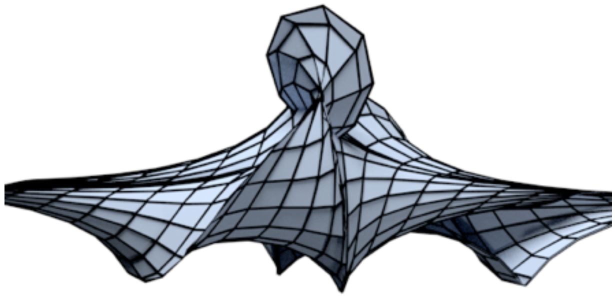

A theory of discrete parametrized surfaces in $\mathbb{R}^3$

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Abstract

In *discrete differential geometry* (DDG) one considers objects from discrete geometry from a differential geometric perspective. Rather than focusing on approximations of the smooth theory, with error vanishing in the continuum limit, DDG focuses on theories that *exactly preserve* geometric quantities and/or the structure of governing equations at every finite resolution.

This thesis is concerned with the DDG of *parametrized* surfaces in three dimensional Euclidean space represented as so-called "quad nets", immersed two dimensional complexes with quadrilateral faces. Our main focus is to obtain analogs of differential geometric notions for quad nets with immersed quadrilaterals that are *nonplanar*. This thesis is split into two parts.

In Part I, we introduce the theory of *edge-constraint nets*, a general discrete surface theory in $\mathbb{R}^3$ that unites the most prevalent versions of discrete analogs of surfaces in special parametrizations. Our theory encapsulates a large class of discrete so-called integrable geometries and, in particular, provides geometric insight into the algebraically constructed discrete analogs of one-parameter associated families of constant curvature surfaces, by introducing notions for both curvature and conformality. Conformal equivalence is introduced for edge-constraint nets using a discrete analog of spin transformations, which is then used to construct discrete Bonnet pairs, two immersed surfaces that are isometric and have the same mean curvature, but are not congruent. Edge-constraint nets are not restricted to integrable geometries, but lay the foundation for a general surface theory that lifts the restriction to special parametrizations.

In Part II, we apply DDG principles to design with inherently discrete materials built from regular grids of inextensible rods, ranging from densely woven wire mesh to sparse elastic gridshells. We model these materials using smooth and discrete Chebyshev nets, a special type of surface parametrization that encodes rod inextensibility. Analytical properties of Chebyshev nets impose counterintuitive global constraints, but taking an applied perspective leads to computational algorithms for interactive design tools and an understanding of the global nature of designing with such materials, whose results suggest a rich design space.

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CHAPTER 1

Overview

Introduction

In *discrete differential geometry* (DDG) one considers objects from discrete geometry from a differential geometric perspective [13, 5, 10, 9]. Rather than focusing on approximations of the smooth theory, with error vanishing in the continuum limit, DDG focuses on theories that *exactly preserve* geometric quantities and/or the structure of governing equations at every finite resolution. Even while taking a structure preserving approach to discretization, multiple perspectives of the same smooth object may arise; in particular, choosing which property to preserve often leads to seemingly, and sometimes provably, disparate discrete theories. Beyond theoretical interest, explorations in DDG have been motivated by applications in computer graphics and architectural design [25], where suitably chosen discrete analogs accelerate computational algorithms and/or allow for material constraints to be maintained during simulation.

This thesis is concerned with the DDG of *parametrized* surfaces in three dimensional Euclidean space represented as so-called "quad nets", immersed two dimensional complexes with quadrilateral faces. Our main focus is to obtain analogs of differential geometric notions for quad nets with immersed quadrilaterals that are *nonplanar*. In this overview chapter we provide context for and a discussion of the main results, which are then detailed in a series of articles in the subsequent chapters [1]. The contributions are split into a theoretical Part I (Chapters 2, 3, and 4) and an applied Part II (Chapters 5 and 6).

In Part I, we introduce the theory of *edge-constraint nets*, a general discrete surface theory in $\mathbb{R}^3$ that unites the most prevalent versions of discrete analogs of surfaces in special parametrizations. Our theory encapsulates a large class of discrete so-called integrable geometries and, in particular, provides geometric insight into the algebraically constructed discrete analogs of one-parameter associated families of constant curvature surfaces, by introducing notions for both curvature and conformality. Conformal equivalence is introduced for edge-constraint nets using a discrete analog of spin transformations, which is then used to construct discrete Bonnet pairs, two immersed surfaces that are isometric and have the same mean curvature, but are not congruent. Edge-constraint nets are not restricted to integrable geometries, but lay the foundation for a general surface theory that lifts the restriction to special parametrizations.

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[1]This overview contains text snippets and figures that are taken directly from these articles.

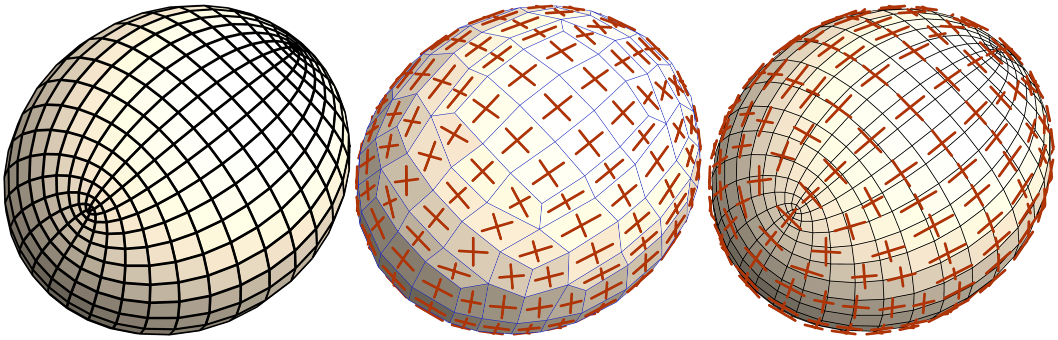

FIGURE 1. Left: Curvature lines (black) of a smooth ellipsoid. Middle: Curvature line field (red) of an ellipsoid edge-constraint net. Right: Overlay of smooth and discrete curvature line fields.

Part I: edge-constraint nets in $\mathbb{R}^3$

Within discrete differential geometry, one considers a discrete parametrized surface as a *quad graph* $\mathbb{G}$, given by a cell decomposition of a two dimensional surface into quadrilateral faces. An immersion map $f : \mathbb{G} \to \mathbb{R}^3$ with nonvanishing straight edges is referred to as a *quad net*. Notice that quad nets generally have nonplanar faces. Quad nets corresponding to particular types of surfaces have been discretized using algebro-geometric approaches for *integrable* geometry—originally using discrete analogs of soliton theory techniques (e.g., discrete Lax representations and finite-gap integration [6, 23, 7]) to construct nets, and more recently using the notion of 3D consistency (encoding discrete Bäcklund–Darboux transformations), which has emerged as an organizing principle of DDG [9]. However, both in the smooth and discrete settings, integrability is bound to specific choices of parameterizations, e.g., asymptotic line or curvature line. Therefore, while these discretizations maintain characteristic integrable properties of their smooth counterparts they often lead to discrete surfaces only in special parametrizations and with seemingly disparate considerations. In other words, different parametrizations of the same smooth surface may result in different discrete analogs.

In this thesis we take a more general approach to quad nets, laying the foundation for a more general discrete parametrized surface theory that lifts the restriction to special parametrizations. Our perspective is to consider a discrete parametrized surface, not as a single quad net, but as a pair of quad nets $f : \mathbb{G} \to \mathbb{R}^3$ and $n : \mathbb{G} \to \mathbb{S}^2$, corresponding to a discrete *immersion* and *Gauß map*, respectively. The fundamental property of our approach is the following *edge-constraint* that couples discrete surface points of the immersion and normals of the Gauß map:

the average normal along an immersed edge is perpendicular to that edge.

This condition arises from a Steiner-type, i.e., offset and mixed area, perspective on curvature and, while elementary, has surprisingly profound consequences for the theory. By introducing a Gauß map for general nonplanar quad nets, our theory of edge-constraint nets builds on basic construction principles of the classical smooth setting (details are provided in Chapter 2). In particular, the edge-constraint guarantees the symmetry of a discrete second fundamental form/shape operator associated to each quad. The resulting principal curvatures are consistent with the mean and Gauß curvature arising from the Steiner perspective and the curvature line fields resemble their smooth counterparts as shown in Figure 1, despite arbitrary vertex valences in the underlying quad graph.

Edge-constraint nets and their curvature theory provide a unifying geometric perspective through which to understand previously defined notions of discrete surfaces in special parametrizations. Moreover, it provides geometric insight into their algebraically generated associated families that arise from an integrable perspective on discretization using a so-called Lax representation. We also find relationships to previously introduced examples of nonintegrable geometries. Moreover, the edge-constraint can be reinterpreted in a quaternionic setting, which gives rise to conformal equivalence between nets using spin transformations. We therefore find that the perspective afforded by edge-constraint nets as discrete parametrized surfaces acts as a first step toward a general discrete parametrized surface theory. However, to appreciate our main results afforded by this more general perspective, one requires the context of the successes of integrable approaches to discrete parametrized surfaces. Therefore, we provide this context first, and then further detail our main results.

Context: discrete parametrizations from integrable geometries. Lax representations and extended moving frames. We briefly review the algebraic method of Lax representations and extended moving frames for smooth integrable geometries in special parametrizations. They result in one-parameter *associated families* of surfaces, and for example, include the associated families of constant mean or Gauß curvature. Extended moving frames naturally carry over to the discrete setting, allowing one to systematically *algebraically* define associated families of quad nets that should correspond to particular smooth classes of surfaces in special parametrizations. For a review of this approach see [8]. In this thesis we show that the perspective of edge-constraint nets helps clarify the geometry of these resulting families of discrete quad nets.

Put briefly, a Lax representation of a parametrized surface in $\mathbb{R}^3$ arises when the compatibility conditions of its moving frame $\Phi \in SU(2)$ exhibit a symmetry that allows them to be satisfied by not a single frame, but a one-parameter *associated family*, of moving frames $\Phi(\lambda)$, depending on a so-called spectral parameter λ. With coordinates (x, y), the linear system $\Phi_x(\lambda) = U(\lambda)\Phi(\lambda), \Phi_y(\lambda) = V(\lambda)\Phi(\lambda)$ determines so-called *Lax transition matrices* $U(\lambda), V(\lambda)$; the compatibility condition for the moving frame is then expressed as

$$U_y(\lambda) - V_x(\lambda) + [U(\lambda), V(\lambda)] = 0. \tag{1}$$

Remarkably, this spectral parameter dependence allows the family of parametrized surfaces f^λ, with Gauß map n^λ, to be found through differentiation (with respect to the parameter λ), as opposed to integration (with respect to the surface coordinates x, y), using the so-called Sym–Bobenko formula [3, 30]:

$$n^\alpha = \Phi^{-1}\Bbbk\Phi, \quad f^\alpha = \Phi^{-1}\frac{d}{d\lambda}\Phi\,. \tag{2}$$

While a general discrete analog of moving frames is not well understood, the Sym–Bobenko formula allows this entire construction to naturally carry over to the discrete setting. Using the structure of a smooth extended frame as an ansatz, one can define discrete Lax representations for a frame $\Phi(\lambda)$ associated to vertices of a quad graph $\mathbb{G}$, with *Lax transition matrices* $U(\lambda), V(\lambda)$ associated to each pair of opposite edges of a quad, respectively. The Sym–Bobenko formula (2) then defines the points of an associated family of discrete quad nets $f^\lambda : \mathbb{G} \to \mathbb{R}^3$, together with an associated "surface normal" from the corresponding unit Gauß map $n^\lambda : \mathbb{G} \to \mathbb{S}^2$. These points are then connected, according to the underlying combinatorics of $\mathbb{G}$, with straight edges in $\mathbb{R}^3$ to form a quad net.

K-surfaces as a motivating example. The relationship between surface theory and integrable systems, however, often only reveals itself in special parametrizations. The most well known example is given by surfaces of constant negative Gauß curvature $\mathcal{K} = -1$, which we refer to as *K-surfaces*. Surfaces of negative Gauß curvature are often parametrized in terms of asymptotic lines along which the normal curvature vanishes. A surface parametrized by asymptotic lines is a K-surface, i.e., has *constant* negative Gauß curvature, if and only if the asymptotic lines form a so-called *weak Chebyshev net*. A weak Chebyshev net parametrization is one in which the coordinates may be simultaneously reparametrized by arc length (with possibly different constants a, b in each direction). The resulting anisotropic Chebyshev nets are built from "infinitesimal parallelograms with constant side "lengths" $a, b > 0$[2]. In these coordinates, the compatibility condition (1) is expressed in terms of the angle between the asymptotic lines $\omega(x, y)$:

$$\omega_{xy} - ab \sin \omega = 0. \tag{3}$$

This so-called sine-Gordon nonlinear partial differential equation is a well known integrable, i.e., soliton, equation from mathematical physics. The invariance of the sine-Gordon equation to the transformation $a \to \lambda a$ and $b \to \lambda^{-1} b$, allows for the insertion of a spectral parameter $\lambda \in \mathbb{R}$, yielding an extended moving frame and associated Lax representation. Varying λ generates an associated family of K-surfaces in asymptotic line parametrization, where the angle between the asymptotic lines is invariant, but the parameters are scaled differently in each coordinate.

K-nets and the development of DDG. In the early 1950s Wunderlich [32] suggested a geometric discretization of K-surfaces by introducing a quad net analog of asymptotic coordinates and then further restricting each quad to a skew rhombus, a discrete analog of Chebyshev nets (when the arc length parameters are equal, i.e., $a = b$). The resulting rhombic nets resembled smooth K-surfaces and are now called *(rhombic) K-nets*. However, this approach did not recover a discrete analog of the sine-Gordon equation (3). In the 1970s, motivated by investigations in mathematical physics, Hirota introduced a nonlinear partial *difference* equation that is a discrete analog of the sine-Gordon equation on a quad (a fourth point is determined by an initial three), but without reference to the geometry of surfaces [17]. In 1996, Bobenko and Pinkall [6] showed that an extended moving frame with Hirota's discrete sine-Gordon equation as compatibility condition generates geometrically defined rhombic K-nets via the discrete Sym–Bobenko formula, and, after relaxing to K-nets built from skew-parallelograms, every K-net arises in this way. This result provided an essential link between previously unrelated geometric and algebraic approaches to structure preserving discretization, laying the foundation for modern discrete differential geometry. Note, however, that this result still does not prioritize a discrete analog of Gauß curvature. We will discuss a second approach to discrete K-surfaces based on curvatures and curvature line coordinates, below.

The role of special parametrizations in extended frames. The associated family of an asymptotic line parametrized K-surface is also parametrized by asymptotic lines. However, expressing a K-surface in, for example, curvature line coordinates does not suggest a natural way to inject a spectral parameter. Using smooth reparametrization one may still write down an associated family, but it no longer satisfies this "parametrization preservation"; in the discrete setting reparametrization alone is challenging. Moreover, even when the injection of a spectral parameter may reveal itself in curvature line coordinates, the associated family is generally no longer

[2]Chebyshev nets built from "infinitesimal rhombi" ($a = b$) play an important role in Part II of this thesis, since this parametrization encodes inextensibility along two parameter directions and can be used to model discrete materials built from a lattice of inextensible rods.

parametrized by curvature lines. For example, a surface of constant mean curvature written in conformal, curvature line coordinates exhibits an extended moving frame whose surfaces in the associated family continue to be conformally parametrized, but no longer by curvature lines. Investigating these more generally parametrized associated families is a central topic of this thesis.

However, many integrable geometries exhibit so-called Bäcklund–Darboux transformations. These transformations allow new solutions to a nonlinear PDE to be constructed from old ones [26]. At the level of parametrized surfaces these results correspond to classical results in differential geometry that preserve special parametrizations and the class of surface, e.g., curvature line parametrizatons map to curvature line parametrizations and K-surfaces map to K-surfaces. In the discrete setting, one often understands the existence of Bäcklund–Darboux transformations in terms of so-called *3D, or multidimensional, consistency.* This notion has become an organizing principle both of modern DDG and the theory of discrete integrable systems [22, 1, 12, 9]. We briefly introduce 3D consistency, and then a second approach to discrete K-surfaces, this time in terms of a well established discrete analog of curvature line coordinates, known as *circular or C-nets.*

3D consistency and C(ircular)-nets An algebraic equation or geometric property defined on a single quad is called 3D consistent if it can be extended onto all faces of a combinatorial cube. As an example, consider a *circular quad*, whose four vertices lie on a common circle (and therefore in a common plane). Consider seven points that define three circular quads meeting at a common vertex. These three quads define the bottom, front, and left faces of a combinatorial cube. As originally pointed out by Cieśliński, Doliwa, and Santini [12], one can prove that the circles defined by each set of three vertices lying on the top, back, and right faces meet at a unique eighth point, closing a combinatorial cube. Therefore, the notion of C(ircular)-nets, where each quad is circular, is called 3D consistent. C-nets are well established as discrete curvature line parametrizations [3]. One understands the relationship to Bäcklund–Darboux as follows. From an initial circular quad—understood as the bottom face of a combinatorial cube—one has freedom to prescribe front and left faces, such that the resulting closed combinatorial cube gives rise to another circular quad on its top face. A discrete Bäcklund–Darboux transformation transforms the bottom circular quad into the top circular quad, mimicking the preservation of curvature line coordinates in the smooth setting. It turns out that a quad property that is 3D consistent is in fact multidimensionally consistency, which in 4D recovers Bianchi's classical result on the permutability of Bäcklund–Darboux transformations. One can prescribe additional constraints on circular quads and still recover 3D and multidimensional consistency, which has allowed more specific types of surfaces to be defined as "consistent reductions" of circular nets. In particular, we review a second approach to discrete K-surfaces.

Steiner offset curvatures for C(ircular)-nets and a second approach to K-surfaces. One can introduce a Gauß map on a C-net by associating unit vectors to each vertex. In analogy to smooth curvature line parametrizations, Schief [27] suggested that the edges of this Gauß map should lie parallel to the corresponding edges of the C-net itself. This condition guarantees that a circular quad, its Gauß map quad, and its "normal" offset quad (defined per vertex by $f + tn$ for offset constant $t \in \mathbb{R}$) are all circular and lie in parallel planes. The areas of each of these planar quads can be measured, thus allowing mean curvature $\mathcal{H}$ and Gauß curvature

[3]There are many other deep reasons to treat the class of circular nets as a discrete analog of the class of curvature line parametrized surfaces [4], e.g., both classes are preserved under Möbius transformations.

$\mathcal{K}$ to be defined analogously to the smooth *Steiner offset formula* (4): for $t > 0$ small enough, the area element of a normal offset of a smooth surface in $\mathbb{R}^3$ is a polynomial in the area element, mean, and Gauß curvature of the original surface. One finds:

$$dA(f + tn) = dA(f)(1 - 2t\mathcal{H} + \mathcal{K}t^2)\,. \tag{4}$$

Discrete K-surfaces in (circular) curvature line parametrization, referred to here as *cK-nets*, can now be defined as a C-net with Gauß map that has constant negative Gauß curvature in this sense [28]. As mentioned above, Bäcklund transformations of K-surfaces preserve curvature lines, and in the discrete setting the reduction of C-nets to cK-nets retains 3D consistency. However, we note that the defining notion of Gauß curvature for a cK-net requires circular—in particular, planar—quads, making the relationship between the theory of cK-nets and the theory of K-nets with nonplanar quads unclear. Moreover, it is unclear how to define an extended moving frame for cK-nets, even though in the smooth setting the two theories are related by reparametrization. Reconciling seemingly disparate theories that arise from extended moving frame, 3D consistency, and curvature approaches is a central topic of this thesis.

Main results. We summarize our main results of Chapters 2, 3, and 4.

(Chapter 2) Curvatures for edge-constraint nets. Our perspective for curvatures to edge-constraint nets also takes a Steiner approach (4). We therefore recover the established notion of curvatures for circular nets with a Gauß map on vertices [27, 11]. Our theory allows to understand the mean and Gauß curvatures of an arbitrary quad net with a Gauß map satisfying the edge-constraint, even though these immersions generally have *nonplanar* faces. Some of our main results emphasize how these curvatures unite previously defined integrable discretizations of surfaces of constant curvature, together with their associated families.

(Chapters 2 and 3) Constant negative Gauß curvature nets. In our brief review of discrete integrable geometries above, we discussed two previous approaches to discrete K-surfaces, the first in discrete asymptotic line parametrization (K-nets) and the second in discrete curvature line parametrization (cK-nets). Since edge-constraint net curvatures agree with those for circular nets, we immediately find that cK-nets are edge-constraint nets of constant negative Gauß curvature. For K-nets, we prove that the moving frame of a rhombic K-net (equal arc length parameters $a = b$), together with its associated family of anisotropic K-nets (constant arc length parameters a, b), yield immersion and Gauß map quad nets that are edge-constraint nets of constant negative Gauß curvature [4]. However, our results extend beyond an understanding of these two approaches at the level of curvature.

The perspective of edge-constraint nets suggested an investigation of extended moving frames in more general parametrizations. This led to a clarification of the relationship between the theory of K-nets and cK-nets; Chapter 3 is entirely devoted to this study. The relationship resembles smooth reparametrization of K-surfaces, where curvature line coordinates arise as the sum and difference of unit arc length parametrized asymptotic lines. We perform a similar "discrete reparametrization" by considering new Lax transition matrices that are the products of a pair of rhombic K-net transition matrices. This leads to an extended moving frame that is tightly linked to the 4D compatibility of K-nets. This extended frame gives rise to cK-nets, and conversely, every cK-net arises in this way. The cK-net extended frame gives rise to an associated family whose members are no longer cK-nets, but turn out to be edge-constraint nets with constant negative Gauß curvature. This also leads to

[4]Weak K-nets, where a and b are not constant, still form edge-constraint nets, but seem to no longer have constant negative Gauß curvature in this sense.

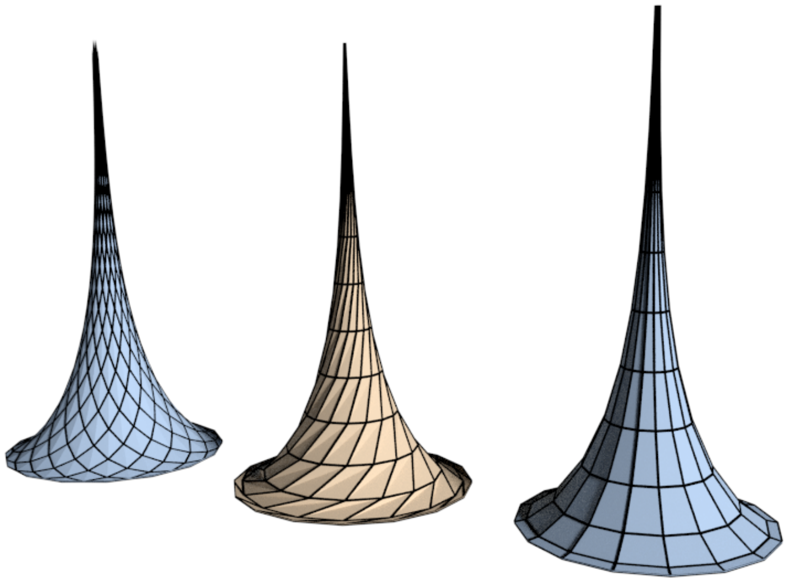

FIGURE 2. Three pseudospheres of revolution which are constant negative Gauß curvature edge-constraint nets. The perspective of edge-constraint nets elucidates the relationship between these discrete analogs. Left: A parametrization by discrete asymptotic lines (K-net). Middle: A parametrization with one discrete asymptotic line and one discrete curvature line. Right: A parametrization by discrete curvature lines (cK-net).

an approach for cK-net Bäcklund transformations that more faithfully resembles the smooth single-step Bäcklund transformation for cK-nets. In [28] it is noted that the analog of a Bäcklund transformation that arises from the 3D consistency of cK-nets in fact resembles a composition of two smooth transformations, rather than a single one. As we show, the combinatorial cube of a cK-net quad, together with its single-step Bäcklund transformation, is unusual since it is not 3D consistent; the top and bottom faces are circular quads while the side faces are skew parallelograms of a weak K-net. Particular double-step Bäcklund transformations recover the usual 3D consistent cube with circular faces.

(Chapter 2) Constant mean curvature nets. Smooth surfaces that exhibit curvature line coordinates that are simultaneously conformal are called isothermic surfaces. Within DDG, discrete isothermic nets are well established as a multidimensionally consistent reduction of circular nets to those whose cross-ratio of each circular quad (understood as a quad in the complex plane) can be "factored" onto the edges in a particular way [7, 16, 8]. Both smooth and discrete isothermic surfaces exhibit so-called Christoffel dual surfaces, and one characterization of a surface having constant mean curvature is that its Christoffel dual is simultaneously a Bäcklund–Darboux transformation. Moreover, this second constant mean curvature surface arises at a constant normal offset from the first one. Bobenko and Pinkall introduced a discrete moving frame using a smooth ansatz for constant nonvanishing mean curvature surfaces in isothermic parametrization. In analogy to all of the above smooth properties, the geometry of the resulting discrete isothermic quad net and its Christoffel dual, lying at a constant normal offset, form a 3D consistent circular net cube with constant mean curvature. However, as remarked in the section on extended moving frames, the associated family of this extended frame leads to constant mean curvature surfaces that are still conformally parametrized, but no longer parametrized by curvature lines. Therefore, the associated families of these nets remained difficult to understand in any of the above senses.

We show that every member of this associated family is an edge-constraint net of constant mean curvature that also exhibits a second constant mean curvature edge-constraint net at a constant offset. Our proof relies on characterizing the geometry of this pair of quad nets and proving that they form a 3D consistent combinatorial cube that generalizes the Bäcklund–Darboux understanding of the discrete isothermic case. In particular, we prove that the vertices of this cube coincide with those of a cube where every face is parallelogram "folded" in the same way. Unexpectedly, this 3D consistent "equally-folded parallelogram cube" yielding the vertices of a pair of associated family constant mean curvature quad nets is the same as the Bianchi permutatibility cube of a discrete so-called Bicycle (or Darboux) transformation of a polygonal curve, which is also known to be integrable in the smooth and discrete settings [31, 18, 24].

(Chapters 2 and 4) Minimal (vanishing mean curvature) nets. Restricting the notion of a discrete isothermic quad net to strictly planar immersions leads to discrete conformal quad nets in the complex plane [7]. These are related to what is now called discrete *nonlinear* complex analysis. These discrete conformal maps exhibit a Weierstrass representation that produces a discrete isothermic quad net in $\mathbb{R}^3$, together with a Gauß map, that has vanishing mean curvature in the circular net curvature theory. In complete analogy to the smooth case, one can extend this representation into an associated family. This corresponds to locally rotating the frame, therefore changing the type of parametrization away from being curvature line (while staying conformal in the smooth setting). In the discrete setting, an analogous algebraic method defines a one-parameter family of discrete quad nets that share a common Gauß map.

We show that every member of this family is in fact an edge-constraint net of vanishing mean curvature. To better understand the conformal equivalence between the members of this family we introduce a notion of discrete conformal equivalence between edge-constraint nets. The key idea to this approach is to introduce a discrete analog of a so-called *spin transformation*.

(Chapters 2 and 4) Discrete Bonnet pairs and conformal nets in $\mathbb{R}^3$. In the smooth setting two conformal immersions for a manifold M, $f, \tilde{f} : M \to \mathbb{R}^3 \cong \Im\mathbb{H}$, are said to be spin-equivalent if there exists a *spin transformation* $\lambda : M \to \mathbb{H}^*$, such that $\mathrm{d}\tilde{f} = \bar{\lambda}\mathrm{d}f\lambda$; the corresponding Gauß maps $n, \tilde{n}$ transform pointwise as $\tilde{n} = \lambda^{-1}n\lambda$. Geometrically, spin transformations correspond to stretch rotations of the tangent plane at every point. Therefore, they are conformal mappings and, for simply connected domains, any two surfaces that are conformally equivalent are related via a spin transformation. Kamberov, Pedit, and Pinkall [19] used spin transformations to classify all Bonnet pairs on a simply connected domain and provide their explicit construction from isothermic surfaces. Bonnet pairs are immersed surfaces that have the same metric and mean curvature but are not rigid body motions of each other.

To define a discrete analog of a spin transformation we observed that the edge-constraint can be reformulated using what we refer to as *normal transport* quaternions. These quaternions have imaginary part equal to the edge of the net and a real part that measures how much the normals at the edge's end points are twisted with respect to each other. These "extended" edges play the role of the smooth differential $\mathrm{d}f$ in our spin transformation. We show that our notion of spin transformation equips nonplanar quads, together with normals, with a so-called *spin-cross-ratio* that is generically different from the quaternionic cross-ratio of its four immersion

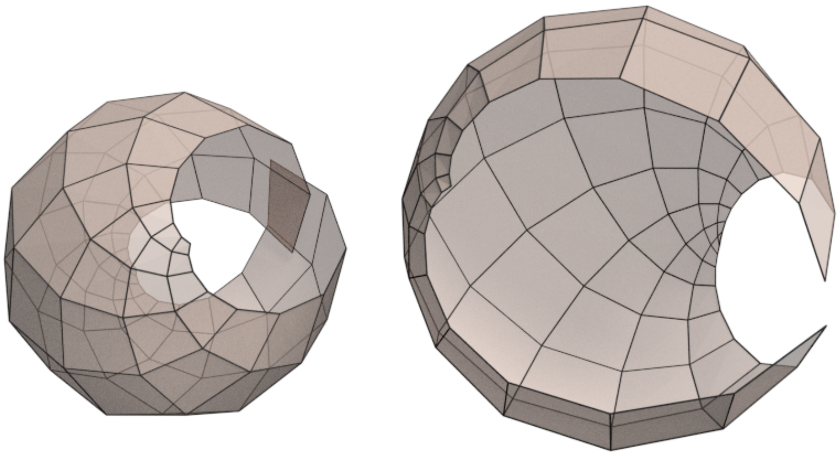

FIGURE 3. A discrete Bonnet pair: two discretely conformally equivalent edge-constraint nets that are isometric and have the same mean curvature per quad, but are not congruent. This pair is constructed from the Gauß map of a discrete isothermic minimal surface.

points, but agrees with the complex cross-ratio in the planar case and for circular nets in $\mathbb{R}^3$. This spin-cross-ratio leads to a naturally defined spin-metric that transforms "conformally," i.e., by a scalar multiple per quad, allowing us to define when two edge-constraint nets are isometric. We then prove our main result, that there exist discrete Bonnet pairs. That is, we explicitly construct spin-equivalent edge-constraint nets that are isometric and have the same mean curvature. Their construction is analogous to the smooth setting, arising as spin transformations of a well established notion of discrete isothermic surfaces. We note that discrete Bonnet pairs generally have nonplanar quads.

(Chapter 2) Developable (vanishing Gauß curvature) nets. Motivated by application, surfaces of planar strips have been considered as discrete developable as they can be unfolded into the plane [20]. We show that such immersions correspond to developable curvature line edge-constraint nets, which are characterized by a discrete analog of parallel framed curves [2]. Examples of developable edge-constraint nets that are not in curvature line parametrization arise from the associated family of a discrete isothermic cylinder; this family contains the well-known Schwarz Lantern [21] as an immersion with vertex normals that coincide with those of the smooth cylinder.

Together, we hope these results motivate further exploration of edge-constraint nets as a general discrete parametrized surface theory in $\mathbb{R}^3$.

Part II: designing with discrete materials from inextensible rods

The applied part of this thesis was motivated by questions posed by computer scientists and experimental physicists interested in designing with certain inherently discrete materials. The resulting collaborations led to the results here. In both instances, we focus on deforming inherently discrete materials whose rest configuration is a planar square grid of inextensbile rods. The grid may be composed of 100s or 1000s of rods that are densely woven to form a wire mesh structure that deforms plastically under an applied load (see Chapter 5 on wire mesh design); or the grid may be formed from only 10s of rods that are held together at the intersections and deform elastically under an applied load (see Chapter 6 on form-finding in elastic gridshells). The commonalities are (i) that the rest state is a connected domain cut from a planar square grid of rods and (ii) that the rods bend during deformation, but cannot change length—they are *inextensible*; the rods are, however, allowed to shear with respect to one another. An idealization of these features results in a model originally introduced in 1878 by Russian mathematician Pafnuty Chebyshev, to model a deformed piece of woven fabric—the special *Chebyshev net* parametrizations we encountered when discussing K-surfaces—whose properties we now review from an applied perspective.

A regularly parametrized surface patch $f(u, v) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a *Chebyshev net* if its parameter lines have unit length, i.e., if $|f_u| = |f_v| = 1$. Physically, this condition corresponds to enforcing that the parameter line "rods" are inextensible. The rods shear during deformation allowing the angles between the parameter lines' tangent vectors to change at each point. While one can locally equip every smooth surface in $\mathbb{R}^3$ with a Chebyshev net [14], this is no longer the case globally without producing singularities. The fundamental global obstruction is given by the so-called formula of Hazzidakis [15], which couples Gauß curvature $\mathcal{K}$ and shearing across the parametrized patch. Let D be an axis aligned (with respective to u, v) rectangular domain. Then

$$\int_D \mathcal{K}(u, v)\mathrm{d}A = 2\pi - \sum_{i=0}^{3} \alpha_i \ , \tag{5}$$

where $\mathrm{d}A$ is the area element on the surface and the α_i are the interior angles of the quadrangle given by the image of the axis-aligned rectangle D under the Chebyshev net f. In other words, Chebyshev nets satisfy the property that the total integrated Gauß curvature of every quadrangle enclosed by parameter lines depends only on the interior angles at its corners; in particular, every such quadrangle cannot have magnitude of integrated curvature more than 2π. If the parameter lines were geodesics, this result would not be surprising. However, we emphasize that the parameter lines of a Chebyshev net are generally *not* geodesics, showing that Chebyshev nets exhibit a global, counterintuitive constraint [5]. In spite of (5), Chebyshev himself proved that there exists an open Chebyshev net on a sphere that contains a closed hemisphere, and Voss showed that there exists a global Chebyshev net on every bounded surface of revolution that does not meet its rotation axis—even if its total Gauß curvature exceeds 2π. The key to these results is that one carefully designs the domain U on which the Chebyshev net is defined, preventing "large" axis aligned rectangles that would contain too much Gauß curvature from closing. From an analytical perspective, it is a delicate problem to extend a local parametrization when a known global obstruction exists. In 2011, Ghys [14] proved that Chebyshev's spherical net containing a hemisphere can be extended to the entire sphere, except for two circular arcs along the south pole. Beyond providing "real world" results, by taking an applied approach, we reveal that the space of Chebyshev nets may be significantly richer than suggested by current analysis.

Our main results are best summarized by the following figure, with discussion below.

[5]This constraint plays a key role in Hilbert's theorem that the complete hyperbolic plane cannot be isometrically embedded into $\mathbb{R}^3$, since the asymptotic coordinates of a K-surface form a Chebyshev net [29]. If such an embedding were to exist, it would contain axis aligned quadrangles of arbitrarily large negative Gauß curvature, which cannot happen.

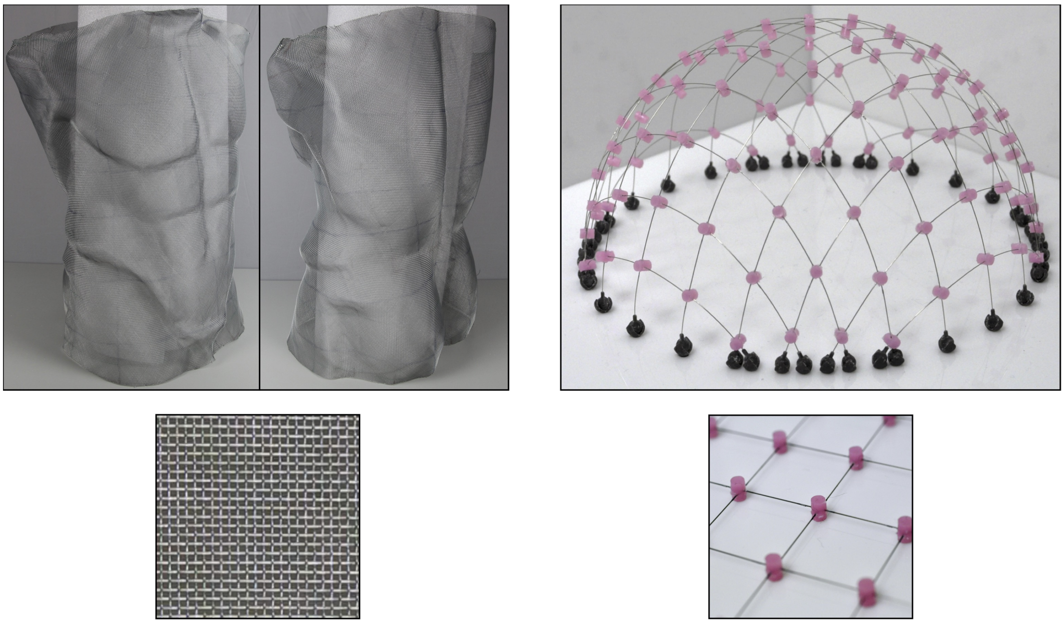

FIGURE 4. Left: Photos of a torso built from our design pipeline out of a single piece of tightly woven *wire mesh* (photos courtesy of Bailin Deng). Right: Photos of a nearly perfect hemispherical *elastic gridshell* we built by loading a suitably chosen subset of a planar grid of inextensible, elastic rods along its boundary (photos courtesy of Changyeob Baek).

In Figure 4 left, we show a wire mesh sculpture that we built using the computational design pipeline introduced in Chapter 5. Our tool finds an approximating discrete Chebyshev net (i.e., a quad net of rhombi) nearby a target shape, which in this instance was a 3D scanned male torso. The key idea is to allow the user to interactively alter the domain of the underlying discrete Chebyshev net by directly adding or removing 3D geometry. Edits are interleaved with a global optimization that heuristically satisfies (5), while remaining close to the target shape. Our tool allows cylindrical topology that enforces rod agreement along the seam, so the design can indeed be constructed from a single piece of wire mesh and then closed. Our tool reveals a rich design space, despite the global constraints imposed by Chebysehv nets—the numerically approximated total Gauß curvature of the shown torso is 124, significantly larger than 2π.

In Figure 4 right, we show an elastic gridshell that is nearly perfectly hemispherical. The shape of its flat domain and actuation boundary (location of the black pinned boundary balls) are sampled from the smooth net for a hemisphere that Chebyshev proposed. The resemblance is surprising, since smooth Chebyshev nets assume a continuum of inextensible rods, and do not account for the elastic buckling of the rods, from which an elastic gridshell derives it shape. This highlights one of our main findings in Chapter 6: despite their sparse structure and elastic response, we show that elastic gridshells can indeed be modeled by smooth Chebyshev nets. This leads to using smooth Chebyshev nets as “ansatzes” for design, while (5) helps explain their nonlocal response. Moreover, we associate a notion of integrated Gauß curvature to each empty quadrangle enclosed by four smooth rods of an elastic gridshell. Since the interior angles of each quadrangle are well defined (where two smooth rods cross), we use the right hand side of (5) as definition. We show that this definition matches our intuition in a variety of examples and experiments.

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Part I. Theoretical contributions: edge-constraint nets in $\mathbb{R}^3$

CHAPTER 2

A Discrete Parametrized Surface Theory in $\mathbb{R}^3$

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CHAPTER 3

A 2×2 Lax Representation, Associated Family, and Bäcklund Transformation for Circular K-Nets

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CHAPTER 4

Discrete Bonnet Pairs and Conformal Nets in $\mathbb{R}^3$

This chapter is joint work with Tim Hoffmann and Max Wardetzky. Unlike the other chapters in this thesis, which are already published, it is still a work-in-progress, and understood as a "snapshot" of our results that will be combined into a forthcoming preprint. Any typos or incorrect statements are purely my own and not the fault of my coauthors.

Discrete Bonnet Pairs and Conformal Nets in $\mathbb{R}^3$

Tim Hoffmann, Andrew O. Sageman-Furnas, Max Wardetzky

September 7, 2017

Abstract

We develop a spin transformation for discrete parametrized surfaces represented as quad nets that leads to a notion of discrete conformal equivalence between quads. As a motivating example we discuss a discrete analog of simply connected Bonnet pairs, two immersed surfaces that have the same metric and mean curvature but are not rigid body motions of each other. We also introduce a discrete analog of the generalized Weierstrass representation for conformal immersions.

1 Introduction

Let $\mathbb{H}$ be the space of quaternions, the four dimensional real vector space generated by $\{\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$, where $\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = -\mathbb{1}$ and $\mathbb{i}\mathbb{j}\mathbb{k} = -\mathbb{1}$. We identify $\mathbb{R}^3$ with $\Im\mathbb{H}$ the space of *imaginary quaternions* via the canonical embedding $(x_1, x_2, x_3) \mapsto x_1\mathbb{i} + x_2\mathbb{j} + x_3\mathbb{k}$. We denote the space of invertible quaternions by $\mathbb{H}^*$.

In the smooth setting, two conformal immersions for a manifold M, $f, \tilde{f} : M \to \mathbb{R}^3 \cong \Im\mathbb{H}$, are said to be spin-equivalent if there exists a *spin transformation* $\lambda : M \to \mathbb{H}^*$, such that $\mathrm{d}\tilde{f} = \bar{\lambda}\mathrm{d}f\lambda$; the surface normal n transforms as $\tilde{n} = \lambda^{-1}n\lambda$. Geometrically, spin transformations correspond to stretch-rotations of the tangent plane at every point. Therefore, they are conformal mappings and for simply connected domains any two surfaces which are conformally equivalent are related via a spin transformation. Kamberov, Pedit, and Pinkall [6] showed using spin transformations that one can classify all Bonnet pairs on a simply connected domain. Bonnet pairs are immersed surfaces that have the same metric and mean curvature but are not rigid body motions of each other.

In this article we discuss a discrete analog of spin transformations for quad nets in $\mathbb{R}^3$. The resulting nets turn out to be so-called *edge-constraint* nets, which have been recently introduced as a discrete analog of a general parametrized surface. Our main result is to construct discrete Bonnet pairs.

1.1 Preliminaries and notations

We consider edge-constraint nets as a discrete analog for parametrized surfaces (see Chapter 2). Their curvature theory is derived in terms of offset surfaces, extending the work of Schief [10] and Bobenko, Pottmann, and Wallner, [4] and encompasses many integrable geometries, such as surfaces of constant curvature together with their associated families.

Within discrete differential geometry, one considers nets defined on a *quad graph* $\mathbb{G}$, i.e., a cell decomposition of a regular surface into quadrilateral faces, where each edge connects distinct vertices and meets at most two faces. To distinguish vertex or edge elements we will use lattice shifts as motived by the case of regular combinatorics $\mathbb{G} = \mathbb{Z}^2$. For example, an arbitrary vertex map is represented by $v = v(k, \ell)$ and indices represent shifts in the corresponding lattice directions: $v_1 = v(k+1, \ell)$, $v_2 = v(k, \ell+1)$, $v_{12} = v(k+1, \ell+1)$. We will often only consider single quads of a net, so this notation will not lead to ambiguity.

Definition 1. *Two maps $f : \mathbb{G} \to \mathbb{R}^3$ and $n : \mathbb{G} \to \mathbb{S}^2$ are said to form an* edge-constraint net *if*

$$f_i - f \perp n_i + n \tag{1}$$

holds for all edges. We further assume that each edge satisfies $f_i - f \neq 0$ and $n_i + n \neq 0$. We refer to f as an immersion *and to n as its* Gauß map.

We begin by reinterpreting the edge-constraint along an arbitrary edge. We will often represent arbitrary edges as shifts in the first lattice direction.

Lemma 2. *Consider two arbitrary points $f_1, f \in \mathbb{R}^3$ with corresponding unit vectors $n_1, n \in \mathbb{S}^2$. The following statements are equivalent*

- *The edge-constraint is satisfied, i.e., $f_1 - f \perp n_1 + n$.*
- *There exists a $\tau \in \mathbb{R}$ extending the immersion edge $f_1 - f$ into a full quaternion $\Phi = \tau + f_1 - f \in \mathbb{H}^*$ that satisfies $\Phi n_1 + n\Phi = 0$.*

Proof. $\Phi n_1 + n\Phi = 0$ is equivalent to $\tau(n_1 + n) = (n_1 - n) \times (f_1 - f)$. □

The condition $\Phi n_1 + n\Phi = 0$ can be rewritten as $n_1 = -\Phi^{-1} n \Phi$, highlighting how Φ can be understood as a *normal transport* quaternion. Normal transports will play a prominent role in our spin transformation for quad nets.

Definition 3. *Consider an edge-constraint net (f, n) with quad graph $\mathbb{G}$. To edges in the first and second lattice directions, respectively, we define the* normal transport *quaternions:*

$$\Phi = \tau + f_1 - f, \quad \text{where } \tau \in \mathbb{R} \text{ is defined by} \quad n_1 = -\Phi^{-1} n \Phi\,. \tag{2}$$

$$\Psi = \eta + f_2 - f, \quad \text{where } \eta \in \mathbb{R} \text{ is defined by} \quad n_2 = -\Psi^{-1} n \Psi\,. \tag{3}$$

Remark. *For clarity in shift notation for these edge based quantities we explicitly write down the shifted normal transport quaternions Φ_2, Ψ_1 for a quad:*

$$\Phi_2 = \tau_2 + f_{12} - f_2, \quad \text{where } \tau_2 \in \mathbb{R} \text{ is defined by} \quad n_{12} = -\Phi_2^{-1} n_2 \Phi_2\,. \tag{4}$$

$$\Psi_1 = \eta_1 + f_{12} - f_1, \quad \text{where } \eta_1 \in \mathbb{R} \text{ is defined by} \quad n_{12} = -\Psi_1^{-1} n_1 \Psi_1\,. \tag{5}$$

Remark. *For an edge-constraint net* (f, n) *with quad graph* $\mathbb{G}$*, the normal offset immersion net* $f^t = f + tn, t \in \mathbb{R}$ *with the same Gauß map* n *is also an edge-constraint net. The normal transports transform as expected:*

$$\Phi^t = \Phi + t(n_1 - n) \text{ and } \Psi^t = \Psi + t(n_2 - n)\,. \tag{6}$$

2 Discrete spin transformations

We briefly recall the notion of a smooth spin transformation before proceeding with a discrete analog. Our presentation closely resembles [6]. Two conformal immersions of a surface M, $f, \tilde{f} : M \to \mathbb{R}^3 \cong \Im\mathbb{H}$, are said to be spin-equivalent if there exists a $\lambda : M \to \mathbb{H}^*$, such that $\mathrm{d}\tilde{f} = \bar{\lambda}\mathrm{d}f\lambda$; the surface normal n transforms as $\tilde{n} = \lambda^{-1}n\lambda$. Geometrically, spin transformations correspond to homotheties (stretch-rotations) in $\mathbb{R}^3$ of the tangent plane at every point. Therefore, they are conformal mappings and for simply connected domains any two surfaces which are conformally equivalent are related via a spin transformation. Using spin transformations, one can (locally) integrate a new conformal immersion $\tilde{f}$ from a reference immersion $f : M \to \mathbb{R}^3$ by solving the integrability condition

$$0 = \mathrm{d}\mathrm{d}\tilde{f} = \mathrm{d}(\bar{\lambda}f\lambda) = \mathrm{d}\bar{\lambda} \wedge \mathrm{d}f\lambda - \lambda\mathrm{d}f \wedge \mathrm{d}\lambda = -2\Im(\bar{\lambda}\mathrm{d}f \wedge \mathrm{d}\lambda)\,. \tag{7}$$

This integrability condition can be rephrased as a reality condition $\bar{\lambda}\mathrm{d}f \wedge \mathrm{d}\lambda = -\rho|\mathrm{d}f|^2$ for a real valued function $\rho : M \to \mathbb{R}$ and surface metric $|\mathrm{d}f|^2$. It turns out that ρ corresponds to the change in so-called mean curvature half-density $\mathcal{H}|\mathrm{d}f|$ under a spin transformation: $\tilde{\mathcal{H}}|\mathrm{d}\tilde{f}| = (\mathcal{H} + \rho)|\mathrm{d}f|$.

Our discrete spin transformation emphasizes the normals and associates a quaternion map λ to the vertices, which transforms its associated Gauß map. Along the edges the normal transport quaternions $\Phi = \tau + f_1 - f$ play the role of a discrete analog to $\mathrm{d}f$. Analogous to the smooth setting, we then define two edge-constraint nets to be spin-equivalent if their normal transport quaternions are "stretch-rotations" of each other.

Definition 4. *Two edge-constraint nets* (f, n) *and* $(\tilde{f}, \tilde{n})$ *with quad graph* $\mathbb{G}$ *are* spin-equivalent *if there exists a vertex map* $\lambda : \mathbb{G} \to \mathbb{H}^*$ *such that the Gauß maps and normal transport quaternions are related via*

$$\tilde{n} = \lambda^{-1}n\lambda, \quad \tilde{\Phi} = \bar{\lambda}\Phi\lambda_1, \quad \text{and} \quad \tilde{\Psi} = \bar{\lambda}\Psi\lambda_2\,. \tag{8}$$

We call $(\tilde{f}, \tilde{n})$ *a* spin transform *of* (f, n).

Remark. *We make a few remarks about this definition.*

- *Spin-equivalence is an equivalence relation: if* $(\tilde{f}, \tilde{n})$ *is a spin transform of* (f, n) *with* λ *then* (f, n) *is a spin transform of* $(\tilde{f}, \tilde{n})$ *with* λ^{-1}*; if* λ *spin transforms* (f, n) *to* $(\tilde{f}, \tilde{n})$ *and* μ *transforms* $(\tilde{f}, \tilde{n})$ *to* $(\hat{f}, \hat{n})$*, then* $\mu\lambda$ *transforms* (f, n) *to* $(\hat{f}, \hat{n})$*.*

- *The map* λ *is constant if and only if* $(\tilde{f}, \tilde{n})$ *and* (f, n) *are related by a Euclidean motion and uniform scaling by* $|\lambda|^2$.
- *The Gauß maps of spin-equivalent edge-constraint nets are related pointwise as in the smooth setting by an* $\mathbb{R}^3$ *rotation. In general, however, the normal transport quaternions of spin-equivalent edge-constraint nets are related by an* $\mathbb{R}^4$ *homothety, not an* $\mathbb{R}^3$ *homothety.*
- *The edge-constraint is encoded in* $\tilde{\Phi}\tilde{n}_1 + \tilde{n}\tilde{\Phi} = \bar{\lambda}(\Phi n_1 + n\Phi)\lambda_1 = 0$.
- *When* $\mathbb{G}$ *is bipartite, the map* λ *has a trivial freedom: one can scale black and white vertices with a factor of* μ *and* $1/\mu$ *respectively, without changing the transformation. We could have therefore defined* λ *to have unit quaternions at the vertices of* $\mathbb{G}$ *and scaling factors along its edges that relate* Φ *to* $\tilde{\Phi}$.

We now compute the discrete integrability condition to construct a spin transform $(\tilde{f}, \tilde{n})$ from a given edge-constraint net (f, n). We require the following definition.

Definition 5. *The* additive holonomy A *of a quad with normal transport quaternions* $\Phi, \Psi, \Phi_2, \Psi_1$ *is* $A = (\Phi + \Psi_1) - (\Psi + \Phi_2)$.

Lemma 6. *Let* (f, n) *be an edge-constraint net with quad graph* $\mathbb{G}$. *For a map* $\lambda : \mathbb{G} \to \mathbb{H}^*$, *consider the transformed Gauß map* $\tilde{n}$ *and quaternions* $\tilde{\Phi}, \tilde{\Psi}$ *as in* (8). *If the additive holonomy of each quad is real, i.e.,* $A \in \mathbb{R}$, *then* $\tilde{n}, \tilde{\Phi}, \tilde{\Psi}$ *are the Gauß map and normal transport quaternions of an edge-constraint net* $(\tilde{f}, \tilde{n})$ *whose immersion edges are given by*

$$\tilde{f}_1 - \tilde{f} = \Im\tilde{\Phi}, \qquad \tilde{f}_2 - \tilde{f} = \Im\tilde{\Psi}, \tag{9}$$

(10)

Proof. It suffices to argue per quad. When $\tilde{A} \in \mathbb{R}$, we have that $\Im\big((\tilde{\Phi} + \tilde{\Psi}_1) - (\tilde{\Psi} + \tilde{\Phi}_2)\big) = 0$. From an initial point $\tilde{f} \in \mathbb{R}^3$ we define $\tilde{f}_1 = \tilde{f} + \Im\tilde{\Phi}$, $\tilde{f}_2 = \tilde{f} + \Im\tilde{\Psi}$, and therefore $\tilde{f}_{12} = \tilde{f}_1 + \Im\tilde{\Psi}_1 = \tilde{f}_2 + \Im\tilde{\Phi}_2$. The edge-constraint for $(\tilde{f}, \tilde{n})$ is automatically satisfied since, as remarked above, $\tilde{\Phi}\tilde{n}_1 + \tilde{n}\tilde{\Phi} = \bar{\lambda}(\Phi n_1 + n\Phi)\lambda_1$, which vanishes since $\Phi n_1 + n\Phi = 0$ for the edge-constraint net (f, n). ☐

The following lemma yields insight into how many spin transformations exist for a given edge-constraint net (f, n).

Lemma 7. *Let* (f, n) *be an edge-constraint net. A spin transformation map* λ *is determined by a quad evolution equation with one real degree of freedom given by the additive holonomy* $\tilde{A}$ *per quad. In other words, from* $\lambda, \lambda_1, \lambda_2$ *and a prescribed real additive holonomy* $\tilde{A}$ *we have that* λ_{12} *is determined by*

$$\lambda_{12} = (\bar{\lambda}_1\Psi_1 - \bar{\lambda}_2\Phi_2)^{-1}(\tilde{A} + \tilde{\Psi} - \tilde{\Phi}). \tag{11}$$

In particular, when $\mathbb{G} = \mathbb{Z}^2$ *all spin transformations are determined by Cauchy data along the axes, together with a real valued function on quads.*

Remark. *In the smooth setting, one can prescribe the real valued function ρ to determine a spin transformation of a reference immersion. Analogously, in the discrete setting, one can prescribe a real additive holonomy $\tilde{A} \in \mathbb{R}$ per quad. However, it is currently unclear how the additive holonomy is related to a notion of discrete mean curvature half-density. Nevertheless, we will see in the next section that this discrete spin transformation leads to discrete Bonnet pairs that have the same mean curvature in the sense of edge-constraint nets.*

3 Discrete spin-metric and conformal equivalence

In the smooth setting spin-equivalence is the same as conformal equivalence for simply connected domains. We therefore make the following definition.

Definition 8. *Let $\mathbb{G}$ be a simply connected quad graph. Two edge-constraint nets (f, n) and $(\tilde{f}, \tilde{n})$ with quad graph $\mathbb{G}$ are* conformally equivalent *if they are spin transformations of each other.*

For the remainder of this section we investigate conformal equivalence in the case when $\mathbb{G}$ is a single quad. Using Lemma 7 the results easily generalize to when $\mathbb{G}$ is a simply connected subset of $\mathbb{Z}^2$. Most likely the results carry over to more general types of quad graphs under mild combinatorial and/or topological restrictions.

Our goal is to define metric quantities for quads of edge-constraint nets that transform appropriately under spin transformation. We show that every edge-constraint net quad can be spin transformed into the plane with upright normals and then characterize when two planar quads are spin-equivalent.

Lemma 9. *Consider an edge-constraint net quad Q with normal transport quaternions $\Phi, \Phi_2, \Psi, \Psi_1$ and Gauß map n, n_1, n_{12}, n_2. Then Q is spin-equivalent to a planar quadrilateral $\tilde{Q}$ lying in the $\mathbb{i}, \mathbb{j}$-plane, with $\tilde{n} = \tilde{n}_1 = \tilde{n}_{12} = \tilde{n}_2 = \mathbb{k}$.*

Proof. There are in fact many ways to spin transform Q to the plane, but we construct one as follows. We want $\tilde{n} = \lambda^{-1} n \lambda = \mathbb{k}$, so we set $\lambda = n + \mathbb{k}$. We want $\tilde{\Phi} = \bar{\lambda}\Phi\lambda_1 = \mathbb{i}$, so we define $\lambda_1 = (\bar{\lambda}\Phi)^{-1}\mathbb{i}$. We want $\tilde{\Psi} = \bar{\lambda}\Psi\lambda_2 = \mathbb{j}$, so we define $\lambda_2 = (\bar{\lambda}\Psi)^{-1}\mathbb{j}$. Note that these choices also lead to $\tilde{n}_1 = \tilde{n}_2 = \mathbb{k}$. Now, the planar quad $\tilde{Q}$ should have zero additive holonomy, so we prescribe $\tilde{A} = 0$. From these initial data, we determine λ_{12} by (11). One verifies that $\tilde{\Psi}_1$ and $\tilde{\Phi}_2$ only have $\mathbb{i}, \mathbb{j}$ components as quaternions, and that $\tilde{n}_{12}$ is also $\mathbb{k}$. ☐

Restricting to planar quads with upright normals, we recover a familiar nonlinear characterization of discrete conformal maps from the complex plane to itself.

Lemma 10. *Let Q and $\tilde{Q}$ be two quadrilaterals in the $\mathbb{i}, \mathbb{j}$-plane with all normals equal to $\mathbb{k}$. Then Q and $\tilde{Q}$ are conformally equivalent (spin-equivalent) if and only if they have the same complex cross-ratio.*

Proof. We will write the vertices of Q with f and the vertices of $\tilde{Q}$ with g. Note that the normal transport quaternions for these quadrilaterals are the immersion edges themselves. Since both quadrilaterals have all normals $\mathbb{k}$, the map λ must satisfy $\mathbb{k} = \lambda^{-1}\mathbb{k}\lambda$ at all four vertices. Therefore, λ only takes values in the span of $\mathbb{1}, \mathbb{k}$.

By definition, Q is spin-equivalent to $\tilde{Q}$ if and only if four equations hold. Each of these equations is of the form $\overline{(w + z\mathbb{k})}(x\mathbb{i} + y\mathbb{j})(w_1 + z_1\mathbb{k})$, where $w, x, y, z \in \mathbb{R}$. Identifying the $\mathbb{i}, \mathbb{j}$-plane with the complex plane, this product is $-(w + zi)(x + iy)(w_1 + z_1 i)$. Therefore, Q and $\tilde{Q}$ are spin-equivalent if and only if (written as complex numbers)

$$\lambda(f_1 - f)\lambda_1 = g_1 - g \text{ and } \lambda(f_2 - f)\lambda_2 = g_2 - g, \tag{12}$$

$$\lambda_1(f_{12} - f_2)\lambda_{12} = g_{12} - g_2 \text{ and } \lambda_2(f_{12} - f_1)\lambda_{12} = g_{12} - g_1. \tag{13}$$

From Lemma 7 we know that $\lambda, \lambda_1, \lambda_2$ can be chosen arbitrarily. There are two equations for λ_{12}

$$\lambda_{12} = (f_{12} - f_1)^{-1}\lambda_1^{-1}(g_{12} - g_1) = (f_{12} - f_1)^{-1}(g_1 - g)^{-1}\lambda(f_1 - f)(g_{12} - g_1), \tag{14}$$

$$\lambda_{12} = (f_{12} - f_2)^{-1}\lambda_2^{-1}(g_{12} - g_2) = (f_{12} - f_2)^{-1}(g_2 - g)^{-1}\lambda(f_2 - f)(g_{12} - g_2), \tag{15}$$

which are satisfied if and only if the complex cross-ratio of Q and $\tilde{Q}$ agree, i.e.,

$$\frac{(f_1 - f)(f_{12} - f_2)}{(f_{12} - f_1)(f_2 - f)} = \frac{(g_1 - g)(g_{12} - g_2)}{(g_{12} - g_1)(g_2 - g)}. \tag{16}$$

□

Combining the previous two lemmas allows us to define a so-called *spin-cross-ratio* for each nonplanar quad in space with corresponding normals that satisfy the edge-constraint. This spin-cross-ratio characterizes spin-equivalent quads.

Definition 11. *The* spin-cross-ratio *of an edge-constraint net quad Q is given by the complex cross-ratio of a spin-equivalent planar quad $\tilde{Q}$ lying in the $\mathbb{i}, \mathbb{j}$-plane (identified with $\mathbb{C} \cong \operatorname{span}\{1, i\}$) with normal $\mathbb{k}$.*

Theorem 12. *Two edge-constraint net quads are conformally equivalent (spin-equivalent) if and only if they have the same spin-cross-ratio.*

We provide a more computable characterization of the spin-cross-ratio.

Theorem 13. *Let Q be an edge-constraint net quad with Gauß map n, n_1, n_{12}, n_2 and normal transport quaternions $\Phi, \Phi_2, \Psi, \Psi_1$. The spin-cross-ratio of Q is given by the quaternion*

$$\operatorname{scr}(Q) = \Phi\bar{\Psi}_1^{-1}\bar{\Phi}_2\Psi^{-1} \tag{17}$$

when understood as a complex number in the plane spanned by $\mathbb{1}, n$. In other words, written in quaternionic polar form, $\operatorname{scr}(Q) = re^{n\theta}$ and the spin-cross-ratio of Q as a complex number is $re^{i\theta}$. If $\tilde{Q}$ and Q are spin-equivalent with map λ then $\operatorname{scr}(\tilde{Q}) = \lambda^{-1}\operatorname{scr}(Q)\lambda = \lambda^{-1}re^{n\theta}\lambda = re^{\tilde{n}\theta}$.

Proof. By the defining properties of normal transport quaternions, we see that $n = \mathrm{scr}(Q)^{-1} n \,\mathrm{scr}(Q)$. Thus, $\mathrm{scr}(Q)$ can be written in the form $w + z\, n$ for real numbers $z, w \in \mathbb{R}$, which written in quaternionic polar form is $re^{n\theta}$ for $r, \theta \in \mathbb{R}$. Now, for spin-equivalent quads Q and $\tilde{Q}$ we compute that $\mathrm{scr}(\tilde{Q}) = \bar{\lambda}\mathrm{scr}(Q)\bar{\lambda}^{-1} = \lambda^{-1}\mathrm{scr}(Q)\lambda$. In other words, the quaternion $\mathrm{scr}(\cdot)$ transforms under a spin transformation by a rotation from the plane spanned by $\mathbb{1}, n$ to the plane spanned by $\mathbb{1}, \tilde{n}$. Finally, we note that for a quad $\tilde{Q}$ lying in the $\mathbb{i}, \mathbb{j}$-plane with normals $\mathbb{k}$, $\mathrm{scr}(\tilde{Q})$ lies in the plane spanned by $\mathbb{1}, \mathbb{k}$ and is equal to the complex cross-ratio of $\tilde{Q}$ when identifying the $\mathbb{i}, \mathbb{j}$-plane with $\mathbb{C} \cong \mathrm{span}\{1, i\}$. □

Remark. *We note a few things about the spin-cross-ratio quaternion.*

- *In the above characterization the Gauß map at the vertex n plays a special role; one can write down similar characterizations using another corner n_i of the quad.*
- *It is important to point out that each vertex of an edge-constraint net is incident to multiple quads. These quads will generically have different spin-cross-ratios, but since they share a Gauß map at their common vertex all of their spin-cross-ratio quaternions* $\mathrm{scr}(\cdot)$ *will lie in the same plane. We do not yet have an interpretation of the relationship between the spin-cross-ratios meeting at a common vertex.*

We comment on the relationship between the spin-cross-ratio and other notions of cross-ratio in discrete differential geometry. The spin-cross-ratio restricted to the plane recovers the normal complex cross-ratio. Therefore, the spin-cross-ratio of Q is real if and only if Q is spin-equivalent to a planar quad with concircular vertices. In general, the spin-cross-ratio of an edge-constraint net quad is not equal to the quaternionic cross-ratio $(f_1 - f)(f_{12} - f_1)^{-1}(f_{12} - f_2)(f_2 - f)^{-1}$ of its four immersion points. However, the two notions agree for so-called *circular nets*, a well established discrete analog of curvature line parametrizations [2].

Definition 14. *An edge-constraint net is called a* circular net *if the four vertices of each immersion quad lie on a circle and the Gauß map satisfies $f_i - f \parallel n_i - n$ along every edge.*

Lemma 15. *A circular net has real spin-cross-ratio on every quad.*

Proof. Consider a quad Q with concircular immersion points whose Gauß map has parallel edges. Since $f_i - f \parallel n_i - n$ the real parts of the normal transport quaternions vanish. Thus, $\mathrm{scr}(Q) = (f_1 - f)(\overline{f_{12} - f_1})^{-1}(\overline{f_{12} - f_2})(f_2 - f)^{-1}$, which is the quaternionic cross-ratio of the concircular immersion points and known to be real. □

Remark. *One can verify that changing the Gauß map of a circular net while maintaining $f_i - f \parallel n_i - n$ and holding its immersion f fixed is a spin transformation.*

We are now able to introduce the notion of a metric [1] for edge-constraint net quads that appropriately transforms under spin transformations.

Definition 16. *Consider an edge-constraint net quad Q with normal transport quaternions $\Phi, \Phi_2, \Psi, \Psi_1$ and spin-cross-ratio $re^{i\theta}$. Its* spin-metric $\mathcal{I}(Q)$ *is defined as*

$$\mathcal{I}(Q) = \begin{pmatrix} |\Phi\Phi_2| & \sin\theta\sqrt{|\Phi\Phi_2||\Psi\Psi_1|} \\ \sin\theta\sqrt{|\Phi\Phi_2||\Psi\Psi_1|} & |\Psi\Psi_1| \end{pmatrix}. \tag{18}$$

Remark. *The spin-metric $\mathcal{I}(Q)$ is diagonal if and only if the quad Q has real spin-cross-ratio; it is a multiple of the identity if Q has spin-cross-ratio -1.*

Remark. *Speaking informally, this definition is motivated by thinking of the normal transports as a discrete analog of* $\mathrm{d}f$*, where opposite sides of the quads each represent a partial derivative, e.g., "$\langle f_x, f_x\rangle$ $|\Phi\Phi_2|$".*

This definition has the following consequence, which motivate its importance in understanding discrete conformality.

Lemma 17. *Let Q and $\tilde{Q}$ be spin-equivalent quads with map λ. Then their spin-metrics are related by*

$$\mathcal{I}(\tilde{Q}) = |\lambda\lambda_1\lambda_{12}\lambda_2|\mathcal{I}(Q). \tag{19}$$

In other words, the spin-metrics of Q and $\tilde{Q}$ are related by a global scaling. In particular, $\mathcal{I}(\tilde{Q}) = \mathcal{I}(Q)$ if the spin transformation map λ has unit length $|\lambda| = 1$ everywhere.

Remark. *The transformation of the spin-metric under discrete spin transformations resembles the smooth setting. The metrics $|\mathrm{d}f|^2$ and $|\mathrm{d}\tilde{f}|^2$, of two smooth immersions $\tilde{f}, f$ that are spin-equivalent via λ, are related by $|\mathrm{d}\tilde{f}|^2 = |\lambda|^4|\mathrm{d}f|^2$. Spin-equivalent immersions are therefore isometric if and only if $|\lambda| = 1$.*

We can now understand when two spin-equivalent edge-constraint nets are isometric, by considering their spin-metrics on corresponding quads.

Definition 18. *Let Q and $\tilde{Q}$ be spin-equivalent quads with map λ. We say that Q and $\tilde{Q}$ are isometric if $\mathcal{I}(Q) = \mathcal{I}(\tilde{Q})$.*

For the remainder of this article we present examples of discrete spin transformations.

[1] This metric is generally different from the "first fundamental form" we introduced for edge-constraint nets in Definition 2.17 in the article presented in Chapter 2.

4 Bonnet pairs

Two conformal immersions $f, \tilde{f} : M \to \mathbb{R}^3$ of a smooth surface M form a Bonnet pair if they are isometric and have the same mean curvature, but are not rigid body motions of each other. In 1998, Kamberov, Pedit, and Pinkall used spin transformations to classify all simply connected Bonnet pairs [6] by showing how they arise from isothermic surfaces. An isothermic surface f is classically defined as a surface that admits conformal, curvature line coordinates (x, y) away from umbilic points, and exhibit a so-called *Christoffel dual* isothermic surface f^* satisfying $\mathrm{d}f^* = -f_x^{-1}\mathrm{d}x + f_y^{-1}\mathrm{d}y$. We recall their construction, slightly reformulated for our present purposes. They also show that each pair arises in this way, leading to the characterization, which we do not formulate.

Theorem 19. *(Kamberov, Pedit, Pinkall, 1998) Consider a simply connected surface M. Let $f : M \to \mathbb{R}^3$ be isothermic with dual $f^* : M \to \mathbb{R}^3$. Choose $\epsilon \in \mathbb{H}^*$ and define $\lambda_+ = (f^* + \epsilon)$ and $\lambda_- = (f^* - \bar{\epsilon})$. Then the spin transforms $f_\pm : M \to \mathbb{R}^3$ given by $\mathrm{d}f_\pm = \bar{\lambda}_\pm \,\mathrm{d}f \lambda_\pm$ form a Bonnet pair.*

Remark. *The smooth integrability condition $\bar{\lambda}\mathrm{d}f \wedge \mathrm{d}\lambda = 0$ for both spin transformations is easily verified, since $\mathrm{d}f \wedge \mathrm{d}f^* = 0$.*

Throughout this section we assume that the quad graph $\mathbb{G}$ of all edge-constraint nets we consider is a simply connected subset of $\mathbb{Z}^2$. We show that an analogous construction as that above yields discrete Bonnet pairs. Explicitly, we will exhibit two edge-constraint nets that are spin-equivalent (conformally equivalent since they are simply connected), whose corresponding quads are isometric and have the same mean curvature in the following sense:

Definition 20. *Let (f, n) be an edge-constraint net. To each quad we associate a unit face normal $N \perp \mathrm{span}\{n_{12} - n, n_2 - n_1\}$. The Gauß and mean curvature are defined by*

$$K := \frac{\det(n_{12} - n, n_2 - n_1, N)}{\det(f_{12} - f, f_2 - f_1, N)} \tag{20}$$

and

$$H := \frac{1}{2}\frac{\det(f_{12} - f, n_2 - n_1, N) + \det(n_{12} - n, f_2 - f_1, N)}{\det(f_{12} - f, f_2 - f_1, N)}\,, \tag{21}$$

respectively.

These mean and Gauß curvatures were introduced with the theory of edge-constraint nets in Chapter 2 of this thesis.

Within discrete differential geometry there is a well understood notion of discrete isothermic nets [1, 2, 3]. We state one definition and will recapitulate facts about them as needed.

Definition 21. *A circular net (f, n) is a* discrete isothermic net *if it exhibits a dual circular net (f^*, n^*) with $n^* = -n$ such that corresponding edges and diagonals of every quad satisfy:*

$$f_i - f \parallel f_i^* - f^*, f_2 - f_1 \parallel f_{12}^* - f^*, \text{ and } f_{12} - f \parallel f_2^* - f_1^*\,. \tag{22}$$

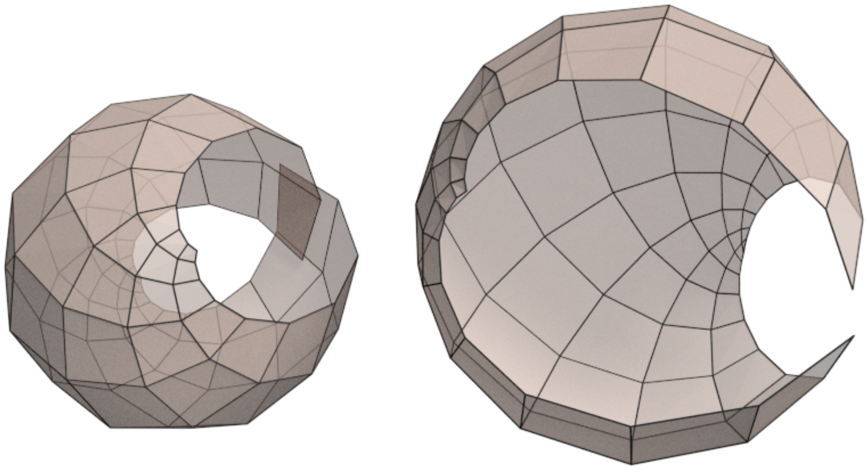

Figure 1: A discrete Bonnet pair: two nets that share the same spin-metric and mean curvature, but are not congruent. They are constructed as in Theorem 22, from the Gauß map of a discrete isothermic minimal surface.

The dual isothermic net (f^*, n^*) is unique (up to scaling and translation).

We can now state and prove our main theorems, which taken together form a discrete analog of Theorem 19 and yield discrete Bonnet pairs.

Theorem 22. *Let (f, n) be a discrete isothermic net with dual isothermic net (f^*, n^*). Let $\epsilon \in \mathbb{H}^*$ be arbitrary. The spin transformations of f with $\lambda_+ = (f^* + \epsilon)$ and $\lambda_- = (f^* - \bar{\epsilon})$ integrate to two edge-constraint nets (f_+, n_+) and (f_-, n_-), respectively.*

Proof. We give the proof for f_+. The proof for f_- is similar.

To show that f_+ exists we need to show that each spin transformed quad closes in $\mathbb{R}^3 \cong \Im\mathbb{H}$. We will show that it in fact closes in $\mathbb{H}$ by showing that the following expression vanishes:

$$\begin{aligned}\overline{(\epsilon + f^*)}(f_1 - f)(f_1^* + \epsilon) + \overline{(\epsilon + f_1^*)}(f_{12} - f_1)(f_{12}^* + \epsilon)\\ -\overline{(\epsilon + f_2^*)}(f_{12} - f_2)(f_{12}^* + \epsilon) - \overline{(\epsilon + f^*)}(f_2 - f)(f_2^* + \epsilon)\end{aligned} \tag{23}$$

Expanding this expression and collecting terms, we can rewrite this expression as a sum: $\bar{\epsilon}E_3\epsilon + \bar{\epsilon}E_2 + E_1\epsilon + E_0$, where

$$\begin{aligned}E_3 =&(f_1 - f) + (f_{12} - f_1) - (f_{12} - f_2) - (f_2 - f),\\ E_2 =&(f_1 - f)f_1^* + (f_{12} - f_1)f_{12}^* - (f_{12} - f_2)f_{12}^* - (f_2 - f)f_2^*,\\ E_1 =&\overline{f^*}(f_1 - f) + \overline{f_1^*}(f_{12} - f_1) - \overline{f_2^*}(f_{12} - f_2) - \overline{f^*}(f_2 - f),\\ E_0 =&\overline{f^*}(f_1 - f)f_1^* + \overline{f_1^*}(f_{12} - f_1)f_{12}^* - \overline{f_2^*}(f_{12} - f_2)f_{12}^* - \overline{f^*}(f_2 - f)f_2^*.\end{aligned} \tag{24}$$

We want to prove that $E_3 = E_2 = E_1 = E_0 = 0$.

Notice that $E_3 = 0$, since f closes a quad in $\mathbb{R}^3$. Now, $E_2 = 0$ and $E_1 = 0$ are equivalent to

$$(f_1 - f)f_1^* - (f_2 - f)f_2^* = -(f_2 - f_1)f_{12}^*, \qquad \text{and} \tag{25}$$
$$\overline{f_1^*}(f_{12} - f_1) - \overline{f_2^*}(f_{12} - f_2) = \overline{f^*}(f_2 - f_1), \tag{26}$$

respectively. Therefore, if we assume that $E_2 = 0 = E_1$, then $E_0 = 0$, since

$$\begin{aligned} E_0 &= \overline{f^*}\Big((f_1 - f)f_1^* - (f_2 - f)f_2^*\Big) + \Big(\overline{f_1^*}(f_{12} - f_1) - \overline{f_2^*}(f_{12} - f_2)\Big)f_{12}^* \\ &= -\overline{f^*}(f_2 - f_1)f_{12}^* + \overline{f^*}(f_2 - f_1)f_{12}^* = 0. \end{aligned} \tag{27}$$

It remains to prove that $E_2 = 0$ and $E_1 = 0$, which are equivalent to proving (25) and (26). We now show that $E_1 = 0$; a similar proof shows that $E_2 = 0$.

We require some facts about discrete isothermic nets (see [3]). Each quad of a discrete isothermic net f and its dual quad of f^* have real cross ratio $\alpha_1^2/\alpha_2^2 \in \mathbb{R}$ and their corresponding edges and diagonals satisfy the respective equations

$$(f_i - f)(f_i^* - f^*) = -\alpha_i^2, \text{ for } i = 1, 2, \text{ and} \tag{28}$$
$$(f_2 - f_1)(f_{12}^* - f^*) = (f_{12} - f)(f_2^* - f_1^*) = (\alpha_1^2 - \alpha_2^2). \tag{29}$$

Now, adding zero in the form of $(f_2 - f)f^* - (f_1 - f)f^* - (f_2 - f_1)f^*$ to E_2, we compute

$$\begin{aligned} E_2 &= (f_1 - f)(f_1^* - f^*) - (f_2 - f)(f_2^* - f^*) + (f_2 - f_1)(f_{12}^* - f^*) \\ &= (-\alpha_1^2) - (-\alpha_2^2) + (\alpha_1^2 - \alpha_2^2) = 0. \end{aligned} \tag{30}$$

Therefore, we have proven that spin transforming f with $\lambda_+ = \epsilon + f^*$ yields an edge-constraint net (f_+, n_+). A similar argument shows that (f_-, n_-) also exists. □

To show that the edge-constraint nets (f_+, n_+) and (f_-, n_-) form a discrete Bonnet pair we relate their geometric properties. In the smooth setting, one can derive how the metric and mean curvature change under spin transformation, which immediately proves that Bonnet pairs indeed have the same metric and mean curvature. In the discrete setting, we know that (f_+, n_+) and (f_-, n_-) have the same spin-metric, since λ_+ and λ_- have the same norm.

Theorem 23. *Let (f, n) be a discrete isothermic net with dual isothermic net (f^*, n^*) and construct the spin-equivalent edge-constraint nets (f_+, n_+) and (f_-, n_-) in the sense of Theorem 22. Then corresponding quads of (f_+, n_+) and (f_-, n_-) are isometric in terms of their spin-metric (Definition 18) and have the same mean curvature in the sense of edge-constraint nets (Definition 20).*

Proof. The proof is a computation that can be verified using symbolic computation in a computer algebra system, e.g., *Mathematica*. □

Remark. *The edge-constraint nets* (f_+, n_+) *and* (f_-, n_-) *also have the same Gauß curvature in the sense of Definition 20. However, spin-equivalent isometric quads do not generally have the same Gauß curvature in this sense, so we left this additional result out of this theorem.* [2] *Although the spin transformation presented here automatically stays within the class of edge-constraint nets, the relationship between the derived geometric quantities of the two approaches is not yet generally understood.*

Remark. *As in the smooth setting, an example of a Bonnet one-parameter* family *is given by discrete isothermic minimal surfaces (given by an analog of Weierstrass data in [1]). Consider a discrete isothermic minimal net* (f, n)*. The immersion* f *has Christoffel dual given by the Gauß map* n*. Spin transforming* (f, n) *with* $\lambda = (\cos\theta/2 - \sin\theta/2 n)$ *for all* $\theta \in [0, 2\pi]$ *generates the associated family of minimal nets. These nets automatically close by the above Bonnet pair comptutation (with a suitably scaled dual net). Since* $|\lambda| = 1$*, we find that the associated family of minimal nets is an isometric deformation.*

5 Generalized Weierstrass representation

In this section we introduce a discrete analog to the generalized Weierstrass representation for surfaces in $\mathbb{R}^3$. For more information on the smooth setting see [7, 5, 8].

We will represent the quaternions $\mathbb{H}$ as a two dimensional complex vector space, with the first copy of $\mathbb{C}$ embedded in the $\mathbb{1}, \mathbb{k}$-plane and the second copy embedded in the $\mathbb{i}, \mathbb{j}$-plane. With this identification the multiplication $(\mathbb{C}\times\mathbb{C})\times(\mathbb{C}\times\mathbb{C}) \to (\mathbb{C}\times\mathbb{C})$ is given by $(a, b)(c, d) \mapsto (ac - b\bar{d}, ad + b\bar{c})$. Note that quaternionic conjugation translates into complex conjugation as $\overline{(a, b)} = (\bar{a}, -b)$. We will often transition between the two notations for quaternions. To avoid some confusion we will continue to use $\Re, \Im$ for the real and imaginary parts of a quaternion, but will use Re, Im for the real and imaginary parts of a complex number (or vector of complex numbers).

The smooth generalized Weierstrass representation is as follows.

Theorem 24. *(Kamberov, Norman, Pedit, Pinkall 2002) Let* M *be a simply connected domain in* $\mathbb{C}$ *with holomorphic coordinate* z *and arbitrary fixed point* $p_0 \in M$*. Every conformal immersion* $f : M \to \mathbb{R}^3$ *with Gauß map* $n : M \to \mathbb{S}^2$ *has the representation*

$$f(p) = \mathrm{Re}\int_{p_0}^{p} (b^2 - a^2, i(b^2 + a^2), 2ab)\mathrm{d}z, \tag{31}$$

$$n(p) = \frac{1}{1 + |\frac{a}{b}|^2}(2\mathrm{Re}\frac{a}{b}, 2\mathrm{Im}\frac{a}{b}, |\frac{a}{b}|^2 - 1). \tag{32}$$

[2] In fact, the Bonnet pairs presented here also possess the same "first fundamental form" in the sense of edge-constraint nets (see Definition 2.17 in the article presented in Chapter 2), which generally does not agree with the spin-metric.

for a and b two complex valued functions satisfying

$$\partial_{\bar z} a = -\frac{\rho}{2}\bar b \quad \text{and} \quad \partial_{\bar z} b = \frac{\rho}{2}\bar a, \tag{33}$$

where $\rho = -\mathcal{H}(|a|^2 + |b|^2)$ is a multiple of the mean curvature function $\mathcal{H}$ of f; the surface metric is $(|a|^2 + |b|^2)^2|dz|^2$.

Conversely, every pair of complex valued functions a and b satisfying (33) with $|a|^2+|b|^2 \neq 0$ define a conformal immersion of M given by (31) with Gauß map given by (32).

This representation arises as a spin transformation of the planar immersion $z \mapsto (0, z)$ with constant Gauß map $(i, 0)$: setting $\lambda = (i\bar a, \bar b) \in \mathbb{H}^*$ the immersion f is given by integrating $\overline{\lambda}(0, \mathrm{d}z)\lambda$ and the Gauß map arises as $\lambda^{-1}(i, 0)\lambda$.

Remark. *We make a few remarks.*

- *The Gauß map is the inverse stereographic projection of the (possibly extended) complex valued function $\frac{a}{b}$.*
- *The sum of the square of the components of the $\mathbb{C}^3$-valued vector $(b^2 - a^2, i(b^2 + a^2), 2ab)$ vanishes, i.e.,*

$$(b^2 - a^2)^2 + (i(b^2 + a^2))^2 + (2ab)^2 = 0\,. \tag{34}$$

- *Using 33 we find*

$$b\partial_{\bar z} a - a\partial_{\bar z} b = -(|a|^2 + |b|^2)\rho = \mathcal{H}(|a|^2 + |b|^2)^2\,. \tag{35}$$

We present the discrete analog of this representation using discrete spin transformations. In this section we restrict to quad graphs that exhibit a so-called complex valued edge labeling, a complex valued function that takes equal values on each pair of opposite edges of each quad of $\mathbb{G}$.

Definition 25. *A quad graph $\mathbb{G}$ exhibits an* edge labeling *in $\mathbb{C}$ if there exists an immersion $p : \mathbb{G} \to \mathbb{C}$ such that the image of each quad is a non degenerate parallelogram. For each quad of $\mathbb{G}$ we label each pair of edges with the complex numbers $\alpha = p_1 - p = p_{12} - p_2$ and $\beta = p_2 - p = p_{12} - p_1$, respectively.*

Remark. *The edge labeling property guarantees that the cross-ratios of the quads α^2/β^2 "factor" onto the edges. This will allow us to argue per quad, with the results naturally generalizing over the entire quad graph $\mathbb{G}$.*

Choosing an edge labeled quad graph corresponds to choosing "holomorphic" coordinates from which to spin transform into a nonplanar edge-constraint net in $\mathbb{R}^3$, yielding a generalized Weierstrass representation. We write the immersion p into $\mathbb{C} \subset \mathbb{R}^3 \subset \mathbb{H}$ as a map $p : \mathbb{G} \to \operatorname{span}\{\mathbb{i}, \mathbb{j}\}$, represented by the pair of complex numbers $(0, p)$. Choosing the constant Gauß map $n = \mathbb{k} = (i, 0)$ extends p into an edge-constraint net. We can change "holomorphic" coordinates by spin transforming the map p to $\tilde p$ while keeping it planar; this preserves the cross-ratio of each quad. We will eventually restrict to initial coordinates $\tilde{p}$ where every quad has real cross-ratio, so $\tilde{p}$ is in fact a circular net contained in the plane. This restriction guarantees that the spin-metric of each quad is diagonal. However, we wish to state our main result in this slightly more general form.

Theorem 26. *Let $\mathbb{G}$ be an edge labeled quad graph with immersion $(0,p) \in \mathbb{H}^*$ and constant Gauß map $(i,0) = \mathbb{k} \in \mathbb{H}$. Spin transforming a single quad whose edges are $(0,\alpha)$ and $(0,\beta)$ with $\lambda = (i\bar{a}, \bar{b})$ yields a quad of an edge-constraint net (f,n) with Gauß map and normal transport quaternions (written as vectors in $\mathbb{R}^4 = \operatorname{span}\{\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$)*

$$n = \tfrac{1}{1+|\frac{a}{b}|^2}(0, 2\mathrm{Re}\tfrac{a}{b}, 2\mathrm{Im}\tfrac{a}{b}, |\tfrac{a}{b}|^2 - 1), \tag{36}$$

$$\Phi = \mathrm{Re}\Big\{\alpha\,(-i(a_1 b - b_1 a), bb_1 - aa_1, i(aa_1 + bb_1), a_1 b + ab_1)\Big\}, \tag{37}$$

$$\Psi = \mathrm{Re}\Big\{\beta\,(-i(a_2 b - b_2 a), bb_2 - aa_2, i(aa_2 + bb_2), a_2 b + ab_2)\Big\}. \tag{38}$$

The discrete integrability condition that the spin transformation closes is expressed as two complex equations, the first represents closing in the $\mathbb{i}, \mathbb{j}$-plane. The second represents closing in the $\mathbb{k}$ coordinate.

$$\alpha aa_1 + \beta a_1 a_{12} - \alpha a_{12} a_2 - \beta aa_2 = \overline{\alpha bb_1 + \beta b_1 b_{12} - \alpha b_{12} b_2 - \beta bb_2}, \text{ and} \tag{39}$$

$$\mathrm{Re}\big\{\alpha(a_1 b + ab_1) + \beta(a_{12} b_1 + a_1 b_{12}) - \alpha(a_{12} b_2 + a_2 b_{12}) - \beta(a_2 b + ab_2)\big\} = 0\,. \tag{40}$$

The additive holonomy is given by

$$A = \mathrm{Im}\big\{\alpha(a_1 b - ab_1) + \beta(a_{12} b_1 - a_1 b_{12}) - \alpha(a_{12} b_2 - a_2 b_{12}) - \beta(a_2 b - b_2 a)\big\}. \tag{41}$$

Remark. *Our current understanding of this representation is only superficial, but we point out a few remarks.*

- *The Gauß map transforms as in the smooth setting, and is the inverse stereographic projection of a/b.*
- *The global scaling of the spin metric is given by*

$$\sqrt{(|a|^2 + |b|^2)(|a_1|^2 + |b_1|^2)(|a_{12}|^2 + |b_{12}|^2)(|a_2|^2 + |b_2|^2)}, \tag{42}$$

  which resembles the smooth setting.
- *The sum of the squares of the $\mathbb{C}^4$ valued vectors corresponding to the normal transport quaternions vanish, e.g., for the complexified Φ given by $(-i(a_1 b - b_1 a), bb_1 - aa_1, i(aa_1 + bb_1), a_1 b + ab_1)$ we find*

$$(-i(a_1 b - b_1 a))^2 + (bb_1 - aa_1)^2 + (i(aa_1 + bb_1))^2 + (a_1 b + ab_1)^2 = 0. \tag{43}$$

  This resembles the smooth conformal immersion of a surface into $\mathbb{R}^4$ (see, e.g., [9]).

- *The* $\mathbb{i}, \mathbb{j}$ *closing condition is that the discrete Hirota equation for* a *is the conjugate of that for* b*. The* $\mathbb{k}$ *closing condition resembles a mixed version of the discrete Hirota equation.*

- *The complexified normal transport quaternion* Φ *can be written as* $\alpha b b_1$ *multiplied by a* C^4 *vector that only depends on the stereographically projected Gauß map points* a/b*. Similarly for* Ψ*, but by facotring out* $\beta b b_2$*. The resulting formulas resemble those found in [5], where conformal immersions in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$ *are characterized by their Gauß maps.*

Remark. *For suitably chosen functions* a *and* b *one can recover the Weierstrass representation for discrete isothermic minimal nets as given in [1].*

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Part II. Applied contributions: designing with discrete materials from inextensible rods

CHAPTER 5

Wire Mesh Design

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CHAPTER 6

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