Coarse Cohomology with twisted Coefficients

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Introduction

This work is intended to give a serious and reasonably complete study of Coarse Geometry via Algebraic Geometry tools. The exposition serves as an introduction to the topic Coarse Geometry and takes a topologist point of view on the subject. There is, nevertheless, also a section on tools from Noncommutative Geometry.

1.1 What is Coarse Geometry?

The topic Coarse Geometry studies metric spaces from a large scale point of view. We want to examine the global structure of metric spaces. One way to approach this problem is by forgetting small scale structure. The coarse category consists of coarse spaces as objects and coarse maps as morphisms.

Now coarse maps preserve the coarse structure of a space in the coarse category. A coarse structure is made of *entourages* which are surroundings of the diagonal. For us metric spaces are the main objects of study. If X is a metric space a subset $E \subseteq X^2$ is an entourage if

$$\sup_{(x,y)\in E} d(x,y) < \infty.$$

The exact opposite of a coarse space and Coarse Geometry of metric spaces are uniform spaces and the Uniform Topology of a metric space. Like coarse spaces uniform spaces are defined via surroundings of the diagonal. Uniform entourages get smaller though while coarse entourages get larger the sharper the point of view.

Many algebraic properties of infinite finitely generated groups are hidden in the geometry of their Cayley graph. To a finitely generated group is associated the word length with regard to a generating set. Note that the metric of the group depends on the choice of generating set while the coarse structure associated to the word length metric is independent of the choice of generating set. Note that group homomorphisms are special cases of coarse maps between groups and group isomorphisms are special cases of coarse equivalences between groups. It is very fruitful to group theory to consider infinite finitely generated groups as coarse objects; these will be a source of examples for us.

Note the examples $\mathbb{R}^n$ and $\mathbb{Z}^n$ both are coarse spaces induced by a metric, for $\mathbb{R}^n$ it is the euclidean metric and for $\mathbb{Z}^n$ the metric is induced by the group $(\mathbb{Z}^n, +)$. Now $\mathbb{Z}^n$ and $\mathbb{R}^n$ look entirely different on small scale they are the same on large scale though. There is a coarse equivalence $\mathbb{Z}^n \to \mathbb{R}^n$.

1.2 Background and related Theories

Nowadays it is hard to embrace all cohomology theory and other theories in the coarse category because of the diversity of the toolsets used. Apart from the controlled K-theory and the Higson corona, wich uses noncommutative tools there are also theories which are topological in nature.

1.2.1 Cohomology theories

A *cohomology theory* assigns an abelian group with a space, in a functorial manner. There are classical examples like Čech cohomology, simplicial homology, ... etc. which all fit in a general framework. The standard choice in the topological category are the Eilenberg-Steenrod axioms. They consist of 5 conditions which characterize singular cohomology on topological spaces. A *generalized cohomology theory* is a sequence of contravariant functors $(H^n)_n$ from the category of pairs of topological spaces (X, A) to the category of abelian groups equipped with natural transformations

$$\delta : H^n(A, \emptyset) \to H^{n+1}(X, A)$$

for $n \in \mathbb{N}$, such that

1. *Homotopy*: If $f_1, f_2 : (X, A) \to (Y, B)$ are homotopic morphisms then they induce isomorphic maps in cohomology.
2. *Excision*: If (X, A) is a pair and $U \subseteq A$ a subset such that $\bar{U} \subseteq A^\circ$ then the inclusion

$$i : (X \setminus U, A \setminus U) \to (X, A)$$

   induces an isomorphism in cohomology.
3. *Dimension*: The cohomology of the point is concentrated in degree 0.
4. *Additivity*: If $X = \bigsqcup_\alpha X_\alpha$ is a disjoint union of topological spaces then

$$H^n(X, \emptyset) = \prod_\alpha H^n(X_\alpha, \emptyset).$$

5. *Exactness*: Every pair of topological spaces (X, A) induces a long exact sequence in cohomology:

$$\begin{aligned} \cdots \to & H^n(X, A) \to H^n(X, \emptyset) \to H^n(A, \emptyset) \\ \to & H^{n+1}(X, A) \to \cdots . \end{aligned}$$

We are interested in theories that are functors on coarse spaces and coarse maps. Let us first recall the standard theories.

There are a number of cohomology theories in the coarse category we present two of them which are the most commonly used ones. We first present the most basic facts about *controlled operator K-theory* and Roe's *coarse cohomology*.

We begin with a covariant invariant $K_*(C^*(\cdot))$ on proper metric spaces called *controlled K-theory*. Note that if a proper metric space B is bounded then it is compact. Then [1, Lemma 6.4.1] shows

$$K_p(C^*(B)) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p = 1. \end{cases}$$

There is a notion of flasque spaces for which controlled K-theory vanishes. An exemplary example is $\mathbb{Z}_+$; in [1, Lemma 6.4.2] it is shown that

$$K_*(C^*(\mathbb{Z}_+)) = 0.$$

The above is used in order to compute the controlled K-theory of $\mathbb{Z}^n$:

$$K_p(C^*(\mathbb{Z}^n)) = \begin{cases} \mathbb{Z} & p \equiv n \mod 2 \\ 0 & p \equiv n+1 \mod 2 \end{cases}$$

which is [1, Theorem 6.4.10]. The notion of Mayer-Vietoris sequence is adapted to this setting: If there are two subspaces A, B of a coarse space and if they satisfy the coarse excisive property which is introduced in [2] then [2, Lemmas 1,2; Section 5] combine to a Mayer-Vietoris sequence in controlled K-theory. There is a notion of homotopy for the coarse category which is established in [3]. Then [3, Theorem 5.1] proves that controlled K-Theory is a coarse homotopy invariant.

Let us now consider *coarse cohomology* $HX^*(\cdot; A)$ which for A an abelian group is a contravariant invariant on coarse spaces. The [4, Example 5.13] notes that if a coarse space B is bounded then

$$HX^q(B; A) = \begin{cases} A & q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now the space $\mathbb{Z}^n$ reappears as an example in [4, Example 5.20]:

$$HX^q(\mathbb{R}^n; \mathbb{R}) = \begin{cases} 0 & q \neq n \\ \mathbb{R} & q = n \end{cases}$$

Whereas another example is interesting: the [4, Example 5.21] shows that if G is a finitely generated group then there is an isomorphism

$$HX^*(G; \mathbb{Z}) = H^*(G; \mathbb{Z}[G]).$$

Here the right side denotes group cohomology. In order to compute coarse cohomology there is one method: We denote by $H_c^*(X; A)$ the cohomology with compact supports of X as a topological space. There is a character map

$$c : HX^q(X; A) \to H_c^q(X; A)$$

By [4, Lemma 5.17] the character map c is injective if X is a proper coarse space which is topologically path-connected. Now [4, Theorem 5.28] states: If R is a commutative ring and X is a uniformly contractible proper coarse space the character map for R-coefficients is an isomorphism.

In the course of this thesis we will design a new cohomology theory on coarse spaces. It has all the pros of the existing coarse cohomology theories and can be compared with them. The main purpose of this work is to design computational tools for the new theory and compute cohomology of a few exemplary examples.

Our main tool will be *sheaf cohomology theory*, which we now recall. If X is a coarse space then $\texttt{Sheaf}(X)$ denotes the abelian category of sheaves of abelian groups on X. Note that $\texttt{Sheaf}(X)$ has enough injectives. Then the global sections functor

$$\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$$

is a left exact functor between abelian categories $\mathtt{Sheaf}(X)$ and $\mathtt{Ab}$, the category of abelian groups. The right derived functors are the sheaf cohomology functors. If $\mathcal{F}$ is a sheaf on X then $\check{H}^*(X, \mathcal{F})$ denotes coarse cohomology with twisted coefficients with values in $\mathcal{F}$.

There are many ways to compute sheaf cohomology. One of them uses acyclic resolutions. Now every sheaf $\mathcal{F}$ on a coarse space X has an injective resolution and injective sheaves are acyclic. Thus there exists a resolution

$$0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots$$

with acyclics $\mathcal{I}_q$, $q \geq 0$. Then the sheaf cohomology groups $\check{H}^q(X, \mathcal{F})$ are the cohomology groups of the following complex of abelian groups

$$0 \to \mathcal{I}_0(X) \to \mathcal{I}_1(X) \to \mathcal{I}_2(X) \to \cdots .$$

We can also compute sheaf cohomology by means of Čech cohomology. If $(U_i)_{i \in I}$ is a *coarse cover* of a subset $U \subseteq X$ and $\mathcal{F}$ an abelian presheaf on X then the group of q-cochains is

$$C^q(\{U_i \to U\}_i, \mathcal{F}) = \prod_{(i_0, \ldots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q})$$

The coboundary operator $d^q : C^q(\{U_i \to U\}_i, \mathcal{F}) \to C^{q+1}(\{U_i \to U\}_i, \mathcal{F})$ is defined by

$$(d^q s)_{i_0, \ldots, i_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu s_{i_0, \ldots, \hat{i}_\nu, \ldots i_{q+1}}|_{i_0, \ldots, i_{q+1}}$$

Then $C^*(\{U_i \to U\}_i, \mathcal{F})$ is a complex and $\check{H}^*(\{U_i \to U\}_i, \mathcal{F})$ is defined to be its cohomology. Now sheaf cohomology can be computed:

$$\check{H}^q(U, \mathcal{F}) = \varinjlim_{\{U_i \to U\}_i} \check{H}^q(\{U_i \to U\}_i, \mathcal{F}).$$

In good circumstances we can compute sheaf cohomology using an acyclic cover. If $(U_i)_{i \in I}$ is a coarse cover of a coarse space X and $\mathcal{F}$ a sheaf on X and if for every nonempty $\{i_1, \ldots, i_n\} \subseteq I$, $q > 0$ we have that

$$\check{H}^q(U_{i_1} \cap \cdots \cap U_{i_n}, \mathcal{F}) = 0$$

then already

$$\check{H}^q(X, \mathcal{F}) = \check{H}^q(\{U_i \to U\}_i, \mathcal{F})$$

for every $q \geq 0$.

Note that homotopy also plays an important part when computing sheaf cohomology.

1.2.2 Boundaries

There are quite number of notions for a boundary on a metric space. In this chapter we are going to discuss properties for three of them. The first paragraph is denoted to the *Higson corona*, in the second paragraph the *space of ends* is presented and in the last paragraph we study the *Gromov boundary*.

First we present the Higson corona. If X is a proper metric space the *Higson corona* νX is the boundary of the Higson compactification hX of X which is a compact topological space that contains the underlying topological space of X as a dense open subset.

If $C(X)$ denotes the bounded continuous functions on X then the so called Higson functions are a subset of $C(X)$. This subset determines a compactification which is called the Higson compactification. By a comment on [4, p. 31] the Higson corona can be defined for any coarse space. The same does not work for the Higson compactification[1]. The [4, Proposition 2.41] implies that the Higson corona is a covariant functor that sends coarse maps modulo closeness to continuous maps. Thus ν is a functor:

$$\nu : \mathtt{Coarse} \to \mathtt{Top}$$

The topology of νX has been studied in [5]. It was shown in [5, Theorem 1] that for every σ-compact subset $A \subseteq \nu X$ the closure $\bar{A}$ of A in νX is equivalent to the Stone-Čech compactification of A. The topology of νX is quite complicated, especially if X is a metric space. It has been noted in [4, Exercise 2.49] that the topology of νX for X an unbounded proper metric space is never second countable. In [6, Theorem 1.1] and [7, Theorem 7.2] it was shown that if the asymptotic dimension $\operatorname{asdim}(X)$ of X is finite then

$$\operatorname{asdim}(X) = \dim(\nu X)$$

where the right side denotes the topological dimension of νX. Note that one direction of the proof uses the notion of coarse covers[2].

Now we present the *space of ends*. If Y is a locally connected, connected and locally compact Hausdorff space then the space of ends of Y is the boundary of the Freudenthal compactification εY. It is totally disconnected and every other compactification of Y that is totally disconnected factors uniquely through εY by [8, Theorem 1]. The points of ΩY are called *endpoints* or *ends*.

Now [8, Theorem 5] shows that if Y is a connected locally finite countable CW-complex every endpoint of Y can be represented by a proper map

$$a : \mathbb{R}_+ \to Y.$$

Two proper maps $a_1, a_2 : \mathbb{R}_+ \to Y$ represent the same endpoint if they are connected by a proper homotopy. Denote by `pTop` the category of topological spaces and proper continuous maps. Then the association $\Omega\cdot$ is a functor:

$$\Omega : \mathtt{pTop} \to \mathtt{Top}$$

If Y is a locally compact Hausdorff space then ΩY can be constructed using a proximity relation which is a relation on the subsets of Y. See [9] for that one.

This section studies the Gromov boundary. If X is a proper Gromov hyperbolic metric space then the *Gromov boundary* ∂X consists of equivalence classes of *sequences that converge to infinity* in X. The topology on ∂X is generated by a basis of open neighborhoods. Loosely speaking two points on the boundary are close if the sequences that represent them stay close for a long time.

By [10, Proposition 2.14] the topological spaces ∂X and $\partial X \cup X$ are compact and by [10, Theorem 2.1] the topology on ∂X is metrizable. If $f : X \to Y$ is a *quasi-isometry* between proper Gromov hyperbolic groups then it extends to a homeomorphism

$$\partial f : \partial X \to \partial Y$$

by [10, Proposition 2.20]. In [11] is studied a notion of morphisms for which the Gromov boundary is a functor: If $f : X \to Y$ is a *visual function* between proper Gromov hyperbolic metric spaces then there is an induced map

$$\partial f : \partial X \to \partial Y$$

[1] for which the topology of X needs to be locally compact which is given if the metric is proper.
[2] but under a different name.

which is continuous by [11, Theorem 2.8].

Now is there a notion of boundary on metric space which is both a functor on coarse spaces and coarse maps and has nice properties such as being Hausdorff and locally compact. As it turns out there is one such functor which is going to be designed in the course of this thesis.

1.3 Main Contributions

The general idea of this work is to transfer toolsets from other topics like Algebraic Topology and Algebraic Geometry and use them in the coarse category. The cohomology theory we are aiming at has its roots in Algebraic Geometry. The space at infinity functor we are going to design has its image in the category of uniform topological spaces.

1.3.1 Sheaf Cohomology on Coarse Spaces

The Chapters 2,3,4 study sheaf cohomology on coarse spaces. They form the core of this thesis. First let us note a few aspects which distinguishes the new theory.

There has been much effort in establishing axioms for cohomology theories in the coarse category. In [12] has been proposed a choice of axioms for coarse cohomology theories. Now we will test our theory against the Eilenberg-Steenrod axiom system. The new theory satisfies similar properties which are going to be discussed in the following list

1. *Homotopy:* In Section 3.2 is designed a homotopy theory for coarse metric spaces. It can be compared with other homotopy theories in the coarse category in that it sees more structure for metric spaces and is automatically reflexive/symmetric/transitive, an equivalence relation on coarse maps. Sheaf cohomology on coarse spaces is a homotopy invariant. In which ways other cohomology theories are homotopy invariant has not been studied yet.

2. *Excision:* Subsection 2.4.5 presents local cohomology in the coarse category.

3. *Dimension:* The space $\mathbb{Z}_+$ can be understood as the coarse equivalent of a point. It is acyclic for constant $\mathbb{Z}/2\mathbb{Z}$- coefficients. If the spaces $\mathbb{Z}^n$ are understood as representatives for dimension then coarse cohomology with twisted coefficients sees dimension.

4. *Additivity:* Sheaf cohomology sees coproducts, see subsection 4.1.2.

5. *Exactness:* Subsection 2.4.4 presents a coarse version of the Mayer-Vietoris sequence.

Now why are there so many powerful results is one of the most natural questions we can ask. The main reason is, that typically sheaf cohomology is a powerful tool in a number of areas. Examples are de Rham cohomology in differential geometry, singular cohomology for nice enough spaces in algebraic topology and étale cohomology in algebraic geometry.

In Chapter 2 the new cohomology theory is introduced. This chapter is taken from [13, Chapters 1-4]. A Grothendieck topology is the least amount of data needed to define sheaves and sheaf cohomology. And that is where we start. We design the Grothendieck topology of coarse covers associated to a coarse space in Definition 58. Then we discover in Lemma 62 that coarse maps give rise to a morphism of topologies. That is all the information that we need to use the powerful machinery of sheaf cohomology.

Then we obtain the first important result: if two coarse maps are close then they induce isomorphic maps in cohomology with twisted coefficients. This is Theorem 72.

Theorem A. *Coarse cohomology with twisted coefficients is a functor on coarse spaces and coarse maps modulo closeness.*

Thus coarsely equivalent coarse spaces have the same cohomology.

The coarse equivalent of a trivial space is either the empty set or a bounded space or both. If B is a bounded space then for every coefficient $\mathcal{F}$ on B:

$$\check{H}^*(B, \mathcal{F}) = 0$$

which is a result of Example 64.

Some computional tools we recognize from algebraic topology can be adopted for our setting. The Chapter 2.4.4 presents a coarse version of Mayer-Vietoris and Chapter 2.4.5 discusses relative cohomology in the coarse category.

In Chapter 3 the homotopy theory is constructed. We present the notions *coarsely proper* and *coarsely geodesic* in a chapter of their own, thereby demonstrating techniques which will be useful later on. The Section 3.1, Section 3.2 are [13, Chapter 5, Chapter 6], respectively.

Before proceeding we design a coarse version of a product of spaces. The coarse version of the point, $\mathbb{Z}_+$ the positive integers, is unfortunately not a final object in the coarse category. Nonetheless we look at the a pullback diagram of coarse spaces

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & \mathbb{Z}_+. \end{tikzcd}$$

The pullback of this diagram exists if the spaces X, Y are nice enough as studied in Lemma 99. Indeed we only need Y to be a coarsely proper coarsely geodesic metric space.

Equipped with this product we can define a coarse version of homotopy. The coarse version of an interval is denoted by $F([0, 1])$. Then a coarse homotopy is defined to be a coarse map

$$H : X * F([0, 1]) \to Y.$$

Here $*$ is the coarse product and X, Y are coarse spaces. There is an equivalent definition of coarse homotopy in Definition 107 using a parameter that varies. We prove in key Theorem 109 that coarse cohomology with twisted coefficients behaves well with regard to coarse homotopy.

Theorem B. *Coarse cohomology with twisted coefficients is a coarse homotopy invariant.*

Now we have enough computational tools to compute actual examples. Chapter 4 applies the new theory; in particular a number of acyclic spaces are constructed which aids in the computation of nontrivial examples. In light of the new cohomology theory we study controlled operator K-theory and compute examples. The first part, Section 4.1 is the same as [13, Chapter 6] and the second part, Section 4.2 is the whole of [14].

First let us note that $\mathbb{Z}_+$ is imperfect as a coarse version of a point as it is not a final object and does not have trivial cohomology. While $\check{H}^q(\mathbb{Z}_+, A) = 0$ for $q \geq 2$ and every constant coefficient A, the cohomology in degree 1,

$$\check{H}^1(\mathbb{Z}_+, \mathbb{Z}) \neq 0$$

is nontrivial for $\mathbb{Z}$-coefficients. If we take a locally finite group, as for example $\mathbb{Z}/2\mathbb{Z}$, as coefficient then

$$\check{H}^q(\mathbb{Z}_+, \mathbb{Z}/2\mathbb{Z}) = 0$$

for $q > 0$. Thus for coefficients $\mathbb{Z}/2\mathbb{Z}$ and more generally for locally finite coefficients the space $\mathbb{Z}_+$ is acyclic and can be used for computations.

Theorem C. *We denote by $\mathbb{Z}/2\mathbb{Z}$ the group with two elements. Then*

$$\check{H}^q(\mathbb{Z}_+, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & q = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

In passing we produce other acyclic spaces in Theorem 124. Then Examples 126,127,128,131 compute the coarse cohomology of some infinite finitely generated groups. Specifically the cohomology of the free abelian groups is

$$\check{H}^q(\mathbb{Z}^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 1, q = 0 \\ \mathbb{Z}/2\mathbb{Z} & n > 1, q = n-1, 0 \\ 0 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$. And the cohomology of the free groups is

$$\check{H}^i(F_n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 2$.

The study of sheaf cohomology on coarse spaces sets foundational frameworks on controlled K-theory. We will study a new excision property for a Mayer-Vietoris exact sequence. By modding out the operators with bounded support we obtain a modified Roe-algebra: If X is a proper metric space then define

$$\hat{C}^*(X) := C^*(X)/\mathbb{K}(\mathcal{H}_X)$$

where $C^*(X)$ denotes the Roe-algebra and $\mathbb{K}(\mathcal{H}_X)$ denotes the compact operators of $\mathbb{B}(\mathcal{H}_X)$. Then we prove the following theorem.

Theorem D. *If U_1, U_2 coarsely cover a subset U of a proper metric space X then there is a six-term Mayer-Vietoris exact sequence*

$$\begin{tikzcd} K_1(\hat{C}^*(U_1 \cap U_2)) \arrow[r] & K_1(\hat{C}^*(U_1)) \oplus K_1(\hat{C}^*(U_2)) \arrow[r] & K_1(\hat{C}^*(U)) \arrow[d] \\ K_0(\hat{C}^*(U)) \arrow[u] & K_0(\hat{C}^*(U_1)) \oplus K_0(\hat{C}^*(U_2)) \arrow[l] & K_0(\hat{C}^*(U_1 \cap U_2)). \arrow[l] \end{tikzcd}$$

1.3.2 Space at Infinity

The second part and most of Chapter 5 prepares and studies the new definition of the *space of ends* functor. In the course of this thesis we will define a functor that associates to every coarse metric space a space at infinity which is a topological space. All this work can be found in [15, Chapter 3].

Based on the observation that twisted coarse cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients of $\mathbb{Z}^n$ is the same as singular cohomology of S^{n-1} with $\mathbb{Z}/2\mathbb{Z}$ coefficients we considered notions of boundary which reflect that observation.

In Definition 159 we define a relation on subsets of a metric space. As it turns out this relation is almost but not quite a proximity relation as noted in Remark 162. The proof of Proposition 161 uses that X is a metric space, it does not work for general coarse spaces. That is why we restrict our study to metric spaces.

Note that while constructing the functor we presuppose which kind of spaces we want to distinguish. Indeed there is a certain class of metric spaces for which the local structure looks boring. The functor that we are going to define, the space of ends functor, is well suited for metric spaces that are coarsely proper coarsely geodesic. That class includes all Riemannian manifolds and finitely generated groups.

While the topology of the space at infinity is immediately defined using coarse covers there are two choices of points which are both solid: If X is a metric space

(A): the endpoints of X are images of coarse maps $\mathbb{Z}_+ \to X$ modulo finite Hausdorff distance or

(B): the points at infinity are subsets of X modulo finite Hausdorff distance.

Note that choice B has been implemented in [16]. The space at infinity with choice A contains strictly less points than choice B. The Proposition 94 guarantees that for choice A there exists at least one endpoint if the space X is coarsely proper coarsely geodesic. The proof of Proposition 94 is similar to the one of Königs Lemma in graph theory.

The space at infinity functor with choice B reflects isomorphisms by [17, Proposition 2.18] and the space at infinity functor with choice A is representable.

In the course of this thesis and in Definition 172 we use choice A, endpoints are images of coarse maps $\mathbb{Z}_+ \to X$. Then we define the topology of the space of ends, $E(X)$, via surroundings of the diagonal in Definition 174. The uniformity on $E(X)$ is generated by a basis $(D_{\mathcal{U}})_{\mathcal{U}}$ of entourages over coarse covers $\mathcal{U}$ of X. If $f : X \to Y$ is a coarse map then it induces a uniformly continuous map $E(f) : E(X) \to E(X)$ between spaces of ends. That way the space of ends $E(\cdot)$ is a functor, we obtain the following result:

Theorem E. *If* $\mathtt{mCoarse}$ *denotes the category of metric spaces and coarse maps modulo closeness and* $\mathtt{Top}$ *the category of topological spaces and continuous maps then* E *is a functor*

$$E : \mathtt{mCoarse} \to \mathtt{Top}.$$

If $\mathtt{Uniform}$ *denotes the category of uniform spaces and uniformly continuous maps then* E *is a functor*

$$E : \mathtt{mCoarse} \to \mathtt{Uniform}.$$

It was nontrivial to show that a subspace in the domain category gives rise to a subspace in the image category. Proposition 184 shows if $Z \subseteq Y$ is a subspace then the inclusion $i : Z \to Y$ induces a uniform embedding $E(i) : E(Z) \to E(Y)$.

The functor $E(\cdot)$ preserves coproducts by Lemma 190. The uniformity on $E(X)$ is totally bounded by Lemma 192 and separating by Proposition 191.

Theorem F. *If* X *is a metric space then* $E(X)$ *is totally bounded and separating.*

We still lack a good study including the most basic properties of the new space of ends functor like compact and metrizable probably because the proofs are more difficult.

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Chapter 2

Coarse Cohomology with twisted Coefficients

Coarse geometry studies coarse spaces up to coarse equivalence. Coarse invariants may help in classifying them.

Our purpose is to pursue an algebraic geometry approach to coarse geometry. We present sheaf cohomology on coarse spaces and study coarse spaces by coarse cohomology with twisted coefficients. The method is based on the theory on Grothendieck topologies.

Note that sheaves on Grothendieck topologies and sheaf cohomology theory have been applied in a number of areas and have lead to many breakthroughs on previously unsolved problems. As stated in [18] one can understand a mathematical problem by

1. finding a mathematical world natural for the problem.
2. Expressing your problem cohomologically.
3. The cohomology of that world may solve your problem.

That way we can apply general theory on sheaf cohomology for tackling previously unsolved problems and studying notions which are quite well known.

2.1 The Coarse Category

The following chapter introduces coarse spaces and coarse maps between coarse spaces. It has been kept as short as possible, giving only the most basic definitions needed for understanding this paper. All this information can be found in [4, Chapter 2].

2.1.1 Coarse Spaces

Definition 1. (**inverse, product**) Let X be a set and let E be a subset of X^2. Then the *inverse* E^{-1} is defined by

$$E^{-1} = \{(y, x)|(x, y) \in E\}.$$

A set E is called *symmetric* if $E = E^{-1}$.

For two subsets $E_1, E_2 \subseteq X^2$ the *product* $E_1 \circ E_2$ is given by

$$E_1 \circ E_2 = \{(x, z)|\exists y : (x, y) \in E_1, (y, z) \in E_2\}.$$

Definition 2. (**coarse structure**) Let X be a set. A *coarse structure* on X is a collection of subsets $E \subseteq X^2$ which will be referred as *entourages* which follow the following axioms:

1. the diagonal $\Delta_X = \{(x,x)|x \in X\}$ is an entourage;
2. if E is an entourage and $F \subseteq E$ a subset then F is an entourage;
3. if F, E are entourages then $F \cup E$ is an entourage;
4. if E is an entourage then the inverse E^{-1} is an entourage;
5. if E_1, E_2 are entourages then their product $E_1 \circ E_2$ is an entourage.

The set X together with the coarse structure on X will be called a *coarse space*.

Definition 3. (**connected**) A coarse space X is *connected* if

6. for every points $x, y \in X$ the set $\{(x,y)\} \subseteq X^2$ is an entourage.

In the course of this paper all coarse spaces are assumed to be connected unless said otherwise.

Definition 4. (**bounded set**) Let X be a coarse space. A subset $B \subseteq X$ is called *bounded* if B^2 is an entourage.

Definition 5. Let X be a set and let $K \subseteq X$ and $E \subseteq X^2$ be subsets. One writes

$$E[K] = \{x|\exists y \in K : (x,y) \in E\}.$$

In case K is just a set containing one point p, we write E_p for $E[\{p\}]$ (called a section).

Lemma 6. *Let X be a coarse space.*

- *If $B_1, B_2 \subseteq X$ are bounded then $B_1 \times B_2$ is an entourage and $B_1 \cup B_2$ is bounded.*
- *For every bounded subset $B \subseteq X$ and entourage E the set $E[B]$ is bounded.*

Proof.

- Fix two points $b_1 \in B_1$ and $b_2 \in B_2$ then (b_1, b_2) is an entourage in X. Thus

$$B_1^2 \circ (b_1, b_2) \circ B_2^2 = B_1 \times B_2$$

is an entourage. Now

$$(B_1 \cup B_2)^2 = B_1^2 \cup B_1 \times B_2 \cup B_2 \times B_1 \cup B_2^2$$

is an entourage, thus $B_1 \cup B_2$ is bounded.

- Note that

$$E \circ B^2 = E[B] \times B$$

is an entourage.

□

Remark 7. Note that an intersection of coarse structures is again a coarse structure.

- If X is a set and δ a collection of subsets of X^2 then the smallest coarse structure ε that contains each element of δ is called the *coarse structure that is generated by δ*. Then δ is called a *subbase for ε*.

- If ε is a coarse structure and $\varepsilon' \subseteq \varepsilon$ a subset such that $E \in \varepsilon$ implies there is some $E' \in \varepsilon'$ with $E \subseteq E'$ then ε' is called a *base for* ε.

Example 8. If X is a set there are two trivial coarse structures on X:

1. the *discrete coarse structure* consists of subset of the diagonal and finitely many off-diagonal points.
2. the *maximal coarse structure* is generated by X^2. Note that in this case each subset of X and in particular X itself is bounded.

Example 9. If X is a metric space with metric d then the *bounded coarse structure* of X consists of those subsets $E \subseteq X^2$ for which

$$\sup_{(x,y)\in E} d(x,y) < \infty.$$

A coarse space X is called metrizable if there is a metric d that can be defined on it such that X carries the bounded coarse structure associated to d. Note that by [4, Theorem 2.55] a coarse space is metrizable if and only if it has a countable base.

Example 10. If X is a paracompact and locally compact Hausdorff space and $\bar{X}$ a compactification of X with boundary ∂X then the topological coarse structure associated to the given compactification consists of subsets $E \subseteq X^2$ such that

$$\partial E \cap \partial X^2 \setminus \Delta_{\partial X} = \emptyset.$$

If the compactification is second countable then by [4, Example 2.53] the topological coarse structure on X is not metrizable.

2.1.2 Coarse Maps

Definition 11. (**close**) Let S be a set and let X a be coarse space. Two maps $f, g : S \to X$ are called *close* if

$$\{(f(s), g(s))|s \in S\} \subseteq X^2$$

is an entourage.

Definition 12. (**maps**) Let $f : X \to Y$ be a map between coarse spaces. Then f is called

- *coarsely proper* if for every bounded set B in Y the inverse image $f^{-1}(B)$ is bounded in X;
- *coarsely uniform* if every entourage E of X is mapped by $f^{\times 2} = f \times f : X^2 \to Y^2$ to an entourage $f^{\times 2}(E)$ of Y;
- a *coarse map* if it is both coarsely proper and coarsely uniform;
- a *coarse embedding* if f is coarsely uniform and for every entourage $F \subseteq Y^2$ the inverse image $(f^{\times 2})^{-1}(F)$ is an entourage.

Definition 13. (**coarsely equivalent**)

- A coarse map $f : X \to Y$ between coarse spaces is a *coarse equivalence* if there is a coarse map $g : Y \to X$ such that $f \circ g : Y \to Y$ is close to the identity on Y and $g \circ f : X \to X$ is close to the identity on X.
- two coarse spaces X, Y are *coarsely equivalent* if there is a coarse equivalence $f : X \to Y$.

Notation 14. We call `Coarse` the category with objects coarse spaces and morphisms coarse maps modulo close. Then coarse equivalences are the isomorphisms in the *coarse category.*

2.2 Coentourages

In this chapter coentourages are introduced. We study the dual characteristics of coentourages to entourages.

2.2.1 Definition

This is a special case of [4, Definition 5.3, p. 71]:

Definition 15. Let X be a coarse space. A subset $C \subseteq X^2$ is called a *coentourage* if for every entourage E there is a bounded set B such that

$$C \cap E \subseteq B^2.$$

The set of coentourages in X is called the *cocoarse structure* of X.

Lemma 16. *The following properties hold:*

1. *Finite unions of coentourages are coentourages.*
2. *Subsets of coentourages are coentourages.*
3. *If $f : X \to Y$ is a coarse map between coarse spaces then for every coentourage $D \subseteq Y^2$ the set $(f^{\times 2})^{-1}(D)$ is a coentourage.*

Proof. 1. Let C_1, C_2 be coentourages. Then for every entourage E there are bounded sets B_1, B_2 such that

$$\begin{aligned}(C_1 \cup C_2) \cap E &= C_1 \cap E \cup C_2 \cap E \\ &\subseteq B_1 \times B_1 \cup B_2 \times B_2 \\ &\subseteq (B_1 \cup B_2)^2.\end{aligned}$$

Now $B_1 \cup B_2$ is bounded because X is connected.

2. Let C be a coentourage and $D \subseteq C$ a subset. Then for every entourage E there is some bounded set B such that

$$\begin{aligned}D \cap E &\subseteq C \cap E \\ &\subseteq B^2.\end{aligned}$$

3. This is actually a special case of [4, Lemma 5.4]. For the convenience of the reader we include the proof anyway.

Let E be an entourage in X. Then there is some bounded set $B \subseteq Y$ such that

$$\begin{aligned}f^2((f^{\times 2})^{-1}(D) \cap E) &\subseteq D \cap f^2(E) \\ &\subseteq B^2.\end{aligned}$$

But then

$$\begin{aligned}(f^{\times 2})^{-1}(D) \cap E &\subseteq (f^{\times 2})^{-1} \circ f^{\times 2}((f^{\times 2})^{-1}(D) \cap E) \\ &\subseteq (f^{\times 2})^{-1}(B^2) \\ &= f^{-1}(B)^2.\end{aligned}$$

□

Example 17. In the coarse space $\mathbb{Z}$ one can see three examples:

- the even quadrants are a coentourage: $\{(x, y) : xy < 0\}$.
- For $n \in \mathbb{Z}$ the set perpendicular to the diagonal with foot (n, n) is a coentourage: $\{(n - x, n + x) : x \in \mathbb{Z}\}$.
- There is another example: $\{(x, 2x) : x \in \mathbb{Z}\}$ is a coentourage.

Example 18. Look at the infinite dihedral group which is defined by

$$D_\infty = \langle a, b : a^2 = 1, b^2 = 1\rangle.$$

In D_∞ the set

$$\{(ab)^n, (ab)^n a : n \in \mathbb{N}\} \times \{(ba)^n, (ba)^n b : n \in \mathbb{N}\}$$

is a coentourage.

2.2.2 A Discussion/ Useful to know

Lemma 19. *Let X be a coarse space. Then for a subset $B \subseteq X$ the set B^2 is a coentourage if and only if B is bounded.*

Proof. If B is bounded then it is easy to see that B^2 is a coentourage.

Conversely suppose B^2 is a coentourage. Then

$$\Delta_X \cap B^2 \subseteq B^2$$

and B^2 is the smallest squared subset of X^2 which contains

$$\{(b, b) : b \in B\}$$

which is $\Delta_X \cap B^2$. Thus B is bounded. □

Definition 20. (**dual structure**) If X is a coarse space let ε and γ be collections of subsets of X^2. Call β the set of bounded sets. We say that

1. ε *detects* γ if for every $D \notin \gamma$ there is some $E \in \varepsilon$ such that $D \cap E \not\subseteq B^2$ for every $B \in \beta$.
2. and ε *is dual to* γ if ε detects γ and γ detects ε.

By definition the collection of coentourages is detected by the collection of entourages. If X is a coarse space such that the cocoarse structure is dual to the coarse structure then X is called *coarsely normal.*

Proposition 21. *Let X be a coarse space with the bounded coarse structure of a metric space*[1] *then X is coarsely normal.*

Proof. Let $F \subseteq X^2$ be a subset which is not an entourage. Then for every entourage there is a point in F that is not in E. Now we have a countable basis for the coarse structure:

$$E_1, E_2, \ldots, E_n, \ldots$$

[1] In what follows coarse spaces with the bounded coarse structure of a metric space will be refered to as metric spaces.

ordered by inclusion. Then there is also a sequence $(x_i, y_i)_i \subseteq X^2$ with $(x_i, y_i) \notin E_i$ and $(x_i, y_i) \in F$. Denote this set of points by f. Then for every i the set

$$E_i \cap f$$

is a finite set of points, thus f is a coentourage. But $F \cap f = f$ is not an entourage, specifically it is not contained in B^2 if B is bounded. $\square$

Proposition 22. *Let X be a paracompact and locally compact Hausdorff space. Let $\bar{X}$ be a compactification of X and equip X with the topological coarse structure associated to the given compactification. Then*

1. *a subset $C \subseteq X^2$ is a coentourage if $\bar{C} \cap \Delta_{\partial X}$ is empty.*
2. *if U, V are subsets of X then $U \times V$ is a coentourage if $\partial U \cap \partial V = \emptyset$.*
3. *X is coarsely normal.*

Proof. easy. $\square$

Example 23. If G is an infinite countable group then there is a canonical coarse structure on G: A subset $E \subseteq G^2$ is an entourage if the set

$$\{g^{-1}h : (g, h) \in E\}$$

is finite. If $U, V \subseteq G$ are two subsets of G then $U \times V$ is a coentourage if

$$U \cap Vg$$

is finite for every $g \in G$.

Lemma 24. *Let X be a coarse space. If $C \subseteq X^2$ is a coentourage and $E \subseteq X^2$ an entourage then $C \circ E$ and $E \circ C$ are coentourages.*

Proof. Let $F \subseteq X^2$ be any entourage. Without loss of generality E is symmetric and contains the diagonal. Now C being a coentourage implies that there is some bounded set $B \subseteq X$ such that

$$C \cap E^{-1} \circ F \subseteq B^2$$

Then

$$\begin{aligned} E \circ C \cap F &\subseteq E \circ (C \cap E^{-1} \circ F) \\ &\subseteq E \circ B^2 \\ &\subseteq (E[B] \cup B)^2 \end{aligned}$$

$\square$

Theorem 25. *Now we are going to characterize coentourages axiomatically. Let γ be a collection of subsets of X^2 such that*

1. *γ is closed under taking subsets, finite unions and inverses;*
2. *we say a subset $B \subseteq X$ is bounded if $B \times X \in \gamma$ and require*

$$X = \bigcup_{B \in \beta} B;$$

3. *for every $C \in \gamma$ there is some bounded set $B \subseteq X$ such that*

$$C \cap \Delta_X \subseteq B^2;$$

4. *If E is detected by γ and $C \in \gamma$ then $E \circ C \in \gamma$.*

Then γ detects a coarse structure.

Proof. Denote by β the collection of bounded sets of X. Note that by points 1 and 2 the system β is a bornology. Now we show that γ detects a coarse structure by checking the axioms in Definition 2.

1. Point 3 guarantees that the diagonal is an entourage.

2. That is because β is a bornology.

3. Same.

4. By point 1 the inverse of an entourage is an entourage.

5. Suppose $E, F \subseteq X^2$ are detected by γ. Without loss of generality E is symmetric and contains the diagonal. Then there is some bounded set B such that

$$F \cap E^{-1} \circ C \subseteq B^2.$$

But then

$$\begin{aligned} E \circ F \cap C &\subseteq E \circ (F \cap E^{-1} \circ C) \\ &\subseteq E \circ B^2 \\ &\subseteq (E[B] \cup B)^2 \end{aligned}$$

and that is bounded because of the first point.

6. this works because of point 2.

□

Notation 26. (**coarsely disjoint**) If $A, B \subseteq X$ are subsets of a coarse space then A is called *coarsely disjoint to B* if

$$A \times B \subseteq X^2$$

is a coentourage. Being coarsely disjoint is a relation on subsets of X.

2.2.3 On Maps

Note that in this chapter every coarse space is assumed to be coarsely normal.

Lemma 27. *Two coarse maps $f, g : X \to Y$ are close if and only if for every coentourage $D \subseteq Y^2$ the set $(f \times g)^{-1}(D)$ is a coentourage.*

Proof. Denote by β the collection of bounded sets. Suppose f, g are close. Let $C \subseteq Y^2$ be a coentourage and $E \subseteq X^2$ an entourage. Set

$$S = (f \times g)^{-1}(C) \cap E.$$

Then there is some bounded set B such that

$$
\begin{aligned}
(f \times g)(S) &= (f \times g) \circ ((f \times g)^{-1}(C) \cap E) \\
&\subseteq (f \times g) \circ (f \times g)^{-1}(C) \cap (f \times g)(E) \\
&\subseteq C \cap (f \times g)(E) \\
&\subseteq B^2.
\end{aligned}
$$

But f and g are coarsely proper thus

$$
\begin{aligned}
S &\subseteq (f^{-1} \times g^{-1}) \circ (f \times g)(S) \\
&\subseteq f^{-1}(B) \times g^{-1}(B)
\end{aligned}
$$

is in β^2.

Now for the reverse direction: Let $C \subseteq Y^2$ be a coentourage. There is some bounded set $B \subseteq X^2$ such that

$$\Delta_X \cap (f \times g)^{-1}(C) \subseteq B^2.$$

Then

$$
\begin{aligned}
(f \times g)(\Delta_X) \cap C &= (f \times g)(\Delta_X) \cap (f \times g) \circ (f \times g)^{-1}(C) \\
&= (f \times g)(\Delta_X \cap (f \times g)^{-1}(C)) \\
&\subseteq (f \times g)(B^2).
\end{aligned}
$$

But f, g are coarsely uniform thus $(f \times g)(B^2) \in \beta^2$. $\square$

Proposition 28. *A map $f : X \to Y$ between coarse spaces is coarse if and only if*

- *for every bounded set $B \subseteq X$ the image $f(B)$ is bounded in Y*
- *and for every coentourage $C \subseteq Y^2$ the reverse image $(f^{\times 2})^{-1}(C)$ is a coentourage in X*

Proof. Suppose f is coarse. By Lemma 16 point 3 the second point holds and by coarsely uniformness the first point holds.

Suppose the above holds. Let $E \subseteq X^2$ be an entourage. For every coentourage $D \subseteq Y^2$ there is some bounded set B such that

$$E \cap (f^{\times 2})^{-1}(D) \subseteq B^2.$$

Then

$$
\begin{aligned}
f^{\times 2}(E) \cap D &= f^{\times 2}(E) \cap f^{\times 2} \circ (f^{\times 2})^{-1}(D) \\
&= f^{\times 2}(E \cap (f^{\times 2})^{-1}(D)) \\
&\subseteq f(B)^2.
\end{aligned}
$$

Because of point 1 we have $f^{\times 2}(B) \in \beta$. By point 2 the reverse image of every bounded set is bounded. $\square$

Definition 29. A map $f : X \to Y$ between coarse spaces is called *coarsely surjective* if one of the following equivalent conditions applies:

- There is an entourage $E \subseteq Y^2$ such that $E[\operatorname{im} f] = Y$.

- there is a map $r : Y \to \operatorname{im} f$ such that

$$\{(y, r(y)) : y \in Y\}$$

  is an entourage in Y.

- The inclusion $\operatorname{im} f \to Y$ is a coarse equivalence.

We will refer to the above map r as the retract of Y to $\operatorname{im} f$. Note that it is a coarse equivalence.

Lemma 30. *Every coarse equivalence is coarsely surjective.*

Proof. Let $f : X \to Y$ be a coarse equivalence and $g : Y \to X$ its inverse. Then $f \circ g : Y \to \operatorname{im} f$ is the retract of Definition 29. □

Lemma 31. *Coarsely surjective coarse maps are epimorphisms in the category of coarse spaces and coarse maps modulo close.*

Proof. Suppose $f : X \to Y$ is a coarsely surjective coarse map between coarse spaces. Then there is an entourage $E \subseteq Y^2$ such that $E[\operatorname{im} f] = Y$. We show f is an epimorphism. Let $g_1, g_2 : Y \to Z$ be two coarse maps to a coarse space such that $g_1 \circ f, g_2 \circ f$ are close. Then the set

$$H := g_1 \circ f \times g_2 \circ f(\Delta_X)$$

is an entourage. Then

$$g_1 \times g_2(\Delta_Y) \subseteq g_1^{\times 2}(E) \circ H \circ g_2^{\times 2}(E^{-1})$$

is an entourage. Thus g_1, g_2 are close. □

Definition 32. A map $f : X \to Y$ between coarse spaces is called *coarsely injective* if for every coentourage $C \subseteq X^2$ the set

$$f^{\times 2}(C)$$

is a coentourage.

Remark 33. Note that every coarsely injective coarse map is a coarse embedding and likewise every coarse embedding is coarsely injective coarse.[2]

Lemma 34. *Let $f : X \to Y$ be a coarse equivalence. Then f is coarsely injective.*

Proof. Let $g : Y \to X$ be a coarse inverse of f. Then there is an entourage

$$F = \{(g \circ f(x), x) : x \in X\}$$

in X. But then $g \circ f$ is coarsely injective because for every coentourage $C \subseteq X^2$ we have

$$g \circ f^{\times 2}(C) \subseteq F \circ C \circ F^{-1}$$

and $F \circ C \circ F^{-1}$ is again a coentourage by Lemma 24. But

$$f^{\times 2}(C) \subseteq (g^{\times 2})^{-1} \circ g^{\times 2} \circ f^{\times 2}(C)$$

is a coentourage, thus f is coarsely injective. □

[2]Although the latter term 'coarse embedding' is in general use and describes the notion more appropriately we will use the former term 'coarsely injective' because adjectives are easier to handle.

Lemma 35. *Coarsely injective coarse maps are monomorphisms in the category of coarse spaces and coarse maps modulo closeness.*

Proof. Suppose $f : X \to Y$ is a coarsely injective coarse map between coarse spaces. We show f is a monomorphism. Let $g_1, g_2 : Z \to X$ be two coarse maps such that $f \circ g_1, f \circ g_2 : Z \to Y$ are close. Then

$$H := f \circ g_1 \times f \circ g_2(\Delta_Z)$$

is an entourage. Now

$$g_1 \times g_2(\Delta_Z) \subseteq (f^{\times 2})^{-1}(H)$$

is an entourage. Thus g_1, g_2 are close. $\square$

Remark 36. Every coarse map can be factored into an epimorphism followed by a monomorphism.

Proposition 37. *If a coarse map $f : X \to Y$ is coarsely surjective and coarsely injective then f is a coarse equivalence.*

Proof. We just need to construct the coarse inverse. Note that the map $r : Y \to \operatorname{im} f$ from the second point of Definition 29 is a coarse equivalence which is surjective. Without loss of generality we can replace f by $\hat{f} = r \circ f$. Now define $g : \operatorname{im} f \to X$ by mapping $y \in \operatorname{im} f$ to some point in $\hat{f}^{-1}(y)$ where the choice is not important. Now we show:

1. g is a coarse map: Let $E \subseteq (\operatorname{im} f)^2$ be an entourage. Then

   $$g^{\times 2}(E) \subseteq (f^{\times 2})^{-1}(E)$$

   is an entourage. And if $B \subseteq X$ is bounded then

   $$g^{-1}(B) \subseteq f(B)$$

   is bounded.

2. $\hat{f} \circ g = id_{\operatorname{im} f}$

3. g is coarsely injective: Let $D \subseteq (\operatorname{im} f)^2$ be a coentourage. Then

   $$g^{\times 2}(D) \subseteq (f^{\times 2})^{-1}(D)$$

   is a coentourage.

4. $g \circ \hat{f} \sim id_X$: we have $g \circ \hat{f} : X \to \operatorname{im} g$ is coarsely injective and thus the retract of Definition 29 with coarse inverse the inclusion $i : \operatorname{im} g \to X$. But

   $$g \circ \hat{f} \circ i = id_{\operatorname{im} g}.$$

$\square$

2.3 Limits and Colimits

The category `Top` of topological spaces is both complete and cocomplete. In fact the forgetful functor `Top` $\to$ `Sets` preserves all limits and colimits that is because it has both a right and left adjoint. We do something similar for coarse spaces.

Note that the following notions generalize the existing notions of product and disjoint union of coarse spaces.

2.3.1 The Forgetful Functor

Definition 38. Denote the category of connected coarse spaces and coarsely uniform maps between them by `DCoarse`.

Theorem 39. *The forgetful functor η : `DCoarse` $\to$ `Sets` preserves all limits and colimits.*

Proof.

- There is a functor δ : `Sets` $\to$ `DCoarse` that sends a set X to the coarse space X with the discrete coarse structure[3]. Then every map of sets induces a coarsely uniform map.
- There is a functor α : `Sets` $\to$ `DCoarse` which sends a set X to the coarse space X with the maximal coarse structure. Again every map of sets induces a coarsely uniform map.
- Let X be a set and Y a coarse space. Then

  $$Hom_{\mathtt{Sets}}(X, \eta Y) = Hom_{\mathtt{DCoarse}}(\delta X, Y)$$

  and

  $$Hom_{\mathtt{Sets}}(\eta Y, X) = Hom_{\mathtt{DCoarse}}(Y, \alpha X)$$

  Thus the forgetful functor is right adjoint to δ and left adjoint to α.
- An application of the [19, Adjoints and Limits Theorem 2.6.10] gives the result.

□

2.3.2 Limits

The following definition is a generalization of [20, Definition 1.21]:

Definition 40. Let X be a set and $f_i : X \to Y_i$ a family of maps to coarse spaces. The *pullback coarse structure of* $(f_i)_i$ on X is generated by $\bigcap_i (f_i^{\times 2})^{-1}(E_i)$ for $E_i \subseteq Y_i$ an entourage for every i. That is, a subset $E \subseteq X^2$ is an entourage if for every i the set $f_i^{\times 2}(E)$ is an entourage in Y_i.

Lemma 41. *The pullback coarse structure is indeed a coarse structure; the maps $f_i : X \to Y_i$ are coarsely uniform.*

Proof.

1. $\Delta_X \subseteq (f_i^{\times 2})^{-1}(\Delta_{Y_i})$ for every i.
2. easy
3. if E_1, E_2 are entourages in X then for every i there are entourages $F_1, F_2 \subseteq Y_i^2$ such that $E_1 \subseteq (f_i^{\times 2})^{-1}(F_1)$ and $E_2 \subseteq (f_i^{\times 2})^{-1}(F_2)$. But then

   $$\begin{aligned} E_1 \cup E_2 &\subseteq (f_i^{\times 2})^{-1}(F_1) \cup (f_i^{\times 2})^{-1}(F_2) \\ &= (f_i^{\times 2})^{-1}(F_1 \cup F_2) \end{aligned}$$

4. if E is an entourage in X then for every i there is an entourage F in Y_i such that $E \subseteq (f_i^{\times 2})^{-1}(F)$. But then

   $$E^{-1} \subseteq (f_i^{\times 2})^{-1}(F^{-1})$$

5. If E_1, E_2 are as above then

   $$E_1 \circ E_2 \subseteq (f_i^{\times 2})^{-1}(F_1 \circ F_2)$$

[3] in which every entourage is the union of a subset of the diagonal and finitely many off-diagonal points

6. If $(x, y) \in X$ then for every i

$$f_i^{\times 2}(x, y) = (f_i(x), f_i(y))$$

is an entourage.

□

Remark 42. Note that it would be ideal if the pullback coarse structure is well-defined up to coarse equivalence and if there is a universal property. We can not use naively the limit in `Sets` and equip it with the pullback coarse structure as the following example shows:

Denote by $\phi : \mathbb{Z} \to \mathbb{Z}$ the map that maps $i \mapsto 2i$ and by $\psi : \mathbb{Z} \to \mathbb{Z}$ the map that maps $i \mapsto 2i + 1$. then both ϕ, ψ are isomorphisms in the coarse category. The pullback of

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, "\phi"] \\ \mathbb{Z} \arrow[r, "\psi"'] & \mathbb{Z} \end{tikzcd}$$

is $\emptyset$ in `Sets` but should be an isomorphism if the diagram is supposed to be a pullback diagram in `Coarse`. See Definition 97 for a sophisticated realization of a pullback diagram.

Proposition 43. *Let X have the pullback coarse structure of $(f_i : X \to Y_i)_i$. A subset $C \subseteq X^2$ is a coentourage if for every i the set $f_i^{\times 2}(C)$ is a coentourage in Y_i. Note that the converse does not hold in general.*

Proof. Let $C \subseteq X^2$ have the above property. If $F \subseteq X^2$ is a subset such that

$$S = C \cap F$$

is not bounded then there is some i such that $f_i^{\times 2}(S)$ is not bounded. Then

$$f_i^{\times 2}(C) \cap f_i^{\times 2}(F) \supseteq f_i^{\times 2}(C \cap F)$$

is not bounded but $f_i^{\times 2}(C)$ is a coentourage in Y_i. Thus $f_i^{\times 2}(F)$ is not an entourage in Y_i, thus F does not belong to the pullback coarse structure on X. Thus C is detected by the pullback coarse structure. □

Example 44. (**Product**) The pullback coarse structure on products agrees with [20, Definition 1.32]: If X, Y are coarse spaces the product $X \times Y$ has the pullback coarse structure of the two projection maps p_1, p_2:

- A subset $E \subseteq (X \times Y)^2$ is an entourage if and only if $p_1^{\times 2}(E)$ is an entourage in X and $p_2^{\times 2}(E)$ is an entourage in Y.
- A subset $C \subseteq (X \times Y)^2$ is a coentourage if and only if $p_1^{\times 2}(C)$ is a coentourage in X and $p_2^{\times 2}(C)$ is a coentourage in Y.

2.3.3 Colimits

Proposition 45. *If $f_i : Y_i \to X$ is a finite family of injective maps from coarse spaces then the subsets*

$$f_i^{\times 2}(E_i)$$

for i an index and $E_i \subseteq Y_i^2$ an entourage are a subbase for a coarse structure; the maps $f_i : Y_i \to X$ are coarse maps.

Proof. Suppose $E_i \subseteq Y_i^2$ is an entourage. Let $C \subseteq X^2$ be an element of the pushout cocoarse structure. Denote

$$S = f_i^{\times 2}(E_i) \cap C.$$

Then

$$\begin{aligned}(f_i^{\times 2})^{-1}(S) &= (f_i^{\times 2})^{-1} \circ f_i^{\times 2}(E_i) \cap (f_i^{\times 2})^{-1}(C) \\ &= E_i \cap (f_i^{\times 2})^{-1}(C)\end{aligned}$$

implies that $f_i^{\times 2}(E_i)$ is an entourage.

Now $E \subseteq X^2$ is an entourage if for every i

$$E \cap (\operatorname{im} f_i)^2$$

is an entourage and if $E \cap (\bigcup_i (\operatorname{im} f_i)^2)^c$ is bounded.

We show that this is indeed a coarse structure by checking the axioms of Definition 2:

1. We show the diagonal in X is an entourage. Let $C \subseteq X^2$ be a subset such that

$$(f_i^{\times 2})^{-1}(C) \subseteq Y_i^2$$

is a coentourage. Denote

$$S = \Delta_X \cap C.$$

Then

$$\begin{aligned}(f_i^{\times 2})^{-1}(\Delta_X \cap C) &= (f_i^{\times 2})^{-1}(\Delta_X) \cap (f_i^{\times 2})^{-1}(C) \\ &= \Delta_{Y_i} \cap (f_i^{\times 2})^{-1}(C) \\ &\subseteq B_i^2\end{aligned}$$

is bounded.

2. easy

3. easy

4. easy

5. If $E_1, E_2 \subseteq X^2$ have the property that for every element $C \subseteq X^2$ of the pushout cocoarse structure and every i:

$$(f_i^{\times 2})^{-1}(E_1) \cap (f_i^{\times 2})^{-1}(C)$$

and

$$(f_i^{\times 2})^{-1}(E_2) \cap (f_i^{\times 2})^{-1}(C)$$

are bounded in Y_i we want to show that $E_1 \circ E_2$ has the same property. Now without loss of generality we can assume that there are ij such that $E_1 \subseteq (\operatorname{im} f_i)^2$ and $E_2 \subseteq (\operatorname{im} f_j)^2$ the other cases being trivial or they can be reduced to that case. Then

$$E_1 \circ (E_2 \cap (\operatorname{im} f_i)^2) \subseteq (\operatorname{im} f_i)^2$$

and

$$(E_1 \cap (\operatorname{im} f_j)^2) \circ E_2 \subseteq (\operatorname{im} f_j)^2$$

are entourages and the other cases are empty.

6. If $(x_1, x_2) \in X^2$ then for every i
$$(f_i^{\times 2})^{-1}(x_1, x_2)$$
is either one point or the empty set in Y_i, both are entourages.

$\square$

Definition 46. Let X be a set and $f_i : Y_i \to X$ a finite family of injective maps from coarse spaces. Then define the *pushout cocoarse structure* on X to be those subsets C of X^2 such that for every i the set
$$(f_i^{\times 2})^{-1}(C) \subseteq Y_i^2$$
is a coentourage.

Example 47. Let A, B be coarse spaces and $A \sqcup B$ their disjoint union. The cocoarse structure and the coarse structure of $A \sqcup B$ look like this:

- A subset $D \subseteq (A \sqcup B)^2$ is a coentourage if $D \cap A^2$ is a coentourage in A and $D \cap B^2$ is a coentourage in B.
- A subset $E \subseteq (A \sqcup B)^2$ is an entourage if $E \cap A^2$ is an entourage of A and $E \cap B^2$ is an entourage of B and $E \cap (A \times B \cup B \times A)$ is contained in $S \times T \cup T \times S$ where S is bounded in A and T is bounded in B. This definition actually agrees with [21, Definition 2.12, p. 277].

Example 48. Let G be a countable group that acts on a set X. We require that for every $x, y \in X$ the set
$$\{g \in G : g.x = y\}$$
is finite. Then the pushout cocoarse structure of the orbit maps
$$\begin{aligned} i_x : G &\to X \\ g &\mapsto g.x \end{aligned}$$
for $x \in X$ is dual to the minimal connected $G-$invariant coarse structure of [4, Example 2.13].

Proof. Note that by the above requirement a subset $B \subseteq X$ is bounded if and only if it is finite. Fix an element $x \in X$ and denote by $X' \subseteq X$ the orbit of x.

For every $C \subseteq G^2$ coentourage
$$E \cap i_x^2(C)$$
being bounded implies that
$$(i_x^{\times 2})^{-1}(E) \cap C \subseteq (i_x^{\times 2})^{-1}(E \cap i_x^{\times 2}(C))$$
is bounded. Thus if $E \subseteq X^2$ is an entourage then $(i_x^{\times 2})^{-1}(E)$ is an entourage.

If $(i_x^{\times 2})^{-1}(E)$ is an entourage then $E = i_x^{\times 2} \circ (i_x^{\times 2})^{-1}(E)$. For every $C \subseteq G^2$ coentourage
$$(i_x^{\times 2})^{-1}(E) \cap C$$
being bounded implies that
$$E \cap i_x^{\times 2}(C)$$
is bounded. Thus E is an entourage.

The $i_x^{\times 2}(E)$ for $E \subseteq G^2$ an entourage are a coarse structure on X' because i_x is surjective on X'.

If x, y are in the same orbit X' then i_x, i_y induce the same coarse structure on X'. $\square$

2.4 Coarse Cohomology with twisted Coefficients

We define a Grothendieck topology on coarse spaces and describe cohomology with twisted coefficients on coarse spaces and coarse maps. We have a notion of Mayer-Vietoris and a notion of relative cohomology.

2.4.1 Coarse Covers

Definition 49. Let X be a coarse space and let $(U_i)_i$ be a finite family of subspaces of X. It is said to *coarsely cover* X if the complement of

$$\bigcup_i U_i^2$$

is a coentourage.

Example 50. The coarse space $\mathbb{Z}$ is coarsely covered by $\mathbb{Z}_-$ and $\mathbb{Z}_+$. An example for a decomposition that does not coarsely cover $\mathbb{Z}$ is $\{x \in \mathbb{Z} : x \text{ is even}\} \cup \{x \in \mathbb{Z} : x \text{ is odd}\}$.

Remark 51. The finiteness condition is important, otherwise $(\{x, y\})_{x,y \in X}$ would coarsely cover X, but if X is not bounded we don't want X to be covered by bounded sets only.

Lemma 52. *A nonbounded coarse space X is coarsely covered by one element U if and only if $X \setminus U$ is bounded.*

Proof. By definition U coarsely covers X if and only if $(U^2)^c$ is a coentourage; now $(U^c)^2 \subseteq (U^2)^c$ thus U^c is bounded by Lemma 19.

Conversely, if U^c is bounded then

$$(U^2)^c = X \times U^c \cup U^c \times X$$

is a coentourage, thus U coarsely covers X. □

Remark 53. If X is coarsely covered by $(U_i)_i$ and they cover X (as sets) then it is the colimit (see Definition 46) of them:

$$X \cong \bigcup_i U_i$$

as a coarse space.

This is going to be useful later:

Proposition 54. *A finite family of subspaces $(U_i)_i$ coarsely covers a metric space X if and only if for every entourage $E \subseteq X^2$ the set*

$$E[U_1^c] \cap \ldots \cap E[U_n^c]$$

is bounded.

Remark 55. This appeared already in [6, Definition 2.1]; wherein $U_1^c, \ldots, U_n^c$ is a finite system of subsets of X that diverges.

Proof. We proceed by induction on the number i of pieces in the cover.

If there is one piece U_1, then by Lemma 52 one subset $U_1 \subseteq X$ coarsely covers X if and only if U_1^c is bounded. By this and Lemma 6 for every entourage $E \subseteq X^2$ the set $E[U_1^c]$ is bounded.

Conversely if $E[U_1^c]$ is bounded for every entourage $E \subseteq X^2$ then U_1^c itself is bounded which implies that U_1 coarsely covers X.

Consider next the case of two subsets U_1, U_2. We first claim that they form a coarse cover if and only if $U_1^c \times U_2^c$ is a coentourage. Indeed $X^2 \backslash (U_1^2 \cup U_2^2) = U_1^c \times U_2^c \cup U_2^c \times U_1^c$, so $X^2 \backslash (U_1^2 \cup U_2^2)$ is a coentourage if and only if both of $U_1^c \times U_2^c$ and $U_2^c \times U_1^c$ are coentourages. Let $E \subseteq X^2$ be an entourage. Now by Lemma 24 this implies that $U_1^c \times E[U_2^c]$ is a coentourage, namely we have that the set $E[U_1^c] \cap E[U_2^c]$ is bounded.

Conversely from the assumption that $E[U_1^c] \cap E[U_2^c]$ is bounded for every entourage $E \subseteq X^2$, we deduce $E[U_1^c] \cap U_2^c$ is a bounded set. This implies that $U_1^c \times U_2^c$ is a coentourage.

Now we consider the inductive step. Suppose $n \geq 1$. Subsets $U_1, \ldots, U_n, U, V$ form a coarse cover of X if and only if $U_1, \ldots, U_n, U \cup V$ is a coarse cover of X and U, V is a coarse cover of $U \cup V$. Let $E \subseteq X^2$ be an entourage. Without loss of generality we can assume E is symetric and contains the diagonal. By the induction hypothesis

$$E[U_1^c] \cap \cdots \cap E[U_n^c] \cap E[(U \cup V)^c]$$

is bounded. And

$$E[U^c \cap V] \cap E[V^c \cap U] \cap (U \cup V)$$

is bounded. Now

$$\begin{aligned}
E[U_1^c] \cap \cdots \cap E[U_n^c] \cap E[U^c] \cap E[V^c] &= E[U_1^c] \cap \cdots \cap E[U_n^c] \cap E[U^c] \cap E[V^c] \cap E[(U \cup V)^c] \\
&\cup E[U_1^c] \cap \cdots \cap E[U_n^c] \cap E[U^c] \cap E[V^c] \cap E[(U \cup V)^c]^c \\
&\subseteq E[U_1^c] \cap \cdots \cap E[U_n^c] \cap E[(U \cup V)^c] \\
&\cup E[U^c \cap V] \cap E[V^c \cap U] \cap (U \cup V)
\end{aligned}$$

is bounded. In the above calculation we use that

$$E[U^c] \cap E[V^c] \cap E[(U \cup V)^c]^c \subseteq E[U^c \cap V] \cap E[V^c \cap U] \cap (U \cup V)$$

by direct calculation.

□

Proposition 56. *If $r : X \to Y$ is a surjective coarse equivalence then $(V_i)_i$ is a coarse cover of Y if and only if $(r^{-1}(V_i))_i$ is a coarse cover of X.*

Proof. Suppose $(V_i)_i$ is a coarse cover of X. then $(\bigcup_i V_i^2)^c$ is a coentourage in Y thus

$$\bigcup_i f^{-1}(V_i)^c = (f^{\times 2})^{-1}((\bigcup_i V_i)^c)$$

is a coentourage. Thus $(f^{-1}(V_i))_i$ is a coarse cover of X.

Conversely suppose $(f^{-1}(V_i))_i$ is a coarse cover of X then

$$(\bigcup_i V_i)^c = f^{\times 2} \circ (f^{\times 2})^{-1}((\bigcup_i V_i)^c)$$

is a coentourage in Y. □

2.4.2 The Coarse Site

Notation 57. In what follows we define a Grothendieck topology on the category of subsets of a coarse space X. What we call a Grothendieck topology is sometimes called a Grothendieck pretopology. We stick to the notation of [22]. If $\mathtt{C}$ is a category a Grothendieck topology T on $\mathtt{C}$ consists of

- the underlying category $Cat(T) = \mathtt{C}$
- the set of coverings $Cov(T)$ which consists of families of morphisms in $\mathtt{C}$ with a common codomain. We write
$$\{U_i \to U\}_i$$
where i stands for the index. They comply with the following rules:
  1. Every isomorphism is a covering.
  2. *Local character:* If $\{U_i \to U\}_i$ is a covering and for every i the family $\{V_{ij} \to U_i\}_j$ is a covering then the composition
  $$\{V_{ij} \to U_i \to U\}_{ij}$$
  is a covering.
  3. *Stability under base change:* For every object $U \in Cat(T)$, morphism $V \to U$ and covering $\{U_i \to U\}_i$ all fibre products $U_i \times_U V$ exist and the family
  $$\{U_i \times_U V \to V\}$$
  is a covering.

In the course of this paper we will mostly (but not always) apply theory on Grothendieck topologies as portrayed in [23, parts I,II].

Definition 58. To a coarse space X is associated a Grothendieck topology X_{ct} where the underlying category of X_{ct} consists of subsets of X, there is an arrow $U \to V$ if $U \subseteq V$. A finite family $(U_i)_i$ covers U if they coarsely cover U, that is, if
$$U^2 \cap (\bigcup_i U_i^2)^c$$
is a coentourage in X.

Lemma 59. *The construction X_{ct}, is indeed a Grothendieck topology.*

Proof. We check the axioms for a Grothendieck topology:

1. if $U \subseteq X$ is a subset the identity $\{U \to U\}$ is a covering

2. Let $\{U_i \to U\}_i$ be a covering and suppose for every i there is a covering $\{U_{ij} \to U_i\}_j$, then:

$$\begin{aligned}
U^2 \cap (\bigcup_{ij} U_{ij}^2)^c &= U^2 \cap \bigcap_i \bigcap_j U_{ij}^{2c} \\
&= \bigcap_i (U^2 \cap \bigcap_j U_{ij}^{2c}) \\
&= \bigcap_i [(U^2 \cap U_i^2 \cap \bigcap_j U_{ij}^{2c}) \cup (U^2 \cap U_i^{2c} \cap \bigcap_j U_{ij}^{2c})] \\
&\subseteq \bigcap_i [(U_i^2 \cap \bigcap_j U_{ij}^{2c}) \cup (U^2 \cap U_i^{2c})] \\
&\subseteq \bigcup_i (U_i^2 \cap \bigcap_j U_{ij}^{2c}) \cup \bigcap_i (U^2 \cap U_i^{2c}) \\
&= \bigcup_i [U_i^2 \cap (\bigcup_j U_{ij}^2)^c] \cup [U^2 \cap (\bigcup_{ij} U_i^2)^c];
\end{aligned}$$

Therefore $U^2 \cap (\bigcup_{ij} U_{i,j}^2)^c$ is a finite union of coentourages, since the index set is finite; so it is a coentourage by Lemma 16.

3. Let $\{U_i \to U\}_i$ be a covering and let $V \subseteq U$ be an inclusion. Then

$$\begin{aligned}
V^2 \cap (\bigcup_i (V \cap U_i)^2)^c &= V^2 \cap \bigcap_i (V \cap U_i)^{2c} \\
&= V^2 \cap \bigcap_i (U_i^{2c} \cup V^{2c}) \\
&= V^2 \cap \bigcap_i U_i^{2c} \\
&= V^2 \cap (\bigcup_i U_i^2)^c \\
&\subseteq U^2 \cap (\bigcup_i U_i^2)^c
\end{aligned}$$

So $\{V \cap U_i \to V\}_i$ is a covering of X_{ct}.

□

Notation 60. If T, T' are two Grothendieck topologies a functor $f : Cat(T) \to Cat(T')$ is called a *morphism of topologies* if

1. if $\{\varphi_i : U_i \to U\}_i$ is a covering in T then $\{f(\varphi_i) : f(U_i) \to f(U)\}_i$ is a covering in T'.
2. if $\{U_i \to U\}_i \in Cov(T)$ and $V \to U$ a morphism in $Cat(T)$ then the canonical morphism

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

is an isomorphism for every i.

Definition 61. Let $f : X \to Y$ be a coarse map between coarse spaces. Then we define a functor

$$\begin{aligned}
f^{-1} : Cat(Y_{ct}) &\to Cat(X_{ct}) \\
U &\mapsto f^{-1}(U)
\end{aligned}$$

Lemma 62. *The functor f^{-1} induces a morphism of Grothendieck topologies $f^{-1}: Y_{ct} \to X_{ct}$.*

Proof. We check the axioms for a morphism of Grothendieck topologies:

1. Let $\{U_i \to U\}_i$ be a covering in Y. Then
$$f^{-1}(U)^2 \cap (\bigcup_i f^{-1}(U_i)^2)^c = (f^{\times 2})^{-1}(U^2 \cap (\bigcup_i U_i^2)^c)$$
is a coentourage. Thus $\{f^{-1}(U_i) \to f^{-1}(U)\}_i$ is a covering in X.

2. for every U, V subsets of X we have
$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

$\square$

Notation 63. Let T be a Grothendieck topology.

- A *presheaf on T with values in $\mathtt{C}$* is defined as a contravariant functor $\mathcal{F}: Cat(T) \to \mathtt{C}$.
- A *morphism $\eta: \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathtt{C}$* is a natural transformation of contravariant functors.
- A presheaf is a *sheaf on T* if for every covering $\{U_i \to U\} \in Cov(T)$ the diagram
$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{ij} \mathcal{F}(U_i \times_U U_j)$$
is an equalizer diagram in $\mathtt{C}$. Exactness at $\mathcal{F}(U)$ means that the first arrow $s \mapsto (s|_{U_i})_i$ is injective (*global axiom*) and exactness at $\prod_i \mathcal{F}(U_i)$ means that the image of the first arrow is equal to the kernel of the double arrow, hence consists of all $(s_i)_i$ such that $s_i|_{U_j} = s_j|_{U_i}$ (*gluing axiom*).
- A *morphism of sheaves* is a morphism of the underlying presheaves.

Example 64. Let B be a space with the indiscrete (maximal) coarse structure. Then B is already covered by the empty covering. But then the equalizer diagram for that covering is
$$\mathcal{F}(B) \to \prod_{\emptyset} \rightrightarrows \prod_{\emptyset}$$
Thus every sheaf on B vanishes.

Proposition 65. ***(Sheaf of Functions)*** *If X, Y are coarse spaces then the assignment $U \subseteq X \mapsto$ (coarse maps $U \to Y$ modulo closeness) is a sheaf on X_{ct}.*

Proof. We check the sheaf axioms:

1. global axiom: Let $f, g: U \to Y$ be two coarse maps and suppose U is coarsely covered by U_1, U_2 and $f|_{U_1} \sim g|_{U_1}$ and $f|_{U_2} \sim g|_{U_2}$. Then
$$f \times g(\Delta_U) = f \times g(\Delta_{U_1}) \cup f \times g(\Delta_{U_2}) \cup f \times g(\Delta_{U \setminus (U_1 \cup U_2)})$$
The first two terms of the union are entourages because f, g are close on U_1 and U_2. The last term is a entourage because $U \setminus (U_1 \cup U_2)$ is bounded. Therefore $(f \times g)(\Delta_U)$ is a union of three entourages, so is itself an entourage. Thus f, g are close on U.

2. gluing axiom: Suppose $U \subseteq X$ is coarsely covered by U_1, U_2 and $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ are coarse maps such that

$$f_1|_{U_2} \sim f_2|_{U_1}.$$

Then there is a global map $f : U \to Y$ defined in the following way:

$$f(x) = \begin{cases} f_1(x) & x \in U_1, \\ f_2(x) & x \in U_2 \setminus U_1, \\ p & x \in U \setminus (U_1 \cup U_2). \end{cases}$$

Here p denotes some point in Y. Now we show f is a coarse map:

We show f is coarsely uniform: If $E \subseteq U^2$ is an entourage then

(a) $f^{\times 2}(E \cap U_1^2) = f_1^{\times 2}(E \cap U_1^2)$ is an entourage;

(b)

$$\begin{aligned} f^{\times 2}(E \cap (U_1 \cap U_2) \times (U_2 \setminus U_1)) &= f_1 \times f_2(E \cap (U_1 \cap U_2) \times (U_2 \setminus U_1)) \\ &\subseteq f_1 \times f_2(\Delta_{U_1 \cap U_2}) \circ f_2^{\times 2}(E \cap (U_1 \cap U_2) \times (U_2 \setminus U_1)) \end{aligned}$$

is an entourage;

(c) $f^{\times 2}(E \cap (U_2 \setminus U_1)^2) = f_2^{\times 2}(E \cap (U_2 \setminus U_1)^2)$ is an entourage;

(d) $E \cap U_1^c \times U_2^c$ and $E \cap U_2^c \times U_1^c$ are already bounded. Now f maps bounded sets to bounded sets because f_1, f_2 and the constant map to p do.

Since

$$U^2 = U_1^2 \cup (U_1 \cap U_2) \times (U_2 \setminus U_1) \cup (U_2 \setminus U_1) \times (U_1 \cap U_2) \cup (U_2 \setminus U_1)^2 \cup (U \setminus (U_1 \cup U_2))^2$$

the set $f^{\times 2}(E)$ is a finite union of entourages and therefore itself an entourage. Thus f is coarsely uniform.

We show f is coarsely proper: If $B \subseteq Y$ is bounded then

$$f^{-1}(B) \subseteq f_1^{-1}(B) \cup f_2^{-1}(B) \cup (U \setminus (U_1 \cup U_2))$$

is bounded.

Thus we showed f is a coarse map.

□

2.4.3 Sheaf Cohomology

Sheaves on the Grothendieck topology X_{ct} give rise to a cohomology theory on coarse spaces and coarse maps:

Notation 66. If T is a Grothendieck topology denote by $\texttt{Presheaf}(T)$ the category of abelian presheaves on T and by $\texttt{Sheaf}(T)$ the category of abelian sheaves on T. The category $\texttt{Sheaf}(T)$ is a full subcategory of $\texttt{Presheaf}(T)$, denote by $i : \texttt{Sheaf}(T) \to \texttt{Presheaf}(T)$ the inclusion functor. The functor i is left exact by [23, Theorem I.3.2.1]. If $U \in Cat(T)$ then denote by $\Gamma(U, \cdot) : \texttt{Presheaf}(T) \to \texttt{Ab}$ the section functor which is an exact functor by [23, Proposition I.2.1.1].

Then $\Gamma(U,\cdot)\circ i$ is additive and a composition of a left exact functor and an exact functor and therefore left exact. The category $\mathtt{Sheaf}(T)$ is an abelian category with enough injectives therefore the right derived functor

$$\check{H}^q(U,\mathcal{F}) = R^q(\Gamma(U,\cdot)\circ i)(\mathcal{F})$$

exists for $\mathcal{F}$ an abelian sheaf on T. See [23, Definition I.3.3.1].

Remark 67. (**coarse cohomology with twisted coefficients**) Let $\mathcal{F}$ be a sheaf of abelian groups on a coarse space X, let $U\subseteq X$ be a subset and let $q\ge 0$ be a number. Then the *qth coarse cohomology group of U with values in $\mathcal{F}$* is

$$\check{H}^q(U,\mathcal{F}),$$

the qth sheaf cohomology of U in X_{ct} with coefficient $\mathcal{F}$.

Remark 68. (**functoriality**) Let $f: X\to Y$ be a coarse map between coarse space. There is a *direct image functor*

$$\begin{aligned} f_*: \mathtt{Sheaf}(X_{ct}) &\to \mathtt{Sheaf}(Y_{ct})\\ \mathcal{F} &\mapsto f_*\mathcal{F}\end{aligned}$$

where

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for every $V\subseteq Y$. The left adjoint functor to f_* exists by [23, Proposition I.3.6.2] and is denoted inverse image functor

$$f^*: \mathtt{Sheaf}(Y)\to \mathtt{Sheaf}(X).$$

Note that f^* is exact. Then there is an edge homomorphism of the Leray spectral sequence[4] of f_* which will also be denoted by f_*: let $U\subseteq Y$ be a subset and let $\mathcal{F}$ be a sheaf on X; then there is a homomorphism

$$f_*: \check{H}^*(f^{-1}U,\mathcal{F})\to \check{H}^*(U,f_*\mathcal{F}).$$

Notation 69. Let T be a Grothendieck topology. By [23, Theorem I.3.1.1] the adjoint to the *inclusion functor* $i: \mathtt{Sheaf}(T)\to\mathtt{Presheaf}(T)$ exists and is denoted by $\#$. If $\mathcal{F}$ is a presheaf then $\mathcal{F}^\#$ is the *sheaf associated to the presheaf* $\mathcal{F}$, also called the *sheafification of* $\mathcal{F}$.

Define for an abelian presheaf $\mathcal{F}$ on T:

$$\mathcal{F}^\dagger(U) = \lim_{\{U_i\to U\}_i\in Cov(T)} H^0(\{U_i\to U\},\mathcal{F})$$

for $U\in Cat(T)$. Here the right side, the term $H^0(\{U_i\to U\},\mathcal{F})$, denotes the 0th Čhech cohomology associated to the covering $\{U_i\to U\}_i$ with values in $\mathcal{F}$. The functor $\mathcal{F}^\dagger$ is a presheaf and

$$\mathcal{F}^\# = (\mathcal{F}^\dagger)^\dagger$$

is the sheaf associated to the presheaf $\mathcal{F}$.

Lemma 70. *Let X be a coarse space and denote by $p: X\times\{0,1\}\to X$ the projection to the first factor. Then*

$$R^q p_* = 0$$

for $q>0$.

[4] This is [23, Theorem I.3.7.6, p. 71]

Proof. In a general setting if $\mathcal{F}$ is a sheaf on a coarse space denote by $\mathcal{H}^q(\mathcal{F})$ the presheaf

$$U \mapsto \check{H}^q(U, \mathcal{F}).$$

Then [23, Proposition I.3.4.3] says that

$$\mathcal{H}^q(\mathcal{F})^\dagger = 0$$

for $q > 0$.

Now [23, Proposition I.3.7.1] implies that for every coarse map $f : X \to Y$ and sheaf $\mathcal{F}$ on X

$$R^q f_*(\mathcal{F}) \cong (f_* \mathcal{H}^q(\mathcal{F}))^\#.$$

Define

$$H = \{((x,i),(x,0)) : (x,i) \in X \times \{0,1\}\} \subseteq (X \times \{0,1\})^2$$

as a subset of $X \times \{0,1\}$ which is an entourage. We identify $X \times 0$ with X. Then $(U_i)_i$ coarsely covers $U \subseteq X$ if and only if $(H[U_i])_i$ coarsely covers $H[U]$.

Let V_1, V_2 be a coarse cover of $U \times \{0,1\}$. Write

$$V_1 = V_1^0 \times 0 \cup V_1^1 \times 1$$

and

$$V_2 = V_2^0 \times 0 \cup V_2^1 \times 1.$$

Note that

$$\begin{aligned} V_i^c &= (V_i^0 \times 0)^c \cap (V_i^1 \times 1)^c \\ &= (V_i^0)^c \times 0 \cup (V_i^1)^c \times 1 \end{aligned}$$

for $i = 1, 2$. But then

$$((V_1^0)^c \cup (V_1^1)^c) \times ((V_2^0)^c \cup (V_2^1)^c)$$

is a coentourage in U. Thus

$$(V_1^0 \cap V_1^1) \times \{0,1\}, (V_2^0 \cap V_2^1) \times \{0,1\}$$

is a coarse cover that refines V_1, V_2.

We show that p_* and $\#$ commute for presheaves $\mathcal{G}$ on X: Let $U \subseteq X$ be a subset then

$$\begin{aligned} (p_*\mathcal{G})^\dagger(U) &= \lim_{\{U_i \to U\}_i \in Cov(X)} H^0(\{U_i \to U\}_i, p_*\mathcal{G}) \\ &= \lim_{\{U_i \to U\}_i \in Cov(X)} H^0(\{H[U_i] \to H[U]\}_i, \mathcal{G}) \\ &= \lim_{\{V_i \to H[U]\}_i \in Cov(X \times \{0,1\})} H^0(\{V_i \to H[U]\}_i, \mathcal{G}) \\ &= \mathcal{G}^\dagger(H[U]) \\ &= p_*\mathcal{G}^\dagger(U) \end{aligned}$$

□

Remark 71. Note that two coarse maps $f, g : X \to Y$ are close if the map $h : X \times \{0,1\} \to Y$ agreeing with f on $X \times 0$ and with g on $X \times 1$ is a coarse map.

Proof. Suppose h is a coarse map we show f,g are close. The set

$$\begin{aligned} f\times g(\Delta_X) &= \{f(x),g(x):x\in X\}\\ &= \{h((x,0),(x,1)):x\in X\}\\ &= h^{\times 2}(\Delta_X\times\{0,1\}) \end{aligned}$$

is an entourage in Y.

□

Theorem 72. ***(close maps)*** *If two coarse maps $f,g:X\to Y$ are close the induced homomorphisms f_*,g_* of coarse cohomology with twisted coefficients are isomorphic.*

Proof. Define a coarse map

$$h:X\times\{0,1\}\to Y$$

by $h|_{X\times 0}=f$ and $h|_{X\times 1}=g$. But the inclusions $i_0:X\times 0\to X\times\{0,1\}$ and $i_1:X\times 1\to X\times\{0,1\}$ are both sections of the projection $p:X\times\{0,1\}\to X$ which by Lemma 70 induces an isomorphism in coarse cohomology with twisted coefficients. Hence the maps $f=h\circ i_0$ and $g=h\circ i_1$ induce maps $f_*=h_*\circ i_{0*}$ and $g_*=h_*\circ i_{1*}$ which is the same map followed by isomorphisms. □

Corollary 73. ***(coarse equivalence)*** *Let $f:X\to Y$ be a coarse equivalence. Then f induces an isomorphism in coarse cohomology with twisted coefficients.*

2.4.4 Mayer-Vietoris Principle

In [24, Section 4.4, p. 24] a Mayer-Vietoris principle for sheaf cohomology on topological spaces is described. it can be translated directly to a Mayer-Vietoris principle for coarse spaces.

Let X be a coarse space and A,B two subsets that coarsely cover X. If $\mathcal{F}$ is a flabby sheaf on X the sequence

$$0\to\mathcal{F}(A\cup B)\to\mathcal{F}(A)\times\mathcal{F}(B)\xrightarrow{\varphi}\mathcal{F}(A\cap B)\to 0$$

is an exact sequence. Here φ sends a pair (s_1,s_2) to $s_1|_{A\cap B}-s_2|_{A\cap B}$. Thus if $\mathcal{G}$ is an arbitrary sheaf on X there is an exact sequence of flabby resolutions of $\mathcal{G}(A\cup B),\mathcal{G}(A)\times\mathcal{G}(B)$ and $\mathcal{G}(A\cap B)$. And thus there is an exact sequence in cohomology:

Theorem 74. ***(Mayer-Vietoris)*** *For two subsets $A,B\subseteq X$ that coarsely cover X there is an exact sequence in cohomology*

$$\begin{aligned} \cdots &\to \check{H}^{i-1}(A\cap B,\mathcal{F})\to\check{H}^i(A\cup B,\mathcal{F})\to\check{H}^i(A,\mathcal{F})\times\check{H}^i(B,\mathcal{F})\\ &\to\check{H}^i(A\cap B,\mathcal{F})\to\cdots \end{aligned}$$

for every sheaf $\mathcal{F}$ on X.

2.4.5 Local Cohomology

Let us define a version of relative cohomology for twisted coarse cohomology. There is already a similar notion for sheaf cohomology on topological spaces described in [25, chapter 1] which is called local cohomology. We do something similar:

Definition 75. (**support of a section**) Let $s \in \mathcal{F}(U)$ be a section. Then the support of s is contained in $V \subseteq U$ if

$$s|_{V^c \cap U} = 0$$

Let X be a coarse space and $Z \subseteq X$ a subspace. Then

$$\Gamma_Z(\mathcal{F}) : U \mapsto \ker(\mathcal{F}(U) \to \mathcal{F}(U \cap Z^c))$$

is a sheaf on X.

Lemma 76. *Let $Z \subseteq X$ be a subspace of a coarse space and let $Y = X \setminus Z$. Then there is a long exact sequence*

$$0 \to \check{H}^0(U, \Gamma_Z(\mathcal{F})) \to \check{H}^0(U, \mathcal{F}) \to \check{H}^0(U, \mathcal{F}|_Y) \to \check{H}^1(U, \Gamma_Z(\mathcal{F})) \to \cdots$$

for every subset $U \subseteq X$ and every sheaf $\mathcal{F}$ on X.

Proof. First we have an exact sequence

$$0 \to \Gamma_Z(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}|_Y$$

and if $\mathcal{F}$ is flabby we can write 0 on the right.

Let $\mathcal{I} = 0 \to \mathcal{F} \to I_0 \to I_1 \to \cdots$ be an injective resolution of $\mathcal{F}$. Note that every injective sheaf is flabby. Then there is an exact sequence of complexes

$$0 \to \Gamma_Z(\mathcal{I}) \to \mathcal{I} \to \mathcal{I}|_Y \to 0$$

which shows what we wanted to show. $\square$

Chapter 3

Coarse Topology of Metric Spaces

We have seen that the new notion *coarse cover* serves as a Grothendieck topology for defining sheaves and sheaf cohomology. And it also serves as an excision property for a Mayer-Vietoris sequence. There is a third application: coarse covers define a uniformity for a space at infinity.

In this Chapter we will make use of the predicted duality between Coarse Geometry and Uniform Topology. We will call what we do Coarse Topology.

The Proposition 161 only works for metric spaces not for general coarse spaces. That is why we restrict our attention to metric spaces only.

3.1 Coarsely proper coarsely geodesic Metric Spaces

This Chapter is denoted to the boring part. We develop the technical preliminaries needed for this and the following studies. We introduce coarsely geodesic coarsely proper metric spaces.

3.1.1 Coarsely Proper:

Notation 77. If X is a metric space we write

$$B(p,r)=\{x\in X: d(x,p)\le r\}$$

for a point $p\in X$ and $r\ge 0$. If we did not specify a coarse space we write

$$E(Y,r)=\{(x,y)\in Y^2: d(x,y)\le r\}$$

for Y a metric space and $r\ge 0$.

This is [26, Definition 3.D.10]:

Definition 78. (**coarsely proper**) A metric space X is called *coarsely proper* if there is some $R_0>0$ such that for every bounded subset $B\subseteq X$ the cover

$$\bigcup_{x\in B} B(x,R_0)$$

of B has a finite subcover.

Remark 79. (**proper**)

- A metric space X is *proper* if the map
$$\begin{aligned} r_p : X &\to \mathbb{R}_+ \\ x &\mapsto d(x,p) \end{aligned}$$
is a proper [1] continuous map for every $p \in X$.
- Every proper metric space is coarsely proper. A coarsely proper metric space is proper if it is complete.
- If X has a proper metric then the topology of X is locally compact.

Lemma 80.
- *If $f : X \to Y$ is a coarse map between metric spaces and $X' \subseteq X$ a coarsely proper subspace then*
$$f(X') \subseteq Y$$
is coarsely proper.
- *being coarsely proper is a coarse invariant.*

Proof.
- Suppose $R_0 > 0$ is such that every bounded subset of X' can be covered by finitely many R_0-balls. Because f is a coarsely uniform map there is some $S_0 > 0$ such that $d(x,y) \leq R_0$ implies $d(f(x), f(y)) \leq S_0$. We show that $f(X')$ is coarsely proper with regard to S_0.

  Let $B \subseteq f(X')$ be a bounded subset. Then $f^{-1}(B)$ is bounded in X thus there is a finite subcover of $\bigcup_{x \in B} B(x, R_0)$ which is
$$f^{-1}(B) = B(x_1, R_0) \cup \cdots \cup B(x_n, R_0).$$
  But then
$$\begin{aligned} B &= f \circ f^{-1}(B) \\ &= f(B(x_1, R_0) \cup \cdots \cup B(x_n, R_0)) \\ &= f(B(x_1, R_0)) \cup \cdots \cup f(B(x_n, R_0)) \\ &\subseteq B(f(x_1), S_0) \cup \cdots \cup B(f(x_n), S_0) \end{aligned}$$
  is a finite cover of B with S_0-balls.
- Suppose $f : X \to Y$ is a coarsely surjective coarse map between metric spaces and that X is coarsely proper. We show that Y is coarsely proper:

  By point 1 the subset $\operatorname{im} f \subseteq Y$ is coarsely proper. Suppose $\operatorname{im} f$ is coarsely proper with regard to $R_0 \geq 0$ and suppose $K \geq 0$ is such that $E(Y, K)[\operatorname{im} f] = Y$, we show that Y is coarsely proper with regard to $R_0 + K$.

  Let $B \subseteq Y$ be a bounded set. Then there are $x_1, \ldots, x_n$ such that
$$B \cap \operatorname{im} f \subseteq B(x_1, R_0) \cup \cdots \cup B(x_n, R_0)$$
  and then
$$B \subseteq B(x_1, R_0 + K) \cup \cdots \cup B(x_n, R_0 + K).$$

□

Example 81. Note that every countable group is a proper metric space.

[1] as in the reverse image of compact sets is compact

3.1.2 Coarsely Geodesic:

The following definition can also be found on [26, p. 10]:

Definition 82. (**coarsely connected**) Let X be a metric space.

- Let $x, y \in X$ be two points. A finite sequence of points $a_0, \ldots, a_n$ in X is called a *$c-$path joining x to y* if $x = a_0, y = a_n$ and $d(a_i, a_{i+1}) \leq c$ for every i.
- then X is called *c-coarsely connected* if for every two points $x, y \in X$ there is a $c-$path between them
- the space X is called *coarsely connected* if there is some $c \geq 0$ such that X is $c-$coarsely connected.

Example 83. Not an example:

$$\{2^n : n \in \mathbb{N}\} \subseteq \mathbb{Z}_+.$$

Lemma 84. *Being coarsely connected is invariant by coarse equivalence.*

Proof. Note that this is [26, Proposition 3.B.7]. The argument for the proof can be found in [26, Proposition 3.B.4]. For the convenience of the reader we recall it:

If $f : X \to Y$ is a coarsely surjective coarse map and X is coarsely connected we will show that Y is coarsely connected. Suppose X is c-coarsely connected. Let y, y' be two points in Y. Note that by coarse surjectivity of f there is some $K \geq 0$ such that $E(Y, K)[\operatorname{im} f] = Y$. And by coarseness of f there is some $d \geq 0$ such that $f^2(E(X, c)) \subseteq E(Y, d)$. Now denote by

$$e = \max(K, d).$$

Choose points $x, x' \in X$ such that $d(y, f(x)) \leq K$ and $d(y', f(x')) \leq K$ and a c-path $x = a_0, a_1, \ldots, a_n = x'$. Then

$$y, f(x), f(a_1), \ldots, f(x'), y'$$

is an e-path in Y joining y to y'. Thus Y is e-coarsely connected which implies that Y is coarsely connected. □

Example 85. By [26, Proposition 4.B.8] a countable group is coarsely connected if and only if it is finitely generated.

This one is [26, Definition 3.B.1(b)]:

Definition 86. (**coarsely geodesic**) A metric space X

- is called $c-$coarsely geodesic if it is c-coarsely connected and there is a function

  $$\Phi(X, c) : \mathbb{R}_+ \to \mathbb{N}$$

  (called the upper control) such that for every $x, y \in X$ there is a $c-$path $x = a_0, \ldots, a_n = y$ such that

  $$n + 1 \leq \Phi(X, c)(d(x, y)).$$

- the space X is called *coarsely geodesic* if there is some $c \geq 0$ such that X is $c-$coarsely geodesic.

Lemma 87. *Being coarsely geodesic is a coarse invariant.*

Proof. Suppose that $f : X \to Y$ is a coarse equivalence between metric spaces and that X is $c-$coarsely geodesic. We proceed as in the proof of Lemma 84, using the same notation:

1. There is a constant $K \geq 0$ such that $E(Y, K)[\mathrm{im}\, f] = Y$;
2. there is a constant $d \geq 0$ such that $f^{\times 2}(E(X, c)) \subseteq E(Y, d)$. By the proof of Lemma 84 the space Y is $e = \max(K, c)-$coarsely connected.
3. For every $r \geq 0$ there is some $s \geq 0$ such that
$$(f^{\times 2})^{-1}(E(Y, r)) \subseteq E(X, s)$$
we store the association $r \mapsto s$ in the map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

Define

$$\Phi(Y, e) : \mathbb{R}_+ \to \mathbb{N}$$
$$r \mapsto \Phi(X, c) \circ \varphi(r + 2K) + 2.$$

Then $\Phi(Y, e)$ is an upper bound for Y: Let $y, y' \in Y$ be two points. Consider the same $e-$path $y, f(x), a_1, \ldots, f(x'), y$ as in the proof of Lemma 84 with the additional condition that $n + 1 \leq \Phi(X, c)$. Then $d(y, y') \leq r$ implies that $d(f(x), f(x')) \leq r + 2K$ which implies $d(x, x') \leq \varphi(r + 2K)$.

Thus Y is $e-$coarsely geodesic which implies that Y is coarsely geodesic. □

Example 88. Note that by [26, Proposition 1.A.1] every finitely generated group is coarsely geodesic.

3.1.3 Geodesification

Lemma 89. *If X is a coarsely proper metric space then there exists a countable R-discrete for some $R > 0$ subset $(x_i)_i \subseteq X$ such that the inclusion $(x_i)_i \to X$ is a coarse equivalence.*

Proof. Suppose X is R_0-coarsely proper. Fix a point $x_0 \in X$. Then

$$X = \bigcup_i B(x_0, i)$$

is a countable union of bounded sets. Now for every i there are $x_{i1}, \ldots, x_{in_i}$ such that

$$B(x_0, i) \subseteq B(x_{i1}, R_0) \cup \cdots \cup B(x_{in_i}, R_0)$$

Then $(x_{ij})_{ij} \subseteq X$ is countable and the inclusion $(x_{ij})_{ij}$ is R_0-coarsely surjective. If $\varepsilon > 0$ small enough then we can inductively choose a subset $S \subseteq (x_{ij})_{ij}$ that is ε-discrete and the inclusion $S \to X$ is $R_0 + \varepsilon$-coarsely surjective. □

This is a variation of [26, Definition 3.B.5]:

Definition 90. ($c-$**geodesification**) Let X be a coarsely proper $c-$coarsely connected metric space. By Lemma 89 we can assume X is countable. Define a total order $<$ on the points in X. We define the $c-$geodesification $\bar{X}^g_c$ of X to be the geometric realization of the following simplicial $1-$complex:

- X_0 consists of every $x \in X$.

- X_1 : there is an edge $e(x, y)$ if $d(x, y) \leq c$ and $x < y$.

Note that $\bar{X}_c^g$ is another name for the 1−skeleton of the c−Rips simplicial complex $\mathsf{Rips}_c^1(X)$ of X.

Lemma 91. *If X is a coarsely proper c-coarsely connected metric space*

- *the map*

$$\varphi_c : \bar{X}_c^g \to X$$
$$t \in e(x, y) \mapsto \begin{cases} x & d(t, x) \leq d(t, y) \\ y & d(t, y) < d(t, x) \end{cases}$$

  is a coarsely surjective coarse map.

- *If X is a $c-$coarsely geodesic metric space then φ_c is a coarse equivalence*

Proof.
- We show φ_c is coarsely surjective coarse: The map φ_c is surjective hence $varphi_c$ is coarsely surjective.

  Now we show φ_c is coarsely uniform: let $n \in \mathbb{N}$ be a number. Then for every $t, s \in \bar{X}^g$ the relation $d(t, s) \leq n$ implies that for the two adjacent vertices x, y ($d(x, t) \leq 1/2$ and $d(y, s) \leq 1/2$) the relation $d(x, y) \leq n + 1$ holds in $\bar{X}_c^g$. But that means there is a $c-$path of length $n + 1$ in X joining x to y. Thus

$$\begin{aligned} d(\varphi_c(t), \varphi_c(s)) &= d(x, y) \\ &\leq (n + 1)c \end{aligned}$$

  in X.

  Now we show the map φ_c is coarsely proper: If $B \subseteq X$ is bounded there are $x \in X$ and $R \geq 0$ such that $B \subseteq B(x, R)$. Choose some n such that $nc \geq R$. Then

$$\varphi_c^{-1}(B) \subseteq B(x, n + 1/2).$$

  is bounded in $\bar{X}_c^g$.

- We show φ_c is coarsely injective: Let $k \geq 0$ be a number. Then $d(x, y) \leq k$ implies that there is a c-path joining x to y with length at most $\Phi(X, c)(k)$. Then for any $s \in \varphi_c^{-1}(x), t \in \varphi_c^{-1}(y)$ the relation

$$d(s, t) \leq \Phi(X, c)(k) + 1$$

  holds.

□

3.1.4 Coarse Rays

In [27] every metric space that is coarsely equivalent to $\mathbb{Z}_+$ is called a coarse ray. We keep with that notation:

Definition 92. (**coarse ray**) If X is a metric space a sequence $(x_i)_i \subseteq X$ is called a *coarse ray in X* if there is a coarsely injective coarse map $\rho : \mathbb{Z}_+ \to X$ such that $x_i = \rho(i)$ for every i.

Lemma 93. *If X is a $c-$coarsely geodesic metric space, $(x_i)_i$ a sequence in X and if for every $i < j$ the sequence*

$$x_i, \dots, x_j$$

is a $c-$path such that $\Phi(X,c)(d(x_i,x_j)) \geq |i-j|+1$ then the association

$$i \mapsto r_i$$

defines a coarsely injective coarse map $\rho : \mathbb{Z}_+ \to X$.

Proof. We show that ρ is coarsely injective coarse:

1. ρ is coarsely uniform: Let $n \in \mathbb{N}$ be a number. Then for every $i,j \in \mathbb{Z}_+$ if $|i-j| \leq n$ then $d(x_i,x_j) \leq cn$.
2. ρ is coarsely injective: Let $k \geq 0$ be a number. Then $d(x_i,x_j) \leq k$ implies $|i-j| \leq \Phi(X,c)(k)-1$ for every ij.

□

Proposition 94. *If X is a coarsely geodesic coarsely proper metric space and*

- *if X is not bounded then there is at least one coarse ray in X.*
- *in fact if $(x_i)_i$ is a sequence in X that is not bounded then there is a subsequence $(x_{i_k})_k$ that is not bounded, a coarse ray $(r_i)_i$ and an entourage $E \subseteq X^2$ such that*

$$(x_{i_k})_k \subseteq E[(r_i)_i].$$

Remark 95. Point 1 is the same as [27, Lemma 4]. The proof is quite different though.

Proof. Suppose X is coarsely proper with regard to R_0 and $c-$coarsely geodesic. We will determine a sequence $(V_i)_i$ of subsets of X and a sequence $(r_i)_i$ of points in X.

Define $r_0 = x_0$ and $V_0 = X$.

Then define for every $y \in X$ the number $d_0(y)$ to be the minimal length of a $c-$path joining x_0 to y.

We define a relation on the points of X: $y \leq z$ if $d_0(y) \leq d_0(z)$ and y lies on a c-path of minimal length joining x_0 to z. This makes $(X, \leq)$ a partially ordered set.

for every $i \in \mathbb{N}$ do:

Denote by

$$C_i = \{x \in X : d_0(x) = i\}.$$

There are $y_1, \dots, y_m \in X$ such that

$$C_i = B(y_1, R_0) \cup \dots \cup B(y_m, R_0).$$

Now $(x_i)_i \cap V_{i-1}$ is not bounded and V_{i-1} is coarsely geodesic. Thus for every $n \in \mathbb{N}$ there is some $x_{n_k} \in V_{i-1}$ with $d_0(x_{n_k}) \geq n+i$.

Then there is one of $j = 1, \dots, m$ such that for infinitely many $n \in \mathbb{N}$: there is $y \in B(y_j, R_0)$ such that $y \leq x_{n_k}$. Then define

$$V_i := \{x \in V_{i-1} : \exists y \in B(y_j, R_0) : y \leq x\}.$$

Note that V_i is coarsely geodesic and that $V_i \cap (x_i)_i$ is not bounded.

Define $r_i := y_j$.

We show that $(r_i)_i$ and $E = E(X, R_0)$ have the desired properties:
The set $(r_i)_i$ is a coarse ray: for every i the sequence

$$r_0, \ldots, r_i$$

is R_0-close to a $c-$path of minimal length which implies that every subsequence is R_0-close to a $c-$path of minimal length.

The set

$$P := \bigcup_i V_i \cap C_i$$

contains infinitely many of the $(x_i)_i$ and $(r_i)_i$ is R_0-coarsely dense in P. Thus the result. □

3.2 Coarse Homotopy

In this chapter we define coarse homotopy. Our coarse cohomology with twisted coefficients is invariant under coarse homotopy.

3.2.1 Asymptotic Product

Lemma 96. *If X is a metric space, fix a point $p \in X$, then*

$$\begin{aligned} r_p : X &\to \mathbb{Z}_+ \\ x &\mapsto \lfloor d(x,p) \rfloor \end{aligned}$$

is a coarse map.

Proof. 1. r_p is coarsely uniform: Let $k \geq 0$. Then for every $(x, y) \in X^2$ with $d(x, y) \leq k$:

$$\begin{aligned} |\lfloor d(x,p) \rfloor - \lfloor d(y,p) \rfloor| &\leq d(x,y) + 2 \\ &\leq k + 2. \end{aligned}$$

2. r_p is coarsely proper: Let $B \subseteq \mathbb{Z}_+$ be a bounded set. Then there is some $l \geq 0$ such that $B \subseteq B(l, 0)$. Then $r_p^{-1}(B(l, 0)) = B(l, p)$ is a bounded set which contains $r_p^{-1}(B)$.

□

Definition 97. (**asymptotic product**) If X is a metric space and Y a coarsely geodesic coarsely proper metric space then the *asymptotic product* [2] $X * Y$ of X and Y is a subspace of $X \times Y$[3]:

- fix points $p \in X$ and $q \in Y$ and a constant $R \geq 0$ large enough.
- then $(x, y) \in X * Y$ if
$$|d_X(x,p) - d_Y(y,q)| \leq R.$$

We define the projection $p_X : X * Y \to X$ by $(x, y) \mapsto x$ and the projection $p_Y : X * Y \to Y$ by $(x, y) \mapsto y$. Note that the projections are coarse maps.

Lemma 98. *The asymptotic product $X * Y$ of two metric spaces where Y is coarsely geodesic coarsely proper is well defined. Another choice of points $p' \in X, q' \in Y$ and constant $R' \geq 0$ large enough gives a coarsely equivalent space.*

[2] We guess this notion first appeared in [7, chapter 3] and kept with the notation.
[3] with the pullback coarse structure defined in Definition 40

Proof. Suppose Y is $c-$coarsely geodesic.

We can rephrase Definition 97 by defining coarse maps

$$\begin{aligned} t : X &\to \mathbb{Z}_+ \\ x &\mapsto d_X(x, p) \end{aligned}$$

and

$$\begin{aligned} s : Y &\to \mathbb{Z}_+ \\ y &\mapsto d_Y(y, q) \end{aligned}$$

and an entourage

$$\begin{aligned} E &= \{(x, y) : |x - y| \leq R\} \\ &\subseteq \mathbb{Z}_+^2. \end{aligned}$$

Then

$$X * Y = (t \times s)^{-1}(E).$$

Another choice of points $p' \in X, q' \in Y$ and constant $R' \geq 0$ defines coarse maps $t' : X \to \mathbb{Z}_+$ and $s' : Y \to \mathbb{Z}_+$ and an entourage $E' \subseteq \mathbb{Z}_+^2$ in much the same way.

Define

$$R'' = d(p, p') + d(q, q') + R$$

If $(x, y) \in (t \times s)^{-1}(E)$ then

$$\begin{aligned} |d(p', x) - d(q', y)| &\leq |d(p', x) - d(p, x)| + |d(p, x) - d(q, y)| + |d(q, y) - d(q', y)| \\ &\leq d(p, p') + R + d(q, q') \\ &= R'' \end{aligned}$$

If $E'' \subseteq \mathbb{Z}_+^2$ is associated to R'' then $(x, y) \in (s' \times t')^{-1}(E'')$. Thus we have shown that $X * Y$ is independent of the choice of points if $X * Y$ is independent of the choice of constant. The second we are going to show now.

Now we can assume R is larger than R' but not by much. Explicitely we require $R \leq 2R' - c - 2$. We show that the inclusion

$$i : (t \times s)^{-1}(E') \to (t \times s)^{-1}(E)$$

is a coarse equivalence. It is a coarsely injective coarse map obviously.

We show i is coarsely surjective. Assume the opposite: there is a sequence $(x_i, y_i)_i \subseteq (s \times t)^{-1}(E)$ such that $(x_i, y_i)_i$ is coarsely disjoint to $(s \times t)^{-1}(E')$. By Proposition 94 there is a coarsely injective coarse map $\rho : \mathbb{Z}_+ \to Y$, a number $S \geq 0$ and subsequences $(i_k)_k \subseteq \mathbb{N}, (l_k)_k \subseteq \mathbb{Z}_+$ such that

$$d(y_{i_k}, \rho(l_k)) \subseteq S$$

for every k. Without loss of generality we can assume that $\rho(0) = q$ and $d(q, \rho(k)) = kc$ for every k. Now for every k:

$$|d(y_{i_k}, q) - d(x_{i_k}, p)| \leq R.$$

Then there is some $z_k \in \mathbb{Z}_+$ such that $|d(y_{i_k}, q) - z_k| \leq R'$ and $|d(x_{i_k}, p) - z_k| \leq R' - c$. Then for every k there is some j_k such that

$$|d(\rho(j_k), q) - z_k| \leq c.$$

Now

$$\begin{aligned} d(\rho(j_k),\rho(l_k)) &= |d(\rho(j_k),q) - d(\rho(l_k),q)| \\ &\le |d(\rho(j_k),q) - z_k| + |z_k - d(\rho(l_k),q)| \\ &\le c + R' + S \end{aligned}$$

for every k. And

$$\begin{aligned} |d(\rho(j_k),q) - d(x_{i_k},p)| &\le |d(\rho(j_k),q) - z_k| + |d(x_{i_k},p) - z_k| \\ &\le c + R' - c \\ &= R'. \end{aligned}$$

Thus $(\rho(j_k),x_{i_k})_k \subseteq (t\times s)^{-1}(E')$ and $d((\rho(j_k),x_{i_k}),(y_{i_k},x_{i_k})) \le c + R' + 2S$ for every k a contradiction to the assumption. □

Lemma 99. *Let X be a metric space and Y a coarsely geodesic coarsely proper metric space. Then*

$$\begin{tikzcd} X * Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "d(\cdot,q)"] \\ X \arrow[r, "d(\cdot,p)"'] & \mathbb{Z}_+ \end{tikzcd}$$

is a limit diagram in `Coarse`*. Note that we only need the diagram to commute up to closeness.*

Proof. Suppose $X * Y$ has constant Q. Let $f : Z \to X$ and $g : Z \to Y$ be two coarse maps from a coarse space Z such that there is some $R \ge 0$ such that

$$|d(f(z),p) - d(g(z),q)| \le R.$$

Assume for a moment there exists a $K \ge 0$ such that for every $z \in Z$ there is $\bar{g}(z) \in Y$ with

1. $|d(f(z),p) - d(\bar{g}(z),q)| \le c$
2. $d(\bar{g}(z),g(z)) \le K$

Then define

$$\begin{aligned} \langle f,g\rangle : Z &\to X * Y \\ z &\mapsto (f(z),\bar{g}(z)). \end{aligned}$$

This map is a coarse map:

- $\langle f,g\rangle$ is coarsely uniform: If $E \subseteq Z^2$ is an entourage then $f^{\times 2}(E) \subseteq X^2, g^{\times 2}(E) \subseteq Y^2$ are entourages. Since $\bar{g}$ is close to g the set $\bar{g}^{\times 2}(E)$ is an entourage. Then $\langle f,g\rangle^{\times 2}(E) \subseteq f^{\times 2}(E) \times \bar{g}^{\times 2}(E)$ is an entourage.
- $\langle f,g\rangle$ is coarsely proper: If $B \subseteq X * Y$ is bounded then $p_1(B), p_2(B)$ are bounded. Then

  $$\langle f,g\rangle^{-1}(B) \subseteq f^{-1}\circ p_1(B) \cup \bar{g}^{-1}\circ p_2(B)$$

  is bounded since f, g and thus $\bar{g}$ are coarsely proper.

Also $p_X \circ \langle f, g \rangle = f$ and $p_Y \circ \langle f, g \rangle \sim g$.

Suppose there is another coarse map $h : Z \to X * Y$ with the property that $p_X \circ h \sim f$ and $p_Y \circ h \sim g$. Then

$$\begin{aligned}\langle f, g \rangle &\sim \langle p_X \circ h, p_Y \circ h \rangle \\ &= h\end{aligned}$$

are close.

Now we prove the above assumption by assuming the opposite: There does not exist a $K \geq 0$ such that for every $z \in Z$ there is $\bar{g}(z)$ with 1. and 2. satisfied. Then there exists an unbounded sequence $(z_i)_i \subseteq Z$ such that $(f(z_i), g(z_i))_i \subseteq X \times Y$ is coarsely disjoint to $X * Y$. Since $(f(z_i), g(z_i))_i$ is a subset of $X * Y$ with constant $R > Q$ and by Lemma 98 the inclusion of $X * Y$ with constant Q to $X * Y$ with constant R is coarsely surjective this leads to a contradiction. □

Lemma 100. *For every metric space X there is a coarse equivalence*

$$X \to X * \mathbb{Z}_+.$$

Proof. easy. □

Lemma 101. *If X, Y are proper metric spaces and Y is coarsely geodesic then $X * Y$ is a proper metric space.*

Proof. We show that $X \times Y$ is a proper metric space. If $B \subseteq X \times Y$ is bounded then the projections B_X of B to X and B_Y of B to Y are bounded. But X, Y are proper thus B_X, B_Y are relatively compact. Then

$$B \subseteq B_X \times B_Y$$

is relatively compact. Thus $X \times Y$ is proper. But $X * Y \subseteq X \times Y$ is a closed subspace. □

3.2.2 Definition

Definition 102. Let T be a metric space then

$$F(T) = T \times \mathbb{Z}_+$$

is a metric space with metric

$$d((x,i),(y,j)) = \sqrt{i^2 + j^2 - (2 - d_T(x,y)^2)ij}.$$

Note that we impose that $\mathbb{Z}_+$ does not contain 0 thus d is a well defined metric.

Remark 103. A countable subset $((x_i, n_i), (y_i, m_i))_i \subseteq F(T)^2$ is an entourage if

1. the set $(n_i, m_i)_i$ is an entourage in $\mathbb{Z}_+$
2. and if $n_i \to \infty$ then there is some constant $c \geq 0$ such that $d(x_i, y_i) \leq c/n_i$

Definition 104. (**coarse homotopy**) Denote by $[0,1]$ the unit interval with the standard euclidean metric $d_{[0,1]}$. Let X be a metric space and Y a coarse space.

- Let $f, g : X \to Y$ be two coarse maps. They are said to be *coarsely homotopic* if there is a coarse map $h : X * F([0,1]) \to Y$ such that f is the restriction of h to $X * F(0)$ and g is the restriction of h to $X * F(1)$.

- A coarse map $f : X \to Y$ is a *coarse homotopy equivalence* if there is a coarse map $g : Y \to X$ such that $f \circ g$ is coarsely homotopic to id_Y and $g \circ f$ is coarsely homotopic to id_X.
- Two coarse spaces X, Y are called *coarsely homotopy equivalent* if there is a coarse homotopy equivalence $f : X \to Y$.

Remark 105. There are other notions of homotopy in `Coarse` but they differ from that one.

Lemma 106. *If two coarse maps $f, g : X \to Y$ between metric spaces are close then they are coarsely homotopic.*

Proof. We define a homotopy $h : X * F([0,1]) \to Y$ between f and g by

$$h(x,(0,i)) = f(x)$$

and for $1 \geq t > 0$:

$$h(x,(t,i)) = g(x).$$

We show that h is a coarse map:

1. h is coarsely uniform: if $t_i \to 0$ in $[0,1]$ such that $d(t_i,0) \leq 1/i$ and $(x_i)_i \subseteq X$ a sequence of points then
$$h((x_i,(t_i,i)),(x_i,(0,i)) = \{(f(x_i), g(x_i)) : i\}$$
is an entourage.
2. h is coarsely proper because f, g are.

□

Definition 107. (**coarse homotopy 2**) Let X, Y be coarsely geodesic coarsely proper metric spaces.

- A *coarse homotopy* is a family of coarse maps $(h_t : X \to Y)_t$ indexed by $[0,1]$ with the property that if $(t_i)_i \subseteq [0,1]$ converges to $t \in [0,1]$ such that there is a constant $c > 0$ such that $|t - t_i| < c/i$ then for every coarsely injective coarse map $\rho : \mathbb{Z}_+ \to X$ the set
$$\{(h_{t_i} \circ \rho(i), h_t \circ \rho(i)) : i \in \mathbb{Z}_+\}$$
is an entourage in Y.
- two coarse maps $f, g : X \to Y$ are *coarsely homotopic* if there is a coarse homotopy $(h_t : X \to Y)_t$ such that $f = h_0$ and $g = h_1$.

Proposition 108. *If X is a coarsely geodesic coarsely proper metric space then Definition 107 of coarse homotopy agrees with Definition 104 of coarse homotopy.*

Proof. Let there be a coarse map $h : X * F([0,1]) \to Y$. First of all for every $x \in X$ choose some i_x such that $(x,(t,i_x)) \in X * F([0,1])$. Then we define

$$h_t(x) = h(x,(t,i_x))$$

for every $t \in [0,1]$. Note that h_t is a coarse map because it is a restriction of h to a subspace and h is a coarse map. Now suppose $(t_i)_i \subseteq [0,1]$ converges to $t \in [0,1]$ such that $|t_i - t| < 1/i$ and $\varphi : \mathbb{Z}_+ \to X$ is a coarse map. Then

$$\begin{aligned}\varphi_i : \mathbb{Z}_+ &\to \mathbb{Z}_+\\ i &\mapsto i_{\varphi(i)}\end{aligned}$$

is a coarse map. But

$$((\varphi(i),(t_i,i_{\varphi(i)})),(\varphi(i),(t,i_{\varphi(i)})))_i$$

is an entourage and h is a coarse map. Thus $(h_t : X \to Y)_t$ is a coarse homotopy.

Let there be a family of coarse maps $(h_t : X \to Y)_t$ with the above properties. Then

$$\begin{aligned} h : X * F([0,1]) &\to Y \\ (x,(t,i)) &\mapsto h_t(x) \end{aligned}$$

is a coarse map: h is coarsely uniform:

Let $((x_n,t_n,i_n),(y_n,s_n,j_n))_n \subseteq (X * F([0,1]))^2$ be a countable entourage. That means both $(x_n,y_n)_n \subseteq X^2$ and $((t_n,i_n),(s_n,j_n))_n \subseteq F([0,1])^2$ are entourages.

Assume the opposite. Then there is a subsequence $(n_k)_k$ such that

$$h^2((x_{n_k},t_{n_k},i_{n_k}),(y_{n_k},s_{n_k},j_{n_k}))_k$$

is an unbounded coentourage. By Proposition 94 there are coarsely injective coarse maps $\rho,\sigma : \mathbb{Z}_+ \to X$ and subsequences $(m_k)_k \subseteq (n_k)_k$ and $(l_k)_k \subseteq \mathbb{N}$ such that $x_{m_k} = \rho(l_k)$, $y_{m_k} = \sigma(l_k)$ and

$$(\rho(l_k),\sigma(l_k))_k$$

is an entourage in X. Note that

$$h^{\times 2}((\rho(l_k),t,m_k),(\sigma(l_k),t,m_k))$$

is an entourage in Y.

Now there is some constant $c > 0$ and $t \in [0,1]$ such that $|t_i - t| \leq c/i$ for every i. Thus by Definition 107 the set

$$h^{\times 2}((\rho(l_k),t_{m_k},i_{m_k}),(\rho(l_k),t,i_{m_k}))_k$$

is an entourage in Y. Similarly there is some constant $d > 0$ such that $|s_i - t| \leq d/i$ for every i. Then

$$h^{\times 2}((\sigma(l_k),t,i_{m_k}),(\sigma(l_k),s_{m_k},i_{m_k}))_k$$

is an entourage in Y.

Combining the two previous arguments the set

$$h^{\times 2}((\rho(l_k),t_{m_k},i_{m_k}),(\sigma(l_k),s_{m_k},i_{m_k}))_k$$

is an unbounded entourage in Y. This is a contradiction to the assumption.

h is coarsely proper:

If $B \subseteq Y$ is bounded then

$$h^{-1}(B) = \bigcup_t (h_t^{-1}(B) * (t \times \mathbb{Z}_+))$$

we show $\bigcup_t h_t^{-1}(B)$ is bounded:

Assume the opposite. Then there is an unbounded sequence $(b_{t_i})_i \subseteq \bigcup_t h_t^{-1}(B)$, here $b_{t_i} \in h_{t_i}^{-1}(B)$ for every t_i. We can assume that every bounded subsequence is finite. By Proposition 94: there is a coarsely injective coarse map $\rho : \mathbb{Z}_+ \to X$ and subsequences $(n_k)_k,(m_k)_k \subseteq \mathbb{N}$ such that $b_{t_{n_k}} = \rho(m_k)$ for every k.

Now there is a subsequence $(l_k)_k \subseteq (n_k)_k$ and some constant $c > 0$ such that $(t_{l_k})_k$ converges to $t \in [0,1]$ and $|t_{l_k} - t| \leq c/k$. By Definition 107 the set

$$(h_{t_{l_k}} \circ \rho(m_k), h_t \circ \rho(m_k))_k$$

is an entourage. Then $h_t(b_{t_{l_k}})$ is not bounded which is a contradiction to the assumption. $\square$

Theorem 109. ***(coarse homotopy invariance)*** *Let $f, g : X \to Y$ be two coarse maps which are coarsely homotopic. Then they induce the same map in coarse cohomology with twisted coefficients.*

Proof. It suffices to show that if $p : X * F([0,1]) \to X$ is the projection to the first factor then $R^q p = 0$ for $q > 0$. We will proceed as in the proof of Lemma 70. Thus we just need to check that if $U \subseteq X$ is a subset and $p^{-1}(U) = U * F([0,1])$ is coarsely covered by V_1, V_2 then there are $U_1, U_2 \subseteq X$ such that $p^{-1}(U_1), p^{-1}(U_2)$ is a coarse cover that refines V_1, V_2. We write

$$V_1 = \bigcup_{t\in[0,1]} V_1^t * (t \times \mathbb{Z}_+)$$

and

$$V_2 = \bigcup_{t\in[0,1]} V_2^t * (t \times \mathbb{Z}_+).$$

We see that

$$\begin{aligned} V_i^c &= \bigcap_t (V_i^t * (t \times \mathbb{Z}_+))^c \\ &= \bigcup_t (V_i^t)^c * (t \times \mathbb{Z}_+). \end{aligned}$$

But then

$$\bigcup_t (V_1^t)^c \times \bigcup_t (V_2^t)^c$$

is a coentourage in X. Which implies that

$$\bigcup_t (V_1^t)^c * F([0,1]) \times \bigcup_t (V_2^t)^c * F([0,1])$$

is a coentourage. Thus $U * F([0,1])$ is coarsely covered by

$$(\bigcap_t V_1^t) * F([0,1]), (\bigcap_t V_2^t) * F([0,1]).$$

□

Corollary 110. *If $f : X \to Y$ is a coarse homotopy equivalence then it induces an isomorphism in coarse cohomology with twisted coefficients.*

Chapter 4

Computing Cohomology

This Chapter is dedicated to computing cohomology.

Already coarse cohomology with constant coefficients sees a lot of structure. Using coarse homotopy we will compute acyclic spaces. We will compute cohomology for free abelian groups of finite type and for free groups of finite type using acyclic covers.

4.1 Constant Coefficients

Now it is time for examples. We compute coarse cohomology with constant coefficients for a few exemplary examples.

4.1.1 Number of Ends

If a space is the coarse disjoint union of two subspaces we have a special case of a coarse cover. In [28] the number of ends of a group were studied; this notion can be generalized in an obvious way to coarse spaces.

Definition 111. A coarse space X is called *oneended* if for every coarse disjoint union $X = \bigsqcup_i U_i$ all but one of the U_i are bounded.

Lemma 112. *The coarse space $\mathbb{Z}_+$ is oneended.*

Proof. Suppose $\mathbb{Z}_+$ is the union of U, V and U, V are not bounded. Without loss of generality we can assume U, V are a disjoint union. Now $(n)_{n\in\mathbb{N}}$ is a sequence where $(n)_{n\in\mathbb{N}} \cap U$ is not bounded and $(n)_{n\in\mathbb{N}} \cap V$ is not bounded.

For every $N \in \mathbb{N}$ there is a smallest $n \in U$ such that $n \geq N$ and there is a smallest $m \in V$ such that $m \geq N$. Without loss of generality n is greater than m, then $(n, n-1) \in U \times V \cap E(\mathbb{Z}_+, 1)$. Here $E(\mathbb{Z}_+, 1)$ denotes the set of all pairs $(x, y) \in \mathbb{Z}_+^2$ with $d(x, y) \leq 1$. This is an entourage. That way there is an infinite number of elements in

$$(U^2 \cup V^2)^c \cap E(\mathbb{Z}_+, 1) = (U \times V \cup V \times U) \cap E(\mathbb{Z}_+, 1)$$

which implies that U, V are not coarsely disjoint. $\square$

Definition 113. Let X be a coarse space. Its number of ends $e(X)$ is at least $n \geq 0$ if there is a coarse cover $(U_i)_i$ of X such that X is the coarse disjoint union of the U_i and n of the U_i are not bounded.

Lemma 114. *If A, B are two coarse spaces and $X = A \sqcup B$ their coarse disjoint union then*

$$e(X) = e(A) + e(B).$$

Proof. Suppose $e(A) = n$ and $e(B) = m$. Then there are coarse disjoint unions $A = A_1 \sqcup \ldots \sqcup A_n$ and $B = B_1 \sqcup \ldots \sqcup B_m$ with nonboundeds. But then

$$X = A_1 \sqcup \ldots \ldots \sqcup A_n \sqcup B_1 \sqcup \ldots \sqcup B_m$$

is a coarse disjoint union with nonboundeds. Thus $e(X) \geq e(A) + e(B)$.

Suppose $e(X) = n$. Then there is a coarse disjoint cover $(U_i)_{i=1,\ldots,n}$ with nonboundeds of X. Thus $(U_i \cap A)_i$ is a coarse disjoint union of A and $(U_i \cap B)_i$ is a coarse disjoint union of B. Then for every i one of $U_i \cap A$ and $U_i \cap B$ is not bounded. Thus

$$e(X) \leq e(A) + e(B).$$

□

Example 115. $e(\mathbb{Z}) = 2$.

Theorem 116. *Let $f : X \to Y$ be a coarsely surjective coarse map and suppose $e(Y)$ is finite. Then*

$$e(X) \geq e(Y).$$

Proof. First we show that $e(X) \geq e(\operatorname{im} f)$: Regard f as a surjective coarse map $X \to \operatorname{im} f$. Suppose that $e(\operatorname{im} f) = n$. Then $\operatorname{im} f$ is coarsely covered by a coarse disjoint union $(U_i)_{i=1,\ldots,n}$ where none of the U_i are bounded. But then $(f^{-1}(U_i))_i$ is a coarse disjoint union of X and because f is a surjective coarse map none of the $f^{-1}(U_i)$ are bounded.

Now we show that $e(Y) = e(\operatorname{im} f)$: Note that there is a surjective coarse equivalence $r : Y \to \operatorname{im} f$. By Proposition 56 a finite family of subsets $(U_i)_i$ is a coarse cover of $\operatorname{im} f$ if and only if $(r^{-1}(U_i))_i$ is a coarse cover of Y. if $(U_i)_i$ is a coarse disjoint union so is $(r^{-1}(U_i))_i$. □

Corollary 117. *The number $e(\cdot)$ is a coarse invariant.*

4.1.2 Definition

Definition 118. Let X be a coarse space and A an abelian group. Then A_X (or just A if the space X is clear) is the sheafification of the constant presheaf which associates to every subspace $U \subseteq X$ the group A.

Lemma 119. *A coarse disjoint union $X = U \sqcup V$ of two coarse spaces U, V is a coproduct in* `Coarse`*.*

Proof. Denote by $i_1 : U \to X$ and $i_2 : V \to X$ the inclusions. We check the universal property: Let Y be a coarse space and $f_1 : U \to Y$ and $f_2 : V \to Y$ coarse maps. But U, V are a coarse cover of X such that $U \cap V$ is bounded. Now we checked that already in Proposition 65. The existence of a map $f : X \to Y$ with the desired properties would be the gluing axiom and the uniqueness modulo closeness would be the global axiom. □

Theorem 120. *Let X be a coarse space and A an abelian group. If X has finitely many ends then*

$$A(X) = A^{e(X)}$$

and if X has infinitely many ends then

$$A(X) = \bigoplus_{\mathbb{N}} A.$$

Here $A(X)$ means the evaluation of the constant sheaf A on X at X.

Proof. By the equalizer diagram for sheaves a sheaf naturally converts finite coproducts into finite products. If X is oneended and U, V a coarse cover of X with nonboundeds then U, V intersect nontrivially. Thus $A(X) = A$ in this case. If X has infinitely many ends then there is a directed system

$$\cdots \to U_1 \sqcup \cdots \sqcup U_n \to U_1 \sqcup \cdots \sqcup U_{n+1} \to$$

in the dual category of $\mathcal{I}_X$ which is the category of coarse covers of X. Here the U_i are nonbounded and constitute a coarse disjoint union in X. Now we use [23, Definition 2.2.5] by which

$$\check{H}^0(X, A) = \varinjlim_{(U_i)_i} H^0((U_i)_i, A).$$

Then we take the direct limit of the system

$$\cdots \to A^n \to A^{n+1} \to A^{n+2} \to \cdots .$$

Thus the result. □

Lemma 121. *If a subset $U \subseteq \mathbb{Z}_+$ is oneended then the inclusion*

$$i : U \to \mathbb{Z}_+$$

is coarsely surjective.

Proof. If the inclusion $i : U \to \mathbb{Z}_+$ is not coarsely surjective then there is an increasing sequence $(v_i)_i \subseteq \mathbb{Z}_+$ such that for every $u \in U$:

$$|u - v_i| > i.$$

Now define

$$A := \{u \in U : v_{2i} < u < v_{2i+1}, i \in \mathbb{N}\}$$

and

$$B := \{u \in U : v_{2i+1} < u < v_{2i}, i \in \mathbb{N}\}.$$

Then for every $a \in A, b \in B$ there is some j such that $a < v_j < b$. Then

$$\begin{aligned} |a - b| &= |a - v_j| + |b - v_j| \\ &> 2j. \end{aligned}$$

If $i \in \mathbb{N}$ then $|a - b| \leq i$ implies $a, b \leq v_i$ Thus A, B are a coarsely disjoint decomposition of U. □

Not for all constant coefficients on $\mathbb{Z}_+$ the cohomology is concentrated in degree 0. For example the constant sheaf $\mathbb{Z}$ on $\mathbb{Z}_+$ has nontrivial cohomology in dimension 1.

Theorem 122. *If A is a locally finite abelian group then*

$$\check{H}^i(\mathbb{Z}_+, A) = \begin{cases} A & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. We will determine a distinguished coarse cover V_1, V_2 of $\mathbb{Z}_+$ and a subset $U \subseteq \mathbb{Z}_+$ with trivial cohomology. Then we show there is a coarse homotopy equivalence $U \to V_1$ and $U \to V_2$. And then we use the Mayer-Vietoris sequence of the coarse cover V_1, V_2 of $\mathbb{Z}_+$ to determine the cohomology of $\mathbb{Z}_+$.

At first follows a description of V_1, V_2: Define increasing sequences $(a_n)_n, (b_n)_n, (c_n)_n, (d_n)_n \subseteq \mathbb{Z}_+$ such that $b_n - c_n = n, d_n - a_{n+1} = n$ and $c_{n+1} - d_n = n, a_{n+1} - b_n = n$ for every $n \in \mathbb{N}$. Now define

$$V_1 = \bigcup_n [a_n, b_n]$$

and

$$V_2 = \bigcup_n [c_n, d_n]$$

Then V_1, V_2 are a coarse cover. Note that

$$V_1 \cap V_2 = \bigcup_n [c_n, b_n] \cup \bigcup_n [a_{n+1}, d_n]$$

Define $U = (a_n)_n$. Note that the constant sheaf A on U is flabby. Thus

$$\check{H}^i(U, A) = \begin{cases} A & i = 0 \\ 0 & i > 0. \end{cases}$$

The maps

$$\begin{aligned} p : V_1 &\to U \\ z \in [a_n, b_n] &\mapsto a_n \end{aligned}$$

and

$$\begin{aligned} i : U &\to V_1 \\ z &\mapsto z \end{aligned}$$

are coarse maps. There is a coarse homotopy joining id_{V_1} to $i \circ p$:

$$\begin{aligned} H : V_1 \times F([0,1]) &\to V_1 \\ (z, (t, i)) &\mapsto \lfloor (1-t)z + t a_n \rfloor \end{aligned}$$

where $z \in [a_n, c_n]$. In the same way there is a coarse homotopy equivalence $V_2 \to U$ and $V_1 \cap V_2 \to U$.

Thus there is a Mayer-Vietoris long exact sequence

$$\begin{aligned} 0 &\to \check{H}^0(\mathbb{Z}_+, A) \to \check{H}^0(V_1, A) \oplus \check{H}^0(V_2, A) \to \check{H}^0(V_1 \cap V_2, A) \\ &\to \check{H}^1(\mathbb{Z}_+, A) \to 0 \end{aligned}$$

It suffices to show that

$$d^0 : \check{H}^0(V_1, A) \oplus \check{H}^0(V_2, A) \to \check{H}^0(V_1 \cap V_2, A)$$
$$(s_1, s_2) \mapsto s_1|_{V_2} - s_2|_{V_1}$$

is surjective. Let $t \in \check{H}^0(V_1 \cap V_2, A)$ be a section. Omitting a bounded set we can assume t is a function taking finitely many values t_1^n, t_2^n on the chunks $[c_n, b_n], [a_{n+1}, d_n]$, $n \in \mathbb{N}$. We will construct $s_1 \in \check{H}^0(V_1, \mathbb{Z}/2\mathbb{Z})$ as a function taking finitely many values s_1^n on chunks $[a_n, b_n]$ and $s_2 \in \check{H}^0(V_2, \mathbb{Z}/2\mathbb{Z})$ as a function taking finitely many values s_2^n on chunks $[c_n, d_n]$ such that $d^0(s_1, s_2) = t$.

Inductively start at $[c_1, b_1]$. Both chunks $[a_1, b_1]$ and $[c_1, d_1]$ restrict to $[c_1, b_1]$. Define $s_1^1 := t_1^1$ and $s_2^1 := 0$. Now start at $[a_2, d_1]$. Both chunks $[a_2, b_2]$ and $[c_1, d_1]$ restrict to $[a_2, d_1]$. Define $s_1^2 := t_2^1$.

Let $n \in \mathbb{N}$. Suppose $s_1^1, \ldots, s_1^n$ and $s_2^1, \ldots, s_2^{n-1}$ have been constructed. Then both chunks $[a_n, b_n]$ and $[c_n, d_n]$ restrict to $[c_n, b_n]$. Define

$$s_2^n := s_1^n - t_1^n.$$

Now suppose $s_1^1, \ldots, s_1^{n-1}$ and $s_2^1, \ldots, s_2^{n-1}$ have been constructed. Then both chunks $[a_n, b_n]$ and $[c_{n-1}, d_{n-1}]$ restrict to $[a_n, d_{n-1}]$. Define

$$s_1^n := t_2^{n-1} + s_2^{n-1}$$

We now check that s_1, s_2 indeed define cochains. It suffices to show that they take finitely many values. Now, by our hypothesis, the t_1^n and t_2^n take finitely many values, say in a finite set S. Then, by our hypothesis that A is locally finite, the group generated by S is also finite and the s_1^n, s_2^n take values in $\langle S \rangle$. We have thus found $s_1 \in \check{H}^0(V_1, A), s_2 \in \check{H}^0(V_2, A)$ such that $d^0(s_1, s_2) = t$. □

4.1.3 Acyclic Spaces

There is a notion of flasque spaces for which most coarse cohomology theories vanish. Let us translate [29, Definition 3.6] into coarse structure notation:

Definition 123. A coarse space X is called *flasque* if there is a coarse map $\phi : X \to X$ such that

- ϕ is close to the identity on X;
- for every bounded set $B \subseteq X$ there is some $N_B \in \mathbb{N}$ such that

  $$\phi^n(X) \cap B = \emptyset$$

  for every $n \geq N_B$.
- For every entourage E the set $\bigcup_n (\phi^n)^2(E)$ is an entourage.

Theorem 124. *If X is a flasque space then there is a coarse homotopy equivalence*

$$\Phi : X \times \mathbb{Z}_+ \to X$$
$$(x, i) \mapsto \phi^i(x).$$

Here ϕ^0 denotes the identity on X.

Proof. We show that the coarse homotopy inverse to Φ is

$$\begin{aligned} i_0 : X &\to X \times \mathbb{Z}_+ \\ x &\mapsto (x, 0). \end{aligned}$$

Now $\Phi \circ i_0 = id_X$.

We show that $i_0 \circ \Phi$ and $id_{X \times \mathbb{Z}_+}$ are coarsely homotopic: Define a map

$$\begin{aligned} h : (X \times \mathbb{Z}_+) * F([0,1]) &\to X \times \mathbb{Z}_+ \\ ((x,i),(t,j)) &\mapsto (\phi^{\lfloor ti \rfloor}(x), \lfloor (1-t)i \rfloor). \end{aligned}$$

We show that h is a coarse map:

1. h is coarsely uniform: let $E \subseteq ((X \times \mathbb{Z}_+) * F([0,1]))^2$ be an entourage. Denote by $p_X : X \times \mathbb{Z}_+ \to X$ and $p_{\mathbb{Z}_+} : X \times \mathbb{Z}_+ \to \mathbb{Z}_+$ the projections to $X, \mathbb{Z}_+$, both $p_X, p_{\mathbb{Z}_+}$ are coarsely uniform maps. We show $p_X^{\times 2} \circ h^{\times 2}(E)$ is an entourage and $p_{\mathbb{Z}_+}^{\times 2} \circ h^{\times 2}(E)$ is an entourage. Note:
$$p_X^{\times 2} \circ h^{\times 2}(E) \subseteq \bigcup_n (\phi^n)^2(E)$$
is an entourage. If $i, j \in \mathbb{Z}_+, t, s \in [0,1]$ then
$$\begin{aligned} |\lfloor (1-t)i \rfloor - \lfloor (1-s)j \rfloor| &\le |(1-t)i - (1-s)j| + 2 \\ &\le |i-j| + 2 \end{aligned}$$
Thus $p_{\mathbb{Z}_+}^{\times 2} \circ h^{\times 2}(E)$ is an entourage.
2. h is coarsely proper: Let $B \subseteq X \times \mathbb{Z}_+$ be a bounded subset. We write
$$B = \bigcup_i B_i \times i$$
which is a finite union. Then for every i there is some N_i such that
$$\phi^n(X) \cap B_i = \emptyset$$
for every $n \ge N_i$. We show $p_X \circ h^{-1}(B)$ and $p_{\mathbb{Z}_+} \circ h^{-1}(B)$ is bounded Then
$$h^{-1}(B) \subseteq (p_X \circ h^{-1}(B) \times p_{\mathbb{Z}_+} \circ h^{-1}(B)) * F([0,1])$$
is bounded in $(X \times \mathbb{Z}_+) * F([0,1])$. Now
$$p_X \circ h^{-1}(B) \subseteq \bigcup_i (\phi^{-0}(B_i) \cup \cdots \cup \phi^{-N_i}(B_i))$$
is bounded in X. If $j \in p_{\mathbb{Z}_+} \circ h^{-1}(B)$ then $\lfloor tj \rfloor \le N_i$ for at least one i. Thus
$$j \le \max_i N_i$$
therefore $p_{\mathbb{Z}_+} \circ h^{-1}(B)$ is bounded in $\mathbb{Z}_+$.

□

Example 125. Note that $\mathbb{Z}_+$ is flasque by

$$\begin{aligned} \phi : \mathbb{Z}_+ &\to \mathbb{Z}_+ \\ n &\mapsto n+1. \end{aligned}$$

Thus there is a coarse homotopy equivalence $\mathbb{Z}_+^2 \to \mathbb{Z}_+$. Now for every n the space $\mathbb{Z}_+^n$ is flasque. As a result $\mathbb{Z}_+^n$ is coarsely homotopy equivalent to $\mathbb{Z}_+$ for every n.

4.1.4 Computing Examples

Example 126. ($\mathbb{Z}$) Now $\mathbb{Z}_+$ is acyclic for constant coefficients $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ is the coarse disjoint union of two copies of it. Thus

$$\begin{aligned}\check{H}^i(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) &= \check{H}^i(\mathbb{Z}_+, \mathbb{Z}/2\mathbb{Z}) \oplus \check{H}^i(\mathbb{Z}_+, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

Example 127. ($\mathbb{Z}^2$) We cover the space $\mathbb{Z}^2$ with five copies of $\mathbb{Z}_+^2$ such that they meet at $(0,0)$ and have nontrivial overlaps (like a cake). Then this gives us a coarse cover of $\mathbb{Z}^2$ with acyclics. Then it is easy to calculate

$$\check{H}^i(\mathbb{Z}^2, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 128. ($\mathbb{Z}^n$) For $n \geq 2$ we can cover $\mathbb{Z}^n$ with copies of $\mathbb{Z}_+^n$ in much the same way as in Example 127. But that is a coarse cover of $\mathbb{Z}^n$ with acyclics with which we can compute

$$\check{H}^i(\mathbb{Z}^n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geq 2$.

Lemma 129. *Let G be a group and $H \leq G$ a subgroup with finite index. Then the inclusion $i : H \to G$ is a coarse equivalence.*

Proof. There are only finitely many right cosets $Hg_1, \ldots, Hg_n$. Then define the coarse inverse to i to be

$$\begin{aligned} r : G &\to H \\ g &\mapsto gg_i^{-1} \text{ if } g \in Hg_i. \end{aligned}$$

□

Proposition 130. *Note the following facts:*

1. *If F_2 is the free group with two generators then for $n \leq 3$ the free group with n generators, F_n, is a subgroup of F_2 with finite index.*
2. *If D_∞ is the infinite dihedral group $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ then it contains $\mathbb{Z}$ as a subgroup with finite index.*
3. *The modular group $\mathbb{Z}_2 * \mathbb{Z}_3$ contains $\mathbb{Z} * \mathbb{Z}$, the free group with two generators, as a subgroup with finite index.*

Proof. 1. This is explained in [30, section 20, chapter 2].

2. easy.

3. This is mentioned in [30, section 22, chapter 2].

□

Example 131. (F_n) Note that F_2 has infinitely many ends. In fact it is a countable coarse disjoint union of copies of $\mathbb{Z}_+$. By Proposition 130 we have

$$\begin{aligned}\check{H}^i(F_n, \mathbb{Z}/2\mathbb{Z}) &= \bigoplus_{\mathbb{N}} \check{H}^i(\mathbb{Z}_+, \mathbb{Z}/2\mathbb{Z}) \\ &= \begin{cases} \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} & i = 0 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

for $n \geq 2$.

Remark 132. Suppose there is a notion of boundary of a coarse space such that

- the boundary of $\mathbb{Z}^n$ is S^{n-1}
- the boundary of F_n is a Cantor set

Then one could try to prove that the singular cohomology of the boundary as a topological space equals local coarse cohomology.

4.2 A twisted Version of controlled K-Theory

Controlled operator $K-$theory is one of the most popular homological invariants on coarse metric spaces. Meanwhile a new cohomological invariant on coarse spaces recently appeared in [13].

The paper [13] studies sheaf cohomology on coarse spaces. Note that cohomology theories that are derived functors are immensely more powerful than those that do not.

In this paper we study the controlled $K-$theory of a proper metric space X which is introduced in [1, Chapter 6.3]. Note that this theory does not appear as a derived functor as far as we know.

The Theorem 145 shows if X is a proper metric space a modified version of the Roe-algebra $C^*(X)$ is a cosheaf on X. This result gives rise to new computational tools one of which is a new Mayer-Vietoris six-term exact sequence which is Corollary 146.

Note that in a general setting cosheaves with values in `Ab` do not give rise to a derived functor. In [31] is explained that the dual version of sheafification, cosheafification, does not work in general.

4.2.1 Cosheaves

We recall [13, Definition 45]:

Definition 133. (**coarse cover**) If X is a coarse space and $U \subseteq X$ a subset a finite family of subsets $U_1, \ldots, U_n \subseteq U$ is said to *coarsely cover* U if for every entourage $E \subseteq U^2$ the set

$$E[U_1^c] \cap \cdots \cap E[U_n^c]$$

is bounded in U. Coarse covers on X determine a Grothendieck topology X_{ct} on X.

Definition 134. (**precosheaf**) A *precosheaf on X_{ct} with values in a category C* is a covariant functor $Cat(X_{ct}) \to C$.

Definition 135. (**cosheaf**) Let C be a category with finite limits and colimits. A precosheaf $\mathcal{F}$ on X_{ct} with values in C is a *cosheaf on X_{ct} with values in C* if for every coarse cover $\{U_i \to U\}_i$ there is a coequalizer diagram:

$$\bigoplus_{ij} \mathcal{F}(U_i \cap U_j) \rightrightarrows \bigoplus_i \mathcal{F}(U_i) \to \mathcal{F}(U)$$

Here the two arrows on the left side relate to the following 2 diagrams:

$$\begin{tikzcd} \bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \arrow[r] & \bigoplus_i \mathcal{F}(U_i) \\ \mathcal{F}(U_i \cap U_j) \arrow[r] \arrow[u] & \mathcal{F}(U_i) \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd} \bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \arrow[r] & \bigoplus_i \mathcal{F}(U_j) \\ \mathcal{F}(U_i \cap U_j) \arrow[r] \arrow[u] & \mathcal{F}(U_j) \arrow[u] \end{tikzcd}$$

where $\bigoplus$ denotes the coproduct over the index set.

Notation 136. If we write

- $\sum_i a_i \in \bigoplus_i \mathcal{F}(U_i)$ then a_i is supposed to be in $\mathcal{F}(U_i)$
- $\sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j)$ then b_{ij} is supposed to be in $\mathcal{F}(U_i \cup U_j)$

Proposition 137. *If $\mathcal{F}$ is a precosheaf on X_{ct} with values in a category C with finite limits and colimits and for every coarse cover $\{U_i \to U\}_i$*

1. *and every $a \in \mathcal{F}(U)$ there is some $\sum_i a_i \in \bigoplus_i \mathcal{F}(U_i)$ such that $\sum_i a_i|_U = a$*
2. *and for every $\sum_i a_i \in \bigoplus_i \mathcal{F}(U_i)$ such that $\sum_i a_i|_U = 0$ there is some $\sum_{ij} b_{ij} \in \bigoplus_{ij} \mathcal{F}(U_i \cap U_j)$ such that $(\sum_j b_{ij} - b_{ji})|_{U_i} = a_i$ for every i.*

then $\mathcal{F}$ is a cosheaf.

Proof. easy. ☐

Remark 138. Denote by `CStar` the category of C^*-algebras. According to [32] all finite limits and finite colimits exist in `CStar`.

4.2.2 Modified Roe-Algebra

Lemma 139. *If X is a proper metric space and $Y \subseteq X$ is a closed subspace then*

- *the subset $I(Y) = \{f \in C_0(X) : f|_Y = 0\}$ is an ideal of $C_0(X)$ and we have*

$$C_0(Y) = C_0(X)/I(Y)$$

- *we can restrict the non-degenerate representation $\rho_X : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$ to a representation*

$$\rho_Y : C_0(Y) \to \mathbb{B}(\mathcal{H}_Y)$$

in a natural way.

- *the inclusion $i_Y : \mathcal{H}_Y \to \mathcal{H}_X$ covers the inclusion $i : Y \to X$.*

Proof.
- This one follows by Gelfand duality.
- We define $\mathcal{H}_{I(Y)} = \overline{\rho_X(I(Y))\mathcal{H}_X}$. Then

$$\mathcal{H}_X = \mathcal{H}_{I(Y)} \oplus \mathcal{H}_{I(Y)}^{\perp}$$

is the direct sum of reducing subspaces for $\rho_X(C_0(X))$. We define

$$\mathcal{H}_Y = \mathcal{H}_{I(Y)}^{\perp}$$

and a representation of $C_0(Y)$ on $\mathcal{H}_Y$ by

$$\rho_Y([a]) = \rho_X(a)|_{\mathcal{H}_Y}$$

for every $[a] \in C_0(Y)$. Note that $\rho_X(\cdot)|_{\mathcal{H}_Y}$ annihilates $I(Y)$ so this is well defined.

- Note that the support of i_Y is

$$\begin{aligned}\operatorname{supp}(i_Y) &= \Delta_Y \\ &\subseteq X \times Y\end{aligned}$$

$\square$

Remark 140. Note that we can not conclude the following: If the representation $\rho_X : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$ is ample and $Y \subseteq X$ is a closed subspace then the induced representation $\rho_Y : C_0(Y) \to \mathbb{B}(\mathcal{H}_Y)$ is ample. There are counterexamples to this claim.

Lemma 141. *If X is a proper metric space, $B \subseteq X$ a compact subset and $T \in C^*(X)$ an operator with*

$$\operatorname{supp} T \subseteq B^2$$

then T is a compact operator.

Proof. Suppose there is a non-degenerate representation $\rho : C_0(X) \to \mathbb{B}(\mathcal{H}_X)$. For every $f \in C_0(B^c), g \in C_0(X)$ the equations $\rho(f)T\rho(g) = 0$ and $\rho(g)T\rho(f) = 0$ hold. This implies $T(I(B)) = 0$ and $\operatorname{im} T \cap I(B) = 0$. Thus $T : \mathcal{H}_B \to \mathcal{H}_B$ is the same map. Thus $T \in C^*(B)$ already. Now T is locally compact, B is compact thus T is a compact operator. $\square$

Definition 142. (**modified Roe-algebra**) Let X be a proper metric space then

$$\hat{C}^*(X) = C^*(X)/\mathbb{K}(\mathcal{H}_X)$$

where $\mathbb{K}(\mathcal{H}_X)$ denotes the compact operators of $\mathbb{B}(\mathcal{H}_X)$ is called the *modified Roe-algebra* of X.

Remark 143. If $U \subseteq X$ is a subset of a proper metric space then U is coarsely dense in $\bar{U}$. We define

$$\hat{C}^*(U) := \hat{C}^*(\bar{U})$$

Note that makes sense because if U_1, U_2 is a coarse cover of U then $\bar{U}_1, \bar{U}_2$ is a coarse cover of $\bar{U}$ also.

Lemma 144. *If $Y \subseteq X$ is a closed subspace and $i_Y : \mathcal{H}_Y \to \mathcal{H}_X$ the inclusion operator of Lemma 139 then*

- *the operator*

$$\begin{aligned} Ad(i_Y) : C^*(Y) &\to C^*(X) \\ T &\mapsto i_Y T i_Y^* \end{aligned}$$

is well defined and maps compact operators to compact operators.

- *Then the induced operator on quotients*

$$\hat{Ad}(i_Y) : \hat{C}^*(Y) \to \hat{C}^*(X)$$

is the dual version of a restriction map.

Proof.
- i_Y covers the inclusion the other statement is obvious.
- easy.

□

Theorem 145. *If X is a proper metric space then the assignment*

$$U \mapsto \hat{C}^*(U)$$

for every subspace $U \subseteq X$ is a cosheaf with values in `CStar`.

Proof. Let $U_1, U_2 \subseteq U$ be subsets that coarsely cover $U \subseteq X$ and $V_1 : \mathcal{H}_{U_1} \to \mathcal{H}_U$ and $V_2 : \mathcal{H}_{U_2} \to \mathcal{H}_U$ the corresponding inclusion operators.

1. Let $T \in C^*(U)$ be a locally compact controlled operator. We need to construct $T_1 \in C^*(U_1), T_2 \in C^*(U_2)$ such that

$$V_1 T_1 V_1^* + V_2 T_2 V_2^* = T$$

modulo compacts. Denote by

$$E = \mathrm{supp}(T)$$

the support of T in U. Define

$$T_1 := V_1^* T V_1$$

and

$$T_2 := V_2^* T V_2$$

then it is easy to check that T_1, T_2 are locally compact and controlled operators, thus elements in $C^*(U_1), C^*(U_2)$. Now $\mathrm{supp}(V_1 T_1 V_1^*) = U_1^2 \cap E$ and $\mathrm{supp}(V_2 T_2 V_2^*) = U_2^2 \cap E$. Thus

$$\begin{aligned} \mathrm{supp}(V_1 T_1 V_1^* + V_2 T_2 V_2^* - T) &= E \cap (U_1^2 \cup U_2^2)^c \\ &\subseteq B^2 \end{aligned}$$

where B is bounded. This implies $T_1|_U + T_2|_U = T$.

2. Suppose there are $T_1 \in C^*(U_1), T_2 \in C^*(U_2)$ such that

$$V_1 T_1 V_1^* + V_2 T_2 V_2^* = 0$$

modulo compacts. That implies that $\mathrm{supp}(V_1T_1V_1^*) \subseteq (U_1 \cap U_2)^2$ modulo bounded sets and $\mathrm{supp}(V_2T_2V_2^*) \subseteq (U_1 \cap U_2)^2$ modulo bounded sets. Also $V_1T_1V_1^* = -V_2T_2V_2^*$ modulo compacts. Denote by $V_{12}^i : \mathcal{H}_{U_1\cap U_2} \to \mathcal{H}_{U_i}$ the inclusion for $i = 1, 2$. Define

$$T_{12} = V_{12}^* T_1 V_{12}$$

Then

$$\begin{aligned} V_{12}^1 T_{12} V_{12}^{1*} &= V_{12}^1 V_{12}^{1*} T_1 V_{12}^1 V_{12}^{1*} \\ &= T_1 \end{aligned}$$

Then $V_1 \circ V_{12}^1 = V_2 \circ V_{12}^2$ implies

$$\begin{aligned} V_{12}^2 T_{12} V_{12}^{2*} &= V_2^* V_2 V_{12}^2 T_{12} V_{12}^{2*} V_2^* V_2 \\ &= V_2^* V_1 V_{12}^1 T_{12} V_{12}^{1*} V_1^* V_2 \\ &= V_2^* V_1 T_1 V_1^* V_2 \\ &= -T_2 \end{aligned}$$

modulo compacts.

□

4.2.3 Computing Examples

Corollary 146. *If U_1, U_2 coarsely cover a subset U of a proper metric space X then there is a six-term Mayer-Vietoris exact sequence*

$$\begin{tikzcd} K_1(\hat{C}^*(U_1 \cap U_2)) \arrow[r] & K_1(\hat{C}^*(U_1)) \oplus K_1(\hat{C}^*(U_2)) \arrow[r] & K_1(\hat{C}^*(U)) \arrow[d] \\ K_0(\hat{C}^*(U)) \arrow[u] & K_0(\hat{C}^*(U_1)) \oplus K_0(\hat{C}^*(U_2)) \arrow[l] & K_0(\hat{C}^*(U_1 \cap U_2)) \arrow[l] \end{tikzcd}$$

Proof. By Theorem 145 there is a pullback diagram of C^*-algebras and $*$-homomorphisms

$$\begin{tikzcd} \hat{C}^*(U_1 \cap U_2) \arrow[r] \arrow[d] & \hat{C}^*(U_2) \arrow[d] \\ \hat{C}^*(U_1) \arrow[r] & \hat{C}^*(U) \end{tikzcd}$$

The result is an application of [1, Exercise 4.10.22]. □

Remark 147. Note that Corollary 146 is applicable if the property ample is preserved by restricting the representation of U to the representations of U_1, U_2.

Remark 148. Now for every proper metric space there is a short exact sequence

$$0 \to \mathbb{K}(\mathcal{H}_X) \to C^*(X) \to \hat{C}^*(X) \to 0$$

which induces a 6-term sequence in K-theoy:

$$\begin{tikzcd} K_0(\mathbb{K}(\mathcal{H}_X)) \arrow[r] & K_0(C^*(X)) \arrow[r] & K_0(\hat{C}^*(X)) \arrow[d] \\ K_1(\hat{C}^*(X)) \arrow[u] & K_1(C^*(X)) \arrow[l] & K_1(\mathbb{K}(\mathcal{H}_X)) \arrow[l] \end{tikzcd}$$

If X is flasque then

$$K_i(\hat{C}^*(X)) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \end{cases}$$

Remark 149. If X is a Riemannian manifold then $L^2(X)$ is a Hilbert space. In Example 150, Example 151 we will use the canonical representations of type $C_0(X) \to \mathbb{B}(L^2(X))$ on $\mathbb{R}, \mathbb{R}^2$ and certain subspaces of them without mentioning it.

Example 150. ($\mathbb{R}$) Now $\mathbb{R}$ is the coarse disjoint union of two copies of $\mathbb{R}_+$ which is a flasque space. By Corollary 146 there is an isomorphism

$$K_i(\hat{C}^*(\mathbb{R})) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \end{cases}$$

Then it is a result of Remark 148 that there is an isomorphism

$$K_i(C^*(\mathbb{R})) = \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \end{cases}$$

no surprise.

Example 151. ($\mathbb{R}^2$) We coarsely cover $\mathbb{R}^2$ with

$$V_1 = \mathbb{R}_+ \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}_+$$

and

$$V_2 = \mathbb{R}_- \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}_-.$$

then again V_1 is coarsely covered by

$$U_1 = \mathbb{R}_+ \times \mathbb{R}$$

and

$$U_2 = \mathbb{R} \times \mathbb{R}_+$$

and V_2 is coarsely covered in a similar fashion. We first compute modified controlled K-theory of V_1 and then of $\mathbb{R}^2$. Note that the inclusion $U_1 \cap U_2 \to U_1$ is split by

$$\begin{aligned} r : \mathbb{R}_+ \times \mathbb{R} &\to \mathbb{R}_+^2 \\ (x, y) &\mapsto (x, |y|) \end{aligned}$$

Thus using Corollary 146 we conclude that

$$K_i(\hat{C}^*(V_j)) = \begin{cases} 0 & i = 0 \\ \mathbb{R} & i = 1 \end{cases}$$

for $j = 1, 2$. Then again using Corollary 146 and that the inclusion $\mathbb{R}_+^2 \to V_i$ is split we can compute

$$K_i(\hat{C}^*(\mathbb{R}^2)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \end{cases}$$

Translating back we get that

$$K_i(C^*(\mathbb{R}^2)) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \end{cases}$$

This one also fits previous computations.

4.3 Remarks

The starting point of this research was the idea to define sheaves on coarse spaces as presented in [33]. And then we noticed that cocontrolled subsets of X^2 which have first been studied in [4] have some topological features.

Finally, after defining coarse covers which depend on the notion of coentourages, we came up with the methods of this paper. Note that coarse cohomology with twisted coefficients is basically just sheaf cohomology on the Grothendieck topology determined by coarse covers.

It would be possible, conversely, after a more thorough examination that coarse (co-)homology theories which are standard tools can be computed using sheaf cohomology tools. As of now a modified version of controlled $K-$theory serves as cosheaf homology and coarse cohomology in dimension 2 is a sheaf on coarse spaces.

We wonder if this result will be of any help with understanding coarse spaces. Note that Remark 132 gave rise to the studies in [15].

However, as of yet, we do not know if coarse covers as defined in this paper are the most natural topology for other classes of spaces than proper geodesic metric spaces.

Chapter 5

Space of Ends

Coarse Geometry of metric spaces studies the large scale properties of a metric space. Meanwhile uniformity of metric spaces is about small scale properties.

Our purpose is to pursue a new version of duality between the coarse geometry of metric spaces and uniform spaces. We present a notion of boundary on coarse metric spaces which is a totally bounded separating uniform space. The methods are very basic and do not require any deep theory.

Note that the topology of metric spaces is well understood and there are a number of topological tools that can be applied on coarse metric spaces which have not been used before. The new discovery may lead to new insight on the topic of coarse geometry.

5.1 Groundwork

5.1.1 Metric Spaces

Definition 152. Let (X, d) be a metric space.

- Then the *bounded coarse structure associated to* d on X consists of those subsets $E \subseteq X^2$ for which
$$\sup_{(x,y)\in E} d(x, y) < \infty.$$
We call an element of the coarse structure *entourage*.

- The *bounded cocoarse structure associated to* d on X consists of those subsets $C \subseteq X^2$ such that every sequence $(x_i, y_i)_i$ in C is either bounded (which means both of the sequences $(x_i)_i$ and $(y_i)_i$ are bounded) or $d(x_i, y_i)_i$ is not bounded. for $i \to \infty$. We call an element of the cocoarse structure coentourage.

- In what follows we assume the metric d to be finite for every $(x, y) \in X^2$.

Remark 153. Note that there is a more general notion of coarse spaces. By [4, Theorem 2.55] a coarse structure on a coarse space X is the bounded coarse structure associated to some metric d on X if and only if the coarse structure has a countable base.

Definition 154. If X is a metric space a subset $B \subseteq X$ is *bounded* if the set B^2 is an entourage in X.

Remark 155. Note the following duality:

- A subset $F \subseteq X^2$ is an entourage if and only if for every coentourage $C \subseteq X^2$ there is a bounded set $A \subseteq X$ such that
$$F \cap C \subseteq A^2.$$
- A subset $D \subseteq X^2$ is a coentourage if and only if for every entourage $E \subseteq X^2$ there is a bounded set B such that
$$E \cap D \subseteq B^2.$$

Definition 156. A map $f : X \to Y$ between metric spaces is called *coarse* if

- $E \subseteq X^2$ being an entourage implies that $f^{\times 2}E$ is an entourage *(coarsely uniform)*;
- and if $A \subseteq Y$ is bounded then $f^{-1}(A)$ is bounded *(coarsely proper)*.

Or equivalently

- $B \subseteq X$ being bounded implies that $f(B)$ is bounded;
- and if $D \subseteq Y^2$ is a coentourage then $(f^{\times 2})^{-1}(D)$ is a coentourage.

Two maps $f, g : X \to Y$ between metric spaces are called *close* if
$$f \times g(\Delta_X)$$
is an entourage in Y. Here Δ_X denotes the diagonal in X.

Notation 157. A map $f : X \to Y$ between metric spaces is called

- *coarsely surjective* if there is an entourage $E \subseteq Y^2$ such that
$$E[\operatorname{im} f] = Y$$
- *coarsely injective* if
  1. for every entourage $F \subseteq Y^2$ the set $(f^{\times 2})^{-1}(F)$ is an entourage in X.
  2. or equivalently if for ever coentourage $C \subseteq X^2$ the set $f^{\times 2}(C)$ is a coentourage in Y.
- two subsets $A, B \subseteq X$ are called *coarsely disjoint* if $A \times B$ is a coentourage.

Remark 158. We study metric spaces up to coarse equivalence. A coarse map $f : X \to Y$ is a *coarse equivalence* if

- There is a coarse map $g : Y \to X$ such that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X.
- or equivalently if f is both coarsely injective and coarsely surjective.

5.1.2 Totally Bounded Uniformity

Definition 159. (**close relation**) Let X be a coarse space. Two subsets $A, B \subseteq X$ are called *close* if they are not coarsely disjoint. We write

$$A \curlywedge B.$$

Then $\curlywedge$ is a relation on the subsets of X.

Lemma 160. *In every metric space X:*

1. *if B is bounded, $B \not\curlywedge A$ for every $A \subseteq X$*
2. *$U \curlywedge V$ implies $V \curlywedge U$*
3. *$U \curlywedge (V \cup W)$ if and only if $U \curlywedge V$ or $U \curlywedge W$*

Proof.
1. easy.
2. easy.
3. easy.

□

Proposition 161. *Let X be a metric space. Then for every subspaces $A, B \subseteq X$ with $A \not\curlywedge B$ there are subsets $C, D \subseteq X$ such that $C \cap D = \emptyset$ and $A \not\curlywedge (X \setminus C)$, $B \not\curlywedge X \setminus D$.*

Proof. Note this is the same as [17, Proposition 4.5] where the same statement was proven in a similar fashion. Let $E_1 \subseteq E_2 \subseteq \cdots$ be a symmetric basis for the coarse structure of X. Then for every $x \in A^c \cap B^c$ there is a least number $n_1(x)$ such that $x \in E_{n_1(x)}[A]$ and a least number $n_2(x)$ such that $x \in E_{n_2(x)}[B]$. Define:

$$V_1 = \{x \in A^c \cap B^c : n_1(x) \leq n_2(x)\}$$

and

$$V_2 = A^c \cap B^c \setminus V_1.$$

Now for every n

$$E_n[V_1] \cap B \subseteq E_{2n}[A] \cap B$$

because for every $x \in V_1$, if $x \in E_n[B]$ then $x \in E_n[A]$. Now define

$$C = A \cup V_1$$

and

$$D = B \cup V_2.$$

□

Remark 162. Compare $\curlywedge$ with the notion of proximity relation [34, chapter 40, pp. 266]. By Lemma 160 and Proposition 161 the close relation satisfies [34, P-1),P-3)-P-5) of Definition 40.1] but not P-2).

Remark 163. If $f : X \to Y$ is a coarse map then whenever $A \curlywedge B$ in X then $f(A) \curlywedge f(B)$ in Y.

We recall [13, Definition 45]:

Definition 164. (**coarse cover**) If X is a metric space and $U \subseteq X$ a subset a finite family of subsets $U_1, \dots, U_n \subseteq U$ is said to *coarsely cover* U if

$$U^2 \cap (\bigcup_i U_i^2)^c$$

is a coentourage in X.

Remark 165. Note that coarse covers determine a Grothendieck topology on X. If $f : X \to Y$ is a coarse map between metric spaces and $(V_i)_i$ a coarse cover of $V \subseteq Y$ then $(f^{-1}(V_i))_i$ is a coarse cover of $f^{-1}(V) \subseteq X$.

Lemma 166. *Let X be a metric space. A finite family $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is a coarse cover if and only if there is a finite cover $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of X as a set such that $V_\alpha \not\curlywedge U_\alpha^c$ for every α.*

Proof. Suppose $\mathcal{U}$ is a coarse cover of X. We proceed by induction on the index of $\mathcal{U}$:

- $n = 1$: a subset U coarsely covers X if and only if U^c is bounded if and only if $U^c \not\curlywedge X$.
- two subsets U_1, U_2 coarsely cover X if and only if $U_1^c \not\curlywedge U_2^c$. Now by Proposition 161 there are $C, D \subseteq X$ with $C^c \cup D^c = X$ and $C^c \not\curlywedge U_1^c$ and $D^c \not\curlywedge U_2^c$. Define $V_1 = C^c, V_2 = D^c$.
- $n+1 \to n+2$: Subsets $U, V, U_1, \dots, U_n$ coarsely cover X if and only if U, V coarsely cover $U \cup V$ and $U \cup V, U_1, \dots, U_n$ coarsely cover X at the same time.

  Suppose $U, V, U_1, \dots, U_n$ coarsely cover X. By induction hypothesis there is a cover of sets V_1', V_2' of $U \cup V$ such that $V_1' \not\curlywedge U^c \cap V$ and $V_2' \not\curlywedge V^c \cap U$ and there is a cover of sets $W, V_1, \dots, V_n$ such that $W \not\curlywedge (U \cup V)^c$ and $V_i \not\curlywedge U_i^c$ for every i. Then $V_1' \cap W \not\curlywedge U^c, V_2' \cap W \not\curlywedge V^c$. Now

  $$B := (U \cup V)^c \cap W$$

  is bounded. Then

  $$V_1' \cap W, V_2' \cap W, V_1, \dots, V_n \cup B$$

  is a finite cover of X with the desired properties.

Suppose $(V_\alpha)_\alpha$ cover X as sets and $V_\alpha \not\curlywedge U_\alpha^c$ for every α. Let $E \subseteq X^2$ be an entourage. Then $E[U_\alpha^c] \cap V_\alpha$ is bounded for every α. Then

$$\begin{aligned}\bigcap_\alpha E[U_\alpha^c] &= \bigcap_\alpha E[U_\alpha^c] \cap (\bigcup_\alpha V_\alpha)\\ &= \bigcup_\alpha (V_\alpha \cap \bigcap_\beta E[U_\beta])\end{aligned}$$

is bounded. $\square$

Remark 167. The [34, Theorem 40.15] states that every proximity relation on a set is induced by some totally bounded uniformity on it. Note that a coarse cover on a metric space X does not precisely need to cover X as a set. Except for that the collection of all coarse covers of a metric space satisfies [34, a),b) of Theorem 36.2]. We can compare coarse covers of X with a base for a totally bounded uniformity on X: the collection of all sets $\bigcup_i U_i^2$ for $(U_i)_i$ a coarse cover satisfies [34, b)-e) of Definition 35.2] but not a). Note that by [34, Definition 39.7] a diagonal uniformity is totally bounded if it has a base consisting of finite covers.

Lemma 168. ***(separation cover)*** *If U_1, U_2 coarsely cover a metric space X (or equivalently if U_1^c, U_2^c are coarsely disjoint) then there exists a coarse cover V_1, V_2 of X such that $V_1 \not\curlywedge U_1^c$ and $V_2 \not\curlywedge U_2^c$.*

Proof. By Proposition 161 there are subsets $C, D \subseteq X$ such that $C \cap D = \emptyset$, $U_1^c \not\curlywedge C^c$ and $U_2^c \not\curlywedge D^c$. Thus U_1, C is a coarse cover of X such that $C \not\curlywedge U_2^c$.

By Proposition 161 there are subsets $A, B \subseteq X$ such that $A \cap B = \emptyset$, $A^c \not\curlywedge U_1^c$ and $B^c \not\curlywedge C^c$. Then B, C are a coarse cover of X such that $B \not\curlywedge U_1^c$.

Then $V_1 = B$ and $V_2 = C$ have the desired properties. □

Notation 169. (**coarse star refinement**) Let $\mathcal{U} = (U_i)_{i \in I}$ be a coarse cover of a metric space X.

1. If $S \subseteq X$ is a subset then
$$\mathsf{cst}(S, \mathcal{U}) = \bigcup \{U_i : S \curlywedge U_i\}$$
is called the *coarse star of S*.

2. A coarse cover $\mathcal{V} = (V_j)_{j \in J}$ of X is called a *coarse barycentric refinement of $\mathcal{U}$* if for every $j_1, \ldots, j_k \in J$ such that there is an entourage $E \subseteq X^2$ such that
$$\bigcap_k E[V_{i_k}]$$
is not bounded then there is some $i \in I$ and entourage $F \subseteq X^2$ such that
$$\bigcup_k V_{i_k} \subseteq F[V_i].$$

3. A coarse cover $\mathcal{V} = (V_j)_{j \in J}$ of X is called a *coarse star refinement of $\mathcal{U}$* if for every $j \in J$ there is some $i \in I$ and entourage $E \subseteq X^2$ such that
$$\mathsf{cst}(V_j, \mathcal{V}) \subseteq E[U_i].$$

Lemma 170. *If $\mathcal{V} = (V_i)_i$ is a coarse star refinement of a coarse cover $\mathcal{U} = (U_i)_i$ of a metric space X then*

- *if $S \subseteq X$ is a subset then there is an entourage $E \subseteq X^2$ such that*
$$\mathsf{cst}(\mathsf{cst}(S, \mathcal{V}), \mathcal{V}) \subseteq E[\mathsf{cst}(S, \mathcal{U})];$$
- *if $f : X \to Y$ is a coarse map between metric spaces, $(U_i)_i$ a coarse cover of Y and $S \subseteq X$ a subset then*
$$f(\mathsf{cst}(S, f^{-1}(\mathcal{U}))) \subseteq \mathsf{cst}(f(S), \mathcal{U}).$$

Proof. • Suppose $E \subseteq X^2$ is an entourage such that for every V_j there is an U_i such that $\mathsf{cst}(V_j, \mathcal{V}) \subseteq U_i$. Note that $S \curlywedge V_j$ implies $S \curlywedge U_i$ in that case. Then
$$\begin{aligned}\mathsf{cst}(\mathsf{cst}(S, \mathcal{V}), \mathcal{V}) &= \mathsf{cst}(\bigcup\{V_i : V_i \curlywedge S\}, \mathcal{V}) \\ &= \bigcup_{V_i \curlywedge S} \mathsf{cst}(V_i, \mathcal{V}) \\ &\subseteq \bigcup_{S \curlywedge U_j} E[U_j] \\ &= E[\mathsf{cst}(S, \mathcal{U})].\end{aligned}$$

- $$\begin{aligned} f(\mathsf{cst}(S, f^{-1}(\mathcal{U}))) &= \bigcup\{f \circ f^{-1}(U_i) : S \curlywedge f^{-1}(U_i)\} \\ &\subseteq \bigcup\{f \circ f^{-1}(U_i) : f(S) \curlywedge f \circ f^{-1}(U_i)\} \\ &\subseteq \bigcup\{U_i : f(S) \curlywedge U_i\} \\ &= \mathsf{cst}(f(S), \mathcal{U}). \end{aligned}$$

□

Lemma 171. *If $\mathcal{U}$ is a coarse cover of a metric space X then there exists a coarse cover $\mathcal{V}$ of X that coarsely star refines $\mathcal{U}$.*

Proof. There are three steps:

If $\mathcal{V} = (V_j)_j$ is a coarse barycentric refinement of $\mathcal{U}$ and $\mathcal{W} = (W_k)_k$ is a coarse barycentric refinement of $\mathcal{V}$ then $\mathcal{W}$ is a coarse star refinement of $\mathcal{U}$:

fix W_k and denote $J = \{j : W_k \curlywedge W_j\}$.

Then for every $j \in J$ there is some V_j and entourage $E_j \subseteq X^2$ such that $W_k \cup W_j \subseteq E_j[V_j]$.

Define $E = \bigcup_j E_j$. Then $\bigcap_j E[V_j] \supseteq W_k$. Thus there is some U_i and entourage $F \subseteq X^2$ such that $\bigcup_j V_j \subseteq F[U_i]$.

For every $j \in J$:

$$\begin{aligned} W_j &\subseteq E[V_j] \\ &\subseteq E \circ F[U_i]. \end{aligned}$$

Thus $\mathsf{cst}(W_k, \mathcal{W}) \subseteq E \circ F[U_i]$.

We show there is a coarse barycentric refinement $\mathcal{V} = (V_i)_i$ of $\mathcal{U}$: First we show if U_1, U_2 is a coarse cover of X then there is a coarse barycentric refinement V_1, V_2, V_3 of U_1, U_2:

By Lemma 168 there is a coarse cover W_1, W_2 of X such that $W_1 \not\curlywedge U_1^c$ and $W_2 \not\curlywedge U_2^c$.

Then W_1^c, U_1 and W_2^c, U_2 are coarse covers of X.

By Proposition 161 there are $C, D \subseteq X$ such that $C \cap D = \emptyset$, $D^c \not\curlywedge U_2^c$, $C^c \not\curlywedge W_2$.

Also there are $A, B \subseteq X$ such that $A \cap B = \emptyset$, $A^c \not\curlywedge U_1^c$, $B^c \not\curlywedge W_1$.

Then

$$V_1 = W_1, V_2 = C \cap B, V_3 = W_2$$

has the desired properties:

$(V_i)_i$ is a coarse cover:

Note that by $B^c \not\curlywedge W_1$ and $W_1^c \not\curlywedge W_2^c$ the sets W_2, B are a coarse cover of X.

Note that by $C^c \not\curlywedge W_2$ and $W_1^c \not\curlywedge W_2^c$ the sets W_1, C coarsely cover X.

Note that $(W_1 \cap W_2) \not\curlywedge (C^c \cup B^c)$. Then, combining items i,ii, we get that

$$W_1, W_2, B \cap C$$

is a coarse cover as required.

$(V_i)_i$ is a coarse barycentric refinement of U_1, U_2:

There is some entourage $E \subseteq X^2$ such that $V_1 \cup V_2 \subseteq E[U_1]$: For W_1 we use that $W_1 \not\curlywedge U_1^c$. For $C \cap B$ we use that $A^c \not\curlywedge U_1^c$ and $B \subseteq A^c$.

There is an entourage $E \subseteq X^2$ such that $V_2 \cup V_3 \subseteq E[U_2]$: For W_2 we use that $W_2 \not\curlywedge U_2^c$. For $C \cap B$ we use that $D^c \not\curlywedge U_2^c$ and $C \subseteq D^c$.

$V_1 \not\curlywedge V_3$: We use $W_1^c \not\curlywedge W_2^c$.

Now we show the general case: Suppose $U_i \subseteq X$ are subsets such that $\mathcal{U} = (U_i)_i$. We show there is a coarse barycentric refinement $\mathcal{V}$ of $\mathcal{U}$.

For every i the sets $U_i, \bigcup_{j\neq i} U_j$ coarsely cover X. By Lemma 168 there are subsets W_1^i, W_2^i that coarsely cover X such that $W_1^i \not\curlywedge U_i^c$ and $W_2^i \not\curlywedge (\bigcup_{j\neq i} U_j)^c$.

Then there is a coarse barycentric refinement V_1^i, V_2^i, V_3^i of W_1^i, W_2^i for every i.

Then we define

$$\mathcal{V} := (\bigcap_i V^i_{\sigma(i)})_\sigma,$$

here $\sigma(i) \in \{1,2,3\}$ is all possible permutations.

We show $\mathcal{V}$ is a coarse cover of X that is a coarse barycentric refinement of $\mathcal{U}$:

$\mathcal{V}$ is a coarse cover: by design.

$\mathcal{V}$ is a coarse barycentric refinement of $\mathcal{U}$: Suppose there is an entourage $E \subseteq X^2$ and a subindex $(\sigma_k)_k$ such that $\bigcap_{\sigma_k} E[\bigcap_{\sigma_k} V^i_{\sigma_k(i)}]$ is not bounded. Then

$$\bigcap_{i,\sigma_k} E[V^i_{\sigma_k(i)}]$$

is not bounded. Then there is an entourage $F \subseteq X^2$ such that for every i:

$$\bigcup_{\sigma_k} V^i_{\sigma_k(i)} \subseteq F[W^i_{l_i}]$$

where l_i is one of $1,2$. Then

$$\begin{aligned}\bigcup_{\sigma_k}\bigcap_i V^i_{\sigma_k(i)} &\subseteq \bigcap_i \bigcup_{\sigma_k} V^i_{\sigma_k(i)}\\ &\subseteq \bigcap_i F[W^i_{l_i}]\end{aligned}$$

if $l_i = 1$ for one i then we are done. Otherwise

$$\bigcup_{\sigma_k}\bigcap_i V^i_{\sigma_k(i)} \subseteq \bigcap_i F[W_2^i]$$

and $F[W_2^i] \not\curlywedge (\bigcup_{j\neq i} U_j)^c$ implies

$$\bigcap_i F[W_2^i] \not\curlywedge \bigcup_i (\bigcup_{j\neq i} U_j)^c$$

which implies that $\bigcap_i F[W_2^i]$ is bounded, a contradiction. □

5.2 Main Part

5.2.1 Definition

We introduce the space of ends of a coarse space which is a functor E from the category of coarse metric spaces to the category of uniform spaces.

Definition 172. (**endpoint**) Let X be a metric space,

- two coarse maps $\phi, \psi : \mathbb{Z}_+ \to X$ are said to represent the same *endpoint in* X if there is an entourage $E \subseteq X^2$ such that

$$E[\psi(\mathbb{Z}_+)] = \phi(\mathbb{Z}_+).$$

- if $\mathcal{U} = (U_i)_i$ is a coarse cover of X and p, q are two endpoints in X which are represented by $\phi, \psi : \mathbb{Z}_+ \to X$. Then q is said to be in a $\mathcal{U}$*−neighborhood of* p, denoted $q \in \mathcal{U}[p]$, if there is an entourage $E \subseteq X^2$ such that

$$E[\mathsf{cst}(\phi(\mathbb{Z}_+),\mathcal{U})] \supseteq \psi(\mathbb{Z}_+)$$

and

$$E[\mathsf{cst}(\psi(\mathbb{Z}_+),\mathcal{U})] \supseteq \phi(\mathbb{Z}_+).$$

Lemma 173. *If $\mathcal{V} \leq \mathcal{U}$ is a refinement of a coarse cover of a metric space X then for every two endpoints p, q of X the relation $q \in \mathcal{V}[p]$ implies the relation $q \in \mathcal{U}[p]$.*

Proof. Suppose $\mathcal{V} = (V_i)_i$ and $\mathcal{U} = (U_i)_i$. If p, q are represented by $\phi, \psi : \mathbb{Z}_+ \to X$ then

$$\begin{aligned}\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{V}) &= \bigcup\{V_i : \phi(\mathbb{Z}_+) \curlywedge V_i\}\\ &\subseteq \bigcup\{U_i : \phi(\mathbb{Z}_+) \curlywedge U_i\}\\ &= \mathsf{cst}(\phi(\mathbb{Z}_+),\mathcal{U})\end{aligned}$$

in the same way $\mathsf{cst}(\psi(\mathbb{Z}_+), \mathcal{V}) \subseteq \mathsf{cst}(\psi(\mathbb{Z}_+),\mathcal{U})$. Then if $q \in \mathcal{V}[p]$ there is some entourage $E \subseteq X^2$ such that

$$\begin{aligned}\psi(\mathbb{Z}_+) &\subseteq E[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{V})]\\ &\subseteq E[\mathsf{cst}(\phi(\mathbb{Z}_+),\mathcal{U})]\end{aligned}$$

and $\phi(\mathbb{Z}_+) \subseteq E[\mathsf{cst}(\psi(\mathbb{Z}_+),\mathcal{U})]$. Thus $q \in \mathcal{U}[p]$. □

Definition 174. (**space of ends**) Let X be a metric space. As a set the *space of ends* $E(X)$ of X consists of the endpoints in X. A subset $U \subseteq E(X)$ is open if for every $p \in U$ there is a coarse cover $\mathcal{U}$ of X such that

$$\mathcal{U}[p] \subseteq U.$$

This defines a topology on $E(X)$.

Remark 175. The topology on the set of endpoints $E(X)$ is generated by a uniformity: If $\mathcal{U}$ is a coarse cover of X then

$$D_{\mathcal{U}} = \{(p, q) : q \in \mathcal{U}[p]\}$$

is the *entourage associated to* $\mathcal{U}$. Then $(D_{\mathcal{U}})_{\mathcal{U}}$ over coarse covers $\mathcal{U}$ of X are a base for a diagonal uniformity on $E(X)$.

Lemma 176. *If X is a metric space then $E(X)$ is indeed a uniform space. Coarse covers of X give rise to a base for the uniform structure.*

Proof. We check that $(D_{\mathcal{U}})_{\mathcal{U}}$ over coarse covers are a base for a uniformity on $E(X)$:

1. If $\mathcal{U}$ is a coarse cover of X then $\Delta \subseteq D_{\mathcal{U}}$, where $\Delta = \{(p, p) : p \in E(X)\}$: $p \in \mathcal{U}[p]$.

2. If $\mathcal{U}, \mathcal{V}$ are coarse covers of X then $D_{\mathcal{U}} \cap D_{\mathcal{V}}$ is an entourage: Suppose $\mathcal{U} = (U_i)_i, \mathcal{V} = (V_i)_i$ then define

$$\mathcal{U} \cap \mathcal{V} := (U_i \cap V_j)_{ij}.$$

Suppose p, q are represented by $\phi, \psi : \mathbb{Z}_+ \to X$. Then $q \in (\mathcal{U} \cap \mathcal{V})[p]$ implies

$$\begin{aligned}\psi(\mathbb{Z}_+) &\subseteq E[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{U} \cap \mathcal{V})]\\ &= E[\bigcup\{U_i \cap V_i : \phi(\mathbb{Z}_+) \curlywedge U_i \cap V_j\}\\ &\subseteq \bigcup_{U_i \cap V_j \curlywedge \phi(\mathbb{Z}_+)} E[U_i] \cap E[V_j]\\ &\subseteq (\bigcup_{U_i \curlywedge \phi(\mathbb{Z}_+)} E[U_i]) \cap (\bigcup_{V_i \curlywedge \phi(\mathbb{Z}_+)} E[V_i])\\ &= E[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{U})] \cap E[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{V})]\end{aligned}$$

In the same way $\phi(\mathbb{Z}_+) \subseteq E[\mathsf{cst}(\psi(\mathbb{Z}_+), \mathcal{U})] \cap E[\mathsf{cst}(\psi(\mathbb{Z}_+), \mathcal{V})]$. Thus $q \in \mathcal{U}[p] \cap \mathcal{V}[p]$. This way we have proven:

$$D_{\mathcal{U} \cap \mathcal{V}} \subseteq D_{\mathcal{U}} \cap D_{\mathcal{V}}.$$

3. If $\mathcal{U}$ is a coarse cover of X then there is a coarse cover $\mathcal{V}$ of X such that $D_{\mathcal{V}} \circ D_{\mathcal{V}} \subseteq D_{\mathcal{U}}$: By Lemma 171 there is a coarse star refinement $\mathcal{V}$ of $\mathcal{U}$. And by Lemma 183 item 2 the uniform cover $(\mathcal{V}[p])_p$ star refines the uniform cover $(\mathcal{U}[p])_p$ thus the result.

4. If $\mathcal{U}$ is a coarse cover then $D_{\mathcal{U}} = D_{\mathcal{U}}^{-1}$.

A subset $D \subseteq E(X)^2$ is an entourage of the uniform structure of $E(X)$ if there is a coarse cover $\mathcal{U}$ of X such that

$$D_{\mathcal{U}} \subseteq D.$$

□

Theorem 177. *If $f : X \to Y$ is a coarse map between metric spaces then the induced map*

$$\begin{aligned}E(f) : E(X) &\to E(Y)\\ [\varphi] &\mapsto [f \circ \varphi]\end{aligned}$$

is a continuous map between topological spaces.

Proof. We show $E(f)$ is well defined: if $\phi, \psi : \mathbb{Z}_+ \to X$ represent the same endpoint in X then there is some entourage $E \subseteq X^2$ such that $E[\psi(\mathbb{Z}_+)] = \phi(\mathbb{Z}_+)$. But then

$$\begin{aligned}f^2(E)[f \circ \psi(\mathbb{Z}_+)] &\supseteq f(E[\psi(\mathbb{Z}_+)])\\ &= f \circ \phi(\mathbb{Z}_+).\end{aligned}$$

Thus $f \circ \phi, f \circ \psi$ represent the same endpoint in Y.

We show $E(f)$ continuous: For that we show that the reverse image of an open set is an open set.

Let $U \subseteq E(Y)$ be open and $p \in E(f)^{-1}(U)$ be a point. Suppose that p is represented by a coarse map $\phi : \mathbb{Z}_+ \to X$. Then $f \circ \phi$ represents $E(f)(p) \in U$. Now there is a coarse cover $\mathcal{U} = (U_i)_i$ of Y such that $\mathcal{U}[E(f)(p)] \subseteq U$. Then $f^{-1}(\mathcal{U}) = (f^{-1}(U_i))_i$ is a coarse cover of X.

If $q \in f^{-1}(\mathcal{U})[p]$ we show that $E(f)(q) \in \mathcal{U}[E(f)(p)]$: Suppose that q is represented by a coarse map $\psi : \mathbb{Z}_+ \to X$. Then there is some entourage $F \subseteq X^2$ such that

$$F[\mathsf{cst}(\phi(\mathbb{Z}_+), f^{-1}(\mathcal{U}))] \supseteq \psi(\mathbb{Z}_+)$$

and

$$F[\mathsf{cst}(\psi(\mathbb{Z}_+), f^{-1}(\mathcal{U}))] \supseteq \phi(\mathbb{Z}_+).$$

By Lemma 170:

$$\begin{aligned} f^2(F)[\mathsf{cst}(f \circ \phi(\mathbb{Z}_+),\mathcal{U})] &\supseteq f^2(F)[f(\mathsf{cst}(\phi(\mathbb{Z}_+), f^{-1}(\mathcal{U})))]) \\ &\supseteq f(F[\mathsf{cst}(\phi(\mathbb{Z}_+), f^{-1}(\mathcal{U}))]) \\ &\supseteq f \circ \psi(\mathbb{Z}_+) \end{aligned}$$

and

$$\begin{aligned} f^2(F)[\mathsf{cst}(f \circ \psi(\mathbb{Z}_+),\mathcal{U})] &\supseteq f^2(F)[f(\mathsf{cst}(\psi(\mathbb{Z}_+), f^{-1}(\mathcal{U})))]) \\ &\supseteq f(F[\mathsf{cst}(\psi(\mathbb{Z}_+), f^{-1}(\mathcal{U}))]) \\ &\supseteq f \circ \phi(\mathbb{Z}_+). \end{aligned}$$

Now $f \circ \psi$ represents $E(f)(q)$ which by the above is in $\mathcal{U}[E(f)(p)]$. □

Remark 178. The proof of Theorem 177 uses the following: if $f : X \to Y$ is a coarse map and $D_{\mathcal{U}}$ the entourage of $E(Y)$ associated to a coarse cover $\mathcal{U}$ of Y then there is an entourage $D_{f^{-1}(\mathcal{U})}$ of $E(X)$ associated to the coarse cover $f^{-1}(\mathcal{U})$ of X such that $(p, q) \in D_{f^{-1}(\mathcal{U})}$ implies $(E(f)(p), E(f)(q)) \in D_{\mathcal{U}}$. Thus $E(f)$ is a uniformly continuous map between uniform spaces $E(X)$ and $E(Y)$.

Lemma 179. *If two coarse maps $f, g : X \to Y$ are close then $E(f) = E(g)$.*

Proof. Let $p \in E(X)$ be a point that is represented by φ. Now f, g are close thus $H := f \times g(\Delta_X)$ is an entourage. But then

$$H[g \circ \varphi(\mathbb{Z}_+)] \supseteq f \circ \varphi(\mathbb{Z}_+)$$

thus $E(f)(p) = E(g)(p)$. □

Corollary 180. *If f is a coarse equivalence then $E(f)$ is a homeomorphism between topological spaces $E(X)$ and $E(Y)$. In fact $E(f)$ is a uniform isomorphism between uniform spaces $E(X)$ and $E(Y)$.*

Corollary 181. *If $\mathtt{mCoarse}$ denotes the category of metric spaces and coarse maps modulo closeness and $\mathtt{Top}$ the category of topological spaces and continuous maps then E is a functor*

$$E : \mathtt{mCoarse} \to \mathtt{Top}.$$

If $\mathtt{Uniform}$ denotes the category of uniform spaces and uniformly continuous maps then E is a functor

$$E : \mathtt{mCoarse} \to \mathtt{Uniform}.$$

Example 182. $E(\mathbb{Z}_+)$ is a point.

5.2.2 Properties

Lemma 183. *If X is a metric space*

- *and $\mathcal{V}$ is a coarse star refinement of a coarse cover $\mathcal{U}$ of X then $q \in \mathcal{V}[p]$ and $r \in \mathcal{V}[q]$ implies $r \in \mathcal{U}[p]$.*

- *if $\mathcal{V}$ coarsely star refines $\mathcal{U}$ then $(\mathcal{V}[p])_p$ star refines $(\mathcal{U}[p])_p$*

Proof.
- Suppose p is represented by $\phi : \mathbb{Z}_+ \to X$, q is represented by $\psi : \mathbb{Z}_+ \to X$ and r is represented by $\rho : \mathbb{Z}_+ \to X$. Then $E[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{V}) \supseteq \psi(\mathbb{Z}_+)$ and $E[\mathsf{cst}(\psi(\mathbb{Z}_+), \mathcal{V}) \supseteq \phi(\mathbb{Z}_+)$, $E[\mathsf{cst}(\rho(\mathbb{Z}_+), \mathcal{V}) \supseteq \psi(\mathbb{Z}_+)$ and $E[\mathsf{cst}(\psi(\mathbb{Z}_+), \mathcal{V}) \supseteq \rho(\mathbb{Z}_+)$. By Lemma 170 there is an entourage $F \subseteq X^2$ such that $\mathsf{cst}(\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{V}), \mathcal{V}) \subseteq F[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{U})]$. Then

$$\begin{aligned} E^{\circ 2} \circ F[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{U})] &\supseteq E^{\circ 2}[\mathsf{cst}(\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{V}), \mathcal{V})] \\ &\supseteq E[\mathsf{cst}(\psi(\mathbb{Z}_+), \mathcal{V})] \\ &\supseteq \rho(\mathbb{Z}_+) \end{aligned}$$

  the other direction works the same way.

- Fix $p \in E(X)$. Then

$$st(\mathcal{V}[p], (\mathcal{V}[p])_p) \subseteq \mathcal{U}[p]$$

  because if $q \in \mathcal{V}[p]$ and $q \in \mathcal{V}[r]$ then $r \in \mathcal{U}[p]$ by Item 1

□

Proposition 184. *If $i : Z \to Y$ is an inclusion of metric spaces then $E(i) : E(Z) \to E(Y)$ is a uniform embedding.*

Proof. That $E(i)$ is injective is easy to see.

Define a map

$$\begin{aligned} \Phi : E(i)(E(Z)) &\to E(Z) \\ E(i)(p) &\mapsto p. \end{aligned}$$

We show Φ is a uniformly continuous map:

If $\mathcal{U} = (U_i)_i$ is a coarse cover of Z we show there is a coarse cover $\mathcal{V}$ of Y such that for every $p, q \in E(Z)$: the relation $E(i)(q) \in \mathcal{V}[E(i)(p)]$ implies $q \in \mathcal{U}[p]$.

Note that for every i the sets $U_i^c \not\curlywedge (\bigcup_{j \neq i} U_j)^c$ are coarsely disjoint in Y. By Lemma 168 there are subsets $W_1^i, W_2^i \subseteq Y$ that coarsely cover Y and $W_1^i \not\curlywedge U_i^c$ and $W_2^i \not\curlywedge (\bigcup_{j \neq i} U_j)^c$. Now define

$$\mathcal{V} := (\bigcap_i W^i_{\sigma(i)})_\sigma$$

with $\sigma(i) \in \{1, 2\}$ all possible permutations. Note that there is some entourage $E \subseteq Z^2$ such that for every σ there is some U_j such that

$$\bigcap_i W^i_{\sigma(i)} \cap Z \subseteq E[U_j].$$

Let $p, q \in E(Z)$ such that $E(i)(q) \in \mathcal{V}[E(i)(p)]$. Suppose p, q are represented by $\phi, \psi : \mathbb{Z}_+ \to Z$. Then there is some entourage $F \subseteq Y^2$ such that

$$F[\mathsf{cst}(\phi(\mathbb{Z}_+), \mathcal{V})] \supseteq \psi(\mathbb{Z}_+)$$

and

$$F[\mathsf{cst}(\psi(\mathbb{Z}_+), \mathcal{V})] \supseteq \phi(\mathbb{Z}_+).$$

Then

$$\begin{aligned} F \circ E[\mathsf{cst}(\phi(\mathbb{Z}_+),\mathcal{U})] &\supseteq F[\mathsf{cst}(\phi(\mathbb{Z}_+),\mathcal{V}) \cap Z] \\ &\supseteq \psi(\mathbb{Z}_+). \end{aligned}$$

The other direction works the same way.
Then $q \in \mathcal{U}[p]$ as we wanted to show. □

Remark 185. By Proposition 184 and Corollary 180 every coarsely injective coarse map $f : X \to Y$ induces a uniform embedding. We identify $E(X)$ with its image $E(f)(E(X))$ in $E(Y)$.

Example 186. There is a coarsely surjective coarse map $\omega : \mathbb{Z}_+ \to \mathbb{Z}^2$. Now $E(\omega) : E(\mathbb{Z}_+) \to E(\mathbb{Z}^2)$ is not a surjective map obviously.

Lemma 187. *If $f : X \to Y$ is a coarse map between metric spaces, Y is a coarsely geodesic coarsely proper metric and $E(f) : E(X) \to E(Y)$ is surjective then f is already coarsely surjective.*

Proof. Assume the opposite. Then $(\operatorname{im} f)^c \subseteq Y$ contains a countable subset $(s_i)_i$ that is coarsely disjoint to $\operatorname{im} f$. Then by [13, Proposition 93] there is a coarse ray $\rho : \mathbb{Z}_+ \to Y$ such that there is an unbounded subsequence $(s_{i_k})_k$ and an entourage $E \subseteq Y^2$ such that

$$(s_{i_k})_k \subseteq E[\rho(\mathbb{Z}_+)]$$

Now ρ represents a point $r \in E(Y)$ and $E(f)$ is surjective. Thus there is some $p \in E(X)$ such that $E(f)(p) = r$. Suppose p is represented by $\varphi : \mathbb{Z}_+ \to X$ then $f \circ \varphi$ represents r and has image in $\operatorname{im} f$. Thus there is an entourage $F \subseteq Y^2$ such that $\rho(\mathbb{Z}_+) \subseteq F[\operatorname{im} f]$. Then

$$(s_{i_k})_k \subseteq E \circ F[\operatorname{im} f]$$

a contradiction to the assumption. □

Remark 188. Note that the coarse map

$$\begin{aligned} \mathbb{Z}_+ &\to \mathbb{Z}_+ \\ n &\mapsto \lfloor \sqrt{n} \rfloor \end{aligned}$$

is not coarsely injective. since every map $E(\mathbb{Z}_+) \to E(\mathbb{Z}_+)$ is an isomorphism we cannot conclude that the functor $E(\cdot)$ reflects isomorphisms.

Lemma 189. *If two subsets U, V coarsely cover a metric space X then*

$$E(U \cap V) = E(U) \cap E(V).$$

Proof. The inclusion $E(U \cap V) \subseteq E(U) \cap E(V)$ is obvious.

We show the reverse inclusion: if $p \in E(U) \cap E(V)$ then it is represented by $\phi : \mathbb{Z}_+ \to U$ in $E(U)$ and $\psi : \mathbb{Z}_+ \to V$ in $E(V)$. Then there is an entourage $E \subseteq X^2$ such that

$$E[\psi(\mathbb{Z}_+)] = \phi(\mathbb{Z}_+).$$

Note that $E \cap V^c \times U^c$ is bounded. Denote by F the set of indices i, j for which

$$E \cap (\phi(i), \psi(j)) \subseteq V^c \times U^c.$$

Now we construct a coarse map $\varphi : \mathbb{Z}_+ \to U \cap V$: for every $i \in \mathbb{N} \setminus F$ do:

1. if $\phi(i) \in V$ then define $\varphi(i) := \phi(i)$;
2. if $\phi(i) \in V^c$ then define $\varphi(i) := \psi(i)$.

Fix a point $x_0 \in U \cap V$ then for every $i \in F$ define: $\varphi(i) = x_0$. Then φ represents p in $E(U \cap V)$. □

Lemma 190. *The functor $E(\cdot)$ preserves finite coproducts.*

Proof. Let $X = A \sqcup B$ be a coarse disjoint union of metric spaces. Without loss of generality we assume that A, B cover X as sets. Fix a point $x_0 \in X$. Then there is a coarse map

$$r : X \to \mathbb{Z}$$
$$x \mapsto \begin{cases} d(x, x_0) & x \in A \\ -d(x, x_0) & x \in B. \end{cases}$$

Note that $E(\mathbb{Z}) = \{-1, 1\}$ is a space which consists of two points with the discrete uniformity. Then $E(r)(E(A)) = 1$ and $E(r)(E(B)) = -1$. Thus $E(X)$ is the uniform disjoint union of $E(A), E(B)$. □

Proposition 191. *Let X be a metric space. The uniformity $E(X)$ is separated.*

Proof. If $p \neq q$ are two points in $E(X)$ we show there is a coarse cover $\mathcal{U}$ such that

$$q \notin \mathcal{U}[p].$$

Suppose p is represented by $\phi : \mathbb{Z}_+ \to X$ and q is represented by $\psi : \mathbb{Z}_+ \to X$. Now there is one of two cases:

1. there is a subsequence $(i_k)_k \subseteq \mathbb{N}$ such that $\phi(i_k)_k \not\curlywedge \psi(\mathbb{Z}_+)$.
2. there is a subsequence $(j_k)_k \subseteq \mathbb{N}$ such that $\psi(j_k)_k \not\curlywedge \phi(\mathbb{Z}_+)$.

Without loss of generality we can assume the first case holds. By Lemma 168 there is a coarse cover $\mathcal{U} = \{U_1, U_2\}$ of X such that $U_1 \not\curlywedge \phi(i_k)_k$ and $U_2 \not\curlywedge \psi(\mathbb{Z}_+)$. Then $q \notin \mathcal{U}[p]$.

Now

$$\begin{aligned} q &\notin \mathcal{U}[p] \\ &= st(p, (\mathcal{U}[r])_r). \end{aligned}$$

Thus the result. □

Lemma 192. *If X is a metric space,*

- *$\mathcal{U} = (U_i)_{i \in I}$ is a coarse cover of X and $p \in E(X)$ is represented by $\phi : \mathbb{Z}_+ \to X$ then define*

  $$I(p) := \{i \in I : \phi(\mathbb{Z}_+) \curlywedge U_i\}.$$

  If $S \subseteq I$ is a subset then define

  $$U(S) = \{p \in E(X) : \phi(\mathbb{Z}_+) \subseteq E[\bigcup_{i \in S} U_i]\}$$

  here $\phi : \mathbb{Z}_+ \to X$ represents p and $E \subseteq X^2$ is an entourage. If $q \in E(X)$ then $q \in \mathcal{U}[p]$ if and only if $q \in U(I(p))$ and $p \in U(I(q))$.

- *Define*
$$\mathcal{U}(S) = \{p \in E(X) : p \in U(S), I(p) \supseteq S\}.$$
Then $q \in \mathcal{U}[p]$ if and only if there is some $S \subseteq I$ such that $p, q \in \mathcal{U}(S)$. The uniform cover
$$(\mathcal{U}(S))_{S \subseteq I}$$
associated to $D_{\mathcal{U}}$ is a finite cover.
- *The uniform space $E(X)$ is totally bounded.*

Proof.
- easy.
- We just need to show: if $q \in \mathcal{U}[p]$ then $p \in \mathcal{U}(I(p) \cap I(q))$. For that it is sufficient to show if $\phi : \mathbb{Z}_+ \to X$ represents p then there is an entourage $E \subseteq X^2$ such that $\phi(\mathbb{Z}_+) \subseteq E[\bigcup_{i \in I(q)} U_i]$. Assume the opposite: there is some subsequence $(i_k)_k \subseteq \mathbb{Z}_+$ such that
$$\phi(i_k)_k \not\curlywedge \bigcup_{i \in I(p) \cap I(q)} U_i.$$
Now $\phi(\mathbb{Z}_+) \not\curlywedge \bigcup_{i \notin I(p)} U_i$ thus
$$\phi(i_k)_k \not\curlywedge (\bigcup_{i \in I(p) \cap I(q)} U_i) \cup (\bigcup_{i \notin I(p)} U_i).$$
And thus $\phi(i_k)_k \not\curlywedge \bigcup_{i \in I(q)} U_i$ a contradiction to the assumption.
- easy

$\square$

Notation 193. If $A, B \subseteq X$ are two subsets of a metric space

- and $x_0 \in X$ a point then define
$$\begin{aligned} \chi_{A,B} : \mathbb{N} &\to \mathbb{R}_+ \\ i &\mapsto d(A \setminus B(x_0, i), B \setminus B(x_0, i)) \end{aligned}$$
if $A \not\curlywedge B$ then $\chi_{A,B}$ is a coarse map. There is a bound
$$\chi_{A,B}(i) \leq 2i.$$
- Now $A \curlywedge B$ if and only if $\chi_{A,B}$ is bounded.
- If $A_1, A_2 \subseteq X$ are subsets with $A_1 = E[A_2]$ then
$$i \mapsto |\chi_{A_1,B}(i) - \chi_{A_2,B}(i)|$$
is bounded.
- If $\chi \in \mathbb{R}_+^{\mathbb{N}}$ is a coarse map then the *class* $m(\chi)$ *of* χ is at least $f \in \mathbb{R}_+^{\mathbb{N}}$ if
$$\chi(i) \geq f(i) + c$$
where $c \leq 0$ is a constant. If two coarse maps $\chi_1, \chi_2 \in \mathbb{R}_+^{\mathbb{N}}$ are close then they have the same class.

Proof.

- easy.
- See [17, Proposition 4.4].
- easy.
- easy.

□

Definition 194. Let X be a metric space. If two endpoints $p, q \in E(X)$ are represented by coarse maps $\phi, \psi : \mathbb{Z}_+ \to X$ then the distance of p to q is at least $f \in \mathbb{R}_+^{\mathbb{N}}$, written $d(p,q) \geq f$, if there is a subsequence $(i_k)_k \subseteq \mathbb{Z}_+$ such that one of the following holds

1. $\phi(\mathbb{Z}_+) \not\curlywedge \psi(i_k)_k$ and $m(\chi_{\phi(\mathbb{Z}_+),\psi(i_k)_k}) \geq f$;
2. $\psi(\mathbb{Z}_+) \not\curlywedge \phi(i_k)_k$ and $m(\chi_{\psi(\mathbb{Z}_+),\phi(i_k)_k}) \geq f$.

We define $d(p,q) = 0$ if and only if $p = q$.

Lemma 195. *If X is a metric space and $\mathcal{U}$ a coarse cover of X then there is a function $f \in \mathbb{R}_+^{\mathbb{N}}$ with $m(f) \neq 0$ such that for every two endpoints $p, q \in E(X)$ the relation $q \notin \mathcal{U}[p]$ implies $d(p,q) \geq f$.*

Proof. By Lemma 192 the uniform space $E(X)$ is totally bounded. Without loss of generality we can fix an endpoint $p \in E(X)$ and study the endpoints $q \in E(X)$ for which $q \notin \mathcal{U}[p]$.

We will define a function $f \in \mathbb{R}_+^{\mathbb{N}}$ as the minimum of a finite collection $f_0, \ldots, f_n \geq 0$ of numbers.

If $q \notin \mathcal{U}[p]$ there are 2 cases:

1. $p \notin U(I(q))$: There is a subset $(i_k)_k \subseteq \mathbb{Z}_+$ such that
$$\phi(i_k)_k \not\curlywedge \bigcup_{i \in I(q)} U_i.$$
Now for every $S \subseteq I$ if $p \notin U(S)$: then there is some subset $(i_k)_k \subseteq \mathbb{Z}_+$ such that $\phi(i_k)_k \not\curlywedge \bigcup_{i \in S} U_i$. Thus if $I(q) = S$ then $d(p,q) > m(\chi_{\phi(i_k)_k, \bigcup_{i \in S} U_i})$. Define
$$f_S := m(\chi_{\phi(i_k)_k, \bigcup_{i \in S} U_i})$$
in this case.
2. $q \notin U(I(p))$: There is some $(i_k)_k \subseteq \mathbb{Z}_+$ such that
$$\psi(i_k)_k \not\curlywedge \bigcup_{i \in I(p)} U_i.$$
Now $\psi(i_k)_k \subseteq \bigcup_{i \notin I(p)} U_i$ and $\phi(\mathbb{Z}_+) \not\curlywedge \bigcup_{i \notin I(p)} U_i$. Then define
$$f_b := m(\chi_{\phi(\mathbb{Z}_+), \bigcup_{i \notin I(p)} U_i}).$$

Then
$$f := \min_S (f_S, f_b)$$
has the desired properties. □

Proposition 196. *If X is a metric space then the uniformity on $E(X)$ is coarser than the uniformity τ_d, induced by d.*

Proof. By Lemma 195 every entourage in $E(X)$ is a neighborhood of an entourage of τ_d on $E(X)$. □

5.2.3 Side Notes

Remark 197. (**large-scale category**) Large-scale geometry[1] (`LargeScale`) studies metric spaces and large-scale maps modulo closeness. Note the following facts:

1. Every large-scale map is already coarsely uniform.
2. Isomorphisms in `LargeScale` are called quasi-isometries.
3. A metric space is coarsely geodesic if and only if it is coarsely equivalent to a geodesic metric space.
4. A metric space is large-scale geodesic if and only if it is quasi-isometric to a geodesic metric space.
5. A coarse map $f : X \to Y$ between large-scale geodesic metric spaces is already large-scale.
6. A coarse equivalence $f : X \to Y$ between large-scale geodesic metric spaces is already a quasi-isometry.

Proof. 1. easy.

2. Definition.
3. See [26, Lemma 3.B.6,(5)];
4. see [26, Lemma 3.B.6,(6)];
5. see [26, Proposition 3.B.9,(1)];
6. see [26, Proposition 3.B.9,(2)].

$\square$

Lemma 198. ***(Higson corona)*** *If X is a metric space then the C^*-algebra that determines the Higson corona is a sheaf. That means exactly that the association*

$$U \mapsto C(\nu U) = B_h(U)/B_0(U)$$

for every subset $U \subseteq X$ is a sheaf with values in `CStar`*. By a sheaf we mean a sheaf on the Grothendieck topology determined by coarse covers on subsets of a coarse space.*

Proof. We recall a few definitions which can be found in [4, p.29,30].

- The algebra of bounded functions that satisfy the Higson condition is denoted by B_h.
- A bounded function $f : X \to \mathbb{C}$ satisfies the Higson condition if for every entourage $E \subseteq X^2$ the function

  $$\begin{aligned} df|_E : E &\to \mathbb{C} \\ (x, y) &\mapsto f(y) - f(x) \end{aligned}$$

  tends to 0 at infinity.
- the ideal of bounded functions that tend to 0 at infinity is called B_0.

[1] The notation is from [26]

- A function $f : X \to \mathbb{C}$ tends to 0 at infinity if for every $\varepsilon > 0$ there is a bounded subset $B \subseteq X$ such that $|f(x)| \geq \epsilon$ implies $x \in B$.

We check the sheaf axioms:

1. global axiom: if U_1, U_2 coarsely cover a subset $U \subseteq X$ and $f \in B_h(X)$ such that $f|_{U_1} \in B_0(U_1)$ and $f|_{U_2} \in B_0(U_2)$ we show that $f \in B_0(U)$ already. Let $\varepsilon > 0$ be a number. Then there are bounded subsets $B_1 \subseteq U_1$ and $B_2 \subseteq U_2$ such that $|f|_{U_i}(x)| \geq \varepsilon$ implies $x \in B_i$ for $i = 1, 2$. Now
$$B := B_1 \cup B_2 \cup (U_1 \cup U_2)^c$$
is a bounded subset of U. Then $|f(x)| \geq \varepsilon$ implies $x \in B$. Thus $f \in B_0(U)$.

2. gluing axiom: if U_1, U_2 coarsely cover a subset $U \subseteq X$ and $f_1 \in B_h(U_1), f_2 \in B_h(U_2)$ are functions such that
$$f_1|_{U_2} = f_2|_{U_1} + g$$
where $g \in B_0(U_1 \cap U_2)$. We show there is a function $f \in B_h(U)$ which restricts to f_1 on U_1 and $f_2 + g$ on U_2. Define:
$$\begin{aligned} f : U &\to \mathbb{C} \\ x &\mapsto \begin{cases} f_1(x) & x \in U_1 \\ f_2(x) + g & x \in U_2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$
then f is a bounded function. We show f satisfies the Higson condition: Let $E \subseteq U^2$ be an entourage and $\varepsilon > 0$ be a number. Then there are bounded subsets $B_1 \subseteq U_1$ and $B_2 \subseteq U_2$ such that $|df_i|_{E \cap U_i^2}(x, y)| \geq \varepsilon$ implies $x \in B_i$ for $i = 1, 2$. There is a bounded subset $B_3 \subseteq U$ such that
$$E \cap (U_1^2 \cup U_2^2)^c \cap U^2 \subseteq B_3^2.$$
Define
$$B := B_1 \cup B_2 \cup B_3$$
then $|df|_E(x)| \geq \varepsilon$ implies $x \in B$. Thus f has the desired properties.

□

Lemma 199. *If X is a proper geodesic metric space denote by $\sim$ the relation on $E(X)$ of belonging to the same uniform connection component in $E(X)$ then there is a continuous bijection*
$$E(X)/\sim \to \Omega(X)$$
where the right side denotes the space of ends of X as a topological space.

Proof. There are several different definitions for the space of ends of a topological space. We use [35, Definition 8.27].

An end in X is represented by a proper continuous map $r : [0, \infty) \to X$. Two such maps r_1, r_2 represent the same end if for every compact subset $C \subseteq X$ there is some $N \in \mathbb{N}$ such that $r_1[N, \infty), r_2[N, \infty)$ are contained in the same path component of $X \setminus C$.

If $r : [0, \infty) \to X$ is an end then there is a coarse map $\varphi : \mathbb{Z}_+ \to X$ and an entourage $E \subseteq X^2$ such that
$$E[r[0, \infty)] = \varphi(\mathbb{Z}_+).$$
We construct φ inductively:

1. $\varphi(0) := r(0)$
2. if $\varphi(i-1) = r(t_{i-1})$ is already defined then $t_i := \min\{t > t_{i-1} : d(\varphi(t_{i-1}), \varphi(t)) = 1\}$. Set $\varphi(i) := r(t_i)$.

By the above construction φ is coarsely uniform. The map φ is coarsely proper because r is proper and X is proper.

Note that every geodesic space is also a length space. If for some compact subset $C \subseteq X$ the space $X \setminus C$ has two path components X_1, X_2 then for every $x_1 \in X_1, x_2 \in X_2$ a path (in particular the shortest) joining x_1 to x_2 contains a point $c \in C$. Thus

$$d(x_1, x_2) = \inf_{c \in C} (d(x_1, c) + d(x_2, c))$$

Then X is the coarse disjoint union of X_1, X_2. On the other hand if X is the coarse disjoint union of subspaces X_1, X_2 then there is a bounded and in particular because X is proper compact subset $C \subseteq X$ such that

$$X \setminus C = X_1' \sqcup X_2'$$

is a path disjoint union and $X_1' \subseteq X_1, X_2' \subseteq X_2$ differ only by bounded sets.

Now we show the association is continuous:

We use [35, Lemma 8.28] in which $\mathcal{G}_{x_0}(X)$ denotes the set of geodesic rays issuing from $x_0 \in X$. Then [35, Lemma 8.28] states that the canonical map

$$\mathcal{G}_{x_0} \to \Omega(X)$$

is surjective. Fix $r \in \mathcal{G}_{x_0}$. Then $\tilde{V}_n \subseteq \mathcal{G}_{x_0}$ denotes the set of proper rays $r' : \mathbb{R}_+ \to X$ such that $r'(n, \infty), r(n, \infty)$ lie in the same path component of $X \setminus B(x_0, n)$. Now [35, Lemma 8.28] states the sets $(V_n = \{[r'] : r' \in \tilde{V}_n\})_n$ form a neighborhood base for $[r] \in \Omega(X)$.

Now to every n we denote by U_1^n the path component of $X \setminus B(x_0, n)$ that contains $r(\mathbb{R}_+)$ and we define $U_2^n := X \setminus U_1^n$. For every $n \in \mathbb{N}$ the sets U_1^n, U_2^n are a coarse cover of X.

Suppose $\rho : \mathbb{Z}_+ \to X$ is a coarse map associated to r and represents $\tilde{r} \in E(X)$. If $s \in \mathcal{G}_0$ suppose $\sigma : \mathbb{Z}_+ \to X$ is the coarse map associated to s and represents $\tilde{s} \in E(X)$. If $[s] \notin V_n$ then $\sigma(\mathbb{Z}_+) \not\prec U_1^n$. This implies $\tilde{s} \notin \{U_1^n, U_2^n\}[\tilde{r}]$.

Thus for every $n \in \mathbb{N}$ there is an inclusion $\{U_1^n, U_2^n\}[\tilde{r}]/\sim \subseteq V_n$ by the association. □

5.3 Remarks

The starting point of this research was an observation in the studies of [13]: coarse cohomology with twisted coefficients looked like singular cohomology on some kind of boundary. We tried to find a functor from the coarse category to the category of topological spaces that would reflect that observation.

And then we noticed that two concepts play an important role: One is the choice of topology on the *space of ends* and one is the choice of points. The points were designed such that

- coarse maps are mapped by the functor to maps of sets
- and the space $\mathbb{Z}_+$ is mapped to a point

If the metric space is Gromov hyperbolic then coarse rays represent the points of the Gromov boundary, thus the Gromov boundary is a subset of the space of ends. The topology was trickier to find. We looked for the following properties:

- coarse maps are mapped to continuous maps
- coarse embeddings are mapped to topological embeddings

Now a proximity relation on subsets of a topological space helps constructing the topology on the space of ends of Freudenthal. We discovered that coarse covers on metric spaces give rise to a totally bounded uniformity and thus used that a uniformity on a space gives rise to a topology.

Finally, after a lucky guess, we came up with the uniformity on the set of endpoints. In which way does the space of ends functor reflect isomorphism classes will be studied in a paper that follows.

It would be possible, conversely, after a more thorough examination to find more applications. Coarse properties on metric spaces may give rise to topological properties on metrizable uniform spaces.

We wonder if this result will be of any help with classifying coarse spaces up to coarse equivalence. However, as of yet, the duality has not been studied in that much detail.

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