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Harmonic analysis on 2-step stratified Lie groups without the Moore-Wolf condition

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Abstract

In this thesis we investigate harmonic analysis on a particular class of sub-Riemannian manifold, namely the 2-step stratified Lie groups $\mathbb{G}$, as well as its applications in partial differential equations. This class consists a breadth of interesting geometric objects such as Heisenberg group and H-type Lie group, which can be seen as a meaningful extension of classical theories.

After reviewing some main definitions and properties in Chapter 2, we start to study the most important representation of $\mathbb{G}$, the so-called Schrödinger representation on $L^2(\mathbb{G})$, and then we prove the Stone-von Neumann theorem for the 2-step stratified Lie groups.

In Chapter 3 we also study the Fourier transforms and define the (λ, ν)-Wigner and (λ, ν)-Weyl transform related to $\mathbb{G}$, we then show some properties of these transforms, which can help us to compute the sub-Laplacian and the λ-twisted sub-Laplacian. Moreover, in this chapter we demonstrate the beautiful interplay between the representation theory on $\mathbb{G}$ and the classical expansions in terms of Hermite functions and Lagueere functions.

As applications, a global calculus of pseudo-differential operator on 2-step stratified Lie groups $\mathbb{G}$ is introduced in the fourth chapter. It relies on the explicit knowledge of the irreducible unitary representations of $\mathbb{G}$, which then allows one to reduce the analysis to study of a rescaled harmonic oscillator on unitary dual $\hat{\mathbb{G}}$. The sub-Laplacian appears as an elliptic operator in this calculus. The explicit formula for the heat kernel of the λ-twisted sub-Laplacian can be also obtained, which gives a closed formula for the heat kernel of the sub-Laplacian on $\mathbb{G}$.

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Contents

1 Introduction

1.1 Background

Harmonic analysis on nilpotent Lie groups is by now classical matter that goes back to the first half of the 20th century (see e.g.[CG90; Rud90] for a self-contained presentation). As is known to all, harmonic analysis on nilpotent Lie groups plays a power role in contemporary investigations of linear PDEs. In fact, it has been realised for a long time that the analysis on nilpotent Lie groups can be effectively used to prove subelliptic estimates for operators such as sums of squares of vector fields on manifolds. Such ideas started coming to light in the works on the construction of parametrices for the Kohn-Laplacian $\Box_b$ (the Laplacian associated to the tangential CR(Cauchy-Riemann) complex on the boundary X of a strictly pseudoconvex domain), which was shown earlier by J. J. Kohn to be hypoelliptic (see e.g. an exposition by Kohn [Koh73] on the analytic and smooth hypoellipticities).

Thus, the corresponding parametrices and subsequent subelliptic estimates have been obtained by Folland and Stein in [FS74] by first establishing a version of the results for a family of sub-Laplacians on the Heisenberg group, and then for the Kohn-Laplacian $\Box_b$ by replacing X locally by the Heisenberg group.

These ideas soon led to powerful generalisations. The general techniques for approximating vector fields on a manifold by left-invariant operators on a nilpotent Lie group have been developed by Rothschild and Stein in [RS76]. A more geometric version of these constructions has been carried out by Folland in [Fol77], see also Goodman [Goo76] for the presentation of nilpotent Lie algebras as tangent spaces (of sub-Riemannian manifolds). The functional analytic background for the analysis in the stratified setting was laid down by Folland in [Fol75]. A general approach to studying geometries appearing from systems of vector fields has been developed by Nigel, Stein and Wainger [NSW85]. Furthermore, in their fundamental book [FS82], Folland and Stein laid down foundations for the anisotropic analysis on general homogeneous groups, i.e., Lie groups equipped with a compatible family of dilations. Such groups are necessarily nilpotent, and the realm of homogeneous groups almost exhausts the whole class of nilpotent Lie groups, including the classes of stratified, and more generally, graded groups. Many other important results have also been obtained, see [FR16; Tay86] and their references.

Consider a locally compact Abelian group $(\mathbb{G},+)$ endowed with a Haar measure μ, and denote by $(\widehat{\mathbb{G}},\cdot)$ the dual group of $(\mathbb{G},+)$ that is the set of characters on $\mathbb{G}$ equipped with the standard multiplication of functions. By definition, the Fourier transform of an integrable function $f:\mathbb{G}\to\mathbb{C}$ is the continuous and bounded function $\mathcal{F}f:\widehat{\mathbb{G}}\to\mathbb{C}$ defined by

$$\forall\gamma\in\widehat{\mathbb{G}}, \mathcal{F}f(\gamma)=\int_{\mathbb{G}} f(x)\overline{\gamma(x)}d\mu(x). \tag{1.1}$$

Being also a locally compact Abelian group, the "frequency space" $\widehat{\mathbb{G}}$ may be endowed with a Haar measure $\widehat{\mu}$. Furthermore, one can normalize $\widehat{\mu}$ so that the following Fourier inversion formula holds true, for all function f in $L^1(\mathbb{G})$ with $\mathcal{F}f$ in $L^1(\widehat{\mathbb{G}})$:

$$\forall x\in\mathbb{G}, f(x)=\int_{\widehat{\mathbb{G}}}\mathcal{F}f(\gamma)\gamma(x)d\widehat{\mu}(\gamma). \tag{1.2}$$

As a consequence, we get the Fourier-Plancherel identity

$$\int_{\mathbb{G}}|f(x)|^2d\mu(x)=\int_{\widehat{\mathbb{G}}}|\mathcal{F}f(\gamma)|^2d\widehat{\mu}(\gamma) \tag{1.3}$$

for all f in $L^1(\mathbb{G})\cap L^2(\mathbb{G})$.

In the Euclidean case $\mathbb{G}=\mathbb{R}^n$ the dual group may be identified to $(\mathbb{R}^n)^*$ through the map $\xi\mapsto \mathrm{e}^{\mathrm{i}\langle\xi,\cdot\rangle}$ (where $\langle\cdot,\cdot\rangle$ stands for the duality bracket between $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$), and the Fourier transform of an integrable function f may thus be seen as the function on $(\mathbb{R}^n)^*$ (usually identified to $\mathbb{R}^n$) given by

$$\mathcal{F}(f)(\xi)=\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\langle\xi,x\rangle}f(x)dx. \tag{1.4}$$

For noncommutative groups, Fourier theory gets wilder, for the dual group is too "small" to keep the definition of the Fourier transform given in (1.1) and still have the inversion formula (1.2). Nevertheless, if the group has "nice" properties, then one can work out a consistent Fourier theory with properties analogous to (1.1),(1.2) and (1.3) (see e.g. [ADBR13; BCD19; CG90; FR16; Tha98] and the references therein).

In that context, the classical definition of the Fourier transform amounts to replacing characters in (1.1) with suitable families of irreducible unitary representations that are valued in Hilbert spaces (see e.g. [BCD19; BFKG16; Lév19] for a detailed presentation). Consequently, the Fourier transform is no longer a complex valued function but rather a family of bounded operators on suitable Hilbert spaces.

The simplest example (apart from $\mathbb{R}^n$) of a nilpotent Lie group is the Heisenberg group, and the harmonic analysis there is a very well researched topic. If we consider the Heisenberg group $\mathbb{H}^d \cong \mathbb{R}^{2d+1}$ as a vector space whose elements $w \in \mathbb{R}^{2d+1}$ can be written $w = (x, y, s)$ with $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, endowed with the following product law:

$$w \cdot w' = (x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' - 2x \cdot y' + 2y \cdot x')$$

where for $x, x' \in \mathbb{R}^d, x \cdot x'$ denotes the Euclidean scalar product between the vectors x and x'. Equipped with the standard differential structure of the manifold $\mathbb{R}^{2d+1}$, the set $\mathbb{H}^d$ is a noncommutative Lie group with identity $(0, 0)$.

As already explained above, as $\mathbb{H}^d$ is noncommutative, in order to have a good Fourier theory, one has to resort to more elaborate irreducible representations than characters. In fact, the group of characters on $\mathbb{H}^d$ is unitary equivalent to the group of characters on $T^\star\mathbb{R}^d$. Roughly, if one defines the Fourier transform according to (1.1) then the information pertaining to the vertical variable s is lost.

Let us recall the Schrödinger representation for $\mathbb{H}^d$, which is the family of group homomorphisms $w \mapsto U_w^\lambda$ (with $\lambda \in \mathbb{R}\backslash\{0\}$) between $\mathbb{H}^d$ and the unitary group of $L^2\left(\mathbb{R}^d\right)$, defined for all $w = (x, y, s)$ in $\mathbb{H}^d$ and u in $L^2\left(\mathbb{R}^d\right)$ by

$$U_w^\lambda u(\xi) = \mathrm{e}^{\mathrm{i}\lambda(s+\langle y, \xi+\frac{1}{2}x\rangle)} u(\xi + x).$$

The classical definition of Fourier transform of integrable functions on $\mathbb{H}^d$ reads as follows:

$$\mathcal{F}_{\mathbb{H}^d} f(\lambda) = \int_{\mathbb{H}^d} f(w) U_{w^{-1}}^\lambda dw, \tag{1.5}$$

and we have the inversion formula:

$$\forall w \in \mathbb{H}^d, f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\mathbb{R}} \operatorname{tr}\left(U_{w^{-1}}^\lambda \mathcal{F}_{\mathbb{H}^d} f(\lambda)\right) |\lambda|^d \,\mathrm{d}\lambda$$

where $\operatorname{tr}(A)$ denotes the trace of the operator A. In particular, the Fourier transform allows to diagonalize the sub-Laplacian $\Delta_{\mathbb{H}^d}$ (see Chapter 2), a property that is based on the following relation that holds true for all functions f and u in $\mathcal{S}\left(\mathbb{H}^d\right)$ and $\mathcal{S}\left(\mathbb{R}^d\right)$, respectively:

$$\mathcal{F}_{\mathbb{H}^d}\left(\Delta_{\mathbb{H}^d} f\right)(\lambda) = 4\mathcal{F}_{\mathbb{H}^d}(f)(\lambda) \circ \Delta_{\mathrm{osc}}^\lambda \quad \text{with} \quad \Delta_{\mathrm{osc}}^\lambda u(x) = \sum_{j=1}^d \partial_j^2 u(x) - \lambda^2 |x|^2 u(x).$$

This indicate that we need study Weyl-Hörmander calculus associated to the harmonic oscillator $\Delta_{\mathrm{osc}}^\lambda$. Recently, the definition of suitable classes of Shubin type for these Weyl-symbols led to another version of the calculus on the Heisenberg group by Bahouri, Fermanian-Kammerer and Gallagher [BFKG12]. The explicit knowledge of the Bargmann-Fock representations of the Heisenberg group allows one to construct the necessary Heisenberg calculus adapted to subelliplic operators in this setting. The approach in this paper is not quite of the same nature as in the works refered to above, as the aim is to define an algebra of operators on functions defined on the Heisenberg group, which contains differential operators and Fourier multipliers, and which has a structure close to that of pseudo-differential operators in the Euclidian space.

Up to now, most of the above works that concern the non-invariant symbolic calculi of operators on nilpotent Lie groups, are restricted to the Heisenberg groups or to manifolds having the Heisenberg group as a local model (except for the calculi which are not symbolic). One of the reasons is that they rely in an essential way on the explicit formulae for representations of the Heisenberg group. Then in this thesis, we try to general the results to more general nilpotent Lie groups, especially 2-step stratified Lie group $\mathbb{G}$. The difficulty is that there is no simple notion of symbols as functions on $\mathbb{G}$, since the Fourier transform is a family of operators on Hilbert spaces depending on some real-valued parameters. Those operators reads in the Schrödinger representation of $\mathbb{G}$ as a family of differential operators belonging to a class of operators of order 1 for the Weyl-Hörmander calculus of the resealed harmonic oscillator. That basic observation is the heart of the matter achieved in this thesis.

1.2 Main results

As we wish for this thesis to be relatively self-contained, the main definitions and properties are covered in Chapter 2 and Appendix. In particular, we first study Lie group $\mathbb{G}$ and the Lie algebra $\mathfrak{g}$ of their left-invariant vector fields. Subsequently, we equip $\mathbb{G}$ with a homogeneous structure by the datum of a well-behaved group of dilations on $\mathbb{G}$. Finally, we introduce the notion of 2-step stratified Lie group and of sub-Laplacian. More specifically we assume that the Lie algebra $\mathfrak{g}$ decomposes into subspaces

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with $\dim \mathfrak{g}_1 = n, \dim \mathfrak{g}_2 = m$ and

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_2 \subseteq \mathfrak{z} = \text{the center of } \mathfrak{g}.$$

A wide number of explicit examples of 2-step stratified Lie group will be also given. Some of them have been known in specialized literature for several years, such as the Heisenberg groups [Cap+07]; the H-type groups [Kap80]; the H-groups in the sense of Métivier [Mét80]. Following [BLU07], we show that these stratified Lie groups are naturally given with the data on $\mathbb{R}^{n+m}$ of m suitable linearly independent and skew-symmetric matrices of order n.

In Chapter 3, we try to develop basic harmonic analysis on 2-step stratified Lie groups. First, we use the orbit method of Kirillov (see [CG90; Ray99]) to describe the explicit construction of irreducible unitary representations. For any $\lambda \in \mathfrak{g}_2^*$ (the dual of $\mathfrak{g}_2$), we define a skew-symmetric bilinear form on $\mathfrak{g}_1$ by

$$B^{(\lambda)}(X,Y) := \lambda([X,Y]) \quad \text{for all } X, Y \in \mathfrak{g}_1.$$

One can find a Zariski-open subset Λ of $\mathfrak{g}_2^*$ such that the number of distinct eigenvalues of $B^{(\lambda)}$ is maximum. We denote by k the dimension of the radical $\mathfrak{r}_\lambda$ of $B^{(\lambda)}$. If $\mathfrak{r}_\lambda = \{0\}$ for each $\lambda \in \Lambda$, then the Lie algebra is called an Moore-Wolf algebra and the corresponding Lie group is called an Moore-Wolf group. In this paper, we will only consider $\mathbb{G}$ to be a 2-step stratified Lie group without Moore-Wolf condition. In this case, the dimension of the orthogonal complement of $\mathfrak{r}_\lambda$ in $\mathfrak{g}_1$ is an even number, which we shall denote by $2d$. Therefore, there exists an orthonormal basis

$$(X_1(\lambda), \dots, X_d(\lambda), Y_1(\lambda), \dots, Y_d(\lambda), R_1(\lambda), \dots, R_k(\lambda))$$

and d continuous functions

$$\eta_j : \mathbb{R}^m \to \mathbb{R}_+, \quad 1 \le j \le d$$

such that $B^{(\lambda)}$ reduces to the form

$$\begin{pmatrix} 0 & \eta(\lambda) & 0 \\ -\eta(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}),$$

where

$$\eta(\lambda) := \operatorname{diag}(\eta_1(\lambda), \dots, \eta_d(\lambda)) \in \mathcal{M}_d(\mathbb{R}),$$

and each $\eta_j(\lambda) > 0$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1, \cdots, \lambda_m)$ and the basis vectors are chosen to depend smoothly on λ in Λ. Decomposing $\mathfrak{g}_1$ as

$$\mathfrak{g}_1 = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda$$

with

$$\mathfrak{p}_\lambda := \operatorname{span}_{\mathbb{R}} \left(X_1(\lambda), \dots, X_d(\lambda) \right),$$
$$\mathfrak{q}_\lambda := \operatorname{span}_{\mathbb{R}} \left(Y_1(\lambda), \dots, Y_d(\lambda) \right),$$
$$\mathfrak{r}_\lambda := \operatorname{span}_{\mathbb{R}} \left(R_1(\lambda), \dots, R_k(\lambda) \right).$$

Then we have the decomposition $\mathfrak{g} = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda \oplus \mathfrak{g}_2$. We denote the element $\exp(X+Y+R+T)$ of $\mathbb{G}$ by (X, Y, R, T) for $X \in \mathfrak{p}_\lambda, Y \in \mathfrak{q}_\lambda, R \in \mathfrak{r}_\lambda, T \in \mathfrak{g}_2$. Further we can write

$$(X, Y, R, T) = \sum_{j=1}^{d} x_j(\lambda) X_j(\lambda) + \sum_{j=1}^{d} y_j(\lambda) Y_j(\lambda) + \sum_{j=1}^{k} r_j(\lambda) R_j(\lambda) + \sum_{j=1}^{m} t_j T_j$$

and denote it by (x, y, r, t) suppressing the dependence of λ which will be understood from the context.

For (λ, ν, w) in $\Lambda \times \mathbb{R}^k \times \mathbb{R}^N$ with

$$w = (x, y, r, t) \in \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^k \oplus \mathbb{R}^m = \mathbb{R}^N,$$

we define the irreducible unitary representations of $\mathbb{R}^N$, equipped with the group law defined above, on $L^2\left(\mathbb{R}^d\right)$

$$\begin{aligned}\left(\pi_{\lambda,\nu}(w)\phi\right)(\xi) &:= \exp\left(i \sum_{j=1}^{m} \lambda_j t_j + i \sum_{j=1}^{k} \nu_j r_j + i \sum_{j=1}^{d} \eta_j(\lambda) \left(y_j \xi_j + \frac{1}{2} x_j y_j \right) \right) \phi(\xi + x) \\ &= e^{i\langle \nu, r\rangle} e^{i\langle \lambda, t\rangle} e^{i \sum_{j=1}^{d} \eta_j(\lambda)\left(y_j\xi_j + \frac{1}{2}x_j y_j\right)} \phi(\xi + x).\end{aligned}$$

We first prove the classic theorem of Stone-von Neumann for the 2-step stratified Lie group, which says in effect that any irreducible unitary representation of $\mathbb{G}$ that is nontrivial on the center is equivalent to some $\pi_{\lambda,\nu}$.

Theorem 1.1. *Let π be any unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, such that for some $\lambda \in \Lambda$, $\pi(0,0,0,t) = e^{i\lambda t} I$. Then $\mathcal{H} = \bigoplus \mathcal{H}_\alpha$ where the $\mathcal{H}_\alpha$ are mutually orthogonal subspaces of $\mathcal{H}$, each invariant under π, such that $\pi|_{\mathcal{H}_\alpha}$ is unitarily equivalent to $\pi_{\lambda,\nu}$ for each α and some $\nu \in \mathbb{R}^k$. In particular, if π is irreducible then π is equivalent to $\pi_{\lambda,\nu}$.*

And then we can study the sub-Laplacian and Fourier transform, which is a family of operators on Hilbert spaces depending on a real-valued parameters λ and ν. We can now define the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ by

$$\mathcal{L} = -\sum_{j=1}^{d} \left(X_j^2 + Y_j^2 \right) - \sum_{l=1}^{k} R_l^2.$$

Explicitly,

$$\mathcal{L} = -\Delta_x - \Delta_y - \Delta_r - \frac{1}{4}\left(|x|^2 + |y|^2\right)\Delta_t + \sum_{s=1}^{m}\sum_{j=1}^{d}\left\{-\left(B_s y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B_s e_j\right)\frac{\partial}{\partial y_j}\right\}\frac{\partial}{\partial t_s},$$

where we use

$$B^{(\lambda)} = \sum_{s=1}^{m}\lambda_s B_s.$$

By taking the Fourier transform of the sub-Laplacian $\mathcal{L}$ with respect to t, we get parametrized λ-twisted sub-Laplacian $\mathcal{L}^\lambda$, $\lambda \in \Lambda$, given by

$$\mathcal{L}^\lambda = -\Delta_x - \Delta_y - \Delta_r + \frac{1}{4}\left(|x|^2 + |y|^2\right)|\lambda|^2 - i\sum_{j=1}^{d}\left\{-\left(B^{(\lambda)}y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B^{(\lambda)}e_j\right)\frac{\partial}{\partial y_j}\right\}.$$

What's more, it is well known from [Won98] that Weyl transforms have intimate connections with analysis with the so-called twisted sub-Laplacian and the Heisenberg group, and the harmonic analysis there is a very well researched topic. Then in Section 3.3, we study the (λ,ν)-Weyl transform $W^{\lambda,\nu}$ and (λ,ν)-Wigner transform $W_{\lambda,\nu}(f,g)$ on 2-step stratified Lie groups $\mathbb{G}$, which should also depend on these parameters.

Theorem 1.2. *For all f_1, g_1, f_2, and g_2 in $\mathcal{S}\left(\mathbb{R}^d\right)$, we have*

$$\left\langle W_{\lambda,\nu}\left(f_1, g_1\right), W_{\lambda,\nu}\left(f_2, g_2\right)\right\rangle = \left\langle f_1, f_2\right\rangle\overline{\left\langle g_1, g_2\right\rangle},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbb{R}^d)$.

Theorem 1.3. *There exists a unique bounded linear operator $Q : L^2(\mathbb{R}^{2d}) \to B(L^2(\mathbb{R}^d))$ such that*

$$\langle (Qa)f, g\rangle = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi)W_{\lambda,\nu}(f,g)(x,\xi)\,dx\,d\xi$$

and

$$\|Qa\|_* \le \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\|a\|_{L^2\left(\mathbb{R}^{2d}\right)}$$

for all f and g in $L^2\left(\mathbb{R}^d\right)$ and a in $L^2\left(\mathbb{R}^{2d}\right)$, where $\|\cdot\|_*$ denotes the norm in $B(L^2(\mathbb{R}^d))$ and $\mathrm{Pf}(\lambda) := \prod_{j=1}^{d}\eta_j(\lambda)$ is the Pfaffian of $B^{(\lambda)}$.

In particular, we can show a relationship between Hilbert-Schmidt pseudo-differential operators on $L^2(\mathbb{G})$ and (λ,ν)-Weyl transforms with symbol in $L^2\left(\mathbb{R}^{2d+k+m}\right)$.

Theorem 1.4. *Let $a \in L^2\left(\mathbb{R}^{2d}\right)$, $K_\lambda : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ be given by (3.29). Then $W_a^{\lambda,\nu}$ is a Hilbert-Schmidt operator with kernel*

$$\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}K_\lambda a.$$

More precisely

$$\left(W_a^{\lambda,\nu} f\right)(x) = \mathrm{Pf}(\lambda)^{\frac{1}{2}} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} K_\lambda a(x,y) f(y) dy, \quad x \in \mathbb{R}^d.$$

We can also give a formula for the product $W_\sigma^{\lambda,\nu} W_\tau^{\lambda,\nu}$ of two Weyl transforms $W_\sigma^{\lambda,\nu}$ and W_τ in terms of a λ-twisted convolution (see Definition 3.38) of σ and τ.

Theorem 1.5. *Let σ and τ be in $L^2\left(\mathbb{R}^{2d}\right)$. Then*

$$W_\sigma^{\lambda,\nu} W_\tau^{\lambda,\nu} = W_\omega^{\lambda,\nu}$$

where $\omega \in L^2\left(\mathbb{R}^{2d}\right)$ and $\mathcal{F}_\lambda \omega = \mathrm{Pf}(\lambda)(2\pi)^{-d} \left(\mathcal{F}_\lambda \sigma\right) *_\lambda \left(\mathcal{F}_\lambda \tau\right))$.

In Section 3.5, we demonstrate the beautiful interplay between the representation theory on $\mathbb{G}$ and the classical expansions in terms of Hermite functions. If $\eta = (\eta_1, \ldots, \eta_d) \in \left(\mathbb{R}_+^*\right)^d$ and $\alpha \in \mathbb{N}^d$, we define the rescaled Hermite function Φ_α^λ by

$$\Phi_\alpha^\lambda := |\,\mathrm{Pf}(\lambda)|^{\frac{1}{4}} \Phi_\alpha \left(\eta_1^{\frac{1}{2}} \cdot, \eta_2^{\frac{1}{2}} \cdot, \cdots, \eta_d^{\frac{1}{2}} \cdot\right),$$

and the special Hermite functions

$$\Phi_{\alpha,\beta}^\lambda(x) = \mathrm{Pf}(\lambda)^{\frac{1}{2}} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\eta(\lambda)\cdot px} \Phi_\alpha^\lambda \left(x + \frac{q}{2}\right) \overline{\Phi_\beta^\lambda \left(x - \frac{q}{2}\right)} dx.$$

In particular, they form an orthonormal basis of $L^2\left(\mathbb{R}^d\right)$ and we have the rescaled harmonic oscillator

$$\mathcal{H}(\lambda)\Phi_\alpha^\lambda := (-\Delta + |\eta \cdot x|^2)\Phi_\alpha^\lambda = \sum_{j=1}^d \eta_j(\lambda)(2\alpha_j + 1)\Phi_\alpha^\lambda,$$

which can help us prove that $\Phi_{\alpha,\beta}^\lambda$ are eigenfunctions of the λ-twisted sub-Laplacian $\mathcal{L}^\lambda$.

Theorem 1.6. *For $\lambda \in \Lambda, \nu \in \mathbb{R}^k$, one has the formula*

$$\mathcal{L}^\lambda(\Phi_{\alpha,\beta}^\lambda) = \left(\sum_{j=1}^d \eta_j(\lambda)(2\alpha_j + 1) + \sum_{j=1}^k \nu_j^2\right) \Phi_{\alpha,\beta}^\lambda.$$

For the Lagueere polynomial $L_\alpha^k(x)$ (see Section 3.6 for details), we can prove the Laguerre formulas for special hermite functions and therefore set up the connection with sub-Laplacian on $\mathbb{G}$.

Theorem 1.7. *For $\alpha \in \mathbb{N}^d$ and any z in $\mathbb{C}^d$,*

$$\Phi_{\alpha,\alpha}^\lambda(z) = \mathrm{Pf}(\lambda)^{\frac{1}{2}} (2\pi)^{-\frac{d}{2}} \prod_{j=1}^d L_{\alpha_j}^0 \left(\frac{1}{2}\eta_j(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}.$$

Theorem 1.8. *For $\alpha \in \mathbb{N}^d, k = 0, 1, \ldots$ and any $z \in \mathbb{C}^d$ we have*

$$(i)\ \Phi^{\lambda}_{\alpha+k,\alpha}(z) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left(\tfrac{\alpha!}{(\alpha+k)!}\right)^{\frac{1}{2}}\left(\tfrac{i}{\sqrt{2}}\right)^{k}\bar{z}^{k}\prod_{j=1}^{k}L^{k_j}_{\alpha_j}(\tfrac{1}{2}\eta_j(\lambda)|z_j|^2)e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}.$$

$$(ii)\ \Phi^{\lambda}_{\alpha,\alpha+k}(z) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left(\tfrac{\alpha!}{(\alpha+k)!}\right)^{\frac{1}{2}}\left(\tfrac{-i}{\sqrt{2}}\right)^{k}z^{k}\prod_{j=1}^{k}L^{k_j}_{\alpha_j}(\tfrac{1}{2}\eta_j(\lambda)|z_j|^2)e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}.$$

In Chapter 4, we give some application for the harmonic analysis theory developed above. First, a global calculus of pseudo-differential operator on 2-step stratified Lie groups $\mathbb{G}$ is introduced. We want to consider the symbol associated with rescaled harmonic oscillator:

$$\mathcal{H}(\lambda) + |\nu|^2 = -\Delta_\xi + |\eta(\lambda)\cdot\xi|^2 + |\nu|^2.$$

Then the Hörmander metric depending on parameters $\lambda \in \mathbb{R}^m$, $\nu \in \mathbb{R}^k$ and $\rho \in (0,1]$ is the metric $g^{(\rho,\lambda,\nu)}$ on $\mathbb{R}^{2d+k}$ defined via

$$g^{(\rho,\lambda)}_{\xi,\theta}(d\xi, d\theta) := \left(\frac{1}{1+|\eta(\lambda)\cdot\xi|^2+|\theta|^2+|\nu|^2}\right)^{\rho}\left(|\eta(\lambda)\cdot\xi|^2 + |d\theta|^2\right).$$

The associated weight function $M^{(\lambda,\nu)}$ on $\mathbb{R}^{2d+k}$ is defined via

$$M^{(\lambda,\nu)}(\xi,\theta,\nu) := \left(1+|\eta(\lambda)\cdot\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{1}{2}}.$$

Similar to Proposition 1.20 in [BFKG12], we have the following results:

Theorem 1.9. *For each $\lambda \in \Lambda$ and $\nu \in \mathbb{R}^k$, the metric $g^{(\rho,\lambda,\nu)}$ is of Hörmander type and the function $M^{(\lambda,\nu)}$ is a $g^{(\rho,\lambda,\nu)}$-weight. Furthermore, if $\rho \in (0,1]$ is fixed, then the structural constants (see Definition* 4.3*) for $g^{(\rho,\lambda,\nu)}$ and for $M^{(\lambda,\nu)}$ can be chosen independent of λ and ν.*

Therefore, in what follows, we shall define a positive, noninteger real number $\varrho \in (0,1)$, which will measure the regularity assumed on the symbols. This number ϱ is fixed from now on and we emphasize that the definitions below depend on ϱ. We have chosen not to keep memory of this number on the notations for the sake of simplicity.

Definition 1.10. Let $\rho \in (0,1]$ be a fixed parameter. For each parameter $\lambda \in \Lambda$ and $\nu \in \mathbb{R}^k$, we define the (λ,ν)-Shubin classes by

$$\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}) := S\left(\left(M^{(\lambda,\nu)}\right)^{\delta}, g^{(\rho,\lambda,\nu)}\right),$$

where we have used the Hörmander notation to define a class of symbols in terms of a metric and a weight. Here this means that $\sum^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$ is the class of functions $a \in C^\infty\left(\mathbb{G}\times\mathbb{R}^{2d+k+m}\right)$ such that for each $N \in \mathbb{N}_0$, the quantity

$$\|a\|_{\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}),N} := \sup_{\substack{|\alpha|+|\beta|+|\gamma|+l\le N\\(\xi,\theta,\nu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^k}} |\eta(\lambda)|^{-\rho\frac{|\alpha|+|\beta|+|\gamma|}{2}}\left(1+|\eta(\lambda)|\left(1+|\xi|^2+|\theta|^2+|\nu|^2\right)\right)^{-\frac{\delta-\rho(|\alpha|+|\beta|+|\gamma|)}{2}}$$
$$\times\left\|(\lambda\partial_\lambda)^l\partial^{\alpha}_{\xi}\partial^{\beta}_{\theta}\partial^{\gamma}_{\nu}a(x,y,r,s,\xi,\theta,\nu,\lambda)\right\|_{C^{\varrho}(\mathbb{G})}$$

is finite. Besides, one additionally requires that the function

$$(w,\xi,\theta,\nu,\lambda)\mapsto\sigma(a)(w,\xi,\theta,\nu,\lambda)\stackrel{\text{def}}{=}a\left(w,\frac{\xi_1}{\eta_1(\lambda)}\dots\frac{\xi_d}{\eta_d(\lambda)},\theta,\nu,\lambda\right)$$

is uniformly smooth close to $\lambda=0$ in the sense that there exists $C>0$ such that $\forall(w,\xi,\theta,\nu)\in\mathbb{G}\times\mathbb{R}^{2d+k},\forall\lambda\in[-1,1]$, $\left\|\partial_\lambda^l\partial_\xi^\alpha\partial_\theta^\beta\partial_\nu^\gamma(\sigma(a))\right\|_{\mathcal{C}^e(\mathbb{G})}\le C_{N,l}\left(1+|\eta(\lambda)|+|\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{\delta-\rho(|\alpha|+|\beta|+\gamma)}{2}}$. In that case we shall write $a\in\Sigma^\delta_{\rho,\lambda,\nu}(\mathbb{G})$.

Theorem 1.11. *To a symbol $a\in\Sigma^\delta_{\rho,\lambda,\nu}(\mathbb{G})$ on $\mathbb{R}^{2d}$ depending on the parameters (w,λ,ν) in $\mathbb{G}\times\Lambda\times\mathbb{R}^k$ and belonging to (λ,ν)-dependent Hörmander class. Then the pseudo-differential operator on $\mathbb{G}$ defined in the following way: for any $f\in\mathscr{S}(\mathbb{G})$ and some constant κ,*

$$\mathrm{Op}(a)f(w)=\kappa\iint_{\Lambda\times\mathbb{R}^k}\operatorname{tr}\left(\pi_{\lambda,\nu}(w^{-1})\mathcal{F}(f)(\lambda,\nu)\,\mathrm{op}^W(a(w,\xi,\theta,\nu,\lambda))\right)\mathrm{Pf}(\lambda)d\lambda d\nu,\ \forall\ w\in\mathbb{G}$$

is well-defined, where op^W is the Weyl quantization defined in (4.1).

For the (λ,ν)-Shubin Sobolev spaces $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ (see Definition 4.17), we have the following properties:

Theorem 1.12. (1) *The space $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ is a Hilbert space endowed with the sesquilinear form*

$$(g,h)_{\mathcal{Q}^{\lambda,\nu}_s}=\left((\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}g,(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}h\right)_{L^2(\mathbb{G})}.$$

We also have

$$L^2(\mathbb{G})=\mathcal{Q}^{\lambda,\nu}_0(\mathbb{G}),$$

and the inclusions

$$\mathscr{S}(\mathbb{G})\subset\mathcal{Q}^{\lambda,\nu}_{s_1}(\mathbb{G})\subset\mathcal{Q}^{\lambda,\nu}_{s_2}(\mathbb{G})\subset\mathscr{S}'(\mathbb{G}),\quad s_1>s_2.$$

(2) *The dual of $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ may be identified with $\mathcal{Q}^{\lambda,\nu}_{-s}(\mathbb{G})$ via the distributional duality form $\langle g,h\rangle=\int_{\mathbb{G}}ghdx$.*

(3) *The complex interpolation between the spaces $\mathcal{Q}^{\lambda,\nu}_{s_0}(\mathbb{G})$ and $\mathcal{Q}^{\lambda,\nu}_{s_1}(\mathbb{G})$ is*

$$\left(\mathcal{Q}^{\lambda,\nu}_{s_0}(\mathbb{G}),\mathcal{Q}^{\lambda,\nu}_{s_1}(\mathbb{G})\right)_\theta=\mathcal{Q}^{\lambda,\nu}_{s_\theta}(\mathbb{G}),\quad s_\theta=(1-\theta)s_0+\theta s_1,\theta\in(0,1).$$

(4) *For any $s\in\mathbb{R}$, $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ coincides with the completion (in $\mathscr{S}'(\mathbb{G})$) of the Schwartz space $\mathscr{S}(\mathbb{G})$ for the norm*

$$\|h\|^{(b)}_{\mathcal{Q}^{\lambda,\nu}_s}=\left\|\mathrm{Op}^W(b^s_\lambda)\,h\right\|_{L^2(\mathbb{G})}$$

where $b^s_\lambda(\xi,\theta,\nu)=\sqrt{1+|\eta(\lambda)\cdot\xi|^2+|\theta|^2+|\nu|^2}$ is (λ,ν)-uniform in $\Psi\Sigma^s_{1,\lambda,\nu}(\mathbb{G})$. The norm $\|\cdot\|^{(b)}_{\mathcal{Q}^{\lambda,\nu}_s}$ extended to $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ is equivalent to $\|\cdot\|_{\mathcal{Q}^{\lambda,\nu}_s}$.

(5) *For any* $s \in \mathbb{R}$, $\lambda \in \mathbb{R}^m$ *and* $\nu \in \mathbb{R}^k$, *the Shubin Sobolev space* $\mathcal{Q}_s^{\lambda,\nu}(\mathbb{G})$ *coincides with the Sobolev space associated with* $g^{(1,\lambda,\nu)}$ *and* $(M^{(\lambda,\nu)})^s$ *(see Definition 4.5)*

$$\mathcal{Q}_s^{\lambda,\nu}(\mathbb{G}) = H\left((M^{(\lambda,\nu)})^s, g^{(\rho,\lambda,\nu)}\right).$$

(6) *For any* $s \in \mathbb{R}$, *the operators* $\mathrm{Op}^W(b^{-s})(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}$ *and* $(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}\mathrm{Op}^W(b^{-s})$ *are bounded and invertible on* $L^2(\mathbb{G})$.

At last, we consider the heat kernel of the rescaled harmonic oscillator $\mathcal{H}(\lambda)$ and the sub-Laplacian $\mathcal{L}$, which are related to the theory of parabolic operators which describes the distribution of heat on a given manifold as well as evolution phenomena and diffusion processes. The solution of an initial value problem for a parabolic partial differential equation depends on its heat kernel, which is the fundamental solution of the associated parabolic operator. Hence the importance of finding explicit formulas for these kernels. We first compute the heat kernel of the rescaled harmonic oscillator as follows.

Theorem 1.13. *The associated heat kernel of the rescaled harmonic oscillator* $\mathcal{H}(\lambda)$ *is*

$$G_\tau(x) = \prod_{j=1}^{d} \frac{1}{2\sinh(\eta_j(\lambda)\tau)} \exp\left\{-\sum_{j=1}^{d} \frac{\eta_j(\lambda)\,|x_j|^2}{2}\coth(\eta_j(\lambda)\tau)\right\},$$

i.e., $G_\tau(x)$ *satisfies the heat equation*

$$\frac{\partial G_\tau}{\partial \tau} + \sum_{j=1}^{d}\left(\eta_j^2(\lambda)x_j^2 - \frac{\partial^2}{\partial x_j^2}\right)G_\tau(x) = 0 \quad \textit{with} \quad \lim_{\tau\to 0}\int_{\mathbb{R}^d} G_\tau(x)f(x)dx = f(0).$$

Now, we consider the initial-value problem given by

$$\begin{cases} \partial_\tau u(\omega,t,\tau) + (\mathcal{L}u)(w,t,\tau) = 0, \\ u(\omega,t,0) = f(\omega,t), \\ \omega = (z,r) \in \mathbb{R}^{2d+k}, t \in \mathbb{R}^m, \tau > 0. \end{cases}$$

By taking the Fourier transform with respect to t and evaluated at λ, we get an initial-value problem for the heat equation governed by the λ-twisted sub-Laplacian $\mathcal{L}^\lambda$, i.e.

$$\begin{cases} \partial_\tau u_\lambda(\omega,\tau) + (\mathcal{L}^\lambda u_\lambda)(\omega,\tau) = 0, \\ u_\lambda(\omega,0) = f_\lambda(\omega), \end{cases}$$

for all $\omega = (z,r) \in \mathbb{R}^{2d+k}, \tau > 0$ and $\lambda \in \Lambda$. With this formula and Theorem 1.13, the heat kernel of $\mathcal{L}$ is given in the following theorem.

Theorem 1.14. *For all* f *in* $L^2(\mathbb{G}), e^{-\tau\mathcal{L}}f = f *_{\mathbb{G}} K_\tau$, *where*

$$K_\tau(\omega,t) = (2\pi)^{-(d+m)}\int_{\mathbb{R}^m} e^{-it\cdot\lambda}e^{-\tau|\nu|^2}\prod_{j=1}^{d}\frac{\eta_j(\lambda)}{2\sinh(\eta_j(\lambda)\tau)}\exp\left\{-\frac{\eta_j(\lambda)\omega_j^2}{2}\coth(\eta_j(\lambda)\tau)\right\}d\lambda$$

for all $(\omega, t) \in \mathbb{G}$.

2 Elementary analysis of stratified Lie groups

In this thesis we deal with a particular class of sub-Riemannian manifold, i.e. stratified Lie groups. Roughly speaking, a sub-Riemannian manifold is a Riemannian manifold together with a constrain on admissible directions of movements. In Riemannian geometry every smoothly embedded curve has locally finite length. In sub-Riemannian geometry, if a curve fails to satisfy the obligation of the constrain, then it has infinite length.

Among sub-Riemannian manifolds, a fundamental role is played by stratified Lie groups, following the terminology of [FS82]. In the literature, the name "Carnot groups" is also used, they seem to owe their name to a paper by Carathéodory [Car09] and was also used in the school of Gromov [Gro96]. In the following we will only use the name "stratified Lie groups" for convenience.

The importance of stratified Lie groups became evident in [Mit85], where it was proved that a suitable blow-up limit of a sub-Riemannian manifold at a generic point is a stratified Lie group. In other words, stratified Lie groups can be seen [Bel96] as the natural "tangent spaces" to sub-Riemannian manifolds, and therefore can be considered as local models of general sub-Riemannian manifolds. Therefore there is a comparison between sub-Riemannian Geometry and Riemannian Geometry: stratified Lie groups are to sub-Riemannian manifolds what Euclidean spaces are to Riemannian manifolds.

This part of this thesis is devoted to an elementary and self-contained introduction to the stratified Lie groups. Our presentation does not require a specialized knowledge neither in algebra nor in differential geometry, which can compare with the formal and abstract approach to the stratified Lie groups commonly used in literature in the Appendix. The approach is intended to be understandable by readers with basic backgrounds only in linear algebra and differential calculus in $\mathbb{R}^N$. We introduce and discuss a wide range of explicit stratified Lie groups of arbitrarily large dimension and step two. It is also played a special attention to the Lie algebras of the groups by stressing their links with second order partial differential operators of Hörmander type (sum of squares of vector fields). All results are already know in the literature, we will take most of the material from [BLU07; CG90].

2.1 Preliminaries on Lie groups

In this section, after giving some notations and the basic definitions concerning with vector fields in $\mathbb{R}^N$, we first study Lie groups $\mathbb{G}$ and the Lie algebra of their left-invariant vector fields. Subsequently, we equip $\mathbb{G}$ with a homogeneous structure by the dilations $\{\delta_\lambda\}_{\lambda>0}$ on $\mathbb{G}$. Finally, we introduce the notion of (homogeneous) stratified Lie groups.

2.1.1 Vector fields in $\mathbb{R}^N$

Vector Fields in $\mathbb{R}^N$

We use any of the notation

$$\partial_j, \quad \partial_{x_j}, \quad \frac{\partial}{\partial x_j}, \quad \partial/\partial x_j$$

to indicate the partial derivative operator with respect to the j-th coordinate of $\mathbb{R}^N$. Let $\Omega \subseteq \mathbb{R}^N$ be an open (and non-empty) set.

Definition 2.1. Given an N-tuple of scalar functions $a_1, \dots, a_N$,

$$a_j : \Omega \to \mathbb{R}, \quad j \in \{1, \dots, N\},$$

the first order linear differential operator

$$X = \sum_{j=1}^{N} a_j \partial_j \tag{2.1}$$

will be called a vector field on Ω with component functions (or simply, components) $a_1, \dots, a_N$.

If $f : \Omega \to \mathbb{R}$ is a differentiable function, we denote Xf the function on Ω by

$$Xf(x) = \sum_{j=1}^{N} a_j(x)\partial_j f(x), \quad x \in \Omega.$$

Occasionally, we shall also use the notation Xf when

$$f : \Omega \to \mathbb{R}^m$$

is a vector-valued function, to mean the component-wise action of X. More precisely, we set

$$Xf(x) = \begin{pmatrix} Xf_1(x) \\ \vdots \\ Xf_m(x) \end{pmatrix} \quad \text{for } f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}.$$

Furthermore, given a differentiable function $f : \Omega \to \mathbb{R}^m$, we shall denote by

$$\mathcal{J}_f(x), \quad x \in \Omega$$

the Jacobian matrix of f at x.

Let $C^\infty(\Omega, \mathbb{R})$ (for brevity, $C^\infty(\Omega)$) be the set of smooth (i.e. infinitely-differentiable) real-valued functions. If the components a_j are smooth, we shall call X a smooth vector field and we shall often consider X as an operator acting on smooth functions,

$$X : C^\infty(\Omega) \to C^\infty(\Omega), \quad f \mapsto Xf.$$

We shall denote by $T(\mathbb{R}^N)$ the set of all smooth vector fields in $\mathbb{R}^N$. Equipped with the natural operations, $T(\mathbb{R}^N)$ is a vector space over $\mathbb{R}$.

We adopt the following notation: I will denote the identity map on $\mathbb{R}^N$ and, if X is the vector field in (2.1), then

$$XI := \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \tag{2.2}$$

will be the column vector of the components of X. This notation is obviously consistent with our definition of the action of X on a vector-valued function. Thus, XI may also be regarded as a smooth map from $\mathbb{R}^N$ to itself.

Often, many authors identify X and XI. Instead, in order to avoid any confusion between a smooth vector field as a function belonging to $C^\infty\left(\mathbb{R}^N, \mathbb{R}^N\right)$ and a smooth vector field as a differential operator from $C^\infty\left(\mathbb{R}^N\right)$ to itself, we prefer to use the different notation XI and X as described in (2.2) and (2.1), respectively.

By consistency of notation, we may write

$$Xf = (\nabla f) \cdot XI,$$

where $\nabla = (\partial_1, \ldots, \partial_N)$ is the gradient operator in $\mathbb{R}^N$, f is any real-valued smooth function on $\mathbb{R}^N$ and $\cdot$ denotes the row $\times$ column product. For example, for the following two vector fields on $\mathbb{R}^3$ (whose points are denoted by $x = (x_1, x_2, x_3)$)

$$X_1 = \partial_{x_1} + 2x_2\partial_{x_3}, \quad X_2 = \partial_{x_2} - 2x_1\partial_{x_3}, \tag{2.3}$$

we have

$$X_1 I(x) = \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix}, \quad X_2 I(x) = \begin{pmatrix} 0 \\ 1 \\ -2x_1 \end{pmatrix}. \tag{2.4}$$

Integral Curves

Definition 2.2. A path $\gamma : \mathcal{D} \to \mathbb{R}^N, \mathcal{D}$ being an interval of $\mathbb{R}$, will be said an integral curve of the smooth vector field X if

$$\dot{\gamma}(t) = XI(\gamma(t)) \quad \text{for every } t \in \mathcal{D}.$$

If X is a smooth vector field, then, for every $x \in \mathbb{R}^N$, the Cauchy problem

$$\begin{cases} \dot{\gamma} = XI(\gamma) \\ \gamma(0) = x \end{cases} \tag{2.5}$$

has a unique solution

$$\gamma_X(\cdot, x) : \mathcal{D}(X, x) \to \mathbb{R}^N.$$

Since X is smooth, $t \mapsto \gamma_X(t, x)$ is a C^∞ function whose n-th Taylor expansion in a neighborhood of $t = 0$ is given by

$$\begin{aligned} \gamma_X(t, x) =& x + tX^{(1)}I(x) + \frac{t^2}{2!}X^{(2)}I(x) + \cdots + \frac{t^n}{n!}X^{(n)}I(x) \\ &+ \frac{1}{n!}\int_0^t (t-s)^n X^{(n+1)} I\left(\gamma_X(s, x)\right) ds. \end{aligned} \tag{2.6}$$

Hereafter, for $k \in \mathbb{N}$, we denote by $X^{(k)}$ the vector field

$$X^{(k)} = \sum_{j=1}^N \left(X^{k-1}a_j\right)\partial_{x_j},$$

being $X^0 = I$ (the identity map) and $X^h, h \geq 1$, the h-th order iterated of X, i.e.

$$X^h := \underbrace{X \circ \cdots \circ X}_{h \text{ times}}.$$

Example 2.3. For example, if X_1 is as in (2.3), since

$$X_1^{(1)}I = \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix}, \quad X_1^{(2)}I = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = X_1^{(k)}I \quad \forall k \geq 3,$$

we have

$$\gamma_{X_1}(t,x) = x + tX_1^{(1)}I(x) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} x_1 + t \\ x_2 \\ x_3 + 2x_2 t \end{pmatrix}.$$

Definition 2.4. Let X be a smooth vector field on $\mathbb{R}^N$. Following all the above notation, we set

$$\exp(tX)(x) := \gamma_X(t,x)$$

where $\gamma_X(\cdot, x)$ is the solution of (2.5).

Then, being X smooth, for every $n \in \mathbb{N}$, we have the expansion

$$\begin{aligned}\exp(tX)(x) =& \sum_{k=0}^{n} \frac{t^k}{k!} X^k I(x) \\ &+ \frac{1}{n!}\int_0^t (t-s)^n X^{n+1} I(\exp(sX)(x))ds.\end{aligned}$$

In particular, for $n=1$,

$$\exp(tX)(x) = x + tX^1 I(x) + \int_0^t (t-s) X^2 I(\exp(sX)(x))ds.$$

Moreover, from the unique solvability of the Cauchy problem related to smooth vector fields we get: $t \in \mathcal{D}(-X,x)$ iff $-t \in \mathcal{D}(X,x)$ and

$$\begin{aligned}&\exp(-tX)(x) := \exp((-t)X)(x) = \exp(t(-X))(x), \\ &\exp(-tX)(\exp(tX)(x)) = x, \\ &\exp((t+\tau)X)(x) = \exp(tX)(\exp(\tau X)(x)), \\ &\exp((t\tau)X)(x) = \exp(t(\tau X))(x),\end{aligned}$$

when all the terms are defined.

Remark 2.5. Let us consider a smooth function $u : \mathbb{R}^N \to \mathbb{R}$ and the vector field in (2.1). Then

$$Xu(x) = \lim_{t\to 0} \frac{u(\exp(tX)(x)) - u(x)}{t} \quad \forall x \in \mathbb{R}^N. \tag{2.7}$$

Indeed, since $\exp(tX)(x) = x + tXI(x) + \mathcal{O}(t^2)$, the limit on the right-hand side of (2.7) is equal to the following one:

$$\lim_{t\to 0} \frac{u(x + tXI(x)) - u(x)}{t} = \nabla u(x) \cdot XI(x) = Xu(x).$$

Lie Brackets of Vector Fields in $\mathbb{R}^N$

Definition 2.6. Given two smooth vector fields X and Y in $\mathbb{R}^N$, we define the Lie bracket $[X, Y]$ as follows

$$[X, Y] := XY - YX.$$

If $X = \sum_{j=1}^{N} a_j \partial_j$ and $Y = \sum_{j=1}^{N} b_j \partial_j$, a direct computation shows that the Lie bracket $[X, Y]$ is the vector field

$$[X, Y] = \sum_{j=1}^{N} (X b_j - Y a_j)\, \partial_j.$$

As a consequence,

$$[X, Y]I = \begin{pmatrix} X b_1 \\ \vdots \\ X b_N \end{pmatrix} - \begin{pmatrix} Y a_1 \\ \vdots \\ Y a_N \end{pmatrix} = \mathcal{J}_{YI} \cdot XI - \mathcal{J}_{XI} \cdot YI.$$

For example, if X_1, X_2 are as in (2.3), we have

$$[X_1, X_2] = (X_1 (-2x_1) - X_2 (2x_2))\, \partial_{x_3} = -4 \partial_{x_3}.$$

It is quite trivial to check that $(X, Y) \mapsto [X, Y]$ is a bilinear map on the vector space $T(\mathbb{R}^N)$ satisfying the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for every $X, Y, Z \in T(\mathbb{R}^N)$.

We shall refer to $T(\mathbb{R}^N)$ (equipped with the above Lie bracket) as the Lie algebra of the vector fields on $\mathbb{R}^N$. Any sub-algebra $\mathfrak{g}$ of $T(\mathbb{R}^N)$ will be called a Lie algebra of vector fields. More explicitly, $\mathfrak{g}$ is a Lie algebra of vector fields if $\mathfrak{g}$ is a vector subspace of $T(\mathbb{R}^N)$ closed with respect to $[,]$, i.e. $[X, Y] \in \mathfrak{g}$ for every $X, Y \in \mathfrak{g}$.

We now introduce some other notation on the algebras of vector fields. Given a set of vector fields $Z_1, \ldots, Z_m \in T(\mathbb{R}^N)$ and a multi-index

$$J = (j_1, \ldots, j_k) \in \{1, \ldots, m\}^k,$$

we set

$$Z_J := \left[Z_{j_1}, \ldots \left[Z_{j_{k-1}}, Z_{j_k} \right] \ldots \right].$$

We say that Z_J is a commutator of length (or height) k of $Z_1, \ldots, Z_m$. If $J = j_1$, we also say that $Z_J := Z_{j_1}$ is a commutator of length 1 of $Z_1, \ldots, Z_m$. A commutator of the form Z_J will also be called nested , in order to emphasize its difference from, e.g. a commutator of the form

$$[[Z_1, Z_2], [Z_3, Z_4]].$$

Definition 2.7 (The Lie algebra generated by a set)**.** If V is any subset of $T(\mathbb{R}^N)$, we denote by $\mathrm{Lie}\{V\}$ the least sub-algebra of $T(\mathbb{R}^N)$ containing V, i.e.

$$\mathrm{Lie}\{V\} := \bigcap \mathfrak{h},$$

where $\mathfrak{h}$ is a sub-algebra of $T(\mathbb{R}^N)$ with $V \subseteq \mathfrak{h}$. We also define

$$\mathrm{rank}(\mathrm{Lie}\{V\}(x)) := \dim_{\mathbb{R}}\{ZI(x) \mid Z \in \mathrm{Lie}\{V\}\}.$$

Example 2.8. Let X_1 and X_2 be as in (2.3). Since $[X_1, X_2] = -4\partial_{x_3}$ and since any commutator involving X_1, X_2 more than twice is identically zero, then $\mathrm{Lie}\,\{X_1, X_2\} = \mathrm{span}\,\{X_1, X_2, [X_1, X_2]\}$, and

$$\mathrm{rank}\,(\mathrm{Lie}\,\{X_1, X_2\}\,(x)) = 3 \qquad \text{for every } x \in \mathbb{R}^3.$$

The following result holds.

Proposition 2.9 (Nested commutators)**.** *Let $V \subseteq T(\mathbb{R}^N)$ be any set of smooth vector fields on $\mathbb{R}^N$. We set*

$$V_1 := \mathrm{span}\{V\}, \quad V_n := \mathrm{span}\,\{[u, v] \mid u \in V, v \in V_{n-1}\}, \quad n \geq 2.$$

Then we have

$$\mathrm{Lie}\{V\} = \mathrm{span}\,\{V_n \mid n \in \mathbb{N}\}.$$

Moreover,

$$[u, v] \in V_{i+j} \qquad \textit{for every } u \in V_i, v \in V_j.$$

We explicitly remark that, from the definition of V_n, the vector fields in V_n are linear combination of nested brackets, i.e. brackets of the type

$$[u_1\,[u_2\,[u_3\,[\cdots[u_{n-1}, u_n]\cdots]]]]$$

with $u_1, \ldots, u_n \in U$. The above proposition then states that every element of $\mathrm{Lie}\{V\}$ is a linear combination of nested brackets.

Corollary 2.10. *Let $Z_1, \ldots, Z_m \in T(\mathbb{R}^N)$ be fixed. Then*

$$\mathrm{Lie}\,\{Z_1, \ldots, Z_m\} = \mathrm{span}\,\left\{Z_J \mid \ \textit{with } J = (j_1, \ldots, j_k) \in \{1, \ldots, m\}^k, k \in \mathbb{N}\right\}.$$

The following notation will be used when dealing with "stratified" (or "graded") Lie algebras. If V_1, V_2 are subsets of $T(\mathbb{R}^N)$, we denote

$$[V_1, V_2] := \operatorname{span}\{[v_1, v_2] \mid v_i \in V_i, i = 1, 2\}.$$

2.1.2 Lie groups on $\mathbb{R}^N$

The Lie Algebra of a Lie Group on $\mathbb{R}^N$

We first recall a well-known definition.

Definition 2.11 (Lie group on $\mathbb{R}^N$)**.** Let $\circ$ be a given group law on $\mathbb{R}^N$, and suppose that the map

$$\mathbb{R}^N \times \mathbb{R}^N \ni (x, y) \mapsto y^{-1} \circ x \in \mathbb{R}^N$$

is smooth. Then $\mathbb{G} := \left(\mathbb{R}^N, \circ\right)$ is called a Lie group on $\mathbb{R}^N$.

Fixed $\alpha \in \mathbb{G}$, we denote by $\tau_\alpha(x) := \alpha \circ x$ the left-translation by α on $\mathbb{G}$. A (smooth) vector field X on $\mathbb{R}^N$ is called left-invariant on $\mathbb{G}$ if

$$X(\varphi \circ \tau_\alpha) = (X\varphi) \circ \tau_\alpha$$

for every $\alpha \in \mathbb{G}$ and for every smooth function $\varphi : \mathbb{R}^N \to \mathbb{R}$. We denote by $\mathfrak{g}$ the set of the left-invariant vector fields on $\mathbb{G}$. It is quite obvious to recognize that

for every $X, Y \in \mathfrak{g}$ and for every $\lambda, \mu \in \mathbb{R}$, we have $\lambda X + \mu Y \in \mathfrak{g}$ and $[X, Y] \in \mathfrak{g}$.

Then, $\mathfrak{g}$ is a Lie algebra of vector fields, sub-algebra of $T(\mathbb{R}^N)$. It will be called the Lie algebra of $\mathbb{G}$.

Example 2.12 (First Heisenberg group $\mathbb{H}^1$)**.** The map

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_2 y_1 - x_1 y_2))$$

endows $\mathbb{R}^3$ with a structure of Lie group. We shall refer to $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$ as the first Heisenberg group on $\mathbb{R}^3$. It is a direct computation to show that the vector fields

$$X_1 = \partial_{x_1} + 2x_2\partial_{x_3}, \quad X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$$

are left invariant w.r.t. $\circ$. Consequently, $X_1, X_2, [X_1, X_2] \in \mathfrak{h}^1$, say, the Lie algebra of $\mathbb{H}^1$. Precisely, $\mathfrak{h}^1 = \operatorname{span}\{X_1, X_2, [X_1, X_2]\} = \operatorname{Lie}\{X_1, X_2\}$.

From the theorem of differentiation of composite functions, we easily get the following characterization of left-invariant vector fields on $\mathbb{G}$.

Proposition 2.13. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. The (smooth) vector field X belongs to $\mathfrak{g}$ if and only if*

$$(XI)(\alpha \circ x) = \mathcal{J}_{\tau_\alpha}(x) \cdot (XI)(x) \quad \forall \alpha, x \in \mathbb{G}. \tag{2.8}$$

As usual, $\mathcal{J}_{\tau_\alpha}(x)$ denotes the Jacobian matrix at the point x of the map τ_α.

Interchanging α with x in (2.8) we obtain

$$(XI)(x \circ \alpha) = \mathcal{J}_{\tau_x}(\alpha) \cdot (XI)(\alpha)$$

for all $\alpha, x \in \mathbb{G}$, so that, when $\alpha = 0$,

$$(XI)(x) = \mathcal{J}_{\tau_x}(0) \cdot (XI)(0) \quad \forall x \in \mathbb{G}. \tag{2.9}$$

This identity says that a left-invariant vector field on $\mathbb{G}$ is completely determined by its value at the origin (and by the Jacobian matrix at the origin of the left-translation).

Proposition 2.14. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. Let η be a fixed vector of $\mathbb{R}^N$, and define the (component functions of the) vector field X as follows*

$$XI(x) := \mathcal{J}_{\tau_x}(0) \cdot \eta, \quad x \in \mathbb{R}^N.$$

Then $X \in \mathfrak{g}$.

Corollary 2.15. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. The vector field X belongs to $\mathfrak{g}$ iff*

$$(XI)(x) = \mathcal{J}_{\tau_x}(0) \cdot (XI)(0) \quad \forall x \in \mathbb{G}.$$

Example 2.16. If $\mathbb{G} = \mathbb{H}^1$, we have

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & 1 \end{pmatrix}.$$

For example, for $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$, we recognize that, for every $x \in \mathbb{H}^1$,

$$(X_1 I)(x) = \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathcal{J}_{\tau_x}(0) \cdot (XI)(0)$$

The same obviously holds, e.g. for the fields $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$ and $[X_1, X_2] = -4\partial_{x_3}$.

From Proposition 2.13 and identity (2.9) it follows that $\mathfrak{g}$ is a vector space of dimension N. Indeed, the following proposition holds.

Proposition 2.17. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. The map*

$$J : \mathbb{R}^N \to \mathfrak{g}, \quad \eta \mapsto J(\eta)$$

with $J(\eta)$ defined by

$$J(\eta)I(x) = \mathcal{J}_{\tau_x}(0) \cdot \eta$$

is an isomorphism of vector spaces. In particular,

$$\dim \mathfrak{g} = N.$$

Example 2.18. The Lie algebra $\mathfrak{h}^1$ of $\mathbb{G} = \mathbb{H}^1$ is given by $\operatorname{span}\{X_1, X_2, [X_1, X_2]\}$. Indeed, $X_1, X_2, [X_1, X_2]$ are three linearly independent left-invariant vector fields and $\dim(\mathfrak{h}^1) = 3$, as stated in Proposition 2.17. Again using the same proposition, we could also argue as follows: $X_1, X_2, [X_1, X_2]$ are the vector fields obtained by multiplying $\mathcal{J}_{\tau_x}(0)$ respectively times the basis of $\mathbb{R}^3$

$$(1,0,0)^T, \quad (0,1,0)^T, \quad (0,0,-4)^T.$$

Proposition 2.19. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. The vector field X belongs to $\mathfrak{g}$ iff there exists $\eta \in \mathbb{R}^N$ such that, for every $\varphi \in C^\infty\left(\mathbb{R}^N, \mathbb{R}\right)$,*

$$(X\varphi)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(x \circ (t\eta)) \quad \forall x \in \mathbb{R}^N.$$

In this case $\eta = XI(0)$.

Given a family of vector fields $X_1, \dots, X_m \in \mathfrak{g}$, the rank of the subset of $\mathbb{R}^N$ spanned by $\{X_1 I(x), \dots, X_m I(x)\}$ is independent of x. More precisely, we have the following result.

Proposition 2.20. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. Let $X_1, \dots, X_m \in \mathfrak{g}$. Then the following statements are equivalent:*

(i) $X_1, \dots, X_m$ *are linearly independent (in $\mathfrak{g}$);*

(ii) $X_1 I(0), \dots, X_m I(0)$ *are linearly independent (in $\mathbb{R}^N$);*

(iii) $\exists x_0 \in \mathbb{R}^N : X_1 I(x_0), \dots, X_m I(x_0)$ *are linearly independent (in $\mathbb{R}^N$);*

(iv) $X_1 I(x), \dots, X_m I(x)$ *are linearly independent for all $x \in \mathbb{R}^N$.*

The Jacobian Basis

Definition 2.21 (Jacobian basis). Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. If $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$ and J is the map defined in Proposition 2.17, we call

$$\{Z_1, \dots, Z_N\}, \quad Z_j := J(e_j)$$

the Jacobian basis of $\mathfrak{g}$.

From the definition of J we obtain

$$Z_j I(x) = \mathcal{J}_{\tau_x}(0) \cdot e_j = j\text{-th column of } \mathcal{J}_{\tau_x}(0) \quad \forall x \in \mathbb{R}^N, \tag{2.10}$$

so that, since $\mathcal{J}_{\tau_x}(0) = \mathbb{I}_N$,

$$Z_j I(0) = e_j.$$

From Proposition 2.19 we also have

$$(Z_j\varphi)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(x \circ te_j) = \left.\frac{\partial}{\partial y_j}\right|_{y=0} \varphi(x \circ y)$$

for every $\varphi \in C^\infty(\mathbb{R}^N)$ and every $x \in \mathbb{G}$.

Consequently, the Jacobian basis $\{Z_1, \dots, Z_N\}$ of $\mathfrak{g}$ is given by the N column of the Jacobian matrix

$$\mathcal{J}_{\tau_x}(0).$$

Moreover, $Z_j|_0 = \left.\frac{\partial}{\partial y_j}\right|_0$ and

$$(Z_j\varphi)(x) = \left.\frac{\partial}{\partial y_j}\right|_{y=0} \varphi(x \circ y) \quad \forall \varphi \in C^\infty(\mathbb{R}^N), x \in \mathbb{G}.$$

Summing up the above results, we have the following equivalent characterizations of the Jacobian basis.

Proposition 2.22. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. Let $j \in \{1, \dots, N\}$ be fixed. Then there exists one and only one vector field in $\mathfrak{g}$, say Z_j, characterized by any of the following equivalent conditions:*

(1) $Z_j|_0 = \left.\frac{\partial}{\partial x_j}\right|_0$, *i.e.*

$$(Z_j\varphi)(0) = \frac{\partial\varphi}{\partial x_j}(0) \quad \textit{for every } \varphi \in C^\infty(\mathbb{R}^N, \mathbb{R});$$

(2) *for every* $\varphi \in C^\infty\left(\mathbb{R}^N, \mathbb{R}\right)$, *it holds*

$$(Z_j\varphi)(x) = \left.\frac{\partial}{\partial y_j}\right|_{y=0} (\varphi(x \circ y)) \quad \textit{for every } x \in \mathbb{G};$$

(3) *if* e_j *denotes the* j*-th vector of the canonical basis of* $\mathbb{R}^N$, *then*

$$Z_j I(0) = e_j;$$

(4) *the column vector of the component functions of* Z_j *is*

$$Z_j I(x) = \mathcal{J}_{\tau_x}(0) \cdot e_j = j\textit{-th column of } \mathcal{J}_{\tau_x}(0);$$

(5) *for every* $x \in \mathbb{G}$, *we have*

$$(Z_j\varphi)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi(x \circ (te_j)) \quad \textit{for every } \varphi \in C^\infty\left(\mathbb{R}^N, \mathbb{R}\right).$$

The system of vector fields $\mathcal{Z} := \{Z_1, \ldots, Z_N\}$ *is a basis of* $\mathfrak{g}$, *the Jacobian basis. The coordinates of* $X \in \mathfrak{g}$ *w.r.t.* $\mathcal{Z}$ *are, orderly, the entries of the column vector* $XI(0)$.

Example 2.23. The Jacobian basis for the Lie algebra of $\mathbb{H}^1$ is given by

$$Z_1 = \partial_{x_1} + 2x_2\partial_{x_3}, \quad Z_2 = \partial_{x_2} - 2x_1\partial_{x_3}, \quad Z_3 = \partial_{x_3},$$

since, in this case, the Jacobian matrix at 0 of the left-translation is

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & 1 \end{pmatrix}.$$

The (Jacobian) Total Gradient

Let $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ be a Lie group on $\mathbb{R}^N$, and let $Z_1, \ldots, Z_N$ be the Jacobian basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$.

For any differentiable function u defined on an open set $\Omega \subseteq \mathbb{R}^N$, we consider a sort of "intrinsic" gradient of u given by $(Z_1u, \ldots, Z_Nu)$ (in the sequel, we shall call it (Jacobian) total gradient). Then it follows from (2.10) that

$$(Z_1u(x), \ldots, Z_Nu(x)) = \nabla u(x) \cdot \mathcal{J}_{\tau_x}(0) \quad \forall x \in \Omega.$$

On the other hand, since $\mathcal{J}_{\tau_x}(0)$ is non-singular and its inverse is given by $\mathcal{J}_{\tau_{x^{-1}}}(0)$, we can write the Euclidean gradient of u in terms of its total gradient in the following way

$$\nabla u(x) = (Z_1u(x), \ldots, Z_Nu(x)) \cdot \mathcal{J}_{\tau_{x^{-1}}}(0) \quad \forall x \in \Omega. \tag{2.11}$$

From (2.11) we immediately obtain the following result.

Proposition 2.24. *Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $Z_1, \dots, Z_N$ be the relevant Jacobian basis (or any basis for $\mathfrak{g}$). Let $\Omega \subseteq \mathbb{R}^N$ be an open and connected set. A function $u \in C^1(\Omega, \mathbb{R})$ is constant in Ω if and only if its total gradient $(Z_1 u, \dots, Z_N u)$ vanishes identically on Ω.*

Example 2.25. When $\mathbb{G} = \mathbb{H}^1$, it indeed holds

$$\begin{aligned}(Z_1 u, Z_2 u, Z_3 u) &= (\partial_{x_1} u + 2x_2 \partial_{x_3} u, \partial_{x_2} u - 2x_1 \partial_{x_3} u, \partial_{x_3} u)\\ &= (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & 1 \end{pmatrix} = \nabla u \cdot \mathcal{J}_{\tau_x}(0)\end{aligned}$$

and, vice versa,

$$(Z_1 u, Z_2 u, Z_3 u) \cdot \mathcal{J}_{\tau_{x^{-1}}}(0) = (Z_1 u, Z_2 u, Z_3 u) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2x_2 & 2x_1 & 1 \end{pmatrix} = \nabla u.$$

The Exponential Map of a Lie Group on $\mathbb{R}^N$

The next lemma will be useful to define the notion of Exponential map from $\mathfrak{g}$ to $\mathbb{G}$, one of the most important tools in the Lie group theory.

Lemma 2.26. *Let $(\mathbb{G}, \circ)$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be its Lie algebra. Let $X \in \mathfrak{g}$, and let $\gamma : [t_0, t_0 + T] \to \mathbb{R}^N$ be an integral curve of X. Then:*

(i) $\alpha \circ \gamma$ is an integral curve of X for every $\alpha \in \mathbb{G}$.

(ii) γ can be continued to an integral curve of X on the interval $[t_0 - T, t_0 + 2T]$.

From assertion (ii) of this Lemma we immediately obtain the following important statement: for every $X \in \mathfrak{g}$, the map

$$(x, t) \mapsto \exp(tX)(x)$$

is well-defined for every $x \in \mathbb{R}^N$ and every $t \in \mathbb{R}$.

The next corollary easily follows from the assertion (i) of Lemma 2.26.

Corollary 2.27. *Let $(\mathbb{G}, \circ)$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be its Lie algebra. Let $X \in \mathfrak{g}$ and $x, y \in \mathbb{G}$. Then*

$$x \circ \exp(tX)(y) = \exp(tX)(x \circ y)$$

for every $t \in \mathbb{R}$. In particular, for $y = 0$,

$$\exp(tX)(x) = x \circ \exp(tX)(0).$$

Definition 2.28 (Exponential map)**.** Let $\mathbb{G}$ be a Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be its Lie algebra. The exponential map of the Lie group $\mathbb{G}$ is defined by

$$\mathrm{Exp} : \mathfrak{g} \to \mathbb{G}, \quad \mathrm{Exp}(X) = \exp(1 \cdot X)(0).$$

More explicitly, $\mathrm{Exp}(X)$ is the value at the time $t = 1$ of the path $\gamma(t)$ solution to

$$\begin{cases} \dot{\gamma(t)} = XI(\gamma(t)), \\ \gamma(0) = 0. \end{cases}$$

Example 2.29. Let us consider once again the first Heisenberg group $\mathbb{H}^1$ on $\mathbb{R}^3$. We have showed that a basis for its Lie algebra $\mathfrak{h}^1$ is given by X_1, X_2, X_3, where $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$, $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$ and $X_3 = [X_1, X_2] = -4\partial_{x_3}$. Let us construct the exponential map. We set, for $\xi \in \mathbb{R}^3$,

$$\begin{aligned} \xi \cdot X &:= \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3 \\ &= \xi_1 \begin{pmatrix} 1 \\ 0 \\ 2x_2 \end{pmatrix} + \xi_2 \begin{pmatrix} 0 \\ 1 \\ -2x_1 \end{pmatrix} + \xi_3 \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ -4\xi_3 + 2\xi_1 x_2 - 2\xi_2 x_1 \end{pmatrix}. \end{aligned}$$

By definition, for fixed $x \in \mathbb{H}^1$, we have $\exp(\xi \cdot X)(x) = \gamma(1)$, where $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is the solution to

$$\begin{cases} \dot{\gamma}(s) = (\xi \cdot X)I(\gamma(s)) = (\xi_1, \xi_2, -4\xi_3 + 2\xi_1\gamma_2(s) - 2\xi_2\gamma_1(s)), \\ \gamma(0) = x. \end{cases}$$

Solving the above system of ODE's, one gets

$$\exp(\xi \cdot X)(x) = \begin{pmatrix} x_1 + \xi_1 \\ x_2 + \xi_2 \\ x_3 - 4\xi_3 + 2\xi_1 x_2 - 2\xi_2 x_1 \end{pmatrix}$$

As a consequence, by definition above, we obtain

$$\mathrm{Exp}(\xi \cdot X) = \exp(\xi \cdot W)(0) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ -4\xi_3 \end{pmatrix}$$

so that Exp is globally invertible and its inverse map is given by

$$\mathrm{Log}(y) := (\mathrm{Exp})^{-1}(y) = \begin{pmatrix} y_1 \\ y_2 \\ -\frac{1}{4}y_3 \end{pmatrix} \cdot X.$$

For example, we have

$$\mathrm{Exp}(-\xi \cdot X) = \begin{pmatrix} -\xi_1 \\ -\xi_2 \\ +4\xi_3 \end{pmatrix} = -\,\mathrm{Exp}(\xi \cdot X) = (\mathrm{Exp}(-\xi \cdot X))^{-1}$$

since the inverse of x in $\mathbb{H}^1$ coincides with $-x$.

Remark 2.30. Let $\{X_1, \dots, X_N\}$ be a basis of $\mathfrak{g}$. Then, for every $X \in \mathfrak{g}$,

$$X = \sum_{j=1}^{N} \xi_j X_j \quad \text{for a suitable} \quad \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N,$$

so that

$$\mathrm{Exp}(X) = \exp\left(\sum_{j=1}^{N} \xi_j X_j\right)(0).$$

From the classical theory of ODE's we know that the map

$$(\xi_1, \dots, \xi_N) \mapsto \exp\left(\sum_{j=1}^{N} \xi_j X_j\right)(0)$$

is smooth. Then we can say that the map $\mathfrak{g} \ni X \mapsto \mathrm{Exp}(X) \in \mathbb{G}$ is smooth. From the Taylor expansion we get

$$\mathrm{Exp}(X) = \sum_{j=1}^{N} \xi_j \eta_j + \mathcal{O}\left(|\xi|^2\right), \quad \text{as } |\xi| \to 0$$

where $\eta_j = X_j I(0)$.

Denote by E the matrix whose column vectors are $\eta_1, \dots, \eta_N$. Then

$$\mathcal{J}_{\mathrm{Exp}}(0) = E.$$

In particular, if $\{X_1, \dots, X_N\} = \{Z_1, \dots, Z_N\}$ is the Jacobian basis of $\mathfrak{g}$, then

$$\mathcal{J}_{\mathrm{Exp}}(0) = \mathbb{I}_N.$$

As a consequence, Exp is a diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $0 \in \mathbb{G}$. Where defined, we denote by Log the inverse map of Exp.

The next proposition is an easy consequence of Corollary 2.27 and shows an important link between the composition law in $\mathbb{G}$ and the exponential map.

Proposition 2.31. *Let* $(\mathbb{G}, \circ)$ *be a Lie group on* $\mathbb{R}^N$*, and let* $\mathfrak{g}$ *be its Lie algebra. Let* $x, y \in \mathbb{G}$*. Assume* $\mathrm{Log}(y)$ *is defined. Then*

$$x \circ y = \exp(\mathrm{Log}(y))(x).$$

Remark 2.32. Suppose that

$$\mathrm{Exp} : \mathfrak{g} \to \mathbb{G} \quad \text{and} \quad \mathrm{Log} : \mathbb{G} \to \mathfrak{g}$$

are globally defined C^∞ maps, inverse to each other. We then define on $\mathfrak{g}$ the operation

$$X \diamond Y := \mathrm{Log}(\mathrm{Exp}(X) \circ \mathrm{Exp}(Y)), \quad X, Y \in \mathfrak{g}.$$

It is immediately seen that $\diamond$ defines a Lie group structure on $\mathfrak{g}$ and

$$\mathrm{Exp} : (\mathfrak{g}, \diamond) \to (\mathbb{G}, \circ)$$

is a Lie-group isomorphism. Indeed, this last fact is obvious from the definition of $\diamond$, whereas the associativity of $\diamond$ on $\mathfrak{g}$ follows immediately from the associativity of o on $\mathbb{G}$.

One of the most striking facts about Lie algebras and Lie groups is that (under suitable hypotheses) the operation $\diamond$ on $\mathfrak{g}$ is well-posed and can be expressed in a somewhat "universal" way as a sum of iterated Lie-brackets of X and Y. For example, the first few terms are

$$X \diamond Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots . \tag{2.12}$$

Example 2.33. When $\mathbb{G} = \mathbb{H}^1$, we saw that Exp is globally invertible. We fix $X \in \mathfrak{h}^1$. If Z_1, Z_2, Z_3 is the Jacobian basis for $\mathfrak{h}^1$, and we set, for brevity, $\xi := XI(0)$, we have

$$X = \xi_1 Z_1 + \xi_2 Z_2 + \xi_3 Z_3 =: \xi \cdot Z.$$

Analogously, if $Y \in \mathfrak{h}^1$, we set $\eta := YI(0)$, so that $Y = \eta \cdot Z$. Thus, we derive

$$\begin{aligned} &\mathrm{Log}(\mathrm{Exp}(X) \circ \mathrm{Exp}(Y)) \\ &\quad = \mathrm{Log}(\mathrm{Exp}(\xi \cdot Z) \circ \mathrm{Exp}(\eta \cdot Z)) \\ &\quad = \mathrm{Log}(\xi \circ \eta) \\ &\quad = \mathrm{Log}\left(\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3 + 2\eta_1\xi_2 - 2\eta_2\xi_1\right) \\ &\quad = \left(\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3 + 2\eta_1\xi_2 - 2\eta_2\xi_1\right) \cdot Z \\ &\quad = \left(\xi_1 + \eta_1\right) Z_1 + \left(\xi_2 + \eta_2\right) Z_2 + \left(\xi_3 + \eta_3 + 2\eta_1\xi_2 - 2\eta_2\xi_1\right) Z_3. \end{aligned} \tag{2.13}$$

On the other hand, we consider (2.12), truncated to the commutators of length two (sine $\mathfrak{h}^1$ is nilpotent of step two!), and we explicitly write down $X \diamond Y$ in our case, thus obtaining

$$\begin{aligned}(\xi \cdot Z) \diamond (\eta \cdot Z) =& \xi \cdot Z + \eta \cdot Z + \frac{1}{2}[\xi \cdot Z, \eta \cdot Z] \\ =& \xi_1 Z_1 + \xi_2 Z_2 + \xi_3 Z_3 + \eta_1 Z_1 + \eta_2 Z_2 + \eta_3 Z_3 \\ & + \frac{1}{2}\left[\xi_1 Z_1 + \xi_2 Z_2 + \xi_3 Z_3, \eta_1 Z_1 + \eta_2 Z_2 + \eta_3 Z_3\right] \\ & \text{(here we use } [Z_1, Z_2] = -4Z_3, [Z_1, Z_3] = [Z_2, Z_3] = 0) \\ =& (\xi_1 + \eta_1) Z_1 + (\xi_2 + \eta_2) Z_2 + (\xi_3 + \eta_3) Z_3 + \frac{1}{2}\left(\left(-4\xi_1\eta_2 + 4\xi_2\eta_1\right) Z_3\right) \\ =& (\xi_1 + \eta_1) Z_1 + (\xi_2 + \eta_2) Z_2 + (\xi_3 + \eta_3 + 2\eta_1\xi_2 - 2\eta_2\xi_1) Z_3\end{aligned}$$

which equals the last term in (2.13). As a consequence, we have proved that in this case it holds

$$\mathrm{Log}(\mathrm{Exp}(X) \circ \mathrm{Exp}(Y)) = X + Y + \frac{1}{2}[X, Y].$$

Homogeneous Lie Groups in $\mathbb{R}^N$

We begin by giving the definition of homogeneous Lie group (see also E.M. Stein [Ste93]).

Definition 2.34 (Homogeneous Lie group (on $\mathbb{R}^N$))**.** Let $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ be a Lie group on $\mathbb{R}^N$. We say that $\mathbb{G}$ is a homogeneous (Lie) group (on $\mathbb{R}^N$) if the following property holds:

(H) There exists an N-tuple of real numbers $\sigma = (\sigma_1, \dots, \sigma_N)$, with $1 \le \sigma_1 \le \cdots \le \sigma_N$, such that the "dilation" $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$, $\delta_\lambda(x_1, \dots, x_N) := (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N)$ is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$.

We shall denote by $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ the datum of a homogeneous Lie group on $\mathbb{R}^N$ with composition law $\circ$ and dilation group $\{\delta_\lambda\}_{\lambda>0}$.

The family of dilations $\{\delta_\lambda\}_{\lambda>0}$ forms a one-parameter group of automorphisms of $\mathbb{G}$ whose identity is

$$\delta_1 = I,$$

the identity map of $\mathbb{R}^N$. Indeed, we have

$$\delta_{rs}(x) = \delta_r(\delta_s(x)) \quad \forall x \in \mathbb{G}, r, s > 0.$$

Moreover, $(\delta_\lambda)^{-1} = \delta_{\lambda^{-1}}$. In the sequel, $\{\delta_\lambda\}_{\lambda>0}$ will be referred to as the dilation group (or group of dilations) of $\mathbb{G}$.

From (H) it follows that

$$\delta_\lambda(x \circ y) = (\delta_\lambda x) \circ (\delta_\lambda y) \quad \forall x, y \in \mathbb{G} \tag{2.14}$$

and, if e denotes the identity of $\mathbb{G}, \delta_\lambda(e) = e$ for every $\lambda > 0$. This obviously implies that $e = 0$. This is consistent with our previous assumption that the origin is the identity of $\mathbb{G}$.

For example, the first Heisenberg group $\mathbb{H}^1$ is a homogeneous Lie group if $\mathbb{R}^3$ is equipped with the dilations $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$.

Remark 2.35. Suppose $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ is a Lie group on $\mathbb{R}^N$ such that there exists an N-tuple of positive real numbers $\sigma = (\sigma_1, \ldots, \sigma_N)$ such that

$$d_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad d_\lambda(x_1, \ldots, x_N) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N)$$

is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$. Then, modulo a permutation of the variables of $\mathbb{R}^N$, it is always not restrictive to suppose that $\sigma_1 \leq \cdots \leq \sigma_N$. Obviously, this permutation of the coordinates does not alter neither (the new permuted) $\mathbb{G}$ being a Lie group on $\mathbb{R}^N$ nor the (relevant permuted) dilation δ_λ satisfying (2.14). Moreover, there exists a group of dilations δ_λ on $\mathbb{G}$ such that

$$\delta_\lambda(x_1, \ldots, x_N) = \left(\lambda^{\tilde{\sigma}_1} x_1, \ldots, \lambda^{\tilde{\sigma}_N} x_N\right)$$

with $1 = \tilde{\sigma}_1 \leq \cdots \leq \tilde{\sigma}_N$. Indeed, it suffices to take (once the σ_j 's have been ordered increasingly)

$$\tilde{\sigma}_j := \sigma_j / \sigma_1 \quad \text{for every } j = 1, \ldots, N$$

With this choice, we have

$$\delta_\lambda \equiv d_{\lambda^{1/\sigma_1}}$$

and $\delta_\lambda(x \circ y) = \delta_\lambda(x) \circ \delta_\lambda(y)$ follows from (2.14), restated for d_λ, with λ replaced by λ^{1/σ_1}.

δ_λ-homogeneous Functions and Differential Operators

Before we continue the analysis of homogeneous Lie groups, we show some basic properties of homogeneous functions and homogeneous differential operators with respect to the family $\{\delta_\lambda\}_\lambda$.

In this subsection, no group law is required on $\mathbb{R}^N$. Here, we only suppose that it is given on $\mathbb{R}^N$ a family of maps δ_λ of the form

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x_1, \ldots, x_N) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N) \tag{2.15}$$

with fixed positive real numbers $\sigma_1, \ldots, \sigma_N$. We set $\sigma := (\sigma_1, \ldots, \sigma_N)$.

A real function a defined on $\mathbb{R}^N$ is called δ_λ-homogeneous of degree $m \in \mathbb{R}$ if a does not vanish identically and, for every $x \in \mathbb{R}^N$ and $\lambda > 0$, it holds

$$a\left(\delta_\lambda(x)\right) = \lambda^m a(x).$$

A non-identically-vanishing linear differential operator X is called δ_λ-homogeneous of degree $m \in \mathbb{R}$ if, for every $\varphi \in C^\infty\left(\mathbb{R}^N\right), x \in \mathbb{R}^N$ and $\lambda > 0$, it holds

$$X\left(\varphi\left(\delta_\lambda(x)\right)\right) = \lambda^m (X\varphi)\left(\delta_\lambda(x)\right).$$

Let a be a smooth δ_λ-homogeneous function of degree $m \in \mathbb{R}$ and X be a linear differential operator δ_λ-homogeneous of degree $n \in \mathbb{R}$. Then Xa is a δ_λ-homogeneous function of degree $m - n$ (unless $Xa \equiv 0$). Indeed, for every $x \in \mathbb{R}^N$ and $\lambda > 0$, we have

$$\lambda^n (Xa)\left(\delta_\lambda(x)\right) = X\left(a\left(\delta_\lambda(x)\right)\right) = X\left(\lambda^m a(x)\right) = \lambda^m (Xa)(x).$$

Given a multi-index $\alpha \in (\mathbb{N} \cup \{0\})^N, \alpha = (\alpha_1, \ldots, \alpha_N)$, we define the δ_λ-length (or δ_λ-height) of α as

$$|\alpha|_\sigma = \langle \alpha, \sigma \rangle = \sum_{i=1}^{N} \alpha_i \sigma_i.$$

Definition 2.36. When $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ is a homogeneous Lie group on $\mathbb{R}^N$ with its given group of dilations $\{\delta_\lambda\}_\lambda$, we shall use the notation $|\alpha|_{\mathbb{G}}$ for the relevant δ_λ-length. In this case, we shall refer to $|\alpha|_{\mathbb{G}}$ as the $\mathbb{G}$-length (or $\mathbb{G}$-height) of α. Moreover, if $p : \mathbb{G} \to \mathbb{R}$ is a polynomial function (the sum below is intended to be finite)

$$p(x) = \sum_{\alpha} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{R},$$

we say that

$$\deg_{\mathbb{G}}(p) := \max\left\{|\alpha|_{\mathbb{G}} : c_\alpha \neq 0\right\}$$

is the $\mathbb{G}$-degree or δ_λ-(homogeneous) degree of p.

Let us now consider a smooth and δ_λ-homogeneous of degree $m \in \mathbb{R}$ function a and a multi-index α. Assume that $D^\alpha a$ is not identically zero. Then, since $D^\alpha a$ is smooth and δ_λ-homogeneous of degree $m - |\alpha|_\sigma$, it has to be $m - |\alpha|_\sigma \geq 0$, i.e. $|\alpha|_\sigma \leq m$. This result can be restated as follows:

$$D^\alpha a \equiv 0 \quad \forall \alpha \text{ such that } |\alpha|_\sigma > m.$$

Thus a is a polynomial function. Let $a(x) = \sum\limits_{\alpha \in \mathcal{A}} a_\alpha x^\alpha$, where $\mathcal{A}$ is a finite set of multi-indices and $a_\alpha \in \mathbb{R}$ for every $\alpha \in \mathcal{A}$. Since a is δ_λ-homogeneous of degree m, we have

$$\sum_{\alpha\in\mathcal{A}} \lambda^m a_\alpha x^\alpha = \lambda^m a(x) = a\left(\delta_\lambda(x)\right) = \sum_{\alpha\in\mathcal{A}} a_\alpha \lambda^{|\alpha|_\sigma} x^\alpha.$$

Hence $\lambda^m a_\alpha = \lambda^{|\alpha|_\sigma} a_\alpha$ for every $\lambda > 0$, so that $|\alpha|_\sigma = m$ if $a_\alpha \neq 0$. Then

$$a(x) = \sum_{|\alpha|_\sigma = m} a_\alpha x^\alpha. \tag{2.16}$$

It is quite obvious that every polynomial function of the form (2.16) is δ_λ-homogeneous of degree m. Thus, we have proved the following proposition.

Proposition 2.37 (Smooth δ_λ-homogeneous functions)**.** *Let δ_λ be as in* (2.15)*. Suppose that $a \in C^\infty\left(\mathbb{R}^N, \mathbb{R}\right)$. Then a is δ_λ-homogeneous of degree $m \in \mathbb{R}$ if and only if a is a polynomial function of the form* (2.16) *with some $a_\alpha \neq 0$. As a consequence, the set of the degrees of the smooth δ_λ-homogeneous (non-vanishing) functions is precisely the set of the nonnegative real numbers*

$$\mathcal{A} = \left\{|\alpha|_\sigma : \alpha \in (\mathbb{N} \cup \{0\})^N\right\},$$

with $|\alpha|_\sigma = 0$ if and only if a is constant.

From the proposition above one easily obtains the following characterization of the smooth δ_λ-homogeneous vector fields.

Proposition 2.38 (Smooth δ_λ-homogeneous vector fields)**.** *Let δ_λ be as in* (2.15)*. Let X be a smooth non-vanishing vector field on $\mathbb{R}^N$,*

$$X = \sum_{j=1}^{N} a_j(x)\partial_{x_j}.$$

Then X is δ_λ-homogeneous of degree $n \in \mathbb{R}$ if and only if a_j is a polynomial function δ_λ-homogeneous of degree $\sigma_j - n$ (unless $a_j \equiv 0$). Hence, the degree of δ_λ- homogeneity of X belongs to the set of real (possibly negative) numbers

$$\mathcal{A}_j = \left\{\sigma_j - |\alpha|_\sigma : \alpha \in (\mathbb{N} \cup \{0\})^N\right\},$$

whenever j is such that a_j is not identically zero.

Corollary 2.39. *Let δ_λ be as in* (2.15)*. Let X be a smooth non-vanishing vector field. Then X is δ_λ-homogeneous of degree $n \in \mathbb{R}$ iff*

$$\delta_\lambda(XI(x)) = \lambda^n XI\left(\delta_\lambda(x)\right).$$

As a straightforward consequence, we have the following simple fact.

Remark 2.40. Let δ_λ be as in (2.15). Let $X \neq 0$ be a smooth vector field on $\mathbb{R}^N$ of the form

$$X = \sum_{j=1}^{N} a_j(x)\partial_{x_j}.$$

If X is δ_λ-homogeneous of degree $n \in \mathbb{R}$, then, for every a_j non-identically zero, we must have $n \leq \sigma_j$. As a consequence, it has to be $n \leq \sigma_N$ (i.e. the set of the δ_λ-homogeneous degrees of the smooth vector fields is bounded above by the maximum exponent of the dilation). Hence, X has the form

$$X = \sum_{j \leq N, \sigma_j \geq n} a_j(x)\partial x_j.$$

From this remark the next proposition straightforwardly follows.

Proposition 2.41. *Let δ_λ be as in* (2.15). *Let $X = \sum_{j=1}^{N} a_j(x)\partial_{x_j}$ be a smooth vector field δ_λ-homogeneous of positive degree. Then its adjoint operator $X^* = -\sum_{j=1}^{N} \partial_j(a_j \cdot)$ satisfies $X^* = -X$ and*

$$X^2 = \operatorname{div}\left(A \cdot \nabla^T\right)$$

where A is the square matrix $(a_i a_j)_{i,j \leq N}$. Finally, X has null divergence.

Vector fields with different degree of δ_λ-homogeneity are linearly independent if they do not vanish at the origin. Indeed, the following proposition holds.

Proposition 2.42. *Let δ_λ be as in* (2.15). *Let $X_1, \ldots, X_k \in T(\mathbb{R}^N)$ be δ_λ-homogeneous vector fields of degree $n_1, \ldots, n_k$, respectively. If $n_i \neq n_j$ for $i \neq j$ and if $X_j I(0) \neq 0$ for every $j \in \{1, \ldots, k\}$, then $X_1, \ldots, X_k$ are linearly independent.*

The following simple proposition will be useful in the sequel.

Proposition 2.43. *Let δ_λ be as in* (2.15). *Let X_1, X_2 be δ_λ-homogeneous vector fields of degree n_1, n_2, respectively. Then $[X_1, X_2]$ is δ_λ-homogeneous of degree $n_1 + n_2$ (unless X_1 and X_2 commute).*

As a consequence, if n_1, n_2 are both positive, then every commutator of X_1, X_2 containing k_1 times X_1 and k_2 times X_2 vanish identically whenever $k_1 n_1 + k_2 n_2 > \sigma_N$.

For example, the differential operators $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}, X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$ on the first Heisenberg group $\mathbb{H}^1$ are homogeneous of degree one with respect to the dilation $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$, and $[X_1, X_2] = -4\partial_{x_3}$ is indeed δ_λ homogeneous of degree two. Moreover, any commutator of X_1, X_2 of length ≥ 3 vanish identically.

Corollary 2.44. *Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. Let $X_1, \ldots, X_k \in \mathfrak{g}$ be non-identically vanishing and δ_λ-homogeneous of degrees $n_1, \ldots, n_k$, respectively. If $n_i \neq n_j$ for $i \neq j$, then $X_1, \ldots, X_k$ are linearly independent.*

Proposition 2.45 (Nilpotence of homogeneous Lie groups on $\mathbb{R}^N$)**.** *Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$. Then $\mathbb{G}$ is nilpotent of step $\leq \sigma_N$, i.e. every commutator of vector fields in $\mathfrak{g}$ containing more than σ_N terms vanishes identically.*

Moreover, if Z_j is the j-th element of the Jacobian basis of $\mathfrak{g}$, Z_j is δ_λ homogeneous of degree σ_j.

The Composition Law of a Homogeneous Lie Group

By using the elementary properties of the homogeneous functions showed in the previous subsection, we shall obtain a structure theorem for the composition law in a homogeneous Lie group $\left(\mathbb{R}^N, \circ, \delta_\lambda\right)$. We first recall two lemmas.

Lemma 2.46. *Let δ_λ be as in* (2.15)*. Let $P : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a smooth nonvanishing function such that*

$$P\left(\delta_\lambda(x), \delta_\lambda(y)\right) = \lambda^{\sigma_j} P(x, y) \quad \forall x, y \in \mathbb{R}^N, \forall \lambda > 0,$$

for some j such that $1 \leq j \leq N$. Assume also that

$$P(x, 0) = x_j, \quad P(0, y) = y_j.$$

Then $P(x, y) = x_1 + y_1$ if $j = 1$ and, if $j \geq 2$,

$$P(x, y) = x_j + y_j + \widetilde{P}\left(x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1}\right),$$

where $\widetilde{P}$ is a polynomial, the sum of mixed monomials in $x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1}$. Moreover, $\widetilde{P}\left(\delta_\lambda(x), \delta_\lambda(y)\right) = \lambda^{\sigma_j} \widetilde{P}(x, y)$. Finally, $P(x, y)$ only depends on the x_k and y_k with $\sigma_k < \sigma_j$.

Lemma 2.47. *Let δ_λ be as in* (2.15)*. Let $Q : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that*

$$Q\left(\delta_\lambda(x), \delta_\lambda(y)\right) = \lambda^m Q(x, y) \quad \forall x, y \in \mathbb{R}^N, \forall \lambda > 0$$

where $m \geq 0$. Then

$$x \mapsto \frac{\partial Q}{\partial y_j}(x, 0)$$

is δ_λ-homogeneous of degree $m - \sigma_j$ (unless it vanishes identically).

Now, we are in the position to give the structure theorem for the composition law of a homogeneous Lie group on $\mathbb{R}^N$.

Theorem 2.48 (Composition of a homogeneous Lie group on $\mathbb{R}^N$)**.** *Let $\left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$. Then $\circ$ has polynomial component functions. Furthermore, we have*

$$(x \circ y)_1 = x_1 + y_1, \quad (x \circ y)_j = x_j + y_j + Q_j(x, y), \quad 2 \leq j \leq N,$$

and the following facts hold:

(1) *Q_j only depends on $x_1, \ldots, x_{j-1}$ and $y_1, \ldots, y_{j-1}$;*

(2) *Q_j is a sum of mixed monomials in x, y;*

(3) *$Q_j\left(\delta_\lambda x, \delta_\lambda y\right) = \lambda^{\sigma_j} Q_j(x, y)$.*

More precisely, $Q_j(x, y)$ only depends on the x_k and y_k with $\sigma_k < \sigma_j$.

Corollary 2.49. *Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$. Let $j \in \{1, \ldots, N\}$. For every $y \in \mathbb{G}$, we have*

$$\left(y^{-1}\right)_j = -y_j + q_j(y),$$

where $q_j(y)$ is a polynomial function in y, δ_λ-homogeneous of degree σ_j, only depending on the y_k with $\sigma_k < \sigma_j$.

Corollary 2.50. *Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$. Let $j \in \{1, \ldots, N\}$. For every $x, y \in \mathbb{G}$, we have*

$$\left(y^{-1} \circ x\right)_j = x_j - y_j + \sum_{k : \sigma_k < \sigma_j} P_k^{(j)}(x, y)\left(x_k - y_k\right),$$

where $P_k^{(j)}(x, y)$ is a polynomial function in x and y only depending on the x_k and y_k with $\sigma_k < \sigma_j$.

The following result describes in a very explicit way the Jacobian matrix at 0 of the left-translation τ_x on a homogeneous Lie group on $\mathbb{R}^N$.

Corollary 2.51 (The Jacobian basis of a homogeneous Lie group)**.** *Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a* *homogeneous Lie group on $\mathbb{R}^N$. Then we have*

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_2^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_N^{(1)} & \cdots & a_N^{(N-1)} & 1 \end{pmatrix}.$$

where $a_i^{(j)}$ is a polynomial function δ_λ-homogeneous of degree $\sigma_i - \sigma_j$. As a consequence, if we let

$$Z_j = \partial_{x_j} + \sum_{i=j+1}^{N} a_i^{(j)} \partial_{x_i} \quad \text{for } 1 \leq j \leq N-1 \text{ and } \quad Z_N = \partial_{x_N}$$

then Z_j is a left-invariant vector field δ_λ-homogeneous of degree σ_j. Moreover,

$$\mathcal{J}_{\tau_x}(0) = (Z_1 I(x) \cdots Z_N I(x))$$

In other words, the Jacobian basis $Z_1, \ldots, Z_N$ for the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is formed by δ_λ-homogeneous vector fields of degree $\sigma_1, \ldots, \sigma_N$, respectively.

Example 2.52. For the first Heisenberg group $\mathbb{H}^1$, we showed that the Jacobian matrix of the left translation on $\mathbb{H}^1$ is

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & 1 \end{pmatrix}.$$

We recognize that the three columns of this matrix give raise to the Jacobian basis $Z_1 = \partial_{x_1} + 2x_2\partial_{x_3}, Z_2 = \partial_{x_2} - 2x_1\partial_{x_3}$ and $Z_3 = \partial_{x_3}$ and these vector fields are homogeneous of degree, respectively, $1, 1, 2$ with respect to $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$.

The structure Theorem 2.48 of the composition law of $(\mathbb{R}^N, \circ, \delta_\lambda)$ implies that the Lebesgue measure on $\mathbb{R}^N$ is invariant under left and right translations on $\mathbb{G}$. Indeed, by Theorem 2.48, the Jacobian matrices of the functions $x \mapsto \alpha \circ x$ and $x \mapsto x \circ \alpha$ have the following lower triangular form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \star & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \cdots & \star & 1 \end{pmatrix}$$

Then, we have proved the following proposition.

Proposition 2.53 (Haar measure on a homogeneous Lie group). *Let $\mathbb{G}$ be a homogeneous Lie group on $\mathbb{R}^N$. Then the Lebesgue measure on $\mathbb{R}^N$ is invariant with respect to the left and the right translations on $\mathbb{G}$.*

If we denote by $|E|$ the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^N$, we then have

$$|\alpha \circ E| = |E| = |E \circ \alpha| \quad \forall \alpha \in \mathbb{G}.$$

We also have that the Lebesgue measure is homogeneous with respect to the dilations $\{\delta_\lambda\}_{\lambda>0}$. More precisely, as a trivial computation shows,

$$|\delta_\lambda(E)| = \lambda^Q |E|,$$

where

$$Q = \sum_{j=1}^{N} \sigma_j.$$

The positive number Q is called the homogeneous dimension of the group $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$.

For example, in the case of the first Heisenberg group $\mathbb{H}^1$, where τ_α is given by

$$\tau_\alpha(x) = (\alpha_1 + x_1, \alpha_2 + x_2, \alpha_3 + x_3 + 2(\alpha_2 x_1 - \alpha_1 x_2)),$$

and $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$, we have

$$\mathcal{J}_{\tau_\alpha}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2\alpha_2 & -2\alpha_1 & 1 \end{pmatrix}, \quad \mathcal{J}_{\delta_\lambda}(x) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix},$$

so that, for every $\alpha, x \in \mathbb{H}^1$ and every $\lambda > 0$, we have

$$\det \mathcal{J}_{\tau_\alpha}(x) = 1, \quad \det \mathcal{J}_{\delta_\lambda}(x) = \lambda^4 = \lambda^Q,$$

since the homogeneous dimension of $\mathbb{H}^1$ is $Q = 1 + 1 + 2 = 4$.

The Lie Algebra of a Homogeneous Lie Group on $\mathbb{R}^N$

Let $\mathbb{G}$ be a homogeneous Lie group on $\mathbb{R}^N$ with Lie algebra $\mathfrak{g}$. From Corollary 2.51 we easily obtain the splitting of $\mathfrak{g}$ as a direct sum of linear spaces spanned by vector fields of constant degree of δ_λ-homogeneity.

More precisely, let us recall that the exponents σ_j in the dilation δ_λ of $\mathbb{G}$ satisfy $\sigma_1 \le \cdots \le \sigma_N$ and can then be grouped together to produce real and natural numbers, respectively, say

$$n_1, \ldots, n_r \quad \text{and} \quad N_1, \ldots, N_r,$$

such that

$$n_1 < n_2 < \cdots < n_r, \quad N_1 + N_2 + \cdots + N_r = N,$$

defined by

$$\begin{cases} n_1 = \sigma_j & \text{for } 1 \leq j \leq N_1 \\ n_2 = \sigma_j & \text{for } N_1 < j \leq N_1 + N_2 \\ \quad \vdots \\ n_r = \sigma_j & \text{for } N_1 + \cdots + N_{r-1} < j \leq N_1 + \cdots + N_{r-1} + N_r \end{cases}$$

Let $Z_1, \ldots, Z_N$ be the Jacobian basis of $\mathfrak{g}$. Define

$$\begin{aligned} &\mathfrak{g}_1 = \operatorname{span}\{Z_j \mid 1 \leq j \leq N_1\} \quad \text{ and, for } i = 2, \ldots, r \\ &\mathfrak{g}_i = \operatorname{span}\{Z_j \mid N_1 + \cdots + N_{i-1} < j \leq N_1 + \cdots + N_{i-1} + N_i\} \end{aligned}$$

By Corollary 2.51, the generators Z_j of $\mathfrak{g}_i$ are δ_λ-homogeneous vector fields of degree $n_i, 1 \leq i \leq r$. Moreover, we obviously have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

We also explicitly notice that a vector field $X \in \mathfrak{g}$ is δ_λ-homogeneous of degree n iff, for a suitable $i \in \{1, \ldots, r\}, n = n_i$ and $X \in \mathfrak{g}_i$.

Example 2.54. The usual additive group $(\mathbb{R}^3, +)$ is a homogeneous Lie group if equipped with the dilation

$$\delta_\lambda(x_1, x_2, x_3) = \left(\lambda^2 x_1, \lambda^\pi x_2, \lambda^4 x_3\right)$$

The decomposition of the Lie algebra is

$$\operatorname{span}\{\partial_{x_1}\} \oplus \operatorname{span}\{\partial_{x_2}\} \oplus \operatorname{span}\{\partial_{x_3}\}.$$

Moreover, $\mathbb{R}^4$ is a homogeneous Lie group if equipped with the group law

$$x \circ y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + 2y_1x_2 - 2y_2x_1 \\ x_4 + y_4 \end{pmatrix}$$

and the dilation

$$\delta_\lambda(x_1, x_2, x_3, x_4) = \left(\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 x_4\right).$$

The decomposition of the Lie algebra is

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \operatorname{span}\{X_1, X_2\} \oplus \operatorname{span}\{\partial_{x_3}, \partial_{x_4}\}$$

where $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}, X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$. Note that

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subsetneqq \mathfrak{g}_2.$$

Observe that the above $(\mathbb{R}^4, \circ)$ is isomorphic to the homogeneous Lie group $(\mathbb{R}^4, *)$ with the composition law

$$\xi * \eta = \begin{pmatrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \xi_3 + \eta_3 \\ \xi_4 + \eta_4 + 2\eta_1\xi_2 - 2\eta_2\xi_1 \end{pmatrix}$$

and the new group of dilations

$$\delta_\lambda \left(\xi_1, \xi_2, \xi_3, \xi_4\right) = \left(\lambda\xi_1, \lambda\xi_2, \lambda\xi_3, \lambda^2 x_4\right).$$

The decomposition of the Lie algebra is

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \operatorname{span}\left\{Z_1, Z_2, \partial_{\xi_3}\right\} \oplus \operatorname{span}\left\{\partial_{\xi_4}\right\},$$

where $Z_1 = \partial_{\xi_1} + 2\xi_2\partial_{\xi_4}, Z_2 = \partial_{\xi_2} - 2\xi_1\partial_{\xi_4}$. Note that this time we have

$$[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2.$$

Definition 2.55 (Dilations on the Lie algebra of a homogeneous Lie group)**.** Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$ with Lie algebra $\mathfrak{g}$ and dilation

$$\delta_\lambda \left(x_1, \ldots, x_N\right) = \left(\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N\right).$$

We define a group of dilations on $\mathfrak{g}$ (which we still denote by δ_λ) as follows: δ_λ is the (only) linear (auto)morphism of $\mathfrak{g}$ mapping the j-th element Z_j of the Jacobian basis for $\mathfrak{g}$ into $\lambda^{\sigma_j} Z_j$.

In other words, if $X \in \mathfrak{g}$ is written w.r.t. the Jacobian basis $Z_1, \ldots, Z_N$ as

$$X = \sum_{j=1}^{N} c_j Z_j, \qquad \text{we then have} \qquad \delta_\lambda(X) = \sum_{j=1}^{N} c_j \lambda^{\sigma_j} Z_j.$$

We immediately recognize that, if $\pi : \mathfrak{g} \to \mathbb{R}^N$ is the map defined by $\pi(X) = XI(0)$, it holds

$$\pi\left(\delta_\lambda(X)\right) = \delta_\lambda(\pi(X)) \quad \forall X \in \mathfrak{g}.$$

Indeed, we have

$$\begin{aligned} \delta_\lambda(\pi(X)) &= \delta_\lambda\left(\pi\left(\sum_{j=1}^{N} c_j Z_j\right)\right) = \delta_\lambda\left(\sum_{j=1}^{N} c_j \pi\left(Z_j\right)\right) = \delta_\lambda\left(\sum_{j=1}^{N} c_j \left(Z_j\right) I(0)\right) \\ &= \delta_\lambda\left(c_1, \ldots, c_N\right) = \left(\lambda^{\sigma_1} c_1, \ldots, \lambda^{\sigma_N} c_N\right) \end{aligned}$$

and, on the other hand,

$$\begin{aligned}\pi\left(\delta_\lambda(X)\right) &= \pi\left(\delta_\lambda\left(\sum_{j=1}^{N} c_j Z_j\right)\right) = \pi\left(\sum_{j=1}^{N} c_j \lambda^{\sigma_j} Z_j\right) \\ &= \sum_{j=1}^{N} c_j \lambda^{\sigma_j} \pi\left(Z_j\right) = \left(\lambda^{\sigma_1} c_1, \ldots, \lambda^{\sigma_N} c_N\right).\end{aligned}$$

The following simple and very useful fact holds.

Proposition 2.56. *Let $\mathbb{G}$ be a homogeneous Lie group on $\mathbb{R}^N$ with Lie algebra $\mathfrak{g}$. The dilation on $\mathfrak{g}$ introduced in Definition* 2.55 *is a Lie algebra automorphism of $\mathfrak{g}$, i.e.*

$$\delta_\lambda([X,Y]) = [\delta_\lambda(X), \delta_\lambda(Y)] \quad \forall X, Y \in \mathfrak{g}.$$

The Exponential Map of a Homogeneous Lie Group

Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$ with Lie algebra $\mathfrak{g}$. The exponential map on $\mathfrak{g}$ has some remarkable properties, due to the homogeneous structure of $\mathbb{G}$. We give such properties in what follows.

Let $Z_1, \ldots, Z_N$ be the Jacobian basis of $\mathfrak{g}$. By Corollary 2.51, Z_j is δ_λ-homogeneous of degree σ_j and takes the form

$$Z_j = \sum_{k=j}^{N} a_k^{(j)}\left(x_1, \ldots, x_{k-1}\right) \partial_{x_k},$$

where $a_k^{(j)}$ is a polynomial function δ_λ-homogeneous of degree $\sigma_k - \sigma_j$ and $a_j^{(j)} \equiv 1$. We now consider on $\mathfrak{g}$ the dilation group introduced in Definition 2.55, i.e. with abuse of notation

$$\delta_\lambda : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \delta_\lambda\left(\sum_{j=1}^{N} \xi_j Z_j\right) := \sum_{j=1}^{N} \lambda^{\sigma_j} \xi_j Z_j. \tag{2.17}$$

The dilation (2.17) is consistent with the one in $\mathbb{G}$. More precisely, if $Z \in \mathfrak{g}$ then, for every $\lambda > 0$, it holds

$$\delta_\lambda(ZI(x)) = (\delta_\lambda Z)\, I\left(\delta_\lambda(x)\right) \quad \forall x \in \mathbb{G}.$$

Lemma 2.57. *Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Lie group on $\mathbb{R}^N$ with Lie algebra $\mathfrak{g}$. Denote also by δ_λ the dilation* (2.17) *on $\mathfrak{g}$. Let $\gamma : [0,T] \to \mathbb{R}^N$ be an integral curve of Z with $Z \in \mathfrak{g}$. Then $\Gamma := \delta_\lambda(\gamma)$ is an integral curve of $\delta_\lambda(Z)$.*

We are now in the position to give the following important theorem.

Theorem 2.58 (Exponential map of a homogeneous Lie group)**.** *Let* $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ *be a homogeneous Lie group with Lie algebra* $\mathfrak{g}$*. Then*

$$\mathrm{Exp} : \mathfrak{g} \to \mathbb{G} \quad \textit{and} \quad \mathrm{Log} : \mathbb{G} \to \mathfrak{g}$$

are globally defined diffeomorphisms with polynomial component functions (provided $\mathfrak{g}$ *is equipped with its vector space structure and any fixed system of linear coordinates).*

Moreover, denote also by δ_λ *the dilation on* $\mathfrak{g}$ *defined in* (2.17)*. Then, for every* $Z \in \mathfrak{g}$ *and* $x \in \mathbb{G}$*, it holds*

$$\mathrm{Exp}\left(\delta_\lambda(Z)\right) = \delta_\lambda(\mathrm{Exp}(Z)) \quad \textit{and} \quad \mathrm{Log}\left(\delta_\lambda(x)\right) = \delta_\lambda(\mathrm{Log}(x)).$$

Corollary 2.59. *For every* $x, y \in \mathbb{G}$*, we have*

$$x \circ y = \exp(\mathrm{Log}(y))(x) \quad \textit{and} \quad x^{-1} = \mathrm{Exp}(-\mathrm{Log}(x)).$$

Theorem 2.58 has many important consequences. We collect some of them in the following remark.

Remark 2.60. From Theorem 2.58 we infer, in particular, that

$$\mathrm{Exp} : \mathfrak{g} \to \mathbb{G} \quad \text{and} \quad \mathrm{Log} : \mathbb{G} \to \mathfrak{g}$$

are globally defined C^∞ maps. Hence, the operation on $\mathfrak{g}$

$$X \diamond Y := \mathrm{Log}(\mathrm{Exp}(X) \circ \mathrm{Exp}(Y)), \quad X, Y \in \mathfrak{g},$$

defines a Lie group structure isomorphic to $(\mathbb{G}, \circ)$. We consider on $\mathfrak{g}$ the dilation (still denoted by δ_λ). We claim that δ_λ is a Lie group automorphism of $(\mathfrak{g}, \diamond)$, i.e.

$$\delta_\lambda(X \diamond Y) = (\delta_\lambda(X)) \diamond (\delta_\lambda(Y)) \quad \forall X, Y \in \mathfrak{g}.$$

Roughly speaking, $(\mathfrak{g}, \diamond, \delta_\lambda)$ is a homogeneous Lie group too.

We now identify $\mathfrak{g}$ with $\mathbb{R}^N$ taking coordinates with respect to the Jacobian basis. In other words, we consider the map

$$\pi : \mathfrak{g} \to \mathbb{R}^N, \quad X \mapsto \pi(X) := XI(0).$$

Again, we transfer the Lie group structure of $(\mathfrak{g}, \diamond)$ into a Lie group $\left(\mathbb{R}^N, *\right)$ in the natural way, by setting

$$\xi * \eta := \pi\left(\pi^{-1}(\xi) \diamond \pi^{-1}(\eta)\right), \quad \xi, \eta \in \mathbb{R}^N.$$

As a consequence, $\left(\mathbb{R}^N, *\right)$ is isomorphic to $(\mathfrak{g}, \diamond)$ and hence to $(\mathbb{G}, \circ)$. We finally consider on $\mathbb{R}^N$ the same dilation δ_λ defined on $\mathbb{G}$ (this makes sense, since the underlying manifold for $\mathbb{G}$

is $\mathbb{R}^N$ too). We claim that

$$\left(\mathbb{R}^N, *, \delta_\lambda\right) \text{ is a homogeneous Lie group.}$$

We can summarize the above remarked facts as follows:

Given a homogeneous Lie group $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$, we can consider a somewhat "more canonical" homogeneous Lie group on $\mathbb{R}^N$

$$\mathrm{CH}(\mathbb{G}) := \left(\mathbb{R}^N, *, \delta_\lambda\right)$$

(which we may call "of Campbell-Hausdorff type") obtained by the natural identification of the Lie algebra of $\mathbb{G}$ (equipped with the Campbell-Hausdorff composition law $\diamond$ to $\mathbb{R}^N$ (via coordinates w.r.t. the Jacobian basis).

2.1.3 Homogeneous stratified Lie groups

We now enter into the core of this chapter by introducing the central definition of stratified Lie group. We give two definitions of stratified Lie groups: the first one is the most convenient for our purposes and it seems very natural in an analysis context; the second one is the classical one from Lie group theory. Then, we will compare the two definitions showing that, up to isomorphism, they are equivalent in the Appendix.

Definition 2.61 (Stratified Lie Group)**.** We say that a Lie group $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$, is a (homogeneous) stratified Lie group or a homogeneous Carnot group, if the following properties hold:

$(C1)$ $\mathbb{R}^N$ can be split as $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$, and the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$

$$\delta_\lambda(x) = \delta_\lambda\left(x^{(1)}, \ldots, x^{(r)}\right) = \left(\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}\right), \quad x^{(i)} \in \mathbb{R}^{N_i},$$

is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$.

Then $\left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ is a homogeneous Lie group on $\mathbb{R}^N$. Moreover, the following condition holds:

$(C2)$ If N_1 is as above, let $Z_1, \ldots, Z_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $Z_j(0) = \left.\frac{\partial}{\partial x_j}\right|_0$ for $j = 1, \ldots, N_1$. Then

$$\operatorname{rank}\left(\operatorname{Lie}\left\{Z_1, \ldots, Z_{N_1}\right\}(x)\right) = N \qquad \text{for every } x \in \mathbb{R}^N.$$

If (C1) and (C2) are satisfied, we shall say that the triple $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ is a (homogeneous) stratified Lie group.

We also say that $\mathbb{G}$ has step r and N_1 generators. The vector fields $Z_1, \ldots, Z_{N_1}$ will be called the (Jacobian) generators of $\mathbb{G}$, whereas any basis for span $\{Z_1, \ldots, Z_{N_1}\}$ is called a system of generators of $\mathbb{G}$.

Definition 2.62. A stratified Lie group (or Carnot group) $\mathbb{G}$ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e. a direct sum decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r \quad \text{such that} \begin{cases} [V_1, V_{i-1}] = V_i & \text{if } 2 \leq i \leq r, \\ [V_1, V_r] = \{0\}. \end{cases}$$

In the sequel, we use the following notation to denote the points of $\mathbb{G}$

$$x = (x_1, \ldots, x_N) = \left(x^{(1)}, \ldots, x^{(r)}\right)$$

with

$$x^{(i)} = \left(x_1^{(i)}, \ldots, x_{N_i}^{(i)}\right) \in \mathbb{R}^{N_i}, \quad i = 1, \ldots, r$$

Furthermore, we shall denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$.

Remark 2.63 (Equivalent definition of stratified Lie group I). Suppose $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ is a Lie group on $\mathbb{R}^N$, and there exist positive real numbers $\tau_1 \leq \cdots \leq \tau_N$ such that $d_\lambda(x) = (\lambda^{\tau_1} x_1, \ldots, \lambda^{\tau_N} x_N)$ is a Lie group morphism of $\mathbb{G}$ for every $\lambda > 0$. Let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$, and let $\mathfrak{g}_1$ be the linear subspace of $\mathfrak{g}$ of the left-invariant vector fields which are d_λ-homogeneous of degree τ_1. If $\mathfrak{g}_1$ Lie-generates the whole $\mathfrak{g}$ (which means that $\operatorname{Lie}(\mathfrak{g}_1) = \mathfrak{g}$), then $\mathbb{G}$ is a stratified Lie group. Precisely, $\mathbb{G}$ has step $r := \tau_N/\tau_1$, it has $m := \dim(\mathfrak{g}_1)$ generators, and it is a homogeneous Lie group with respect to the dilation

$$\delta_\lambda = d_{\lambda^{1/\tau_1}}$$

Also, set $\sigma_j := \tau_j/\tau_1$, then $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ are consecutive integers starting from 1 up to r.

Proof. As we observed in Remark 2.35, δ_λ is a morphism of $(\mathbb{G}, \circ)$, i.e. $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ is a homogeneous Lie group on $\mathbb{R}^N$. Obviously, $X \in \mathfrak{g}_1$ if and only if X is δ_λ homogeneous of degree 1 . Let v be the maximum of the integers k such that $\sigma_k = 1$. Let us denote by $\{Z_1, \ldots, Z_N\}$ the Jacobian basis related to $\mathbb{G}$ and observe that (by Proposition 2.45), for every $j \leq N, Z_j$ is δ_λ-homogeneous of degree σ_j. We claim that

$$(\star) \quad v = \dim(\mathfrak{g}_1) =: m, \quad \text{and} \{Z_1, \ldots, Z_m\} \quad \text{is a basis for} \mathfrak{g}_1.$$

Indeed, let $X \in \mathfrak{g}_1$. Then $X = \xi_1 Z_1 + \cdots + \xi_N Z_N$ for suitable scalars ξ_j. Since X is δ_λ-

homogeneous of degree 1 , by Corollary 2.44 and the definition of v, it holds $\xi_j = 0$ for every $j > v$. Hence, $\mathfrak{g}_1$ is spanned by $\{Z_1, \dots, Z_\nu\}$ whence (this system of vectors being linearly independent) the claimed $(\star)$ holds.

By the assumption $\mathrm{Lie}\,(\mathfrak{g}_1) = \mathfrak{g}$ and $(\star)$, it follows

$$(\star\star) \quad \mathrm{Lie}\,(Z_1, \dots, Z_m) = \mathfrak{g}.$$

For every $j \in \mathbb{N}, j \geq 2$, let us set $\mathfrak{g}_j := [\mathfrak{g}_1, \mathfrak{g}_{j-1}]$. By Proposition 2.45, $\mathfrak{g}_j = \{0\}$ for every $j > r := \sigma_N$. Also, by Proposition 2.43, any $X \in \mathfrak{g}_j$ is δ_λ-homogeneous of degree j. Let now $j \in \{m+1, \dots, N\}$ be fixed. Then, by $(\star\star)$, Z_j is a linear combination of nested commutators of $Z_1, \dots, Z_m$. But any such commutator is δ_λ-homogeneous of an integer degree in $1, \dots, r$. This proves that σ_j (the δ_λ-homogeneous degree of Z_j) is integer and (again from Corollary 2.44) $\sigma_j \in \{1, \dots, r\}$. As a consequence, we have the splitting of $\mathbb{R}^N$, as requested in $(C1)$ of Definition 2.61, with $N_1 = m$.

Finally, let us prove that $(C2)$ holds too. This is obvious thanks to $(\star\star)$, since

$$\mathrm{rank}(\mathfrak{g}(x)) \geq \mathrm{rank}\,(Z_1 I(x), \dots, Z_N I(x)) = \mathrm{rank}\,(Z_1 I(0), \dots, Z_N I(0)) = N$$

for every $x \in \mathbb{G}$ (see Proposition 2.20). $\square$

Remark 2.64 (Equivalent definition of stratified Lie group *II*). A stratified Lie group is a connected and simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ admits a (vector space) decomposition of the type

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r,$$

where

$$\begin{cases} [V_i, V_j] \subseteq V_{i+j} & \forall i, j : i + j \leq r \\ [V_i, V_j] = 0 & \forall i, j : i + j > r \end{cases} \tag{2.18}$$

and V_1 generates all $\mathfrak{g}$. This means that every element of $\mathfrak{g}$ can be written as a linear combination of iterated Lie brackets of various elements of V_1.

Proof. In fact, in (2.18), it holds $[V_i, V_j] = V_{i+j}$ if $i + j \leq r$. If $\mathbb{G}$ is a stratified Lie group according to Definition 2.62, then (setting $V_i := \{0\}$ if $i > r$) it holds

$$V_i = \underbrace{[V_1, \cdots [V_1, V_1]]}_{i \text{ times}} \qquad \text{for every } i \in \mathbb{N}.$$

In particular, V_1 Lie-generates all the V_i 's (whence it generates also $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$). This also gives

$$\begin{aligned}[V_i, V_j] &= [\underbrace{[V_1, \cdots [V_1, V_1]]}_{i \text{ times}}, \underbrace{[V_1, \cdots [V_1, V_1]]}_{j \text{ times}}] \\ &\subseteq \underbrace{[V_1, \cdots [V_1, V_1]]}_{i+j \text{ times}} = V_{i+j}.\end{aligned}$$

In particular, (2.18) holds. Vice versa, let $\mathbb{G}$ satisfy the above hypothesi. Set $W_1 := V_1$ and

$$W_i := [W_1, W_{i-1}] = \underbrace{[V_1, \cdots [V_1, V_1]]}_{i \text{ times}} \quad \text{for } i \geq 2.$$

Prove that condition (2.18) implies that $W_i \subseteq V_i$ for every $1 \leq i \leq r$ and $W_i = \{0\}$ for every $i > r$. Moreover, the second hypothesis i.e. V_1 Lie-generates $\mathfrak{g}$ ensures that $\mathfrak{g} = W_1 + \cdots + W_r$. Now, a simple linear algebra argument shows that the following conditions

$$W_1 + \cdots + W_r = \mathfrak{g} = V_1 \oplus \cdots \oplus V_r, \quad W_i \subseteq V_i \quad \forall i \leq r$$

are sufficient to derive that $W_i = V_i$ for every $1 \leq i \leq r$. As a consequence, we have $[V_1, V_j] = [W_1, W_j] = W_{j+1} = V_{j+1}$ whenever $1 + j \leq r$, and $[V_1, V_j] = [W_1, W_j] = \{0\}$ whenever $1 + j > r$, so that $\mathbb{G}$ is a stratified Lie group according to Definition 2.62. □

Example 2.65. The first Heisenberg group $\mathbb{H}^1$ is a stratified Lie group of step two and two generators. Indeed, it is a homogeneous Lie group (with dilations $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$). Moreover (since the first two vector fields of the Jacobian basis are $Z_1 = \partial_{x_1} + 2x_2\partial_{x_3}$ and $Z_2 = \partial_{x_2} - 2x_1\partial_{x_3}$), we have

$$\operatorname{rank}(\operatorname{Lie}\{Z_1, Z_2\}(x)) = 3 \quad \text{for every } x \in \mathbb{R}^3.$$

Thus, the above properties $(C1)$ and $(C2)$ are fulfilled.

Example 2.66. Stratified Lie groups must be homogeneous Lie groups. However the opposite is not true. We now give an example of a homogeneous Lie group which is not a stratified Lie group. Let us consider the following composition law on $\mathbb{R}^2$

$$(x_1, x_2) \circ (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + x_1 y_1).$$

It can be readily verified that $\mathbb{G} = (\mathbb{R}^2, \circ)$ is a Lie group (here $(x_1, x_2)^{-1} = (-x_1, -x_2 + x_1^2)$). Moreover, $\mathbb{G}$ is a homogeneous group, if equipped with the dilation $\delta_\lambda(x_1, x_2) := (\lambda x_1, \lambda^2 x_2)$. Hence (C1) is satisfied. However, (C2) is not. Indeed, if $Z_1 = \partial_{x_1} + x_1\partial_{x_2}$ is the first vector field of the Jacobian basis, we have

$$\operatorname{rank}(\operatorname{Lie}\{Z_1\}(x)) = 1 \neq 2 \quad \text{for every } x \in \mathbb{R}^2.$$

Hence $\mathbb{G}$ is not a homogeneous stratified group.

Remark 2.67. Let us remark that the triple $(\mathbb{R}^2, +, \delta_\lambda)$ is a homogeneous stratified Lie group if $\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$, whereas if $\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2), (\mathbb{R}^2, +, \delta_\lambda)$ is a homogeneous Lie group but not a stratified one.

From properties $(C1)$ and $(C2)$ of Definition 2.61 and the results on the homogeneous Lie groups we immediately get the assertions contained in the following remarks.

Remark 2.68. Let $\left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a stratified Lie group. Then $\circ$ has polynomial component functions. Moreover, denoting $x \circ y$ by $\left((x \circ y)^{(1)}, \ldots, (x \circ y)^{(r)}\right)$, we have

$$(x \circ y)^{(1)} = x^{(1)} + y^{(1)}, \quad (x \circ y)^{(i)} = x^{(i)} + y^{(i)} + Q^{(i)}(x, y), \quad 2 \le i \le r,$$

where

(1) $Q^{(i)}$ only depends on $x^{(1)}, \ldots, x^{(i-1)}$ and $y^{(1)}, \ldots, y^{(i-1)}$;

(2) the component functions of $Q^{(i)}$ are sums of mixed monomials in x, y;

(3) $Q^{(i)}(\delta_\lambda x, \delta_\lambda y) = \lambda^i Q^{(i)}(x, y)$.

Remark 2.69. Let $\left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a stratified Lie group. Then we have

$$\mathcal{J}_{\tau_x}(0) = \begin{pmatrix} \mathbb{I}_{N_1} & 0 & \cdots & 0 \\ J_2^{(1)}(x) & \mathbb{I}_{N_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ J_r^{(1)}(x) & \cdots & J_r^{(r-1)}(x) & \mathbb{I}_{N_r} \end{pmatrix},$$

where $\mathbb{I}_n$ is the $n \times n$ identity matrix, whereas $J_j^{(i)}(x)$ is a $N_j \times N_i$ matrix whose entries are δ_λ-homogeneous polynomials of degree $j - i$. In particular, if we let

$$\mathcal{J}_{\tau_x}(0) = \left(Z^{(1)}(x) \cdots Z^{(r)}(x)\right)$$

where $Z^{(i)}(x)$ is a $N \times N_i$ matrix, then the column vectors of $Z^{(i)}(x)$ define δ_λ homogeneous vector fields of degree $i:$ those of the relevant Jacobian basis.

Remark 2.70. Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a stratified Lie group with Lie algebra $\mathfrak{g}$. Let $Z_1, \ldots, Z_N$ be the Jacobian basis of $\mathfrak{g}$, i.e.

$$Z_j \in \mathfrak{g} \quad \text{and} \quad Z_j(0) = \partial_{x_j}\big|_0, \quad j = 1, \ldots, N.$$

We shall also denote the Jacobian basis by

$$Z_1^{(1)}, \ldots, Z_{N_1}^{(1)}; \ldots; Z_1^{(r)}, \ldots, Z_{N_r}^{(r)}.$$

Obviously, $Z_j^{(1)} = Z_j$ for $1 \leq j \leq N_1$. By Corollary 2.51, $Z_j^{(i)}$ is δ_λ-homogeneous of degree i and takes the form

$$Z_j^{(i)} = \partial/\partial x_j^{(i)} + \sum_{h=i+1}^{r} \sum_{k=1}^{N_h} a_{j,k}^{(i,h)} \left(x^{(1)}, \dots, x^{(h-i)}\right) \partial/\partial x_k^{(h)},$$

where $a_{j,k}^{(i,h)}$ is a δ_λ-homogeneous polynomial function of degree $h-i$. In particular, the Jacobian generators of $\mathbb{G}$, i.e. the vector fields $Z_1^{(1)}, \dots, Z_{N_1}^{(1)}$ are δ_λ-homogeneous of degree 1 .

Remark 2.71. With the notation of the above remark, the Lie algebra $\mathfrak{g}$ is generated by $Z_1, \dots Z_{N_1}$,

$$\mathfrak{g} = \operatorname{Lie}\{Z_1, \dots Z_{N_1}\}.$$

Indeed, the inclusion $\operatorname{Lie}\{Z_1, \dots Z_{N_1}\} \subseteq \mathfrak{g}$ is obvious. Since $\dim(\mathfrak{g}) = N$, in order to show the opposite inclusion, it is enough to prove that

$$\dim\left(\operatorname{Lie}\{Z_1, \dots Z_{N_1}\}\right) = N.$$

By condition (C2), there exists $X_1, \dots, X_N \in \operatorname{Lie}\{Z_1, \dots Z_{N_1}\}$ such that $X_1 I(0), \dots, X_N I(0)$ are linearly independent vectors in $\mathbb{R}^N$. Then $X_1, \dots, X_N$ are linearly independent in $\mathfrak{g}$. Hence

$$N \geq \dim\left(\operatorname{Lie}\{Z_1, \dots Z_{N_1}\}\right) \geq N,$$

and this ends the proof.

Remark 2.72 (Stratification of the algebra of a stratified Lie group). Let us denote by $W^{(k)}$ the vector space spanned by the commutators of length k of $Z_1, \dots, Z_{N_1}$,

$$W^{(k)} := \operatorname{span}\left\{Z_J \mid J \in \{1, \dots, N_1\}^k\right\}.$$

Obviously, $W^{(k)} \subseteq \mathfrak{g}$, and every $Z \in W^{(k)}$ is δ_λ-homogeneous of degree k. Then $W^{(k)} = \{0\}$ if $k > r$, while

$$W^{(k)} \subseteq \operatorname{span}\left\{Z_1^{(k)}, \dots, Z_{N_k}^{(k)}\right\} \qquad \text{if } 2 \leq k \leq r. \tag{2.19}$$

Then, if we agree to let

$$W^{(1)} = \operatorname{span}\{Z_1, \dots, Z_{N_1}\} = \operatorname{span}\left\{Z_1^{(1)}, \dots, Z_{N_1}^{(1)}\right\}$$

we have

$$\dim\left(W^{(k)}\right) \leq N_k \qquad \text{for any } k \in \{1, \dots, r\}. \tag{2.20}$$

On the other hand, by Proposition 2.9,

$$\operatorname{span}\left\{W^{(1)}, \dots, W^{(r)}\right\} = \operatorname{Lie}\left\{Z_1^{(1)}, \dots, Z_{N_1}^{(1)}\right\}.$$

Thus, by Remark 2.71,

$$\mathfrak{g} = \operatorname{span}\left\{W^{(1)}, \dots, W^{(r)}\right\},$$

so that, since $W^{(h)} \cap W^{(k)} = \{0\}$ if $h \neq k$, we have

$$\mathfrak{g} = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(r)}.$$

As a consequence,

$$\dim(\mathfrak{g}) = \sum_{k=1}^{r} \dim\left(W^{(k)}\right).$$

On the other hand, $\dim(\mathfrak{g}) = N = \sum_{k=1}^{r} N_k$. Then, by (2.19),

$$\dim\left(W^{(k)}\right) = N_k \quad \text{ for any } k \in \{1, \dots, r\},$$

and, by (2.20),

$$W^{(k)} = \operatorname{span}\left\{Z_1^{(k)}, \dots, Z_{N_k}^{(k)}\right\} \quad \text{ if } 1 \leq k \leq r.$$

We also have

$$\left[W^{(1)}, W^{(i-1)}\right] = W^{(i)} \quad \text{ for } 2 \leq k \leq r, \tag{2.21}$$

and

$$\left[W^{(1)}, W^{(r)}\right] = \{0\}. \tag{2.22}$$

Indeed, let us put $V_1 := W^{(1)}$ and

$$V_i := [V_1, V_{i-1}] \quad \text{ for } i = 2, \dots, r.$$

By the definition of $W^{(k)}$ and Proposition 2.9, $V_i \subseteq W^{(i)}$ for $i = 2, \dots, r$. Then $\dim(V_i) \leq \dim\left(W^{(i)}\right) = N_i$. On the other hand, by Proposition 2.45, $[V_1, V_r] = \{0\}$, and, by Proposition 2.9,

$$\mathfrak{g} = \operatorname{Lie}\left\{Z_1^{(1)}, \dots, Z_{N_1}^{(1)}\right\} = \operatorname{span}\{V_1, V_2, \dots, V_r\}.$$

Then $N = \sum\limits_{i=1}^{r} \dim(V_i) \leq \sum\limits_{i=1}^{r} N_i = N$. This implies $\dim(V_i) = N_i$ for every $i \in \{1, \dots, r\}$. As a consequence, $V_i = W^{(i)}$ for every $i \in \{1, \dots, r\}$, and (2.21) and (2.22) hold.

Summing up, we have proved the "stratification" of the Lie algebra g, i.e. the decomposition

$$\mathfrak{g} = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(r)},$$

with

$$\left[W^{(1)}, W^{(i-1)}\right] = W^{(i)} \quad \text{for } 2 \le k \le r,$$
$$\left[W^{(1)}, W^{(r)}\right] = \{0\},$$

where

$$W^{(k)} = \text{span}\left\{Z_1^{(k)}, \ldots, Z_{N_k}^{(k)}\right\} \quad \text{if } 1 \le k \le r.$$

2.2 The sub-Laplacians on stratified Lie groups

We begin with a central definition.

Definition 2.73. If $Z_1, \ldots, Z_{N_1}$ are the Jacobian generators of the stratified Lie group $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$, the second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{N_1} Z_j^2$$

is called the canonical sub-Laplacian on $\mathbb{G}$. Any operator

$$\mathcal{L} = \sum_{j=1}^{N_1} Y_j^2$$

where $Y_1, \ldots, Y_{N_1}$ is a basis of span $\{Z_1, \ldots, Z_{N_1}\}$, is simply called a sub-Laplacian on $\mathbb{G}$. The vector valued operator

$$\nabla_{\mathbb{G}} = (Z_1, \ldots, Z_{N_1})$$

will be called the canonical (or horizontal) G-gradient.

Finally, the notation $\nabla_{\mathcal{L}} = (Y_1, \ldots, Y_{N_1})$ will be used to denote the $\mathcal{L}$-gradient (or horizontal $\mathcal{L}$-gradient).

Example 2.74. The canonical sub-Laplacian of the first Heisenberg group $\mathbb{H}^1$ is

$$\begin{aligned}\Delta_{\mathbb{H}^1} &= \{\partial_{x_1} + 2x_2\partial_{x_3}\}^2 + \{\partial_{x_2} - 2x_1\partial_{x_3}\}^2 \\ &= (\partial_{x_1})^2 + (\partial_{x_2})^2 + 4\left(x_1^2 + x_2^2\right)(\partial_{x_3})^2 + 4x_2\partial_{x_1,x_3} - 4x_1\partial_{x_2,x_3}.\end{aligned}$$

A (non-canonical) sub-Laplacian on $\mathbb{H}^1$ is, for example,

$$\begin{aligned}\mathcal{L} &= \{(\partial_{x_1} + 2x_2\partial_{x_3}) - (\partial_{x_2} - 2x_1\partial_{x_3})\}^2 + \{\partial_{x_2} - 2x_1\partial_{x_3}\}^2 \\ &= (\partial_{x_1})^2 + 2(\partial_{x_2})^2 + 4\left(x_1^2 + (x_1 + x_2)^2\right)(\partial_{x_3})^2 \\ &\quad - 2\partial_{x_1,x_2} + 4(x_1 + x_2)\partial_{x_1,x_3} - 4(x_1 + (x_1 + x_2))\partial_{x_2,x_3}.\end{aligned}$$

Remark 2.75. For the sub-Laplacians on groups of step two, provided the inverse map on the group is $-x$, then any sub-Laplacian on a 2-step stratified Lie group contains only second order coordinate partial derivatives.

We would like to list some basic properties of the sub-Laplacians, straightforward consequences of the properties of the vector fields $Z_1, \ldots, Z_{N_1}$. In what follows $\mathcal{L} = \sum_{j=1}^{N_1} Y_j^2$ will denote any sub-Laplacian on $\mathbb{G}$.

(1) $\mathcal{L}$ is hypoelliptic, i.e. every distributional solution to $\mathcal{L}u = f$ is of class C^∞ whenever f is of class C^∞.

(2) $\mathcal{L}$ is invariant with respect to the left translations on $\mathbb{G}$, i.e. for every fixed $\alpha \in \mathbb{G}$,

$$\mathcal{L}(u(\alpha \circ x)) = (\mathcal{L}u)(\alpha \circ x) \quad \text{for every } x \in \mathbb{G} \text{ and every } u \in C^\infty\left(\mathbb{R}^N\right).$$

(3) $\mathcal{L}$ is δ_λ-homogeneous of degree two, i.e. for every fixed $\lambda > 0$,

$$\mathcal{L}\left(u\left(\delta_\lambda(x)\right)\right) = \lambda^2 (\mathcal{L}u)\left(\delta_\lambda(x)\right) \quad \text{for every } x \in \mathbb{G} \text{ and every } u \in C^\infty\left(\mathbb{R}^N\right).$$

(4) $\mathcal{L}$ can be written as

$$\mathcal{L} = \operatorname{div}\left(A(x)\nabla^T\right)$$

where div denotes the divergence operator in $\mathbb{R}^N, \nabla = (\partial_1, \ldots, \partial_N), A$ is the $N \times N$ symmetric matrix

$$A(x) = \sigma(x)\sigma(x)^T$$

and $\sigma(x)$ is the $N \times N_1$ matrix whose columns are $Y_1 I(x), \ldots, Y_{N_1} I(x)$. We also have the expression of $\mathcal{L}$ with respect to the usual coordinate partial derivatives,

$$\mathcal{L} = \sum_{k=1}^{N_1} Y_k^2 = \sum_{i,j=1}^{N} a_{i,j}(x)\partial_{i,j} + \sum_{j=1}^{N} b_j(x)\partial_j$$

where

$$a_{i,j}(x) = \sum_{k=1}^{N_1} \left(Y_k I\right)_i(x) \left(Y_k I\right)_j(x), \quad b_j(x) = \sum_{k=1}^{N_1} Y_k\left(\left(Y_k I\right)_j(x)\right).$$

(5) If $x \in \mathbb{G}$ is fixed and $A(x)$ is the matrix, then the quadratic form in $\xi \in \mathbb{R}^N$

$$q_{\mathcal{L}}(x, \xi) := \langle A(x)\xi, \xi\rangle$$

is called the characteristic form of $\mathcal{L}$. We have

$$q_{\mathcal{L}}(x, \xi) = \sum_{j=1}^{N_1} \langle Y_j I(x), \xi\rangle^2$$

so that $q_{\mathcal{L}}(x,\cdot)$ is obtained by formally replacing in $\mathcal{L}$ the coordinate derivatives $\partial_1,\ldots,\partial_N$ by $\xi_1,\ldots,\xi_N$.

(6) The sub-Laplacian $\mathcal{L}$ is the second order partial differential operator related to the Dirichlet form

$$u \mapsto \int |\nabla_{\mathcal{L}} u|^2 \, dx.$$

More precisely, let $\Omega \subseteq \mathbb{R}^N$ be an open set, and consider the functional

$$C^\infty(\Omega,\mathbb{R}) \ni u \mapsto J(u) = \frac{1}{2}\int_\Omega |\nabla_{\mathcal{L}} u|^2 \, dx, \quad |\nabla_{\mathcal{L}} u|^2 = \sum_{j=1}^{N_1} (Y_j u)^2 .$$

Denoting by $\langle , \rangle$ the inner product in $\mathbb{R}^{N_1}$, we have

$$J(u+h) - J(u) = \int_\Omega \langle \nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} h\rangle \, dx + J(h)$$

for every $h \in C_0^\infty(\Omega,\mathbb{R})$. We call critical point of J any function $u \in C^\infty(\Omega,\mathbb{R})$ such that

$$\int_\Omega \langle \nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} h\rangle \, dx = 0 \quad \forall h \in C_0^\infty(\Omega,\mathbb{R}).$$

Then, given $u \in C^\infty(\Omega,\mathbb{R})$, we have u is a critical point of J if and only if $\mathcal{L}u = 0$ in Ω. Indeed, since $Y_j^* = -Y_j$, an integration by parts gives

$$\begin{aligned}\int_\Omega \langle \nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} h\rangle \, dx &= \sum_{j=1}^{N_1} \int_\Omega Y_j u Y_j h \ \mathrm{d}x = -\sum_{j=1}^{N_1} \int_\Omega \left(Y_j^2 u\right) h \ \mathrm{d}x \\ &= -\int_\Omega (\mathcal{L}u) h \ \mathrm{d}x\end{aligned}$$

for every $u \in C^\infty(\Omega,\mathbb{R})$ and $h \in C_0^\infty(\Omega,\mathbb{R})$.

Remark 2.76. The sub-Laplacian $\mathcal{L}$ is a second order differential operator in divergence form with polynomial coefficients. The characteristic form of $\mathcal{L}$ is positive semi-definite. If the step of $\mathbb{G}$ is ≥ 2, then $\mathcal{L}$ is not elliptic at any point of $\mathbb{G}$. If the step of $\mathbb{G}$ is 1 , then $\mathcal{L}$ is an elliptic operator with constant coefficients.

We end this section with some useful results on the horizontal $\mathcal{L}$-gradient.

Proposition 2.77. *Let $\mathcal{L} = \sum_{j=1}^{N_1} X_j^2$ be a sub-Laplacian on the stratified Lie group $\mathbb{G}$. Let $u \in C^\infty(\mathbb{G},\mathbb{R})$ be such that $X_j u$ is a polynomial function of $\mathbb{G}$-degree not exceeding m for every $j = 1,\ldots,N_1$. Then u is a polynomial function of $\mathbb{G}$-degree not exceeding $m+1$.*

Corollary 2.78. *Let $u \in C^\infty(\mathbb{G},\mathbb{R})$ be such that*

$$X^\beta u = 0 \quad \forall \beta : |\beta| = m$$

for a suitable integer $m \geq 1$. Then u is a polynomial function on $\mathbb{G}$ of $\mathbb{G}$-degree not exceeding $m-1$.

Proposition 2.79. *Let Ω be an open and connected subset of the stratified Lie group $\mathbb{G}$. Let $\mathcal{L}$ be any sub-Laplacian on $\mathbb{G}$. Then a function $u \in C^1(\Omega, \mathbb{R})$ is constant in Ω if and only if the relevant horizontal $\mathcal{L}$-gradient $\nabla_{\mathcal{L}} u$ vanishes identically on Ω.*

2.3 Stratified Lie groups of step two

In this section, we study a special class of stratified Lie groups, which has step two ($r = 2$ in Definition 2.61). In particular, we have the following fact:

A (finite dimensional) nilpotent Lie algebra $\mathfrak{g}$ of step two is necessarily stratified.

Indeed, let us set $V_2 = [\mathfrak{g}, \mathfrak{g}]$ and choose any V_1 such that $\mathfrak{g} = V_1 \oplus V_2$: then it also holds $[V_1, V_1] = V_2$ and $[V_1, V_2] = \{0\}$. We will observe in next chapter that the explicit construction of irreducible unitary representations for 2-step stratified Lie groups is much simpler. The aim of this section is to collect some results and many explicit examples of stratified Lie groups of step two and n generators, $n \geq 2$. In particular, we show that they are naturally given with the data on $\mathbb{R}^{n+m}$ of m skew-symmetric matrices of order n.

2.3.1 Characterization of 2-step stratified groups

Let $m, n \in \mathbb{N}$. Set $\mathbb{R}^N := \mathbb{R}^n \times \mathbb{R}^m$ and denote its points by $z = (x, t)$ with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^m$. Given an m-tuple $B^{(1)}, \ldots, B^{(m)}$ of $n \times n$ matrices with real entries, let

$$(x, t) \circ (\xi, \tau) = \left(x + \xi, t + \tau + \frac{1}{2}\langle Bx, \xi \rangle \right). \tag{2.23}$$

Here $\langle Bx, \xi \rangle$ denotes the m-tuple

$$\left(\langle B^{(1)}x, \xi \rangle, \ldots, \langle B^{(m)}x, \xi \rangle\right) \quad \left(\text{also written as } \sum_{i,j=1}^{n} B_{i,j} x_j \xi_i \right)$$

and $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$. One can easily verify that $\left(\mathbb{R}^N, \circ\right)$ is a Lie group whose identity is the origin and where the inverse is given by

$$(x, t)^{-1} = (-x, -t + \langle Bx, x \rangle).$$

We highlight that the inverse map is the usual $-(x,t)$ if and only if, for every $k = 1, \dots, m$, it holds

$$\left\langle B^{(k)}x, x\right\rangle = 0 \quad \forall x \in \mathbb{R}^n,$$

i.e. iff the matrices $B^{(k)}$ are skew-symmetric. It is also quite easy to recognize that the dilation

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x,t) = \left(\lambda x, \lambda^2 t\right) \tag{2.24}$$

is an automorphism of $\left(\mathbb{R}^N, \circ\right)$ for any $\lambda > 0$. Then $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ is a homogeneous Lie group.

We explicitly remark that the composition law of any Lie group in $\mathbb{R}^n \times \mathbb{R}^m$, homogeneous w.r.t. the dilations $\{\delta_\lambda\}_\lambda$ as in (2.24), takes the form (2.23) (see Theorem 2.48).

The Jacobian matrix at $(0,0)$ of the left translation $\tau_{(x,t)}$ takes the following block form

$$\mathcal{J}_{\tau_{(x,t)}}(0,0) = \left(\begin{array}{c|c} \mathbb{I}_n & 0 \\ \hline \frac{1}{2}Bx & \mathbb{I}_m \end{array}\right)$$

where, if $B^{(k)} = \left(b_{i,j}^{(k)}\right)_{i,j\le m}$ for $k = 1, \dots, n$, Bx denotes the matrix

$$\left(\sum_{j=1}^n b_{i,j}^{(k)} x_j\right)_{k\le m, i\le n}.$$

More explicitly, we have

$$\mathcal{J}_{\tau_{(x,t)}}(0,0) = \left(\begin{array}{ccc|c} & \mathbb{I}_n & & 0_{n\times m} \\ \hline \frac{1}{2}\sum_{j=1}^n b_{1,j}^{(1)}x_j & \cdots & \frac{1}{2}\sum_{j=1}^n b_{n,j}^{(1)}x_j & \\ \vdots & & \vdots & \mathbb{I}_m \\ \frac{1}{2}\sum_{j=1}^n b_{1,j}^{(m)}x_j & \cdots & \frac{1}{2}\sum_{j=1}^n b_{n,j}^{(m)}x_j & \end{array}\right).$$

Then the Jacobian basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$, is given by

$$\begin{aligned} X_i &= \partial x_i + \frac{1}{2}\sum_{k=1}^m \left(\sum_{l=1}^n b_{i,l}^{(k)} x_l\right) \partial t_k \\ &= \partial x_i + \frac{1}{2}\left\langle (Bx)_i, \nabla_t\right\rangle, \quad i = 1, \dots, n, \\ T_k &= \partial t_k, \quad k = 1, \dots, m. \end{aligned} \tag{2.25}$$

Here, we briefly denoted by $(Bx)_i$ the vector of $\mathbb{R}^m$

$$\left(\left(B^{(1)}x\right)_i, \dots, \left(B^{(m)}x\right)_i\right)$$

where $\left(B^{(k)}x\right)_i$ is the i-th component of $B^{(k)}x$. An easy computation shows that

$$[X_j, X_i] = \sum_{k=1}^{m} \frac{1}{2}\left(b_{i,j}^{(k)} - b_{j,i}^{(k)}\right)\partial_{t_k} =: \sum_{k=1}^{m} c_{i,j}^{(k)}\partial_{t_k}.$$

We have denoted by $C^{(k)} = \left(c_{i,j}^{(k)}\right)_{i,j\leq n}$ the skew-symmetric part of $B^{(k)}$, i.e.

$$C^{(k)} = \frac{1}{2}\left(B^{(k)} - \left(B^{(k)}\right)^T\right)$$

Let us now assume that $C^{(1)}, \ldots, C^{(m)}$ are linearly independent. This implies that the $n^2 \times m$ matrix

$$\begin{pmatrix} C_{1,1}^{(1)} & \cdots & C_{1,1}^{(m)} \\ C_{1,2}^{(1)} & \cdots & C_{1,2}^{(m)} \\ \vdots & \cdots & \vdots \\ C_{1,n}^{(1)} & \cdots & C_{1,n}^{(m)} \\ C_{2,1}^{(1)} & \cdots & C_{2,1}^{(m)} \\ \vdots & \cdots & \vdots \\ C_{n,n}^{(1)} & \cdots & C_{n,n}^{(m)} \end{pmatrix}$$

has rank equal to m. As a consequence,

$$\operatorname{span}\left\{[X_j, X_i] \mid i,j = 1,\ldots,n\right\} = \operatorname{span}\left\{\partial_{t_1}, \ldots, \partial_{t_m}\right\}.$$

Therefore,

$$\begin{aligned} &\operatorname{rank}\left(\operatorname{Lie}\left\{X_1,\ldots,X_n\right\}(0,0)\right) \\ &\quad = \dim\left(\operatorname{span}\left\{\partial_{x_1},\ldots,\partial_{x_n},\partial_{t_1},\ldots,\partial_{t_m}\right\}\right) = n+m. \end{aligned}$$

This shows that $\mathbb{G}$ is a stratified Lie group of step two and Jacobian generators $X_1, \ldots, X_n$.

We explicitly remark that the linear independence of the matrices

$$C^{(1)}, \ldots, C^{(m)}$$

is also necessary for $\mathbb{G}$ to be a stratified Lie group. Then, we have proved the following proposition.

Proposition 2.80. *Every stratified Lie group $\mathbb{G}$ on $\mathbb{R}^N$, homogeneous with respect to the dilation*

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_\lambda(x,t) = \left(\lambda x, \lambda^2 t\right)$$

(where $x \in \mathbb{R}^n, t \in \mathbb{R}^m$ and $N = n+m$), is equipped with the composition law

$$(x,t)\circ(\xi,\tau) = \left(x+\xi, t_1+\tau_1+\frac{1}{2}\left\langle B^{(1)}x,\xi\right\rangle, \ldots, t_m+\tau_m+\frac{1}{2}\left\langle B^{(m)}x,\xi\right\rangle\right)$$

for m suitable $n \times n$ matrices $B^{(1)}, \dots, B^{(m)}$.

Moreover, a characterization of stratified Lie groups of step two and n generators is given by the above $\mathbb{G} = (\mathbb{R}^{n+m}, \circ, \delta_\lambda)$, where the skew-symmetric parts of the $B^{(k)}$ are linearly independent.

We remark that the above arguments show that there exist stratified Lie groups of any dimension $n \in \mathbb{N}$ of the first layer and any dimension

$$m \leq \frac{n(n-1)}{2}$$

of the second layer: it suffices to choose m linearly independent matrices $B^{(1)}, \dots, B^{(m)}$ in the vector space of the skew-symmetric $n \times n$ matrices (which has dimension $\frac{n(n-1)}{2}$) and then define the composition law as in (2.23).

By (2.25), we can write explicitly the canonical sub-Laplacian of the Lie group $\mathbb{G} = \left(\mathbb{R}^N, \circ\right)$ with $\circ$ as in (2.23). It is given by

$$\Delta_{\mathbb{G}} = \Delta_x + \frac{1}{4} \sum_{h,k=1}^{m} \left\langle B^{(h)}x, B^{(k)}x \right\rangle \partial_{t_h t_k} + \sum_{k=1}^{n} \left\langle B^{(k)}x, \nabla_x \right\rangle \partial_{t_k} + \frac{1}{2} \sum_{k=1}^{m} \operatorname{trace}\left(B^{(k)}\right) \partial_{t_k}. \tag{2.26}$$

Here, we denoted

$$\Delta_x = \sum_{i=1}^{n} \partial_{x_i, x_i} \quad \text{and} \quad \nabla_x = (\partial_{x_1}, \dots, \partial_{x_n}).$$

We recognize that $\Delta_{\mathbb{G}}$ contains partial differential terms of second order only if $\operatorname{trace}\left(B^{(k)}\right) = 0$ for every $k = 1, \dots, m$. This happens, for example, if the $B^{(k)}$ are skew-symmetric, i.e. if the inverse map on $\mathbb{G}$ is $x \mapsto -x$.

Example 2.81. Following all the above notation, let us take $n = 3, m = 2$ and

$$B^{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then the composition law on $\mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2$ as in (2.23) becomes (denoting $(x,t) = (x_1, x_2, x_3, t_1, t_2)$ and analogously for (ξ, τ))

$$(x,t) \circ (\xi, \tau) = \begin{pmatrix} x_1 + \xi_1 \\ x_2 + \xi_2 \\ x_3 + \xi_3 \\ t_1 + \tau_1 + \frac{1}{2}\left(x_1\xi_1 + \xi_1 x_2 - \xi_2 x_1\right) \\ t_2 + \tau_2 + \frac{1}{2}\left(x_2\xi_2 - \xi_1 x_3 - \xi_3 x_1\right) \end{pmatrix}$$

and the dilation is

$$\delta_\lambda (x_1, x_2, x_3, t_1, t_2) = \left(\lambda x_1, \lambda x_2, \lambda x_3, \lambda^2 t_1, \lambda^2 t_2\right).$$

Then $\mathbb{G} = (\mathbb{R}^5, \circ, \delta_\lambda)$ is a stratified Lie group, for the skew-symmetric parts of $B^{(1)}$ and $B^{(2)}$ are linearly independent,

$$\frac{1}{2}\left(B^{(1)} - \left(B^{(1)}\right)^T\right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{2}\left(B^{(2)} - \left(B^{(2)}\right)^T\right) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In fact, we can compute the first three vector fields of the Jacobian basis and verify that they are Lie-generators for the whole Lie algebra,

$$\begin{aligned}
&X_1 = \partial_{x_1} + \frac{1}{2}(x_1 + x_2)\,\partial_{t_1} - \frac{1}{2} x_3 \partial_{t_2}, \\
&X_2 = \partial_{x_2} - \frac{1}{2} x_1 \partial_{t_1} + \frac{1}{2} x_2 \partial_{t_2}, \\
&X_3 = \partial_{x_3} + \frac{1}{2} x_1 \partial_{t_2}, \\
&[X_1, X_2] = -\partial_{t_1}, [X_1, X_3] = \partial_{t_2}, [X_2, X_3] = \frac{1}{2}\partial_{t_2}.
\end{aligned}$$

The related canonical sub-Laplacian is

$$\begin{aligned}
\Delta_{\mathbb{G}} =& \partial_{x_1,x_1} + \partial_{x_2,x_2} + \partial_{x_3,x_3} \\
&+ \frac{1}{4}\left\{\left\{(x_1 + x_2)^2 + (-x_1)^2\right\}\partial_{t_1,t_1} + \left\{(-x_3)^2 + (x_2)^2 + (x_1)^2\right\}\partial_{t_2,t_2}\right. \\
&\left.+2\left\{(x_1 + x_2)(-x_3) + (-x_1)(x_2)\right\}\partial_{t_1,t_2}\right\} \\
&+ \left\{(x_1 + x_2)\,\partial_{x_1} - x_1 \partial_{x_2}\right\}\partial_{t_1} + \left\{-x_3 \partial_{x_1} + x_2 \partial_{x_2} + x_1 \partial_{x_3}\right\}\partial_{t_2} \\
&+ \frac{1}{2}\partial_{t_1} + \frac{1}{2}\partial_{t_2},
\end{aligned}$$

$\Delta_{\mathbb{G}}$ contains first order terms, for $\operatorname{trace}\left(B^{(1)}\right) \neq 0 \neq \operatorname{trace}\left(B^{(2)}\right)$. On the contrary, if

$$B^{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then the composition law on $\mathbb{R}^5$ given by

$$(x, t) \circ (\xi, \tau) = \begin{pmatrix} x_1 + \xi_1 \\ x_2 + \xi_2 \\ x_3 + \xi_3 \\ t_1 + \tau_1 + \frac{1}{2}(x_1 \xi_1 + \xi_1 x_2 - \xi_2 x_1) \\ t_2 + \tau_2 + \frac{1}{2}(x_2 \xi_2 - 2\xi_1 x_2 + 2\xi_2 x_1) \end{pmatrix}$$

does not define a stratified Lie group, because the skew-symmetric parts of $B^{(1)}$ and $B^{(2)}$ are linearly dependent,

$$\frac{1}{2}\left(B^{(1)}-\left(B^{(1)}\right)^T\right)=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \frac{1}{2}\left(B^{(2)}-\left(B^{(2)}\right)^T\right)=\begin{pmatrix}0 & -2 & 0\\ 2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

In fact, the only admissible dilation would be

$$\delta_\lambda(x,t)=\left(\lambda x_1,\lambda x_2,\lambda x_3,\lambda^2 t_1,\lambda^2 t_2\right)$$

but the first three vector fields of the related Jacobian basis are not Lie-generators for the whole Lie algebra, since

$$\begin{aligned}
X_1&=\partial_{x_1}+\frac{1}{2}\left(x_1+x_2\right)\partial_{t_1}-x_2\partial_{t_2},\\
X_2&=\partial_{x_2}-\frac{1}{2}x_1\partial_{t_1}+\left(\frac{1}{2}x_2+x_1\right)\partial_{t_2},\\
X_3&=\partial_{x_3},\\
[X_1,X_2]&=-\partial_{t_1}+2\partial_{t_2},\\
[X_1,X_3]&=[X_2,X_3]=0.
\end{aligned}$$

2.3.2 Some examples

The aim of this subsection is to collect some explicit examples of stratified Lie groups of step two. To begin with, we present the most studied (and by far one of the most important) among stratified Lie groups, the Heisenberg group. Then, we turn our attention to general stratified Lie groups of step two such as free step-two stratified Lie groups, Heisenberg-type groups and Métivier groups.

The Heisenberg Group

Let us consider in $\mathbb{C}^n\times\mathbb{R}$ (whose points we denote by (z,t) with $t\in\mathbb{R}$ and $z=(z_1,\dots,z_n)\in\mathbb{C}^n$) the following composition law

$$(z,t)\circ(z',t')=\left(z+z',t+t'+2\,\mathrm{Im}\left(z\cdot\overline{z'}\right)\right). \tag{2.27}$$

In (2.27), we have set (i obviously denotes the imaginary unit) $\mathrm{Im}(x+iy)=y$ $(x,y\in\mathbb{R})$, whereas $z\cdot\overline{z'}$ denotes the usual Hermitian inner product in $\mathbb{C}^n$,

$$z\cdot\overline{z'}=\sum_{j=1}^n\left(x_j+iy_j\right)\left(x'_j-iy'_j\right).$$

Hereafter we agree to identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and to use the following notation to denote the points of $\mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$:

$$(z,t) \equiv (x,y,t) = (x_1,\ldots,x_n,y_1,\ldots,y_n,t)$$

with $z = (z_1,\ldots,z_n)\,, z_j = x_j + iy_j$ and $x_j, y_j, t \in \mathbb{R}$. Then, the composition law $\circ$ can be explicitly written as

$$(x,y,t)\circ(x',y',t') = (x+x', y+y', t+t'+2\langle y,x'\rangle - 2\langle x,y'\rangle)\,, \tag{2.28}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$. It is quite easy to verify that $(\mathbb{R}^{2n+1},\circ)$ is a Lie group whose identity is the origin and where the inverse is given by $(z,t)^{-1} = (-z,-t)$. Let us now consider the dilations

$$\delta_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}, \quad \delta_\lambda(z,t) = \big(\lambda z, \lambda^2 t\big)\,.$$

A trivial computation shows that δ_λ is an automorphism of $(\mathbb{R}^{2n+1},\circ)$ for every $\lambda > 0$. Then $\mathbb{H}^n = (\mathbb{R}^{2n+1}, 0, \delta_\lambda)$ is a homogeneous group. It is called the Heisenberg group in $\mathbb{R}^{2n+1}$.

For example, when $n=1$, the first Heisenberg group $\mathbb{H}^1$ in $\mathbb{R}^3$ is equipped with the composition law

$$(x,y,t)\circ(x',y',t') = (x+x', y+y', t+t'+2\,(yx'-xy'))\,,$$

while, when $n=2$, the Heisenberg group $\mathbb{H}^2$ in $\mathbb{R}^5$ is equipped with the composition law

$$\begin{aligned}&(x_1,x_2,y_1,y_2,t)\circ(x_1',x_2',y_1',y_2',t')\\ &\quad = (x_1+x_1', x_2+x_2', y_1+y_1', y_2+y_2', t+t'+2\,(y_1x_1'+y_2x_2'-x_1y_1'-x_2y_2'))\,.\end{aligned}$$

The Jacobian matrix at the origin of the left translation $\tau_{(z,t)}$ is the following block matrix

$$\mathcal{J}_{\tau_{(z,t)}}(0,0) = \begin{pmatrix} \mathbb{I}_n & 0 & 0 \\ 0 & \mathbb{I}_n & 0 \\ 2y^T & -2x^T & 1 \end{pmatrix},$$

where I_n denotes the $n\times n$ identity matrix, while $2y^T$ and $-2x^T$ stand for the $1\times n$ matrices $(2y_1\cdots 2y_n)$ and $(-2x_1\cdots -2x_n)$, respectively. Then, the Jacobian basis of $\mathfrak{h}_n$, the Lie algebra of $\mathbb{H}^n$, is given by

$$X_j = \partial_{x_j} + 2y_j\partial_t, \quad Y_j = \partial_{y_j} - 2x_j\partial_t, \quad j=1,\ldots,n, \quad T = \partial_t.$$

Since $[X_j, Y_j] = -4\partial_t$, we have

$$\begin{aligned}&\operatorname{rank}\left(\operatorname{Lie}\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}(0,0)\right)\\ &\quad = \dim\left(\operatorname{span}\{\partial_{x_1},\ldots,\partial_{x_n},\partial_{y_1},\ldots,\partial_{y_n},-4\partial_t\}\right) = 2n+1.\end{aligned}$$

This shows that $\mathbb{H}^n$ is a stratified Lie group with the following stratification

$$\mathfrak{h}_n = \operatorname{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \oplus \operatorname{span}\{\partial_t\} \tag{2.29}$$

The step of $(\mathbb{H}^n, 0)$ is $r = 2$ and its Jacobian generators are the vector fields $X_j, Y_j (j = 1, \ldots, n)$. The canonical sub-Laplacian on $\mathbb{H}^n$ (also referred to as Kohn Laplacian) is then given by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right).$$

An explicit formula for $\Delta_{\mathbb{H}^n}$ can be found in Example 2.74. Finally, we exhibit the explicit form of the exponential map for $\mathbb{H}^n$. It is given by

$$\operatorname{Exp}((\xi, \eta, \tau) \cdot Z) = (\xi, \eta, \tau)$$

Here we have set $(\xi, \eta, \tau) \cdot Z = \sum\limits_{j=1}^{n} (\xi_j X_j + \eta_j Y_j) + \tau T$.

Heisenberg-type group

Consider the homogeneous Lie group

$$\mathbb{H} = \left(\mathbb{R}^{n+m}, \circ, \delta_\lambda\right)$$

with composition law as

$$(x, t) \circ (\xi, \tau) = \left(x + \xi, t_1 + \tau_1 + \frac{1}{2}\left\langle B^{(1)}x, \xi\right\rangle, \ldots, t_m + \tau_m + \frac{1}{2}\left\langle B^{(m)}x, \xi\right\rangle\right)$$

where $B^{(1)}, \ldots, B^{(m)}$ are fixed $n \times n$ matrices, and dilations as in (2.24). Let us also assume that the matrices $B^{(1)}, \ldots, B^{(m)}$ have the following properties:

(1) $B^{(j)}$ is an $n \times n$ skew-symmetric and orthogonal matrix for every $j \leq m$;

(2) $B^{(i)}B^{(j)} = -B^{(j)}B^{(i)}$ for every $i, j \in \{1, \ldots, m\}$ with $i \neq j$.

If all these conditions are satisfied, $\mathbb{H}$ is called a group of Heisenberg-type, in short, a H-type group.

A H-type group is a stratified Lie group, since conditions (1) and (2) imply the linear independence of $B^{(1)}, \ldots, B^{(m)}$. Indeed, if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^m \backslash \{0\}$, then

$$\frac{1}{|\alpha|} \sum_{s=1}^{m} \alpha_s B^{(s)}$$

is orthogonal (hence non-vanishing), as the following computation shows,

$$\begin{aligned}
&\left(\frac{1}{|\alpha|}\sum_{s=1}^{m}\alpha_s B^{(s)}\right)\cdot\left(\frac{1}{|\alpha|}\sum_{s=1}^{m}\alpha_s B^{(s)}\right)^T\\
&=-\frac{1}{|\alpha|^2}\sum_{r,s\le m}\alpha_r\alpha_s B^{(r)}B^{(s)}\\
&=-\frac{1}{|\alpha|^2}\sum_{r\le m}\alpha_r^2\left(B^{(r)}\right)^2-\frac{1}{|\alpha|^2}\sum_{r,s\le m, r\neq s}\alpha_r\alpha_s B^{(r)}B^{(s)}\\
&=\mathbb{I}_n.
\end{aligned}$$

Here we used the following facts: $\left(B^{(r)}\right)^2=-\mathbb{I}_n$, since $B^{(r)}$ is skew-symmetric and orthogonal; $B^{(r)}B^{(s)}=-B^{(s)}B^{(r)}$ according to condition (2).

The generators of $\mathbb{H}$ are the vector fields (see (2.25))

$$X_i=\partial_{x_i}+\frac{1}{2}\sum_{k=1}^{m}\left(\sum_{l=1}^{n}b_{i,l}^{(k)}x_l\right)\partial_{t_k},\quad i=1,\dots,n.$$

Moreover, if we set

$$T_k:=\partial t_k,\quad k=1,\dots,m,$$

then we know that

$$\{X_1,\dots,X_n;T_1,\dots,T_m\}$$

is the Jacobian basis for $\mathbb{H}$.

A direct computation shows that the canonical sub-Laplacian $\Delta_{\mathbb{H}}=\sum_{i=1}^{n}X_i^2$ can be written as follows

$$\Delta_{\mathbb{H}}=\Delta_x+\frac{1}{4}\sum_{h,k=1}^{m}\left\langle B^{(h)}x,B^{(k)}x\right\rangle\partial_{t_h t_k}+\sum_{k=1}^{n}\left\langle B^{(k)}x,\nabla_x\right\rangle\partial_{t_k}+\frac{1}{2}\sum_{k=1}^{m}\operatorname{trace}\left(B^{(k)}\right)\partial_{t_k}.$$

On the other hand, by conditions (1) and (2),

$$\left\langle B^{(h)}x,B^{(h)}x\right\rangle=|x|^2,$$

while, for $h\neq k$,

$$\left\langle B^{(h)}x,B^{(k)}x\right\rangle=0$$

since

$$\left\langle B^{(h)}x,B^{(k)}x\right\rangle=-\left\langle B^{(k)}B^{(h)}x,x\right\rangle=\left\langle B^{(h)}B^{(k)}x,x\right\rangle=-\left\langle B^{(k)}x,B^{(h)}x\right\rangle.$$

We also have $\operatorname{trace}\left(B^{(k)}\right)=0$, since $B^{(k)}$ is skew-symmetric. Then $\Delta_{\mathbb{H}}$ takes the very compact form

$$\Delta_{\mathbb{H}} = \Delta_x + \frac{1}{4}|x|^2\Delta_t + \sum_{k=1}^{n} \left\langle B^{(k)}x, \nabla_x \right\rangle \partial_{t_k}.$$

Remark 2.82. From (2.26) one obtains

$$|\nabla_{\mathbb{H}}u|^2 = |\nabla_x u|^2 + \frac{1}{4}\sum_{i=1}^{n} \langle (Bx)_i, \nabla_t u\rangle^2 + \sum_{i=1}^{n} \langle (Bx)_i, \nabla_t u\rangle \, \partial_{x_i} u, \quad u \in C^\infty.$$

On the other hand,

$$\sum_{i=1}^{n} \langle (Bx)_i, \nabla_t u\rangle^2 = \sum_{h,k=1}^{m} \left\langle B^{(h)}x, B^{(k)}x \right\rangle \partial_{t_h} u \partial_{t_k} u = |x|^2 \, |\nabla_t u|^2$$

and

$$\sum_{i=1}^{n} \langle (Bx)_i, \nabla_t u\rangle \, \partial_{x_i} u = \sum_{k=1}^{n} \left\langle B^{(k)}x, \nabla_x u \right\rangle \partial_{t_k} u.$$

Thus, for every smooth real-valued function u, it holds

$$|\nabla_{\mathbb{H}}u|^2 = |\nabla_x u|^2 + \frac{1}{4}|x|^2 \, |\nabla_t u|^2 + \sum_{k=1}^{m} \left\langle B^{(k)}x, \nabla_x u \right\rangle \partial_{t_k} u.$$

Remark 2.83. The first layer of a H-type group has even dimension n. Indeed, if B is a $n \times n$ skew-symmetric orthogonal matrix, we have $\mathbb{I}_n = B \cdot B^T = -B^2$, whence $1 = (-1)^n(\det B)^2$.

Remark 2.84. With the previous notation, if $\mathbb{H} = (\mathbb{R}^{n+m}, \circ, \delta_\lambda)$ is a H-type group, then

$$\mathfrak{z} = \{(0,t) \mid t \in \mathbb{R}^m\}$$

is the center of $\mathbb{H}$. Indeed, let $(y,t) \in \mathbb{H}$ be such that

$$(x,s) \circ (y,t) = (y,t) \circ (x,s) \quad \text{ for every } (x,s) \in \mathbb{H}.$$

This holds iff

$$\left\langle B^{(k)}x, y \right\rangle = \left\langle B^{(k)}y, x \right\rangle$$

for any $x \in \mathbb{R}^n$ and any $k \in \{1,\dots,m\}$. Then, since $\left(B^{(k)}\right)^T = -B^{(k)}$,

$$\left\langle B^{(k)}y, x \right\rangle = 0 \quad \forall x \in \mathbb{R}^n, \ \forall k \in \{1,\dots,m\},$$

so that $y = 0$ because $B^{(k)}$ is orthogonal (hence non-singular).

Remark 2.85. The classical Heisenberg group $\mathbb{H}^k$ on $\mathbb{R}^{2k+1}$ is canonically isomorphic to a H-type group. Precisely, it is isomorphic to the H-type group $(\mathbb{H}, *)$ corresponding to the case $n = 2k, m = 1$ and

$$B^{(1)} = \begin{pmatrix} 0 & -\mathbb{I}_k \\ \mathbb{I}_k & 0 \end{pmatrix}.$$

The isomorphism $\varphi : (\mathbb{R}^{2k+1}, *) \to (\mathbb{H}^k, \circ)$ is given by

$$\varphi(\xi, \eta, \tau) = (\xi, \eta, -4\tau)$$

Moreover, $\mathbb{H}$ is in its turn isomorphic to the H-type group with $n = 2k$, $m = 1$ and

$$\widetilde{B}^{(1)} = \operatorname{diag}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}, \quad \text{the block occurring k times.}$$

This type of Heisenberg-group is the only (up to isomorphism) H-type group with one-dimensional center.

Remark 2.86. Groups of Heisenberg type with center of dimension $m \geq 2$ do exist. For example, the following two matrices

$$B^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

satisfy conditions $(1)-(2)$ and hence they define in $\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{R}^2$ a H-type group whose center has dimension 2. The composition law is

$$(x, t) \circ (\xi, \tau) = \begin{pmatrix} x_1 + \xi_1 \\ x_2 + \xi_2 \\ x_3 + \xi_3 \\ x_4 + \xi_4 \\ t_1 + \tau_1 + \frac{1}{2}\left(-x_2\xi_1 + x_1\xi_2 - x_4\xi_3 + x_3\xi_4\right) \\ t_2 + \tau_2 + \frac{1}{2}\left(x_3\xi_1 - x_4\xi_2 - x_1\xi_3 + x_2\xi_4\right) \end{pmatrix}.$$

The above matrices $B^{(1)}$ and $B^{(2)}$, together with

$$B^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

define in $\mathbb{R}^7 = \mathbb{R}^4 \times \mathbb{R}^3$ a H-type group whose center has dimension 3.

Remark 2.87. The groups of Heisenberg-type were introduced by A. Kaplan in [Kap80]. He also shows the following result. Let n, m be two positive integers. Then there exists a H-type group of dimension $n+m$ whose center has dimension m if and only if it holds $m < \rho(n)$, where

ρ is the so-called Hurwitz-Radon function, i.e.

$$\rho : \mathbb{N} \to \mathbb{N}, \quad \rho(n) := 8p + q, \qquad \text{where } n = (\text{odd}) \cdot 2^{4p+q}, \quad 0 \leq q \leq 3.$$

We explicitly remark that if n is odd, then $\rho(n) = 0$, whence the first layer of any H-type group has even dimension (as we already proved in Remark 2.83).

Métivier group

Following G. Métivier [Mét80], we give the following definition.

Definition 2.88. Let $\mathfrak{g}$ be a (finite-dimensional real) Lie algebra, and let us denote by $\mathfrak{z}$ its center. We say that $\mathfrak{g}$ is of Métivier Lie algebra if it admits a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \begin{cases} [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2, \\ \mathfrak{g}_2 \subseteq \mathfrak{z}, \end{cases}$$

with the following additional property: for every $\eta \in \mathfrak{g}_2^*$ (the dual space of $\mathfrak{g}_2$)), the skew-symmetric bilinear form on $\mathfrak{g}_1$ defined by

$$B_\eta : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{R}, \quad B_\eta(X, X') := \eta([X, X'])$$

is non-degenerate whenever $\eta \neq 0$.

We say that a Lie group is a Métivier group, if its Lie algebra is of Métivier Lie algebra.

Remark 2.89. First, a Métivier Lie algebra is obviously nilpotent of step two. Moreover, we have

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1 + \mathfrak{g}_2, \mathfrak{g}_1 + \mathfrak{g}_2] \subseteq [\mathfrak{g}_1, \mathfrak{g}_1] \quad (\text{since } \mathfrak{g}_2 \subseteq \mathfrak{g}),$$
$$[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq [\mathfrak{g}, \mathfrak{g}] \quad (\text{since } \mathfrak{g}_1 \subseteq \mathfrak{g}).$$

Consequently, it holds

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1]. \tag{2.30}$$

Finally, we claim that

$$\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]. \tag{2.31}$$

Indeed, from (2.30) we first derive that $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2$. We are left to show that $\mathfrak{g}_2 \subseteq [\mathfrak{g}, \mathfrak{g}]$. Suppose to the contrary that there exists $Z \in \mathfrak{g}_2$ such that $Z \notin [\mathfrak{g}, \mathfrak{g}]$. This implies, in particular, that $Z \neq \{0\}$. Moreover, since both $Z \in \mathfrak{g}_2$ and $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_2$, there certainly exists $\eta \in \mathfrak{g}_2^*$ such that $\mathfrak{g}_2(Z) \neq 0$ (whence $\eta \neq 0$) and η vanishes identically on $[\mathfrak{g}, \mathfrak{g}]$ (here, we are using the fact that $Z \notin [\mathfrak{g},\mathfrak{g}]$). But this implies that, for every $X, X' \in \mathfrak{g}_1$, we have

$$B_\eta(X,X') = \eta([X,X']) = 0$$

for

$$[X,X'] \in [\mathfrak{g}_1,\mathfrak{g}_1] = [\mathfrak{g},\mathfrak{g}] \quad \text{and} \quad \eta|_{[\mathfrak{g},\mathfrak{g}]} \equiv 0.$$

This is in contradiction with the non-degeneracy of B_η.

Collecting together (2.30) and (2.31), we see that a Métivier Lie algebra is stratified: indeed we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \text{with } [\mathfrak{g}_1,\mathfrak{g}_1] = \mathfrak{g}_2 \text{ and } [\mathfrak{g}_1,\mathfrak{g}_2] = \{0\}.$$

As a consequence, a Métivier group is a stratified Lie group of step two.

Collecting the above results, we have proved the following proposition.

Proposition 2.90. *A Métivier group is a stratified Lie group $\mathbb{G}$ of step two such that if*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad ([\mathfrak{g}_1,\mathfrak{g}_1] = \mathfrak{g}_2, \quad [\mathfrak{g}_1,\mathfrak{g}_2] = \{0\})$$

is any stratification of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, then the following property holds: for every non-vanishing linear map η from $\mathfrak{g}_2$ to $\mathbb{R}$, the (skew-symmetric) bilinear form B_η on $\mathfrak{g}_1$ defined by

$$B_\eta(X,X') := \eta([X,X']), \quad X, X' \in \mathfrak{g}_1,$$

is non-degenerate.

When $\mathbb{G}$ is expressed in its logarithmic coordinates, the above definition is easily re-written as follows. We consider a homogeneous Lie group of step two $\mathbb{G} = (\mathbb{R}^{n+m}, \circ, \delta_\lambda)$ with the composition law as in (2.24), i.e.

$$(x,t)\circ(\xi,\tau) = \left(x+\xi, t_1+\tau_1+\frac{1}{2}\left\langle B^{(1)}x,\xi\right\rangle,\dots,t_m+\tau_m+\frac{1}{2}\left\langle B^{(m)}x,\xi\right\rangle\right),$$

where $B^{(1)},\dots,B^{(m)}$ are fixed $n\times n$ matrices, and the group of dilations is $\delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$. For the sake of simplicity, we may also suppose that the matrices $B^{(k)}$ are skew-symmetric.

Now, if η is a linear map from $\mathfrak{g}_2$ to $\mathbb{R}$, there exist m scalars $\eta_1,\dots,\eta_m \in \mathbb{R}$ such that

$$\eta : \mathfrak{g}_2 \to \mathbb{R}, \quad \eta(\partial_{t_i}) = \eta_i \quad \text{for all } i = 1,\dots,m.$$

In particular, the map B_η can be explicitly written as follows

$$\text{if } X = \sum_{i=1}^n v_i X_i \text{ and } X' = \sum_{i=1}^n v_i' X_i, \text{ then } B_\eta(X,X') = \sum_{i,j=1}^m \left(-\sum_{k=1}^m \eta_k B_{i,j}^{(k)}\right) v_i v_j'.$$

In other words, the matrix representing the (skew-symmetric) bilinear map B_η w.r.t. the basis $X_1, \dots, X_m$ of $\mathfrak{g}_1$ is the matrix

$$\eta_1 B^{(1)} + \cdots + \eta_n B^{(n)}.$$

Hence, to ask for B_η to be non-degenerate (for every $\eta \neq 0$) is equivalent to ask that any linear combination of the matrices $B^{(k)}$ is non-singular, unless it is the null matrix. We have thus obtained the following proposition.

Proposition 2.91. *Let $\mathbb{G} = (\mathbb{R}^{n+m}, \circ)$ be a stratified Lie group of step two, with the composition law*

$$(x,t) \circ (\xi, \tau) = \left(x + \xi, t_1 + \tau_1 + \frac{1}{2} \left\langle B^{(1)} x, \xi \right\rangle, \dots, t_m + \tau_m + \frac{1}{2} \left\langle B^{(m)} x, \xi \right\rangle \right),$$

where $B^{(1)}, \dots, B^{(m)}$ are $n \times n$ skew-symmetric linearly independent matrices. Then $\mathbb{G}$ is a Métivier group if and only if every non-vanishing linear combination of the matrices $B^{(k)}$ is non-singular.

In particular, if the above $\mathbb{G}$ is a Métivier group, then the $B^{(k)}$ are all non-singular $n \times n$ matrices, but since the $B^{(k)}$ are also skew-symmetric, this implies that n is necessarily even.

Remark 2.92 (Any H-type group is a Métivier group). Indeed, as it can be seen from the definition of H-type group that, for every $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n, \eta \neq 0$, we proved that $\sum_{k=1}^m \eta_k B^{(k)}$ is $|\eta|$ times an orthogonal matrix, hence (in particular) $\sum_{k=1}^m \eta_k B^{(k)}$ is non-singular. The converse is not true. For example, consider the group on $\mathbb{R}^5$ (the points are denoted by $(x,t), x \in \mathbb{R}^4$, $t \in \mathbb{R}$) with the composition law

$$(x,t) \circ (\xi, \tau) = \left(x + \xi, t + \tau + \frac{1}{2} \langle Bx, \xi \rangle \right),$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Then $\mathbb{G}$ is obviously a Métivier group, for B is a non-singular skew-symmetric matrix. But $\mathbb{G}$ is not a H-type group, for B is not orthogonal.

Free 2-Step Stratified Lie Groups

In this subsection, we fix a particular set of matrices $B^{(k)}$ and consider the relevant stratified Lie group $(\mathbb{F}_{n,2}, \star)$, which will serve as prototype for what we shall call free stratified Lie group of step two and n generators. Throughout the section, $n \geq 2$ is a fixed integer.

Let $i, j \in \{1, \dots, n\}$ be fixed with $i > j$, and let $S^{(i,j)}$ be the $n \times n$ skew-symmetric matrix whose entries are -1 in the position (i, j), $+1$ in the position (j, i) and 0 elsewhere. For example, if $n = 3$, we have

$$S^{(2,1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S^{(3,1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, S^{(3,2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then, we agree to denote by $(\mathbb{F}_{n,2}, \star)$ the stratified Lie group on $\mathbb{R}^N$ associated to these $\frac{n(n-1)}{2}$ matrices according to (2.23) of the previous section. We set

$$m := \frac{n(n-1)}{2}, N = n + m = \frac{n(n+1)}{2}, \mathcal{I} := \{(i, j) \mid 1 \leq j < i \leq n\}.$$

We observe that the set $\mathcal{I}$ has exactly m elements.

In the sequel of this section, we denote the points of $\mathbb{F}_{n,2}$ by (x, γ), where $x = (x_1, \dots, x_m) \in \mathbb{R}^m, \gamma \in \mathbb{R}^n$, and the coordinates of γ are denoted by

$$\gamma_{i,j} \quad \text{where } (i, j) \in \mathcal{I}.$$

Here we have ordered $\mathcal{I}$ in an arbitrary (henceforth) fixed way. Then, the composition law $\star$ is given by

$$(x, \gamma) \star (x', \gamma') = \begin{pmatrix} x_h + x'_h, \quad h = 1, \dots, n \\ \gamma_{i,j} + \gamma'_{i,j} + \frac{1}{2}\left(x_i x'_j - x_j x'_i\right), \quad (i, j) \in \mathcal{I} \end{pmatrix}.$$

For example, when $n = 3$, we have

$$(x, \gamma) \circ (x', \gamma') = \begin{pmatrix} x_1 + x'_1 \\ x_2 + x'_2 \\ x_3 + x'_3 \\ \gamma_{2,1} + \gamma'_{2,1} + \frac{1}{2}(x_2 x'_1 - x_1 x'_2) \\ \gamma_{3,1} + \gamma'_{3,1} + \frac{1}{2}(x_3 x'_1 - x_1 x'_3) \\ \gamma_{3,2} + \gamma'_{3,2} + \frac{1}{2}(x_3 x'_2 - x_2 x'_3) \end{pmatrix}.$$

By (2.25), we can compute the Jacobian basis

$$X_h, \quad h = 1, \dots, n, \quad \Gamma_{i,j}, \quad (i, j) \in \mathcal{I},$$

of $\mathfrak{f}_{n,2}$, the Lie algebra of $\mathbb{F}_{n,2}$: it holds

$$X_h = \partial x_h + \frac{1}{2} \sum_{1\le j<i\le n} \left(\sum_{l=1}^{n} S_{h,l}^{(i,j)} x_l \right) \partial\gamma_{i,j}$$

$$= \begin{cases} \partial_{x_1} + \frac{1}{2} \sum\limits_{1<i\le n} x_i \partial\gamma_{i,1} \\ \partial_{x_h} + \frac{1}{2} \sum\limits_{h<i\le n} x_i \partial\gamma_{i,h} - \frac{1}{2} \sum\limits_{1\le j<n} x_j \partial\gamma_{h,j} & \text{if } 1<h<n, \\ \partial_{x_n} + \frac{1}{2} \sum\limits_{1\le j<n} x_j \partial\gamma_{n,j} & \text{if } h=n, \end{cases}$$

$$\Gamma_{i,j} = \partial\gamma_{i,j}, \quad (i,j)\in\mathcal{I}.$$

Moreover, for every $(i,j)\in\mathcal{I}$, we have the commutator identities

$$[X_j, X_i] = \sum_{1\le k<h\le n} S_{i,j}^{(h,k)} \partial\gamma_{h,k} = \partial\gamma_{j,i} - \partial\gamma_{i,j}$$

whence we recognize that the algebra $f_{n,2}$ is "the most non-Abelian as possible" (as it is allowed for an algebra with n generators and step two).

This is the reason why we shall refer to (any algebra isomorphic to) $f_{n,2}$ as a free Lie algebra with n generators and step two. For example, when $n=3$, we have

$$\begin{aligned} X_1 &= \partial x_1 + \frac{1}{2}\left(x_2\partial\gamma_{2,1} + x_3\partial\gamma_{3,1}\right) \\ X_2 &= \partial x_2 + \frac{1}{2}\left(x_3\partial\gamma_{3,2} - x_1\partial\gamma_{2,1}\right) \\ X_3 &= \partial x_3 - \frac{1}{2}\left(x_1\partial\gamma_{3,1} - x_2\partial\gamma_{3,2}\right) \\ \Gamma_{2,1} &= \partial\gamma_{2,1}, \quad \Gamma_{3,1} = \partial\gamma_{3,1}, \quad \Gamma_{3,2} = \partial\gamma_{3,2}. \end{aligned}$$

From (2.26), we derive the explicit expression for the canonical sub-Laplacian of $\mathbb{F}_{3,2}$,

$$\begin{aligned} \Delta_{\mathbb{F}_{3,2}} = {} & (\partial x_1)^2 + (\partial x_2)^2 + (\partial x_3)^2 \\ & + \frac{1}{4}\left\{\left(x_2^2+x_1^2\right)(\partial\gamma_{2,1})^2 + \left(x_3^2+x_1^2\right)(\partial\gamma_{3,1})^2 + \left(x_3^2+x_2^2\right)(\partial\gamma_{3,2})^2\right\} \\ & + \frac{1}{2}x_2x_3\left(\partial\gamma_{2,1}\partial\gamma_{3,1}\right) - \frac{1}{2}x_1x_3\left(\partial\gamma_{2,1}\partial\gamma_{3,2}\right) + \frac{1}{2}x_1x_2\left(\partial\gamma_{3,1}\partial\gamma_{3,2}\right) \\ & + \left(x_2\partial x_1 - x_1\partial x_2\right)\partial\gamma_{2,1} + \left(x_3\partial x_1 - x_1\partial x_3\right)\partial\gamma_{3,1} \\ & + \left(x_3\partial x_2 - x_2\partial x_3\right)\partial\gamma_{3,2}. \end{aligned}$$

3 Harmonic analysis on stratified Lie groups of step two

In this chapter we study some basics of harmonic analysis on 2-step stratified Lie groups to make the paper self contained. In particular, we use the orbit method of Kirillov (see [CG90] for details) to describe the explicit construction of irreducible unitary representations. As in [CG90], the following are the important steps:

(i) To parametrize the coadjoint orbits of $\mathfrak{g}^*$ or at least to parametrize a set of coadjoint orbits which is of full Plancherel measure.

(ii) Given $\lambda \in \mathfrak{g}^*$, to construct a maximal subalgebra $\mathfrak{h}$ subordinate to λ, that is $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$.

For the explicit expression of the Plancherel measure , see, e.g., [CG90, Theorem 4.3.9].

For general nilpotent Lie groups (i) and (ii) have explicit answers by Chevalley-Rosenlicht theorem and Vergne polarizations (see [CG90]), but as is only to be expected, on 2-step stratified Lie groups both (i) and (ii) turn out to be much simpler (see [Ray99]). After this we will go to explicit construction of irreducible unitary representations of $\mathbb{G}$. In Kirillov theory the representations arise as induced representation, but as we will see, for the two step case they come directly from the Stone-von Neumann theorem. And then we can study the sub-Laplacian and Fourier transform. A complete account of Fourier analysis for connected, simply connected step two nilpotent Lie groups can also be found in [Lév19].

In the sequel, we will restrict our attention to stratified Lie group of step two, which means the left-invariant Lie algebra $\mathfrak{g}$ is endowed with a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with $\dim \mathfrak{g}_1 = n, \dim \mathfrak{g}_2 = m$ and

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_2 \subseteq \mathfrak{z} = \text{the center of } \mathfrak{g}.$$

Then, there exists a bilinear, antisymmetric map

$$\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$$

such that, for $Z, Z' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}^m$,

$$[(Z,t),(Z',t')] = (0, \sigma(Z,Z'))$$

and

$$(Z,t)\cdot(Z',t') = \left(Z + Z', t + t' + \frac{1}{2}\sigma(Z,Z')\right). \tag{3.1}$$

The map σ and the integers n, m are determined by the group law and dimension. Conversely, for any integers n, m and any bilinear, antisymmetric map $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, one may define a Lie group of step two by the formula (3.1).

3.1 Orbit method on stratified Lie group of step two

In this section we give the detail construction of irreducible unitary representations on 2-step stratified Lie groups without the Moore-Wolf condition. All results are already know in the literature, we will take most of the material from [Ray99].

3.1.1 Parametrization of coadjoint orbits

Let $\mathbb{G}$ be a stratified Lie group with Lie algebra $\mathfrak{g}$, and denote the dual of $\mathfrak{g}$ by $\mathfrak{g}^*$. Then $\mathbb{G}$ acts on $\mathfrak{g}^*$ by the coadjoint action, that is

$$\begin{aligned} \mathrm{Ad}^* : \mathbb{G} \times \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \\ (g, \lambda) &\longrightarrow \mathrm{Ad}^*_g \lambda \end{aligned}$$

which is given by

$$\begin{aligned} \mathrm{Ad}^*_g \lambda(X) &= \lambda\left(\mathrm{Ad}(g^{-1})(X)\right), \quad g \in \mathbb{G}, \lambda \in \mathfrak{g}^*, X \in \mathfrak{g}, \\ &= \lambda(\mathrm{Ad}(\exp Y)(X)) \\ &= \lambda\left(e^{\mathrm{ad} Y}(X)\right) \\ &= \lambda(X) + \lambda([Y, X]), \end{aligned}$$

where Y is the unique element in $\mathfrak{g}$ corresponding to g. We need to parametrize the orbits under this action. For this it is important to consider the structure of these orbits. Let us fix some notation first. Let $\lambda \in \mathfrak{g}^*$, then

- O_λ = The coadjoint orbit of λ.
- B_λ = The skew symmetric matrix corresponding to λ, that is, given a basis through the center of $\mathfrak{g}$, namely $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_N\}$, we consider the matrix

$$B_\lambda = (B_\lambda(i,j)) = (\lambda([X_i, X_j]).$$

- r_λ = The radical of the bilinear form B_λ, that is,

$$r_\lambda = \{X \in \mathfrak{g} : \lambda([X,Y]) = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

  Clearly r_λ is an ideal of $\mathfrak{g}$ and $\mathfrak{z} \subset r_\lambda$.

- $\tilde{r}_\lambda = \operatorname{span}_{\mathbb{R}} \{X_1, \ldots, X_n\} \cap r_\lambda$.
- $\tilde{B}_\lambda = B_\lambda \mid_{\mathbb{R}^n \times \mathbb{R}^n}$ that is restriction of B_λ on the complement of the center of $\mathfrak{g}$.

It follows trivially for two step nilpotent Lie groups that all the coadjoint orbits are hyperplanes ([CG90; LR96]). In fact we have from the above, the following result:

Theorem 3.1. *Let $\lambda \in \mathfrak{g}^*$. Then $O_\lambda = \lambda + r_\lambda^\perp$ where $r_\lambda^\perp = \{h \in \mathfrak{g}^* : h \mid_{r_\lambda} = 0\}$.*

Proof. Let $\lambda' \in O_\lambda$. Then $\lambda' = \lambda \circ Ad(\exp X)$ for some $X \in \mathfrak{g}$. Then for $Y \in r_\lambda$

$$(\lambda - \lambda')(Y) = \lambda(Y) - \lambda'(Y) = \lambda(Y) - \lambda(Y) - \lambda([X,Y]) = 0.$$

Thus $\lambda' = \lambda + (\lambda' - \lambda) \in \lambda + r_\lambda^\perp$. Hence $O_\lambda \subseteq \lambda + r_\lambda^\perp$.

Let $\{X_1, \ldots, X_k, X_{k+1}, \ldots, X_N\}$ be a basis of $\mathfrak{g}$ passing through r_λ in the sense that

$$\operatorname{span}_{\mathbb{R}} \{X_{k+1}, \ldots, X_N\} = r_\lambda.$$

Let $\lambda' \in \lambda + r_\lambda^\perp$ and $\lambda'(X_i) = \lambda'_i$, $\lambda(X_i) = \lambda_i, 1 \le i \le N$. We want to get hold of an $X \in \mathfrak{g}$ such that

$$\lambda(X_i) + \lambda([X, X_i]) = \lambda'(X_i), \quad 1 \le i \le k,$$

that is

$$\lambda([X, X_i]) = \lambda'_i - \lambda_i, \quad 1 \le i \le k.$$

Expressing $X = \sum_{j=1}^{k} \alpha_j X_j + \sum_{j=k+1}^{N} \alpha_j X_j$, we are looking for the solutions of

$$\sum_{j=1}^{k} \alpha_j \lambda([X_j, X_i]) = \lambda'_i - \lambda_i, \quad 1 \le i \le k,$$

which is a system of k linear equations in k unknowns. Since the matrix $L = (L_{ij}) = (\lambda([X_i, X_j]))$ is just the matrix of the bilinear form corresponding to the linear functional $\bar{\lambda}$ on $\mathfrak{g}/r_\lambda$, L is invertible. So the above system has a unique solution $(\alpha_1, \ldots, \alpha_k)$ say. Then for any $Y \in r_\lambda$, we have

$$\exp\left(Y + \sum_{j=1}^{k} \alpha_j X_j\right)^{-1} \cdot \lambda = \lambda'.$$

So $\lambda' \in O_\lambda$ and hence $O_\lambda = \lambda + r_\lambda^\perp$. This completes the proof. □

Remark 3.2. By Theorem 3.1, $\lambda' \in O_\lambda$ if and only if $r_\lambda = r_{\lambda'}$ and $\lambda\,|_{r_\lambda} = \lambda'\,|_{r_{\lambda'}}$.

Let $\mathcal{B} = \{X_1, \ldots, X_n, X_{n+1}, \ldots, X_N\}$ be a basis of $\mathfrak{g}$ such that

$$\operatorname{span}_{\mathbb{R}}\{X_{n+1}, \ldots, X_{n+m}\} = \text{ center of } \mathfrak{g} = \mathfrak{z}.$$

So B_λ is the $N \times N$ matrix whose (i,j)-th entry is $\lambda([X_i, X_j])$, $1 \leq i, j \leq N$. Let $\mathcal{B}^* = \{X_1^*, \ldots, X_N^*\}$ be the dual basis of $\mathfrak{g}^*$. This is a Jordan-Hölder basis, that is $\mathfrak{g}^* = \operatorname{span}_{\mathbb{R}}\left\{X_1^*, \ldots, X_j^*\right\}$ is $\mathrm{Ad}^*(\mathbb{G})$ stable for $1 \leq j \leq N$. Let $\lambda \in \mathfrak{g}^*$ and $X_i \in \mathcal{B}$.

Definition 3.3. The term i is called a jump index for λ if the rank of the $i \times N$ submatrix of B_λ, consisting of first i rows, is strictly greater than the rank of the $(i-1) \times N$ submatrix of B_λ, consisting of first $(i-1)$ rows.

Since an alternating bilinear form has even rank the number of jump indices must be even. The set of jump indices are denoted by $J = \{j_1, \ldots, j_{2d}\}$. The subset of $\mathcal{B}$ corresponding to J is then $\{X_{j_1}, \ldots, X_{j_{2d}}\}$. Notice that if i is a jump index then rank $B_\lambda^i = \operatorname{rank} B_\lambda^{i-1} + 1$ where B_λ^i is the submatrix of B_λ consisting of first i row's.

Remark 3.4. These jump indices depend on λ and on the order of the basis as well. But ultimately we will restrict ourselves to 'generic linear functionals' and they will have the same jump indices.

Now we are going spell out what we mean by generic linear functionals. This is also a basis dependent definition. We work with the basis $\mathcal{B}$ chosen above. Let us fix some notations. Let $R_i(\lambda) = \operatorname{rank} B_\lambda^i$ and $R_i = \operatorname{Max}\{R_i(\lambda) : \lambda \in \mathfrak{g}^*\}$.

Definition 3.5. A linear functional $\lambda \in \mathfrak{g}^*$ is called generic if $R_i(\lambda) = R_i$ for all $i, 1 \leq i \leq N$.

Let $\mathcal{U} = \{\lambda \in \mathfrak{g}^* : \lambda \text{ is generic }\}$.

Example 3.6. Consider the free 2-step stratified Lie groups $\mathfrak{f}_{3,2} = \operatorname{span}_{\mathbb{R}}\{X_1, \ldots, X_6\}$ with nontrivial brackets

$$[X_4, X_5] = X_1, \quad [X_4, X_6] = X_2, \quad [X_5, X_6] = X_3.$$

Thus $\mathfrak{z} = \operatorname{span}_R \{X_1, X_2, X_3\}$. Let $\lambda = \sum_{i=1}^{6} \lambda_i X_i^* \in \mathfrak{g}^*$. Then

$$B_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & 0 & -\lambda_1 & 0 & \lambda_3 \\ 0 & 0 & 0 & -\lambda_2 & -\lambda_3 & 0 \end{pmatrix}.$$

Thus $R_1(\lambda) = R_2(\lambda) = R_3(\lambda) = 0. R_4(\lambda) = 1$ if $l_\lambda \neq 0$ or $\lambda_2 \neq 0$.

$$R_5(\lambda) = \begin{cases} 2 & \text{if } \lambda_1 \neq 0 \\ 0 \text{ or } 1 & \text{if } \lambda_1 = 0 \end{cases}$$

and

$$R_6(\lambda) = \begin{cases} 2 & \text{if } \lambda_1 \neq 0 \\ 0 \text{ or } 2 & \text{if } \lambda_1 = 0 \end{cases}.$$

Thus $R_1 = R_2 = R_3 = 0, R_4 = 1, R_5 = R_6 = 2$. Hence $\mathcal{U} = \{\lambda \in \mathfrak{g}^* : \lambda_1 = \lambda(X_1) \neq 0\}$ and $4, 5$ are jump indices.

Remark 3.7. If $\lambda \in \mathfrak{g}^*$ is such that $\tilde{B}_\lambda$ is an invertible matrix, then $r_\lambda = \mathfrak{z}$ and then $1, \ldots, n$ are jump indices and then

$$\mathcal{U} = \Big\{\lambda \in \mathfrak{g}^* : \tilde{B}_\lambda \text{ is an invertible matrix}\}.$$

Clearly, if codimension of $\mathfrak{z}$ in g is odd then this cannot happen. Following [MW73] and [MR96], we call the 2-step nilpotent Lie algebras, Moore-Wolf algebras (MW algebra) if there exist $\lambda \in \mathfrak{g}^*$ such that $\tilde{B}_\lambda$ is non-degenerate (or the corresponding matrix is invertible). It is obvious that Heisenberg algebras and Métivier algebras are MW algebras. Recall that a set $U \subseteq \mathbb{R}^m$ is Zariski-open if it is a union of sets $\{x \in \mathbb{R}^m : P(x) \neq 0\}$, where P is a polynomial.

Remark 3.8. Since for any $\lambda \in \mathfrak{g}^*$, we have $\mathrm{Ad}_g^* \lambda|_{\mathfrak{z}} = \lambda|_{\mathfrak{z}}$, we get $R_i(\lambda) = R_i(\mathrm{Ad}_g^* \lambda), 1 \leq i \leq N$ and hence,

(*i*) $\mathcal{U}$ is a $\mathbb{G}$-invariant Zariski open subset of $\mathfrak{g}^*$. So $\mathcal{U}$ is union of orbits.

(ii) If j is a jump index for some $\lambda \in \mathcal{U}$, then j is a jump index for all $\lambda \in \mathcal{U}$.

(iii) Let $\lambda \in \mathcal{U}$, then the number of jump indices for λ is the same as the dimension of O_λ (as a manifold). For, the rank of the matrix B_λ is equal to the number of jump indices ($= 2d$) and the dimension of the radical r_λ is the nullity of the matrix of B_λ, which is $N - 2d$. Since $\mathfrak{g}/r_\lambda$ is diffeomorphic to O_λ (see [CG90]), we have $\dim O_\lambda = 2d$.

(iv) Every orbit in $\mathcal{U}$ is of maximum dimension though every maximum dimensional orbit may not be in $\mathcal{U}$.

Our aim is to parametrize the orbits in $\mathcal{U}$. We will see that they constitute a set of full Plancherel measure. We again describe some notation

$T = \{n_1, \dots n_r, n+1, \cdots, n+m\} \subset \{1, \dots, N\}$ is the complement of J in $\{1, \dots, N\}$

$V_J = \operatorname{span}_{\mathbb{R}} \{X_{j_i} : 1 \leq i \leq 2d, j_i \in J\}$,

$V_T = \operatorname{span}_{\mathbb{R}} \{X_{n_i}, X_{n+1}, \dots, X_{n+m} : 1 \leq i \leq r, n_i \in T\}$,

$V_J^* = \operatorname{span}_{\mathbb{R}} \left\{X_{j_1}^*, \dots, X_{j_{2d}}^*\right\}$,

$V_T^* = \operatorname{span}_{\mathbb{R}} \left\{X_{n_i}^*, X_{n+1}^*, \dots, X_{n+m}^* : n_i \in T\right\}$,

$\tilde{V}_T^* = \operatorname{span}_{\mathbb{R}} \{X_{n_i} : n_i \in T\}$.

The following theorem shows that there exist a vector subspace of $\mathfrak{g}^*$ which intersects almost every orbit contained in $\mathcal{U}$ at exactly one point (see [CG90]). In the two step case one can easily prove it using Theorem 3.1 (see [Ray99]). We give the proof for convenience here.

Theorem 3.9. *(i) V_T^* intersects every orbit in $\mathcal{U}$ at a unique point.*

(ii) There exist a birational homeomorphism $\Psi : (V_T^* \cap \mathcal{U}) \times V_J^* \to \mathcal{U}$.

Proof. (i) Let $\lambda \in \mathcal{U}$. We first try to describe $\bar{r}_\lambda$. Denoting by $\bar{\rho}_i(\lambda)$ the i-th row of the matrix $\tilde{B}_\lambda$, every vector $\bar{\rho}_{n_i}(\lambda), n_i \in T$, is a unique linear combination of j_s-th rows of $\tilde{B}_\lambda, 1 \leq s \leq 2d$ that is

$$\tilde{\rho}_{n_i}(\lambda) = \sum_{s=1}^{2d} c_s^i(\lambda) \tilde{\rho}_{j_s}(\lambda),$$

where the scalars $c_s^i(\lambda)$ depend rationally on λ, in fact they depend only on $\lambda|_{\mathfrak{z}}$. Also if $j_s >$

$n_i, c_s^i(\lambda) = 0$. Thus

$$\begin{aligned}
\tilde{\rho}_{n_i}(\lambda) &= \left(\lambda\left([X_{n_i}, X_1]\right), \ldots, \lambda\left([X_{n_i}, X_n]\right)\right) \\
&= \sum_{s=1}^{2d} c_s^i(\lambda)\left(\lambda\left([X_{j_s}, X_1]\right), \ldots, \lambda\left([X_{j_s}, X_n]\right)\right) \\
&= \left(\lambda\left([\sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}, X_1]\right), \ldots, \lambda\left([\sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}, X_n]\right)\right).
\end{aligned}$$

So

$$\left(\lambda\left(\left[X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}, X_1\right]\right), \ldots, \lambda\left(\left[X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}, X_n\right]\right)\right) = 0.$$

Hence $\tilde{X}_{n_i} = X_{n_i} - \sum\limits_{s=1}^{2d} c_s^i(\lambda) X_{j_s} \in \tilde{r}_\lambda$. Since $\left\{\bar{X}_{n_i} : 1 \le i \le r\right\}$ are linearly independent vectors in $\tilde{r}_\lambda$ we have $\tilde{r}_\lambda = \operatorname{span}_{\mathbb{R}}\left\{\tilde{X}_{n_i} : i \le i \le r\right\}$. We need to exhibit a unique $\bar{\lambda} \in V_T^*$ (that is, $\bar{\lambda}\left(X_{j_i}\right) = 0, 1 \le j \le 2d\big)$ such that $\bar{\lambda} \in O_\lambda$; so $\bar{\lambda}$ has to satisfy $r_\lambda = r_{\bar{\lambda}}$ and $\lambda\left|_{r_\lambda} = \bar{\lambda}\right|_{r_{\bar{\lambda}}}$.

We define $\bar{\lambda}\big|_{\mathfrak{z}} = \lambda\,|_{\mathfrak{z}}$. For any such $\bar{\lambda}$

$$r_{\bar{\lambda}} = r_\lambda = \operatorname{span}_{\mathbb{R}}\left\{X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}, X_{n+1} \ldots, X_{n+m}, : 1 \le j \le r\right\}.$$

We also define

$$\bar{\lambda}\left(X_{j_i}\right) = 0, \quad 1 \le i \le 2d,$$

and

$$\bar{\lambda}\left(X_{n_i}\right) = \lambda\left(X_{n_i}\right) - \sum_{s=1}^{2d} c_s^i(\lambda) \lambda\left(X_{j_s}\right), \quad 1 \le i \le r.$$

Thus $\bar{\lambda}\,|r_{\bar{\lambda}} = \lambda|\,r_\lambda$. So $\bar{\lambda} \in O_\lambda$.

Suppose there exist $\lambda' \in \mathfrak{g}^*$ such that $\lambda'\left(X_{j_i}\right) = 0, 1 \le i \le 2d$ and $r_{\lambda'} = r_\lambda$ with $\lambda'\,|r_{\lambda'} = \lambda|\,r_\lambda$. Then $\lambda'|_{\mathfrak{z}} = \lambda|_{\mathfrak{z}} = \bar{\lambda}|_{\mathfrak{z}}$. Now for all i, $1 \le i \le r$,

$$\begin{aligned}
\lambda'\left(X_{n_i}\right) &= \lambda'\left(X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}\right) \qquad \text{as } \lambda'\left(X_{j_i}\right) = 0 \\
&= \lambda\left(X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}\right) \qquad \text{as } \lambda'\left|_{r_{\lambda'}} = \lambda\right|_{r_\lambda} \\
&= \bar{\lambda}\left(X_{n_i}\right) \qquad \text{by definition of } \bar{\lambda}.
\end{aligned}$$

This completes the proof of (i).

(ii) Let $(\lambda_T, \lambda_J) \in (V_T \cap \mathcal{U}) \times V_J^*$ where

$$\lambda_N = \sum_{i=n+1}^{N} \lambda_i X_i^* + \sum_{i=1}^{r} \lambda_{n_i} X_{n_i}^* \quad \text{and} \quad \lambda_J = \sum_{i=1}^{2d} \lambda_{j_i} X_{j_i}^*.$$

Since $\lambda_T \in \mathcal{U}$, there exist constants $c_s^i(\lambda_T) = c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m})$ such that

$$\tilde{r}_{\lambda_T} = \operatorname{span}_{\mathbb{R}} \left\{ X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda_{n+1} \dots, \lambda_{n+m}) \, \mathrm{X}_{j_s} : 1 \le i \le r \right\}.$$

Now we define Ψ by putting

$$\begin{aligned} &\Psi(\lambda_T, \lambda_J)(X_i) = \lambda_i, \quad n+1 \le i \le n+m, \\ &\Psi(\lambda_T, \lambda_J)(X_{j_i}) = \lambda_{j_i}, \quad 1 \le i \le 2d, \\ &\Psi(\lambda_T, \lambda_I)(X_{n_i}) = \lambda_{n_i} + \sum_{s=1}^{2d} c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m}) \lambda_{j_s}. \end{aligned}$$

As

$$\tilde{r}_{\Psi(\lambda_T,\lambda_J)} = \operatorname{span}_{\mathbb{R}} \left\{ X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m}) X_{j_s} : 1 \le i \le r \right\} = \bar{r}_{\lambda_T},$$

and

$$\begin{aligned} &\Psi(\lambda_T, \lambda_J) \left(X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m}) X_{j_s} \right) \\ =&\lambda_{n_i} + \sum_{s=1}^{2d} c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m}) \lambda_{j_s} - \sum_{s=1}^{2d} c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m}) \lambda_{j_s} \\ =&\lambda_T \left(X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m}) X_{j_s} \right), \end{aligned}$$

it follows that $\Psi(\lambda_T, \lambda_J) \in O_{\lambda_T} \subseteq \mathcal{U}$. Thus Ψ is well defined. It is easy to describe $\Psi^{-1} : \mathcal{U} \to (V_T^* \cap U) \times V_J$. Let $\lambda \in \mathcal{U}$ with $\lambda(X_i) = \lambda_i, n+1 \le i \le N$, $\lambda(X_{n_i}) = \lambda_{n_i}, 1 \le i \le r$ and $\lambda(X_{j_i}) = \lambda_j, i \le i \le 2d$. Then $\Psi^{-1}(\lambda) = (\lambda_T, \lambda_J)$ is defined by the conditions

$$\begin{aligned} &\lambda_N(X_i) = \lambda_i, \quad n+1 \le i \le N, \\ &\lambda_N(X_{n_i}) = \lambda_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda_{n+1}, \dots, \lambda_{n+m}) \lambda_{j_s}, \quad 1 \le i \le r, \\ &\lambda_J(X_{j_i}) = \lambda_{j_i}, \quad 1 \le i \le 2d. \end{aligned}$$

Clearly Ψ is birational. This completes the proof. □

Example 3.10. Let $\mathbb{G} = \mathbb{F}_{3,2}$. Then $n = \{1,2,3,6\}$, $J = \{4,5\}$, $\mathcal{U} = \{\lambda \in \mathfrak{f}_{3,2}^* : \lambda_1 = \lambda(X_1) \neq 0\}$, $V_T^* \cap \mathcal{U} = \left\{\lambda \in \mathfrak{f}_{3,2}^* : \lambda_1 = \lambda(X_1) \neq 0, \lambda_4 = \lambda(X_4) = \lambda_5 = \lambda(X_5) = 0\right\}$, $n_1 =$ the first jump index outside center $= 6$.

$$\tilde{\rho}_6(\lambda) = -\frac{\lambda_3}{\lambda_1}\tilde{\rho}_4(\lambda) + \frac{\lambda_2}{\lambda_1}\tilde{\rho}_5(\lambda).$$

Thus $c_4^6(\lambda) = -\frac{\lambda_3}{\lambda_1}, c_5^6(l) = \frac{\lambda_2}{\lambda_1}$. So

$$\Psi((\lambda_1, \lambda_2, \lambda_3, \lambda_6), (\lambda_4, \lambda_5)) = \left(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 - \frac{\lambda_3}{\lambda_1}\lambda_4 + \frac{\lambda_2}{\lambda_1}\lambda_5\right),$$

where, as before $(\lambda_1, \lambda_2, \lambda_3, \lambda_6) \in V_T^* \cap \mathcal{U}$ and $(\lambda_4, \lambda_5) \in V_J^*$.

Remark 3.11. For each coadjoint orbit in $\mathcal{U}$, we choose their representatives from $V_T^* \cap \mathcal{U}$. Notice that $V_T^* \cap \mathcal{U}$ can be identified with the Cartesian product of $\tilde{V}_T^*$ and a Zariski open subset Λ of $\mathfrak{z}^*$, where $\Lambda = \{\lambda \in \mathfrak{z}^* : R_i(\lambda) = R_i, n+1 \leq i \leq n+m\}$. In the next section our aim will be to construct irreducible unitary representations corresponding to elements in $V_T^* \cap \mathcal{U}$ by the orbit method of Kirillov. In particular, we will identify $\tilde{V}_T^*$ with $\mathbb{R}^k$ in the following for simplification.

3.1.2 Polarization and unitary representation

In this section, we first give a brief discussion of Kirillov theory, for details see [CG90] and then we use it to the 2-step stratified Lie group. All the results in this section can be found in [Ray99].

Let $\mathbb{G}$ be a connected, simply connected stratified Lie group with Lie algebra $\mathfrak{g}$. $\mathbb{G}$ acts on $\mathfrak{g}^*$ by the coadjoint action. Given any $\lambda' \in O_\lambda$, the coadjoint orbit of λ, there exist a subalgebra $\mathfrak{h}_{\lambda'}$ of $\mathfrak{g}$ which is maximal with respect to the property

$$\lambda'\left([\mathfrak{h}_{\lambda'}, \mathfrak{h}_{\lambda'}]\right) = 0. \tag{3.2}$$

Thus we have a character $\chi_{\lambda'} : \exp\left(\mathfrak{h}_{\lambda'}\right) \to \mathbb{T}$ given by

$$\chi_{\lambda'}(\exp X) = e^{i\lambda'(X)}, X \in \mathfrak{h}_{\lambda'},$$

here we omit 2π for convenience.

Let $\pi_{\lambda'} = \mathrm{ind}_{\exp(\mathfrak{h}_{\lambda'})}^G \chi_{\lambda'}$(induced representation). Then

(1) $\pi_{\lambda'}$ is an irreducible unitary representation of $\mathbb{G}$.

(2) If $\mathfrak{h}'$ is another subalgebra maximal with respect to the property $\lambda'\left([\mathfrak{h}', \mathfrak{h}']\right) = 0$, then $\mathrm{ind}_{\exp(\mathfrak{h}')}^G \chi_\lambda \cong \mathrm{ind}_{\exp(\mathfrak{h}')}^G \chi_{\lambda'}$

(3) $\pi_{\lambda_1} \cong \pi_{\lambda_2}$ if and only if λ_1 and λ_2 belong to the same coadjoint orbit.

(4) Any irreducible unitary representation π of $\mathbb{G}$ is equivalent to π_λ for some $\lambda \in \mathfrak{g}^*$.

So we have a map $\kappa : \mathfrak{g}^*/\mathrm{Ad}^*(\mathbb{G}) \to \hat{\mathbb{G}}$, which is a bijection. A subalgebra corresponding to $\lambda \in \mathfrak{g}^*$, maximal with respect to (3.2) is called a polarization. It is known that the maximality of $\mathfrak{h}$ with respect to (3.2) is equivalent to the following dimension condition

$$\dim \mathfrak{h} = \frac{1}{2}\left(\dim \mathfrak{g} + \dim r_\lambda\right).$$

Now suppose $\mathfrak{g}$ is a 2-step stratified Lie algebra and $\lambda \in \mathfrak{g}^*$. The following technique for construction of a polarization corresponding to λ, seems to be standard: we consider the bilinear form $\bar{B}_\lambda$ on the complement of the center, we restrict $\tilde{B}_\lambda$ on its nondegenerate subspace, then on that subspace one can choose a basis with respect to which $\bar{B}_\lambda$ is the canonical symplectic form. With a little modification the basis can be chosen to be orthonormal as well. This is essentially what was done in [BJR90; MR96; Par95]. We will set down the basis change explicitly follows from [Ray99]:

Lemma 3.12. *Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a nondegenerate, alternating, bilinear form. Then there exist an orthonormal basis $\{X_i, Y_i : 1 \leq i \leq d\}$ of $\mathbb{R}^n$ such that $B(X_i, Y_j) = \delta_{i,j}\eta_j(B)$, $B(X_i, X_j) = B(Y_i, Y_j) = 0, 1 \leq i, j \leq d, n = 2d$ where $\pm i\eta_j(B)$ are eigenvalues of the matrix of B. Moreover, we can write $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_n$.*

As a consequence we have the following result.

Corollary 3.13. *Let $\lambda \in \mathfrak{g}^*$. Then there exist an Jacobian basis*

$$\{X_1(\lambda), \ldots, X_d(\lambda), Y_1(\lambda), \ldots, Y_d(\lambda), R_1(\lambda), \ldots, R_k(\lambda), T_1(\lambda), \ldots, T_m(\lambda)\}, \tag{3.3}$$

of $\mathfrak{g}$ such that

(i) $r_\lambda = \operatorname{span}_{\mathbb{R}} \{R_1(\lambda), \ldots, R_k(\lambda), T_1(\lambda), \ldots, T_m(\lambda)\}$.

(ii) $\lambda([X_i(\lambda), Y_j(\lambda)]) = \delta_{ij}\eta_j(\lambda), 1 \leq i, j \leq d$ *and*

$$\lambda([X_i(\lambda), X_j(\lambda)]) = \lambda([Y_i(\lambda), Y_j(\lambda)]) = 0, 1 \leq i, j \leq d.$$

(iii) $\operatorname{span}_{\mathbb{R}} \{Y_1(\lambda), \ldots, Y_d(\lambda), R_1(\lambda), \ldots, R_k(\lambda), T_1, \ldots, T_m\} = \mathfrak{h}$ *is a polarization for* λ.

Proof. We choose a basis $\mathcal{B} = \{X_1, \ldots, X_n, X_{n+1}, \ldots, X_N\}$ $(N = n + m)$ of $\mathfrak{g}$ such that

$$\operatorname{span}_{\mathbb{R}} \{X_{n+1}, \ldots, X_{n+m}\} = \mathfrak{z}.$$

We define the Euclidean inner product on $\mathfrak{g}$ such that $\mathcal{B}$ is an orthonormal basis. Let $\lambda \in \mathfrak{g}^*$ and suppose $\dim r_\lambda = m + k$ and $\dim O_\lambda = 2d = N - m - k$. We get hold of

$$r_\lambda = \operatorname{span}_{\mathbb{R}} \left\{ X_{n+1}, \ldots, X_{n+m}, \tilde{X}_{n_i} = X_{n_i} - \sum_{s=1}^{2d} c_s^i(\lambda) X_{j_s}, 1 \leq i \leq k \right\}.$$

We use Gram-Schmidt orthogonalization on r_λ to get an orthonormal basis

$$\{R_1(\lambda), \ldots, R_k(\lambda), T_1(\lambda), \ldots, T_m(\lambda)\}.$$

On $r_\lambda^\perp$, the orthogonal complement of $r_\lambda, \bar{B}_\lambda$ is nondegenerate. By Lemma 3.12 we get an orthonormal basis $\{X_1(\lambda), \ldots, X_d(\lambda), Y_1(\lambda), \ldots, Y_d(\lambda)\}$ of $r_\lambda^\perp$ such that $\lambda\left([X_i(\lambda), Y_j(\lambda)]\right) = \delta_{ij}\eta_j\left(B_\lambda\right)$ and $\lambda\left([X_i(\lambda), X_j(\lambda)]\right) = \lambda\left([Y_i(\lambda), Y_j(\lambda)]\right) = 0$. If we define $\eta_j\left(B_\lambda\right) = \eta_j(\lambda), 1 \le j \le d$ then (i) and (ii) follow. (iii) follows by observing that $\mathfrak{h}$ satisfies (3.2) and the dimension condition. □

Remark 3.14. we call the above basis an almost symplectic basis. Civen $X \in \mathfrak{g}$ and a basis (3.3) we write

$$X = \sum_{j=1}^{d} x_j X_j(\lambda) + \sum_{j=1}^{d} y_j Y_j(\lambda) + \sum_{j=1}^{k} r_j R_j(\lambda) + \sum_{j=1}^{m} t_j T_j(\lambda) = (x, y, r, t).$$

Since we are going to use induced representations we need to describe nice sections of $\mathbb{G}/\mathbb{H}$ and a $\mathbb{G}$-invariant measure on $\mathbb{G}/\mathbb{H}$. In our situation we will always have that $\mathbb{H}$ is a normal subgroup of $\mathbb{G}$. We identify $\mathbb{G}$ and $\mathfrak{g}$, via the exponential map. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ containing $\mathfrak{z}$ and $\mathbb{H} = \exp \mathfrak{h}$.

We take $\{X_1, \ldots, X_d, X_{d+1}, \cdots, X_N\}$ a basis of $\mathfrak{g}$ such that

$$\mathfrak{z} = \operatorname{span}_{\mathbb{R}}\left\{X_{n+1}, \ldots, X_{n+m}\right\}, \quad \mathfrak{h} = \operatorname{span}_{\mathbb{R}}\left\{X_{d+1}, \ldots, X_N\right\}.$$

If $L_g(x) = g^{-1}x$ and $R_g(x) = xg, x, g \in \mathbb{G}$, then it is clear from the group multiplication that the Jacobian matrix for either of the transformations is upper triangular with diagonal entries 1. Thus we have the following lemma whose proof can be found in [CG90].

Lemma 3.15. *Let $\{X_1, \ldots, X_d, X_{d+1}, \ldots, X_N\}$ be a basis of $\mathfrak{g}$. Then*

(i) $dx_1 \ldots dx_N$ is a left and right invariant measure on $\mathbb{G}$.

(ii) $\sigma : \mathbb{G}/\mathbb{H} \to \mathbb{G}$ given by

$$\sigma\left(\exp\left(\sum_{i=1}^{N} t_i X_i\right)\mathbb{H}\right) = \exp\left(\sum_{i=1}^{2d} t_i X_i\right)$$

is a section for $\mathbb{G}/\mathbb{H}$.

(iii) $dx_1 \ldots dx_{2d}$ is a left $\mathbb{G}$-invariant measure on $\mathbb{G}/\mathbb{H}$.

Now we come to the construction of representations corresponding to $\lambda \in V_T^* \cap \mathcal{U}$. Let $\dim r_\lambda = m + k$ and $\dim O_\lambda = 2d$, so $m + k + 2d = N$. We choose an almost symplectic basis (3.3) of $\mathfrak{g}$ corresponding to λ and get hold of $\mathfrak{h}_\lambda$ as in Corollary 3.13. On $\mathbb{H}_\lambda = \exp\left(\mathfrak{h}_\lambda\right)$ we have the character $\chi_\lambda : \mathbb{H}_\lambda \to \mathbb{T}$. Let $\pi_\lambda = \operatorname{ind}_{H_\lambda}^{G}\chi_\lambda$. We do not use the standard model for the induced representation as given in Chapter 2 of [CG90], rather using the continuous section σ given in Lemma 3.15 and computing the unique splitting of a typical group element

$$(x, y, r, t) = (x, 0, 0, 0)\left(0, y, r, t - \frac{1}{2}[(x, 0, 0, 0), (0, y, r, 0)]\right)$$

corresponding to σ, the representation π_λ is realised on $L^2\left(\mathbb{R}^d\right)$ and is given by

$$(\pi_\lambda(x, y, r, t)f)(\xi) \quad f \in L^2\left(\mathbb{R}^d\right) = e^{i\left(\lambda(t)+\lambda(r)+\sum\limits_{i=1}^{d}\eta_i(\lambda)(\frac{1}{2}y_i x_i+\xi_i y_i)\right)} f(\xi + x) \tag{3.4}$$

for almost every $\xi \in \mathbb{R}^d$. At this point we indulge ourselves a little to stop to show that, for 2-step stratified Lie groups, the Kirillov theory can be totally bypassed. The conclusions (3) and (4) listed at the beginning of the section can be reached through a straight forward application of the Stone-von Neumann theorem. This fact is most likely known to experts, our justification for including it here is that we know of no source pointing it out clearly.

Suppose π' is an irreducible unitary representation of $\mathbb{G}$ acting on the Hilbert space $\mathcal{H}_{\pi'}$, with the condition that $\pi'(\exp X) = e^{i\lambda(X)}$ where $X \in \mathfrak{z}$ and $\lambda \in \mathfrak{z}^*$. As before we get hold of an almost symplectic basis (3.3) (note that r_λ is actually determined by $\lambda\mid_{\mathfrak{z}}$). We again write elements of the Lie algebra and the group as well by (x, y, r, t). Then it is easy to show that π' has to satisfy the following properties:

(*a*) $\pi'(0, 0, r, 0)\pi'(0, 0, r_1, 0) = \pi'(0, 0, r + r_1, 0)$,

(*b*) $\pi'(0, 0, r, 0)\pi'(x, y, 0, 0) = \pi'(x, y, 0, 0)\pi'(0, 0, r, 0)$,

(*c*) $\pi'(0, y, 0, 0)\pi'(0, y_1, 0, 0) = \pi'(0, y + y_1, 0, 0)$,

(*d*) $\pi'(x, 0, 0, 0)\pi'(x_1, 0, 0, 0) = \pi'(x + x_1, 0, 0, 0)$,

(*e*) $\pi'(x, 0, 0, 0)\pi'(0, y, 0, 0) = e^{i\sum\limits_{i=1}^{d} x_i y_i \eta_i(\lambda)}\pi'(0, y, 0, 0)\pi'(x, 0, 0, 0)$.

From (a) and (b), it follows by Schur's lemma that,

$$\pi'(0, 0, r, 0) = e^{i\nu(r)} \quad \nu \in \operatorname{span}_{\mathbb{R}}\{R_1(\lambda), \ldots, R_k(\lambda)\}^*.$$

By $(c) - (e)$ and Stone-von Neumann theorem $\mathcal{H}_{\pi'}$ is unitarily equivalent to $L^2\left(\mathbb{R}^d\right)$ and

$$(\pi'(x, 0, 0, 0)f)(\xi) = f(\xi + x), \quad f \in L^2\left(\mathbb{R}^d\right),$$

$$(\pi'(0, y, 0, 0)f)(\xi) = e^{i\sum\limits_{i=1}^{d}\xi_i y_i \eta_i(\lambda)} f(\xi),$$

for almost every $\xi \in \mathbb{R}^d$. Then by using the fact that

$$(x, y, r, t)$$
$$= (x,0,0,0)(0,y,0,0)(0,0,r,0)\left(0,0,0,t-\frac{1}{2}[(0,0,r,0),(0,y,0,0)]-\frac{1}{2}[(0,y,r,0),(x,0,0,0)]\right)$$

we get that for almost every $\xi \in \mathbb{R}^d$

$$(\pi'(x,y,r,w)f)(\xi) = e^{i\left[\lambda(t)+\nu(r)+\sum_{i=1}^{d} y_i\xi_i\eta_i(\lambda)+\frac{1}{2}\sum_{i=1}^{d} x_iy_i\eta_i(\lambda)\right]} f(\xi + x).$$

If $\nu = \lambda' \mid_{\mathrm{span}_{\mathbb{R}}\{R_1(\lambda),\ldots,R_r(\lambda)\}}$ then it follows from (3.4) that $\pi' = \mathrm{ind}_{H_\lambda}^{G} \chi_{\lambda'}$ where $\lambda' \in \mathfrak{g}^*$ is such that

$$\lambda' \mid_{\mathrm{span}_{\mathbb{R}}\{W_1(\lambda),\ldots,W_m(\lambda)\}} = \lambda,$$
$$\lambda' \mid_{\mathrm{span}_{\mathbb{R}}\{X_1(\lambda),\ldots,Y_k(\lambda)\}} = 0.$$

We have noted above that every unitary irreducible representation of $\mathbb{G}$ is of the form (3.4). The assertion about equivalences among the representations now is an immediate consequence of the uniqueness of the Stone-von Neumann theorem (see Section 3.4). For, if π_{λ_1} and π_{λ_2} are given by (3.4), then the analysis $(a)-(e)$ on π_{λ_1} and π_{λ_2} would show that $\pi_{\lambda_1} \cong \pi_{\lambda_2}$ if and only if λ_1 and λ_2 belong to the same coadjoint orbit.

3.2 The Fourier analysis

3.2.1 Irreducible unitary representations

In this section, we first rewrite the results above by a more analysis language, and then we give some examples for 2-step stratified Lie groups to describe the explicit construction of irreducible unitary representations.

Let $\mathbb{G}$ be a two step connected simply connected stratified Lie group so that its Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2$ is contained in the center of $\mathfrak{g}$ and $\mathfrak{g}_1$ is any subspace of $\mathfrak{g}$ complementary to $\mathfrak{g}_2$. We choose an inner product on $\mathfrak{g}$ such that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are orthogonal. Fix an Jacobian basis $\mathcal{B} = \{X_1, X_2 \cdots, X_n, X_{n+1}, \cdots, X_{n+m}\}$ so that $\mathfrak{g}_1 = \mathrm{span}_{\mathbb{R}}\{X_1, X_2 \cdots, X_n\}$ and $\mathfrak{g}_2 = \mathrm{span}_{\mathbb{R}}\{X_{n+1}, \cdots, X_{n+m}\}$. Since $\mathfrak{g}$ is nilpotent the exponential map is an analytic diffeomorphism. We can identify $\mathbb{G}$ with $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and write $(X+T)$ for $\exp(X+T)$ and denote it by (X, T) where $X \in \mathfrak{g}_1$ and $T \in \mathfrak{g}_2$. The product law on $\mathbb{G}$ is given by (3.1).

Now, given $\lambda \in \mathbb{R}^m$, we define the matrix $B^{(\lambda)} \in \mathcal{M}_n(\mathbb{R})$ as follows. For any $Z, Z' \in \mathbb{R}^n$, there holds

$$\langle \lambda, \sigma(Z, Z') \rangle = \left\langle Z, B^{(\lambda)} \cdot Z' \right\rangle.$$

If $(\iota_1, \ldots, \iota_m)$ denotes an orthonormal basis of $\mathbb{R}^m$, we also define $B_k \in \mathcal{M}_n(\mathbb{R})$ by

$$\langle \iota_k, \sigma(Z, Z') \rangle = \langle Z, B_k \cdot Z' \rangle.$$

Then for $\lambda = \sum\limits_{k=1}^{m} \lambda_k \iota_k$, we get

$$B^{(\lambda)} = \sum_{k=1}^{m} \lambda_k B_k.$$

Conversely, the map σ may be defined from $(B_k)_{1 \leq k \leq m}$ thanks to the equality

$$\sigma(Z, Z') = (\langle Z, B_k \cdot Z' \rangle)_{1 \leq k \leq m}.$$

Notice that the map $\lambda \mapsto B^{(\lambda)}$ is linear, with its image spanned by $(B_k)_{1 \leq k \leq m}$. As $B^{(\lambda)}$ is an antisymmetric matrix, its rank is an even number. We define the integer d by

$$2d := \max_{\lambda \in \mathbb{R}^m} \operatorname{rank} B^{(\lambda)}.$$

The set $\Lambda := \left\{ \lambda \in \mathbb{R}^m \mid \operatorname{rank} B^{(\lambda)} = 2d \right\}$ is then a nonempty Zariski-open subset of $\mathbb{R}^m$. We denote by k the dimension of the radical r_λ of $B^{(\lambda)}$. If $r_\lambda = \{0\}$ for each $\lambda \in \Lambda$, then the Lie algebra is called an MW algebra and the corresponding Lie group is called an MW group. In this paper, we will only consider $\mathbb{G}$ to be a 2-step stratified Lie group without MW-condition.

For

$$(X, T) = \exp\left(\sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{m} t_j X_{n+j} \right), \quad x_j, t_j \in \mathbb{R},$$

the map

$$\begin{aligned}(x_1, \cdots, x_n, t_1 \cdots, t_m) &\longrightarrow \sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{m} t_j X_{n+j} \\ &\longrightarrow \exp\left(\sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{m} t_j X_{n+j} \right)\end{aligned}$$

takes Lebesgue measure $dx_1 \cdots dx_n, dt_1 \cdots dt_m$ of $\mathbb{R}^{n+m}$ to Haar measure on $\mathbb{G}$. Any measurable function f on $\mathbb{G}$ will be identified with a function on $\mathbb{R}^{n+m}$.

Therefore, there exists an orthonormal basis

$$(X_1(\lambda), \ldots, X_d(\lambda), Y_1(\lambda), \ldots, Y_d(\lambda), R_1(\lambda), \ldots, R_k(\lambda))$$

and d continuous functions

$$\eta_j : \mathbb{R}^m \to \mathbb{R}_+, \quad 1 \leq j \leq d$$

such that $B^{(\lambda)}$ reduces to the form

$$\begin{pmatrix} 0 & \eta(\lambda) & 0 \\ -\eta(\lambda) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$$

where

$$\eta(\lambda) := \operatorname{diag}\left(\eta_1(\lambda), \ldots, \eta_d(\lambda)\right) \in \mathcal{M}_d(\mathbb{R})$$

and each $\eta_j(\lambda) > 0$ is smooth and homogeneous of degree 1 in $\lambda = (\lambda_1, \cdots, \lambda_m)$ and the basis vectors are chosen to depend smoothly on λ in Λ. Decomposing $\mathfrak{g}_1$ as

$$\mathfrak{g}_1 = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda$$

with

$$\mathfrak{p}_\lambda := \operatorname{span}_{\mathbb{R}}\left(X_1(\lambda), \ldots, X_d(\lambda)\right), \mathfrak{q}_\lambda := \operatorname{span}_{\mathbb{R}}\left(Y_1(\lambda), \ldots, Y_d(\lambda)\right), \mathfrak{r}_\lambda := \operatorname{span}_{\mathbb{R}}\left(R_1(\lambda), \ldots, R_k(\lambda)\right).$$

Then we have the decomposition $\mathfrak{g} = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda \oplus \mathfrak{r}_\lambda \oplus \mathfrak{g}_2$. We denote the element $\exp(X+Y+R+T)$ of $\mathbb{G}$ by (X, Y, R, T) for $X \in \mathfrak{p}_\lambda, Y \in \mathfrak{q}_\lambda, R \in \mathfrak{r}_\lambda, T \in \mathfrak{g}_2$. Further we can write

$$(X, Y, R, T) = \sum_{j=1}^{d} x_j(\lambda) X_j(\lambda) + \sum_{j=1}^{d} y_j(\lambda) Y_j(\lambda) + \sum_{j=1}^{k} r_j(\lambda) R_j(\lambda) + \sum_{j=1}^{m} t_j T_j$$

and denote it by (x, y, r, t) suppressing the dependence of λ which will be understood from the context.

For (λ, ν, w) in $\Lambda \times \mathbb{R}^k \times \mathbb{R}^N$ with

$$w = (x, y, r, t) \in \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^k \oplus \mathbb{R}^m = \mathbb{R}^N,$$

we define the irreducible unitary representations of $\mathbb{R}^N$ (we will prove this fact in Section 3.4), equipped with the group law of the nilpotent group defined above, on $L^2\left(\mathbb{R}^d\right)$

$$\begin{aligned}\left(\pi_{\lambda,\nu}(w)\phi\right)(\xi) &:= \exp\left(i\sum_{j=1}^{m} \lambda_j t_j + i\sum_{j=1}^{k} \nu_j r_j + i\sum_{j=1}^{d} \eta_j(\lambda)\left(y_j\xi_j + \frac{1}{2}x_j y_j\right)\right)\phi(\xi + x) \\ &= e^{i\langle \nu, r\rangle} e^{i\langle \lambda, t\rangle} e^{i\sum\limits_{j=1}^{d} \eta_j(\lambda)\left(y_j\xi_j + \frac{1}{2}x_j y_j\right)} \phi(\xi + x).\end{aligned}$$

3.2.2 Examples

Let us give a few examples of well-known stratified Lie groups with a two step stratification:

The Heisenberg group:

The Heisenberg group $\mathbb{R}^d$ is defined as the space $\mathbb{R}^{2d+1}$ whose elements can be written

$w = (x, y, s)$ with $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, endowed with the product law

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' - 2\langle x, y'\rangle + 2\langle y, x'\rangle)$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product on $\mathbb{R}^d$. In that case the center consists of the points of the form $(0, 0, s)$ and is of dimension 1. The Lie algebra of left-invariant vector fields is generated by

$$X_j := \partial_{x_j} + 2y_j\partial_s, \quad Y_j := \partial_{y_j} - 2x_j\partial_s \quad \text{for } 1 \le j \le d; \quad S := \partial_s = \frac{1}{4}\left[Y_j, X_j\right].$$

Regarding the choice of suitable bases, let $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ be a basis of $\mathbb{R}^{2d}$ in which the matrix of σ_c assumes the form

$$\begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \in \mathcal{M}_{2d}(\mathbb{R}).$$

For $\lambda > 0$, we choose $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ as a basis of $\mathbb{R}^{2d}$, while for $\lambda < 0$ this choice becomes $(y_1, \ldots, y_d, x_1, \ldots, x_d)$. Hence, for any $\lambda \in \mathbb{R}^*$, we have, as desired,

$$B^{(\lambda)} = \begin{bmatrix} 0 & 4|\lambda| I_d \\ -4|\lambda| I_d & 0 \end{bmatrix} \in \mathcal{M}_{2d}(\mathbb{R}).$$

Its radical reduces to $\{0\}$ with $\Lambda = \mathbb{R}^*$, and $|\eta_j(\lambda)| = 4|\lambda|$ for all $j \in \{1, \ldots, d\}$.

H-type group:

These groups are canonically isomorphic to $\mathbb{R}^{n+m}$ and are a multidimensional version of the Heisenberg group. The group law is of the form

$$\left(x^{(1)}, x^{(2)}\right) \cdot \left(y^{(1)}, y^{(2)}\right) := \begin{pmatrix} x_j^{(1)} + y_j^{(1)}, & j = 1, \ldots, n \\ x_k^{(2)} + y_k^{(2)} + \frac{1}{2}\left\langle x^{(1)}, U^{(k)} y^{(1)}\right\rangle, & k = 1, \ldots, m \end{pmatrix}$$

where $U^{(j)}$ are $n \times n$ linearly independent, orthogonal, skew-symmetric matrices satisfying the property

$$U^{(r)}U^{(s)} + U^{(s)}U^{(r)} = 0$$

for every $r, s \in \{1, \ldots, m\}$ with $r \neq s$. In that case the center is of dimension m and may be identified with $\mathbb{R}^m$, and the radical of the canonical skew-symmetric form associated with the frequencies λ is again $\{0\}$. For example, the Iwasawa subgroup of semisimple Lie groups of split rank 1 (see [Kor85]) is of this type. On H-type groups, n is an even number, which we denote by $2l$, and the Lie algebra of left-invariant vector fields is spanned by the following vector fields, where we have written $z=(x,y)$ in $\mathbb{R}^l \times \mathbb{R}^l$: for $j=1,\dots,l$ and $k=1,\dots,m$,

$$X_j := \partial_{x_j} + \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{2l} z_l U_{l,j}^{(k)} \partial_{t_s}, \quad Y_j := \partial_{y_j} + \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{2l} z_l U_{l,j+l}^{(k)} \partial_{t_s} \quad \text{and} \quad \partial_{t_s}.$$

In that case, we have $\Lambda = \mathbb{R}^m \backslash \{0\}$ with $\eta_j(\lambda) = \sqrt{\lambda_1^2+\cdots+\lambda_m^2}$ for all $j \in \{1,\dots,l\}$.

Diamond groups:

These groups, which occur in crystal theory (for more details, see [Lud95; Pog99]), are of the type $\Sigma \ltimes \mathbb{H}^d$, where Σ is a connected Lie group acting smoothly on $\mathbb{H}^d$. One can find examples for which the radical of the canonical skew-symmetric is of any dimension k, $0 \leq k \leq d$. For example, one can take for Σ the k-dimensional torus, acting on $\mathbb{H}^d$ by

$$\theta(w) := (\theta \cdot z, s) := \left(\mathrm{e}^{i\theta_1} z_1, \dots, \mathrm{e}^{i\theta_k} z_k, z_{k+1}, \dots, z_d, s\right), \quad w=(z,s)$$

where the element $\theta=(\theta_1,\dots,\theta_k)$ corresponds to the element $\left(\mathrm{e}^{i\theta_1},\dots,\mathrm{e}^{i\theta_k}\right)$ of $\mathbb{T}^k$. Then the product law on $G=\mathbb{T}^k \ltimes \mathbb{H}^d$ is given by

$$(\theta, w)\cdot(\theta', w') = (\theta+\theta', w\cdot(\theta(w')))$$

where $w.(\theta(w'))$ denotes the Heisenberg product of w by $\theta(w')$. As a consequence, the center of G is of dimension 1 , since it consists of the points of the form $(0,0,s)$ for $s \in \mathbb{R}$. Let us choose for simplicity $k=d=1$; the algebra of left-invariant vector fields is generated by the vector fields $\partial_\theta, \partial_s$ $\Gamma_{\theta,x}$ and $\Gamma_{\theta,y}$, where

$$\Gamma_{\theta,x} = \cos\theta\partial_x + \sin\theta\partial_y + 2(y\cos\theta - x\sin\theta)\partial_s,$$

$$\Gamma_{\theta,y} = -\sin\theta\partial_x + \cos\theta\partial_y - 2(y\sin\theta + x\cos\theta)\partial_s.$$

It is not difficult to check that the radical of B_λ is of dimension 1 . In the general case, where $k \leq d$, the algebra of left-invariant vector fields is generated by the vector fields ∂_s, the $2(d-k)$ vectors

$$X_l = \partial_{x_l} + 2y_l\partial_s \quad \text{and} \quad Y_l = \partial_{yl} - 2x_l\partial_s$$

and the $3k$ vectors defined for $1 \leq j \leq k$ by $\partial_{\theta_j}, \Gamma_{\theta_j,x_j}$ and Γ_{θ_j,y_j}, where

$$\Gamma_{\theta_j,x_j} = \cos\theta_j\partial_{x_j} + \sin\theta_j\partial_{y_j} + 2\left(y_j\cos\theta_j - x_j\sin\theta_j\right)\partial_s,$$

$$\Gamma_{\theta_j,y_j} = -\sin\theta_j\partial_{x_j} + \cos\theta_j\partial_{y_j} - 2\left(y_j\sin\theta_j + x_j\cos\theta_j\right)\partial_s,$$

and this provides an example with a radical of dimension k.

The product of Heisenberg groups:

Consider $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$, the set of elements (w_1,w_2) in $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ that can be written as

$(w_1, w_2) = (x_1, y_1, s_1, x_2, y_2, s_2)$ in $\mathbb{R}^{2d_1+1} \times \mathbb{R}^{2d_2+1}$, equipped with the product law

$$(w_1, w_2) \cdot (w_1', w_2') = (w_1 \cdot w_1', w_2 \cdot w_2')$$

where $w_1 \cdot w_1'$ and $w_2 \cdot w_2'$ denote the product in $\mathbb{H}^{d_1}$ and $\mathbb{H}^{d_2}$, respectively. Clearly $\mathbb{H}^{d_1} \otimes \mathbb{H}^{d_2}$ is a 2-step stratified Lie group with center of dimension 2 and radical index null. Moreover, for $\lambda = (\lambda_1, \lambda_2)$ in the dual of the center, the canonical skew bilinear form $B(\lambda)$ has radical $\{0\}$ with $\Lambda = \mathbb{R}^* \times \mathbb{R}^*$, and one has $\eta_1(\lambda) = 4\,|\lambda_1|$ and $\eta_2(\lambda) = 4\,|\lambda_2|$.

The product of H-type groups:

The group $\mathbb{R}^{m_1+p_1} \otimes \mathbb{R}^{m_2+p_2}$ is easily verified to be a 2-step stratified Lie group with center of dimension p_1+p_2, radical index null and a skew bilinear form $B(\lambda)$ defined on $\mathbb{R}^{m_1+m_2}$ with $m_1 = 2l_1$ and $m_2 = 2l_2$. The Zariski-open set associated with B is given by $\Lambda = (\mathbb{R}^{p_1} \setminus \{0\}) \times (\mathbb{R}^{p_2} \setminus \{0\})$ and, for $\lambda = (\lambda_1, \ldots, \lambda_{p_1+p_2})$, we have

$$\eta_j(\lambda) = \sqrt{\lambda_1^2 + \cdots + \lambda_{p_1}^2} \quad \text{for all } j \in \{1, \ldots, l_1\}\,,$$
$$\eta_j(\lambda) = \sqrt{\lambda_{p_1+1}^2 + \cdots + \lambda_{p_1+p_2}^2} \text{ for all } j \in \{l_1+1, \ldots, l_1+l_2\}\,.$$

3.2.3 The Fourier transform

The stratified Lie groups being noncommutative, then the Fourier transform on $\mathbb{G}$ is defined using irreducible unitary representations of $\mathbb{G}$. We devote this section to the introduction of the basic concepts that will be needed in the sequel. For (λ, ν, w) in $\Lambda \times \mathbb{R}^k \times \mathbb{R}^N$ with

$$w = (x, y, r, t) \in \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^k \oplus \mathbb{R}^m = \mathbb{R}^N,$$

we define the irreducible unitary representations of $\mathbb{R}^N$, equipped with the group law of the nilpotent group defined above, on $L^2\left(\mathbb{R}^d\right)$

$$\left(\pi_{\lambda,\nu}\phi\right)(\xi) = e^{i\langle\nu,r\rangle} e^{i\langle\lambda,t\rangle} e^{i\sum\limits_{j=1}^{d} \eta_j(\lambda)\left(y_j\xi_j + \frac{1}{2}x_j y_j\right)} \phi(\xi + x).$$

In the case of the first Heisenberg group $\mathbb{H}^1$, we have $k = 0$ and $\eta(\lambda) = d = m = 1$; hence, for $\xi, x, y, \lambda \in \mathbb{R}$ and ϕ in $L^2(\mathbb{R})$, we have

$$\left(\pi_\lambda\phi\right)(\xi) = e^{i\lambda\left(t + y\left(\xi + \frac{x}{2}\right)\right)} \phi(\xi + x)$$

which, up to a factor of -2π in front of the x variable, the well-known formula for the Heisenberg's representations found e.g. in [BCD19; BFKG12].

With these notations, the Fourier transform of an integrable function of $\mathbb{G}$ is defined as follows:

Definition 3.16. The Fourier transform of the function $f \in L^1(\mathbb{G})$ at the point

$$(\lambda, \nu) \in \Lambda \times \mathbb{R}^k$$

is a unitary operator acting on $L^2(\mathbb{G})$ with

$$\mathcal{F}(f)(\lambda, \nu) = (\hat{f}(\lambda, \nu) := \int_{\mathbb{G}} f(w)\pi_{\lambda,\nu}(w)^* dw.$$

Here $\pi_{\lambda,\nu}(w)$ are the Schrödinger representations and the integral is a Bochner integral taking values in the Hilbert space $L^2\left(\mathbb{R}^d\right)$. If ψ is another function in $L^2\left(\mathbb{R}^d\right)$, then

$$(\hat{f}(\lambda, \nu)\varphi, \psi) = \int_{\mathbb{G}} f(w)\left(\pi_{\lambda,\nu}(w)^* \varphi, \psi\right) dw.$$

Since $\pi_{\lambda,\nu}(w)$ are unitary operators, it follows that

$$|(\pi_{\lambda,\nu}(w)\varphi, \psi)| \leq \|\varphi\|_2 \|\psi\|_2$$

and consequently

$$|(\hat{f}(\lambda, \nu)\varphi, \psi)| \leq \|\varphi\|_2 \|\psi\|_2 \|f\|_1.$$

This shows that $\hat{f}(\lambda, \nu)$ is a bounded operator on $L^2\left(\mathbb{R}^d\right)$ and the operator norm satisfies $\|\hat{f}(\lambda, \nu)\| \leq \|f\|_1$. In summary, we have the following proposition:

Proposition 3.17. *The Fourier transformation is continuous in all its variables, in the following sense.*

- *For any $\lambda \in \Lambda$ and $\nu \in \mathbb{R}^k$, the map*

$$\mathcal{F}(\cdot)(\lambda, \nu) : L^1\left(\mathbb{R}^d\right) \longrightarrow \mathcal{L}\left(L^2\left(\mathbb{R}^d\right)\right)$$

  is linear and continuous, with norm bounded by 1.

- *For any $u \in L^2\left(\mathbb{R}^d\right)$ and $f \in L^1\left(\mathbb{R}^d\right)$, the map*

$$\mathcal{F}(f)(\cdot, \cdot)(u) : \Lambda \times \mathbb{R}^k \longrightarrow L^2\left(\mathbb{R}^d\right)$$

  is continuous.

Further, the Fourier transform can be extended to an isometry from $L^2(\mathbb{G})$ onto the Hilbert space of two-parameter families $A = \{A(\lambda, v)\}_{(\lambda,v)\in\Lambda\times\mathbb{R}^k}$ of operators on $L^2\left(\mathbb{R}^d\right)$ which are Hilbert-Schmidt for almost every $(\lambda, v) \in \Lambda \times \mathbb{R}^k$, with $\|A(\lambda, v)\|_{\mathrm{HS}\left(L^2\left(\mathbb{R}^d\right)\right)}$ measurable and with norm

$$\|A\| := \left(\iint_{\Lambda\times\mathbb{R}^k} \kappa \|A(\lambda, v)\|^2_{\mathrm{HS}\left(L^2\left(\mathbb{R}^d\right)\right)} \operatorname{Pf}(\lambda) d\nu d\lambda\right)^{\frac{1}{2}} < \infty,$$

where $\kappa > 0$ is a constant depending only on the choice of the group, $\mathrm{Pf}(\lambda) := \prod_{j=1}^{d} \eta_j(\lambda)$ is the Pfaffian of $B^{(\lambda)}$. We have the following Fourier-Plancherel formula:

Proposition 3.18. *There exists some constant $\kappa > 0$ depending only on the choice of the group such that, for any $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, there holds*

$$\int_{\mathbb{G}} |f(w)|^2 dw = \kappa \iint_{\Lambda \times \mathbb{R}^k} \|\mathcal{F}(f)(\lambda,\nu)\|^2_{HS\left(L^2\left(\mathbb{R}^d\right)\right)} \mathrm{Pf}(\lambda) d\lambda d\nu.$$

Proof. The proof is standard as in [CG90; Ray99], we provide proof for completeness. We recall, for $(\lambda,\nu) \in \Lambda \times \mathbb{R}^k$, we get hold of an almost symplectic basis (3.3) and because of the orthonormal basis change, $dxdydrdt$ is the normalized Haar measure, where

$$(x,y,r,t) = \sum_{i=1}^{d} x_i X_i(\lambda) + \sum_{i=1}^{d} y_i Y_i(\lambda) + \sum_{i=1}^{k} r_i R_i(\lambda) + \sum_{i=1}^{m} t_i T_i.$$

Let $\lambda\big|_{\mathfrak{z}} = (\lambda_1, \ldots, \lambda_m)$ and $\lambda\ |_{\mathrm{span}_{\mathbb{R}}\{R_1(\lambda),\ldots,R_k(\lambda)\}} = (\nu_1,\ldots,\nu_k)$ and $d\nu = d\nu_1 \ldots d\nu_k$ denotes the usual Lebesgue measure on $\mathbb{R}^k$. We first prove the following results:

$$\kappa \int_{\mathbb{R}^k} \|\mathcal{F}(f)(\lambda,\nu)\|^2_{HS\left(L^2\left(\mathbb{R}^d\right)\right)} \mathrm{Pf}(\lambda) d\nu = \int_{\mathbb{R}^{2d+k}} \left|\mathcal{F}_1 f\left(x,y,r_1,\ldots,r_k,\lambda_1,\ldots,\lambda_m\right)\right|^2 dxdyd\nu \tag{3.5}$$

where

$$\mathcal{F}_1 f\left(x,y,r_1,\ldots,r_k,\lambda_1,\ldots,\lambda_m\right) = \kappa^{-1} \int_{\mathbb{R}^m} f\left(x,y,r_1,\ldots,r_r,t_1,\ldots,t_m\right) e^{-i\sum_{j=1}^{m} \lambda_j t_j} dt_1 \ldots dt_m$$

and $\lambda\left(T_i\right) = \lambda_i, 1 \le i \le m$. Let $\phi \in L^2\left(\mathbb{R}^d\right)$. Then from (3.4)

$$\begin{aligned}
&\mathcal{F}(f)(\lambda,\nu)\phi(\xi) \\
&= \int_{\mathbb{R}^N} f(x,y,r,t)\pi_{\lambda,\nu}\phi(\xi+x)dxdydrdt \\
&= \int_{\mathbb{R}^{2d+k+m}} f(x,y,r,t)e^{-i\langle\nu,r\rangle}e^{-i\langle\lambda,t\rangle}e^{-i\left\langle\eta(\lambda)\cdot(\xi+\frac{1}{2}x),y\right\rangle}\phi(\xi+x)dxdydrdt \\
&= \int_{\mathbb{R}^{2d+k+m}} f(x-\xi,y,r,t)e^{-i\langle\nu,r\rangle}e^{-i\langle\lambda,t\rangle}e^{-i\sum_{j=1}^{d}\eta_j(\lambda)y_j\xi_j - i\frac{1}{2}\sum_{j=1}^{d}\eta_j(\lambda)(x_j-\xi_j)y_j}\phi(x)dxdydrdt \\
&= \int_{\mathbb{R}^{2d+k+m}} f(x-\xi,y,r,t)e^{-i\langle\nu,r\rangle}e^{-i\langle\lambda,t\rangle}e^{-i\frac{1}{2}\sum_{j=1}^{d}\eta_j(\lambda)x_jy_j - i\frac{1}{2}\sum_{j=1}^{d}\eta_j(\lambda)\xi_jy_j}\phi(x)dxdydrdt \\
&= \int_{\mathbb{R}^{2d+k+m}} f(x-\xi,y,r,t)e^{-i\langle\nu,r\rangle}e^{-i\langle\lambda,t\rangle}e^{-i\frac{1}{2}\sum_{j=1}^{d}(x_j+\xi_j)\eta_j(\lambda)y_j}\phi(x)dxdydrdt.
\end{aligned}$$

Let

$$K(x,\xi) = \int_{\mathbb{R}^{d+k+m}} f(x-\xi,y,r,t)e^{-i\langle\nu,r\rangle}e^{-i\langle\lambda,t\rangle}e^{-i\frac{1}{2}\sum_{j=1}^{d}(x_j+\xi_j)\eta_j(\lambda)y_j} dydrdt.$$

Since $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, it follows that $K \in L^2\left(\mathbb{R}^d \times \mathbb{R}^d\right)$. Then

$$K(x,\xi) = \kappa^{-1}\mathcal{F}_{234} f\left(x-\xi, \frac{x_1+\xi_1}{2}\eta_1(\lambda), \ldots, \frac{x_d+\xi_d}{2}\eta_d(\lambda), \nu_1, \ldots, \nu_k, \lambda_1, \ldots, \lambda_m\right)$$

where $\mathcal{F}_{234}$ stands for the partial Fourier (Euclidean) transform in the variables y, r, t. Thus $\mathcal{F}(f)(\lambda,\nu)$ is a Hilbert-Schmidt operator on $L^2\left(\mathbb{R}^d\right)$ with the kernel $K(x,\xi)$. If we do the change of variables

$$v_i = \frac{x_i+\xi_i}{2}\eta_i(\lambda), 1 \leq i \leq d,$$
$$u_i = x_i - \xi_i, \qquad 1 \leq i \leq d.$$

then the modulus of the Jacobian determinant is $|\eta_1(\lambda)\ldots\eta_d(\lambda)|$ and the above integral reduces to

$$\kappa^{-1}\left|\eta_1(\lambda)\ldots\eta_d(\lambda)\right|^{-1}\left(\int_{\mathbb{R}^{2d}} \left|\mathcal{F}_{234} f\left(u, v, \nu_1, \ldots \nu_r, \lambda_1, \ldots, \lambda_m\right)\right|^2 dudv\right)$$

where $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$. By applying the Euclidean Plancherel theorem in the variable u we get

$$\|\mathcal{F}(f)(\lambda,\nu)\|^2_{HS\left(L^2\left(\mathbb{R}^d\right)\right)} = \kappa^{-1}\left|\eta_1(\lambda)\ldots\eta_d(\lambda)\right|^{-1}\int_{\mathbb{R}^{2d}} \left|\mathcal{F}_{12} f\left(u, v, \nu_1, \ldots \nu_r, \lambda_1, \ldots, \lambda_m\right)\right|^2 dudv.$$

If we integrate both sides of the above equation on $\mathbb{R}^k$ with respect to the usual Lebesgue measure and use change of variables by the map ϕ defined by

$$\begin{aligned} &\phi : \mathbb{R}^k \to \mathbb{R}^k \\ &\phi\left(\lambda_{n_1}, \ldots, \lambda_{n_r}\right) = \left(\nu_1, \ldots, \nu_r\right). \end{aligned} \tag{3.6}$$

We need to find the modulus of the Jacobian determinant of ϕ, which states:

Claim:

$$|\det J_\phi| = \frac{|\operatorname{Pf}(\lambda)|}{\eta_1(\lambda)\eta_2(\lambda)\ldots\eta_d(\lambda)}$$

where J_ϕ is the Jacobian matrix of ϕ.

We restrict ourselves only to the complement of the center, because it is there that the change of basis takes place. We define

$$A_1 : \{X_1, X_2, \ldots, X_n\} \to \{X_{j_1}, \ldots, X_{j_{2d}}, X_{n_1}, \ldots, X_{n_k}\},$$
$$A_2 : \{X_{j_1}, \ldots, X_{J_{2d}}, X_{n_1}, \ldots, X_{n_k}\} \to \left\{X_{j_1}, \ldots, X_{j_{2k}}, \tilde{X}_{n_1}, \ldots, \tilde{X}_{n_k}\right\},$$
$$A_3 : \left\{X_{j_1}, \ldots, X_{j_{2d}}, \tilde{X}_{n_1}, \ldots, \tilde{X}_{n_k}\right\} \to \{X_1(\lambda), \ldots, X_d(\lambda), Y_1(\lambda), \ldots, Y_d(\lambda), R_1(\lambda), \ldots, R_k(\lambda)\},$$

where $\tilde{X}_{n_i} = X_{n_i} - \sum\limits_{s=1}^{2d} c_s^i(l)X_{j_s}, 1 \leq i \leq k$. A_1 is just a rearrangement of basis and hence is given by an orthogonal matrix. A_2 is clearly given by a lower triangular matrix with diagonal entries equal to one. The matrix of A_3 looks like

$$\begin{pmatrix} A' & C' \\ 0 & D' \end{pmatrix}$$

where A' is a $k \times k$ matrix, C' is a $k \times 2d$ matrix and D' is a $2d \times 2d$ matrix, because A_3 is obtained from the following operations:

(i) Gram-Schmidt orthogonalisation of $\left\{\tilde{X}_{n_i} : 1 \leq i \leq k\right\}$

(ii) Finding the orthogonal complement of the span of $\left\{\tilde{X}_{n_i} : 1 \leq i \leq k\right\}$

(iii) Choosing an almost symplectic basis on the nondegenerate subspace of $\tilde{B}_\lambda$

Notice that for $\lambda \in \tilde{V}_T^*, \lambda\left(X_{j_i}\right) = 0, 1 \leq i \leq 2d$; thus $\lambda\left(\tilde{X}_{n_i}\right) = \lambda\left(X_{n_i}\right), 1 \leq i \leq k$. Hence

$$|\det J_\phi| = |\det A'|\,.$$

Since $|\det A_1 \cdot \det A_2 \cdot \det A_3| = 1$, we have $|\det A_3| = 1$. But

$$|\det A_3| = |\det A'||\det D'|.$$

So

$$|\det J_\phi| = |\det D'|^{-1}\,.$$

If we write $\tilde{B}_\lambda$ in terms of the basis $\left\{X_{j_1}, \ldots, X_{j_{2d}}, \tilde{X}_{n_1}, \ldots, \tilde{X}_{n_k}\right\}$, then the matrix of $\tilde{B}_\lambda$ looks like

$$\begin{pmatrix} 0 & 0 \\ 0 & B'_\lambda \end{pmatrix}$$

where $\left(B'_\lambda\right)_{is} = \lambda\left(\left[X_{j_i}, X_{j_s}\right]\right)$. Thus clearly

$$|\det B'_\lambda| = |\operatorname{Pf}(\lambda)|^2.$$

Because of A_3 the above matrix changes to

$$\begin{pmatrix} 0 & 0 \\ 0 & D'B'_\lambda\left(D'\right)^t \end{pmatrix}$$

which is nothing but the matrix in (3.6). So

$$|\det D'|^2 = \frac{|\eta_1(\lambda)\ldots\eta_d(\lambda)|^2}{|\operatorname{Pf}(\lambda)|^2} \Rightarrow |\det D'| = \frac{|\eta_1(\lambda)\ldots\eta_d(\lambda)|}{|\operatorname{Pf}(\lambda)|}.$$

Thus

$$|\det J_\phi| = \frac{|\operatorname{Pf}(\lambda)|}{|\eta_1(\lambda)\ldots\eta_d(\lambda)|}$$

as claimed.

Thus, we get

$$\begin{aligned}&\int_{\mathbb{R}^k} \|\mathcal{F}(f)(\lambda,\nu)\|^2_{HS\left(L^2\left(\mathbb{R}^d\right)\right)} d\nu_1\dots d\nu_k\\ &= |\eta_1(\lambda)\dots\eta_d(\lambda)|^{-1}\frac{|\eta_1(\lambda)\dots\eta_d(\lambda)|}{|\operatorname{Pf}(\lambda)|}\int_{\mathbb{R}^{2d+k}}|\mathcal{F}_{12}f\left(u,v,\nu_1,\dots\nu_k,\lambda_1,\dots,\lambda_m\right)|^2\,dudvd\nu.\end{aligned}$$

Then by applying the Euclidean Plancherel theorem on the variables $(\nu_1,\dots\nu_k)\in\mathbb{R}^k$, we obtain (3.5).

We integrate both sides of (3.5) with respect to the standard Lebesgue measure on $\mathbb{R}^m$ to get

$$\begin{aligned}&\int_{\Lambda\times\mathbb{R}^k}\|\mathcal{F}(f)(\lambda,\nu)\|^2_{HS\left(L^2\left(\mathbb{R}^d\right)\right)}|\operatorname{Pf}(\lambda)|d\lambda\\ &=\int_\Lambda\left(|\operatorname{Pf}(\lambda)|\int_{\mathbb{R}^k}\|\mathcal{F}(f)(\lambda,\nu)\|^2_{HS\left(L^2\left(\mathbb{R}^d\right)\right)}d\nu_1\dots d\nu_k\right)d\lambda_1\dots d\lambda_m\\ &=\kappa^{-1}\int_\Lambda\left(\int_{\mathbb{R}^{2d+k}}|\mathcal{F}_1f\left(x,y,r_1,\dots,r_k,\lambda_1,\dots,\lambda_m\right)|^2\,dxdyd\nu\right)d\lambda_1\dots d\lambda_m\\ &=\kappa^{-1}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2d+k}}|f\left(x,y,r_1,\dots,r_k,t_1,\dots,t_m\right)|^2\,dxdydrdt\end{aligned}$$

by using the Euclidean Plancherel theorem in the outer integral. The last integral is, of course, $\|f\|^2_{L^2(\mathbb{G})}$ and the proof is complete. □

Remark 3.19. The situation is simpler if we consider the case of MW groups. In this case $\mathbb{R}^k=\emptyset$ and the representation π_λ is given by

$$\left(\pi_\lambda(x,y,t)f\right)(\xi)=e^{i\lambda t+\sum\limits_{j=1}^{d}\xi_jy_j\eta_j(\lambda)+\frac{1}{2}\sum\limits_{j=1}^{d}x_jy_j\eta_j(\lambda)}f(x+\xi),$$

where $\xi\in\mathbb{R}^d$, $f\in L^2\left(\mathbb{R}^d\right)$ and $\dim\mathfrak{g}/\mathfrak{z}=2d$. Then it follows from the calculations above that

$$\kappa\,\|\mathcal{F}(f)(\lambda)\|^2_{HS\left(L^2\left(\mathbb{R}^d\right)\right)}=\frac{1}{|\eta_1(\lambda)\dots\eta_d(\lambda)|}\int_{\mathbb{R}^{2d}}|\mathcal{F}_3f\left(x,y,\lambda_1,\dots,\lambda_m\right)|^2\,dxdy.$$

Clearly $|\eta_1(\lambda)\dots\eta_d(\lambda)|=|\operatorname{Pf}(\lambda)|$, since $\tilde{B}_l$ is nondegenerate. The Plancherel theorem again follows. So the change of variables through the map ϕ is not needed for MW groups.

On the Heisenberg group $\mathbb{H}^d$, the Pfaffian is simply $\operatorname{Pf}(\lambda)=|\lambda|^d$ and the value of κ is known, namely

$$\kappa\left(\mathbb{H}^d\right)=\frac{2^{d-1}}{\pi^{d+1}}.$$

In this context, we have an inversion formula as stated in the following proposition:

Proposition 3.20. *For $f\in L^1\left(\mathbb{R}^N\right)$ and almost every $w\in\mathbb{R}^N$, the following inversion formula* *holds:*

$$f(w) = \kappa \iint_{\Lambda \times \mathbb{R}^k} \operatorname{tr}\left(\left(\pi_{\lambda,\nu}(w)\right) \mathcal{F}(f)(\lambda,\nu)\right) \operatorname{Pf}(\lambda) d\lambda d\nu$$

with the same constant $\kappa > 0$.

Finally, the Fourier transform exchanges as usual convolution and product, in the following sense.

Proposition 3.21. *For any $f_1, f_2 \in L^1\left(\mathbb{R}^N\right)$ and $(\lambda,\nu) \in \Lambda \times \mathbb{R}^k$, we have, denoting by $\cdot$ the operator composition on $\mathcal{L}\left(L^2\left(\mathbb{R}^d\right)\right)$,*

$$\mathcal{F}\left(f_1 * f_2\right)(\lambda,\nu) = \mathcal{F}\left(f_1\right)(\lambda,\nu) \cdot \mathcal{F}\left(f_2\right)(\lambda,\nu).$$

3.2.4 The sub-Laplacian operator

Let $\mathfrak{g}$ be a 2-step stratified Lie algebra with a basis $\mathcal{B}$ as before. Now we consider elements of $\mathfrak{g}$ as left invariant differential operators acting on $C^\infty(\mathbb{G})$, that is given $X \in \mathfrak{g}$ and $f \in C^\infty(\mathbb{G})$, the differential operator X acts on f by the rule

$$(Xf)(g) = \left.\frac{d}{ds}\right|_{s=0} f(g \exp sX). \tag{3.7}$$

We define the sub-Laplacian of $\mathbb{G}$ by

$$\mathcal{L} = -\sum_{i=1}^{n} X_i^2.$$

It is a self-adjoint operator which is independent of the orthonormal basis $(X_1,\ldots,X_n)$, and homogeneous of degree 2 with respect to the dilations in the sense that

$$\delta_\lambda^{-1}\mathcal{L}\delta_\lambda = \lambda^2\mathcal{L}.$$

To write its expression in Fourier space, we analysis the left-invariant vector fields as follows. Let $\mathfrak{g}$ be the Lie algebra of all left-invariant vector fields on $\mathbb{G}$. For $j = 1,2,\ldots,d$, let $\gamma_{1,j} : \mathbb{R} \to \mathbb{G}$ and $\gamma_{2,j} : \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{1,j}(\tau) = (\tau e_j, 0, 0, 0)$$

and

$$\gamma_{2,j}(\tau) = (0, \tau e_j, 0, 0)$$

for all $\tau \in \mathbb{R}$, where e_j is the standard unit vector in $\mathbb{R}^d$. For all $l = 1, 2, \ldots, k$ and $s = 1, 2, \ldots, m$, let $\gamma_{3,l} : \mathbb{R} \to \mathbb{G}$ and $\gamma_{4,s} : \mathbb{R} \to \mathbb{G}$ be curves in $\mathbb{G}$ given by

$$\gamma_{3,l}(\tau) = (0, 0, \tau e_l, 0)$$

and

$$\gamma_{4,k}(\tau) = (0, 0, 0, \tau e_s)$$

for all $\tau \in \mathbb{R}$, where e_l is the standard unit vector in $\mathbb{R}^k$ and e_s is the standard unit vector in $\mathbb{R}^m$. Then we define the left-invariant vector fields X_j, Y_j and $R_l, T_s, j = 1, 2, \ldots, d, l = 1, 2, \ldots, k, s = 1, 2, \ldots, m$, on $\mathbb{G}$ as follows. Let $f \in C^\infty(\mathbb{G})$. Then for all $j = 1, 2, \ldots, d$, we define X_j and Y_j by

$$\begin{aligned}
(X_j f)(x, y, r, t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left((x, y, r, t) \cdot \gamma_{1,j}(\tau)\right)\Big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left(x + \tau e_j, y, r, \left(t_s + \frac{1}{2}(B_s y, \tau e_k)\right)_{s=1}^{m}\right)\Bigg|_{\tau=0} \\
&= \frac{\partial}{\partial x_j} f(x, y, r, s) + \frac{1}{2}\sum_{s=1}^{m} (B_s y, e_j)\frac{\partial}{\partial t_s} f(x, y, r, s)
\end{aligned}$$

and

$$\begin{aligned}
(Y_j f)(x, y, r, t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left((x, y, r, t) \cdot \gamma_{2,j}(\tau)\right)\Big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left(x, y + \tau e_j, r, \left(t_s - \frac{1}{2}(x, \tau B_k e_j)\right)_{s=1}^{m}\right)\Bigg|_{\tau=0} \\
&= \frac{\partial}{\partial y_j} f(x, y, r, s) - \frac{1}{2}\sum_{s=1}^{m} (x, B_k e_j)\frac{\partial}{\partial t_s} f(x, y, r, s)
\end{aligned}$$

for all $(x, y, r, s) \in \mathbb{G}$. Similarly, for $l = 1, 2, \ldots, k$ and $s = 1, 2, \ldots, m$, the function $R_l f$ and $T_s f$ are defined by

$$\begin{aligned}
(R_l f)(x, y, r, t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left((x, y, r, t) \cdot \gamma_{3,l}(\tau)\right)\Big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left(x, y, r + \tau e_l, t\right)\Big|_{\tau=0} \\
&= \frac{\partial}{\partial r_l}(x, y, r, t)
\end{aligned}$$

and

$$\begin{aligned}
(T_s f)(x, y, r, t) &= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left((x, y, r, t) \cdot \gamma_{4,s}(\tau)\right)\Big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau} f\left(x, y, r, t + \tau e_s\right)\Big|_{\tau=0} \\
&= \frac{\partial}{\partial t_s} f(x, y, r, t)
\end{aligned}$$

for all $(x, y, r, t) \in \mathbb{G}$. We can easily check that

$$[X_i, Y_j] = -\frac{1}{4}\sum_{s=1}^{m} (B_k)_{ij} T_s, \quad i, j = 1, 2, \ldots, N$$

and the other commutators are zero.

Theorem 3.22. *The Lie algebra $\mathfrak{g}$ is generated by $\{X_i, Y_j, R_l, [X_i, Y_j] : i, j = 1, 2, \ldots, d, l = 1, 2, \ldots, k\}$.*

Proof. It is enough to show that

$$\operatorname{span}\{T_1, T_2, \ldots, T_m\} = \operatorname{span}\{[X_i, X_j] : i, j = 1, 2, \ldots, d\}$$

Let

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_m \end{pmatrix}$$

and

$$Z = \begin{pmatrix} [X_1, Y_1] \\ [X_1, Y_2] \\ \vdots \\ [X_1, Y_n] \\ [X_2, Y_1] \\ [X_2, Y_2] \\ \vdots \\ [X_d, Y_d] \end{pmatrix}.$$

For $1 \leq s \leq m$ and $1 \leq i, j \leq d$, let $(B_k)_{ij}$ be the entry of the matrix B_k in the i th row and j th column. Consider the $d^2 \times m$ matrix

$$C = \begin{bmatrix} (B_1)_{11} & (B_2)_{11} & \ldots & (B_m)_{11} \\ (B_1)_{12} & (B_2)_{12} & \ldots & (B_m)_{12} \\ \vdots & \vdots & \ddots & \vdots \\ (B_1)_{1d} & (B_2)_{1d} & \ldots & (B_m)_{1n} \\ (B_1)_{21} & (B_2)_{21} & \ldots & (B_m)_{21} \\ (B_1)_{22} & (B_2)_{22} & \ldots & (B_m)_{22} \\ \vdots & \vdots & \vdots & \vdots \\ (B_1)_{dd} & (B_2)_{dd} & \ldots & (B_m)_{dd} \end{bmatrix}$$

Then $CT = -Z$. Since C has full column rank, it follows that there exists an $m \times d^2$ matrix (left inverse) D such that

$$DC = I$$

where I is the $m \times m$ identity matrix. Therefore $T = -DZ$. □

We can now define the sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$ by

$$\mathcal{L} = -\sum_{j=1}^{d} \left(X_j^2 + Y_j^2\right) - \sum_{l=1}^{k} R_l^2.$$

Explicitly,

$$\mathcal{L} = -\Delta_x - \Delta_y - \Delta_r - \frac{1}{4}\left(|x|^2 + |y|^2\right)\Delta_t + \sum_{s=1}^{m}\sum_{j=1}^{d}\left\{-\left(B_s y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B_s e_j\right)\frac{\partial}{\partial y_j}\right\}\frac{\partial}{\partial t_s}.$$

By taking the Fourier transform of the sub-Laplacian $\mathcal{L}$ with respect to t, we get parametrized λ-twisted sub-Laplacian $\mathcal{L}^\lambda, \lambda \in \mathbb{R}^m$, given by

$$\mathcal{L}^\lambda = -\Delta_x - \Delta_y - \Delta_r + \frac{1}{4}\left(|x|^2 + |y|^2\right)|\lambda|^2 - i\sum_{j=1}^{d}\left\{-\left(B^{(\lambda)}y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B^{(\lambda)}e_j\right)\frac{\partial}{\partial y_j}\right\},$$

where we use

$$B^{(\lambda)} = \sum_{s=1}^{m}\lambda_s B_s.$$

For $j = 1, 2, \ldots, d$, we define the linear partial differential operators Z_j^λ and $\bar{Z}_j^\lambda$ by

$$Z_j^\lambda = \partial_{z_j} + \frac{1}{2}i\lambda\sum_{s=1}^{m}(B_s)_j\bar{z}_j,$$

and

$$\bar{Z}_j^\lambda = \partial_{\bar{z}_j} - \frac{1}{2}i\lambda\sum_{s=1}^{m}(B_s)_j z_j.$$

Then

$$\begin{aligned}\mathcal{L}^\lambda &= -\frac{1}{2}\sum_{j=1}^{d}\left(Z_j^\lambda\bar{Z}_j^\lambda + \bar{Z}_j^\lambda Z_j^\lambda\right) - \sum_{l=1}^{k}R_l^2 \\ &= -\Delta_z - \Delta_r + \frac{1}{4}|z|^2|\lambda|^2 - iN,\end{aligned}$$

where N is the operator

$$N = i\sum_{j=1}^{d}\left\{-\left(B^{(\lambda)}y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B^{(\lambda)}e_j\right)\frac{\partial}{\partial y_j}\right\}.$$

3.3 (λ, ν)-Weyl transforms

In the standard formulation of quantum mechanics the probability density $\rho(x)$ in position space x is given by the square of the magnitude of the wave function, $\rho(x) = |\psi(x)|^2$. Thus knowing $\psi(x)$ it is easy to visualize the distribution $\rho(x)$. Obtaining the distribution in momentum p is also straightforward. The wave function in p is found by

$$\varphi(p) = \frac{1}{\sqrt{h}} \int e^{-ixp/\hbar} \psi(x) dx = \langle p, \psi \rangle,$$

where all integrations are understood to be over the entire space. The quantity $|\varphi(p)|^2$ gives the probability density in the momentum variable. Although straightforward, the momentum distribution is difficult to visualize if one only has $\psi(x)$. It would be desirable to have a function that displays the probability distribution simultaneously in the x and p variables. The Wigner function, introduced by Wigner in 1932 [Wig32] does just that. Wigner's original goal was to find quantum corrections to classical statistical mechanics where the Boltzmann factors contain energies which in turn are expressed as functions of both x and p. As is well known from the Heisenberg uncertainty relation, there are constraints on this distribution and thus on the Wigner function.

When using Wigner functions the expectation values are obtained in conjunction with the closely associated Weyl transforms of the operators corresponding to physical observables. As shown in [Cas08] the correct Weyl transform is critical for obtaining the spread of the energy of a state; without it, the Wigner function is little more than a visual aid for understanding quantum states.

In fact, the classical Weyl transform was first envisaged in [Wey50] by Hermann Weyl arising in quantum mechanics. The theory of Weyl transform is a vast subject of remarkable interest both in mathematical analysis and physics. In the theory of partial differential equations, Weyl operators have been studied as a particular type of pseudo-differential operators. They have proved to be a useful technique in a quantity of problems like elliptic theory, spectral asymptotics, regularity problems, etc. [Won98].

What's more, it is well known from [Won98] that Weyl transforms have intimate connections with analysis with the so-called λ-twisted sub-Laplacian and the Heisenberg group, and the harmonic analysis there is a very well researched topic. Then in this section, we will study the Weyl transforms and Wigner transforms on 2-step stratified Lie groups $\mathbb{G}$, which should also depend on these parameters and can help us to compute the sub-Laplacian and the λ-twisted sub-Laplacian.

3.3.1 (λ,ν)-Fourier-Wigner transform

In this section, we want to define the (λ,ν)-Weyl transform. As we have known, a basic tool we use in the study of the (λ,ν)-Weyl transform is the (λ,ν)-Wigner transform. And now we find it convenient to introduce first a related transform, which we call the (λ,ν)-Fourier-Wigner transform.

Let $p,q\in\mathbb{R}^d$ and let $\lambda\in\Lambda,\nu\in\mathbb{R}^k$. Then, for every measurable function ϕ on $\mathbb{R}^d$, the function $\pi^{\lambda,\nu}(p,q)\phi$ on $\mathbb{R}^d$ is defined by

$$\pi^{\lambda,\nu}(p,q)\phi(x)=e^{i\sum\limits_{i=1}^{d}\eta_j(\lambda)(p_jx_j+\frac{1}{2}p_jq_j)}\phi(x+q),\quad p,q\in\mathbb{R}^d \tag{3.8}$$

where $\pi^{\lambda,\nu}(p,q)$ stands for $\pi_{\lambda,\nu}(p,q,0,0)$.

Proposition 3.23. *$\pi^{\lambda,\nu}(p,q):L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is a unitary operator for all p and q in $\mathbb{R}^d$.*

Proof. We only need to prove that

$$\|\pi^{\lambda,\nu}(p,q)f\|_{L^2(\mathbb{R}^d)}=\|f\|_{L^2(\mathbb{R}^d)},\quad f\in L^2\left(\mathbb{R}^d\right)$$

and $\pi^{\lambda,\nu}(p,q)$ is onto for all p and q in $\mathbb{R}^d$. Indeed, it follows from (3.8) that

$$\begin{aligned}\|\pi^{\lambda,\nu}(p,q)f\|^2_{L^2(\mathbb{R}^d)}&=\int_{\mathbb{R}^d}\left|e^{i\sum\limits_{i=1}^{d}\eta_j(\lambda)(p_jx_j+\frac{1}{2}p_jq_j)}f(x+q)\right|^2dx\\&=\int_{\mathbb{R}^d}|f(x+q)|^2dx\\&=\int_{\mathbb{R}^d}|f(x)|^2dx\\&=\|f\|^2_{L^2(\mathbb{R}^d)},\quad f\in L^2\left(\mathbb{R}^d\right),\end{aligned}$$

for all p and q in $\mathbb{R}^d$. To prove that $\pi^{\lambda,\nu}(p,q)$ is onto, we let $g\in L^2\left(\mathbb{R}^d\right)$ and define the function f on $\mathbb{R}^d$ by

$$f(x)=e^{-i\sum\limits_{i=1}^{d}\eta_j(\lambda)(p_jx_j-\frac{1}{2}p_jq_j)}g(x-q),\quad x\in\mathbb{R}^d \tag{3.9}$$

Then f is obviously in $L^2\left(\mathbb{R}^d\right)$, and by (3.8) and (3.9),

$$\begin{aligned}(\pi^{\lambda,\nu}(p,q)f)(x)&=e^{i\sum\limits_{i=1}^{d}\eta_j(\lambda)(p_jx_j+\frac{1}{2}p_jq_j)}f(x+q)\\&=e^{i\sum\limits_{i=1}^{d}\eta_j(\lambda)(p_jx_j+\frac{1}{2}p_jq_j)}e^{-i\sum\limits_{i=1}^{d}\eta_j(\lambda)(p_j(x_j+q_j)-\frac{1}{2}p_jq_j)}g(x)\\&=g(x),\quad x\in\mathbb{R}^d.\end{aligned}$$

□

Definition 3.24. For $f, g \in L^2\left(\mathbb{R}^d\right)$, the (λ,ν)-Fourier-Wigner transform of f and g is defined by

$$V_{\lambda,\nu}(f,g)(p,q) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left\langle \pi^{\lambda,\nu}(p,q)f, g\right\rangle,$$

where $\langle , \rangle$ is the inner product in $L^2\left(\mathbb{R}^d\right)$.

Remark 3.25. The (λ,ν)-Fourier-Wigner transform $V_{\lambda,\nu} : \mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d})$ is a bilinear mapping, and we have the following symmetric form

$$\begin{aligned} V_{\lambda,\nu}(f,g)(p,q) &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left\langle \pi^{\lambda,\nu}(p,q)f, g\right\rangle \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\sum\limits_{i=1}^{d}\eta_j(\lambda)(p_jx_j+\frac{1}{2}p_jq_j)} f(x+q)\overline{g(x)}dx \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\eta(\lambda)\cdot px} f\left(x+\frac{q}{2}\right)\overline{g\left(x-\frac{q}{2}\right)}dx. \end{aligned}$$

Moreover, it is easy to see that the (λ,ν)-Fourier-Wigner transform is related to the ordinary Fourier-Wigner transform by

$$V_{\lambda,\nu}(f,g)(p,q) = V(f,g)(\eta(\lambda)\cdot p, q).$$

3.3.2 (λ,ν)-Wigner transform

Next, we introduce the (λ,ν)-Wigner transform and study some of its very basic properties. The original Wigner transform $W(f)$ of a function f in $L^2\left(\mathbb{R}^d\right)$, introduced by Wigner in [Wig32], is a tool for the study of the nonexisting joint probability distribution of position and momentum in the state f. To do this, we begin by computing the Fourier transform of the (λ,ν)-Fourier-Wigner transform.

Definition 3.26. We define the Fourier transform by

$$\left(\mathcal{F}_\lambda(f)\right)(y) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} f(x)e^{-i\sum\limits_{j=1}^{d}\eta_j(\lambda)x_jy_j}dx, \quad y\in\mathbb{R}^d,$$

where $\lambda\in\Lambda$, $f\in L^1\left(\mathbb{R}^d\right)$ and the inverse Fourier transform is defined by

$$\left(\mathcal{F}_\lambda^{-1}(f)\right)(x) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} f(x)e^{i\sum\limits_{j=1}^{d}\eta_j(\lambda)x_jy_j}dy \quad x\in\mathbb{R}^d.$$

Theorem 3.27. *Let f and g be in $\mathcal{S}\left(\mathbb{R}^d\right)$. Then*

$$\left(\mathcal{F}_\lambda\left(V_{\lambda,\nu}(f,g)\right)\right)(x,\xi) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-i\sum\limits_{j=1}^{d}\eta_j(\lambda)q_j\xi_j} f\left(x+\frac{q}{2}\right)\overline{g\left(x-\frac{q}{2}\right)}dq. \tag{3.10}$$

Proof. For any positive number ε, we define the function I_ε on $\mathbb{R}^{2d}$ by

$$\begin{aligned} I_\varepsilon(x,\xi) =& \mathrm{Pf}(\lambda)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2|p|^2}{2}} e^{-i\langle \eta(\lambda), x\cdot p+\xi\cdot q\rangle} V_{\lambda,\nu}(f,g)(p,q)dpdq \\ =& \mathrm{Pf}(\lambda)^{\frac{3}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2|p|^2}{2}} \times e^{-i\langle \eta(\lambda), x\cdot p+\xi\cdot q - p\cdot y\rangle} f\left(y+\frac{q}{2}\right)\overline{g\left(y-\frac{q}{2}\right)}dqdpdy \\ =& \mathrm{Pf}(\lambda)\int_{\mathbb{R}^d} e^{-i\sum\limits_{j=1}^{d}\eta_j(\lambda)q_j\xi_j}\int_{\mathbb{R}^d}\varepsilon^{-d}\times e^{\frac{|\eta(\lambda)|^2|x-y|^2}{2\varepsilon^2}} f\left(y+\frac{q}{2}\right)\overline{g\left(y-\frac{q}{2}\right)}dpdy. \end{aligned} \tag{3.11}$$

Now, for each q in $\mathbb{R}^d$, we define the function F_q on $\mathbb{R}^d$ by

$$F_q(y) = f\left(y+\frac{q}{2}\right)\overline{g\left(y-\frac{q}{2}\right)}, \quad y\in\mathbb{R}^d. \tag{3.12}$$

Then, by (3.11) and (3.12),

$$I_\varepsilon(x,\xi) = \mathrm{Pf}(\lambda)\int_{\mathbb{R}^d} e^{-i\sum_{j=1}^d \eta_j(\lambda)q_j\xi_j}\left(F_q*\varphi_\varepsilon\right)(x)dq, \quad x,\xi\in\mathbb{R}^d \tag{3.13}$$

where

$$\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi\left(\frac{\eta(\lambda)\cdot x}{\varepsilon}\right),\ \varphi(x) = e^{-\frac{|x|^2}{2}}, \quad x\in\mathbb{R}^d. \tag{3.14}$$

Note that, for each fixed q in $\mathbb{R}^d$,

$$F_q*\varphi_\varepsilon \to \left(\int_{\mathbb{R}^d}\varphi(x)dx\right)F_q = \mathrm{Pf}(\lambda)(2\pi)^{\frac{d}{2}}F_q \tag{3.15}$$

uniformly on compact subsets of $\mathbb{R}^d$ as $\varepsilon\to 0$. Let N be any positive integer. Then, by (3.12) and (3.14), there exists a positive constant C_N such that

$$\begin{aligned} \left|\left(F_q*\varphi_\varepsilon\right)(x)\right| &\le \|F_q\|_{L^\infty(\mathbb{R}^d)}\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^d)} \\ &= \|F_q\|_{L^\infty(\mathbb{R}^d)}\|\varphi\|_{L^1(\mathbb{R}^d)} \\ &\le \mathrm{Pf}(\lambda)(2\pi)^{\frac{d}{2}}\sup_{y\in\mathbb{R}^d}\left|f\left(y+\frac{q}{2}\right)g\left(y-\frac{q}{2}\right)\right| \\ &\le C_N\left(1+|q|^2\right)^{-N}, \quad x,q\in\mathbb{R}^d, \end{aligned} \tag{3.16}$$

for all positive numbers ε. So, by (3.13), (3.15) and (3.16), and the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon\to 0} I_\varepsilon(x,\xi) = \mathrm{Pf}(\lambda)^2(2\pi)^{\frac{d}{2}}\int_{\mathbb{R}^d} e^{-i\sum_{j=1}^d\eta_j(\lambda)q_j\xi_j} f\left(x+\frac{q}{2}\right)\overline{g\left(x-\frac{q}{2}\right)}dq, \quad x,\xi\in\mathbb{R}^d \tag{3.17}$$

But, using (3.11) and again the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{\varepsilon\to 0} I_\varepsilon(x,\xi) &= \mathrm{Pf}(\lambda)^{\frac{3}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\langle\eta(\lambda),x\cdot p+\xi\cdot q\rangle}V_{\lambda,\nu}(f,g)(p,q)dpdq \\ &= \mathrm{Pf}(\lambda)^{\frac{3}{2}}(2\pi)^d\left(\mathcal{F}_\nu\left(V_{\lambda,\nu}(f,g)\right)\right)(x,\xi). \end{aligned} \tag{3.18}$$

So, by (3.17) and (3.18),(3.10) is valid. ☐

Definition 3.28. The (λ,ν)-Wigner transform $W_{\lambda,\nu}(f,g)$ for all $f,g \in L^2\left(\mathbb{R}^d\right)$ is defined by

$$\begin{aligned} W_{\lambda,\nu}(f,g)(x,\xi) &= \left(\mathcal{F}_\lambda\left(V_{\lambda,\nu}(f,g)\right)\right)(x,\xi) \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\sum\limits_{j=1}^{d}\eta_j(\lambda)q_j\xi_j} f\left(x+\frac{q}{2}\right)\overline{g\left(x-\frac{q}{2}\right)}dq. \end{aligned}$$

Theorem 3.29 (The Moyal Identity)**.** *For all f_1, g_1, f_2, and g_2 in $\mathcal{S}\left(\mathbb{R}^d\right)$, we have*

$$\left\langle W_{\lambda,\nu}\left(f_1,g_1\right), W_{\lambda,\nu}\left(f_2,g_2\right)\right\rangle = \left\langle f_1,f_2\right\rangle \overline{\left\langle g_1,g_2\right\rangle} \tag{3.19}$$

Proof. We define $\tilde{W} : \mathcal{S}\left(\mathbb{R}^{2d}\right) \to \mathcal{S}\left(\mathbb{R}^{2d}\right)$ by

$$(\tilde{W}F)(x,\xi) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\sum\limits_{j=1}^{d}\eta_j(\lambda)q_j\xi_j} F\left(x+\frac{q}{2}, x-\frac{q}{2}\right)dq, \quad x,\xi \in \mathbb{R}^d \tag{3.20}$$

for all F in $\mathcal{S}\left(\mathbb{R}^{2d}\right)$. Then, by (3.20) and the Plancherel theorem,

$$\begin{aligned} \left\langle \tilde{W}F_1, \tilde{W}F_2\right\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\tilde{W}F_1\right)(x,\xi)\overline{\left(\tilde{W}F_2\right)(x,\xi)}dxd\xi \\ &= \int_{\mathbb{R}^d}\left\{\int_{\mathbb{R}^d} \left(\tilde{W}F_1\right)(x,\xi)\overline{\left(\tilde{W}F_2\right)(x,\xi)}d\xi\right\}dx \\ &= \int_{\mathbb{R}^d}\left\{\int_{\mathbb{R}^d} F_1\left(x+\frac{q}{2}, x-\frac{q}{2}\right)\overline{F_2\left(x+\frac{q}{2}, x-\frac{q}{2}\right)}dq\right\}dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_1\left(x+\frac{q}{2}, x-\frac{q}{2}\right)\overline{F_2\left(x+\frac{q}{2}, x-\frac{q}{2}\right)}dqdx \end{aligned} \tag{3.21}$$

for all F_1 and F_2 in $\mathcal{S}\left(\mathbb{R}^{2d}\right)$. Let $u = x+\frac{q}{2}$ and $v = x-\frac{q}{2}$. Then, by (3.21), we get

$$\begin{aligned} \left\langle \tilde{W}F_1, \tilde{W}F_2\right\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_1(u,v)\overline{F_2(u,v)}dudv \\ &= \left\langle F_1, F_2\right\rangle, \quad F_1, F_2 \in \mathcal{S}\left(\mathbb{R}^{2d}\right). \end{aligned}$$

Now, let f_1, g_1, f_2, and g_2 be in $\mathcal{S}\left(\mathbb{R}^d\right)$. Let F_1 and F_2 be functions on $\mathbb{R}^{2d}$ defined by

$$F_1(u,v) = f_1(u)\overline{g_1(v)}, \quad u,v \in \mathbb{R}^d$$

and

$$F_2(u,v) = f_2(u)\overline{g_2(v)}, \quad u,v \in \mathbb{R}^d$$

Therefore we have

$$\begin{aligned}\left\langle W_{\lambda,\nu}\left(f_1,g_1\right), W_{\lambda,\nu}\left(f_2,g_2\right)\right\rangle &= \left\langle \tilde{W}F_1, \tilde{W}F_2\right\rangle = \left\langle F_1, F_2\right\rangle \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F_1(u,v)\overline{F_2(u,v)}dudv \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_1(u)\overline{g_1(v)f_2(u)}g_2(v)dudv \\ &= \left(\int_{\mathbb{R}^d} f_1(u)\overline{f_2(u)}du\right)\left(\int_{\mathbb{R}^d}\overline{g_1(v)}g_2(v)dv\right) \\ &= \left\langle f_1, f_2\right\rangle\overline{\left\langle g_1, g_2\right\rangle}.\end{aligned}$$

□

Corollary 3.30. $W_{\lambda,\nu} : \mathcal{S}\left(\mathbb{R}^d\right) \times \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^{2d}\right)$ *can be extended uniquely to a bilinear operator*

$$W_{\lambda,\nu} : L^2\left(\mathbb{R}^d\right) \times L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^{2d}\right)$$

such that

$$\|W_{\lambda,\nu}(f,g)\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2\left(\mathbb{R}^d\right)}\|g\|_{L^2\left(\mathbb{R}^d\right)}$$

for all f and g in $L^2\left(\mathbb{R}^d\right)$.

Corollary 3.31. *The Moyal identity and preceding corollary are also true for the (λ,ν)-Fourier-Wigner transform: For all $f_1, g_1, f_2,$ and g_2 in $\mathcal{S}\left(\mathbb{R}^d\right)$, we have*

$$\left\langle V_{\lambda,\nu}\left(f_1,g_1\right), V_{\lambda,\nu}\left(f_2,g_2\right)\right\rangle = \left\langle f_1, f_2\right\rangle\overline{\left\langle g_1, g_2\right\rangle}. \tag{3.22}$$

3.3.3 (λ, ν)-Weyl transform

We can now introduce the (λ,ν)-Weyl transform and explicate its beautiful connection with the Wigner transform.

Definition 3.32. Let a be a function in the Schwartz space $\mathcal{S}\left(\mathbb{R}^{2d}\right)$. For $\lambda \in \Lambda$ and $\nu \in \mathbb{R}^k$, we define $W_a^{\lambda,\nu}$ to be the (λ,ν) -Weyl transform associated to the function a by

$$\begin{aligned}\left\langle W_a^{\lambda,\nu} f, g\right\rangle &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}(2\pi)^{-d/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi)W_{\lambda,\nu}(f,g)(x,\xi)dxd\xi \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\mathcal{F}_\lambda a\right)(p,q)V_{\lambda,\nu}(f,g)(p,q)dpdq \\ &= \mathrm{Pf}(\lambda)(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\mathcal{F}_\lambda a\right)(p,q)\left\langle \pi^{\lambda,\nu}(p,q)f, g\right\rangle dpdq.\end{aligned}$$

Thus we can also write

$$W_\sigma^{\lambda,\nu} = \mathrm{Pf}(\lambda)(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\mathcal{F}_\lambda a\right)(p,q)\pi^{\lambda,\nu}(p,q)dpdq. \tag{3.23}$$

Theorem 3.33. *There exists a unique bounded linear operator* $Q : L^2\left(\mathbb{R}^{2d}\right) \to B\left(L^2\left(\mathbb{R}^d\right)\right)$ *such that*

$$\langle (Qa)f, g\rangle = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi) W_{\lambda,\nu}(f,g)(x,\xi)\, dx d\xi \tag{3.24}$$

and

$$\|Qa\|_* \leq \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \|a\|_{L^2\left(\mathbb{R}^{2d}\right)} \tag{3.25}$$

for all f *and* g *in* $L^2\left(\mathbb{R}^d\right)$ *and* a *in* $L^2\left(\mathbb{R}^{2d}\right)$*, where* $\|\cdot\|_*$ *denotes the norm in* $B\left(L^2\left(\mathbb{R}^d\right)\right)$*.*

Proof. Let $a \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$. Then, for any f in $\mathcal{S}\left(\mathbb{R}^d\right)$, we define $(Qa)f$ by

$$(Qa)f = W_a^{\lambda,\nu} f.$$

Then for all f and g in $\mathcal{S}\left(\mathbb{R}^d\right)$, by (3.23) we have

$$\begin{aligned} \langle (Qa)f, g\rangle &= \left\langle W_a^{\lambda,\nu} f, g\right\rangle \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi) W_{\lambda,\nu}(f,g)(x,\xi)\, dx d\xi. \end{aligned} \tag{3.26}$$

Therefore, by Theorem 3.29 and (3.26),

$$\begin{aligned} |\langle (Qa)f, g\rangle| &\leq \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \|a\|_{L^2\left(\mathbb{R}^{2dn}\right)} \|W_{\lambda,\nu}(f,g)\|_{L^2\left(\mathbb{R}^{2d}\right)} \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \|a\|_{L^2\left(\mathbb{R}^{2d}\right)} \|f\|_{L^2\left(R^d\right)} \|g\|_{L^2\left(\mathbb{R}^d\right)}. \end{aligned} \tag{3.27}$$

Hence we have

$$\|(Qa)f\|_{L^2\left(\mathrm{R}^d\right)} \leq \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \|a\|_{L^2\left(\mathbb{R}^{2d}\right)} \|f\|_{L^2\left(\mathbb{R}^d\right)}$$

and

$$\|Qa\|_* \leq \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \|a\|_{L^2\left(\mathrm{R}^{2d}\right)}, \quad a \in \mathcal{S}\left(\mathbb{R}^{2d}\right). \tag{3.28}$$

Now, let $a \in L^2\left(\mathbb{R}^{2d}\right)$. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of functions in $\mathcal{S}\left(\mathbb{R}^{2d}\right)$ such that $a_k \to a$ in $L^2\left(\mathbb{R}^{2d}\right)$ as $k \to \infty$. Then, by (3.28),

$$\|Qa_k - Qa_l\|_* \leq \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \|a_k - a_l\|_{L^2\left(\mathbb{R}^{2d}\right)} \to 0$$

as $k, l \to \infty$. Thus, $\{Qa_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $B\left(L^2\left(\mathbb{R}^d\right)\right)$. We define Qa to be the limit in $B\left(L^2\left(\mathbb{R}^d\right)\right)$ of the sequence $\{Qa_k\}_{k=1}^{\infty}$. This definition is independent of the choice of the sequence $\{a_k\}_{k=1}^{\infty}$. Indeed, let $\{\tau_k\}_{k=1}^{\infty}$ be another sequence of functions in $\mathcal{S}\left(\mathbb{R}^{2d}\right)$ such that $\tau_k \to a$ in $L^2\left(\mathbb{R}^{2d}\right)$ as $k \to \infty$. Then, again, by (3.28),

$$\|Qa_k - Q\tau_k\|_* \leq \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \|a_k - \tau_k\|_{L^2(\mathbb{R}^{2n})} \to 0$$

as $k \to \infty$. Thus, the limits in $B\left(L^2\left(\mathbb{R}^d\right)\right)$ of $\{Qa_k\}_{k=1}^{\infty}$ and $\{Q\tau_k\}_{k=1}^{\infty}$ are equal. Next, let

$a \in L^2\left(\mathbb{R}^{2d}\right)$, and let $\{a_k\}_{k=1}^{\infty}$ be a sequence of functions in $\mathcal{S}\left(\mathbb{R}^{2d}\right)$ such that $a_k \to a$ in $L^2\left(\mathbb{R}^{2d}\right)$ as $k \to \infty$. Then, by (3.28)

$$\|Qa\|_* = \lim_{k\to\infty} \|Qa_k\|_* \leq \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \lim_{k\to\infty} \|a_k\|_{L^2\left(\mathbb{R}^{2d}\right)} = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\|a\|_{L^2\left(\mathbb{R}^{2d}\right)}$$

and (3.25) is proved. Now, if f and g are in $\mathcal{S}\left(\mathbb{R}^d\right)$, then

$$\begin{aligned}
\langle (Qa)f, g\rangle &= \lim_{k\to\infty} \langle (Qa_k)\, f, g\rangle \\
&= \lim_{k\to\infty} \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a_k(x,\xi) W_{\lambda,\nu}(f,g)(x,\xi)\,dxd\xi \\
&= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi) W_{\lambda,\nu}(f,g)(x,\xi)\,dxd\xi.
\end{aligned}$$

Finally, let f and g be in $L^2\left(\mathbb{R}^d\right)$. Then we pick sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{S}\left(\mathbb{R}^d\right)$ such that $f_k \to f$ in $L^2\left(\mathbb{R}^d\right)$ and $g_k \to g$ in $L^2\left(\mathbb{R}^d\right)$ as $k \to \infty$. We have

$$\begin{aligned}
\langle (Qa)f, g\rangle &= \lim_{k\to\infty} \langle (Qa) f_k, g_k\rangle \\
&= \lim_{k\to\infty} \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi) W_{\lambda,\nu}(f_k,g_k)(x,\xi)\,dxd\xi \\
&= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi) W_{\lambda,\nu}(f,g)(x,\xi)\,dxd\xi
\end{aligned}$$

It is obvious that $Q : L^2\left(\mathbb{R}^{2d}\right) \to B\left(L^2\left(\mathbb{R}^d\right)\right)$ is the only bounded linear operator satisfying (3.24) for all f and g in $L^2\left(\mathbb{R}^d\right)$ and σ in $L^2\left(\mathbb{R}^{2d}\right)$. □

For $a \in L^2\left(\mathbb{R}^{2d}\right)$, we define $D_{\mathrm{Pf}(\lambda)}a(x,\xi) = a\left(x_1\eta_1(\lambda), \cdots, x_d\eta_d(\lambda), \xi\right)$. Then the (λ,ν)-Weyl transform also can be expressed in terms of the dialation $D_{\mathrm{Pf}(\lambda)}$, and the Fourier transform on a 2-step stratified Lie group is in fact a (λ,ν)-Weyl transform on $\mathbb{R}^d$, which are proved in the following propositions.

Proposition 3.34. *[VS21] Let $a \in \mathcal{S}\left(\mathbb{R}^d \times \mathbb{R}^d\right)$. Then the (λ,ν)-Weyl transform $W_a^{\lambda,\nu}$ is given by*

$$W_a^{\lambda,\nu} = W_{D_{\mathrm{Pf}(\lambda)^{-1}}a}.$$

Proposition 3.35. *[VS21] Let $f \in L^1(G)$. Then*

$$\widehat{f}(\mu,\nu) = \mathrm{Pf}(\lambda)^{-1}(2\pi)^d W^{\lambda,\nu}_{\mathcal{F}_\lambda^{-1}\left(f^{\lambda,\nu}\right)}$$

for every $\lambda \in \Lambda$ and $\nu \in \mathbb{R}^t$, where $f^{\lambda,\nu}$ is defined by

$$f^{\lambda,\nu}(x,y) = \int_{\mathbb{R}^k}\int_{\mathbb{R}^m} e^{-i\langle\nu,r\rangle} e^{-i\langle\lambda,t\rangle} f(x,y,r,t)\,drdt.$$

We end this section by showing a relationship between Hilbert-Schmidt pseudo-differential operators on $L^2(\mathbb{G})$ and (λ,ν)-Weyl transforms with symbol in $L^2\left(\mathbb{R}^{2d+k+m}\right)$. The twisting operator $T : L^2\left(\mathbb{R}^{2d}\right) \to L^2\left(\mathbb{R}^{2d}\right)$ is defined by

$$(Tf)(x,y) = f\left(x+\frac{y}{2}, x-\frac{y}{2}\right), \quad x,y \in \mathbb{R}^d,\ \forall f \in L\left(\mathbb{R}^{2d}\right).$$

Clearly, T is a unitary operator and its the inverse is given by

$$\left(T^{-1}f\right)(x,y) = f\left(\frac{x+y}{2}, x-y\right), x,y \in \mathbb{R}^d.$$

Let us define the operator $K_\lambda : L^2\left(\mathbb{R}^{2d}\right) \to L^2\left(\mathbb{R}^{2d}\right)$ by

$$\left(K_\lambda f\right)(x,y) = \left(T^{-1}\mathcal{F}_\lambda^2 f\right)(y,x), x,y \in \mathbb{R}^d \tag{3.29}$$

where $\mathcal{F}_\lambda^2$ is the Fourier transform with respect to the second variable.

We need the following proposition whose proofs can be found in [Won98, Proposition 6.7].

Proposition 3.36. *The linear operator K_λ on $L^2\left(\mathbb{R}^{2d}\right)$ defined by* (3.29) *has the following properties:*

(i) $K_\lambda : L^2\left(\mathbb{R}^{2d}\right) \to L^2\left(\mathbb{R}^{2d}\right)$ *is a unitary operator.*

(ii) $K_\lambda = T^{-1}(\mathcal{F}_\lambda^2)^{-1}$.

(iii) $K_\lambda \bar{f} = (K_\lambda f)^*, \quad f \in L^2\left(\mathbb{R}^{2d}\right)$.

(iv) $W_{\lambda,\nu}(f,g) = K_\lambda^{-1}(f \otimes \bar{g}), \quad f,g \in L^2\left(\mathbb{R}^N\right)$.

We can now give the following important property of the (λ,ν)-Weyl transform.

Theorem 3.37. *Let $a \in L^2\left(\mathbb{R}^{2d}\right)$. Then $W_a^{\lambda,\nu}$ is a Hilbert-Schmidt operator with kernel*

$$\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}K_\lambda a.$$

More precisely

$$\left(W_a^{\lambda,\nu}f\right)(x) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} K_\lambda a(x,y) f(y)dy, \quad x \in \mathbb{R}^d.$$

Proof. Let f and g be in $L^2\left(\mathbb{R}^d\right)$. Then, by Theorem 3.33 and Proposition 3.36, we get

$$\begin{aligned}\left\langle W_a^{\lambda,\nu}f, g\right\rangle &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x,\xi)W_{\lambda,\nu}(f,g)(x,\xi)dxd\xi \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\langle W_{\lambda,\nu}(f,g), \bar{a}\rangle \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left\langle K_\lambda^{-1}(f\otimes\bar{g}), \bar{a}\right\rangle \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left\langle f\otimes\bar{g}, (K_\lambda a)^*\right\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle W_a^{\lambda,\nu} f, g\right\rangle &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\bar{g}(y)(K_\lambda a)(y,x)dxdy \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\left\{\int_{\mathbb{R}^d}(K_\lambda a)(y,x)f(x)dx\right\}\overline{g(y)}dy \\ &= \langle S_k f, g\rangle, \quad f,g\in L^2\left(\mathbb{R}^d\right)\end{aligned}$$

where we define the integral operator $S_k : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$S_k f(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy, \quad x\in\mathbb{R}^d,\ \forall f\in L^2(\mathbb{R}^d),$$

and k is the function on $\mathbb{R}^{2d}$ defined by

$$k(x,y) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}(K_\lambda a)(x,y), \quad x,y\in\mathbb{R}^d$$

and the proof is complete. □

3.3.4 The λ-twisted convolution

The aim of this section is to express the symbol of the product of two (λ,ν)-Weyl transforms with symbols in $L^2\left(\mathbb{R}^{2d}\right)$ in terms of a twisted convolution, which we now define.

Definition 3.38. Let f and g be functions in $L^2\left(\mathbb{C}^d\right)$. Then the λ-twisted convolution $f *_\lambda g$ of f and g is the function on $\mathbb{C}^d$ defined by

$$\left(f *_\lambda g\right)(z) = \iint_{\mathbb{C}^d} f(z-w)g(w)e^{\frac{i}{2}\lambda\sigma(z,w)}dzdw, \quad z\in\mathbb{C}^d, \tag{3.30}$$

where $\sigma(z,w)$ is the bilinear, antisymmetric map of z and w, provided that the integral exists.

Proposition 3.39. *Let f and g be measurable functions on $\mathbb{C}^d$ such that $\left(f *_\lambda g\right)(z)$ exists at the point z in $\mathbb{C}^d$. Then $\left(g *_{-\lambda} f\right)(z)$ exists, and*

$$\left(f *_\lambda g\right)(z) = \left(g *_{-\lambda} f\right)(z)$$

Proof. In (3.30), we change the variable of integration from w to ζ by $w = z-\zeta$. Then we get

$$\left(f *_\lambda g\right)(z) = \int_{\mathbb{C}^n} g(z-\zeta)f(\zeta)e^{\frac{i}{2}\lambda\sigma(z,z-\zeta)}d\zeta \tag{3.31}$$

By (3.30) and (3.31), we get

$$\begin{aligned}\left(f *_\lambda g\right)(z) &= \int_{\mathbb{C}^n} g(z-\zeta)f(\zeta)e^{-\frac{i}{2}\lambda\sigma(z,\zeta)}d\zeta \\ &= \left(g *_{-\lambda} f\right)(z).\end{aligned}$$

□

Remark 3.40. It is clear from Proposition 3.39 that the twisted convolution is, in general, noncommutative.

We can now give a formula for the product $W_\sigma^{\lambda,\nu}W_\tau^{\lambda,\nu}$ of two Weyl transforms $W_\sigma^{\lambda,\nu}$ and W_τ in terms of a twisted convolution of σ and τ.

Theorem 3.41. *Let σ and τ be in $L^2\left(\mathbb{R}^{2d}\right)$. Then*

$$W_\sigma^{\lambda,\nu}W_\tau^{\lambda,\nu} = W_\omega^{\lambda,\nu}$$

where $\omega \in L^2\left(\mathbb{R}^{2d}\right)$ and $\mathcal{F}_\lambda\omega = \mathrm{Pf}(\lambda)(2\pi)^{-d}\left(\mathcal{F}_\lambda\sigma\right) *_\lambda \left(\mathcal{F}_\lambda\tau\right)$

Proof. We begin with the case when both σ and τ are in $\mathcal{S}\left(\mathbb{R}^{2d}\right)$. To do this, let φ and ψ be in $\mathcal{S}\left(\mathbb{R}^{d}\right)$. Then, by Definition 3.24, 3.28 and Theorem 3.27, Fubini's theorem and the adjoint formula in the theory of the Fourier transform,

$$\begin{aligned}
\left\langle W_\sigma^{\lambda,\nu}\varphi, \psi\right\rangle &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\mathcal{F}_\lambda\sigma)(p,q)\langle\pi_{\lambda,\nu}(p,q)\varphi,\psi\rangle dqdp\\
&= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\mathcal{F}_\lambda\sigma)(p,q)\left\{\int_{\mathbb{R}^d}(\pi_{\lambda,\nu}(p,q)\varphi)(x)\overline{\psi(x)}dx\right\}dqdp\\
&= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-d}\int_{\mathbb{R}^d}\overline{\psi(x)}\left\{\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\sigma)(z)(\pi_{\lambda,\nu}(z)\varphi)(x)dz\right\}dx,
\end{aligned}$$

and hence

$$\left(W_\sigma^{\lambda,\nu}\varphi\right)(x) = \mathrm{Pf}(\lambda)^{1/2}(2\pi)^{-d}\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\sigma)(z)(\pi_{\lambda,\nu}(z)\varphi)(x)dz, \quad x\in\mathbb{R}^d \tag{3.32}$$

for all φ in $\mathcal{S}\left(\mathbb{R}^{d}\right)$. But by the irreducible unitary representations and (3.32),

$$\begin{aligned}
\left(\pi_{\lambda,\nu}(z)\left(W_\tau^{\lambda,\nu}\varphi\right)\right)(x) &= e^{i\langle\eta(\lambda)\cdot p, x+\frac{1}{2}q\rangle}\left(W_\tau^{\lambda,\nu}\varphi\right)(x+q)\\
&= e^{i\langle\eta(\lambda)\cdot p, x+\frac{1}{2}q\rangle}\,\mathrm{Pf}(\lambda)^{1/2}(2\pi)^{-d}\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\tau)(w)(\pi_{\lambda,\nu}(w)\varphi)(x+q)dw \qquad (3.33)\\
&= \mathrm{Pf}(\lambda)^{1/2}(2\pi)^{-d}\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\tau)(w)(\pi_{\lambda,\nu}(z)\pi_{\lambda,\nu}(w)\varphi)(x)dw, \quad x\in\mathbb{R}^d,
\end{aligned}$$

for all φ in $\mathcal{S}\left(\mathbb{R}^{d}\right)$. Thus, by (3.32) and (3.33),

$$\left(W_\sigma^{\lambda,\nu}W_\tau^{\lambda,\nu}\varphi\right)(x) = \mathrm{Pf}(\lambda)(2\pi)^{-2d}\int_{\mathbb{C}^d}\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\sigma)(z)(\mathcal{F}_\lambda\tau)(w)(\pi^{\lambda,\nu}(z)\pi^{\lambda,\nu}(w)\varphi)(x)dzdw, \quad x\in\mathbb{R}^d. \tag{3.34}$$

Now, by the definition of irreducible unitary representation on $\mathbb{G}$, we get

$$\begin{aligned}
\pi^{\lambda,\nu}(z)\pi^{\lambda,\nu}(w) &= \pi_{\lambda,\nu}(z,0)\pi_{\lambda,\nu}(w,0) = \pi_{\lambda,\nu}(z+w,\sigma(z,w))\\
&= \pi^{\lambda,\nu}(z+w)e^{-\frac{i}{2}\lambda\sigma(z,w)}, \quad z,w\in\mathbb{C}^d.
\end{aligned} \tag{3.35}$$

So, by (3.34) and (3.35),

$$\left(W_\sigma^{\lambda,\nu}W_\tau^{\lambda,\nu}\varphi\right)(x)=\mathrm{Pf}(\lambda)(2\pi)^{-2d}\int_{\mathbb{C}^d}\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\sigma)(z)(\mathcal{F}_\lambda\tau)(w)(\pi^{\lambda,\nu}(z+w)\varphi)(x)e^{-\frac{i}{2}\lambda\sigma(z,w)}dzdw \tag{3.36}$$

for all φ in $\mathcal{S}\left(\mathbb{R}^d\right)$. Now, in (3.36), we change the variable z to ζ by $z=\zeta-w$. Then, by (3.31), we get

$$\begin{aligned}\left(W_\sigma^{\lambda,\nu}W_\tau^{\lambda,\nu}\varphi\right)(x)&=\mathrm{Pf}(\lambda)(2\pi)^{-2d}\int_{\mathbb{C}^d}\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\sigma)(\zeta-w)(\mathcal{F}_\lambda\tau)(w)(\pi^{\lambda,\nu}(\zeta)\varphi)(x)e^{-\frac{i}{2}\lambda\sigma(\zeta-w,w)}d\zeta dw\\&=\mathrm{Pf}(\lambda)(2\pi)^{-2d}\int_{\mathbb{C}^d}\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\sigma)(\zeta-w)(\mathcal{F}_\lambda\tau)(w)(\pi^{\lambda,\nu}(\zeta)\varphi)(x)e^{-\frac{i}{2}\lambda\sigma(\zeta,w)}d\zeta dw\\&=\mathrm{Pf}(\lambda)(2\pi)^{-2d}\int_{\mathbb{C}^d}\left\{\int_{\mathbb{C}^d}(\mathcal{F}_\lambda\sigma)(\zeta-w)(\mathcal{F}_\lambda\tau)(w)e^{-\frac{i}{2}\lambda\sigma(\zeta,w)}dw\right\}(\pi^{\lambda,\nu}(\zeta)\varphi)(x)d\zeta\\&=\mathrm{Pf}(\lambda)(2\pi)^{-2d}\int_{\mathbb{C}^d}\left((\mathcal{F}_\lambda\sigma)*_\lambda(\mathcal{F}_\lambda\tau)\right)(\zeta)(\pi^{\lambda,\nu}(\zeta)\varphi)(x)d\zeta,\quad x\in\mathbb{R}^d.\end{aligned} \tag{3.37}$$

Hence, by (3.32)and (3.37),

$$W_\sigma^{\lambda,\nu}W_\tau^{\lambda,\nu}=W_\omega^{\lambda,\nu} \tag{3.38}$$

where

$$\mathcal{F}_\lambda\omega=\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-d}\left((\mathcal{F}_\lambda\sigma)*_\lambda(\mathcal{F}_\lambda\tau)\right). \tag{3.39}$$

Now let $\{\sigma_k\}_{k=1}^\infty$ and $\{\tau_k\}_{k=1}^\infty$ be sequences of functions in $\mathcal{S}\left(\mathbb{R}^{2d}\right)$ such that

$$\sigma_k\to\sigma\quad\text{and}\quad\tau_k\to\tau \tag{3.40}$$

in $L^2\left(\mathbb{R}^{2d}\right)$ as $k\to\infty$. Then, by (3.38)and (3.39)

$$W_{\sigma_k}^{\lambda,\nu}W_{\tau_k}^{\lambda,\nu}=W_{\omega_k}^{\lambda,\nu} \tag{3.41}$$

where

$$\mathcal{F}_\lambda\omega_k=\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-d}\left((\mathcal{F}_\lambda\sigma)_k*_\lambda(\mathcal{F}_\lambda\tau)_k\right) \tag{3.42}$$

for $k=1,2,\ldots$. Now, by Theorem 3.33, and (3.40)

$$W_{\sigma_k}^{\lambda,\nu}\to W_\sigma^{\lambda,\nu}\quad\text{and}\quad W_{\tau_k}^{\lambda,\nu}\to W_\tau^{\lambda,\nu} \tag{3.43}$$

in $B\left(L^2\left(\mathbb{R}^d\right)\right)$ as $k\to\infty$. So, by (3.43)

$$W_{\omega_k}^{\lambda,\nu}=W_{\sigma_k}^{\lambda,\nu}W_{\tau_k}^{\lambda,\nu}\to W_\sigma^{\lambda,\nu}W_\tau^{\lambda,\nu} \tag{3.44}$$

in $B\left(L^2\left(\mathbb{R}^d\right)\right)$ as $k \to \infty$. By (3.31) and (3.40)

$$\begin{aligned}\left(\mathcal{F}_\lambda\sigma\right)_k *_\lambda \left(\mathcal{F}_\lambda\tau\right)_k\right)(z) &= \int_{\mathbb{C}^d} (\mathcal{F}_\lambda\sigma)_k(z-w)(\mathcal{F}_\lambda\tau)_k(w)e^{-\frac{i}{2}\lambda\sigma(z,w)}dw \\ &\to \int_{\mathbb{C}^d} (\mathcal{F}_\lambda\sigma)(z-w)(\mathcal{F}_\lambda\tau)(w)e^{-\frac{i}{2}\lambda\sigma(z,w)}dw \\ &= \left(\mathcal{F}_\lambda\sigma\right) *_\lambda \left(\mathcal{F}_\lambda\tau\right)\right)(z)\end{aligned} \tag{3.45}$$

for almost all z in $\mathbb{C}^d$ as $k \to \infty$. On the other hand, we get

$$\begin{aligned}&\|(\mathcal{F}_\lambda\sigma)_k *_\lambda (\mathcal{F}_\lambda\tau)_k - (\mathcal{F}_\lambda\sigma)_j *_\lambda (\mathcal{F}_\lambda\tau)_j\|_{L^2\left(\mathbb{R}^{2d}\right)} \\ &= \|(\mathcal{F}_\lambda\sigma)_k *_\lambda (\mathcal{F}_\lambda\tau)_k - (\mathcal{F}_\lambda\sigma)_k *_\lambda (\mathcal{F}_\lambda\tau)_j + (\mathcal{F}_\lambda\sigma)_k *_\lambda (\mathcal{F}_\lambda\tau)_j - (\mathcal{F}_\lambda\sigma)_j *_\lambda (\mathcal{F}_\lambda\tau)_j\|_{L^2\left(\mathbb{R}^{2d}\right)} \\ &\leq \|(\mathcal{F}_\lambda\sigma)_k *_\lambda (\mathcal{F}_\lambda(\tau_k-\tau_j))\|_{L^2\left(\mathbb{R}^{2d}\right)} + \|(\mathcal{F}_\lambda(\sigma_k-\sigma_j)) *_\lambda (\mathcal{F}_\lambda\tau)_j\|_{L^2\left(\mathbb{R}^{2d}\right)} \\ &= \mathrm{Pf}(\lambda)^{\frac{3}{2}}(2\pi)^{\frac{3d}{2}}\left\|W^{\lambda,\nu}_{\sigma_k}W^{\lambda,\nu}_{\tau_k-\tau_j}\right\|_{HS} + \mathrm{Pf}(\lambda)^{\frac{3}{2}}(2\pi)^{\frac{3d}{2}}\left\|W^{\lambda,\nu}_{\sigma_k-\sigma_j}W^{\lambda,\nu}_{\tau_j}\right\|_{HS} \\ &\leq \mathrm{Pf}(\lambda)^{\frac{3}{2}}(2\pi)^{\frac{3d}{2}}\left\|W^{\lambda,\nu}_{\sigma_k}\right\|_{HS}\left\|W^{\lambda,\nu}_{\tau_k-\tau_j}\right\|_{HS} + \mathrm{Pf}(\lambda)^{\frac{3}{2}}(2\pi)^{\frac{3n}{2}}\left\|W^{\lambda,\nu}_{\sigma_k-\sigma_j}\right\|_{HS}\left\|W^{\lambda,\nu}_{\tau_j}\right\|_{HS} \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{\frac{d}{2}}\left(\|\sigma_k\|_{L^2\left(\mathbb{R}^{2d}\right)}\|\tau_k-\tau_j\|_{L^2\left(\mathbb{R}^{2d}\right)} + \|\sigma_k-\sigma_j\|_{L^2\left(\mathbb{R}^{2d}\right)}\|\tau_j\|_{L^2\left(\mathbb{R}^{2d}\right)}\right) \\ &\to 0\end{aligned}$$

as $k, j \to \infty$. Hence, by the Plancherel theorem, there exists a function ω such that

$$(\mathcal{F}_\lambda\sigma)_k *_\lambda (\mathcal{F}_\lambda\tau)_k \to \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^d(\mathcal{F}_\lambda\omega) \tag{3.46}$$

in $L^2\left(\mathbb{R}^{2d}\right)$ as $k \to \infty$. Therefore, by (3.46), there exists a subsequence $\{(\mathcal{F}_\lambda\sigma)_{k'} *_\lambda (\mathcal{F}_\lambda\tau)_{k'}\}_{k'=1}^\infty$ of $\{(\mathcal{F}_\lambda\sigma)_k *_\lambda (\mathcal{F}_\lambda\tau)_k\}_{k=1}^\infty$ such that

$$(\mathcal{F}_\lambda\sigma)_{k'} *_\lambda (\mathcal{F}_\lambda\tau)_{k'} \to (2\pi)^d(\mathcal{F}_\lambda\omega) \tag{3.47}$$

a.e. on $\mathbb{R}^{2d}$ as $k' \to \infty$. Thus, by (3.45) and (3.47)

$$(2\pi)^d(\mathcal{F}_\lambda\omega) = (\mathcal{F}_\lambda\sigma) *_\lambda (\mathcal{F}_\lambda\tau) \tag{3.48}$$

a.e. on $\mathbb{R}^{2d}$. By (3.42), (3.46), and the Plancherel theorem,

$$\omega_k \to \omega \tag{3.49}$$

in $L^2\left(\mathbb{R}^{2d}\right)$ as $k \to \infty$. Thus, by Theorem 3.33 and (3.49),

$$W^{\lambda,\nu}_{\omega_k} \to W^{\lambda,\nu}_{\omega} \tag{3.50}$$

in $B\left(L^2\left(\mathbb{R}^d\right)\right)$ as $k \to \infty$. So, by (3.44), (3.48) and (3.50), the proof of the theorem is complete.

□

3.4 Stone-von Neumann theorem

In this section, we prove the famous Stone-von Neumann theorem for 2-step nilpotent Lie groups without the Moore-Wolf condition, which means that the unitary irreducible representation (3.4) is uniqueness. We first prove that the representations introduced above are actually irreducible.

Theorem 3.42. *The representations $\pi_{\lambda,\nu}(p,q,r,t)$ are irreducible for any $\lambda \in \Lambda$.*

Proof. Suppose $M \subset L^2\left(\mathbb{R}^d\right)$ is invariant under all $\pi_{\lambda,\nu}(p,q,r,t)$. If $M \neq \{0\}$ we will show that $M = L^2\left(\mathbb{R}^d\right)$ proving the irreducibility of $\pi_{\lambda,\nu}$.

If M is a proper subspace of $L^2\left(\mathbb{R}^d\right)$ invariant under $\pi_{\lambda,\nu}(p,q,r,t)$ for all $(p,q,r,t) \in \mathbb{G}$, then there are nontrivial functions f and g in $L^2\left(\mathbb{R}^d\right)$ such that $f \in M$ and g is orthogonal to $\pi^{\lambda,\nu}(p,q)f$ for all $p,q \in \mathbb{R}^d$. This means that $(\pi^{\lambda,\nu}(p,q)f,g) = 0$. But by above proposition

$$\|V_{\lambda,\nu}(f,g)\|_2 = \|f\|_2\|g\|_2,$$

this is a contradiction since both f and g are nontrivial. Hence M has to be the whole of $L^2(\mathbb{R}^d)$ and this proves that $\pi_{\lambda,\nu}$ is irreducible. □

We now prove the classic theorem of Stone and von Neumann for the 2-step stratified Lie group, which says in effect that any irreducible unitary representation of $\mathbb{G}$ that is nontrivial on the center is equivalent to some $\pi_{\lambda,\nu}$. Since the irreducible representations that are trivial on the center are easily described, as we shall see below, we shall obtain a complete classification of the irreducible unitary representations of $\mathbb{G}$.

We first establish some technical results, which we will mainly use to prove the Stone-von Neumann theorem. The integrated representations of Gaussian functions are of particular interest and importance in this context.

Lemma 3.43. *For $a,b,c,d \in \mathbb{R}^d$ we have*

$$V_{\lambda,\nu}(\pi^{\lambda,\nu}(a,b)f, \pi^{\lambda,\nu}(c,d)g)(p,q) = e^{\frac{1}{2}i\eta(\lambda)\cdot(qa+qc+bc-pb-pd-ad)}V_{\lambda,\nu}(f,g)(p+a-c,q+b-d).$$

Proof. The claim follows from the identity

$$V_{\lambda,\nu}(\pi^{\lambda,\nu}(a,b)f, \pi^{\lambda,\nu}(c,d)g)(p,q) = \langle \pi^{\lambda,\nu}(-c,-d)\pi^{\lambda,\nu}(p,q)\pi^{\lambda,\nu}(a,b)f, g\rangle$$

and the fact that

$$\begin{aligned}
&(-c,-d,0,0)(p,q,0,0)(a,b,0,0) \\
&= (-c,-d,0,0)(p+a,q+b,0,\frac{1}{2}\eta(\lambda)\cdot(aq-pb)) \\
&= (p+a-c,q+b-d,0,\frac{1}{2}\eta(\lambda)\cdot(aq-pb)+\frac{1}{2}\eta(\lambda)\cdot(-d(p+a)+c(q+b))) \\
&= (p+a-c,q+b-d,0,\frac{1}{2}\eta(\lambda)\cdot(qa+qc+bc-pb-pd-ad)).
\end{aligned}$$

□

Corollary 3.44. *The following three identities are special cases of Lemma* 3.43.

1. $V_{\lambda,\nu}(\pi^{\lambda,\nu}(a,b)f,g)(p,q) = e^{\frac{1}{2}i\eta(\lambda)\cdot(qa-pb)}V_{\lambda,\nu}(f,g)(p+a,q+b)$,
2. $V_{\lambda,\nu}(f,\pi^{\lambda,\nu}(c,d)g)(p,q) = e^{\frac{1}{2}i\eta(\lambda)\cdot(qc-dp)}V_{\lambda,\nu}(f,g)(p-c,q-d)$,
3. $V_{\lambda,\nu}(\pi^{\lambda,\nu}(a,b)f,\pi^{\lambda,\nu}(a,b)g)(p,q) = e^{i\eta(\lambda)\cdot(qa-pb)}V_{\lambda,\nu}(f,g)(p,q)$.

The matrix elements of the integrated representation can also be expressed in terms of the (λ,ν)-Fourier-Wigner transform. Indeed, we have

$$\begin{aligned}
\langle \pi^{\lambda,\nu}(F)f,g\rangle &= \iint F(p,q)\langle \pi^{\lambda,\nu}(p,q)f,g\rangle dpdq \\
&= \mathrm{Pf}(\lambda)^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\iint F(p,q)V_{\lambda,\nu}(f,g)(p,q)dpdq.
\end{aligned} \tag{3.51}$$

An interesting thing happens when we use the conjugate of a (λ,ν)-Fourier-Wigner transform as input for the representation $\pi^{\lambda,\nu}$:

Lemma 3.45. *If $\phi,\psi\in L^2(\mathbb{R}^d)$ and $\Phi = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\overline{V_{\lambda,\nu}(\phi,\psi)}$ then*

$$\pi^{\lambda,\nu}(\Phi)f = \langle f,\phi\rangle\psi \quad \textit{for} \quad f\in L^2(\mathbb{R}^d).$$

Proof. By (3.22) and (3.51), we have

$$\begin{aligned}
\langle \pi^{\lambda,\nu}(\Phi)f,g\rangle &= \int \overline{V_{\lambda,\nu}(\phi,\psi)}V_{\lambda,\nu}(f,g) \\
&= \langle V_{\lambda,\nu}(f,g),V_{\lambda,\nu}(\phi,\psi)\rangle \\
&= \langle f,\phi\rangle\overline{\langle g,\psi\rangle} \\
&= \langle f,\phi\rangle\langle \psi,g\rangle
\end{aligned}$$

whence the result is immediate. □

Lemma 3.46. *Let $\varphi(x) := \pi^{-\frac{d}{4}}e^{-\frac{x^2}{2}}$, a scalar multiple of the Gaussian probability distribution, let $\Phi := \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}V_{\lambda,\nu}(\varphi,\varphi)$ and $\Phi^{ab} := \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}V_{\lambda,\nu}(\varphi,\pi(a,b)\varphi)$. Then, we have*

(*i*) $\Phi(p,q) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}e^{-\frac{p^2+q^2}{4}}$, *(ii)* $\Phi^{ab}(p,q) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} e^{\frac{1}{2}i\eta(\lambda)\cdot(qa-pb)} e^{-\frac{(p-a)^2+(q-b)^2}{4}}$,

(iii) $\pi^{\lambda,\nu}(\Phi)\pi^{\lambda,\nu}(a,b)\pi^{\lambda,\nu}(\Phi) = e^{-\frac{a^2+b^2}{4}}\pi^{\lambda,\nu}(\Phi)$.

Proof. To begin with, note that for $\gamma(x) := e^{-x^2/2}$ we have

$$\begin{aligned}(2\pi)^{-\frac{d}{2}}\mathcal{F}\gamma(x) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\gamma(y)e^{-ixy}dy \\ &= (2\pi)^{\frac{d}{2}}\mathcal{F}^{-1}\gamma(x) = e^{-\frac{x^2}{2}}.\end{aligned}$$

(i) It follows that the Fourier transform for Gaussians:

$$\begin{aligned}\Phi(p,q) &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}V_{\lambda,\nu}(\varphi,\varphi)(p,q) \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\pi^{-\frac{d}{2}}\,\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\eta(\lambda)\cdot px}\varphi(x+\frac{q}{2})\overline{\varphi(x-\frac{q}{2})}dx \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\pi^{-\frac{d}{2}}2^{-\frac{d}{2}}e^{-\frac{q^2}{4}}(2\pi)^d\mathcal{F}^{-1}\gamma(p/\sqrt{2}) \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\pi^{-\frac{d}{2}}2^{-\frac{d}{2}}e^{-\frac{q^2}{4}}(2\pi)^{\frac{d}{2}}e^{-\frac{p^2}{2}} \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}e^{-\frac{q^2+p^2}{4}}.\end{aligned}$$

(ii) Once we have (i), an application of Corollary 3.44 (ii) gives:

$$\begin{aligned}\Phi^{ab}(p,q) &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}V_{\lambda,\nu}(\varphi,\pi^{\lambda,\nu}(a,b)\varphi) \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}e^{\frac{1}{2}i\eta(\lambda)\cdot(qa-pb)}V_{\lambda,\nu}(\varphi,\varphi)(p-a,q-b) \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}e^{\frac{1}{2}i\eta(\lambda)\cdot(qa-pb)}\Phi(p-a,q-b) \\ &= \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}e^{\frac{1}{2}i\eta(\lambda)\cdot(qa-pb)}e^{-\frac{(p-a)^2+(q-b)^2}{4}}.\end{aligned}$$

Identity (iii) is due to (i) and repeated use of Lemma 3.45:

$$\begin{aligned}\pi^{\lambda,\nu}(\Phi)\pi^{\lambda,\nu}(a,b)\pi^{\lambda,\nu}(\Phi)f &= \pi^{\lambda,\nu}(\Phi)\langle f,\varphi\rangle\pi^{\lambda,\nu}(a,b)\varphi = \langle f,\varphi\rangle\langle\pi^{\lambda,\nu}(a,b)\varphi,\varphi\rangle\varphi \\ &= V_{\lambda,\nu}(\varphi,\varphi)(a,b)\langle f,\varphi\rangle\varphi = e^{-\frac{a^2+b^2}{4}}\pi^{\lambda,\nu}(\Phi)f.\end{aligned}$$

□

Theorem 3.47. *Let π be any unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, such that for some $\lambda \in \Lambda$, $\pi(0,0,0,t) = e^{i\lambda t}I$. Then $\mathcal{H} = \bigoplus \mathcal{H}_\alpha$ where the $\mathcal{H}_\alpha$ are mutually orthogonal subspaces of $\mathcal{H}$, each invariant under π, such that $\pi|_{\mathcal{H}_\alpha}$ is unitarily equivalent to $\pi_{\lambda,\nu}$ for each α and some $\nu \in \mathbb{R}^k$. In particular, if π is irreducible then π is equivalent to $\pi_{\lambda,\nu}$.*

Proof. The proof is similar to the proof for the Heisenberg group in [Fol89], we give it here for completeness. The key tools in this proof are the (λ,ν)-Fourier-Wigner transform and Gaussian functions as well as their analogues for general unitary representations of $\mathbb{G}$ on any Hilbert space

$\mathcal{H}$. Treating these objects, we adopt the notation from above:

$$
\begin{aligned}
&\varphi(x) := \pi^{-\frac{d}{4}} e^{-\frac{x^2}{2}}, \\
&\varphi^{ab}(x) := \pi^{\lambda,\nu}(a,b)\varphi(x) = \pi^{-\frac{d}{4}} e^{i\eta(\lambda)\cdot(ax+\frac{1}{2}ab)} e^{-\frac{(x+b)^2}{2}}, \\
&\Phi := \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} V_{\lambda,\nu}(\varphi,\varphi), \\
&\Phi^{ab} := \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} V_{\lambda,\nu}(\varphi, \pi^{\lambda,\nu}(a,b)\varphi) = e^{\frac{1}{2}i\eta(\lambda)\cdot(qa-bp)} e^{-\frac{(p-a)^2+(q-b)^2}{4}}.
\end{aligned} \tag{3.52}
$$

Let π be an arbitrary unitary representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$. First we set $\pi(p,q) = \pi(p,q,0,0)$ and we have

$$
\pi(p,q)\pi(r,s) = \pi(p+r, q+s, \frac{1}{2}\eta(\lambda)\cdot(ps-qr)) = e^{\frac{1}{2}i\eta(\lambda)\cdot(ps-qr)}\pi(p+r,q+s).
$$

We consider the integrated version of π,

$$
\pi(F) = \iint F(p,q)\pi^{\lambda,\nu}(p,q)dpdq, \quad F \in L^1\left(\mathbb{R}^{2d}\right),
$$

then we have

$$
\pi(F)\pi(a,b) = \pi(G) \;\text{ where }\; G(p,q) = e^{\frac{1}{2}i\eta(\lambda)\cdot(aq-bp)}F(p-a,q-b), \tag{3.53}
$$

and

$$
\pi(a,b)\pi(F) = \pi(H) \;\text{ where }\; H(p,q) = e^{\frac{1}{2}i\eta(\lambda)\cdot(bp-aq)}F(p-a,q-b). \tag{3.54}
$$

Moreover, π is faithful on $L^1(\mathbb{R}^{2d})$. Indeed, if $\pi(F) = 0$ then, by (3.53) and (3.54), for any $u, v \in \mathcal{H}$ and $a, b \in \mathbb{R}^d$

$$
0 = \langle \pi(a,b)\pi(F)\pi(-a,-b)u, v\rangle = \iint e^{\frac{1}{2}i\eta(\lambda)\cdot(bp-aq)}F(p,q)\langle \pi(p,q)u,v\rangle dpdq.
$$

Thus by the Fourier inversion theorem,

$$
F(p,q)\langle \pi(p,q)u,v\rangle = 0 \quad \text{for a.e. } (p,q)
$$

and since u and v are arbitary, $F = 0$ a.e.

Now let us take F to be the function Φ defined above. By (3.51)-(3.54),

$$
\pi(\Phi)\pi(a,b)\pi(\Phi) = e^{-\frac{(a^2+b^2)}{4}}\pi(\Phi).
$$

In particular, taking $a = b = 0$ we obtain $\pi(\Phi)^2 = \pi(\Phi)$, and since Φ is even and real it is easily seen that $\pi(\Phi)$ is self-adjoint. Thus $\pi(\Phi)$ is an orthogonal projection which is nonzero since $\Phi \neq 0$ and π is faithful.

Let $\mathcal{R}$ denote the range of $\pi(\Phi)$. If $u, v \in \mathcal{R}$ then $u = \pi(\Phi)u$ and $v = \pi(\Phi)v$, so

$$\begin{aligned}\langle \pi(p,q)u, \pi(r,s)v\rangle &= \langle \pi(-r,-s)\pi(p,q)\pi(\Phi)u, \pi(\Phi)v\rangle \\ &= e^{\frac{1}{2}i\eta(\lambda)\cdot(ps-qr)}\langle \pi(\Phi)\pi(p-r,q-s)\pi(\Phi)u, v\rangle \\ &= e^{\frac{1}{2}i\eta(\lambda)\cdot(ps-qr)}e^{-\frac{(p-r)^2+(q-s)^2}{4}}\langle u, v\rangle.\end{aligned} \tag{3.55}$$

Let $\{v_\alpha\}$ be an orthonormal basis for $\mathcal{R}$, and let $\mathcal{H}_\alpha$ be the closed linear span of $\left\{\pi(p,q)v_\alpha : p, q \in \mathbb{R}^d\right\}$. By (3.55), $\mathcal{H}_\alpha \perp \mathcal{H}_\beta$ for $\alpha \neq \beta$, and $\mathcal{H}_\alpha$ is invariant under π by definition. Hence $\mathcal{N} = \left(\bigoplus \mathcal{H}_\alpha\right)^\perp$ is also invariant under π, and we have $\pi(\Phi)|_{\mathcal{N}} = 0$. But this implies that $\mathcal{N} = \{0\}$, for otherwise we could apply the above reasoning to $\pi|_{\mathcal{N}}$ to conclude that $\pi(\Phi)|_{\mathcal{N}}$ were a nonzero orthogonal projection.

We claim that $\pi|_{\mathcal{H}_\alpha}$ is equivalent to $\pi^{\lambda,\nu}$ for all α. Indeed, fix an α and let $v^{pq} = \pi(p,q)v_\alpha$. Then by (3.55)

$$\langle v^{pq}, v^{rs}\rangle = \langle \phi^{pq}, \phi^{rs}\rangle \qquad \text{for all } p, q, r, s.$$

It follows that if $u = \sum a_{jk}v^{p_jq_k}$ and $f = \sum a_{jk}\phi^{p_jq_k}$ then $\|u\|_{\mathcal{H}} = \|f\|_2$, and in particular $u = 0$ iff $f = 0$. Therefore the correspondence $v^{pq} \to \phi^{pq}$ extends by linearity and continuity to a unitary map from $\mathcal{H}_\alpha$ to $L^2\left(\mathbb{R}^d\right)$ that intertwines $\pi|_{\mathcal{H}_\alpha}$ and ρ. $\square$

3.5 Hermite and special Hermite functions

The seminal work by Fourier, published in 1822 [Fou88], about the solution of the heat equation had a deep impact in physics and mathematics as is well known. Roughly speaking, the Fourier method decomposes functions into a superposition of “special functions” [AAR99; Fol89]. In addition, the Fourier method makes use of different types of special functions; each of these types is often related with a group. For Euclidean space it is Bessel functions. The interplay between the properties of Bessel functions and the Euclidean harmonic analysis is beautifully described in Stein and Weiss [SW71]. For noncompact Rank one symmetric spaces it is Legendre and Jacobi functions, which is given in [AT17]. For the Heisenberg group it is the Lagueere and Hermite polynomials, which can be found in [Fol89] and [Tha93]. Based on the methods for the Heisenberg group, it is nature to develop harmonic analysis on stratified Lie groups by the Lagueere and Hermite functions.

In this section we introduce and study some properties of the Hermite and special Hermite functions. For the 2-step stratified Lie group, these functions are eigenfunctions of the rescaled harmonic oscillator and the λ-twisted Laplacian, respectively. As we will see later, the two operators are directly related to the sub-Laplacian on the 2-step stratified Lie group.

We start with the definition of the Hermite polynomials. For $\alpha = 0, 1, 2, \ldots$, and $x \in \mathbb{R}$ we define $H_\alpha(x)$ by the equation

$$H_\alpha(x) = (-1)^\alpha \left(\frac{d^\alpha}{dt^\alpha} \left\{ e^{-x^2} \right\} e^{x^2} \right).$$

The Hermite functions are then defined by

$$h_\alpha(x) = H_\alpha(x) e^{-\frac{1}{2}x^2}.$$

First of all we have the following generating function identity for the Hermite polynomials. If $|r| < 1$, then we have

$$\sum_{\alpha=0}^{\infty} \frac{H_\alpha(x)}{\alpha!} r^\alpha = e^{2xr - r^2}. \tag{3.56}$$

It follows from (3.56) that

$$H'_\alpha(x) = 2\alpha H_{\alpha-1}(x), \qquad H_\alpha(x) = 2x H_{\alpha-1}(x) - H'_{\alpha-1}(x).$$

Defining the creation operator

$$A = -\frac{d}{dx} + x$$

and the annihilation operator

$$A^* = \frac{d}{dx} + x.$$

Then we have

$$A h_\alpha(x) = h_{\alpha+1}(x) \quad \text{and} \quad A^* h_\alpha(x) = 2\alpha h_{\alpha-1}(x).$$

Now, an easy calculation shows that $H = -\frac{d^2}{dx^2} + x^2$, the harmonic oscillator, can be written in the form

$$H = \frac{1}{2}(AA^* + A^*A).$$

The Hermite functions h_α are then eigenfunctions of this operator and

$$H(h_\alpha) = (2\alpha + 1) h_\alpha.$$

Note that $\int_{-\infty}^{\infty} h_\alpha(x)^2 dx = 2^\alpha \alpha! \sqrt{\pi}$, then the normalised Hermite functions are defined by

$$e_\alpha(x) = \left(2^\alpha \sqrt{\pi} \alpha! \right)^{-\frac{1}{2}} H_\alpha(x) e^{-\frac{1}{2}x^2}.$$

These functions form an orthonormal basis for $L^2(\mathbb{R})$. The higher dimensional Hermite functions denoted by Φ_α are then obtained by taking tensor products. Thus for any multi-index α

and $x \in \mathbb{R}^d$, we define

$$\Phi_\alpha(x) = \prod_{j=1}^{d} e_{\alpha_j}(x_j).$$

The family $\{\Phi_\alpha\}$ is then an orthonormal basis for $L^2(\mathbb{R}^d)$ and in particular that for any $i, j \in \mathbb{N}^d$,

$$\left(\Phi_{\alpha_i}, \Phi_{\alpha_j}\right)_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \Phi_{\alpha_i}(x)\Phi_{\alpha_j}(x)dx = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.57}$$

Furthermore, the definition of the Hermite functions entails that for any $\alpha \in \mathbb{N}^d$ and $1 \leq j \leq d$, there holds

$$A_j e_\alpha = \sqrt{2(\alpha_j + 1)} e_{\alpha+\delta_j}$$

and by duality, we get

$$A_j^* e_\alpha = \sqrt{2\alpha_j} e_{\alpha-\delta_j},$$

where $\alpha \pm \delta_j := (\alpha_1, \cdots, \alpha_j \pm 1, \cdots, \alpha_n)$. Also, combining the action of A_j and A_j^* gives, for $\alpha \in \mathbb{N}^d$ and $1 \leq j \leq d$, the harmonic oscillator

$$H := \frac{1}{2}\sum_{j=1}^{d}(A_j A_j^* + A_j^* A_j) = -\Delta + |x|^2,$$

and we have

$$H\Phi_\alpha = (2|\alpha| + d)\Phi_\alpha,$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

Now, if $\eta = (\eta_1, \dots, \eta_d) \in \left(\mathbb{R}_+^*\right)^d$ and $\alpha \in \mathbb{N}^d$, we define the rescaled Hermite function Φ_α^λ by

$$\Phi_\alpha^\lambda := |\operatorname{Pf}(\lambda)|^{\frac{1}{4}} \Phi_\alpha\left(\eta_1^{\frac{1}{2}}\cdot, \eta_1^{\frac{1}{2}}\cdot, \cdots, \eta_d^{\frac{1}{2}}\cdot\right).$$

These functions satisfy identities similar to those of the usual Hermite functions. In particular, they also form an orthonormal basis of $L^2(\mathbb{R}^d)$ and for $\alpha \in \mathbb{N}^d, \eta = (\eta_1, \dots, \eta_d) \in \left(\mathbb{R}_+^*\right)^d$, we have the rescaled harmonic oscillator

$$\mathcal{H}(\lambda)\Phi_\alpha^\lambda := (-\Delta + |\eta \cdot x|^2)\Phi_\alpha^\lambda = \sum_{j=1}^{d} \eta_j(\lambda)(2\alpha_j + 1)\Phi_\alpha^\lambda.$$

Now let us study the action of the Fourier transform on derivatives. Straightforward computations show that

$$\mathcal{F}(Z_j f)(\lambda, \nu) = \mathcal{F}(f)(\lambda, \nu)Q_j^\lambda, \quad \mathcal{F}\left(\bar{Z}_j f\right)(\lambda, \nu) = \mathcal{F}(f)(\lambda, \nu)\bar{Q}_j^\lambda$$

where the operators Q_j^λ and $\bar{Q}_j^\lambda$ are defined by

$$Q_j^\lambda = \partial_{\xi_j} - \eta_j(\lambda)\xi_j \quad \text{and} \quad \bar{Q}_j^\lambda = \partial_{\xi_j} + \eta_j(\lambda)\xi_j.$$

We therefore can write

$$\mathcal{F}(\mathcal{L}f)(\lambda,\nu)(u) = \mathcal{F}(f)(\lambda,\nu)\left(\mathcal{H}(\lambda) + |\nu|^2\right)(u).$$

3.5.1 Mehler's formula for the rescaled harmonic oscillator

In this section, we discuss the Mehler's formula for the rescaled harmonic oscillator with the parameter $|r| = 1$. We begin with the following formulas:

Lemma 3.48. *For α in $\mathbb{C}$ with $|\arg\alpha| \leq \frac{\pi}{2}, \operatorname{Re}\alpha \geq 0$ and $\alpha \neq 0$,*

$$\int_{\mathbb{R}} e^{-\alpha x^2} dx = \pi^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \tag{3.58}$$

which is the improper integration in the sense of Riemann.

Immediately, Lemma 3.48 gives the following lemma, which can be regarded as the Fourier transform of tempered distribution $e^{-\alpha x^2}$ with $\operatorname{Re}\alpha \geq 0$:

Lemma 3.49. *For all ξ in $\mathbb{R}$ and α in $\mathbb{C}$ with $\operatorname{Re}\alpha = 0$,*

$$(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} e^{-\alpha x^2} dx = \begin{cases} (2\alpha)^{-\frac{1}{2}} e^{-\frac{\xi^2}{4\alpha}}, & \alpha \neq 0, \\ (2\pi)^{\frac{1}{2}} \delta(\xi), & \alpha = 0, \end{cases}$$

which is Fourier transform in the sense of tempered distribution $\mathcal{S}'(\mathbb{R})$ and δ is Dirac's delta function. For all ξ in $\mathbb{R}$ and α in $\mathbb{C}$ with $\operatorname{Re}\alpha > 0$,

$$(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-ix\xi} e^{-\alpha x^2} dx = (2\alpha)^{-\frac{1}{2}} e^{-\frac{\xi^2}{4\alpha}}$$

which is the improper integration in the sense of Riemann.

For all λ in Λ and $\eta(\lambda) = (\eta_1(\lambda), \eta_2(\lambda), \cdots, \eta_d(\lambda))$, we now define rescaled harmonic oscillators on $\mathbb{R}$ by

$$e_k^\lambda(x) = |\eta_k(\lambda)|^{\frac{1}{4}} e_k\left(|\eta_k(\lambda)|^{\frac{1}{2}} x\right), \quad k = 0, 1, 2 \ldots.$$

We put $\alpha \in \mathbb{N}^d$ and

$$\Phi_\alpha^\lambda(x) = e_{\alpha_1}^\lambda(x_1) \cdots e_{\alpha_d}^\lambda(x_d)$$

for $x = (x_1, \ldots, x_d)$ in $\mathbb{R}^d$. Lemma 3.49 gives to verify the following Mehler's formula with the parameter $|r| = 1$.

Theorem 3.50. *For all λ in Λ, x and y in $\mathbb{R}^d$ and all $r \in \mathbb{C}$ with $|r| = 1$ and $r \neq \pm 1$,*

$$M_d^\lambda(x, y, r) = \sum_\alpha \Phi_\alpha^\lambda(x)\Phi_\alpha^\lambda(y)r^{|\alpha|} = \frac{\mathrm{Pf}(\lambda)^{1/2}}{\pi^{\frac{d}{2}}(1-r^2)^{\frac{d}{2}}} e^{-\frac{1}{2}|\eta(\lambda)|\frac{1+r^2}{1-r^2}\left(x^2+y^2\right)+\frac{2|\eta(\lambda)|r}{1-r^2}x\cdot y} \tag{3.59}$$

Proof. We prove in $d = 1$ case. It follows from

$$g(x) = e^{-x^2} = \frac{1}{\pi^{\frac{1}{2}}}\int_{\mathbb{R}} e^{-u^2+2ixu}du, \quad \xi \in \mathbb{R}$$

that for all x in $\mathbb{R}$

$$\begin{aligned} \frac{d^k g}{dx^k}(x) &= \frac{1}{\pi^{\frac{1}{2}}}\int_{\mathbb{R}} \left(\frac{d}{dx}\right)^k e^{-u^2+2ixu}du \\ &= \frac{1}{\pi^{\frac{1}{2}}}\int_{\mathbb{R}} (2iu)^k e^{-u^2+2ixu}du. \end{aligned} \tag{3.60}$$

So, the definition of rescaled harmonic oscillators and (3.60) gives

$$\begin{aligned} M_1^\lambda(x, y, r) &= \sum_{k=0}^\infty e_k^\lambda(x)e_k^\lambda(y)r^k \\ &= \sum_{k=0}^\infty \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} e^{\frac{|\eta(\lambda)|}{2}\left(x^2+y^2\right)} \left(2^k k!\right)^{-1} \left\{\frac{d^k g}{dx^k}\left(|\eta(\lambda)|^{\frac{1}{2}}x\right)\right\}\left\{\frac{d^k g}{dy^k}\left(|\eta(\lambda)|^{\frac{1}{2}}y\right)\right\} r^k \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} e^{\frac{|\eta(\lambda)|}{2}\left(x^2+y^2\right)} \sum_{k=0}^\infty \left(2^k k!\right)^{-1} \frac{1}{\pi^{\frac{1}{2}}} \\ &\quad \times \int_{\mathbb{R}} (2iu)^k e^{-u^2+2|\eta(\lambda)|^{\frac{1}{2}}xui}du \left\{\frac{d^k g}{dy^k}\left(|\eta(\lambda)|^{\frac{1}{2}}y\right)\right\} r^k \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi} e^{\frac{|\eta(\lambda)|}{2}\left(x^2+y^2\right)} \sum_{k=0}^\infty \int_{\mathbb{R}}^\infty \frac{(uri)^k}{k!}\left\{\frac{d^k g}{dy^k}\left(|\eta(\lambda)|^{\frac{1}{2}}y\right)\right\} e^{-u^2+2|\eta(\lambda)|^{\frac{1}{2}}xui}du. \end{aligned}$$

Now, using Taylor's theorem, we have

$$g\left(|\eta(\lambda)|^{\frac{1}{2}}y - uri\right) = \sum_{k=0}^\infty \frac{(uri)^k}{k!}\left\{\frac{d^k g}{dy^k}\left(|\eta(\lambda)|^{\frac{1}{2}}y\right)\right\}.$$

Since $r \in \mathbb{C}$ with $|r| = 1$ and $r \neq \pm 1$, we have $\mathrm{Re}\,(1 - r^2) > 0$. Then we have

$$\begin{aligned} &\left|\int_{\mathbb{R}} \sum_{k=0}^\infty \frac{(uri)^k}{k!}\left\{\frac{d^k g}{dy^k}\left(|\eta(\lambda)|^{\frac{1}{2}}y\right)\right\} e^{-u^2+2|\eta(\lambda)|^{\frac{1}{2}}xui}du\right| \\ &= \left|\int_{\mathbb{R}} e^{-\left(|\eta(\lambda)|^{\frac{1}{2}}y-uri\right)^2} e^{-u^2+2|\eta(\lambda)|^{\frac{1}{2}}xui}du\right| \\ &\le \int_{\mathbb{R}} \left|e^{-|\eta(\lambda)|y^2-\left(1-r^2\right)u^2}\right| du \\ &\le e^{-|\eta(\lambda)|y^2}\frac{\pi^{\frac{1}{2}}}{(1-\mathrm{Re}\,r^2)^{\frac{1}{2}}}. \end{aligned}$$

Therefore, Lebesgue's dominated convergence theorem gives

$$\begin{aligned} M_1^\lambda(x,y,r) &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi} e^{\frac{|\eta(\lambda)|}{2}\left(x^2+y^2\right)} \int_{\mathbb{R}} e^{-\left(|\eta(\lambda)|^{\frac{1}{2}}y-uri\right)^2} e^{-u^2+2|\eta(\lambda)|^{\frac{1}{2}}xui} du \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi} e^{\frac{|\eta(\lambda)|}{2}\left(x^2-y^2\right)} \int_{\mathbb{R}} e^{-\left(1-r^2\right)u^2+2|\eta(\lambda)|^{\frac{1}{2}}(x-yr)ui} du \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi} e^{\frac{|\eta(\lambda)|}{2}\left(x^2-y^2\right)} e^{-\frac{|\eta(\lambda)|(x-yr)^2}{1-r^2}} \int_{\mathbb{R}} e^{-\left(1-r^2\right)\left(u-\frac{|\eta(\lambda)|^{\frac{1}{2}}(x-yr)}{1-r^2}i\right)^2} du. \end{aligned}$$

Hence, using Cauchy integral theorem and Lemma 3.49, we have

$$\begin{aligned} M_1^\lambda(x,y,r) &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi} e^{\frac{|\eta(\lambda)|}{2}\left(x^2-y^2\right)} e^{-\frac{|\eta(\lambda)|(x-yr)^2}{1-r^2}} \int_{\mathbb{R}} e^{-\left(1-r^2\right)u^2} du \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} e^{\frac{|\eta(\lambda)|}{2}\left(x^2-y^2\right)-\frac{|\eta(\lambda)|}{1-r^2}(x-yr)^2} \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} e^{-\frac{\eta_j(\lambda)}{2}\frac{1+r^2}{1-r^2}\left(x^2+y^2\right)+\frac{2|\eta(\lambda)|r}{1-r^2}xy} \end{aligned} \tag{3.61}$$

and the proof is complete. □

Proposition 3.51. *For all $\lambda \in \Lambda, \varphi \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$ and $r \in \mathbb{C}$ with $|r| = 1$ and $r \neq \pm 1$,*

$$\lim_{\substack{r\to\pm1\\|r|=1}} \left\langle \sum_\alpha \Phi_\alpha^\lambda(x)\Phi_\alpha^\lambda(y)r^{|\alpha|}, \varphi \right\rangle = \langle \delta(x \mp y), \varphi\rangle$$

where $\delta(x \pm y) = \delta\left(x_1 \pm y_1\right)\cdots\delta\left(x_d \pm y_d\right)$.

Proof. We consider 2-dimensional case. Let r be in $\mathbb{C}$ with $|r| = 1$ and $r \neq \pm 1$. By (3.61), for all $\varphi \in \mathcal{S}\left(\mathbb{R}^2\right)$,

$$\begin{aligned} \left\langle M_1^\lambda(x,y,r), \varphi(x,y)\right\rangle &= \left\langle \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} e^{-\frac{|\eta(\lambda)|}{2}\frac{1+r^2}{1-r^2}\left(x^2+y^2\right)+\frac{2|\eta(\lambda)|r}{1-r^2}xy}, \varphi(x,y) \right\rangle \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} \left\langle e^{-\frac{|\eta(\lambda)|}{4}\left\{\frac{1+r}{1-r}(x-y)^2+\frac{1-r}{1+r}(x+y)^2\right\}}, \varphi(x,y)\right\rangle \end{aligned}$$

where $\frac{1+r}{1-r}, \frac{1-r}{1+r} \in i\mathbb{R}$ by $|r| = 1$. Putting $2s = x - y$ and $2t = x + y$,

$$\left\langle M_1^\lambda(x,y,r), \varphi(x,y)\right\rangle = \frac{|\eta(\lambda)|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} \iint_{\mathbb{R}^2} e^{-|\eta(\lambda)|\left\{\frac{1+r}{1-r}s^2+\frac{1-r}{1+r}t^2\right\}} \varphi(t+s, t-s) 2dsdt.$$

If we put $u = \frac{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}{(1-r)^{\frac{1}{2}}} s$, then we have $u^2 \in i\mathbb{R}$ and

$$\begin{aligned} \left\langle M_1^\lambda(x,y,r), \varphi(x,y)\right\rangle =& \frac{2}{\pi^{\frac{1}{2}}(1+r)} \iint_{\mathbb{R}^2} e^{-u^2-\frac{||(1-r)}{1+r}t^2} \varphi \\ & \times \left(t + \frac{(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}u, t - \frac{(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}u\right) dudt. \end{aligned}$$

Then we take the limit as r tends to 1 for $|r| = 1$, by Lebesgue's dominated convergence theorem and Lemma 3.48, we have

$$\lim_{r\to+1}\left\langle M_1^\lambda(x,y,r),\varphi(x,y)\right\rangle=\lim_{r\to+1}\frac{2}{\pi^{\frac{1}{2}}(1+r)}\iint_{\mathbb{R}^2}e^{-u^2-\frac{|\eta(\lambda)|\frac{1-r}{4}}{1+r}t^2}\varphi$$
$$\times\left(t+\frac{(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}u,t-\frac{(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}u\right)dudt$$
$$=\frac{1}{\pi^{\frac{1}{2}}}\iint_{\mathbb{R}^2}e^{-u^2}\varphi(t,t)dudt=\int_{\mathbb{R}}\varphi(t,t)dt=\langle\delta(x-y),\varphi(x,y)\rangle.$$

On the other hand, if we put $v=\frac{|\eta(\lambda)|^{\frac{1}{2}}(1-r)^{\frac{1}{2}}}{(1+r)^{\frac{1}{2}}}t$, we have

$$\left\langle M_1^\lambda(x,y,r),\varphi(x,y)\right\rangle=\frac{2}{\pi^{\frac{1}{2}}(1-r)}\iint_{\mathbb{R}^2}e^{-\frac{|\eta(\lambda)|(1+r)}{1-r}s^2-v^2}\varphi$$
$$\times\left(\frac{(1+r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1-r)^{\frac{1}{2}}}v+s,\frac{(1+r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1-r)^{\frac{1}{2}}}v-s\right)dvds$$

Similarly, taking the limit as r tends to -1,

$$\lim_{r\to-1}\left\langle M_1^\lambda(x,y,r),\varphi(x,y)\right\rangle=\langle\delta(x+y),\varphi(x,y)\rangle$$

and the proof is complete. □

3.5.2 Special Hermite functions

In this section, we define and prove some important properties of special Hermite functions for the rescaled harmonic oscillator. For each $\alpha,\beta\in\mathbb{N}^d$ and $z\in\mathbb{C}^d$, we define the special Hermite functions $\Phi_{\alpha,\beta}^\lambda$ by

$$\Phi_{\alpha,\beta}^\lambda(z)=V_{\lambda,\nu}(\Phi_\alpha^\lambda,\Phi_\beta^\lambda)(p,q)$$
$$=\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{i\eta(\lambda)\cdot px}\Phi_\alpha^\lambda\left(x+\frac{q}{2}\right)\overline{\Phi_\beta^\lambda\left(x-\frac{q}{2}\right)}dx.$$

Thus $\Phi_{\alpha,\beta}^\lambda(z)$ are the (λ,ν)-Fourier-Wigner transforms of the Hermite functions Φ_α^λ and Φ_β^λ.

The connection of $\left\{\Phi_{\alpha,\beta}^\lambda:\alpha,\beta=0,1,2,\ldots\right\}$ with $\{\Phi_{\alpha,\beta}:\alpha,\beta=0,1,2,\ldots\}$ is given by the following formula.

Theorem 3.52. *For $\lambda\in\Lambda$ and $\alpha,\beta=0,1,2,\ldots$,*

$$\Phi_{\alpha,\beta}^\lambda(p,q)=\mathrm{Pf}(\lambda)^{\frac{1}{2}}\Phi_{\alpha,\beta}\left(\frac{\eta(\lambda)\cdot p}{\sqrt{|\eta(\lambda)|}},\sqrt{|\eta(\lambda)|}q\right),\quad p,q\in\mathbb{R}^d.$$

Proof. For $\lambda \in \Lambda$ and $\alpha, \beta = 0, 1, 2, \ldots$,

$$\begin{aligned}
&\Phi^\lambda_{\alpha,\beta}(p,q)\\
=&V_{\lambda,\nu}(\Phi^\lambda_\alpha, \Phi^\lambda_\beta)(p,q)\\
=&\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\eta(\lambda)\cdot px}\Phi^\lambda_\alpha\left(x+\frac{q}{2}\right)\overline{\Phi^\lambda_\beta\left(x-\frac{q}{2}\right)}dx\\
=&(2\pi)^{-\frac{d}{2}}\,\mathrm{Pf}(\lambda)\int_{\mathbb{R}^d} e^{i\eta(\lambda)\cdot px}\Phi_\alpha\left(\sqrt{|\eta(\lambda)|}\left(x+\frac{q}{2}\right)\right)\overline{\Phi_\beta\left(\sqrt{|\eta(\lambda)|}\left(x-\frac{q}{2}\right)\right)}dx\\
=&\mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\eta(\lambda)\cdot p\frac{x}{\sqrt{|\eta(\lambda)|}}}\Phi_{\alpha_j}\left(x+\sqrt{|\eta(\lambda)|}\frac{q}{2}\right)\overline{\Phi_{\beta_k}\left(x-\sqrt{|\eta(\lambda)|}\frac{q}{2}\right)}dx\\
=&\mathrm{Pf}(\lambda)^{\frac{1}{2}}\Phi_{\alpha,\beta}\left(\frac{\eta(\lambda)\cdot p}{\sqrt{|\eta(\lambda)|}}, \sqrt{|\eta(\lambda)|}q\right)
\end{aligned}$$

for all p and q in $\mathbb{R}^d$. ☐

The Mehler's formula (Theorem 3.50) with the parameter $|r| = 1$ and Lemma 3.49 give the following proposition:

Proposition 3.53. *For $|r| = 1$ with $r \neq \pm 1$, we have*

$$\sum_{k=0}^\infty \Phi^\lambda_{k,k}(p,q)r^k = \frac{\mathrm{Pf}(\lambda)^{\frac{1}{2}}}{(2\pi)^{\frac{d}{2}}(1-r)}e^{-\frac{|\eta(\lambda)|}{4}\frac{1+r}{1-r}(p^2+q^2)}$$

Proof. We only consider the 1-dimension case and first consider in the case of $\eta(\lambda) = 1$. By (3.61), for $|r| = 1$,

$$\begin{aligned}
\sum_{k=0}^\infty e_k\left(x+\frac{q}{2}\right)e_k\left(x-\frac{q}{2}\right)r^k &= \frac{1}{\pi^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}}}e^{-\frac{1}{2}\frac{1+r^2}{1-r^2}\left(x^2+\frac{q^2}{4}\right)+\frac{2r}{1-r^2}\left(x^2-\frac{q^2}{4}\right)}\\
&= \frac{1}{\pi^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}}}e^{-\frac{1-r}{1+r}x^2-\frac{1}{4}\frac{1+r}{1-r}q^2}.
\end{aligned}$$

So we have

$$\begin{aligned}
\sum_{k=0}^\infty e_{k,k}(p,q)r^k &= \sum_{k=0}^\infty\left\{\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx}e_k\left(x+\frac{q}{2}\right)e_k\left(x-\frac{q}{2}\right)dx\right\}r^k\\
&= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx}\left\{\sum_{k=0}^\infty e_k\left(x+\frac{q}{2}\right)e_k\left(x-\frac{q}{2}\right)r^k\right\}dx\\
&= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx}\frac{1}{\pi^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}}}e^{-\frac{1-r}{1+r}y^2-\frac{1}{4}\frac{1+r}{1-r}q^2}dx\\
&= \frac{1}{\pi^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}}}e^{-\frac{1}{4}\frac{1+r}{1-r}q^2}\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx}e^{-\frac{1-r}{1+r}x^2}dx.
\end{aligned}$$

But if $|r| = 1$, $\frac{1-r}{1+r}$ is in $i\mathbb{R}$. So, by Lemma 3.49,

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx}e^{-\frac{1-r}{1+r}x^2}dx = 2^{-\frac{1}{2}}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}e^{-\frac{1}{4}\frac{1+r}{1-r}p^2}.$$

Then we have

$$\sum_{k=0}^{\infty} e_{k,k}(p,q)r^k = \frac{1}{(2\pi)^{\frac{1}{2}}(1-r)} e^{-\frac{1}{4}\frac{1+r}{1-r}\left(p^2+q^2\right)}.$$

Because of Theorem 3.52, we have

$$\begin{aligned}\sum_{k=0}^{\infty} e^{\lambda}_{k,k}(p,q)r^k &= \sum_{k=0}^{\infty} |\eta(\lambda)|^{\frac{1}{2}} e_{k,k}\left(\frac{\eta(\lambda)\cdot p}{|\eta(\lambda)|^{\frac{1}{2}}}, |\eta(\lambda)|^{\frac{1}{2}} q\right) r^k \\ &= \frac{|\eta(\lambda)|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}(1-r)} e^{-\frac{|\eta(\lambda)|}{4}\frac{1+r}{1-r}\left(p^2+q^2\right)}\end{aligned}$$

as asserted. □

We also consider to take the limit as $r \to \pm 1$.

Proposition 3.54. *For all* $\varphi \in \mathcal{S}\left(\mathbb{R}^{2d}\right)$ *and* $\lambda \in \Lambda$,

$$\lim_{r\to 1}\left\langle \sum_{k=0}^{\infty} \Phi^{\lambda}_{k,k}(p,q)r^k, \varphi(p,q)\right\rangle = \frac{(2\pi)^{\frac{d}{2}}}{\mathrm{Pf}(\lambda)^{\frac{1}{2}}}\langle \delta(p,q), \varphi(p,q)\rangle,$$

and

$$\lim_{r\to -1}\sum_{k=0}^{\infty} \Phi^{\lambda}_{k,k}(p,q)r^k = \frac{\mathrm{Pf}(\lambda)^{\frac{1}{2}}}{(2\pi)^{\frac{d}{2}}}.$$

Proof. We still consider the case $d=1$. Putting $p = \frac{2(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}u$ and $q = \frac{2(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}v$, we have

$$\begin{aligned}&\left\langle \sum_{k=0}^{\infty} \Phi^{\lambda}_{k,k}(p,q)r^k, \varphi(q,p)\right\rangle \\ =&\frac{|\eta(\lambda)|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}(1-r)}\frac{4(1-r)}{|\eta(\lambda)|(1+r)}\iint_{\mathbb{R}^2} e^{-u^2-v^2}\varphi\times\left(\frac{2(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}u, \frac{2(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}v\right) dudv \\ =&\frac{2^{3/2}}{|\eta(\lambda)|^{\frac{1}{2}}\pi^{\frac{1}{2}}(1+r)}\iint_{\mathbb{R}^2} e^{-u^2-v^2}\varphi\times\left(\frac{2(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}u, \frac{2(1-r)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}(1+r)^{\frac{1}{2}}}v\right) dudv.\end{aligned}$$

Then it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}\lim_{r\to 1}\left\langle \sum_{k=0}^{\infty} \Phi^{\lambda}_{k,k}(p,q)r^k, \varphi(q,p)\right\rangle &= \frac{2^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}\pi^{\frac{1}{2}}}\varphi(0,0)\iint_{\mathbb{R}^2} e^{-u^2-v^2}dudv \\ &= \frac{2^{\frac{1}{2}}\pi^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}}\varphi(0,0) = \frac{(2\pi)^{\frac{1}{2}}}{|\eta(\lambda)|^{\frac{1}{2}}}\langle \delta(q,p), \varphi(q,p)\rangle\end{aligned}$$

On the other hand, according to Proposition 3.53, simple computation complete the proof. □

We now show that $\left\{\Phi^{\lambda}_{\alpha,\beta}\right\}$ is an orthonormal basis for $L^2\left(\mathbb{C}^d\right)$.

Proposition 3.55. *The special Hermite functions form a complete orthonormal system for* $L^2\left(\mathbb{C}^d\right)$.

Proof. The orthonormality follows from the properties of the (λ,ν)-Fourier-Wigner transform. To prove completeness, we use the Plancherel theorem for the (λ,ν)-Weyl transform. Suppose $f \in L^2\left(\mathbb{C}^d\right)$ is orthogonal to all $\Phi^\lambda_{\alpha,\beta}$. Using the definition of $\Phi^\lambda_{\alpha,\beta}$ this means that

$$\int_{\mathbb{C}^d} \bar{f}(z)\left(\pi^{\lambda,\nu}(z)\Phi^\lambda_\alpha, \Phi^\lambda_\beta\right) dz = \left(W^{\lambda,\nu}(\bar{f})\Phi^\lambda_\alpha, \Phi^\lambda_\beta\right) = 0.$$

The completeness of $\left\{\Phi^\lambda_\alpha\right\}$ in $L^2\left(\mathbb{R}^d\right)$ proves that $W^{\lambda,\nu}(\bar{f}) = 0$ which implies $f = 0$ in view of the Plancherel theorem for the (λ,ν)-Weyl transform. □

3.5.3 Eigenvalue problems of the λ-twisted sub-Laplacian

We now show that our special Hermite functions are eigenfunctions of the λ-twisted operator $\mathcal{L}^\lambda$ as in Section 3.2.4. For $j = 1, 2, \ldots, d$, we define the linear partial differential operators Z^λ_j and $\bar{Z}^\lambda_j$ by

$$Z^\lambda_j = \partial_{z_j} + \frac{1}{2} i B^{(\lambda)} \bar{z}_j,$$

and

$$\bar{Z}^\lambda_j = \partial_{\bar{z}_j} - \frac{1}{2} i B^{(\lambda)} z_j.$$

Then

$$\begin{aligned}\mathcal{L}^\lambda &= -\frac{1}{2}\sum_{j=1}^{d}\left(Z^\lambda_j \bar{Z}^\lambda_j + \bar{Z}^\lambda_j Z^\lambda_j\right) - \sum_{l=1}^{k} R_l^2 \\ &= -\Delta_z - \Delta_r + \frac{1}{4}|z|^2|\lambda|^2 - iN,\end{aligned}$$

where N is the operator

$$N = i\sum_{j=1}^{d}\left\{-\left(B^{(\lambda)}y, e_j\right)\frac{\partial}{\partial x_j} + \left(x, B^{(\lambda)}e_j\right)\frac{\partial}{\partial y_j}\right\},$$

We now prove that $\Phi^\lambda_{\alpha,\beta}$ are eigenfunctions of the operator $\mathcal{L}^\lambda$.

Theorem 3.56. *For $j = 1, 2, \ldots, d$, one has the formulas*

(1) $\left(Z^\lambda_j \Phi^\lambda_{\alpha,\beta}\right) = i\eta_j(\lambda)^{\frac{1}{2}}(2\alpha_j)^{\frac{1}{2}}\Phi^\lambda_{\alpha,\beta-e_j}$;

(2) $\left(\bar{Z}^\lambda_j \Phi^\lambda_{\alpha,\beta}\right) = i\eta_j(\lambda)^{\frac{1}{2}}(2\alpha_j+2)^{\frac{1}{2}}\Phi^\lambda_{\alpha,\beta+e_j}$;

(3) $\mathcal{L}^\lambda(\Phi^\lambda_{\alpha,\beta}) = \left(\sum\limits_{j=1}^{d}\eta_j(\lambda)(2\alpha_j+1) + \sum\limits_{j=1}^{k}\nu_j^2\right)\Phi^\lambda_{\alpha,\beta}$.

Proof. As the functions $\Phi^\lambda_{\alpha,\beta}(z)$ are products of $\Phi^\lambda_{\alpha_j,\beta_j}(z_j)$, so we consider the functions

$$\Phi^\lambda_{j,l}(p,q) = \eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} e^{i\eta_j(\lambda)\cdot px}\Phi^\lambda_j(x+\frac{q}{2})\Phi^\lambda_l(x-\frac{q}{2})dx.$$

Differentiating with respect to p and writing $2x = (x + \frac{q}{2}) + (x - \frac{q}{2})$ we have

$$\begin{aligned}\frac{\partial \Phi^\lambda_{j,l}}{\partial p}(p,q) &= i\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \eta_j(\lambda) x e^{i\eta_j(\lambda)\cdot px} \Phi^\lambda_j \left(x + \frac{q}{2}\right) \Phi^\lambda_l \left(x - \frac{q}{2}\right) dx \\ &= J^{(+)}(p,q) + J^{(-)}(p,q),\end{aligned}$$

where

$$J^{(\pm)}(p,q) = \frac{1}{2} i\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \eta_j(\lambda) e^{i\eta_j(\lambda)\cdot px} \left(x \pm \frac{q}{2}\right) \Phi^\lambda_j \left(x + \frac{q}{2}\right) \Phi^\lambda_l \left(x - \frac{q}{2}\right) dx.$$

Next, for all p and q in $\mathbb{R}$

$$i\frac{\partial \Phi^\lambda_{j,l}}{\partial q}(p,q) = K^{(1)}(p,q) - K^{(2)}(p,q)$$

where

$$K^{(1)}(p,q) = \frac{i}{2}\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} (\Phi^\lambda_j)' \left(x + \frac{q}{2}\right) \Phi^\lambda_l \left(y - \frac{q}{2}\right) dx$$

and

$$K^{(2)}(p,q) = \frac{i}{2}\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} \Phi^\lambda_j \left(x + \frac{q}{2}\right) (\Phi^\lambda_l)' \left(x - \frac{q}{2}\right) dx.$$

Now, by the construction at beginning we get, for $l = 0, 1, 2, \ldots$,

$$\left(x - \frac{d}{dx}\right) e_l(x) = (2l+2)^{\frac{1}{2}} e_{l+1}(x), \quad x \in \mathbb{R},$$

and,

$$\left(x + \frac{d}{dx}\right) e_l(x) = (2l)^{\frac{1}{2}} e_{l-1}(x), \quad x \in \mathbb{R}.$$

So, for $j = 0, 1, 2, \ldots$ and $l = 1, 2, \ldots$ we get

$$\begin{aligned}\frac{\partial \Phi^\lambda_{j,l}}{\partial z}(z) &= \left(J^{(+)}(p,q) - K^{(1)}(p,q)\right) + \left(J^{(-)}(p,q) + K^{(2)}(p,q)\right) \\ &= \frac{1}{2} i\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} (2\alpha_j + 2)^{\frac{1}{2}} \eta_j(\lambda)^{\frac{1}{2}} \Phi^\lambda_{j+1} \left(x + \frac{q}{2}\right) \Phi^\lambda_l \left(x - \frac{q}{2}\right) dx \right. \\ &\qquad \left. + \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} (2\alpha_l)^{\frac{1}{2}} \eta_j(\lambda)^{\frac{1}{2}} \Phi^\lambda_j \left(x + \frac{q}{2}\right) \Phi^\lambda_{l-1} \left(x - \frac{q}{2}\right) dx \right\} \\ &= \frac{1}{2} i\eta_j(\lambda)^{\frac{1}{2}} \left\{ (2\alpha_j + 2)^{\frac{1}{2}} \Phi^\lambda_{j+1,l}(z) + (2\alpha_l)^{\frac{1}{2}} \Phi^\lambda_{j,l-1}(z) \right\}, \quad z = p + iq \in \mathbb{C}.\end{aligned} \tag{3.62}$$

We can also obtain

$$
\begin{aligned}
\frac{\partial \Phi_{j,l}^{\lambda}}{\partial \bar{z}}(z) =& \left(J^{(+)}(p,q) + K^{(1)}(p,q)\right) + \left(J^{(-)}(p,q) - K^{(2)}(p,q)\right) \\
=& \frac{1}{2} i\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} \eta_j(\lambda)^{\frac{1}{2}} (2\alpha_j)^{\frac{1}{2}} \Phi_{j-1}^{\lambda}\left(x + \frac{q}{2}\right) \Phi_l^{\lambda}\left(x - \frac{q}{2}\right) dx \right. \\
& \left. + \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} \eta_j(\lambda)^{\frac{1}{2}} (2\alpha_l + 2)^{\frac{1}{2}} \Phi_j^{\lambda}\left(x + \frac{q}{2}\right) \Phi_{l+1}^{\lambda}\left(x - \frac{q}{2}\right) dx \right\} \\
=& \frac{1}{2} i\eta_j(\lambda)^{\frac{1}{2}} \left\{ (2\alpha_j)^{\frac{1}{2}} \Phi_{j-1,l}^{\lambda}(z) + (2\alpha_l + 2)^{\frac{1}{2}} \Phi_{j,l+1}^{\lambda}(z) \right\}, \quad z = p + iq \in \mathbb{C}.
\end{aligned}
\tag{3.63}
$$

Writing $\eta_j(\lambda)pe^{i\eta_j(\lambda)px} = -i\frac{\partial}{\partial x} e^{i\eta_j(\lambda)\cdot px}$ and integrating by parts we get

$$
\begin{aligned}
\frac{1}{2}\eta_j(\lambda)p\Phi_{j,l}^{\lambda}(p,q) &= -\frac{i}{2}(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial x} e^{i\eta_j(\lambda)\cdot px} \right\} \Phi_j^{\lambda}\left(x + \frac{q}{2}\right) \Phi_l^{\lambda}\left(x - \frac{q}{2}\right) dx \\
&= K^{(1)}(p,q) + K^{(2)}(p,q), \quad p, q \in \mathbb{R}.
\end{aligned}
\tag{3.64}
$$

Using the formula

$$
q = \left(x + \frac{q}{2}\right) - \left(x - \frac{q}{2}\right),
$$

we also have

$$
\frac{i}{2} q\eta_j(\lambda)\Phi_{j,l}^{\lambda}(p,q) = J^{(+)}(p,q) + J^{(-)}(p,q), \quad p, q \in \mathbb{R}. \tag{3.65}
$$

So, by (3.64) and (3.65), we get, for $j = 1, 2, \ldots$ and $l = 0, 1, 2, \ldots,$

$$
\begin{aligned}
\frac{1}{2}\eta_j(\lambda)z\Phi_{j,l}^{\lambda}(z) =& \frac{i}{2}\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \left\{ \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} \eta_j(\lambda)^{\frac{1}{2}} (2\alpha_j)^{\frac{1}{2}} \Phi_{j-1}^{\lambda}\left(x + \frac{q}{2}\right) \Phi_l^{\lambda}\left(x - \frac{q}{2}\right) dy \right. \\
& \left. - \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} \eta_j(\lambda)^{\frac{1}{2}} (2\alpha_l + 2)^{\frac{1}{2}} \Phi_j^{\lambda}\left(x + \frac{q}{2}\right) \Phi_{l+1}^{\lambda}\left(x - \frac{q}{2}\right) dy \right\} \\
=& \frac{i}{2}\eta_j(\lambda)^{\frac{1}{2}} \left\{ (2\alpha_j)^{\frac{1}{2}} \Phi_{j-1,l}^{\lambda}(z) - (2\alpha_l + 2)^{\frac{1}{2}} \Phi_{j,l+1}^{\lambda}(z) \right\}
\end{aligned}
\tag{3.66}
$$

and

$$
\begin{aligned}
\frac{1}{2}\eta_j(\lambda)\bar{z}\Phi_{j,l}^{\lambda}(z) =& \frac{i}{2}\eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \left\{ - \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} \eta_j(\lambda)^{\frac{1}{2}} (2\alpha_j + 2)^{\frac{1}{2}} \Phi_{j+1}^{\lambda}\left(x + \frac{q}{2}\right) \Phi_l^{\lambda}\left(x - \frac{q}{2}\right) dy \right. \\
& \left. + \int_{-\infty}^{\infty} e^{i\eta_j(\lambda)\cdot px} \eta_j(\lambda)^{\frac{1}{2}} (2\alpha_l)^{\frac{1}{2}} \Phi_j^{\lambda}\left(x + \frac{q}{2}\right) \Phi_{l-1}^{\lambda}\left(x - \frac{q}{2}\right) dy \right\} \\
=& \frac{i}{2}\eta_j(\lambda)^{\frac{1}{2}} \left\{ (2\alpha_l)^{\frac{1}{2}} \Phi_{j,l-1}^{\lambda}(z) - (2\alpha_j + 2)^{\frac{1}{2}} \Phi_{j+1,l}^{\lambda}(z) \right\}
\end{aligned}
\tag{3.67}
$$

for all z in $\mathbb{C}$. Therefore, by (3.62) and (3.67),

$$
Z_j^{\lambda}\Phi_{j,l}^{\lambda} = i\eta_j(\lambda)^{\frac{1}{2}}(2\alpha_l)^{\frac{1}{2}}\Phi_{j,l-1}^{\lambda}, \quad j = 0, 1, 2, \ldots, l = 1, 2, \ldots
$$

and by (3.63) and (3.66),

$$
\bar{Z}_j^{\lambda}\Phi_{j,l}^{\lambda} = i\eta_j(\lambda)^{\frac{1}{2}}(2\alpha_j + 2)^{\frac{1}{2}}\Phi_{j,l+1}^{\lambda}, \quad j = 1, 2, \ldots, l = 0, 1, 2, \ldots.
$$

Now, for $j = 0, 1, 2, \ldots$ and $l = 1, 2, \ldots$, we have

$$Z_j^\lambda \bar{Z}_j^\lambda \Phi_{j,l}^\lambda = i\eta_j(\lambda)^{\frac{1}{2}}(2\alpha_l + 2)^{\frac{1}{2}} Z_j^\lambda \Phi_{j,l+1}^\lambda = -(2\alpha_l + 2)\eta_j(\lambda)\Phi_{j,l}^\lambda,$$

and

$$\bar{Z}_j^\lambda Z_j^\lambda \Phi_{j,l}^\lambda = i\eta_j(\lambda)^{\frac{1}{2}}(2\alpha_l)^{\frac{1}{2}} Z_j^\lambda \Phi_{j,l-1}^\lambda = -2\alpha_l\eta_j(\lambda)\Phi_{j,l}^\lambda.$$

Then the third one follows from (1) and (2) and the definition of $\mathcal{L}^\lambda$. □

3.6 Laguerre functions

3.6.1 Laguerre polynomials

In this section, we recall some properties for the Laguerre polynomials, which can be found in [Tha93; Won98], we give some proofs here for completeness. Let $\delta > -1$, Laguerre polynomial of degree k and order δ are defined by the formula

$$L_k^\delta(x) = \frac{x^{-\delta}e^x}{k!}\left(\frac{d}{dx}\right)^k \left(e^{-x}x^{\delta+k}\right).$$

Here $x > 0$ and $k = 0, 1, 2, \ldots$. If we write out $L_k^\delta(x), x > 0$, in detail, then we get

$$L_k^\delta(x) = \frac{x^{-\delta}e^x}{k!}\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}(-1)^j e^{-x}\left(\frac{d}{dx}\right)^{k-j}\left(x^{\delta+k}\right), \quad x > 0.$$

Thus,

$$\begin{aligned} L_k^\delta(x) &= \frac{(-1)^k}{k!}x^k + \sum_{j=0}^{k-1}\frac{(\delta+k)(\delta+k-1)\cdots(\delta+j+1)}{(k-j)!j!}(-x)^j \\ &= \sum_{j=0}^{k}\frac{\Gamma(k+\delta+1)}{\Gamma(k-j+1)\Gamma(j+\delta+1)}\frac{(-x)^j}{j!} \end{aligned}.$$

Lemma 3.57. *The Laguerre polynomials satisfy the orthogonality properties*

$$\int_0^\infty L_k^\delta(x)L_j^\delta(x)e^{-x}x^\delta dx = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)}\delta_{jk}.$$

Proof. Let f be any polynomial and consider

$$\int_0^\infty f(x)L_k^\delta(x)e^{-x}x^\delta dx = \frac{1}{k!}\int_0^\infty f(x)\frac{d^k}{dx^k}\left(e^{-x}x^{k+\delta}dx\right).$$

Integrating by parts we see that

$$\int_0^\infty f(x)L_k^\delta(x)e^{-x}x^\delta dx = \frac{(-1)^k}{k!}\int_0^\infty f^{(k)}(x)e^{-x}x^{k+\delta}dx.$$

If f is a polynomial of degree $j < k$ then it follows that

$$\int_0^\infty f(x) L_k^\delta(x) e^{-x} x^\delta dx = 0.$$

In particular this proves that when $k \neq j$

$$\int_0^\infty L_k^\delta(x) L_j^\delta(x) e^{-x} x^\delta dx = 0.$$

And when $k = j$, taking $f(x) = L_k^\delta(x)$ we observe that $f^{(k)}(x) = (-1)^k$ so that

$$\int_0^\infty L_k^\delta(x) L_k^\delta(x) e^{-x} x^\delta dx = \frac{1}{k!} \int_0^\infty x^{k+\delta} e^{-x} dx$$

which proves the Lemma. □

Therefore, if we define $\mathcal{L}_k^\delta(x)$ by

$$\mathcal{L}_k^\delta(x) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} \right)^{\frac{1}{2}} e^{-x} x^{\frac{\delta}{2}} L_k^\delta(x),$$

we have the following result.

Theorem 3.58. *$\mathcal{L}_k^\delta(x)$ is an orthonormal basis for $L^2(0,\infty)$.*

Proof. By Lemma 3.57, we only need to prove that if $g \in L^2(0,\infty)$ is such that

$$\left\langle g, L_k^\delta \right\rangle = 0, \quad k = 0, 1, 2, \ldots$$

then $g = 0$ a.e. on $(0,\infty)$. Now, for $k = 0, 1, 2, \ldots$, we get, by [Won98, Lemma 18.5],

$$x^k = \sum_{j=0}^{k} c_j L_j^\delta(x), \quad x > 0$$

where $c_0, c_1, c_2, \ldots, c_k$ are constants. Thus, for $k = 0, 1, 2, \ldots$, we have

$$\int_0^\infty g(x) x^k x^\delta e^{-x} dx = \sum_{j=0}^{k} c_j \int_0^\infty g(x) L_j^\delta(x) x^\delta e^{-x} dx = 0. \tag{3.68}$$

Let $x = y^2$ in (3.68), we get, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} 2\int_0^\infty g\left(y^2\right) y^{2k} y^{2\delta+1} e^{-y^2} dy = 0 \Rightarrow 2 &\int_0^\infty g\left(y^2\right) y^{2k} |y|^{2\delta+1} e^{-y^2} dy = 0 \\ \Rightarrow &\int_{-\infty}^\infty g\left(y^2\right) y^{2k} |y|^{2\delta+1} e^{-y^2} dy = 0 \end{aligned}.$$

Hence,

$$\int_{-\infty}^\infty g\left(y^2\right) y^k |y|^{2\delta+1} e^{-y^2} dy = 0, \quad k = 0, 1, 2, \ldots.$$

Let P be any polynomial of degree j. Then

$$P(y) = \sum_{k=0}^{j} a_k y^k, \quad y \in \mathbb{R}$$

where $a_0, a_1, a_2, \ldots, a_j$ are constants. And

$$\int_{-\infty}^{\infty} g\left(y^2\right) P(y)|y|^{2\delta+1} e^{-y^2} dy = 0.$$

Also, for all ξ in $\mathbb{R}$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left|g(y^2)\right| |y|^{2\delta+1} e^{|y\xi|} e^{-y^2} dy \\
&\leq \left\{\int_{-\infty}^{\infty} \left|g\left(y^2\right)\right|^2 |y|^{2\delta+1} e^{-y^2} dy\right\}^{\frac{1}{2}} \left\{\int_{-\infty}^{\infty} |y|^{2\delta+1} e^{2|y\xi|} e^{-y^2} dy\right\}^{\frac{1}{2}}. \\
&= \left\{\int_{0}^{\infty} |g(x)|^2 x^{\delta} e^{-x} dx\right\}^{\frac{1}{2}} \left\{\int_{-\infty}^{\infty} |y|^{2\delta+1} e^{-2|y\xi|} e^{-y^2} dy\right\}^{\frac{1}{2}} \\
&< \infty
\end{aligned}$$

Thus, by [Won98, Lemma 18.6], we have $g(y^2)|y|^{2\delta+1} = 0$ for almost all y in $\mathbb{R}$. Therefore, $g = 0$ a.e. on $(0, \infty)$. $\square$

Theorem 3.59. *For each fixed positive number x,*

$$\sum_{k=0}^{\infty} L_k^{\delta}(x) r^k = \frac{e^{-\frac{xr}{1-r}}}{(1-r)^{\delta+1}}, \quad |r| < 1,$$

where the series is uniformly and absolutely convergent on every compact subset of $\{r \in \mathbb{C} : |r| < 1\}$. We call $e^{-\frac{xr}{1-r}}(1-r)^{-\delta-1}$ the generating function of the Laguerre polynomials $L_k^{\delta}, k = 0, 1, 2, \ldots$.

Proof. Let γ be a circle with center at x and radius l, and lying inside the right half plane. Now,

$$\begin{aligned}
\sum_{k=0}^{\infty} L_k^{\delta}(x) r^k &= \sum_{k=0}^{\infty} \frac{x^{-\delta} e^x}{k!} \left(\frac{d}{dx}\right)^k \left(e^{-x} x^{\delta+k}\right) r^k \\
&= \frac{x^{-\delta} e^x}{2\pi i} \sum_{k=0}^{\infty} r^k \int_{\gamma} \frac{e^{-\zeta} \zeta^{\delta+k}}{(\zeta - x)^{k+1}} d\zeta
\end{aligned} \tag{3.69}$$

where the principal branch of $\zeta^{\delta+k}$ is taken, i.e.,

$$\zeta^{\delta+k} = e^{(\delta+k) \operatorname{Log}_{-\pi} \zeta}$$

and

$$\operatorname{Log}_{-\pi} \zeta = \ln|\zeta| + i \operatorname{Arg}_{-\pi} \zeta, \quad -\pi < \operatorname{Arg}_{-\pi} \zeta < \pi.$$

Next, for $k = 1, 2, \ldots$

$$\begin{aligned}\left|\frac{e^{-\zeta}\zeta^{\delta+k}}{(\zeta-x)^{k+1}}\right| &= \frac{e^{-\operatorname{Re}\zeta}e^{(\delta+k)\ln|\zeta|}}{l^{k+1}} \\ &\leq \frac{e^{-(x-l)}(x+l)^{\delta+k}}{l^{k+1}} \\ &= e^{-(x-l)}\frac{(x+l)^{\delta}}{l}\left(\frac{x+l}{l}\right)^{k}.\end{aligned}$$

Then, for all r in $\mathbb{C}$ with $|r| < \frac{l}{x+l}$, the series $\sum_{k=0}^{\infty}\frac{r^k e^{-\zeta}\zeta^{\delta+k}}{(\zeta-x)^{k+1}}$ is uniformly and absolutely convergent with respect to r on $\{r \in \mathbb{C} : |r| < r_x\}$ and ζ on γ, where r_x is any number in $\left(0, \frac{l}{x+l}\right)$. Therefore, by (3.69)

$$\sum_{k=0}^{\infty} L_k^{\delta}(x)r^k = \frac{x^{-\delta}e^x}{2\pi i}\int_{\gamma}\frac{e^{-\zeta}\zeta^{\delta}}{\zeta-x}\sum_{k=0}^{\infty}\left(\frac{r\zeta}{\zeta-x}\right)^k d\zeta$$

for $|r| < r_x$. Note that

$$\left|\frac{r\zeta}{\zeta-x}\right| \leq \frac{|r|(x+l)}{l} < 1$$

and hence,

$$\begin{aligned}\sum_{k=0}^{\infty} L_k^{\delta}(x)r^k &= \frac{x^{-\delta}e^x}{2\pi i}\int_{\gamma}\frac{e^{-\zeta}\zeta^{\delta}}{\zeta-x}\frac{1}{1-\frac{r\zeta}{\zeta-x}}d\zeta = \frac{x^{-\delta}e^x}{2\pi i}\int_{\gamma}\frac{e^{-\zeta}\zeta^{\delta}}{\zeta-x-r\zeta}d\zeta \\ &= \frac{x^{-\delta}e^x}{2\pi i}\int_{\gamma}\frac{e^{-\zeta}\zeta^{\delta}}{(1-r)\zeta-x}d\zeta = \frac{x^{-\delta}e^x}{1-r}\frac{1}{2\pi i}\int_{\gamma}\frac{e^{-\zeta}\zeta^{\delta}}{\zeta-\frac{x}{1-r}}d\zeta\end{aligned} \tag{3.70}$$

for $|r| < r_x$. For sufficiently small r, $\frac{x}{1-r}$ is inside γ. So, by (3.70)

$$\sum_{k=0}^{\infty} L_k^{\delta}(x)r^k = \frac{x^{-\delta}e^x}{1-r}e^{-\frac{x}{1-r}}\left(\frac{x}{1-r}\right)^{\delta} = \frac{e^{-\frac{xr}{1-r}}}{(1-r)^{\delta+1}}$$

for sufficiently small r. Now, $\frac{e^{-\frac{xr}{1-r}}}{(1-r)^{\delta+1}}$ is an analytic function on $\{r \in \mathbb{C} : |r| < 1\}$. Thus, by the principle of analytic continuation,

$$\sum_{k=0}^{\infty} L_k^{\delta}(x)r^k = \frac{e^{\frac{-xr}{1-r}}}{(1-r)^{\delta+1}}, \quad |r| < 1,$$

and the theorem is proved. □

3.6.2 Laguerre formulas for special Hermite functions

Now we give the formula expressing the special Hermite functions $\Phi_{\alpha,\alpha}^{\lambda}$ in terms of Laguerre polynomials.

Theorem 3.60. *For $\alpha \in \mathbb{N}^d$ and any z in $\mathbb{C}^d$,*

$$\Phi^{\lambda}_{\alpha,\alpha}(z) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \prod_{j=1}^{d} L^0_{\alpha_j}\left(\frac{1}{2}\eta_j(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}.$$

Proof. We only consider the 1-dimension case and first consider the case of $\eta(\lambda) = 1$. By Mehler's formula in Theorem 3.50,

$$\sum_{k=0}^{\infty} e_k\left(x+\frac{q}{2}\right) e_k\left(x-\frac{q}{2}\right) r^k = \frac{1}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} e^{-\frac{1-r}{1+r}x^2 - \frac{1}{4}\frac{1+r}{1-r}q^2}.$$

So we have

$$\begin{aligned}\sum_{k=0}^{\infty} e_{k,k}(p,q) r^k &= \sum_{k=0}^{\infty}\left\{\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx} e_k\left(x+\frac{q}{2}\right) e_k\left(x-\frac{q}{2}\right) dx\right\} r^k \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx}\left\{\sum_{k=0}^{\infty} e_k\left(x+\frac{q}{2}\right) e_k\left(x-\frac{q}{2}\right) r^k\right\} dx \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx}\frac{1}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} e^{-\frac{1-r}{1+r}y^2 - \frac{1}{4}\frac{1+r}{1-r}q^2} dx \\ &= \frac{1}{\pi^{\frac{1}{2}}\left(1-r^2\right)^{\frac{1}{2}}} e^{-\frac{1}{4}\frac{1+r}{1-r}q^2}\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx} e^{-\frac{1-r}{1+r}x^2} dx.\end{aligned}$$

By Lemma 3.49, we have

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{ipx} e^{-\frac{1-r}{1+r}x^2} dx = 2^{-\frac{1}{2}}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}} e^{-\frac{1}{4}\frac{1+r}{1-r}p^2}.$$

Then

$$\sum_{k=0}^{\infty} e_{k,k}(p,q) r^k = \frac{1}{(2\pi)^{\frac{1}{2}}(1-r)} e^{-\frac{1}{4}\frac{1+r}{1-r}\left(p^2+q^2\right)}.$$

Because of Theorem 3.52, we have

$$\begin{aligned}\sum_{k=0}^{\infty} e^{\lambda}_{k,k}(p,q) r^k &= \sum_{k=0}^{\infty} \eta_j(\lambda)^{\frac{1}{2}} e_{k,k}\left(\frac{\eta_j(\lambda)\cdot p}{\eta_j(\lambda)^{\frac{1}{2}}}, \eta_j(\lambda)^{\frac{1}{2}} q\right) r^k \\ &= \frac{\eta_j(\lambda)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}(1-r)} e^{-\frac{\eta_j(\lambda)}{4}\frac{1+r}{1-r}\left(p^2+q^2\right)}.\end{aligned}$$

So, by Theorem 3.59

$$\sum_{k=0}^{\infty} e^{\lambda}_{k,k}(z_j) r^k = \frac{\eta_j(\lambda)^{\frac{1}{2}}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} L^0_k\left(\frac{1}{2}\eta_j(\lambda)|z_j|^2\right) r^k e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}$$

and hence

$$e^{\lambda}_{k,k}(z_j) = \eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} L^0_k\left(\frac{1}{2}\eta_j(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}, \quad z_j \in \mathbb{C}.$$

□

The following lemma will be used later.

Lemma 3.61. *For $\delta > -1$ and $k = 1, 2, \ldots,$*

$$\frac{d}{dx}\left(L_k^{\delta}(x)\right) = -L_{k-1}^{\delta+1}(x), \quad x > 0.$$

Proof. By definition of Laguerre polynomials,

$$\begin{aligned}
\frac{d}{dx}\left(L_k^{\delta}(x)\right) &= \frac{d}{dx}\sum_{j=0}^{k}\frac{\Gamma(k+\delta+1)}{\Gamma(k-j+1)\Gamma(j+\delta+1)}\frac{(-x)^j}{j!} \\
&= -\sum_{j=1}^{k}\frac{\Gamma(k+\delta+1)}{\Gamma(k-j+1)\Gamma(j+\delta+1)}\frac{(-x)^{j-1}}{(j-1)!} \\
&= -\sum_{l=0}^{k-1}\frac{\Gamma(k-1+\delta+1+1)}{\Gamma(k-1-l+1)\Gamma(l+\delta+1+1)}\frac{(-x)^l}{l!} \\
&= -L_{k-1}^{\delta+1}(x), \quad x > 0.
\end{aligned}$$

□

If α and k are multi-index we write $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_d!$ and $z^k = z_1^{k_1}\cdots z_d^{k_d}$. With these notations we have the following formulas expressing Hermite functions in terms of Laguerre polynomials.

Theorem 3.62. *For $\alpha \in \mathbb{N}^d, k = 0, 1, \ldots$ and any $z \in \mathbb{C}^d$ we have*

(i) $\Phi_{\alpha+k,\alpha}^{\lambda}(z) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left(\frac{\alpha!}{(\alpha+k)!}\right)^{\frac{1}{2}}\left(\frac{i}{\sqrt{2}}\right)^k \bar{z}^k\prod_{j=1}^{k} L_{\alpha_j}^{k_j}(\frac{1}{2}\eta_j(\lambda)|z_j|^2)e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}.$

(ii) $\Phi_{\alpha,\alpha+k}^{\lambda}(z) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}\left(\frac{\alpha!}{(\alpha+k)!}\right)^{\frac{1}{2}}\left(\frac{-i}{\sqrt{2}}\right)^k z^k\prod_{j=1}^{k} L_{\alpha_j}^{k_j}(\frac{1}{2}\eta_j(\lambda)|z_j|^2)e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}.$

Proof. Again we only consider the one dimension case. From the definition it follows that $\Phi_{\alpha,\beta}^{\lambda}(z) = \overline{\Phi}_{\alpha,\beta}^{\lambda}(-z)$. Thus, if part (i) of Theorem is true, then,

$$\begin{aligned}
e_{j,j+k}^{\lambda}(z_j) &= \eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}\left\{\frac{j!}{(j+k)!}\right\}^{\frac{1}{2}}\left(-\frac{i}{\sqrt{2}}\right)^k(-z_j)^k L_j^k\left(\frac{1}{2}\eta_j(\lambda)|z_j|^2\right)e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2} \\
&= \eta_j(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}\left\{\frac{j!}{(j+k)!}\right\}^{\frac{1}{2}}\left(\frac{i}{\sqrt{2}}\right)^k z_j^k L_j^k\left(\frac{1}{2}\eta_j(\lambda)|z_j|^2\right)e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}, \quad z_j \in \mathbb{C}.
\end{aligned}$$

Thus, to prove this Theorem, we only need to prove part (i).

By Theorem 3.60, the formula is true if $k = 0$. Suppose that the formula is true for all nonnegative integers j and all nonnegative integers k with $k \leq l$. Then, by Theorem 3.56,

$$e_{j+k+1,j} = -i\eta_{j+1}(\lambda)^{-\frac{1}{2}}(2j+2)^{-\frac{1}{2}}Z_j^{\lambda}e_{j+k+1,j+1}. \tag{3.71}$$

Now, by the induction hypothesis, we have, for all $z_j \in \mathbb{C}$,

$$e_{j+k+1,j+1}(z) = (2\pi)^{-\frac{1}{2}}\eta_{j+1}(\lambda)^{\frac{1}{2}}\left\{\frac{(j+1)!}{(j+k+1)!}\right\}^{\frac{1}{2}}\left(\frac{i}{\sqrt{2}}\right)^k(\bar{z})^k L_{j+1}^k\left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right)e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2}.$$

Let f_j be the function on $\mathbb{C}$ defined by

$$f_j(z) = (\bar{z})^k L_{j+1}^k \left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2}, \quad z_j \in \mathbb{C}. \tag{3.72}$$

Then, for $k \geq 1$,

$$\begin{aligned} \frac{\partial f_j}{\partial p}(z) =&(\bar{z})^k \left(\partial L_{j+1}^k\right) \left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right) \eta_{j+1}(\lambda) p e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z|^2} \\ &+ (\bar{z})^k L_{j+1}^k \left(\frac{1}{2}\eta_{j+1}(\lambda)|z|^2\right) \left(-\frac{1}{2}\eta_{j+1}(\lambda)p\right) e^{-\frac{1}{4}|z|^2} \\ &+ k(\bar{z})^{k-1} L_{j+1}^k \left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2}, \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} i\frac{\partial f_j}{\partial q}(z) =&(\bar{z})^k \left(\partial L_{j+1}^k\right) \left(\frac{1}{2}\eta_{j+1}(\lambda)|z|^2\right) i\eta_{j+1}(\lambda) q e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z|^2} \\ &+ (\bar{z})^k L_{j+1}^k \left(\frac{1}{2}\eta_{j+1}(\lambda)|z|^2\right) \left(-\frac{i}{2}\eta_{j+1}(\lambda)q\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2} \\ &+ k(\bar{z})^{k-1} L_{j+1}^k \left(\frac{1}{2}\eta_{j+1}(\lambda)|z|^2\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z|^2}. \end{aligned} \tag{3.74}$$

So, by (3.72)-(3.74),

$$\left(Z_j^\lambda f_j\right)(z_j) = (\bar{z_j})^{k+1} \left(\partial L_{j+1}^k\right) \left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2}, \quad z_j \in \mathbb{C}. \tag{3.75}$$

It is easy to see that (3.75) is also true for $k = 0$. Thus, by (3.71)-(3.75),

$$\begin{aligned} e_{j+k+1,j}(z_j) =&(2\pi)^{-\frac{1}{2}} \eta_{j+1}(\lambda)^{\frac{1}{2}} (-i)(2j+2)^{-\frac{1}{2}} \left\{\frac{(j+1)!}{(j+k+1)!}\right\}^{\frac{1}{2}} \\ &\times \left(\frac{i}{\sqrt{2}}\right)^k (\bar{z_j})^{k+1} \left(\partial L_{j+1}^k\right) \left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2} \end{aligned}$$

for all z_j in $\mathbb{C}$. It follows from Lemma 3.61 that

$$\begin{aligned} &e_{j+k+1,j}(z_j) \\ &= (2\pi)^{-\frac{1}{2}} \eta_{j+1}(\lambda)^{\frac{1}{2}} i(2j+2)^{-\frac{1}{2}} \left\{\frac{(j+1)!}{(j+k+1)!}\right\}^{\frac{1}{2}} \left(\frac{i}{\sqrt{2}}\right)^k (\bar{z_j})^{k+1} L_j^{k+1} \left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2} \\ &= (2\pi)^{-\frac{1}{2}} \eta_{j+1}(\lambda)^{\frac{1}{2}} \left\{\frac{j!}{(j+k+1)!}\right\}^{\frac{1}{2}} \left(\frac{i}{\sqrt{2}}\right)^{k+1} (\bar{z_j})^{k+1} L_j^{k+1} \left(\frac{1}{2}\eta_{j+1}(\lambda)|z_j|^2\right) e^{-\frac{1}{4}\eta_{j+1}(\lambda)|z_j|^2}, \quad z_j \in \mathbb{C}, \end{aligned}$$

and the proof is complete. □

Defining P_k to be the projection onto the kth eigenspace, we conclude thie paper with the following result which connects the Weyl transform, the Hermite projection operator P_k and the Laguerre function φ_k which is defined by

$$\varphi_k(z) = L_k^{d-1}(\frac{1}{2}|\eta(\lambda)||z|^2) e^{-\frac{1}{4}|\eta(\lambda)||z|^2},$$

where L_k^{d-1} beging a Laguerre polynomial to type $(d-1)$.

Theorem 3.63.

$$W(\varphi_k) = \mathrm{Pf}(\lambda)^{-\frac{1}{2}}(2\pi)^{\frac{d}{2}} P_k.$$

Proof. For $f, g \in L^2(\mathbb{R})$, it follows from the properties of the (λ, ν)-Fourier-Wigner transform that

$$\begin{aligned}\big(W(\Phi^\lambda_{\alpha,\alpha})f, g\big) &= \int_{\mathbb{R}^{2d}} \Phi^\lambda_{\alpha,\alpha}(z)(\pi^{\lambda,\nu}(z)f, g)dz \\ &= \mathrm{Pf}(\lambda)^{-\frac{1}{2}}(2\pi)^{\frac{d}{2}} \big(V_{\lambda,\nu}(f,g), V_{\lambda,\nu}(\Phi^\lambda_\alpha, \Phi^\lambda_\alpha)\big) \\ &= \mathrm{Pf}(\lambda)^{-\frac{1}{2}}(2\pi)^{\frac{d}{2}}(f, \Phi^\lambda_\alpha)(\Phi^\lambda_\alpha, g).\end{aligned}$$

Then we have

$$W(\Phi^\lambda_{\alpha,\alpha})f = \mathrm{Pf}(\lambda)^{-\frac{1}{2}}(2\pi)^{\frac{d}{2}}(f, \Phi^\lambda_\alpha)\Phi^\lambda_\alpha.$$

As $P_k f = \sum_{|\alpha|=k} \left(f, \Phi^\lambda_\alpha\right) \Phi^\lambda_\alpha$ it is enough to show that

$$\varphi_k(z) = \mathrm{Pf}(\lambda)^{-\frac{1}{2}}(2\pi)^{\frac{d}{2}} \sum_{|\alpha|=k} \Phi^\lambda_{\alpha,\alpha}(z). \tag{3.76}$$

The Laguerre functions φ_k satisfy the generating function

$$\sum_{k=0}^\infty \varphi_k(z) r^k = (1-r)^{-d} e^{-\frac{1}{2}\frac{1+r}{1-r}|\eta(\lambda)||z|^2}. \tag{3.77}$$

On the other hand, Theorem 3.60 gives

$$\Phi^\lambda_{\alpha,\alpha}(z) = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}} \prod_{j=1}^d L^0_{\alpha_j}\left(\frac{1}{2}\eta_j(\lambda)\left|z_j\right|^2\right) e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2} \tag{3.78}$$

and each $L^0_{\alpha_j}\left(\frac{1}{2}\eta_j(\lambda)\left|z_j\right|^2\right) e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2}$ satisfy the relation

$$\sum_{k=0}^\infty L^0_k\left(\frac{1}{2}\eta_j(\lambda)\left|z_j\right|^2\right) e^{-\frac{1}{4}\eta_j(\lambda)|z_j|^2} r^k = (1-r)^{-1} e^{-\frac{1}{2}\frac{1+r}{1-r}\eta_j(\lambda)|z_j|^2}. \tag{3.79}$$

From (3.78) and (3.79) it is clear that

$$\sum_{k=0}^\infty \left(\sum_{|\alpha|=k} \Phi^\lambda_{\alpha,\alpha}(z)\right) r^k = \mathrm{Pf}(\lambda)^{\frac{1}{2}}(2\pi)^{-\frac{d}{2}}(1-r)^{-n} e^{-\frac{1}{2}\frac{1+r}{1-r}|\eta(\lambda)||z|^2}.$$

Comparing this with (3.77) we obtain (3.76). $\square$

4 Applications

4.1 Weyl-Hörmander calculus

4.1.1 Weyl-Hörmander calculus on $\mathbb{R}^n$

We first recall some elements of the Weyl-Hörmander pseudo-differential calculus and the associated Sobolev spaces that will be relevant for our analysis. For more details on the underlying general theory, we can refer, for instance, to [Ler10].

We consider $\mathbb{R}^n$ and identify its cotangent bundle $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$. The canonical symplectic form on $\mathbb{R}^{2n}$ is ω defined by

$$\omega(T, T') = x \cdot \xi' - x' \cdot \xi, \quad T = (\xi, x), T' = (\xi', x') \in \mathbb{R}^{2n}.$$

Definition 4.1. If q is a positive quadratic form on $\mathbb{R}^{2n}$, then we define its conjugate q^ω by

$$\forall\, T \in \mathbb{R}^{2n} \quad q^\omega(T) := \sup_{T' \in \mathbb{R}^{2n} \setminus \{0\}} \frac{|\omega(T, T')|^2}{q(T')},$$

and its gain factor by

$$\Lambda_q := \inf_{T \in \mathbb{R}^{2n} \setminus \{0\}} \frac{q^\omega(T)}{q(T)}.$$

Definition 4.2. We shall say that the metric g is of Hörmander type if it is a family of positive quadratic forms

$$g = \left\{ g_X, X \in \mathbb{R}^{2n} \right\}$$

depending smoothly on $X \in \mathbb{R}^{2n}$ and satisfies:

- The metric g is uncertain, i.e. $\forall X \in \mathbb{R}^{2n}, \Lambda_{g_X} \geq 1$.
- The metric g is slowly varying, i.e. there exists a constant $\bar{C} > 0$ such that we have for any $X, X' \in \mathbb{R}^{2n}$

$$g_X(X - X') \leq \bar{C}^{-1} \Longrightarrow \sup_{T \in \mathbb{R}^{2n} \setminus \{0\}} \left(\frac{g_X(T)}{g_{X'}(T)} + \frac{g_{X'}(T)}{g_X(T)} \right) \leq \bar{C}.$$

- The metric g is temperate, i.e. there are constants $\bar{C} > 0$ and $\bar{N} > 0$ such that we have for any $X, X' \in \mathbb{R}^{2n}$ and $T \in \mathbb{R}^{2n}\backslash\{0\}$:
$$\frac{g_X(T)}{g_{X'}(T)} \leq \bar{C}\left(1 + g^{\omega}_X\left(X - X'\right)\right)^{\bar{N}}.$$
In the following any constant depending only on $\bar{C} > 0$ and $\bar{N} > 0$ will be called a structural constant.

We also define a weight as a positive function on $\mathbb{R}^{2n}$ satisfying the same type of conditions as a Hörmander metric.

Definition 4.3. Let g be a metric of Hörmander type. A positive function M defined on $\mathbb{R}^{2n}$ is a g-weight when there are structural constants $\bar{C}'$ and $\bar{N}'$ satisfying for any $X, Y \in \mathbb{R}^{2n}$:
$$g_X(X - Y) \leq \bar{C}'^{-1} \Longrightarrow \frac{M(X)}{M(Y)} + \frac{M(Y)}{M(X)} \leq \bar{C}'$$
and
$$\frac{M(X)}{M(Y)} \leq \bar{C}\left(1 + g^{\omega}_X(X - Y)\right)^{\bar{N}'}.$$

Definition 4.4 (Hörmander symbol class $S(M, g)$)**.** Let g be a metric of Hörmander type and M a g-weight on $\mathbb{R}^{2n}$. The symbol class $S(M, g)$ is the set of functions $a \in C^{\infty}\left(\mathbb{R}^{2n}\right)$ such that for each integer $\ell \in \mathbb{N}_0$, the quantity
$$\|a\|_{S(M,g),\ell} := \sup_{\ell' \leq \ell, X \in \mathbb{R}^{2n}} \frac{\left|\partial_{T_1} \ldots \partial_{T_{\ell'}} a(X)\right|}{M(X)}$$
is finite. Here $\partial_T a$ denotes the quantity (da, T).

Now, if a is a symbol in $S(M, g)$, then its Weyl quantization is the operator which associates to $u \in \mathscr{S}\left(\mathbb{R}^n\right)$ the function $\mathrm{op}^W(a)f$ defined by
$$\forall x, y \in \mathbb{R}^n, \quad \left(\mathrm{op}^W(a)u(x)\right) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \mathrm{e}^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \tag{4.1}$$

Let us mention that the operator op $\mathrm{op}^W(a)$ has a kernel $K\left(x, y\right)$ defined by
$$K\left(x, y\right) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathrm{e}^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) d\xi,$$
which is linked to its symbol through
$$a(x, \eta) = \int_{\mathbb{R}^n} \mathrm{e}^{-iy\xi} K\left(x + \frac{y}{2}, x - \frac{y}{2}\right) dy.$$

Let us also point out that a concept of Sobolev space $H(M, g)$ was introduced by R. Beals in [Bea81]. We will use the following characterization of those spaces.

Definition 4.5 (Sobolev spaces $H(M,g)$)**.** Let g be a metric of Hörmander type and M a g-weight on $\mathbb{R}^{2n}$. We denote by $H(M,g)$ the set of all tempered distributions u on $\mathbb{R}^n$ such that for any symbol $a \in S(M,g)$ we have $\mathrm{op}^W(a)u \in L^2(\mathbb{R}^n)$.

The following properties are well known [Ler10, Chapters 1 and 2]:

Proposition 4.6. *Let g be a metric of Hörmander type on $\mathbb{R}^{2n}$.*

- *The space $H(1,g)$ coincides with $L^2(\mathbb{R}^n)$. Furthermore, there exist a structural constant $C>0$ and a structural integer $\ell \in \mathbb{N}_0$ such that for any symbol $a \in S(1,g)$, we have*

$$\left\|\mathrm{op}^W(a)\right\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leq C\|a\|_{S(1,g),\ell}.$$

- *Let M_1, M_2 be g-weights. For any $a \in S(M_1,g)$, the operator $\mathrm{op}^W(a)$ maps continuously $H(M_2,g)$ to $H\left(M_2M_1^{-1},g\right)$. Furthermore, there exist a constant $C>0$ and an integer $\ell \in \mathbb{N}_0$ such that*

$$\left\|\mathrm{op}^W(a)\right\|_{\mathscr{L}\left(H(M_2,g),H\left(M_2M_1^{-1},g\right)\right)} \leq C\|a\|_{S(M_1,g),\ell}$$

  The constant C and the integers ℓ may be chosen to depend only on the structural constants of g, M_1, M_2 and to be independent of g, M and a.

Proposition 4.7. *Let g be a metric of Hörmander type and let M, M_1, M_2 be g weights. Then:*

- *The symbol class $S(M,g)$ is a vector space endowed with a Fréchet topology via the family of seminorms $\|\cdot\|_{S(M,g),\ell}$, $\ell \in \mathbb{N}_0$.*

- *If $a \in S(M,g)$ then the symbol b defined by*

$$\mathrm{op}^W b = \left(\mathrm{op}^W a\right)^*$$

  is in $S(M,g)$ as well. Furthermore, for any $\ell \in \mathbb{N}_0$ there exist a constant $C>0$ and a integer $\ell' \in \mathbb{N}_0$ such that

$$\|b\|_{S(M,g),\ell} \leq C\|a\|_{S(M,g),\ell'}$$

  The constant C and the integer ℓ' may be chosen to depend on ℓ and on the structural constants and to be independent of g, M and a.

- *If $a_1 \in S(M_1,g)$ and $a_2 \in S(M_2,g)$ then the symbol b defined by*

$$\mathrm{op}^W b = \left(\mathrm{op}^W a_1\right)\left(\mathrm{op}^W a_2\right)$$

is in $S(M_1M_2, g)$*. Furthermore, for any* $\ell \in \mathbb{N}_0$ *there exist a constant* $C > 0$ *and two integers* $\ell_1, \ell_2 \in \mathbb{N}_0$ *such that*

$$\|b\|_{S(M_1M_2,g),\ell} \leq C \|a_1\|_{S(M_1,g),\ell_1} \|a_2\|_{S(M_2,g),\ell_2}$$

The constant C *and the integers* ℓ_1, ℓ_2 *may be chosen to depend on* ℓ *and on the structural constants and to be independent of* g, M_1, M_2 *and* a_1, a_2*.*

As pointed out in Chapter 3, it is natural to base the quantization of symbols on $\mathbb{R}^n$ on the calculus related to the harmonic oscillator. In that case the metric

$$\frac{d\xi^2 + d\theta^2}{(1 + |\theta|^2 + |\xi|^2)^\rho}$$

is of Hörmander type with corresponding weights $(1 + |\theta|^2 + |\xi|^2)^{\frac{\delta}{2}}$ for $\delta \in \mathbb{R}$. For $\delta \in \mathbb{R}$ and $\rho \in (0, 1]$, we denote by $\Sigma^\delta_\rho(\mathbb{R}^n)$ the corresponding symbol class, often called the Shubin classes of symbols on $\mathbb{R}^n$:

$$\Sigma^\delta_\rho(\mathbb{R}^n) := S\left(\left(1 + |\theta|^2 + |\xi|^2\right)^{\frac{\delta}{2}}, \frac{d\xi^2 + d\theta^2}{(1 + |\theta|^2 + |\xi|^2)^\rho}\right).$$

This means that a symbol $a \in C^\infty(\mathbb{R}^{2n})$ is in $\Sigma^\delta_\rho(\mathbb{R}^n)$ if and only if for any $\alpha, \beta \in \mathbb{N}^n_0$ there exists a constant $C = C_{\alpha,\beta} > 0$ such that

$$\forall (\xi, \theta) \in \mathbb{R}^{2n} \quad \left|\partial^\alpha_\xi \partial^\beta_\theta a(\xi, \theta)\right| \leq C \left(1 + |\xi|^2 + |\theta|^2\right)^{\frac{\delta - \rho(|\alpha| + |\beta|)}{2}}.$$

The class $\Sigma^\delta_\rho(\mathbb{R}^n)$ is a vector subspace of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which becomes a Fréchet space when endowed with the family of seminorms:

$$\|a\|_{\Sigma^\delta_\rho,N} = \sup_{(\xi,\theta) \in \mathbb{R}^n \times \mathbb{R}^n} \left(1 + |\xi|^2 + |\theta|^2\right)^{-\frac{\delta - \rho(|\alpha| + |\beta|)}{2}} \left|\partial^\alpha_\xi \partial^\beta_\theta a(\xi, \theta)\right|,$$

where $N \in \mathbb{N}_0$. We denote by

$$\Psi\Sigma^\delta_\rho(\mathbb{R}^n) := \mathrm{Op}^W\left(\Sigma^\delta_\rho(\mathbb{R}^n)\right)$$

the corresponding class of operators and by $\|\cdot\|_{\Psi\Sigma^\delta_\rho,N}$ the corresponding seminorms. We have the inclusions

$$\rho_1 \geq \rho_2 \quad \text{and} \quad \delta_1 \leq \delta_2 \Longrightarrow \Psi\Sigma^{\delta_1}_{\rho_1}(\mathbb{R}^n) \subset \Psi\Sigma^{\delta_2}_{\rho_2}(\mathbb{R}^n).$$

The following is well known and can be viewed more generally as a consequence of the Weyl-Hörmander calculus.

Proposition 4.8. • *The class of operators* $\bigcup_{\delta \in \mathbb{R}} \Psi\Sigma^\delta_\rho(\mathbb{R}^n)$ *forms an algebra of operators sta-*

ble by taking the adjoint. Furthermore, the operations

$$\Psi\Sigma_\rho^\delta(\mathbb{R}^n) \longrightarrow \Psi\Sigma_\rho^\delta(\mathbb{R}^n)$$
$$A \longmapsto A^*$$

and

$$\Psi\Sigma_\rho^{\delta_1}(\mathbb{R}^n) \times \Psi\Sigma_\rho^{\delta_2}(\mathbb{R}^n) \longrightarrow \Psi\Sigma_\rho^{\delta_1+\delta_2}(\mathbb{R}^n)$$
$$(A, B) \longmapsto AB$$

are continuous.

- *The operators in* $\Psi\Sigma_\rho^0(\mathbb{R}^n)$ *extend boundedly to* $L^2(\mathbb{R}^n)$. *Furthermore, there exist* $C > 0$ *and* $N \in \mathbb{N}$ *such that if* $A \in \Psi\Sigma_\rho^0(\mathbb{R}^n)$ *then*

$$\|A\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leq C\|A\|_{\Psi\Sigma_\rho^\delta, N}.$$

Example 4.9. The operators $\partial_{u_j} = \mathrm{Op}^W(i\xi_j), j = 1, \ldots, n$, or multiplication by $u_k = \mathrm{Op}^W(u_k), k = 1, \ldots, n$, are two operators in $\Psi\Sigma_1^1(\mathbb{R}^n)$.

Example 4.10. For each $\delta \in \mathbb{R}$, the symbol b^δ, where

$$b(\xi, \theta) = \sqrt{1 + |\theta|^2 + |\xi|^2}$$

is in $\Sigma_1^\delta(\mathbb{R}^n)$.

From Example 4.9, it follows that the (positive) harmonic oscillator

$$A := \sum_{j=1}^n \left(-\partial_{u_j}^2 + u_j^2\right)$$

is in $\Psi\Sigma_1^2(\mathbb{R}^n)$.

4.1.2 The (λ, ν)-Shubin classes $\sum_{\rho,\lambda,\nu}^\delta(\mathbb{G})$

Now, we will prove our main results. It follows from Chapter 3 that we want to consider the symbol associated with rescaled harmonic oscillator:

$$-\Delta_{osc,\eta(\lambda)} + |\nu|^2 = -\Delta_\xi + |\eta(\lambda) \cdot x|^2 + |\nu|^2,$$

where $\eta(\lambda) > 0$ is smooth and homogeneous of degree 1 in λ.

Then the Shubin metric depending on parameters $\lambda \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^k$ is the metric $g^{(\rho,\lambda,\nu)}$ on $\mathbb{R}^{2d+k}$ defined via

$$g_{\xi,\theta,\nu}^{(\rho,\lambda,\nu)}(d\xi, d\theta) := \left(\frac{1}{1 + |\eta(\lambda) \cdot \xi|^2 + |\theta|^2 + |\nu|^2}\right)^\rho \left(|\eta(\lambda) \cdot d\xi|^2 + |d\theta|^2\right).$$

The associated positive function $M^{(\lambda,\nu)}$ on $\mathbb{R}^{2d+k}$ is defined via

$$M^{(\lambda,\nu)}(\xi,\theta,\nu) := \left(1+|\eta(\lambda)\cdot\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{1}{2}}.$$

For the Heisenberg group with $\eta(\lambda)=\lambda$ and $\nu=0$, these λ-families of metrics and weights were first introduced in [BFKG12] in the case $\rho=1$. Similar to Proposition 1.20 in [BFKG12], we have the first result as follows:

Lemma 4.11. *For each $\lambda\in\mathbb{R}^m$ and $\nu\in\mathbb{R}^k$, the metric $g^{(\rho,\lambda,\nu)}$ is of Hörmander type and the function $M^{(\lambda,\nu)}$ is a $g^{(\rho,\lambda,\nu)}$-weight. Furthermore, if $\rho\in(0,1]$ is fixed, then the structural constants for $g^{(\rho,\lambda,\nu)}$ and for $M^{(\lambda,\nu)}$ can be chosen independent of λ and ν.*

Now, motivated by the examples on the Heisenberg group studied in [BFKG12], we shall give a definition of symbols, and pseudo-differential operators, on two step nilpotent Lie groups. Therefore, in what follows, we shall define a positive, noninteger real number ϱ, which will measure the regularity assumed on the symbols. This number ϱ is fixed from now on and we emphasize that the definitions below depend on ϱ. We have chosen not to keep memory of this number on the notations for the sake of simplicity.

Definition 4.12. Let $\rho\in(0,1]$ be a fixed parameter. For each parameter $\lambda\in\mathbb{R}^m$ and $\nu\in\mathbb{R}^k$, we define the (λ,ν)-Shubin classes by

$$\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}) := S\left(\left(M^{(\lambda,\nu)}\right)^{\delta}, g^{(\rho,\lambda,\nu)}\right),$$

where we have used the Hörmander notation to define a class of symbols in terms of a metric and a weight. Here this means that $\sum^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$ is the class of functions $a\in C^\infty\left(\mathbb{G}\times\mathbb{R}^{2d+k+m}\right)$ such that for each $N\in\mathbb{N}_0$, the quantity

$$\|a\|_{\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}),N} := \sup_{\substack{|\alpha|+|\beta|+|\gamma|+l\le N\\ (\xi,\theta,\nu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^k}} |\eta(\lambda)|^{-\rho\frac{|\alpha|+|\beta|+|\gamma|}{2}}\left(1+|\eta(\lambda)|\left(1+|\xi|^2+|\theta|^2+|\nu|^2\right)\right)^{-\frac{\delta-\rho(|\alpha|+|\beta|+|\gamma|)}{2}}$$
$$\times\left\|(\lambda\partial_\lambda)^l\partial_\xi^\alpha\partial_\theta^\beta\partial_\nu^\gamma a(x,y,r,s,\xi,\theta,\nu,\lambda)\right\|_{C^\varrho(\mathbb{G})}$$

is finite. Besides, one additionally requires that the function

$$(w,\xi,\theta,\nu,\lambda)\mapsto\sigma(a)(w,\xi,\theta,\nu,\lambda) \overset{\text{def}}{=} a\left(w,\frac{\xi_1}{\eta_1(\lambda)}\dots\frac{\xi_d}{\eta_d(\lambda)},\theta,\nu,\lambda\right) \tag{4.2}$$

is uniformly smooth close to $\lambda=0$ in the sense that there exists $C>0$ such that

$$\left\|\partial_\lambda^l\partial_\xi^\alpha\partial_\theta^\beta\partial_\nu^\gamma(\sigma(a))\right\|_{C^\varrho(\mathbb{G})}\le C_{N,l}\left(1+|\eta(\lambda)|+|\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{\delta-\rho(|\alpha|+|\beta|+\gamma)}{2}},$$

where $\forall(w,\xi,\theta,\nu)\in\mathbb{H}^d\times\mathbb{G}\times\mathbb{R}^{2d+k},\forall\lambda\in[-1,1]$. In that case we shall write $a\in\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$.

Remark 4.13. The additional assumption (4.2) on $\sigma(a)$ is necessary in order to guarantee that pseudo-differential operators associated with those symbols are continuous on $\mathscr{S}(\mathbb{G})$ (see [BFKG12, Proposition 2.6]). It is also required to obtain that the space of pseudo-differential operators is an algebra.

The class of symbols $\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$ is a vector subspace of $C^{\infty}\left(\mathbb{G}\times\mathbb{R}^{2d+k+m}\right)$ becomes a Fréchet space when endowed with the family of seminorms $\|\cdot\|_{\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}),N}$, $N\in\mathbb{N}_0$. We denote by

$$\Psi\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}) := \mathrm{Op}^{W}\left(\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})\right)$$

the corresponding class of operators, and by $\|\cdot\|_{\Psi\Sigma^{\delta}_{\rho,\lambda,\nu},N}$ the corresponding norms on the Fréchet space $\Psi\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$.

By the definition of symbol class, Lemma 4.11 has important consequences which are stated below.

Corollary 4.14. *Let a be a symbol in $\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$. Then for any $w\in\mathbb{G}$, $\lambda\in\mathbb{R}^m$ and $\nu\in\mathbb{R}^k$, the operator $\mathrm{Op}^w(a(w,\lambda,\nu))$ is continuous from $H\left(M, g^{(\rho,\lambda,\nu)}\right)$ into $H\left(M\left(M^{(\lambda)}\right)^{-\delta}, g^{(\rho,\lambda,\nu)}\right)$ for any $g^{(\rho,\lambda,\nu)}$ weight M, and the constant of continuity is uniform with respect to λ , ν and w. In particular for $\delta=0$, the operator $\mathrm{Op}^w(a(w,\lambda,\nu))$ maps $L^2(\mathbb{G})$ into itself uniformly with respect to w , λ and ν.*

Let us now mention some properties of the function $\sigma(a)$ defined in (4.2).

Proposition 4.15. *A function a belongs to $\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$ if and only if $\sigma(a)\in\mathcal{C}^{\infty}\left(\mathbb{G}\times\mathbb{R}^{2d+k+m}\right)$ satisfies: for all $l,N\in\mathbb{N}$, there exists a constant $C_{N,l}>0$ such that for any $\alpha,\beta,\gamma\in\mathbb{N}^d$ satisfying $|\alpha|+|\beta|+|\gamma|\le N$, and for all $(w,\xi,\theta,\nu,\lambda)\in\mathbb{G}\times\mathbb{R}^{2d+k+m}$,*

$$\left\|\partial^l_{\lambda}\partial^{\alpha}_{\xi}\partial^{\beta}_{\theta}\partial^{\gamma}_{\nu}(\sigma(a))\right\|_{\mathcal{C}^{\varrho}(\mathbb{G})} \le C_{N,l}\left(1+|\eta(\lambda)|+|\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{\delta-\rho(|\alpha|+|\beta|+\gamma)}{2}}(1+|\lambda|)^{-l}. \tag{4.3}$$

Proof. For any multi-index β satisfying $|\beta|\le N$, we have

$$\begin{aligned}\left|\partial^{\alpha}_{\xi}\partial^{\beta}_{\theta}\partial^{\gamma}_{\nu}(\sigma(a)(w,\xi,\theta,\nu,\lambda))\right| &= ||\eta(\lambda)|^{-\rho\frac{|\alpha|+|\beta|+|\gamma|}{2}}\left(\partial^{\alpha}_{\xi}\partial^{\beta}_{\theta}\partial^{\gamma}_{\nu}a\right)\left(w,\frac{\xi_1}{\eta_1(\lambda)}\cdots\frac{\xi_d}{\eta_d(\lambda)},\lambda\right)| \\ &\le \|a\|_{\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}),N}\left(1+|\eta(\lambda)|+|\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{\delta-\rho(|\alpha|+|\beta|+|\gamma|)}{2}}.\end{aligned} \tag{4.4}$$

Besides, there exists a constant $C>0$ such that for $\lambda\in\mathbb{R}$

$$\begin{aligned}&\left|(\lambda\partial_{\lambda})^l\left(\sigma(a)(w,\xi,\theta,\nu,\lambda)\right)\right| \le C\left|\left((\lambda\partial_{\lambda})^l a\right)\left(w,\frac{\xi_1}{\eta_1(\lambda)}\cdots\frac{\xi_d}{\eta_d(\lambda)},\lambda\right)\right| \\ &+C\sum_{|\alpha|+|\beta|+|\gamma|=l}|\eta(\lambda)|^{-\frac{l}{2}}\left(|\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{l}{2}}\left|\left(\partial^{\alpha}_{\xi}\partial^{\beta}_{\theta}\partial^{\gamma}_{\nu}a\right)\left(w,\frac{\xi_1}{\eta_1(\lambda)}\cdots\frac{\xi_d}{\eta_d(\lambda)},\lambda\right)\right| \\ &\le C\|a\|_{\Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G}),N}\left(1+|\eta(\lambda)|+|\xi|^2+|\theta|^2+|\nu|^2\right)^{\frac{\delta}{2}}.\end{aligned}$$

The converse inequalities come easily: one has $a \in \Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$ if and only if for all $l, N \in \mathbb{N}$, there exists a constant $C_{N,l}$ such that for any $\alpha, \beta, \gamma \in \mathbb{N}^d$ satisfying $|\alpha| + |\beta| + |\gamma| \leq N$ and for all $(w, \xi, \theta, \nu, \lambda)$ belonging to $\mathbb{G} \times \mathbb{R}^{2d+k+m}$,

$$\left\| (\lambda\partial_\lambda)^l \, \partial_\xi^\alpha \partial_\theta^\beta \partial_\nu^\gamma (\sigma(a)) \right\|_{\mathcal{C}^\varrho(\mathbb{G})} \leq C_{N,l} \left(1 + |\eta(\lambda)| + |\xi|^2 + |\theta|^2 + |\nu|^2\right)^{\frac{\delta - \rho(|\alpha|+|\beta|+|\gamma|)}{2}}. \tag{4.5}$$

We then remark that if $|\eta(\lambda)| \leq 1$, the smoothness of $\sigma(a)$ yields that (4.4) implies on the compact $\{|\eta(\lambda)| \leq 1\}$

$$(1 + |\lambda|)^l \left\| \partial_\lambda^l \partial_\xi^\alpha \partial_\theta^\beta \partial_\nu^\gamma (\sigma(a)) \right\|_{\mathcal{C}^\varrho(\mathbb{G})} \leq C_{N,l} \left(1 + |\eta(\lambda)| + |\xi|^2 + |\theta|^2 + |\nu|^2\right)^{\frac{\delta - \rho(|\alpha|+|\beta|+|\gamma|)}{2}}.$$

Besides, for $|\eta(\lambda)| \geq 1$, (4.5) gives

$$\left\| \partial_\lambda^l \partial_\xi^\alpha \partial_\theta^\beta \partial_\nu^\gamma (\sigma(a)) \right\|_{\mathcal{C}^\varrho(\mathbb{G})} \leq C_{N,l} \left(1 + |\eta(\lambda)| + |\xi|^2 + |\theta|^2 + |\nu|^2\right)^{\frac{\delta - \rho(|\alpha|+|\beta|+|\gamma|)}{2}} (1 + |\lambda|)^{-l}.$$

This ends the proof of the proposition. □

Theorem 4.16. *To a symbol $a \in \Sigma^{\delta}_{\rho,\lambda,\nu}(\mathbb{G})$ on $\mathbb{R}^{2d}$ depending on the parameters (w, λ, ν) in $\mathbb{G} \times \mathbb{R}^p \times \mathbb{R}^k$ and belonging to (λ, ν)-dependent Hörmander class. Then the pseudo-differential operator on $\mathbb{G}$ defined in the following way: for any $f \in \mathscr{S}(\mathbb{G})$,*

$$\mathrm{Op}(a)f(w) = \kappa \iint_{\Lambda \times \mathbb{R}^k} \mathrm{tr}\left(u^{\lambda,\nu}_{w^{-1}} \mathcal{F}(f)(\lambda,\nu) \, \mathrm{op}^W(a(w,\xi,\theta,\nu,\lambda)) \right) \mathrm{Pf}(\lambda) d\lambda d\nu, \ \forall \ w \in \mathbb{G}$$

is well-defined.

4.1.3 (λ, ν)-Shubin Sobolev spaces

In this subsection, we study (λ, ν)-Shubin Sobolev spaces for the rescaled harmonic oscillator:

$$\mathcal{H}(\lambda, \nu) = -\Delta_x + |\eta(\lambda) \cdot x|^2 + |\nu|^2,$$

which is the diagonal operator defined on $L^2(\mathbb{R}^d)$ by

$$\mathcal{H}(\lambda, \nu)\Phi^\lambda_\alpha = \left(\sum_{j=1}^d \eta_j(\lambda)(2\alpha_j + 1) + \nu^2 \right) \Phi^\lambda_\alpha.$$

The rescaled harmonic oscillator $\mathcal{H}(\lambda, \nu)$ is a positive (unbounded) operator on $L^2\left(\mathbb{R}^d\right)$. Its spectrum is

$$\left\{ \left(\sum_{j=1}^d \eta_j(\lambda)(2\alpha_j + 1) + \nu^2 \right), n_j \in \mathbb{N}_0 \right\}.$$

The eigenfunctions associated with the eigenvalues $\left(\sum_{j=1}^d \eta_j(\lambda)(2n_j + 1) + \nu^2 \right)$ are the Hermite functions Φ^λ_α. Therefore, the functions Φ^λ_α form an orthonormal basis of $L^2\left(\mathbb{R}^d\right)$. For each

$s \in \mathbb{R}$, we define the operator $(\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}}$ using the functional calculus, that is, in this case, the domain of $(\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}}$ is the space of functions

$$\mathrm{Dom}(\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}} = \left\{ h \in L^2\left(\mathbb{R}^d\right) : \sum_{n \in \mathbb{N}_0^d} \left(\sum_{j=1}^{d} \eta_j(\lambda)(2\alpha_j + 1) + \nu^2 \right)^s \left| \left(\Phi_\alpha^\lambda, h\right)_{L^2\left(\mathbb{R}^d\right)} \right|^2 < \infty \right\}$$

and if $h \in \mathrm{Dom}(\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}}$ then

$$(\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}} h = \sum_{n \in \mathbb{N}_0^d} \left(\sum_{j=1}^{d} \eta_j(\lambda)(2\alpha_j + 1) + \nu^2 \right)^{\frac{s}{2}} \left(\Phi_\alpha^\lambda, h\right)_{L^2\left(\mathbb{R}^d\right)} \Phi_\alpha^\lambda.$$

Many of their properties, especially their equivalent characterisations, are well known for $\eta(\lambda) = 1$ and $\nu = 0$. Our starting point will be the following definition for the (λ, ν)-Shubin Sobolev spaces:

Definition 4.17. Let $s \in \mathbb{R}$. The $(\lambda, \nu)-$Shubin Sobolev space $\mathcal{Q}_s^{\lambda,\nu}(\mathbb{G})$ is the subspace of $\mathscr{S}'(\mathbb{G})$ which is the completion of $\mathrm{Dom}(\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}}$ for the norm

$$\|h\|_{\mathcal{Q}_s^{\lambda,\nu}} := \left\|(\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}} h\right\|_{L^2\left(\mathbb{R}^d\right)}.$$

Theorem 4.18. *We have the following properties:*

(1) *The space $\mathcal{Q}_s^{\lambda,\nu}(\mathbb{G})$ is a Hilbert space endowed with the sesquilinear form*

$$(g, h)_{\mathcal{Q}_s^{\lambda,\nu}} = \left((\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}} g, (\mathrm{I} + \mathcal{H}(\lambda, \nu))^{\frac{s}{2}} h\right)_{L^2(\mathbb{G})}.$$

We also have

$$L^2(\mathbb{G}) = \mathcal{Q}_0^{\lambda,\nu}(\mathbb{G}),$$

and the inclusions

$$\mathscr{S}(\mathbb{G}) \subset \mathcal{Q}_{s_1}^{\lambda,\nu}(\mathbb{G}) \subset \mathcal{Q}_{s_2}^{\lambda,\nu}(\mathbb{G}) \subset \mathscr{S}'(\mathbb{G}), \quad s_1 > s_2.$$

(2) *The dual of $\mathcal{Q}_s^{\lambda,\nu}(\mathbb{G})$ may be identified with $\mathcal{Q}_{-s}^{\lambda,\nu}(\mathbb{G})$ via the distributional duality form $\langle g, h\rangle = \int_{\mathbb{G}} ghdx$.*

(3) *The complex interpolation between the spaces $\mathcal{Q}_{s_0}^{\lambda,\nu}(\mathbb{G})$ and $\mathcal{Q}_{s_1}^{\lambda,\nu}(\mathbb{G})$ is*

$$\left(\mathcal{Q}_{s_0}^{\lambda,\nu}(\mathbb{G}), \mathcal{Q}_{s_1}^{\lambda,\nu}(\mathbb{G})\right)_\theta = \mathcal{Q}_{s_\theta}^{\lambda,\nu}(\mathbb{G}), \quad s_\theta = (1-\theta)s_0 + \theta s_1, \theta \in (0,1).$$

(4) *For any $s \in \mathbb{R}$, $\mathcal{Q}_s^{\lambda,\nu}(\mathbb{G})$ coincides with the completion (in $\mathscr{S}'(\mathbb{G})$) of the Schwartz space $\mathscr{S}(\mathbb{G})$ for the norm*

$$\|h\|_{\mathcal{Q}_s^{\lambda,\nu}}^{(b)} = \left\|\mathrm{Op}^W\left(b_\lambda^s\right) h\right\|_{L^2(\mathbb{G})}$$

where $b^s_\lambda(\xi,\theta,\nu) = \sqrt{1+|\eta(\lambda)\cdot\xi|^2+|\theta|^2+|\nu|^2}$ *is* (λ,ν)*-uniform in* $\Psi\Sigma^s_{1,\lambda,\nu}(\mathbb{G})$*. The norm* $\|\cdot\|^{(b)}_{\mathcal{Q}^{\lambda,\nu}_s}$ *extended to* $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ *is equivalent to* $\|\cdot\|_{\mathcal{Q}^{\lambda,\nu}_s}$*.*

(5) *For any* $s\in\mathbb{R}$*,* $\lambda\in\mathbb{R}^m$ *and* $\nu\in\mathbb{R}^k$*, the Shubin Sobolev space* $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ *coincides with the Sobolev space associated with* $g^{(1,\lambda,\nu)}$ *and* $(M^{(\lambda,\nu)})^s$*(see Definition* 4.5*)*

$$\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G}) = H\left((M^{(\lambda,\nu)})^s, g^{(\rho,\lambda,\nu)}\right).$$

(6) *For any* $s\in\mathbb{R}$*, the operators* $\mathrm{Op}^W(b^{-s})(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}$ *and* $(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}\mathrm{Op}^W(b^{-s})$ *are bounded and invertible on* $L^2(\mathbb{G})$*.*

Proof. From Definition 4.17, it is easy to prove that the space $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ is a Hilbert space, that it is included in $\mathscr{S}'(\mathbb{G})$ and that $\mathcal{Q}^{\lambda,\nu}_0(\mathbb{G}) = L^2(\mathbb{G})$. The proofs for the dual of $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ is $\mathcal{Q}^{\lambda,\nu}_{-s}(\mathbb{G})$ via the distributional duality and that the spaces $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ decrease with $s\in\mathbb{R}$ are standard, we omit them here (Part 1 and 2).

Let us prove the complex interpolation property of Part (3). We may assume $s_1>s_0$. For $h\in\mathcal{Q}^{\lambda,\nu}_{s_\theta}(\mathbb{G})$, we consider the function

$$f(z) := (\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{-(zs_1+(1-z)s_0)+s_\theta}{2}}h,$$

and we check easily that

$$f(\theta)=h, \quad \|f(iy)\|_{\mathcal{Q}^{\lambda,\nu}_{s_0}} = \|f(1+iy)\|^{\lambda,\nu}_{\mathcal{Q}_{s_1}} = \|h\|_{\mathcal{Q}_{s_\theta}} \quad \forall y\in\mathbb{G}.$$

This shows that $\mathcal{Q}^{\lambda,\nu}_{s_\theta}(\mathbb{G})$ is continuously included in $\left(\mathcal{Q}^{\lambda,\nu}_{s_0}(\mathbb{G}),\mathcal{Q}^{\lambda,\nu}_{s_1}(\mathbb{G})\right)_\theta$. By duality of the complex interpolation and of the $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$, we obtain the reverse inclusion and Part (3) is proved.

Let us prove Part (4). For any $s\in\mathbb{R}$, the operator $\mathrm{Op}^W(b^s)$ maps $\mathscr{S}(\mathbb{G})$ to itself and the mapping $\|\cdot\|^{(b)}_{\mathcal{Q}^{\lambda,\nu}_s}$ as defined in Part (4) is a norm on $\mathscr{S}(\mathbb{G})$. We denote its completion in $\mathscr{S}'(\mathbb{G})$ by $\mathcal{Q}^{(b)}_s(\mathbb{G})$. From the properties of the calculus it is standard that the dual of $\mathcal{Q}^{(b)}_s(\mathbb{G})$ is $\mathcal{Q}^{(b)}_{-s}(\mathbb{G})$ via the distributional duality and that the spaces $\mathcal{Q}^{(b)}_s(\mathbb{G})$ decrease with $s\in\mathbb{R}$.

We claim the following property about interpolation between the $\mathcal{Q}^{(b)}(\mathbb{G})$ spaces which is analogous to Part(3):

$$\left(\mathcal{Q}^{(b)}_{s_0}(\mathbb{R}^n),\mathcal{Q}^{(b)}_{s_1}(\mathbb{G})\right)_\theta = \mathcal{Q}^{(b)}_{s_\theta}(\mathbb{G}), \quad s_\theta=(1-\theta)s_0+\theta s_1, \theta\in(0,1) \tag{4.6}$$

Indeed we may assume $s_1>s_0$. For $h\in\mathcal{Q}^{(b)}_{s_\theta}(\mathbb{G})$, we consider the function

$$f(z)=e^{z(s_z-s_\theta)}\mathrm{Op}^W\left(b^{-s_z+s_\theta}\right)h \quad \text{where} \quad s_z=(1-z)s_0+zs_1.$$

Clearly $f(\theta) = h$. Furthermore,

$$\begin{aligned}\|f(iy)\|^{(b)}_{\mathcal{Q}_{s_1}} &= \left|e^{iy(s_{iy}-s_\theta)}\right| \left\|\mathrm{Op}^W\left(b^{s_1}\right)\mathrm{Op}^W\left(b^{-s_{iy}+s_\theta}\right)h\right\|_{L^2(\mathbb{G})} \\ &\le e^{-y^2(s_1-s_0)}\left\|\mathrm{Op}^W\left(b^{s_1}\right)\mathrm{Op}^W\left(b^{-s_{iy}+s_\theta}\right)\mathrm{Op}^W\left(b^{-s_\theta}\right)\right\|_{\mathscr{L}(L^2(\mathbb{G}))}\|h\|^{(b)}_{\mathcal{Q}_{s_\theta}}\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\|f(1+iy)\|^{(b)}_{\mathcal{Q}_{s_0}} &= \left|e^{(1+iy)(s_{1+iy}-s_\theta)}\right| \left\|\mathrm{Op}^W\left(b^{s_0}\right)\mathrm{Op}^W\left(b^{-s_{1+iy}+s_\theta}\right)h\right\|_{L^2(\mathbb{G})} \\ \le e^{s_1-s_\theta-y^2(s_1-s_0)} &\left\|\mathrm{Op}^W\left(b^{s_0}\right)\mathrm{Op}^W\left(b^{-s_{1+iy}+s_\theta}\right)\mathrm{Op}^W\left(b^{-s_\theta}\right)\right\|_{\mathscr{L}(L^2(\mathbb{G}))}\|h\|^{(b)}_{\mathcal{Q}_{s_\theta}}.\end{aligned} \tag{4.8}$$

From the calculus we obtain that the two operator norms on $L^2(\mathbb{G})$ in (4.7) and (4.8) are bounded by a constant of the form $C(1+|y|)^N$ where $C > 0$ and $N \in \mathbb{N}_0$ are independent of y. This shows that $\mathcal{Q}^{(b)}_{s_\theta}$ is continuously included in $\left(\mathcal{Q}^{(b)}_{s_0}(\mathbb{G}), \mathcal{Q}^{(b)}_{s_1}(\mathbb{G})\right)_\theta$. By duality of the complex interpolation and of the spaces $\mathcal{Q}_s(\mathbb{G})$, we obtain the reverse inclusion and (4.6) is proved.

Let us show that the spaces $\mathcal{Q}^{(b)}_s(\mathbb{G})$ and $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ coincide. First let us assume $s \in 2\mathbb{N}_0$. We have for any $h \in \mathcal{Q}^{(b)}_s(\mathbb{G})$

$$\|h\|_{\mathcal{Q}^{\lambda,\nu}_s} \le \left\|(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}\mathrm{Op}^W\left(b^{-s}\right)\right\|_{\mathcal{L}(L^2(\mathbb{G}))}\|h\|^{(b)}_{\mathcal{Q}_s}.$$

As $\mathcal{H}(\lambda,\nu) \in \Psi\Sigma^2_1(\mathbb{G})$, the operator $(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}\mathrm{Op}^W(b^{-s})$ is in $\Psi\Sigma^0_1$ and thus is bounded on $L^2(\mathbb{G})$. We have obtained a continuous inclusion of $\mathcal{Q}^{(b)}_s(\mathbb{G})$ into $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$. Conversely, we have for any $h \in \mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$ that

$$\|h\|^{(b)}_{\mathcal{Q}_s} \le \left\|\mathrm{Op}^W\left(b^s\right)(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{-\frac{s}{2}}\right\|_{\mathcal{L}(L^2(\mathbb{G}))}\|h\|_{\mathcal{Q}^{\lambda,\nu}_s}.$$

The inverse of $\mathrm{Op}^W(b^s)(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{-\frac{s}{2}}$ is $(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}\left(\mathrm{Op}^W(b^s)\right)^{-1}$ since the operators $\mathrm{I}+\mathcal{H}(\lambda,\nu)$ and $\mathrm{Op}^W(b^s)$ are invertible. Moreover, for the same reason as above, $(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{\frac{s}{2}}\left(\mathrm{Op}^W(b^s)\right)^{-1}$ is bounded on $L^2(\mathbb{G})$. By the inverse mapping theorem, $\mathrm{Op}^W(b^s)(\mathrm{I}+\mathcal{H}(\lambda,\nu))^{-\frac{s}{2}}$ is bounded on $L^2(\mathbb{G})$. This shows the reverse continuous inclusion. We have proved

$$\mathcal{Q}^{(b)}_s(\mathbb{G}) = \mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$$

with equivalence of norms for $s \in 2\mathbb{N}_0$ and this implies that this is true for any $s \in \mathbb{R}$ by the properties of duality and interpolation for $\mathcal{Q}^{(b)}_s(\mathbb{G})$ and $\mathcal{Q}^{\lambda,\nu}_s(\mathbb{G})$. This shows Part (4) and implies Parts (5) and (6). $\square$

4.2 Heat kernels of sub-Laplacians

The development of mathematics in the past few decades has witnessed an unprecedented rise in the usage of the notion of heat kernel in the diverse and seemingly remote sections of mathematics. The special role of exponential functions $t \longmapsto e^{at}$ has been seen in the first analysis courses. No wonder a far-reaching generalization of exponential functions—heat semigroups $\{e^{-At}\}_{t\geq 0}$, where A is a positive definite linear operator which plays an integral role in mathematics and physics, not least because it solves the associated heat equation $\partial_t u + Au = 0$. If the operator A acts in the function space, the action of the semigroup e^{-At} is usually given by the integral operator whose kernel is called the heat kernel of A.

As we have known that if in additional the operator A is Markovian, i.e. generates a Markov process (for example, the case where A is a second-order elliptic differential operator), then the information about the heat kernel can be used to answer the question about the process itself ([GH09]). What's more, upper and/or lower bound estimates about heat kernel that can also help solve various problems related to operator A and its spectrum, solutions to heat equations, and properties of the underlying space ([HLM02]).

The culmination of this work was the proof by Li and Yau [LY86] in 1986 of the parabolic Harnack inequality and the heat kernel two-sided estimates on complete manifolds of non-negative Ricci curvature, which stimulated further research on heat kernel estimates by many authors. Apart from the general wide influence on geometric analysis, the gradient estimates of Li and Yau motivated Richard Hamilton in his program on Ricci flow that eventually lead to the resolution of the Poincaré conjecture by Grigory Perel'man, which can be viewed as a most spectacular application of heat kernels in geometry. On the other hand, an interesting application of heat kernels is the heat equation approach to the Atiyah-Singer index theorem (see [ABP73]). Then the last purpose of this thesis is to consider the sub-Laplacian and the heat kernel on 2-step stratified Lie group without the Moore-Wolf condition.

4.2.1 Heat kernels of $\mathcal{H}(\lambda)$

In this section, we consider the heat kernel of the rescaled harmonic oscillator:

$$\mathcal{H}(\lambda) := -\Delta_x + |\eta(\lambda) \cdot x|^2 = \sum_{j=1}^{d} \left(\eta_j^2(\lambda) x_j^2 - \frac{\partial^2}{\partial x_j^2} \right).$$

As we have known, the heat kernel plays an important role in many problems in harmonic analysis and partial differential equations. An explicit expression for the heat kernel on the Heisenberg group was obtained in [Gav77; Hul76; Sta03]. Gaveau [Gav77] also obtained the heat kernel for free nilpotent Lie groups of two step. Cygan [Sta03] obtained the heat kernel for all nilpotent Lie groups of two step. But neither Gaveau's expression for free nilpotent Lie groups nor Cygan's expression for arbitrary nilpotent Lie groups of two step were as explicit as those in the cases of Heisenberg groups and quaternionic Heisenberg groups. Our results revise and generalize the results in [CT05] to the 2-step stratified Lie group, where we can give explicitly all irreducible unitary representations, which can help us to give a explicit expression for the kernel and fundamental solution for the sub-Laplacian on 2-step stratified Lie group, which can be found in next section. We state our main results as follows.

Theorem 4.19. *The associated heat kernel of the rescaled harmonic oscillator $\mathcal{H}(\lambda)$ is*

$$G_\tau(x) = \prod_{j=1}^{d} \frac{1}{2\sinh(\eta_j(\lambda)\tau)} \exp\left\{ -\sum_{j=1}^{d} \frac{\eta_j(\lambda)\,|x_j|^2}{2} \coth\left(\eta_j(\lambda)\tau\right) \right\},$$

i.e., $G_\tau(x)$ satisfies the heat equation

$$\frac{\partial G_\tau}{\partial \tau} + \sum_{j=1}^{d} \left(\eta_j^2(\lambda) x_j^2 - \frac{\partial^2}{\partial x_j^2} \right) G_\tau(x) = 0 \quad \textit{with} \quad \lim_{\tau \to 0} \int_{\mathbb{R}^d} G_\tau(x) f(x) dx = f(0).$$

Proof. We are looking for a distribution $K(x,y)$ such that

$$\sum_{j=1}^{d} \left(\eta_j^2(\lambda) x_j^2 - \frac{\partial^2}{\partial x_j^2} \right) K(x,y) = \delta(x-y). \tag{4.9}$$

We find $K(x) = K(x,0)$, i.e., the fundamental solution with singularity at the origin. Taking the Fourier transform on $\mathbb{R}^d$

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx$$

to the rescaled harmonic oscillator and applying the formulae

$$\mathcal{F}\left(\frac{\partial f}{\partial x_j}\right) = i\xi_j \mathcal{F}(f)(\xi) \text{ and } \mathcal{F}\left(x_j f(x)\right) = i\frac{\partial}{\partial \xi_j}(\mathcal{F}(f))(\xi)$$

then, when $y = 0$, equation (4.9) becomes

$$\left(|\xi|^2 - \sum_{j=1}^{d} \eta_j^2(\lambda) \frac{\partial^2}{\partial \xi_j^2} \right) \widehat{K}(\xi) = 1.$$

Next, for $\mathbf{k} = (k_1, \ldots, k_d)$ we define the d-tuple Hermite function $\Psi_{\mathbf{k}}(\xi) = \prod_{j=1}^{d} \psi_{k_j,\eta_j(\lambda)} =$

$\prod_{j=1}^{d} \psi_{k_j}\left(\xi_j \sqrt{\eta_j(\lambda)}\right)$ and let

$$\widehat{K}(\xi) = \sum_{|\mathbf{k}|=0}^{\infty} c_{\mathbf{k}} \Psi_{\mathbf{k}}(\xi), \qquad \text{where } |\mathbf{k}| = k_1 + \cdots + k_d.$$

Then we apply the operator $\left(|\xi|^2 - \sum_{j=1}^{d} \eta_j^2(\lambda) \frac{\partial^2}{\partial \xi_j^2}\right)$ to $\widehat{K}(\xi)$ and obtain:

$$\left(|\xi|^2 - \sum_{j=1}^{d} \eta_j^2(\lambda) \frac{\partial^2}{\partial \xi_j^2}\right) \widehat{K}(\xi) = \sum_{j=1}^{d} \eta_j(\lambda)(2k_j+1) \sum_{|\mathbf{k}|=0}^{\infty} c_{\mathbf{k}} \Psi_{\mathbf{k}}(\xi).$$

Next we use the orthogonality property (3.57) to find $c_{\mathbf{k}}$. It is easy to see that

$$\sum_{j=1}^{d} \eta_j(\lambda)(2k_j+1) \sum_{|\mathbf{k}|=0}^{\infty} c_{\mathbf{k}} \Psi_{\mathbf{k}}(\xi) = 1,$$

implies

$$\sum_{j=1}^{d} \eta_j(\lambda)(2k_j+1) c_{\mathbf{k}} \langle \Psi_{\mathbf{k}}, \Psi_{\mathbf{k}} \rangle = \langle 1, \Psi_{\mathbf{k}} \rangle .$$

Here $\langle \Psi_{\mathbf{k}}, \Psi_{\mathbf{m}} \rangle$ is the usual inner product in $L^2(\mathbb{R})$. Since

$$\langle \Psi_{\mathbf{k}}, \Psi_{\mathbf{k}} \rangle = \prod_{j=1}^{d} \sqrt{\eta_j(\lambda)\pi} 2^{k_j} k_j!, \quad \langle 1, \Psi_{2\mathbf{k}+1} \rangle = 0 \quad \text{and} \quad \langle 1, \Psi_{2\mathbf{k}} \rangle = \prod_{j=1}^{d} \sqrt{\eta_j(\lambda)\pi} \frac{(2k_j)!}{k_j!},$$

we have $c_{2\mathbf{k}+1} = 0$ for $\mathbf{k} \in (\mathbb{Z}_+)^d$ and

$$\begin{aligned} c_{2\mathbf{k}} &= \frac{\langle 1, \Psi_{2\mathbf{k}} \rangle}{\sum_{j=1}^{d} \eta_j(\lambda)(2k_j+1) \langle \Psi_{2\mathbf{k}}, \Psi_{2\mathbf{k}} \rangle} \\ &= \frac{1}{\sum_{j=1}^{d} \eta_j(\lambda)(2k_j+1)} \frac{\prod_{j=1}^{n} \sqrt{\eta_j(\lambda)\pi} \frac{(2k_j)!}{k_j!}}{\prod_{j=1}^{d} \sqrt{\eta_j(\lambda)\pi} 2^{2k_j} (2k_j)!} \\ &= \frac{1}{\sum_{j=1}^{d} \eta_j(\lambda)(2k_j+1)} \cdot \frac{1}{\prod_{j=1}^{d} 2^{2k_j} k_j!}. \end{aligned}$$

Hence

$$\widehat{K}(\xi) = \sum_{|\mathbf{k}|=0}^{\infty} c_{2\mathbf{k}} \Psi_{2\mathbf{k}} = \sum_{|\mathbf{k}|=0}^{\infty} \frac{1}{\sum_{j=1}^{d} \eta_j(\lambda)(2k_j+1)} \prod_{j=1}^{d} \frac{\psi_{2k_j}\left(\sqrt{\eta_j(\lambda)}\xi_j\right)}{2^{2k_j} k_j!}.$$

Next we apply

$$\frac{1}{A} = \int_0^{\infty} e^{-As} ds \text{ for } A = \sum_{j=1}^{d} \eta_j(\lambda)\left(2k_j+1\right)$$

and obtain

$$\begin{aligned}
\widehat{K}(\xi) &= \sum_{|\mathbf{k}|=0}^{\infty} \int_0^{\infty} \prod_{j=1}^{d} \frac{\psi_{2k_j}\left(\sqrt{\eta_j(\lambda)}\xi_j\right)}{2^{2k_j} k_j!} e^{-(2k_j+1)\eta_j(\lambda)s} ds \\
&= \int_0^{\infty} \prod_{j=1}^{d} e^{-\eta_j(\lambda)s} \sum_{k_j=0}^{\infty} \frac{\psi_{2k_j}\left(\sqrt{\eta_j(\lambda)}\xi_j\right)}{2^{2k_j} k_j!} e^{-2k_j\eta_j(\lambda)s} ds \\
&= \int_0^{\infty} \prod_{j=1}^{d} e^{-\eta_j(\lambda)s} g_j\left(\sqrt{\eta_j(\lambda)}\xi_j, s\right) ds
\end{aligned}$$

with

$$g_j\left(\sqrt{\eta_j(\lambda)\xi_j}, s\right) = \sum_{k_j=0}^{\infty} \frac{\psi_{2k_j}\left(\sqrt{\eta_j(\lambda)}\xi_j\right)}{2^{2k_j} k_j!} e^{-2k_j\eta_j(\lambda)s}.$$

To sum up with respect to k_j in $g_j\left(\sqrt{\eta_j(\lambda)}\xi_j, s\right)$, we apply the relationship between the Hermite function and Laguerre polynomial (see Chapter 3) to get

$$g_j(x,s) = \sum_{k_j=0}^{\infty} e^{-\frac{x^2}{2}} L_{k_j}^{(0)}\left(x^2\right) e^{-2k_j\eta_j(\lambda)s} = e^{-\frac{x^2}{2}} \sum_{k_j=0}^{\infty} L_{k_j}^{(0)}\left(x^2\right) \left(e^{-2\eta_j(\lambda)s}\right)^{k_j}. \tag{4.10}$$

The Laguerre polynomials are defined by their generating formula (see Theorem 3.59):

$$\sum_{k=0}^{\infty} L_k^{(\beta)}(w) z^k = \frac{1}{(1-z)^{\beta+1}} \exp\left\{\frac{wz}{z-1}\right\}.$$

Now we may apply the generating formula of the Laguerre polynomials to sum up the series (4.10) and find $g_j(x,s)$

$$\begin{aligned}
g_j(x,s) &= \frac{e^{-\frac{x^2}{2}}}{1-e^{-2\eta_j(\lambda)s}} \exp\left\{\frac{x^2 e^{-2\eta_j(\lambda)s}}{e^{-2\eta_j(\lambda)s}-1}\right\} \\
&= \frac{1}{1-e^{-2\eta_j(\lambda)s}} \exp\left\{-\frac{x^2}{2}\left[1-\frac{2e^{-2\eta_j(\lambda)s}}{e^{-2\eta_j(\lambda)s}-1}\right]\right\} \\
&= \frac{1}{1-e^{-2\eta_j(\lambda)s}} \exp\left\{-\frac{x^2}{2} \cdot \frac{-1-e^{-2\eta_j(\lambda)s}}{e^{-2\eta_j(\lambda)s}-1}\right\}.
\end{aligned}$$

Hence

$$\widehat{K}(\xi) = \int_0^{\infty} \prod_{j=1}^{d} \frac{e^{-\eta_j(\lambda)s}}{1-e^{-2\eta_j(\lambda)s}} \exp\left\{-\sum_{j=1}^{d} \frac{\eta_j(\lambda)\left|\xi_j\right|^2}{2} \cdot \frac{1+e^{-2\eta_j(\lambda)s}}{1-e^{-2\eta_j(\lambda)s}}\right\} ds$$

We may rewrite the above formula in terms of hyperbolic functions:

$$\widehat{K}(\xi) = \int_0^{\infty} \prod_{j=1}^{d} \frac{1}{2\sinh(\eta_j(\lambda)s)} \exp\left\{-\sum_{j=1}^{d} \frac{\eta_j(\lambda)\left|\xi_j\right|^2}{2} \coth\left(\eta_j(\lambda)s\right)\right\} ds,$$

where we use

$$\sinh \eta_j(\lambda)s = \frac{1-e^{-2\eta_j(\lambda)s}}{2e^{-\eta_j(\lambda)s}},$$

and

$$\coth(-\eta_j(\lambda)s) = -\coth(\eta_j(\lambda)s) = \frac{1+e^{-2\eta_j(\lambda)s}}{1-e^{-2\eta_j(\lambda)s}}.$$

Let

$$G(\xi,\tau) = \prod_{j=1}^{d} \frac{1}{2\sinh(\eta_j(\lambda)\tau)} \exp\left\{-\sum_{j=1}^{d} \frac{\eta_j(\lambda)\,|\xi_j|^2}{2} \coth\left(\eta_j(\lambda)\tau\right)\right\}$$

be the integrand of the above integral. We can prove directly that

$$\left[\sum_{j=1}^{d}\left(\xi_j^2 - \eta_j^2(\lambda)\frac{\partial^2}{\partial\xi_j^2}\right)\right]\widehat{K}(\xi) = 1$$

by showing that the function $G(\xi,\tau)$ satisfies the heat equation

$$\frac{\partial G}{\partial\tau} + \left[\sum_{j=1}^{d}\left(\xi_j^2 - \eta_j^2(\lambda)\frac{\partial^2}{\partial\xi_j^2}\right)\right] G(\xi,\tau) = 0 \quad \text{and} \quad \lim_{\tau\to 0^+} G(\xi,\tau) = 1. \tag{4.11}$$

Then the fundamental theorem of calculus yields

$$\begin{aligned}\left[\sum_{j=1}^{d}\left(\xi_j^2 - \eta_j^2(\lambda)\frac{\partial^2}{\partial\xi_j^2}\right)\right]\widehat{K}(\xi) &= \int_0^\infty \left[\sum_{j=1}^{d}\left(\xi_j^2 - \eta_j^2(\lambda)\frac{\partial^2}{\partial\xi_j^2}\right)\right] G(\xi,\tau)d\tau \\ &= \int_0^\infty \left(-\frac{\partial G}{\partial\tau}\right) d\tau = G(0) = 1.\end{aligned}$$

The fact that $G(\xi,\tau)$ satisfies the heat equation (4.11) can be proved directly by simple differentiation. Therefore

$$\frac{\partial G}{\partial\tau} + \sum_{j=1}^{d}\left(\xi_j^2 - \eta_j^2(\lambda)\frac{\partial^2}{\partial\xi_j^2}\right) G(\xi,\tau) = 0.$$

This shows that $G(\xi,\tau)$ is the heat kernel of the rescaled harmonic oscillator $\sum_{j=1}^{d}\left(\xi_j^2 - \eta_j^2(\lambda)\frac{\partial^2}{\partial\xi_j^2}\right)$ with $G(\xi,0) = 1$. □

4.2.2 Heat kernels of $\mathcal{L}$

In this section, we are interested in finding the heat kernel of $\mathcal{L}$, which are related to the kernel of the integral operator $e^{-\tau\mathcal{L}}, \tau > 0$. We need the following proposition that follows from the Theorem 3.29 and Theorem 3.41.

Proposition 4.20. *For all multi-indices α, β, μ and γ*

$$e^\lambda_{\alpha,\gamma} *_\lambda e^\lambda_{\beta,\mu} = \mathrm{Pf}(\lambda)^{-\frac{1}{2}}(2\pi)^{\frac{d}{2}}\delta_{\beta,\gamma}e^\lambda_{\alpha,\mu}$$

where $\delta_{\beta,\alpha}$ is the Kronecker delta function.

Theorem 4.21. *For all $f \in L^2\left(\mathbb{R}^{2d+k}\right)$ and all $\tau > 0$,*

$$e^{-\tau\mathcal{L}^\lambda} f = k_\tau^\lambda *_{-\lambda} f$$

where

$$k_\tau^\lambda(z,r) = e^{-\tau|\nu|^2} \prod_{j=1}^d (2\pi)^{-\frac{1}{2}} \frac{\eta_j(\lambda)}{2\sinh\left(\eta_j(\lambda)\tau\right)} \exp\left\{-\frac{\eta_j(\lambda) z_j^2}{2} \coth\left(\eta_j(\lambda)\tau\right)\right\}$$

for all $z \in \mathbb{R}^{2d+k}$.

Proof. Let $f \in \mathcal{S}\left(\mathbb{R}^{2d+k}\right)$ and $\tau > 0$. Then by Theorem 3.56,

$$e^{-\tau\mathcal{L}^\lambda} f = \sum_\beta e^{-\tau\left(\sum\limits_{j=1}^d \eta_j(\lambda)(2\beta_j+1)+|\nu|^2\right)} \sum_\alpha \left(f, e_{\alpha,\beta}^\lambda\right)_{L^2\left(\mathbb{R}^{2d}\right)} e_{\alpha,\beta}^\lambda.$$

By Proposition 4.20,

$$\begin{aligned}
f *_\lambda e_{\beta,\beta}^\lambda &= \sum_\alpha \sum_\gamma \left(f, e_{\alpha,\gamma}^\lambda\right)_{L^2\left(\mathbb{R}^{2d+k}\right)} e_{\alpha,\gamma}^\lambda *_\lambda e_{\beta,\beta}^\lambda \\
&= \mathrm{Pf}(\lambda)^{-\frac{1}{2}} (2\pi)^{\frac{d}{2}} \sum_\alpha \sum_\gamma \left(f, e_{\alpha,\gamma}^\lambda\right)_{L^2\left(\mathbb{R}^{2d+k}\right)} \delta_{\gamma,\beta} e_{\alpha,\beta}^\lambda \\
&= \mathrm{Pf}(\lambda)^{-\frac{1}{2}} (2\pi)^{\frac{d}{2}} \sum_\alpha \left(f, e_{\alpha,\beta}^\lambda\right)_{L^2\left(\mathbb{R}^{2d+k}\right)} e_{\alpha,\beta}^\lambda
\end{aligned}$$

for all $\beta \in \mathbb{N}_0^d$. Thus,

$$\begin{aligned}
e^{-\tau\mathcal{L}^\lambda} f &= \mathrm{Pf}(\lambda)^{\frac{1}{2}} (2\pi)^{-\frac{d}{2}} \sum_\beta e^{-\tau\left(\sum\limits_{j=1}^d \eta_j(\lambda)(2\beta_j+1)+|\nu|^2\right)} f *_\lambda e_{\beta,\beta}^\lambda \\
&= \mathrm{Pf}(\lambda)^{\frac{1}{2}} (2\pi)^{-\frac{d}{2}} \sum_\beta e^{-\tau\left(\sum\limits_{j=1}^d \eta_j(\lambda)(2\beta_j+1)+|\nu|^2\right)} e_{\beta,\beta}^\lambda *_{-\lambda} f.
\end{aligned}$$

To compute

$$\sum_\beta e^{-\tau\left(\sum\limits_{j=1}^d \eta_j(\lambda)(2\beta_j+1)+|\nu|^2\right)} e_{\beta,\beta}^\lambda,$$

we first to compute

$$\sum_\beta e^{-\tau\left(\sum\limits_{j=1}^d \eta_j(\lambda)(2\beta_j+1)\right)} e_{\beta,\beta}^\lambda$$

by Mehler's formula. In fact,

$$e_{\beta,\beta}^\lambda(p,q) = \mathrm{Pf}(\lambda)^{\frac{1}{2}} \prod_{j=1}^d e_{\beta_j,\beta_j}\left(\frac{\left((B^{(\lambda)})^t p\right)_j}{\sqrt{|\eta(\lambda)|}}, \sqrt{|\eta(\lambda)|} q_j\right), \quad p,q \in \mathbb{R}^d,$$

where e_{β_j,β_j} is the ordinary Fourier-Wigner transform of the Hermite functions e_{β_j}. Hence for all $(p,q) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\sum_{\beta} e^{-\tau\left(\sum_{j=1}^{d} \eta_j(\lambda)(2\beta_j+1)\right)} e^{\lambda}_{\beta,\beta}(p,q) = \mathrm{Pf}(\lambda)^{\frac{1}{2}} \prod_{j=1}^{d} \left(\sum_{\beta_j=0}^{\infty} e^{-\tau} e_{\beta_j,\beta_j} \left(\frac{\left((B^{(\lambda)})^t p\right)_j}{\sqrt{|\eta(\lambda)|}}, \sqrt{|\eta(\lambda)|} q_j \right) \right).$$

Now, by Theorem 4.19

$$\sum_{\beta_j=0}^{\infty} e^{-\tau\left(\sum_{j=1}^{d} \eta_j(\lambda)(2\beta_j+1)\right)} e_{\beta_j,\beta_j}(p_j,q_j) = \frac{1}{\sqrt{2\pi}} \frac{1}{2\sinh(\eta_j(\lambda)\tau)} \exp\left\{ -\frac{\eta_j(\lambda)\left((p_j)^2+(q_j)^2\right)}{2} \coth(\eta_j(\lambda)\tau) \right\}$$

for all (p_j,q_j) in $\mathbb{R}\times\mathbb{R}$. So,

$$\sum_{\beta} e^{-\tau\left(\sum_{j=1}^{d} \eta_j(\lambda)(2\beta_j+1)+|\nu|^2\right)} e^{\lambda}_{\beta,\beta}(p,q) = e^{-\tau|\nu|^2} \prod_{j=1}^{d} (2\pi)^{-\frac{1}{2}} \frac{\eta_j(\lambda)^{1/2}}{2\sinh(\eta_j(\lambda)\tau)} \exp\left\{ -\frac{\eta_j(\lambda) z_j^2}{2} \coth(\eta_j(\lambda)\tau) \right\}.$$

□

Now, we use the heat kernel κ^{λ}_{τ} of the λ-twisted sub-Laplacian to find the heat kernel of the sub-Laplacian by taking the Fourier transform with respect to the parameter λ. To do this, we need some preparation. The group convolution of two measurable functions f and g on $\mathbb{G}$ is defined by

$$(f *_{\mathbb{G}} g)(z,s) = \int_{\mathbb{G}} f\left((z,s)(w,s)^{-1}\right) g(w,s)\,dw\,ds, \quad z,w \in \mathbb{R}^{2d+k}, s \in \mathbb{R}^m$$

if the integral exists. Moreover, we denote by f_λ the ordinary Fourier transform of f with respect to the s variable evaluated at the point $\lambda \in \mathbb{R}^m$. More precisely,

$$f_\lambda(z) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-is\cdot\lambda} f(z,s)\,ds, \quad z \in \mathbb{R}^{2d+k}.$$

We need the following theorem.

Theorem 4.22. *Let f and g be functions in $L^1(\mathbb{G})$. Then for all nonzero $\lambda \in \mathbb{R}^m$,*

$$(f *_{\mathbb{G}} g)_\lambda = (2\pi)^{\frac{m}{2}} f_\lambda *_{-\lambda} g_\lambda.$$

Proof. For all $z \in \mathbb{R}^{2d+k}$

$$\begin{aligned}(f *_{\mathbb{G}} g)_\lambda &= (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-is\cdot\lambda} (f *_\lambda g)(z,s)\,ds \\ &= (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-is\cdot\lambda} \left(\int_{\mathbb{R}^{2d+k}} \int_{\mathbb{R}^m} f\left(z-w, s-l-\frac{1}{2}\sigma(z,w)\right) g(w,l)\,dw\,ds \right) dl.\end{aligned}$$

Let $s' = s - \frac{1}{2}\sigma(z,w)$. Then

$$(f *_{\mathbb{G}} g)_\lambda = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2d+k}} e^{-is'\cdot\lambda} f(z-w, s'-l)\, g(w,l) e^{-\frac{i}{2}\lambda\sigma(z,w)}\,dw\,dl\,ds'.$$

On the other hand, for all z in $\mathbb{R}^{2d+k}$, we get

$$\begin{aligned}(f_\lambda *_{-\lambda} g_\lambda)(z) &= \int_{\mathbb{R}^{2d+k}} f_\lambda(z-w) g_\lambda(w) e^{-\frac{i}{2}\lambda\sigma(z,w)} dw \\ &= (2\pi)^{-m} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^{2d+k}} e^{-is\cdot\lambda} f(z-w, s-l) g(w,l) e^{-\frac{i}{2}\lambda\sigma(z,w)} dwdlds,\end{aligned}$$

and the proof is complete. □

Now, we consider the initial-value problem given by

$$\begin{cases} \partial_\tau u(\omega,t,\tau) + (\mathcal{L}u)(w,t,\tau) = 0, \\ u(\omega,t,0) = f(\omega,t), \\ \omega = (z,r) \in \mathbb{R}^{2d+k}, t \in \mathbb{R}^m, \tau > 0. \end{cases}$$

By taking the Fourier transform with respect to t and evaluated at λ, we get an initial-value problem for the heat equation governed by the λ-twisted sub-Laplacian $\mathcal{L}^\lambda$, i.e.

$$\begin{cases} \partial_\tau u_\lambda(\omega,\tau) + (\mathcal{L}^\lambda u_\lambda)(\omega,\tau) = 0, \\ u_\lambda(\omega,0) = f_\lambda(\omega), \end{cases}$$

for all $\omega = (z,r) \in \mathbb{R}^{2d+k}, \tau > 0$ and $\lambda \in \mathbb{R}^m \backslash \{0\}$. By Theorem 4.21,

$$u_\lambda(\omega,\tau) = \left(k_\tau^\lambda *_{-\lambda} f_\lambda\right)(\omega), \quad \omega \in \mathbb{R}^{2d+k}, \tau > 0$$

for all $\lambda \in \Lambda$. Therefore by taking the inverse Fourier transform with respect to λ and evaluated at s, and using Theorem 4.22, we get the solution of the initial-value problem governed by the sub-Laplacian given by

$$u(\omega,t,\tau) = (2\pi)^{-\frac{m}{2}} \left(f *_{\mathbb{G}} K_\tau\right)(\omega,t), \quad \omega \in \mathbb{R}^{2d+k}, t \in \mathbb{R}^m, \tau > 0$$

where K_τ is the Fourier transform of the heat kernel of k_τ^λ with respect to λ and evaluated at t. So, the heat kernel of $\mathcal{L}$ is given in the following theorem.

Theorem 4.23. *For all f in $L^2(\mathbb{G}), e^{-\tau\mathcal{L}}f = f *_{\mathbb{G}} K_\tau$, where*

$$K_\tau(\omega,t) = (2\pi)^{-(d+m)} \int_{\mathbb{R}^m} e^{-it\cdot\lambda} e^{-\tau|\nu|^2} \prod_{j=1}^{d} \frac{\eta_j(\lambda)}{2\sinh(\eta_j(\lambda)\tau)} \exp\left\{-\frac{\eta_j(\lambda)\omega_j^2}{2}\coth(\eta_j(\lambda)\tau)\right\} d\lambda$$

for all $(\omega,t) \in \mathbb{G}$.

Our results can be seen as a generalization of Heisenberg group and H-type group. In fact, if we take $k=0, m=1$, then the step two stratified Lie group is the Heisenberg group H^{2d+1}, and our result cover the one in [Dua13]. If we take $k=0$, then the step two stratified Lie group is the H-type group, and some results can be found in [Cyg79; DW15; MW18; MR03; YZ08]. Also, there are some results in this direction by different methods, i.e. [AG16; CKW21; LZ19].

To the best of our knowledge, this is the first result on the Weyl transform and Heat kernel for sub-Laplacian on general 2-step stratified Lie groups, especially we consider the case $k \neq 0$.

5 Appendix

The aim of this appendix is to prove that, up to a canonical isomorphism, the classical definition of stratified Lie group (Definition 2.62) coincides with our Definition 2.61, as given in Section 2.1.3. To this aim, we begin by recalling some basic facts about abstract Lie groups, providing all the terminology and the main results about manifolds, tangent vectors, left-invariant vector fields, Lie algebras, homomorphisms, the exponential map. We take most of the material from [BLU07].

5.1 Abstract Lie groups

Let $N \in \mathbb{N}$, and let us define, for $i = 1, \ldots, N$, the coordinate projections on $\mathbb{R}^N$ (whose points will be denoted by $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ with $\xi_1, \ldots, \xi_N \in \mathbb{R}$)

$$\pi_i : \mathbb{R}^N \longrightarrow \mathbb{R}, \quad \pi_i(\xi) := \xi_i.$$

Definition 5.1 (N-dimensional locally Euclidean space)**.** An N-dimensional locally Euclidean space M is a Hausdorff topological space such that every point of M has a neighborhood in M homeomorphic to an open subset of $\mathbb{R}^N$. If φ is a homeomorphism between a connected open set $U \subseteq M$ and an open subset of $\mathbb{R}^N$, we say that $\varphi : U \to \mathbb{R}^N$ is a coordinate map,

$$x_i := \pi_i \circ \varphi : U \to \mathbb{R}$$

is *a* coordinate function, and the pair (U, φ) (sometimes also denoted by $(U, x_1, \ \ldots, x_N)$) is *a* coordinate system or a chart. If $m \in U$ and $\varphi(m) = 0$, we say that the coordinate system is centered at m.

Definition 5.2 (Differentiable manifold)**.** A C^∞ differentiable structure $\mathcal{F}$ on *a* locally Euclidean space M is a collection of coordinate systems

$$\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$$

with the following properties:

- $\bigcup_{\alpha \in A} U_\alpha = M$;
- $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^∞ for every $\alpha, \beta \in A$ (whenever it is defined);
- $\mathcal{F}$ is maximal w.r.t. the second property in the sense that if (U, φ) is a coordinate system such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are C^∞ for every $\alpha \in A$, then $(U, \varphi) \in \mathcal{F}$.

An N-dimensional C^∞ differentiable manifold is a couple $(M, \mathcal{F})$, where M is a second countable N-dimensional locally Euclidean space and $\mathcal{F}$ is a C^∞ differentiable structure.

As usual, when we say "M is an N-dimensional C^∞ differentiable manifold", we leave unsaid that M is equipped with the fixed datum of a C^∞ differentiable structure $\mathcal{F}$ on M.

Definition 5.3 (Tangent vector, space and bundle)**.** Let M be an N-dimensional C^∞ differentiable manifold. A tangent vector $\mathbf{v}$ at $m \in M$ is a linear functional, defined on the collection of the real-valued functions C^∞ in some neighborhood of m, such that

$$\mathbf{v}(f) = 0$$

whenever f is horizontal in m.

We denote by M_m the set of the tangent vectors at $m \in M$, and we say that M_m is the tangent space to M at m. We finally set

$$T(M) := \bigcup_{m \in M} \{m\} \times M_m = \{(m, \mathbf{v}) : m \in M, \mathbf{v} \in \mathbf{M_m}\}.$$

$T(M)$ is called the tangent bundle to M.

Proposition 5.4. *Let M be an N-dimensional C^∞ differentiable manifold. Then* $\dim(M_m) = N = \dim M$.

Definition 5.5 (Partial derivatives on M)**.** Let M be an N-dimensional C^∞ differentiable manifold. Let (U, φ) be a coordinate system with coordinate functions $x_1, \ldots, x_N$ ($x_i := \pi_i \circ \varphi$), and let $m \in U$. For every $i \in \{1, \ldots, N\}$ we define *a* tangent vector, denoted

$$\left.\frac{\partial}{\partial x_i}\right|_m \in M_m$$

by setting

$$\left.\frac{\partial}{\partial x_i}\right|_m (f) := \left.\frac{\partial}{\partial \xi_i}\right|_{\varphi(m)} \left(f \circ \varphi^{-1}\right)(\xi)$$

for every C^∞ function f defined in a neighborhood of m.

Definition 5.6 (Differential at a point)**.** Let $\psi : M \longrightarrow M'$ be a C^∞ map between two differentiable manifolds, and let $m \in M$. The differential of ψ at m is the linear map

$$\mathrm{d}_m\psi : M_m \longrightarrow M'_{\psi(m)}$$

defined as follows: if $\mathbf{v} \in M_m$, $\mathrm{d}_m\psi(\mathbf{v})$ is the tangent vector in $M'_{\psi(m)}$ acting in the following way: if f is aC^∞ function in a neighborhood of $\psi(m)$, we set

$$(\mathrm{d}_m\psi(\mathbf{v}))\,(f) := \mathbf{v}(f \circ \psi).$$

Definition 5.7 ($d\psi$ as a map on the tangent bundle)**.** Let $\psi : M \to M'$ be a C^∞ map between two differentiable manifolds M, M'. We set

$$\mathrm{d}\psi : T(M) \to T\left(M'\right), \quad d\psi(m, \mathbf{v}) := (\psi(m), d_m\psi(\mathbf{v}))\,.$$

Note that, whereas $\mathrm{d}_m\psi$ is a map from M_m to $M'_{\psi(m)}$ (for any fixed $m \in M$) , $d\psi$ is a map from $T(M)$ to $T\left(M'\right)$.

Definition 5.8 (Vector field)**.** Let $\Omega \subseteq M$ be an open subset of a differentiable manifold M. A vector field X on Ω is an application

$$X : \Omega \longrightarrow T(M)$$

such that,

$$X(m) = (m, \mathbf{v}(m)) \in T(M) \quad \forall m \in \Omega.$$

Equivalently, we have

$$X(m) = (m, \mathbf{v}(m)), \qquad \text{where } \mathbf{v}(m) \in M_m \text{ for every } m \in \Omega.$$

If $T(M)$ is the tangent bundle of a differentiable manifold M, and, for every $m \in M, \mathbf{v} \in M_m$, we set $\pi(m, \mathbf{v}) := \mathbf{v}$, then the following map is well posed on $T(M)$:

$$\pi : T(M) \to \bigcup_{m \in M} M_m, \quad (m, \mathbf{v}) \mapsto \mathbf{v}.$$

In the sequel, if X is a vector field on an open set $\Omega \subseteq M$, we shall use the notation $X(m)$ for the map

$$X : \Omega \to T(M), \quad m \mapsto X(m),$$

whereas X_m will denote the map

$$\Omega \to \bigcup_{m \in M} M_m, \quad m \mapsto X_m := (\pi \circ X)(m).$$

So, the above positions can be summarized as

$$X(m) = (m, X_m) \qquad \text{for every } m \in M.$$

Finally, if f is a C^∞ function on Ω and X is a vector field on Ω, we shall denote (with an abuse of notation) by $X(f)$ or shortly Xf the function on Ω whose value at m is $X_m(f)$, i.e.

$$Xf : \Omega \to \mathbb{R}, \quad (Xf)(m) := X_m(f). \tag{5.1}$$

Definition 5.9 (Smooth vector field)**.** Let X be a vector field defined on a manifold M. We say that X is C^∞ (or smooth) if, for every open set $\Omega \subseteq M$ and every smooth real-valued function f on Ω, the function Xf as defined in (5.1) is smooth on Ω.

Remark 5.10 (Smooth vector fields as operators on $C^\infty(M, \mathbb{R})$). Let X be a smooth vector field on a differentiable manifold M. Besides a map from M to $T(M)$, it is possible to identify X with the map

$$X : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \quad f \mapsto Xf$$

where

$$Xf : M \to \mathbb{R}, \quad m \mapsto (Xf)(m) = X_m f.$$

We denote by $\mathcal{X}(M)$ the set of the smooth vector fields considered as linear operators (i.e. endomorphisms) on $C^\infty(M, \mathbb{R})$. Note that $\mathcal{X}(M)$ is a vector space over $\mathbb{R}$.

In what follows, we introduce an important definition. The adjectives "regular" and "smooth" will always mean "of class C^∞ ".

Definition 5.11 (Tangent vector to a curve)**.** Let $\mu : [a, b] \to M$ be a regular curve. The tangent vector to the curve μ at time t is defined by

$$\dot{\mu}(t) := \mathrm{d}_t\mu \left(\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=t} \right) \in M_{\mu(t)}.$$

Hence, fixed $t \in [a, b]$, if f is C^∞ near $\mu(t)$, we have

$$\dot{\mu}(t)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=t} (f(\mu(r))).$$

Definition 5.12 (Integral curve)**.** Let X be a smooth vector field on the differentiable manifold M. A regular curve $\mu : [a, b] \longrightarrow M$ is called an integral curve of X if

$$\dot{\mu}(t) = X_{\mu(t)} \qquad \text{for every } t \in [a, b]. \tag{5.2}$$

More explicitly, (5.2) means that

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=t} (f(\mu(r))) = X(f)(\mu(t))$$

for every smooth function f on M and every $t \in [a, b]$.

Definition 5.13 (Complete vector field)**.** Let X be a smooth vector field on the differentiable manifold M. We say that X is complete if, for every $m \in M$, the integral curve μ of X such that $\mu(0) = m$ is defined on the whole $\mathbb{R}$ (i.e. its maximal interval of definition is $\mathbb{R}$).

In the sequel, we denote by $C^\infty(M, \mathbb{R})$ or, shortly, $C^\infty(M)$ the set of the smooth real-valued functions defined on a differentiable manifold M. It is immediate to observe that if X is a smooth vector field on M and $f \in C^\infty(M, \mathbb{R})$, we have $Xf \in C^\infty(M, \mathbb{R})$. We explicitly recall that, here and in the sequel, we use the notation in (5.1):

$$Xf : M \to \mathbb{R}, \quad (Xf)(m) = X_m(f).$$

As a consequence, the following definition is well posed.

Definition 5.14 (Commutators)**.** Let X and Y be smooth vector fields on a differentiable manifold M. We define a vector field on M (called the commutator of X and Y) in the following way:

$$[X, Y] : M \to T(M), \quad [X, Y](m) := (m, [X, Y]_m),$$

where

$$[X, Y]_m(f) := X_m(Yf) - Y_m(Xf)$$

for every $m \in M$ and every $f \in C^\infty(M, \mathbb{R})$.

Definition 5.14 is well posed as it follows from (i) in the proposition below.

Proposition 5.15. *If X, Y and Z are smooth vector fields on M, we have:*

(i) $[X, Y]$ *is a smooth vector field on M;*

(ii) $[X, Y]_m = -[Y, X]_m$ *for every $m \in M$;*

(iii) $[[X, Y], Z]_m + [[Y, Z], X]_m + [[Z, X], Y]_m = 0$ *for every $m \in M$.*

Remark 5.16. Consider the alternative definition of smooth vector field as an element of $\mathcal{X}(M)$. The commutator operation rewrites as an operation on $\mathcal{X}(M)$ in the following way: Given $X, Y \in \mathcal{X}(M)$, we consider the operator on $C^\infty(M, \mathbb{R})$ defined by

$$[X, Y] : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}), \quad f \mapsto [X, Y]f,$$

where

$$([X,Y]f)(m) := [X,Y]_m f = X_m(Yf) - Y_m(Xf)$$

Then, obviously, $[X,Y] \in \mathcal{X}(M)$ is the operator on $C^\infty(M,\mathbb{R})$ related to the (usual) vector field $[X,Y]$.

With this meaning of the commutation, Proposition 5.15 rewrites as: If X, Y and Z belong to $\mathcal{X}(M)$, we have:

(i) $[X,Y] \in \mathcal{X}(M)$;

(ii) $[X,Y] = -[Y,X]$;

(iii) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

Definition 5.17 (Lie group)**.** A Lie group $\mathbb{G}$ is a differentiable manifold $\mathbb{G}$ along with a group law $* : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$ such that the applications

$$\mathbb{G} \times \mathbb{G} \ni (x,y) \mapsto x * y \in \mathbb{G}, \quad \mathbb{G} \ni x \mapsto x^{-1} \in \mathbb{G}$$

are smooth.

In the following, we shall always denote by e the identity of $(\mathbb{G}, *)$. Moreover, fixed $\sigma \in \mathbb{G}$, we denote by τ_σ the left translation on $\mathbb{G}$ by σ, i.e. the map

$$\mathbb{G} \ni x \mapsto \tau_\sigma(x) := \sigma * x \in \mathbb{G}.$$

Definition 5.18 (Lie algebra)**.** A (real) Lie algebra is a real vector space $\mathfrak{g}$ with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ (called (Lie) bracket) such that, for every $X, Y, Z \in \mathfrak{g}$, we have:

1. (anti-commutativity) $[X,Y] = -[Y,X]$;
2. (Jacobi identity) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

A very remarkable fact is that, given any Lie group, there exists a certain finite dimensional Lie algebra such that the group properties are reflected into properties of the algebra. For instance, any connected and simply connected Lie group is completely determined (up to isomorphism) by its Lie algebra. Therefore, the study of a Lie group is often reduced to the study of its Lie algebra.

Remark 5.19. If $X_1, \dots, X_m$ are elements of an (abstract) Lie algebra, then a system of generators of Lie$\{X_1, \dots, X_m\}$ is given by the commutators

$$X_I := \left[X_{i_1}, \left[X_{i_2}, \left[X_{i_3}, \dots \left[X_{i_{k-1}}, X_{i_k}\right] \dots\right]\right]\right],$$

where $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ and $I = (i_1, i_2, \ldots, i_k)\,, k \in \mathbb{N}$.

5.2 Left invariant vector fields and the Lie algebra

Definition 5.20 (Left invariant vector fields)**.** Let $\mathbb{G}$ be a Lie group. A smooth vector field X on $\mathbb{G}$ is called left invariant if, for every $\sigma \in \mathbb{G}, X$ is τ_σ-related to itself, i.e.

$$\mathrm{d}\tau_\sigma \circ X = X \circ \tau_\sigma. \tag{5.3}$$

Here $\mathrm{d}\tau_\sigma$ is intended as a map from $T(\mathbb{G})$ to itself. Condition (5.3) is equivalent to the following one:

$$(\mathrm{d}_x\tau_\sigma)\,(X_x) = X_{\sigma * x} \quad \forall x, \sigma \in \mathbb{G}. \tag{5.4}$$

Applying (5.4) at the identity e, it follows immediately that if X is a left invariant vector field, we have

$$\mathrm{d}_e\tau_\sigma\,(X_e) = X_\sigma \quad \forall \sigma \in \mathbb{G},$$

which proves that a left invariant vector field is determined by its action at the origin. Moreover, (5.4) can also be written as

$$X_x\,(f \circ \tau_\sigma) = X_{\sigma * x}(f) \qquad \text{for every } x, \sigma \in \mathbb{G} \text{ and every } f \in C^\infty(\mathbb{G}, \mathbb{R})$$

or again as (the most commonly used)

$$X_x(y \mapsto f(\sigma * y)) = (Xf)(\sigma * x).$$

Before giving the following central Definition, we pause a moment in order to recall the multiple ways a smooth vector field can be thought of. A smooth vector field on $\mathbb{G}$ is a map $X : \mathbb{G} \to T(\mathbb{G})$ such that, for every $x \in \mathbb{G}$, it holds $X(x) = (x, X_x)$, where $X_x \in \mathbb{G}_x$ for every $x \in \mathbb{G}$ and such that, for every $f \in C^\infty(\mathbb{G}, \mathbb{R})$, the function $x \mapsto X_x(f)$ is smooth on $\mathbb{G}$. A smooth vector field can be identified to the operator

$$\begin{aligned} X : C^\infty(\mathbb{G}, \mathbb{R}) &\longrightarrow C^\infty(\mathbb{G}, \mathbb{R}), \\ f &\mapsto Xf : \mathbb{G} \to \mathbb{R}, \\ &\qquad x \mapsto X_x f. \end{aligned}$$

The set of the vector fields, as the above described operators, is denoted by $\mathcal{X}(\mathbb{G})$. Obviously, the set of the left invariant operators on $\mathbb{G}$ gives rise to a relevant subset in $\mathcal{X}(\mathbb{G})$, following the above identification. We are ready to give the following central definition.

Definition 5.21 (Algebra of a Lie group)**.** Let $\mathbb{G}$ be a Lie group. Then the subset of $\mathcal{X}(\mathbb{G})$ of the smooth left invariant vector fields on $\mathbb{G}$ is called the (Lie) algebra of $\mathbb{G}$. It will be denoted by $\mathfrak{g}$.

More precisely, following Remark 5.10, we henceforth identify a left invariant vector field X on $\mathbb{G}$ with the following operator

$$X : C^\infty(\mathbb{G}, \mathbb{R}) \to C^\infty(\mathbb{G}, \mathbb{R})$$

such that, for every $f \in C^\infty(\mathbb{G}, \mathbb{R})$, the function Xf on $\mathbb{G}$ is defined by

$$(Xf)(x) := X_x f \quad \forall x \in \mathbb{G}.$$

Hence, $\mathfrak{g}$ is a (linear) set of endomorphisms on $C^\infty(\mathbb{G}, \mathbb{R})$,

$$\mathfrak{g} \subseteq \mathcal{X}(\mathbb{G}).$$

Note that, from the left invariance of $X \in \mathfrak{g}$, we have

$$(Xf)(x) = X\left(f \circ \tau_x\right)(e) \quad \forall x \in \mathbb{G}\ \forall f \in C^\infty(\mathbb{G}, \mathbb{R}).$$

Along with the above definition of the algebra of a Lie group, there is a wide commonly used identification of $\mathfrak{g}$ with $\mathbb{G}_e$ described in the following theorem.

Theorem 5.22 (The Lie algebra of a Lie group)**.** *Let $\mathbb{G}$ be a Lie group and $\mathfrak{g}$ be its algebra. Then we have:*

(i) $\mathfrak{g}$ *is a vector space, and the map*

$$\alpha : \mathfrak{g} \longrightarrow \mathbb{G}_e,$$
$$X \mapsto \alpha(X) := X_e$$

is an isomorphism between $\mathfrak{g}$ and the tangent space $\mathbb{G}_e$ to $\mathbb{G}$ at the identity e of $\mathbb{G}$. As a consequence, $\dim \mathfrak{g} = \dim \mathbb{G}_e = \dim \mathbb{G}$;

(ii) The commutator of smooth left invariant vector fields is a smooth left invariant vector field;

(iii) $\mathfrak{g}$ *with the commutation operation is a Lie algebra.*

Example 5.23 (The Lie algebra of $(\mathbb{R}, +)$)**.** It is obvious that the Lie algebra $\mathfrak{r}$ of the usual Euclidean Lie group $(\mathbb{R}, +)$ is

$$\operatorname{span}\left\{\frac{\mathrm{d}}{\mathrm{d}r}\right\}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}r} : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto f'.$$

With the usual formalism X_x for vector fields, this rewrites as

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_t f = f'(t) \quad \text{for every } t \in \mathbb{R}.$$

Definition 5.24 (Homomorphisms)**.** Let $(\mathbb{G}, \bullet)$ and $(\mathbb{H}, *)$ be Lie groups. A map $\varphi : \mathbb{G} \longrightarrow \mathbb{H}$ is a homomorphism of Lie groups if it is C^∞ and if

$$\varphi(x \bullet y) = \varphi(x) * \varphi(y) \quad \forall x, y \in \mathbb{G}.$$

A map φ is an isomorphism of Lie groups if it is a homomorphism of Lie groups and a diffeomorphism of differentiable manifolds. An isomorphism of $\mathbb{G}$ onto itself is called an automorphism of $\mathbb{G}$.

Let $(\mathfrak{g}, [\cdot, \cdot]_1)$ and $(\mathfrak{h}, [\cdot, \cdot]_2)$ be Lie algebras. *A* map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a homomorphism of Lie algebras if it is linear and if

$$\varphi\left([X, Y]_1\right) = [\varphi(X), \varphi(Y)]_2 \quad \forall X, Y \in \mathfrak{g}.$$

A map φ is an isomorphism of Lie algebras if it is a bijective homomorphism of Lie algebras. An isomorphism of $\mathfrak{g}$ onto itself is called an automorphism of $\mathfrak{g}$.

We recall that, according to Definition 5.13, a smooth vector field X on a Lie group $\mathbb{G}$ is complete if, for every $x \in \mathbb{G}$, the integral curve μ of X such that $\mu(0) = x$ is defined on the whole $\mathbb{R}$.

Proposition 5.25 (Completeness of the left invariant vector fields)**.** *The left invariant vector fields on a Lie group* $\mathbb{G}$ *are complete.*

Definition 5.26 (The exponential curve $\exp_X(t)$)**.** Let $\mathbb{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$ be fixed. By Proposition 5.25, the integral curve $\mu(t)$ of X passing through the identity of $\mathbb{G}$ when $t = 0$ is defined on the whole $\mathbb{R}$. We set

$$\exp_X(t) := \mu(t).$$

By the Definition 5.12 of integral curve, we have

$$\exp_X : \mathbb{R} \to \mathbb{G} \quad \text{with} \quad \begin{cases} \exp_X(0) = e_{\mathbb{G}}, \\ \mathrm{d}_t \exp_X\left(\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=t}\right) = X_{\exp_X(t)} \quad \forall t \in \mathbb{R}. \end{cases} \tag{5.5}$$

In terms of functionals on $C^\infty(\mathbb{G},\mathbb{R})$, (5.5) can be written more explicitly as

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)_{r=t}\{f\left(\exp_X(r)\right)\} = X_{\exp_X(t)}(f) \quad \forall f \in C^\infty(\mathbb{G},\mathbb{R}).$$

In particular, when $t=0$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)_{r=0}\{f\left(\exp_X(r)\right)\} = X_e(f) \quad \forall f \in C^\infty(\mathbb{G},\mathbb{R}).$$

Again from (5.5) with $t=0$ we infer

$$\mathrm{d}_0 \exp_X\left\{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0\right\} = X_e.$$

For future reference, we collect some other useful formulas for $\exp_X(t)$, immediate consequence of the facts proved above.

Theorem 5.27. *Let $(\mathbb{G},*)$ be a Lie group with algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$. Then:*

(i) $\exp_X(r+s) = \exp_X(r) * \exp_X(s)$ *for every $r,s \in \mathbb{R}$;*

(ii) $\exp_X(-t) = \left(\exp_X(t)\right)^{-1}$ *for every $t \in \mathbb{R}$;*

(iii) $\exp_X(0) = e$*;*

(iv) $\mathbb{R} \ni t \mapsto \exp_X(t) \in \mathbb{G}$ *is a smooth curve;*

(v) $\exp_X(t)$ *is the unique integral curve of X passing through the identity at time zero, so that, for every $x \in \mathbb{G}$,*

$$t \mapsto x * \left(\exp_X(t)\right)$$

is the unique integral curve of X passing through x at time zero.

We are ready to give the fundamental definition.

Definition 5.28 (Exponential map)**.** Let $(\mathbb{G},*)$ be a Lie group with Lie algebra $\mathfrak{g}$. Following the notation in Definition 5.26, we set

$$\mathrm{Exp}: \mathfrak{g} \longrightarrow \mathbb{G},$$
$$X \mapsto \mathrm{Exp}(X) := \exp_X(1).$$

Exp is called the exponential map (related to the Lie group $\mathbb{G}$).

The following results hold.

Proposition 5.29. *Let $(\mathbb{G},*)$ be a Lie group with Lie algebra $\mathfrak{g}$. For every $X \in \mathfrak{g}$, we have* (*i*) $\mathrm{Exp}(tX) = \exp_X(t)$ *for every* $t \in \mathbb{R}$*;*

(*ii*) $\mathrm{Exp}((r+s)X) = \mathrm{Exp}(rX) * \mathrm{Exp}(sX)$ *for every* $r, s \in \mathbb{R}$*;*

(*iii*) $\mathrm{Exp}(-tX) = (\mathrm{Exp}(tX))^{-1}$, *for every* $t \in \mathbb{R}$.

Theorem 5.30. *Let* $\mathbb{G}$ *and* $\mathbb{H}$ *be Lie groups with associated algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$*. We denote by* $\mathrm{Exp}_{\mathbb{G}}$ *and* $\mathrm{Exp}_{\mathbb{H}}$ *the exponential maps related to* $\mathbb{G}$ *and to* $\mathbb{H}$*, respectively. Finally, let* $\varphi : \mathbb{G} \longrightarrow \mathbb{H}$ *be a Lie group homomorphism. Then the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{G} \arrow[r, "\phi"] & \mathbb{H} \\ \mathfrak{g} \arrow[u, "\mathrm{Exp}_{\mathbb{G}}"] \arrow[r, "d\varphi"'] & \mathfrak{h} \arrow[u, "\mathrm{Exp}_{\mathbb{H}}"'] \end{tikzcd} .$$

5.3 Nilpotent Lie groups

In this section we discuss nilpotent Lie algebras and groups in the spirit of Folland and Stein's book [FS82] as well as introduce homogeneous (Lie) groups. For more analysis and details in this direction we refer to the recent open access books [FR16].

Definition 5.31 (Graded Lie algebras and groups)**.** A Lie algebra $\mathfrak{g}$ is called graded if it is endowed with a vector space decomposition (where all but finitely many of the V_k's are 0)

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j \quad \text{such that} \quad [V_i, V_j] \subset V_{i+j}.$$

Consequently, a Lie group is called graded if it is a connected and simply connected Lie group whose Lie algebra is graded.

The condition that the group is connected and simply connected is technical but important to ensure that the exponential mapping is a global diffeomorphism between the group and its Lie algebra.

Definition 5.32 (Stratified Lie algebras and groups)**.** A graded Lie algebra $\mathfrak{g}$ is called stratified if V_1 generates $\mathfrak{g}$ an algebra. In this case, if $\mathfrak{g}$ is nilpotent of step r we have

$$\mathfrak{g} = \oplus_{j=1}^{r} V_j, \quad [V_j, V_1] = V_{j+1}$$

and the natural dilations of $\mathfrak{g}$ are given by

$$\delta_\lambda \left(\sum_{k=1}^{r} X_k \right) = \sum_{k=1}^{r} \lambda^k X_k, \quad (X_k \in V_k)\,.$$

Consequently, a Lie group is called stratified if it is a connected simply-connected Lie group whose Lie algebra is stratified.

Definition 5.33 (Homogeneous groups)**.** Let δ_λ be dilations on $\mathbb{G}$. We say that a Lie group $\mathbb{G}$ is a homogeneous group if:

(a) It is a connected and simply-connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_\lambda\}$.

(b) The maps $\exp \circ \delta_\lambda \circ \exp^{-1}$ are group automorphism of $\mathbb{G}$.

Remark 5.34. The exponential mapping exp is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, it induces the corresponding family on $\mathbb{G}$ which we may still call the dilations on $\mathbb{G}$ and denote by δ_λ. Thus, for $x \in \mathbb{G}$ we will write $\delta_\lambda(x)$ or abbreviate it writing simply λx.

Lemma 5.35. *Graded Lie algebras are naturally equipped with dilations. If a Lie algebra $\mathfrak{g}$ has a family of dilations such that the weights are all rational, then $\mathfrak{g}$ has a natural gradation.*

Proposition 5.36. *The following holds:*

(i) *A Lie algebra equipped with a family of dilations is nilpotent.*

(ii) *A homogeneous Lie group is a nilpotent Lie group.*

Remark 5.37. A gradation over a Lie algebra is not unique: the same Lie algebra may admit different gradations. For example, any vector space decomposition of $\mathbb{R}^n$ yields a graded structure on the group $(\mathbb{R}^n, +)$. More convincingly, we can decompose the 3 dimensional Heisenberg Lie algebra $\mathfrak{h}_1$ as

$$\mathfrak{h}_1 = \bigoplus_{j=1}^{3} V_j \quad \text{with} \quad V_1 = \mathbb{R}X_1, V_2 = \mathbb{R}Y_1, V_3 = \mathbb{R}T$$

This example can be easily generalised to find several gradations on the Heisenberg groups $\mathbb{H}_{n_o}, n_o = 2, 3, \ldots,$ which are not the classical ones. Another example would be

$$\mathfrak{h}_1 = \bigoplus_{j=1}^{8} V_j \quad \text{with} \quad V_3 = \mathbb{R}X_1, V_5 = \mathbb{R}Y_1, V_8 = \mathbb{R}T$$

and all the other $V_j = \{0\}$.

Remark 5.38. A gradation may not even exist. The first obstruction is that the existence of a gradation implies nilpotency; in other words, a graded Lie group or a graded Lie algebra are nilpotent. Even then, a gradation of a nilpotent Lie algebra may not exist. As a curiosity, let us mention that the (dimensionally) lowest nilpotent Lie algebra which is not graded is the seven dimensional Lie algebra given by the following commutator relations:

$$[X_1, X_j] = X_{j+1} \text{ for } j = 2, \ldots, 6,$$
$$[X_2, X_3] = X_6,$$
$$[X_2, X_4] = [X_5, X_2] = [X_3, X_4] = X_7.$$

They define a seven dimensional nilpotent Lie algebra of step 6 (with basis $\{X_1, \ldots, X_7\}$). It is the (dimensionally) lowest nilpotent Lie algebra which is not graded.

If $\mathbb{H}$ is a stratified Lie group, its Lie algebra admits at least a stratification, but it can also have more than one. For example, if $\mathbb{H} = \mathbb{H}^1$ is the Heisenberg group on $\mathbb{R}^3$, its Lie algebra admits the stratifications

$$\operatorname{span}\{X_1, X_2\} \oplus \operatorname{span}\{[X_1, X_2]\},$$
$$\operatorname{span}\{X_1 - 3[X_1, X_2], X_2\} \oplus \operatorname{span}\{[X_1, X_2]\},$$
$$\operatorname{span}\{X_1 + X_2, 3X_1 + [X_1, X_2]\} \oplus \operatorname{span}\{[X_1, X_2]\}.$$

Definition 5.39 (Basis adapted to the stratification)**.** Let $\mathbb{H}$ be a stratified Lie group. Let $\mathcal{V} = (V_1, \ldots, V_r)$ be a fixed stratification of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$. We say that a basis $\mathcal{B}$ of $\mathfrak{h}$ is adapted to $\mathcal{V}$ if

$$\mathcal{B} = \left(E_1^{(1)}, \ldots, E_{N_1}^{(1)}; \ldots; E_1^{(r)}, \ldots, E_{N_r}^{(r)}\right)$$

where, for $i = 1, \ldots, r$, we have $N_i := \dim V_i$, and

$$\left(E_1^{(i)}, \ldots, E_{N_i}^{(i)}\right) \text{ is a basis for } V_i.$$

Obviously, every stratified Lie group admits an adapted basis to any of its stratifications.

Proposition 5.40. *Let $\mathbb{H}$ be a stratified Lie group. Suppose that $(V_1, \ldots, V_r)$ and $\left(\widetilde{V}_1, \ldots, \widetilde{V}_{\widetilde{r}}\right)$ be any two stratifications of the algebra of $\mathbb{H}$. Then $r = \widetilde{r}$ and $\dim(V_i) = \dim\left(\widetilde{V}_i\right)$ for every $i = 1, \ldots, r$. Moreover, the algebra of $\mathbb{H}$ is a nilpotent Lie algebra of step r. Hence, the natural number*

$$Q := \sum_{i=1}^{r} i \dim(V_i)$$

depends only on the stratified nature of $\mathbb{H}$ and not on the particular stratification. Q is called the homogeneous dimension of $\mathbb{H}$.

Lemma 5.41 (The two-stratification lemma)**.** *Let $\mathbb{H}$ be a stratified Lie group with Lie algebra $\mathfrak{h}$. Suppose $V := (V_1, \ldots, V_r)$ and $W := (W_1, \ldots, W_r)$ are two stratifications of $\mathfrak{h}$.*

Then, for every couple of bases $\mathcal{V}$ and $\mathcal{W}$ of $\mathfrak{h}$ respectively adapted to the stratifications V and W, the transition matrix between the two bases is non-singular and has the block-triangular form

$$\begin{pmatrix} M^{(1)} & 0 & \cdots & 0 \\ \star & M^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \cdots & \star & M^{(r)} \end{pmatrix}$$

where, for every $i = 1, \dots, r$, the block $M^{(i)}$ is a$N_i \times N_i$ non-singular matrix (N_i being the common value of $\dim(V_i) = \dim(W_i)$).

The following proposition shows that "to be a stratified Lie group" is an invariant under isomorphism of Lie groups.

Proposition 5.42. *Let $\mathbb{H}$ be a stratified Lie group. Suppose $\mathbb{G}$ is a Lie group isomorphic to $\mathbb{H}$. Then $\mathbb{G}$ is a stratified Lie group too. Moreover, $\mathbb{H}$ and $\mathbb{G}$ have the same step, the same number of generators and even the dimensions of the layers of the relevant stratifications are preserved. Also, $\mathbb{H}$ and $\mathbb{G}$ have the same homogeneous dimension Q.*

More precisely, suppose $\varphi : \mathbb{H} \to \mathbb{G}$ is a Lie group isomorphism and that $(V_1, \dots, V_r)$ is a stratification of $\mathfrak{h}$, the algebra of $\mathbb{H}$. Then, if $\mathfrak{g}$ is the algebra of $\mathbb{G}$, a stratification for $\mathfrak{g}$ is given by $(\mathrm{d}\varphi(V_1), \dots, d\varphi(V_r))$, where $\mathrm{d}\varphi$ is the differential of φ which is an isomorphism of Lie algebras (and of vector spaces).

We recall the following result, which also gives the well known Campbell-Hausdorff formula.

Theorem 5.43. *Let $(\mathbb{H}, *)$ be a connected and simply connected Lie group. Suppose that the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$ is nilpotent. Then $\diamond$ defines a Lie group structure on $\mathfrak{h}$ and $\mathrm{Exp} : (\mathfrak{h}, \diamond) \to (\mathbb{H}, *)$ is a group-isomorphism. In particular, we have*

$$\mathrm{Exp}(X) * \mathrm{Exp}(Y) = \mathrm{Exp}(X \diamond Y) \quad \forall X, Y \in \mathfrak{h}.$$

Theorem 5.44 (The third fundamental theorem of Lie)**.** *Let $\mathfrak{h}$ be a finite-dimensional Lie algebra. Then there exists a connected and simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{h}$.*

Collecting the above two theorems, we obtain the following result.

Corollary 5.45. *Let* $\mathfrak{h}$ *be a finite-dimensional nilpotent Lie algebra. Then* $\diamond$ *defines a Lie group structure on* $\mathfrak{h}$*. Moreover, the Lie algebra associated to the Lie group* $(\mathfrak{h}, \diamond)$ *is isomorphic to the algebra* $\mathfrak{h}$*.*

5.4 Abstract and homogeneous stratified Lie groups

We now aim to prove that, up to isomorphism, the definitions of classical and homogeneous stratified Lie group are equivalent. To begin with, we prove the following simple fact:

Proposition 5.46 (Homogeneous $\Rightarrow$ stratified)**.** *A homogeneous stratified Lie group in Definition* 2.61 *is a stratified group of Definition* 2.62*.*

Proof. Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, \delta_\lambda\right)$ be a homogeneous Carnot group. Clearly, $\mathbb{G}$ is connected and simply connected. Let $\mathfrak{g}$ be the algebra of $\mathbb{G}$.

For $i = 1, \ldots, r$ and $j = 1, \ldots, N_i$, let $Z_j^{(i)}$ be the vector field of $\mathfrak{g}$ agreeing with $\partial/\partial x_j^{(i)}$ at the origin. We set

$$V_i := \operatorname{span}\left\{Z_1^{(i)}, \ldots, Z_{N_i}^{(i)}\right\}$$

Remark 2.72 proves that $(V_1, \ldots, V_r)$ is a stratification of $\mathfrak{g}$, as in Definition 2.62. This ends the proof. $\square$

Proposition 5.47 (Stratified $\overset{\text{isom.}}{\Longrightarrow}$ homogeneous)**.** *Let* $\mathbb{H}$ *be a stratified Lie group, according to Definition* 2.62*. Then there exists a homogeneous stratified Lie group* $\mathbb{H}^*$ *(according to our Definition* 2.61*) which is isomorphic to* $\mathbb{H}$*.*

We can choose as $\mathbb{H}^*$ *the Lie algebra* $\mathfrak{h}$ *of* $\mathbb{H}$ *(identified to* $\mathbb{R}^N$ *by a suitable choice of an adapted basis of* $\mathfrak{h}$*) equipped with the composition law* $\diamond$ *defined by the Campbell-Hausdorff operation. In this case, a group isomorphism from* $\mathbb{H}^*$ *to* $\mathbb{H}$ *is the exponential map*

$$\operatorname{Exp} : (\mathfrak{h}, \diamond) \to (\mathbb{H}, *).$$

Proof. Let $(\mathbb{H}, *)$ be as in Definition 2.62. Let $\mathfrak{h}$ be the algebra of $\mathbb{H}$. Let $\mathfrak{h} = V_1 \oplus \cdots \oplus V_r$ be a fixed stratification of $\mathfrak{h}$. By Proposition 5.40, $\mathfrak{h}$ is nilpotent of step r.

Then Theorem 5.43 yields that

$$\operatorname{Exp} : (\mathfrak{h}, \diamond) \to (\mathbb{H}, *) \text{ is a Lie-group isomorphism.}$$

We now prove that $(\mathfrak{h}, \diamond)$ is a homogeneous stratified Lie group according to Definition 2.61.

We fix a basis for $\mathfrak{h}$ adapted to its stratification: for $i = 1, \dots, r$, set $N_i := \dim V_i$, and let $\left(E_1^{(i)}, \dots, E_{N_i}^{(i)}\right)$ be a basis for V_i. Then consider the basis for $\mathfrak{h}$ given by

$$\mathcal{E} = \left(E_1^{(1)}, \dots, E_{N_1}^{(1)}; \dots; E_1^{(r)}, \dots, E_{N_r}^{(r)}\right).$$

By means of this basis, we fix a coordinate system on $\mathfrak{h}$, and we identify $\mathfrak{h}$ with $\mathbb{R}^N$, where $N := N_1 + \cdots + N_r$. More precisely, we consider the map

$$\pi_{\mathcal{E}} : \mathfrak{h} \to \mathbb{R}^N, \quad E \cdot \xi := \sum_{i=1}^{r} \sum_{j=1}^{N_i} \xi_j^{(i)} E_j^{(i)} \mapsto \left(\xi^{(1)}, \dots, \xi^{(r)}\right),$$

where $\xi^{(i)} = \left(\xi_1^{(i)}, \dots, \xi_{N_i}^{(i)}\right) \in \mathbb{R}^{N_i}$ for every $i = 1, \dots, r$. Next, we set

$$\Psi := \operatorname{Exp} \circ (\pi_{\mathcal{E}})^{-1} : \mathbb{R}^N \to \mathbb{H}, \quad \Psi(\xi) = (\operatorname{Exp}(E \cdot \xi)).$$

Notice that, more explicitly,

$$\Psi(\xi) = \operatorname{Exp}\left(\sum_{i=1}^{r} \sum_{j=1}^{N_i} \xi_j^{(i)} E_j^{(i)}\right) \quad \forall \xi \in \mathbb{R}^N.$$

Finally, we equip $\mathbb{R}^N$ with the composition law $\diamond_{\mathcal{E}}$ defined by

$$\xi \diamond_{\mathcal{E}} \eta := \Psi^{-1}(\Psi(\xi) * \Psi(\eta)), \quad \xi, \eta \in \mathbb{R}^N.$$

We define a family of dilations $\{\Delta_\lambda\}_{\lambda>0}$ on the Lie algebra $\mathfrak{h}$ as follows:

$$\Delta_\lambda : \mathfrak{h} \to \mathfrak{h}, \quad \Delta_\lambda\left(\sum_{i=1}^{r} X_i\right) := \sum_{i=1}^{r} \lambda^i X_i, \qquad \text{where } X_i \in V_i.$$

Obviously,

$$\Delta_\lambda \text{ is a vector-space automorphism of } \mathfrak{h}.$$

And Δ_λ turns into a family of dilations $\{\delta_\lambda\}_{\lambda>0}$ on $\mathbb{R}^N$ via Ψ by setting

$$\delta_\lambda := \pi_{\mathcal{E}} \circ \Delta_\lambda \circ \pi_{\mathcal{E}}^{-1}. \tag{5.6}$$

We claim that $\mathbb{H}^* := \left(\mathbb{R}^N, \diamond\mathcal{E}, \delta_\lambda\right)$ is a homogeneous stratified Lie group (of step r and N_1 generators) isomorphic to $(\mathbb{H}, *)$ via the Lie group isomorphism Ψ.

To prove the claim, we split the proof in steps.

Step 1. By the definition of $\diamond_{\mathcal{E}}$ and Ψ, we have

$$\Psi(\xi \diamond \mathcal{E}\eta) = \Psi(\xi) * \Psi(\eta) \quad \forall \xi, \eta \in \mathbb{R}^N,$$

which, in turn, is equivalent to

$$\pi^{-1}(\xi \diamond \mathcal{E}\eta) = \pi_{\mathcal{E}}^{-1}(\xi) \diamond \pi_{\mathcal{E}}^{-1}(\eta) \quad \forall \xi, \eta \in \mathbb{R}^N, \tag{5.7}$$

or, equivalently,

$$\pi_{\mathcal{E}}(X \diamond Y) = \pi_{\mathcal{E}}(X) \diamond_{\mathcal{E}} \pi_{\mathcal{E}}(Y) \quad \forall X, Y \in \mathfrak{h}. \tag{5.8}$$

Which means that

$$\left(\mathbb{R}^N, \diamond_{\mathcal{E}}\right), \quad (\mathfrak{h}, \diamond), \quad (\mathbb{H}, *)$$

are isomorphic Lie groups via the Lie-group isomorphisms

$$\left(\mathbb{R}^N, \diamond\mathcal{E}\right) \xrightarrow{\pi_{\mathcal{E}}^{-1}} (\mathfrak{h}, \diamond) \xrightarrow{\operatorname{Exp}} (\mathbb{H}, *).$$

In particular,

$$\Psi = \operatorname{Exp} \circ \pi_{\mathcal{E}}^{-1} : \left(\mathbb{R}^N, \diamond_{\mathcal{E}}\right) \to (\mathbb{H}, *) \text{ is a Lie-group isomorphism.} \tag{5.9}$$

Step 2. We now investigate the dilation δ_λ. The stratified notation

$$\mathfrak{h} \ni E \cdot \xi = \sum_{i=1}^{r} \sum_{j=1}^{N_i} \xi_j^{(i)} E_j^{(i)}$$

for an arbitrary vector of $\mathfrak{h}$ and the fact that

$$\pi_{\mathcal{E}}(E \cdot \xi) = \xi$$

suggests the notation

$$\mathbb{R}^N \ni \xi = \left(\xi^{(1)}, \ldots, \xi^{(r)}\right)$$

for the points in $\mathbb{R}^N$. We claim that, with the above notation, δ_λ introduced in (5.6) has the form

$$\delta_\lambda\left(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(r)}\right) = \left(\lambda\xi^{(1)}, \lambda^2\xi^{(2)}, \ldots, \lambda^r\xi^{(r)}\right).$$

Indeed,

$$\begin{aligned}
\delta_\lambda(\xi) &= \left(\pi_{\mathcal{E}} \circ \Delta_\lambda \circ \pi_{\mathcal{E}}^{-1}\right)(\xi) = \pi_{\mathcal{E}}\left(\Delta_\lambda(E \cdot \xi)\right) = \pi_{\mathcal{E}}\left(\Delta_\lambda\left(\sum_{i=1}^{r}\sum_{j=1}^{N_i} \xi_j^{(i)} E_j^{(i)}\right)\right) \\
&= \pi_{\mathcal{E}}\left(\left(\sum_{i=1}^{r}\sum_{j=1}^{N_i} \xi_j^{(i)} \Delta_\lambda\left(E_j^{(i)}\right)\right)\right) = \pi_{\mathcal{E}}\left(\left(\sum_{i=1}^{r}\sum_{j=1}^{N_i} \xi_j^{(i)} \lambda^i E_j^{(i)}\right)\right) \\
&= \pi_{\mathcal{E}}\left(E \cdot \left(\lambda\xi^{(1)}, \ldots, \lambda^r\xi^{(r)}\right)\right) = \left(\lambda\xi^{(1)}, \ldots, \lambda^r\xi^{(r)}\right).
\end{aligned}$$

Next, we proceed by showing that Δ_λ is an automorphism of the Lie-group $(\mathfrak{h}, \diamond)$, i.e.

$$\Delta_\lambda(X \diamond Y) = \Delta_\lambda(X) \diamond \Delta_\lambda(Y) \quad \forall X, Y \in \mathfrak{h}, \quad \forall \lambda > 0.$$

In fact, it is enough to prove that

$$\Delta_\lambda([X, Y]) = [\Delta_\lambda(X), \Delta_\lambda(Y)] \qquad \text{for every } X, Y \in \mathfrak{h}. \tag{5.10}$$

If $X=\sum_{i=1}^{r} X_i$ and $Y=\sum_{i=1}^{r} Y_i$, where $X_i, Y_i \in V_i$, we have $[X_i, Y_j] \in V_{i+j}$, whence

$$\begin{aligned}\Delta_\lambda([X,Y]) &= \sum_{i,j=1}^{r} \Delta_\lambda\left([X_i,Y_j]\right) = \sum_{i,j=1}^{r} \lambda^{i+j}\left[X_i,Y_j\right]\\ &= \sum_{i,j=1}^{r}\left[\lambda^i X_i, \lambda^j Y_j\right] = \sum_{i,j=1}^{r}\left[\Delta_\lambda\left(X_i\right), \Delta_\lambda\left(Y_j\right)\right] = \left[\Delta_\lambda(X), \Delta_\lambda(Y)\right].\end{aligned}$$

Now, a joint application of (5.7),(5.8) and (5.10) prove that δ_λ is a Lie-group automorphism of $\left(\mathbb{R}^N, \diamond\mathcal{E}\right)$, i.e.

$$\delta_\lambda\left(\xi \diamond_{\mathcal{E}} \eta\right) = \delta_\lambda(\xi) \diamond_{\mathcal{E}} \delta_\lambda(\eta) \quad \forall \xi, \eta \in \mathbb{R}^N, \quad \forall \lambda > 0.$$

Step 3. Thus, $\mathbb{H}^* := \left(\mathbb{R}^N, \diamond_{\mathcal{E}}, \delta_\lambda\right)$ is a homogeneous Lie group on $\mathbb{R}^N$. Let now $\mathfrak{h}^*$ be the Lie algebra of $\mathbb{H}^*$. Dealing with a Lie group on $\mathbb{R}^N$ (and the fixed Cartesian coordinates ξ 's on $\mathbb{R}^N$), the Jacobian basis related to the composition $\diamond_{\mathcal{E}}$ is well-posed. We denote by

$$\mathcal{Z} = \left(Z_1^{(1)}, \ldots, Z_{N_1}^{(1)}; \ldots; Z_1^{(r)}, \ldots, Z_{N_r}^{(r)}\right)$$

this Jacobian basis, i.e. $Z_k^{(i)}$ is the vector field in $\mathfrak{h}^*$ agreeing at the origin with $\partial/\partial\xi_k^{(i)}$. The proof is complete if we show that the Lie algebra generated by $Z_1, \ldots, Z_{N_1}$ coincides with the whole $\mathfrak{h}^*$.

To this end, we first observe that, thanks to (5.9), $d\Psi : \mathfrak{h}^* \to \mathfrak{h}$ is an algebra isomorphism. Furthermore, we have

$$\mathrm{d}\Psi = (\mathrm{d}\,\mathrm{Exp}) \circ \left(\mathrm{d}\left(\pi_{\mathcal{E}}^{-1}\right)\right).$$

Moreover, since $E_1^{(1)}, \ldots, E_{N_1}^{(1)}$ is a system of Lie-generators for $\mathfrak{h}$ (by the very definition of stratification!), it is enough to prove that

$$\mathrm{d}\Psi\left(Z_k^{(i)}\right) = E_k^{(i)} \quad \text{for every } i = 1, \ldots, r \text{ and every } k = 1, \ldots, N_i. \tag{5.11}$$

In order to prove (5.11), we recall that a left-invariant vector field is determined by its value at the identity. Hence, (5.11) will follow if we show that

$$\left(\mathrm{d}\Psi\left(Z_k^{(i)}\right)\right)_e = \left(E_k^{(i)}\right)_e.$$

For every $f \in C^\infty(\mathbb{H}, \mathbb{R})$, we have

$$\begin{aligned}
\left(\mathrm{d}\Psi\left(Z_k^{(i)}\right)\right)_e (f) &= \left(\mathrm{d}_0\Psi\left(Z_k^{(i)}\right)_0\right)(f) = \left(Z_k^{(i)}\right)_0 (f \circ \Psi) \\
&= \left(\partial/\partial\xi_k^{(i)}\right) |_{\xi=0}\, f\left(\mathrm{Exp}\left(\sum_{i=1}^{r}\sum_{j=1}^{N_i} \xi_j^{(i)} E_j^{(i)}\right)\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\mathrm{Exp}\left(tE_k^{(i)}\right)\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(\exp_{E_k^{(i)}}(t)\right) \\
&= \left(E_k^{(i)}\right)_e (f).
\end{aligned}$$

The proposition is thus completely proved. □

At last, we furnish some properties in the following proposition which collect several already proved facts.

Proposition 5.48. *Let $\mathbb{H}$ be a stratified Lie group with Lie algebra $\mathfrak{h}$ and exponential map $\mathrm{Exp}_{\mathbb{H}} : \mathfrak{h} \to \mathbb{H}$. Let also $\diamond$ be the Campbell-Hausdorff operation on $\mathfrak{h}$. Let $V_1 \oplus \cdots \oplus V_r$ be a stratification of $\mathfrak{h}$. Let $\mathcal{E}$ be any basis for $\mathfrak{h}$ adapted to the stratification. Set $N := \dim(\mathfrak{h})$, consider the map $\pi_{\mathcal{E}} : \mathfrak{h} \to \mathbb{R}^N$, where, for every $X \in \mathfrak{h}, \pi_{\mathcal{E}}(X)$ is the N-tuple of the coordinates of X w.r.t. $\mathcal{E}$.*

Then the binary operation on $\mathbb{R}^N$ defined by

$$x \diamond \mathcal{E} y = \pi_{\mathcal{E}}\left(\left(\pi_{\mathcal{E}}^{-1}(x)\right) \diamond \left(\pi_{\mathcal{E}}^{-1}(y)\right)\right) \quad \forall x, y \in \mathbb{R}^N$$

has the following properties:

(1) $\mathbb{G} := \left(\mathbb{R}^N, \diamond_{\mathcal{E}}\right)$ *is a Lie group on* $\mathbb{R}^N; \mathbb{G}$ *is isomorphic to* $\mathbb{H}$ *via the map* $\Psi = \mathrm{Exp}_{\mathbb{H}} \circ \pi_{\mathcal{E}}^{-1}$ *and to* $(\mathfrak{h}, \diamond)$ *via* $\pi_{\mathcal{E}}$*, whence* $(\mathbb{G}, \diamond_{\mathcal{E}})$ *and* $(\mathfrak{h}, \diamond)$ *are stratified Lie groups.*

(2) *Let* $\mathcal{Z} = \{Z_1, \ldots, Z_N\}$ *be the Jacobian basis related to* $\mathbb{G}$*; then, denoting the adapted basis by* $\mathcal{E} = \{E_1, \ldots, E_N\}$*, we have*

$$\mathrm{d}\Psi\left(Z_i\right) = E_i \quad \text{for every } i = 1, \ldots, N$$

or, equivalently,

$$Z_i(f \circ \Psi) \equiv E_i(f) \circ \Psi \text{ on } \mathbb{G}$$

for every $f \in C^\infty(\mathbb{H}, \mathbb{R})$*. Moreover, if* $\mathfrak{g}$ *is the algebra of* $\mathbb{G}$*, the exponential map* $\mathrm{Exp}_{\mathbb{G}} : \mathfrak{g} \to \mathbb{G}$ *is a linear map and it sends* Z_i *in the i-th element of the standard basis of* $\mathbb{G} \equiv \mathbb{R}^N$, *whence*

$$\mathrm{Exp}_{\mathbb{G}}\left((x_1,\dots,x_N)_{\mathcal{Z}}\right) = (x_1,\dots,x_N)$$

being $(x_1,\dots,x_N)_{\mathcal{Z}} = x_1 Z_1 + \cdots + x_N Z_N$.

(3) *The inversion on* $\mathbb{G}$ *is the Euclidean inversion* $-x$.

(4) *For every* $i \in \{1,\dots,N\}$, *we have*

$$(x \diamond_{\mathcal{E}} y)_i = x_i + y_i + \mathcal{R}_i(x,y),$$

where $\mathcal{R}_i(x,y)$ *is a polynomial function depending on the* x_k *and* y_k *with* $k < i$, *and* $\mathcal{R}_i(x,y)$ *can be written as a sum of polynomials each containing a factor of the following type*

$$x_h y_k - x_k y_h \qquad \text{with } h \neq k \text{ and } h,k < i.$$

(5) *Let* Δ_λ *be the linear map on* h *such that, for every* $i = 1,\dots,r$,

$$\Delta_\lambda(X) = \lambda^i X \qquad \text{whenever } X \in V_i.$$

Let $\delta_\lambda := \pi_{\mathcal{E}} \circ \Delta_\lambda \circ \pi_{\mathcal{E}}^{-1}$. *Then* $\left(\mathbb{R}^N, \diamond_{\mathcal{E}}, \delta_\lambda\right)$ *is a (homogeneous) stratified Lie group of the same step and number of generators as* $\mathbb{H}$.

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Index

Index of Symbols