



On Newton-Okounkov bodies, linear series and
positivity.

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CHAPTER 1

Introduction and overview of the dissertation

Let $X \hookrightarrow \mathbb{P}^n$ be a d -dimensional projective variety. A *Newton-Okounkov body* $\Delta(L) \subseteq \mathbb{R}^d$ is a convex compact real set associated to a line bundle $\pi: L \rightarrow X$. Its main feature is the property that the volume of the body $\text{vol}_{\mathbb{R}^d}(\Delta(L))$ equals the volume of the line bundle L multiplied by $d!$. Recall that the volume of the line bundle L is defined as

$$\text{vol}(L) := \lim_{k \rightarrow \infty} \frac{h^0(X, L^{\otimes k})}{k^d/d!}.$$

Let us now consider the Serre-Twisting sheaf $\mathcal{O}_X(1)$ (which is the dual of the tautological line bundle) corresponding to the given embedding of X . The volume $\text{vol}(\mathcal{O}_X(1))$ is equal to the classical notion of the *degree* of a projective variety. Hence, we have given the degree of a variety $X \hookrightarrow \mathbb{P}^n$ a convex geometric interpretation as the volume of a convex real body $\Delta(\mathcal{O}_X(1))$. This is the starting point for the far reaching bridge between algebraic geometry and convex geometry via the theory of Newton-Okounkov bodies.

1. Historical background

The idea that one can associate polytopes Δ to a variety X which carries information about X goes back to the Russian school in the mid 70's (Bernstein, Khovanskii and Kushnirenko). In particular, they considered Newton polytopes of multivariate polynomials. Let $f = \sum_{m \in M_f} a_m x^m$ be a polynomial in n variables. Then the *Newton polytope* of f is given by

$$\Delta(f) = \text{conv}(\{m \mid m \in M_f\}).$$

The famous theorems by Bernstein and Kushnirenko answer the following question. First fix n distinct finite subsets $M_i \subseteq \mathbb{N}^n$ and define L_i as the vector spaces of polynomials which are generated by the monomials x^m for $m \in M_i$.

QUESTION. *How many solutions in $(\mathbb{C}^*)^n$ has the system of equations*

$$P_1 = \dots = P_n = 0$$

where the $P_i \in L_i$ are generic?

They gave the following answer.

THEOREM 1.1 ([B75],[K76]). *The number of solutions of the above system of equations is equal to the mixed volume $V(\Delta_1, \dots, \Delta_n)$, multiplied by $n!$. In particular if $\Delta_1 = \dots = \Delta_n =: \Delta$, then the number of solutions is equal to $\text{vol}(\Delta)$, multiplied by $n!$.*

Khovanskii studied similar correspondences in [Kh77] and [Kh78]. For example, he considered curves $X \subset (\mathbb{C}^*)^2$ defined by a generic polynomial P and its associated Newton polytope $\Delta(P)$. The general flavour of its studies is that one can interpret many discrete invariants of X as invariants of the polytope $\Delta(P)$. One example of such an interpretation is that the genus of X is equal to the number of integral points in the interior of $\Delta(P)$.

The above mentioned work lay the foundation for the correspondence of toric varieties, introduced by Demazure in 1970, and polytopes. A normal variety X is *toric* if it contains $T = (\mathbb{C}^*)^n$ as an open subset and the action of T on itself extends to X . These varieties are completely described by a fan Σ and the polytopes P admitting Σ as its normal fan correspond to the embeddings $X \hookrightarrow \mathbb{P}^n$ into projective space. This leads to a complete dictionary of algebraic properties of toric varieties and combinatorial/convex geometric properties of fans/polytopes. Its translations gave rise to new results on both sides and has been an active area of research ever since.

However, all the above correspondences between algebraic geometry and convex geometry work only for particular classes of projective varieties. There was no correspondence for an arbitrary projective variety X . It was Okounkov who in his papers [O96] and [O00] gave a construction of a convex body associated to an embedded variety, carrying its degree as the volume. It took ten years after Okounkov's first paper on this construction before two independent foundational manuscripts ([KK12] and [LM09]) were published in which his ideas were developed into a systematic theory.

In the following we want to present the state of the art in the theory of Newton-Okounkov bodies from the starting point of [KK12] resp. [LM09] and explain our contributions to the field.

2. Construction and first properties of Newton-Okounkov bodies

In this section we want to give a very brief overview of the construction of Newton-Okounkov bodies and state some elementary facts about them. We will mainly follow the lines of [LM09]

2.1. Construction. Let X be a projective variety of dimension d over an algebraically closed field k of characteristic 0. Let D be a (Cartier) divisor on X and consider the corresponding line bundle $L = \mathcal{O}_X(D)$. We choose a flag

$$Y_\bullet: X = Y_0 \supset Y_1 \supset \dots \supset Y_d = \{pt\}$$

where $Y_i \subset X$ is a closed subvariety of X of codimension i which is smooth at the point $\{pt\}$. Next, we construct a map

$$\nu_{Y_\bullet}: H^0(X, L^{\otimes k}) \setminus \{0\} \rightarrow \mathbb{Z}^d \quad s \mapsto (\nu_1(s), \dots, \nu_d(s))$$

for $k \in \mathbb{N}$ as follows. Let $s \in H^0(X, L^{\otimes k}) = H^0(X, \mathcal{O}_X(kD))$ be a global section. We choose an open set U around $\{pt\}$ on which s defines a regular function $f = s|_U$ and Y_1 is given by the zero set of a regular function g . Then we define

$$\nu_1(s) = \text{ord}_{Y_1}(s)$$

which is defined as the maximal integral k such that f is divisible by g^k in the ring of regular functions. We can then define a section $\tilde{s}_1 \in H^0(X, \mathcal{O}_X(kD - \nu_1(s)Y_1))$ which does not vanish at Y_1 and is given on U by $f/g^{\nu_1(s)}$. We denote by s_1 the

restriction of this section to Y_1 . Now, by choosing an open set U_1 on Y_1 , we can in the same manner define the number

$$\nu_2(s) = \text{ord}_{Y_2}(s_1).$$

Iterating this procedure d -times defines the map ν_{Y_\bullet} .

The two essential properties of ν_{Y_\bullet} are:

- ordering \mathbb{Z}^d lexicographically, we have

$$\nu_{Y_\bullet}(s_1 + s_2) \geq \min\{\nu_{Y_\bullet}(s_1), \nu_{Y_\bullet}(s_2)\}$$

for any $s_1, s_2 \in H^0(X, L^{\otimes k}) \setminus \{0\}$

- given two non zero sections $s \in H^0(X, L^{\otimes k}) \setminus \{0\}$ and $t \in H^0(X, L^{\otimes l}) \setminus \{0\}$ then

$$\nu_{Y_\bullet}(s \otimes t) = \nu_{Y_\bullet}(s) + \nu_{Y_\bullet}(t).$$

We call a map ν which satisfies the above properties a *valuation-like function*.

Now we can define the following semigroup associated to a divisor D

$$\Gamma_{Y_\bullet}(D) := \{(\nu_{Y_\bullet}(s), k) : s \in H^0(X, \mathcal{O}_X(kD)) \setminus \{0\}, k \in \mathbb{N}\} \subseteq \mathbb{N}^{d+1}.$$

Then the *Newton-Okounkov body of D corresponding to the flag Y_\bullet* is given by

$$\Delta_{Y_\bullet}(D) := \overline{\text{Cone}(\Gamma_{Y_\bullet}(D))} \cap (\mathbb{R}^d \times \{1\}).$$

2.2. First properties. By far the most interesting property about the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is the following.

THEOREM 2.1 ([LM09]). *Let X be a projective variety, Y_\bullet an admissible flag and D a big divisor. Then*

$$\text{vol}_{\mathbb{R}^d}(\Delta_{Y_\bullet}(D)) = \frac{1}{d!} \cdot \text{vol}(D)$$

where

$$\text{vol}(D) := \lim_{k \rightarrow \infty} \frac{\dim H^0(X, \mathcal{O}_X(kD))}{k^d/d!}.$$

We make several remarks.

- REMARK 2.2.**
- *The construction and the shape of $\Delta_{Y_\bullet}(D)$ depend on the choice of the flag Y_\bullet . However, the above theorem shows that the volume of $\Delta_{Y_\bullet}(D)$ is independent of the choice.*
 - *The Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is a convex compact set in \mathbb{R}^d . However, it is not necessarily a polytope. It might nevertheless happen that for some flags it is a polytope, while for others it is not.*
 - *If X is a toric variety, D a T -invariant divisor and Y_\bullet is a T -invariant flag. Then, up to translation, the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ recovers the usual correspondence for divisors and polytopes in toric geometry.*

In general, Newton-Okounkov bodies are hard to compute. However, one major tool for doing so is to consider *restricted linear series*. For this purpose consider a closed subvariety $X \subset Y$ and a divisor D on Y . Then the restriction morphism of global sections

$$\text{rest}: H^0(Y, \mathcal{O}_Y(D)) \rightarrow H^0(X, \mathcal{O}_X(D))$$

is not necessarily surjective. We define the vector space $H^0(Y, \mathcal{O}_Y(D))|_X$ as the image of the above restriction map.

Then the *restricted Newton-Okounkov body* is defined by

$$\Delta_{Y|X}(D) = \overline{\text{Cone}(\{(\nu_{Y_\bullet}(s), k) : s \in H^0(Y, \mathcal{O}_Y(kD))|_X \setminus \{0\}, k \in \mathbb{N}\})} \cap (\mathbb{R}^d \times \{1\}).$$

The idea is now that we can interpret the slices $\Delta_{Y_\bullet}(D)_{\nu_1=t} := \Delta_{Y_\bullet}(D) \cap (\{t\} \times \mathbb{R}^d)$ as restricted Newton-Okounkov bodies of dimension $d-1$. More concretely, we have the following theorem.

THEOREM 2.3 ([LM09]). *Let X be a projective variety, Y_\bullet an admissible flag such that Y_1 is a divisor and D a big divisor. Assume furthermore that $Y_1 \notin \mathbf{B}_+(D)$. Let $\mu \in \mathbb{R}$ be the maximal number such that $D - \mu Y_1$ is effective. Then for all $0 \leq t < \mu$, we have*

$$\Delta_{Y_\bullet}(D)_{\nu_1=t} = \Delta_{X|Y_1}(D - tY_1).$$

Here, $\mathbf{B}_+(D)$ is the *augmented base locus* defined by $\mathbf{B}_+(D) = \mathbf{B}(D - A)$ for any small enough ample \mathbb{Q} -divisor A . Note that the rather technical condition $Y_1 \notin \mathbf{B}_+(D)$ makes sure that the slice $\Delta_{Y_\bullet}(D)_{\nu_1=t}$ is a $d-1$ dimensional body.

2.3. Global Newton-Okounkov body. Having defined the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$, we are interested in the question how $\Delta_{Y_\bullet}(D)$ changes when varying the divisor D . For this purpose we work in the *Néron-Severi* space $N^1(X)$ which is the group of divisors $\text{Div}(X)$ modulo numerical equivalence. This means that two divisors D_1 and D_2 are identified with each other if for all curves C the intersection products $(D_1 \cdot C) = (D_2 \cdot C)$ coincide. This group has the advantage of being finitely generated and consequently the associated vector space $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is finite dimensional. It turns out that $\Delta_{Y_\bullet}(D)$ only depends on the numerical class of D , so that it makes sense to talk of the Newton-Okounkov body of a class $\xi \in N^1(X)$. The answer to the question how the $\Delta_{Y_\bullet}(\xi)$ vary as we vary $\xi \in N^1(X)_{\mathbb{R}}$ can be answered by the following theorem.

THEOREM 2.4 ([LM09]). *There exists a closed convex cone $\Delta_{Y_\bullet}(X) \subset \mathbb{R}^d \times N^1(X)_{\mathbb{R}}$ such that for each $\xi \in N^1(X)$ which is big we have*

$$pr_2^{-1}(\xi) \cap \Delta_{Y_\bullet}(X) = \Delta_{Y_\bullet}(\xi).$$

We call $\Delta_{Y_\bullet}(X)$ the *global Newton-Okounkov body* of X corresponding to Y_\bullet .

The above theorem shows that there is a universal convex object which carries information about all the Newton-Okounkov bodies corresponding to the flag Y_\bullet .

2.4. Generic Newton-Okounkov body. Similarly as in the previous section, we can ask the following question. How does $\Delta_{Y_\bullet}(D)$ change when varying the flag Y_\bullet ? One major difference between this question and the one posed in the last section is that the geometry of the parameter space, i.e. the moduli space of admissible flags of X , is a lot more complex. However, suppose we have the following situation. Let T be a parameter space and $\pi: X_T \rightarrow T$ a flat family such that for each $t \in T$ the fiber $\pi^{-1}(t) = X_t$ is an irreducible subvariety of dimension d . Suppose furthermore we have a partial flag

$$\mathcal{Y}_\bullet: X_T \supset \mathcal{Y}_1 \supset \cdots \supset \mathcal{Y}_d$$

such that the fiber over t defines an admissible flag $Y_{t,\bullet}$ for each $t \in T$. Let D be a divisor on X_T . Then we can for each $t \in T$ define the Newton-Okounkov body $\Delta_{Y_{t,\bullet}}(D_t)$ on X_t . In this situation the following holds.

THEOREM 2.5. *Let $\pi: X_T \rightarrow T$, \mathcal{Y}_\bullet and D be as described above. Then for a very general choice of $t \in T$, the Newton-Okounkov bodies $\Delta_{Y_{t,\bullet}}(D_t)$ all coincide.*

This theorem allows to define a generic Newton-Okounkov body on X for a divisor D without the choice of a flag Y_\bullet . It shows that the following construction gives rise to a canonical Newton-Okounkov body. Let $x \in X$ be a very general point. Consider the blow-up $\pi: Bl_x(X) \rightarrow X$ of X in x . Let $E_x \cong \mathbb{P}(T_x X) \cong \mathbb{P}^{n-1}$ be the exceptional divisor and let

$$T_x X \supset V_1 \supset V_2 \supset \cdots \supset V_{d-1} \supset \{0\}$$

be a very general flag of subspaces. Taking the projectivization of the above flag, induces an admissible flag Y_\bullet on $Bl_x(X)$ and the above theorem shows that the Newton-Okounkov body $\Delta_{Y_\bullet}(\pi^* D)$ is canonically defined.

Although we have a canonically defined Newton-Okounkov body, the above theorem gives us no tool for constructing it. In general, it is almost never possible to compute such a canonically defined Newton-Okounkov body.

3. Positivity and Newton-Okounkov bodies

Positivity is a central concept in algebraic geometry.

Given a line bundle $L = \mathcal{O}_X(D)$ on a projective variety X one can associate a rational map

$$h_D: X \rightarrow \mathbb{P}^{n-1}$$

which is locally just given as follows. Let $s_1, \dots, s_n \in H^0(X, L)$ be a basis of the global sections. For each $x \in X$ such that $(s_1(x), \dots, s_n(x)) \neq 0$ we define $h_D(x) = (s_1(x) : \cdots : s_n(x)) \in \mathbb{P}^{n-1}$.

The positivity of a line bundle is a measure of ‘how many global sections’ the line bundle L admits. Another way of interpreting positivity is that the corresponding rational map h_D should, in some sense, be well behaved, e.g. a regular map, birational map, closed immersion etc.. In the following we summarize the most important positivity properties:

- A line bundle $L = \mathcal{O}_X(D)$ is called *effective* if it admits a non-zero global section $0 \neq s \in H^0(X, L)$.
- A line bundle $L = \mathcal{O}_X(D)$ is called *big* if $\text{vol}(L) = \lim_{k \rightarrow \infty} \frac{h^0(X, L^{\otimes k})}{k^d/d!} > 0$, where d is the dimension of X . Equivalently, this means that the rational map $h_D: X \rightarrow \mathbb{P}^{n-1}$ defined by $\mathcal{O}_X(kD)$ for some $k \gg 0$ is birational onto its image.
- A line bundle $L = \mathcal{O}_X(D)$ is called *base point-free* if the map h_D is regular. This means that for each $x \in X$, there is a section $s \in H^0(X, L)$ which does not vanish at x .
- A line bundle $L = \mathcal{O}_X(D)$ is called *very ample* if the map h_D defines a closed immersion.

It turns out that a lot of positivity properties such as bigness and ampleness are actually numerical properties, i.e. if $D_1 \equiv_{\text{num}} D_2$ then D_1 has property P if and

only if D_2 has property P . Hence, we will define the following cones in the finite dimensional vector space $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

- $\text{Big}(X) = \text{Cone}(\{[D] \in N^1(X) \mid \mathcal{O}_X(D) \text{ is a big line bundle}\})$, which we call the big cone .
- $\text{Amp}(X) = \text{Cone}(\{[D] \in N^1(X) \mid \mathcal{O}_X(D) \text{ is a very ample line bundle}\})$, which we call the ample cone.

Note that the big and the ample cone are open. We call its closure the pseudo-effective, resp. the nef cone and denote it by $\overline{\text{Eff}}(X)$, resp. $\text{Nef}(X)$.

3.1. Previous work on Newton-Okounkov bodies and positivity. The connection between positivity and Newton-Okounkov bodies stems from the following observation by S.-Y. Jow.

THEOREM 3.1 ([J10]). *Let X be a normal projective variety. Two big divisors D_1, D_2 are numerical equivalent if for all admissible flags Y_{\bullet} we have*

$$\Delta_{Y_{\bullet}}(D_1) = \Delta_{Y_{\bullet}}(D_2).$$

Note that the reverse direction was already proven in [LM09]. Philosophically, the above theorem tells us that we can read off all numerical properties of a big divisor D from the set of Newton-Okounkov bodies $\Delta_{Y_{\bullet}}(D)$. Thus, we should be able to translate algebro geometric properties into properties of real convex bodies. This is in particular possible for positivity properties. In [KL14] and [KL17] Küronya and Lozovanu translate the properties of being nef/ample into conditions on Newton-Okounkov bodies. More concretely, they prove the following.

THEOREM 3.2 ([KL17]). *Let L be a line bundle on a projective variety X .*

- $L = \mathcal{O}_X(D)$ is nef if and only if for all admissible flags Y_{\bullet} the corresponding Newton-Okounkov body $\Delta_{Y_{\bullet}}(D)$ contains the origin O .
- $L = \mathcal{O}_X(D)$ is ample if and only if for all admissible flags Y_{\bullet} the corresponding Newton-Okounkov body $\Delta_{Y_{\bullet}}(D)$ contains a standard simplex $\Delta_{\varepsilon} := \{(x_1, \dots, x_d) \in \mathbb{R}_{\geq 0}^d \mid \sum_{i=1}^d x_i \leq \varepsilon\}$ for some $\varepsilon > 0$.

In fact, they prove a more refined version of this by considering the numerical base loci $\mathbf{B}_+(D)$ and $\mathbf{B}_-(D)$. We refer to [ELMNP09] for an introduction on these loci. What they actually prove is the following.

- $x \notin \mathbf{B}_-(D)$ if and only if for all admissible flags Y_{\bullet} such that $Y_d = \{x\}$ the origin is contained in $\Delta_{Y_{\bullet}}(D)$
- $x \notin \mathbf{B}_+(D)$ if and only if for all admissible flags Y_{\bullet} such that $Y_d = \{x\}$, there is an $\varepsilon > 0$ such that $\Delta_{\varepsilon} \subset \Delta_{Y_{\bullet}}(D)$.

Theorem 3.2 now follows from the above together with the fact that $\mathbf{B}_-(D) = \emptyset \leftrightarrow D$ is nef and $\mathbf{B}_+(D) = \emptyset \leftrightarrow D$ is ample

From the above described analysis and Jow's theorem on numerical equivalence the following question seems to be natural.

QUESTION. *Let $x \in X$. Suppose D_1 and D_2 are two divisors such that for all admissible flags Y_{\bullet} for which $Y_d = \{x\}$ we have $\Delta_{Y_{\bullet}}(D_1) = \Delta_{Y_{\bullet}}(D_2)$. How are D_1 and D_2 related?*

For surfaces X the above question was answered by Roé in terms of the Zariski decomposition of a divisor. If D is a divisor on a surface X , then the Zariski decomposition of D is given by $D = P(D) + N(D)$ where $P(D)$ is nef and $N(D)$

has only negative curves in its support. Building upon the Zariski decomposition, Roé defines a refinement of this decomposition. For each point $x \in X$ we decompose the negative part $N(D) = N_x(D) + N_x^c(D)$ into components which go through the point x and those which do not. Having defined this refinement, he proves the following.

THEOREM 3.3 ([Roé16]). *Let X be a smooth projective surface and $x \in X$. Then $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$ for all admissible flags Y_\bullet centered at x if and only if $P(D_1) \equiv_{num} P(D_2)$ and $N_x(D_1) = N_x(D_2)$.*

He ends his paper with raising the following question.

QUESTION. *Can the above theorem be generalized to projective varieties of higher dimension?*

3.2. Our contribution: generalizing Roé’s theorem. In our article [BM18] we answer the question above, posed by Roé. Note that Zariski decompositions do not necessarily exist in higher dimension. However, we consider a similar decomposition, namely, the Nakayama σ -decomposition introduced in [N04]. We write $D = P_\sigma(D) + N_\sigma(D)$ for this decomposition. The main difference between the Nakayama σ -decomposition and the Zariski decomposition is the fact that $P_\sigma(D)$ does not have to be nef, but rather has the property that the base locus $\mathbf{B}_-(D)$ is small, i.e. of codimension at least two. We can analogously as in the Zariski case consider a refinement of the σ -decomposition by decomposing $N_\sigma(D) = N_{\sigma,x}(D) + N_{\sigma,x}^c(D)$. Then we are able to prove a natural generalization of Theorem 3.3, which answers the above question.

THEOREM 3.4. *Let X be a smooth projective variety. Let $x \in X$. Then $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$ for all admissible flag Y_\bullet centered over x if and only if $P_\sigma(D_1) \equiv_{num} P(D_2)$ and $N_{\sigma,x}(D_1) = N_{\sigma,x}(D_2)$.*

Note that the ‘ \leftarrow ’ direction is the easier direction and can be proved rather directly. The more complicated and technical direction is ‘ \rightarrow ’. For this direction Roé’s ideas cannot be used since they are too surface-specific. The way we proceed instead is to analyze the proof of Jow’s theorem step by step and make sure that his proof still works by considering just flags where Y_d is fixed.

As a byproduct of this analysis we obtain a new criterion for a big divisor to be nef via restricted volumes. More concretely, we prove the following.

THEOREM 3.5. *Let D be a big divisor on X . Then the following conditions are equivalent.*

- D is nef.
- For all $Y \notin \mathbf{B}_+(D)$ we have

$$\text{vol}_{X|Y}(D) = \text{vol}_Y(D|_Y).$$

The restricted volume $\text{vol}_{X|Y}(D)$ is defined as follows. Let D be a divisor on the projective variety X . Let furthermore $Y \subseteq X$ be a closed subvariety of dimension d . Then we define

$$\text{vol}_{X|Y}(D) = \lim_{k \rightarrow \infty} \frac{\dim(H^0(X, \mathcal{O}_X(D))|_Y)}{k^d/d!}$$

where $H^0(X, \mathcal{O}_X(D))|_Y$ denotes the image of the restriction morphism

$$\text{rest}: H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(Y, \mathcal{O}_Y(D)).$$

This volume is equal to the volume of the restricted Newton-Okounkov body $\Delta_{X|Y}(D)$ multiplied by $d!$.

4. Rational polyhedrality of N.-O. bodies and toric degenerations

In the construction of Newton-Okounkov bodies $\Delta_{Y_\bullet}(D)$ the choice of a flag Y_\bullet is crucial. It seems to be desirable to find those flags Y_\bullet such that the shape of $\Delta_{Y_\bullet}(D)$ is not too complicated. One of the fundamental objects of study in combinatorial convex geometry is the notion of a lattice polytope. A slightly more general notion is a rational polytope, i.e. a polytope with rational extreme points. If we could find a flag Y_\bullet such that $\Delta_{Y_\bullet}(D)$ is a rational polytope we would have a convex combinatorial description of the divisor D on X . Thus, we ask the following question.

QUESTION. *Given a divisor D on X . Does there exist a flag Y_\bullet such that $\Delta_{Y_\bullet}(D)$ is rational polyhedral?*

Note that there are divisors D such that its volume $\text{vol}(D)$ is an irrational number. Hence, the answer to the above question cannot always be yes. However, if we assume that the algebra of sections $R(X, D) = \bigoplus_{k \in \mathbb{N}} H^0(X, \mathcal{O}_X(kD))$ is finitely generated, the volume is indeed rational. We call D finitely generated in this case. Note that in particular all free divisors are finitely generated. For finitely generated divisors, Anderson, Küronya and Lozovanu gave an affirmative answer to the above question.

THEOREM 4.1 ([AKL12]). *Let X be a projective normal variety and D a big divisor on X which is finitely generated. Then there exists a flag Y_\bullet such that $\Delta_{Y_\bullet}(D)$ is a rational simplex.*

Although this theorem answers the above question, the resulting Newton-Okounkov bodies are all simplices of length one except for the first coordinate. This implies that apart from the volume of the divisor they do not contain any particular information about the divisor in question. An explanation for this circumstance is that $\Delta_{Y_\bullet}(D)$ is in some sense not universal enough since the choice of Y_\bullet heavily depends on the divisor D . Thus, it seems to be more interesting to ask if there is a universal flag Y_\bullet such that $\Delta_{Y_\bullet}(D)$ is rational polyhedral for all divisors on X . In general, the divisors on X will not all be finitely generated. However, the finite generation of divisors is a natural property for Mori dream spaces, introduced in [HuKe00]. Hence, the following question posed in [LM09] seems natural.

QUESTION. *Let X be a Mori dream space. Does there always exist a flag Y_\bullet such that the global Newton-Okounkov body $\Delta_{Y_\bullet}(X)$ is rational polyhedral?*

In the surface case, it was proven in [KLM12] that for all choices of flags Y_\bullet the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is a finite rational polygon.

Moreover, an affirmative answer to the above question was given by Schmitz and Seppänen. In fact, they give a proof of the following more general statement.

THEOREM 4.2 ([SS16]). *Let X be a surface which admits a rational polyhedral pseudo-effective cone, then for a general flag Y_\bullet the global Newton-Okounkov body is rational polyhedral.*

Apart from proving the above theorem, they also provide a concrete set of generators. Other classes of varieties for which the above question could be answered affirmatively are complexity one varieties ([P11]) and Bott-Samelson varieties ([SS17]). Closely related to the rational polyhedrality of $\Delta_{Y_\bullet}(D)$ is the finite generation of the semigroup $\Gamma_{Y_\bullet}(D)$, which is a stronger property. The importance of the finite generation of $\Gamma_{Y_\bullet}(D)$ is due to the following observation by Anderson.

THEOREM 4.3 ([A13]). *Let X be a projective variety and A be an ample divisor on X . Let Y_\bullet be an admissible flag such that $\Gamma_{Y_\bullet}(A)$ is finitely generated. Then there exists a toric degeneration of X to the toric variety $X_0 = \text{Proj}(k[\Gamma_{Y_\bullet}(A)])$ whose normalization is the normal toric variety corresponding to the polytope $\Delta_{Y_\bullet}(A)$ via the usual correspondence.*

The above theorem states that, under the assumption that $\Gamma_{Y_\bullet}(A)$ is finitely generated, there is a deep connection between X and the toric variety X_0 . Indeed, they share the same Hilbert polynomial, which means that their dimension, degree, genus, etc. coincide.

4.1. Our contributions on normal toric degenerations. In this section we present our results from the article [M18].

Anderson's Theorem gives a connection between X and the toric variety whose normalization corresponds to $\Delta_{Y_\bullet}(A)$. However, this connection is rather implicit since it involves taking the normalization of $X_0 = \text{Proj}(k[\Gamma_{Y_\bullet}(A)])$. We can omit this problem if we can make sure that X_0 is already normal. But this is connected to the normality of the semigroup $\Gamma_{Y_\bullet}(A)$, which means that

$$\text{Cone}(\Gamma_{Y_\bullet}(A)) \cap \mathbb{Z}^d = \Gamma_{Y_\bullet}(A).$$

So in [M18], we deal with the answer of the following question.

QUESTION. *Given a divisor D , when is the semigroup $\Gamma_{Y_\bullet}(D)$ normal finitely generated for a flag Y_\bullet ?*

The answer to this question involves the notion of the *Ehrhart polynomial*. For a lattice polytope $\Delta \subseteq \mathbb{R}^d$ the corresponding Ehrhart polynomial P_Δ is given by

$$P_\Delta(k) = |k \cdot \Delta \cap \mathbb{Z}^d|.$$

The Hilbert function of D is given by $h_D(k) = h^0(X, \mathcal{O}_X(kD))$.

The following is a first quite general answer.

THEOREM 4.4. *Let X be a projective variety, Y_\bullet an admissible flag and D a very ample divisor such that $\Delta_{Y_\bullet}(D)$ is a rational polytope. Then $\Gamma_{Y_\bullet}(D)$ is normal and finitely generated if and only if the Hilbert function and the Ehrhart polynomial define the same function.*

We always have $h_D \leq P_{\Delta_{Y_\bullet}(D)}$. If D is very ample and $\Delta_{Y_\bullet}(D)$ is a lattice polytope, then h_D resp. $P_{\Delta_{Y_\bullet}(D)}$ are both polynomials of degree $\dim X$ with its first coefficient equal to $\text{vol}(D)/d!$ resp. $\text{vol}(\Delta_{Y_\bullet}(D))$. But from the theory of Newton-Okounkov bodies we know that these numbers are equal. However, also the second coefficient of the Ehrhart polynomial has a geometric meaning. It is half the sum of the induced surface area of the facets of $\Delta_{Y_\bullet}(D)$. We call this number the *normalized surface area* $S(D, \nu_{Y_\bullet})$. Hence, a necessary condition for $\Gamma_{Y_\bullet}(D)$ to be normal is that this number is minimal. This enables us to view the problem of finding a flag Y_\bullet for a fixed divisor D as a minimization problem.

From now on we will focus on surfaces. In this situation the condition that $S(D, \nu_{Y_\bullet})$ is minimal means that the number of lattice points on the boundary of $\Delta_{Y_\bullet}(D)$ is minimal.

We prove that under some mild condition on X , which we call $(*)$, this minimum does indeed exist.

THEOREM 4.5. *Let X be a smooth surface satisfying condition $(*)$. Let D be a big divisor on X . Then there exists an admissible flag Y_\bullet such that the normalized surface area $S(D, \nu_{Y_\bullet})$ is minimal.*

If we additionally assume that the positive and negative parts of the Zariski decomposition of an integral divisor are integral, we present a concrete algorithm how to find flags on which the minimum is attained. We will apply this algorithm in the case of del Pezzo surfaces and find for several divisors flags which admit normal toric degenerations.

In the second part of the article we give a more concrete answer to the above question about normal toric degenerations for smooth (weak) del Pezzo surfaces. These are surfaces such that their anticanonical bundle $-K_X$ is ample (nef). Smooth del Pezzo surfaces are up to isomorphism equal to the blow-up of \mathbb{P}^2 in up to eight points in general position. Weak del Pezzo surfaces are, roughly speaking, characterized by blow-ups of \mathbb{P}^2 of up to eight points in almost general position, i.e. there are more constellation allowed than for del Pezzo surfaces. We prove the following.

THEOREM 4.6. *Suppose one of the following situations.*

- $X = X_r$ is the blow-up of $1 \leq r \leq 6$ points in \mathbb{P}^2 general position and Y_\bullet is an admissible flag such that Y_1 is negative.
- $X = L_3$ is the blow-up of four points, where three of them are on a line or $X = S_6$ is the blow-up of six points on a conic in \mathbb{P}^2 . Let Y_\bullet be an admissible flag such that Y_1 is the unique (-2) -curve on X .

Then for each big divisor D the semigroup $\Gamma_{Y_\bullet}(D)$ is finitely generated normal.

Note that in the first case the variety X is a del Pezzo surface and in the second case it is a weak del Pezzo surface.

Finally, we consider global Newton-Okounkov bodies on (weak) del Pezzo surfaces and their corresponding semigroups $\Gamma_{Y_\bullet}(X)$. We generalize Theorem 4.2 to arbitrary flags Y_\bullet and compute some examples of global Newton-Okounkov bodies.

In the construction of the global Newton-Okounkov body one has to take the closed convex cone of the following semigroup

$$\Gamma_{Y_\bullet}(X) := \{(\nu(s), [D]) \mid s \in H^0(X, \mathcal{O}_X(D)), D \in \text{Pic}(X)\}.$$

It is now a natural question whether $\Gamma_{Y_\bullet}(X)$ is normal and finitely generated. By analyzing the semigroups $\Gamma_{Y_\bullet}(D)$ for effective (i.e. not necessarily big) divisors, we are able to prove the following.

THEOREM 4.7. *Suppose one of the following situations.*

- $X = X_r$ is the blow-up of $1 \leq r \leq 6$ points in general position and Y_\bullet is an admissible flag such that Y_1 is negative.
- $X = L_3$ is the blow-up of four points, where three of them are on a line or $X = S_6$ is the blow-up of six points on a conic. Let Y_\bullet be an admissible flag such that Y_1 is the unique (-2) -curve on X .

Then the global semigroup $\Gamma_{Y_\bullet}(X)$ is finitely generated normal.

4.2. Our contributions to Mori chambers and Newton-Okounkov bodies of Bott-Samelson varieties. In our article [MSS17] we investigate in the study of Mori chambers and Newton-Okounkov bodies on Bott-Samelson varieties. These are desingularization of Schubert/ flag varieties. For an introduction to Bott-Samelson varieties we refer to [LT04]. We will try to omit as many technicalities as possible and just summarize the most important properties of Bott-Samelson varieties which we will need for our purposes. In the following let X be an n -dimensional Bott-Samelson variety. Then we have $\text{Pic}(X) \cong \mathbb{Z}^n \cong N^1(X)$ and there are effective prime divisors E_1, \dots, E_n which generate $\text{Pic}(X)$ as a group. Moreover, the classes of the E_i are the generators of the pseudo-effective cone. We call E_1, \dots, E_n the *effective basis* of X . There are furthermore divisors D_1, \dots, D_n which also generate $\text{Pic}(X)$ and its classes generate the nef cone. We call this basis the $\mathcal{O}(1)$ *basis*. It was proven in [SS17] that Bott-Samelson varieties are Mori dream spaces, which means that the Cox ring

$$\text{Cox}(X) = \bigoplus_{L \in \text{Pic}(X)} H^0(X, L)$$

is finitely generated as an algebra. We start our article by proving the fundamental result that Bott-Samelson varieties admit Zariski decompositions, like in the surface case. More concretely, we prove the following.

THEOREM 4.8. *Let $X = X_w$ be a Bott-Samelson variety corresponding to a reduced sequence w . Then every movable divisor on X is base point-free and hence*

$$\text{Mov}(X) = \text{Nef}(X).$$

The above theorem shows that the earlier mentioned Nakayama σ -decomposition, which is a decomposition of a divisor into movable and fixed part, is indeed a Zariski decomposition, i.e. $P_\sigma(D)$ is nef. This means that we can decompose each divisor $D = P + N$ into a nef part P and a fixed part N , such that all global sections of a high enough multiple of $\mathcal{O}_X(kD)$ come from global sections of the nef bundle $\mathcal{O}_X(kP)$. Having proved this, we can decompose the effective cone into, so called, Zariski chambers inside which the positive part P and the negative part N varies linearly if we vary the divisor linearly. This decomposition is a generalization of the one introduced in [BKS04] in the surface case. We explicitly describe these chambers with the help of the effective- and the $\mathcal{O}(1)$ -basis. Furthermore we compare these Zariski chambers of X with the Mori chambers of X defined in [HuKe00]. We derive the following.

THEOREM 4.9. *Let $X = X_w$ be a Bott-Samelson variety for a reduced word w . Then each Zariski chamber defines a Mori chamber and vice versa.*

We also illustrate these results with explicit three and four dimensional examples.

In the second part of our article, we investigate Newton-Okounkov bodies on Bott-Samelson varieties. Note that the study of Newton-Okounkov bodies on Bott-Samelson varieties has recently become an active field of research. In [A13] a particular Bott-Samelson variety is considered as an example. A more thorough analysis of Newton-Okounkov bodies for Bott-Samelson varieties was initiated by Kaveh in [Ka15]. In [HaY15] the authors describe Newton-Okounkov bodies of Bott-Samelson varieties for divisors D satisfying a certain condition. In contrast to

Kaveh's work, they use a flag to define the valuation, which we will call the 'horizontal' flag. In particular, they prove the finite generation of the value semigroup in this context. In [SS17] the rational polyhedrality of the global Newton-Okounkov with respect to the, so scalled, 'vertical' flag was proven. In our work we want to combine the properties derived in [SS17] and [HaY15]. More concretely, we will use the 'vertical flag' as a tool for proving the existence of Zariski decompositions on Bott-Samelson varieties. The existence will then help us to prove the finite generation of the value semigroup for all divisors on the 'horizontal flag'. Thus, we generalize the results in [HaY15]. Moreover, we also prove the rational polyhedrality of the global Newton-Okounkov body for the 'horizontal flag'. On top of that we prove the finite generation of the global semigroup $\Gamma_{Y_\bullet}(X)$, which was already considered in the previous section.

THEOREM 4.10. *Let $X = X_w$ be a Bott-Samelson variety for a reduced word w and let Y_\bullet be the horizontal flag. Then, the semigroup*

$$\Gamma_{Y_\bullet}(X_w) := \{(\nu(s), [D]) \mid D \in \text{Pic}(X_w), s \in H^0(X, \mathcal{O}_X(D)) \setminus \{0\}\}$$

is finitely generated.

Like in the previous section, we are concerned with the normality of $\Gamma_{Y_\bullet}(X)$. We do not expect this to hold in general, however we give a criterion in terms of the corresponding Zariski decomposition on X . Namely, the additional property that the Zariski decomposition is an integral decomposition, i.e. for an integral divisor $D = P + N$ the positive part P as well as the negative part N should also be integral. We finish the article by computing the global Newton-Okounkov body for a three-dimensional example and prove that in this case the integrality of the Zariski decompositions holds.

5. Newton-Okounkov bodies for graded linear series

In the above introduction to Newton-Okounkov bodies we associated a convex body to a given divisor D on a projective variety X . However, the construction given in [LM09] is actually more general. In their article they associate a Newton-Okounkov body to a *graded linear series* corresponding to a divisor D . Let us explain what we mean by this. The set $S_\bullet = \{S_k\}_{k \in \mathbb{N}_0}$ is called a graded linear series if for each $k > 0$

$$S_k \subseteq H^0(X, \mathcal{O}_X(kD))$$

is a finite dimensional subspace, $S_0 = k$ and the following inclusion

$$S_k \cdot S_l \subseteq S_{k+l}$$

is satisfied. Here, the product on the left is the image of $S_k \otimes S_l$ under the multiplication map

$$H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(lD)) \rightarrow H^0(X, \mathcal{O}_X((k+l)D)).$$

Then we can define the associated graded linear algebra $R(S_\bullet) = \bigoplus_{k \in \mathbb{N}_0} S_k \subseteq R(X, D)$.

The construction of the corresponding Newton-Okounkov body is as follows. We define the semigroup

$$\Gamma_{Y_\bullet}(S_\bullet) := \{(\nu_{Y_\bullet}(s), k) \mid k \in \mathbb{N}, s \in S_k \setminus \{0\}\}$$

and the Newton-Okounkov body as

$$\Delta_{Y_\bullet}(S_\bullet) = \overline{\text{Cone}(\Gamma_{Y_\bullet}(S_\bullet))} \cap (\mathbb{R}^d \times \{1\}).$$

One of the most interesting examples of graded linear series is the restricted linear series. Let D be a divisor on Y and $X \subseteq Y$ a closed subvariety. Then the restricted linear series S_\bullet of D to X is given via $S_k = H^0(Y, \mathcal{O}_Y(kD))|_X$.

The following is now a natural question.

QUESTION. *Which properties of Newton-Okounkov bodies of divisors do still hold for more general graded linear series or rather which assumption do we need to make on S_\bullet such that these properties still hold?*

The first property that comes to mind is the volume formula of Theorem 2.1. So the question is whether the following equality still holds

$$\text{vol}_{\mathbb{R}^d}(\Delta_{Y_\bullet}(S_\bullet)) = \frac{1}{d!} \cdot \lim_{k \rightarrow \infty} \frac{\dim S_k}{k^d/d!} =: \frac{1}{d!} \cdot \text{vol}(S_\bullet) ?$$

For arbitrary graded linear series this is not true. However, if we assume that the group generated by $\Gamma_{Y_\bullet}(S_\bullet)$ is equal to \mathbb{Z}^d this formula holds. Furthermore, in [LM09] two conditions are stated such that this holds. The first one, is that S_\bullet is a *birational graded linear series*. This means that for $k \gg 0$ the rational map

$$h_{S_k}: X \rightarrow \mathbb{P}^N$$

defined by the linear series $|S_k|$ is birational onto its image. Then the above volume formula holds for all flags Y_\bullet such that the point Y_d lies in the open set of points where ϕ_k defines an isomorphism. A stronger condition which makes sure that the volume formula holds for all admissible flags is that S_\bullet *contains an ample series*. This means the following. Let S_\bullet be a graded linear series corresponding to D . It contains the ample series A if we can decompose $D = A + E$ into ample plus effective such that

$$H^0(X, \mathcal{O}_X(kA)) \subseteq S_k \subseteq H^0(X, \mathcal{O}_X(kD))$$

for all k divisible enough. Note that the left inclusion is given via the multiplication map with a defining section of E . In particular, for the restricted linear series this condition holds whenever $X \subseteq Y$ is not contained in the locus of h_D where it is not an isomorphism, i.e. the exceptional locus of h_D .

5.1. Our work on Newton-Okounkov bodies of graded linear series.

Apart from the volume formula, which was considered in [LM09] for graded linear series S_\bullet , there are many interesting properties of Newton-Okounkov bodies which were only considered for the case of a complete graded linear series $R(S_\bullet) = R(X, D)$. The main features which were left open are generalizations of slicing theorems such as in Theorem 2.3 and generalizations of the existence of generic Newton-Okounkov bodies such as in Theorem 2.5. In our article [M18.2] we derive generalizations of both theorems for certain graded linear series.

If we choose the graded linear series to be arbitrary, there is little hope that properties concerning the geometry of X can hold in general. This is why it makes sense to pose conditions on S_\bullet which contain at least some information about the geometry of X . The conditions we have in mind were already introduced in the previous section, namely either that S_\bullet contains an ample series or the weaker property that S_\bullet is birational. Certainly, the additional property of the finite generation of the

algebra $R(S_\bullet)$ seems convenient. It turns out that a key point for the theory of graded linear series is to understand the connection between the volume of S_\bullet and its stable base locus $\mathbf{B}(S_\bullet)$, i.e. the subspace of X where all sections of S_\bullet vanish. We derive the following characterization.

THEOREM 5.1. *Let $S_\bullet \subseteq T_\bullet$ be two finitely generated graded linear series corresponding to a big divisor D . Then the following two conditions are equivalent*

- (a) $\text{vol}(S_\bullet) = \text{vol}(T_\bullet)$.
- (b)
 - The rational map $h_{S_\bullet}: X \rightarrow \text{Proj}(S_\bullet)$ is birational and
 - $\mathbf{B}(S_\bullet) = \emptyset$ on $\text{Proj}(R(T_\bullet))$.

With the help of the above theorem we are able to derive the following slice formula.

THEOREM 5.2. *Let S_\bullet be a graded linear series containing the ample series $D - E$. Let Y_\bullet be an admissible flag such that the divisorial component Y_1 is not contained in E and $Y_d \notin \mathbf{B}(S_\bullet)$. Then we have*

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{X|Y_1}(S_\bullet).$$

This can be seen as a generalization of Theorem 2.3, with the additional restriction for the point of the flag Y_\bullet .

For the question of the existence of generic Newton-Okounkov bodies it turns out that a sufficient condition for the graded linear series S_\bullet is that it is birational. For such graded linear series we prove the following.

THEOREM 5.3. *Let X_T, T and \mathcal{Y}_\bullet be as described in Section 2.4. Let S_\bullet be a birational graded linear series on X_T . Then for a very general choice of $t \in T$ all the Newton-Okounkov bodies $\Delta_{Y_{i,\bullet}}(S_\bullet)$ coincide.*

Analogously as described in Section 2.4, this will enable us to define canonical Newton-Okounkov bodies $\Delta(S_\bullet)$ for a birational graded linear series S_\bullet .

Bibliography

- [A13] Anderson, D., *Okounkov bodies and toric degenerations*. Math. Ann. **356** (3) (2013), 1183–1202
C
- [AKL12] Anderson, D., Küronya, A., Lozovanu, V., *Okounkov bodies of finitely generated divisors*, International Mathematics Research Notices 2013; doi: 10.1093/imrn/rns286
- [B75] Bernstein, D. N., *The number of roots of a system of equations*, Funkcial. Anal. i Priložhen. 1975. V. 9, No **3**, 1–4.
- [BKS04] Bauer, T., Küronya, A., Szemberg, T., *Zariski chambers, volumes, and stable base loci*, J. Reine Angew. Math. **576** (2004), 209–233
- [BM18] Blum, H., Merz, G., *Local positivity and Newton-Okounkov bodies in higher dimension*, unpublished
- [ELMNP09] Ein, L., Lazarsfeld, R., Mustață, M., Nakamaye, M., Popa, M., *Restricted Volumes and Base Loci of Linear Series* American Journal of Mathematics **131**(2009), No. 3 , 607–651
- [HaY15] Harada, M., Yang, J., *Newton-Okounkov bodies of Bott-Samelson varieties and Grossberg-Karshon twisted cubes*, preprint 2015, arXiv:1504.00982, Accepted for publication in J. of Korean Math. Soc
- [HuKe00] Hu, Y., Keel, S., *Mori Dream Spaces and GIT*, Michigan Math. J. **48** (2000), 331–348
- [J10] Jow, Shin-Yao, *Okounkov bodies and restricted volumes along very general curves*, Adv. Math. **223**, (2010), 4, 1356–1371
- [K76] Kusnirenko, A. G., *Polyedres de Newton et nombres de Milnor*, Invent. Math. 1976. V. 32, No **1**, 1–31
- [Kh77] Khovanskii A., *Newton polyhedra, and toroidal varieties*, Funct. Anal. Appl. 1977. **V. 11**, No 4, 289–296
- [Kh78] Khovanskii, A., *Newton Polyhedra and the genus of complete intersections*, Funct. Anal. Appl. 1978. **V. 12**, No 1, 38–46
- [KK12] Kaveh, K., Khovanskii, A.G., *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), 2, 925–978
- [Ka15] Kaveh, K., *Crystal bases and Newton-Okounkov bodies*, Duke Math. J. **164** (2015), 13, 2461–2506
- [Ki15] Kiritchenko, V., *Newton-Okounkov polytopes of flag varieties*, preprint, arxiv.org/abs/1506.00362
- [KLM12] Küronya, A., Lozovanu, V., Maclean, C., *Convex bodies appearing as Okounkov bodies of divisors* Advances in Mathematics **229** (2012), 2622–2639
- [KL14] Küronya, A., Lozovanu, V., *Local Positivity of linear Series on Surfaces*, accepted in Algebra & Number Theory, arXiv:1411.6205
- [KL17] Küronya, A., Lozovanu, V., *Positivity of line bundles and Newton-Okounkov bodies*, Documenta Math., **22** (2017), pp. 1285–1302
- [LT04] Lauritzen, N., Thomsen, J.F., *Line bundles on Bott-Samelson varieties*, J. Alg. Geom. **13** (2004), 461–473
- [LM09] Lazarsfeld, R., Mustață, M., *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 783–835
- [M18] Merz, G., *Newton-Okounkov bodies and normal toric degenerations*, preprint arXiv:1801.04125
- [M18.2] Merz, G., *On Newton-Okounkov bodies of graded linear series*, preprint arXiv:1801.04611
- [MSS17] Merz, G., Schmitz, D., Seppänen, H., *On the Mori theory and Newton-Okounkov bodies of Bott-Samelson varieties*, preprint arXiv:1709.09910

- [N04] Nakayama, N., *Zariski-decomposition and abundance*, Mathematical Society of Japan, Tokyo (2004)
- [O96] Okounkov, A., *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996), 405–411
- [O00] Andrei Okounkov, *Why would multiplicities be log-concave?*, The orbit method in geometry and physics, Progr. Math. **213**, 2003, 329–347
- [O16] Okawa, S., *On images of Mori dream spaces*, Math. Ann. **364** (2016), no. 3–4, 1315–1342
- [P11] Petersen, L., *Okounkov Bodies of Complexity-One T-Varieties*, preprint, 2011, arXiv:1108.0632
- [Roe16] J. Roé. *Local positivity in terms of Newton-Okounkov bodies*. Adv. Math. **301** (2016), 486–498.
- [SS16] Schmitz, D., Seppänen, H., *On the polyhedrality of global Okounkov bodies*, Adv. Geom. **16** (2016),1, 83–91
- [SS17] Schmitz, D., Seppänen, H., *Global Okounkov bodies for Bott-Samelson varieties*, Journal of Algebra **490** (2017), 518–554

CHAPTER 2

Dissertation Articles

We will now present the articles which constitute this dissertation.

- *Local Positivity and Newton-Okounkov Bodies in higher Dimension* by Harold Blum and Georg Merz.

ABSTRACT. We extend a result of Roé concerning Newton-Okounkov bodies and local positivity on surfaces to all dimensions. Specifically, we show that the set of all Newton-Okounkov bodies of a big divisor with respect to flags centered at a fixed point determines and is determined by the numerical class of the divisor up to negative components in the σ -decomposition that do not pass through the fixed point.

- *Newton-Okounkov Bodies and normal toric Degenerations* by Georg Merz.

ABSTRACT. Anderson proved that the finite generation of the value semigroup $\Gamma_{Y_\bullet}(D)$ in the construction of the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ induces a toric degeneration of the corresponding variety X to some toric variety X_0 . In this case the normalization of X_0 is the normal toric variety corresponding to the rational polytope $\Delta_{Y_\bullet}(D)$. Since X_0 is not normal in general this correspondence is rather implicit. In this article we investigate in conditions to assure that X_0 is normal, by comparing the Hilbert polynomial with the Ehrhart polynomial. In the case of del Pezzo surfaces this will result in an algorithm which outputs for a given divisor D a flag Y_\bullet such that the value semigroup in question is indeed normal. Furthermore, we will find flags on del Pezzo surfaces and on some particular weak del Pezzo surfaces which induce normal toric degenerations for all possible divisors at once. We will prove that in this case the global value semigroup $\Gamma_{Y_\bullet}(X)$ is finitely generated and normal.

- *On the Mori Theory and Newton-Okounkov Bodies on Bott-Samelson Varieties* by Georg Merz, David Schmitz and Henrik Seppänen.

ABSTRACT. We prove that on a Bott-Samelson variety X every movable divisor is nef. This enables us to consider Zariski decompositions of effective divisors, which in turn yields a description of the Mori chamber decomposition of the effective cone. This amounts to information on all possible birational morphisms from X . Applying this result, we prove the rational polyhedrality of the global Newton-Okounkov body of a Bott-Samelson variety with respect to the so called ‘horizontal’ flag. In fact, we prove the stronger property of the finite generation of the corresponding global value semigroup.

- *On Newton-Okounkov Bodies of graded linear Series* by Georg Merz.

ABSTRACT. We generalize the theory of Newton-Okounkov bodies of big divisors to the case of graded linear series. One of the results is the generalization of slice formulas and the existence of generic Newton-Okounkov bodies for birational graded linear series. We also give a characterization of graded linear series which have full volume in terms of their base locus.

LOCAL POSITIVITY AND NEWTON-OKOUNKOV BODIES IN HIGHER DIMENSION

HAROLD BLUM AND GEORG MERZ

ABSTRACT. We extend a result of Roé concerning Newton-Okounkov bodies and local positivity on surfaces to all dimensions. Specifically, we show that the set of all Newton-Okounkov bodies of a big divisor with respect to flags centered at a fixed point determines and is determined by the numerical class of the divisor up to negative components in the σ -decomposition that do not pass through the fixed point.

1. INTRODUCTION

Let D be a big divisor on a smooth projective variety X of dimension d . The *Newton-Okounkov body* of D serves as a tool for studying positivity properties of D . The construction of the Newton-Okounkov body was first introduced in the work of Okounkov [Oko96] and independently developed in the work of Lazarsfeld and Mustața [LM09] and Kaveh and Khovanskii [KK12]. The construction is dependent on a flag

$$Y_\bullet = \{X = Y_0 \supset Y_1 \supset \cdots \supset Y_d = \{p\}\}$$

such that each Y_i is smooth at p . The *Newton-Okounkov body of D along Y_\bullet* is a convex set $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^d$ and encodes information on sections of $H^0(\mathcal{O}_X(mD))$ that vanish along Y_\bullet .

Theorem 1.1. [LM09] [Jow10] *Let D_1 and D_2 be big divisors on a smooth projective variety X . The following are equivalent.*

- (1) *For all admissible flags Y_\bullet on X , we have $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$.*
- (2) *The divisors D_1 and D_2 are numerically equivalent.*

Philosophically, Theorem 1.1 implies that all numerical properties of a divisor D are encoded in the convex geometry of Newton-Okounkov bodies of D . For example, the volume of a divisor D is $d!$ times the euclidean volume of $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^d$.

In [KL14], [KL15] and [KL17] it was shown that local positivity at some point $O \in X$ is related to the Newton-Okounkov bodies of D with respect to admissible flags centered at the point O . Motivated by these ideas, Roé asks the following.

Question 1.2. Let D_1 and D_2 be big divisors on X such that $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$ for all admissible flags Y_\bullet centered at O . How are D_1 and D_2 related?

When X is a surface, Roé gives an elegant answer to this question [Roe16]. First, he introduces the following definition.

Key words and phrases. Local positivity, Newton-Okounkov bodies.

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Definition 1.3. Let D be a big divisor on a normal surface X and write $D = P(D) + N(D)$ for the Zariski Decomposition of D into positive and negative components. Next, fix a point $O \in X$ and write $N(D) = N_O(D) + N_O^c(D)$ for the decomposition of $N(D)$ into components containing O and disjoint from O .

We say that two big divisor D_1 and D_2 are *locally numerically equivalent at O* if

$$P(D_1) \equiv P(D_2) \text{ and } N_O(D_1) = N_O(D_2).$$

Note that local numerical equivalence at all points of X implies numerical equivalence.

Roughly speaking, two divisors are locally numerically equivalent at a point O if the divisors are numerically equivalent modulo fixed components of D that do not pass through O . With this definition, Roé proves the following.

Theorem 1.4. [Roe16] *Let D_1 and D_2 be big divisors on a normal surface X and $O \in X$ a closed point. The following are equivalent:*

- (1) *The divisors D_1 and D_2 are locally numerically equivalent at O .*
- (2) *For all admissible flags Y_\bullet centered over O , we have $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$.*

Roé leaves the generalization of Theorem 1.4 to higher dimensions open. A key obstacle in extending the theorem to higher dimensions is that Zariski decompositions of big divisors do not always exist in dimensions three and higher. However, Nakayama introduced a weaker analogue of the Zariski decomposition called the σ -decomposition [N04]. Such decompositions always exist for big divisors.

We extend Roé's definition of local numerical equivalence to higher dimensions by replacing the Zariski decomposition in the definition with the σ -decomposition (see Section 2.3). With this definition, we prove the following generalization of Theorem 1.4 to higher dimensions.

Theorem 1.5. *Let D_1 , and D_2 be two big divisors on a smooth projective variety X and $O \in X$ a closed point. The following are equivalent.*

- (1) *The divisors D_1 and D_2 are locally numerically equivalent at O .*
- (2) *For all admissible flags Y_\bullet centered over O , we have $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$.*

While working on this article we learnt that another group consisting of Sung Rak Choi, Jinhyung Park and Joonyeong Won were working independently on similar generalization results of Roé's work.

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2. PRELIMINARIES

2.1. **Conventions.** For the purposes of this paper, all varieties are defined over \mathbb{C} .

2.2. **Asymptotic base loci.** Let X be a smooth projective variety and D a divisor on X . Recall that the *base locus* of D is the subscheme $\text{Bs}(D) \subseteq X$ defined by the image of the evaluation map

$$H^0(X, \mathcal{O}_X(D)) \otimes_{\mathbb{C}} \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X.$$

The *stable base locus* of D is

$$\mathbf{B}(D) := \bigcap_{m \geq 1} \mathbf{Bs}(mD)_{\text{red}}.$$

Since $\mathbf{B}(D) = \mathbf{B}(pD)$ for $p \geq 1$, the stable base locus can naturally be defined for \mathbf{Q} -divisors. If D is a \mathbf{Q} -divisor, we set $\mathbf{B}(D) := \mathbf{B}(kD)$ where k is a positive integer such that kD is integral.

Since the stable base locus satisfies various pathologies (e.g. it is not a numerical invariant), it is natural to consider the following notions. The *augmented base locus* and *restricted base locus* of a \mathbf{Q} -divisor D are given by

$$\mathbf{B}_+(D) := \bigcap_A \mathbf{B}(D - A) \text{ and } \mathbf{B}_-(D) := \bigcup_A \mathbf{B}(D + A) \quad (1)$$

where the union and intersection are taken over all ample \mathbf{Q} -divisors A . We refer to [ELMNP06] and [ELMNP08] for basic properties of the augmented and restricted base loci.

2.3. The divisorial Zariski decomposition. We recall Nakayama's *divisorial Zariski decomposition*.

Definition 2.1. Let X be a smooth projective variety and D a big \mathbf{R} -divisor. For a prime divisor Γ on X , we define

$$\sigma_\Gamma(D) := \lim_{m \rightarrow \infty} \frac{\text{ord}_\Gamma | \lfloor mD \rfloor |}{m},$$

where $\text{ord}_\Gamma |F|$ denotes the coefficient of Γ in a general element of the complete linear system $|F|$. We set

$$N_\sigma(D) := \sum_\Gamma \sigma_\Gamma(D) \cdot \Gamma \text{ and } P_\sigma(D) := D - N_\sigma(D).$$

The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is the *divisorial Zariski decomposition* of D . Note that even when D is a \mathbf{Z} -divisor, $P_\sigma(D)$ and $N_\sigma(D)$ are \mathbf{R} -divisors and may not be \mathbf{Q} -divisors.

The following proposition records basic properties of the divisorial Zariski decomposition.

Proposition 2.2. *Let X be a smooth variety and D, D_1, D_2 be big \mathbf{R} -divisors. The following hold.*

(1) *The natural map*

$$H^0(X, \mathcal{O}_X(\lfloor mP_\sigma(D) \rfloor)) \longrightarrow H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$$

is an isomorphism.

(2) *If $D_1 \equiv D_2$, then $N_\sigma(D_1) = N_\sigma(D_2)$ and $P_\sigma(D_1) \equiv P_\sigma(D_2)$.*

(3) *The support of $N_\sigma(D)$ equals the divisorial part of $\mathbf{B}_-(D)$.*

Next, we introduce the notion of local numerical equivalence. Roughly speaking, two big divisors are locally numerically equivalent at a fixed point $O \in X$ if their positive parts are numerically equivalent and their negatives parts agree up to components passing through P .

Definition 2.3. Let D be a big divisor on a smooth projective variety X and $O \in X$ a closed point. We write

$$N_\sigma(D) = N_{\sigma, O}(D) + N_{\sigma, O}^c(D)$$

for the decomposition of $N_\sigma(D)$ into components containing $O \in X$ and the components that do not contain $O \in X$. Let D_1 and D_2 be two big divisors on X . We say that D_1 and D_2 are *locally numerically equivalent at O* if

$$P_\sigma(D_1) \equiv P_\sigma(D_2) \text{ and } N_{\sigma,O}(D_1) = N_{\sigma,O}(D_2).$$

If X is a surface, the above definition agrees with Roé's definition [Roe16]. This follows from the fact that on a surface the divisorial Zariski decomposition is the same as the Zariski decomposition. [N04, Remark III.1.17].

2.4. Restricted volumes and augmented base loci. Let D be a divisor D on a smooth projective variety X . The restricted volume of D along a subvariety $Y \subseteq X$ measures how many global sections of $\mathcal{O}_X(D)|_Y$ come from sections on X . More precisely,

$$\text{vol}_{X|Y}(D) := \limsup_{m \rightarrow \infty} \frac{\dim(\text{Im}(H^0(X, \mathcal{O}_X(mD)) \rightarrow H^0(Y, \mathcal{O}_Y(mD))))}{m^d/d!}, \quad (2)$$

where d is the dimension of Y . The augmented base locus is related to restricted volumes in the following way.

Theorem 2.4. [ELMNP08] *If D is a divisor on X , then $\mathbf{B}_+(D)$ is the union of all positive dimensional subvarieties Y such that $\text{vol}_{X|Y}(D) = 0$.*

In particular this implies that if $Y \not\subseteq \mathbf{B}_+(D)$, then $\text{vol}_{X|Y}(D) > 0$.

3. NEWTON-OKOUNKOV BODIES

We now proceed to recall the construction and some relevant properties of Newton-Okounkov bodies. Let X be a projective variety of dimension d . We call Y_\bullet an *admissible flag on X* if Y_\bullet is a flag on X , where

$$Y_\bullet = \{X = Y_0 \supset Y_1 \supset \cdots \supset Y_d = \{p\}\}$$

such that each Y_i is an irreducible subvariety of codimension i and is smooth at the point p . We now proceed to define the *Newton-Okounkov body* associated to a big divisor D on X and an admissible flag Y_\bullet on X . Given a divisor F on X , there is a valuation map

$$\nu_{Y_\bullet} = \nu : H^0(X, \mathcal{O}_X(F)) \setminus \{0\} \longrightarrow \mathbb{Z}_{\geq 0}^d$$

that measures order of vanishing of sections along Y_\bullet . (see [LM09] for the definition of ν_{Y_\bullet}). The *Newton-Okounkov body of D along Y_\bullet* is the convex body

$$\Delta_{Y_\bullet}(D) := \text{closed convex hull} \left(\bigcup_{m \geq 1} \frac{1}{m} \nu_{Y_\bullet}(mD) \right).$$

It will also be useful for us to consider flags that do not live on X . We say that Y_\bullet is an *admissible flag over X* if there exists a proper birational morphism $\pi : \tilde{X} \rightarrow X$ and Y_\bullet is an admissible flag on \tilde{X} . Given a closed point $p \in X$ we say that Y_\bullet is an *admissible flag over p* if Y_\bullet is an admissible flag over X and the image of Y_d on X is p . Given a big divisor D on X and Y_\bullet an admissible flag over X as above, we set

$$\Delta_{Y_\bullet}(D) := \Delta_{Y_\bullet}(\pi^*D).$$

In this paper, it will be necessary to consider the Newton-Okounkov body of a big \mathbf{R} -divisor D . See [KL15] for its definition.

The following statement is (1) \implies (2) of our main theorem.

Proposition 3.1. *Let D_1 and D_2 be two big divisors on a smooth projective variety X and $O \in X$ a closed point. If D_1 and D_2 are locally numerically equivalent at O , then*

$$\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2).$$

for all flags Y_\bullet centered over O .

Proof. Let $\pi: \tilde{X} \rightarrow X$ be a proper birational morphism and Y_\bullet an admissible flag on \tilde{X} centered at the point $P \in \pi^{-1}(O)$.

We claim that

$$\Delta_{Y_\bullet}(\pi^*D_i) = \Delta_{Y_\bullet}(P_\sigma(\pi^*D_i) + N_{\sigma,P}(\pi^*D_i)). \quad (3)$$

Indeed, [KL15, Theorem 4.2 (3)] implies

$$\Delta_{Y_\bullet}(\pi^*D_i) = \Delta_{Y_\bullet}(P_\sigma(\pi^*D_i)) + \nu_{Y_\bullet}(N_{\sigma,P}(\pi^*D_i) + N_{\sigma,P}^c(\pi^*D_i)).$$

Since $\nu_{Y_\bullet}(N_{\sigma,P}(\pi^*D_i) + N_{\sigma,P}^c(\pi^*D_i)) = \nu_{Y_\bullet}(N_{\sigma,P}(\pi^*D_i))$ the claim is complete.

Since D_1 and D_2 are locally numerically equivalent at O and $\pi(P) = O$, it follows that $P_\sigma(\pi^*D_1) \equiv P_\sigma(\pi^*D_2)$ and $N_{\sigma,P}(\pi^*D_1) = N_{\sigma,P}(\pi^*(D_2))$. Combining the previous statement with (3) completes the proof. \square

In order to prove the reverse direction of the main theorem, it is necessary to read off numerical data from the Newton-Okounkov body. One step in this direction is the following lemma. While the statement is well known (see [Jow10, Roe16]), we include it for sake of completeness.

Proposition 3.2. *Let D_1 and D_2 be big \mathbf{R} -divisors on X and $E \subseteq X$ a prime divisor on X . If there exists a point $O \in E$ such that E is a smooth at O and $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$ for all admissible flags Y_\bullet centered at O , then*

$$\sigma_E(D_1) = \sigma_E(D_2).$$

Proof. By our assumption on O , we may choose an admissible flag Y_\bullet on X such that $Y_1 = E$. We claim that $\sigma_{Y_1}(D_i)$ can be read off from $\Delta_{Y_1}(D_i)$. Indeed, $\sigma_{Y_1}(D_i)$ is the minimum value of the projection of $\Delta_{Y_1}(D_i)$ onto its first coordinate. Since $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$, we conclude $\sigma_{Y_1}(D_1) = \sigma_{Y_1}(D_2)$. \square

4. VERY GENERAL FLAGS

In this section we extend some results of [Jow10, Section 3] to flags passing through a fixed point. In order to prove such results, we must assume that the base locus of our big divisor is “well behaved” at x (see Condition 4.1).

4.1. Construction of flags. Let X be a smooth projective variety of dimension d , $P \in X$ a closed point, and D a big divisor on X . Additionally, we fix very ample divisors L_1, \dots, L_{d-1} on X .

Next, we construct a flag on X centered at P . Let A_i be an element of $|L_i - P|$, where $|L_i - P|$ denotes the linear series of divisors in $|L_i|$ passing through P . For each $r \in \{1, \dots, d-1\}$, we set

$$Y_r := A_1 \cap \dots \cap A_r.$$

By a Bertini Type Theorem [Zha09, Theorem 2.5], if each A_i is a *general* element of $|L_i - P|$, then each Y_i is smooth. Therefore,

$$Y_\bullet := \{X = Y_0 \supset Y_1 \supset \dots \supset Y_d = \{P\}\}$$

is an admissible flag on X with center P .

Our goal will be to relate the intersection number $Y_{d-1} \cdot P_\sigma(D)$ to the convex body $\Delta_{Y_\bullet}(D)$. To prove such a relationship, we impose restrictions on the intersection of $\text{Bs}(mD)$ and our flag Y_\bullet for m divisible enough.

Condition 4.1. *We say that a big divisor D on X satisfies Condition 4.1 at $P \in X$ if for all m divisible enough $\text{Bs}(mD)$ is purely codimension 1 and smooth at P .*

The above criterion can be stated as follows. For each m divisible enough, there exists an open set $U_m \subseteq X$ containing P and an effective divisor F_m on U_m such that the scheme $\text{Bs}(mD) \cap U_m$ is defined by the ideal $\mathcal{O}_{U_m}(-F_m)$ and $\text{Supp}(F_m)$ is smooth.

Denote by $C_\bullet(X, D)$ the complete linear series associated to D and by $C_\bullet(X, D)|_Y$ the restricted linear series for a subvariety $Y \subseteq X$.

Proposition 4.2. *Let $P \in X$, Y_\bullet , and D be as above. Assume that D satisfies Condition 4.1 at P . Let E_1, \dots, E_n be the $(d-1)$ -dimensional components of $\mathbf{B}(D)$.*

(1) *If the A_i 's are general, then*

$$\mathbf{B}(C_\bullet(X, D)|_{Y_{d-1}}) = Y_{d-1} \cap \mathbf{B}(D) = \bigcup_{i=1}^n (Y_{d-1} \cap E_i).$$

(2) *If the A_i 's are very general, then for all m divisible enough*

$$\text{ord}_p(C_m(X, D)|_{Y_{d-1}}) = \text{ord}_{E_i}(|mD|)$$

for every $p \in Y_{d-1} \cap E_i$ with $i \in \{1, \dots, n\}$.

Proof. We first prove (1). If the A_i 's are general, then $Y_{d-1} \not\subseteq \text{Bs}(mD)_{\text{red}}$ for all m divisible enough. Therefore, $\mathbf{B}(C_\bullet(X, D)|_{Y_{d-1}}) = Y_{d-1} \cap \mathbf{B}(D)$. For the next equality, we choose our A_i 's such that Y_i intersects $Y_{i-1} \cap \mathbf{B}(D)$ very properly in Y_{i-1} for all $i \in \{1, \dots, d-1\}$ in the sense of [Jow10, Definition 3.1]. This assumption on the intersections implies that $Y_{d-1} \cap \mathbf{B}(D) = \bigcup_{i=1}^n (Y_{d-1} \cap E_i)$. (Note that it is impossible for $Y_d = P$ to intersect $Y_{d-1} \cap \mathbf{B}(D)$ very properly if $P \in \mathbf{B}(D)$. While Jow assumes this to prove (1), he is only using that Y_i intersects $Y_{i-1} \cap \mathbf{B}(D)$ very properly in Y_{i-1} for $i \in \{1, \dots, d-1\}$.)

We move on to (2). We claim that if the A_i 's are very general, then the curve Y_{d-1} intersects each of the E_i 's and none of these intersection points lies in an embedded component of $\text{Bs}(mD)$. The claim relies on the fact that D satisfies Condition 4.1 at D . Now, if Y_{d-1} satisfies the above property, it follows that $\text{ord}_p(C_m(X, D)|_{Y_{d-1}}) = \text{ord}_{E_i}(|mD|)$ for all $p \in Y_{d-1} \cap E_i$. □

Corollary 4.3. *If the very ample divisors A_1, \dots, A_{d-1} are very general so that conclusion of Proposition 4.2 hold, then*

$$\text{vol}_{X|Y_{d-1}}(D) = Y_{d-1} \cdot D - \sum_{i=1}^n \sum_{p \in Y_{d-1} \cap E_i} \sigma_{E_i}(D) = Y_{d-1} \cdot P_\sigma(D).$$

Proof. The proof is identical to [Jow10, Corollary 3.3]. □

For the next theorem we define

$$\Delta_{Y_\bullet}(D)|_{\mathbf{0}^{d-1}} = \Delta_{Y_\bullet}(D) \cap \underbrace{(\{0\} \times \dots \times \{0\})}_{d-1 \text{ times}} \times \mathbb{R}.$$

Theorem 4.4. *If the very ample divisors A_1, \dots, A_{d-1} are very general so that conclusion of Proposition 4.2 holds, then*

$$\mathrm{vol}_{\mathbf{R}^1}(\Delta_{Y_\bullet}(D)|_{\mathbf{0}^{d-1}}) = \mathrm{vol}_{X|Y_{d-1}}(D).$$

Proof. The proof is identical to that of [Jow10, Theorem 3.4.b]. \square

4.2. Relation to numerical equivalence. In this section, we prove an analogue of [Jow10, Theorem A]. We follow Jow's technique, but use results from the previous section.

Proposition 4.5. *Let X be a smooth projective variety, $P \in X$ a closed point, and D_1, D_2 big divisors on X satisfying Condition 4.1 at P . If $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$, for all admissible flags Y_\bullet on X with center equal to P , then $P_\sigma(D_1) \equiv P_\sigma(D_2)$.*

Before proving the statement, we recall the following result of Jow.

Lemma 4.6. [Jow10, Lemma 3.5] *Let X be a smooth projective variety of dimension d and $Y \subseteq X$ be a transversal complete intersection of $(d-2)$ very ample divisors. If D_1, \dots, D_ρ are ample divisors on X whose numerical classes form a basis of $N^1(X)_{\mathbf{Q}}$, then the curve classes*

$$\{C_j := Y \cdot D_j | j = 1, \dots, \rho\}$$

form a basis of $N_1(X)_{\mathbf{Q}}$.

Proof of Proposition 4.5. We can use Lemma 4.6 to construct ρ admissible flags $Y_\bullet^{(j)}$ for $j = 1, \dots, \rho$ centered at P which are sufficiently general so that the conclusion of Proposition 4.2 is satisfied with respect to both D_1 and D_2 , and the curves $Y_{d-1}^{(1)}, \dots, Y_{d-1}^{(\rho)}$ form a basis of $N_1(X)_{\mathbf{Q}}$.

Applying Corollary 4.3 and Theorem 4.4, we see

$$\mathrm{vol}_{\mathbf{R}^1}(\Delta_{Y_\bullet^{(j)}}(D_i)|_{\mathbf{0}^{d-1}}) = P_\sigma(D_i) \cdot Y_{d-1}^{(j)}$$

for $i \in \{1, 2\}$ and $j \in \{1, \dots, d-2\}$. Since each $Y_\bullet^{(j)}$ is a flag with center equal to P , our assumption on D_1, D_2 implies $\Delta_{Y_\bullet^{(j)}}(D_1) = \Delta_{Y_\bullet^{(j)}}(D_2)$. Note that Lemma 4.6 implies the curve classes $Y_{d-1}^{(1)}, \dots, Y_{d-1}^{(\rho)}$ form a basis of $N_1(X)_{\mathbf{Q}}$. Thus, $P_\sigma(D_1) \equiv P_\sigma(D_2)$. \square

5. PROOF OF THEOREM 1.5

In this section we prove Theorem 1.5. Before proving the theorem, we note the following elementary lemma.

Lemma 5.1. *Let X be a smooth projective variety, $P \in X$ a closed point, and D a big divisor on X . Let $\pi : Y \rightarrow X$ denote the blowup of X at P with exceptional divisor E . If \tilde{P} is a very general closed point of E , then $\pi^*(D)$ satisfies Condition 4.1 at \tilde{P} .*

Remark 5.2. Note that if P is not in $Bs(|mD|)$ for m divisible enough, then it is trivially true that D satisfies Condition 4.1 at P . Additionally, the condition is satisfied for $\pi^*(D)$ at all points $\tilde{P} \in E$.

Proof. For each $m \in \mathbf{Z}$ such that $h^0(mD) \neq 0$, we set

$$U_m := Y \setminus Bs(\pi^*(mD) - d_m E),$$

where $d_m := \mathrm{ord}_E(|\pi^*(mD)|)$. Note that $E \cap U_m$ is nonempty, since E is not contained in the base locus of $|\pi^*(mD) - d_m E|$. Additionally, the subscheme $Bs(\pi^*(mD) - d_m E)$ of

X is defined by the ideal $\mathcal{O}_Y(-d_m E)$ in the set U_m . Thus, D satisfies Condition 4.1 for all points in

$$\bigcap_{m|h^0(mD) \neq 0} (U_m \cap E).$$

□

Proof of Theorem 1.5. The implication (1) implies (2) is precisely Theorem 3.1. We now prove that (2) implies (1). Let D_1 and D_2 be two big divisors on X such that $\Delta_{Y_\bullet}(D_1) = \Delta_{Y_\bullet}(D_2)$ for all admissible flags over Y_\bullet over X with center P . Let $\pi : \tilde{X} \rightarrow X$ denote the blowup of X at P with exceptional divisors E . By the previous lemma, we may choose a very general point $\tilde{P} \in E$ so that $\pi^*(D_1)$ and $\pi^*(D_2)$ both satisfy Condition 4.1 at \tilde{P} .

By Lemma 3.2, it follows that $\sigma_E(\pi^*D_1) = \sigma_E(\pi^*D_2)$ and $N_{\sigma,P}(D_1) = N_{\sigma,P}(D_2)$. It is left to show that $P_\sigma(D_1) \equiv P_\sigma(D_2)$. By Proposition 4.5, we have that

$$P_\sigma(\pi^*(D_1)) \equiv P_\sigma(\pi^*(D_2)).$$

Since

$$P_\sigma(\pi^*D_i) = \pi^*P_\sigma(D_i) - \sigma_E(\pi^*D_i)E,$$

it follows that $\pi^*P_\sigma(D_1) \equiv \pi^*P_\sigma(D_2)$. Thus, $P_\sigma(D_1) \equiv P_\sigma(D_2)$.

□

6. CHARACTERIZATION OF BIG AND NEF DIVISORS VIA RESTRICTED VOLUMES

In this section, we want to apply the above discussion in order to derive a characterization for a big divisor D to be nef in terms of restricted volumes. More concretely, we have the following.

Theorem 6.1. *Let D be a big divisor on X . Then the following two assertions are equivalent.*

- D is nef.
- For all $Y \not\subseteq \mathbf{B}_+(D)$ we have

$$\text{vol}_{X|Y}(D) = \text{vol}_Y(D|_Y).$$

Proof. “ \Rightarrow ” This follows from [ELMNP08, Corollary 2.17].

“ \Leftarrow ” Suppose that D is not nef. Then $\mathbf{B}_-(D) \neq \emptyset$. Choose a point $O \in \mathbf{B}_-(D)$ and consider the blow up $\pi : \tilde{X} \rightarrow X$ of X in O . Then a very general point P in $E := \pi^{-1}(O)$ satisfies Condition 4.1. By Lemma 5.1, we can find very general A_i which go through P for $i = 1, \dots, d-1$ such that Y_{d-1} satisfies Corollary 4.3. Let us additionally assume that Y_{d-1} does not lie in $\mathbf{B}_+(D)$ as well as not in E . By construction, $\sigma_E(\pi^*D) > 0$, and we can deduce from Corollary 4.3 that

$$\text{vol}_{\tilde{X}|Y_{d-1}}(\pi^*D) < (\pi^*D \cdot Y_{d-1}) = \text{vol}_{Y_{d-1}}(\pi^*D|_{Y_{d-1}}).$$

Consider now the image $C := \pi(Y_{d-1})$, which defines an irreducible curve in X . [ELMNP08, Lemma 2.4] implies that $\text{vol}_{\tilde{X}|Y_{d-1}}(\pi^*D) = \text{vol}_{X|C}(D)$. On the other hand, $\pi|_{Y_{d-1}} : Y_{d-1} \rightarrow C$ is birational so that we have $\text{vol}(\pi^*D|_{Y_{d-1}}) = \text{vol}(D|_C)$. This implies

$$\text{vol}_{X|C}(D) < \text{vol}_C(D|_C),$$

which is a contradiction to the second assumption. □

REFERENCES

- [CHPW] S. R. Choi, Y. Hyun, J. Park, and J. Won. *Asymptotic base loci via Okounkov bodies*. Adv. Math. **323** (2018), 784–810.
- [ELMNP06] L. Ein, R. Lazarsfeld, M. Mustață, M. Nakamaye, M. Popa. *Asymptotic invariants of base loci*. Ann. Inst. Fourier (Grenoble) **56**(2006), 1701–1734.
- [ELMNP08] L. Ein, R. Lazarsfeld, M. Mustață, M. Nakamaye, M. Popa. *Restricted volumes and base loci of linear series*. Amer. J. Math. **131** (2009), 607–651.
- [Jow10] S. Y. Jow. *Okounkov bodies and restricted volumes along very general curves*. Adv. Math. **223** (2010), 1356–1371.
- [KL14] A. Küronya and V. Lozovanu. *Local positivity of linear series on surfaces*. accepted in Algebra & Number Theory, [arXiv:1411.6205](https://arxiv.org/abs/1411.6205).
- [KL15] A. Küronya and V. Lozovanu. *Positivity of line bundles and Newton-Okounkov bodies*. Documenta Math., **22** (2017), 1285–1302.
- [KL17] A. Küronya and V. Lozovanu. *Infinitesimal Newton-Okounkov bodies and jet separation*. Duke Math. J. **166** (2017), 1349–1376.
- [KK12] K. Kaveh and A. G. Khovanskii. *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory*. Ann. of Math. **176** (2012), 925–978.
- [Laz04] Lazarsfeld, R.: *Positivity in Algebraic Geometry I*. Springer-Verlag, Berlin, (2004).
- [LM09] R. Lazarsfeld and M. Mustață. *Convex bodies associated to linear series*. Ann. Sci. Éc. Norm. Supér **42** (2009), 783–835.
- [N04] N. Nakayama. *Zariski-decomposition and abundance*, Mathematical Society of Japan, Tokyo (2004).
- [Oko96] A. Okounkov. *Brunn-Minkowski inequality for multiplicities*. Invent. Math. **125** (1996), 405–411.
- [Roe16] J. Roé. *Local positivity in terms of Newton-Okounkov bodies*. Adv. Math. **301** (2016), 486–498.
- [Zha09] J. Zhang. *Bertini Type Theorems*. [arXiv:0910.4105](https://arxiv.org/abs/0910.4105).

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NEWTON-OKOUNKOV BODIES AND NORMAL TORIC DEGENERATIONS

GEORG MERZ

ABSTRACT. Anderson proved that the finite generation of the value semigroup $\Gamma_{Y_\bullet}(D)$ in the construction of the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ induces a toric degeneration of the corresponding variety X to some toric variety X_0 . In this case the normalization of X_0 is the normal toric variety corresponding to the rational polytope $\Delta_{Y_\bullet}(D)$. Since X_0 is not normal in general this correspondence is rather implicit. In this article we investigate in conditions to assure that X_0 is normal, by comparing the Hilbert polynomial with the Ehrhart polynomial. In the case of del Pezzo surfaces this will result in an algorithm which outputs for a given divisor D a flag Y_\bullet such that the value semigroup in question is indeed normal. Furthermore, we will find flags on del Pezzo surfaces and on some particular weak del Pezzo surfaces which induce normal toric degenerations for all possible divisors at once. We will prove that in this case the global value semigroup $\Gamma_{Y_\bullet}(X)$ is finitely generated and normal.

1. INTRODUCTION

Newton-Okounkov bodies are convex bodies associated to linear series on a projective variety. They were introduced by Okounkov [O96] and further systematically studied by Lazarsfeld-Mustață [LM09] and Kaveh-Khovanskii [KK12]. Newton-Okounkov bodies of a linear series $|V|$ are not unique but depend upon the choice of a valuation on the graded algebra of sections $R(|V|)$. In the special case of X being toric and D a torus invariant divisor, one can define a valuation such that the associated Newton-Okounkov body is, up to translation, the polytope $\Delta(D)$ corresponding to D in the sense of the usual toric correspondence (see [LM09, Proposition 6.1]). In general, for an arbitrary projective variety X and a valuation ν the Newton-Okounkov body does not need to be rational polyhedral. However, if $\Delta_\nu(D)$ is rational polyhedral, one can ask the following question.

Question. *Assume $\Delta_\nu(D)$ is rational polyhedral. What is the connection between X and the toric variety corresponding to $\Delta_\nu(D)$?*

The answer to this question was given by D. Anderson. He showed the following.

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Theorem ([A13]). *Let X be a projective variety, D a very ample (Cartier) divisor, and ν a valuation-like function. Assume that the semigroup $\Gamma = \Gamma_\nu(D) = \{(\nu(s), k) \mid s \in H^0(X, \mathcal{O}_X(kD)), k \in \mathbb{N}\}$ is finitely generated. Then there exists a toric degeneration of X w.r.t. D to the toric variety $X_0 := \text{Proj}(\mathbb{K}[\Gamma])$. Moreover, the normalization of X_0 is the normal toric variety corresponding to the polytope $\Delta_\nu(D)$.*

Anderson's Theorem can be seen as a generalization of the theory of SAGBI bases (see [S96, Chapter 11] for an introduction). In the SAGBI case, one of the prerequisites is that the coordinate ring $\mathbb{K}[X]$ of the corresponding variety X needs to be contained in a polynomial ring $\mathbb{K}[T_1, \dots, T_n]$. But this is quite a strong constraint, which we can omit by considering the valuation ν .

However, the connection between the variety X and the normal toric variety corresponding to the polytope $\Delta_\nu(D)$ is rather implicit, since we need to normalize the variety we degenerate to. Hence, we can raise the following question.

Question. *Under which circumstances, does there exist a degeneration of X to the normal toric variety corresponding to $\Delta_\nu(D)$?*

In order to answer this question, we need to determine when the variety $\text{Proj}(\mathbb{K}[\Gamma])$ is normal. This is the case if and only if there is a $k \in \mathbb{N}$ such that the semigroup $k \cdot \Gamma$ is normal, i.e. $\text{Cone}(k \cdot \Gamma) \cap \mathbb{Z}^{d+1} = k \cdot \Gamma$ (see also Section 2.1 for more details).

We will see that the property of inducing a normal toric degeneration can indeed be checked by considering the shape of $\Delta_{Y_\bullet}(D)$, or more concretely the Ehrhart polynomial of $\Delta_{Y_\bullet}(D)$. We will define the difference between the Ehrhart polynomial of the Newton-Okounkov body and the Hilbert polynomial of D as the *normal defect*. It is then not difficult to prove that this difference is zero if and only if $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration. This gives the following answer.

Answer. $\Delta := \Delta_\nu(D)$ induces a normal toric degeneration if the Ehrhart polynomial corresponding to Δ is equal to the Hilbert polynomial corresponding to D .

This observation enables us to view the problem of finding a flag for a given divisor which induces a normal toric degeneration as an optimization problem. We will evolve this idea further in the case where X is a surface. It turns out that one can formulate this optimization problem in the following form:

Given a divisor D , find a flag Y_\bullet such that the number of integral points on the boundary of $\Delta_{Y_\bullet}(D)$ is minimal.

We will indeed prove that under some condition (e.g. if X is a Mori dream surface) such a flag always exists (see Theorem 4.9). If we additionally assume that the Zariski decomposition of X is integral (e.g. for del Pezzo surfaces), we will give a concrete algorithm in order to find such flags.

Finally, we focus on (weak) del Pezzo surfaces. It will turn out that in this situation, negative curves are good candidates for flags inducing normal toric degeneration. More concretely, we prove the following statement.

Theorem. *Suppose one of the following situations.*

- $X = X_r$ is the blow-up of $1 \leq r \leq 6$ points in \mathbb{P}^2 in general position and Y_\bullet is an admissible flag such that Y_1 is negative.
- $X = L_3$ is the blow-up of four points, where three of them are on a line or $X = S_6$ is the blow-up of six points on a conic in \mathbb{P}^2 . Let Y_\bullet be an admissible flag such that Y_1 is the unique (-2) -curve on X .

Then the global semigroup

$$\Gamma_{Y_\bullet}(X) = \{(\nu_{Y_\bullet}(s), D) \mid D \in \text{Pic}(X) = N^1(X), s \in H^0(X, \mathcal{O}(D))\}$$

is finitely generated normal.

In order to prove such a statement, one first needs to prove the finite generation and normality of the value semigroup $\Gamma_{Y_\bullet}(D)$ for all big divisors D . One main ingredient of such a proof is the fact that the divisors which occur in the construction of Newton-Okounkov bodies with respect to the above flags, admit integral Zariski decompositions. Another one is the fact that $-K_X + Y_1$ is big and nef, which shows that the restriction morphism of every nef divisor on X to the curve Y_1 is surjective. After one has established such a fact, it is necessary to consider what happens when D moves to the boundary of the effective cone. We will prove that the numerical and the valuative Newton-Okounkov body in this case coincide. Then the above statement will follow from Gordan's lemma.

We end the article with two examples which illustrate our results.

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2. PRELIMINARIES

In this section we briefly describe normal affine semigroups, the construction of Newton-Okounkov bodies, its connection to toric degenerations and introduce the notion of Ehrhart polynomials. Note that all varieties mentioned in this article will be defined over an algebraically closed field \mathbb{K} of characteristic 0. Moreover, a divisor will always mean a Cartier divisor.

2.1. Normal affine semigroups. In this article we will only consider graded semigroups contained in $\mathbb{N}^d \times \mathbb{N}$ where the grading is induced from the last factor. So whenever we talk about a semigroup, we mean a set $\Gamma \subset \mathbb{N}^{d+1}$ which is closed under addition and induces a grading from the last \mathbb{N} factor. We define $\Gamma_m := \{(a_1, \dots, a_{d+1}) \in \Gamma \mid a_{d+1} = m\}$ as well as the semigroup $m\Gamma = \bigcup_{k \in \mathbb{N}} \Gamma_{mk}$ which is \mathbb{N} graded by considering the isomorphism of semigroups $\mathbb{N} \cong m\mathbb{N}$. An *affine semigroup* is a semigroup which

is finitely generated. We denote the group generated by a semigroup Γ by $G(\Gamma)$. We call the semigroup Γ a *normal semigroup* if for all $g \in G(\Gamma)$ and $n \in \mathbb{N}$ such that $n \cdot g \in \Gamma$, it follows that $g \in \Gamma$. Equivalently this means that $\text{Cone}(\Gamma) \cap G(\Gamma) = \Gamma$. For more details on normal semigroups we refer to [BG09, 2.B].

When D is a big divisor on a d -dimensional variety, and Y_\bullet is an admissible flag on X , we know that $G(\Gamma_{Y_\bullet}(D)) = \mathbb{Z}^{d+1}$ (see [LM09, Lemma 2.2]). In this case, $\Gamma_{Y_\bullet}(D)$ is normal if all integral points of $\text{Cone}(\Gamma_{Y_\bullet}(D))$ are valuation points.

The connection to algebraic geometry comes with the fact that an affine semigroup Γ is normal if and only if the algebra $\mathbb{K}[\Gamma]$ is normal (see [BG09, Lemma 4.39]). Furthermore, the projective variety $X = \text{Proj}(\mathbb{K}[\Gamma])$ is projectively normal if $\mathbb{K}[\Gamma]$, and thus Γ , is normal. However, X is normal if and only if there is an $m \in \mathbb{N}$ such that the $\mathbb{K}[\Gamma]^{(m)} := \bigoplus_{k \in \mathbb{N}} \mathbb{K}[\Gamma]_{mk}$ is normal. But one can easily see that $\mathbb{K}[\Gamma]^{(m)} = \mathbb{K}[m\Gamma]$. Thus $\text{Proj}(\mathbb{K}[\Gamma])$ is normal if and only if there is an integer m such that $m\Gamma$ is normal.

Again, if $\Gamma = \Gamma_{Y_\bullet}(D)$, the variety $\text{Proj}(\mathbb{K}[\Gamma])$ is normal if and only if after passing to an m -th multiple of D , all the integral points of $\text{Cone}(\Gamma_{Y_\bullet}(mD))$ are valuation points.

2.2. Newton-Okounkov bodies. Let X be a d -dimensional projective variety and D a big divisor. We consider \mathbb{Z}^d as an ordered group by choosing the lexicographical order. Let

$$\nu: \bigsqcup_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}(D)) \setminus \{0\} \rightarrow \mathbb{Z}^d$$

be a *valuation-like function*. This is a function having the following properties:

- $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ for $f, g \in H^0(X, \mathcal{O}_X(kD))$
- $\nu(f \otimes g) = \nu(f) + \nu(g)$ for $f \in H^0(X, \mathcal{O}_X(m_1D))$ and $g \in H^0(X, \mathcal{O}_X(m_2D))$.

Additionally, we also pose the following conditions on ν .

- ν has one dimensional leaves (see [KK12, Section 2] for more details)
- The group generated by $\{(\nu(f), k) \mid k \in \mathbb{N}, f \in H^0(X, \mathcal{O}_X(kD))\}$ is equal to \mathbb{Z}^{d+1} .

Then we define the semigroup

$$\Gamma_\nu(D) := \{(\nu(f), k) \mid k \in \mathbb{N}, f \in H^0(X, \mathcal{O}_X(kD))\} \subseteq \mathbb{Z}^d \times \mathbb{N}.$$

The *Newton-Okounkov body* of D with respect to ν is given by

$$\Delta_\nu(D) = \overline{\text{Cone}(\Gamma_\nu(D))} \cap (\mathbb{R}^d \times \{1\}).$$

In this article, we are mainly interested in valuation-like functions induced by flags Y_\bullet , which we denote by ν_{Y_\bullet} . For details on their construction we refer to [LM09].

2.3. Toric degenerations. The connection between toric degenerations and Newton-Okounkov bodies was first established in [A13]. Before phrasing the main result of interest let us make explicit what we mean by a toric degeneration.

Definition 2.1. Let X be projective variety. Let D be a very ample divisor on X . We say that X admits a toric degeneration with respect to D if there is a flat projective family $p: \mathcal{X} \rightarrow \mathbb{A}^1$ such that the zero fiber $X_0 := p^{-1}(0)$ is a toric variety and $\mathcal{X} \setminus X_0$ is isomorphic to $X \times (\mathbb{A}^1 \setminus \{0\})$. Furthermore, there is a divisor \mathcal{D} on \mathcal{X} such that it restricts on fibers $X_t \cong X$ for $t \neq 0$ to the divisor D and on X_0 to an ample divisor D_0 . We call it a *normal toric degeneration* if X_0 is normal. We call it a *projectively normal toric degeneration* if D_0 is very ample and X_0 is projectively normal with respect to the embedding given by D_0 .

The main result in [A13] can be summarized in the following theorem.

Theorem 2.2 ([A13]). *Let X be a projective variety, D a very ample divisor, and ν a valuation-like function. Assume that the semigroup $\Gamma = \Gamma_\nu(D) = \{(\nu(s), k) \mid s \in H^0(X, \mathcal{O}_X(kD)), k \in \mathbb{N}\}$ is finitely generated. Then there exists a toric degeneration of X with respect to D to the toric variety $X_0 := \text{Proj}(\mathbb{K}[\Gamma])$. Moreover, the normalization of X_0 is the normal toric variety corresponding to the polytope $\Delta_\nu(D)$.*

For the sake of clarity we want to make it precise what it means that a Newton-Okounkov body induces a normal toric degeneration.

Definition 2.3. Let X be projective variety. Let D be a big divisor on X and ν a valuation-like function. We say that $\Delta_\nu(D)$ induces a toric degeneration if $\Gamma_\nu(D)$ is finitely generated. We say it induces a *normal toric degeneration* if in addition $\text{Proj}(\mathbb{K}[\Gamma_\nu])$ is normal.

2.4. Ehrhart theory. Let $\Delta \subseteq \mathbb{R}^d$ be a convex body with non empty interior. We define the *Ehrhart function* $h_\Delta: \mathbb{N} \rightarrow \mathbb{N}$ by setting

$$h_\Delta(k) := |k\Delta \cap \mathbb{Z}^d|.$$

Now, let $\Delta \subseteq \mathbb{R}^d$ be a lattice polytope, i.e. a polytope with integral extreme points. Then there is a polynomial $P_\Delta = \sum_{i=0}^d a_i t^i \in \mathbb{C}[t]$ such that $P_\Delta(k) = h_\Delta(k)$ for all $k \in \mathbb{N}$. We call P_Δ the *Ehrhart polynomial* corresponding to Δ . Some basic facts are the following:

- The degree of P_Δ is d .
- a_d is equal to $\text{vol}(\Delta)$.
- We have $a_0 = 1$.
- Let F be a facet of Δ , and let L_F be the induced lattice on that facet. Let furthermore $\text{vol}(F)$ be the volume of F with respect to the lattice L_F . Then a_{d-1} is equal to half the sum of $\text{vol}(F)$ over all facets F of Δ .

3. NEWTON-OKOUNKOV BODIES AND NORMAL TORIC DEGENERATIONS

In this section we want to establish the connection between the Ehrhart polynomial of $\Delta_{Y_\bullet}(D)$ and normal toric degenerations induced by $\Delta_{Y_\bullet}(D)$.

3.1. Normal defect. As we have already mentioned, the toric variety $X_0 = \text{Proj}(\mathbb{K}[\Gamma])$ is not necessarily normal. In order to measure the failure of normality, we introduce the following.

Definition 3.1. Let X be a projective variety, D a big divisor on X and ν a valuation-like function. Let $h_D: \mathbb{N} \rightarrow \mathbb{N}$ be the Hilbert function of D , i.e. $h_D(k) = \dim(H^0(X, \mathcal{O}_X(kD)))$ for $k > 0$. Let $\Delta_\nu(D)$ be the Newton-Okounkov body, and $h_{\Delta_\nu(D)}: \mathbb{N} \rightarrow \mathbb{N}$ the corresponding Ehrhart function, i.e. $h_{\Delta_\nu(D)}(k) = |k\Delta_\nu(D) \cap \mathbb{Z}^d|$. We call the function

$$\text{Def}_{\nu,D} := (h_{\Delta_\nu(D)} - h_D): \mathbb{N} \rightarrow \mathbb{N}.$$

the *normal defect*.

The next theorem justifies the name normal defect.

Theorem 3.2. *Let X be a projective variety, D a very ample divisor on X and ν a valuation-like function. Then a rational polyhedral Newton-Okounkov body $\Delta_\nu(D)$ induces a normal toric degeneration if and only if $\text{Def}_{\nu,kD} = 0$ for $k \gg 0$ divisible enough.*

Proof. Suppose first that $\Delta_\nu(D)$ induces a normal toric degeneration. This means in particular that the semigroup $\Gamma := \Gamma_\nu(D)$ is finitely generated. Suppose Γ is generated in degree k . Hence, we can compute $\Delta_\nu(kD)$ by taking the convex hull of Γ_k . By increasing k even more, we might assume that $k\Gamma = \Gamma_\nu(kD)$ is a normal affine semigroup. This means that all integral points in $C := \text{Cone}(k\Gamma)$ are indeed valuation points, i.e. lie in $k\Gamma$. Consider all the integral points of C at level m . They can be identified with integral points in $m\Delta_\nu(kD)$. There exists $h_{\Delta_\nu(kD)}(m)$ many of them. However, the number of different valuation points in $k\Gamma$ of level m is equal to $\dim(H^0(X, \mathcal{O}_X(mkD))) = h_{kD}(m)$. By the assumption that $k\Gamma$ is normal, they both agree. This proves the vanishing of the normal defect.

Now let $k \in \mathbb{N}$ such that the normal defect $\text{Def}_{\nu,kD}$ is zero. As in the previous case it follows that for each $m \in \mathbb{N}$, there are $h_{\Delta_\nu(kD)}(m) = \dim H^0(X, \mathcal{O}_X(kmD))$ integral points in the m -th level of $k\Gamma$. This proves that all these integral points are valutive, i.e. $\text{Cone}(k\Gamma) \cap (\mathbb{Z}^d \times \{m\}) = (k\Gamma)_m$. Hence, by Gordan's lemma, $k\Gamma$ is a normal affine semigroup. This proves the claim. \square

Let us now denote by P_D the Hilbert polynomial corresponding to the ample divisor D . This means that P_D is the polynomial such that $P_D(k) = h_D(k)$ for $k \gg 0$.

Corollary 3.3. *Let X , be a projective variety, D a very ample divisor on X and ν a valuation-like function. Then an integral polyhedral Newton-Okounkov body $\Delta_\nu(D)$ induces a normal toric degeneration if and only if $P_{\Delta_\nu(D)} = P_D$.*

Proof. This follows from the above Theorem and the fact that $h_D(k) = P_D(k)$ and $h_{\Delta_\nu(D)}(k) = P_{\Delta_\nu(D)}(k)$ for $k \gg 0$. \square

The next two corollaries demonstrate that the condition that $\Delta_\nu(D)$ induces a normal toric degeneration is completely determined by the class of D and the shape of $\Delta_\nu(D)$.

Corollary 3.4. *Let X be a projective variety, Y_\bullet an admissible flag, and D and D' be two numerically equivalent ample line bundles on X . Then $\Delta_\nu(D)$ induces a normal toric degeneration if and only if $\Delta_\nu(D')$ does.*

Proof. First of all, the Newton-Okounkov body of a divisor depends only on its class [LM09, Proposition 4.1]. Moreover, it follows from the Hirzebruch-Riemann-Roch that the Hilbert polynomial of an ample divisor also depends only on the numerical class. Hence, the normal defect of kD , does only depend on the numerical class for $k \gg 0$. \square

Corollary 3.5. *Let X be a projective variety, ν and ν' be valuation-like functions, and D an ample divisor on X . Suppose $\Delta_\nu(D) = \Delta_{\nu'}(D)$. Then $\Delta_\nu(D)$ induces a normal toric degeneration if and only if $\Delta_{\nu'}(D)$ does.*

Proof. This also follows from the equality of defects $\text{Def}_{\nu, kD} = \text{Def}_{\nu', kD}$ for each $k \in \mathbb{N}$. \square

Remark 3.6. *The above corollary a posteriori legitimates to say that $\Delta_\nu(D)$ induces a normal toric degeneration, instead of $\Gamma_\nu(D)$.*

3.2. Normalized surface area. Despite the characterization of normal toric degenerations in terms of the normal defect, it is not quite practical, since it involves knowing the Hilbert polynomial of a line bundle, as well as the Ehrhart polynomial. In this section we want to omit both problems, but still find a necessary condition to induce normal toric degenerations.

Let us fix an ample divisor D on X . Our aim is to find a valuation-like function ν which induces a normal toric degeneration. The idea is to regard this problem as an optimization problem of the shape of $\Delta_\nu(D)$.

For this purpose consider the following definitions. Let P be a lattice polytope in \mathbb{Z}^d . Then denote by $A(P)$ the surface area of P i.e. the sum of the volume of each facet F with respect to the induced sublattice on F .

Definition 3.7. Let X be a projective variety of dimension d , D a very ample divisor on X and ν a valuation-like function. Let furthermore $\Delta_\nu(D)$ be rational polyhedral. Let $k \in \mathbb{N}$ be an integer such that $k\Delta_\nu(D)$ is an integral polyhedron. Then we call

$$S(D, \nu) := \frac{A(\Delta_\nu(kD))}{k^{d-1}}$$

the *normalized surface area* of $\Delta_\nu(D)$. If $\Delta_\nu(D)$ is not rational polyhedral, we define $S(D, \nu) = \infty$.

It is not a priori clear that the above definition is well defined. So let k, k' be two integers such that $k\Delta_\nu(D)$ and $k'\Delta_\nu(D)$ are integral polyhedra. Consider both Ehrhart polynomials $P_{k\Delta_\nu(D)} = \sum_{i=0}^d a_i t^i$ and $P_{k'\Delta_\nu(D)} = \sum_{i=0}^d a'_i t^i$. From our discussion of Ehrhart theory it follows that

$$A(\Delta_\nu(kD)) = 2 \cdot a_{d-1} \quad A(\Delta_\nu(k'D)) = 2 \cdot a'_{d-1}$$

Trivially, $P_{k\Delta_\nu(D)}(k') = P_{k'\Delta_\nu(D)}(k)$. Let us consider the Ehrhart polynomial $P_{k, k'\Delta_\nu(D)} = \sum_{i=0}^d b_i t^i$.

Comparing coefficients, we deduce that $b_{d-1} = a'_{d-1} \cdot k^{d-1} = a_{d-1} \cdot (k')^{d-1}$. This proves that

$$\frac{A(\Delta_\nu(kD))}{k^{d-1}} = \frac{A(\Delta_\nu(k'D))}{(k')^{d-1}}.$$

Theorem 3.8. *Suppose that $\Delta_\nu(D)$ induces a normal toric degeneration. Then the normalized surface area $S(D, \nu)$ is minimal, i.e. for all valuation-like functions ν' we have*

$$S(D, \nu') \geq S(D, \nu).$$

Proof. Suppose $\Delta_\nu(D)$ induces a normal toric degeneration. Let ν' be another valuation-like function. By Theorem 3.2, there is a $k \in \mathbb{N}$ such that the normal defect $\text{Def}(kD, \nu) = 0$. We can assume that $\Delta_{\nu'}(D)$ is rational polyhedral, since otherwise $S(D, \nu) = \infty$. Assume furthermore without loss of generality that $\Delta_\nu(kD)$ and $\Delta_{\nu'}(kD)$ are integral polyhedra. Since $\text{Def}(kD, \nu') \geq 0$ we can follow that

$$\sum_{i=0}^d a_i t^i = P_{\Delta_{\nu'}(kD)} \geq P_{\Delta_\nu(kD)} = \sum_{i=0}^d b_i t^i$$

The first coefficients a_d and b_d of the above polynomials are both equal to $\text{vol}(\Delta_\nu(kD)) = \text{vol}(\Delta_{\nu'}(kD)) = d! \cdot k^d \text{vol}(D)$. Thus, we have $a_{d-1} \geq b_{d-1}$, which in turn implies $S(D, \nu') \geq S(D, \nu)$. □

4. NORMAL TORIC DEGENERATIONS ON SURFACES

In this section we want to apply the above discussions to the case where X is a surface. We will also restrict our attention to valuations coming from flags. One reason why the surface case in a lot of situations works particularly well is that we have a Zariski decomposition of divisors. In our case this leads to a nice characterization of Newton-Okounkov bodies, which makes things more explicit to handle.

Before we dive into normal toric degenerations, we give an overview of the main facts about Zariski decomposition and Newton-Okounkov bodies on surfaces in the first two paragraphs. After that we will prove that for

surfaces satisfying condition (*) (see Definition 4.4) there exists a flag Y_\bullet such that its normalized surface area is minimal with respect to all admissible flags. If we make some more assumptions on the surface X , we will establish an algorithm that computes for a given divisor D a flag Y_\bullet which induces a Newton-Okounkov body with minimal normalized surface area with respect to all valuations coming from flags. Hence, if there exist flags which induce normal toric degenerations, this algorithm will indeed find them.

In the following let X always denote a smooth surface.

4.1. Zariski decomposition. Let X be a smooth surface. Then the Zariski decomposition of a pseudo-effective \mathbb{Q} -divisor is given by $D = P + N$ where P and N are \mathbb{Q} -divisors such that

- (a) P is nef
- (b) the support of $N = \sum_{i=1}^N a_i C_i$ consists of negative curves such that $P \cdot C_i = 0$ for all $i = 1, \dots, N$ and
- (c) the intersection matrix $(C_i \cdot C_j)_{i,j=1,\dots,N}$ is negative-definite.

A decomposition with the above prescribed property is unique and we call P the positive and N the negative part of D . One consequence of the above properties is that for $k \in \mathbb{N}$ divisible enough such that kD as well as kP are integral divisors the natural morphism

$$H^0(X, \mathcal{O}_X(kP)) \rightarrow H^0(X, \mathcal{O}_X(kD))$$

is an isomorphism. That means that, after passing to a multiple, all sections of kD are induced by sections of a nef divisor. Zariski's original proof relied on the construction of the negative part, which was rather complicated. An easier approach was introduced by Bauer [B09], whose idea was to construct the positive part of an effective divisor D as the maximal nef subdivisor of D . This reduces the problem of finding the Zariski decomposition of a given divisor to solving a linear program. More concretely, if we write $D = \sum a_i C_i$ as a positive combination of prime divisors, one finds $P = \sum b_i C_i$, where the b_i are chosen such that $\sum b_i$ is maximal under the constraints that $0 \leq b_i \leq a_i$, and $\sum b_i C_i$ is nef.

Remark 4.1. *Note that even if D is an integral divisor the Zariski decomposition $D = P + N$ is still a decomposition of \mathbb{Q} -divisors, i.e. P and N are not necessarily integral.*

However, in [BPS15] the authors give an upper bound for the size of the denominators occurring in terms of the negativity of N . In the proof of Theorem 2.2 they show the following:

Theorem 4.2 ([BPS15]). *Let X be a smooth projective surface with Picard number $\rho(X)$, let D be a divisor and $N = \sum a_i \cdot C_i$ be its negative part, with $a_i > 0$ and C_i prime divisors. Let furthermore d be the denominator of N , i.e. the smallest natural number d such that $d \cdot N$ is integral, and b be the*

maximum of the negative numbers $(C_i)^2$. Then we have

$$d \leq b^{\rho(X)-1}.$$

Another very important feature about the Zariski decomposition, is that it induces a decomposition of the big cone into chambers \mathcal{C}_i ; the so called *Zariski chambers*. This chamber decomposition was introduced in [BKS04]. We summarize some facts about this decomposition:

- The support of the negative parts of $D \in \mathcal{C}_i$ for a fixed i is constant.
- The \mathcal{C}_i are locally polyhedral and form a locally finite decomposition of the big cone
- Inside the closure of each Zariski chamber \mathcal{C}_i the Zariski decomposition varies linearly.

4.2. Newton-Okounkov bodies on surfaces. Newton-Okounkov bodies are in general difficult to compute. However, on a surface with a valuation-like function coming from a flag, we can give a rather explicit description. Let $Y_\bullet = (X \supseteq C \supseteq \{P\})$ be an admissible flag, i.e. P is a point and C is an irreducible curve which is smooth at P . Then we can define a valuation-like function ν_{Y_\bullet} , by setting for a section $s \in H^0(X, \mathcal{O}_X(D))$

$$\nu_1(s) = \text{ord}_C(s) \quad \nu_2(s) = \text{ord}_{\{P\}}(\tilde{s})$$

where \tilde{s} is the restriction of the section $s/(s_C)^{\nu_1(s)}$ to the curve C and s_C is a defining section of C .

In order to describe the Newton-Okounkov body of a big divisor D with respect to a flag $C \supset \{P\}$ we fix the following notation:

- $\nu := \text{ord}_C(N)$
- $\mu := \sup\{t \in \mathbb{R}_{\geq 0} \mid D - tC \text{ is effective}\}$
- For $t \in [0, \mu]$ we define $D_t := D - tC = P_t + N_t$ where the latter is its Zariski decomposition.
- We define the functions $\alpha, \beta: [\nu, \mu] \rightarrow \mathbb{R}_{\geq 0}$ by setting

$$\alpha(t) := \text{ord}_P(N_{t|C}) \quad \beta(t) := \alpha(t) + (P_t \cdot C).$$

Moreover, we write α_D, β_D if we want to stress that we consider the divisor D .

Finally, we present the description of Newton-Okounkov bodies in the following theorem, which is based on the discussions in [LM09, Section 6.2] and [KLM12, Section 2].

Theorem 4.3. *The Newton-Okounkov body of a big divisor D with respect to an admissible flag Y_\bullet on a surface X is given by*

$$\Delta_{Y_\bullet}(D) = \{(t, y) \in \mathbb{R}^2 \mid t \in [\nu, \mu], y \in [\alpha(t), \beta(t)]\}.$$

Moreover, $\Delta_{Y_\bullet}(D)$ is a finite polygon, with all extremal points rational except for possibly $(\mu, \alpha(\mu))$ and $(\mu, \beta(\mu))$.

The proof of the above theorem uses the fact that the Zariski decomposition varies linearly inside the Zariski chambers. The fact that it is a finite polygon follows by showing that the set of divisors D_t for $t \in [\nu, \mu]$ only meets finitely many chambers. Additionally, it follows from the proof that the extreme points of $\Delta_{Y_\bullet}(D)$ are all of the following form:

- $(\nu, \alpha(\nu)), (\nu, \beta(\nu))$
- $(\mu, \alpha(\mu)), (\mu, \beta(\mu))$
- $(t, \alpha(t)), (t, \beta(t))$ for $t \in (\nu, \mu)$ such that D_t lies on the boundary of a Zariski chamber.

4.3. Existence of Newton-Okounkov bodies with minimal normalized surface area. In this paragraph we will prove that for a given divisor D there exists a flag Y_\bullet such that the normalized surface area of $\Delta_{Y_\bullet}(D)$ is minimal with respect to all admissible flags.

We will now consider surfaces with the following constraints.

Definition 4.4. We say that a smooth projective surface X satisfies condition (*) if it satisfies the following conditions:

- (a) Every pseudo-effective divisor D is semi-effective, i.e. a multiple of D is effective.
- (b) X contains only finitely many negative curves.

Remark 4.5. A large class of examples which satisfy condition (*) are Mori dream surfaces.

One necessary condition on the curve of the flag to induce a normal toric degeneration is the following.

Lemma 4.6. Let $Y_\bullet = (C \supseteq \{P\})$ be an admissible flag such that $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration. Then the genus of C is zero, i.e. $C \cong \mathbb{P}^1$.

Proof. Choose a rational $t \in \mathbb{Q}$ such that the slice $\{t\} \times \mathbb{R}$ meets the interior of $\Delta_{Y_\bullet}(D)$. Let then $k \in \mathbb{N}$ be such that $kD_t = kP_t + kN_t$ is a decomposition of integral divisors and kt is integral. It follows from Theorem 4.3 that the slice $\Delta_{Y_\bullet}(kD)_{\nu_1=kt}$ contains $k(P_t \cdot C) + 1$ integral points. The valuation points having kt as first coefficient are given by the image of

$$\text{ord}_P: H^0(X, \mathcal{O}(kD_t))|_C \rightarrow \mathbb{Z}$$

and the number of valuation points is given by

$$h^0(X, \mathcal{O}(kD_t))|_C = h^0(X, \mathcal{O}(kP_t))|_C \leq h^0(C, \mathcal{O}_C(kP_t)).$$

However, it follows from Riemann-Roch on curves that for $k \gg 0$ we can compute

$$h^0(C, \mathcal{O}_C(kP_t)) = k(P_t \cdot C) + 1 - g$$

where g is the genus of C . But since all integral points of $\Delta_{Y_\bullet}(kD)$ are valutive for $k \gg 0$ it follows that $g = 0$ and thus $C \cong \mathbb{P}^1$. \square

We continue by proving two helpful lemmata.

Lemma 4.7. *Let D be a big divisor on X and $[C] \in N^1(X)$ be the numerical class of an irreducible curve C . Then the set of Newton-Okounkov bodies $\Delta_{Y_\bullet}(D)$, where Y_\bullet is a flag such that $[Y_1] = [C]$, is finite.*

Proof. Consider the negative part N_μ of the pseudo-effective divisor $D_\mu = D - \mu C$. Let C_1, \dots, C_l be the irreducible curves in the support of N_μ . It follows from [KLM12, Proposition 2.1] that the irreducible components of N_t for $t \in [\nu, \mu]$ is a subset of $\{C_1, \dots, C_l\}$, and that C is not equal to C_i for all $i = 1, \dots, l$. Let $\nu \leq t_1 \leq \dots \leq t_r \leq \mu$ be all rational numbers in $[\nu, \mu]$ such that D_{t_i} lies on the boundary of some Zariski chamber. By the discussion in section 4.2, these are indeed finitely many. Consider the negative parts $N_\nu, N_{t_1}, \dots, N_{t_r}, N_\mu$. By replacing D with kD for $k \gg 0$, we may assume without loss of generality that all these negative parts are integral divisors and the numbers t_1, \dots, t_r are integral. For each $P \in C$ we have

$$\alpha(\mu) = \text{ord}_P N_{\mu|C} \leq \sum_{x \in C \cap N_\mu} \text{ord}_x(N_{\mu|C}) = (N_\mu \cdot C).$$

By [KLM12, Theorem B], the function α is increasing, and piecewise linear with possible breaking points at t_1, \dots, t_r . This shows that for a fixed class $[C]$ the function α is bounded by some constant independent from the point P . By construction, α takes integer values at the points $\nu, t_1, \dots, t_r, \mu$. Varying the point P , there are only finitely many possibilities for α since it is uniquely defined by its values on ν, t_1, \dots, t_r and μ . But, by definition, the same holds for β . However, we have seen in Theorem 4.3 that $\Delta_{Y_\bullet}(D)$ is determined by α and β . This shows the claim. \square

Lemma 4.8. *Let X be a smooth surface that satisfies condition (*). Let D be a big and nef divisor on X . Then the set*

$$H_k := \{[D'] \in N^1(X)_{\mathbb{R}} : D' \text{ is nef and } (D' \cdot D) = k\}$$

is compact for all $k \in \mathbb{N}$.

Proof. Suppose that H_k is not compact. It is easy to check that H_k is closed. This means H_k is not bounded. But since it is also convex, there exists for every point in H_k a half line through the given point which is completely contained in H_k . For this purpose fix any ample class $[A] \in N^1(X)$ and consider the \mathbb{Q} -divisor $A' = \frac{k}{(D \cdot A)} A$ which lies in H_k . Since H_k is not bounded there is a divisor class $[F] \in N^1(X)_{\mathbb{R}}$ such that for all $\lambda > 0$, the class $[A' + \lambda F]$ lies in H_k . We claim that F is nef. Indeed, suppose that F is not nef. Then for $\lambda \gg 0$ the divisor $A' + \lambda F$ is not nef as well and thus does not lie in H_k .

For a given $\lambda > 0$, we have

$$(A' + \lambda F)^2 = (A')^2 + 2\lambda(A' \cdot F) + (\lambda F)^2 \geq (A')^2 + 2\lambda(A' \cdot F).$$

But A' is ample and F nef, hence semi-effective by condition (*). This implies that $A' \cdot F > 0$ and enables us to find a $\lambda > 0$ such that

$$\sqrt{(A' + \lambda F)^2} > k/\sqrt{(D)^2}.$$

Here, we used the fact that D is big and nef and thus $(D^2) > 0$. As D and $A + \lambda F$ are both nef, we can use the Hodge index theorem to deduce

$$(D \cdot (A' + \lambda F)) \geq \sqrt{(D)^2 \cdot (A' + \lambda F)^2} > k.$$

This shows that $A' + \lambda F$ does not lie in H_k , which is a contradiction. Hence, H_k is compact. \square

Theorem 4.9. *Let X be a smooth surface satisfying condition (*). Let D be a big divisor on X . Then there exists an admissible flag $Y_\bullet = (C \supset \{x\})$ such that its normalized surface area $S(D, \nu_{Y_\bullet})$ is minimal, i.e. for any admissible flag Y'_\bullet we have $S(D, \nu_{Y_\bullet}) \leq S(D, \nu_{Y'_\bullet})$.*

Proof. By scaling and considering the positive part in the Zariski decomposition of D , we can without loss of generality assume that D is big and nef. The idea of the proof is to show that only a finite number of classes of curves C have to be tested. Together with Lemma 4.7 we can then prove the claim.

First of all if C is an irreducible curve, then its class $[C]$ is either nef or it is a negative curve depending on whether $C^2 \geq 0$ or $C^2 < 0$. Since X satisfies condition (*), we have to test only finitely many negative curves. Hence, we can restrict our attention to the case that C is nef. Let Y_\bullet be any admissible flag such that Y_1 is nef, and set $M := S(D, \nu_{Y_\bullet})$. Then for all C' , and any point $P' \in C'$ such that $(D \cdot C') > M$, we know that there are already more than $M + 1$ integral points on the boundary of $\Delta_{C' \supset \{P'\}}(D)$, namely $(0, \alpha(0)), (0, \alpha(0) + 1), \dots, (0, \beta(0))$. However, this implies $S(D, \nu_{Y_\bullet}) > M$. Hence, we have limited the candidates to nef irreducible curves C' such that $(D \cdot C') \leq M$. But Lemma 4.8 implies there are only finitely many integral nef classes of curves which satisfy this condition. Moreover, each class of a curve $[C]$ we have to test, has finitely many different Newton-Okounkov bodies, when varying the point P by Lemma 4.7. This proves the claim. \square

4.4. Algorithm for finding a flag with minimal normalized surface area. In this paragraph we want to introduce and discuss an algorithm, which outputs for a given big divisor D on a surface X a flag Y_\bullet such that $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration if such a flag exists. In Theorem 4.9, we have limited the possible candidates for flags which induce normal toric degenerations to finitely many classes of curves $[Y_1]$. However, it is a rather difficult task to describe what possible points $P \in C$ can occur and how the function α from Section 4.2 varies. The idea of this section is to show that it is possible to reduce to a general point on the chosen curve. Then $\alpha = 0$ and $\beta(t) = (P_t \cdot C)$. It follows that the corresponding Newton-Okounkov body only depends on the numerical class of the curve C and in

this situation Theorem 4.9 gives rise to a rather explicit algorithmic way of finding a class of a curve which is the best candidate for defining a flag Y_\bullet such that $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration.

The price we pay for being able to choose a general point is the following constraint.

Definition 4.10. We say that a smooth projective surface X satisfies condition (**) if it satisfies condition (*) and the Zariski decomposition is a decomposition of integral divisors, i.e. for each integral divisor D its positive part $P(D)$ as well as its negative part $N(D)$ is integral.

Remark 4.11. *It follows from Theorem 4.2 that a surface having only negative curves with self intersection -1 induces integral Zariski decompositions for all divisors. An example for this situation would be smooth del Pezzo surfaces (more details follow in the next section).*

The following lemma is the key for reducing to the case of a general point P on C .

Lemma 4.12. *Let X be a smooth surfaces satisfying condition (**). Suppose D is a big divisor and $Y_\bullet = (C \supset \{P\})$ is an admissible flag such that $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration. Then for each point $P' \in C$ consider the flag $Y'_\bullet = (C \supset \{P'\})$. Then $\Delta_{Y'_\bullet}(D)$ induces a normal toric degeneration as well.*

Proof. We will do this by proving that the Ehrhart polynomial $P_{\Delta_{Y_\bullet}(kD)}$ is independent of the point P for $k \gg 0$. Let $k \gg 0$ such that $\Delta_{Y_\bullet}(kD)$ is an integral polytope. Define for integral m, t the divisor $D_{m,t} := mD - tC =: P_{m,t} + N_{m,t}$.

Since X satisfies condition (**), the function $\alpha_{kD}(t) = \text{ord}_P(N_{k,t}|_C)$ and $\beta_{kD}(t) = \alpha_{kD}(t) + (P_{k,t} \cdot C)$ admit integral values for each integral t . From this we can deduce

$$|k\Delta_{Y_\bullet}(D) \cap \mathbb{Z}^2| = \sum_{t=k\nu}^{k\mu} ((P_{k,t} \cdot C) + 1).$$

But the right hand side does not depend on the choice of the point $P \in C$. Hence, the result follows from Theorem 3.2. \square

Remark 4.13. *The condition that X admits integral Zariski decompositions is indeed necessary for the above lemma.*

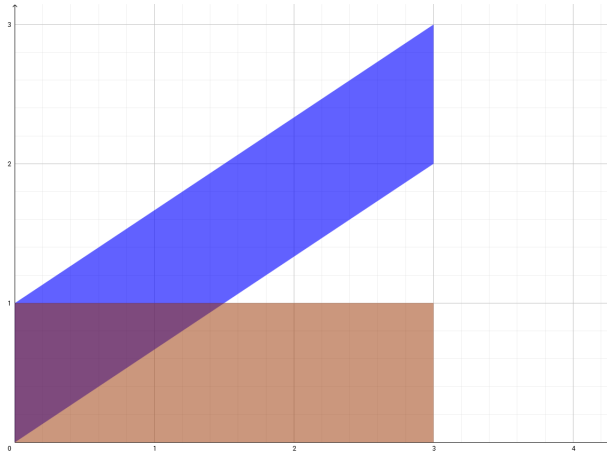


FIGURE 1. Newton-Okounkov body of toric variety

For a counterexample consider the blue (dark) polytope in Figure 1. Then, by the discussion in [LM09, Section 6.1], there is a toric variety X and a divisor D such that with respect to a certain flag Y_\bullet , the corresponding Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is equal to the above blue (dark) polytope. It also follows from this discussion that $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration. If we change the point Y_2 of the flag and pick a general one instead, the resulting Newton-Okounkov body equals the red (light) polytope in Figure 1. However, the number of integral boundary points on the red polytope (8) is bigger than the boundary points on the blue polytope (4). This proves that for a general point, the corresponding Newton-Okounkov body does not induce a normal toric degeneration even though this holds for a special point.

As we have seen above, the Ehrhart polynomial of $\Delta_{Y_\bullet}(D)$ only depends on the numerical class $[Y_1]$ if X satisfies condition (**). Hence, also the normalized surface area $S(D, \nu_{Y_\bullet})$ only depends on the numerical class $[Y_1]$. Therefore, we will just write $S(D, [Y_1])$ instead of $S(D, \nu_{Y_\bullet})$.

We are now able to describe an algorithm which will give us for a given divisor D an optimal class of a curve $[C]$.

Theorem 4.14. *Let X be a projective surface satisfying condition (**). Let D be a very ample divisor. Let us assume that there is a flag Y_\bullet such that $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration. Then the output of Algorithm 1 gives a list of all classes of curves C_i which give rise to flags such that the corresponding Newton-Okounkov bodies induce normal toric degenerations.*

Proof. This follows from Theorem 3.8, the proof of Theorem 4.9 and Lemma 4.12. \square

Algorithm 1: Algorithm for finding normal toric degenerations

Input: a big divisor D

Result: optimal class of a curve C

$D = P + N$ // compute Zariski decomposition

$D := P$ // replace the divisor D by its positive part

for *negative classes of curves* N_i **do**

\lfloor compute $S(D, [N_i])$

$optimum := \min_{N_i} S(D, [N_i])$

$optimalcurve := \operatorname{argmin}_{N_i} S(D, [N_i])$

for all $\xi \in N^1(X)$ s.t. $(D \cdot \xi) < optimum + 1$ **do**

if $\xi == [C]$ for some irreducible curve C **then**

if $optimum \leq S(D, \xi)$ **then**

$optimum = S(D, \xi)$

$optimalcurve := \xi$

else if $optimum == S(D, \xi)$ **then**

$optimalcurve.append(\xi)$

Output: $optimalcurve$

5. NORMAL TORIC DEGENERATIONS OF (WEAK) DEL PEZZO SURFACES

In this paragraph we will use our previous findings and additional ideas to construct normal toric degenerations of *(weak) del Pezzo surfaces*.

5.1. Normal toric degeneration of smooth del Pezzo surfaces. Let us first present some basic facts about smooth del Pezzo surfaces.

Definition 5.1. We call X a del Pezzo surface if it is a surface and its anticanonical divisor $-K_X$ is ample.

Before we give the characterization of smooth del Pezzo surfaces, let us define what we mean by points in general position.

Definition 5.2. We say that $1 \leq r \leq 8$ distinct points p_1, \dots, p_8 in \mathbb{P}^2 are in *general position* if:

- No three of them lie on a line.
- No six of them lie on a conic.
- No eight of them lie on a cubic with a singularity at some of the p_i .

We can now state the well known characterization of smooth del Pezzo surfaces.

Theorem 5.3. *Up to isomorphy the smooth del Pezzo surfaces are given by $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of \mathbb{P}^2 in $0 \leq r \leq 8$ points in general position.*

Let X_r be the smooth del Pezzo surface obtained by blowing up r points in general position. In the following we collect some more facts, we want to use:

(a) We have

$$\text{Pic}(X) \cong N^1(X) \cong \mathbb{Z}^{r+1} \cong \mathbb{Z}[H] \oplus \bigoplus_{i=1}^r \mathbb{Z}[E_i]$$

where H is the total transform of a line in \mathbb{P}^2 and E_i are the exceptional divisors.

- (b) The intersection form on $N^1(X)$ is determined by the identities $(H)^2 = 1$, $(H \cdot E_i) = 0$, $(E_i \cdot E_j) = -\delta_{ij}$.
- (c) The anticanonical divisor is given by

$$-K_{X_r} = 3H - E_1 - \dots - E_r.$$

- (d) Every irreducible curve with negative self intersection number is a (-1) -curve, and it is, up to permutation of indices, linear equivalent to one of the following divisors:

$$\begin{aligned} &E_1 \\ &H - E_1 - E_2 \\ &2H - E_1 - \dots - E_5 \\ &3H - 2E_1 - E_2 - \dots - E_7 \\ &4H - 2E_1 - 2E_2 - 2E_3 - E_4 - \dots - E_8 \\ &5H - 2E_1 - \dots - 2E_6 - E_7 - E_8 \\ &6H - 3E_1 - 2E_2 - \dots - 2E_8. \end{aligned}$$

- (e) Let $N = \{C_1, \dots, C_N\}$ be the set of (-1) -curves. The effective cone $\text{Eff}(X)$ is generated by the negative curves C_i in N . The nef cone is determined by the supporting hyperplanes

$$C_i^\perp := \{[D] \in N^1(X)_{\mathbb{R}} \mid D \cdot C_i = 0\}.$$

- (f) The Zariski chambers of X_r are also determined by the chamber decomposition of $\text{Eff}(X)$ induced from the hyperplanes C_i^\perp .
- (g) Suppose $r = 1, \dots, 6$. A divisor class $D \in \text{Pic}(X_r)$ contains an irreducible curve $C \in |D|$ if and only if D is either (a) one of the (-1) -curves in N or (b) D is big and nef or (c) D is a conic (i.e. $D \cdot (-K_{X_r}) = 2$) and $D^2 = 0$.
- (h) Let $C \subset X_r$ be an irreducible smooth curve such that $C \equiv_{\text{lin}} aH - b_1E_1 - \dots - b_rE_r$. Then the genus of C is given by

$$g(C) = 1/2(a-1)(a-2) - 1/2 \sum_{i=1}^r b_i(b_i-1).$$

As a reference, we refer to [H77, V.4] for (a), (b), (c), (g), to [ADH14, Chapter 5] for properties (d), (e). Property (f) is derived in [BKS04, Proposition 3.4]. Furthermore, (h) is an easy calculation using Riemann-Roch.

Let us now apply Algorithm for a specific del Pezzo surface.

Example 5.4. Let X_5 be the blow-up of \mathbb{P}^2 in five general points. That means that no three of them lie on a line. In this case, the negative curves are of the form

$$E_1, \dots, E_5, \text{ and } H - E_i - E_j \text{ for } i, j = 1, \dots, 5, i \neq j.$$

For a given divisor D and a curve C , we have all the necessary information to compute the Newton-Okounkov body $\Delta_{C \supset \{P\}}(D)$ for a very general point $P \in C$. With the help of a computer we can thus use our algorithm to compute the set of optimal curves, and the optimal normalized surface area for a given divisor D . We can use [SX10, Example 1.3] to efficiently compute the Hilbert polynomial of a given divisor D as the Ehrhart polynomial of some polytope. Hence, we can compare the second coefficient of the Hilbert polynomial with the normalized surface area. If they agree the given Newton-Okounkov body with respect to the curves found by the algorithm induce normal toric degenerations. Running the algorithm for some randomly chosen divisors gives the following result. Note that all the divisor classes are represented by the basis H, E_1, \dots, E_5 .

D	optimal curves		min. $S(D, [C])$	2nd coef. of $h_D(t)$
	nef curves	negative curves		
$(6, -1, -1, -2, -3, -4)$	$(4, -2, 0, -2, -2, -2),$ $(4, -2, 0, -2, -2, -2),$ $(3, -1, -1, -1, -1, -2),$ $(2, -1, -1, -1, 0, -1),$ $(2, -1, -1, 0, -1, -1),$ $(2, -1, 0, -1, -1, -1),$ $(4, 0, -2, -2, -2, -2),$ $(2, 0, -1, -1, -1, -1),$ $(2, 0, 0, -1, -1, -1),$ $(1, 0, 0, -1, 0, 0),$ $(1, 0, 0, 0, -1, 0),$ $(1, 0, 0, 0, 0, -1)$	$(0, 1, 0, 0, 0, 0),$ $(0, 0, 1, 0, 0, 0),$ $(0, 0, 0, 1, 0, 0),$ $(0, 0, 0, 0, 1, 0),$ $(0, 0, 0, 0, 0, 1),$ $(1, -1, -1, 0, 0, 0),$ $(1, 0, -1, -1, 0, 0),$ $(1, 0, 0, -1, -1, 0),$ $(1, -1, 0, -1, 0, 0),$ $(1, -1, 0, 0, -1, 0),$ $(1, 0, -1, 0, -1, 0),$ $(1, 0, 0, 0, -1, -1),$ $(1, 0, 0, -1, 0, -1),$ $(1, 0, -1, 0, 0, -1),$ $(1, -1, 0, 0, 0, -1),$ $(2, -1, -1, -1, -1, -1)$	6	6
$(6, -1, -3, -1, -2, -3)$	$(2, -1, -1, -1, 0, -1),$ $(2, -1, -1, -1, 0, -1),$ $(2, -1, -1, 0, -1, -1),$ $(2, 0, -1, -1, -1, -1),$ $(1, 0, -1, 0, 0, 0),$ $(1, 0, 0, 0, 0, -1)$	$(0, 1, 0, 0, 0, 0),$ $(0, 0, 1, 0, 0, 0),$ $(0, 0, 0, 1, 0, 0),$ $(0, 0, 0, 0, 1, 0),$ $(0, 0, 0, 0, 0, 1),$ $(1, -1, -1, 0, 0, 0),$ $(1, 0, -1, -1, 0, 0),$ $(1, 0, 0, -1, -1, 0),$ $(1, -1, 0, -1, 0, 0),$ $(1, -1, 0, 0, -1, 0),$ $(1, 0, -1, 0, -1, 0),$ $(1, 0, 0, 0, -1, -1),$ $(1, 0, 0, -1, 0, -1),$ $(1, 0, -1, 0, 0, -1),$ $(1, -1, 0, 0, 0, -1),$ $(2, -1, -1, -1, -1, -1)$	8	8

D	optimal curves		min. $S(D, [C])$	2nd coef. of $h_D(t)$
	nef curves	negative curves		
(8, -3, -2, -2, -2, -3)	\emptyset	(0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1), (1, -1, -1, 0, 0, 0), (1, 0, -1, -1, 0, 0), (1, 0, 0, -1, -1, 0), (1, -1, 0, -1, 0, 0), (1, -1, 0, 0, -1, 0), (1, 0, -1, 0, -1, 0), (1, 0, 0, 0, -1, -1), (1, 0, 0, -1, 0, -1), (1, 0, -1, 0, 0, -1), (1, -1, 0, 0, 0, -1), (2, -1, -1, -1, -1, -1)	12	12
	nef curves	negative curves		
(4, -1, -1, -1, 0, -1)	(2, -1, -1, -1, 0, -1), (2, -1, -1, -1, 0, -1), (1, -1, 0, 0, 0, 0), (1, 0, -1, 0, 0, 0), (1, 0, 0, -1, 0, 0), (1, 0, 0, 0, 0, -1)	(0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1), (1, -1, -1, 0, 0, 0), (1, 0, -1, -1, 0, 0), (1, 0, 0, -1, -1, 0), (1, -1, 0, -1, 0, 0), (1, -1, 0, 0, -1, 0), (1, 0, -1, 0, -1, 0), (1, 0, 0, 0, -1, -1), (1, 0, 0, -1, 0, -1), (1, 0, -1, 0, 0, -1), (1, -1, 0, 0, 0, -1), (2, -1, -1, -1, -1, -1)	8	8
	nef curves	negative curves		
(7, -4, 0, -2, -3, -3)	(3, -2, -1, -1, -1, -1), (3, -2, -1, -1, -1, -1), (2, -1, -1, -1, -1, 0), (2, -1, -1, -1, 0, -1), (2, -1, -1, 0, -1, -1), (4, -2, 0, -2, -2, -2), (3, -2, 0, -1, -1, -1), (2, -1, 0, -1, -1, -1), (2, -1, 0, -1, -1, 0), (2, -1, 0, -1, 0, -1), (2, -1, 0, 0, -1, -1), (1, -1, 0, 0, 0, 0), (1, 0, 0, -1, 0, 0), (1, 0, 0, 0, -1, 0), (1, 0, 0, 0, 0, -1)	(0, 1, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1), (1, -1, -1, 0, 0, 0), (1, 0, -1, -1, 0, 0), (1, 0, 0, -1, -1, 0), (1, -1, 0, -1, 0, 0), (1, -1, 0, 0, -1, 0), (1, 0, -1, 0, -1, 0), (1, 0, 0, 0, -1, -1), (1, 0, 0, -1, 0, -1), (1, 0, -1, 0, 0, -1), (1, -1, 0, 0, 0, -1), (2, -1, -1, -1, -1, -1)	9	9

We can make several conjectures from this example. First of all in each example the optimal normalized surface area is equal to the second coefficient of the Hilbert polynomial of D . Thus, in each example we do indeed get normal toric degenerations. Moreover, in each example all negative curves are optimal. We will see later that this is true for all varieties X_r for $r = 1, \dots, 6$.

The next theorem describes some conditions on X and on the flag Y_\bullet which make sure that $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration.

Theorem 5.5. *Let X be a smooth surface, admitting integral Zariski decompositions. Let $Y_\bullet = (C \supseteq \{P\})$ be an admissible flag such that $C \cong \mathbb{P}^1$ and $-K_X - C$ defines a big and nef class. Let D be a big divisor on X . Then $\Delta_{Y_\bullet}(D)$ induces a normal toric degeneration if and only if $\Delta_{Y_\bullet}(D)$ is rational polyhedral.*

Proof. Let us first make some observations. If D is a nef divisor, then by assumption $D - C - K_X$ is big and nef. We can therefore use the Kawamata-Viehweg vanishing theorem to deduce that $H^1(X, \mathcal{O}_X(D - C)) = 0$. This implies that the restriction morphism $H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(C, \mathcal{O}_C(D))$ is surjective for every nef divisor D . Our aim is to show that we have an equality

$$\Gamma_k(D) = k \cdot \Delta_{Y_\bullet}(D) \cap \mathbb{Z}^2.$$

Then the statement follows by using Gordan's lemma. We will do this by considering the vertical t -slices of $k\Delta_{Y_\bullet}(D)$, i.e. points such that the first coordinate is equal to a fixed integer $t \in [k\nu, k\mu]$. The second coordinate of the valuation points $\Gamma_k(D)$ in the t -slice are given by the valuation points of the restricted linear series $H^0(X, \mathcal{O}_X(kD - tC))|_C$ of the valuation ord_P . Define $D_{k,t} := kD - tC$, $P_{k,t} := P(D_{k,t})$ and $N_{k,t} := N(D_{k,t})$. By Theorem 4.12, we can without loss of generality assume that the point P is not contained in the support of the negative part $N_{k,t}$. Since X admits an integral Zariski decomposition, we can replace the restricted linear series $H^0(X, \mathcal{O}_X(D_{k,t}))|_C$ with the linear series $H^0(X, \mathcal{O}_X(P_{k,t}))|_C = H^0(C, \mathcal{O}_C(P_{k,t}))$. As $C \cong \mathbb{P}^1$, we can apply Riemann-Roch to deduce that

$$\dim H^0(C, \mathcal{O}_C(P_{t,k}|_C)) = (P_{t,k} \cdot C) + 1.$$

Hence, the valuation points in the t -slice are exactly all the points (t, s) where $s \in \{0, \dots, (P_{k,t} \cdot C)\}$. These are all the integer points in the t -slice of $k\Delta_{Y_\bullet}(D)$. \square

We can use the above theorem to prove the following.

Theorem 5.6. *Let X_r be the blow-up of r general points in \mathbb{P}^2 for $r = 1, \dots, 6$. Let D be a big divisor on X_r , $C \subset X_r$ a negative curve and $P \in C$ an arbitrary point. Then $\Delta_{C \supset \{P\}}(D)$ induces a normal toric degeneration.*

Proof. Since X_r has a rational polyhedral effective cone, the Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is rational polyhedral for all big divisors and admissible flags Y_\bullet . It follows from Theorem 4.2 and the fact that the only negative curves in X_r are (-1) -curves, that X_r admits integral Zariski decompositions. The negative curves of X_r are either exceptional divisors E_i , lines $H - E_i - E_j$ or conics $2H - E_{i_1} - \dots - E_{i_4}$. All of them are rational. In addition, a calculation shows that the divisor $-K_X - C$ is big and nef for all possible negative curves C . Now, we can use Theorem 5.5, which proves the claim. \square

Remark 5.7. *Note that for X_r , where $r = 7, 8$ the assumptions of Theorem 5.5 are not fulfilled for all negative curves. Consider for example the negative curve $C = 3H - 2E_1 - E_2 - \dots - E_7$ on X_7 . Then $-K_{X_7} - C = E_1$ which is clearly not big and nef.*

5.2. Normal toric degeneration on weak del Pezzo surfaces. In this paragraph we want to discuss examples of weak del Pezzo surfaces which induce normal toric degenerations.

Definition 5.8. We call X a *weak del Pezzo surface* if it is a surface and its anticanonical divisor $-K_X$ is nef and big.

The characterization of smooth weak del Pezzo surfaces is a bit more complex. Roughly speaking, more constellation of points to blow-up are allowed. One of the main differences to del Pezzo surfaces is that no longer only (-1) -curves occur as negative curves but also (-2) -curves.

We will focus on two examples. First, the blow-up of six points on a conic and second, the blow-up of four points where three of them lie on a line.

5.2.1. Blow-up of six points on a conic. Consider the variety S_6 which is given as the blow-up of six points in \mathbb{P}^2 such that no three of them are collinear but all six lie on a single conic. The negative curves are:

- (a) E_1, \dots, E_6 the exceptional divisors
- (b) $H - E_i - E_j$ for $i \neq j, i, j \in \{1, \dots, 6\}$ the strict transforms of the lines through two points.
- (c) $2H - E_1 - \dots - E_6$ the strict transform of the conic through all the six points.

The first two types of curves are (-1) -curves and the last one is a (-2) -curve.

Theorem 5.9. *Let D be a big divisor on S_6 . Let furthermore C be the strict transform of the conic going through the six chosen points, and $P \in C$ an arbitrary point. Then $\Delta_{C \supset \{P\}}(D)$ induces a normal toric degeneration.*

Proof. The proof works similarly as before with the only difference that there are also (-2) -curves occurring. This means that it is not clear whether S_6 admits integral Zariski decompositions.

However, a computation shows that $-K_X - C$ is big and nef. We know that C is not contained in the support of N_t for $\nu \leq t \leq \mu$ (see proof of Proposition 2.1 in [KLM12]).

Since C is the only (-2) -curve in S_6 , the support of the divisors of N_t only consists of (-1) -curves. We can thus use Theorem 4.2 to deduce that if D_t is integral then also P_t and N_t are integral. Then the proof works exactly as in Theorem 5.5. \square

In order to be able to compute Newton-Okounkov bodies, we need to know the Zariski chambers of the effective cone of S_6 . Note that unlike in the case of del Pezzo surfaces, the decomposition of Zariski chambers is not necessarily given by the decomposition induced from the hyperplanes C^\perp where C is in the set of negative curves. This is a consequence of the fact that there exists a (-2) -curve.

In general it is quite difficult to describe this decomposition. However, in order to compute Newton-Okounkov bodies of a divisor D with respect to the curve C given by the conic going through the six points, we just need to compute the wall crossings of the segment $D - tC$ for $t \in [\nu, \mu]$. The next lemma describes these crossing points.

Lemma 5.10. *Let D be a big divisor on S_6 . Then the intersections of the divisors $D - tC$ for $t \in (\nu, \mu)$ with the boundary of the Zariski chambers all lie in the set $\bigcup C_i^\perp$ where the union is taken over all (-1) -curves.*

Proof. The proof is very similar to Proposition 3.4 in [BKS04]. It is shown in the mentioned proof that if N is a negative divisor whose support contains only (-1) -curves, then all the irreducible components of N are orthogonal. Let us now suppose $D_t := D - tC$ for $t \in (\nu, \mu)$ lies on the boundary of a Zariski chamber. If we define for a divisor D the sets

$$\begin{aligned} \text{Null}(D) &= \{C \mid \text{irreducible with } (C \cdot D) = 0\} \\ \text{Neg}(D) &= \{C \mid \text{irreducible component of } N(D)\}, \end{aligned}$$

then according to [BKS04, Proposition 1.5], this means that

$$\text{Null}(P_t) \setminus \text{Neg}(D_t) \neq \emptyset.$$

Let C' be a curve which lies in $\text{Null}(P_t)$ but not in $\text{Neg}(D_t)$. Then C' is a negative curve and $N_{D_t} + C'$ is a negative divisor according to [BKS04, Lemma 4.3]. We want to show that $C' \neq C$. Suppose that they are equal. Then $(P_t \cdot C) = 0$. We know from the choice of t that the slice $\Delta_{C \subset \{P\}}(D)_{\nu_1=t}$ has length bigger than 0 for $t \in (\nu, \mu)$. However, this is a contradiction to $(P_t \cdot C) = 0$. Hence, $C' \neq C$ and thus C' is a (-1) -curve. It follows that the support of $N_t + C'$ consists of (-1) -curves and we conclude $(C' \cdot N_t) = 0$ which implies that $(D_t \cdot C) = 0$. This shows that $D_t \in C^\perp$. \square

We are now able to present an example of a Newton-Okounkov body on S_6 which induces a normal toric degeneration.

Example 5.11. Let us consider the divisor $D = 4H - E_1 - \dots - E_6$. This is an ample divisor. The corresponding Newton-Okounkov body with respect to the curve $C = 2H - E_1 - \dots - E_6$ and a general point P is illustrated in Figure 2.

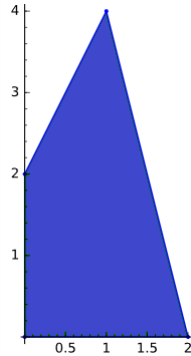


FIGURE 2. N.-O. body of $D = 4H - E_1 - \dots, E_6$ on S_6

The Hilbert polynomial, which is equal to the Ehrhart polynomial of $\Delta_{Y_\bullet}(D)$, is given by

$$P_D(t) = 5t^2 + 3t + 1.$$

5.3. Blow-up of four points three of them on a line. Let L_3 be the blow-up of \mathbb{P}^2 of four points where three points lie on a line. This is again a weak del Pezzo surface. The negative curves are:

- (a) E_1, E_2, E_3, E_4 the exceptional divisors.
- (b) $H - E_4 - E_2, H - E_4 - E_3, H - E_4 - E_1$ the strict transforms of the lines through two points.
- (c) $H - E_1 - E_2 - E_3$ the strict transform of the line through the three collinear points.

The first two types of curves are (-1) - and the last one is a (-2) -curve. Analogously as in the previous section, we get the following result.

Theorem 5.12. *Let D be a big divisor on L_3 . Let furthermore $C = H - E_1 - E_2 - E_3$ be the line through the three chosen points, and $P \in C$ an arbitrary point. Then $\Delta_{C \supset \{P\}}(D)$ induces a normal toric degeneration.*

□

Since $H - E_1 - E_2 - E_3$ is the only (-2) -curves, we can use an analog of Lemma 5.10 in order to compute Newton-Okounkov bodies.

Example 5.13. Let us consider the divisor $D = 4H - E_1 - E_2 - E_3 - E_4$. This is an ample divisor. We want to compute the Newton-Okounkov body with respect to the curve $C = H - E_1 - E_2 - E_3$ and a very general point P on C .

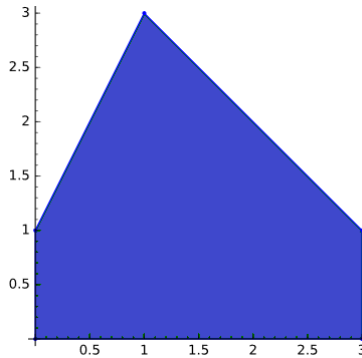


FIGURE 3. N.-O. body of $D = 4H - E_1 - \dots - E_4$ on L_3

The Hilbert polynomial of D is given by:

$$P_D(t) = 6t^2 + 4t + 1.$$

6. GLOBAL NEWTON-OKOUNKOV BODIES ON SURFACES

In this section we want to use our previous findings in order to compute global Newton-Okounkov bodies on (weak) del Pezzo surfaces. We will see that under good conditions the global semigroup $\Gamma_{Y_\bullet}(X)$ is finitely generated. Moreover, we will see how the generators of this semigroup give rise to generators of the Cox ring. We will illustrate our results for the varieties X_5 and L_3 .

6.1. Generators of the global Newton-Okounkov body on surfaces.

In this section we want to generalize results from [SS16] to arbitrary admissible flags.

Let us start by defining what we mean by a global Newton-Okounkov body.

Definition 6.1. Let X be a projective variety. Let Y_\bullet be an admissible flag on X . Then we define the *global Newton-Okounkov body of X* with respect to Y_\bullet as the closure of

$$\text{Cone}(\{(\nu_{Y_\bullet}(s), [D]) \mid s \in H^0(X, \mathcal{O}_X(D)), D \in \text{Pic}(X)\})$$

in $\mathbb{R}^d \times N^1(X)_{\mathbb{R}}$. We denote it by $\Delta_{Y_\bullet}(X)$.

Note that for any big divisor D , we have the following identity

$$\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(X) \cap (\mathbb{R}^d \times \{[D]\}).$$

We will now focus on the case where X is a smooth surface. Moreover, let us assume that X admits a rational polyhedral pseudo-effective cone, e.g. if X is a Mori dream surface. Let C be a curve on X and $P \in C$ a smooth point on C . Let

$$D_1 = P_1 + N_1, \dots, D_r = P_r + N_r$$

be the set of generators of the Zariski chambers with the property that C is not contained in the support of the negative parts N_1, \dots, N_r . The following is a generalization of [SS16, Theorem 3.2] to arbitrary flags. Note that the proof works quite similarly as the mentioned one.

Theorem 6.2. *Consider the notation introduced above. The generators of the global Newton-Okounkov body $\Delta_{C \supset \{P\}}(X)$ are given by*

- $(1, 0, [C])$
- $(0, \text{ord}_P(N_{i|C}), [D_i])$ for $i = 1, \dots, r$
- $(0, \text{ord}_P(N_{i|C}) + (P_i \cdot C), [D_i])$ for $i = 1, \dots, r$.

Proof. It is not hard to see that all the above points are contained in $\Delta_{C \supset \{P\}}(X)$.

Let us now show that all points in $\Delta_{C \supset \{P\}}(X)$ are positive linear combinations of the above points. It is enough to show that all valuation points are of this kind. Let D be a big divisor and $s \in H^0(X, \mathcal{O}(D))$ an arbitrary section. Define $a := \text{ord}_C(s)$, consider $D' := D - aC$ and set $\xi := s/s_C^a$ where s_C is a defining section of C . We have

$$(\nu(s), [D]) = a \cdot (1, 0, [C]) + (\nu(\xi), [D']).$$

Therefore, it is enough to show that $(\nu(\xi), [D'])$ is a positive linear combination of the above points. Let D_{i_1}, \dots, D_{i_s} be the generators of the unique Zariski chamber which contains the divisor D' . Then we can write

$$D' = \sum_{k=1}^s t_k \cdot D_{i_k}.$$

Furthermore, for the negative part $N' := N(D') = \sum t_k N_{i_k}$ and $P' := P(D') = \sum t_k P_{i_k}$. By definition of D' , we get that C is not contained in the support of the negative part $N(D')$. This also shows that C is not contained in the negative parts of the D_{i_k} .

Let us now choose $m \in \mathbb{Z}$ such that mN' and mP' are both integral. We can decompose $\xi^m = \zeta\sigma$ for $\zeta \in H^0(X, \mathcal{O}_X(mP'))$ and $\sigma \in H^0(X, \mathcal{O}_X(mN'))$. Then

$$m \cdot (\nu(\xi), [D']) = (\nu(\zeta) + \nu(\sigma), [mP' + mN']) = (\nu(\zeta), [mP']) + (\nu(\sigma), [mN']).$$

Furthermore,

$$\nu(\sigma) = (0, \text{ord}_P(\sigma|_C)) = m \cdot \sum_{k=1}^s t_k \cdot (0, \text{ord}_P(N_{i_k|C})).$$

On the other hand, we have

$$\nu(\zeta) = (0, bm) \text{ where } b \in [0, P' \cdot C]$$

Thus there is a $c \in [0, 1]$ such that

$$\nu(\zeta) = cm \cdot \sum_{k=1}^s t_k \cdot (0, 0) + (1-c)m \sum_{k=1}^s t_k \cdot (0, P_k \cdot C).$$

Putting everything together we get

$$\begin{aligned}
m \cdot (\nu(\xi), [D]) &= \left(mc \cdot \sum_{k=1}^s t_k(0, \text{ord}_P(N_{i_k|C})) + m(1-c) \left(\sum_{k=1}^s t_k(0, \text{ord}_P(N_{i_k|C})) \right. \right. \\
&\quad \left. \left. + P_k \cdot C \right), \left[\sum_{k=1}^s t_k \cdot D_{i_k} \right] \right) = \\
&= mc \cdot \sum_{k=1}^s t_k(0, \text{ord}_P(N_{i_k|C}), [D_{i_k}]) + \\
&\quad + m(1-c) \cdot \sum_{k=1}^s t_k(0, \text{ord}_P(N_{i_k|C}) + (P_{i_k} \cdot C), [D_{i_k}]).
\end{aligned}$$

This proves the claim. \square

The next proposition gives a more concrete characterization of the above mentioned divisors D_i .

Proposition 6.3. *Let D be a divisor which spans an extremal ray of the closure of a Zariski chamber $\overline{\Sigma_P}$. Then D spans an extremal ray of either the pseudo-effective cone or the nef cone of X .*

Proof. Let P be a big and nef divisor. Then we define

$$\begin{aligned}
\text{Face}(P) &:= \bigcap_{C \in \text{Null}(P)} C^\perp \cap \text{Nef}(X) \\
V^{\geq 0}(\text{Null}(P)) &:= \text{Cone}(\text{Null}(P)).
\end{aligned}$$

Then by [BKS04, Proposition 1.8], we have

$$\text{Big}(X) \cap \overline{\Sigma_P} = \text{Big}(X) \cap \text{Face}(P) + V^{\geq 0}(\text{Null}(P)).$$

Hence, the extremal rays of $\overline{\Sigma_P}$ are either extremal rays of $\text{Face}(P)$ or of $V^{\geq 0}(\text{Null}(P))$. However, since $\text{Face}(P)$ is a face of the Nef cone, the first set of extremal rays lies inside the set of extremal rays of the Nef cone. The extremal rays of $V^{\geq 0}(\text{Null}(P))$ are all negative, and thus extremal rays of the pseudo-effective cone. \square

Remark 6.4. *Proposition 6.3 combined with Theorem 6.2 is in some sense surprising. It shows that in order to calculate the global Newton-Okounkov body on a surface X , it is not necessary to know the exact structure of the Zariski chambers. It is not even necessary to compute any Zariski decomposition at all. However, in order to derive the structure of the generators of the global Newton-Okounkov body we heavily relied on the fact that Zariski decomposition as well as the Zariski chamber decomposition does exist.*

6.2. Finite generation of the global semigroup. We have seen above, that a smooth surface X with a rational polyhedral pseudo-effective cone admits rational polyhedral global Newton-Okounkov bodies with respect to all admissible flags. In this section we want to prove a stronger property, namely finite generation of the global semigroup appearing in the construction of global Newton-Okounkov bodies. We will prove this property for the examples we have dealt with so far. In order to prove such a statement, we need to consider Newton-Okounkov bodies of effective but not big divisors. There are two different ways of defining these Newton-Okounkov bodies which both coincide for big divisors. One way is to define it via taking a fiber of the global Newton-Okounkov body. The corresponding body is called the *numerical Newton-Okounkov body*. More concretely, we have

$$\Delta_{Y_\bullet}^{num}(D) := \Delta_{Y_\bullet}(X) \cap (\mathbb{R}^d \times \{[D]\}).$$

Another way to associate a convex body to an effective divisor is to just use the same definition as for big divisors. The resulting body is called the *valuative Newton-Okounkov body*. More, concretely we define

$$\Delta_{Y_\bullet}^{val}(D) := \overline{\text{Cone}(\Gamma_{Y_\bullet}(D))} \cap (\mathbb{R}^d \times \{1\})$$

where

$$\Gamma_{Y_\bullet}(D) := \{(\nu_{Y_\bullet}(s), k) \mid k \in \mathbb{N}, s \in H^0(X, \mathcal{O}(kD)) \setminus \{0\}\}.$$

In general, we have $\Delta_{Y_\bullet}^{val}(D) \neq \Delta_{Y_\bullet}^{num}(D)$. However, if D is big the mentioned equality holds.

Lemma 6.5. *Let X be a smooth Mori dream surface, D an effective divisor on X and Y_\bullet an admissible flag such that $-K_X - Y_1$ is big and nef. Then $\Delta_{Y_\bullet}(D)^{num} = \Delta_{Y_\bullet}(D)^{val}$.*

Proof. Without loss of generality we may assume that D is nef. Following [CPW17], there are two different cases for $\Delta_{Y_\bullet}^{num}(D)$. The first case is that $\mu := \sup\{t : D - tY_1 \text{ is effective}\}$ is equal to 0. Then from [CPW17], we get

$$\Delta_{Y_\bullet}^{num}(D) = \{(0, x) \mid x \in [0, D \cdot Y_1]\}.$$

Since $\mu = 0$, we can deduce that

$$\Delta_{Y_\bullet}^{val}(D) = \{0\} \times \Delta_{X|Y_1}^{val}(D) = \{0\} \times \Delta_{Y_1}^{val}(D).$$

Note that for the last identity we have used the fact that $H^1(X, \mathcal{O}_X(D - Y_1)) = 0$, which means that the restriction morphism $H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(Y_1, \mathcal{O}_{Y_1}(D))$ is surjective. However, it easily follows that $\Delta_{Y_1}^{val}(D) = [0, D \cdot Y_1]$. This proves $\Delta_{Y_\bullet}^{val}(D) = \Delta_{Y_\bullet}^{num}(D)$ in the case $\mu = 0$.

Suppose now that $\mu > 0$. Then $\Delta_{Y_\bullet}^{num}(D)$ is given by a line segment $\text{Conv}\{(0, 0), (\mu, Q)\}$ for some number $Q \in \mathbb{R}_{\geq 0}$. Since $\Delta_{Y_\bullet}^{val}(D) \subseteq \Delta_{Y_\bullet}^{num}(D)$ it is enough to prove that there are sections $s_1, s_2 \in H^0(X, \mathcal{O}(kD))$ such that $\nu_{Y_\bullet}(s_1) = (0, 0)$, and $\nu_{Y_\bullet}(s_2) = k(\mu, Q)$. However, since D is nef and thus

semi ample, the first assertion is clear. Moreover, since X is a Mori dream space $D - \mu C$ is semi-effective. This proves the second assertion. \square

Lemma 6.6. *Let X be a smooth del Pezzo surface. Let Y_\bullet be an admissible flag such that $-K_X - Y_1$ is big and nef, and let Y_1 be rational, i.e. of genus 0. Then for all effective divisors D , the semigroup $\Gamma_{Y_\bullet}(D)$ is finitely generated normal.*

Proof. This proof works similarly as the proof of Theorem 5.5. \square

Remark 6.7. *The above lemma is also valid for the varieties L_3 and S_6 , if we take as Y_1 the single (-2) -curve.*

Theorem 6.8. *Suppose one of the following situations.*

- $X = X_r$ is the blow-up of $1 \leq r \leq 6$ points in \mathbb{P}^2 in general position and Y_\bullet is an admissible flag such that Y_1 is negative.
- $X = L_3$ or $X = S_6$ and Y_\bullet is an admissible flag such that Y_1 is the corresponding single (-2) -curve.

Then the global semigroup

$$\Gamma_{Y_\bullet}(X) = \{(\nu_{Y_\bullet}(s), D) \mid D \in N^1(X) = \text{Pic}(X), s \in H^0(X, \mathcal{O}(D))\}$$

is finitely generated normal.

Proof. We know, by Theorem 6.2, that $\Delta_{Y_\bullet}(X) = \overline{\text{Cone}(\Gamma_{Y_\bullet}(X))}$ is rational polyhedral. We want to prove that $\overline{\text{Cone}(\Gamma_{Y_\bullet}(X))} \cap (\mathbb{Z}^2 \times N^1(X)) = \Gamma_{Y_\bullet}(X)$. Then the result follows from Gordan's lemma. Consider $(a, D) \in \overline{\text{Cone}(\Gamma_{Y_\bullet}(X))}$ for $a \in \mathbb{Z}^2$ and D an integral effective divisor in $N^1(X)$. This means that

$$a \in \Delta_{Y_\bullet}^{\text{num}}(D) = \Delta_{Y_\bullet}^{\text{val}}(D) = \overline{\text{Cone}(\Gamma_{Y_\bullet}(D))} \cap (\mathbb{R}^2 \times \{1\}).$$

But by Lemma 6.6 and Remark 6.7, $\Gamma_{Y_\bullet}(D)$ is normal. Thus, there is a section $s \in H^0(X, \mathcal{O}(D))$ such that $\nu_{Y_\bullet}(s) = a$. This proves that $(a, D) \in \Gamma_{Y_\bullet}(X)$. \square

The finite generation of the global semigroup $\Gamma_{Y_\bullet}(X)$ has the following consequences for the Cox ring $\text{Cox}(X)$.

Theorem 6.9. *Let X be a \mathbb{Q} -factorial variety with $N^1(X) = \text{Pic}(X)$. Let Y_\bullet be an admissible flag. Suppose $\Gamma_{Y_\bullet}(X)$ is finitely generated by*

$$(\nu_{Y_\bullet}(s_1), D_1), \dots, (\nu_{Y_\bullet}(s_N), D_n).$$

Then the Cox ring $\text{Cox}(X)$ is generated by the sections s_1, \dots, s_N .

Proof. Let R be the \mathbb{C} -algebra which is generated by the sections s_1, \dots, s_N . Let D be any effective divisor in X . Let $k = h^0(X, \mathcal{O}_X(D)) = |\nu_{Y_\bullet}(H^0(X, \mathcal{O}_X(D)) \setminus \{0\})|$. Since the $(\nu_{Y_\bullet}(s_1), D_1), \dots, (\nu_{Y_\bullet}(s_N), D_n)$ generate $\Gamma_{Y_\bullet}(X)$, it follows that there are $f_1, \dots, f_k \in R \cap H^0(X, \mathcal{O}_X(D)) \setminus \{0\}$ which all have a different valuation. But then it follows from [KK12, Proposition 2.3] that

f_1, \dots, f_k are linearly independent. This proves that they form a basis of $H^0(X, \mathcal{O}_X(D))$ and that every section $s \in H^0(X, \mathcal{O}(D))$ lies in the algebra R . This show that $R \cong \text{Cox}(D)$. \square

6.3. Examples of global Newton-Okounkov bodies and global semi-groups. In this last paragraph we want to consider two concrete examples and compute their global Newton-Okounkov bodies. In the second example we also present generators of the global semigroup and use them to find generators of the Cox ring.

Example 6.10. First of all we consider the del Pezzo surface X_5 , which is the blow-up of five points in general position in \mathbb{P}^2 . As a flag, we take the negative curve $C := H - E_1 - E_2$, and a general point on it. According to Theorem 6.2, we need to compute all ray generators of Zariski chambers, whose support of the negative part does not contain the negative curve C . Using Proposition 6.3, these are given by all the negative curves except the curve C and the generators of the extremal rays of the nef cone.

With the help of a computer calculation, we compute the global Newton Okounkov body and present the resulting hyperplane representation in $\mathbb{R}^2 \times N^1(X_5)_{\mathbb{R}} \cong \mathbb{R}^8$. Choosing H, E_1, \dots, E_5 as a basis for $N^1(X_5)_{\mathbb{R}}$ we get the following representation for $\Delta_{Y_\bullet}(X_5)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & 2 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 2 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 1 & 1 & 2 & 1 & 1 \\ -1 & -1 & 3 & 1 & 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & 0 & 1 & 0 & 1 & 1 \\ -1 & -1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 2 & 1 & 0 & 1 & 0 & 1 \\ -2 & -1 & 2 & 0 & 0 & 1 & 0 & 1 \\ -2 & -1 & 2 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 1 & 0 & 0 & 1 & 1 \\ -2 & -1 & 2 & 0 & 0 & 0 & 1 & 1 \\ -2 & -1 & 3 & 1 & 0 & 1 & 1 & 2 \\ -2 & -1 & 3 & 1 & 0 & 1 & 2 & 1 \\ -2 & -1 & 3 & 1 & 0 & 2 & 1 & 1 \\ -2 & -1 & 3 & 0 & 1 & 1 & 1 & 2 \\ -2 & -1 & 3 & 0 & 1 & 1 & 2 & 1 \\ -2 & -1 & 3 & 0 & 1 & 2 & 1 & 1 \\ -2 & -1 & 4 & 1 & 1 & 2 & 2 & 2 \\ -1 & 0 & 2 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 \\ -3 & -1 & 4 & 0 & 1 & 2 & 2 & 2 \\ -3 & -1 & 4 & 1 & 0 & 2 & 2 & 2 \\ -4 & -1 & 4 & 0 & 0 & 2 & 2 & 2 \\ -3 & -1 & 3 & 0 & 0 & 2 & 1 & 1 \\ -3 & -1 & 3 & 0 & 0 & 1 & 1 & 2 \\ -3 & -1 & 3 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \cdot (x_1, \dots, x_8)^T \leq 0.$$

It is a convex cone in \mathbb{R}^8 which is defined by a minimal number of 39 inequalities or a minimal number of 22 rays. Note that the above equations give an Ehrhart type formula for the Hilbert polynomial of a given divisor $D = (x_3, \dots, x_8)$ similar to the one derived in [SX10, Example 1.3].

Example 6.11. Consider now L_3 , which is the blow-up of four points such that three of them lie on a line. Let us suppose that P_1, \dots, P_3 lie on a line. We choose H, E_1, \dots, E_4 as our basis for $N^1(L_3)_{\mathbb{R}}$. Then the global Newton-Okounkov body of L_3 with respect to the curve $C = H - E_1 - E_2 - E_3$ and a general point on it is given by the following linear inequalities:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot (x_1, \dots, x_7)^T \geq 0.$$

The ray generators of the global Newton-Okounkov body are

$$\begin{aligned} & (0, 0, E_4), (0, 0, E_3), (0, 0, E_2), (0, 0, E_1), (0, 0, H - E_1 - E_4), \\ & (0, 0, H - E_2 - E_4), (0, 0, H - E_3 - E_4), (0, 1, H - E_4), \\ & (1, 0, H - E_1 - E_2 - E_3). \end{aligned}$$

A calculation shows that these generators, form a Hilbert basis, so that they are a generating set of the global semigroup $\Gamma_{Y_{\bullet}}(L_3)$. It follows from Theorem 6.9, that $\text{Cox}(L_3)$ is generated by the following sections:

- the negative curves
- the strict transform of a general line going through P_4 .

REFERENCES

- [A13] Anderson, D., *Okounkov bodies and toric degenerations* Math. Ann. **356** (2013), no. 3, 1183–1202
- [ADH14] Arzhantsev, I., Derenthal, U., Hausen, J., *Cox Rings*, Cambridge University Press, 2014
- [B09] Bauer, T., *A simple proof for the existence of Zariski decompositions on surfaces*, J. Algebraic Geom. **18** (4), (2009), 789–793
- [BKS04] Bauer, T., Küronya, A., Szemberg, T., *Zariski chambers, volumes, and stable base loci*, J. Reine Angew. Math. **576**, (2004), 209–233
- [BG09] Bruns, W., Gubeladze, J., *Polytopes, rings, and K-theory*, Springer, Dordrecht, (2009).
- [BPS15] Bauer, T., Pokora, P., Schmitz, D., *On the boundedness of the denominators in the Zariski decomposition on surfaces*, J. Reine Angew. Math., (2015)

- [CPW17] Choi, S., Park, J., Won, J., *Okounkov bodies and Zariski decompositions on surfaces* arXiv:1603.00618, to appear in Bull. Korean Math. Soc.
- [H77] Hartshorne, R., *Algebraic Geometry*, Springer, 1977
- [KK12] Kaveh, K., Khovanskii, A.G., *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), 2, 925–978
- [LM09] Lazarsfeld, R., Mustaa, M., *Convex bodies associated to linear series*, Ann. Sci. c. Norm. Supr. **42** (2009), 783–835
- [O96] Okounkov, A., *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996), 405–411
- [SS16] Schmitz, D., Seppnen, H., *On the polyhedrality of global Okounkov bodies*, Adv. Geom. **16** (2016),1, 83–91
- [KLM12] Kronya, A., Lozovanu, V., Maclean, C., *Convex bodies appearing as Okounkov bodies of divisors* Advances in Mathematics **229** (2012), 2622–2639
- [HK15] Harada, M., Kaveh, K., *Integrable systems, toric degenerations and Okounkov bodies* Inventiones mathematicae **202** (2015) Issue 3, 927–985
- [S96] Sturmfels, B., *Grbner bases and convex polytopes*, University Lecture Series **8**, Amer. Math. Soc., Providence, 1996.
- [SX10] Sturmfels, B., Xu, Z. *Sagbi bases of Cox-Nagata rings*, J. Eur. Math. Soc. (JEMS), **12** (2010), 429–459

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ON THE MORI THEORY AND NEWTON-OKOUNKOV BODIES OF BOTT-SAMELSON VARIETIES

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ABSTRACT. We prove that on a Bott-Samelson variety X every movable divisor is nef. This enables us to consider Zariski decompositions of effective divisors, which in turn yields a description of the Mori chamber decomposition of the effective cone. This amounts to information on all possible birational morphisms from X . Applying this result, we prove the rational polyhedrality of the global Newton-Okounkov body of a Bott-Samelson variety with respect to the so called ‘horizontal’ flag. In fact, we prove the stronger property of the finite generation of the corresponding global value semigroup.

1. INTRODUCTION

Bott-Samelson varieties arise naturally from the study of flag varieties as resolutions of singularities of Schubert varieties (see e.g. [D74]). Their line bundles have been studied by Lauritzen and Thomsen ([LT04]), and by Anderson ([A15]). If X_w is a Bott Samelson variety corresponding to a reduced word w , an explicit finite set of generators for the cone of effective/nef divisors is described by the first two authors.

In addition to the rational polyhedrality of the effective/nef cone, Anderson ([A15]) and the last two authors of this article ([SS17]) showed independently that a Bott-Samelson variety $X = X_w$ is log-Fano. In particular, X is a Mori dream space in the sense of [HuKe00]. Consequently, the Cox ring $\text{Cox}(X) = \bigoplus_{D \in \text{Div}(X)} H^0(X, \mathcal{O}_X(D))$ is a finitely generated \mathbb{C} -algebra. Moreover, there are only finitely many contracting birational maps from X . These contractions correspond to a decomposition of the effective cone into finitely many subcones, the so called Mori chambers. More concretely, in [HuKe00] two big divisors D_1 and D_2 are called *Mori-equivalent*, if their induced maps $X \dashrightarrow \text{Proj}(\bigoplus_m H^0(X, \mathcal{O}_X(mD_i)))$ agree. This can be rephrased by saying that their respective Minimal Model Programs (MMP) coincide. The closures in the Néron-Severi vector space $N^1(X)_{\mathbb{R}}$ of the equivalence classes are the aforementioned Mori chambers. Despite their straightforward definition, these chambers are very hard to determine in almost all concrete cases. Especially the existence of small contractions still poses a plethora of challenges one of which the two last authors faced when studying global Newton-Okounkov bodies of Mori dream spaces in [SS17].

Key words and phrases. Effective cone, Bott-Samelson variety, Mori dream space, Newton-Okounkov body.

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In their work the main complication consisted in the fact that restricted volumes of divisors with small base loci, i.e. of codimension higher than one, behave unpredictably compared to those of nef divisors. Another consequence of the appearance of small base loci is the fact that in order to define Zariski decompositions of effective divisors, one has to consider higher birational models on which these base loci are resolved. In practice, it is usually very difficult to control these resolutions.

In this article, we apply results about Newton-Okounkov bodies from [KüL15] and [SS17] to show that in the case of Bott-Samelson varieties such problems need not concern us. Concretely, we prove the following.

Theorem A. *Let $X = X_w$ be a Bott-Samelson variety corresponding to a reduced sequence w . Then every movable divisor on X is base-point-free, and hence*

$$\text{Mov}(X) = \text{Nef}(X).$$

At a first glance it may seem that, by realizing this fact, the struggle in [SS17] to deal with small modifications on Bott-Samelson varieties was in vain. Note however that in order to prove the above theorem we rely on a result from [SS17] for which the existence of small contractions could not be ruled out from the outset.

Once Theorem A is established, we have access to Zariski decompositions, which then just consist of the decomposition of a big divisor into its (\mathbb{Q}) -movable and (\mathbb{Q}) -fixed parts. This enables us to describe the pseudo-effective cone of a Bott-Samelson variety in detail and in particular to give criteria for divisor classes to span a common Mori chamber. It will turn out that Mori chambers are uniquely determined by stable base loci occurring in their interior. More concretely, we prove the following.

Theorem B. *Let $X = X_w$ be a Bott-Samelson variety for a reduced word w . Then each Zariski chamber defines a Mori chamber and vice versa.*

As an application of Theorem A, we turn to Newton-Okounkov bodies on Bott-Samelson varieties and the question of finite generation of the semigroups of valuation vectors coming up in the construction.

The question of finite generation of this semigroup has been intensely studied in the last years since D. Anderson's observation in [A13] that the finite generation of the value semigroup of an ample divisor implies the existence of a toric degeneration and the construction of a related integrable system in this situation in [HaKa15].

Also the study of Newton-Okounkov bodies on Bott-Samelson varieties has recently become an active field of research. In [A13] a particular Bott-Samelson variety is considered as an example. A more thorough analysis of Newton-Okounkov bodies for Bott-Samelson varieties was initiated by Kaveh in [Ka15], where he showed that Littelmann's string polytopes (cf. [Li98]) can be realized as Newton-Okounkov bodies for divisors with respect to a certain valuation. This valuation is however not defined by a flag of subvarieties in terms of order of vanishing. In [HaY15], the authors describe Newton-Okounkov bodies of Bott-Samelson varieties for divisors D satisfying a certain condition. In contrast to Kaveh's work, they use a flag

to define the valuation, which we will call the ‘horizontal’ flag. In particular they prove the finite generation of the value semigroup in this context. In [SS17], the rational polyhedrality of the global Newton-Okounkov with respect to the so-called ‘vertical’ flag was proven.

Although both the ‘vertical’- and the ‘horizontal’ flag consists of Bott-Samelson varieties, their embeddings into X are very different: whereas the divisor Y_1 in the ‘vertical’ flag is a fibre of a bundle $X \rightarrow \mathbb{P}^1$, and thus moves in a natural family, the divisor in the horizontal flag is fixed, i.e., it is the only element in its linear system.

In this article, we consider the ‘horizontal’ flag and generalize the results of [HaY15] to all effective divisors D . We prove the rational polyhedrality of the global Newton-Okounkov body, similarly as in [SS17]. Note, however, that our proof will be substantially less technical since we can make use of Theorem A and the fact, derived in [LT04] (see Lemma 6.2), that the restriction morphism of global sections of nef divisors to Y_1 is surjective. This property is significantly stronger than the corresponding identity of restricted volumes $\text{vol}_{X|Y_1}(D) = \text{vol}_{Y_1}(D|Y_1)$ which holds for the ‘vertical’ flag. Indeed, it will give us the following result.

Theorem C. *Let $X = X_w$ be a Bott-Samelson variety for a reduced word w , and let Y_\bullet be the horizontal flag. Then, the semigroup*

$$\Gamma_{Y_\bullet}(X_w) := \{(\nu(s), D) \mid D \in \text{Pic}(X_w), s \in H^0(X, \mathcal{O}_X(D)) \setminus \{0\}\}$$

is finitely generated.

To our best knowledge, apart from the toric case, no known examples of varieties admitting a finitely generated global semigroup $\Gamma_{Y_\bullet}(X)$ have been studied in the literature so far. It also remains unclear to us whether the above theorem also holds with the ‘vertical’ flag.

Note also that our result goes in line with the recent work of Postingshel and Urbinati ([PU16]). Apart from also showing that for each Mori dream space X there is a flag on a birational model of X such that the corresponding global Newton-Okounkov body is rational polyhedral, they prove the finite generation of the value semigroup $\Gamma(D)$ of any big divisor D . However, their work does not imply that the global semigroup $\Gamma(X)$ is finitely generated.

It is a well-known fact that the finite generation of the value semigroup, does not induce a toric degeneration to a *normal* toric variety. This is directly related to the fact that the value semigroup itself need not be normal despite being finitely generated. Hence, it would be desirable to find a criterion for normality of value semigroups. We prove a sufficient criterion in the case of Bott-Samelson varieties. Namely, if the Zariski decomposition of any integral effective divisor on X is integral, then the normality of the global value semigroup $\Gamma_{Y_\bullet}(X)$ with respect to the ‘horizontal’ flag follows. Thus, in this case any ample divisor yields a degeneration to a normal toric variety.

We can generalize the picture described so far to the setting of flag varieties and Schubert varieties contained therein. Given a parabolic subgroup $P \subset G$ containing B , and a reduced expression $w = (s_1, \dots, s_n)$ such that

we have a birational resolution $p : X_w \rightarrow Z_{\bar{w}}$ of the Schubert variety corresponding to \bar{w} , the ‘horizontal’ flag on X_w induces a \mathbb{Z}^n -valued valuation-like function ν on $Z_{\bar{w}}$. Then, the following is a consequence of Theorem C.

Theorem D. *Let $Z_{\bar{w}} \subseteq G/P$ be the Schubert variety for the reduced word w , then the global semigroup $\Gamma_\nu(Z_{\bar{w}})$ is finitely generated. In particular, $\Delta_\nu(Z_{\bar{w}})$ is rational polyhedral.*

Consequently, for any partial flag variety G/P the global semigroup $\Gamma_\nu(G/P)$ is finitely generated.

In order to illustrate the results of this paper, we apply them to two concrete Bott-Samelson varieties given as incidence varieties. Their Mori chamber structure is described in Sections 5, whereas the generators of their global value semigroups and their global Newton-Okounkov bodies are determined in Section 7.

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2. PRELIMINARIES AND NOTATION

2.1. Bott-Samelson varieties. Let G be a connected and simply connected reductive complex linear group, let $B \subseteq G$ be a Borel subgroup, and let W be the Weyl group of G . Then for a sequence $w = (s_1, \dots, s_n)$ in W , we can associate the Bott-Samelson variety X_w as follows. Let P_i be the minimal parabolic subgroup containing B corresponding to the simple reflection s_i . Let $P_w := P_1 \times \dots \times P_n$ be the product of the corresponding parabolic subgroups, and consider the right action of B^n on P_w given by

$$(p_1, \dots, p_n)(b_1, \dots, b_n) := (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3, \dots, b_{n-1}^{-1} p_n b_n).$$

The Bott-Samelson variety X_w is the quotient

$$X_w := P_w / B^n = P_1 \times^B (P_2 \times^B \dots \times^B P_n).$$

We can represent points in X_w by tuples $[(p_1, \dots, p_n)]$ for $p_i \in P_i$ and the square brackets denote taking the class in the quotient. For more details on this construction we refer to [LT04] or [SS17].

Originally, Damazure constructed these varieties for a sequence w which is reduced [D74]. In this case, he proved that X_w is a desingularization of the Schubert variety $Z_{\bar{w}}$. In this article, whenever we talk about a Bott-Samelson variety X_w , the sequence w will be assumed to be reduced.

Let now $X = X_w$ be a Bott-Samelson variety of dimension n . Then $\text{Pic}(X) \cong \mathbb{Z}^n$. There are two important bases of $\text{Pic}(X)$. The first one is called the *effective basis*. It consists of prime divisors E_1, \dots, E_n , which can be defined inductively. Justifying its name, the cone spanned by this basis in $N^1(X)_{\mathbb{R}}$ is the cone of effective divisor classes. The second basis, which we call the *$\mathcal{O}(1)$ -basis*, will be denoted by D_1, \dots, D_n . These divisors also generate $\text{Pic}(X)$ as a group, whereas the cone they span coincides with the nef cone $\text{Nef}(X)$. Note that since $\text{Pic}(X) = N^1(X)$, we will not explicitly distinguish between the divisor D and its class $[D]$.

2.2. Newton-Okounkov bodies. For the theory of Newton-Okounkov bodies we follow the notation and conventions of [LM09]. In particular, a flag of irreducible subvarieties $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{pt\}$ is called *admissible* if Y_n is a smooth point on each Y_i . Each admissible flag Y_\bullet gives rise to a valuation-like function

$$\nu_{Y_\bullet} : \bigsqcup_{D \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \longrightarrow \mathbb{Z}^d.$$

Then for a big divisor D , we can define the value semigroup

$$\Gamma_{Y_\bullet}(D) = \{(\nu_{Y_\bullet}(s), k) \mid k \in \mathbb{N}, s \in H^0(X, \mathcal{O}(kD)) \setminus \{0\}\}.$$

The *Newton-Okounkov body* of D with respect to the flag Y_\bullet as

$$\Delta_{Y_\bullet}(D) = \overline{\text{Cone}(\Gamma_{Y_\bullet}(D))} \cap (\mathbb{R}^d \times \{1\}).$$

It is proven in [LM09] that $\Delta_{Y_\bullet}(D)$ only depends on the numerical class of D in $N^1(X)$.

Similarly, if Y is a closed subvariety, then $\Delta_{X|Y}(D)$ denotes the Newton-Okounkov body of the graded linear system W_\bullet with

$$W_k = \text{Im}(H^0(X, \mathcal{O}_X(kD)) \longrightarrow H^0(Y, \mathcal{O}_Y(kD)))$$

with respect to a fixed flag on Y .

Furthermore, there exists a closed convex cone

$$\Delta_{Y_\bullet}(X) \subset \mathbb{R}^n \times N^1(X)_\mathbb{R}$$

such that for each big divisor D the fibre of the second projection over $[D]$ is exactly $\Delta_{Y_\bullet}(D)$. We call $\Delta_{Y_\bullet}(X)$ the *global Newton-Okounkov body*.

In case we have $\text{Pic}(X) = N^1(X)$, e.g. if X is a Bott-Samelson variety, we can define the global Newton-Okounkov body using the *global semigroup*

$$\Gamma_{Y_\bullet}(X) := \{(\nu(s), D) \mid D \in \text{Pic}(X) = N^1(X), s \in H^0(X, \mathcal{O}_X(D)) \setminus \{0\}\}$$

Then, the global Newton-Okounkov is given by $\Delta_{Y_\bullet}(X) = \overline{\text{Cone}(\Gamma_{Y_\bullet}(X))}$.

Now we consider Newton-Okounkov bodies of effective but not necessarily big divisors. There are two different ways to define Newton-Okounkov bodies in this situation. One way is to just define them via the valuation-like function ν_{Y_\bullet} . More concretely, for an effective \mathbb{Q} -divisor D on X , we define the *valuative Newton-Okounkov body* as

$$\Delta_{Y_\bullet}^{val}(D) := \frac{1}{k} \cdot \overline{\text{Cone}(\Gamma_{Y_\bullet}(kD))} \cap (\mathbb{R}^n \times \{1\})$$

where $k \in \mathbb{Z}$ is chosen such that kD is integral. Note that $\Delta_{Y_\bullet}^{val}(D)$ is in general not well-defined for numerical classes and does indeed depend on the linear equivalence class. On the other hand, we can also define a Newton-Okounkov body by considering the global Newton-Okounkov body and then taking a fibre over a divisor. So, we define the *numerical Newton-Okounkov body* as

$$\Delta_{Y_\bullet}^{num}(D) := \Delta_{Y_\bullet}(X) \cap (\mathbb{R}^n \times \{D\}).$$

Note that, in general $\Delta_{Y_\bullet}^{num}(D) \neq \Delta_{Y_\bullet}^{val}(D)$. However, if D is big, both definitions coincide and we just write $\Delta_{Y_\bullet}(D)$.

3. THE MOVABLE CONE

Before we are able to prove Theorem A, we need the following.

Lemma 3.1. *Let D be a big divisor on a Mori dream space X . Let furthermore $\iota : Y \hookrightarrow X$ be a closed subvariety of X which is itself a Mori dream space and assume that ι^*D is big and nef. Let us denote by W_\bullet the restricted graded linear series of D to Y . If the identity of volumes $\text{vol}_{X|Y}(D) = \text{vol}_Y(\iota^*D)$ holds, then the stable base loci also agree, i.e.*

$$\mathbb{B}(W_\bullet) = \mathbb{B}(\iota^*D) = \emptyset.$$

Proof. Since Y is a Mori dream space, ι^*D is semiample, i.e. $\mathbb{B}(\iota^*D) = \emptyset$. Let us assume that $\mathbb{B}(W_\bullet)$ is not empty and choose $P \in \mathbb{B}(W_\bullet)$. Let Y_\bullet be an admissible flag on Y which is centered at the point P . By [KüL15, Theorem A], it follows that the origin is contained in the Newton-Okounkov body $\Delta_{Y_\bullet}(\iota^*D)$. Clearly, $\Delta_{Y_\bullet}(W_\bullet)$ is contained in $\Delta_{Y_\bullet}(\iota^*D)$. Since we have an equality of volumes, $\text{vol}_{X|Y}(D) = \text{vol}_Y(\iota^*D)$, this inclusion is in fact an equality, i.e.,

$$\Delta_{Y_\bullet}(\iota^*D) = \Delta_{Y_\bullet}(W_\bullet) = \Delta_{X|Y}(D).$$

As W_\bullet is the restricted graded linear series of the finitely generated divisor D , it is finitely generated. We can assume without loss of generality that it is generated in degree 1. Seeing that 0 lies in $\Delta_{X|Y}(D)$, it follows from the construction that there is sequence $(m_k)_{k \in \mathbb{N}}$ of natural numbers, with $\lim_{k \rightarrow \infty} m_k = \infty$, and a sequence of sections $(s_k)_{k \in \mathbb{N}}$ such that $s_k \in W_{m_k}$ and

$$1/m_k \cdot \nu(s_k) = 1/m_k \cdot (\nu_1(s_k), \dots, \nu_n(s_k)) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty.$$

This implies that $1/m_k \sum_{i=1}^n \nu_i(s_k) \longrightarrow 0$. However, by [KüL15, Lemma 2.4],

$$1/m_k \cdot \text{ord}_P(s_k) \leq 1/m_k \cdot \sum_{i=1}^n \nu_i(s_k).$$

Since W_\bullet is generated in degree one, all sections $s \in W_{m_k}$ vanish at the point P to order at least m_k . As a consequence, the left hand side is bounded from below by 1. This, however, contradicts the fact that the right hand side tends to 0. Hence, P cannot lie in $\mathbb{B}(W_\bullet)$. \square

Theorem 3.2. *Let $X = X_w$ be a Bott-Samelson variety. Then every movable divisor on X is base-point-free, and hence*

$$\text{Mov}(X) = \text{Nef}(X).$$

Proof. We prove the claim by induction on $n = \dim X$. If $n = 1$, then $X = \mathbb{P}^1$ and the claim obviously holds.

Assume now that it holds for $n - 1$, and write X as the fibre bundle $X = P \times^B Y$ with projection

$$\pi : X \longrightarrow P/B = \mathbb{P}^1, \quad \pi([p, y]) := pB,$$

where $Y = \pi^{-1}(pB)$ is a Bott-Samelson variety of dimension $n - 1$. The group P acts on X by

$$(p, [p', y]) := [pp', y],$$

and the projection π is clearly P -equivariant.

Let D be a movable divisor on X . Since P acts naturally on the line bundle $\mathcal{O}_X(D)$, and hence on the section space $H^0(X, \mathcal{O}_X(D))$ and the section ring $R(X, \mathcal{O}_X(D))$ (cf. [SS17]), the stable base locus $\mathbb{B}(D)$ is P -invariant. We therefore have

$$(1) \quad \mathbb{B}(D) = P(\mathbb{B}(D) \cap Y),$$

and

$$\text{codim}(\mathbb{B}(D), X) = \text{codim}(\mathbb{B}(D) \cap Y, Y).$$

In particular, the restriction $D|_Y$ is a movable divisor on Y , and hence $D|_Y$ is base point-free by induction, i.e.,

$$\mathbb{B}(D|_Y) = \emptyset.$$

We now assume that D is both movable and big. By [SS17, Proposition 3.1] we then have the identity of volumes

$$\text{vol}_Y(D|_Y) = \text{vol}_{X|Y}(D),$$

where the right hand side denotes the volume of the restricted linear series. From this, we deduce by Lemma 3.1 that

$$\mathbb{B}(D) \cap Y = \mathbb{B}(D|_Y) = \emptyset.$$

The identity (1) now shows that D is base-point-free, which implies that D is nef.

Finally, if D is merely movable, an approximation by big and movable divisors yields that D is nef. \square

4. MORI CHAMBER DECOMPOSITION OF BOTT-SAMELSON VARIETIES

In this section we give an explicit description of the Mori chamber decomposition of a Bott-Samelson variety.

4.1. Zariski decomposition. We have seen in the previous section that for a Bott-Samelson variety X_w the nef cone and the movable cone coincide. One consequence of this fact is that Nakayama's σ -decomposition of pseudo-effective divisors D as $D = P_\sigma(D) + N_\sigma(D)$ into movable and fixed part (c.f. [N04]) indeed gives a Zariski decomposition, i.e., the positive part $P_\sigma(D)$ is automatically nef. In this situation, we say that X *admits Zariski decompositions* and we write $D = P(D) + N(D)$ for the positive and negative part, respectively.

Remark 4.1. The negative part of the σ -decomposition is characterized by being the minimal subdivisor $N(D)$ of D such that $D - N(D)$ is movable [N04, Proposition 1.14 (2)]. If X admits Zariski decompositions, this means that the negative part is the minimal subdivisor $N(D)$ of D such that $D - N(D)$ is nef. Or differently stated, the positive part $P(D)$ of D is the maximal subdivisor of D such that $P(D)$ is nef.

4.2. Zariski chambers. In this section we define and describe Zariski chambers on a Bott-Samelson variety. The definition of Zariski chambers is analogous to the surface case introduced in [BKS04]. Note, however, that in order to prevent complications on the boundary, we pass to the closure of equivalence classes. This choice is not essential in the remainder, as in order to identify Zariski chambers with Mori chambers we will have to consider closures in any case.

Definition 4.2. Suppose X admits Zariski decompositions. We say that two effective divisors D and D' on X are *Zariski equivalent*, if

$$\text{supp}(N(D)) = \text{supp}(N(D')).$$

We denote the closure of the equivalence class of D by Σ_D . In case that Σ_D contains an open set, we call it a *Zariski chamber*.

Remark 4.3. Note that most of the pleasant characteristics of Zariski chambers discovered in [BKS04] in the surface case carry over to general X admitting Zariski decompositions. In particular, on the interior of each chamber the augmented base locus $\mathbb{B}_+(D)$ of a divisor D equals the support of the negative part $N(D)$ and thus these loci are constant on the interior of a chamber. Furthermore, just as in the surface case, the volume of a big divisor is given by the top self-intersection of its positive part and therefore the volume varies polynomially on the interior of each Zariski chamber.

On the other hand, we do not claim that in general Zariski chambers should be locally polyhedral on the big cone, or even convex. In fact there are examples of varieties that admit Zariski decompositions but have a Zariski chamber which is not convex. As an example consider the blowup of \mathbb{P}^3 in two intersecting lines ℓ_1, ℓ_2 and blow up further along the strict transform of the line in E_1 corresponding to ℓ_2 . Then every movable divisor is nef and the exceptional divisor E_3 is the exceptional locus of two different contractions (corresponding to the different rulings). Its Zariski chamber is the non-convex cone over the classes $E_3, H, H - E_1, H - E_2, 2H - E_1 - E_2 + E_3$.

In order to describe the Zariski chambers on Bott-Samelson varieties, we need the following sequence of lemmata. For these, we first recall that a divisor D is called *fixed* if all sufficiently divisible multiples of D are effective and constitute their complete linear series, i.e., $H^0(X, \mathcal{O}_X(mD)) = \mathbb{C} \cdot s_{mD}$, or $|mD| = mD$.

Lemma 4.4. *Let X be a Mori dream space. For each face F of the effective cone $\text{Eff}(X)$ either all divisors of F are fixed, or all divisors of $\text{relint}(F)$ are not fixed.*

Proof. Let $D \in \text{relint}(F)$ be a divisor which is not fixed. Let M be an arbitrary divisor in $\text{relint}(F)$. Then there are positive integers k and ℓ such that $L := k \cdot M - \ell D$ still lies in F . We can write $k \cdot M = L + \ell D$ and assume that L is effective (otherwise a positive multiple is). As the sum of a non fixed effective divisor with any effective divisor is clearly not fixed, this proves the claim. \square

In the next lemma we use the following common convex-geometric terminology.

Definition 4.5. Let $C \subseteq \mathbb{R}^d$ be a closed convex cone, and let $P \in \mathbb{R}^d$ be a point. Let H be a supporting hyperplane of C , and H^+ be the closed half-space defined by H which contains C . We say that H separates C from P if P is not contained in H^+ .

Lemma 4.6. Let $X = X_w$ be a Bott-Samelson variety. Let E be a fixed extremal divisor of the effective cone, and let H be a supporting hyperplane of the nef cone which separates the nef cone from E . Let P be a nef divisor on H . Then $D_{k\ell} := kP + \ell E$ is a Zariski decomposition for all $k, \ell \geq 0$, i.e. $P(D_{k\ell}) = kP$ and $N(D_{k\ell}) = \ell E$.

Proof. Since kP is a nef subdivisor of $D_{k\ell}$, it follows from Remark 4.1 that $kP \leq P(D_{k\ell})$. Since $P(D_{k\ell})$ is the maximal subdivisor of $D_{k\ell}$ it is of the form $kP + mE$ for some $0 \leq m \leq n$. Let now H^+ be the closed half space corresponding to H which contains the nef cone, and $H_{<0}^- := \mathbb{R}^n \setminus H^+$ its complementary open half space. Since H separates the nef cone from E , we have $E \in H^-$ as well as, by assumption, $P \in H$. It follows that $kP + mE$ lies in H^- , which means that it is not nef, unless $m = 0$. Therefore, the maximal nef subdivisor of $D_{k\ell}$ is $P(D_{k\ell})$. \square

The next lemma says that all but the first extremal rays of the effective cone are indeed fixed. We use the notation from [LT04] (see Section 2).

Lemma 4.7. Let X_w be a Bott-Samelson variety and let $\{E_1, \dots, E_n\}$ be the associated effective basis. Then, E_1 is nef, and E_2, \dots, E_n are fixed.

Proof. Since $\mathcal{O}_X(E_1)$ is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ by the morphism $\pi : X_w \rightarrow \mathbb{P}^1$, E_1 is nef.

We prove the fixedness of E_2, \dots, E_n by induction on the dimension of X_w . For this, let the given reduced w be $w = (s_{i_1}, \dots, s_{i_n})$, where the s_{i_j} are simple reflections, associated to the simple roots α_{i_j} . Then, $s_{i_j} \neq s_{i_{j+1}}$ for $j = 1, \dots, n-1$, since w is reduced. Let $w[1] := (s_{i_1}, \dots, s_{i_{n-1}})$, and consider the $(n-1)$ -dimensional Bott-Samelson variety $X_{w[1]}$ and the fibre bundle $\pi_1 : X_w \rightarrow X_{w[1]}$, as well as the embedding $\iota : X_{w[1]} \hookrightarrow X_w$ with image E_n . For any divisor D on $X_{w[1]}$ we have $H^0(X_w, \pi_1^* \mathcal{O}_{X_{w[1]}}(D)) \cong H^0(X_{w[1]}, \mathcal{O}_{w[1]}(D))$. From this, we can conclude by induction that the divisors E_2, \dots, E_{n-1} are fixed.

Now, being an extremal generator of $\text{Eff}(X)$, if E_n were not fixed, it would be nef. Hence, its restriction to $X_{w[1]} \cong E_n$ would also be nef. However, [HaY15, Lemma 3.6.] shows that the restriction of $\mathcal{O}_{X_w}(E_n)$ to E_n has a negative D_{n-1} -coefficient with respect to the nef basis for $\text{Pic}(E_n)$. This finishes the induction step, and hence the proof. \square

The next lemma is the key to the explicit description of the Zariski chambers. It gives a correspondence between the extremal fixed divisors and the facets of the nef cone.

Lemma 4.8. Let $X = X_w$ be a Bott-Samelson variety. For each fixed extremal ray $E_i, i = 2, \dots, n$, of the effective cone of X there is a unique facet F_i of the nef cone such that its supporting hyperplane H_i separates the extremal ray from the nef cone.

Proof. Let $n = \dim X$. Then there are $n - 1$ fixed extremal rays of the effective cone and n facets of the nef cone. We first claim that there is one facet, say F_1 , such that its supporting hyperplane coincides with a supporting hyperplane of a facet of the effective cone. This can be seen as follows. Let us suppose that $X = X_w$ for a reduced expression w . Then, E_1, \dots, E_{n-1} , resp. D_1, \dots, D_{n-1} , define the effective cone, resp. the nef cone of $X_{w[1]}$. This proves that D_1, \dots, D_{n-1} lie in the linear space defined by E_1, \dots, E_{n-1} . Hence, choosing E_1, \dots, E_n as a basis for $\text{Pic}_{\mathbb{R}} \cong \mathbb{R}^n$, the facet defined by E_1, \dots, E_{n-1} and D_1, \dots, D_{n-1} have the supporting hyperplane $H_1 = \{x_n = 0\}$.

This facet, F_1 , does not separate any extremal fixed divisor E_i from the nef cone. However, it is clear that for any fixed extremal divisor E_i there is at least on facet F_j of the nef cone such that its supporting hyperplane does separate the nef cone and E_i . Furthermore, we claim that there are no supporting hyperplanes of the facets F_i such that two distinct extremal rays E_ℓ and E_k are separated from the nef cone simultaneously. Suppose there is, then the interior of the cones $F_i + \text{Cone}(E_\ell)$ and $F_i + \text{Cone}(E_k)$ do intersect. But this contradicts the uniqueness of the Zariski decomposition and Lemma 4.6. Altogether, we have $n - 1$ extremal fixed divisors to which we can individually associate at least one facet. But, as we have seen, one facet cannot correspond to more than one extremal divisor. Since there exist only n facets, from which one facet does not correspond to any extremal ray, the claim is proven. \square

We now use the above lemma to introduce the following notation. For a Bott-Samelson variety X_w of dimension n , we define F_i , $i = 2, \dots, n$, as the facets of the nef cone which correspond to the fixed divisors E_i , $i = 2, \dots, n$, according to the above lemma. Furthermore, we call F_1 the remaining facet, which is just the facet spanned by the divisors D_1, \dots, D_{n-1} . Let H_1, \dots, H_n be the supporting hyperplanes corresponding to the facets F_i and denote by H_i^+ the closed half spaces corresponding to H_i which contain the nef cone. Having the notation fixed, we are now in a position to explicitly describe the Zariski chambers of Bott-Samelson varieties.

Theorem 4.9. *Let E be a fixed divisor with support $E_{i_1} \cup \dots \cup E_{i_\ell}$. Then Σ_E is given by*

$$\Sigma_E = \Pi_E := (F_{i_1} \cap \dots \cap F_{i_\ell}) + \text{Cone}(\{E_{i_1}, \dots, E_{i_\ell}\}).$$

Moreover, Σ_E defines a Zariski chamber which is an n -dimensional simplex.

Proof. Let us first prove that $\Pi_E \subseteq \Sigma_E$. Choose D in the relative interior of Π_E . So we can write $D = P + \sum_{k=1}^{\ell} \lambda_k E_{i_k}$ for $P \in F_{i_1} \cap \dots \cap F_{i_\ell}$, and $\lambda_k \geq 0$. We want to prove that this is already the Zariski decomposition, i.e., $P(D) = P$. Since P lies in $\bigcap_{k=1, \dots, \ell} F_{i_k}$ it follows that $P + \varepsilon E_{i_k}$ for $\varepsilon > 0$ and $k = 1, \dots, \ell$ is not nef. This proves the maximality of P and shows $P(D) = P$ as well as $N(D) = \sum_{k=1}^{\ell} \lambda_k E_{i_k}$. Hence, $D \in \Sigma_E$ and by the closedness of Σ_E the inclusion follows.

We now prove the reverse inclusion $\Sigma_E \subseteq \Pi_E$. Let D be any effective divisor such that its Zariski decomposition is given by $D = P + \sum_{k=1}^{\ell} \lambda_k E_{i_k}$ for $\lambda_k > 0$. Then in order to prove that D lies in Π_E we need to show that

$P \in F_{i_1} \cap \cdots \cap F_{i_\ell}$. Suppose P does not lie in F_{i_k} for some $k = 1, \dots, \ell$. That means P lies in the open half space $(H_{i_k}^+)_{>0} := H_{i_k}^+ \setminus H_{i_k}$. Then for $\varepsilon > 0$ small enough, we have $P + \varepsilon E_{i_k}$ still lies in $H_{i_k}^+$. By Lemma 4.8, E_{i_k} does lie in H_j^+ for $j \neq k$. This proves that $P + \varepsilon E_{i_k}$ lies in H_j^+ for all $j = 1, \dots, n-1$. Hence, $P + \varepsilon E_{i_k}$ is nef, contradicting the maximality of P . We have thus shown that $P \in F_{i_k}$. Again taking the closedness of Π_E into account, we obtain the reverse inclusion.

Let us now show that Σ_E defines an n -dimensional simplex. Let $F := F_{i_1} \cap \cdots \cap F_{i_r}$ and denote the generators of F by $D_{j_1}, \dots, D_{j_{n-r}}$. If Σ_E is of dimension n we are done. Suppose that it is of dimension less than n . That means the points $E_{i_1}, \dots, E_{i_r}, D_{j_1}, \dots, D_{j_{n-r}}$ are linearly dependent. Hence, there are $\lambda_i, \mu_i \geq 0$ for $i = 1, \dots, n$ with $\lambda_i \neq \mu_i$ for all $i = 1, \dots, n$ such that

$$\sum_{k=1}^{n-r} \lambda_{r+k} D_{j_k} + \sum_{k=1}^r \lambda_k E_{i_k} = \sum_{k=1}^{n-r} \mu_{r+k} D_{j_k} + \sum_{k=1}^r \mu_k E_{i_k}.$$

However, we have seen in the first part of the proof that the above decomposition is actually a Zariski decomposition. Since this decomposition is unique we get

$$\sum_{k=1}^{n-r} \lambda_{r+k} D_{j_k} = \sum_{k=1}^{n-r} \mu_{r+k} D_{j_k}$$

and

$$\sum_{k=1}^r \lambda_k E_{i_k} = \sum_{k=1}^r \mu_k E_{i_k}.$$

But both the D_i 's and the E_i 's are part of a basis of $N_{\mathbb{R}}^1(X_w)$. Hence, it follows that $\lambda_i = \mu_i$ for all $i = 1, \dots, n$. This contradicts the linear dependence of $E_{i_1}, \dots, E_{i_r}, D_{j_1}, \dots, D_{j_{n-r}}$. \square

4.3. Mori chambers. In this subsection we prove that the previously defined Zariski chambers coincide with the Mori chambers defined in [HuKe00]. Let us first recall what we mean by a Mori chamber. First of all, we call two divisors D_1 and D_2 on a Mori dream space Mori equivalent if there is an isomorphism $\text{Proj}(R(X, D_1)) \cong \text{Proj}(R(X, D_2))$ such that the obvious diagram

$$\begin{array}{ccc} X & \dashrightarrow & \text{Proj}(R(X, D_1)) \\ & \searrow & \downarrow \cong \\ & & \text{Proj}(R(X, D_2)) \end{array}$$

commutes. Then, Mori chambers are the closure of Mori equivalence classes which have non-empty interior.

Theorem 4.10. *Let X be a Bott-Samelson variety. Then each Zariski chamber defines a Mori chamber and vice versa.*

Proof. For the Mori chamber $\mathcal{C} = \text{Nef}(X)$ this is clear. Let us assume that two divisors D_1 and D_2 lie in the interior of a Mori chamber $\mathcal{C} \neq \text{Nef}$. Then by [O16], D_1 and D_2 are strongly Mori equivalent and in particular, their stable base loci coincide. But this just means $\text{supp}(N(D_1)) = \text{supp}(N(D_2))$, which in turn shows that D_1 and D_2 lie in the same Zariski chamber.

Let us assume they lie in $\Sigma_E = F + \text{Cone}(E_{i_1}, \dots, E_{i_\ell})$. By the description in [HuKe00, Proposition 1.11], we know that each Mori chamber is the Minkowski sum of some $g_i^* \text{Nef}(Y_i)$, for a birational contraction $g_i: X \dashrightarrow Y_i$, and the cone generated by some extremal fixed prime divisor. However, since $\text{Mov}(X) = \text{Nef}(X)$, g_i is actually a regular birational contraction. Thus, $g_i^* \text{Nef}(Y_i) \subset \text{Nef}(X)$, and since two Mori chambers intersect along a common face it follows that $g_i^* \text{Nef}(Y_i)$ actually is a face of $\text{Nef}(X)$. However, since \mathcal{C} lies in Σ_E , the only way to generate a chamber with non-empty interior is to take F as the face of the nef cone and $E_{i_1}, \dots, E_{i_\ell}$ as our extremal fixed divisors. This proves that $\mathcal{C} = \Sigma_E$. \square

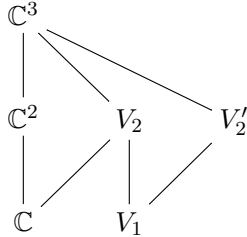
5. EXAMPLES OF MORI CHAMBER DECOMPOSITION

In this section we give two examples where we compute the Mori chamber decomposition of the effective cone. Note that all necessary computations were done on a computer, using Sage.

5.1. A 3-dimensional incidence variety. We start with the three-dimensional incidence variety Y which is described in [SS17, Example 2]. It consists of tuples of linear subspaces (V_1, V_2, V_2') of \mathbb{C}^3 such that V_1 is one-dimensional and V_2, V_2' are two-dimensional. Furthermore the following incidences hold:

$$\mathbb{C} \subseteq V_2, \quad V_1 \subseteq V_2, \quad V_1 \subseteq V_2'.$$

These incidences can be illustrated in the following diagram:

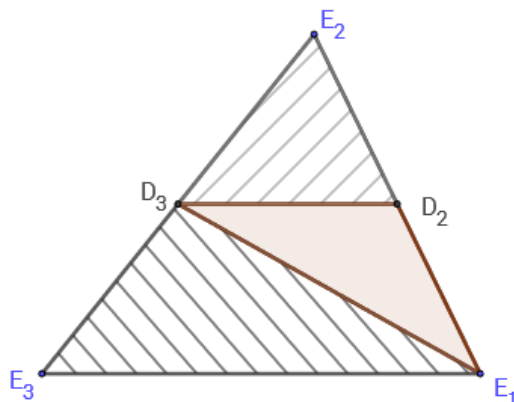


In [SS17] the relations between the divisors E_1, E_2, E_3 and D_1, D_2, D_3 were computed and are given by

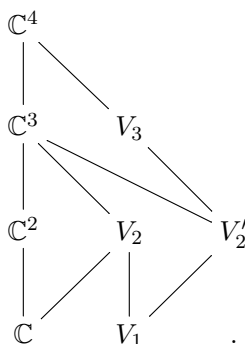
$$\begin{aligned} D_1 &= E_1 \\ D_2 &= E_2 + E_1 \\ D_3 &= E_3 + E_2. \end{aligned}$$

Cutting the effective/nef cone with a generic hyperplane, we get the following picture:

We can see from the picture that there are exactly three Mori chambers, which are given by the nef cone, $\text{Nef}(X) = \text{Cone}(E_1, D_2, D_3)$, and the two cones $\text{Cone}(E_2, D_2, D_3)$ and $\text{Cone}(E_3, D_1, D_3)$.

FIGURE 1. The Mori decomposition of Y

5.2. A 4-dimensional incidence variety. Let us now consider a four-dimensional incidence variety X which consists of tuples of linear subspaces (V_1, V_2, V'_2, V_3) such that V_1 is one-dimensional, V_2, V'_2 are two-dimensional, and V_3 is three-dimensional. Furthermore, the incidences are described in the following diagram:



We define a map $q: X \rightarrow Y$ by $(V_1, V_2, V'_2, V_3) \mapsto (V_1, V_2, V'_2)$. This makes X a Bott-Samelson variety, given as a \mathbb{P}^1 bundle over Y .

We proceed by describing the new occurring divisors E_4 and D_4 . The divisor E_4 is the image of the embedding of Y into X by mapping

$$(V_1, V_2, V'_2) \mapsto (V_1, V_2, V'_2, \mathbb{C}^3).$$

Denote by $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}'_2, \mathcal{V}_3$ the tautological vector bundles with fibres V_1, V_2, V'_2, V_3 over the point $(V_1, V_2, V'_2, V_3) \in X$. Then D_4 is equal to $\det(\mathcal{V}_3)^*$.

We can describe the divisor E_4 as the zero set of the section

$$\begin{aligned} s_{E_4} &\in H^0(X, \text{Hom}(\mathbb{C}^3/\mathcal{V}'_2, \mathbb{C}^4/\mathcal{V}_3)) = H^0(X, (\mathbb{C}^3/\mathcal{V}'_2 \otimes (\mathbb{C}^4/\mathcal{V}_3)^*) \\ s_{E_4}(V_1, V_2, V'_2, V_3) &: (v + V'_2) \mapsto v + V_3. \end{aligned}$$

Furthermore we have the following identifications, which can be readily checked:

$$\mathbb{C}^4/\mathcal{V}'_2 \cong (\det \mathcal{V}'_2)^*$$

$$\mathbb{C}^4/\mathcal{V}_3 \cong (\det \mathcal{V}_3)^*$$

From this we can conclude $E_4 = D_4 - D_3$, which leads to

$$D_4 = E_4 + E_3 + E_2.$$

Let us now determine all the fixed faces of the effective cone. We already know from the above picture that $\text{Cone}(E_2, E_3)$ is not fixed. This implies that the face $\text{Cone}(E_2, E_3, E_4)$ is not fixed either. The only left faces which are not known to be fixed are $\text{Cone}(E_3, E_4)$ and $\text{Cone}(E_2, E_4)$. Let us prove that $E_2 + E_4$ is fixed. Indeed, if it were not fixed, then there would exist a subdivisor $P = \lambda E_2 + \mu E_4$, for $0 \leq \lambda, \mu \leq 1$, which is nef. But

$$\lambda E_2 + \mu E_4 = \lambda(D_2 - D_1) + \mu(D_4 - D_3)$$

which is never nef as long as $\max(\lambda, \mu) > 0$. But $\max(\lambda, \mu) = 0$ implies $P = 0$. Thus $E_2 + E_4$ is fixed, and therefore all the divisors in $\text{Cone}(E_2, E_4)$ are. Similarly, we can prove that $\text{Cone}(E_3, E_4)$ is fixed. This shows that $\overline{\text{Eff}}(X)$ decomposes into six Mori chambers, corresponding to the fixed divisors $E_2, E_3, E_4, E_2 + E_4, E_3 + E_4$, and the nef cone.

The following table displays which facets of the nef cone correspond to which extremal rays. Here, we fix the basis (E_1, \dots, E_n) .

Facet generators(Nef Cone)	Supp. Half-space	Opposite extr. ray
D_1, D_2, D_3	$x_4 \geq 0$	does not separate
D_1, D_2, D_4	$x_3 - x_4 \geq 0$	E_4
D_1, D_3, D_4	$x_2 - x_3 \geq 0$	E_3
D_2, D_3, D_4	$x_1 - x_2 + x_3 \geq 0$	E_2

FIGURE 2. Correspondence facets-extremal rays of X

This leads to the following notation for the facets:

$$F_1 = \text{Cone}(D_1, D_2, D_3)$$

$$F_2 = \text{Cone}(D_2, D_3, D_4)$$

$$F_3 = \text{Cone}(D_1, D_3, D_4)$$

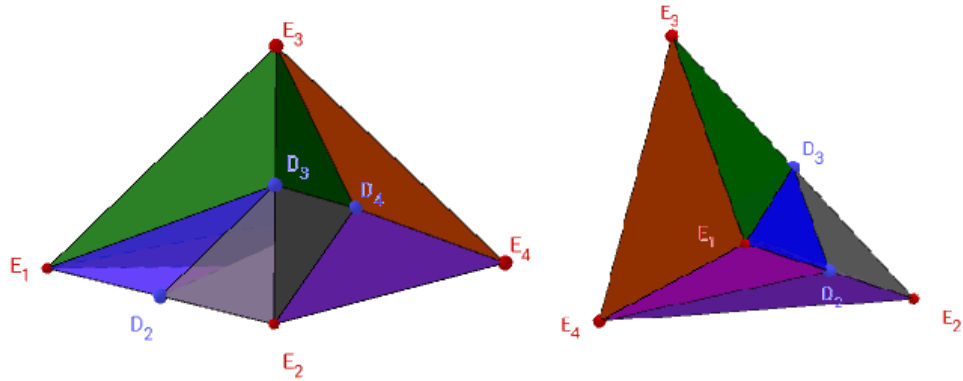
$$F_4 = \text{Cone}(D_1, D_2, D_4).$$

Now, we have all the necessary information to explicitly describe the Mori chambers of X :

Negative Support	Mori chamber	Color
\emptyset	$\text{Cone}(D_1, D_2, D_3, D_4)$	●
E_2	$\text{Cone}(E_2) + F_2 = \text{Cone}(E_2, D_2, D_3, D_4)$	●
E_3	$\text{Cone}(E_3) + F_3 = \text{Cone}(E_3, D_1, D_3, D_4)$	●
E_4	$\text{Cone}(E_4) + F_4 = \text{Cone}(E_4, D_1, D_2, D_4)$	●
$E_2 \cup E_4$	$\text{Cone}(E_2, E_4) + F_2 \cap F_4 = \text{Cone}(E_2, E_4, D_2, D_4)$	●
$E_3 \cup E_4$	$\text{Cone}(E_3, E_4) + F_3 \cap F_4 = \text{Cone}(E_3, E_4, D_1, D_4)$	●

FIGURE 3. Mori chambers of X

Finally, let us illustrate the Mori decomposition of X by plotting a slice of the chamber decomposition with a generic hyperplane. For a better overview we display the decomposition from two different perspectives.

FIGURE 4. Mori chamber decomposition of X

6. GLOBAL NEWTON-OKOUNKOV BODIES ON BOTT-SAMELSON VARIETIES AND THE GLOBAL VALUE SEMIGROUP

In this section, we consider (global) Newton-Okounkov bodies with respect to the so-called ‘horizontal’ flag. Furthermore, we show that the global semigroup $\Gamma_{Y_\bullet}(X)$ is finitely generated.

6.1. The horizontal flag. We start with a description of the *horizontal flag*. Let $w = (s_1, \dots, s_n)$ be a reduced word and denote by X_w the corresponding Bott-Samelson variety. Let furthermore E_1, \dots, E_n be the effective basis, satisfying $E_i \cong X_{w(i)}$ for $w(i) = (s_1, \dots, \hat{s}_i, \dots, s_n)$. Moreover, let us define, for $i = 1, \dots, n$, the truncated sequence $w[i] = (s_1, \dots, s_{n-i})$. Then,

if w is a reduced sequence, the sequence $w[i]$ is still reduced and $X_{w[i]} \subseteq X_w$ is a closed subvariety of codimension i which is again a Bott-Samelson variety. We can also write $X_{w[i]} = E_n \cap \cdots \cap E_{n-i+1}$ and represent $X_{w[i]}$ as a closed subvariety of X_w as follows

$$X_{w[i]} = \{(p_1, \dots, p_n) \in X_w \mid p_n = \cdots = p_{n-i+1} = e\}.$$

We define the *horizontal flag* Y_\bullet as follows. For $i = 1, \dots, n$, we set

$$Y_i := X_{w[i]}.$$

We write $Y_{k\bullet} = (Y_k \supseteq Y_{k+1} \supseteq \cdots \supseteq Y_n)$ for the induced flag on Y_k . Note that the effective basis of Y_i is given by $(E_1)|_{Y_i}, \dots, (E_{n-i})|_{Y_{n-i}}$, and the $\mathcal{O}(1)$ -basis of Y_i is given by $(D_1)|_{Y_i}, \dots, (D_{n-i})|_{Y_{n-i}}$. For the sake of simplifying the notation, we shall omit the restrictions and simply write E_1, \dots, E_{n-i} and D_1, \dots, D_{n-i} for divisors on Y_i whenever no confusion should arise.

Remark 6.1. Note that Newton-Okounkov bodies with respect to the horizontal flag were already studied in [HaY15]. In the mentioned article they show the finite generation of the semigroup $\Gamma(X_w, D)$ under a condition which they called condition **(P)**. In the following, we shall generalize this result to all divisors D on X_w . It is also worth to note that the techniques used in the above mentioned article substantially differ from ours. While their approach relied on representation theory and combinatorics, we mainly use our earlier established Mori-theoretic properties of Bott-Samelson varieties and results from [LT04].

6.2. Rational polyhedrality of global Newton-Okounkov bodies.

The key to proving the finite generation of the semigroup, as well as the rational polyhedrality of the global Newton-Okounkov body, is the following lemma.

Lemma 6.2. *Let D be a nef divisor on X_w . Then the restriction map*

$$H^0(X, \mathcal{O}_X(D)) \longrightarrow H^0(E_n, \mathcal{O}_{E_n}(D))$$

is surjective.

Proof. In [LT04, Thm 7.4] it was proved in particular that for a nef divisor D , the first cohomology group $H^1(X, \mathcal{O}(D - E_n))$ vanishes. This shows that the restriction morphism:

$$H^0(X, \mathcal{O}_X(D)) \longrightarrow H^0(E_n, \mathcal{O}_{E_n}(D))$$

is surjective if D is nef. □

Theorem 6.3. *Let $X = X_w$ be a Bott-Samelson variety and Y_\bullet be the horizontal flag. Then the global Newton-Okounkov body $\Delta_{Y_\bullet}(X_w)$ is rational polyhedral.*

Proof. The proof is based on results already established in [SS17]. Recall that the divisors D_1, \dots, D_n form the $\mathcal{O}(1)$ -basis on X_w . Denote by $\Gamma(D_1, \dots, D_n)$ the semigroup generated by D_1, \dots, D_n in $N^1(X)$. We define the semigroup

$$S(D_1, \dots, D_n) := \{(\nu_{Y_\bullet}(s), D) \in \mathbb{N}^n \times \Gamma(D_1, \dots, D_n) \mid s \in H^0(X_w, \mathcal{O}(D)), \nu_1(s) = 0\}.$$

It follows from [SS17, Theorem 3.3] that $\Delta_{Y_\bullet}(X_w)$ is rational polyhedral if $\text{Cone}(S(D_1, \dots, D_n))$ is rational polyhedral. Consider now the semigroup

$$S_1(D_1, \dots, D_n) := \{(\nu_{Y_1}(s), D) \in \mathbb{N}^{n-1} \times \Gamma(D_1|_{Y_1}, \dots, D_n|_{Y_1}) \mid s \in H^0(Y_1, \mathcal{O}_{Y_1}(D))\}.$$

We define a natural map

$$q_0 : S(D_1, \dots, D_n) \rightarrow S_1(D_1, \dots, D_n) \\ (\nu(s), D) \mapsto ((\nu_2(s), \dots, \nu_n(s)), D \cdot Y_1),$$

which extends to the linear map

$$(2) \quad q : \mathbb{R}^n \oplus N^1(X_w)_{\mathbb{R}} \longrightarrow \mathbb{R}^{n-1} \oplus N^1(Y_1)_{\mathbb{R}}, \\ ((x_1, \dots, x_n), D) \mapsto ((x_2, \dots, x_n), D \cdot Y_1).$$

We now use the following fact which we shall prove in the below lemma:

$$\overline{\text{Cone}(S(D_1, \dots, D_n))} = \\ q^{-1}(\overline{\text{Cone}(S_1(D_1, \dots, D_n))}) \cap (\{0\} \times \mathbb{R}_{\geq 0}^{n-1} \times \text{Cone}(D_1, \dots, D_n)).$$

This identity shows that $\overline{\text{Cone}(S(D_1, \dots, D_n))}$ is rational polyhedral if $\overline{\text{Cone}(S_1(D_1, \dots, D_n))}$ is rational polyhedral.

Now, we proceed by induction on the dimension n of X_w . If $n = 1$, then $X_w \cong \mathbb{P}^1$. It can be easily checked that the global Newton-Okounkov body of \mathbb{P}^1 with respect to any admissible flag is rational polyhedral.

Assume now that the assertion is true for $n-1$. Then, $\Delta_{Y_1 \bullet}(Y_1)$ is rational polyhedral, and we have

$$\overline{\text{Cone}(S_1(D_1, \dots, D_n))} = pr_2^{-1} \left(\overline{\text{Cone}(D_1|_{Y_1}, \dots, D_n|_{Y_1})} \right) \cap \Delta_{Y_1 \bullet}(Y_1).$$

But this implies that $\overline{\text{Cone}(S_1(D_1, \dots, D_n))}$ and $\overline{\text{Cone}(S(D_1, \dots, D_n))}$ are rational polyhedral. Finally, this proves that $\Delta_{Y_\bullet}(X_w)$ is rational polyhedral. \square

Lemma 6.4. *With the notation introduced above, we have*

$$\overline{\text{Cone}(S(D_1, \dots, D_n))} = \\ q^{-1}(\overline{\text{Cone}(S_1(D_1, \dots, D_n))}) \cap (\{0\} \times \mathbb{R}_{\geq 0}^{n-1} \times \text{Cone}(D_1, \dots, D_n)).$$

Proof. In order to show the inclusion ' \subseteq ', it is enough to show this inclusion for the semigroup $S(D_1, \dots, D_n)$ since both sides are closed convex cones. So, let $(0, a_2, \dots, a_n, D) \in S(D_1, \dots, D_n)$. Then this is clearly a preimage of $(a_2, \dots, a_n, D \cdot Y_1)$ under q and lies in $(\{0\} \times \mathbb{R}_{\geq 0}^{n-1} \times \text{Cone}(D_1, \dots, D_n))$. This shows the first inclusion.

For the second inclusion ' \supseteq ', note that both sides are closed sets. Hence, it is enough to show that the inclusion holds for rational points in the interior, and—since both sides are also cones—it is enough to show the inclusion for integral points in the interior. Let therefore

$$(0, a, D) \in q^{-1}(\overline{\text{Cone}(S_1(D_1, \dots, D_n))}) \cap (\{0\} \times \mathbb{R}_{\geq 0}^{n-1} \times \text{Cone}(D_1, \dots, D_n))$$

be an integral point in the interior. By definition,

$$(a, D|_{Y_1}) \in \text{Cone}(S_1(D_1, \dots, D_n)).$$

After scaling appropriately, we can assume $(ka, kD \cdot Y_1) \in S_1(D_1, \dots, D_n)$. This means that there is a section $s \in H^0(Y_1, \mathcal{O}_{Y_1}(kD))$ such that $\nu_{Y_1 \bullet}(s) = ka$. By Lemma 6.2, we can lift the section s to a section $\tilde{s} \in H^0(X, \mathcal{O}_X(kD))$ such that $\tilde{s}|_{Y_1} = s$. Then, we clearly have $\nu_{Y_1 \bullet}(\tilde{s}) = (0, ka)$. This proves that $(0, ka, kD) \in S(D_1, \dots, D_n)$, which implies $(0, a, D) \in \overline{\text{Cone}(S(D_1, \dots, D_n))}$. \square

6.3. Value semigroups. We need the following notation: for a fixed flag Y_\bullet on X and an effective divisor $D \neq 0$, we define the following semigroup

$$\Gamma_{Y_\bullet}(D) = \bigsqcup_{k \in \mathbb{N}} \Gamma_{Y_\bullet}(D)_k := \{(\nu_{Y_\bullet}(s), k) \mid k \in \mathbb{N}, s \in H^0(X, \mathcal{O}_X(kD)) \setminus \{0\}\}$$

as well as

$$\Gamma_{Y_\bullet}(D)_{\nu_1=a} = \bigsqcup_{k \in \mathbb{N}} (\Gamma_{Y_\bullet}(D)_{\nu_1=a})_k = \{(\nu_1(s), \dots, \nu_n(s), k) \mid k \in \mathbb{N}, s \in H^0(X, \mathcal{O}_X(kD)), \nu_1(s) = ak\}.$$

If D is a \mathbb{Q} -divisor such that kD is integral for a given $k \in \mathbb{N}$, we define

$$\Gamma_{Y_\bullet}(D)_k = \{(\nu(s), k) \mid s \in H^0(X, \mathcal{O}_X(kD))\}.$$

Furthermore, for $a > 0$ we abbreviate

$$D_a := D - aY_1,$$

as well as

$$(3) \quad P_a := P(D_a) \quad \text{and} \quad N_a := N(D_a)$$

Proposition 6.5. *Let $X = X_w$ be an n -dimensional Bott-Samelson variety. Let Y_\bullet be an admissible flag with $Y_1 = X_{w[1]} = E_n$. Let D be an effective divisor. Then, the identity*

$$(\Gamma_{Y_\bullet}(D)_{\nu_1=a})_k = \{ak\} \times (\Gamma_{Y_1 \bullet}(P_a|_{Y_1})_k + k \cdot \nu_{Y_1 \bullet}(N_a|_{Y_1}))$$

holds for all $a \in \mathbb{Q}$ such that $\Gamma_{Y_\bullet}(D)_{\nu_1=a} \neq \emptyset$, and $k > 0$ such that kP_a and kN_a are integral.

Proof. We claim that N_a does not contain Y_1 in its support. Suppose it does, then

$$D = P_a + (N_a + tY_1)$$

is the Zariski decomposition of D , i.e. $N(D) = N_a + tY_1$. However, this proves $\nu_1(s) > ak$ for each $s \in H^0(X, \mathcal{O}_X(kD))$, which contradicts the fact that $\Gamma_{Y_\bullet}(D)_{\nu_1=a} \neq \emptyset$.

For each $a \in \mathbb{Q}$, we choose $k \in \mathbb{N}$ such that kP_a and kN_a are integral. Let us now show the inclusion ‘ \subseteq ’. Let $s \in H^0(X, \mathcal{O}_X(kD)) \setminus \{0\}$ such that $\nu_1(s) = ak$. Then consider $\tilde{s} := s/s_{Y_1}^{ka}$. We can decompose this section $\tilde{s} = s_P \otimes s_N$. By construction,

$$(4) \quad \nu_{Y_\bullet}(s) = (ak, \nu_{Y_1 \bullet}(\tilde{s}|_{Y_1})) = (ak, \nu_{Y_1 \bullet}(s_P|_{Y_1}) + k \cdot \nu_{Y_1 \bullet}(N_a|_{Y_1})).$$

This proves the desired inclusion.

Now, in order to prove the reverse inclusion ‘ \supseteq ’, consider an arbitrary section $\tilde{s}_P \in H^0(Y_1, \mathcal{O}_{Y_1}(kP_a))$. By Lemma 6.2, there is a section $s_P \in H^0(X, \mathcal{O}_X(kP_a))$ which coincides with \tilde{s}_P on Y_1 . Define $\tilde{s} := s_P \otimes s_N$, where

s_N is a section of $\mathcal{O}(kN_a)$, and finally, set $s := \tilde{s} \otimes s_{Y_1}^{ak} \in H^0(X, \mathcal{O}_X(kD))$. Then the reverse inclusion follows from equation (4). \square

Corollary 6.6. *Let D be a big divisor on X_w , and Y_\bullet be the horizontal flag. Then for every $a \in \mathbb{Q}$ such that $D - aY_1$ is big and $(D - aY_1)|_{Y_1}$ is big, we have*

$$\Delta_{Y_\bullet}(D)_{\nu_1=a} = \Delta_{Y_1_\bullet}(P_a|_{Y_1}) + \nu_{Y_1_\bullet}(N_a|_{Y_1}).$$

Proof. If $a > 0$ then it follows from the fact that $D - aY_1$ is big that $\{a\} \times \mathbb{R}^{n-1}$ meets the interior of $\Delta_{Y_\bullet}(D)$. But then we have

$$\text{Cone}(\Gamma_{Y_\bullet}(D)_{\nu_1=a}) = \text{Cone}(\Gamma_{Y_\bullet}(D))_{\nu_1=a}$$

which was established in [LM09, (4.8)] and proves the proposition for $a > 0$. For $a = 0$, replace D by $D + \varepsilon A$, for an ample class A , and let $\varepsilon \rightarrow 0$. \square

6.4. Numerical and valuative Newton-Okounkov bodies. In this subsection we consider Newton-Okounkov bodies of effective (not necessarily big) divisors.

Lemma 6.7. *Let X_w be an n -dimensional Bott-Samelson variety. Let Y_\bullet be the horizontal flag. Let D be an effective divisor. Then, the inclusion*

$$\Delta_{Y_\bullet}^{val}(D)_{\nu_1=a} \supseteq \{a\} \times \left(\Delta_{Y_1_\bullet}^{val}(P_a|_{Y_1}) + \nu_{Y_1_\bullet}(N_a|_{Y_1}) \right)$$

holds for all $a \in \mathbb{Q}$ such that $\Delta_{Y_\bullet}^{val}(D)_{\nu_1=a} \neq \emptyset$.

Proof. The above inclusion follows from Proposition 6.5 and the fact that $\text{Cone}(\Gamma_{Y_\bullet}(D)_{\nu_1=a}) \subseteq \text{Cone}(\Gamma_{Y_\bullet}(D))_{\nu_1=a}$ for all rational $a \geq 0$. \square

Lemma 6.8. *Let D be an effective divisor on X_w . Let Y_\bullet be the horizontal flag. Then,*

$$\Delta_{Y_\bullet}^{num}(D)_{\nu_1=a} \subseteq \{a\} \times \left(\Delta_{Y_1_\bullet}^{num}(P_a|_{Y_1}) + \nu_{Y_1_\bullet}(N_a|_{Y_1}) \right)$$

for all $a \in \mathbb{Q}$ such that $\Delta_{Y_\bullet}^{num}(D)_{\nu_1=a} \neq \emptyset$.

Proof. We now fix an ample divisor A . Note that it follows from the assumption on a that $D - aY_1$ is effective, and that Y_1 is not contained in the support of the negative part of D . Therefore, the divisor $D - aY_1 + 1/k \cdot A$, as well as its restriction to Y_1 , is big.

Now, we can use Corollary 6.6 to deduce that

$$\begin{aligned} \Delta_{Y_\bullet}^{num}(D)_{\nu_1=a} &= \left(\bigcap_{k \in \mathbb{N}} \Delta_{Y_\bullet}(D + 1/k \cdot A) \right) \cap (\{a\} \times \mathbb{R}^{n-1}) \\ &= \bigcap_{k \in \mathbb{N}} ((\Delta_{Y_\bullet}(D + 1/k \cdot A) \cap (\{a\} \times \mathbb{R}^{n-1}))) \\ &= \{a\} \times \left(\bigcap_{k \in \mathbb{N}} (\Delta_{Y_1_\bullet}(P_{a,1/k}|_{Y_1}) + \nu_{Y_1_\bullet}(N_{a,1/k}|_{Y_1})) \right). \end{aligned}$$

where $P_{a,b} := P(D + bA - aY_1)$ and $N_{a,b} := N(D + bA - aY_1)$.

Let us now show that $\nu_{Y_\bullet}(N_{a,1/k}) \rightarrow \nu_{Y_\bullet}(N_a)$ as $k \rightarrow \infty$. Indeed, there is an integer $K > 0$ such that $K \cdot D_a + A$ and D lie in the same Zariski chamber. Then, for each $k > K$, we have

$$N(kD_a + A) = N((k - K)D_a + (KD_a + A)) = (k - K)N(D_a) + N(KD_a + A).$$

Dividing by k and applying the valuation ν_{Y_\bullet} yields the result.

The fact that

$$\bigcap_{k \in \mathbb{N}} (\Delta_{Y_1 \bullet}(P_{a,1/k}|_{Y_1})) \subseteq \Delta_{Y_1 \bullet}^{num}(P_a|_{Y_1}).$$

follows by observing that $P_{a,1/k}|_{Y_1}$ converges to $P_a|_{Y_1}$ as $k \rightarrow \infty$. Hence, the result follows. \square

Theorem 6.9. *Let D be an effective divisor on a Bott-Samelson variety $X = X_w$, and Y_\bullet the horizontal flag. Then*

$$\Delta_{Y_\bullet}^{num}(D) = \Delta_{Y_\bullet}^{val}(D).$$

Moreover, we have

$$\Delta_{Y_\bullet}(D)_{\nu_1=a} = \{a\} \times (\Delta_{Y_1 \bullet}(P_a|_{Y_1}) + \nu_{Y_1 \bullet}(N_a|_{Y_1}))$$

for all $a \in \mathbb{Q}$ such that $\Delta_{Y_\bullet}(D)_{\nu_1=a} \neq \emptyset$.

Proof. We prove this by induction over the dimension n of X_w . For $n = 1$ this is trivial since on the curve $X_w \cong \mathbb{P}^1$, the only effective non-big divisor is the zero divisor.

Let us now suppose that equality holds for Bott-Samelson varieties of dimension $n - 1$. Let D be an effective divisor on X_w and A and arbitrary ample class. Let $a \in \mathbb{Q}$ be such that $\Gamma_X(D)_{\nu_1=a} \neq \emptyset$. Then we have

$$\begin{aligned} \Delta_{Y_\bullet}^{num}(D)_{\nu_1=a} &\subseteq \{a\} \times (\Delta_{Y_1 \bullet}^{num}(P_a|_{Y_1}) + \nu_{Y_1 \bullet}(N_a|_{Y_1})) \\ &= \{a\} \times (\Delta_{Y_1 \bullet}^{val}(P_a|_{Y_1}) + \nu_{Y_1 \bullet}(N_a|_{Y_1})) \\ &\subseteq \Delta_{Y_\bullet}^{val}(D)_{\nu_1=a}. \end{aligned}$$

This proves $\Delta_{Y_\bullet}^{num}(D) \subseteq \Delta_{Y_\bullet}^{val}(D)$. Since the reverse inclusion is always true, this proves the claim. \square

Corollary 6.10. *Let D be an effective divisor on X_w . Let Y_\bullet be the horizontal flag and $b = (b_1, \dots, b_n) \in \Delta_{Y_\bullet}(D)$ be a rational point. Then there is an integer $k \in \mathbb{N}$ such that $k \cdot b \in \Gamma_{Y_\bullet}(D)_k$.*

Proof. We prove this by induction on the dimension n of X_w . If $n = 1$, this means that $X_w = \mathbb{P}^1$, the statement is easy to check.

Let now X_w be of dimension n and suppose $(b_1, \dots, b_n) \in \Delta_{Y_\bullet}(D)$. Hence,

$$(b_1, b_2, \dots, b_n) \in \Delta_{Y_\bullet}(D)_{\nu_1=b_1} = \{b_1\} \times (\Delta_{Y_1 \bullet}^{num/val}(P_{b_1}|_{Y_1}) + \nu_{Y_1 \bullet}(N_{b_1}|_{Y_1})).$$

Now, we can use the induction hypothesis to deduce that

$$k \cdot (b_1, \dots, b_n) \in \{b_1 k\} \times ((\Gamma_{Y_1 \bullet}(P_{b_1}|_{Y_1}))_k + k \cdot \nu_{Y_1 \bullet}((N_{b_1}|_{Y_1}))).$$

But by Proposition 6.5, we have

$$k \cdot (b_1, \dots, b_n) \in (\Gamma_{Y_\bullet}(D)_{\nu_1=b_1})_k \subseteq \Gamma_{Y_\bullet}(D)_k.$$

This proves the claim. \square

Theorem 6.11. *Let $X = X_w$ be a Bott-Samelson variety, Y_\bullet the horizontal flag. Then the global semigroup*

$$\Gamma_{Y_\bullet}(X_w) := \{(\nu_{Y_\bullet}(s), D) \mid D \in \text{Pic}(X_w), s \in H^0(X_w, \mathcal{O}_{X_w}(D))\}$$

is finitely generated.

Proof. It was proved in Theorem 6.3 that $\Delta_{Y_\bullet}(X_w) = \overline{\text{Cone}(\Gamma_{Y_\bullet}(X_w))}$ is rational polyhedral. First note that $\text{Cone}(\Gamma_{Y_\bullet}(X_w))$ is already closed: let $(a, D) \in \Delta_{Y_\bullet}(X_w)$ be a rational point. This means that $a \in \Delta_{Y_\bullet}^{\text{num}}(D)$. Then, by Corollary 6.10 we can deduce that $ka \in \Gamma_{Y_\bullet}(D)_k$. But this clearly proves that $(ka, kD) \in \Gamma_{Y_\bullet}(X_w)$, and therefore $(a, D) \in \text{Cone}(\Gamma_{Y_\bullet}(X_w))$.

It follows then from [BG09, Corollary 2.10] that $\Gamma_{Y_\bullet}(X)$ is finitely generated.

□

We end this section with a sufficient condition for the global semigroup $\Gamma_{Y_\bullet}(X)$ to be normal, namely with the existence of an integral Zariski decomposition.

Definition 6.12. We say that X admits *integral Zariski decompositions* if it admits a Zariski decomposition, and both the divisors $N(D)$ and $P(D)$ of an integral divisor D are integral.

Proposition 6.13. *Let X_w be a Bott-Samelson variety that admits integral Zariski decompositions. Let Y_\bullet be the horizontal flag. Then $\Gamma_{Y_\bullet}(X_w)$ is a normal semigroup.*

Proof. First of all, we prove that for an effective divisor D the semigroup $\Gamma_{Y_\bullet}(D)$ is normal. We proceed by induction on the dimension.

Let $\dim X_w = 1$. Then $X_w \cong \mathbb{P}^1$, and it is not difficult to see that $\Gamma_P(D)$ is a normal semigroup for every point $P \in X_w$. Now let us suppose the claim holds in dimension $n-1$. In order to use the induction hypothesis we need to prove that Y_1 admits integral Zariski decompositions. Let D be an effective divisor on Y_1 . Let $\iota: Y_1 \rightarrow X_w$ be the closed embedding of Y_1 into X_w , and let $\pi: X_w \rightarrow Y_1 = X_{w[1]}$ be the projection to the first $n-1$ coordinates. It then follows that $\text{id} = \pi \circ \iota$. We claim that $D = \iota^*P(\pi^*D) + \iota^*N(\pi^*D)$ is the Zariski decomposition of D . This proves that Y_1 admits integral Zariski decompositions.

Since $\iota^*P(\pi^*D)$ is nef and $\iota^*N(\pi^*D)$ is effective, we have $P(D) \geq \iota^*P(\pi^*D)$. Consider now $\pi^*D = \pi^*P(D) + \pi^*N(D)$. Again, since $\pi^*P(D)$ is nef and $\pi^*N(D)$ is effective, we conclude $\pi^*P(D) \leq P(\pi^*D)$. Applying ι^* on both sides, we get $P(D) \leq \iota^*P(\pi^*D)$. This proves the claim

Now suppose that $((ma_1, \dots, ma_n), mk) \in (\Gamma_X(D))_{mk}$ for a tuple of non-negative integers $(a_1, \dots, a_n, k) \in \mathbb{N}^{n+1}$. The divisor $mkD_{a_1/k}$ is integral. Since X_w admits integral Zariski decompositions, the divisors $mkP_{a_1/k}$ and $mkN_{a_1/k}$ are integral as well. We can thus use Proposition 6.5 to deduce that

$$((ma_2, \dots, ma_n), mk) \in \Gamma_{Y_1}(P_{a_1/k}|_{Y_1})_{mk} + mk \cdot \nu_{Y_1}(N_{a_1/k}|_{Y_1}).$$

Put $(b_2, \dots, b_n) := k \cdot \nu_{Y_1}(N_{a_1/k}|_{Y_1}) \in \mathbb{Z}^{n-1}$. Then,

$$m \cdot (a_2 - b_2, \dots, a_n - b_n, k) \in \Gamma_{Y_1}(P_{a_1/k}|_{Y_1})_{mk}.$$

Hence, we can use the induction hypothesis for Y_1 and $kP_{a_1/k}|_{Y_1}$ to conclude that

$$(a_2 - b_2, \dots, a_n - b_n, k) \in \Gamma_{Y_1}(P_{a_1/k}|_{Y_1})_k.$$

We again use the fact that X_w induces integral Zariski decompositions to deduce that $kP_{a_1/k}$ and $kN_{a_1/k}$ are integral. Hence, we can use Proposition 6.5 to deduce that $(a_1, \dots, a_n, k) \in \Gamma_{Y_\bullet}(D)_k$. This proves that $\Gamma_{Y_\bullet}(D)$ is a normal semigroup. But then it follows easily that $\Gamma_{Y_\bullet}(X_w)$ is normal. \square

6.5. Connection between the global semigroup and the Cox ring.

The finite generation of the global semigroup is connected to the finite generation of the Cox ring. More precisely we have the following theorem.

Theorem 6.14. *Let X be a \mathbb{Q} -factorial variety with $N^1(X) = \text{Pic}(X)$. Let Y_\bullet be an admissible flag. Suppose $\Gamma_{Y_\bullet}(X)$ is finitely generated by*

$$(\nu_{Y_\bullet}(s_1), D_1), \dots, (\nu_{Y_\bullet}(s_N), D_n).$$

Then X is a Mori dream space, and the Cox ring $\text{Cox}(X)$ is generated by the sections s_1, \dots, s_N .

Proof. Let R be the \mathbb{C} -algebra which is generated by the sections s_1, \dots, s_N . Let D be any effective divisor in X . Let

$$k := h^0(X, \mathcal{O}_X(D)) = |\nu_{Y_\bullet}(H^0(X, \mathcal{O}_X(D)) \setminus \{0\})|.$$

Since the $(\nu_{Y_\bullet}(s_1), D_1), \dots, (\nu_{Y_\bullet}(s_N), D_n)$ generate $\Gamma_{Y_\bullet}(X)$, it follows that there are sections $f_1, \dots, f_k \in R \cap H^0(X, \mathcal{O}_X(D)) \setminus \{0\}$ with distinct values, and it then follows from [KaKo12, Proposition 2.3] that f_1, \dots, f_k are linearly independent. This proves that they yield a basis for $H^0(X, \mathcal{O}_X(D))$ and that every section $s \in H^0(X, \mathcal{O}_X(D))$ belongs to the algebra R . This proves that $R \cong \text{Cox}(D)$. \square

In particular, the above theorem shows that $\Gamma_{Y_\bullet}(X)$ cannot be finitely generated unless X is a Mori dream space.

6.6. Newton-Okounkov bodies of Schubert varieties. We can now use our results on Bott-Samelson varieties to deduce some consequences for Schubert varieties.

Let $P \subseteq G$ be any parabolic subgroup containing B , and let $w = (s_1, \dots, s_n)$ be a reduced expression for which there is a birational morphism

$$p: X_w \longrightarrow Z_{\bar{w}}$$

with $Z_{\bar{w}}$ denoting the Schubert variety corresponding to $\bar{w} := s_1 \cdots s_n$ in the partial flag variety G/P . In particular, one special case is $Z_{\bar{w}} = G/P$.

As $Z_{\bar{w}}$ is normal, for every effective divisor D on $Z_{\bar{w}}$ we have

$$H^0(Z_{\bar{w}}, \mathcal{O}_{Z_{\bar{w}}}(D)) \cong H^0(X_w, p^* \mathcal{O}_{Z_{\bar{w}}}(D)).$$

Hence, we can use the horizontal flag on X_w to define a valuation

$$\nu_{Y_\bullet}: \bigsqcup_{D \in \text{Pic}(Z_{\bar{w}})} H^0(Z_{\bar{w}}, \mathcal{O}_{Z_{\bar{w}}}(D)) \setminus \{0\} \longrightarrow \mathbb{N}^n$$

and a corresponding (global) Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ (resp. $\Delta_{Y_\bullet}(Z_{\bar{w}})$).

We can now use our previous findings to deduce the following.

Theorem 6.15. *Let $Z_{\bar{w}} \subseteq G/P$ be the Schubert variety for the reduced word w . Let $\nu_{Y_{\bullet}}$ be the above described valuation-like function on $Z_{\bar{w}}$. Then the global semigroup $\Gamma_{Y_{\bullet}}(Z_{\bar{w}})$ is finitely generated. Hence, $\Delta_{Y_{\bullet}}(Z_{\bar{w}})$ is rational polyhedral.*

In particular, the global semigroup $\Gamma_{Y_{\bullet}}(G/P)$ for any partial flag variety G/P is finitely generated.

Proof. Let D_1, \dots, D_k be the generators of the effective cone $\text{Eff}(Z_{\bar{w}})$. Then $\Delta_{Y_{\bullet}}(Z_{\bar{w}}) = \Delta_{Y_{\bullet}}(X_w) \cap (\mathbb{R}^n \times \text{Cone}(p^*D_1, \dots, p^*D_n))$ is clearly rational polyhedral. Furthermore, we have

$$\Gamma_{Y_{\bullet}}(Z_{\bar{w}}) = \{(\nu_{Y_{\bullet}}(s), [D]) \mid D \in \Gamma(p^*D_1, \dots, p^*D_k), s \in H^0(X_w, \mathcal{O}_{X_w}(D))\}.$$

It follows then, completely analogously to the proof of Theorem 6.11, that $\Gamma_{Y_{\bullet}}(Z_{\bar{w}})$ is finitely generated. \square

Remark 6.16. In [FeFoL17], Feigin, Fourier, and Littelmann show that partial flag varieties G/P for the groups SL_n, Sp_n , and G_2 admit rational polyhedral local Okounkov bodies with respect to a valuation defined in local coordinates. The global description of their valuation seems to us to amount to considering the blow-up $Bl_x(X_w)$ at a point $x \in X_w$ of a Bott-Samelson resolution X_w of G/P and choosing a suitable linear flag in the projective space $\mathbb{P}(T_x(X_w))$.

7. EXAMPLE OF A GLOBAL NEWTON-OKOUNKOV BODY

Let us consider the 3-dimensional incidence variety Y from Section 5.1 again. In this section we compute the global Newton-Okounkov body of Y with respect to the horizontal flag as well as the global semigroup $\Gamma_{Y_{\bullet}}(Y)$. The necessary computations were facilitated by the use of Sagemath.

7.1. Integrality of the Zariski decomposition. First of all, we note that Y admits an integral Zariski decomposition. This can be deduced as follows. It can be checked by hand that the three different triples of generators of the Mori chambers (D_3, D_2, E_2) , (D_1, D_2, D_3) and (D_3, E_3, E_1) each form a \mathbb{Z} -basis of $\text{Pic}_{\mathbb{Z}}(X_w)$. Hence every integral effective divisor D can be written as a \mathbb{N} -linear combination of the generators of its corresponding Mori chamber. But this induces the Zariski decomposition of D , which proves that it is integral.

7.2. Global Newton-Okounkov body of the surface E_3 . By Proposition 6.13, we have $\Gamma_{Y_{\bullet}}(Y) = \text{Cone}(\Gamma_{Y_{\bullet}}(Y)) \cap \mathbb{Z}^6$. It suffices therefore to compute $\Delta_{Y_{\bullet}}(Y) = \text{Cone}(\Gamma_{Y_{\bullet}}(Y))$ in order to determine $\Gamma_{Y_{\bullet}}(Y)$. We start with computing the global Newton-Okounkov body of the surface E_3 , with respect to the induced horizontal flag. The divisor E_3 is isomorphic to the Blowup X of \mathbb{P}^2 in one point. Since E_2 is an extremal ray of the effective cone which is not nef, it is the exceptional divisor. Since $D_1 = E_1$ is a nef divisor which is an extremal divisor of the effective cone, it is linear equivalent to the strict transform of a line going through the blown up point. Furthermore, it follows that D_2 is linear equivalent to the pullback of a line of \mathbb{P}^2 to X . Hence, we get

$$(E_1)^2 = 0 \quad (E_1 \cdot D_2) = 1 \quad (E_2 \cdot D_2) = 0 \quad (E_1 \cdot E_2) = 1.$$

Now, it follows with the help of [SS16] that $\Delta_{Y_\bullet}(Y_1)$ is generated by the following vectors

$$(1, 0, E_2), (0, 0, D_1), (0, 0, D_2), (0, 1, D_1).$$

7.3. Global Newton-Okounkov body of Y . In order to compute $C(S_1) := \text{Cone}(S_1(D_1, \dots, D_3))$, we need to intersect $\Delta_{Y_\bullet}(Y_1)$ with $\mathbb{R}^2 \times \text{Cone}(\{D_1, D_2\})$. A computation yields to the following generators of $C(S_1)$

$$(0, 0, D_1), (0, 0, D_2), (0, 1, D_1), (1, 0, D_2), (1, 1, D_2).$$

Choosing the basis E_1, E_2, E_3 for $N^1(Y)$, the following are the defining inequalities of $C(S_1)$

$$\begin{aligned} -x_1 + x_4 &\geq 0, & x_1 &\geq 0, & x_3 - x_4 &\geq 0, \\ x_2 &\geq 0, & x_1 - x_2 + x_3 - x_4 &\geq 0. \end{aligned}$$

Now we consider the restriction morphism $q: \text{Cone}(S(D_1, D_2, D_3)) \rightarrow C(S_1)$ which induces a linear morphism on the corresponding linear spaces. By realizing that $E_3|_{E_3} = E_1 - E_2$, the morphism q can be written as

$$q: \mathbb{R}^6 \rightarrow \mathbb{R}^4 \quad (a_1, \dots, a_6) \mapsto (a_2, a_3, a_4 + a_6, a_5 - a_6).$$

Hence, the defining inequalities of $q^{-1}(C(S_1))$ are given by

$$\begin{aligned} -x_2 + x_5 - x_6 &\geq 0, & x_2 &\geq 0, & x_4 - x_5 + 2 \cdot x_6 &\geq 0, \\ x_3 &\geq 0, & x_2 - x_3 + x_4 - x_5 + 2x_6 &\geq 0. \end{aligned}$$

The set $\text{Cone}(S(D_1, D_2, D_3))$ is given by $q^{-1}(C(S_1)) \cap (\{0\} \times \text{Cone}(D_1, D_2, D_3))$. The ray generators are then computed as

$$\begin{aligned} (0, 0, 0, D_3), (0, 0, 0, D_1), (0, 0, 0, D_2), (0, 0, 1, D_3), \\ (0, 0, 1, D_1), (0, 1, 0, D_2), (0, 1, 1, D_2). \end{aligned}$$

In order to obtain the generators of $\Delta_{Y_\bullet}(Y)$ we simply need to add the ray $(1, 0, 0, E_3)$ as well as $(0, 1, 0, E_2)$. This yields to the following minimal set of generators of $\Delta_{Y_\bullet}(Y)$:

$$\begin{aligned} (0, 0, 0, D_3), (0, 0, 0, D_1), (0, 0, 0, D_2), (0, 0, 1, D_3), \\ (0, 0, 1, D_1), (0, 1, 0, E_2), (1, 0, 0, E_3). \end{aligned}$$

7.4. Generators of the global semigroup/Cox ring. In fact, the above generators of $\Delta_{Y_\bullet}(Y)$ are actually a Hilbert basis and consequently they generate the global semigroup $\Gamma_{Y_\bullet}(Y)$. Furthermore, we can use the above generator and Theorem 6.14 to deduce that the following sections generate the Cox ring $\text{Cox}(Y)$

- a general section of D_i for $i = 1, 2, 3$
- the generating section of E_i for $i = 2, 3$
- a section of D_i for $i = 1, 3$ which vanishes exactly once at the chosen point of the horizontal flag and not on $E_3 \cap E_2$.

7.5. Connections to the flag variety $Fl(\mathbb{C}^3)$. The birational morphism $p: X_w \rightarrow Y_{\bar{w}}$ in our example is the following:

$$p: Y \rightarrow Fl(\mathbb{C}^3) \quad (V_1, V_2, V_2') \mapsto (V_1, V_2').$$

The divisors $D_2 = \det(\mathcal{V}_1)^*$ and $D_3 = \det(\mathcal{V}_2')^*$ generate the effective cone of $Fl(\mathbb{C}^3)$. Hence, we can compute the global Newton-Okounkov body as

$$\Delta_{Y_\bullet}(Fl(\mathbb{C}^3)) = \Delta_{Y_\bullet}(Y) \cap (\mathbb{R}^3 \times \text{Cone}(D_2, D_3)).$$

A computation shows that the following points are generators of the global Newton-Okounkov body $\Delta_{Y_\bullet}(Fl(\mathbb{C}^3))$:

$$(0, 0, 0, D_2), (0, 0, 0, D_3), (0, 1, 1, D_2), (0, 0, 1, D_3), \\ (1, 1, 0, D_3), (0, 1, 0, D_2).$$

As in the previous case for Y it turns out that the above generators are indeed a Hilbert basis for the global semigroup $\Gamma_{Y_\bullet}(Y_{\bar{w}})$. Moreover, with the help of Theorem 6.14, we can deduce that the following sections generate the Cox ring $\text{Cox}(Fl(\mathbb{C}^3))$:

- A general section s_i of D_i for $i = 2, 3$.
- A section s'_3 of D_3 which vanishes exactly once at a fixed point and not on $C = E_2 \cap E_3$.
- The section of $s_{E_2} \otimes s_{E_3}$ in D_3 where s_{E_i} is a defining section of E_i for $i = 2, 3$.
- A section of the form $s_{E_2} \otimes s'_1$ where s'_1 is a section in D_1 which vanishes exactly once at a fixed point and not on $C = E_2 \cap E_3$.

7.6. Computation of an example for a (local) Newton-Okounkov body. We can now use the description of the global Newton-Okounkov body to compute Newton-Okounkov bodies corresponding to special divisors. Let us, for example, fix the divisor $D = D_1 + D_2 + D_3$. Then $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(Y) \cap (\mathbb{R}^3 \times \{D\})$. A computation shows that the above Newton-Okounkov body is the convex hull of the vertices

$$(1, 0, 0), (1, 2, 0), (1, 2, 2), (0, 1, 3), \\ (0, 1, 0), (0, 0, 2), (0, 0, 0).$$

This Newton-Okounkov body has been computed in [HaY15, Example 4.1] by different methods. Since $\Gamma_{Y_\bullet}(D)$ is normal, we can compute the Hilbert polynomial of D as the Ehrhart polynomial of this polytope. Note that the Ehrhart polynomial P of a lattice polytope $\Delta \subset \mathbb{R}^d$ is the polynomial function given on integers $k \in \mathbb{N}$ by:

$$P(k) = \#(k\Delta \cap \mathbb{Z}^d).$$

It is given by

$$P_D(t) = 5/2t^3 + 11/2t^2 + 4t + 1.$$

8. OPEN PROBLEMS AND CONJECTURES

We end this article with some open questions and conjectures.

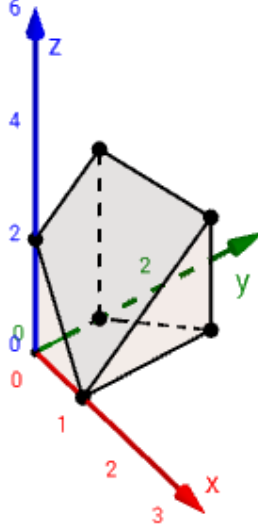


FIGURE 5. Newton-Okounkov body of $D = D_1 + D_2 + D_3$

8.1. Equality of moving cone and nef cone. One of the main reasons which lead us to nice characterizations of the Mori chambers and Newton-Okounkov bodies was the fact that we have a Zariski decomposition on Bott Samelson varieties X_w . This is a consequence of the fact that $\text{Mov}(X) = \text{Nef}(X)$. Besides for surfaces there are not many varieties known which have this property. Therefore, we raise the following question.

Question 8.1. For which varieties X do the cones $\text{Mov}(X)$ and $\text{Nef}(X)$ agree?

8.2. Finite generation of the global semigroup. We have seen that the finite generation of the global semigroup $\Gamma_{Y_\bullet}(X)$ was quite restrictive, i.e. we really needed a lot of nice properties (like existence of Zariski decomposition, vanishing of cohomology) in order to establish this result. However, it was proven in [PU16], that for a Mori dream space X , there always exist a flag Y_\bullet such that $\Delta_{Y_\bullet}(X)$ is rational polyhedral. It was also proven in this article that for an ample divisor A , the semigroup $\Gamma_{Y_\bullet}(A)$ is finitely generated. We can now pose the following problem.

Question 8.2. Let X be a Mori dream space. Does there always exist a flag Y_\bullet such that the corresponding semigroup $\Gamma_{Y_\bullet}(X)$ is finitely generated?

8.3. Toric degenerations. As the finite generation of the global semigroup $\Gamma_{Y_\bullet}(X)$ is an interesting question per se, we believe that, in analogy to the fact that the finite generation of the semigroups $\Gamma_{Y_\bullet}(D)$ induce toric degenerations of X to $\Delta_{Y_\bullet}(D)$ ([A13]), the following holds.

Conjecture 8.3. Let X be a Mori dream space. Then the finite generation of $\Gamma_{Y_\bullet}(X)$ induces a degeneration of $\text{Spec}(\text{Cox}(X))$, which is compatible with the toric degenerations of $\Gamma_{Y_\bullet}(D)$, considered in [A13].

8.4. Normality of the semigroup. We have seen in Proposition 6.13 that the normality of the semigroup $\Gamma_{Y_\bullet}(X_w)$ is connected to the existence of integral Zariski decompositions. We have also seen in the last section that for our example the variety Y induces integral Zariski decompositions. It is also not difficult to prove that the four dimensional example X from Section 5.2, induces integral Zariski decompositions. It is now natural to ask the following question.

Question 8.4. Under which circumstances does the Bott-Samelson variety X_w induce integral Zariski decompositions?

REFERENCES

- [A13] Anderson, D., *Okounkov bodies and toric degenerations*. Math. Ann. **356** (3) (2013), 1183–1202
- [A15] Anderson, D., *Effective divisors on Bott-Samelson varieties* preprint 2015, arXiv:1501.00034
- [BG09] Bruns, W., Gubeladze, J., *Polytopes, rings, and K-theory*, Springer, Dordrecht, (2009).
- [BKS04] Bauer, T., Küronya, A., Szemberg, T., *Zariski chambers, volumes, and stable base loci*, J. Reine Angew. Math. **576** (2004), 209–233
- [D74] Demazure, M., *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. Éc. Norm. Sup. **7** (1974), 53–88
- [FeFoL17] Feigin, E., Fourier, G., Littelmann, P., *Favourable modules: Filtration, polytopes, Newton-Okounkov bodies and flat degenerations*, Transformation Groups **22** (2017), 2, 321–352
- [Fu15] Fujita, N., *Newton-Okounkov bodies for Bott-Samelson varieties and string polytopes for generalized Demazure modules*, preprint 2015, arXiv:1503.08916
- [HaKa15] Harada, M., Kaveh, K., *Integrable systems, toric degenerations and Okounkov bodies*, Invent. Math. **202**(3) (2015), 927–985
- [HaY15] Harada, M., Yang, J., *Newton-Okounkov bodies of Bott-Samelson varieties and Grossberg-Karshon twisted cubes*, preprint 2015, arXiv:1504.00982
- [HuKe00] Hu, Y., Keel, S., *Mori Dream Spaces and GIT*, Michigan Math. J. **48** (2000), 331–348
- [Ka15] Kaveh, K., *Crystal bases and Newton-Okounkov bodies*, Duke Math. J. **164** (2015), 13, 2461–2506
- [KaKo12] Kaveh, K., Khovanskii, A.G., *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), 2, 925–978
- [Ki15] Kiritchenko, V., *Newton-Okounkov polytopes of flag varieties*, preprint, arxiv.org/abs/1506.00362
- [KüL15] Küronya, A., Lozovanu, V., *Positivity of line bundles and Newton-Okounkov bodies*, preprint 2015, arXiv:1506.06525
- [KM98] Kollár J., Mori, Sh., *Birational Geometry of Varieties*, Cambridge Tracts Math. 134, Cambridge University Press, 1998
- [LT04] Lauritzen, N., Thomsen, J.F., *Line bundles on Bott-Samelson varieties*, J. Alg. Geom. **13** (2004), 461–473
- [La04] Lazarsfeld, R., *Positivity in Algebraic Geometry I*, Springer, 2004
- [LM09] Lazarsfeld, R., Mustață, M., *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 783–835
- [Li98] Littelmann, P., *Cones, crystals, and patterns*, Transform. Groups **3** (1998), 2, 145–179
- [N04] Nakayama, N., *Zariski-decomposition and abundance*, Mathematical Society of Japan, Tokyo (2004)
- [O16] Okawa, S., *On images of Mori dream spaces*, Math. Ann. **364** (2016), no. 3–4, 1315–1342

- [PU16] Postingshel, E., Urbinati, S., *Newton-Okounkov bodies and toric generations of Mori dream spaces by tropical compactifications*, preprint, 2016, arXiv:1612.03861
- [SS16] Schmitz, D., Seppänen, H., *On the polyhedrality of global Okounkov bodies*, Adv. Geom. **16** (2016),1, 83–91
- [SS17] Schmitz, D., Seppänen, H., *Global Okounkov bodies for Bott-Samelson varieties*, Journal of Algebra **490** (2017), 518–554

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ON NEWTON-OKOUNKOV BODIES OF GRADED LINEAR SERIES

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ABSTRACT. We generalize the theory of Newton-Okounkov bodies of big divisors to the case of graded linear series. One of the results is the generalization of slice formulas and the existence of generic Newton-Okounkov bodies for birational graded linear series. We also give a characterization of graded linear series which have full volume in terms of their base locus.

1. INTRODUCTION

The beautiful paper [O96] by Andrei Okounkov began the theory of Newton-Okounkov bodies. Originally, he was interested in asymptotic multiplicities of group representations on line bundles. To study this, he constructed a compact convex set Δ and noticed that the volume of this body can be interpreted as the asymptotic multiplicity of the given representation. Inspired by that work, Kaveh and Khovanskii in [KK12] and independently Lazarsfeld and Mustaa in [LM09] realized that Okounkov's construction gave rise to a completely new approach for studying the asymptotics of linear series on a projective variety. They used Okounkov's construction to associate a convex body $\Delta(S_\bullet)$ to a graded linear series S_\bullet on a projective variety X . The construction of this body does not only depend on S_\bullet , but also on the choice of a flag Y_\bullet consisting of closed irreducible subvarieties. However, the main feature of the convex body $\Delta_{Y_\bullet}(S_\bullet)$ is that, under "mild" conditions, its euclidean volume gives a geometric interpretation of the classical notion of the volume of a graded linear series S_\bullet [LM09, Theorem A]. Indeed, for a graded linear series S_\bullet on a variety X of dimension d one has

$$\text{vol}(\Delta_{Y_\bullet}(S_\bullet)) = \lim_{k \rightarrow \infty} \frac{\dim S_k}{k^n} = d! \cdot \text{vol}(S_\bullet).$$

A posteriori, one concludes that the volume of $\Delta_{Y_\bullet}(S_\bullet)$ is independent of the choice of the flag Y_\bullet .

For a complete graded linear series S_\bullet equal to the full section algebra

$$R_\bullet(X, D) := \bigoplus_{k \in \mathbb{N}} H^0(X, \mathcal{O}_X(kD))$$

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associated to a big divisor D , we get a very nice correspondence between algebraic geometry and convex geometry. Although, a priori, the Newton-Okounkov body of a divisor D is well-defined up to linear equivalence of D , Lazarsfeld and Mustață [LM09, Proposition 4.1] showed that the construction of the Newton-Okounkov body does only depend on the numerical equivalence class. Conversely, Jow showed in [J10] that two divisors D and D' are numerical equivalent if the Newton-Okounkov bodies $\Delta_{Y_\bullet}(D)$ and $\Delta_{Y_\bullet}(D')$ coincide for all flags Y_\bullet . Philosophically, this means that we can interpret a numerical equivalence class of a divisor D as a collection of real convex bodies parametrized by the set of all flags. Therefore, in principle, it should be possible to translate all numerical properties of a divisor into properties of convex geometry and vice versa. An outline of this approach is given in recent works of Küronya and Lozovanu (see [KL14],[KL15] and [KL17]).

Even though Lazarsfeld and Mustață tried to build up their theory of Newton-Okounkov bodies in a general setting based on graded linear series, many statements are only formulated for complete graded linear series corresponding to a big divisor D , i.e. for $R(S_\bullet) = R_\bullet(X, D)$. The reason why this case is significantly easier to understand is due to the following facts which are not shared by arbitrary graded linear series.

- The algebra $R_\bullet(X, D)$ is induced by the locally free sheaves $\mathcal{O}_X(kD)$.
- The body $\Delta_{Y_\bullet}(D)$ is well defined for a numerical equivalence class of D which can be interpreted as a point in a finite dimensional vector space over \mathbb{R} , the Néron-Severi space.
- There exists a global Newton-Okounkov body which characterizes all Newton-Okounkov bodies $\Delta_{Y_\bullet}(D)$ at once [LM09, Theorem B].

The two main features of Newton-Okounkov bodies which were proved in the case $S_\bullet = R_\bullet(X, D)$ but were left open for arbitrary graded linear series are the following .

- (a) *Slice formulas for Newton-Okounkov bodies:* Let $t \geq 0$ be a rational number. Let Y_\bullet be a flag such that Y_1 is a Cartier divisor. Suppose D is a big Cartier divisor such that $Y_1 \not\subseteq \mathbf{B}_+(D)$ and $D - t \cdot Y_1$ is big. Then the t -slice

$$\Delta_{Y_\bullet}(D)_{\nu_1=t} := \Delta_{Y_\bullet}(D) \cap \left(\{t\} \times \mathbb{R}^{d-1} \right)$$

is equal to the Newton-Okounkov body of the restricted linear series [LM09, Theorem 4.24]

$$R_\bullet(X, D - tY_1)|_{Y_1}.$$

- (b) *Existence of a generic Newton-Okounkov body:* If we have a family of Newton-Okounkov bodies $\Delta_{Y_{t,\bullet}}(X_t, D_t)$ where all the relevant data move in flat families over T , then for a very general choice of $t \in T$ the Newton-Okounkov bodies all coincide [LM09, Theorem 5.1].

Question. *Is there a natural generalization of the above properties for more general graded linear series S_\bullet ?*

We will prove that property (a) for rational $t > 0$ does hold for a completely arbitrary graded linear series S_\bullet corresponding to a big divisor.

Theorem A. *Let S_\bullet be a graded linear series. Let Y_\bullet be an admissible flag and $t = a/b > 0$ for $(a, b) = 1$ a rational number such that $\{t\} \times \mathbb{R}^{d-1}$ meets the interior of $\Delta_{Y_\bullet}(S_\bullet)$. Then*

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=t} = 1/b \cdot \Delta_{X|Y_1}(S_\bullet^{(b)} - aY_1)$$

via the identification of $\{t\} \times \mathbb{R}^{d-1} \cong \mathbb{R}^{d-1}$.

However, for $t = 0$ we need some more constraints on S_\bullet . It turns out that a natural assumption in order to treat S_\bullet “like” a divisor D is to assume that their volumes are equal, i.e. $\text{vol}(S_\bullet) = \text{vol}(D)$. Another assumption which is necessary to prevent S_\bullet from being too wild is that it should be finitely generated as an algebra. We will prove the following characterization of such graded linear series.

Theorem B. *Let $S_\bullet \subseteq R(X, D)$ be finitely generated graded linear series corresponding to a big divisor D . Then the following two conditions are equivalent*

- (a) $\text{vol}(S_\bullet) = \text{vol}(D)$.
- (b)
 - The rational map $h_{S_\bullet}: X \dashrightarrow \text{Proj}(S_\bullet)$ is birational and
 - $\mathbf{B}(S_\bullet) = \emptyset$ on $\text{Proj}(R(X, D))$.

The above result enables us to derive a slice formula for $t = 0$ for such graded linear series (see Theorem 4.18). However, we are even able to derive the following slice theorem for graded linear series containing an ample series.

Theorem C. *Let S_\bullet be a graded linear series containing the ample series $D - E$. Let Y_\bullet be an admissible flag such that the divisorial component Y_1 is not contained in E and $Y_d \notin \mathbf{B}(S_\bullet)$. Then we have*

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{X|Y_1}(S_\bullet).$$

The existence of a generic Newton-Okounkov body, according to Part (b), can be generalized to the case of birational graded linear series S_\bullet . More precisely, we derive the following theorem.

Theorem D. *Let X, T and \mathcal{Y}_\bullet be as in Section 5.1. Let S_\bullet be a birational graded linear series. Then for a very general choice of $t \in T$ all the Newton-Okounkov bodies $\Delta_{Y_{t,\bullet}}(S_\bullet)$ coincide.*

Note that we do not need any flatness hypothesis for the above theorem. The idea of the proof is that we replace the graded linear series S_\bullet by a possibly larger one which is induced by coherent sheaves and then use the generic flatness theorem to make sure all the involved data is flat. In addition

to proving the existence of generic Newton-Okounkov bodies, we give several examples of how to construct such families.

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2. PRELIMINARIES

2.1. Notation. We work over the field \mathbb{C} . Whenever not otherwise stated, X denotes a projective variety over \mathbb{C} of dimension d and D a big Cartier divisor over X . Moreover, when we talk of a divisor, we will always refer to an integral Cartier divisor. By an admissible flag Y_\bullet of X we mean an ordered set of irreducible subvarieties:

$$Y_\bullet: X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{d-1} \supseteq Y_d = \{pt\}$$

where d is the dimension of X such that $\text{codim}_X(Y_i) = i$ and each Y_i is non-singular at the point Y_d .

2.2. Graded linear series. Let D be a divisor on X and let $V \subseteq H^0(X, \mathcal{O}_X(D))$ be a non-zero vector subspace. Then the projective space of one dimensional vector subspaces of V , which we denote by $|V| := \mathbb{P}(V)$, is called a linear series. In case $V = H^0(X, \mathcal{O}_X(D))$ we call it the complete linear series and write $|D|$. Often we are interested in the asymptotic behaviour of $|mD|$ as $m \rightarrow \infty$. In order to generalize this for non complete linear series we need the following definition.

Definition 2.1. A *graded linear series* on X corresponding to a divisor D consists of a collection

$$S_\bullet = \{S_k\}_{k \geq 0}$$

of finite dimensional vector subspaces $S_m \subseteq H^0(X, \mathcal{O}_X(mD))$ and $S_0 = \mathbb{C}$. These subspaces are required to satisfy the property

$$S_k \cdot S_l \subseteq S_{k+l} \quad \text{for all } k, l \geq 0$$

where $S_k \cdot S_l$ denotes the image of $S_k \otimes S_l$ under the homomorphism

$$H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(lD)) \rightarrow H^0(X, \mathcal{O}_X((k+l)D)).$$

We call S_\bullet a complete linear series if $S_k = H^0(X, \mathcal{O}_X(kD))$ for all $k \geq 0$.

Given a graded linear series S_\bullet we can define the *graded algebra*

$$R(S_\bullet) := \bigoplus_{k=0}^{\infty} S_k.$$

If S_\bullet is a complete graded linear series corresponding to D , we write $R_\bullet(X, D)$ for the graded algebra of sections. We say that S_\bullet is *finitely generated* if $R(S_\bullet)$ is finitely generated as a \mathbb{C} -algebra.

A linear series $|V|$ corresponds to a rational morphism

$$h_{|V|}: X \dashrightarrow \mathbb{P}^N.$$

Definition 2.2. We say that a graded linear series S_\bullet is *birational* if the rational map corresponding to the linear series $|S_k|$

$$h_{|S_k|}: X \dashrightarrow \mathbb{P}^N$$

is birational onto its image for $k \gg 0$.

If S_\bullet is finitely generated, then S_\bullet is birational if and only if the induced rational map

$$h_{S_\bullet}: X \dashrightarrow \text{Proj}(S_\bullet) := \text{Proj}(R(S_\bullet))$$

is birational.

If S_\bullet and T_\bullet are two graded linear series, we write $S_\bullet \subseteq T_\bullet$ if $S_k \subseteq T_k$ for all $k \geq 0$.

Finally, note that many notions as the volume or the stable base locus can be defined for graded linear series completely analogously as in the case of complete graded linear series (see [L04, 2.4] for more details).

2.3. Construction of Newton-Okounkov bodies. In this section we want to give a very brief overview of the construction of Newton-Okounkov bodies and state some elementary facts about them. For a detailed overview see e.g. [LM09].

First of all we fix an admissible flag Y_\bullet and a graded linear series S_\bullet of X . Now by an iterative procedure, taking the order of vanishing along the given Y_i into account, we construct for each $k \in \mathbb{N}$ a valuation map

$$\nu_{Y_\bullet}: S_k \setminus \{0\} \rightarrow \mathbb{Z}^d.$$

The two essential properties of ν_{Y_\bullet} are:

- ordering \mathbb{Z}^d lexicographically, we have

$$\nu_{Y_\bullet}(s_1 + s_2) \geq \min\{\nu_{Y_\bullet}(s_1), \nu_{Y_\bullet}(s_2)\}$$

for any $s_1, s_2 \in S_k \setminus \{0\}$

- given two non zero sections $s \in S_k$ and $t \in S_l$ then

$$\nu_{Y_\bullet}(s \otimes t) = \nu_{Y_\bullet}(s) + \nu_{Y_\bullet}(t).$$

The valuation function gives rise to the semigroup

$$\Gamma(S_\bullet) := \{(\nu_{Y_\bullet}(s), k) : s \in S_k \setminus \{0\}, k \in \mathbb{N}\} \subseteq \mathbb{N}^{d+1}.$$

Then the Newton-Okounkov body of S_\bullet corresponding to the flag Y_\bullet is given by

$$\Delta_{Y_\bullet}(S_\bullet) := \overline{\text{Cone}(\Gamma(S_\bullet))} \cap (\mathbb{R}^d \times \{1\}).$$

In [LM09] it was shown that for a graded linear series S_\bullet corresponding to a big divisor D which has the additional property that the semigroup $\Gamma(S_\bullet)$ generates \mathbb{Z}^{d+1} as a group, we have

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta_{Y_\bullet}(S_\bullet)) = \frac{1}{d!} \cdot \mathrm{vol}(S_\bullet)$$

where

$$\mathrm{vol}(S_\bullet) := \lim_{k \rightarrow \infty} \frac{\dim S_k}{k^d/d!}.$$

However, the more general case was treated in [KK12]. They showed that for an arbitrary graded linear series S_\bullet , we have

$$(1) \quad \frac{\mathrm{vol}(\Delta_{Y_\bullet}(S_\bullet))}{\mathrm{ind}(S_\bullet)} = \frac{\mathrm{vol}(S_\bullet)}{d!}$$

where $\mathrm{ind}(S_\bullet)$ is the index of the group generated by $\Gamma(S_\bullet)$ in \mathbb{Z}^d .

So for arbitrary graded linear series the volume of the Newton-Okounkov body does indeed depend on the choice of the flag.

3. VOLUME AND BASE LOCUS OF GRADED LINEAR SERIES

In this section we want to analyze the correspondence between the volume of a graded linear series and its base locus. We first focus on the case where the graded linear series S_\bullet corresponding to D has full volume, i.e.

$$\mathrm{vol}(S_\bullet) = \mathrm{vol}(D).$$

In this case we have a characterization of finitely generated graded linear series S_\bullet given by Theorem 3.8. This characterization will help us to make sense of the sheafification of a graded linear series, which will be necessary for Section 5, as well as for deriving slice formulas in the following section.

3.1. Stable base locus and volume of finitely generated graded linear series. The aim of this paragraph is to show that two finitely generated graded linear series which have the same volume also have the same stable base locus.

The following proposition will be helpful.

Proposition 3.1. *Let S_\bullet be a graded linear series and $x \in X$. Consider the induced graded linear series W_\bullet defined by*

$$W_k := \{s \in S_k : \mathrm{ord}_x(s) \geq \lceil kr \rceil\}$$

for some fixed $r > 0$. Then for all admissible flags Y_\bullet centered at x , the origin 0 does not lie in the Newton-Okounkov body $\Delta_{Y_\bullet}(W_\bullet)$.

Proof. Let us assume that 0 lies in the Newton-Okounkov body. Then it must be an extreme point. Thus there exists a series of sections $s^k \in W_k$

such that $1/k \cdot \nu(s^k) = 1/k \cdot (\nu_1(s^k), \dots, \nu_d(s^k))$ converges to 0 as k tends to infinity. By [KL15, Lemma 2.4], we have

$$\text{ord}_x(s^k) \leq \sum_{i=1}^d \nu_i(s^k).$$

Dividing by k leads to

$$1/k \cdot \text{ord}_x(s^k) \leq 1/k \cdot \sum_{i=1}^d \nu_i(s^k).$$

As k tends to infinity the right hand side goes to 0, but the left hand side is lower bounded by $r > 0$, which gives a contradiction. Thus 0 does not lie in the Newton-Okounkov body $\Delta_{Y_\bullet}(W_\bullet)$. \square

The next theorem is an analog of [KL15] Theorem A for finitely generated graded linear series.

Theorem 3.2. *Let X be smooth and let S_\bullet be a finitely generated graded linear series. Then the following conditions are equivalent:*

- (a) $x \notin \mathbf{B}(S_\bullet)$
- (b) $0 \in \Delta_{Y_\bullet}(S_\bullet)$ for each admissible flag Y_\bullet centered at $\{x\}$
- (c) There exists an admissible flag Y_\bullet centered at x such that $0 \in \Delta_{Y_\bullet}(S_\bullet)$.

Proof. (a) \rightarrow (b) is trivial, since $\nu_{Y_\bullet}(s) = 0$ for all sections $s \in S_k$ that do not vanish at x . (b) \rightarrow (c) is also trivial.

Let us prove (c) \rightarrow (a). Let 0 lie in the Newton-Okounkov body $\Delta_{Y_\bullet}(S_\bullet)$ and let us assume that $x \in \mathbf{B}(S_\bullet)$. Let s_1, \dots, s_n be homogeneous generators of $R(S_\bullet)$ and denote by N the maximum of the degrees of these generators. By assumption, all these generators vanish at x to order at least one. Thus, we have an inclusion $S_\bullet \subseteq W_\bullet$ of graded linear series, where

$$W_k := \{s \in H^0(X, \mathcal{O}_X(kD)) : \text{ord}_x(s) \geq \lceil k/N \rceil\}.$$

But by the previous proposition, $0 \notin \Delta(W_k)$. This contradicts the fact that $0 \in \Delta_{Y_\bullet}(S_\bullet) \subseteq \Delta_{Y_\bullet}(W_\bullet)$. Thus $x \notin \mathbf{B}(S_\bullet)$. \square

Lemma 3.3. *Let $S_\bullet \subseteq T_\bullet$ be two graded linear series. Then $\text{vol}(S_\bullet) = \text{vol}(T_\bullet) \neq 0$ implies that $\Delta_{Y_\bullet}(S_\bullet) = \Delta_{Y_\bullet}(T_\bullet)$ for all admissible flags Y_\bullet .*

Proof. First of all, we show that for all admissible flags the volume of the Newton-Okounkov bodies coincide. We will do this by showing that the indices of the semigroups $\Gamma(S_\bullet)$ and $\Gamma(T_\bullet)$ are equal. Clearly, $\Gamma(S_\bullet) \subseteq \Gamma(T_\bullet)$ and hence $\text{ind}(S_\bullet) \geq \text{ind}(T_\bullet)$. On the other hand, the volume formula for Newton-Okounkov bodies yields the equality

$$d! \cdot \text{vol}(S_\bullet) = \frac{\text{vol}(\Delta_{Y_\bullet}(S_\bullet))}{\text{ind}(S_\bullet)} = \frac{\text{vol}(\Delta_{Y_\bullet}(T_\bullet))}{\text{ind}(T_\bullet)} = d! \cdot \text{vol}(T_\bullet).$$

From this equality and the fact that $\text{vol}(\Delta_{Y_\bullet}(S_\bullet)) \leq \text{vol}(\Delta_{Y_\bullet}(T_\bullet))$ we get that $\text{ind}(S_\bullet) \leq \text{ind}(T_\bullet)$, which implies $\text{ind}(S_\bullet) = \text{ind}(T_\bullet)$. Again from

the volume formula, we deduce that the volume of the Newton-Okounkov bodies are equal. Let us now assume that there is a flag Y_\bullet such that $\Delta_{Y_\bullet}(S_\bullet) \subsetneq \Delta_{Y_\bullet}(T_\bullet)$. Then there is a point P in $\Delta_{Y_\bullet}(T_\bullet)$ which does not lie in $\Delta_{Y_\bullet}(S_\bullet)$. Since $\Delta_{Y_\bullet}(S_\bullet)$ is closed and convex, the point P has a positive distance to $\Delta_{Y_\bullet}(S_\bullet)$. Hence, there is a d -dimensional ball $B(0, \varepsilon)$ around the origin which does not intersect $\Delta_{Y_\bullet}(S_\bullet)$. The intersection of $B(0, \varepsilon)$ with $\Delta_{Y_\bullet}(T_\bullet)$ has positive volume. This shows that we cannot have $\text{vol}(\Delta_{Y_\bullet}(S_\bullet)) = \text{vol}(\Delta_{Y_\bullet}(T_\bullet))$. \square

For the next lemma we will need the definition of a pulled back linear series. Let $\pi: X \rightarrow Y$ be a morphism of projective varieties and S_\bullet a graded linear series on Y . Then we can define π^*S_\bullet by $\pi^*S_k := \{\pi^*s : s \in S_k\}$.

Lemma 3.4. *Let $\pi: X \rightarrow Y$ be a surjective morphism of projective varieties. Let S_\bullet be a graded linear series on Y . Then we have*

$$\pi(\mathbf{B}(\pi^*S_\bullet)) = \mathbf{B}(S_\bullet)$$

Proof. Let $x \in \mathbf{B}(\pi^*S_\bullet)$ this is equivalent to

$$\pi^*s(x) = s(\pi(x)) = 0$$

for all $k \geq 0$ and $s \in S_k$. But this is equivalent to $\pi(x) \in \mathbf{B}(S_\bullet)$. Hence, we have the desired result. \square

Theorem 3.5. *Let $S_\bullet \subseteq T_\bullet$ be two graded linear series and let S_\bullet be finitely generated. Then $\text{vol}(S_\bullet) = \text{vol}(T_\bullet)$ implies that $\mathbf{B}(S_\bullet) = \mathbf{B}(T_\bullet)$.*

Proof. Let us first assume that X is smooth. It is obvious that $\mathbf{B}(T_\bullet) \subseteq \mathbf{B}(S_\bullet)$. Let us show the other inclusion. Let therefore $x \in \mathbf{B}(S_\bullet)$ and assume that x does not lie in $\mathbf{B}(T_\bullet)$. Then for all admissible flags Y_\bullet centered at $\{x\}$ we have that $0 \in \Delta_{Y_\bullet}(T_\bullet)$. By Theorem 3.2, we know that $0 \notin \Delta_{Y_\bullet}(S_\bullet)$. Using Lemma 3.3 implies that $\Delta_{Y_\bullet}(S_\bullet) = \Delta_{Y_\bullet}(T_\bullet)$, which gives a contradiction. Thus x does lie in $\mathbf{B}(T_\bullet)$.

Now consider the case where X is not necessarily smooth and $\pi: \tilde{X} \rightarrow X$ is a resolution of singularities. Since we have bijection of sections $\pi^*S_k \cong S_k$ and $\pi^*T_k \cong T_k$ we conclude $\text{vol}(\pi^*T_\bullet) = \text{vol}(\pi^*S_\bullet)$. Since \tilde{X} is smooth we can deduce that $\mathbf{B}(\pi^*S_\bullet) = \mathbf{B}(\pi^*T_\bullet)$ and from the above lemma the desired result follows. \square

The next example illustrates that the assumption for S_\bullet to be finitely generated is indeed necessary.

Example 3.6. Let S_\bullet be an arbitrary graded linear series such that $\text{vol}(S_\bullet) > 0$. Then choose any point $x \in X \setminus \mathbf{B}(S_\bullet)$ and consider the graded linear series S_\bullet^x defined by

$$S_k^x := \{s \in S_k : s(x) = 0\}.$$

Clearly, we have $\mathbf{B}(S_\bullet) \neq \mathbf{B}(S_\bullet^x)$ since x is not contained in the first set but is contained in the latter by construction. We will nevertheless show

that $\text{vol}(S_\bullet) = \text{vol}(S_\bullet^x)$, and can therefore conclude that S_\bullet^x is never finitely generated, even though S_\bullet might be.

In order to prove the equality of volumes, we first show equality of Newton-Okounkov bodies. Let Y_\bullet be any admissible flag. We surely have an inclusion $\Delta_{Y_\bullet}(S_\bullet^x) \subseteq \Delta_{Y_\bullet}(S_\bullet)$. Let now P be a point in $\Delta_{Y_\bullet}(S_\bullet)$. Using the fact that the valuation points in the Newton-Okounkov body are dense (see Theorem 4.5), there is a series of sections $(\xi^{m_i})_{i \in \mathbb{N}}$ such that $\xi^{m_i} \in S_{m_i}$ and

$$\frac{\nu_{Y_\bullet}(\xi^{m_i})}{m_i} \rightarrow P \quad \text{as } m_i \rightarrow \infty.$$

Now let $s \in S_k$ be any section such that $s(x) = 0$. The existence of such a section follows from the fact that $\text{vol}(S_\bullet) > 0$. Suppose otherwise that no section $s \in S_k$ vanishes at $x \in X$, then $\Delta_{Y_\bullet}(S_\bullet) = \{0\}$ is the origin for all flags Y_\bullet centered at $\{x\}$.

Consider the series $(s \otimes \xi^{m_i})_{i \in \mathbb{N}}$ for which we have:

$$\frac{\nu_{Y_\bullet}(s \otimes \xi^{m_i})}{k + m_i} = \frac{\nu_{Y_\bullet}(s)}{k + m_i} + \frac{\nu_{Y_\bullet}(\xi^{m_i})}{k + m_i} \rightarrow P \quad \text{as } m_i \rightarrow \infty.$$

But since $s \otimes \xi^{m_i} \in S_{m_i+k}^x$ we conclude the equality of the Newton-Okounkov bodies $\Delta_{Y_\bullet}(S_\bullet^x) = \Delta_{Y_\bullet}(S_\bullet)$. In order to derive the equality of volumes, we will show that for both graded linear series the group generated by the semigroup of valuation points coincide. We trivially have an inclusion $G(\Gamma(S_\bullet^x)) \subseteq G(\Gamma(S_\bullet))$. Now let $a \in G(\Gamma(S_\bullet))$ be an arbitrary element. We can write it as

$$a = (\nu_{Y_\bullet}(\xi^1), k_1) - (\nu_{Y_\bullet}(\xi^2), k_2)$$

for some $\xi^i \in S_{k_i}$, $i = 1, 2$. Choose again a section $s \in S_k$ such that $s(x) = 0$ and note that $s \otimes \xi^i \in S_{k_i+k}^x$ for $i = 1, 2$. Then we can write

$$\begin{aligned} a' &:= (\nu_{Y_\bullet}(s \otimes \xi^1), k + k_1) - (\nu_{Y_\bullet}(s \otimes \xi^2), k + k_2) \\ &= (\nu_{Y_\bullet}(s), k) + (\nu_{Y_\bullet}(\xi^1), k_1) - ((\nu_{Y_\bullet}(s), k) + (\nu_{Y_\bullet}(\xi^2), k_2)) \\ &= a. \end{aligned}$$

But $a' \in G(\Gamma(S_\bullet^x))$ which implies the equality of both groups and in particular the equality of both indices $\text{ind}(S_\bullet^x) = \text{ind}(S_\bullet)$. Applying the volume formula (1), we get the desired equality of volumes.

3.2. Characterization of finitely generated graded linear series with full volume. In this paragraph we want to classify all finitely generated graded linear series S_\bullet corresponding to a big divisor D such that $\text{vol}(S_\bullet) = \text{vol}(D)$.

We will need the following lemma.

Lemma 3.7. *Let $f: X \rightarrow Y$ be a dominant finite morphism of varieties. Then f is of degree one if and only if f is birational.*

Proof. If f is birational, then there are open subsets $U = \text{Spec } B \subseteq X$ and $V = \text{Spec } A \subseteq Y$ such that $f|_U: U \rightarrow V$ is an isomorphism. This means that $A \cong B$ and thus $\mathbb{C}(X) = \text{Quot}(B) \cong \text{Quot}(A) = \mathbb{C}(Y)$. Hence, f is of degree one.

Now let f be finite of degree one. Consider an open affine subscheme $V = \text{Spec } A \subseteq Y$ such that $f^{-1}(V) = U = \text{Spec } B$. Then $f|_U: \text{Spec } B \rightarrow \text{Spec } A$ corresponds to an injective morphism of rings $\phi: A \rightarrow B$. Since f has degree one, ϕ induces an isomorphism

$$\text{Quot}(\phi): \text{Quot}(A) \rightarrow \text{Quot}(B).$$

Suppose $b_1, \dots, b_n \in B$ is a set of generators of B as an A -module. Let $a'_1, \dots, a'_n \in A$ be the set of denominators of the preimages of the b_i under the isomorphism $\text{Quot}(\phi)$. Let $a' = a'_1 \cdots a'_n$ be the product of the denominators and $A_{a'}$ be the corresponding localization. Next, we consider the morphism $\text{Quot}(\phi)$ restricted to the A -module A_a

$$\text{Quot}(\phi)|_{A_{a'}}: A_{a'} \rightarrow \text{Quot}(B).$$

By construction, this restriction gives an isomorphism of $A_{a'}$ to its image which is exactly $B_{a'}$. Applying the Spec functor again gives an isomorphism of schemes

$$f|_{U'}: U' = \text{Spec } B_{a'} \rightarrow V' = \text{Spec}(A_{a'}) = D(a').$$

Hence, f is a birational morphism. \square

Theorem 3.8. *Let $S_\bullet \subseteq T_\bullet$ be finitely generated graded linear series corresponding to D . Let T_\bullet be birational. Then the following two conditions are equivalent*

- (a) $\text{vol}(S_\bullet) = \text{vol}(T_\bullet)$.
- (b)
 - The rational map $h_{S_\bullet}: X \dashrightarrow \text{Proj}(S_\bullet)$ is birational and
 - $\mathbf{B}(S_\bullet) = \emptyset$ on $\text{Proj}(T_\bullet)$.

Proof. Consider the rational morphism corresponding to the section ring $R(T_\bullet)$

$$h_{T_\bullet}: X \dashrightarrow \text{Proj}(T_\bullet) =: Y.$$

This is a rational contraction and we have $h_{T_\bullet}^*(\mathcal{O}_Y(1)) = D$ as well as $R(T_\bullet) \cong R_\bullet(Y, \mathcal{O}_Y(1))$. Via this bijection, we may regard S_\bullet as a finitely generated graded linear subseries of $R_\bullet(Y, \mathcal{O}_Y(1))$ on Y . Since $\text{vol}(T_\bullet) = \text{vol}(\mathcal{O}_Y(1))$, we can deduce from Theorem 3.5, that $\text{vol}(S_\bullet) = \text{vol}(T_\bullet)$ implies that S_\bullet is base point free on Y . Therefore, we just need to show that (a) is equivalent to the first part of (b) under the assumption that $\text{vol}(S_\bullet) = \text{vol}(T_\bullet)$. So let us assume that S_\bullet is base point free on Y . We want to show that the inclusion $\phi: R(S_\bullet) \rightarrow R(T_\bullet)$ induces a morphism $Y \rightarrow \text{Proj}(S_\bullet) =: Z$. Due to [GW, Remark 13.7], the previous inclusion gives us a morphism $G(\phi) \rightarrow \text{Proj}(S_\bullet)$ where

$$G(\phi) := \bigcup_{s \in S_k, k > 0} D_+(s) \subseteq Y.$$

But by the base point freeness of S_\bullet on Y we have $G(\phi) = Y$. Hence, we get a globally defined morphism

$$j: Y \rightarrow \text{Proj}(S_\bullet) = Z,$$

which fits into the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h_{T_\bullet}} & Y \\ & \searrow h_{S_\bullet} & \downarrow j \\ & & Z. \end{array}$$

Since h_{T_\bullet} is birational, the above diagram implies that j is birational if and only if h_{S_\bullet} is birational. The morphism j is by construction affine and projective and thus finite. By Lemma 3.7 and taking the above equivalence into account, we get that h_{S_\bullet} is birational if and only if j is finite of degree one. But this is equivalent to

$$\text{vol}(T_\bullet) = \text{vol}(j^* \mathcal{O}_Z(1)) = \text{vol}(\mathcal{O}_Z(1)) = \text{vol}(S_\bullet).$$

□

Corollary 3.9. *Let D be a semi ample big divisor on X and S_\bullet be a finitely generated graded linear series corresponding to D . Then the following two conditions are equivalent*

- (a) $\text{vol}(S_\bullet) = \text{vol}(D)$
- (b)
 - $\mathbf{B}(S_\bullet) = \mathbf{B}(D) = \emptyset$
 - $h_{S_\bullet}: X \rightarrow \text{Proj}(S_\bullet)$ is birational.

Proof. The only thing which we need to prove in order to use the above Theorem is that b) implies that $\mathbf{B}(S_\bullet) = \emptyset$ on $\text{Proj}(R_\bullet(X, D))$ but this follows from Lemma 3.4.

□

3.3. Volume and base ideal. In this paragraph we want to take the scheme structure of the base locus into account. Hence, we will be interested in the connection between the volume and the base ideal of a graded linear series. The main motivation for this is [J10, Theorem C], which states that we can compute the volume of a birational graded linear series by passing to the base point free linear series on the blow-up along the base ideal. We will give a short introduction into base ideals and then derive some variations of Jow's statement.

Let $|V|$ be a linear series corresponding to a divisor D on X . Let $s_1, \dots, s_N \in V$ be global sections which induce a basis on $|V|$. Then we define \mathcal{F} as the coherent sheaf generated by the s_i . More explicitly, \mathcal{F} is the image sheaf of the following morphism:

$$\mathcal{O}_X^N \rightarrow \mathcal{O}_X(D)$$

which is given on open sets $U \subseteq X$ by

$$(\lambda_1, \dots, \lambda_N) \mapsto \sum_{j=1}^N \lambda_j \cdot s_j|_U.$$

Consider an open cover $X = \bigcup_{i \in I} U_i$ and trivializations $\psi_i: \mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}_{U_i}$. Then $\psi_i(\mathcal{F}|_{U_i})$ is an ideal of \mathcal{O}_{U_i} , which does not depend on the choice of ψ_i . Gluing these ideals together induces a well defined ideal sheaf \mathcal{I} such that $\mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$. Clearly, the support of \mathcal{I} is $\text{Bs}(|V|)$ and we call it the base ideal of $|V|$. This gives the set $\text{Bs}(|V|)$ the structure of a scheme. For a graded linear series S_\bullet , we denote the base ideal of $|S_k|$ by \mathfrak{b}_{S_k} .

Theorem 3.10. *Let S_\bullet and T_\bullet be birational graded linear series corresponding to a divisor D . Suppose $\mathfrak{b}_{S_k} = \mathfrak{b}_{T_k}$ for all $k \gg 0$. Then*

$$\text{vol}(S_\bullet) = \text{vol}(T_\bullet).$$

Proof. An alternate formulation of [J10, Theorem C] is given in [H13, Proposition 3.7]. The volume of S_\bullet can be calculated in the following way. Let $\pi_k: X_k \rightarrow X$ be the blow-up of X along \mathfrak{b}_{S_k} . Let $M_k := \pi_k^*D - E_k$, where E_k is the exceptional divisor of the blow-up. Then

$$\text{vol}(S_\bullet) = \lim_{k \rightarrow \infty} \frac{(M_k)^d}{k^d}.$$

Hence, the volume of S_\bullet just depends on the base ideals \mathfrak{b}_{S_k} for $k \gg 0$ and the divisor D . Therefore we get the same volume for T_\bullet as for S_\bullet . \square

3.4. Rationality properties of finitely generated graded linear series. The volume of a graded linear series can in general behave very wildly (see for example [KLM13]). Without any restrictions, it certainly can be irrational. Indeed, it is an easy consequence of [L04, Ex. 2.4.14] that actually all non-negative real numbers occur as the volume of some graded linear series. However, for a finitely generated divisor D it is shown in [AKL12] that there exists a flag Y_\bullet such that the corresponding Newton-Okounkov body $\Delta_{Y_\bullet}(D)$ is a rational simplex. We will generalize this result to the case of finitely generated birational graded linear series. For finitely generated graded linear series which are not necessarily birational, we recover the well known fact that its volume is rational

Theorem 3.11. *Let S_\bullet be a birational graded linear series generated in degree one. Let $\pi: \tilde{X} \rightarrow X$ be the blow-up of X along the base ideal \mathfrak{b}_{S_1} and E be the exceptional divisor. Let \tilde{Y}_\bullet be an admissible flag of \tilde{X} centered at $\{\tilde{x}\} = \pi^{-1}(x)$ where $x \in X \setminus \mathbf{B}(S_\bullet)$. Let Y_\bullet be the admissible flag centered at $\{x\}$ given as the image of \tilde{Y}_\bullet under π . Then*

$$\Delta_{Y_\bullet}(S_\bullet) = \Delta_{\tilde{Y}_\bullet}(\pi^*D - E).$$

Proof. First of all we show that the volume of both graded linear series are equal. Consider the graded linear series $\pi^*S_\bullet - E$ generated by $\pi^*S_1 -$

$E := \{s/s_E^k : s \in S_1\}$ corresponding to the divisor $\pi^*D - E$. By construction, $\pi^*S_\bullet - E$ is base point free and birational. Hence, due to Theorem 3.10, we have

$$\text{vol}(S_\bullet) = \text{vol}(\pi^*S_\bullet - E) = \text{vol}(\pi^*D - E).$$

The valuation ν_{Y_\bullet} respectively $\nu_{\tilde{Y}_\bullet}$ are both defined locally around x respectively $\tilde{x} = \pi^{-1}(x)$. Since π defines an isomorphism around $\{x\}$, we get $\nu_{Y_\bullet}(s) = \nu_{\tilde{Y}_\bullet}(\pi^*s)$ for all $s \in S_k$. As E lies away from x , we also have $\nu_{\tilde{Y}_\bullet}(\pi^*s/s_E^k) = \nu_{\tilde{Y}_\bullet}(s)$ for all $s \in S_k$ and s_E a defining section of E . This shows that

$$\Delta_{Y_\bullet}(S_\bullet) = \Delta_{\tilde{Y}_\bullet}(\pi^*S_\bullet - E) \subseteq \Delta_{\tilde{Y}_\bullet}(\pi^*D - E).$$

Combining this with the above equality of volumes, we get the desired result. \square

Corollary 3.12. *Let S_\bullet be a finitely generated birational graded linear series. Then there is an admissible flag Y_\bullet such that $\Delta_{Y_\bullet}(S_\bullet)$ is a rational simplex.*

Proof. Using the notation of Theorem 3.11, the only thing we need to prove is that there exists an admissible flag \tilde{Y}_\bullet centered at some point $\tilde{x} \notin E$ such that corresponding Newton Okounkov body of the globally generated divisor $\pi^*D - E$ is a rational simplex. But the existence of such a flag is proven in [AKL12, Proposition 7]. Note, that we used the fact that $\pi^*D - E$ is finitely generated since it is by construction free. \square

If S_\bullet is not birational but still finitely generated, we are not able to prove any rational polyhedrality property of the corresponding Newton-Okounkov body yet. However, the next theorem shows that the volume will nevertheless be rational.

Theorem 3.13. *Let S_\bullet be a finitely generated graded linear series. Then the volume of S_\bullet is a rational number.*

Proof. We may without loss of generality assume that S_\bullet is generated in degree one. The volume of S_\bullet is equal to the volume of the free graded linear series $\pi^*S_\bullet - E$ where π is the blow-up of X along the base ideal of S_1 . Hence, it suffices to show that the volume of a free graded linear series is rational. So let us assume without loss of generality that S_\bullet is a free finitely generated graded linear series generated in degree one on X corresponding to a base point free divisor D which is also generated in degree one. If S_\bullet is free on X , then by Lemma 3.4, S_\bullet is also free on $\text{Proj}(R_\bullet(X, D))$. As in the proof of Theorem 3.8, we conclude that the inclusion $R(S_\bullet) \subseteq R_\bullet(X, D)$ induces a finite morphism:

$$j: Y := \text{Proj}(R_\bullet(X, D)) \rightarrow \text{Proj}(S_\bullet) =: Z.$$

Let k be the degree of j . Then we have

$$\text{vol}(D) = \text{vol}(\mathcal{O}_Y(1)) = \text{vol}(j^*\mathcal{O}_Z(1)) = k \cdot \text{vol}(\mathcal{O}_Z(1)) = k \cdot \text{vol}(S_\bullet).$$

So the rationality of $\text{vol}(S_\bullet)$ follows from the rationality of the volume of the free divisor D . \square

Note, that the fact that the volume of a finitely generated graded linear series, can also be established by realizing that its Hilbert polynomial has indeed rational coefficients.

3.5. Sheafification of graded linear series. The aim of this section is to replace a birational graded linear series S_\bullet by a possibly larger one coming from global sections of coherent subsheaves of $\mathcal{O}_X(kD)$. This sheafification process has the desirable feature that it does not change the volume of the graded linear series and hence not the Newton-Okounkov body.

Let S_\bullet be a birational graded linear series. Denote by \mathfrak{b}_{S_k} the base ideal of the linear series $|S_k|$.

Definition 3.14. The *sheafification* of S_\bullet is given by the sheaf $\mathcal{S}_\bullet = (\mathcal{S}_k)_{k \geq 0}$ where

$$\mathcal{S}_k = \mathfrak{b}_{S_k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(kD).$$

The *sheafified linear series* \tilde{S}_\bullet is defined by

$$\tilde{S}_k = H^0(X, \mathcal{S}_k).$$

Remark 3.15. The sheaf \mathcal{S}_k is equal to \mathcal{F} considered in Section 3.3 for $V = S_k$.

Theorem 3.16. Let S_\bullet be a birational graded linear series. Then the volume of S_\bullet and the volume of the sheafified linear series \tilde{S}_\bullet are equal i.e.

$$\text{vol}(S_\bullet) = \text{vol}(\tilde{S}_\bullet).$$

Proof. This is a consequence of Theorem 3.10. The sheaf $\mathcal{S}_k = \mathfrak{b}_{S_k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(kD)$ is by construction globally generated and therefore the base ideal of \tilde{S}_k is equal to \mathfrak{b}_{S_k} . Thus, the base ideal of \tilde{S}_k and of S_k are both equal for every $k > 0$, from which the equality of volumes follows. \square

Corollary 3.17. Let S_\bullet be a birational graded linear series. Then for all admissible flags Y_\bullet we have:

$$\Delta_{Y_\bullet}(S_\bullet) = \Delta_{Y_\bullet}(\tilde{S}_\bullet)$$

\square

Example 3.18. Consider

$$X = \mathbb{P}^2 = \text{Proj } \mathbb{C}[X_1, X_2, X_3]$$

and $L = \mathcal{O}(2)$. We can identify

$$H^0(X, L) \cong \mathbb{C}[X_1, X_2, X_3]_2.$$

Now let S_\bullet be the graded linear series which is generated by all monomials of degree two except X_2X_3 , i.e. generated by

$$S_1 := \text{span}_{\mathbb{C}}(X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3)$$

Clearly S_\bullet is base point free and $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}(S_\bullet)$, for the standard flag Y_\bullet such that

$$Y_1 := V(X_1), Y_2 := V(X_1) \cap V(X_2) \text{ and } Y_3 := \{[0 : 0 : 1]\}.$$

Next, we want to show that the semigroup $\Gamma(S_\bullet)$ generates \mathbb{Z}^3 as a group. We have

$$\begin{aligned} v_1 &:= (1, 1, 1) = (\nu(X_1 X_2), 1) \in \Gamma(S_\bullet), \\ v_2 &:= (1, 0, 1) = (\nu(X_1 X_3), 1) \in \Gamma(S_\bullet), \end{aligned}$$

and

$$v_3 := (0, 2, 1) = (\nu(X_2), 1) \in \Gamma(S_\bullet).$$

This shows that $e_2 := v_1 - v_2 \in G(\Gamma(S_\bullet))$, $e_3 = v_3 - 2e_2 \in G(\Gamma(S_\bullet))$ and $e_1 = v_1 - e_2 - e_3 \in G(\Gamma(S_\bullet))$. Hence, $G(\Gamma(S_\bullet)) = \mathbb{Z}^3$. From this fact and the equality of Newton-Okounkov bodies, we deduce that $\text{vol}(S_\bullet) = \text{vol}(D)$. Applying Theorem 3.10 gives us that S_\bullet is birational and that $\mathfrak{b}_{S_1} = \mathcal{O}_X$. Hence, from the above theorem we expect the following equality for the sheafified linear series.

$$R(\tilde{S}_\bullet) = R_\bullet(X, L).$$

Indeed, we will show by hand how the missing global section $X_2 X_3 \in H^0(X, L)$ can be glued locally from sections in S_1 .

In

$$U_i := \text{Spec } \mathbb{C}[X_1/X_i, X_2/X_i, X_3/X_i] \subseteq \mathbb{P}^2$$

we have $s_i = (X_2 X_3)|_{U_i} = \frac{X_2 X_3}{X_i^2} \in H^0(U_i, \mathcal{O}_X)$.

Since $H^0(U_i, \mathcal{S}_1)$ is a $H^0(U_i, \mathcal{O}_X)$ -module, we have

$$s_i = \lambda_i \cdot (t_i)|_{U_i} \in H^0(U_i, \mathcal{S}_1)$$

for $\lambda_i = s_i$ and $t_i = X_i^2$.

4. SLICE FORMULA FOR GRADED LINEAR SERIES

Slice formulas are one of the most important tools to analyze the shape of Newton-Okounkov bodies. They are the main ingredient for the characterization of Newton-Okounkov bodies on surfaces [LM09, Theorem 6.4], for Jow's Theorem [J10] and for most of the rational polyhedrality properties (see e.g. [AKL12], or [SS17]).

By a slice formula we mean the following: Let S_\bullet be a graded linear series and Y_\bullet be a flag. A slice of the Newton-Okounkov body $\Delta_{Y_\bullet}(S_\bullet)$ is given by intersecting it with some affine hypersurface $\{t\} \times \mathbb{R}^{d-1} \subseteq \mathbb{R}^d$ for $t \geq 0$. We denote this slice by $\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=t}$. Suppose there is a graded linear series W_\bullet on Y_1 such that

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=t} = \Delta_{Y'_\bullet}(W_\bullet)$$

where Y'_\bullet is the restriction of the flag Y_\bullet on Y_1 . Then the above equality will be called a slice formula.

Slice formulas are very useful since the set of all slices $\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=t}$ determines $\Delta_{Y_\bullet}(S_\bullet)$ and thus we are able to reduce the calculation of $\Delta_{Y_\bullet}(S_\bullet)$ to the calculation of Newton-Okounkov bodies of dimension one less. This will give us in some cases the possibility to argue inductively on the dimension of X .

In all cases we will consider, the graded linear series W_\bullet will be a certain *restricted linear series*. Let us quickly recall basic notions of restricted linear series. Let S_\bullet be a graded linear series on X corresponding to D and let $Y \subseteq X$ be an irreducible subvariety. Let

$$\text{rest}_k: H^0(X, \mathcal{O}_X(kD)) \rightarrow H^0(Y, \mathcal{O}_Y(kD))$$

be the restriction of global sections of kD to Y . Then we define the restricted graded linear series $S_{|Y, \bullet}$ on Y , by

$$S_{|Y, k} := \text{rest}_k(S_k).$$

If $R(S_\bullet) = R_\bullet(X, D)$, we write $R_\bullet(X, D)_{|Y}$ for the graded algebra of sections of the restricted linear series of S_\bullet . We will write $\text{vol}_{X|Y}(S_\bullet) := \text{vol}_Y(S_{|Y, \bullet})$ for the volume of the restricted linear series, as well as $\text{vol}_{X|Y}(D)$ if $R(S_\bullet) = R_\bullet(X, D)$. Let Y_\bullet be a flag on X . We write

$$\Delta_{X|Y_1}(S_\bullet) := \Delta_{Y'_1}(S_{|Y_1, \bullet})$$

for Y'_1 defined as above. In a lot of situations we want to assume that Y_1 is a Cartier divisor. Therefore, we make the following definition.

Definition 4.1. Let X be a projective variety and Y_\bullet an admissible flag. We call Y_\bullet *very admissible* if Y_1 defines a Cartier divisor.

Note that if X is smooth then every admissible flag is very admissible.

The existence of slice formulas is directly connected to the distribution of valutive points in $\Delta_{Y_\bullet}(S_\bullet)$. Therefore, before deriving slice formulas, we will analyze this distribution.

4.1. Valuation points. A different construction of the Newton-Okounkov body than the one we mentioned is the following. First one constructs the set of normalized valuations $\Sigma := \bigcup_{m>0} 1/m \cdot \Gamma_m(S_\bullet)$ and then one takes the closed convex hull of this set. This is completely equivalent to the earlier mentioned definition of Newton-Okounkov bodies. A priori, it might happen that taking the convex hull destroys a lot of information about Σ . In this paragraph we will see that this is not the case and indeed it is not even necessary to take the convex hull in the first place.

Definition 4.2. Let S_\bullet be a graded linear series. Let Y_\bullet be an admissible flag. Then a point $P \in \Delta_{Y_\bullet}(D)$ is called a *valuation point* or *valutive* if there is a section $s \in S_k$ such that $\nu_{Y_\bullet}(s)/k = P$ for some $k \in \mathbb{N}$.

Lemma 4.3. *Let S_\bullet be a graded linear series on X . Let Y_\bullet be an admissible flag. Let P and Q be two valutive points of $\Delta_{Y_\bullet}(S_\bullet)$. Then all rational points in the line segment PQ are also valutive.*

Proof. Let $s \in S_k$ and $t \in S_m$ be sections such that $P = \nu_{Y_\bullet}(s)/k$ and $Q = \nu_{Y_\bullet}(t)/m$. Let $A := (a/b) \cdot P + (1 - a/b) \cdot Q$ for integers a, b such that $0 \leq a < b$ be an arbitrary rational point in the line segment PQ . Define the section

$$s^{ma} \otimes t^{k(b-a)} \in S_{kbm}.$$

Then we have

$$\begin{aligned} \frac{\nu_{Y_\bullet}(s^{ma} \otimes t^{k(b-a)})}{kbm} &= ma \cdot \frac{\nu_{Y_\bullet}(s)}{kbm} + k(b-a) \cdot \frac{\nu_{Y_\bullet}(t)}{kbm} \\ &= a/b \cdot P + (1 - a/b) \cdot Q \\ &= A. \end{aligned}$$

This shows that every rational point in the segment PQ is valutive. \square

Lemma 4.4. *Let S_\bullet be a graded linear series on X . Let Y_\bullet be an admissible flag. Let P_1, \dots, P_n be valutive points of $\Delta_{Y_\bullet}(S_\bullet)$. Then all rational points in the relative interior of the convex hull of P_1, \dots, P_n are also valutive.*

Proof. We will prove this by induction on the number of points n . The case $n = 2$ was done in the previous lemma. Let us prove the claim for n , assuming that it holds for integers $\leq n - 1$. Let P be a rational point in the interior of the convex hull. The induction hypothesis tells us that all rational points on the facets of the convex hull are valutive. Now, consider the line going through P_1 and P . This line intersects the boundary of the convex hull in P_1 and in one more point, which we will call Q . Clearly the point Q is rational and hence, by the induction hypothesis, valutive. Now, by construction, P is in the convex hull of the two valutive points P_1 and Q . Again, using the induction hypothesis, the point P is valutive. \square

Theorem 4.5. *Let S_\bullet be a graded linear series on X . Let Y_\bullet be an admissible flag. Then all rational points in the relative interior of the Newton-Okounkov body $\Delta_{Y_\bullet}(S_\bullet)$ are valutive.*

Proof. Let P be a rational point in the interior of $\Delta_{Y_\bullet}(S_\bullet)$. By construction of $\Delta_{Y_\bullet}(S_\bullet)$ this means that

$$P \in \text{conv} \left(\bigcup_{m>0} 1/m \cdot \nu_{Y_\bullet}(S_m \setminus \{0\}) \right).$$

So there are finitely many sections $s_i \in S_{m_i}$ and coefficients $\lambda_i \geq 0$ for $i = 1, \dots, N$ such that $\sum_{i=1}^N \lambda_i = 1$ and

$$P = \sum_{i=1}^N \lambda_i \cdot \nu_{Y_\bullet}(s_i)/m_i.$$

Define $K := \prod_{i=1}^N m_i$. Then $s_i^{K/m_i} \in S_K$ and we can write

$$P = \sum_{i=1}^N \lambda_i \cdot \nu_{Y_\bullet}(s_i^{K/m_i})/K.$$

This implies that

$$P \in \text{conv}(1/K \cdot \nu_{Y_\bullet}(S_K \setminus \{0\})),$$

and due to the previous lemma it follows that P is valutive. \square

Remark 4.6. *The fact that the valutive points are dense in the Newton-Okounkov body was already proven in [KL14] for the divisorial case $S_\bullet = R_\bullet(X, D)$ and in the general case of graded linear series in [KMS12, Lemma 2.6].*

Corollary 4.7. *Let D be a big divisor on X and let Y_\bullet be a very admissible flag such that $Y_1 \not\subseteq \mathbf{B}_+(D)$. Then all the rational points in the relative interior of $\Delta_{Y_\bullet}(D)_{\nu_1=0}$ are valutive.*

Proof. By construction of the Newton-Okounkov body, the valutive points of the slice $\Delta_{Y_\bullet}(D)_{\nu_1=0}$ are of the form $(0, P)$ where P is a valutive point of the Newton-Okounkov body of the restricted linear series $R_\bullet(X, D)|_{Y_1}$, which we denote by $\Delta_{X|Y_1}(D)$. But due to the slice formula in [LM09, Theorem 4.24 b)] we have an equality

$$\Delta_{X|Y_1}(D) = \Delta_{Y_\bullet}(D)_{\nu_1=0}.$$

Combining this with the above theorem yields the desired result. \square

The fact that some points on the boundary of the Newton-Okounkov body are valutive and some may just be limits of valutive points corresponds to the fact that the semigroup $\Gamma(S_\bullet)$ may be not finitely generated. Finite generation of the semigroup is a very pleasant property. It was shown in [A13] that if $\Gamma(S_\bullet)$ is finitely generated then there exists a corresponding flat degeneration of X to the toric variety whose normalization corresponds to the polytope $\Delta_{Y_\bullet}(S_\bullet)$

The connection to the existence of valutive points is given by the following theorem.

Theorem 4.8. *Let S_\bullet be a graded linear series, Y_\bullet be an admissible flag of X and $\Gamma(S_\bullet)$ be finitely generated. Then all rational points of $\Delta_{Y_\bullet}(S_\bullet)$ are valutive. If $\Delta_{Y_\bullet}(S_\bullet)$ is rational polyhedral, then $\Gamma(S_\bullet)$ is finitely generated if and only if all rational points of $\Delta_{Y_\bullet}(S_\bullet)$ are valutive.*

Proof. Let $\Gamma(S_\bullet)$ be finitely generated and $(\nu_{Y_\bullet}(s_1), k_1), \dots, (\nu_{Y_\bullet}(s_N), k_N)$ be the generators. Then $\Delta_{Y_\bullet}(S_\bullet)$ is equal to the convex hull of the points $P_1 = 1/k_1 \cdot \nu_{Y_\bullet}(s_1), \dots, P_N = 1/k_N \cdot \nu_{Y_\bullet}(s_N)$. Due to Lemma 4.4, all points in $\Delta_{Y_\bullet}(S_\bullet)$ are valutive.

Let now $\Delta_{Y_\bullet}(S_\bullet)$ be rational polyhedral. It remains to prove that $\Gamma(S_\bullet)$ is finitely generated if all rational points are valutive. However, this follows from [BG09, Corollary 2.10]. \square

Remark 4.9. *Note that in the second part of the above theorem the condition that $\Delta_{Y_\bullet}(S_\bullet)$ is rational polyhedral is necessary. For a counterexample consider [LM09, Proposition 1.17] for any K which is not rational polyhedral. In this case it is clear from the construction that all rational points are valutive. However, the corresponding semigroup can never be finitely generated unless its Newton-Okounkov body is rational polyhedral.*

The next theorem says that for a surface X the semigroup $\Gamma(D)$ is almost never finitely generated.

Theorem 4.10. *Let X be a smooth surface. Let S_\bullet be a graded linear series corresponding to a big divisor D such that $\text{vol}(S_\bullet) = \text{vol}(D)$. Let $C \subseteq X$ be a curve of genus $g > 0$. Then for a general point $x \in C$ the semigroup $\Gamma_{C \supseteq \{x\}}(S_\bullet)$ is not finitely generated.*

Proof. Without loss of generality, we may replace D by kD and can therefore assume that there is a non-negative integer $t \in \mathbb{N}$ such that $\text{vol}_{X|C}(D - tC) > 0$. Consider the Zariski decomposition of $D - tC$:

$$D_t := D - tC = P_t + N_t,$$

and choose $x \in C$ very general such that the semigroup $\{(k, \text{ord}_x(s)) \mid s \in H^0(C, \mathcal{O}_C(P_t))\}$ is not finitely generated [LM09, Example 1.7]. If we set $\Delta_{\{x\}}(P_t|_C) = [0, c]$, then the failure of finite generation just means that c is not a valutive point of $\Delta_{\{x\}}(P_t|_C)$. But since $\text{vol}_{X|C}(D_t) = \text{vol}_{X|C}(P_t) > 0$, we deduce that $C \not\subseteq \mathbf{B}_+(P_t)$ and we have $\text{vol}_{X|C}(P_t) = \text{vol}(P_t|_C)$ [ELMNP09, Corollary 2.17]. Thus c is not a valutive point of the restricted Newton-Okounkov body $\Delta_{X|C}(P_t)$. The valutive points of $\Delta_{X|C}(P_t)$ correspond to the valutive points of the restricted Newton-Okounkov body $\Delta_{X|C}(D_t)$ up to a translation of $\text{ord}_x(N_t|_C)$. But each valutive point Q of $\Delta_{X|C}(D_t)$ corresponds one to one to the valutive point (t, Q) of $\Delta_{C \supseteq \{x\}}(D)_{\nu_1=t}$. This shows that $(t, c + \text{ord}_x(N_t|_C)) \in \Delta_{C \supseteq \{x\}}(D)$ is not a valutive point and thus surely it is not a valutive point of $\Delta_{C \supseteq \{x\}}(S_\bullet)$. Applying Theorem 4.8 gives then the desired failure of finite generation. \square

Example 4.11. Let X be a smooth Mori dream surface, let D be a big divisor on X and $Y_\bullet: X \supset C \supset \{x\}$ be an admissible flag on X consisting of a curve C on X which is not contained in $\mathbf{B}_+(D)$. We use [KLM12, Theorem B] for describing the Newton-Okounkov body of a big divisor on a surface: There are piecewise linear functions with rational slopes and rational breaking points $\alpha, \beta: [\nu, \mu] \rightarrow \mathbb{R}^+$ such that the Newton-Okounkov-body is given by:

$$\Delta_{Y_\bullet}(D) = \{(t, y) \in \mathbb{R}^2 : \nu \leq t \leq \mu, \quad \text{and } \alpha(t) \leq y \leq \beta(t)\}$$

Moreover, the number ν is rational and μ is given by

$$\mu := \sup\{s > 0 : D - sC \text{ is big}\}.$$

Since in our case X is a Mori dream space and thus the Big cone is rational polyhedral, the number μ is rational as well. In this situation we have a quite good understanding of the valuative points of the Newton-Okounkov body. The following points are valuative:

- (a) the rational points in the interior of $\Delta_{Y_\bullet}(D)$
- (b) points of the form (ν, y) for rational $y \in [\alpha(\nu), \beta(\nu))$
- (c) points of the form $(t, \alpha(t))$ for all rational $t \in [\nu, \mu)$.

Let us prove that all the above listed points are indeed valuative.

Part (a) follows from Theorem 4.5. Part (b) follows by using the slice formula [LM09, Theorem 4.24 b)] which states that $\Delta_{Y_\bullet}(D)_{\nu_1=t} = \Delta_{X|C}(D - tC)$ for all $t \in [\nu, \mu)$. Indeed, for such rational t the valuative points of the latter Newton-Okounkov body correspond to the valuative points of $\Delta_{Y_\bullet}(D)$ with first coordinate equal to t . Hence, again by Theorem 4.5, all the rational points of the form (ν, t_1) and for $t_1 \in (\alpha(\nu), \beta(\nu))$ are valuative. Part (c) follows from the following fact: Let S_\bullet be a finitely generated graded linear series on a curve C and let P be a smooth point on C . Let $\Delta(S_\bullet) = [b, c]$. Then b is a valuative point. To prove this we can without loss of generality assume that S_\bullet is generated in degree one. Let $s_1, \dots, s_l \in S_1$ be the generators of S_\bullet . Now suppose b is not a valuative point. Then $\nu(s_i) \geq b + 1$ for all $i = 1, \dots, l$. Consider $s \in S_k$ which can be written as $s = \sum_{\alpha \in \mathbb{N}^l} c_\alpha \bar{s}^\alpha$ where $\bar{s} = (s_1, \dots, s_l)$. Then

$$\begin{aligned} \nu(s) &= \nu\left(\sum_{\alpha \in \mathbb{N}^l} c_\alpha \bar{s}^\alpha\right) \\ &\geq \min(\nu(\bar{s}^\alpha)) \\ &\geq k(b + 1) \end{aligned}$$

which implies that b does not lie in $\Delta(S_\bullet)$ inducing a contradiction. Using this fact for the restricted graded linear series of $D - tC$ to C which is indeed finitely generated since X is a Mori dream space, gives us the valuativity of the remaining listed points.

If C is a curve of genus $g > 0$ and x is a very general point in C , then we can say even more. The points of the form $(t, \beta(t))$ for $t \in [\nu, \mu)$ are not valuative if $\beta(\nu) > \alpha(\nu)$. If $\alpha(\nu) = \beta(\nu)$, then this holds for $t \in (\nu, \mu)$. In order to prove this, we make use of the proof of Theorem 4.10. There we showed that for rational $t \in [\mu, \nu)$ such that $\text{vol}(\Delta_{X|C}(D - tC)) > 0$ for a general choice of $x \in C$ the point $(t, \beta(t))$ is not valuative. Since t varies in a countable set, we conclude that for a very general choice of $x \in C$ this holds for all considered rational t at once.

In this situation the only points where we do not know whether they are valuative or not are the rational points of the form (μ, y) for $y \in [\alpha(\mu), \beta(\mu)]$. The situation is summarized in Figure 1.

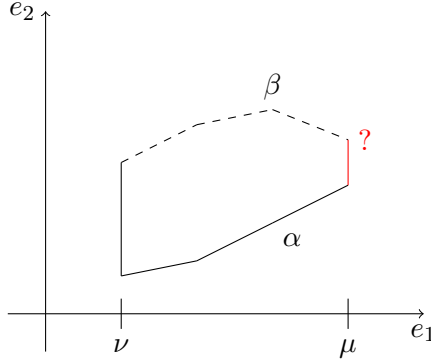


FIGURE 1. Valuation points of NO-body on a surface

4.2. Slice Formula. In this paragraph we generalize the slice formula given in [LM09, Theorem 4.2.4] to graded linear series S_\bullet . Let us first state the content of the theorem: Let D be a big divisor, Y_\bullet be an admissible flag such that Y_1 is an effective Cartier divisor for which $Y_1 \not\subseteq \mathbf{B}_+(D)$ and $\mu := \sup\{t \in \mathbb{R}^+ | (D - tY_1) \text{ is big}\}$. Then we have for all $0 \leq t < \mu$

$$\Delta_{Y_\bullet}(D)_{\nu_1=t} = \Delta_{X|Y_1}(D - tY_1).$$

The following definition will be useful for the generalization.

Definition 4.12. Let S_\bullet be a graded linear series on X . Let $Y \subseteq X$ be an irreducible subvariety of codimension 1 which defines a Cartier divisor and ε be a non-negative rational number. Then we define the graded linear series $S_\bullet - \varepsilon Y$ by setting

$$(S_\bullet - \varepsilon Y)_k = \{s/s_Y^{[\varepsilon \cdot k]} : s \in S_k \quad \text{ord}_Y(s) \geq [\varepsilon \cdot k]\} \subseteq H^0(X, \mathcal{O}_X(kD - [\varepsilon \cdot k]Y)).$$

Using the above definition, we are able to formulate our first slice formula for slices which meet the interior of the corresponding Newton-Okounkov body.

Theorem 4.13. *Let S_\bullet be a graded linear series. Let Y_\bullet be a very admissible flag and ε a positive rational number such that $\{\varepsilon\} \times \mathbb{R}^{d-1}$ meets the interior of $\Delta_{Y_\bullet}(S_\bullet)$. Then*

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=\varepsilon} = \Delta_{X|Y_1}(S_\bullet - \varepsilon Y_1)$$

via the identification of $\{\varepsilon\} \times \mathbb{R}^{d-1} \cong \mathbb{R}^{d-1}$.

Proof. By considering the k -th Veronese $S_\bullet^{(k)}$ of the graded linear series S_\bullet for a high enough multiple, i.e. $S_\bullet^{(k)}$ defined by $S_l^{(k)} = S_{l \cdot k}$, we can without loss of generality assume that ε is an integer. We will now show that the rational points in the interior of both Newton-Okounkov bodies are indeed equal, from which the statement will follow by Theorem 4.5.

Consider first the rational points in the interior of $\Delta_{X|Y_1}(S_\bullet - \varepsilon Y_1)$. By construction these are given in degree k by

$$\begin{aligned} 1/k \cdot \Gamma_k((S_\bullet - \varepsilon Y_1)|_{Y_1}) &= \{1/k \cdot (\nu_2(s), \dots, \nu_d(s)) \mid s \in S_k \text{ s. t. } \nu_1(s) = \varepsilon \cdot k\} \\ &\cong \{1/k \cdot (\varepsilon \cdot k, \nu_2(s), \dots, \nu_d(s)) \mid s \in S_k \text{ s.t. } \nu_1(s) = \varepsilon \cdot k\} \\ &= (1/k \cdot \Gamma_k(S_\bullet)) \cap (\{\varepsilon\} \times \mathbb{R}^{d-1}). \end{aligned}$$

But the last set are just the valuative points of $\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=\varepsilon}$ in degree k . This finishes the proof. \square

Corollary 4.14. *Let S_\bullet be a graded linear series. Let Y_\bullet be a very admissible flag and ε a positive rational number such that $\{\varepsilon\} \times \mathbb{R}^{d-1}$ meets the interior of $\Delta_{Y_\bullet}(S_\bullet)$. Then*

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1 \geq \varepsilon} := \Delta_{Y_\bullet}(S_\bullet) \cap [\varepsilon, \infty) \times \mathbb{R}^{d-1} = \Delta_{Y_\bullet}(S_\bullet - \varepsilon Y_1) + (\varepsilon, 0, \dots, 0).$$

Proof. This follows by realizing that the slices of both sides agree for all rational vertical slices. Indeed, we have for all $\delta > 0$ such that $\{\delta + \varepsilon\} \times \mathbb{R}^{d-1}$ meets the interior of $\Delta_{Y_\bullet}(S_\bullet)$.

$$\begin{aligned} \Delta_{Y_\bullet}(S_\bullet - \varepsilon Y_1)_{\nu_1 = \delta} &= \Delta_{X|Y_1}(S_\bullet - (\varepsilon + \delta)Y_1) \\ &= \Delta_{Y_\bullet}(S_\bullet)_{\varepsilon + \delta}. \end{aligned}$$

\square

The above theorem shows that for $t > 0$ the slice formula of [LM09, Theorem 4.2.4] completely generalizes to the case of arbitrary graded linear series without any restrictions. However, the reduction to the case $t = 0$ does not work as in [LM09]. The idea of the proof was to replace the divisor D by some small perturbation $D + \varepsilon Y_1$ and thus reduce the question to the case $t > 0$. However, for a graded linear series it is not clear how to generalize this construction. Therefore, we need some additional properties for the graded linear series S_\bullet in order to recover more of the geometry of X and the corresponding divisor D . We would like to assume that S_\bullet as well as the restricted series $S_{|Y_\bullet}$ are birational. In order to make sure that the restricted series has this property, we pose a stronger condition on S_\bullet , namely, that it contains an ample series. (This corresponds to condition (C) in [LM09], see Definition 4.16).

In addition, we will start with the case that S_\bullet is also finitely generated and $\text{vol}(S_\bullet) = \text{vol}(D)$. After that we will reduce the general case to the special case by using Fujita approximation.

Lemma 4.15. *Let $S_\bullet \subseteq T_\bullet$ be two birational finitely generated graded linear series such that the map of projective spectra $\text{Proj}(T_\bullet) \rightarrow \text{Proj}(S_\bullet)$ defined by the inclusion of graded linear algebras $R(S_\bullet) \subseteq R(T_\bullet)$ is globally defined. Then for each closed subvariety $Y \subseteq X$, the induced map $\text{Proj}(T_{|Y_\bullet}) \rightarrow \text{Proj}(S_{|Y_\bullet})$ is also globally defined.*

Proof. Consider the following diagram of morphisms of graded algebras

$$\begin{array}{ccc} R(S_{\bullet}) & \xrightarrow{r_S} & R(S_{|Y, \bullet}) \\ \downarrow \iota & & \downarrow \iota_Y \\ R(T_{\bullet}) & \xrightarrow{r_T} & R(T_{|Y, \bullet}) \end{array}$$

where the horizontal mappings are just the restriction of sections and the vertical maps are given by inclusion. For a graded algebra $U = \bigoplus_{k \in \mathbb{N}} U_k$, define $U^+ := \bigoplus_{k > 0} U_k$. We want to show that the inclusion ι_Y defines a global morphism of corresponding projective spectra. Therefore, we need to check if the preimage under ι_Y of each relevant homogeneous prime ideal $\mathfrak{p} \subset R(T_{|Y, \bullet})^+$ is still relevant. So suppose that the preimage is not relevant, i.e. $R(S_{|Y, \bullet})^+ \subseteq \iota_Y^{-1}(\mathfrak{p})$. Then by definition of the restriction morphism, we get:

$$R(S_{\bullet})^+ = r_S^{-1}(R(S_{|Y, \bullet})^+) \subseteq r_S^{-1}(\iota_Y^{-1}(\mathfrak{p})).$$

This means that the ideal on the right hand side is not relevant. Due to the commutativity of the above diagram, the right hand side is equal to $\iota^{-1}(r_T^{-1}(\mathfrak{p}))$. However, the ideal $r_T^{-1}(\mathfrak{p})$ is relevant since r_T is surjective and therefore the ideal $\iota^{-1}(r_T^{-1}(\mathfrak{p}))$ is relevant as well since, by assumption, ι induces a global morphism of projective spectra. Hence, we get a contradiction, which shows the claim \square

Let us now define what it means to contain an ample series.

Definition 4.16. Let S_{\bullet} be a graded linear series on X corresponding to D . We say that S_{\bullet} contains the ample series $A = D - E$ if

- (a) $S_k \neq 0$ for $k \gg 0$ and
- (b) there is a decomposition of \mathbb{Q} -divisors $D = A + E$ where A is ample and E is effective such that

$$H^0(X, \mathcal{O}(kA)) \subseteq S_k \subseteq H^0(X, \mathcal{O}(kD))$$

for all k divisible enough. Note that the inclusion of the outer groups is given by the multiplication of a defining section of kE .

As it was already pointed out in [J10], it is not difficult to show that a graded linear series containing an ample series is birational. This follows from the birationality of the ample series.

Lemma 4.17. Let S_{\bullet} be a graded linear series corresponding to D which contains the ample series $D - E$. Let $Y \subseteq X$ be a closed irreducible subvariety such that $Y \not\subseteq \text{Supp}(E)$. Then the restricted linear series $S_{|Y, \bullet}$ contains an ample series corresponding to the decomposition $D|_Y = A|_Y + E|_Y$.

Proof. The restriction of an ample divisor to a closed subvariety is ample. Since $Y \not\subseteq E$ we conclude that $E|_Y$ is effective. Hence $D|_Y = A|_Y + E|_Y$ is a decomposition into ample and effective. Furthermore, the stable base locus

of S_\bullet is contained in E , by the assumption that S_\bullet contains the ample series $D - E$. Hence, there is a $k \gg 0$ such that $S_{|Y,k} \neq 0$. For k divisible enough, we conclude, by Serre vanishing, that $H^0(Y, \mathcal{O}_Y(kA)) = H^0(X, \mathcal{O}_X(kA))|_Y$. From this identity we deduce the desired inclusion

$$H^0(Y, \mathcal{O}_Y(kA)) \subseteq S_{|Y,\bullet} \subseteq H^0(Y, \mathcal{O}_Y(kD)).$$

□

Since $\mathbf{B}_+(D) = \bigcap_{D=A+E} \text{Supp}(E)$, we recover the fact deduced in [LM09], that for a big divisor D on X and $Y \not\subseteq \mathbf{B}_+(D)$, the restricted linear series contains an ample series (satisfies condition (C)).

Now, we are able to prove our first slice formula for $t = 0$ under the condition that S_\bullet has full volume. It will follow as a corollary of the following.

Theorem 4.18. *Let $S_\bullet \subseteq T_\bullet$ be two finitely generated graded linear series such that $\text{vol}(S_\bullet) = \text{vol}(T_\bullet) > 0$. Suppose furthermore that S_\bullet contains the ample series $D - E$. Then for all closed irreducible subvarieties $Y \not\subseteq \text{Supp}(E)$ we have*

$$\text{vol}_{X|Y}(S_\bullet) = \text{vol}_{X|Y}(T_\bullet).$$

Proof. From the equality of volumes and the birationality of the maps h_{S_\bullet} and h_{T_\bullet} , we can conclude, as in Theorem 3.8, that the inclusion $R(S_\bullet) \subseteq R(T_\bullet)$ gives rise to a globally defined regular map:

$$\text{Proj}(T_\bullet) \rightarrow \text{Proj}(S_\bullet).$$

Due to Lemma 4.15, we arrive at the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{h_{T_{|Y,\bullet}}} & \text{Proj}(T_{|Y,\bullet}) \\ & \searrow h_{S_{|Y,\bullet}} & \downarrow j \\ & & \text{Proj}(S_{|Y,\bullet}). \end{array}$$

By Lemma 4.17, the restricted series $S_{|Y,\bullet}$ and $T_{|Y,\bullet}$ contain an ample series. Hence, the maps $h_{T_{|Y,\bullet}}$ and $h_{S_{|Y,\bullet}}$ are both birational. Then we can conclude, as in Theorem 3.8, that $\text{vol}_{X|Y}(S_\bullet) = \text{vol}_{X|Y}(T_\bullet)$. □

Corollary 4.19. *Let X be a normal projective variety. Let S_\bullet be a finitely generated graded linear series corresponding to a finitely generated divisor D such that $\text{vol}(S_\bullet) = \text{vol}(D)$. Suppose furthermore that S_\bullet contains the ample series $D - E$. Then for all very admissible flags Y_\bullet such that Y_1 does not contain the support of E we have:*

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{X|Y_1}(S_\bullet).$$

Proof. From the above theorem we conclude that $\text{vol}_{X|Y_1}(D) = \text{vol}_{X|Y_1}(S_\bullet)$, which implies an equality $\Delta_{X|Y_1}(D) = \Delta_{X|Y_1}(S_\bullet)$. We have $\mathbf{B}_+(D) \subseteq E$

and therefore $Y_1 \not\subseteq \mathbf{B}_+(D)$. Hence, we can use the slice formula [LM09, Theorem 4.2.4] to conclude

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{Y_\bullet}(D)_{\nu_1=0} = \Delta_{X|Y_1}(D) = \Delta_{X|Y_1}(S_\bullet).$$

□

Now, we want to get rid of the assumption that S_\bullet has to be finitely generated with full volume. However, the price we pay for this reduction is the additional constraint that the point Y_d of the flag be not contained in the stable base locus $\mathbf{B}(S_\bullet)$.

Theorem 4.20. *Let X be a normal projective variety. Let S_\bullet be a graded linear series containing the ample series $D - F$. Let Y_\bullet be a very admissible flag such that Y_1 is not contained in the support of F and $Y_d \notin \mathbf{B}(S_\bullet)$. Then we have*

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{X|Y_1}(S_\bullet).$$

Proof. Let us first treat the case where S_\bullet is a finitely generated graded linear series generated by S_1 . Let $\pi: X' \rightarrow X$ be the blow-up of the base ideal \mathfrak{b}_{S_1} , E the exceptional divisor and let $S'_\bullet := \pi^*S_\bullet - E$, as well as $D_E := \pi^*D - E$. Consider the decomposition $D = A + F$ into ample plus effective. There is a $k \gg 0$ such that $\pi^*A - kE$ is very ample, we get an induced decomposition $\pi^*D = (\pi^*A - 1/k \cdot E) + (1/k \cdot E + \pi^*F)$. Then it is easy to see that S'_\bullet contains the ample series $\pi^*A - 1/k \cdot E = \pi^*D - (1/k \cdot E + \pi^*F)$. Let \tilde{Y}_i be the strict transform of Y_i and \tilde{Y}_\bullet be the corresponding flag on \tilde{X} . Clearly, the strict transform \tilde{Y}_1 is not contained in the support of $1/k \cdot E + \pi^*F$ since $Y_1 \not\subseteq \text{Supp}(F)$ and $\mathbf{B}(S_\bullet) \subset \text{Supp}(F)$. Now we have

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{\tilde{Y}_\bullet}(S'_\bullet)_{\nu_1=0} = \Delta_{\tilde{X}|\tilde{Y}_1}(S'_\bullet)$$

where the second equality follows from Corollary 4.19. To finish the first part of the proof we need to show that $\Delta_{\tilde{X}|\tilde{Y}_1}(S'_\bullet) = \Delta_{X|Y_1}(S_\bullet)$. We have

$$\Delta_{X|Y_1}(S_\bullet) \subseteq \Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{\tilde{X}|\tilde{Y}_1}(S'_\bullet).$$

The other inclusion follows from the fact that

$$\text{vol}_{\tilde{X}|\tilde{Y}_1}(S'_\bullet) = \text{vol}_{\tilde{X}|\tilde{Y}_1}(\pi^*S_\bullet) = \text{vol}_{X|Y_1}(S_\bullet)$$

where the first equality follows from the bijection $S'_{1|Y_1} \cong \pi^*S_1$ given by multiplication with the restriction of a defining section s_E of E to Y'_1 and the last equality follows from the property that $(\pi^*s)_{|\tilde{Y}_1} = (\pi_{|\tilde{Y}_1})^*(s_{|Y_1})$. This proves the theorem for S_\bullet being finitely generated.

Finally, we want to treat the case when S_\bullet is not necessary finitely generated. We will use Fujita approximation to reduce the statement to finitely generated graded linear series. Define the graded linear series $V_{\bullet,p}$ by

$$V_{k,p} := \text{Im}(\text{Sym}^k(S_p) \rightarrow S_{kp}).$$

From [LM09, Theorem 3.5], we deduce that for each $\varepsilon > 0$, we can find p_0 such that for all $p \geq p_0$

$$\text{vol}(1/p \cdot \Delta_{Y_\bullet}(V_{\bullet,p})) \geq \text{vol}(\Delta_{Y_\bullet}(S_\bullet)) - \varepsilon.$$

Combining this with the easy fact that $1/p \cdot \Delta_{Y_\bullet}(V_{\bullet,p}) \subseteq 1/p' \cdot \Delta_{Y_\bullet}(V_{\bullet,p'})$ for $p \leq p'$ we have

$$\Delta_{Y_\bullet}(S_\bullet) = \overline{\bigcup_{p \geq 0} 1/p \cdot \Delta_{Y_\bullet}(V_{\bullet,p})}.$$

Analogously, we get

$$\Delta_{X|Y_1}(S_\bullet) = \overline{\bigcup_{p \geq 0} 1/p \cdot \Delta_{X|Y_1}(V_{\bullet,p})}.$$

Combining these two properties leads to

$$\begin{aligned} \Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} &= \overline{\bigcup_{p \geq 0} (1/p \cdot \Delta_{Y_\bullet}(V_{\bullet,p}) \cap (\{0\} \times \mathbb{R}^{d-1}))} = \\ &= \overline{\bigcup_{p \geq 0} (1/p \cdot \Delta_{Y_\bullet}(V_{\bullet,p}) \cap (\{0\} \times \mathbb{R}^{d-1}))} = \\ &= \overline{\bigcup_{p \geq 0} (\Delta_{X|Y_1}(V_{\bullet,p}))} = \\ &= \Delta_{X|Y_1}(S_\bullet). \end{aligned}$$

Note that in the second equality we used the slice formula for finitely generated graded linear series. This finishes the proof. \square

We can apply the above theorem to the case of a restricted graded linear series. This enables us to get a generalization of [J10, Theorem B], which states that for a divisor D and a curve C which is constructed from intersecting $d-1$ very general very ample effective divisors A_i on X . We have that the restricted volume of $\text{vol}_{X|C}(D)$ is equal to the length of $\Delta_{Y_\bullet}(D)_{\nu_1=0, \dots, \nu_{d-1}=0}$ where $Y_i := A_1 \cap \dots \cap A_i$.

Corollary 4.21. *Let D be a divisor on X and Y_\bullet an admissible flag centered at $\{x\} \notin \mathbf{B}(D)$, such that Y_i defines a Cartier divisor in Y_{i+1} . Then for $Y_i \not\subseteq \mathbf{B}_+(D)$ we have*

$$\Delta_{X|Y_i}(D) = \Delta_{Y_\bullet}(D)_{\nu_1=0, \dots, \nu_i=0}.$$

\square

The last slice formula does not make any assumptions on the centered point $\{x\}$ of the flag, but has more constraints on the divisorial component Y_1 of the chosen flag Y_\bullet .

Theorem 4.22. *Let S_\bullet be a graded linear series that contains the ample series $A = D - E$. Let Y_\bullet be a very admissible flag such that Y_1 is not*

contained in the support of E and is not fixed, i.e. there is a natural number $k \in \mathbb{N}$ such that $h^0(X, \mathcal{O}_X(k \cdot Y_1)) > 1$. Then we have

$$\Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} = \Delta_{X|Y_1}(S_\bullet).$$

Proof. Let \mathcal{S}_\bullet be the sheafification of S_\bullet . Consider the graded linear series T_\bullet corresponding to the divisor $D+Y_1$ defined by the sheaves $\mathcal{S}_k \otimes_{\mathcal{O}_X} \mathcal{O}_X(k \cdot Y_1)$, i.e. defined by $T_k := H^0(X, \mathcal{S}_k \otimes_{\mathcal{O}_X} \mathcal{O}_X(k \cdot Y_1))$. We want to show first that the restricted linear series $T_{|Y_1, \bullet}$ is not equal to zero. This follows from the fact that there is a non zero section $s \in H^0(X, \mathcal{O}_X(k \cdot Y_1))$ which does not vanish at Y_1 . Indeed, such a section exists. Let s be a section in $H^0(X, \mathcal{O}_X(k \cdot Y_1))$ which is not equal to a power of s_{Y_1} up to a constant. Let a be the order of vanishing of s along Y_1 . By definition of s_{Y_1} , we have $a < k$ and $s/s_{Y_1}^{\otimes a} \in H^0(X, \mathcal{O}_X((k-a) \cdot Y_1))$ does not vanish at Y_1 .

Since Y_1 is not contained in the support of E , it is in particular not contained in the stable base locus of S_\bullet . Thus, we can pick a non-zero section $s' \in S_k$ which does not vanish at Y_1 . Hence, the section $s'' := s^{\otimes k} \otimes s' \in T_k$, does not vanish at Y_1 . This implies that $\nu_1(s'') = 0$. Moreover, we can choose a section $s \in S_k$ such that $\nu_1(s) > 0$. Then for the section $\tilde{s} := s_E^{\otimes k} \otimes s \in T_k$ we have $\nu_1(\tilde{s}) > 1$. It follows from the above results on

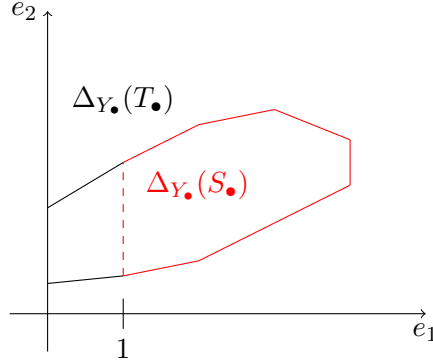


FIGURE 2. Newton-Okounkov body $\Delta_{Y_\bullet}(T_\bullet)$

valuation vectors that the slice $\{1\} \times \mathbb{R}^{d-1}$ meets the interior of the Newton-Okounkov body $\Delta_{Y_\bullet}(T_\bullet)$. By construction of T_\bullet , we have an isomorphism of sections $(T_\bullet - Y_1)_k \cong \tilde{S}_k$ where \tilde{S}_\bullet is the sheafified graded linear series of S_\bullet . With the help of Theorem 4.13, Corollary 4.14 and Corollary 3.17 we deduce:

$$\begin{aligned} \Delta_{Y_\bullet}(S_\bullet)_{\nu_1=0} &= \Delta_{Y_\bullet}(\tilde{S}_\bullet)_{\nu_1=0} = \Delta_{Y_\bullet}(T_\bullet)_{\nu_1=1} \\ &= \Delta_{X|Y_1}(\tilde{S}_\bullet) = \Delta_{X|Y_1}(S_\bullet). \end{aligned}$$

Note that the last equality is due to Theorem 4.18. \square

5. GENERIC NEWTON-OKOUNKOV BODIES

In this section we want to generalize the discussion in Chapter 5 of [LM09] to the case of birational graded linear series S_\bullet . In order to define Newton-Okounkov bodies, we have to fix the variety X , the flag Y_\bullet and a graded linear series S_\bullet , respectively a big divisor D . It was established in [LM09, Theorem 5.1] that if we vary all the different data X, Y_\bullet and D in a flat family, the resulting bodies all coincide for a very general choice of these parameters. This allows to define generic Newton-Okounkov bodies, called the infinitesimal Newton-Okounkov bodies, which do no longer depend on the choice of a flag Y_\bullet . The proof, which is presented in [LM09], relies heavily on the fact that D induces a locally free sheaf $\mathcal{O}_X(D)$. Hence, in order to generalize their results, we need to make use of the sheafification process considered in Section 3.5. However, the resulting coherent sheaves \mathcal{S}_k are not locally free, which also leads to technical difficulties to take into account. Finally, we will also get rid of the flatness hypothesis by using the theorem of generic flatness [GW, Corollary 10.84].

5.1. Family of Newton-Okounkov bodies. Let us start by fixing the notation. Let T be a (not necessarily projective) irreducible variety. This will be our parameter space. Let

$$\pi_T: X_T \rightarrow T$$

be a family, such that for all $t \in T$ the fibers

$$X_t := X_T \times_T k(t)$$

are projective varieties of dimension d . Let $S_{T,\bullet}$ be a graded linear series corresponding to a divisor D_T on X_T which is induced by a graded series of coherent sheaves $\mathcal{S}_{T,k} \subseteq \mathcal{O}_{X_T}(k \cdot D_T)$. Furthermore, denote by $S_{t,\bullet}$ the graded linear series which is defined by taking the global sections of the pulled back sheaves $\mathcal{S}_{T,k}|_{X_t}$. Additionally, we want to assume that $S_{t,\bullet}$ is a graded linear series corresponding to the divisor $D_t := D_T|_{X_t}$ as well as $\mathcal{S}_{t,k}$ are subsheaves of $\mathcal{O}_{X_t}(k \cdot D_t)$.

Let \mathcal{Y}_\bullet be a partial flag of subvarieties

$$X_T = \mathcal{Y}_0 \supseteq \mathcal{Y}_1 \supseteq \cdots \supseteq \mathcal{Y}_d$$

with the following additional properties. Denote the fibers of the flag \mathcal{Y}_\bullet over $t \in T$ by

$$Y_{i,t} := \mathcal{Y}_i \cap X_t.$$

The additional properties are:

- (a) Each $Y_{\bullet,t}$ is an admissible flag on X_t .
- (b) The variety \mathcal{Y}_{i+1} is a Cartier divisor in \mathcal{Y}_i .

To summarize the above discussion, we give the following definition.

Definition 5.1. Let $\pi_T: X_T \rightarrow T$, $S_{T,\bullet}$ and \mathcal{Y}_\bullet be given such that all the above prescribed assumptions are fulfilled. Then we call $(X_T, S_{T,\bullet}, \mathcal{Y}_\bullet)$ an *admissible family of Newton-Okounkov bodies over T* .

Suppose we are given an admissible family of Newton-Okounkov bodies over T . Then for each $t \in T$, the following Newton-Okounkov body on X_t is well defined:

$$\Delta_{Y_{t,\bullet}}(S_{t,\bullet}).$$

The next lemma will be of help in the last section where we want to construct examples of admissible families of Newton-Okounkov bodies.

Lemma 5.2. *Let $p: X \rightarrow Y$ be a morphism of varieties. Let S_\bullet be a graded linear series corresponding to a divisor D on Y such that $S_1 \neq \{0\}$. Suppose that S_\bullet is given by taking global sections of a graded series of coherent sheaves $\mathcal{S}_k \subseteq \mathcal{O}_Y(k \cdot D)$. Then the pullback of the graded series of sheaves $p^*\mathcal{S}_k$ are coherent subsheaves of $\mathcal{O}_X(k \cdot p^*D)$. Furthermore, by taking its global sections, it defines a graded linear series on X (which we also denote by p^*S_\bullet) if one of the following conditions are fulfilled:*

- $p: X \rightarrow Y$ is flat, or
- $p: X \rightarrow Y$ is birational with Y normal such that the image of the exceptional locus is away from $\text{Bs}(S_1)$.

Moreover, if p is a morphism of projective varieties and satisfies the second condition, then $\text{vol}(p^*S_\bullet) = \text{vol}(S_\bullet)$

Proof. Let us first suppose that $p: X \rightarrow Y$ is flat. By the flatness of p , the sheaf $p^*\mathcal{S}_k$ is a coherent \mathcal{O}_X -module which is contained in $p^*\mathcal{O}_X(kD)$. Furthermore, for each non-negative pair of integers k, l the injection $\mathcal{S}_k \otimes \mathcal{S}_l \rightarrow \mathcal{S}_{k+l}$ pulls back to an injection

$$p^*\mathcal{S}_k \otimes p^*\mathcal{S}_l \rightarrow p^*\mathcal{S}_{k+l}.$$

Therefore it is easy to see that p^*S_\bullet defines a graded linear series.

Now, let $p: X \rightarrow Y$ be birational, Y normal and suppose there is an open subset $V \subseteq Y$ such that $\text{Bs}(S_k) \subseteq \text{Bs}(S_1) \subseteq V$ which induces an isomorphism $p|_{p^{-1}(V)}: p^{-1}(V) \rightarrow V$. Now we claim that the induced canonical morphism

$$\kappa: p_*p^*\mathcal{S}_k \rightarrow \mathcal{S}_k$$

is an isomorphism. We will prove this by showing that for each $y \in Y$ we find an open subset U_y such that the induced morphism of sections $\kappa(U_y)$ is an isomorphism. For $y \in V$ choose $U_y \subseteq V$. Then the induced morphism of sections is an isomorphism since $p|_{p^{-1}(U_y)}$ is an isomorphism. If $y \notin V$ we can find an open neighborhood U_y such that $U_y \subseteq Y \setminus \text{Bs}(S_k)$. But on U_y the coherent sheaf $\mathcal{S}_k|_{U_y}$ is invertible. Hence, we have an isomorphism

$$\mathcal{S}_k|_{U_y} \cong \mathcal{O}_Y(kD)|_{U_y}.$$

However, for the locally free sheaf $\mathcal{O}_Y(kD)$ the canonical morphism $p_*p^*\mathcal{O}_X(kD) \cong \mathcal{O}_X(kD)$ is an isomorphism. This follows by using Zariski's Main theorem and the projection formula. Hence, the canonical morphism $\kappa(U_y)$ is an isomorphism which completes the proof of the fact that κ is an isomorphism. We have the following commutative diagram of coherent sheaves on Y

$$\begin{array}{ccc} p_*p^*\mathcal{S}_k & \xrightarrow{\cong} & \mathcal{S}_k \\ \downarrow & & \downarrow \\ p_*p^*\mathcal{O}_X(kD) & \xrightarrow{\cong} & \mathcal{O}_X(kD). \end{array}$$

Taking global sections of the left vertical map induces an injection

$$H^0(X, p^*\mathcal{S}_k) \rightarrow H^0(X, p^*\mathcal{O}_X(kD)).$$

It remains to prove that the global sections define a graded algebra. We can prove exactly as before that we have an isomorphism of \mathcal{O}_Y -modules $p_*p^*(\mathcal{S}_k \otimes \mathcal{S}_l) \cong \mathcal{S}_k \otimes \mathcal{S}_l$. Again, we have a commutative diagram of coherent sheaves on Y given by

$$\begin{array}{ccc} p_*p^*(\mathcal{S}_l \otimes \mathcal{S}_k) & \xrightarrow{\cong} & \mathcal{S}_l \otimes \mathcal{S}_k \\ \downarrow & & \downarrow \\ p_*p^*\mathcal{S}_{l+k} & \xrightarrow{\cong} & \mathcal{S}_{l+k}. \end{array}$$

Taking global sections of this diagram gives us an injection

$$H^0(X, p^*\mathcal{S}_l \otimes p^*\mathcal{S}_k) \rightarrow H^0(X, p^*\mathcal{S}_{l+k}).$$

This implies that $p^*\mathcal{S}_\bullet$ defines a graded linear series. Now we want to prove that $p^*\mathcal{S}_k \subseteq \mathcal{O}_X(p^*D)$. This can again be checked by case distinction of open sets. Let U be open such that $p|_U$ induces an isomorphism, then clearly

$$H^0(U, p^*\mathcal{S}_k) \cong H^0(p(U), \mathcal{S}_k) \subseteq H^0(p(U), \mathcal{O}_X(k \cdot D)) \cong H^0(U, \mathcal{O}_Y(k \cdot p^*D)).$$

If $U \subseteq X \setminus p^{-1}(\text{Bs}(\mathcal{S}_k))$, consider the induced morphism $p: U \rightarrow Y \setminus \text{Bs}(\mathcal{S}_k) := W$. Let $\mathfrak{b}_{\mathcal{S}_k}$ be the ideal sheaf of \mathcal{S}_k , then $\mathcal{S}_k|_W = (\mathcal{O}_X(kD) \otimes \mathfrak{b}_{\mathcal{S}_k})|_W = \mathcal{O}_X(kD)|_W$. But this shows that $H^0(U, p^*\mathcal{S}_k) = H^0(U, p^*\mathcal{O}_Y(kD))$.

The equality of volumes follows by taking global sections of the canonical isomorphism κ which gives an isomorphism

$$H^0(X, p^*\mathcal{S}_k) \cong H^0(Y, \mathcal{S}_k).$$

□

Corollary 5.3. *Let $p: X \rightarrow Y$ be a morphism of projective varieties satisfying the properties of the second statement in the above lemma. Let Y .*

be an admissible flag on X and S_\bullet be birational graded linear series on Y which is induced by the graded linear series of sheaves \mathcal{S}_\bullet . Then

$$\Delta_{Y_\bullet}(p^*S_\bullet) = \Delta_Y(p^*S_\bullet).$$

Proof. By Theorem 3.16, we can without loss of generality assume that S_\bullet is induced by the graded series of sheaves \mathcal{S}_\bullet . Since $\text{vol}(p^*S_\bullet) = \text{vol}(S_\bullet) = \text{vol}(p^*\mathcal{S}_\bullet)$ it is enough to show one inclusion. But by construction $p^*\mathcal{S}_k$ contains all the elements p^*s for $s \in S_k$. This shows that $\Delta_{Y_\bullet}(p^*S_\bullet) \subseteq \Delta_Y(p^*\mathcal{S}_\bullet)$ and proves the claim. \square

Remark 5.4. Note that by p^*S_\bullet we denote the graded linear series, defined by considering the pullback of sections of S_k . However, by $p^*\mathcal{S}_\bullet$ we denote the graded linear series, given by taking the global sections of the pullback of coherent sheaves of $p^*\mathcal{S}_k$.

5.2. Partial sheafification of a graded linear series. Let X be a (not necessarily projective) variety, D a divisor and \mathcal{S} a coherent subsheaf of $\mathcal{O}_X(D)$. In this paragraph we want to generalize the discussion in [LM09, Rem 1.4/1.5] to the sheaf \mathcal{S} . Let Y_\bullet be a partial flag of X of length r such that Y_{i+1} is a Cartier divisor in Y_i . Analogously as in the definition of Newton-Okounkov bodies, this partial flag defines a valuation map

$$\nu_{Y_\bullet}: H^0(X, \mathcal{S}) \setminus \{0\} \rightarrow \mathbb{Z}^r.$$

If we fix a tuple $\sigma = (\sigma_1, \dots, \sigma_r) \in \mathbb{Z}^r$, we can define a subsheaf of \mathcal{S} by setting for each open $U \subset X$ such that the induced flag $\mathcal{Y}_\bullet|_U$ is of length $r' \leq r$

$$(2) \quad H^0(U, \mathcal{S}^{\geq(\sigma)}) := \{s \in H^0(U, \mathcal{S}) \mid \nu_{Y_\bullet|_U}(s) \geq (\sigma_1, \dots, \sigma_{r'})\}.$$

Here, the map $\nu_{Y_\bullet|_U}$ is the valuation map corresponding to the restricted flag given by $Y_{i|U} := Y_i \cap U$ on U .

For the next theorem it is practical to make the following two abbreviations:

$$\begin{aligned} \mathcal{S}(\sigma_1, \dots, \sigma_r) &:= \mathcal{S}|_{Y_r} \otimes_{Y_r} \mathcal{O}_X(-\sigma_1 Y_1)|_{Y_r} \otimes_{Y_r} \cdots \otimes_{Y_r} \mathcal{O}_{Y_{r-1}}(-\sigma_r Y_r)|_{Y_r} \\ \mathcal{S}(\sigma_1, \dots, \sigma_{r+1})|_{Y_r} &:= \mathcal{S}(\sigma_1, \dots, \sigma_r) \otimes_{Y_r} \mathcal{O}_{Y_r}(-\sigma_{r+1} Y_{r+1}). \end{aligned}$$

Note that these sheaves are both defined over Y_r . However, by a slight abuse of notation we will also consider them as sheaves over X without writing them as pushforwards of the inclusion map $Y_r \hookrightarrow X$.

Theorem 5.5. Let \mathcal{S} be a coherent subsheaf of $\mathcal{O}_X(D)$. Then for each partial flag Y_\bullet of X of length r and $\sigma \in \mathbb{Z}^r$ there exists a coherent sheaf $\mathcal{S}^{\geq(\sigma)}$ such that (2) holds and it induces a surjective morphism

$$q_r: \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r)} \rightarrow \mathcal{S}(\sigma_1, \dots, \sigma_r).$$

Proof. We will prove this using induction on r . Let $r = 1$. Then sections of $\mathcal{S}^{\geq(\sigma_1)}$ are those sections of \mathcal{S} which vanish locally along Y_1 at least σ_1

times. These are given by the image of the following injection of coherent sheaves

$$\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\sigma_1 Y_1) \rightarrow \mathcal{S}.$$

On an open set $U \subseteq X$ the above map of sections is defined by multiplication with a defining section $s_{Y_1|U}$ of Y_1 to the power of σ_1 . Since $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\sigma_1 Y_1)$ is a coherent sheaf, this proves the claim for $r = 1$.

Now let us suppose we have already defined $\mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r)}$ as a coherent sheaf. Then we need to construct the sheaf $\mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r, \sigma_{r+1})}$. From the construction of the valuation ν_{Y_\bullet} we get a morphism of coherent sheaves

$$q_r: \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r)} \rightarrow \mathcal{S}(\sigma_1, \dots, \sigma_r).$$

We claim that this morphism is surjective. We will prove this using the same induction on r . For $r = 1$, this map is just the restriction map to the closed subvariety Y_1

$$q_1: \mathcal{S}^{\geq(\sigma_1)} \cong \mathcal{S} \otimes_X \mathcal{O}_X(-\sigma_1 Y_1) \rightarrow \mathcal{S}|_{Y_1} \otimes_{\mathcal{O}_{Y_1}} \mathcal{O}_X(-\sigma_1 Y_1)|_{Y_1}$$

and hence surjective. Now, let us additionally assume that q_r is surjective. We have the following natural inclusion

$$\iota_r: \mathcal{S}(\sigma_1, \dots, \sigma_{r+1})|_{Y_r} \hookrightarrow \mathcal{S}(\sigma_1, \dots, \sigma_r)$$

given by multiplication with a defining section of Y_{r+1} in Y_r . Now we define $\mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})}$ as the preimage of $\mathcal{S}(\sigma_1, \dots, \sigma_{r+1})|_{Y_r}$ under the morphism q_r , yielding the following diagram:

$$(3) \quad \begin{array}{ccc} \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})} & \xrightarrow{p_r} & \mathcal{S}(\sigma_1, \dots, \sigma_{r+1})|_{Y_r} \\ \downarrow & & \downarrow \iota_r \\ \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r)} & \xrightarrow{q_r} & \mathcal{S}(\sigma_1, \dots, \sigma_r). \end{array}$$

By the construction of the valuation, the sections of the coherent sheaf $\mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})}$ are exactly the ones which satisfy equation (2). It remains to show that the morphism

$$q_{r+1}: \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})} \rightarrow \mathcal{S}(\sigma_1, \dots, \sigma_{r+1})$$

is surjective. But q_{r+1} is just the composition of the surjection p_r with the surjective restriction morphism. Hence, the surjectivity follows. \square

5.3. Generic Newton-Okounkov Body. Let $(X_T, \mathcal{S}_{T_\bullet}, \mathcal{Y}_\bullet)$ be an admissible family of Newton-Okounkov bodies over T . In this paragraph we want to prove that for a very general choice of $t \in T$ the Newton-Okounkov bodies $\Delta_{Y_\bullet, t}(X_t, \mathcal{S}_{t_\bullet})$ all coincide. The idea of the proof is to show that for a very general choice $t \in T$ the dimension of the space of global sections

$$H^0(X_t, (\mathcal{S}_{k,t})^{\geq(\sigma)})$$

is independent from t . The main issue of the proof is to show that we have an equality of coherent sheaves $(\mathcal{S}_{T,k}^{\geq(\sigma)})_t = (\mathcal{S}_{t,k})^{\geq(\sigma)}$. Once we have

established this equality, we can use the theorem of generic flatness to deduce the constancy of the dimension.

We first need some helpful lemmata.

Lemma 5.6. *The commutative diagram constructed in (3) gives rise to the following commutative diagram, where the rows are exact:*

$$(4) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})} & \rightarrow & \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})} & \xrightarrow{p_r} & \mathcal{S}(\sigma_1, \dots, \sigma_{r+1})|_{Y_r} \rightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \iota_r \\ 0 & \rightarrow & \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})} & \rightarrow & \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r)} & \xrightarrow{q_r} & \mathcal{S}(\sigma_1, \dots, \sigma_r) \rightarrow 0. \end{array}$$

Proof. The only thing left to prove is the identity of the kernels of the horizontal maps q_r and p_r . We want to show that the induced map of sections on the lower row is exact. Let us first assume that $U \subseteq X$ is an open subset such that the induced flag $\mathcal{Y}_{\bullet}|_U$ is of length $r' < r$. Then

$$H^0(U, \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})}) = H^0(U, \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r'})}) = H^0(U, \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r)}).$$

Furthermore, $H^0(U, \mathcal{S}(\sigma_1, \dots, \sigma_r)) = \{0\}$, since it is supported on \mathcal{Y}_r . This proves the exactness of sections on such U . Now let U be chosen such that $\mathcal{Y}_{\bullet}|_U$ is of maximal length r . We calculate the kernel of $q_r(U)$. Let $s \in H^0(U, \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})})$. By definition of the valuation and the construction of the map q_r , the section s will be sent to zero. However, if $\nu_{\mathcal{Y}_{\bullet}|_U}(s) = (\sigma_1, \dots, \sigma_r)$, then it is also clear that the image of $s \neq 0$ under q_r does not vanish. This shows that $\ker(q_r|_U) = \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})}$. The kernel of p_r can be calculated using diagram (3) as

$$\ker p_r = \ker q_r \cap \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_r, \sigma_{r+1})} = \mathcal{S}^{\geq(\sigma_1, \dots, \sigma_{r+1})}.$$

□

The next lemma seems to be common folklore knowledge. However, as a matter of a missing reference, we will prove this anyway.

Lemma 5.7. *Let S be a noetherian scheme and $i: Z \rightarrow X$ be a closed immersion of noetherian S -schemes such that Z is flat over S . Let T be another S -scheme and $i_T: Z \times_S T \rightarrow X \times_S T$ be the closed immersion given by the following fiber diagram*

$$(5) \quad \begin{array}{ccc} Z \times_S T & \xrightarrow{p_Z} & Z \\ \downarrow i_T & & \downarrow i \\ X \times_S T & \xrightarrow{p_X} & X. \end{array}$$

Then for each coherent \mathcal{O}_Z -module E we have a functorial isomorphism

$$i_{T*} p_Z^* E \cong p_X^* i_* E.$$

Proof. This question is local on X , T and S . So let us set, without loss of generality, $X = \text{Spec } A$, $T = \text{Spec } B$ and $S = \text{Spec } R$. Then there is an ideal $I \subseteq A$ such that $Z = \text{Spec}(A/I)$. Consider the short exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0.$$

Since A/I is flat over R , we may tensor this sequence by $\otimes_R B$ and get

$$0 \rightarrow I \otimes_R B \rightarrow A \otimes_R B \rightarrow A/I \otimes_R B \rightarrow 0.$$

Hence, diagram (5) gives rise to the following commutative diagram of rings

$$\begin{array}{ccc} A/I & \longrightarrow & (A \otimes_R B)/(I \otimes_R B) \\ \uparrow & & \uparrow \\ A & \longrightarrow & A \otimes_R B. \end{array}$$

Let M be the A/I -module such that $\tilde{M} = E$. Then we have

$$p_X^* i_* E = ({}_A M \otimes_R A \otimes_R B)^\sim = ({}_A M \otimes_A B)^\sim.$$

Clearly, $I \cdot {}_A M = 0$ and from this we deduce $(I \otimes_A B)({}_A M \otimes_A B) = 0$. But this means that $p_X^* i_* E$ can be viewed as a sheaf over $(A \otimes_R B)/(I \otimes_R B)$. So there is a coherent $\mathcal{O}_{Z \otimes_S T}$ -module L such that $i_{T*} L \cong p_X^* i_* E$. Taking i_T^* of this isomorphism, gives us an isomorphism on $X \otimes_S T$

$$i_T^* i_{T*} L \cong i_T^* p_X^* i_* E = p_Y^* i^* i_* E.$$

Since the canonical morphisms $i_T^* i_{T*} L \cong L$ and $i^* i_* E \cong E$ are isomorphisms, we conclude that $L \cong p_Y^* E$. Taking i_{T*} of this isomorphism then gives the desired result. \square

Lemma 5.8. *Let $(X_T, S_{T,\bullet}, \mathcal{Y}_\bullet)$ be an admissible family of Newton-Okounkov bodies and $(\sigma_1, \dots, \sigma_r) \in \mathbb{N}^r$. Let furthermore \mathcal{Y}_i be flat over T for all $i = 1, \dots, d$. Consider the natural map*

$$\iota: \mathcal{S}_{T,k}(\sigma_1, \dots, \sigma_{r+1})|_{\mathcal{Y}_r} \hookrightarrow \mathcal{S}_{T,k}(\sigma_1, \dots, \sigma_r).$$

Viewing this as a map of coherent sheaves on X_T , for $t \in T$ the map ι pulls back via the closed immersion $i_t: X_t \hookrightarrow X_T$ to the natural map

$$i_t^* \iota: \mathcal{S}_{t,k}(\sigma_1, \dots, \sigma_{r+1})|_{\mathcal{Y}_{t,r}} \hookrightarrow \mathcal{S}_{t,k}(\sigma_1, \dots, \sigma_r)$$

viewed as a morphism of coherent sheaves on X_t .

Proof. We consider the following fiber diagram

$$\begin{array}{ccc} Y_{r,t} = \mathcal{Y}_r \times_T k(t) & \longrightarrow & \mathcal{Y}_r \\ \downarrow & & \downarrow \\ X_t = X_T \times_T k(t) & \longrightarrow & X_T \end{array}$$

By the previous Lemma 5.7, the restriction of the coherent sheaf $\mathcal{S}_{T,\bullet}(\sigma_1, \dots, \sigma_r)$ on X_T to X_t is the same as the restriction of $\mathcal{S}_{T,\bullet}(\sigma_1, \dots, \sigma_r)$ (viewed as a sheaf on \mathcal{Y}_r) to $Y_{r,t}$ and then viewing it as a sheaf on X_t . This means that

$$\mathcal{S}_{T,\bullet}(\sigma_1, \dots, \sigma_r)|_{X_t} = \mathcal{S}_{T,\bullet}(\sigma_1, \dots, \sigma_r)|_{Y_{r,t}}.$$

For $i \leq r + 1$ we have

$$(\mathcal{O}_{X_T}(-\sigma_i \mathcal{Y}_i)|_{\mathcal{Y}_r})|_{Y_{r,t}} = (\mathcal{O}_{X_T}(-\sigma_i \mathcal{Y}_i)|_{X_t})|_{Y_{r,t}} = \mathcal{O}_{X_t}(-\sigma_i Y_{i,t})|_{Y_{r,t}}.$$

Since also $(\mathcal{S}_{T,k})|_{Y_{r,t}} = \mathcal{S}_{t,k}|_{Y_{r,t}}$ we can follow that

$$(\mathcal{S}_{T,k}(\sigma_1, \dots, \sigma_r))|_{X_t} \cong \mathcal{S}_{t,k}(\sigma_1, \dots, \sigma_r).$$

Similarly, we have

$$((\mathcal{S}_{T,k}(\sigma_1, \dots, \sigma_{r+1}))|_{\mathcal{Y}_r})|_{X_t} \cong \mathcal{S}_{t,k}(\sigma_1, \dots, \sigma_{r+1})|_{Y_{t,r}}$$

Finally, the restricted morphism $i_t^* \iota$ is given by multiplication with a defining section of Y_r to the power of σ_{r+1} and hence is injective. \square

Lemma 5.9. *Let $(X_T, \mathcal{S}_{T,\bullet}, \mathcal{Y}_\bullet)$ be an admissible family of Newton-Okounkov bodies over T . For a very general point $t \in T$ we have for every $k \in \mathbb{N}$ and $\sigma \in \mathbb{Z}^r$*

$$(\mathcal{S}_{T,k}^{\geq(\sigma)})_t = (\mathcal{S}_{t,k})^{\geq(\sigma)}.$$

Proof. First let us fix the number k and abbreviate $\mathcal{S} := \mathcal{S}_k$. We will prove the lemma using induction on r . Let $r = 1$. Then we can identify $\mathcal{S}_T^{\geq(\sigma_1)} \cong \mathcal{S}_T \otimes_{X_T} \mathcal{O}_{X_T}(-\sigma_1 \mathcal{Y}_1)$. Pulling this back to the fiber X_t leads to

$$\begin{aligned} (\mathcal{S}_T \otimes_{X_T} \mathcal{O}_{X_T}(-\sigma_1 \mathcal{Y}_1))|_{X_t} &= (\mathcal{S}_T)|_{X_t} \otimes_{X_t} \mathcal{O}_{X_t}(-\sigma_1 \mathcal{Y}_1)|_{X_t} = \\ &= \mathcal{S}_t \otimes_{X_t} \mathcal{O}_{X_t}(-\sigma_1 Y_{1,t}) \cong \mathcal{S}_t^{\geq(\sigma_1)}. \end{aligned}$$

This proves the lemma for $r = 1$.

Now we prove the lemma for $r + 1$ assuming that it holds for r . Using the theorem of generic flatness [GW, Corollary 10.85] we can find open subsets $V_{\sigma,k} \subset T$ such that all coherent sheaves occurring in diagram (4) as well as the cokernels of the vertical morphisms are flat over T . Furthermore, we can also assume that all the \mathcal{Y}_i for $i = 1, \dots, r + 1$ are flat over $V_{\sigma,k}$. Then we let t be in $\bigcap_{\sigma \in \mathbb{Z}^{r+1}, k \in \mathbb{N}} V_{\sigma,k}$. Due to the flatness, the induction hypothesis and Lemma 5.8, we can pull back the right hand square of diagram (4) and obtain the following square:

$$\begin{array}{ccc} (\mathcal{S}_T^{\geq(\sigma_1, \dots, \sigma_{r+1})})_t & \twoheadrightarrow & \mathcal{S}_t(\sigma_1, \dots, \sigma_{r+1})|_{Y_{r,t}} \\ \downarrow & & \downarrow \iota_r \\ \mathcal{S}_t^{\geq(\sigma_1, \dots, \sigma_r)} & \xrightarrow{q_r} & \mathcal{S}_t(\sigma_1, \dots, \sigma_r). \end{array}$$

The above diagram implies that we get an injection from $(\mathcal{S}_T^{\geq(\sigma_1, \dots, \sigma_{r+1})})_t$ to the inverse image of $\mathcal{S}_t(\sigma_1, \dots, \sigma_{r+1})|_{Y_{r,t}}$ under the map q_r , which is by construction equal to $\mathcal{S}_t^{\geq(\sigma_1, \dots, \sigma_{r+1})}$. We also have an isomorphism $\mathcal{S}_t(\sigma_1, \dots, \sigma_{r+1}) \cong (\mathcal{S}_T(\sigma_1, \dots, \sigma_{r+1}))_t$ and therefore we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & (\mathcal{S}_T)_t^{\geq(\sigma_1, \dots, \sigma_{r+1}+1)} & \rightarrow & (\mathcal{S}_T)_t^{\geq(\sigma_1, \dots, \sigma_{r+1})} & \rightarrow & (\mathcal{S}_T(\sigma_1, \dots, \sigma_{r+1}))_t \rightarrow 0 \\ & & \downarrow \iota_t & & \downarrow \iota_t & & \downarrow \cong \\ 0 & \rightarrow & (\mathcal{S}_t)^{\geq(\sigma_1, \dots, \sigma_{r+1}+1)} & \rightarrow & (\mathcal{S}_t)^{\geq(\sigma_1, \dots, \sigma_{r+1})} & \rightarrow & \mathcal{S}_t(\sigma_1, \dots, \sigma_{r+1}) \rightarrow 0. \end{array}$$

We will use a second induction argument on σ_{r+1} . Let $\sigma_1, \dots, \sigma_r$ as well as $t \in T$ be fixed. Let $\sigma_{r+1} = 0$. Then we have $\mathcal{S}_t^{\geq(\sigma_1, \dots, \sigma_r, 0)} = \mathcal{S}_t^{\geq(\sigma_1, \dots, \sigma_r)}$ as well as $\mathcal{S}_T^{\geq(\sigma_1, \dots, \sigma_r, 0)} = \mathcal{S}_T^{\geq(\sigma_1, \dots, \sigma_r)}$. Hence, for $\sigma_{r+1} = 0$ the desired identity follows from the induction hypothesis on r . Now let us assume, we know that the desired identity of sheaves is true for σ_{r+1} . Then we want to prove it is true for $\sigma_{r+1} + 1$. However, this follows by using the above commutative diagram and the Five lemma. Indeed, by our induction hypothesis, the middle vertical morphism is an isomorphism. Hence, the left vertical morphism must be an isomorphism as well. This proves the claim. \square

Theorem 5.10. *Let $(X_T, \mathcal{S}_T, \bullet, \mathcal{Y}_\bullet)$ be an admissible family of Newton-Okounkov bodies over T . Then for a very general $t \in T$ the Newton-Okounkov bodies*

$$\Delta_{Y_{t,\bullet}}(S_{t,\bullet})$$

all coincide.

Proof. For a fixed $k \in \mathbb{N}$ and $\sigma \in \mathbb{N}^d$, there is an open subset $U_{\sigma,k}$ such that $\mathcal{S}_k^{\geq(\sigma)}$ is flat over $U_{\sigma,k}$ due to the theorem of generic flatness. Furthermore, by the the semicontinuity theorem, we can shrink $U_{\sigma,k}$ even more and have for all $t \in U_{\sigma,k}$ that the dimension of

$$h^0(X, (\mathcal{S}_{T,k}^{\geq(\sigma)})_t)$$

is independent from t . For a very general $t \in \bigcap_{k,\sigma} U_{\sigma,k}$ the constancy of the above dimension holds for every k and σ . Furthermore for $t \in \bigcap_{\sigma,k} U_{\sigma,k} \cap \bigcap_{\sigma,k} V_{\sigma,k}$, we have that $h^0(X, (\mathcal{S}_{T,k}^{\geq(\sigma)})_t) = h^0(X, \mathcal{S}_{t,k}^{\geq(\sigma)})$ are independent from t . But for a fixed $t \in T$ the dimensions of all the $H^0(X, \mathcal{S}_{t,k}^{\geq(\sigma)})$ completely determine the valuation points of $\Delta_{Y_{t,\bullet}}(S_{t,\bullet})$ and thus also the body $\Delta_{Y_{t,\bullet}}(S_{t,\bullet})$. From this observation it follows that for a very general t all Newton-Okounkov bodies coincide. \square

5.4. Examples of generic Newton-Okounkov bodies. In this paragraph we want to construct some admissible families of Newton-Okounkov bodies over T , in order to illustrate how to make use of Theorem 5.10 to get generic Newton-Okounkov bodies. We will give three construction how

to realize this. The first two examples are families where we just vary the flag Y_\bullet . In the third construction we will also vary the varieties X_t and the graded linear series $S_{t,\bullet}$ in a family. For the sake of simplicity, we assume in this paragraph that all the varieties X occurring are smooth.

5.4.1. *Variation of the flag.* Let S_\bullet be a birational graded linear series in X corresponding to a divisor D and let T be an irreducible (not necessarily projective) variety. Since S_\bullet is birational, we can consider the sheafification \mathcal{S}_\bullet of S_\bullet . By Corollary 3.17, we may replace S_\bullet by \tilde{S}_\bullet and may therefore assume that S_\bullet is induced by the family of sheaves \mathcal{S}_\bullet . Consider the variety $X_T := X \times_{\mathbb{C}} T$ and the projection

$$\pi_X: X_T \rightarrow X.$$

We define the family of sheaves $\mathcal{S}_{T,\bullet}$ as the pullback of \mathcal{S}_\bullet under the projection π_X . Since π_X is flat, we can use Lemma 5.2 to see that the family of sheaves $\mathcal{S}_{T,\bullet}$ defines a graded linear series on X_T and that the sheaves $\mathcal{S}_{T,k}$ are subsheaves of $\mathcal{O}_{X_T}(k\pi_X^*D)$. In order to get a family of Newton-Okounkov bodies, it remains to choose an admissible flag \mathcal{Y}_\bullet of X_T . So let us suppose we have fixed such a flag \mathcal{Y}_\bullet . For a point $t \in T$, the fiber over t of X_T induces an isomorphism $X_t \cong X$. It is not hard to see that the following composition of morphism is an isomorphism

$$X \cong X_t \hookrightarrow X \times_{\mathbb{C}} T \xrightarrow{\pi_X} X$$

where the first map is the natural inclusion of the fiber. This shows that $\mathcal{S}_{t,\bullet} \cong \mathcal{S}_\bullet$ and hence $(X_T, \mathcal{S}_{T,\bullet}, \mathcal{Y}_\bullet)$ defines an admissible family of Newton-Okounkov bodies.

Example 5.11 (Variation of the point Y_d). Let us suppose we have a birational graded linear series S_\bullet on a smooth projective variety X and a partial admissible flag of smooth subvarieties

$$Y_1 \supseteq \cdots \supseteq Y_{d-1}$$

fixed. Then set $T := Y_{d-1}$ and consider the variety $X_T := X \times_{\mathbb{C}} T$ and the partial flag \mathcal{Y}_\bullet defined by $\mathcal{Y}_i := Y_i \times_{\mathbb{C}} Y_{d-1}$ for $i = 1, \dots, d-1$ and $\mathcal{Y}_d := Y_{d-1}$ which is embedded in $\mathcal{Y}_{d-1} = Y_{d-1} \times_{\mathbb{C}} Y_{d-1}$ via the diagonal embedding. Then for each $x \in T = Y_{d-1}$, the flag $\mathcal{Y}_{x,\bullet}$ is just the partial flag Y_1, \dots, Y_{d-1} with the additional component $Y_{x,d} = \{x\}$. Hence, Theorem 5.10 implies that for a very general point x in Y_{d-1} the Newton-Okounkov bodies $\Delta_{\{Y_1 \supseteq \cdots \supseteq \{x\}\}}(S_\bullet)$ all coincide.

We will now show that for the special case of a surface X and a finitely generated birational graded linear series this result can be established in a more direct way and also holds for a general choice of points of the flag. Let X be a smooth surface, D a big divisor on X and C a smooth curve. Then for each $x \in C$ we obtain an admissible flag $X \supseteq C \supseteq \{x\}$. We have the following description of the Newton-Okounkov body on surfaces (see also

4.11):

$$\Delta_{C \supset \{x\}}(D) = \{(t, y) \in \mathbb{R}^2 : a \leq t \leq \mu, \text{ and } \alpha(t) \leq y \leq \beta(t)\}.$$

Without loss of generality we may replace D by $D - aC$ and assume that C is not contained in the support of the negative part of D . Then $\alpha(t) = \text{ord}_x(N_t|C)$ and $\beta(t) = \text{ord}_x(N_t|C) + (C \cdot P_t)$, where $D_t := D - tC = P_t + N_t$ is the Zariski decomposition. However, it is an easy consequence of [KLM12, Proposition 2.1] that, the support of all N_t is contained in a finite union of closed subvarieties. Hence, we can choose a general point $x \in C$, such that $x \notin \text{supp}(N_t)$ for each $t \in [0, \mu]$. Then $\text{ord}_x(N_t|C) = 0$ and $\alpha(t)$ as well as $\beta(t)$ do not depend on the general point $x \in C$. This shows that for a general choice of $x \in C$, the Newton-Okounkov body $\Delta_{\{C \supset \{x\}\}}(D)$ is independent from x .

Now let S_\bullet be a finitely generated birational graded linear series on X . Without loss of generality we may assume that it is finitely generated in S_1 . Let $\pi: X' \rightarrow X$ be the blow-up of X along \mathfrak{b}_{S_1} and let \tilde{C} be the strict transform of C . Without loss of generality, we may assume that X' and \tilde{C} are smooth. If this does not hold we can pass to a resolution of singularities, without changing the Newton-Okounkov body. By Theorem 3.11, we have for all $x \in \tilde{C} \setminus \mathbf{B}(\pi^*S_\bullet)$:

$$\Delta_{\{\tilde{C} \supset \{x\}\}}(\pi^*D - E) = \Delta_{\{C \supset \{x\}\}}(S_\bullet).$$

But the above discussion shows that the left hand side does not depend on \tilde{x} for a general choice. Hence, also the right hand side does not.

Example 5.12 (Flags of complete intersection of very ample divisors). In this example we want to consider flags which are defined by complete intersections corresponding to global sections of a fixed very ample divisor. We will see that the family of such flags induces an admissible flag. Thus we can define a generic Newton-Okounkov body corresponding to a birational graded linear series S_\bullet on X , which just depends on the choice of a very ample divisor A .

Consider the variety $S' := \mathbb{P}(H^0(X, \mathcal{O}_X(A)))^{d-1}$. By Bertini's Theorem, there is an open subvariety $S \subset S'$ such that for all $([s_1], \dots, [s_{d-1}]) \in S'$, the variety cut out by the s_1, \dots, s_i

$$Y_i = \{x \in X \mid s_1(x) = \dots = s_i(x) = 0\}$$

for $i = 1, \dots, d-1$ are smooth of codimension i in X . Consider the variety

$$T := \{(x, s_1, \dots, s_{d-1}) \in X \times_{\mathbb{C}} S \mid s_1(x) = \dots = s_{d-1}(x) = 0\}.$$

as our parameter space, as well as $X_T = X \times_{\mathbb{C}} T$ as our total space. Note that T is irreducible since, it surjects into S which is irreducible and the fibers T_s are irreducible curves for each $s \in S$.

Then we can define the partial flag \mathcal{Y}_\bullet by setting

$$\mathcal{Y}_i := \{(x, y, [s_1], \dots, [s_{d-1}]) \in X_T \subseteq X \times_{\mathbb{C}} X \times_{\mathbb{C}} S \mid s_1(x) = \dots = s_i(x) = 0\}.$$

From the construction it follows that for each $t = (y, [s_1], \dots, [s_{d-1}]) \in T$, the induced flag $Y_{t, \bullet}$ consists of the smooth varieties $Y_{t,i}$ defined above for $i = 1, \dots, d-1$ and $Y_{t,d} = \{y\}$. Now, we want to show that the \mathcal{Y}_i are Cartier divisors in \mathcal{Y}_{i-1} . We may without loss of generality replace the variety T by an open subset $U \subseteq T$. Then we can assume that T is smooth and all the \mathcal{Y}_i are flat over T . Since all the fiber $Y_{i,t}$ for $t \in T$ are smooth, we can use [GW, Proposition 14.57] to deduce that \mathcal{Y}_i is smooth as well. Hence, all the \mathcal{Y}_i can be considered as Cartier divisors in \mathcal{Y}_{i-1} .

We have shown that $(X_T, S_{T, \bullet}, \mathcal{Y}_\bullet)$ is an admissible family of Newton-Okounkov bodies and can therefore use Theorem 5.10 to get a generic Newton-Okounkov body $\Delta_A(S_\bullet)$ corresponding to the very ample line bundle A and the birational graded linear series S_\bullet .

5.4.2. Infinitesimal Newton-Okounkov body. Finally we do not just want to vary the flag Y_\bullet but also the variety X by considering blow-ups at various points on a variety. So let us fix a birational graded linear series S_\bullet on a smooth variety X . Then if we choose a point $x \in X$, we denote by X_x the blow-up of X at x . Let $E = \mathbb{P}(T_x X)$ be the exceptional divisor and $\pi: X_x \rightarrow X$ the corresponding blow-up morphism. Then for each choice of flags of vector spaces

$$T_x X = V_0 \supseteq V_1 \supseteq \dots \supseteq V_{d-1} \supseteq \{0\}$$

we get an induced linear flag $\mathbb{P}(V_\bullet)$ defined by

$$\mathbb{P}(T_x X) = E = \mathbb{P}(V_0) \supseteq \mathbb{P}(V_1) \supseteq \dots \supseteq \mathbb{P}(V_{d-1}) \supseteq \{pt\}$$

on X_x starting with E . Hence, we can define

$$\Delta_{F(x, V_\bullet)} := \Delta_{V_\bullet}(\pi^* S_\bullet),$$

which we call an infinitesimal Newton-Okounkov body (see also [LM09, Section 5.2]). We want to see that this construction varies in an admissible family of Newton-Okounkov bodies. The following lemma is a first step.

Lemma 5.13. *Let X be a smooth projective variety. There is a smooth projective variety B and a projection $p: B \rightarrow X$ such that for each $x \in X$ the fiber B_x is isomorphic to X_x which is the blow-up of X at the point x .*

Proof. Consider the diagonal closed embedding $X \hookrightarrow X \times_{\mathbb{C}} X$. Let $\pi: B := Bl_X(X \times_{\mathbb{C}} X) \rightarrow X \times_{\mathbb{C}} X$ be the blow-up of the closed variety X inside $X \times_{\mathbb{C}} X$ with respect to the above embedding. We consider B as a family over X by

$$B \xrightarrow{\pi} X \times_{\mathbb{C}} X \xrightarrow{p_2} X$$

where p_2 is the projection on the second factor. Let $x \in X$ be a closed point. Then we make the following abbreviations:

$$\{x\} := \text{Spec } k(x) \quad X \times_{\mathbb{C}} \{x\} := X \times_{\mathbb{C}} X \times_X \text{Spec } k(x).$$

Now we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & B & \xrightarrow{\pi} & X \times_{\mathbb{C}} X & \xrightarrow{p_2} & X \\
 & & \uparrow p_1 & & \uparrow & & \uparrow \\
 & & B \times_{X \times X} (X \times_{\mathbb{C}} \{x\}) & \xrightarrow{p_2} & X \times_{\mathbb{C}} \{x\} & \xrightarrow{p_2} & \{x\} \\
 & \nearrow q' & & & & & \\
 Bl_{(x,x)}(X \times_{\mathbb{C}} \{x\}) & \xrightarrow{q} & & & & &
 \end{array}$$

Note that the two rightmost boxes are Cartesian and the map q is given by the universal property of fiber products. We want to prove that the morphism q is an isomorphism. We will do this by constructing the inverse map, using the universal property of the blow-up map

$$Bl_{(x,x)}(X \times_{\mathbb{C}} \{x\}) \rightarrow X \times_{\mathbb{C}} \{x\}.$$

This is constructed by showing that the inverse image of $(x, x) \in X \times_{\mathbb{C}} \{x\}$ under the map

$$p_2: B \times_{X \times X} (X \times_{\mathbb{C}} \{x\}) \rightarrow X \times_{\mathbb{C}} \{x\}$$

is an effective Cartier divisor. But this inverse image is the same as the the inverse image of the exceptional divisor $\pi^{-1}(X) \subset B$ under the map p_1 . However, one can easily see that that the image of p_1 is not contained in the exceptional divisor $\pi^{-1}(X)$ and hence, we can pull back the Cartier divisor by just pulling back the local equation. Consequently, we have defined a map q' which fits into the above commutative diagram. Now, by the universal property of the blow-up and the universal property of the fiber product in the middle box, we deduce that q' is an inverse map of q .

Thus, the fiber B_x which is just $B \times_{X \times X} (X \times_{\mathbb{C}} \{x\})$ is isomorphic to $Bl_{(x,x)}(X \times_{\mathbb{C}} \{x\})$ which can be interpreted as the blow-up $Bl_x(X)$ of X in the point $\{x\}$. □

Let us denote by $N := N_{X/X \times X}$ the normal bundle of the diagonal embedding, viewed as a vector bundle over X . The projectivization $\mathbb{P}(N)$ is the exceptional divisor of the blow-up B . Its fibers $\mathbb{P}(N)_x$ are isomorphic to the exceptional divisors E_x of X_x . Let $T := Fl(N) \xrightarrow{p} X$ be the flag bundle of N over X . By the splitting principle, there exists a filtration of vector bundles of $p^*N = N \times_X T$:

$$p^*N = N \times_X T \supset \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_{d-2} \supset \{0\}.$$

We can also consider the projectivized filtration:

$$\mathbb{P}(N) \times_X T \supset \mathbb{P}(\mathcal{V}_1) \supset \cdots \supset \mathbb{P}(\mathcal{V}_{d-2}).$$

Now we define $X_T := B \times_X T$, as well as $\mathcal{Y}_1 := \mathbb{P}(N) \times_X T$ which is a subset of codimension one in X_T since $T \xrightarrow{p} X$ is flat. Furthermore, we define

$\mathcal{Y}_i := \mathbb{P}(\mathcal{V}_{i+1})$. The so defined flag \mathcal{Y}_\bullet is admissible and has the desired properties on the fibers over $t \in T$ since $\mathbb{P}(N) \times_X k(t)$ is isomorphic to the exceptional divisor of the blow-up of X in $p(t)$.

In order to define a graded linear series induced by a graded series of sheaves, we need to shrink T a bit more. Let S_\bullet be a graded linear series on X . Let $Z := \mathbf{B}(S_\bullet)$ be the corresponding base locus and define $U = X \setminus Z$.

Then we consider the base change $T' := T \times_X U \xrightarrow{p'} U$ which is an open subset of T and define $X_{T'} := X_T \times_X T' = B \times_X T \times_X U$. Now consider the following composition of morphisms of varieties

$$X_{T',\mu'} \hookrightarrow X_{T'} \rightarrow (B \times_X U) \rightarrow (X \times X) \times_X U \xrightarrow{p_1} X.$$

Let \mathcal{S}_\bullet be the sheafification of S_\bullet . We can now use Lemma 5.2 to deduce that the pullback of S_\bullet to $X_{T'}$, which we define as $\mathcal{S}_{T',\bullet}$, defines a graded linear series since it factors as the pullback of a flat morphism composed with a birational morphism which has the prescribed property of Lemma 5.2, again composed with a flat morphism. Furthermore, the composed map $X_{T',\mu'} \rightarrow X$ is just the blow-up morphism of X in $p'(t) \in U$. Again we can use Lemma 5.2 to deduce that $\mathcal{S}_{t,\bullet}$ defines a graded linear series on $X_{T',\mu'}$ and $\mathcal{S}_{t,k} \subseteq \mathcal{O}_{X_x}(D|_{X_x})$. Corollary 5.3, then says that the Newton-Okounkov body of $\mathcal{S}_{t,\bullet}$ is the same as for the graded linear series π^*S_\bullet which is given by pulling back the global sections. We can also replace the flag \mathcal{Y}_\bullet with the flat base change $\mathcal{Y}'_\bullet := (\mathcal{Y} \times_X U)_\bullet$. Then it follows from our discussion that $(X_{T'}, \mathcal{Y}'_\bullet, \mathcal{S}_{T',\bullet})$ is an admissible family of Newton-Okounkov bodies over T' . Let us summarize what we have shown.

Theorem 5.14. *Let X be a smooth variety and S_\bullet be a birational graded linear series. Then for a very general choice of points $p \in X$ and a linear flag V_\bullet starting with $E_x := \mathbb{P}(T_x(X)) \cong \mathbb{P}^{d-1}$, the corresponding Newton-Okounkov bodies $\Delta_{F(x,V_\bullet)}(S_\bullet)$ all coincide.*

□

REFERENCES

- [A13] Anderson, D., *Okounkov bodies and toric degenerations* Math. Ann. **356** (2013), no. 3, 1183-1202
- [AKL12] Anderson, D., Küronya, A., Lozovanu, V., *Okounkov bodies of finitely generated divisors*, International Mathematics Research Notices 2013; doi: 10.1093/imrn/rns286
- [BG09] Bruns, W., Gubeladze, J., *Polytopes, rings, and K-theory*, Springer, Dordrecht, (2009).
- [ELMNP09] Ein, L., Lazarsfeld, R., Mustața, M., Nakamaye, M., Popa, M., *Restricted Volumes and Base Loci of Linear Series* American Journal of Mathematics **131**(2009), No. 3, 607-651
- [GW] Görtz, U., Wedhorn, T., "Algebraic Geometry: Part I: Schemes. With Examples and Exercises (Advanced Lectures in Mathematics)", Vieweg+Teubner Verlag, 2010
- [H13] Hisamoto, T., *On the volume of graded linear series and Monge Ampre mass*, Math. Z. **275** (2013), 233-243

- [J10] Jow, Shin-Yao, *Okounkov bodies and restricted volumes along very general curves*, Adv. Math. **223**, (2010), 4, 1356–1371
- [KK12] Kaveh, K., Khovanskii, A.G., *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. **176** (2012), 925–978.
- [KL14] Küronya, A., Lozovanu, V., *Local Positivity of linear Series on Surfaces*, accepted in Algebra & Number Theory, arXiv:1411.6205
- [KL15] Küronya, A., Lozovanu, V., *Positivity of line bundles and Newton-Okounkov bodies*, Documenta Math., **22** (2017), pp. 1285–1302
- [KL17] A. Küronya and V. Lozovanu. *Infinitesimal Newton-Okounkov bodies and jet separation*. Duke Math. J. **166** (2017), 1349–1376.
- [KLM12] Küronya, A., Lozovanu, V., Maclean, C., *Convex bodies appearing as Okounkov bodies of divisors* Advances in Mathematics **229** (2012), 2622-2639
- [KLM13] Küronya, A., Lozovanu, V., Maclean, C., *Volume functions of linear series*, Mathematische Annalen **356** (2013), 635–652
- [KMS12] Küronya, A., Maclean, C., Szemberg, T., *Functions on Okounkov bodies coming from geometric valuations (with an appendix by Sébastien Boucksom)* preprint, arXiv:1210.3523
- [L04] Lazarsfeld, R., “Positivity in Algebraic Geometry I”, Springer, 2004
- [LM09] Lazarsfeld, R., Mustață, M., *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 783–835
- [O96] Okounkov, A., *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996), 405–411
- [SS17] Schmitz, D., Seppänen, H., *Global Okounkov bodies for Bott-Samelson varieties*, Journal of Algebra **490** (2017), 518–554

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Anhänge

Erklärung gemäß §8.2

Ich versichere hiermit, dass ich die Dissertation selbstständig und ohne unerlaubte Hilfe angefertigt habe. Ich habe anderweitig keine Promotion beantragt, für die die hier vorgelegte Dissertation oder Teile hiervon eingereicht worden sind.



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