb{Z} \rtimes_{\bar{\rho}} \mathcal{G}; \sigma) \rtimes_{\lambda} G$, we have $\hat{P}$ and P' are isomorphic and H' is the third cohomology class corresponding to $\sigma^\vee$ under the above isomorphisms (1.36) i.e. we get that the third and the fourth relations hold.

Remark 1.37. We do hope to construct an equivalence between Daenzer's approach and other approaches explicitly. Unfortunately, twisting classes in Daenzer's model are not well understood yet and so we can't find a way to interpret it appropriately. Therefore we the relations we get here is not exactly an equivalence.

1.8 Examples of Topological T-duality

In section 1.4, we give some simple examples of topological T-duality. In this section we will give some more examples. Also we will do some computations of twisted K-groups using spectral sequence. The computations are all based on the following Atiyah-Hirzebuch spectral sequence:

Theorem 1.38 ([2]). *Let P be a infinite dimensional projective Hilbert bundle over X. Then there is a spectral sequence whose abutment is $K_P^*(X)$ with*

$$E_2^{pq} = H^p(X, K^q(*)).$$

Also, the following result in [3] is also useful in our computation.

Proposition 1.39. In the Atiyah-Hirsebruch spectral sequence for the functor K_P the differential d_3 is given by:

$$d_3(x) = Sq_{\mathbb{Z}}^3(x) - \eta \cup x, \tag{1.37}$$

where η is the class of P in $H^3(X, \mathbb{Z})$.

Example 1.40. Let $\pi : S^3 \to S^2$ be the Hopf bundle with fiber S^1. If we choose the trivial twist with S^3, then the T-dual space is just the product $S^2 \times S^1$. Since the first Chern class of the Hopf bundle is the generator of $H^2(S^2, \mathbb{Z})$ which we denote by x here, therefore the T-dual twist in $S^2 \times S^1$ is the generator of $H^3(S^2 \times S^1)$ i.e. $x \cup \theta$, which we denote by H. Here θ is the generator of $H^1(S^1, \mathbb{Z})$. Now we consider the twisted K-groups $K^i(S^2 \times S^1, H)$. The E_2 terms are

$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & if 0 \leqslant p \leqslant 3, q \; is \; even \\ 0. & otherwise \end{cases}$$

Therefore the differential d_2 is trivial. According to Proposition (1.39), $d_3(x) = Sq_{\mathbb{Z}}^3(x) - H \cup x$. Since $S^2 \times S^1$ is 3-dimensional, so we only need to consider the differential

$$d_3 : E_3^{2k,0} \to E_3^{(2k-2),3}.$$

We compute $Sq_{\mathbb{Z}}^3(x \cup x)$ first, where x is the generator of $H^0(S^2 \times S^1, \mathbb{Z})$.

$$\begin{aligned} Sq_{\mathbb{Z}}^3(x \cup x) &= \sum_{i+j=3} (Sq_{\mathbb{Z}}^i x) \cup (Sq_{\mathbb{Z}}^j x) \\ &= x \cup Sq_{\mathbb{Z}}^3 x + Sq_{\mathbb{Z}}^3 x \cup x + Sq_{\mathbb{Z}}^1 x \cup Sq_{\mathbb{Z}}^2 x + Sq_{\mathbb{Z}}^2 x \cup Sq_{\mathbb{Z}}^1 x \\ &= x \cup Sq_{\mathbb{Z}}^3 x + Sq_{\mathbb{Z}}^3 x \cup x. \end{aligned}$$

Therefore we have $Sq^3_{\mathbb{Z}}x$ is trivial. Then we get $d_3(x) = -H \cup x$. In this case, we have d_3 is an isomorphism. Since $H^i(X, \mathbb{Z})$ is trivial we get that all higher differentials are trivial.

$$E_\infty^{p,q} = \begin{cases} \mathbb{Z}, & if\ p = 1\,2\ and\ q\ is\ even\ or\ q = p = 0; \\ 0, & otherwise. \end{cases}$$

Therefore we have $K^i(S^2 \times S^1, H) \cong \mathbb{Z}$. While on the other side, we have $K^i(S^3) \cong \mathbb{Z}$. So we can see that the corresponding twisted K-groups are isomorphic.

Example 1.41. According to the computations in the above example, we can get the following results:

Lemma 1.42. *If M is a 3-dimensional manifold and η is the twisting class, then the third differential operator d_3 for Atiyah-Segal spectral sequence is*

$$d_3(x) = -\eta \cup x. \tag{1.38}$$

Using this lemma, we can compute more general twisted K-groups. For example, consider the principal circle bundles P_m over a genus g surface Σ_g with first Chern class mx, where $m \in \mathbb{Z}$ and x is the generator of $H^2(\Sigma_g, \mathbb{Z})$. First of all, we compute the cohomology of the total space. If the principal S^1-bundle is trivial, then we can get the cohomology using the K*ü*nneth theorem.

$$H^0(P, \mathbb{Z}) \cong H^2(P, \mathbb{Z}) \cong \mathbb{Z}, H^1(P, \mathbb{Z}) \cong \mathbb{Z}^{2g+1}.$$

If the principal S^1-bundle is not trivial, then consider the Leray-Serre sequence of the principal bundle $\pi : P_m \to \Sigma_g$. The E_2 terms are

$$E_2^{p,q} = \begin{cases} \mathbb{Z}, & if\ p = 0\ or\ 2\ and\ q = 0\ or\ 1; \\ \mathbb{Z}^{2g}, & if\ p = 1\ and\ q = 0\ or\ 1; \\ 0, & otherwises \end{cases}$$

Therefore the only non-trivial differential is d_2, which is exactly do the cup product with first Chern class here. Therefore we have that

$$\begin{aligned} H^0(P_m, \mathbb{Z}) &\cong H^3(P_m, \mathbb{Z}) \cong \mathbb{Z}; \\ H^1(P_m, \mathbb{Z}) &\cong \mathbb{Z}^{2g},\ H^2(P_m, \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_m. \end{aligned}$$

If we denote the generator of $H^3(P_m, \mathbb{Z})$ as H and choose nH as the twisting class, we can do the similar calculations above and get the twisted K-groups as follows and still get the only nontrivial differential is $d_3 : E_3^{2k,0} \to E_3^{2k-2,3}$, which is do the cup product with the twisting class. Therefore we have for trivial principal S^1-bundle,

$$K^0(P, nH) \cong \mathbb{Z}^{2g+1}, \ K^1(P, nH) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n\mathbb{Z}. \tag{1.39}$$

For nontrivial principal S^1-bundles,

$$K^0(P_m, nH) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/m\mathbb{Z}, \ K^1(P_m, nH) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n\mathbb{Z}. \tag{1.40}$$

From the results of the calculations, we can guess that the pairs (P_m, nH) and (P_n, mH) are T-dual to each other. It is a good exercise to check that the two pairs satisfy (1.1).

Example 1.43. In this example we consider principal S^1-bundles $\pi : P_m \to CP^n$ with generator mX, where X is the generator of $H^2(CP^n, \mathbb{Z})$ and we assume that m is not 0. First we compute the cohomology of P_m using the Leray-Serre spectral sequence. The E_2 terms are as follows

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & ... & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & ... & \mathbb{Z} & 0 \\ & 0 & 1 & 2 & 3 & ... & 2n & ... \end{pmatrix}$$

The only nontrivial differentials are $d_2 : E_2^{2k,1} \to E^{2k+2,0}$, which are all exactly doing cup product with the first Chern class. Then we get the E_3 terms

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & ... & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & \mathbb{Z}/m\mathbb{Z} & 0 & ... & \mathbb{Z}/m\mathbb{Z} & 0 \\ & 0 & 1 & 2 & 3 & ... & 2n & ... \end{pmatrix}$$

Since all of the higher differentials are trivial, therefore we can obtain that

$$\begin{aligned} H^0(P_m, \mathbb{Z}) &\cong H^{2n+1}(P_m, \mathbb{Z}) \cong \mathbb{Z}; \\ H^2(P_m, \mathbb{Z}) &\cong H^4(P_m, \mathbb{Z}) \cong \cdots \cong H^{2n}(P_m, \mathbb{Z}) \cong \mathbb{Z}_m. \end{aligned}$$

From this results we can see that the twisting class over P_m must be trivial. Therefore the T-dual space is $CP^n \times S^1$ and the T-dual twist is $mX \cup \theta$. We compute the corresponding twisted K-groups now.

First we consider $K^*(CP^n \times S^1, mX \cup \theta)$. By definition $E_2^{p,q} = \mathbb{Z}$ if $0 \leqslant p \leqslant 2n+1$ and q is even. Otherwise $E_2^{p,q}$ is 0. The differential d_2 is trivial. And the differential d_3 is doing cup product with $-mX \cup \theta$. Then we find that it is hard to compute it using the Atiyah-Hirzebruch spectral sequence since it is complicated to get the information of the differentials d_{2n}. Fortunately, using topological T-duality. We can transform this question to a simple space i.e. its T-duality pair, whose twisted K-groups we can compute. This could be seen as an application of topological T-duality.
For $K^*(P_m)$, we use the Atiyah-Hirzebruch spectral sequence. Since all of the differentials are trivial, therefore we have $E_\infty^{p,q} = H^p(P_m, K^q(*))$. Then we get that $K^0(P_m) \cong \mathbb{Z}$. And $K^1(P_m) \cong \mathbb{Z} \oplus G_{m^n}$. Here G_{m^n} is an abelian group with m^n elements and with successive quotients $\mathbb{Z}_m$.

Example 1.44. Now we consider the principal S^1-bundle $\pi : P \to CP^1 \times CP^1$ with $n(x_1 + x_2)(n \neq 0)$ as the first Chern class. Here x_1 and x_2 are the generators of the two copies of $H^2(CP^1, \mathbb{Z})$ respectively. First of all, we consider the Leray-Serre spectral sequence and compute the cohomology of P.

$$\begin{aligned} H^0(P,\mathbb{Z}) &\cong H^3(P,\mathbb{Z}) \cong H^5(P,\mathbb{Z}) \cong \mathbb{Z}, H^1(P,\mathbb{Z}) \cong 0 \\ H^2(P,\mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, H^2(P,\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}. \end{aligned}$$

Here the twist can be non-trivial. We choose mH as the twist, where H is the generator of $H^3(P,\mathbb{Z})$ and m is not 0. Then the T-dual space is the principal S^1-bundle over $CP^1 \times CP^1$ with first Chern class $m(x_1 + x_2)$. We denote it by $\hat{\pi} : \hat{P} \to CP^1 \times CP^1$. Also, we can compute the cohomology groups of $\hat{P}$:

$$\begin{aligned} H^0(\hat{P},\mathbb{Z}) &\cong H^3(\hat{P},\mathbb{Z}) \cong H^5(\hat{P},\mathbb{Z}) \cong \mathbb{Z}; H^1(\hat{P},\mathbb{Z}) \cong 0 \\ H^2(\hat{P},\mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}, H^4(\hat{P},\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}. \end{aligned}$$

Denote the generator of $H^3(\hat{P},\mathbb{Z})$ by $\hat{H}$. Then the T-dual twisting class is just $n\hat{H}$. We just compute the twisted K-theory of (P, H) here, while the T-dual part is similar. Using Atiyah-Hirzebruch spectral sequence, we can get the E_4-terms as follows

$$\begin{pmatrix} 2 & 0 & 0 & \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} & \mathbb{Z}/m\mathbb{Z} & \mathbb{Z}/n\mathbb{Z} & \mathbb{Z}/m\mathbb{Z} & 0 \\ 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} & \mathbb{Z}/m\mathbb{Z} & \mathbb{Z}/n\mathbb{Z} & \mathbb{Z}/m\mathbb{Z} & 0 \\ & 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Then we can see that all higher differentials are trivial. Therefore we get that

$$K^0(P, mH) \cong \mathbb{Z} \oplus G_{n^2},\ K^1(P, mH) \cong G_{m^2}$$

Here G_{n^2} is a abelian group with n^2 elements and with successive quotient $\mathbb{Z}/n\mathbb{Z}$.

Example 1.45. Consider S^{2n+1} as the unit sphere of $\mathbb{C}^{n+1}$. Assume $(k_1, k_2, ..., k_n)$ is equal to 1. Define an S^1-action over $\mathbb{C}^{n+1}$ as follows:

$$\rho : S^1 \times \mathbb{C}^{n+1} \mapsto \mathbb{C}^{n+1} \tag{1.41}$$

$$\rho(e^{i\theta}, (z_1, z_2, ..., z_{n+1})) = (e^{ik_1\theta}z_1, e^{ik_2\theta}z_2, ..., e^{ik_{n+1}\theta}z_{n+1}). \tag{1.42}$$

This action induces a free S^1-action over S^{2n+1}. The quotient space Q is called weighted projective space. It is easy to see that the T-dual of $(S^{2n+1}, 0)$ over Q is $(Q \times S^1, c \cup [\theta])$. Here c is the first Chern class of the principal S^1-bundle $S^{2n+1} \to Q$ and $[\theta]$ is the positive generator of $H^1(S^1, \mathbb{Z})$.

Now we give the computation of twisted K-groups for an infinite dimensional manifold, which we will use later.

Example 1.46. Consider the infinite dimensional complex projective space CP^∞, we know that the $\mathbb{Z}$-coefficient cohomology of CP^∞ is the polynomial ring $\mathbb{Z}[x]$. Here x is the generator of $H^2(CP^\infty, \mathbb{Z})$. Therefore for every n, the n-th $\mathbb{Z}$-coefficient cohomology group is $\mathbb{Z}$ with the generator x^k (if $n = 2k$) or $x^k \cup z$ (if $n = 2k+1$). Here z is the generator of $H^1(S^1, \mathbb{Z})$. We choose the H-flux to be the generator of $H^3(S^1 \times CP^\infty, \mathbb{Z})$. Then we can write down the E_2-term of the Atiyah-Hirzebruch spectral sequence as follows:

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} \arrow[ddrrr, "d_3"'] & \mathbb{Z} \arrow[ddrrr, "d_3"'] & \mathbb{Z} \arrow[ddrrr, "d_3"'] & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \cdots
\end{tikzcd}$$

Obviously, d_2 is trivial. The first nontrivial differential is d_3, which is $\cup H$ in this case.

Since $z \cup z$ is trivial, so we have the E_4-term is that

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & & & & & & \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \cdots \\
 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & & & & & & \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \cdots
\end{array}$$

Therefore we can read that $K^0(S^1 \times CP^\infty, H)$ is trivial and $K^1(S^1 \times CP^\infty, H) \cong \mathbb{Z}$.

Lemma 1.47. *For each positive integral n, the inclusion $i_n : S^1 \times CP^n \hookrightarrow S^1 \times CP^\infty$ induces an isomorphism:*

$$i_n^* : K^1(S^1 \times CP^\infty, H) \cong K^1(S^1 \times CP^n, H_n). \tag{1.43}$$

Here H_n is the pullback of H along i_n.

Proof. We just need to show that i_n^* is injective. To get this we just compare the Atiyah-Hirzebruch two spectral sequences for $(S^1 \times CP^n, H_n)$ and $(S^1 \times CP^\infty, H)$ and then we can see the inclusion induces an injection between the two $E_2^{0,1}$-entries. From the above calculations we can see that $E_2^{0,1}$ gives K^1-group in both case. Therefore we get the conclusion. □

2 Geometric Twisted K-homology and T-duality

2.1 Definitions of geometric twisted K-homology

In this chapter we will discuss different models of geometric twisted K-homology and the T-duality transformation of geometric cycles. Given a space X and a twist $\alpha : X \to K(\mathbb{Z}, 3)$, a direct way to construct a twisted K-homology is to use the K-homology of the associated continuous trace C^*-algebra of $C^*(X, \alpha)$ (here $C^*(X, \alpha)$ is the algebra of sections of the pullback $\mathcal{K}$-bundle $\alpha^*(\mathfrak{K})$ and $\mathfrak{K}$ is the universal $\mathcal{K}$-bundle over $K(\mathbb{Z}, 3)$). We denote this twisted K-homology group by $K_*^a(X, \alpha)$. The drawback of the definition is the same clear as its advantage. It is very difficult to see the K-cycles geometrically. In [7] and [67], more topological and geometric models are constructed. Let $\mathfrak{K}$ be the complex K-theory spectrum and $\mathcal{P}_\alpha(\mathfrak{K})$the corresponding bundle of based spectra over X. In [67], the topological twisted K-homology group is defined to be

$$K_n^t(X, \alpha) := \lim_{k\to\infty}[S^{n+2k}, \mathcal{P}_\alpha(\Omega^{2k}\mathfrak{K})/X]. \tag{2.1}$$

This definition comes from the classical definition of homology theory by spectra, which is automatically a homology theory. In [67], B.L. Wang also gives a version of geometric twisted K-homology. Before giving his constructions, we first review the definition of geometric cycles of K-homology. A geometry K-cycle on a pair of space (X, Y) $(Y \subset X)$ is a triple (M, f, E), such that

- M is a spin^c-manifold(possibly with boundary);
- f is a continuous map from M to X such that $f(\partial M) \subset Y$;
- E is a vector bundle over M.

Now we give the definition of twisted geometric K-cycles in [67].

Definition 2.1. Let X be a locally compact CW-complex and Y be a subcomplex of X. $\alpha : X \to K(\mathbb{Z}, 3)$ is a twist over X. A geometric cycle for (X, Y, α) is a quintuple $(M, \iota, \upsilon, \eta, E)$ such that

- M is an α-twisted spin^c manifold, i.e. M is a compact oriented manifold which admits the following diagram.

$$\begin{tikzcd} M \arrow[rr, "\upsilon"] \arrow[d, "\iota"] & & \mathbf{BSO} \arrow[d, "W_3"] \arrow[dll, "\eta"'] \\ X \arrow[rr, "\alpha"] & & K(\mathbb{Z}, 3) \end{tikzcd} \tag{2.2}$$

  Here υ is the classifying map of the stable normal bundle of M, W_3 the third integer Stiefel-Whitney class and η is a homotopy between $\alpha \circ \iota$ and $W_3 \circ \upsilon$. We require $\iota(\partial M) \subset Y$.

- $[E]$ is an element class of $K^0(X)$ which is represented by a $\mathbb{Z}_2$-graded vector bundle E.

Let $\Gamma(X, \alpha)$ be the collections of all geometric cycles for (X, α). To get geometric twisted K-homology, one still needs to impose some equivalence relation on $\Gamma(X, \alpha)$, which is generated by the following basic relations:

- **Direct sum - disjoint union** If $(M, \iota, \upsilon, \eta, E_1)$ and $(M, \iota, \upsilon, \eta, E_2)$ are geometric cycles with the same α-twisted $Spin^c$ structure, then

$$(M, \iota, \upsilon, \eta, E_1) \cup (M, \iota, \upsilon, \eta, E_2) \sim (M, \iota, \upsilon, \eta, E_1 \oplus E_2); \tag{2.3}$$

- **Bordism** Let $(M, \iota, \upsilon, \eta, E_1)$ and $(M, \iota, \upsilon, \eta, E_2)$ be two α-twisted geometric cycles over X. If there exists a α-twisted spin^c manifold $(W, \iota, \upsilon, \eta)$ and $[E] \in K^0(W)$ such that

$$\delta(W, \iota, \upsilon, \eta) = -(M_1, \iota_1, \upsilon_1, \eta_1) \cup (M_2, \iota_2, \upsilon_2, \eta_2) \tag{2.4}$$

  and $\delta(E) = E_1 \cup -E_2$. then we have

$$(M, \iota, \upsilon, \eta, E_1) \sim (M, \iota, \upsilon, \eta, E_2). \tag{2.5}$$

  Here $-(M_1, \iota_1, \upsilon_1, \eta_1)$ denotes the manifold M_1 with the opposite α-twisted spin^c-structure;

- Spin^c **vector bundle modification** Given a geometric cycle $(M, \iota, \upsilon, \eta, E)$ and a spin^c-vector bundle V over M with even dimensional fibers, we can choose a Riemannian metric on $V \oplus \mathbb{R}$ and get the sphere bundle $\hat{M} = S(V \oplus \mathbb{R})$. Then the vertical tangent bundle $T^v(\hat{M})$ admits a natural spin^c structure. Let S_V^+ be the associated positive spinor bundle and $\rho : \hat{M} \to M$ be the projection. Then

$$(M, \iota, \upsilon, \eta, E) \sim (\hat{M}, \iota \circ \rho, \upsilon \circ \rho, \eta \circ \rho, \rho^* E \otimes S_V^+). \tag{2.6}$$

Definition 2.2 (Wang [67]). $K_*^g := \Gamma(X, \alpha)/\sim$. Addition is given by disjoint union. Let $K_0^g(X, \alpha)$ (respectively $K_1^g(X, \alpha)$) be the subgroup of $K_*^g(X, \alpha)$ determined by all geometric cycles with even (respectively odd) dimensional α-twisted Spin^c manifolds.

There is a natural isomorphism μ between $K_{0/1}^g(X, \alpha)$ and $K_{0/1}^a(X, \alpha)$:

$$\mu(M, \iota, \upsilon, \eta, E) = \iota_* \circ \eta_* \circ I^* \circ PD([E]). \tag{2.7}$$

Here $PD : K^i(M) \to K_{n+i}(M, W_3 \circ \tau)$ is the Poincaré duality map between K-group and K-homology group (see [21]), ι_* is the push-forward map induced by ι, η_* is the canonical isomorphism induced by η

$$\eta_* : K_{0/1}^a(M, W_3 \circ \upsilon) \to K_{1/0}^a(M, \alpha \circ \iota) \tag{2.8}$$

and $I : K_{0/1}^a(M, W_3 \circ \tau) \to K_{0/1}^a(M, W_3 \circ \upsilon)$ is a natural isomorphism, whose explicit construction can be found in Section 3 of [67]. The following theorem states that μ is an isomorphism.

Theorem (**Theorem 6.4** in [67]). *The assignment* $(M, \iota, \upsilon, \eta, [E]) \to \mu(M, \iota, \upsilon, \eta, [E])$, *called the assembly map, defines a natural homomorphism*

$$\mu : K_{0/1}^g(X, \alpha) \to K_{0/1}^a(X, \alpha),$$

which is an isomorphism for any smooth manifold X with a twisting $\alpha : X \to K(\mathbb{Z}, 3)$.

In [7], another version of geometric twisted K-homology is constructed. Before giving the definitions of the associated cycles in [7], we need to point out that they use another description of twists here. Let X be a second countable locally compact Hausdorff topological space. Then a twisting datum on X in [12] is a locally trivial bundle $\mathcal{A}$ of elementary C^*-algebras on X, i.e. each fiber of $\mathcal{A}$ is an elementary C^*-algebra and with the structure group the automorphism group of $\mathcal{K}(\mathbb{H})$ for some Hilbert space $\mathbb{H}$. We first give definition of topological twisted K-cycles in [7].

Definition 2.3. An $\mathcal{A}$-twisted K-cycles on X is a triple (M, σ, ψ) where

- M is a compact spin^c-manifold without boundary;
- $\phi : M \to X$ is a continuous map;
- $\sigma \in K_0(\Gamma(M, \phi^* \mathcal{A}^{op}))$.

Let $E(X, \alpha)$ be all of the K-cycles over $(X, \mathcal{A})$. Then the topological $\mathcal{A}$-twisted K-homology group over $(X, \mathcal{A})$ is defined by

$$K_*^{top}(X, \mathcal{A}) := E(X, \mathcal{A}) / \sim \tag{2.9}$$

Here $\sim$ is a similar equivalence generated by disjoint unions, bordisom and vector bundle modification. Moreover, a more geometric version of K-cycle is also given in [7], which is closer to D-branes in string theory and therefore is call D-cycle. In order to introduce D-cycle, we still also need the following definition for the geometric K-cycles.

Definition 2.4. A spinor bundle for a twisting $\mathcal{A}$ is a vector bundle S of Hilbert spaces on X together with a given isomorphism of twisting data:

$$\mathcal{A} \cong \mathcal{K}(S). \tag{2.10}$$

Remark 2.5. A fact we need to know about spinor bundles is that a spinor bundle for $\mathcal{A}$ exists if and only if $DD(\mathcal{A}) = 0$.

Now we can give the definition of D-cycles in [7].

Definition 2.6 (Baum, Carey and Wang [7]). A D-cycle for $(X, \mathcal{A})$ is a 4-tuple (M, E, ϕ, S) such that

- M is a compact oriented C^∞ Riemannian manifold;
- E is a complex vector bundle on M;
- ϕ is a continuous map from M to X;
- S is a spinor bundle for $Cliff_{\mathbb{C}}^+(TM) \otimes \phi^* \mathcal{A}^{op}$.

Two D-cycles (M, E, ϕ, S), (M', E', ϕ', S') for $(X, \mathcal{A})$ are isomorphic if there is an orientation preserving isometric diffeomorphism $f : M \to M$ such that the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\phi"'] & & M' \arrow[dl, "\phi'"] \\ & X & \end{tikzcd} \tag{2.11}$$

commutes, and $f^*E' \cong E, f^*S' \cong S$. Moreover, we require that the isomorphism $f^*S' \cong S$ is compatible with the given isomorphisms in the definition of spinor bundles. More explicitly, we require that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{K}(S') \arrow[r, "f^*"] \arrow[d, "\gamma'"] & \mathcal{K}(S) \arrow[d, "\gamma"] \\ Cliff_{\mathbb{C}}^+(TM') \otimes \phi'^*\mathcal{A}^{op} \arrow[r, "f^*"] & Cliff_{\mathbb{C}}^+(TM) \otimes \phi^*\mathcal{A}^{op} \end{tikzcd} \tag{2.12}$$

in which $\gamma : \mathcal{K}(S) \cong Cliff_{\mathbb{C}}^+(TM) \otimes \phi^*\mathcal{A}^{op}$ and $\gamma' : \mathcal{K}(S') \cong Cliff_{\mathbb{C}}^+(TM') \otimes \phi'^*\mathcal{A}^{op}$ are the corresponding chosen isomorphisms in the definition of spinor bundles.

Let $\Gamma^D(X, \mathcal{A})$ be the collection of all D-cycles over $(X, \mathcal{A})$. Similarly, we can impose an equivalence generated by disjoint union, bordism and vector bundle modification on $\Gamma^D(X, \mathcal{A})$ and get the geometric $\mathcal{A}$-twisted K-homology in [7], which we denote by $K_*^{geo}(X, \mathcal{A})$. However, they don't give an explicit construction of the equivalence $\sim$ in [7]. But some part of the construction is not so completely evident, so we give an explicit statement as follows:

- **Direct sum - disjoint union** If (M, E_1, ι, S) and (M, E_2, ι, S) are D-cycles for (X, α) then
$$(M, E_1, \iota, S) \cup (M, E_2, \iota, S) \sim (M, E_1 \oplus E_2, \iota, S); \tag{2.13}$$
- **Bordism** Given two D-cycles (M_0, E_0, ι_0, S_0) and (M_1, E_1, ι_1, S_1) , if there exists a 4-tuple (W, E, Φ, S) such that W is a compact oriented Riemannian manifold with boundary, E is a complex vector bundle over W , Φ is a continuous map from W to X and
$$(\delta W, E|\delta W, \Phi|\delta W, S^+|\delta W) \cong (M_0, E_0, \iota_0, S_0) \cup -(M_1, E_1, \iota_1, S_1), \tag{2.14}$$

in which the isomorphism is compatible with the given isomorphisms of spinor bundles. then

$$(M_0, E_0, \iota_0, S_0) \sim (M_1, E_1, \iota_1, S_1). \tag{2.15}$$

Here $S^+|\delta$ is the positive part of S. It is S itself when W is odd dimensional.

- Spin^c **vector bundle modification** Use the notations in the definition of $K^g(X,\alpha)$. Given a D cycle (M, E, ι, S) and a spin^c vector bundle V over M with even dimensional fibers. Let S_V be the spinor bundle of V. Then

$$(M, E, \iota, S) \sim (\hat{M}, S_V^+ \otimes \rho^* E, \iota \circ \rho, \rho^*(S)). \tag{2.16}$$

In [7] they give a natural charge map $h : K_*^{geo}(X, \mathcal{A}) \to K_*^{top}(X, \mathcal{A})$ as follows: Let (M, E, ϕ, S) be a D-cycle and choose a normal bundle of M with even dimensional fibers, then

$$h(M, E, \phi, S) := (S(\nu \oplus \underline{\mathbb{R}}), \phi \circ \pi, \sigma) \tag{2.17}$$

Here σ is defined as the image of E under the composition of $s_!$ and χ:

1. Let $s : M \to S(\upsilon \oplus \underline{\mathbb{R}})$ be the canonical section of unite section on the trivial real line bundle. More explicitly, s is given by the unit section of $\underline{\mathbb{R}}$, i.e. for any $x \in M$

$$s(x) = (x, (0, 0, ..., 0, 1)). \tag{2.18}$$

   For simplicity, we denote the total space of sphere bundle $S(\upsilon \oplus \underline{\mathbb{R}})$ by M and the bundle map by ρ. Then $s_! : K^0(M) \to K_0(\Gamma(\hat{M}, \rho^*(Cliff_{\mathbb{C}}(\upsilon))))$ is the Gysin homomorphism induced by s.

2. $\chi : K_0(\Gamma(\hat{M}, \rho^*(Cliff_{\mathbb{C}}(\upsilon)))) \to K_0(\Gamma(\hat{M}, (\phi \circ \rho)^* \mathcal{A}^{op}))$ is the isomorphism induced by the trivialization of $TM \oplus \upsilon$ and the given spinor bundle S.

In [7] they propose a question: is h an isomorphism? We will discuss this problem in the Section 2.3.

2.2 Equivalence between the two versions of geometric twisted K-homology

We first list the following theorem, which gives us characteristic classes of Clifford bundles.

Theorem 2.7 ([53]). *Let E be a vector bundle over a space X and $\tilde{E}$ be the Clifford bundle of E. Let $W_3(E)$ be the third integer Stiefel-Whitney class of E. Then we have:*

$$W_3(E) = \begin{cases} DD(\tilde{E}) & if\ E\ has\ even\ dimension; \\ DD(\tilde{E}^+) & if\ E\ has\ odd\ dimension. \end{cases} \tag{2.19}$$

Remark 2.8. Through the above theorem we can get a piece of idea why the two versions of geometric twisted K-cycles in last section are equivalent. The existence of a spinor bundle implies the triviality of Dixmier-Douady class of DD(Cliff$^+(TM) \otimes \phi^*(\mathcal{A}^{op})$), so the choice of the spinor bundle S for Cliff$^+(TM) \otimes \phi^*\mathcal{A}^{op}$ in the definition of D-cycle corresponds to the choice of the homotopy η in the definition of geometric K-cycle in [67].

Let X be a locally finite CW-complex and $\alpha : X \to K(\mathbb{Z}, 3)$ be a twist in the sense of [67]. Since $BPU(\mathbb{H})$ is a model of $K(\mathbb{Z}, 3)$, α gives a principal $PU(\mathbb{H})$-bundle over X ,which has an associated bundle $\mathcal{A}$ with fiber $\mathcal{K}(\mathbb{H})$. While $\mathcal{A}$ is exactly a twist in the sense of [7]. Therefore, we get the basic set up data of the two definitions of geometric twisted K-cycles are equivalent. First we give two lemmas.

Lemma 2.9. *Denote the projection from $X \times I$ to X by p. For every $\mathcal{K}$-bundle $\mathcal{A}$ over $X \times I$, there exists a $\mathcal{K}$-bundle $\mathcal{A}'$ over X such that $\mathcal{A} \cong p^*(\mathcal{A}')$.*

Proof. Denote the Dixmier-Douady class of $\mathcal{A}$ by δ. Since p induces an isomorphism between $H^3(X\times[0,1], \mathbb{Z})$ and $H^3(X, \mathbb{Z})$, so there exists a $\delta' \in H^3(X, \mathbb{Z})$ such that $p^*(\delta') = \delta$. Choose a $\mathcal{K}$-bundle $\mathcal{A}'$ over X with δ' as its Dixmier-Dourady class. Then we obtain that $\mathcal{A} \cong p^*(\mathcal{A}')$. □

Lemma 2.10. *Let $\mathcal{A}$ be a $\mathcal{K}$-bundle over X, then there is a canonical spinor bundle for $\mathcal{A} \otimes \mathcal{A}^{op}$.*

Proof. The set of Hilbert-Schmidt operators on a separable Hilbert space $\mathbb{H}$ forms a Hilbert space under the inner product given by traces. More explicitly, given two Hilbert-Schmidt operators T_1, T_2, the inner product is given by

$$< T_1, T_2 >= Trace(T_1 T_2^*).$$

Moreover, the set of Hilbert-Schmidt operators is also a two-sided idea in the algebra of compact operators. For each fiber $\mathcal{A}_x$ (which is isomorphic to $\mathcal{K}(\mathbb{H})$), if we denote the corresponding Hilbert space of Hilbert-Schmidt operators by $(\mathcal{A}_x)_{H-S}$, then the left multiplication and right multiplication of $\mathcal{A}_x$ gives a left action of $\mathcal{K}(\mathbb{H}) \otimes \mathcal{K}^{op}(\mathbb{H})$ on $\mathbb{H}_{H-S}$, which also identifies $\mathcal{K}(\mathbb{H}) \otimes \mathcal{K}^{op}(\mathbb{H})$ with the compact operators over $\mathbb{H}_{H-S}$. We denote the natural isomorphism by $c_x : \mathcal{A}_x \otimes \mathcal{A}_x^{op} \cong \mathcal{K}((\mathcal{A}_x)_{H-S})$. Let S be a Hilbert bundle over X whose fiber is $(\mathcal{A}_x)_{H-S}$. Then the c_xs imply a fiberwise isomorphism between $\mathcal{A} \otimes \mathcal{A}^{op}$ and $\mathcal{K}(S)$. Therefore S is a spinor bundle of $\mathcal{A} \otimes \mathcal{A}^{op}$. □

Now we give a construction which is useful in the proof of the main theorem in this section. Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are two $\mathcal{K}$-bundles over X and $\lambda : \mathcal{K}_1 \to \mathcal{K}_2$ is an isomorphism. Then we can get a $\mathcal{K}$-bundle over $X \times [0, 1]$ as follows. First we can see that $\mathcal{K}_1 \times [0, 1/2]$ and $\mathcal{K}_2 \times [1/2, 1]$ are $\mathcal{K}$-bundle over $X \times [0, 1/2]$ and $X \times [1/2, 1]$ respectively. Then we can glue $\mathcal{K}_1 \times [0, 1/2]$ and $\mathcal{K}_2 \times [1/2, 1]$ at $\{1/2\} \times X$ via λ and get a $\mathcal{K}$-bundle $\mathcal{K}_0$ over $X \times [0, 1]$. Now we give the main theorem of this section.

Theorem 2.11. *Let X be a locally finite CW-complex and $\alpha : X \to K(\mathbb{Z}, 3)$ be a twist over X. Moreover we denote $\mathcal{A}$ to be the pullback $\mathcal{K}$-bundles along α. There exists an isomorphism $\mathcal{F} : K^g(X, \alpha) \to K^{geo}(X, \mathcal{A})$.*

To make the proof easier to read, we give the basic idea first. The idea is that we transform spinor bundles over underlying manifold M in D-cycles to $\mathcal{K}$-bundles over $M \times [0, 1]$. Then we use these $\mathcal{K}$-bundles to define homotopies in α-twisted spin^c-manifolds and also define F below. And we reverse the procedure to prove that F is surjective. The injectivity of F is essentially implied by the compatibility of F with $\sim$. Now we start the proof.

Proof. Let $[x]$ be a class in $K^{geo}(X, \mathcal{A})$ and (M^g, E, ϕ, S) be a D-cycle representing $[x]$. By definition S is a spinor bundle of $Cliff_{\mathbb{C}}^+ \otimes \phi^*\mathcal{A}^{op}$. And we denote the chosen isomorphism between $\mathcal{K}(S)$ and $Cliff_{\mathbb{C}}^+ \otimes \phi^*\mathcal{A}^{op}$ by λ. We define $F([x])$ in $K^g(X, \alpha)$ to be $[(M, \phi, \upsilon, \eta, E)]$, in which

- M is the underlying manifold of M^g, ϕ and E are the same map and bundle in the D-cycle;
- υ is the classifying map of the stable normal bundle of M;
- η is a homotopy between $W_3 \circ \upsilon$ and $\alpha \circ \phi$.

We only need to explain how to construct η from (M^g, E, ϕ, S). By Lemma 2.10 we know that there exists a canonical Hilbert bundle V over M and a canonical isomorphism c between $\mathcal{K}(V) \cong \phi^*\mathcal{A} \otimes \phi^*\mathcal{A}^{op}$. Combine c and h we get an isomorphism between

$\mathcal{K}(S) \otimes \phi^*\mathcal{A}$ and $Cliff^+_{\mathbb{C}}(TM) \otimes \mathcal{K}(V)$. According to the discussion before the theorem we can glue $\mathcal{K}(S)\otimes\phi^*\mathcal{A}\times[0,1/2]$ and $Cliff^+_{\mathbb{C}}(TM)\otimes\mathcal{K}(V)\times[1/2,1]$ to obtain a $\mathcal{K}$-bundle $\mathcal{W}$ over $M\times I$ such that $\mathcal{W}_{M\times\{0\}} \cong \mathcal{K}(S)\otimes\phi^*\mathcal{A}$ and $\mathcal{W}_{M\times\{1\}} \cong Cliff^+_{\mathbb{C}}(TM)\otimes\mathcal{K}(V)$. Since $BPU(\mathbb{H})$ is a classifying space of $\mathcal{K}$-bundles, therefore there exists an $\eta : X \times [0,1] \to K(\mathbb{Z},3)$ such that $(\eta \circ (\phi \times Id))^*(\mathfrak{R})$ is isomorphic to $\mathcal{W}$. Moreover, we get two maps α_0, $\alpha_1 : X \to K(\mathbb{Z},3)$ by restricting η to $X \times \{0\}$ and $X \times \{1\}$ respectively, which give the following isomorphisms:

$$(\alpha_0 \circ \phi)^*(\mathfrak{R}) \cong \mathcal{K}(S) \otimes \mathcal{A}, \quad (\alpha_1 \circ \phi)^*(\mathfrak{R}) \cong Cliff^+_{\mathbb{C}}(TM) \otimes \mathcal{K}(V) \tag{2.20}$$

Different choices of η are homotopic to each other, so they represent the same class in $K^g(X,\alpha)$ via the **Bordism** relation. To show that F is well defined, we still need to check that it is compatible with the relations which define $\sim$.

- From the definition of F we can see

$$F([(M^g, E_1, \phi, S)] \cup [(M^g, E_2, \phi, S)]) = [(M, E_1 \oplus E_2, \phi, S)],$$
$$F([M^g, E_1, \phi, S]) \cup F([M^g, E_2, \phi, S]) = [(M, E_1 \oplus E_2, \phi, S)],$$

  i.e. F respects the **disjointunion** relation.

- Let (M^g, E, ϕ, S) be a bordism between $(M^g_0, E_0, \phi_0, S_0)$ and $(M^g_1, E_1, \phi_1, S_1)$. Denote the associated isomorphisms of the spinor bundles by h, h_0 and h_1. Denote the chosen representing cycles of the image of their homology classes under F by $(M, \phi, \upsilon, \eta, E)$, $(M_0, \phi_0, \upsilon_0, \eta_0, E_0)$ and $(M_1, \phi_1, \upsilon_1, \eta_1, E_1)$ respectively. Choosing a tubular neighborhood of M_0 in M we can get that $TM|_{M_0} \cong TM_0 \otimes \underline{\mathbb{R}}$, so we get the stable normal bundle of M_0 agrees with the restriction of the stable normal bundle of M to M_0 i.e. $\upsilon|_{M_0}$ is homotopic to υ_0. Similarly we can get υ_{M_1} is homotopic to υ_1.

  Let $\mathcal{W}$, $\mathcal{W}_0$ and $\mathcal{W}_1$ be the three $\mathcal{K}$-bundles (which we see above in the construction of η) over $M \times [0,1]$, $M_0 \times [0,1]$ and $M_1 \times [0,1]$ respectively. Since the isomorphism c in Lemma 2.10 is natural and $h|_{\mathcal{K}(S_i)} = h_i$ $(i = 0, 1)$, therefore we get that $\mathcal{W}|_{M_i\times[0,1]}$ is isomorphic to $\mathcal{W}_i$ $(i = 0, 1)$. This implies that $\eta|_{M\times\{i\}}$ is homotopic to η_i $(i = 0, 1)$. So F is compatible with the **Bordism** relation.

- Use the notations in Section 2.1 and denote $F([(\hat{M}^g, S^+_V \otimes \rho^*E, \phi \circ \rho, \rho^*S)])$ by $[\hat{M}, S^+_V \otimes \rho^*E, \phi \circ \rho, \upsilon \circ \rho, \eta \circ \rho]$. We only need to prove that η' is homotopic tov

$\eta \circ \rho$ i.e. we only need to show the corresponding $\mathcal{K}$-bundles $\mathcal{V}'$ and $\mathcal{V}''$ over $\hat{M} \times [0,1]$ defined by η' and $\eta \circ \rho$ respectively are isomorphic, which follows from the fact that their Dixmier-Dourady classes are the same. Therefore we get that F is compatiable with the **Vectorbundlemodification** relation.

So we get that F is a well defined homomorphism. Now we turn to the injectivity of F. If a D-class $[x]$ is mapped to 0 by F, we can assume that $F(x)$ is a trivial geometric cycle without loss of generality i.e. $F(x) = (pt, i, 0, 0, 0)$. Otherwise we can use the three relations to transfer it to a trivial cycle. According to the definition of F, we get that $x = (pt, 0, i, \mathcal{H})$, which is obviously a trivial D-cycle. The left is to show that F is surjective. For any class $[y] \in K^g(X, \alpha)$, choose a geometric cycle $(M, \phi, \upsilon, \eta, E)$ to represent it. η induces a $\mathcal{K}$-bundle over $M \times [0,1]$, which we denote by $\mathcal{X}$. By the definition of η we have

$$\mathcal{X}|_{M\times\{0\}} \cong Cliff_{\mathbb{C}}^{+}(TM) \otimes \mathcal{K},\ \mathcal{X}|_{M\times\{1\}} \cong (\alpha \circ \phi)^*(\mathfrak{K}).$$

Let l be an isomorphism between $Cliff_{\mathbb{C}}^{+}(TM) \otimes \mathcal{K}$ and $(\alpha \circ \phi)^*(\mathfrak{K})$ and $\mathcal{L}$ be a $\mathcal{K}$-bundle constructed via gluing $Cliff_{\mathbb{C}}^{+}(TM) \otimes \mathcal{K} \times [0, 1/2]$ and $(\alpha \circ \phi)^*(\mathfrak{K}) \times [1/2, 1]$ by l at $M \times \{1/2\}$. Then we have that $\mathcal{L}$ is isomorphic $\mathcal{X}$ since they have the same Dixmier-Dourady class. Combining l with the canonical isomorphism in Lemma 2.10, we get a spinor bundle S for $Cliff_{\mathbb{C}}^{+}(TM) \otimes \phi^*\mathcal{A}^{op}$. The D-cycle (M^g, E, ϕ, S) satisfies that $F([M^g, E, \phi, S]) = [y]$. □

Remark 2.12. The above theorem shows that twisted K-homology via geometric K-cycles in [7] and D-cycles in [67] are equivalent to each other. Therefore we can choose any one of them to construct T-duality isomorphism for twisted geometric K-homology. Moreover, since $K^g(X, \alpha)$ is isomorphic to $K^a(X, \alpha)$ we could get that $K^{geo}(X, \alpha)$ is isomorphic to $K^a(X, \alpha)$ and therefore isomorphic to $K^{top}(X, \alpha)$. However, this does not answer the question in Section 8 of [7] as we still don't know if this isomorphism can be realized by the charge map $h : K_*^{geo}(X, \mathcal{A}) \to K_*^{top}(X, \mathcal{A})$, which is our topic in the next section.

2.3 The charge map is an isomorphism

As we discussed in the remark in the end of the last section, we can't directly get that the charge map h is an isomorphism, we will show it by considering the following diagram in this section:

$$\begin{tikzcd} K^{geo}(X,\mathcal{A}) \arrow[r, "h"] \arrow[d, "F"] & K^{top}(X,\mathcal{A}) \arrow[d, "\eta"] \\ K^{g}(X,\alpha) \arrow[r, "\mu"] & K^{a}(X,\alpha) \end{tikzcd} \tag{2.21}$$

Here μ is the analytic index map in [67] and $\eta : K_*^{top}(X,\mathcal{A}) \to K_*^{a}(X,\mathcal{A})$ is the natural map in [7] , which is defined as follows:

$$\eta(M,\phi,\sigma) = \phi_*(PD(\sigma)). \tag{2.22}$$

Moreover, we know that μ is an isomorphisms for smooth manifolds and η is an isomorphism for locally finite CW-complexes. And we proved that F is an isomorphism for any locally finite CW-complexes. If we can show that diagram (2.21) is commutative, then we will get that h must also be an isomorphism.

Proposition 2.13. For any smooth manifold X, the diagram (2.21) is commutative.

Proof. If we write the formula of the map in diagram (2.21) for a D-cycle (M,E,ι,S) over (X,α) using the notation before, we get

$$\eta\circ h(M,E,\iota,S) = \iota_*\circ\rho_*\circ PD\circ\chi\circ s_!([E]), \tag{2.23}$$

$$\mu\circ F(M,E,\iota,S) = \iota_*\circ\eta_*\circ I_*\circ PD([E]). \tag{2.24}$$

Therefore the commutativity of diagram (2.21) is equivalent to

$$\iota_*\circ\rho_*\circ PD\circ\chi\circ s_! = \iota_*\circ\eta_*\circ I_*\circ PD, \tag{2.25}$$

i.e. equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} K^0(E) \arrow[r, "PD"] \arrow[d, "s_!"] & K_*(M, W_3\circ\tau) \arrow[d, "I_*"] \\ K^*(\hat{M}, W_3\circ\upsilon\circ\varrho)) \arrow[d, "\chi"] & K_*(M, W_3\circ\upsilon) \arrow[ddd, "\eta_*"] \\ K^*(\hat{M}, -\alpha\circ\iota\circ\rho) \arrow[d, "PD"] & \\ K_*(\hat{M}, \alpha\circ\iota\circ\rho) \arrow[r, "\rho_*"] & K_*(M, \alpha\circ\iota) \end{tikzcd} \tag{2.26}$$

Here $PD : K^*(\hat{M}, (\iota \circ \rho)^*(\mathcal{A}^{op}) \to K_*(\hat{M}, (\iota \circ \rho)(\mathcal{A}))$ is the Poincaré duality map between twisted K-theory and twisted K-homology in [25] and [64]. By the naturality of Poincaré duality (one can see Corollary 3.8 in [21]), we have that the commutative diagram below

$$\begin{tikzcd} K^0(M) \arrow[r, "PD"] \arrow[d, "s_!"] & K_*(M, W_3 \circ \tau) \arrow[d, shift left, "s_*"] \\ K^0(\hat{M}, W_3 \circ \upsilon \circ \rho) \arrow[r, "PD"] & K_*(\hat{M}, W_3 \circ \tau \circ \rho) \arrow[u, shift left, "\rho_*"] \end{tikzcd} \tag{2.27}$$

The twist of the lower right K-homology group is $W_3 \circ \tau \circ \rho$ since $\hat{M}$ admits a spin^c-structure. Since $\rho \circ s = Id$, we have $\rho_* \circ s_* = id$, therefore we can get

$$PD = \rho_* \circ s_* \circ PD = \rho_* \circ PD \circ s_!. \tag{2.28}$$

To prove the whole proposition, we first give the following lemma.

Lemma 2.14. *Denote the map on twisted K-homology groups induced by changing twists by $\tilde{\chi} : K_*(\hat{M}, W_3 \circ \tau \circ \rho) \to K_*(\hat{M}, \alpha \circ \iota \circ \rho)$, which can be defined similar to χ in the end of Section 2.1. Then the following diagram is commutative*

$$\begin{tikzcd} K^*(\hat{M}, W_3 \circ \upsilon \circ \rho) \arrow[r, "PD"] \arrow[d, "\chi"] & K_*(\hat{M}, W_3 \circ \tau \circ \rho) \arrow[r, "\rho_*"] \arrow[d, "\tilde{\chi}"] & K_*(M, W_3 \circ \tau) \arrow[d, "\eta_* \circ I_*"] \\ K^*(\hat{M}, -\alpha \circ \iota \circ \rho) \arrow[r, "PD"] & K_*(\hat{M}, \alpha \circ \iota \circ \rho) \arrow[r, "\rho_*"] & K_*(M, \alpha \circ \iota) \end{tikzcd} \tag{2.29}$$

Here υ and τ are the classifying maps of the stable normal bundle and the tangent bundle respectively.

If we combine the above lemma and diagram (2.27), we can get that diagram (2.21) is commutative. □

Corollary 2.15. *If X is an smooth manifold and $\mathcal{A}$ is a twisting on X, then the charge map $h : K_*^{geo}(X, \alpha) \to K_*^{top}(X, \alpha)$ is an isomorphism.*

Proof of (2.14).
- **Step 1** We prove the first square in diagram ((2.29)) is commutative. First of all, we review the definition of Poincaré duality $PD : K^*(\hat{M}, \mathcal{A}) \to K_*(\hat{M}, \mathcal{A}^{op})$ for twisted K-theory in Lemma 2.1 of [25]:

$$PD(x) = \sigma_{C(\hat{M}, \mathcal{A}^{op})}(x). \tag{2.30}$$

Here $\sigma_{C(\hat{M},\mathcal{A}^{op})} : KK(C(\hat{M},\mathcal{A})\hat{\otimes}A, B) \to KK(A, C(\hat{M},\mathcal{A}^{op})\hat{\otimes}B)$ is a canonical isomorphism for any C^*-algebra A and B, which is given by tensoring with $C(\hat{M},\mathcal{A}^{op})$. If we choose A and B both to be $\mathbb{C}$ and $C(\hat{M},\mathcal{A})$ to be $C(\hat{M},(W_3\circ\upsilon\circ\rho)^*(\mathfrak{K}))$, then we get the Poincaré duality PD on the top of the first square . If we choose A and B to be $\mathbb{C}$ and $C(\hat{M},\mathcal{A})$ to be $C(\hat{M},(-\alpha\circ\iota\circ\rho)^*(\mathfrak{K}))$, then we get the Poincaré duality PD on the bottom of the first square. Then the commutativity of the first square follows from that PD is natural over $C(\hat{M},\mathcal{A}^{op})$.

- **Step 2** From the definition of $\tilde{\chi}$, we know that it is induced by changing the twist from $W_3\circ\tau\circ\rho$ to $\alpha\circ\iota\circ\rho$ using the trivialization given by the spinor bundle ρ^*S and the canonical trivialization of $TM\oplus\upsilon$. On the other hand, we know that I_* is the changing twist map induced by the canonical trivialization of $TM\oplus\upsilon$ and η_* is the map induced by the homotopy η, which is induced by the spinor bundle S. So the commutativity of the second square follows from the fact that the changing twist map is natural.

□

Remark 2.16. In section 2.5 we will prove that the geometric twisted K-homology group defined in [7] is homotopy invariant. Therefore we have that the charge map is an isomorphism for any finite CW-complex homotopic to a smooth manifold. For countable locally finite CW-complexes, one idea to prove that h is an isomorphism is constructing the Milnor's $\varprojlim^1$-exact sequence for geometric twisted K-homology and using the Five Lemma. Unfortunately, we only prove the Milnor's exact sequence for some special cases here.

2.4 Bundle Gerbes and Twisted K-homology

In this section we give another version of geometric twisted K-homology using bundle gerbes. First of all, we give the definition of bundle gerbes.

Definition 2.17. Given a space B (which we assume to be a finite CW-complex in this chapter), a bundle gerbe over B is a pair (P, Y), where $\pi : Y \to B$ is a locally split map and P is a principal $U(1)$-bundle over $Y\times_M Y$ with an associative product, i.e. for every point $(y_1,y_2),(y_2,y_3)\in Y\times_M Y$, there is an isomorphism

$$P_{(y_1,y_2)}\otimes_{\mathbb{C}} P_{(y_2,y_3)} \to P_{(y_1,y_3)} \tag{2.31}$$

and the following diagram commutes

$$\begin{tikzcd}
P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} \arrow[r] \arrow[d] & P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \arrow[d] \\
P_{(y_1,y_2)} \otimes P_{(y_2,y_4)} \arrow[r] & P_{(y_1,y_4)}
\end{tikzcd} \tag{2.32}$$

For every principal $U(1)$-bundle J over B we can define a bundle gerbe $\delta(J)$ by $\delta(J)_{(y_1,y_2)} = J_{y_1} \otimes J^*_{y_2}$. The product is induced by the pairing between J^* and J. A bundle gerbe (P, Y) is called trivial if there is a Hermitian line bundle J such that $P \cong \delta(J)$. In this case, J (with this fixed isomorphism $P \cong \delta(J)$) is called a trivialization of (P, Y). For each bundle gerbe (P, Y) over M we can associate a third integer cohomology class $d(P) \in H^3(M, \mathbb{Z})$ to describe the non-triviality of the bundle gerbe, which is called Dixmier-Douady class (see [47]).

Definition 2.18. Two bundle gerbes (P, Y) and (Q, Z) are stable isomorphic to each other if there is a trivialization of $p_1^*(P) \otimes p_2^*(Q)^*$. Here $p_1 : Y \times_B Z \to Y$ and $p_2 : Y \times_B Z \to Z$ are the natural projections. And the trivialization is called a stable isomorphism between (P, Y) and (Q, Z).

The following theorem gives the relation between stable isomorphism classes and Dixmier-Douady classes.

Theorem 2.19. *Two bundle gerbes are stable isomorphic to each other iff their Dixmier-Douady classes are the same. Moreover, the Dixmier-Douady class defines a bijection between stable isomorphic classes of bundle gerbes over M and $H^3(M, \mathbb{Z})$.*

Now we give another definition which is important in our construction of geometric twisted K-homology.

Definition 2.20. Let (P, Y) be a bundle gerbe over B. A finite dimensional Hermitian bundle E over Y is called a (P, Y)-module if there is a complex vector bundle isomorphism

$$\phi : P \otimes \pi_1^*(E) \cong \pi_2^*(E),$$

which is compatible with the bundle gerbe product, i.e. the following diagram is commutative

$$\begin{tikzcd}
P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes E_{y_3} \arrow[r] \arrow[d] & P_{(y_1,y_3)} \otimes E_{y_3} \arrow[d] \\
P_{(y_1,y_2)} \otimes E_{y_2} \arrow[r] & E_{y_1}
\end{tikzcd}$$

where π_i ($i = 1, 2$) are two projections from $Y \times_B Y$ to Y. Moreover, the Grothendieck group of isomorphism classes of bundle gerbe modules over (P, Y) is called the K-group of (P, Y).

Now we give the construction of geometric twisted K-cycles.

Definition 2.21. Let B be a space , $H \in H^3_{tor}(X, \mathbb{Z})$ and (P, Y) be a bundle gerbe over B with $-H$ as Dixmier-Douady class. A geometric twisted K-cycle is a triple (M, f, E) where

- M is a compact $spin^c$-manifold;
- $f : M \to B$ is continuous;
- $[E]$ is an isomorphism class of $f^*(P, Y)$-module.

We denote the whole twisted K-cycles over (B, H) by $\Gamma_{(P,Y)}(B)$.

To give twisted K-homology group, we also need to define an equivalence $\sim$ on these geometric cycles as follows:

- **Direct sum-disjoint union** For any two cycles (M, f, E_1) and (M, f, E_2) over $(B, (P, Y))$, then we have

$$(M, f, [E_1]) \cup (M, f, [E_2]) \sim (M, f, [E_1] + [E_2]); \tag{2.33}$$

- **Bordism** Given two cycles (M_0, f_0, E_0) and (M_1, f_1, E_1) over $(B, (P, Y))$, if there exists a cycle (M, f, E) over $(B, (P, Y))$ such that

$$\delta M = -M_0 \cup M_1, \tag{2.34}$$

and $E_{\delta M} = -[E_0] \cup [E_1]$, then

$$(M_0, f_0, E_0) \sim (M_1, f_1, E_1); \tag{2.35}$$

- **Spin^c-vector bundle modification** Given a cycle (M, f, E) over $(B, (P, Y))$ and an even dimensional spin^c-vector bundle V over M. Let $\hat{M}$ to be the sphere bundle of $V \oplus \underline{\mathbb{R}}$. Denote the bundle map by $\rho : \hat{M} \to M$ and the positive spinor bundle of $T^v(\hat{M})$ by S_V^+. The vector bundle $S_V^+ \otimes \rho^*(E)$ over $\hat{M}$ is a $(\rho \circ f)^*(P, Y)$ module. Then

$$(M, f, [E]) \sim (\hat{M}, \rho \circ f, [S_V^+ \otimes \rho^*(E)]). \tag{2.36}$$

Definition 2.22. For any space B and bundle gerbe (P, H) over B. We define $K^{gg}_{i,(P,Y)}(B, H) = \Gamma_i(B, (P, Y))/\sim$ (i=0, 1). The parity depends on the dimension of the spin^c-manifold in a twisted K-cycle.

Proposition 2.23. If (P, Y) and (Q, Z) are two bundle gerbes over B with the same Dixmier-Douady class $-H$, then we have $K^{gg}_{i,(P,Y)}(B, H)$ is isomorphic to $K^{gg}_{i,(Q,Z)}(B, H)$.

Proof. Let R be a stable isomorphism between (P, Y) and (Q, Z) i.e. a trivialization of $p_1^*(P) \otimes p_2^*(Q)$. Without loss of generality we can just assume that $Z = Y$. Otherwise we can consider the bundle gerbe $(p_1^*P, Y \times_B Z)$ and $(p_2^*Q, Y \times_B Z)$ instead. Let $(M, f, [E]) \in \Gamma^i(B, (P, Y))$. Since $Q \cong P \otimes R$, therefore $(M, f, [E] \otimes L_R)$ (here L_R is the natural associated line bundle of R) is a twisted geometric K-cycle over $(B, (Q, Y))$. So we get a homomorphism from $\Gamma(B, (P, Y))$ to $\Gamma(B, (Q, Z))$, which we denote by r. A tedious check tells us that r respects disjoint union, bordism and *spin^c*-bundle modification. Therefore r induces a homomorphism from $K_i^{gg}(B, (P, Y))$ to $K_i^{gg}(B, (Q, Y))$. If we change the roles of (P, Y) and (Q, Z) in the above construction, then we get an inverse of r. Therefore r is an isomorphism. □

Let (P, Y) be a bundle gerbe over B with Dixmier-Douady class H. According to **Proposition 6.4** in [6], the i-th K-group of bundle gerbe (P, Y) is isomorphic to the i-th twisted K-group $K^i(X, -H)$. Then the definitions of $K_*^{gg}(X, H)$ and $K_*^{top}(X, \alpha)$ implies the following proposition:

Proposition 2.24. Let X be a finite CW-complex and $H \in H^3_{torsion}(X, \mathbb{Z})$. Then we have

$$K_i^{gg}(X, (P, Y)) \cong K_i^{top}(X, \alpha) \cong K_i^{geo}(X, \alpha). \tag{2.37}$$

2.5 Properties of Geometric Twisted K-homology

In this section, we establish some properties of geometric twisted K-homology i.e. homotopy invariance, excision isomorphism, additivity and six-term exact sequence (not completely). As a consequence, we get the Mayer-Vietoris sequence and $\varprojlim^1$-exact sequence of geometric twisted K-homology groups. In this section, we always assume X to be a locally finite CW-complex and $\alpha : X \to K(\mathbb{Z}, 3)$ such that there exists a α-twisted spin^c-manifold over X. Before going to the Eilenberg-Streenrod axioms, we first give a simple lemma:

Lemma 2.25. *If $f : Y \to X$ is a continuous map, then f induces a homomorphism $f_* : K^g(Y, \alpha \circ f) \to K^g(X, \alpha)$.*

Proof. Given a twisted geometric K-cycle $(M, \phi, \upsilon, \eta, E)$ over $(Y, \alpha \circ f)$, we define f_* by

$$f_*(M, \phi, \upsilon, \eta, E) = (M, f \circ \phi, \upsilon, \eta, E). \tag{2.38}$$

We need to show that f_* is compatible with disjoint union, bordism and $Spin^c$-vector bundle modification.

- Given two geometric K-cycles $(M, \phi, \upsilon, \eta, E_i)$ $(i = 1, 2)$, we have

  $f_*((M, \phi, \upsilon, \eta, E_1) \cup (M, \phi, \upsilon, eta, E_2)) = f_*((M, \phi, \upsilon, \eta, E_1 \oplus E_2)) = (M, \phi \circ f, \upsilon, \eta, E_1 \oplus E_2)$;

- If $(M, \phi, \upsilon, \eta, E)$ gives a bordism between $(M_1, \phi_1, \upsilon_1, \eta_1, E_1)$ and $(M_2, \phi_2, \upsilon_2, \eta_2, E_2)$, then clearly $(M, f \circ \phi, \upsilon, \eta, E)$ gives a bordism between $(M_1, f \circ \phi_1, \upsilon_1, \eta_1, E_1)$ and $(M_2, f \circ \phi_2, \upsilon_2, \eta_2, E_2)$;

- Since $f((\hat{M}, \phi \circ \rho, \upsilon \circ \rho, \eta \circ (\rho \times Id), \rho^* E \otimes S_V^+))$ is $(\hat{M}, f \circ \phi \circ \rho, \upsilon \circ \rho, \eta \circ (\rho \times Id), \rho^* E \otimes S_V^+)$, which is exactly the spin^c-vector bundle modification of $(M, f \circ \phi, \upsilon, \eta, E)$, so we get that f_* respects spin^c-vector bundle modification.

□

Theorem 2.26 (Homotopy). *If $f : Y \to X$ is a homotopy equivalence, then the induced map $f_* : K_*^g(Y, \alpha \circ f) \to K_*^g(X, \alpha)$ is an isomorphism.*

Proof. We first show that if $g : Y \to X$ is homotopic to f, then $K_*^g(Y, \alpha \circ f) \cong K_*^g(Y, \alpha \circ g)$. Let $H : Y \times [0, 1] \to X$ be a homotopy from f to g i.e. $H(y, 0) = f(y)$ and $H(y, 1) = g(y)$. Given a twisted geometric K-cycle $(M, \phi, \upsilon, \eta, E)$ over $(Y, \alpha \circ f)$, we get a twisted geometric K-cycle over $(Y, \alpha \circ g)$ as follows: $(M, \phi, \upsilon, \eta', E)$. Here $\eta' : M \times [0, 1] \to K(\mathbb{Z}, 3)$ is defined by

$$\eta'(m, t) = \begin{cases} \eta(m, 2t) & 0 \le t \le 1/2; \\ \alpha \circ H(m, 2t - 1) \circ (\phi \times Id) & 1/2 \le t \le 1. \end{cases}$$

It is not hard to check that the above map is compatible with the disjoint union, bordism and spin^c-vector bundle modification. We skip the details here since they are similar to the proof of the above lemma. Therefore we get a homomorphism H_* from $K_*^g(Y, \alpha \circ f)$ to $K_*^g(Y, \alpha \circ g)$. Similarly we can get the inverse of H_* by using $H(1-t, y)$ as a homotopy from g to f. So we get that H_* is an isomorphism. Clearly, we have that $f_* = g_* \circ H_*$. Let $q : X \to Y$ be a homotopy inverse of $f : Y \to X$ i.e. $f \circ q$ is homotopic to id_X and $q \circ f$ is homotopic to id_Y. Denote the associated homotopies by H_1 and H_2 respectively. Then we have that $(f \circ q)_* = (H_1)_*$ and $(q \circ f)_* = (H_2)_*$. Since $(H_1)_*$ and $(H_2)_*$ are both isomorphisms, we get that f_* is an isomorphism as well. □

Theorem 2.27 (Excision). *Let (X, Y) be a pair of locally finite CW-spaces, U is an open set of X such that $\overline{U} \in Y$. Then we the inclusion $i : (X - U, Y - U) \hookrightarrow (X, Y)$ induces an isomorphism*

$$K_*^g(X - U, Y - U; \alpha \circ i) \cong K_*^g(X, Y; \alpha). \tag{2.39}$$

Proof. By Lemma 2.25 the inclusion i induces a homomorphism i_*. We need to show that it is injective and surjective. We first prove that i_* is surjective. For any $y \in K_*^g(X, Y; \alpha)$, we choose a geometric cycle $(M, \phi, \upsilon, \eta, E)$ to represent it. By the Urysohn's Lemma, there exists a Morse function $f : X \to \mathbb{R}$ which seperates $\bar{U}$ and $X - Y$ which satisfies that $sup_{x \in \bar{U}} f(x) = a < inf_{x \in X - Y} f(x) = b$. Let $c \in [a, b]$ be a regular value of f and we denote $f^{-1}(-\infty, c)$ by M'. Then the restriction of M to M' gives a geometric cycle over $(X - U, Y - U; \alpha_{X-U})$ which we denote by $(M', \phi', \upsilon', \eta', E')$ and obviously it is mapped to $(M, \phi, \upsilon, \eta, E)$ under i^*. For the injectivity, if $(M, \phi, \upsilon, \eta, E)$ is a geometric cycle over $(X, Y; \alpha)$ inducing by a geometric cycle $(M', \phi', \upsilon', \eta', E')$ over $(X - U, Y - U; \alpha_{X-U})$. We can define a map $j : K^*(X, Y; \alpha) \to K^*(X - U, Y - U; \alpha_{X-U})$ by choosing a regular value c of f and construct a similar geometric cycle $(M', \phi', \upsilon', \eta', E')$. It is not hard to see that $(M', \phi', \upsilon', \eta', E')$ is equivalent to the geometric cycle mapped to $(M, \phi, \upsilon, \eta, E)$ under i^*, which implies that $j \circ i^* = Id$. □

Another important axiom in Eilenberg-Steenrod axioms is the long exact sequence. Before moving on to the long exact sequence of geometric twisted K-homology, we first introduce a new definition.

Definition 2.28. A twist $\alpha : X \to K(\mathbb{Z}, 3)$ is called representable if there exists an oriented real vector bundle V over X such that $W_3(V) = [\alpha]$. Here $[\alpha]$ is the pullback of the generator of $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$ along α.

Example 2.29. 1. The trivial twist is representable since we can just choose a trivial vector bundle to represent it;

2. If X is a smooth oriented manifold, then the twist corresponding to $W_3(X)$ is representable by the tangent bundle of X.

Remark 2.30. Not each $\mathbb{Z}/2\mathbb{Z}$-torsion twist over X is representable. Assume that a twist α can be represented by a real oriented bundle V. Then $w_4(V \oplus V) = w_2(V) \cup w_2(V)$, moreover we have $\rho(p_1(V)) = w_2(V) \cup w_2(V)$ (here ρ is the coefficient reduction homomorphism). Therefore $w_2(V) \cup w_2(V)$ is a reduction of an integral class. However, this condition is not always satisfied for any $\mathbb{Z}/2\mathbb{Z}$-cohomology class. The counterexample can be found in [28]. For this remark I would like to thank Dr. Mark. Grant pointing out the necessary condition of representability and the counterexample.

Theorem 2.31 (Six-term exact sequence). *Let Y be a sub-space of X and i be the inclusion map from Y to X and α be a representable twist over X. Then we have the six-term exact sequence:*

$$\begin{array}{ccccc} K_0^g(Y, \alpha \circ i) & \xrightarrow{i_*} & K_0^g(X, \alpha) & \xrightarrow{j_*} & K_0^g(X, Y; \alpha) \\ \delta \uparrow & & & & \downarrow \delta \\ K_1^g(X, Y; \alpha) & \xleftarrow[j_*]{} & K_1^g(X, \alpha) & \xleftarrow[i_*]{} & K_1^g(Y, \alpha \circ i) \end{array}$$

Here the boundary operator is given by

$$\delta([M, \phi, \upsilon, \eta, E]) = [(\delta M, \phi|_{\delta M}, \upsilon|_{\delta M}, \eta|_{\delta M \times [0,1]}, E|_{\delta M})]. \tag{2.40}$$

To prove this theorem, we first prove two lemmas as a preparation.

Lemma 2.32. *Let $\theta = (M, \iota, \upsilon, \eta, E)$ be a geometric K-cycle over (X, α) and E_i $(i = 1, 2)$ be spin^c-vector bundles over M with even dimensional fibers. Denote the vector bundle modification of θ with a spin^c-vector bundle F by θ_F. Then we have that $\theta_{E_1 \oplus E_2}$ is bordant to $(\theta_{E_1})_{p_1^* E_2}$, in which p_1 is the projection from the sphere bundle $S(E_1 \otimes \underline{\mathbb{R}})$ to M.*

Proof. Assume the dimension of the fiber of E_i is n_i and write $\theta_{E_1 \oplus E_2}$ and $(\theta_{E_1})_{p_1^* E_2}$ explicitly as $(V, \iota_V, \upsilon_V, \eta_V, E_V)$ and $(W, \iota_W, \upsilon_W, \eta_W, E_W)$ respectively. Then the fibers of V and W are $S^{n_1+n_2}$ and $S^{n_1} \times S^{n_2}$ respectively. We can embed both of the two bundles over M

into the vector bundle $E_1 \oplus E_2 \oplus \mathbb{R}$ as follows. First we choose a Riemannian metric over $E_1 \oplus E_2 \oplus \mathbb{R}$ and embed V into $E_1 \oplus E_2 \oplus \mathbb{R}$ as the standard unit sphere bundle. We embed W into $E_1 \oplus E_2 \oplus \underline{\mathbb{R}}$ such that in each fiber over $p \in M$ it is embedded as follows:

$$((x, s), (y, t)) \mapsto ((5 - t)(x, s), y),$$

in which $x \in (E_1)_p$, $y \in (E_2)_p$ and $s, t \in (\underline{\mathbb{R}})_p$. A careful check tells us that this indeed induces an embedding of W into $E_1 \oplus E_2 \oplus \underline{\mathbb{R}}$ and we still denote the image of the embedding by W. We embed V into $E_1 \oplus E_2 \oplus \underline{\mathbb{R}}$ with a scaling such that the radius of each fiber is 10. Denote the standard disk bundle with radius 10 of $E_1 \oplus E_1 \oplus \underline{\mathbb{R}}$ by $D^{n_1+n_2+1}(10)$ and the solid torus bundle bounds by V by $\bar{V}$. Then we have that $D^{n_1+n_2+1}(10) - \bar{V}$ (which we denote by Z) gives rise to a bordism between V and W. $(Z, \iota \circ p_Z, \upsilon_Z, \eta \circ (p_Z \times id), E_Z)$ gives a bordism between $(V, \iota_V, \upsilon_V, \eta_V, E_V)$ and $(W, \iota_W, \upsilon_W, \eta_W, E_W)$. □

Lemma 2.33. *Let $(M, \iota, \upsilon, \eta, E_M)$ be a geometric cycle over (X, α) and α be representable. And $(\delta M, \iota_{\delta M}, \upsilon_{\delta M}, \eta_{\delta M}, E_{\delta M})$ be its restriction to the boundary of M. If a spin^c-vector bundle modification with vector bundle E of $(\delta M, \iota_{\delta M}, \upsilon_{\delta M}, \eta_{\delta M}, E_{\delta M})$ is bordant to trivial cycle, then there exists a spin^c-vector bundle V over δM such that the spin^c-vector bundle modification with V is bordant to the trivial cycle and V can be extended to a spin^c-vector bundle over M.*

Proof. Denote the spin^c-vector bundle modification of $(\partial M, \iota_{\partial M}, \upsilon_{\partial M}, \eta_{\partial M}, E_{\partial M})$ with vector bundle E by $(Q, \iota_{\partial M} \circ \pi, \upsilon_Q, \eta_Q, E_Q)$, which is bordant to the trivial cycle via a bordism of $(W, \iota_W, \upsilon_W, \eta_W, E_W)$. There exists a normal bundle F over W, whose restriction to Q is also a normal bundle of TQ. On the other hand, by the construction of spin^c-vector bundle modification we can observe that there exists a normal bundle of TQ such that it is isomorphic to the pullback (along π) of the direct sum of a normal bundle of $T\partial M$ (which we denote by $N(T\partial M)$) and a normal bundle of E (which we denote by $N(E)$). Consider the spin^c-modification of $(W, \iota_W, \upsilon_W, \eta_W, E_W)$ with $F \oplus \iota_W^*(\mathbf{V})$ (here $\mathbf{V}$ is the vector bundle over X with $W_3(V) = [\alpha]$). It gives a bordism from the spin^c-modification of $(Q, \iota_{\partial M} \circ \pi, \upsilon_Q, \eta_Q, E_Q)$ with $(F \oplus \iota_W^*(\mathbf{V}))_{|Q}$ and the trivial cycle. According to Lemma 2.32 and the observation before we can see that the spin^c-modification of $(Q, \iota_{\partial M} \circ \pi, \upsilon_Q, \eta_Q, E_Q)$ with $(F \oplus \iota_W^*(\mathbf{V}))_{|Q}$ is bordant to the spin^c-modification of $(\partial M, \iota_{\partial M}, \upsilon_{\partial M}, \eta_{\partial M}, E_{\partial M})$ with $E \oplus N(T\partial M) \oplus N(E) \oplus \iota_W^*(\mathbf{V})_{|Q}$. While $E \oplus N(E)$ is trivial, we get that the spin^c-modification of $(\partial M, \iota_{\partial M}, \upsilon_{\partial M}, \eta_{\partial M}, E_{\partial M})$ with $N(T\partial M) \oplus \iota_{\partial M}^*(\mathbf{V})$ is bordant to a trivial cycle. The normal bundle on the boundary can be extended to the whole manifold obviously and $\iota_{\partial M}^*(\mathbf{V})$ can be extended to a vector bundle $\iota_M^*(\mathbf{V})$. So we get our statement. □

Now we start the proof of Theorem 2.31.

proof of Theorem 2.31. We will show the exactness at $K_0^g(X,\alpha)$, $K_0^g(X,Y;\alpha)$ and $K_1^g(Y,\alpha)$. The proof of the rest part is similarly.

- For any $[y] \in K_0^g(Y,\alpha)$, we choose a twisted geometric K-cycle $(M_0,\phi_0,\upsilon_0,\eta_0,E_0)$ to represent it. Its image under $j_*\circ i_*$ is $(M_0, i\circ\phi_0,\upsilon_0,\eta_0,E_0)$, which is bordant to a trivial K-cycle relative Y in X. Therefore we have $j_*\circ i_*([y])$ is trivial. Assume that $[x] \in K_0^g(X,\alpha)$ and $j^*([x]) = 0$. We still choose a twisted geometric K-cycle $(M_1,\phi_1,\upsilon_1,\eta_1,E_1)$ to represent $[x]$. Since $j^*([x])$ is trivial, we obtain that if we do several times of spin^c-vector bundle modification for $(M_1, j\circ\phi_1,\upsilon_1,\eta_1,E_1)$ relative to Y in X we get a trivial K-cycle relative to Y i.e. if we denote the results of spin^c-vector bundle modifications by $(\hat{M}_1, j\circ\phi_1\circ\rho,\upsilon_1',\eta_1',E_1')$, then $(\hat{M}_1, j\circ\phi_1\circ\rho,\upsilon_1',\eta_1',E_1')$ satisfies that $(j\circ\phi_1\circ\rho(\hat{M}_1))\subset Y$, which also implies that $j\circ\phi_1(M)\subset Y$. Therefore we get that $[x]\in \mathrm{im}\ i_*$.

- First of all we need to point out that δ is well defined i.e. it is compatible with disjoint union, bordism and spin^c-vector bundle modification. It is a tedious check from the definition of δ, which we skip here. By the definition of δ we can see that $\delta\circ j_* = 0$. To show that $ker\delta \subset imj_*$, we choose a twisted geometric K-cycle $(M_2,\phi_2,\upsilon_2,\eta_2,E_2)$ such that $\delta[(M_2,\phi_2,\upsilon_2,\eta_2,E_2)]$ is trivial i.e. several times of $Spin^c$-vector bundle modification for $(\delta M_2, \phi|_{\delta M_2}, \upsilon|_{\delta M_2}, \eta|_{\delta M_2}, E|_{\delta M_2})$ is bordant to trivial K-cycle over Y. By Lemma 2.33 we can assume that each spin^c-vector bundle over the boundary of a manifold can be extended to a spin^c-vector bundle over the whole manifold, therefore we get that if we do the spin^c-vector bundle modifications for $(M_2,\phi_2,\upsilon_2,\eta_2,E_2)$, then the resulting twisted spin^c-manifold is bordant to a twisted spin^c-manifold without boundary over X. Finally we obtain that $(M_2,\phi_2,\upsilon_2,\eta_2,E_2)$ is equivalent to a twisted geometric K-cycle whose underling twisted spin^c-manifold is closed, which implies that $[(M_2,\phi_2,\upsilon_2,\eta_2,E_2)]$ lies in the image of j_*.

- Let $(M_3,\phi_3,\upsilon_3,\eta_3,E_3)$ be a geometric α-twisted K-cycle over $(X,Y;\alpha)$. Then $[(\delta M_3, \phi_3|_{\delta M_3}, \upsilon_3|_{\delta M_3}, \eta_3|_{\delta M_3}, E_3|_{\delta M_3})$ is clearly bordant to a trivial twisted geometric K-cycle over X i.e. $i_*\circ\delta = 0$. Let $[(M_4,\phi_4,\upsilon_4,\eta_4,E_4)] \in K_1^g(Y,\alpha\circ i)$ be a class which lies in the kernel of i_*. A similar strategy leads us to get that the underling twisted spin^c-manifold of several times of spin^c-vector bundle modification of $[(M_4,\phi_4,\upsilon_4,\eta_4,E_4)]$

is a boundary of a twisted spin^c-manifold over X , from which we can easily get that $[(M_4, \phi_4, \upsilon_4, \eta_4, E_4)] \in im\delta$.

□

Remark 2.34. The condition of representability of the twist is essential for the proof here. In general, a twist is not always representable. We leave the six-term exact sequence of geometric twisted K-homology for general twists as a further question to be investigated.

Theorem 2.35 (**Additivity**). *Let $(X_i)_{i\in I}$ be a family of locally finite CW-complexes and $\alpha_i : X_i \to K(\mathbb{Z}, 3)$ be a twist over X_i for each i. Moreover, we require that there exists an α_i-twisted spin^c-manifold over X_i for each i. Denote X to be the disjoint union of X_i and α is a twist over X such that the restriction of X to each X_i is α_i. Then we have the following isomorphism:*

$$K^g_*(X, \alpha) \cong \oplus_i K^g_*(X_i, \alpha_i). \tag{2.41}$$

Proof. Denote the inclusion of X_i into X by j^i. $j^i_* : K^g_*(X_i, \alpha_i) \to K^g_*(X, \alpha)$ is an injective homomorphism. Therefore $j_* = \oplus_i j^i_* : \oplus_i K^g_*(X_i, \alpha_i) \to K(X, \alpha)$ is an injective homomorphism. We only need to show j_* is surjective now. Let $[x] \in K^g_*(X, \alpha)$ be a twisted K-homology class, which is represented by a twisted geometric $(M, \phi, \upsilon, \eta, E)$. We assume that M is connected, then M is mapped to one of the X_is, which we denote by X_k. Then $[(M, \phi, \upsilon, \eta, E)] \in im j^k_*$. If M is not connected, we consider their connected components one by one. So we prove that j_* is surjective. □

The above theorem implies the Mayer-Vietoris sequence and $\varprojlim^1$-exact sequence of geometric twisted K-homology.

Theorem 2.36 (**Mayer-Vietoris sequence**). *Assume two open set U and V of X satisfies $X = U \bigcup V$ and the twist α is representable, we have the Mayer-Vietoris sequence of twisted K-homology:*

$$\begin{tikzcd}
K^g_1(X,\alpha) \arrow[r, "\delta"] & K^g_0(U\bigcap V, \alpha\circ i_{U\bigcap V}) \arrow[r, "(j_U)_*\oplus(j_V)_*"] & K^g_0(U,\alpha\circ i_U)\oplus K^g_0(V,\alpha\circ i_V) \arrow[d, "(i_U)_*-(i_V)_*"] \\
K^g_1(U,\alpha\circ i_U)\oplus K^g_1(V,\alpha\circ i_V) \arrow[u, "(i_U)_*-(i_V)_*"] & K^g_1(U\bigcap V,\alpha\circ i_{U\bigcap V}) \arrow[l, "(j_U)_*\oplus(j_V)_*"'] & K^g_0(X,\alpha) \arrow[l, "\delta"']
\end{tikzcd}$$

Proof. Let Z be the disjoint union of U and V, Y be $U \cap V$. Then consider the six-term exact sequence for the pair (Z, Y) and use the excision isomorphism $K_*^g(Z, Y; \alpha) \cong K_*^g(X, \alpha)$ we can get the Mayer-Vietoris sequence for twisted geometric K-homology groups. □

2.6 Some Constructions about Geometric Twisted K-cycles

In order to give the T-duality transformation of geometric twisted K-homology we will give the construction of analogue maps of induced map , wrong way map and changing twist map for twisted K-homology.

1. **Induced map** Assume $f : (X_1, Y_1) \to (X_2, Y_2)$ is a continuous map between two pairs of topological spaces. Then the induced map $f_* : K_i^g(X_1, Y_1; \alpha \circ f) \to K_i^g(X_2, Y_2; \alpha)$ is defined as follows:

$$f_*([M, \phi, \upsilon, \eta, E]) = [(M, \phi \circ f, \upsilon, \eta, E)]; \tag{2.42}$$

2. **Wrong way map** Let $f : P \to N$ be a K-oriented map between smooth manifolds and $\alpha : N \to K(\mathbb{Z}, 3)$ be a twist over N. we define the wrong way map $f^! : K_{i-1}^g(N, \alpha) \to K_i^g(P, \alpha \circ f)$ to be

$$\pi^!([M, \phi, \upsilon, \eta, E]) = [(\tilde{M}, \tilde{\phi}, \tilde{\upsilon}, \tilde{\eta}, \tilde{\pi}^*(E))]. \tag{2.43}$$

Here $\tilde{M}$ is the fiber product $M \times_N P$, $\tilde{\upsilon}$ is the stable normal bundle of $\tilde{M}$ and $\tilde{\eta}$ is a homotopy induced by η as follows.

Remark 2.37. As f is K-oriented, therefore $W_3(\tilde{\upsilon} \oplus f^*(\upsilon))$ is trivial, which implies that $W_3 \circ \tilde{\upsilon}$ is homotopic to $\alpha \circ f \circ \tilde{\phi}$ via a fixed (given by the K-orientation of f) homotopy λ. $\tilde{\eta} : \tilde{M} \times [0, 1] \to K(\mathbb{Z}, 3)$ is given by the combination of λ and η as follows:

$$\tilde{\eta}(x, t) = \begin{cases} \lambda(x, 2t), & 0 \leq t \leq 1/2; \\ \eta \circ (f' \times id)(x, 2t - 1), & 1/2 \leq t \leq 1, \end{cases}$$

in which $f' : \tilde{M} \to M$ is the canonical projection to M. Therefore we get that $(\tilde{M}, \tilde{\phi}, \tilde{\upsilon}, \tilde{\eta}, \tilde{\pi}^*(E))$ is a twisted geometric cycle over $(P, \alpha \circ f)$. In particular, when f is the bundle map for a principal S^1-bundle, $\tilde{M}$ is the pullback S^1-bundle along f.

3. **Changing twist map** Let $\pi : P \to B$ and $\hat{\pi} : \hat{P} \to B$ be principal S^1-bundles over B and $\alpha : P \to K(\mathbb{Z}, 3)$, $\hat{\alpha} : \hat{P} \to K(\mathbb{Z}, 3)$ are two twists over P and $\hat{P}$ respectively. Denote the corresponding integer cohomology classes by $[\alpha]$ and $[\hat{\alpha}]$. Assume that $(\hat{P}, [\hat{\alpha}])$ is T-dual to $(P, [\alpha])$ in the sense of Bunke-Schick. Then we can choose a homotopy h between $\pi \circ \alpha$ and $\hat{\pi} \circ \hat{\alpha}$ such that the restriction of h to each fiber of $P \times_B \hat{P}$ corresponds to the cohomology class $\theta \cup \hat{\theta} \in H^2(P \times_B \hat{P}_b, \mathbb{Z})$. Here θ and $\hat{\theta}$ are generators of the first cohomology group of the two copies of S^1 of a fiber. Then for a geometric cycle δ of $(P \times_B \hat{P}, \pi \circ \alpha)$ one can define the changing twist map $u : K_i^g(P \times_B \hat{P}, \pi \circ \alpha) \to K_i^g(P \times_B \hat{P}, \hat{\pi} \circ \hat{\alpha})$ as follows:

$$u([M, \phi, \upsilon, \eta, E]) = [(M, \phi, \upsilon, \hat{\eta}, E)]. \tag{2.44}$$

Here $\hat{\eta}$ is induced by the following diagram:

$$W_3 \circ \upsilon \xrightarrow{\quad \eta \quad} \pi \circ \alpha \circ \phi \xrightarrow{h \circ (\phi \times id)} \hat{\pi} \circ \hat{\alpha} \circ \phi$$

More explicitly, the $\hat{\eta}$ is given by the composition of η and $h \circ (\phi \times id)$, which we denote by $(h \circ (\phi \times id)) * \eta$

$$(h \circ (\phi \times id))(\eta)(x, t) = \begin{cases} \eta(x, 2t), & 0 \le t \le 1/2; \\ h \circ (\phi \times id)(x, 2t - 1), & 1/2 \le t \le 1. \end{cases}$$

It is worthwhile to point out that different choices of h are homotopic to each other, so the corresponding geometric cycles are equivalent to each other via **Bordism**.

Lemma 2.38. *The induced map, wrong way map and changing twist map above are all compatible with disjoint union, bordism and spin^c-vector bundle modification.*

Proof. We have proved the induced map part in Lemma 2.25 and it is not hard to see that they all respect the disjoint union. We do the rest here.

- Let $(M, \phi, \upsilon, \eta, E)$ be a bordism between $(M_1, \phi_1, \upsilon_1, \eta_1, E_1)$ and $(M_2, \phi_2, \upsilon_2, \eta_2, E_2)$ over X. Denote $p^!(M_i, \phi_i, \upsilon_i, \eta_i, E_i)$ by $(\tilde{M}_i, \tilde{\phi}_i, \tilde{\upsilon}_i, \tilde{\eta}_i, \tilde{\pi}^* E_i)$. Since the boundary of a pullback space is the pullback of the original boundary, therefore $(\tilde{M}, \tilde{\phi}, \tilde{\upsilon}, \tilde{\eta}, \tilde{\pi}^* E)$ gives a bordism between $(\tilde{M}_1, \tilde{\phi}_1, \tilde{\upsilon}_1, \tilde{\eta}_1, \tilde{\pi}^* E_1)$ and $(\tilde{M}_2, \tilde{\phi}_2, \tilde{\upsilon}_2, \tilde{\eta}_2, \tilde{\pi}^* E_2)$.

  Let V be an even-dimensional spin^c-vector bundle and use the notation in Section 2.1. $p^!(\hat{M}, \phi \circ \rho, \upsilon', \eta', S_V^+ \otimes \rho^* E)$ is $(\tilde{\hat{M}}, \phi \circ \rho \circ \tilde{\hat{\pi}}, \tilde{\upsilon}', \tilde{\eta}', \tilde{\pi}^*(S_V^+ \otimes \rho^* E))$. On the other hand, $\tilde{\pi}^*V$ is also a spin^c-vector bundle over $\tilde{M}$. The associated spin^c-vector bundle modification of $(\tilde{M}, \tilde{\phi}, \tilde{\upsilon}, \tilde{\eta}, \tilde{\pi}^*E)$ is $(\tilde{\hat{M}}, \tilde{\phi}\circ\tilde{\rho}, \upsilon'', \tilde{\eta}', \tilde{\pi}^*(S_V^+ \otimes \rho^* E))$ (Be careful that here $\tilde{\eta}'$ and $\widetilde{\eta'}$ are two different homotopies! We use this potential confusing notation to describe different orders of operations). The maps appearing in the above twisted geometric K-cycles are shown in the following diagram:

$$
\begin{tikzcd}
 & \tilde{\hat{M}} \arrow[dl, "\tilde{\rho}"'] \arrow[dr, "\tilde{\hat{\pi}}"] & \\
\tilde{M} \arrow[dr, "\tilde{\pi}"'] & & \hat{M} \arrow[dl, "\rho"] \\
 & M &
\end{tikzcd}
\tag{2.45}
$$

By the commutativity of the above diagram, we have that $\phi \circ \rho \circ \tilde{\hat{\pi}} = \phi \circ \tilde{\pi} \circ \tilde{\rho} = \tilde{\phi} \circ \tilde{\rho}$. Moreover, $\widetilde{\upsilon'}$ and υ'' are homotopic to each other because they are both the classifying map of the stable normal bundle of $\tilde{\hat{M}}$. Together with the construction of $\tilde{\eta}$ in Remark 2.40, it is not hard to see that $\widetilde{\eta'}$ and $\tilde{\eta}'$ are homotopic to each other since we can choose a homotopy between $\widetilde{\upsilon'}$ and υ''. So the wrong way map respects the spin^c-vector bundle modification construction;

- Use the notation above and $(M, \phi, \upsilon, \eta, E)$ gives a bordism between $(M_0, \phi_0, \upsilon_0, \eta_0, E_0)$ and $(M_1, \phi_1, \upsilon_1, \eta_1, E_1)$. From the definition of u we can see other entries in a geometric cycle are invariant under u except the homotopy η. So we only need to check the homotopies. By the natual of T-duality pairs we can see that the restriction of a chosen homotopy h to $M_i \times I$ $(i = 0, 1)$ gives us a required h_i. So the restriction of $\hat{\eta}$ to $M_i \times I$ $(i = 0,\ 1)$ gives us a $\hat{\eta}_i$, which is exactly what we need to check. Since the composition of homotopies are associative up to homotopy, we get that u respects the spin^c-vector bundle modification construction.

□

2.7 T-duality for Twisted Geometric K-homology

Theorem ((0.1))**.** *Let B be a finite CW-complex and (P, H) and $(\hat{P}, \hat{H})$ are T-dual to each other over B.*

$$\begin{tikzcd} & P \times_B \hat{P} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\ & B & \end{tikzcd}$$

Moreover, we assume that $\alpha : P \to K(\mathbb{Z}, 3)$ and $\hat{\alpha} : \hat{P} \to K(\mathbb{Z}, 3)$ are representable and they satisfy that $\alpha^*([\Theta]) = H$ and $\hat{\alpha}^*([\Theta]) = \hat{H}$ (Here $[\Theta]$ is the positive generator of $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$. Then the map $T = \hat{p}_* \circ u \circ p^! : K^g_*(X, \alpha) \mapsto K^g_{*+1}(\hat{X}, \hat{\alpha})$ is an isomorphism.

The proof of theorem (0.1) depends on the following lemma

Lemma 2.39. *T is compatible with the boundary operator and the induced map in the Mayer-Vietoris sequence.*

Proof. We first prove the compatibility with the induced map. Assume we have a map $f : X \mapsto Y$ and we have the associated T-duality diagrams over Y and pullback it to X. f induces maps by $F : P_X \mapsto P_Y$, $\hat{F} : \hat{P}_X \to \hat{P}_Y$ and $G : P_X \times_X \hat{P}_X \to P_Y \times_Y \hat{P}_Y$. Then we have the following identities:

$$\begin{aligned} \hat{F}_* \circ T_X = & \quad \hat{F}_* \circ (\hat{p}_X)_* \circ u_X \circ p^!_X \\ = & \quad (\hat{p}_Y)_* \circ G_* \circ u_X \circ p^!_X \\ = & \quad (\hat{p}_Y)_* \circ u_Y \circ (G \circ p_Y)^! \circ F_* \\ = & \quad T_Y \circ F_*. \end{aligned} \tag{2.46}$$

Now we turn to the compatibility with the boundary map, in the Mayer-Vietoris sequence of the boundary operator$\delta : K^g_*(X, \alpha) \to K^g_{*+1}(U \cap V, \alpha \circ i_{U \cap V})$ is given as follows: Choose a continuous map $f : X \to [0, 1]$ such that $f_{U - U \cap V}$ is 0 and $f_{V - U \cap V}$ is 1. Without loss of generality we assume that $f \circ \phi : M \to [0, 1]$ is a smooth function and $1/2$ is a regular point of $f \circ \phi$. For any twisted geometric K-cycle $x = (M, \phi, \upsilon, \eta, E)$, define $\delta x = (f^{-1}(1/2), \phi \circ i, \upsilon \circ i, \eta \circ (i \times id), i^* E)$. By this formula, we get that δ is compatible with induced map. Also the homotopies $(h \circ (phi \circ i \times id)) * (\eta \circ (i \times id))$ and $(\eta * (h \circ (\phi \times id)) \circ (i \times id)$

are homotopic to each other, which implies that $u \circ \delta = \delta \circ u$. The remainder is to show that $\hat{p}^! \circ \delta = \delta \circ \hat{p}^!$. We write both sides explicitly first: Given a principal S^1-bundle $\pi : P \to B$ and a twisted geometric cycle $(M, \phi, \upsilon, \eta, E)$ over P

$$\hat{p}^! \circ \delta(M, \phi, \upsilon, \eta, E) = (((f \circ \phi)^{\tilde{-1}}(1/2)), \phi \circ \tilde{\pi} \circ i, \tilde{\upsilon} \circ i, \eta \circ ((\tilde{\pi} \circ i) \times id), (i \circ \tilde{\pi})^* E);$$

$$\delta \circ \hat{p}^!(M, \phi, \upsilon, \eta, E) = ((f \circ \phi \circ \tilde{\pi})^{-1}(1/2), \phi \circ \tilde{\pi} \circ i, \tilde{\upsilon} \circ i, \eta \circ ((\tilde{\pi} \circ i) \times id), (i \circ \tilde{\pi})^* E).$$

Since $f \tilde{\circ} \phi^{-1}(1/2)$ is exactly $(f \circ \phi \circ \tilde{\pi})^{-1}(1/2)$, we get that $\hat{p}^! \circ \delta = \delta \circ \hat{p}^!$. Finally, we have that

$$T \circ \delta = (\hat{p}^! \circ u \circ p_*) \circ \delta = \delta \circ (\hat{p}^! \circ u \circ p_*) = \delta \circ T.$$

□

Proof of Theorem 0.1. We do the proof by induction on the number of cells. Assume X is a point, then P and $\hat{P}$ are both S^1 and the correspondence space is $S^1 \times S^1$. We first prove that T is an isomorphism in this case. Here all of the twists are trivial and therefore the involving K-homology groups are untwisted, which are (natural) isomorphic to the associated K-groups (via Poincare duality). We know from Chapter 1 that the T-duality transformations of twisted K-groups are isomorphisms. Therefore we get that T is an isomorphism when X is a point.

Assume T is an isomorphism when the number of cells is no more than n,then we adjoin another cell σ_{n+1} to X and we choose open set $U = X \cup \sigma_{n+1} - pt, V = \sigma_{n+1} - \bar{pt}$ and we can get the Mayer-Vietoris sequence of geometric twisted K-homology groups. Then the conclusion of the theorem is impled by induction and the Five-Lemma. □

Remark 2.40. The construction of T-duality transformation of geometric twisted K-homology can be easily generalized to T-dual pairs of higher dimensional torus bundles.

3 Uniqueness of the T-duality Isomorphism

3.1 Introduction and Notations

In this section we give an introduction to T-duality triples and some new notations which will be used in this chapter. For simplicity, we will use Bunke and Schick's T-duality triple as the basic object in this chapter. When we say a pair $(P, \mathcal{A})$ over a space B in this chapter (we will always assume that B is a finite CW-complex in this chapter) we mean that P is a principal $\mathbb{T}^n$-bundle P over B and $\mathcal{A}$ is a $\mathcal{K}$ (compact operator algebra)-bundle over P. Recall the definition of T-duality triples i.e. Definition 0.2. A triple $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ over B contains two pairs $(P, \mathcal{A})$, $(\hat{P}, \hat{\mathcal{A}})$ and an isomorphism u between the twists $j^*\mathcal{A}$ and $\hat{j}^*(\hat{\mathcal{A}})$ which satisfies the condition in Definition 0.2.

Given a T-dual triple $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$, one can get the following diagram, which we call a T-duality diagram over B below together with the twists information over the spaces.

$$\begin{tikzcd}
 & P \times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\
P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\
 & B &
\end{tikzcd}$$

We will denote $P \times_B \hat{P}$ by B' and the composition of j and π by f. We can pullback the associated bundles along f and get a T-duality diagram over B':

$$\begin{tikzcd}
 & P' \times_{B'} \hat{P}' \arrow[dl, "i"'] \arrow[dr, "\hat{i}"] & \\
P' \arrow[dr, "\pi'"'] & & \hat{P}' \arrow[dl, "\hat{\pi}'"] \\
 & B' &
\end{tikzcd}$$

And we denote the pullback map at every vertex of the diagram by $f : B' \to B$, $F : P' \to P$, $\hat{F} : \hat{P} \to \hat{P}'$, $f' : P' \times_{B'} \hat{P}' \to P \times_B \hat{P}$ respectively. Moreover, denote the last map by $f' : B'' \to B'$ briefly and T-isomorphism over B (or B') by T_B (or $T_{B'}$) below.

In this chapter, we will work in the category of finite CW-complexes i.e. all of the base spaces and the corresponding principal $\mathbb{T}^n$-bundles are all finite CW-complexes. Here are some notions we will use in this chapter.

Definition 3.1.
- Let $\mathbf{T-triple}_n$ to be the category of all of T-duality triples $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$. Here P and $\hat{P}$ are principal $\mathbb{T}^n$-bundles over a base space. Morphisms are pullbacks of T-duality triples induced by .
- We define $\mathbf{Pair}_n$ to be the category of all of the pairs $(P, \mathcal{A})$ which admit an extension to a T-duality triple. Morphisms are pullbacks of pairs induced by maps between base spaces.

3.2 Main Theorem

As we have seen in Chapter 1, T-duality isomorphisms in the three approaches ([43], [22] and [16]) all give an isomorphism of twisted K-groups for each object in $\mathbf{T-pair}_n$. Let us start from the simple case i.e. $\mathbf{T-pair}_1$. Before going into complicated cases, we first have a look at T-duality isomorphisms for the most simple case i.e. the case in which the base space is a point. It is well known that $K^0(S^1) \cong K^1(S^1) \cong \mathbb{Z}$. We choose $f(x) = 1$ and $g(x) = x$ to represent two positive oriented generators of $K^0(S^1)$ and $K^1(S^1)$ respectively.

Lemma 3.2. *Denote the positive oriented generators by e_0 and e_1 respectively. Then we have the map*

$$T_{pt} := \hat{j}_! \circ u \circ j^* : K^i(S^1) \to K^{i+1}(S^1), \tag{3.1}$$

which takes e_0 to e_1 and e_1 to e_0 respectively. Moreover, we get that two-fold composition of T_{pt} maps e_0 to e_0 and e_1 to e_1 i.e. $T_{pt} \circ T_{pt}$ is the identity map.

Proof. In this case, we know that $K^i(S^1 \times S^1) \cong \oplus_{i=m+n} K^m(S^1) \otimes K^n(S^1)$ and the push-forward map $\hat{j}^* : K^i(S^1 \times S^1) \to K^{i-1}(S^1)$ is the projection from $K^1(S^1) \otimes K^{i-1}(S^1)$ to $K^1(S^1)$. More explicitly, we use e_0 ,e_1 to represent positive generators of the K-groups of S^1 and use $\hat{e_0}$, $\hat{e_1}$ to represent positive generators of the K-groups of the other copy of S^1 in $S^1 \times S^1$. Then the positive generators of $K^0(S^1 \times S^1)$ is $e_0 \otimes \hat{e_0}$, $e_1 \otimes \hat{e_1}$ and the positive generators of $K^1(S^1 \times S^1)$ is $e_1 \otimes \hat{e_0}$, $e_0 \otimes \hat{e_1}$. Then we get that $\hat{j}_!$ maps $e_1 \otimes \hat{e_0}$ and $e_1 \otimes \hat{e_1}$ to $\hat{e_0}$ and $\hat{e_1}$ respectively. We need to explain how u works. In this case, u is defined by tensoring with the K-element of the Poincare line bundle L over $S^1 \times S^1$. First we recall that in the twisted de-Rham cohomology cases, it is given by $\wedge \exp(c_1(L))$. Let θ and $\hat{\theta}$ be positive generators of the first cohomology groups of $S^1 \times S^1$. Then $c_1(L) = \theta \otimes \hat{\theta}$. Now we compute how u works here.

$$u(1 \otimes \hat{1}) = 1 \otimes \hat{1} + \theta \otimes \hat{\theta}, \;\; u(1 \otimes \hat{\theta}) = 1 \otimes \hat{\theta},$$
$$u(\theta \otimes \hat{1}) = \theta \otimes \hat{1}, \;\; u(\theta \otimes \hat{\theta}) = \theta \otimes \hat{\theta}.$$

Since the Chern character provides an isomorphism between K-groups (tensoring with $\mathbb{R}$) and $\mathbb{Z}/2\mathbb{Z}$-graded (twisted) de-Rham cohomology, therefore we have

$$u(e_0 \otimes \hat{e_0}) = e_0 \otimes \hat{e_0} + e_1 \otimes \hat{e_1}, \;\; u(e_1 \otimes \hat{e_0}) = e_1 \otimes \hat{e_0},$$
$$u(e_0 \otimes \hat{e_1}) = e_0 \otimes \hat{e_1}, \;\; u(e_1 \otimes \hat{e_1}) = e_1 \otimes \hat{e_1}.$$

Then we have that

$$\begin{aligned} T(e_0) &= \hat{j}_! \circ u \circ j^*(e_0) = \hat{j}_! \circ u \circ (e_0 \otimes \hat{e_0}) \\ &= \hat{j}_!(e_1 \otimes \hat{e_1}) = \hat{e_1}, \\ T(e_1) &= \hat{j}_! \circ u \circ j^*(e_1) = \hat{j}_! \circ u \circ (e_1 \otimes \hat{e_0}) \\ &= \hat{j}_!(e_1 \otimes \hat{e_0}) = \hat{e_0}. \end{aligned}$$

If we forget differences between e_0, e_1 and $\hat{e_0}$, $\hat{e_1}$, then we get the first assertion. For the second one, we can continue the above computations

$$\begin{aligned} T \circ T(e_0) &= j_! \circ u' \circ \hat{j}^*(\hat{e_1}) = j_! \circ u'(e_0 \otimes \hat{e_1}) \\ &= j_!(e_0 \otimes \hat{e_1}) = e_0, \\ T \circ T(e_1) &= j_! \circ u' \circ \hat{j}^*(\hat{e_0}) = j_! \circ u'(e_0 \otimes \hat{e_0}) \\ &= j_!(e_1 \otimes \hat{e_1}) = e_1. \end{aligned}$$

Here u' is changing twist map in the second T-duality isomorphism. □

Next we move on to the study of the trivial case via C^*-algebra approach. This case is much more complicated. First of all, we give some facts and notions which we will use in our proof. It is well known that the space, $\mathfrak{F}$, of Fredholm operators is a classifying space for K^0 by [2]. In [5] they proved that the component, $\mathfrak{F}^s$, of the self adjoint Fredholm operators is a classifying space for K^1. In [11], another description of K^0 and K^1 is given as follows. Let B be a C^*-algebra, then

$$K_0(B) \cong KK(\mathbb{C}, B) \cong \{[T] : T \in M^s(B),\ TT^* - 1,\ T^*T - 1 \in B \otimes \mathcal{K}\};$$
$$K_1(B) \cong KK^1(\mathbb{C}, B) \cong \{[T] : T \in M^s(B),\ T - T^*, T^2 - 1 \in B \otimes \mathcal{K}\}.$$

in which $M^s(B)$ is the stable multiplier algebra of B. In particular, when $B = C(S^1)$, we get that $K^1(S^1)$ can be represented by an S^1-family of of self adjoint operators satisfies the condition $T^2 - 1 \in B \otimes \mathcal{K}$ above. These two descriptions are equivalent, which we will not give the details here. In this section we discuss the positive generators of $K^0(S^1)$. In the Lemma 3.2, we use the unitary $f(x) = x \in C(S^1)$ to represent to the positive generator of $K^1(S^1)$. According to the above discussion, we can also use an S^1-family of self adjoint Fredholm operators to represent the positive generator of $K^1(S^1)$. To determine the positivity of a generator in this approach, we need the notion of spectral flows.

Definition 3.3. Let $B : [0, 1] \to \mathfrak{F}^s$ be a continuous path. The spectral flow of $(B_t)_{t\in[0,1]}$ is the net of number of eigenvalues (counted with multiplicity) which pass through 0 in the positive direction as t goes from 0 to 1.

In [51] another version of spectral flows is given, which is equivalent to this one but more complicated for our computations here. We give some properties of spectral flows which are given in [51].

Proposition 3.4.

1. The Spectral flow is homotopy invariant;

2. sf induces an isomorphism from $\pi_1(\mathfrak{F}^s)$ to $\mathbb{Z}$;

3. Let $\hat{F}^\infty$ be a subspace of $\mathfrak{F}^s$ given by $\hat{F}^\infty = \{B \in \mathfrak{F}^s |\ ||B|| = 1, sp(B)$ is finite, and $sp(\pi(B)) = \{1, -1\}$. Here $\pi : B \to B/\mathcal{K}$ is the natural projection. The following

diagram is a commuting square of isomorphisms

$$\begin{tikzcd}
\pi_1(\mathfrak{F}^s) \arrow[r, "sf"] \arrow[d, "i_*"] & \mathbb{Z} \\
\pi_1(\hat{F}^\infty) \arrow[r] & \pi_1(U(\infty)) \arrow[u]
\end{tikzcd}$$

where the map $\pi_1(U(\infty)) \to \mathbb{Z}$ is the winding number of the determinant.

Remark 3.5. From the last assertion of the above proposition we can get that the positive generator of $K^1(S^1)$ given by the unitary $f(x) = x$ is equivalent to the positive generator given by an S^1-family of self adjoint Fredholm operator with spectral flow equal to 1.

Lemma 3.6. *Denote the Connes-Thom isomorphism by $C_{pt} : K_i(C(S^1)) \to K_{i+1}(C(S^1) \rtimes \mathbb{R})$ and the isomorphism between $K_i(C(S^1) \rtimes \mathbb{R})$ and $K_i(S^1)$ in Theorem 3.7 of [62] by S_{pt}. Use the notions above, Then we have that $a \circ S_{pt} \circ C_{pt}$ maps positive generators of $K^*(S^1)$ to positive generators of $K^*(S^1)$. Here $a : K_*(C(S^1, \mathcal{K})) \to K_*(C(S^1))$ is the isomorphism induced by the canonical Morita equivalence between $\mathcal{K}$ and $\mathbb{C}$.*

Proof. We will use KK-cycles in the proof. According to the discussion on crossed product C^*-algebra and the Connes-Thom isomorphism in Appendix C.3. We get that the Thom element $(C(S^1) \rtimes \mathbb{R}, \phi, F_f) \in KK^1(C(S^1), C(S^1) \rtimes \mathbb{R})$ represents the Connes-Thom isomorphism. Without loss of generality we can choose f to be $f(s) = \frac{1}{\pi}\int_{-s}^{s} \frac{sint}{t} dt$. Composing $(C(S^1) \rtimes \mathbb{R}, \phi, F_f)$ with S we get another KK-cycle $(C(S^1, \mathcal{K}(L^2(\mathbb{Z}))), \phi \circ S, B) \in KK^1(C(S^1), C(S^1, \mathcal{K}(L^2(\mathbb{Z}))))$. Here we need to describe the operator B carefully. By the construction in [62], F_f is mapped to a $\mathbb{R}$-family of Hilbert-Schmidt operators B is an S^1-family of self adjoint Fredholm operators on $L^2(\mathbb{Z})$ given by the following formula

$$B(t\mathbb{Z}) = U(t)\bar{f}(t)U(t)^{-1}.$$

Here $\bar{f}(t)$ is an $\mathbb{R}$-family of operators given by

$$\bar{f}(t)x(n) = \sum_k f(t)\delta(k)x(n-k),$$

in which $x \in L^2(\mathbb{Z})$, $n \in \mathbb{Z}$ and δ is the Dirac function over $L^2(\mathbb{Z})$. More explicitly, if we choose the canonical orthogonal base of $L^2(\mathbb{Z})$, then $\bar{f}$ is a family of diagonal operators. Moreover, it is self adjoint for each t. The eigenvalues at t is $(..., f(t -$

$n), ..., f(t-1), f(t), f(t+1), ..., f(t+n), ...,)$. Let t goes from 0 to 1, we can compute the spectrum flow of $\bar{f}$. The function f only has one intersection with x-axis, so we get $sp(\bar{f}) = 1$ by Definition 3.3. This implies that $sp(B)$ is also 1. Composing $(C(S^1, \mathcal{K}(L^2(\mathbb{Z}))), \phi \circ S, B)$ with the canonical Morita equivalence between $\mathcal{K}(L^2(\mathbb{Z}))$ and $\mathbb{C}$, we get a KK-cycle $(C(S^1) \otimes L^2(S^1), \varphi, B) \in KK^1(C(S^1), C(S^1))$. Here the left $C(S^1)$-action is given by pointwise multiplication on $L^2(S^1)$. The inclusion of $\mathbb{C}$ into $C(S^1)$ induces a homomorphism from $KK^1(C(S^1), C(S^1))$ to $KK^1(\mathbb{C}, C(S^1))$ and this homomorphism makes $(C(S^1) \otimes L^2(S^1), \varphi, B)$ to be a KK-cycle in $KK^1(\mathbb{C}, C(S^1))$, which is isomorphic to $K^1(C(S^1))$. By Remark 3.5 we get that $(C(S^1) \otimes L^2(S^1), \varphi, B)$ represents the positive generator of $KK^1(\mathbb{C}, C(S^1))$. On the other hand, the evaluation map $C(S^1) \to \mathbb{C}$ induces a homomorphism from $KK^1(C(S^1), C(S^1)) \to KK^1(C(S^1), C)$. $(C(S^1) \otimes L^2(S^1), \varphi, B)$ is mapped to $(L^2(S^1), \varphi, B(1))$. Let $D = i\frac{d}{d\theta}$ be the canonical Dirac operator on S^1 and we know the eigenvalues of $\frac{D}{\sqrt{1+DD^*}}$ are $(..., -\frac{n}{\sqrt{1+n^2}}, ..., -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, ..., \frac{n}{1+n^2}, ...)$. Therefore we get that $B(1) - \frac{D}{\sqrt{1+DD^*}}$ is compact (see the Remark A.15 in the appendix A) and the corresponding KK-cycle are equivalent. Therefore $(L^2(S^1), \varphi, B(1))$ is the positive generator of $KK^1(C(S^1), \mathbb{C}) \cong K^1(C(S^1)) \cong K^0(S^1)$. If we translate the above result of KK-cycles to homomorphism between K-groups, we get the conclusion of the lemma holds by the universal coefficient theorem. $\square$

Now we go on with our discussion of T-duality isomorphisms.

Proposition 3.7. Each of the constructions of the T-duality isomorphisms in [43], [22] and [16] gives a T-duality isomorphism for any object in $\mathbf{T-triple}_1$, which satisfies the following axioms.

- **Axiom 1** When the base space is a point pt, T_{pt} satisfies the following equalities:

$$T_{pt}(e_0) = e_1, T_{pt}(e_1) = e_0; \tag{3.2}$$

  Here e_0 and e_1 are the positive generators of $K^0(S^1)$ and $K^1(S^1)$ respectively.

- **Axiom 2** If $g : X \to Y$ is a continuous map, and we pullback the T-duality diagram over Y to X, the T isomorphisms T_X and T_Y satisfy the following naturality condition:

$$T_X \circ F^* = \hat{F}^* \circ T_Y; \tag{3.3}$$

- **Axiom 3** Given a T-duality diagram (0.1) over B, we can get the T-duality diagram as follows:

$$\begin{tikzcd}
 & S^1 \times P \times_B \hat{P} \arrow[dl, "id_{S^1} \times j"'] \arrow[dr, "id_{S^1} \times \hat{j}"] & \\
S^1 \times P \arrow[dr, "id_{S^1} \times \pi"'] & & S^1 \times \hat{P} \arrow[dl, "id_{S^1} \times \hat{\pi}"] \\
 & S^1 \times B &
\end{tikzcd}$$

Then we have that:

$$T_{S^1 \times B} = Id_{K^*(S^1)} \otimes T_B. \tag{3.4}$$

Proof. The proof contains three parts.

part 1 First of all we consider the T-isomorphism in [16]. Use the notations in section 3.1. We have the pullback map $j^* : K^*(P, \mathcal{A}) \to K^*(B', j^*(\mathcal{A}))$. Since $\hat{j} : B' \to \hat{\mathcal{A}}$ is a principal S^1-bundle, there exists a push-forward map by [19]:

$$\hat{j}_! : K^*(B', \hat{j}^*(\hat{\mathcal{A}})) \to K^{*-1}(\hat{P}, \hat{\mathcal{A}}).$$

The remainder for the preparation for the construction of T-isomorphism for twisted K-theory is the changing twist isomorphism u, which we have explained before.

Then we the T-isomorphism is given as the composition of the above three maps:

$$T = \hat{j}_! \circ u \circ j^* : K^*(P, \mathcal{A}) \to K^{*+1}(\hat{P}, \hat{\mathcal{A}}) \tag{3.5}$$

Now we show that this T-isomorphism satisfies the three axioms above: The first axiom holds for the above T by Lemma 3.2.

For the second one, it follows from some commutation relations between push-forward map and pullback map. Explicitly, given any continuous map $f : X \to Y$, and a $T-$dual diagram over Y, we get a pullback $T-$dual diagram over X as follows:

$$
\begin{tikzcd}
 & P_X \times_X \hat{P}_X \arrow[dl, "j_X"'] \arrow[dr, "\hat{j}_X"] \arrow[rrrr, "F'"] & & & & P_Y \times \hat{P}_Y \arrow[dl, "j_Y"'] \arrow[dr, "\hat{j}_Y"] & \\
P_X \arrow[dr, "\pi_X"'] \arrow[rrrr, bend left=10, "F"] & & \hat{P}_X \arrow[dl, "\hat{\pi}_X"] \arrow[rrrr, bend right=10, "\hat{F}"] & & P_Y \arrow[dr, "\pi_Y"'] & & \hat{P}_Y \arrow[dl, "\hat{\pi}_Y"] \\
 & X \arrow[rrrr, "f"'] & & & & Y &
\end{tikzcd}
$$

Then we have

$$
\begin{aligned}
T_X \circ F^* &= (\hat{j}_! \circ t_X \circ j^*) \circ F^* \\
&= \hat{j}_! \circ t_X \circ (j^*) \circ F^*) \\
&= \hat{j}_! \circ t_X \circ F'^* \circ i^* \\
&= \hat{j}_! \circ F'^* \circ t_Y \circ i^* \\
&= \hat{F}^* \circ \hat{i}_! \circ t_Y \circ i^* \\
&= \hat{F}^* \circ T_Y.
\end{aligned}
$$

For the third axiom, we will prove a more general statement which implies the third axiom immediately.

Proposition 3.8. Given any T-duality diagram over B, and any smooth manifold M with a twisting $\mathcal{K}$-bundle $\mathcal{B}$. If we do the product with M at every vertex of the T-duality diagram over M and get another T-duality diagram over $B \times M$ below:

$$
\begin{tikzcd}
 & (P \times_B \hat{P}) \times M \arrow[dl, "j \times id_M"'] \arrow[dr, "\hat{j} \times id_M"] & \\
P \times M \arrow[dr, "\pi"'] & & \hat{P} \times M \arrow[dl, "\hat{\pi}"] \\
 & B \times M &
\end{tikzcd}
$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} K^*(P,\mathcal{A})\otimes K^*(M,\mathcal{B}) & \xrightarrow{\alpha} & K^*(P\times M, i^*(\mathcal{A})\otimes j^*(\mathcal{B})) \\ T_B\otimes Id \Big\downarrow & & T_{B\times M}\Big\downarrow \\ K^*(\hat{P},\hat{\mathcal{A}})\otimes K^*(M,\mathcal{B}) & \xrightarrow[\hat{\alpha}]{} & K^*(\hat{P}\times M, \hat{i}^*(\hat{\mathcal{A}})\otimes \hat{j}^*(\mathcal{B})) \end{array}$$

Here i and j are projections from $P\times M$ to P and M respectively, $\hat{i}$ and $\hat{j}$ are projections from $\hat{P}\times M$ to $\hat{P}$ and M respectively.

Proof of (3.8). Since α is actually defined by the intersection product, i.e.

$$\alpha : KK(\mathbb{C},A)\times KK(\mathbb{C},B)\to KK(\mathbb{C},A\otimes B).$$

As we will see in the next section, the T-isomorphism T_B here corresponds to a KK-element, which we denote by $[T_B]$. Then the commutativity of the first square is equivalent to

$$x * y * [T_{B\times M}] = x * (y * [T_B]). \tag{3.6}$$

Here $x\in K^*(M,\mathcal{B}), y\in K^*(P,\mathcal{A})$ and $*$ denote the intersection product of KK-elements. By the definition of $T_{B\times M}$ we get that $[T_{B\times M}] = [id_{K^*(M,\mathcal{B})}] * [T_B]$. To see this we write down the KK-element of $[T_B]$ and $[T_{B\times M}]$ explicitly as follows:

$$[id_{K^*(M,\mathcal{B})}] * [T_B] = [id_{K^*(M,\mathcal{B})}] * [p^*] * [u] * [\hat{p}_!],$$

$$[T_{B\times M}] = [(p\times id)^*] * [u\times id] * [(\hat{p}\times id)_!].$$

Since the $[id_{K^*(M,\mathcal{B})}]$ is $(C(M,\mathcal{B}), m, 0)$, so the two products are the same according to the definition of intersection products.

$$x * y * [T_{B\times M}] = x * (y * [T_{B\times M}]) = x * (y * [id_{K^*(M,\mathcal{B})*[T_B]} = x * (y * [T_B]).$$

□

When M is S^1, we get the third axiom since $K^*(S^1\times P, i^*(\mathcal{A}))\cong K^*(S^1)\otimes K^*(P,\mathcal{A})$.
part 2 Next we show that for the second version of T-duality isomorphism, i.e. the composition of the Coones-Thom isomorphism and Schneider's isomorphism.

Lemma 3.6 implies that $S\circ C$ satisfies the first axiom. For the second one. Assume that $f : X\to Y$ is a continuous map and there is a T−duality diagram over Y. According to [26], we know that Connes-Thom isomorphism satisfies some naturality, i.e.

$$F^*_{C(P_Y,H_Y)\rtimes\mathbb{R}}\circ C_{C(P_Y,H_Y)} = C_{C(P_X,H_X)}\circ F_{C(P_Y,H_Y)}.$$

So we only need to prove the naturality for S, i.e.

$$F^*_{C(\hat{P}_Y,\hat{H}_Y)} \circ S_{C(P_Y,H_Y)\rtimes\mathbb{R}} = S_{C(P_X,H_X)\rtimes\mathbb{R}} \circ F^*_{C(P_Y,H_Y)\rtimes\mathbb{R}}.$$

First of all, let us recall the construction of Schneider's isomorphism in his thesis. He defined it locally. More explicitly, if we choose a good open covering (U_i) of the base space Y, then an element of $C(P_Y, H_Y) \rtimes \mathbb{R}$ can be represented by a family of functions from U_i to $C_c(\mathbb{R} \times S^1, \mathcal{K})$. Then

$$S_i : C(U_i, C_c(\mathbb{R} \times S^1, \mathcal{K}(\mathcal{H}))) \to C(U_i \times \hat{\mathbb{R}}/\mathbb{Z}^\perp, \mathbb{K}(L^2(\mathbb{R}\hat{/}\mathbb{Z}, \mathcal{H})))$$

is defined by

$$S_i f_i(u, \kappa\mathbb{Z}^\perp) := (T_{\bar{\mu}_i(u)} f_i(u))(\kappa\mathbb{Z}^\perp) = Ad(\Lambda(\kappa)) f_i(u)^{\bar{\mu}_i(u)}(\kappa). \tag{3.7}$$

Here $\Lambda : \hat{\mathbb{R}} \to U(L^2(\mathbb{Z}))$ is a continuous homomorphism and $\bar{\mu}_i : U_i \to Z^1_{Bor}(\mathbb{R}, L^\infty(\mathbb{R}/\mathbb{Z}, \mathcal{H}))$ are unitary Borel cocycles. Assume we have a continuous map $f : X \to Y$ and we use $(f^{-1}(U_i))$ as the open covering for X. Then we have that:

$$F^*_{i_X} S_{i_Y} g_i(\tilde{u}, \kappa\mathbb{Z}^\perp) = (T_{\bar{\mu}_i(f(\tilde{u}))} g_i(f(\tilde{u})))(\kappa\mathbb{Z}^\perp)$$

and

$$(S_{i_X} \circ F^*_{i_Y}) g_i(\tilde{u}, \kappa\mathbb{Z}^\perp) = (T_{\bar{\mu}_i(f(\tilde{u}))} g_i(f(\tilde{u})))(\kappa\mathbb{Z}^\perp).$$

Then we get that naturality holds for $S \circ C$.

For the third axiom, the idea is similar to the second one. According to [26], we get

$$C_{S^1 \times M} = C_{S^1} \times id_M \tag{3.8}$$

as there is no $\mathbb{R}$-action over M by assumption. By the definition of S, it also involves only the one admitting the $\mathbb{R}$-action, which means that

$$\begin{aligned} (S \circ C)_{S^1 \times M} &= S \circ (C_{S^1} \otimes id_M) = (S \circ C)_{S^1} \otimes id_M \\ &= T_* \otimes id_M. \end{aligned}$$

part 3 At last we need to show that the axioms hold for the T-duality isomorphism constructed from C. Daenzer's groupoid approach. Actually we will show that the KK-element of the T-isomorphism in [22] satisfies (3.4). First of all, let us first review the construction of T-isomorphism of twisted K-theory in [22]. It is the composition of the following maps:

$$\begin{aligned} K^*(C^*(G/N \rtimes_{\bar\rho} \mathcal{G};\sigma)) \to K^{*+1}(C^*(G/N \rtimes_{\bar\rho} \mathcal{G};\sigma) \rtimes G) \to K^{*+1}(C^*(N \rtimes^{\delta\rho} \mathcal{G};\iota^*\varphi) \\ \to K^{*+1}(C^*(\hat N \rtimes_{\bar\lambda} \mathcal{G};\sigma^\vee)) \end{aligned} \tag{3.9}$$

The first map is the Connes-Thom isomorphism C, and the third map is the inverse of the Fourier-type transformation $\mathfrak{F}$ in **Theorem 8.3** [22]. Here is the explicit formula of $\mathfrak{F}$:

$$\mathfrak{F}(a)(\phi,\gamma) := \int_{g\in G} a(g,\gamma)\phi(g^{-1}). \tag{3.10}$$

Here $a \in C^*(\hat N \rtimes_{\bar\lambda} \mathcal{G};\sigma^\vee)$, $\phi \in \hat N$ and $\gamma \in \mathcal{G}$.
The second map in (3.9) is a little bit complicated. In short, it is induced by a Morita equivalent between the groupoids $N \rtimes^{\delta\rho} \mathcal{G}$ and $G \ltimes (G/N \rtimes_\rho \mathcal{G})$. According to **Proposition 10.5** in [22], there exists an inclusion :

$$\iota : N \rtimes^{\delta\bar\rho} \mathcal{G} \hookrightarrow G \ltimes (G/N \rtimes_\rho \mathcal{G}), (n,\gamma) \mapsto (n\rho(\gamma), eN, \gamma). \tag{3.11}$$

Here $\rho : \mathcal{G} \to G$ is a continuous lifting of $\bar\rho : \mathcal{G} \to G/N$. The Morita $N \rtimes^{\delta\rho} \mathcal{G}$, $G \ltimes (G/N \rtimes_\rho \mathcal{G})$-bimodule is $\mathcal{G}_0 \times_{G/N\times\mathcal{G}_0} G \times G/N \times \mathcal{G}_1$. Moreover, by the construction in Section 12 of [22] we know that the corresponding 2-cocycles are cohomologous. By **Theorem9.1** in [22] we get that the $U(1)$-gerbes $U(1) \rtimes^{\iota^*\phi} N \rtimes^{\delta\rho} \mathcal{G}$ and $U(1) \rtimes^{\phi} (G \ltimes (G/N \rtimes_{\bar\rho} \mathcal{G})$.

Theorem 3.9 ([46])**.** *Suppose that $(\mathcal{G},\lambda)$ and $(\mathcal{H},\beta)$ are second countable locally compact groupoids with Haar systems λ and β. Then for any $(\mathcal{G},\mathcal{H})$-equivalence Z. $C_c(Z)$ can naturally be completed into a $C^*(G,\lambda) - C^*(\mathcal{H},\beta)$ imprimitivity bimodule. In particular, $C^*(\mathcal{G},\lambda)$ and $C^*(\mathcal{H},\beta)$ are strongly Morita equivalent.*

Therefore we get the second map in (3.9) is realized by the $U(1) \rtimes^{\iota^*\phi} N \rtimes^{\delta\rho} \mathcal{G} - U(1) \rtimes^{\phi}$ $(G \ltimes (G/N \rtimes_{\bar\rho} \mathcal{G})$ bimodule: $\mathcal{G}_0 \times_{G/N\times\mathcal{G}_0} U(1) \times G \times G/N \times \mathcal{G}_1$.
Now we continue our proof. We denote the corresponding KK-elements of the maps in (3.9) by $[c], [m]$ and $[f]$ respectively. Then we need to show that $[c] * [m] * [f]$ satisfies the three axioms in (3.4).

In [26], they showed the intersection product with $[c]$ maps the positive generator of $K^0(*)$ to the positive generator of $K^1(\mathbb{R})$ and doing intersection product with $[c]$ is natural and satisfies the product formula. Therefore we need to deal with $[m] * [f]$. We divides the remaining proof into three steps.

Step 1 First we show that the map of doing intersection product with $[m] * [f]$ maps the positive generator of $K^i(C(S^1) \rtimes \mathbb{R})$ to the positive generator of $K^i(C(S^1))$. When G is $\mathbb{R}$, N is $\mathbb{Z}$ and the groupoid $\mathcal{G}$ is trivial groupoid $* \rightrightarrows *$. Then we get that the $C_c(\mathbb{R} \times S^1) - C(\mathbb{Z})$ bimodule we mentioned in the above remark is just $C_c(\mathbb{R} \times S^1)$. Here $C(\mathbb{Z})$ is the groupoid C^*−algebra of the groupoid $\mathbb{Z} \rightrightarrows *$. We choose $(L^2(\mathbb{R}) \otimes \mathcal{H}_{C(S^1)}, i, -i\frac{d}{dt} \otimes id)$ and $(L^2(\mathbb{R}), i, -i\frac{d}{dt})$ to represent the positive generator of $K^i(\mathbb{R} \times S^1)$ $(i = 0, 1)$ respectively. We do the intersection product with $(L^2(\mathbb{R}) \otimes \mathcal{H}_{C(S^1)}, i, -i\frac{d}{dt} \otimes id)$. We get

$$(L^2(\mathbb{R}) \otimes \mathcal{H}_{C(S^1)}, i, -i\frac{d}{dt} \otimes id) * (L^2(\mathbb{R}) \otimes \mathcal{H}_{(C(S^1)}, i, -i\frac{d}{dt} \otimes id) \cong (L^2(\mathbb{R}^2) \times \mathcal{H}_{C(S^1 \times S^1)}, i, -b \otimes id),$$

$$(L^2(\mathbb{R}), i, -i\frac{d}{dt}) * (L^2(\mathbb{R}) \otimes \mathcal{H}_{C(S^1)}, i, -i\frac{d}{dt} \otimes id) \cong (L^2(\mathbb{R}^2) \otimes \mathcal{H}_{C(S^1)}, i, b \otimes id).$$

Here b is the Bott element. According to Bott periodicity, $(L^2(\mathbb{R}^2) \times \mathcal{H}_{C(S^1 \times S^1)}, i, -b \otimes id)$ and $(L^2(\mathbb{R}^2) \otimes \mathcal{H}_{C(S^1)}, i, b \otimes id)$ are equivalent to the positive generator of $K^1(S^1)$ and $K^0(S^1)$ respectively we give in **part 1**. Therefore under the Fourier transformation they are also the positive generators of $K^1(\hat{N})$ and $K^0(\hat{N})$, which implies that doing intersection product with $[m] * [f]$ maps the positive generator of $K^i(C(S^1) \rtimes \mathbb{R})$ to the positive generator of $K^i(C(S^1))$.

Step 2 To show that doing intersection product with $[m]$ is nature and the product formula is satisfied, we only need to show that the $U(1) \rtimes^{\iota^*\phi} N \rtimes^{\delta\rho} \mathcal{G} - U(1) \rtimes^{\phi} (G \ltimes (G/N \rtimes_{\bar{\rho}} \mathcal{G})$ bimodule $\mathcal{G}_0 \times_{G/N \times \mathcal{G}_0} U(1) \times G \times G/N \times \mathcal{G}_1$ is nature over $\mathcal{G}$ and satisfies the product formula. Since all of the constructions are nature over $\mathcal{G}$ we get naturality of $[m]$. For product formula, let M be another manifold, then we get the associated bimodule is that $\mathcal{G}_0 \times M \times_{G/N \times \mathcal{G}_0 \times M} U(1) \times G \times G/N \times \mathcal{G}_1 \times M$ i.e. $\mathcal{G}_0 \times_{G/N \times \mathcal{G}_0} U(1) \times G \times G/N \times \mathcal{G}_1 \times M$. Therefore we have the associated KK-element $[m_{\mathcal{G} \times M}]$is isomorphic to $[m_{\mathcal{G}}] \times [id_M]$.

Step 3 At last we show that doing intersection product with $[f]$ is nature and the product formula is also satisfied. We first show that the Fourier transformation $\mathfrak{F}$ in [22] is nature and satisfies the product formula.

Assume there exists a groupoid morphism $g : \mathcal{H} \to \mathcal{G}$, it induces a C^*-algebra map $g : C^*(G \rtimes \mathcal{G}; \sigma) \to C^*(G \rtimes \mathcal{H}; \sigma)$. To show $\mathfrak{F}$ is natural we need to show that the following diagram is commutative:

$$\begin{tikzcd}
C^*(G \rtimes_\rho \mathcal{G}; \sigma) \arrow[r, "\mathfrak{F}"] \arrow[d, "g_G"] & C^*(\hat{G} \rtimes^f \mathcal{G}; \tau) \arrow[d, "g_{\hat{G}}"] \\
C^*(G \rtimes_{\rho \circ g} \mathcal{H}; g^*(\sigma)) \arrow[r, "\mathfrak{F}"] & C^*(\hat{G} \rtimes^{f \circ g} \mathcal{H}; g^*(\tau))
\end{tikzcd} \tag{3.12}$$

Concretely, for any $a \in C^*$, $\phi \in \hat{G}$ and $\gamma \in \mathcal{H}$, we have

$$g_{\hat{G}} \circ \mathfrak{F}(a)(\phi, \gamma) = \int_{x \in G} a(x, g(\gamma)\phi(x^{-1})).$$

On the other side, we have

$$\mathfrak{F} \circ g_G(a)(\phi, \gamma) = \int_{x \in G} a(x, g(\gamma))\phi(x^{-1}).$$

Now we show the product formula for $\mathfrak{F}$, i.e., given another manifold M, we can have an product groupoid $\mathcal{G} \times M$, therefore we can define a Fourier transformation for this groupoid

$$\mathfrak{F}_{\mathcal{G}\times M} : C^*(G \rtimes_{\rho\times 0} (\mathcal{G} \times M; \sigma)\; C^*(\hat{G} \rtimes^{f\times 0} (\mathcal{G} \times M); \tau), \tag{3.13}$$

$$a \otimes f \mapsto \mathfrak{F}_{\mathcal{G}\times M}(a)(\phi, \gamma) = \int_{x \in g} a(x, \gamma)\phi(x^{-1}) \otimes f. \tag{3.14}$$

i.e. we have that

$$\mathfrak{F}_{\mathcal{G}\times M} = \mathfrak{F}_{\mathcal{G}} \otimes id_{C^*(M)}. \tag{3.15}$$

□

Next we give the main theorem of this chapter.

Theorem (Theorem 0.2)**.** *There exists a unique T-duality isomorphism which satisfies the following axioms for each object in the category* $\mathbf{T-pair}_1$ *i.e. for any space B and any T-duality triple $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ over B, there is a unique way to assign a T-duality isomorphism between the corresponding twisted K-groups $K^*(P, \mathcal{A})$ and $K^{*+1}(\hat{P}, \hat{\mathcal{A}})$ such that the following axioms are satisfied.*

- ***Axiom 1*** *When the base space is a point pt, T_{pt} satisfies the following equalities:*

$$T_{pt}(e_0) = e_1, T_{pt}(e_1) = e_0. \tag{3.16}$$

Here e_0 and e_1 are the positive generators of $K^0(S^1)$ and $K^1(S^1)$ respectively.

- ***Axiom 2*** *If $g : X \to Y$ is a continuous map, and we pullback the T−duality triple over Y to X. The T isomorphisms T_X and T_Y satisfy the following naturality condition:*

$$T_X \circ F^* = \hat{F}^* \circ T_Y; \tag{3.17}$$

- ***Axiom 3****Let* $((P,\mathcal{A}),(\hat{P},\hat{\mathcal{A}}),u)$ *be a T-duality triple over B.* $((P\times S^1, j^*\mathcal{A}),(\hat{P}\times S^1,\hat{j}^*\hat{\mathcal{A}}),u)$ *gives a T-duality triple over* $B\times S^1$.

$$\begin{tikzcd}
 & S^1\times P\times_B \hat{P} \arrow[dl, "id_{S^1}\times j"'] \arrow[dr, "id_{S^1}\times \hat{j}"] & \\
S^1\times P \arrow[dr, "id_{S^1}\times \pi"'] & & \hat{S^1}\times \hat{P} \arrow[dl, "id_{S^1}\times \hat{\pi}"] \\
 & S^1\times B &
\end{tikzcd}$$

Then the following identity holds:

$$T_{S^1\times B} = Id_{K^*(S^1)}\otimes T_B. \tag{3.18}$$

The existence is clear from Proposition 3.7. We only need to prove the uniqueness. First of all, we give a lemma which is important for the proof of the main theorem.

Lemma 3.10. *Let* $((P,\mathcal{A}),(\hat{P},\hat{\mathcal{A}}))$ *be a T-duality pair over B.*

$$\begin{tikzcd}
 & P\times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\
P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\
 & B &
\end{tikzcd} \tag{3.19}$$

Using the notations in the last section. Then $F^*: K^*(P,H)\to K^*(P',F^*(H))$ *is injective, i.e., there exists a left inverse of* F^*. *And the same conclusion holds for* $\hat{F}^*$ *as well.*

Proof. We give an simple observation first. Since we know that

$$T=\hat{j}_!\circ u\circ j^*: K^*(P,H)\to K^{*+1}(\hat{P},\hat{H})$$

is an isomorphism, therefore j^* is injective and the left inverse is given by $T^{-1}\circ\hat{j}_!\circ u$. To prove the lemma, let us consider the following diagram:

$$\begin{tikzcd}
 & P \times_B P \arrow[dl, "\alpha"'] \arrow[dr, "\alpha"] & & P' \arrow[ll, "\beta"] \arrow[dr, "\hat{\beta}"] & \\
P \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] & & P \times_B \hat{P} \arrow[ll, "\hat{\alpha}"] \\
 & B & & &
\end{tikzcd} \tag{3.20}$$

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We can see that $F = \alpha \circ \beta$. Because $(P \times_B P, \alpha^*(H))$ and $(P \times_B \hat{P}, \hat{j}^*(\hat{H}))$ are T-dual to each other, we get that $\beta^* : K^*(P \times_B P, \alpha^*(H)) \to K^*(P', F^*(H))$ is injective using the observation. Moreover, as $\alpha : P \times_B P \to P$ is a trivial principal S^1 bundle we get $\alpha^* : K^*(P, H) \to K^*(P \times_B P, \alpha^*(H))$ is also injective. Therefore we get F^* is injective. $\square$

proof of Theorem 0.2. Use the notations above. By the second axiom, we get

$$\hat{F}^* \circ T_B = T_{B'} \circ F^*. \tag{3.21}$$

From (3.10) we know that $\hat{F}$ has a left inverse which we denote by $\hat{F}^{-1}$. Then we get

$$T_B = (\hat{F}^*)-1 \circ T_{B'} \circ F^*. \tag{3.22}$$

As the pullback principal S^1-bundle along itself is always trivial, all of the principal bundles in the pullback T-duality diagram over B' are trivial.By axiom 3, we have

$$T_{B'} = T_* \circ id_{B'}. \tag{3.23}$$

Therefore we only need to prove the uniqueness for the point case. In this case, the universal coefficient theorem implies $\mathrm{Hom}(K^*(S^1), K^{*+1}(S^1)) \cong KK^{*+1}(C(S^1), C(S^1))$. While $\mathrm{Hom}(K^*(S^1), K^{*+1}(S^1))$ has two generators which we denote by f_1 and f_2. Here

$$f_1(e_0) = e_1,\ f_1(e_1) = 0, \tag{3.24}$$

$$f_2(e_1) = e_0,\ f_2(e_0) = 0. \tag{3.25}$$

By axiom 1, the T-isomorphism is determined uniquely by the sum of the two generators in $KK^{*+1}(C(S^1), C(S^1))$. $\square$

Then we get the following corollary,

Corollary 3.11. *The T-duality isomorphisms of twisted K-groups constructed from the topological approach and the C^*-algebra approach are the same.*

Remark 3.12. With the above corollary and the conclusion from A.Schneider's thesis, we get that the topological T-duality construction from topological approach and C^*-algebra approach are equivalent not only on the level of object, but also on the level of T-duality isomorphism.

3.3 Higher Dimensional Cases

In this section we generalize the Theorem 0.2 to topological T-duality of higher dimensional principal torus bundles. According to Chapter 1, we know that there are two differences for the higher dimensional cases:

- The T-dual pair of a given pair does not always exist in higher dimensional torus bundle cases. There are some restrictions on the H-flux. Particularly, one can see Theorem 10.2 in [14];

- When the base space is a point, $Hom(K^*(T^n), K^{*+1}(T^n))$ have more generators than 1-dimensional case.

For the first obstacle, our strategy is to use the notion of T-duality triple in [17] here. For the second one, we need to make some specifications below. First of all we have the following lemma.

Lemma 3.13. *Let $\mathbb{T}^n$ be a trivial n-torus bundle over a point and $H \in H^3(\mathbb{T}^n, \mathbb{Z})$ be a twist of $\mathbb{T}^n$. If H is not trivial, then $(\mathbb{T}^n, H)$ does not admit an extension to a T-duality triple.*

Proof. By (1.14), we need to prove that H does not lie in $\mathcal{F}^2H^3(\mathbb{T}^n, \mathbb{Z})$. When $n = 1$ or 2, H must be trivial. Therefore we assume that $n \geqslant 3$, in which case $\mathcal{F}^2H^3(\mathbb{T}^n, \mathbb{Z})$ is trivial. So we get the nontrivial H can't lie in $\mathcal{F}^2H^3(\mathbb{T}^n, \mathbb{Z})$. □

Therefore we can assume that in (3.26) the twist is trivial and we the following diagram

$$\begin{tikzcd} & \mathbb{T}^n \times \hat{\mathbb{T}}^n \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\ \mathbb{T}^n \arrow[dr, "\pi"'] & & \hat{\mathbb{T}}^n \arrow[dl, "\hat{\pi}"] \\ & * & \end{tikzcd} \tag{3.26}$$

Use the construction in [12] and [16] we can define a natural T isomorphism for twisted K-theory:

$$\mathbf{T^n_*} := \hat{j}_! \circ u \circ j^* : K^*(\mathbb{T}^n) \to K^{*+1}(\hat{\mathbb{T}}^n). \tag{3.27}$$

Also, we first discuss the T-duality isomorphism for the trivial case. We first give some notations. By the Kunneth theorem, we get that $K^i(\mathbb{T}^n) \cong \otimes_{k=1}^n K^{i_k}(S^1)$. If we use the notion of last section, we can write the generators of $K^i(\mathbb{T}^n)$ by $e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_n}$. Here i_k is 0 or 1. We denote the generators of $K^0(\mathbb{T}^n)$ by $f_0, f_1, ..., f_{2^{n-1}}$. If we change each factor of f_i from e_0 to e_1 or from e_1 to e_0 in f_i, we get a positive generator in $K^1(\mathbb{T}^n)$ when n is odd, which we denote by g_i $(i = 0,, 1, ... \, 2^{n-1})$.

Lemma 3.14. *Use the above notations. Then the T-duality isomorphism*

$$T^n_{pt} := \hat{j}_! \circ u \circ j^* : K^i(\mathbb{T}^n) \to K^{i+n}(\mathbb{T}^n)$$

can be decomposed as $\otimes^n T^1_{pt}$. Here T^1_{pt} is the T-duality isomorphism for 1-dimensional trivial case. Moreover, two-fold composition of T_{pt} maps f_i to f_i and g_i to g_i

Proof. By the Kunneth theorem, we get that $K^i(\mathbb{T}^n) \cong \otimes_{k=1}^n K^{i_k}(S^1)$. The computations is similar to that in Lemma 3.2. Here u is still given by tensoring with Poincare line bundle over $\mathbb{T}^n \times \hat{\mathbb{T}}^n$. According to the definition of pullback, pushforward and changing twists map, we get the first assertion holds. The second assertion follows from the first one and Lemma 3.2. □

For higher dimensional cases of Lemma (3.6), we just need to notice the fact that $C(\mathbb{T}^n) \rtimes \mathbb{R}^n$ is isomorphic to n-copies of $C(S^1) \rtimes \mathbb{R}$. By K*ü*nneth theorem, we have that positive generators can be represented by tensor products of positive generators of $K^*(C(S^1 \rtimes \mathbb{R}))$. Therefore Lemma (3.6) implies the following lemma.

Lemma 3.15. *The composition of the n-fold Connes-Thom isomorphism $C^n_{pt} : K^*(C(\mathbb{T}^n)) \to K^{*+n}(C(\mathbb{T}^n) \rtimes \mathbb{R}^n)$, Schneider's isomorphism $S^n_{pt} : K^*(C(\mathbb{T}^n) \rtimes \mathbb{R}^n) \to K^*(C(\mathbb{T}^n, \mathcal{K}))$ and the canonical isomorphism $m : K^*(C(\mathbb{T}^n, \mathcal{K})) \to K^*(C(\mathbb{T}^n))$ maps positive generators of $K^*(C(\mathbb{T}^n))$ to the associated positive generators of $K^{*+n}(C(\mathbb{T}^n))$.*

Now we the uniqueness theorem for higher principal torus bundles:

Theorem 3.16. *There exists a unique T-isomorphism for twisted K-theory for each object in the category* $\mathbf{T-triple}_n$*, which satisfies the following three axioms:*

- ***Axiom 1*** *When the base space is a point pt,*

$$T^n_{pt} = \otimes T^1_{pt}; \tag{3.28}$$

- ***Axiom 2*** *If $g : X \to Y$ is a continuous map, and we pullback the $T-$duality triple over Y to X, the T-duality isomorphisms T_X and T_Y satisfy the following naturality condition:*

$$T_X \circ F^* = \hat{F}^* \circ T_Y; \tag{3.29}$$

- ***Axiom 3*** *Let $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}), u)$ be a T-duality triple over B. $((P \times S^1, j^*\mathcal{A}), (\hat{P} \times S^1, \hat{j}^*\hat{\mathcal{A}}), u)$ gives a T-duality triple over $B \times S^1$.*

$$\begin{tikzcd}
 & S^1 \times P \times_B \hat{P} \arrow[dl, "id_{S^1} \times j"'] \arrow[dr, "id_{S^1} \times \hat{j}"] & \\
S^1 \times P \arrow[dr, "id_{S^1} \times \pi"'] & & \widehat{S^1} \times \hat{P} \arrow[dl, "id_{S^1} \times \hat{\pi}"] \\
 & S^1 \times B &
\end{tikzcd}$$

Then we have that:

$$T_{S^1 \times B} = Id_{K^*(S^1)} \otimes T_B. \tag{3.30}$$

Proof. For a T-duality diagram

$$\begin{tikzcd} & P \times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\ P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\ & B & \end{tikzcd}$$

we can define a T-isomorphism using a similar formula as (3.27):

$$T_B := \hat{j}_! \circ u \circ j^*. \tag{3.31}$$

The proof is almost the same as the proof of Theorem 0.2. The only difference lies on the point case which we have done in Lemma 3.6 and Lemma 3.14. □

According to theorem 0.2, we can define a category $\mathbf{T}-\mathbf{Pair}_n$. The objects of $\mathbf{T}-\mathbf{Pair}_n$ are pairs $(\mathfrak{D}, T_{\mathfrak{D}})$. Here each $\mathfrak{D}$ is a T-duality pair over base space B and each $T_{\mathfrak{D}}$ is a T-duality isomorphism between the twisted K-groups of T-duality pairs in $\mathfrak{D}$. Moreover, we require $T_{\mathfrak{D}}$ satisfies the axioms in (0.2). The morphisms of $\mathfrak{T}-\mathfrak{Pair}$ are also the pull-backs induced by the continuous maps between base spaces. Then Theorem 0.2 can also be stated as follows

Theorem 3.17. $\mathbf{T}-\mathbf{pair}_n$ *and* $\mathbf{T}-\mathbf{Pair}_n$ *are equivalent to each other.*

3.4 KK-version

In this section we discuss the uniqueness of T-isomorphism using KK-theory. First of all, we give a construction of a KK-element from the construction of T-duality isomorphism.

In Bunke-Schick construction, the T-duality isomorphism T is defined by the composition of three maps, in which each map naturally corresponds to a KK-cycle in the following way:

- j^* gives a KK-cycle $(\mathcal{H}_{C(P\times_B\hat{P},j^*\mathcal{A})}, j^*, 0)$ in $KK^0(C(P,\mathcal{A}), C(P\times_B \hat{P}, j^*\mathcal{A}))$, which we denote by $[j^*]$. Here $\mathcal{H}_{C(P\times_B\hat{P},j^*\mathcal{A})}$ is the Hilbert module constructed from $C(P\times_B \hat{P}, j^*\mathcal{A})$, whose construction one can find in Appendix 3;

- u gives a KK-cycle $(\mathcal{H}_{C(P\times_B\hat{P},\hat{j}^*\hat{\mathcal{A}})}, u, 0)$ in $KK^0(C(P\times_B\hat{P}, j^*\mathcal{A}), C(P\times_B\hat{P}, \hat{j}^*\hat{\mathcal{A}}))$. We denote the KK-cycle by $[u]$;

- $\hat{j}_!$ is defined by a KK-cycle $(\mathcal{P}, \phi, P) \in KK^n(C(P\times_B\hat{P}, \hat{j}^*\hat{\mathcal{A}}), C(\hat{P}, \hat{\mathcal{A}}))$ in [11]. Here $\mathcal{P}$ is a family of Hilbert spaces $\mathcal{H}_p = L^2(\hat{j}^{-1}(p), S_p)$ parameterized by P and S_p is the Spin^c structure over $\hat{j}^{-1}(p)$, ϕ is the pointwise action of $C(\hat{P}, \hat{A})$ on $\mathcal{P}$ and P is the operator determined by the family of Dirac operators $D_{\hat{j}^{-1}(p)}$. And we denote this KK-cycle by $[\hat{j}_!]$.

Given the above constructions, we know that the intersection product of the three KK-cycle gives a KK-cycle in $KK^n(C(P,\mathcal{A}), C(\hat{P},\hat{\mathcal{A}}))$, which we denote by $[T_B]$ below.

In [11], he defines a KK-cycle (which he calls the Thom element)

$$t_\alpha = (A \rtimes_\alpha \mathbb{R}, id, F_f), \tag{3.32}$$

which gives an element of $KK^1(A, A \rtimes_\alpha \mathbb{R})$ and corresponds to the Connes-Thom isomorphism. Here f is a continuous $\mathbb{C}$-valued function on $\mathbb{R}$ for which $lim_{t\to+\infty}f(t) = 1$ and $lim_{t\to-\infty}f(t) = -1$ and $F_f \in M(A \rtimes_\alpha \mathbb{R})$ is called a Thom operator on $A \rtimes_\alpha \mathbb{R}$. For general cases, we can first construct the Thom element in $KK^1(A \rtimes_\alpha \mathbb{R}^i, A \rtimes_\alpha \mathbb{R}^i \rtimes \mathbb{R})$ one by one as above and then their intersection product gives a KK^1-cycle which gives an element in $KK^n(A, A \rtimes \mathbb{R}^n)$. The isomorphism in Theorem 3.7 of [62] also gives a KK^0-cycle $[S]$. And the intersection product of the Thom element and $[S]$ gives another KK^n-cycle $[T_B]'$.

Similarly, if we consider the T-duality isomorphism in Section 12 of [22], which is given by the composition of the Connes-Thom isomorphism, generalized Mackey-Rieffel imprimitivity (which gives a Morita equivalence of C^*-algebra) and Pontryagin duality (which gives an isomorphism of C^*-algebra), then we get another KK^n-cycle $[T_B]''$.

Now we prove that the KK-elements $[T_B]$, $[T_B]'$ and $[T_B]''$ that we construct above are all invertible KK-elements. According to the Universal Coefficient Theorem [11] for any C^*-algebras A, B which belong to some special class of C^*-algebra N,we have the following exact sequence:

$$0 \to Ext_1(A, B) \to KK^*(A, B) \to Hom(K^*(A), K^*(B)) \to 0 \tag{3.33}$$

Here N is the class of C^*-algebras which satisfy the following conditions

- **N1** N contains $\mathbb{C}$;

- **N2** N is closed under countable inductive limits;

- **N3** If
$$0 \to A \to D \to B \to 0$$
is an exact sequence, and two of the terms are in N, then so is the third;

- **N4** N is closed under KK-equivalence.

Remark 3.18. According to [11], N is essentially the class of C^*-algebras which are KK-equivalent to the commutative C^*-algebras.

Proposition 3.19. The KK-elements $[T_B]$, $[T_B]'$ and $[T_B]''$ are all invertible KK-elements.

Proof. The proof is clear from the following proposition and lemma. □

We state the following proposition without proof here. The proof is in [11].

Proposition 3.20. Let A and B belong to N', and $x \in KK(A, B)$ with the property that $\gamma(x)$ is an isomorphism in $Hom(K(A), K(B))$, then x is a KK-equivalence.

Here N' is the class of C^*-algebras A for which the Universal Coefficient Theorem holds for (A, D) for any D. Clearly, we have N' contains N.

Remark 3.21. Using the same method in the proof of the proposition above we can get that if $x \in KK(A, B)$ and $\gamma(x)$ is injective (surjective) in $Hom(K^*(A), K^*(B))$, then x has an left (right) inverse.

Now we prove the following lemma:

Lemma 3.22. *N contains all of the continuous trace C^*-algebras.*

Unfortunately we didn't find a direct way to prove this. But according to Theorem 6.1.11 in [50] we know that continuous trace C^*-algebras are all type I C^*-algebra and such algebras satisfies the Universal Coefficient Theorem [60].

Now we turn to the uniqueness of the KK-element which corresponds to T-duality isomorphism . Similarly, we have the following KK version of our main theorem.

Theorem 3.23. *For each object $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}))$ in* $\mathbf{T-pair}_n$*, there exists a unique element $[T_B] \in KK^n(C(P, \mathcal{A}), C(\hat{P}, \hat{\mathcal{A}}))$ which satisfies that :*

- *If the base space of the T-duality pair is a point, then the corresponding KK-element $[T^n_{pt}]$ satisfies*
$$[T^n_{pt}] = \prod_{i=1}^{n} [T^1_{pt}]_i, \tag{3.34}$$
in which $[T^1_{pt}]_i \in KK^1(C(S^1), C(S^1))$ and the $\prod$ is the intersection product;

- *Let $f : X \to Y$ be a continuous map and we use the notations in Section 3.1, then we have:*

$$[\hat{F}^*] * [T_Y] = [T_X] * [F^*];$$

- *For any spaces B and X, $G \in H^{(}X, \mathbb{Z})$ we have*

$$[T_{B\times X}] = [T_B] \otimes [Id_{C(X,G)}].$$

Proof. The idea of the proof here is similar to the proof of the main theorem above. First of all, we get the following identity from the second axiom

$$[\hat{F}^*] * [T_B] = [T_{B'}] * [F^*]. \tag{3.35}$$

By (3.21), as $\hat{F}^*$ is injective, $[\hat{F}^*]$ has a left inverse $[\gamma]$. Then we have

$$[T_B] = [\gamma] * [T_{B'}] * [F^*]. \tag{3.36}$$

By the third axiom, we have

$$[T_{B'}] = [T_*] \otimes [Id_{B'}]. \tag{3.37}$$

Therefore,

$$[T_B] = [\gamma] * ([T_*] \otimes [Id_{B'}]) * [F^*]. \tag{3.38}$$

Then we get the uniqueness of $[T_B]$. □

3.5 2-Fold Composition of T-duality Isomorphisms

For a T-duality pair $((P, \mathcal{A}), (\hat{P}, \hat{\mathcal{A}}))$ in $\mathbf{T-pair}_n$, we construct a T-duality isomorphism T_B from $K^*(P, \mathcal{A})$ to $K^{*+n}(\hat{P}, \hat{\mathcal{A}})$. Similarly, we can also construct a T-duality isomorphism T'_B from $K^*(\hat{P}, \hat{\mathcal{A}})$ to $K^{*+n}(P, \mathcal{A})$. If we compose these two maps we get an isomorphism from $K^*(P, \mathcal{A})$ to itself. The construction of the composition can be showed in the following diagram.

$$\begin{tikzcd}
 & P \times_X \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & & \hat{P} \times P \arrow[dl, "\hat{j}"'] \arrow[dr, "j"] & \\
P \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \arrow[dr, "\hat{\pi}"'] & & P \arrow[dl, "\pi"] \\
 & B & & B &
\end{tikzcd}$$

As we have pointed out in Section 1.1, in [12] they stated that the composition of the two T-duality isomorphism above is the identity map. However, they didn't give a solid proof. In this section we use the similar method which we used in this chapter before to prove that the composition of two T-duality isomorphisms on twisted K-theory is indeed identity.

Proposition 3.24. For any T-duality isomorphism T which satisfies the axioms in section 2, we have

$$T_B' \circ T_B = id_{K^*(P,\mathcal{A})}. \tag{3.39}$$

The above proposition is implied by the following theorem.

Theorem ((0.4))**.** *For each object $(P,\mathcal{A})$ in the category **Pair**$_n$, there exists a unique isomorphism $\tau_B : K^*(P,\mathcal{A}) \to K^*(P,\mathcal{A})$ (here B is the base space of P) which satisfies the axioms below.*

- *(**Axiom 1**) When B is a point, $\tau_{pt} = Id$;*
- *(**Axiom 2**) If there is a map $l : X \to B$, then $L^* \circ \tau_B = \tau_X \circ L^*$. Here $L : l^*P \to P$ is the map induced by l;*
- *(**Axiom 3**) If we do product with S^1 for both entry in the T-duality triple, we get another isomorphism $\tau_{B\times S^1}$ for the new pair $(P \times S^1, i^*(\mathcal{A}))$. It satisfies*

$$\tau_{B\times S^1} = \tau_B \otimes Id_{K^*(S^1)}. \tag{3.40}$$

In particular, we get the 2-fold composition of T-isomorphism of twisted K-group is the identity map.

Proof. Clearly, if we just choose τ to be the identity map all the time, the axioms are satisfied automatically. Now we prove the uniqueness part. Use the notations in section 3.1. For any τ satisfying the above axioms, we can pullback the T-duality diagram using the map $f : B' \to B$. By Lemma 3.10 , we have that $F^* : K^*(P,\mathcal{A}) \to K^*(P', F^*(H))$ is injective and we still denote its left inverse by $(F^*)_l^{-1}$ By **Axiom 2**, we have

$$F^* \circ \tau_B = \tau_{B'} \circ F^*$$

or equivalently, we can write

$$\tau_B = (F^*)_l^{-1} \circ \tau_{B'} \circ F^*.$$

Since all of the principal S^1-bundles in the pullback T-duality diagram are trivial, we can use **Axiom 3** to get

$$\tau_{B'} = \tau_* \otimes Id = Id.$$

Finally we get that τ_B equals to the identity map. □

Now we give the proof of Proposition 3.24.

Proof. Let $\tau_B = T_B \circ T'_B$, we first show that $\tau_* = Id$. This is true as the first axiom in section 2 implies that τ_* maps the positive generators to themselves. For **Axiom 2**, for any map $l : X \to B$ we have :

$$L^* \circ \tau_B = L^* \circ T_B \circ T'_B = T_B \circ \hat{L}^* \circ T'_B = T_B \circ T'_B \circ L^* = \tau_B \circ L^*.$$

For the product formula, since $T_{B\times S^1} = T_B \circ Id_{K^*(S^1)}$ and $T'_{B\times S^1} = T'_B \circ Id_{K^*(S^1)}$, therefore we have

$$\tau_{B\times S^1} = T_{B\times S^1} \circ T'_{B\times S^1} = (T_B \otimes Id_{K^*(S^1)}) \circ (T'_B \otimes Id_{K^*(S^1)}) = \tau_B \otimes Id_{K^*(S^1)}.$$

□

We can also use KK-elements to represent T-isomorphisms and get a KK-version of the above proposition.

Proposition 3.25. Use the notations in Section 3.4. And we denote the KK-element corresponding to T'_B by $[T'_B]$. Then we get

$$[T_B] * [T'_B] = 1 \in KK^0(C^*(P, H), C^*(P, H)).$$

Proof. The proof is similar to the proof above. We first use the language of KK-theory to give a similar list of axioms below,

- (**Axiom 1**) When B is a point, $[\tau_{pt}] = [Id] \in KK^0(C^*(P, H), C^*(P, H))$;
- (**Axiom 2**) If there is a map $l : X \to B$, then $[L^*] * [\tau_B] = [\tau_X] * [L^*]$. Here $L : l^*P \to P$ is the map induced by l;
- (**Axiom 3**)If we do the product with M at every element of a T-duality diagram over B, we get another T-duality diagram over $B \times M$, and

$$[\tau_{B\times S^1}] = [\tau_B] * [Id_{K^*(S^1)}]. \tag{3.41}$$

then we show that the KK-elements satisfying these axioms and unique, finally we show that $[T_B] * [T'_B]$ satisfies these axioms. We skip these procedures here since we can just replace the homomorphisms by the corresponding KK-elements in the above proof and get the proof here. □

3.6 Further discussion

In [15]J. Brodzki, V. Mathai, J. Rosenberg and R. Szabo give a definition of K-theoretic T-duality as follows:

Definition 3.26. Let $\underline{\Im}$ be a suitable category of separable C^*-algebras, possible equipped with some extra structure(such as the $\mathbb{R}^n$-action above). Elements of $\underline{\Im}$ are called T-dualizable algebras, with the following properties:

1. There is a covariant functor $T : \underline{\Im} \to \underline{\Im}$ which sends an algebra A to an algebra $T(A)$ called its T-dual;
2. There is a functorial map $A \mapsto \gamma_A \in KK_n(A, T(A))$ such that γ_A is a KK-equivalence;
3. The pair $(A, T(T(A)))$ are Morita equivalent to each other, and moreover the Kasparov product $\gamma_A \otimes_{T(A)} \gamma_{T(A)}$ is the KK-equivalence associated to this Morita equivalence

For the first property, in Bunke-Schick model as the object are topological spaces with twists, the functor T becomes a covariant functor. In other models the first property is satisfied. In the final part of Section 3.5 we prove a stronger version of the third property, i.e., $\gamma_A \otimes_{T(A)} \gamma_{T(A)}$ is not only a Morita equivalence but actually the identity element.
In the proof of the main theorem, we only use the condition that T_B is always an isomorphism for any B. Therefore it is natural to consider that this proof can be generalized to another twisted (co)homology theory which are T-admissible(see [16]), such as twisted de-Rham cohomology. If we check the proof of Theorem 0.2, then we get the fact that T is an isomorphism is the only essential factor here. Therefore we get that for any T-admissible twisted cohomology theory h, we have the following theorem

Theorem 3.27. *Let h be a T-admissible twisted cohomology theory which is defined over* ***Pair**$_n$*. *Then for each object in the category* $\mathbf{T-pair}_n$ *, there exists a unique T-duality* *isomorphism for h which satisfies the following axioms i.e. for any space B and any T-duality pair* $\mathbf{D}$ *over B, there is a unique way to assign a T-duality isomorphism between the corresponding h-groups of T-duality pairs in* $\mathbf{D}$ *such that the following axioms are satisfied.*

- ***Axiom 1*** *When the base space is a point , a given isomorphism on* $h^*(*)$ *determines* T_{pt}*;*
- ***Axiom 2*** *If* $g : X \to Y$ *is a continuous map, and we pullback the* T*−duality diagram over Y to X, the T-duality isomorphisms* T_X *and* T_Y *satisfy the following naturality condition:*
$$T_X \circ F^* = \hat{F}^* \circ T_Y; \tag{3.42}$$
- ***Axiom 3*** *Given a T-duality diagram like* (0.1)*, we can get another T-duality diagram by doing product with* S^1 *for each element as follows*

$$\begin{tikzcd}
 & P \times_B \hat{P} \times S^1 \arrow[dl, "j \times id_{S^1}"'] \arrow[dr, "\hat{j} \times id_{S^1}"] & \\
P \times S^1 \arrow[dr, "\pi \times id_{S^1}"'] & & \hat{P} \times S^1 \arrow[dl, "\hat{\pi} \times id_{S^1}"] \\
 & B \times S^1 &
\end{tikzcd}$$

Then we have that:
$$T_{B\times S^1} = T_B \otimes Id_{K^*(S^1)}. \tag{3.43}$$

If we think a little bit on the first axiom in Theorem 0.2, we may wonder if there exists a T-duality isomorphism for twisted K-groups which maps the positive generator of $K^*(S^1)$ to the negative generator of $K^{*-1}(S^1)$. It turns out to be possible. In the construction of the changing twist map u for point case, we can replace the Poincare line bundle L by its adjoint bundle L^* whose first Chern class is $-c_1(L)$ and define a changing twist map u^*. Then we can get another T-duality transformation

$$T^* := \hat{j}_! \circ u^* \circ j^* : K^*(S^1) \to K^{*-1}(S^1). \tag{3.44}$$

A similar calculation in Lemma 3.2 leads us to the following lemma.

Lemma 3.28. *Use the notions in Lemma 3.2. Then we have*

$$T^*(e_0) = -e_1, \;\; T^*(e_1) = e_0.$$

Proof. We still first compute how u^* works on twisted de-Rham cohomology. Use the notions in the proof of Lemma 3.2 we get that

$$u^*(1) = 1 - \theta \cup \hat{\theta}, \; u^*(\theta) = \theta,$$

which implies that

$$u^*(e_0 \otimes \hat{e}_0) = e_0 \otimes \hat{e}_0 - e_1 \otimes \hat{e}_1, \;\; u^*(e_1 \otimes \hat{e}_0) = e_1 \otimes \hat{e}_0.$$

Therefore we have

$$\begin{aligned} T^*(e_0) &= \hat{j}_! \circ u^*(e_0 \otimes \hat{e}_0) = -\hat{e}_1, \\ T^*(e_1) &= \hat{j}_! \circ u^*(e_1 \otimes \hat{e}_0) = \hat{e}_0. \end{aligned}$$

□

The above lemma implies that

$$\begin{aligned} T^* \circ T^*(e_0) &= -e_0, & T^* \circ T^*(e_1) &= -e_1, \\ T^* \circ T^* \circ T^* \circ T^*(e_0) &= e_0, & T^* \circ T^* \circ T^* \circ T^*(e_1) &= e_1. \end{aligned}$$

If we generalized the above construction of T^* to each object in $\mathbf{T-pair}_1$, we can get a T-duality isomorphism which satisfies the following axioms:

- **Axiom 1'** When the base space is a point , T_{pt} satisfies the following equalities:

  $$T_{pt}(e_0) = -e_1, T_{pt}(e_1) = e_0. \tag{3.45}$$

  Here e_0 and e_1 are the positive generators of $K^0(S^1)$ and $K^1(S^1)$ respectively.

- **Axiom 2** If $g : X \to Y$ is a continuous map, and we pullback the $T-$dual diagram over Y to X, the T isomorphisms T_X and T_Y satisfy the following naturality condition:

  $$T_X \circ F^* = \hat{F}^* \circ T_Y, \tag{3.46}$$

- **Axiom 3** Given a T-duality diagram like (0.1). We can get another T-duality diagram by doing product with S^1 for each element in the diagram as follows,

$$
\begin{tikzcd}
& P \times_B \hat{P} \times S^1 \arrow[dl, "j \times id_{S^1}"'] \arrow[dr, "\hat{j} \times id_{S^1}"] & \\
P \times S^1 \arrow[dr, "\pi \times id_{S^1}"'] & & \hat{P} \times S^1 \arrow[dl, "\hat{\pi} \times id_{S^1}"] \\
& B \times S^1 &
\end{tikzcd}
$$

Similarly, we can get analog results of Theorem 0.2 and Proposition 3.24.

Theorem 3.29. *There exists a unique T-duality satisfies **Axiom 1', Axiom 2** and **Axiom 3** above for each object in the category of* $\mathbf{T-pair}_1$.

Theorem 3.30. *The 4-fold composition of the T-duality isomorphism T^* in Theorem 3.29 is identity.*

Without any difficulty, we can generalize Theorem 3.29 and Theorem 3.30 to more general category $\mathbf{T-pair}_n$. For simplicity, we will not do it here.
In [49] A. Pande constructed a T-duality model for the semi-free S^1-actions on manifold. In [45], they gave a model of T-duality for general smooth S^1-action on manifold. In Chapter 5 we will use C. Daenzer's approach to describe topological T-duality for general smooth S^1-actions on smooth manifolds. Therefore another future problem is that we can consider the uniqueness of T-isomorphism for these more general version of T-duality isomorphism, which we will discuss in the next chapter.

4 T-duality for Circle Actions

In this chapter we generalize our discussions of topological T-duality to manifolds with proper S^1-actions. If an S^1-action is free, then it degenerates to the case we discuss before. While if the action is not free, it becomes complicated to deal with. Non-free cases are also important in mathematics physics since it corresponds to some singularies or monopoles of space-time manifolds in physics. There have been several results on this topic and we start from the review and comparison of them. After that we move on and discuss other possible approaches. In this chapter, we assume all of the group actions are proper.

4.1 Mathai and Wu's Construction

In this section we give a rough introduction to the construction of V.Mathai and S.Y.Wu. Their paper([45]) deals with the T-duality for a manifold with an S^1-action(which can be not free). The idea is to use the Borel construction to transform non-free cases to free cases which we have understood very well. Instead of cohomology theory or K-theory they use S^1-equivariant cohomology or S^1-equivariant K-theory to describe the T-duality isomorphism.

Let X be a connected manifold which admits an S^1-action and $H \in \Omega^3_c(X)$ be a closed 3-form whose corresponding class lies in the image of $H^3(X, \mathbb{Z})$ in $H^3_{dR}(X)$. In addition we require that H is preserved by the S^1-action (the S^1-invariance is essential in Mathai and Wu's constuction since they use this to construct the equivariant twisted cohomology groups). If we consider the natural S^1-action over $X \times ES^1$ induced by the previous S^1-action, then we get a principal S^1-bundle $\pi : X \times ES^1 \longrightarrow X_{S^1}$. We denote the first Chern class of this principal S^1-bundle by e_{S^1}. Then according to Proposition 4.1 there exists an equivariant principal S^1-bundle over X whose equivariant first Chern class $c_1^{S^1}(\hat{X})$ equals to $\pi_!([H])$.

Proposition 4.1 ([36])**.** Let X be a connected S^1-manifold. The equivariant first Chern class $c_1^{S^1}$ gives rise to a one-to-one correspondence between equivalence classes of S^1-equivariant circle bundles over X and elements of $H^2_{S^1}(X, \mathbb{Z})$.

According to the above arguments, we get a diagram as follows which is quite similar to the classical one in topological T-duality:

$$\begin{tikzcd} & \hat{X} \times S^\infty \arrow[dl, "p \times id"'] \arrow[dr, "\hat{p}"] & \\ X \times S^\infty \arrow[dr, "\pi"'] & & \hat{X}_{S^1} \arrow[dl, "\hat{\pi}"] \\ & X_{S^1} & \end{tikzcd} \tag{4.1}$$

The Gysin sequence of equivariant cohomology can give us a background flux $\hat{H}$ over $\hat{X}_{S^1}$ which is uniquely determined by the following two conditions:

$$\hat{\pi}_!([\hat{H}]) = e_{S^1} \in H^2_{S^1}(X, \mathbb{Z});$$

$$(p \times id)^*([H]) = \hat{p}^*([\hat{H}]) \in H^3(\hat{X}, \mathbb{Z}).$$

Using these constructions, Mathai and Wu get the following results in their paper:

Theorem 4.2 ([45])**.** *Let X be a connected smooth manifold and H be a background H-flux i.e. H be a 3-differential form with integral period. Moreover, there is a smooth S^1-action over X which preserves H. Then there exists a T-duality pair $(\hat{X}, \hat{H})$ of (X, H) such that*

$$c_1^{S^1}(\hat{X}) = \pi_*([H]), \ \ \hat{\pi}_*([\hat{H}]) = e_{S^1}. \tag{4.2}$$

Here $\hat{X}$ is an S^1-equivariant principal S^1-bundle $\hat{X}$ over X and $\hat{H}$ is a closed 3-form with integral periods and invariant under the S^1-action (which is induced by the S^1-action over X) over $\hat{X}$. Moreover there exists a T-duality isomorphism between the twisted equivariant cohomology groups:

$$H^\bullet(X, H) \cong H^{\bullet+1}_{S^1}(\hat{X}, \hat{H}). \tag{4.3}$$

Remark 4.3. Mathai and Wu discussed T-duality isomorphism of twisted K-theory via the Connes-Thom isomorphism for σ-C^*-algebra. They get that $K^*(X, H) \cong K^*_{S^1}(\hat{X}, \hat{H})^\sim$.

Here $K^*_{S^1}(\hat{X},\hat{H})^\sim$ is the $I(S^1)$-adic completion of $K^*_{S^1}(\hat{X},\hat{H})$. According to the Atiyah-Segal completion theorem for twisted K-theory in ([37]) they get :

$$K^\bullet_{S^1}(\hat{X},\hat{H})^\sim \cong RK^\bullet(\hat{X}_{S^1},\hat{H}). \tag{4.4}$$

From this isomorphism we can see that we get a similar T-duality isomorphism of twisted K-theory. The difference is that the T-dual space X_{S^1} is infinite dimensional. Therefore twisted representable K-theory is used instead of twisted K-theory. In the next section we will deal with this situation in another approach.

4.2 T-duality for infinite CW-complexes

In this section we give a construction of topological T-duality for countable infinite dimensional CW-complexes. In particular, this implies the case in the last section and helps us to understand some relations between different (twisted) K-groups.

4.2.1 Milnor's Exact Sequence for Twisted K-groups

We first revisit the definition of twisted K-theory given by Atiyah ([3]).

Definition 4.4. Each infinite CW-complex X and $H \in H^3(X,\mathbb{Z})$ determines a unique (up to isomorphism) principal $PU(\mathcal{H})$-bundle $\mathfrak{P}$ over X (see Appendix A.1). Consider the associated bundle $Fred(\mathfrak{P})$ of $\mathcal{P}$ with fiber $Fred(\mathcal{H})$, we define $K^0(X,H)$ to be the set of homotopy classes of continuous sections of $Fred(\mathfrak{P})$. Actually, this definition works for any space X. For higher degree twisted K-groups, we can choose iterated loop spaces $\Omega^n_X Fred(\mathcal{H})$ and the rest is similar. Let $U \subset X$. We can define the relative twisted K-group $K^0(X,U;H)$ to be the homotopy classes of sections of $Fred(\mathcal{P})$ relative to U. The higher degree twisted K-groups are defined similarly.

Remark 4.5. In [3] they showed that the above definition agrees with the algebraic definition using C^*-algebra when X is compact. However, when X is not compact, the continuous functions over X do not form a C^*-algebra. If X is countable infinite CW-complex, algebraists can use the representable K-theory of σ-C^*-algebra to give an alternative definition of twisted K-theory. However, the push-forward map for the representable K-theory is still not well understood yet. It will be interesting to investigate the analogous representable KK-theory for σ-C^*-algebras. We will not discuss this in this thesis. Therefore we use the topological definition of twisted K-theory above in this section.

For this version of twisted K-groups, we have the following properties.

Proposition 4.6. Let $\{X_n\}_{n=1,2...}$ be a sequence of CW complexes, and $H_n \in H^3(X_n, \mathbb{Z})$. Let $X = \sqcup X_n$ and $H = \sum i_n^*(H_n)$. Then we have σ-additivity property, i.e., the inclusions i_n induce an isomorphism

$$K^*(X, H) \cong \prod K^*(X_n, H_n). \tag{4.5}$$

Proof. For each $i_n : X_n \hookrightarrow X$, we denote the corresponding associated bundles by $Fred(\mathcal{P})_n$ and $Fred(\mathcal{P})$ respectively. For every section $T \in Fred(\mathcal{P})$, $(i_n)^*(T)$ gives a section of $Fred(\mathcal{P})_n$. Therefore $(i_n)^*$ gives a homomorphism from $K^*(X, H)$ to $K^*(X_n, H_n)$. We then show that $\prod(i_n)^*$ is surjective and injective.

- For every element x_n in $K^*(X_n, H_n)$, we can represent it by a section $s_n \in \Gamma(Fred(\mathcal{P})_n) = \Gamma((i_n)^*(Fred(\mathcal{P})))$. Therefore there is a section $s \in \Gamma(Fred(\mathcal{P}))$ such that $s_n = (i_n)^*(s)$ i.e. $(i_n)^*([s]) = x_n$.

- For any element $x \in K^*(X, H)$, if $i_n(x)$ are trivial for all n and x is not trivial, then the representative section of x must be not homotopic to trivial section over at least one component X_k. However, this implies that $(i_k)^*(x)$ is not trivial. Contradiction!

□

Theorem 4.7. *Let X be any countable infinite CW-complex , $H \in H^3(X, \mathbb{Z})$ and U, V are two subcomplexes of X whose interiors cover X. Denote $H_U = i_U^*(H)$ and $H_V = i_V^*(H)$. Then we have the following six-term exact sequence:*

$$\begin{tikzcd}
K^1(U \cap V, H_{U\cap V}) \arrow[r, "\delta"] & K^0(X, H) \arrow[r, "i_U^* \oplus i_V^*"] & K^0(U, H_U) \oplus K^0(V, H_V) \arrow[d, "j_U^* - j_V^*"] \\
K^1(U, H_U) \oplus K^1(V, H_V) \arrow[u, "j_U^* - j_V^*"] & K^1(X, H) \arrow[l, "i_U^* \oplus i_V^*"'] & K^0(U \cap V, H_{U\cap V}) \arrow[l, "\delta"']
\end{tikzcd}$$

Proof. We first give the definitions of the two boundary maps. The boundary map $\delta : K^1(U \cap V, H_{U\cap V}) \to K^0(X, H)$ is defined as follows. Denote $X_1 = X/U^\circ$, $X_2 = X/V^\circ$. By Urysohn's lemma we can choose a continuous function $\phi : X \to [0, 1]$ such that ϕ is 0 over X_1 and 1 over X_2. For any section s over $\Omega Fred(\mathcal{P})$, we can define δs to be $s(x)$ evaluated over $\phi(x)$ over a point $x \in U \cap V$. For the other points in X we define $s(x)$ to be the based point. The boundary map $\delta : K^0(X, H) \to K^1(U \cap V, H_{U\cap V})$ is defined similarly by the Bott periodicity $K^0(X, H) \cong K^2(X, H)$. Now we show the exactness of the diagram.

- we first show the exactness at $K^0(U, H_U) \oplus K^0(V, H_V)$. Obviously $im(i_U \oplus i_V) \subset ker(j_U - j_V)$. For any $(x_1, x_2) \in K^0(U, H_U) \oplus K^0(V, H_V)$, if $j_U x_1 = j_V x_2$. Then we have $(x_1)_{U\cap V} = (x_2)_{U\cap V}$. We choose continuous sections s_1 and s_2 to represent x_1 and x_2 respectively. Since $j_U x_1 = j_V x_2$, we get that $s_1|_{U\cap V}$ and $s_2|_{U\cap V}$ are homotopic to each other i.e. there exists a homotopy $g : U \cap V \times [0, 1] \to Fred(\mathfrak{P})$ between s_1 and s_2. Since $U \cap V$ is a sub-complex of U, so there exists a homotpoty $G : U \times [0, 1] \to Fred(\mathfrak{P})$ such that $G(x, t) = g(x, t)$ when $x \in U \cap V$. Moreover, $G(x, 0) = s_1(x)$. $G(x, 1)$ gives another section which represents x_1 and $G(x, 1)|_{U\cap V} = (s_2)|_{U\cap V}$. We glue $G(x, 1)$ and s_2 together we get a section $s : X \to Fred(\mathfrak{P})$. Denote the corresponding class of s by x. Then we have $(i_U)^* \oplus (i_V)^*(x) = (x_1, x_2)$. So we get the exactness. The proof of exactness at $K^0(U, H_U) \oplus K^0(V, H_V)$ is similar;

- Then we show the exactness at $K^0(X, H)$. It follows from the definition of δ that $(i_U \oplus i_V) \circ \delta$ is trivial since the elements in the image of δ are always homotopic to trivial section when restricted to U or V. Conversely, if $i_U x = 0$ and $i_V x = 0$, i.e. the representing section s of $x \in K^0(X, H)$ is homotopic to trivial section when restricted to U or V, then there are two homotopies from $s(p)$ to Id. For the given map $\phi : X \to [0, 1]$, we compose the two homotopies to form a loop $l(p)$ at p such that $l(p)(\phi(p)) = s(p)$. The family of $l(p)$s gives a section l of $\Omega Fred(\mathcal{P})_{U\cap V}$ and we see that $\delta[l] = x$. The exactness of $K^1(X, H)$ can be proved similarly;

- The rest is the exactness at $K^i(U \cap V, H_{U\cap V})$ for $i = 0, 1$. We still only give the proof for $i = 1$ and the proof of the other case is similar. From the definition of δ we get that $\delta \circ j_U$ and $\delta \circ j_V$ are both trivial since the restriction of sections over $\Omega Fred(\mathcal{P})_U$ and $\Omega Fred(\mathcal{P})_V$ over $U \cap V$ can be both homotopic to trivial section. For the converse part, if δx is trivial, then there exist a trivial extension of the representing section $s \in \Omega Fred(\mathcal{P})_{U\cap V}$ to a section of $\Omega Fred(\mathcal{P})_U$ or $\Omega Fred(\mathcal{P})_V$. Therefore we have $x \in im j_U$ or $x \in im j_V$.

□

Similarly, we can get the Mayer-Vietoris sequence for relative twisted K-groups.

Theorem 4.8. *Use the notations in the above theorem. Moreover, let Y be a subspace of X and $A \subset U$, $B \subset V$ be two subspaces of Y whose interiors cover Y. Then we have the* *following diagram:*

$$\begin{tikzcd}
K^1(U\cap V, A\cap B; H_{U\cap V}) \arrow[r, "\delta"] & K^0(X,Y;H) \arrow[r, "i_U^*\oplus i_V^*"] & K^0(U,A;H_U)\oplus K^0(V,B;H_V) \arrow[d, "j_U^*-j_V^*"] \\
K^1(U,A;H_U)\oplus K^1(V,B;H_V) \arrow[u, "j_U^*-j_V^*"] & K^1(X,Y;H) \arrow[l, "i_U^*\oplus i_V^*"'] & K^0(U\cap V, A\cap B; H_{U\cap V}) \arrow[l, "\delta"']
\end{tikzcd}$$

Theorem 4.9. *Given an infinite CW-complex X and a twisting class $H \in H^3(X,\mathbb{Z})$. We denote the n-skeleton of X to be X_n and the inclusion $X_n \hookrightarrow X$ to be i_n. Let $H_n = i_n^*(H)$. Then we have the following exact sequence*

$$1 \to \varprojlim{}^1 K^{*-1}(X_n, H_n) \to K^*(X,H) \to \varprojlim K^*(X_n,H_n) \to 1 \tag{4.6}$$

Proof. To prove this, we need to use the telescope construction. Denote $telX_n = \cup_n X_n \times [n, n+1]$ with the identifications given by i_ns at the ends of cylinders. Choose $0 < \varepsilon < 1$ and define

$$U = X_0 \times [0,1] \bigsqcup \Big(\bigsqcup_{i\geq 1} X_{2i-1} \times [2i-\varepsilon, 2i] \cup X_{2i} \times [2i, 2i+1]\Big)$$

and

$$V = \bigsqcup_{i\geq 0} X_{2i} \times [2i+1-\varepsilon, 2i+1] \cup X_{2i+1} \times [2i+1, 2i+2].$$

Then we get $C = U \cap V = \bigsqcup_{i\geq 0} X_i \times [i+1-\varepsilon, i+1]$. Since $telX_n$ is homotopic to X, and if we denote the corresponding twisting class over $telX_n$ by H_{tel}, we get $K^i(X,H) \cong K^i(telX_n, H_{tel})$. Use Mayer-Vietoris sequence for $(telX_n, U, V)$ we get the following exact sequence:

$$\begin{tikzcd}
K^1(U\cap V, H_{U\cap V}) \arrow[r, "\delta"] & K^0(telX_n, H_{tel}) \arrow[r, "i_U^*\oplus i_V^*"] & K^0(U,H_U)\oplus K^0(V,H_V) \arrow[d, "j_U^*-j_V^*"] \\
K^1(U,H_U)\oplus K^1(V,H_V) \arrow[u, "j_U^*-j_V^*"] & K^1(telX_n, H_{tel}) \arrow[l, "i_U^*\oplus i_V^*"'] & K^0(U\cap V, H_{U\cap V}) \arrow[l, "\delta"']
\end{tikzcd}$$

Since $K^i(X,H) \cong K^i(telX_n, H_{tel})$, $K^i(U,H_U) \cong \oplus_k K^i(X_{2k}, H_{2k})$, $K^i(V,H_V) \cong \oplus_k K^i(X_{2k+1}, H_{2k+1})$

and $K^i(U \cap V, H_{U\cap V}) \cong \oplus_k K^i(X_k, H_k)$, so we get the above diagram can be rewritten as

$$\begin{array}{ccccc} \oplus_k K^i(X_k,H_k) & \xrightarrow{\alpha_0} & K^0(X,H) & \xrightarrow{\beta_1} & \oplus_k K^i(X_k,H_k) \\ \Phi_1 \uparrow & & & & \downarrow \Phi_0 \\ \oplus_k K^i(X_k,H_k) & \xleftarrow[\beta_1]{} & K^1(X,H) & \xleftarrow[\alpha_1]{} & \oplus_k K^i(X_k,H_k) \end{array}$$

Therefore we get the exact sequence

$$1 \to coker\Phi_1 \xrightarrow{\alpha_0} K^*(X,H) \xrightarrow{\beta_0} Ker\Phi_0 \to 1$$

From the definition of inverse limit we can get $Ker\Phi_i \cong \varprojlim K^i(X_k, H_k)$ and $\varprojlim^1 K^{i-1}(X_k, H_k) \cong Coker\Phi_i$. Therefore we get the Milnor $\varprojlim^1$-exact sequence

$$1 \to \varprojlim^1 K^{*-1}(X_n, H_n) \to K^*(X,H) \to \varprojlim K^*(X_n, H_n) \to 1 \tag{4.7}$$

□

4.2.2 Push-forward Map

In this part we give the construction of push-forward map of twisted K-theory for spin^c-bundles.

Let $\pi : P \to B$ be a bundle with fiber F and F is a finite dimensional spin^c-manifold. Here we assume that B to be a countable infinite CW-complex. Let $\mathcal{H}$ be an infinite separable Hilbert space and $\mathcal{P}$ be a principal $PU(\mathcal{H})$-bundle over B with $H \in H^3(B, \mathbb{Z})$ as its characteristic class (see Appendix A.1) and $Fred(\mathcal{P})$ be the associated bundle of $\mathcal{P}$ with fiber $Fred(\mathcal{H})$. Now we begin to construct the push-forward map $\pi_! : K^*(P, \pi^*(H)) \to K^{*-dimF}(B, H)$.

Assume that the dimension of F is even. For each point $b \in B$, the fiber $\pi^{-1}(b)$ is an even dimensional $spin^c$-manifold. Let S_F be the canonical spinor bundle over $\pi^{-1}(b)$ and $\mathcal{H}_F$ be the Hilbert space $L^2(S_F^+) \oplus L^2(S_F^-)$. Let $D_b : L^2(S_F^{\pm}) \to L^2(S_F^{\mp})$ be the Dirac operator over P_b. Using D_b we can construct a family of Fredholm operators $V_b = \frac{D_b}{\sqrt{1+D_b^*D_b}}$. Let T be an element of $K^0(P, \pi^*(H))$. It can be represented by a section of $\pi^*(Fred(\mathcal{H}))$. More explicitly, if we choose a good open cover $\{U_i\}$ of B which gives a trivialization of $Fred(\mathcal{P})$ (without loss of generality we assume that each U_i is contractible) , then the open cover $\{\pi^{-1}(U_i)\}$ gives a trivialization of $\pi^*(Fred(\mathcal{P}))$. Therefore T can be represented by

$T_i : U_i \times F \to Fred(\mathcal{H})$ and $T_j = (\pi^* g_{ij}) T_i (\pi^* g_{ij})^{-1}$. Here g_{ij} is the transition function of $Fred(\mathcal{P})$. Then the section $\pi_!(T)_i = V_b \otimes Id_{\mathcal{H}_F} + Id_{\mathcal{H}} \otimes T_b : U_i \to Fred(\mathcal{H} \otimes \mathcal{H}_F)$ and satisfy the condition $\pi_!(T)_j = (g_{ij} \times Id)\pi_!(T)_i (g_{ij} \otimes Id)^{-1}$. Therefore they give rise to an element of $K^0(B, H)$, which we define to be $\pi_!(T)$.
For elements in $K^1(P, H)$, we can consider the F-bundle $\pi \times id : P \times \mathbb{R} \to B \times \mathbb{R}$ and use the Thom-Connes isomorphism (for the K-groups of σ C^*-algebras)to transfer $K^1(P, H)$ to $K^0(P \times \mathbb{R}, H)$.
When the dimension of the fiber F is odd, we can consider the $F \times \mathbb{R}$-bundle $P \times \mathbb{R} \to B$ instead.

4.2.3 T-duality Isomorphism

Let $\pi : P \to B$ be a principal S^1-bundle over a countable CW-complex B and let $H \in H^3(P, \mathbb{Z})$. Like the finite CW-complex case, we can construct a T-duality pair $(\hat{\pi} : \hat{P} \to B, \hat{H})$ s.t $\pi_!(H) = c_1(\hat{P})$ and $\hat{\pi}_!(\hat{H}) = c_1(P)$ via Gysin sequence. Therefore we can still get a T-duality diagram like diagram (0.1). The pullback map and the changing twist map can be defined in the same way as finite CW-complexes. We have given the push-forward map construction above. Therefore we can define the T-duality isomorphism :

$$T := \hat{p}_! \circ u \circ p^* : K^*(P, H) \to K^{*-1}(\hat{P}, \hat{H}). \tag{4.8}$$

We denote the n-th skeleton of P by P_n and the associated twisting class over P_n by H_n. By the naturalness of T we get the following commutative diagram from the $\varprojlim^1$-exact sequence of twisted K-groups.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \varprojlim^1 K^{i-1}(P_n, H_n) \arrow[r] \arrow[d, "T_n"'] & K^i(P, H) \arrow[r] \arrow[d, "\mathrm{T}"'] & \varprojlim K^i(P_n, H_n) \arrow[r] \arrow[d, "T_n"'] & 1 \arrow[d] \\
1 \arrow[r] & \varprojlim^1 K^i(\hat{P}_n, \hat{H}_n) \arrow[r] & K^{i-1}(\hat{P}, \hat{H}) \arrow[r] & \varprojlim K^i(P_n, H_n) \arrow[r] & 1
\end{tikzcd}$$

We have already know that T_n are isomorphisms for each n. Therefore by the five lemma we get the following theorem:

Theorem 4.10. *With the notations above, T defined by 4.8 is a natural isomorphism.*

We can see that the T-duality diagram 4.1 is a special case of the above theorem.

$$\begin{tikzcd} & (P \times S^{\infty}) \times_B \hat{P} \arrow[dl, "j"'] \arrow[dr, "\hat{j}"] & \\ P \times S^{\infty} \arrow[dr, "\pi"'] & & \hat{P} \arrow[dl, "\hat{\pi}"] \\ & B & \end{tikzcd} \tag{4.9}$$

Therefore we get the following theorem.

Theorem 4.11. *Let P be a countable infinite CW-complexes and there is a proper S^1-action over P. Let H be a third integral cohomology class over P. Then we can get the diagram* (4.9) *which gives a pair of principal S^1-bundles over the quotient space B. Moreover, we get the associated T-duality transformation $T := \hat{p}_! \circ u \circ p^*$ gives an isomorphism between $K^*(P, H)$ and $K^{*-1}(\hat{P}, \hat{H})$.*

Remark 4.12. We can also use Bunke-Schick construction to give the definition of T-duality pairs here and even extended the above construction to higher dimensional cases. From the viewpoint of category, we have extended T-duality pairs from the finite CW-complexes to countable infinite CW-complexes.

Until now we have constructed topological T-duality for principal S^1-bundles over countable CW-complexes, we will generalize these to higher dimensional torus bundles in this part. We still use the notion of T-dual triple here. In this part we will always assume that the spaces are countable CW-complexes.

Definition 4.13. An n-dimensional T-duality over a space B triple is a triple

$$((P, H), (\hat{P}, \hat{H}), u)$$

consisting of T^n-bundles $\pi : P \to B$, $\hat{\pi} : \hat{P} \to B$, where the twists H and $\hat{H}$ lies in the second filtration step of the Leray-Serre spectral sequence filtration and their leading parts satisfy

$$[H]^{2,1} = [\sum_{i=1}^{n} y_i \otimes \hat{c}_i] \in^{\pi} E_{\infty}^{2,1} \tag{4.10}$$

and

$$[\hat{H}]^{2,1} = [\sum_{i=1}^{n} \hat{y}_i \otimes c_i] \in^{\hat{\pi}} E_{\infty}^{2,1}, \tag{4.11}$$

respectively, and an isomorphism $u : \hat{j}^*(\hat{H}) \to j^*H$ which satisfy the following condition: When we restrict the T-duality diagram to a point b of B, u is the isomorphism corresponding to $[\sum_{i=1}^{n} y_i \cup \hat{y}_i] \in H^2(\mathbb{T}^n_b \times \hat{\mathbb{T}}^n{}_b, \mathbb{Z})$. Here $y_i \in H^1(\mathbb{T}^n, \mathbb{Z}), \hat{y}_i \in H^1(\hat{\mathbb{T}}^n, \mathbb{Z})$ are respectively the ith generators.

Similarly we can prove the proposition which tells us when a pair (P, H) over B could be extended to a T-duality pair.

Theorem 4.14. *The pair (P, H) admits an extension to a T-duality triple* $((P, H), (\hat{P}, \hat{H}), u)$ *if and only if $H \in \mathcal{F}^2 H^3(P, \mathbb{Z})$.*

Also, we get that for a T-dual triple above, the T-duality isomorphism gives an isomorphism of the associated twisted K-theory, whose proof is similar to the 1-dimensional case.

4.2.4 Uniqueness Theorem

Like the finite CW-complex case, we can also give a uniqueness theorem for the T-duality isomorphism of locally finite CW-complex case. More explicitly, we have the following theorem:

Theorem 4.15. *Let $\mathcal{CW}$ be the category of countable infinite CW-complexes and* $\mathbf{T-pair}_n$ *be the category of T-duality pairs with base spaces in $\mathcal{CW}$. Then there still exists a unique T-duality isomorphism T_* between the corresponding twisted K-groups for each T-duality pair which satisfies the following axioms:*

- *If B is a point, then T_{pt} maps the positive generators of $K^i(\mathbb{T}^n)$ to the positive generators of $K^{i-1}(\mathbb{T}^n)$ as in Theorem 3.27;*
- *If $g : X \to Y$ is a continuous map, and we pullback the $T-$dual diagram over Y to X, the T isomorphisms T_X and T_Y satisfy the following naturality condition:*

$$T_X \circ F^* = \hat{F}^* \circ T_Y; \tag{4.12}$$

- *Given a space M with twisting $G \in H^3(M, \mathbb{Z})$, We can get another $T-$duality dia-*

gram over $* \times M$ as follows:

$$
\begin{tikzcd}
 & S^1 \times_* \hat{S}^1 \times M \arrow[dl, "j \times id_M"'] \arrow[dr, "\hat{j} \times id_M"] & \\
S^1 \times M \arrow[dr, "\pi \times id_M"'] & & \hat{S}^1 \times M \arrow[dl, "\hat{\pi} \times id_M"] \\
 & * \times M &
\end{tikzcd}
$$

Then we have that:

$$T_{*\times M} = T_* \otimes id_M. \tag{4.13}$$

Proof. The proof here is the same as the proof of Theorem 0.2, which we will skip here. □

Similarly, we can also get the following proposition for two fold of T-duality isomorphism.

Proposition 4.16. Use the notion in the above theorem. Then we have $T \circ T$ is the identity map of $K^i(X, H)$ for each T-duality diagram.

4.3 Groupoid Approach

In this section we use groupoid theory to give another description of T-duality of manifolds with general S^1-actions. We have discussed the groupoid approach in Section 1.6. We first give more details about this here.

Assume that X is a compact smooth manifold, $\bar{\alpha} : S^1 \times X \to X$ is a smooth S^1-action over X (which can be lifted to a $\mathbb{R}$-action $\alpha : \mathbb{R} \times X \to X$ over X) and σ is a $U(1)$-valued Čech 2-cocycle over X (which admits a lift $[(\sigma, \lambda, 1)]$ to the equivariant cohomology group $H^*_{\mathbb{R}}(X, U(1))$ (see section 6 of [22]). Denote the action groupoid associated to the S^1-action over X by

$$\mathcal{G} := S^1 \times X \rightrightarrows X. \tag{4.14}$$

Define a map $\bar{\rho} : \mathcal{G} \to S^1$ as $\bar{\rho}(\theta, x) = \theta$. Then we have the following lemma:

Lemma 4.17. *$\bar{\rho}$ is a groupoid homomorphism.*

Proof. $\bar{\rho}((\theta_2, x_2) \circ (\theta_1, x_1)) = \bar{\rho}(\theta_2\theta_1, x_1) = \theta_2\theta_1 = \bar{\rho}((\theta_2, x_2)) \circ \bar{\rho}((\theta_1, x_1))$. □

Therefore we get a generalized principal S^1-bundle $S^1 \rtimes_{\bar{\rho}} \mathcal{G}$ and $(S^1 \rtimes_{\bar{\rho}} \mathcal{G}, (\sigma, \lambda, 1))$ gives the data of the original part in C.Daenzer's approach. As we discussed in the end of Section 1.6, we can use the cocycle condition of $(\sigma, \lambda, 1)$ and the commutativity of S^1 and $\mathbb{R}$ to get that λ induces a homomorphism $\bar{\lambda} : \mathcal{G} \to \hat{\mathbb{Z}}$. Then we do the same constructions in Section 12 of [22] and get the T-dual of $(S^1 \rtimes_{\bar{\rho}} \mathcal{G}, (\sigma, \lambda, 1))$ as follows:

$$(S^1 \rtimes_{\bar{\lambda}} \mathcal{G}, (\sigma^\vee, \rho, 1)).$$

Here $\sigma^\vee \in Z^2(S^1 \rtimes_{\bar{\lambda}} \mathcal{G}, U(1))$ is given by the following formula:

$$\sigma^\vee(\phi, \gamma_1, \gamma_2) = \sigma(e, \gamma_1, \gamma_2)\lambda(\rho(\gamma_1), \gamma_2) < \phi\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) > .$$

Remark 4.18. $< \cdot, \cdot >$ means the Pontryatin duality between $\hat{\mathbb{Z}}$ and $\mathbb{Z}$. The constructions in Section 12 of [22] are quite different from classical topological T-duality. C.Daenzer uses some particular constructions for groupoids such as Pontryagin duality for groupoids (Theorem 8.2 [22]) and generalized Mackey-Riffel imprimitivity (Theorem 11.1 [22])

Similarly he gets a T-duality isomorphism between twisted K-groups of twisted groupoids

$$K^{\cdot}(S^1 \rtimes_{\bar{\rho}} \mathcal{G}, \sigma) \cong K^{\cdot+1}(\hat{\mathbb{Z}} \rtimes \mathcal{G}, \sigma^\vee). \tag{4.15}$$

The twisted K-theory here is defined by the K-theory of the twisted groupoid algebra below.

Definition 4.19. Given a twisted groupoid $(\mathcal{G}, \sigma \in Z^2(\mathcal{G}, U(1)))$, the associated twisted groupoid algebra $C^*(\mathcal{G}, \sigma)$, is the C^*-algebra completion of the compactly supported functions on $\mathcal{G}_1$, with σ-twisted multiplication

$$a * b(\gamma) := \int_{\gamma_1\gamma_2=\gamma} a(\gamma_1)b(\gamma_2)\sigma(\gamma_1, \gamma_2), a, b \in C_c(\mathcal{G}_1) \tag{4.16}$$

and involution $a \mapsto a^*(\gamma) := \overline{a(\gamma^{-1})\sigma(\gamma, \gamma^{-1})}$. Here functions are $\mathbb{C}$-valued and the overline means complex conjugation. It is easy to see that when σ is trivial this definition reduces to the definition of groupoid algebras.

Now we discuss a little bit about the twisting class $(\sigma, \lambda, 1)$ in this groupoid approach. Consider the projection

$$\begin{aligned} \Pi : Z^2_G(G/N \rtimes_\rho \mathcal{G}, U(1)) &\to Z^2(G/N \rtimes_\rho \mathcal{G}, U(1)) \\ (\sigma, \lambda, 1) &\to \sigma. \end{aligned} \tag{4.17}$$

The kernel of Π is $\{\lambda \in C^1(G, C^1(G/N \rtimes_\rho \mathcal{G}, U(1)))| \, d\lambda = \delta\lambda = 1\}$, which is not trivial. In other words, if a 2-cocycle $\sigma \in Z^2(G/N \rtimes_{rho} \mathcal{G}, U(1))$ admits an extension to a twisting triple $(\sigma, \lambda, 1)$, the extension is not unique. In particular, if $G = \mathbb{R}$ and $N = \mathbb{Z}$, and $\mathcal{G}$ is the action groupoid we use in this section, then $d\lambda = 1$ implies

$$\lambda(t_1 + t_2, \theta, \gamma) = (t_1 \cdot \lambda(t_2, \theta, \gamma))(\lambda(t_1, \theta, \gamma)),$$

in which $t_1, t_2 \in \mathbb{R}$, $\theta \in S^1$ and $\gamma \in \mathcal{G}$ i.e.

$$\lambda(t_1 + t_2, \theta, \gamma) = \lambda(t_1, \theta, \gamma)\lambda(t_2, \exp 2i\pi t_1 \cdot \theta, \gamma).$$

Since $\mathbb{R}$ is commutative, therefore we can get:

$$\lambda(t_1 + t_2, \theta, \gamma) = \lambda(t_2, \theta, \gamma)\lambda(t_1, \exp 2i\pi t_2 \cdot \theta, \gamma).$$

The above two identities implies that

$$\frac{\lambda(t_1, \theta, \gamma)}{\lambda(t_1, \exp 2i\pi t_2 \cdot \theta, \gamma)} = \frac{\lambda(t_2, \theta, \gamma)}{\lambda(t_2, \exp 2i\pi t_1 \cdot \theta, \gamma)}. \tag{4.18}$$

If we fix γ, then we get a continuous $U(1)$-valued function

$$f_\lambda(t_1, t_2, \theta) = \frac{\lambda(t_1, \theta, \gamma)}{\lambda(t_1, \exp 2i\pi t_2 \cdot \theta, \gamma)}, \tag{4.19}$$

which satisfies the following properties

1. $f_\lambda(t_1, t_2, \theta) = f_\lambda(t_2, t_1, \theta)$;

2. $f_\lambda(t_1 + z, t_2, \theta) = f_\lambda(t_1, t_2, \theta)$ for any $z \in \mathbb{Z}$.

Clearly, if λ_1, λ_2 are cochains satisfying $d\lambda_1 = d\lambda_2 = 1$, then we have $f_{\lambda_1\lambda_2} = f_{\lambda_1} f_{\lambda_2}$.

4.4 Connections with Mathai and Wu's Construction

We want to construct an equivalence between the groupoid picture and Mathai and Wu's picture. Unfortunately we can't make it here, a reason is that the model of twists given in Daenzer's paper is not well understood. Here we just give part of the project and construct some connections between Mathai and Wu's approach and groupoid approach in this and next section. The idea is to construct the connections between the original part and the T-dual part respectively. The most difficulty part is to connect twisting classes $\sigma^\vee$ and $\hat{H}$. We leave the discussion on this to next section.

First of all, we list the connections we want to construct as follows:

1.
$$S^1 \rtimes_{\bar{\rho}} \mathcal{G} \leftrightarrow M; \tag{4.20}$$

2.
$$\sigma \leftrightarrow H; \tag{4.21}$$

3.
$$S^1 \rtimes_{\bar{\lambda}} \mathcal{G} \leftrightarrow \hat{X} \times ES^1/S^1; \tag{4.22}$$

4.
$$\sigma^\vee \leftrightarrow \hat{H}. \tag{4.23}$$

Before moving on to constructions, we need to introduce some definitions.

Definition 4.20. Let $\mathcal{G}$ be a groupoid. A **left $\mathcal{G}$-module** is a space P with a continuous map $P \xrightarrow{\varepsilon} \mathcal{G}_0$ called the **moment map** and a continuous map:

$$\mathcal{G} \times_{\mathcal{G}_0} P \to P;\ (\gamma, p) \mapsto \gamma p,$$

where the fiber product $\mathcal{G} \times_{\mathcal{G}_0} P \to P$ is the set $\{(\gamma, p)|s(\gamma) = \varepsilon(p)\}$. In addition, for any composed pair $\gamma_1,\ \gamma_2 \in \mathcal{G}$, we need $\gamma_1(\gamma_2 p) = (\gamma_1\gamma_2)p$. The **right $\mathcal{G}$-module** can be defined similarly. The difference is that we will use the fiber product $\mathcal{G} \times_{t,\varepsilon,\mathcal{G}_0} P$ to give the action map. A $\mathcal{G}$-module is called **principle** if the $\mathcal{G}$ action is both free and proper.

Next we come to the notion of Morita equivalence for groupoids.

Definition 4.21. Two groupoids $\mathcal{G}$ and $\mathcal{H}$ are **Morita equivalent** if there exists a **Morita equivalence** ($\mathcal{G} - \mathcal{H}$)-**bimodule** i.e. there exist a principal left $\mathcal{G}$-module and a principal right $\mathcal{H}$-module structures over P. Moreover we need the two module structures are commutative to each other and satisfy the following conditions:

- The quotient space $\mathcal{G}/P$ (with its quotient topology) is homeomorphic to $\mathcal{H}_0$ in a way that identified the right moment map $P \to \mathcal{H}_0$ with the quotient map $P \to \mathcal{G}$ P.
- The quotient space $P/\mathcal{H}$ (with its quotient topology) is homeomorphic to $\mathcal{G}_0$ in a way that identifies the left moment map $P \to \mathcal{G}_0$ with the quotient map $P \to P/\mathcal{H}$.

Now we start to construct the first connection (4.20), which is implied by the following lemma.

Lemma 4.22. *$S^1 \rtimes_{\bar{\rho}} \mathcal{G}$ is Morita equivalent to the initial groupoid of X.*

Proof. The source and target map of $S^1 \rtimes_{\bar{\rho}} \mathcal{G} : S^1 \times S^1 \times X \rightrightarrows S^1 \times X$ are :

$$s(\theta_1, \theta_2, x) = (\theta_1\theta_2, x),\ t(\theta_1, \theta_2, x) = (\theta_1, \theta_2 \cdot x).$$

We choose the $X \rightrightarrows X/S^1 \rtimes_{\bar{\rho}} \mathcal{G}$ bimodule P to be $S^1 \times X$. The left action is the trivial action with the projection to X as the left moment map ε_l. The right moment map ε_r is the identity map. Denote by R the fiber product $S^1 \times S^1 \times X \times_{t,\varepsilon_r,S^1\times X} S^1 \times X$ here. The right action ϱ_r is given by

$$\varrho_r : R \mapsto S^1 \times X \tag{4.24}$$

$$\varrho_r(\gamma, p) = s(\gamma), \tag{4.25}$$

where $(\gamma, p) \in R$ i.e. $t(\gamma) = \varepsilon_r(p)$. Given any composable pair $\gamma_1, \gamma_2 \in S^1 \times S^1 \times X$ and any $p \in P$ we have that

$$(p\gamma_2)\gamma_1 = s(\gamma_1) = p(\gamma_1\gamma_2). \tag{4.26}$$

Therefore P is an $X \rightrightarrows X/S^1 \rtimes_{\bar{\rho}} \mathcal{G}$ bimodule.
According to the definitions of the actions, we get the two module structures commute with each other and they are both principal. Since the left action is trivial we get that $X \rightrightarrows X/P$ is homeomorphic to $S^1 \times X$ by the identity map. On the other hand, if we write the right action explicitly, we have

$$\varrho_r((\theta_1, \theta_2, \cdot x), (\theta_1, \theta_2 \cdot x)) = (\theta_1\theta_2, x). \tag{4.27}$$

Therefore we get that $P/S^1 \rtimes_{\bar{\rho}} \mathcal{G}$ is homeomorphic to X by the projection to X. So $S^1 \rtimes_{\bar{\rho}} \mathcal{G}$ is Morita equivalent to $X \rightrightarrows X$. □

The connection (4.21) is clear under the isomorphism between $H^k(X, U(1))$ and $H^{k+1}(X, \mathbb{Z})$. So we get the set-up data of these two approaches are equivalent. The remainder is to show the connections between the data of T-dual parts. We deal with (4.22) in the remainder of this section and leave the last one (4.23) to the next section.
We may want to show that the two groupoids $S^1 \rtimes_{\bar{\lambda}} \mathcal{G}$ and the action groupoid associated to the S^1-action on $\hat{X} \times ES^1$ are Morita equivalent. Unfortunately this is not true in general. To see this we can just consider the case when the S^1-action over X is free. In this case, the generalized principal bundle $S^1 \rtimes_{\bar{\lambda}} \mathcal{G}$ is a principal bundle $\hat{P}$ over $B := X/S^1$. While $X \times ES^1/S^1$ is not $\hat{P}$, which shows that the two groupoids are not Morita equivalent in this case. So we have to use other method to characterize their relations here.

In [48], the theory of classifying spaces for topological groupoids (and topological stacks) is discussed. Here we want to show that the classifying space of $S^1 \rtimes_{\bar{\lambda}} \mathcal{G}$ is homomorphic to $(\hat{X} \times S^\infty)/S^1$. Of course, to prove this we will need to make the assumptions on $\bar{\lambda}$. First we give the definition of the (Haefliger-Milnor) **classifying space** $B\mathcal{G}$ and the **universal bundle** $E\mathcal{G}$ of a topological groupoid $\mathcal{G}$.

Definition 4.23. An element in $E\mathcal{G}$ is a sequence $(t_0\alpha_0, t_1\alpha_1, ..., t_n\alpha_n, ...)$, where $\alpha_i \in \mathcal{G}_1$ are such that $s(\alpha_i)$ are equal to each other, and $t_i \in [0, 1]$ are such that all but finitely many of them are zero and $\sum t_i = 1$. Let $t_i : E\mathcal{G} \to [0, 1]$ denote the map $(t_0\alpha_0, t_1\alpha_1, ..., t_n\alpha_n, ...) \mapsto t_i$ by t_i and let $\alpha_i : t_i^{-1}(0, 1] \to \mathcal{G}_1$ denote the map $(t_0\alpha_0, t_1\alpha_1, ..., t_n\alpha_n, ...) \mapsto \alpha_i$. The topology on $E\mathcal{G}$ is the weakest topology in which $t_i^{-1}(0, 1]$ are all open and t_i and α_i are all continuous.

The classifying space $B\mathcal{G}$ is defined to be the quotient of $E\mathcal{G}$ under the following equivalence relation. We say two elements $(t_0\alpha_0, t_1\alpha_1, ..., t_n\alpha_n, ...)$ and $(t'_0\alpha'_0, t'_1\alpha'_1, ..., t'_n\alpha'_n, ...)$ of $E\mathcal{G}$ are equivalent, if $t_i = t'_i$ for all i, and if there is an element $\gamma \in \mathcal{G}_1$ such that $\alpha_i = \gamma\alpha'_i$. Let $p : E\mathcal{G} \to B\mathcal{G}$ be the projection map.

Lemma 4.24. *The projection map $p : E\mathcal{G} \to B\mathcal{G}$ can be naturally made into a $\mathcal{G}$-tosor. (see Section B.1)*

The proof of this lemma can be found in Section 4.1 of [48]

Lemma 4.25. *Let $\mathcal{G}$ to be the action groupoid $X \times S^1 \rightrightarrows X$, then the (Haefliger-Milnor) classifying space of $\mathcal{G}$ is homomorphic to $X \times ES^1/S^1$.*

Proof. We first identity elements in $E\mathcal{G}$ with elements in $X \times ES^1$. Let $(t_0\alpha_0, t_1\alpha_1, ..., t_n\alpha_n, ...)$ be an element in $E\mathcal{G}$ and $s(\alpha_i) = x \in X$. We can represent each α_i by $(\theta_i, x) \in S^1 \times X$. Then we define a map from $E\mathcal{G}$ to $X \times ES^1$ as follows:

$$c((t_0\alpha_0, t_1\alpha_1, ..., t_n\alpha_n, ...)) = (x, (t_0\theta_0, t_1\theta_1, ..., t_n\theta_n, ...)). \tag{4.28}$$

From the definition we can see that c is a bijection and continuous for the topology of $E\mathcal{G}$ and $X \times ES^1$. Let $(t_0\alpha_0, t_1\alpha_1, ..., t_n\alpha_n, ...)$ and $(t'_0\alpha'_0, t'_1\alpha'_1, ..., t'_n\alpha'_n)$ be equivalent to each other, i.e. $t_i = t'_i$ and there exists an $\gamma \in \mathcal{G}$ such that $\alpha_i = \gamma\alpha'_i$. We write $\alpha'_i = (\gamma \cdot x, \theta'_i)$. Then clearly we have that $(t_0\theta_0, t_1\alpha_1, ..., t_n\alpha_n, ...)$ is equivalent to $(t'_0\theta'_0, t'_1\theta'_1, ..., t'_n\theta'_n, ...)$. Therefore we get c can be reduced to a bijective continuous map from $B\mathcal{G}$ to $X \times BS^1$. Similarly, we can define the inverse of c by

$$c^{-1}(x, (t_0\theta_0, t_1\theta_1, ..., t_n\theta_n, ...)) = (t_0(x, \theta_0), t_1(x, \theta_1), ..., t_n(x, \theta_n), ...). \tag{4.29}$$

And similarly we can prove it is bijective and continuous. Therefore we have $B\mathcal{G}$ is homomorphic to $X \times ES^1/S^1$. □

Lemma 4.26. *The classifying space of $S^1 \rtimes_{\bar{\lambda}} \mathcal{G}$ is a principal S^1-bundle over $B\mathcal{G}$.*

Proof. We need to define a free S^1-action over $B(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$ and show that the quotient space is $B\mathcal{G}$. We first define an S^1-action over $E(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$:

$$a : S^1 \times E(S^1 \rtimes_{\bar{\lambda}} \mathcal{G}) \to E(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$$

$$a(\theta, (t_0(\theta_0, u_0, x), t_1(\theta_1, u_1, x), ..., t_n(\theta_n, u_n, x), ...) = (t_0(\theta\theta_0, u_0, x), t_1(\theta\theta_1, u_1, x), ..., t_n(\theta\theta_n, u_n, x), ...).$$

It is not hard to check that a is compatible with the action is compatible with the equivalence relation in the definition of classifying space. Therefore a can be reduced to an S^1-action over $B(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$. Assume $\theta \in S^1$ and $a(\theta, p) = p$ for any $p \in E(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$ i.e. there exists a $\gamma \in S^1 \rtimes_{\bar{\lambda}} \mathcal{G}$ such that $(\theta\theta_i, u_i, x) = \gamma \cdot (\theta_i, u_i, x)$, then we get γ is an element like $(1, 1, x)$ i.e. a is a free action. The canonical projection $\hat{\pi} : S^1 \rtimes_{\bar{\lambda}} \mathcal{G} \to \mathcal{G}$ induces a projection from $B(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$ to $B\mathcal{G}$. Explicitly, the projection is given by

$$\hat{\pi}([(t_0(\theta_0, u_0, x), t_1(\theta_1, u_1, x), ..., t_n(\theta_n, u_n, x), ...)]) = [(t_0(u_0, x), t_1(u_1, x), ..., t_n(u_n, x), ...)]. \tag{4.30}$$

Obviously, $\hat{\pi}$ is S^1-equivariant. Therefore it reduces to a map from $B(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$ to $B\mathcal{G}$, which we still denote by $\hat{\pi}'$. By definition $\hat{\pi}'$ is surjective and open. To show that $\hat{\pi}'$ is a homomorphism we only need to show its injectivity. Let $p, p' \in B(S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$ and $\hat{\pi}(p) = \hat{\pi}(p')$. We get that $[(t_0(u_0, x), t_1(u_1, x), ..., t_n(u_n, x), ...)] = [(t'_0(u'_0, x'), t'_1(u'_1, x'), ..., t'_n(u'_n, x'), ...)]$. Therefore $t_i = t'_i$ and there exists a $\alpha \in \mathcal{G}$ such that $(u_0, x) = \alpha(u'_0, x')$, which implies that $(\theta_i, u_i, x) = (1, \alpha) \cdot (\theta_i \bar{\lambda}(\alpha), u'_i, x')$. On the other hand, we have $\bar{\lambda}(u'_i, x')\theta'_i = \bar{\lambda}(u'_j, x')\theta'_j = \theta'$ and $\bar{\lambda}(u_i, x)\theta_i = \bar{\lambda}(u_j, x)\theta_j = \theta$ for any nonnegative integers i, j. Then we obtain that $[(\theta_i, u_i, x)] = (\theta'/\theta) \cdot [(\theta'_i, u'_i, x')]$ i.e. $\hat{\pi}'$ is injective. □

From the proof of the above two lemmas we can get $B(S^1 \rtimes_{\bar{\rho}} \mathcal{G})$ is also a principal S^1-bundle over $B\mathcal{G}$ and $B(S^1 \rtimes_{\bar{\rho}} \mathcal{G})$ is homomorphic to $X \times S^\infty/S^1$. If we assume that the pullback of $\bar{\lambda}$ along π gives the first Chern class of $j : \hat{X} \times S^\infty \to X \times S^\infty$, then we get that $\hat{X} \times S^\infty$ is homomorphic to the classifying space of $S^1 \rtimes_{\bar{\rho}} (S^1 \rtimes_{\bar{\lambda}} \mathcal{G})$. And the classifying space of $S^1 \rtimes_{\bar{\lambda}} \mathcal{G}$ gives the T-dual space in Mathai and Wu's approach. Unfortunately, we can't prove the assumption here. Since it is still unknown how to construct an explicit transformation from $\bar{\lambda}$ and the difficulties lie in the transformation between groupoid cocyles and de-Rham cohomology classes. Even when the groupoid is a Cech groupoid of a manifold, this transformation is far from known to us. Moreover, there is not an explicit Gysin map for groupoid cohomology or even Cech cohomology. In the next section we will discuss some possible approach to understand the relation between the two models of T-dual twists.

Remark 4.27. Now we get the third relation we list in the beginning of this section. It is not a Morita equivalence but a homomorphism of the associated classifying spaces. According to [48], cohomology theories of topological stacks can be represented by cohomology theories on the associated classifying spaces, so the above proposition implies that the associated cohomology theories in these two differentiable stacks are isomorphic.

4.5 Connections between T-dual Twisting Classes

In this section we discuss the connection between the T-dual twisting classes of the two approaches. In Mathai and Wu's approach the T-dual twisting class is uniquely determined by two conditions:

$$(1) \quad \hat{\pi}_!([\hat{H}]) = e_{S^1} \in H^2_{S^1}(X, \mathbb{Z}); \tag{4.31}$$

$$(2) \quad (p \times id)^*([H]) = i^*([\hat{H}]) \in H^3(\hat{X}, \mathbb{Z}), \tag{4.32}$$

where $i : \hat{X} \to \hat{X}_{S^1}$ is the inclusion map. In order to show the connection we will show that $\sigma^\vee$ satisfies two similar conditions. Before we start to compare the two twisting classes we first give a construction of product on the Čech cochains.

Definition 4.28. Assume $\alpha \in C^k(X, U(1)), \beta \in C^l(X, U(1))$ and we choose liftings of α,β: $\tilde{\alpha} \in C^k(X, \mathbb{R}), \tilde{\beta} \in C^l(X, \mathbb{R})$. Then we define a $(k+l+1)$ cochain $\alpha * \beta$ by the following formula:

$$\begin{aligned}(\alpha * \beta)_{i_0 i_1 \ldots i_{k+l+1}} &= \ < (\tilde{\alpha})_{i_{l+1} i_{l+2} \ldots i_{k+l+1}}, (\delta\tilde{\beta})_{i_0 i_1 \ldots i_{l+1}} > \\ &= e^{i(\tilde{\alpha})_{i_{l+1} i_{l+2} \cdots i_{k+l+1}} \cdot (\delta\tilde{\beta})_{i_0 i_1 \cdots i_{l+1}}}.\end{aligned}$$

Here $< \cdot, \cdot >$ is the paring associated to the Pontryagin dual of $\mathbb{R}$.

Remark 4.29. It is not hard to see we can generalize this construction to general $U(1)$-valued groupoid cochains. But the definition here is enough for our discussion, so we will not give generalization here. The other thing we need to point out is that this product construction depends on the choices of the liftings. However, we will see that this dependence disappear once we go to the cohomology class level.

The following lemma gives some properties of this product.

Lemma 4.30. *Assume $\alpha \in C^k(X, U(1))$, $\beta \in C^l(X, U(1))$ and $\gamma \in C^n(X, U(1))$*

$$(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma); \tag{4.33}$$

$$\delta(\alpha * \beta) = (\delta\alpha * \beta) \cdot (\alpha * \delta\beta)^{\pm 1}; \tag{4.34}$$

Proof. (1)We choose the $\tilde{\alpha} \cdot \delta\tilde{\beta}$ and $\tilde{\beta} \cdot \delta\tilde{\gamma}$ as the liftings of $\alpha * \beta$ and $\beta * \gamma$ respectively.Then we have:

$$(\alpha * \beta) * \gamma = e^{i\tilde{\alpha}\cdot\delta\tilde{\beta}\cdot\delta\tilde{\gamma}}$$

and

$$\alpha * (\beta * \gamma) = e^{i\tilde{\alpha}\cdot\delta(\widetilde{\beta*\gamma})} = e^{i\tilde{\alpha}\cdot\delta\tilde{\beta}\cdot\delta\tilde{\gamma}}.$$

(2)

$$\begin{aligned}
\delta(\alpha * \beta)_{i_0 i_1 \dots i_{k+l+2}} &= \prod_n (\alpha * \beta)^{(-1)^n}_{i_0 i_1 \dots \hat{i}_n \dots i_{k+l+2}} \\
&= e^{\sum_{n \le l+2} (-1)^n \tilde{\alpha}_{i_{l+2} \dots i_{k+l+2}}} (\delta\tilde{\beta})_{i_0 i_1 \dots \hat{i}_n \dots i_{l+2}} + \sum_{n \ge l+1} (-1)^n \tilde{\alpha}_{i_{l+1} \dots \hat{i}_n \dots i_{k+l+2}} \cdot \delta(\tilde{\beta})_{i_0 i_1 \dots i_{l+1}} \\
&= \delta(\alpha) * \beta \cdot (\alpha * \delta(\beta))^{(-1)^{l+1}}.
\end{aligned}$$

□

According to this lemma, we get the following proposition.

Proposition 4.31. The product $*$ induces a product on $U(1)$-valued Čech cohomology. Moreover, this product coincides with the cup product of $\mathbb{Z}$-valued Čech cohomology over X under the isomorphism between $H^k(X, U(1))$ and $H^{k+1}(X, \mathbb{Z})$.

Proof. Given a $U(1)$-valued Čech k-cocycle $\alpha \in Z^k(X, U(1))$, we can choose a lifting $\tilde{\alpha}$ of α. Then $\alpha \mapsto \delta\tilde{\alpha}$ induces the isomorphism between $H^k(X, U(1))$ and $H^{k+1}(X, \mathbb{Z})$ and this isomorphism is independent of the choice of lifting. The statements of the above proposition is given by the above lemma. □

Now we introduce an interesting notion for generalized principal $U(1)$-bundle, which is quite important for our computations below.

Definition 4.32. Given a generalized principal bundle $S^1 \rtimes_{\bar{\rho}} \mathcal{G}$, we define the **canonical connection** of the generalized principal S^1-bundle to be the $U(1)$-valued 0-cochain Φ below:

$$\begin{aligned} \Phi : S^1 \times \mathcal{G}_0 &\mapsto S^1 \\ (\phi, x) &\to \phi^{-1}. \end{aligned}$$

Remark 4.33. Φ is called a connection is because of the following identity: For any $(\phi, \gamma) \in S^1 \times \mathcal{G}_1$,

$$\delta\Phi(\phi, \gamma) = \Phi(s(\phi, \gamma))^{-1} \cdot \Phi(t(\phi, \gamma)) = \bar{\rho}(\gamma).$$

When $\mathcal{G}$ is a Čech groupoid, a generalized principal S^1-bundle is indeed a principal S^1-bundle. And we know that $\bar{\rho}$ gives a $U(1)$-valued Čech 1-cocycle which determines the principal S^1-bundle. Moreover $\bar{\rho}$ gives the first Chern class of the principal S^1-bundle under the isomorphism $H^*(X, S^1) \cong H^{*+1}(X, \mathbb{Z})$. If we denote a connection over the principal S^1-bundle by A and its curvature by F, then the relation $\bar{\rho}(\gamma) = \delta\Phi(\phi, \gamma)$ corresponds to the relation $F = dA$ in the classical Chern-Weil theory.

Now we turn to the real topic of this section. The idea here is to show that the twisting class $\sigma^\vee$ satisfies two similar conditions which determines $\hat{H}$ uniquely in the beginning of this section. The first condition is that $\hat{\pi}_*([\hat{H}]) = e_{S^1}$ which we call the **push-forward condition** below. There is no explicit push-forward map for groupoid cohomology. Therefore we will try to show that the image of the class representing $\sigma^\vee$ in de Rham cohomology group satisfies the push-forward condition. The other condition is that $(p \times id)^*([H]) = i^*([\hat{H}])$, which we call the **pullback condition** below. We will try to compare the quotient the pullback of σ and $\sigma^\vee$ and show that it is exact, which implies the corresponding de Rham cohomology class satisfies the **pullback condition**. Because we don't know how to write the explicit image of $\sigma^\vee$ in the de Rham cohomology, we will explore the construction of the T-dual class in [12] and apply the similar construction to $\sigma^\vee$. First of all we list Prop 10.9 in [22].

Proposition 4.34. The groupoid cohomology $H^*(\mathbb{R} \ltimes (\mathbb{R} \rtimes_\rho \mathcal{G}); B)$ is a direct summand of the equivariant cohomology $H^*_{\mathbb{R}}(\mathbb{R} \rtimes_\rho \mathcal{G}; B)$. Here $\mathbb{R} \ltimes (\mathbb{R} \rtimes_\rho \mathcal{G})$ is the crossed product of $\mathbb{R} \rtimes_\rho \mathcal{G}$ and $H^*_{\mathbb{R}}(\mathbb{R} \rtimes_\rho \mathcal{G}; B)$ is the cohomology group in Definition 1.33 in section 1.6.

The image of $[(\sigma, \lambda, 1)] \in H^2_{\mathbb{R}}(\mathbb{R} \rtimes_\rho \mathcal{G}, U(1))$ in $H^2(\mathbb{R} \ltimes (\mathbb{R} \rtimes_\rho \mathcal{G}); U(1))$ is given by $[\sigma(e, \gamma_1, \gamma_2)\lambda(\rho(\gamma_1), \gamma_2)]$ (see Proposition 6.1 in [22]). Denote the corresponding de-Rham cohomology class of $[\sigma(e,\gamma_1,\gamma_2)\lambda(\rho(\gamma_1),\gamma_2)]$ by $[\omega_1] \in H^3(\hat{X}\times S^\infty/S^1)$. Then $[\omega_1] \in ker(\hat{\pi}_*)$ as $H^*(\mathbb{R}\ltimes(\mathbb{R}\rtimes_\rho \mathcal{G}); U(1))$ belongs to the image of $\hat{\pi}^*$. Therefore from now on we only need to consider the rest part in the formula of $\sigma^\vee$ i.e. $<\phi\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1.\gamma_2)>$. We will show that the push-forward image of the corresponding class for $<\phi\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1.\gamma_2)>$ in de-Rham cohomology is the first Chern class of $\hat{\pi}:\hat{X}\to B$ i.e. $\bar{\rho}$ (we denote its image in $H^2_{dR}(X)$ by c_1).

Let us start with a simple case when $\bar{\lambda}$ is trivial or the principal S^1-bundle is trivial. In this case $<\phi, \delta\rho(\gamma_1,\gamma_2)>=\Phi*\bar{\rho}$, where Φ is the canonical connection in (4.32). According to the Proposition 4.31 we get $\Phi*\bar{\rho}$ is mapped to $d\theta\wedge c_1$, which is mapped to c_1 under the push-forward map of the trivial S^1-bundle.
In general case, it is more complicated. We first remind us how to construct the twisting class in classic topological T-duality. Let A be connection 1-form and $y\in\Omega^3(B)$ satisfying $dy = F\wedge\hat{F}$, then we have $\hat{\pi}_!(A\wedge\hat{\pi}^*c_1 - \hat{\pi}^*\alpha) = c_1$. Then we state that there are relations as follows:

$$<\phi, \delta\rho(\gamma_1,\gamma_2)>\leftrightarrow -A\wedge\hat{\pi}^*(c_1); <\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1,\gamma_2)>\leftrightarrow y.$$

The first relation is obvious from the definition of $*$ product and its properties. For the second one, we have the following lemma.

Lemma 4.35.

$$\delta(<\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1,\gamma_2)>)(\gamma_0,\gamma_1,\gamma_2) = (\bar{\lambda}*\bar{\rho})(\gamma_0,\gamma_1,\gamma_2).$$

Proof. Denote $<\bar{\lambda}(\cdot)\bar{\lambda}(\cdot), \delta\rho(\cdot,\cdot)>$ by $\tau(\cdot,\cdot)$. Then we have:

$$\begin{aligned}
&(\delta\tau)(\gamma_0,\gamma_1,\gamma_2)\\
&= \gamma_0\bullet\tau(\gamma_1,\gamma_2)\cdot\tau(\gamma_0\gamma_1,\gamma_2)^{-1}\cdot\tau(\gamma_0,\gamma_1\gamma_2)\cdot\tau(\gamma_0,\gamma_1)^{-1}\\
&= [\bar{\lambda}(\gamma_0)\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2)]^{\delta\rho(\gamma_1,\gamma_2)}[\bar{\lambda}(\gamma_0)\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2)]^{-\delta\rho(\gamma_0\gamma_1,\gamma_2)+\delta\rho(\gamma_0,\gamma_1\gamma_2)}\cdot[\bar{\lambda}(\gamma_0)\bar{\lambda}(\gamma_1)]^{-\delta\rho(\gamma_0,\gamma_1)}\\
&= [\bar{\lambda}(\gamma_2)]^{\delta\rho(\gamma_0,\gamma_1)} = \bar{\lambda}*\bar{\rho}(\gamma_0,\gamma_1,\gamma_2).
\end{aligned}$$

□

According to the properties of $*$ product we get that $\bar{\lambda}*\bar{\rho}$ corresponds to $F\wedge\hat{F}$. Then we get that $\sigma^\vee$ satisfies the **push-forward conditions**.
The other condition in section 4.1 is that $(p\times id)^*([H]) = \hat{p}^*([\hat{H}])$. In the classical topological T-duality we also need similar conditions to fix the twisting class uniquely. Consider

the diagram in topological T-duality:

$$\begin{tikzcd}
 & X \times_B \hat{X} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\
X \arrow[dr, "\pi"'] & & \hat{X} \arrow[dl, "\hat{\pi}"] \\
 & B &
\end{tikzcd}$$

Denote the local connection 1-forms of X and $\hat{X}$ by A and $\hat{A}$ respectively. Then the twisting class in the T-dual part satisfies the following condition: define $\Theta = p^*A \wedge \hat{p}^*\hat{A}$, then:

$$d\Theta = -p^*(H) + \hat{p}^*(\hat{H}).$$

We call this condition the **pull-back condition** below. The push-forward condition and pull-back condition together determine the twisting class in the T-dual part uniquely. Now we turn to show that $\sigma^\vee$ satisfies the pull-back condition. First of all we have the following T-dual diagram in groupoid approach:

$$\begin{tikzcd}
 & S^1 \rtimes_{\bar{\lambda}} (S^1 \rtimes_{\bar{\rho}} \mathcal{G}) \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\
S^1 \rtimes_{\bar{\rho}} \mathcal{G} \arrow[dr, "\pi"'] & & S^1 \rtimes_{\bar{\lambda}} \mathcal{G} \arrow[dl, "\hat{\pi}"] \\
 & \mathcal{G} &
\end{tikzcd} \tag{4.35}$$

Remark 4.36. Since we will use the groupoid $S^1 \rtimes_{\bar{\lambda}} (S^1 \rtimes_{\bar{\rho}} \mathcal{G})$, we give its source and range maps here:

$$\begin{aligned}
S(\phi', \phi, \gamma) &= (\phi'\bar{\lambda}(\gamma), \phi\bar{\rho}(\gamma), s(\gamma)); \\
R(\phi', \phi, \gamma) &= (\phi', \phi, s(\gamma)).
\end{aligned}$$

The correspondence "space" in the above diagram can also be written as $S^1 \rtimes_{\bar{\lambda}} (S^1 \rtimes_{\bar{\rho}} \mathcal{G})$ since these two groupoids are isomorphic.

Denote $I = \frac{p*(\sigma)(\phi,\phi',\gamma_1,\gamma_2)}{\hat{p}^*(\sigma^\vee)(\phi,\phi',\gamma_1,\gamma_2)}$. According to the cocycle conditions of $(\sigma, \lambda, 1)$:

$$\delta\sigma = 1, \ \ d\sigma = \delta\lambda, \ \ d\lambda = 1, \tag{4.36}$$

$d\sigma = \delta\lambda$ implies that $d\sigma(\phi) = \sigma(e, \gamma_1, \gamma_2)\sigma(\phi, \gamma_1, \gamma_2)^{-1}$, so we have

$$\begin{aligned} I &= \frac{\sigma(\phi, \gamma_1, \gamma_2)}{\sigma^{\vee}(\phi', \gamma_1, \gamma_2)} \\ &= \frac{d\sigma(\phi)(\gamma_1, \gamma_2)^{-1}\sigma(e, \gamma_1, \gamma_2)}{\sigma(e, \gamma_1, \gamma_2)\lambda(\rho(\gamma_1), \gamma_2) < \phi\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) >} \\ &= \frac{\delta\lambda(\phi)(\gamma_1, \gamma_2)^{-1}}{\lambda(\rho(\gamma_1), \gamma_2) < \phi'\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) >} \end{aligned}$$

We need to show that I is an exact cochain. Obviously $\delta\lambda(\phi')(\gamma_1, \gamma_2)$ is exact. Therefore we only need to show that $\lambda(\rho(\gamma_1), \gamma_2) < \phi\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) >$ is exact. Before we move on to show that I is exact, we first define two canonical connections over $S^1 \rtimes_{\bar{\lambda}} (S^1 \rtimes_{\bar{\rho}} \mathcal{G})$:

$$\begin{aligned} \Phi : S^1 \times S^1 \times \mathcal{G}_0 &\mapsto S^1 \\ (\phi', \phi, x) &\to \phi^{-1} \\ \Phi' : S^1 \times S^1 \times \mathcal{G}_0 &\mapsto S^1 \\ (\phi', \phi, x) &\to \phi'^{-1}. \end{aligned}$$

Like the remark (4.33) we have that:

$$\delta\Phi(\phi', \phi, \gamma) = \bar{\rho}(\gamma), \quad \delta\Phi'(\phi', \phi, \gamma) = \bar{\lambda}(\gamma). \tag{4.37}$$

Then we prove a lemma which gives us an explicit expression of λ.

Lemma 4.37. *Let $(\sigma, \lambda, 1)$ be a 2-cocycle in $Z^2_{\mathbb{R}}(S^1 \rtimes_\rho \mathcal{G}, U(1))$ and $f_\lambda = 1$ (see (4.19)). Then for any $g \in \mathbb{R}$ and $\gamma \in S^1 \rtimes \mathcal{G}$, we have*

$$\lambda(g, \gamma) = \bar{\lambda}(\gamma)^g,$$

Proof. Since $f_\lambda = 1$ i.e. the $\mathbb{R}$ action over λ is trivial. So $d\lambda = 1$ implies $\lambda(g_1 + g_2, \gamma) = \lambda(g_1, \gamma) \cdot \lambda(g_2, \gamma)$. Then the conclusion follows from the continuity of λ. □

Now we prove that $\lambda(\rho(\gamma_1), \gamma_2) < \phi\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) >$ is exact:

Lemma 4.38.

$$\delta(\Phi' * \Phi)(\phi', \phi, \gamma_1, \gamma_2) = \bar{\lambda}(\gamma_2)^{\rho(\gamma_1)} \cdot < \phi'\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) > .$$

Proof.

$$
\begin{aligned}
& \delta(\Phi' * \Phi)(\phi', \phi, \gamma_1, \gamma_2) \\
= \;& (\delta\Phi') * \Phi(\phi', \phi, \gamma_1, \gamma_2) \cdot (\Phi') * (\delta\Phi)(\phi', \phi, \gamma_1, \gamma_2)^{-1} \\
= \;& \delta\Phi'(\phi'\bar{\lambda}(\gamma_1), \phi\bar{\rho}(\gamma_1), \gamma_2)^{\widetilde{\delta\Phi}(\phi', \phi, \gamma_1)} \cdot (\Phi')(S(\phi', \phi, \gamma_1\gamma_2))^{\widetilde{\delta\delta(\rho)}(\phi', \phi, \gamma_1, \gamma_2)} \\
= \;& \bar{\lambda}(\gamma_2)^{\widetilde{\delta\Phi'}(\phi', \phi, \gamma_1)} \cdot (\phi'\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2))^{\delta\rho(\gamma_1, \gamma_2)} \\
= \;& \bar{\lambda}(\gamma_2)^{\rho(\gamma_1)} \cdot < \phi'\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) > \\
= \;& \lambda(\rho(\gamma_1), \gamma_2) < \phi'\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) > \\
= \;& \bar{\lambda}(\gamma_2)^{\rho(\gamma_1)} \cdot < \phi'\bar{\lambda}(\gamma_1)\bar{\lambda}(\gamma_2), \delta\rho(\gamma_1, \gamma_2) >
\end{aligned}
$$

The last two identities are from Lemma 4.37. □

Combining the above discussions we get

Proposition 4.39. If $f_\lambda = 1$ (see (4.19)), then $I = \hat{p}^*(\sigma^\vee)/p^*(\sigma)$ is an exact 3-cochain over $S^1 \rtimes_{\bar{\lambda}} (S^1 \rtimes_{\bar{\rho}} \mathcal{G})$. And with all of the discussions above together we get that $\sigma^\vee$ satisfies the **pullback** and **pushforward condition**.

Remark 4.40. To remove the condition $f_\lambda = 1$, one approach is to study the kernel of Π ((4.17)) and prove that for every $\sigma \in Z^2(S^1 \rtimes_\rho \mathcal{G}, U(1))$ which admits an extension $(\sigma, \lambda, 1)$ we can always find a λ such that $(\sigma, \lambda, 1)$ is an extension with $f_\lambda = 1$. However, we will not prove it here and just list it as a further question:

Question 1. Let $\mathcal{G}$ be a Lie groupoid and $S^1 \rtimes_\rho \mathcal{G}$ be a generalized principal S^1-bundle over $\mathcal{G}$. If $\sigma \in Z^2(S^1 \rtimes_\rho \mathcal{G}, U(1))$ admits an extension to a 2-cocycle $(\sigma, \lambda, 1) \in Z^2_{\mathbb{R}}(S^1 \rtimes_\rho \mathcal{G}, U(1))$, then can we always find another extension $(\sigma, \lambda', 1)$ of σ such that $f_{\lambda'} = 1$?

With all of the above discussions together, partially we get a connection between the T-dual twists (4.23). Now we give a definiton of the T-duality for groupoids.

Definition 4.41. Let $\mathcal{G}$ be a topological groupoid. A groupoid pair over $\mathcal{G}$ is a pair $(S^1 \rtimes_\rho \mathcal{G}, (\sigma, \lambda, 1))$ in which $S^1 \rtimes_\rho \mathcal{G}$ is a generalized principal S^1-bundle over $\mathcal{G}$ and $(\sigma, \lambda, 1) \in Z^2_{\mathbb{R}}(G, U(1))$. Another groupoid pair $(S^1 \rtimes_{\hat{\rho}} \mathcal{G}, (\hat{\sigma}, \hat{\lambda}, 1))$ over $\mathcal{G}$ are called T-dual to $(S^1 \rtimes_\rho \mathcal{G}, (\sigma, \lambda, 1))$ if the following conditions are satisfied:

1. The homomorphism $\bar{\lambda} : \mathcal{G} \to U(1)$ indued by $(\sigma, \lambda, 1)$ (see Section 1.6) and $\hat{\rho}$ are cohomologous as cocycles in $Z^1(\mathcal{G}, U(1))$. Similarly, the homomorphism $\bar{\hat{\lambda}}$ induced by $(\hat{\sigma}, \hat{\lambda}, 1)$ and ρ are cohomologous.

2.

$$\delta < \bar{\lambda}(\cdot)\bar{\lambda}(\cdot)), \delta\rho(\cdot,\cdot) >= \bar{\lambda} * \bar{\rho}; \tag{4.38}$$

3. The pullback condition is satisfied. If we use the notion in diagram 4.35, then $p^*(\sigma)/\hat{p}^*(\hat{\sigma})$ is exact.

Combining the discussions in this section and last section, we can get that the above definition gives a generalization of topological T-duality over spaces and it is at least partially compatiable with C.Daenzer's constructions. It will be interesting to investigate more on the T-duality isomorphism for such a T-duality pair and the associated uniqueness results.

4.6 T-duality pairs via Differentiable Stacks

In this section we discuss other possible approaches to the topological T-duality with singularities. The basic definitions about differential stacks and the construction of the cohomology of differential stacks are given in the Appendix B. We will use them directly here.

4.6.1 Construction of T-duality Pairs

If X is a manifold which admits a smooth S^1-action and $H \in H^3(X,\mathbb{Z})$, we can consider X as a principal S^1-bundle over the quotient stack $[X/S^1]$. According to [10] , we know that a principal S^1-bundle over a differentiable stack is also determined (up to an isomorphism) by a second integer cohomology class of the differentiable stack which is also called first Chern class in [10]. Moreover, an S^1-gerbe over a differential stack also has a characteristic class in its third integer cohomology group, which is also called its Dixmier-Douady class. In [66], they define a version of twisted K-theory for differentiable stacks with S^1-gerbes as twists. Until now, the data here is very similar to the pair (P, H) in classical topological T-duality. Generally, we can define a pair as an S^1-bundle $\mathfrak{P}$ over a differentiable stack $\mathfrak{B}$ with a third integral cohomology class $\mathfrak{H}$ over $\mathfrak{P}$. Therefore (X, H) can be interpreted as a pair over the quotient stack $[X/S^1]$.

If we check the construction of T-duality pair in [12], we would always get the calculations and constructions always works if we have the Gysin sequence holds. Fortunately, in [30] G. Ginot and B. Noohi establish the Gysin sequence for S^1-bundles over differentiable stacks. Therefore we can completely follow the constructions in [12] and get the following proposition:

Proposition 4.42. Given a pair $(\mathfrak{P}, \mathfrak{H})$ over a differentiable stack $\mathfrak{B}$, there exists an S^1-bundle $\hat{\pi} : \hat{\mathfrak{P}} \to \mathfrak{B}$ and a third integral cohomology class $\hat{\mathfrak{H}}$ over $\hat{\mathfrak{B}}$ which satisfy

$$\pi_!(\mathfrak{H}) = c_1(\hat{\mathfrak{P}}), \tag{4.39}$$

$$\hat{\pi}_!(\hat{\mathfrak{H}}) = c_1(\mathcal{P}). \tag{4.40}$$

Here $\pi_! : H^{n+1}(\mathfrak{P}, \mathbb{Z}) \to H^n(\mathfrak{B}, \mathbb{Z})$ is the push-forward map of cohomology groups and so is $\hat{\pi}_!$, for their definition we list in the appendix or one can find in Section 8 of [30].

Proof. Since the isomorphism classes of S^1-bundles over a differentiable stack $\mathfrak{B}$ are classified by $H^2(\mathfrak{B}, \mathbb{Z})$, therefore there exists an S^1-bundle $\hat{\pi} : \hat{\mathfrak{P}} \to \mathfrak{B}$ with first Chern class $\pi_!(\mathfrak{H})$. The left is to find the twist $\hat{\mathfrak{H}}$ over $\hat{\mathcal{P}}$ which satisfies the above identities in the proposition. We can still do the fiber product of differentiable stacks. And we have the following lemma:

Lemma 4.43. *The fiber product $\mathfrak{P} \times_{\mathfrak{B}} \hat{\mathcal{P}}$ with the natural projections $\mathfrak{p}$ to $\mathfrak{P}$ and $\hat{\mathfrak{p}}$ to $\hat{\mathfrak{P}}$ are both S^1-bundles.*

We leave the proof of the lemma to the end of this section. Now we use the Gysin sequence of differentiable stacks for the S^1-bundle $\pi : \mathfrak{P} \to \mathfrak{B}$ .i.e.

$$\to H^{i+1}(\mathfrak{P}, \mathbb{Z}) \to H^i(\mathfrak{B}, \mathbb{Z}) \to H^{i+2}(\mathfrak{B}, \mathbb{Z}) \to H^{i+2}(\mathfrak{P}, \mathbb{Z}) \to \tag{4.41}$$

Since $c_1(\hat{\mathfrak{P}}) = \pi_!(\mathfrak{H})$, we have that $c_1(\hat{\mathfrak{P}}) \cup c_1(\mathfrak{P})$ is trivial. Now we write down the Gysin sequence for the S^1-bundle $\hat{\pi} : \hat{\mathfrak{P}} \to \mathfrak{B}$:

$$\to H^{i+1}(\hat{\mathfrak{P}}, \mathbb{Z}) \to H^i(\mathfrak{B}, \mathbb{Z}) \to H^{i+2}(\mathfrak{B}, \mathbb{Z}) \to H^{i+2}(\hat{\mathfrak{P}}, \mathbb{Z}) \to \tag{4.42}$$

Since $c_1(\hat{\mathfrak{P}}) \cup c_1(\mathfrak{P})$ is trivial, by the exactness of the sequence we have that there exists a third integral cohomology class $\hat{\mathfrak{H}}$ whose image under $\hat{\surd}$ is $c_1(\mathfrak{P})$. □

Remark 4.44. According to the above proposition, we know that we can construct stack version of T-duality pairs. Besides these, we can consider the Bunke-Schick construction for this picture. But here we will not discuss these items. All we need is a special case of the above proposition. i.e. for an S^1-manifold with a twist, we can construct a T-duality pair which lives in the category of differentiable stacks, which gives another approach to understand the topological T-duality with singularities.

proof of (4.43). We only prove that $\mathfrak{p} : \mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}}$ is an S^1-bundle. The other part is the same. To prove this we first need to show $\mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}}$ is an S^1-differential stack. Since $\hat{\mathfrak{P}}$ is an S^1-differential stack, therefore it induces an S^1-action over $\mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}}$. The other thing we need to show is that for any manifold U the following 2-commutative diagram gives an S^1-bundle over U via the pullback along $U \to \mathfrak{P}$:

$$\begin{tikzcd}
\mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}} \times S^1 \arrow[r, "\mu"] \arrow[d, "p"'] & \mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}} \arrow[d, "\mathfrak{p}"] \arrow[dl, twoheadrightarrow] \\
\mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}} \arrow[r, "\mathfrak{p}"'] & \mathfrak{P}
\end{tikzcd} \tag{4.43}$$

Denote the map $U \to \mathfrak{P}$ by i_U. According to the definition of fiber product of differential stacks (B.5), we get $i_U^*(\mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}})$ is an S^1-bundle over U, the S^1-action is induced by the S^1-action over $\mathfrak{P} \times_{\mathfrak{B}} \hat{\mathfrak{P}}$. □

4.6.2 Push-forward Map for Twisted K-theory of Differentiable Stacks

In the section we construct a push-forward map for twisted K-groups along the projection of S^1-bundles over differentiable stacks. We first give the definition of twisted K-theory of differentiable stacks. For any differentiable stack $\mathfrak{X}$, one can always find a Lie groupoid Γ such that $\mathfrak{X}$ is the cateogory of all Γ-torsors (see Appendix B.1). The following theorem (**Theorem 2.26** in [10]) gives the relation between differentiable stacks and Lie groupoids.

Theorem 4.45. *Let $X_\bullet$ and $Y_\bullet$ are two Lie groupoids. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the associated differentiable stacks, i.e. $\mathfrak{X}$ is the stack of $X_\bullet$-torsors and $\mathfrak{Y}$ is the stack of $Y_\bullet$-torsors. Then the following are equivalent*

1. *$\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic;*
2. *$X_\bullet$ and $Y_\bullet$ are Morita equivalent;*
3. *There exists an $X_\bullet$-$Y_\bullet$-bitorsor, i.e. there exists a manifold Q together with two smooth maps $f : Q \to X_0$ and $g : Q \to Y_0$ and commuting $X_\bullet$ and Y_{bullet} actions such that Q is a $Y_\bullet$-torsor over X_0 (via f) and a $X_\bullet$-torsor over Y_0 (via g).*

Given the above theorem, one can define twisted K-theory of a differentiable stack using the associated Lie groupoid. Now we give the exact definition in [66].

Definition 4.46 ([66])**.** Let Γ be a Lie groupoid and $S^1 \to R \to \Gamma \rightrightarrows M$ an S^1-central extension. Let $C_c(R)^{S^1} = \xi \in C_c(R) \mid \xi(\lambda r) = \lambda^{-1}\xi(r), \forall \lambda \in S^1, r \in R$. One can easily check that $C_c(R)^{S^1}$ is stable under both the convelution and the adjoint. So $C_c(R)^{S^1}$ gives a C^*-algebra.

According to the following proposition, the C^*-algebra we define above is Morita invariant.

Proposition 4.47 ([66])**.** Let $R_i \to \Gamma_i \rightrightarrows M_i$ $(i = 1, 2)$ be Morita equivalent S^1-central extensions. Then $C^*_r(R_1)^{S^1}$ and $C^*_r(R_2)^{S^1}$ are Morita equivalent C^*-algebras.

Definition 4.48 ([66])**.** Let Γ be a Lie groupoid and $\alpha \in H^2(\Gamma_\bullet, S^1)$. We define the twisted K-theory as

$$K^i_\alpha(\Gamma_\bullet) = K_{-i}(C^*_r(R)^{S^1}),$$

where $S^1 \to R \to \Gamma \rightrightarrows M'$ is any central extension of Γ realizing the class α.

From (4.47) we know that the above definition is well defined i.e. it does not depend on the choice of the S^1-extensions.

Example 4.49. (1) When Γ is a manifold $M \rightrightarrows M$ and $\alpha \in H^3(M, \mathbb{Z})$, the above definition gives the twisted K-theory of manifolds in [58].

(2)When a Lie group G acts properly on a manifold M and $\alpha \in H^3_G(M, \mathbb{Z})$ Then we can use the action groupoid $G \times M \rightrightarrows M$ to define twisted equivariant K-theory

$$K^i_{G,\alpha}(M) := K_{-i}(C^*_r(R)^{S^1}),$$

where $S^1 \to R \to G \times M \rightrightarrows M$ is any S^1-central extension realizing the class α.

The following theorem gives another description for twisted K-theory of differentiable stacks.

Theorem 4.50 ([66])**.** *Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid, $S^1 \to R \to \Gamma$ an S^1-central extension and denote by α its class in $H^2(\Gamma^\bullet, S^1)(\cong H^3(\Gamma^\bullet, \mathbb{Z}))$. Then*

$$K^i_\alpha(\Gamma^\bullet) = \{[T] | T \in \mathcal{F}^i_\alpha\}, \tag{4.44}$$

where $[T]$ denotes the homotopy class of T.

Here $\mathcal{F}^0_\alpha$ is the set of $\Gamma_\bullet$-equivariant sections of Fredlhom bundles over M. Explicitly, let $\mathcal{L}(\tilde{H})$ be the associated bundle of the principal $PU(\mathbb{H})$-bundle determined by α with fiber $L(\mathbb{H})$ (bounded linear operators over $\mathbb{H}$), then $\mathcal{F}^0_\alpha$ is the space of $T \in C_b(M, \mathcal{L}(\hat{\mathbb{H}})^\Gamma$ such that there exists $S \in C_b(M, \mathcal{L}(\hat{\mathcal{H}}))^\Gamma$ satisfying :

1. T_x and S_x are Fredholm for all x, and the sections $x \mapsto T_x$ and $x \mapsto S_x$ are $*$-strongly continuous and Γ-invariant;

2. $1 - T_xS_x$, $1 - S_xT_x$ are compact operators for all x and the sections $x \mapsto 1 - T_xS_x$, $x \mapsto 1 - S_xT_x$ are norm-continuous and vanish at ∞ in M/Γ.

Denote by $\mathcal{F}^1_\alpha$ the space of self-adjoint elements in $\mathcal{F}^0_\alpha$. We will use this description of twisted K-theory of differentiable stacks to give the push-forward map.

First of all, let us consider the push-forward map for bundles of differentiable stacks with even dimensional spin^c-bundle. Let $\pi : \mathfrak{P} \to \mathfrak{B}$ be a fiber bundle of differentiable stacks with fiber F. Here we assume that F is $B_\bullet$-invariant spin^c-structure. Let $[B] \to \mathfrak{B}$ be an atlas of $\mathfrak{B}$. Then the associated Lie groupoid of $\mathfrak{B}$ is $B_1 \rightrightarrows B$. Moreover, $P = B \times_{\mathfrak{B}} \mathfrak{P}$ gives an atlas of $\mathfrak{P}$ and it is an S^1-bundle over B. And $P_1 \rightrightarrows P$ is the associated Lie groupoid of $\mathfrak{P}$. It is easy to see that P_1 is an F-bundle over B_1. Assume that $\alpha \in H^2(B_\bullet, S^1)$, then we can do the pullback map along π and get a cohomology class $\pi^*(\alpha) \in H^2(P_\bullet, S^1)$. Now we start to construct the push-forward map:

$$\pi : K^*_{\pi^*(\alpha)}(P_\bullet) \to K^{*-1}_\alpha(B_\bullet). \tag{4.45}$$

Assume that the dimension of F is even. For each point $b \in B$, the fiber $\pi^{-1}(b)$ is an even dimensional $B_\bullet$-equivariant *spin*^{c}-manifold. Let S_F be the canonical $B_\bullet$-spinor bundle over $\pi^{-1}(b)$ and $\mathcal{H}_F$ be the Hilbert space $L^2(S^+_F) \oplus L^2(S^-_F)$. Let $D_b : L^2(S^\pm_F) \to L^2(S^\mp_F)$ be the $B_\bullet$-equivariant Dirac operator over P_b. Using D_b we can construct a family of Fredholm operators $V_b = \frac{D_b}{\sqrt{1+D^*_bD_b}}$. Let T be an element of $K^0_{\pi^*(\alpha)}(P)$. It can be represented by a section of $\pi^*(Fred(\mathcal{H}))$. More explicitly, if we choose a good open cover $\{U_i\}$ of B which gives a trivialization of $Fred(\mathcal{P})$ (without loss of generality we assume that each U_i is contractible) , then the open cover $\{\pi^{-1}(U_i)\}$ gives a trivialization of $\pi^*(Fred(\mathcal{P}))$. Therefore T can be represented by $T_i : U_i \times F \to Fred(\mathcal{H})$ and $T_j = (\pi^*g_{ij})T_i(\pi^*g_{ij})^{-1}$. Here g_{ij} is the transition function of $Fred(\mathcal{P})$. Then the section $\pi_!(T)_i = V_b \otimes Id_{\mathcal{H}_F} + Id_{\mathcal{H}} \otimes T_b : U_i \to Fred(\mathcal{H} \otimes \mathcal{H}_F)$ and satisfy the condition $\pi_!(T)_j = (g_{ij} \times Id)\pi_!(T)_i(g_{ij} \otimes Id)^{-1}$. Therefore they give rise to an element of $K^0_\alpha(B_\bullet)$, which we define to be $\pi_!(T)$. For the odd degree twisted K group, one can use Bott perodicity (**Proposition 3.7** in [66]) to transfer to the even degree case. For the fiber bundles with odd dimensional fibers, one can first consider the new Lie groupoid $P_1 \times S^1 \rightrightarrows P \times S^1$ with source and target map:

$$s'(p, x) = (s(p), x), \quad t'(p, x) = (t(p), x),$$

where $p \in P^1$, $x \in \mathbb{R}$ and s, t are the source and target map of $P^\bullet$ respectively. Assuming that $B_\bullet$ acts trivial on the S^1 part, then $P \times \mathbb{R}$ becomes a $B_\bullet$ torsor with even dimensional fiber and we can use this and Bott periodicity ([66]) to transfer this to the even dimension case.

Remark 4.51. The push-forward map (or Gysin map) we give above has been discussed in [65] using KK-theory. Here we give another more geometric description. The idea is from [19]. It is worthwhile to point out that here we need that the spin^c-structure is $B_\bullet$-invariant to choose a family $B_\bullet$-invariant Dirac operators in the construction of push-forward map. Otherwise it may be not possible to define this push-forward map in this way.

Remark 4.52. Once we have the push-forward map, we can do similar constructions like the classical case and get a T-homomorphism between $K^*(X, H)$ and $K^*(\hat{X}, \hat{H})$. One would expect that this is also an isomorphism. However, it turns out to be probably negative. Since in [45] they showed that $K^*(X, H) \to RK^*(\hat{X}_{\mathbb{T}}, \hat{H})$, while on the other hand according to Atiyah-Segal completion theorem [4] implies that $RK^*(\hat{X}_{\mathbb{T}}, \hat{H})$ is generally not isomorphic to the twisted K-theory of the T-dual differentiable stacks. However, if we use Borel twisted K-theory instead of the twisted K-theory in [66] we can still get an isomorphism of twisted K-groups, which is the same as the results in section 4.2.

4.7 Other Approaches

Besides the approaches we discussed before, there are some other possible approaches which could become candidates for the solution to the Missing T-dual problem. For example, A. Pande gave a construction of topological T-duality for semi-free S^1-actions in [49]. Now we give a geometric construction here which could be viewed as a geometrical interpretation of his approach.

For a semi-free S^1-action we mean that the orbits of the action is either S^1 or a point. Let M be a compact smooth manifold which admits a semi-free action. Denote the fixed

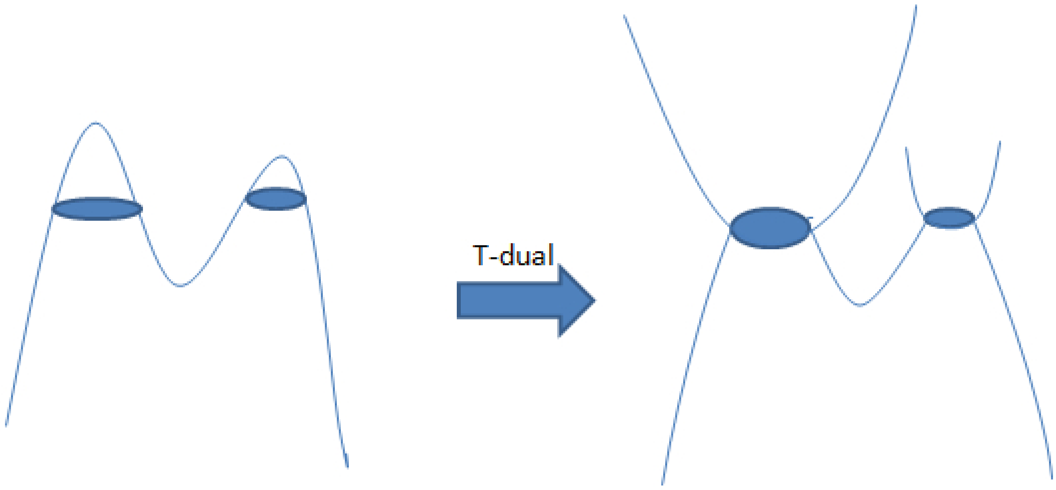

Figure 4.1: T-dual via Gluing

points by $(x_1, x_2, ..., x_n)$. Assume that U_i is an equivariant contractible open neighborhood of x_i for each i and $U_i \cup U_j = \emptyset$. We can assume that each U_i is diffeomorphic to a open disk. Write $P = M - \cup_{i=1,...,n} U_i$. Then the action is free over P which induces a principal S^1-bundle $\pi : P \to B$. Therefore we can apply our constructions of the classical topological T-duality and get the T-dual pair of $(P, 0)$: $(\hat{P}, \hat{H})$. Both P and $\hat{P}$ are manifolds with boundaries. Actually, $(\delta P, 0)$ and $(\delta\hat{P}, \hat{H}_{\delta\hat{P}})$ are also T-dual to each other as a pair over δB. Now here comes the trick. M can be seen as gluing disks U_i along the boundary of P. We hope that we can glue some spaces along the boundary of $\hat{P}$ and get a T-dual space of M. The idea can be expressed in the figure (4.1).

The problem is which space should be glued on. First of all, we need to see that what does the boundaries of $\hat{P}$ looks like? As we know that the boundary of P are spheres. The only free S^1-actions over a sphere is induced by the S^1-actions over $\mathbb{C}^n$, i.e. the dimension of the boundary of P must be odd. According to Example 1.45, the T-dual of the sphere S^{2n+1} is a weighted projective space W product with S^1, which we denote by N. Now we can see that the candidate of the gluing object is the joint product $\mathbb{R} * N$.

As we pointed out in the introduction chapter, T-duality starts from the change radius of the circle r to $1/r$. Therefore as r approaches 0, we will expect that the T-dual circle approaches ∞. The advantage of this model is that we can really see the process of this limit phenomenon. The disadvantage is that there is not an obvious isomorphism of K-theory. We may see two examples to get some feeling of this approach.

Example 4.53. Consider the rotation S^1-action over a closed half sphere D, the pole P is the only fixed point. Choose an open neighborhood U of P. We can assume that U is contractible and S^1-equivariant. Then $D - U$ is homomorphic to $S^1 \times I$, whose T-dual is $S^1 \times I$ itself. If we choose a point $\hat{p}$ over one component S^1 of the boundary and denote the resulting space $S^1 \times I/\hat{p}$ by $\hat{D}$. Using Mayer-Vietoris sequence we can get that $K^0(\hat{D})$ is trivial and $K^1(\hat{D}) \cong \mathbb{Z}$, which satisfies the isomorphisms $K^0(D) \cong K^1(\hat{D})$ and $K^1(D) \cong K^1(\hat{D})$.

Example 4.54. Consider the rotation action of S^1 over S^2. The north pole N and the south pole S are the only two fixed points. If we cut off small open neighborhoods of N and S respectively, we get a cylinder. Its T-dual is still a cylinder. After gluing we get that a T-dual space $\hat{M}$ which is a hyperbolic paraboloid. Actually this can be seen as we attach another $S^1 \times (-\infty, 0]$ to the other component of the boundary of $\hat{D}$ in the last example. Still using the Mayer-Vietoris sequence, we can get that $K^0(\hat{M})$ is trivial and $K^1(\hat{M}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

In general, we have that the boundary of the "good" part is union of some odd dimensional spheres S^{2n+1}. Therefore the boundary of T-dual space is the union of $W \times S^1$ with twist $\hat{H}$ (Here W is weighted projective space and $\hat{H}$ is the generator of its third cohomology group). For the most simple case, i.e. the case in which there is only one fixed point, we can get the following exact sequence of the associated (twisted) K-groups.

$$\begin{tikzcd}
K^0(S^{2n+1}) \arrow[r] & K^0(P) \oplus \mathbb{Z} \arrow[r] & K^0(M) \arrow[d] \\
K^1(S^{2n+1}) \arrow[u] & K^1(M) \arrow[l] & K^1(P) \arrow[l]
\end{tikzcd} \tag{4.46}$$

For the T-dual part, we have

$$\begin{tikzcd}
K^1(W \times S^1, \hat{H}) \arrow[r] & K^1(\hat{P}, \hat{H}) \oplus \mathbb{Z} \arrow[r] & K^1(\hat{M}, \hat{H}) \arrow[d] \\
K^0(W \times S^1, \hat{H}) \arrow[u] & K^0(\hat{P}, \hat{H}) \arrow[l] & K^1(\hat{M}) \arrow[l]
\end{tikzcd} \tag{4.47}$$

From the above two diagrams we may expect that $K^0(M) \cong K^1(\hat{M}, \hat{H})$ and $K^1(M) \cong K^0(\hat{M}, \hat{H})$ since the groups on other nodes of the diagrams are isomorphic. Unfortunately, we can't prove it here. We list this as a question which is definitely worthwhile to understand.

Question 2. Does the isomorphisms we expect above hold? Or more generally, for any space admitting semi-free S^1-actions, does the above construction gives a model of T-duality pair for which the T-isomorphisms of twisted K-theory hold?

A Appendix

A.1 Principal Bundles

In this section we list some basic definitions and facts about principal torus bundles and principal $PU(\mathbb{H})$-bundles.

Definition A.1. Let G be a topological Lie group and X be a topological space. A **principle G-bundle** over X is a fiber bundle $\pi : P \to X$ together with a right G-action $G \times P \to P$ such that the action preserves the fibers of P and acts freely and transitively on the fibers.

Example A.2. A product space $X \times G$ with the canonical G-action is a (trivial) principal G-bundle.

Example A.3. Let G be a Lie group and T is the maximal torus of G, then $G \to G/T$ is a principal T-bundle over G/T. The action $T \times G \to G$ is induced by the right multiplication in G.

Example A.4. If $\pi : V \to M$ is a smooth vector bundle over M with a Riemiannian metric g, then the structure bundle of V is a (smooth) principal $SO(n)$-bundle. Here n is the dimension of the fiber of V.

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $U(\mathcal{H})$ be the group of unitary operators over $\mathcal{H}$. There is a natural S^1-action over $U(\mathcal{H})$ by multiplication and we call the quotient group $U(\mathcal{H})/S^1$ projective unitary group of $\mathcal{H}$ and denote it by $PU(\mathcal{H})$.

Theorem A.5 ([42])**.** *For an infinite dimensional separable Hilbert space $\mathcal{H}$, the unitary group $U(\mathcal{H})$ with strong topology is contractible.*

Combining with the long exact sequence of homotopy groups, the above theorem implies that $PU(\mathcal{H})$ is a model of Eilenberg-Maclane space $K(\mathbb{Z}, 2)$. Moreover, we get that the classifying space of principal $PU(\mathcal{H})$-bundles is a model of $K(\mathbb{Z},3)$. According to a classical result in algebraic topology below, we get that there exists a one-to-one correspondence between the isomorphism classes of principal $PU(\mathcal{H})$-bundles over X and the third integer cohomology classes of X.

Theorem A.6. *Let G be an abelian group and n is a positive integer. For any topological space X, there is a natural bijection between $[X, K(G,n)]$ and $H^n(X,G)$. Here $[X, K(G,n)]$ is the set of homotopy classes of continuous maps from X to $K(G,n)$.*

We put all of the discussions above together and obtain the classification of principal $PU(\mathcal{H})$-bundles.

Theorem A.7. *There is a natural bijection between the set of isomorphism classes of principal $PU(\mathcal{H})$-bundles over a space X and $H^3(X,\mathbb{Z})$.*

Definition A.8. According to the above theorem, there exists a principal $PU(\mathcal{H})$-bundle $\mathfrak{P}$ over X with characteristic class H for any $H \in H^3(X,\mathbb{Z})$. The adjoint $PU(\mathcal{H})$-action over the set of Fredlhom operators over $\mathcal{H}$ induces an associated bundle $Fred(\mathfrak{P})$. We can define the 0-th twisted K-group of (X,H) to be the homotopy equivalence classes of the continuous sections of $Fred(\mathfrak{P})$. For degree n twisted K-groups $K^n(X,H)$, we can use the corresponding n-fold based loop spaces of $Fred(\mathcal{H})$ instead of $Fred(\mathcal{H})$.

There is another version of twisted K-theory using C^*-algebras.

Definition A.9. Use the notations in the above definition. The $PU(\mathcal{H})$-action over the compact operators $\mathcal{K}(\mathcal{H})$ also gives an associated bundle of $\mathfrak{P}$, which we denote by $\mathcal{A}$. The continuous sections of $\mathcal{A}$ forms a C^*-algebra. We denote it by $C(P,\mathcal{A})$. Then $K^i(P,H)$ is defined to be $K_i(C(P,\mathcal{A}))$.

Remark A.10. The above two versions of twisted K-theory is equivalent.

Now we give the definition of Brauer group as the end of this section.

Definition A.11. Let X be a second countable Hausdorff space. The elements of Brauer group over X are Morita equivalence classes $[A]$ of continuous-trace algebras A with spectrum X. The multiplication is given by the balanced C^*-algebaic tensor product $[A][B] = [A \otimes_{C(X)} B]$, the identity is $[C_0(X)]$ and the inverse of $[A]$ is represented by the conjugate algebra $\bar{A}$.

Let G be a locally compact transformation group given by a homomorphism $a : G \to Homeo(X)$. The elements of equivariant Brauer group $Br_G(X)$ are (A, α) (here A is a continuous-trace C^*-algebra with spectrum X and α is an action of G on A which induces the given action of G on X). The group operation is given by $[A, \alpha][B, \beta] = [A \otimes_{C(X)} B, \alpha \otimes_{C(X)} \beta]$, the identity is $[C_0(X), \tau]$, where $\tau_s(f)(x) = f(s^{-1} \cdot x)$, and the inverse of $[A, \alpha]$ is $[\bar{A}, \bar{\alpha}]$, where $\bar{\alpha}(\bar{a}) = \overline{\alpha(a)}$.

A.2 Fredholm Operators and Compact Operators

Let $\mathcal{H}$ be a Hilbert space, we give two kinds of operators on $\mathcal{H}$ and some of their properties.

Definition A.12. An operator $T : \mathcal{H} \to \mathcal{H}$ over $\mathcal{H}$ is called compact if and only if for every bounded sequence $\{x_n\} \subset \mathcal{H}$, $T(x_n)$ has a subsequence convergent in $\mathcal{H}$. More generally, we can also define the compact operators between two Banach spaces X and Y to be the operators which takes bounded sets in X into precompact sets in Y. An operator F over $\mathcal{H}$ is called a Fredholm operator if its kernel and cokernel spaces are both finite dimensional. Given a Fredholm operator over $\mathcal{H}$, one can define its index $indexF = dim\ kerF - dim\ cokerF$.

Next proposition gives a relation between Fredholm operators and compact operators.

Proposition A.13. A bounded operator F over $\mathcal{H}$ is a Fredholm operator if and only if there is another bounded operator G such that $Id - FG$ and $Id - GF$ are both compact operators.

Its proof can be found in [11]. We give some properties on the compact operators, whose proof can be found in [57].

Theorem A.14.
1. *If $\{T_n\}$ are compact and $T_n \to T$ in the norm topology, then T is compact;*
2. *T is compact iff T^* is compact;*
3. *If S is a bounded operator over $\mathcal{H}$ and T is compact, then ST and TS are both compact;*
4. *If $\mathcal{H}$ is separable, then every compact operator on $\mathcal{H}$ is the norm limit of a sequence of operators of finite rank;*

5. *Let A be a self-adjoint compact operator on $\mathcal{H}$. Then there is a complete orthogonal basis for $\mathcal{H}$ such that A is diagonal on this basis.*

Remark A.15. In particular, if T is diagonal in some basis of $\mathcal{H}$ and the eigenvalues $\{\lambda_n\}$ of T converges to 0 when n goes to ∞, then the first conclusion in the above theorem implies that T is compact.

In the end we give two theorems which give classifying spaces for K^0 and K^1 respectively. The first theorem is given in [2] and the second one is given in [5].

Theorem A.16. *Let $\mathfrak{F}$ be the spaces of Fredholm operators over $\mathcal{H}$. Then for any compact space we have a natural isomorphism*

$$index : [X, \mathfrak{F}] \to K(X). \tag{A.1}$$

Theorem A.17. *Let $\mathfrak{F}^s$ be the spaces of self-adjoint Fredholm operators over $\mathcal{H}$. Then $\mathfrak{F}^s$ is a classifying space for K^1.*

B Appendix

In this part we give a brief introduction to differentiable stack.

B.1 Definition of Topological and Differentiable Stacks

Recall the definition of differentiable stack in [31] and [10] Let $Diff$ be the category of smooth manifolds whose objects are smooth manifolds and morphisms are smooth maps between smooth manifolds.

Definition B.1. *A category fibered in groupoids* over $Diff$ is a category $\mathfrak{X}$ together with a functor $\pi : \mathfrak{X} \to Diff$ satisfying:

1. for every arrow $V \to U$ in $Diff$, and every object x of $\mathfrak{X}$ lying over U, there exists an arrow $y \to x$ in $\mathfrak{X}$ lying over $V \to U$;

2. for every commutative triangle in $Diff$
$$\begin{tikzcd} V \arrow[dr, "f"] \arrow[dd, "h"'] & \\ & U \arrow[dl, "g"] \\ W & \end{tikzcd}$$
and $F : x \to z, G : y \to z$ are lifting of f and g, then there is a unique morphism $H : x \to y$ lying over $h : x \to y$ such that the below triangle commutes
$$\begin{tikzcd} x \arrow[dr, "F"] \arrow[dd, "H"'] & \\ & z \arrow[dl, "G"] \\ y & \end{tikzcd}$$

Sometimes a *category fibered in groupoids* over $Diff$ is also called a pre-stack. For any $U \in Diff$ we define a category $\mathfrak{X}(U)$ with objects $x|\pi(x) = (U), x \in \mathfrak{X}$ and with morphisms $Mor_{\mathfrak{X}(U)}(x, y) = Mor_{\mathfrak{X}}(x, y)$. A category fibered in groupoids $\mathfrak{X}$ is called a stack if for any $M \in Diff$ and any open cover $(U_i)_{i\in I}$ of M, the following two conditions are satisfied:

1. Given two objects A, B over M and any family $(\phi_i : A_{|U_i} \to B_{|U_i})$ of maps such that $(\phi_i)_{|U_i \cap U_j} = (\phi_j)_{|U_i \cap U_j}$, there exists a unique morphism $\phi : A \to B \in \mathfrak{X}$ over $id : M \to M$ such that $\phi_{|U_i} = \phi_i$ for any i;

2. Assume we have objects A_i lying over U_i for any i, together with isomorphisms $\varphi_{ij} : (A_j)_{|U_i \cap U_j} \to (A_i)_{|U_i \cap U_j}$ lying over $id : U_i \cap U_j \to U_i \cap U_j$, which satisfy the cocyle condition:

$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$$

on $U_i \cap U_j \cap U_k$ for any i, j and k. Then there exists an object A lying over M, together with isomorphisms $\varphi_i : A_{|U_i} \to A_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$.

If we replace the category $Diff$ by the category of topological spaces Top, we get the definition of topological stacks.

Remark B.2. All of the categories fibered in groupoids (or pre-stacks) over $Diff$ can be naturally organized into a 2-category. A morphism f between two pre-stacks $\pi_{\mathfrak{X}} : \mathfrak{X} \to Diff$ and $\pi_{\mathfrak{Y}} : \mathfrak{Y} \to Diff$ is a functor $f : \mathfrak{X} \to \mathfrak{Y}$ between the underlying categories and $\pi_{\mathfrak{Y}} \circ f = \pi_{\mathfrak{X}}$. Given two morphisms f and g, a 2-morphism $\varphi : f \Rightarrow g$ between them is a natural transformation of functors φ from f to g such that $\pi_{\mathfrak{Y}} \circ \varphi\varphi$ is the identity transformation from π_{χ} to itself. We denote this two category by **Pre − Stack**. Similar, we can get its subcategory with stacks as objects, which we denote by **Stack**.

We give two simple examples of stacks. For more example, one can find in [31] and [10].

Example B.3. Let M be a manifold. We define the category $[M]$ with objects all smooth maps $f : U \to M$. The morphisms between $f : U \to M$ and $g : V \to M$ are smooths maps $\psi : U \to V$ s.t $f = g \circ \phi$. The functor from $[M]$ to $Diff$ maps $U \to M$ to U.

Example B.4. Let G be a Lie group. Define the category BG with objects principal G-bundles over smooth manifolds. Morphisms are bundle morphisms between principal G-bundles. The functor from BG to $Diff$ maps a principal G-bundle $P \to M$ to the base space M.

Given two pre-stack morphisms $F : \mathfrak{X} \to \mathfrak{Z}$ and $G : \mathfrak{Y} \to \mathfrak{Z}$, we can define a fiber product of pre-stacks as follows:

Definition B.5. The fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is defined to be the category fibered in groupoids with objects all triples:

$$\{(x, y, \alpha : F(x) \to G(y)) | U \in Diff, x \in \mathfrak{X}(U), y \in \mathfrak{Y}(U), \alpha \in \mathfrak{Z}(U)\} \tag{B.1}$$

together with morphisms

A morphism between (x, y, α) and (x', y', α') in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is a pair (u, v), in which

- $u : x \to x'$ is a morphism in $\mathfrak{X}$ and $v : y \to y'$ is a morphism in $\mathfrak{Y}$;
- The following diagram commutes

$$\begin{tikzcd}
F(x) \arrow[r, "F(u)"] \arrow[d, "\alpha"] & F(x') \arrow[d, "\alpha'"] \\
G(y) \arrow[r, "G(v)"] & G(y')
\end{tikzcd}$$

The functor $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to Diff$ maps (x, y, α) to U. When the corresponding pre-stacks are all stacks, the fiber product is also a stack.

Now we can give the definition of differentiable stack

Definition B.6. A stack $\mathfrak{X}$ over $Diff$ is called a *differentiable stack* if there exists a morphism $p : [X] \to \mathfrak{X}$, where X is a manifold and $[X]$ is the stack corresponding to X s.t for all maps $[U] \to \mathfrak{X}$, where $U \in Diff$, the fiber product $[X] \times_{\mathfrak{X}} [U]$ is isomorphic to a manifold and the induced map $X \times_{\mathfrak{X}} U \to U$ is a surjective submersion in $Diff$. $[X] \to \mathfrak{X}$ is called an **atlas** for $\mathfrak{X}$.

Note that the fiber product of two differentiable stacks is still a differentiable stack and the two examples we give above are both differentiable stacks.

The following definition gives an analogue of the concept of principal bundle for Lie groupoid.

Definition B.7. Let $\mathcal{G}$ be a Lie groupoid and M be a manifold. A $\mathcal{G}$-torsor over M is a manifold P together with a surjective submersion $\pi : P \to M$ and a right $\mathcal{G}$-action over P such that for all p, $p' \in P$ with $\pi(p) = \pi(p')$, there exists a unique $\gamma \in \mathcal{G}$ such that $p = \gamma \cdot p'$.

For a Lie group G, we have already seen that the category of all principal G-bundles is a differentiable stack. For a Lie groupoid Γ, the following proposition shows that the category of all Γ torsors $B\Gamma$ is also a differentiable stack.

Proposition B.8 (**Proposition 2.21** in [10])**.** For every Lie groupoid Γ, the category of all Γ torsors $B\Gamma$ is a differentiable stack.

B.2 Cohomology of Differentiable Stacks

In [10], they construct a dictionary between differentiable stacks and Lie groupoids. In brief, the classifying space of any Lie groupoid is a differentiable stack and any differentiable stack is isomorphic to a classifying space of a Lie groupoid. Roughly, one can say that there is a bijection between isomorphisms classed of differentiable stacks and Morita equivalent classes of Lie groupoids. According to this dictionary, one can define the cohomology of differentiable stacks using cohomology of their corresponding Lie groupoids.

Assume that χ is a differentiable stack with chart X. Denote the associated Lie groupoid by $X_1 \rightrightarrows X$. For any sheaves of abelian groups $\mathfrak{F}$ over χ (for the associated definitions see [31]) or [34], the sheaf cohomology group of the differentiable stack $H^*(\chi, \mathfrak{F})$ is defined as the derived functor of the global section functor. If $\mathfrak{F}$ is cartesian, then we can use the following proposition in [34] to compute the cohomology of differentiable stacks.

Proposition ([34])**.** *Let $\mathfrak{F}$ be a cartesian sheaf of abelian groups on a stack $\mathfrak{M}$. Let $X \to \mathfrak{M}$ be a atlas and $\mathfrak{F}_\bullet$ the induced sheaf on the simplicial space $X_\bullet$ then there is an E_1 spectral sequence:*

$$E_1^{p,q} = H^q(X_p, \mathfrak{F}_p) \Rightarrow H^{p+q}(\mathfrak{M}, \mathfrak{F}) \tag{B.2}$$

The spectral sequence is functorial with respect to morphisms $\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, Rightarrow] \\ \mathfrak{M} \arrow[r] & \mathfrak{N} \end{tikzcd}$ *for any atalas X, Y of $\mathfrak{M}, \mathfrak{N}$.*

Especially, Behrend and Xu discussed the cohomology of the de-Rham sheaf over differentiable stacks in [10]. Denote the n-th fiber product of X_1 by X_n and the differential operator for the Lie groupoid cohomology by δ, then the de-Rham cohomology of the differentiable stack $\mathfrak{X}$ is the total cohomology of double complex

$$(\Omega^p(X_q), d, \delta). \tag{B.3}$$

B.3 Circle Bundles over Differentiable Stacks and the Gysin Sequence

For any differentiable stack $\mathfrak{X}$ and Lie group G, a left G-action on $\mathfrak{X}$ is a triple (μ, α, a) where $\mu : G \times \mathfrak{X} \to \mathfrak{X}$ is a morphism and α and a are 2-morphism as in the diagrams

$$\begin{tikzcd} G \times G \times \mathfrak{X} \arrow[r, "m \times id_{\mathfrak{X}}"] \arrow[d, "id_G \times \mu"'] & G \times \mathfrak{X} \arrow[d, "\mu"] \\ G \times \mathfrak{X} \arrow[r, "\mu"'] \arrow[ur, Rightarrow, "\alpha"'] & \mathfrak{X} \end{tikzcd} \qquad \begin{tikzcd} G \times \mathfrak{X} \arrow[r, "\mu"] & \mathfrak{X} \\ \mathfrak{X} \arrow[u, "1 \times id_{\mathfrak{X}}"] \arrow[ur, "id_{\mathfrak{X}}"'] \arrow[u, phantom, shift left=3, "\Rightarrow a"] & \end{tikzcd} .$$

Definition B.9. An S^1-bundle over differentiable stack $\mathfrak{X}$ is an S^1-differentiable stack $\mathfrak{P}$ with a stack morphism $\pi : \mathfrak{P} \to \mathfrak{X}$ and the following 2-commutative diagram

$$\begin{tikzcd} \mathfrak{P} \times S^1 \arrow[r, "\mu"] \arrow[d, "proj"'] & \mathfrak{P} \arrow[d, "\pi"] \arrow[dl, Rightarrow] \\ \mathfrak{P} \arrow[r, "\pi"] & \mathfrak{X} \end{tikzcd}$$

such that for any submersion from a manifold U, the pullback via $U \to \mathfrak{X}$ is an S^1-bundle over U.

Like S^1-bundles over manifolds, we can also construct pullback bundles for S^1-bundle over differentiable stacks.

Proposition B.10. Let $\pi : \mathfrak{P} \to \mathfrak{B}$ be an S^1-bundle over differentiable stack $\mathfrak{B}$ and $F : \mathfrak{X} \to \mathfrak{B}$ is a morphism of differentiable stack. Then the fiber product $\mathfrak{X} \times_{\mathfrak{B}} \mathfrak{P}$ is an S^1-bundle over $\mathfrak{X}$ with the obvious bundle map, which we call the pullback bundle of $\mathfrak{P}$ via F and denote by $F^*(\mathfrak{P})$.

Proof. First of all we need to construct the S^1-action over the fiber product $\mathfrak{X} \times_{\mathfrak{B}} \mathfrak{P}$. Denote the S^1-action over $\mathfrak{P}$ by (μ, α, a). We define:

$$\mu' : S^1 \times \mathfrak{X} \times_{\mathfrak{B}} \mathfrak{P} \mapsto \mathfrak{X} \times_{\mathfrak{B}} \mathfrak{P} \tag{B.4}$$

$$\mu'(\theta, (x, p, f)) = (x, \mu(\theta, p), \tilde{f}); \tag{B.5}$$

$$\mu'(\theta, (u, v)) = (u, \tilde{v}) \tag{B.6}$$

Denote the natural transformation in the 2-diagram of (B.9) by β and $\mathfrak{X} \times_{\mathfrak{B}} \mathfrak{P}$ by $\mathfrak{Y}$. Then $\tilde{f}$ is $\beta_{\theta,p} \circ f$ and $\tilde{v}$ is $\mu(\theta, v)$. For any $(\theta_1, \theta_2, (x, p, f)) \in S^1 \times S^1 \times \mathfrak{Y}$, we have

$$\mu' \circ m \times id(\theta_1, \theta_2, (x, p, f)) = (x, \mu(\theta_1\theta_2, p), \beta_{\theta_1\theta_2,p} \circ f), \tag{B.7}$$

$$\mu' \circ (id \times \mu')(\theta_1, \theta_2, (x, p, f)) = (x, \mu(\theta_1, \mu(\theta_2, p)), \beta_{\theta_1,\mu(\theta_2,p)} \circ \beta_{\theta_2,p} \circ f). \tag{B.8}$$

Since $\beta_{\theta_1\theta_2,p} = \alpha(\theta_1,\theta_2,p) \circ \beta_{\theta_1,\mu\theta_2,p} \circ \beta_{\theta_2,p}$, we have there exists a map α':

$$\begin{aligned}&\alpha'_{\theta_1,\theta_2,y}(x,\mu(\theta_1,\mu(\theta_2,p)),\beta_{\theta_1,\mu(\theta_2,p)} \circ \beta_{\theta_2,p} \circ f)\\ =\ &(x,\alpha_{\theta_1,\theta_2,p}\mu(\theta_1,\mu(\theta_2,p)),\alpha_{\theta_1,\theta_2,p}\circ)\beta_{\theta_1,\mu\theta_2,p} \circ \beta_{\theta_2,p} \circ f.\end{aligned}$$

Since α is natural, we have α' is the natural transformation from $\mu' \circ (id \times \mu')$ to $\mu' \circ (m \times id_{\mathfrak{Y}})$. Thirdly, we need to define a natural transformation a' from $id_{\mathfrak{Y}}$ to $\mu' \circ (1 \times id_{\mathfrak{Y}})$. For any $y \in \mathfrak{Y}$, we define a' as follows

$$a'(\mu'(1,y)) = (x, a_p \circ mu(1,p), a_p \circ \beta_{1,p} \circ f). \tag{B.9}$$

Similarly we get a' is the natural transformation between $id_{\mathfrak{Y}}$ and $\mu' \circ (1 \times id_{\mathfrak{Y}})$. The bundle map $\tilde{\pi} : \mathfrak{Y} \to \mathfrak{X}$ is defined to be the projection to $\mathfrak{X}$. According to the construction of μ', naturally we have the following 2-commutative diagram:

$$\begin{tikzcd} S^1 \times \mathfrak{Y} \arrow[r, "\mu'"] \arrow[d, "Proj"'] & \mathfrak{Y} \arrow[d, "\tilde{\pi}"] \\ \mathfrak{Y} \arrow[r, "\tilde{\pi}"'] \arrow[ur, Rightarrow] & \mathfrak{X} \end{tikzcd}$$ □

Note that the S^1-bundles over a differentiable stack $\mathfrak{X}$ are determined by $H^1(\mathfrak{X}, \mathcal{S}^\infty) \cong H^2(\mathfrak{X}, \mathbb{Z})$. Here $\mathcal{S}^\infty$ is the sheaf of S^1-valued function over $\mathfrak{X}$. Therefore we can still define the notion of first Chern class for S^1-bundle over a differentiable stack. For more details, one can see [10]. In [30], they gave the Gysin sequence of homology groups for a S^1-stack $\mathfrak{P}$. Following their proof it is easy to prove the cohomology version.

Proposition B.11. Let $\mathfrak{P}$ be an S^1-bundle over $\mathfrak{B}$. There is a (natural with respect to S^1-equivariant maps of stacks) long exact sequence in cohomology

$$\to H^i(\mathfrak{P}) \overset{\pi_!}{\to} H^{i-1}(\mathfrak{B}) \overset{\cup c}{\to} H^{i+1}(\mathfrak{B}) \overset{\pi^*}{\to} H^{i+1}(\mathfrak{P}) \to \tag{B.10}$$

Where $\pi_!$ is the Gysin map and c is the first Chern class of $\mathfrak{P}$.

Now we give an explicit construction of push-forward map for de-Rham cohomology groups of differential stack.

Let $\mathcal{P} : P_1 \rightrightarrows P_0$ and $\mathcal{B} : B_1 \rightrightarrows B_0$ be the Lie groupoids corresponding to $\mathfrak{P}$ and $\mathfrak{B}$ respectively. Then we have that $\pi_0 : P_0 \to B_0$ is an S^1-bundle. And so is $\pi_1 : P_1 \to B_1$. We know that the de-Rham cohomology of the differentiable stack $\mathfrak{P}$ is defined to be the total cohomology of the double complex $(\Omega^j(P_i), d, \delta)$. Since $\pi_i : P_i \to B_i$ are all S^1-bundles, we get an integration map over the fiber $\int_{S^1} : \Omega^j(P_i) \to \Omega^{j-1}(B_i)$ for each i and j according to the Proposition VIII and Proposition X of chapter XII in [29]. If $\pi : P \to X$

be a principal S^1-bundle over a manifold X and $f : Y \to X$ be a smooth map. Then the Gysin map $\int_{S^1} : \Omega^j_{dR}(P) \to \Omega^{j-1}_{dR}(B)$ satisfies the following identities:

$$\int_{S^1} \circ F^* = f^* \circ \int_{S^1},$$
$$d \circ \int_{S^1} = \int_{S^1} \circ d.$$

According to the above discussion these integration maps $\int_{S^1}$ give a double complex homomorphism between $(\Omega^j(P_i), d, \delta)$ and $(\Omega^j(B_i), d, \delta)$. Therefore they induce a homomorphism of the corresponding cohomology group which we call the push-forward map of de-Rham cohomology group for the S^1-bundle over differentiable stack.

C Appendix

We give a very brief introduction to KK-theory in this appendix.

C.1 Definition of KK-theory and KK-equivalence

To define KK-theory, we first give the definition of Hilbert module.

Definition C.1. Let A be a C^*-algebra. A pre-Hilbert A-module is a right A-module E equipped with a function $< \cdot, \cdot >: E \times E \to A$, with the following properties:

- (1) $< \cdot, \cdot >$ is sequilinear;
- (2)
$$< x, ya >=< x, y > a$$
for all $x, y \in E, a \in A$;
- (3)
$$< y, x >=< x, y >^*$$
for all $x, y \in E$;
- (4) if $< x, x >= 0$, then $x = 0$.

For $x \in E$, put $\| x \|=< x, x >^{1/2}$. It gives a norm of E. If E is complete, E is called a Hilbert A-module.

Example C.2. 1. All of the Hilbert spaces are Hilbert $\mathbb{C}$-module;

2. A is a Hilbert A-module itself with $< a, b >= a^*b$. More generally, any closed right ideal of A is a Hilbert A-module;

3. Let $\mathcal{H}_A$ be the completion of the direct sum of countable copies of A, for any $(a_n),(b_n) \in \mathcal{H}_A$, $<(a_n),(b_n)>=\sum_n a_n^* b_n$. Then $\mathcal{H}_A$ is a Hilbert A-module with this "inner product".

Now we give the definition of bounded operators on Hilbert modules.

Definition C.3. Let E be a Hilbert A-module. $\mathbb{B}_E$ is the set of all module homomorphisms $T : E \to E$ for which there is an adjoint module homomorphism $T^* : E \to E$ with $< Tx, y >=< x, T^*y >$ for all $x, y \in E$.

From now on, let us assume that A and B be graded C^*-algebras. Let $\mathbb{E}(A, B)$ be the set of triples (E, ϕ, F). Here E is a countable generated Hilbert B-module, $\phi : A \to \mathbb{B}(E)$ is a graded $*$-homomorphism and F is an operator in $\mathbb{B}(E)$ of degree 1, such that $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$, and $(F - F^*)\phi(a)$ are all in $\mathbb{K}(E)$ for all $a \in A$. The elements of $\mathbb{E}(A, B)$ are called Kasparov modules for (A, B). $\mathbb{D}(A, B)$ is the set of triples in $\mathbb{E}(A, B)$ for which $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$, and $(F - F^*)\phi(a)$ are all in 0 for all $a \in A$.

Like the homotopy of topological spaces, we also have the notion of homotopy between different Kasparov (A, B)-modules. Given two (A, B)-modules (E_0, ϕ_0, F_0) and (E_1, ϕ_1, F_1), a homotopy connecting them is an element (E, ϕ, F) in $\mathbb{E}(A, IB)$ for which $(E\hat{\otimes}_{f_i}B, f_i \circ \phi, f_{i^*}(F)) \approx_u (E_i, \phi_i, F_i)$, here f_i $(i = 0, 1)$ is the evaluation homomorphism from IB to B and $\approx_u$ means unitary equivalent. Homotopy equivalence is denoted by $\sim_h$ below. Now we can give the definition of $KK(A, B)$.

Definition C.4. $KK(A, B)$ is the set of equivalence classes of $\mathbb{E}(A, B)$ under $\sim_h$. More generally, we define $KK^n(A, B) = KK(A, B\hat{\otimes}\mathbb{C}_n)$. Here $\mathbb{C}_n$ is the complex Clifford algebra.

Proposition C.5. $KK(A, B)$ is an abelian group for any graded C^*-algebras A and B under the direct sum operation.

Proof. see [11] Proposition 17.3.3. □

To see what is a KK-equivalence, we still need to know the construction of Kasparov product. The whole construction is too technical and we will just give the basic definition here. For the details, one can find in [11]. Given graded C^*-algebras A, B and D. We will define a map

$$\hat{\otimes}_D : KK(A, D) \times KK(D, B) \to KK(A, B). \tag{C.1}$$

If $x \in KK(A, D)$ and $y \in KK(D, B)$, we will write $x\hat{\otimes}_D y$ for the product. Before we give the construction of the product, we give a useful definition first:

Definition C.6. Suppose E_1 is a countably generated Hilbert D-module, E_2 a countably generated Hilbert B-module, $\phi : D \to B(E_2)$ is a graded $*$-homomorphism and $F_2 \in B(E_2)$ has the property that $[F_2, \phi(D)] \subset K(E_2)$. An operator $F \in \mathbb{B}(E)$ is called an F_2-connection for E_1 if , for any $x \in E_1$,

$$T_x \circ F_2 - (-1)^{\delta x \delta F_2} F \circ T_x \subseteq \mathbb{K}(E_2, E),$$

$$F_2 \circ T_x^* - (-1)^{\delta x \delta F_2} T_x^* \circ F \subseteq \mathbb{K}(E, E_2).$$

Here $T_x \in L(E_2, E)$ is a bounded operator defined by $T_x(y) = x\hat{\otimes}y$ for each $x \in E_1$. T_x^* is defined by $T_x^*(z\hat{\otimes}y) = \phi(< x, z >)y$.

Now we give the definition of Kasparov product,

Definition C.7. For a Kasparov (A, D)-module (E_1, ϕ_1, F_1) and a (D, B)-module (E_2, ϕ_2, F_2), $(E_1\hat{\otimes}_{\phi_2}E_2, \phi_1\hat{\otimes}_{\phi_2}1, F)$ is called their Kasparov product if

- (a) F is an F_2-connection on E;
- (b) $(E_1\hat{\otimes}_{\phi_2}E_2, \phi_1\hat{\otimes}_{\phi_2}1, F)$ is a Kasparov (A, B)-module;
- (c) For all $a \in A$, $\phi(a)[F_1\hat{\otimes}1, F]\phi(a)^* \geq 0 \bmod \mathbb{K}(E)$.

Finally ,we turn to the definition of KK-equivalence.

Definition C.8. An element $\mathbf{x} \in KK(A, B)$ is a KK-equivalence if there is $\mathbf{y} \in KK(B, A)$ with $\mathbf{xy} = 1_A$, $\mathbf{yx} = 1_B$. A and B are KK-equivalent if there exists a KK-equivalence in $KK(A, B)$.

C.2 UCT and Kunneth Theorem

First of all, we introduce a special class of C^*-algebra we denote by N, which will be used in the statement of the UCT and K*ü*nneth Theorem. We list the conditions for N as follows:

- **N1** N contains $\mathbb{C}$;
- **N2** N is closed under countable inductive limits;

- **N3** If
$$0 \to A \to D \to B \to 0$$
is an exact sequence, and two of the terms are in N, then so is the third;

- **N4** N is closed under KK-equivalence.

Remark C.9. According to [11], N is essentially the class of C^*-algebras which are KK-equivalent to the commutative C^*-algebras.

Next we give some notations below, for any separable C^*-algebras A and B, there are maps:

$$\alpha : K_*(A) \otimes K_*(B) \to K_*(A \otimes B), \tag{C.2}$$

$$\beta : K^*(A) \otimes K_*(B) \to KK^*(A, B), \tag{C.3}$$

$$\gamma : KK^*(A, B) \to Hom(K_*(A), K_*(B)), \tag{C.4}$$

which are natural in A and B. Now we state the theorems without proof below.

Theorem C.10. *(Universal Coefficient Theorem) Assume $A \in N$, then there is a short exact sequence*

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\rho} KK^*(A, B) \xrightarrow{\gamma} Hom(K_*(A), K_*(B)) \to 0 \tag{C.5}$$

The map γ has degree 0 *and ρ has degree* 1. *The sequence is natural in each variable, and splits unnaturally. So if $K_*(A)$ is free or $K_*(B)$ is divisible, then γ is an isomorphism.*

Theorem C.11. *Künneth Theorem Assume $A \in N$, and $K_*(A)$ or $K_*(B)$ is finite generated. Then there is a short exact sequence*

$$0 \to K_*(A) \otimes K_*(B) \xrightarrow{\beta} KK^*(A, B) \xrightarrow{\rho} Tor^{\mathbb{Z}}_1(K^*(A), K^*(B)) \to 0 \tag{C.6}$$

The map β has degree 0 *and ρ has degree* 1. *The sequence is natural in each variable, and splits unnaturally. So if $K^*(A)$ or $K^*(B)$ is torsion-free, β is an isomorphism.*

C.3 Crossed Products and the Thom-Connes Isomorphism

Let A be a C^*-algebra, G a locally compact group, and $\alpha : G \to Aut(A)$ is a continuous homomorphism. Then the triple (A, G, α) is called a covariant system. A covariant representation of (A, G, α) is a pair of representations (π, ρ) of A and G on the same Hilbert space such that $\rho(g)\pi(a)\rho(g)^* = \pi(\alpha_g(a))$. Denote the twisted convolution algebra of (A, G, α) by $C_c(G, A)$, whose elements are continuous maps from G to A with compact support and compositions are given by convolution. Each covariant representation of (A, G, α) gives a representation of $C_c(G, A)$ by integration, and hence a pre C^*-norm on $C_c(G, A)$. The supremum of all these norms is a C^*-norm, and the completion of $C_c(G, A)$ is called the crossed product of A by G, which we denoted by $A \rtimes G$. There is also a formal definition of crossed product from [55]

Definition C.12. A crossed product for a covariant system (A, G, α) is a C^*-algebra B together with a homomorphism $i_A : A \to M(B)$ and a strictly continuous homomorphism $i_G : G \to UM(B)$ (here $UM(B)$ is the unitary elements of the multiplier algebra $M(B)$) satisfying

- $i_A(\alpha_s(a)) = i_G(s)i_A(a)i_G(s)^*$ for $a \in A, s \in G$;
- for every covariant representation (π, U) of (A, G, α), there is a non-degenerate representation $\pi \times U$ of B with $\pi = (\pi \times U) \circ i_A$ and $U = (\pi \times U) \circ i_G$;
- the span of $i_A(a)i_G(z) : a \in A, z \in C_c(G)$ is dense in B, where i_G is the extension from G to $C_c(G)$.

When G is the group of real numbers with addition $\mathbb{R}$, the well-known Thom-Connes isomorphism holds. Here we explain a little bit about this theorem and this isomorphism. Here we use the language of KK-theory.

If α is a continuous action of $\mathbb{R}$ on A, then there are canonical homomorphisms ϕ and φ from A and $C^*(\mathbb{R})$ into $M(A \rtimes_\alpha \mathbb{R})$ as follows.

$$\phi(a)(x)(t) = \alpha_t(a) \cdot x(t);$$
$$\varphi(f)(x)(t) = \int \hat{f}(s)\alpha_s x(t-s)ds,$$

in which $a \in A$, $f \in C^*(\mathbb{R})$, $x \in C(S^1) \rtimes_\alpha \mathbb{R}$ and $\hat{f}$ is the Fourier transformation of f. If f is any bounded complex-valued functions over $\mathbb{R}$, then f defines a multiplier F_f of $A \rtimes \mathbb{R}$ canonically. Now we give a notion in chapter 19 of [11].

Definition C.13. If f is a continuous complex-valued function on $\mathbb{R}$ for which $lim_{t\to+\infty} f(t) = 1$ and $lim_{t\to-\infty} = -1$, the corresponding element $F_f \in M(A \rtimes \mathbb{R})$ is called a Thom operator on $A \rtimes \mathbb{R}$. The triple $(A \rtimes \mathbb{R}, \phi, F_f)$ defines a KK^1-cycle in $KK^1(A, A \rtimes \mathbb{R})$, which is called the Thom element of (A, α) and denote by $\mathbf{t}_\alpha$.

Using the above notations, the Thom-Connes isomorphism can be stated as follows.

Theorem C.14 ([11])**.** *If A is a separable trivially graded C^*-algebra with a continuous action α of $\mathbb{R}$, then $A \rtimes_\alpha \mathbb{R}$ is KK-equivalent to SA. The element $\mathbf{t}_\alpha$ gives the isomorphism of the corresponding K-groups.*

In particular, we can choose f to be $f(s) = \frac{1}{\pi}\int_{-s}^{s} \frac{sint}{t} dt$, whose Fourier transformation is

$$g(t) = \begin{cases} \frac{i}{t} & |t| \leq 1; \\ 0 & |t| > 1. \end{cases}$$

C.4 Representable K-theory

Representable K-theory is a generalization of K-theory for more general spaces or algebras. For example, if a space X is not compact, the classical definition of K-theory does not apply. $C_0(X)$ is always a C^*-algebra but it can be zero for non-compact spaces. One can also consider the algebra of all compact support continuous functions $C_c(X)$ or the whole complex continuous functions over X. However, both of these algebras are not C^*-algebra but σ-C^*-algebras i.e. countable inverse limits of C^*-algebras. For these σ-C^*-algebras the corresponding generalized cohomology theory is representable K-theory, which was developed in [52].

Definition C.15 ([52])**.** Let A a σ-C^*-algebra. Then the stable multiplier algebra of A is the σ-C^*-algebra $M(\mathbb{K} \otimes A)$. The outer stable multiplier algebra $Q(A)$ of A is the σ-C^*-algebra $M(\mathbb{K} \otimes A)/(\mathbb{K} \otimes A)$. If A is a unital σ-C^*-algebra. Then we define its 0-th representable K-group $RK_0(A)$ by $UQ(A)/U_0Q(A)$. Here $U_0Q(A)$ is the path component of identity. If A is not unital and A^+ is its unitization, then the 0-th representable K group $RK_0(A)$ is defined to be the kernel of the map $\phi_* : RK_0(A^+) \to RK_0(C)$ induced by the canonical map $\phi : A^+ \to \mathbb{C}$. Higher degree representable K-groups are defined by $RK_i(A) = RK_0(S^iA)$

Remark C.16. Representable K-theory is quite similar to K-theory. Actually, when A is a C^*-algebras, the definition of representable K-theory coincides with the definition of K-theory. Moreover, it is homotopy invariant. As a generalized cohomology theory, the long exact sequence and Mayer-Vietoris sequence also holds for representable K-theory. For locally compact Hausdorff space X, the representable K-group of $C(X)$ is isomorphic to the K-group of its Stone-Čech compactfication βX

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ESI, Vienna, Austria June 2012

Awards and Achievements

Funding from RTG 1493 for my Ph.D project *July 2011- Dec 2014*

Scholarship of Peking University *Sep 2009- June 2011*

Guanghua Scholarship *2010*

Scholarship of Zhengzhou University *2006*

Teaching

Teaching Assistant for Caculus I, II in Peking University *Sep 2009 - June 2010*

Teaching Assistant for Mathematical Methods in Classical Mechanics *Sep 2010- Jan 2011*

Language

Chinese *(native),* ***English****(fluent),* ***German****(basic)*

Skills

Latex*,* ***Excel****,* ***Word****,* ***Matlab****(basic) and* ***C*** *(basic)*