Novikov-Shubin Invariants of Nilpotent Lie Groups

Relations to Random Walks and Fibre Bundles

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Chapter 0

Introduction

0.1 The Heat Equation

Novikov-Shubin invariants are homotopy invariants related to the Laplace operators on non-compact manifolds. They can be viewed as describing the heat decay on these manifolds. In this introduction, we briefly review the classical theory of heat diffusion on compact manifolds. Then, we will consider the same question on non-compact manifolds and discuss the problems that arise. This will lead us to the definition of Novikov-Shubin invariants.

0.1.1 Heat Diffusion on Compact Manifolds

Let us consider the following problem: Given a compact Riemannian manifold M and a function $g\colon M \to \mathbb{R}$, where g can be interpreted as a function assigning to $x \in M$ the amount of heat energy $g(x)$ at x at the current point of time. Can we find $f\colon M \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, such that $f(x,0) = g(x)$ is the current amount of heat energy at x and $f(x,t)$ describes the amount of heat energy at x after time t? This problem is well-studied in physics. The solution f can be described in terms of g, let us say $g \in \mathcal{C}^\infty(M)$, by a partial differential equation. This equation is called the heat equation[1]:

$$\text{(Heat Equation)} \quad \begin{cases} f(x,0) &= g(x) \\ \frac{\partial}{\partial t} f(x,t) &= -\Delta f_t(x). \end{cases}$$

Here, $f_t(x)$ denotes the function $x \mapsto f(x,t)$ and Δ is the Laplace operator of the Riemannian manifold M.[2] Starting here, many questions can be asked. The one most important to this thesis is the following one: Given some initial heat distribution g and the corresponding solution f to the heat equation, does the limit $f_\infty(x) = \lim_{t\to\infty} f(x,t)$ exist and how quickly is the convergence $f_t \to f_\infty$ (for example in the supremum norm $\|\cdot\|_\infty$)?
From the heat equation one sees directly that the limit f_∞ has to satisfy

$$-\Delta f_\infty(x) = \frac{\partial}{\partial t} f_\infty(x) = 0.$$

[1]It should be noted that if $\partial M \neq \emptyset$ one should demand extra conditions at the boundary. This will not be important later on and is therefore omitted here. The experienced reader can think about Dirichlet or Neumann boundary conditions. Otherwise, one can think about closed manifolds instead of compact manifolds.

[2]For example, if $M = \mathbb{R}^n$, then $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

So $\Delta f_\infty = 0$, meaning that f_∞ is an eigenfunction of Δ to the eigenvalue 0. These eigenfunctions are called harmonic functions. On closed connected manifolds, harmonic functions are constant. This agrees with the expectation that heat will spread out evenly over time. For compact manifolds with boundary, one needs to demand extra conditions on the boundary which might lead to a different space of harmonic functions.
To find how quickly f_t converges to such a harmonic function f_∞, we study the spectrum $\sigma(\Delta)$ of the Laplace operator Δ. It is a self-adjoint non-negative operator, so its spectrum is contained in $\mathbb{R}_{\geq 0} \subset \mathbb{C}$. On compact manifolds, the spectrum is discrete. If the manifold is additionally connected, we always have $0 \in \sigma(\Delta)$ with multiplicity one coming from the one-dimensional space of harmonic functions. Thus, the spectrum of Δ is given by a discrete subset

$$\sigma(\Delta) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\} \quad \subset \mathbb{R}_{\geq 0}.$$

Corresponding eigenfunctions $f_0, f_1, f_2, \dots$ can be chosen such that they form an orthonormal basis of $L^2(M)$. We can therefore write

$$g(x) = \sum_{i=0}^{\infty} C_i(g) f_i(x)$$

for some constant factors $C_i(g)$ depending on g but not on $x \in M$. Functional calculus then gives us the solution to the heat equation as

$$\begin{aligned} f(x,t) = e^{-t\Delta} g(x) &= \sum_{i=0}^{\infty} C_i(g) e^{-\lambda_i t} f_i(x) \\ &= C_0(g) f_0(x) + \sum_{i=1}^{\infty} C_i(g) e^{-\lambda_i t} f_i(x) \\ &= C_0(g) f_0(x) + \mathcal{O}(e^{-\lambda_1 t}) \qquad \text{as } t \to \infty. \end{aligned}$$

From this, we see that the limit $f_\infty(x)$ is given by the harmonic function

$$f_\infty(x) = C_0(g) f_0(x)$$

and $f_t(x) \to f_\infty(x)$ as quickly as $e^{-\lambda_1 t} \to 0$ for $t \to \infty$. Notice that the smallest non-zero eigenvalue λ_1 of Δ plays a crucial role. The larger λ_1 is, the quicker is the convergence.
There are many results connecting λ_1 to other areas of mathematics. A classical result is an inequality given by J. Cheeger [Che70]. This result relates λ_1 for a closed Riemannian manifold M to a geometric quantity known as the Cheeger isoperimetric constant,

$$h(M) = \inf_{N \subset M} \left\{ \frac{\operatorname{area}(N)}{\min\{\operatorname{vol}(M_1), \operatorname{vol}(M_2)\}} \right\},$$

where the infimum runs over all submanifolds $N \subset M$ of codimension one that separate M into two pieces, $M = M_1 \cup M_2$, with $M_1 \cap M_2 = N$. Informally speaking, this measures how easily M can be cut into two pieces of roughly the same size. Indeed, $h(M)$ becomes small, if there is a small submanifold $N \subset M$ and M_1 and M_2 are roughly of the same size such that the minimum is as large as possible. This quantity relates to λ_1 by the following theorem.

Theorem 0.1 (Cheeger's Inequality, 1970). *Let M be a closed Riemannian manifold, then*

$$\lambda_1 \geq \frac{1}{4} h(M)^2.$$

An upper bound for λ_1 in terms of $h(M)$ cannot be found in general, as P. Buser showed in [Bus78]. However, if there is a lower bound on the Ricci curvature of M one can give an upper bound known as Buser's inequality [Bus82]. In this sense, the geometry of M can give bounds on the speed of heat diffusion and vice versa. Another result by S. Kakutani [Kak45] relates the heat equation to Brownian motions. In this setting, λ_1 is linked to the mixing time associated to this Brownian motion.

0.1.2 Heat Decay on Non-Compact Manifolds

If we consider a non-compact complete Riemannian manifold M instead, we can still study the heat equation as formulated above, for example for $g \in L^2(M)$.
However, while $\sigma(\Delta) \subset \mathbb{R}_{\geq 0}$ still holds true, the spectrum now contains contributions from the continuous spectrum. For example, in the case of the flat space $\mathbb{R}^n$ with $n \geq 1$ we obtain $\sigma(\Delta_{\mathbb{R}^n}) = \mathbb{R}_{>0}$. This makes it harder to find the speed of convergence of $f_t \to f_\infty$. As before, we ignore $0 \in \sigma(\Delta)$ if it appears, as it corresponds to the harmonic functions and does not impact the rate of convergence. Now, two things can happen.

1. It is possible that there is a spectral gap $\lambda_1 > 0$ such that $(0, \lambda_1) \cap \sigma(\Delta) = \emptyset$. Here, as in the compact case, $f_t \to f_\infty$ at least as quickly as $e^{-t\lambda_1} \to 0$ for $t \to \infty$.

2. However, it can also happen that $(0, \varepsilon) \cap \sigma(\Delta) \neq \emptyset$ for all $\varepsilon > 0$, so there are arbitrarily small non-zero values in $\sigma(\Delta)$. Here, it is not clear what λ_1 is supposed to be. Indeed, the speed of convergence might no longer be exponential but potentially polynomial.

The Novikov-Shubin invariant $\alpha_0^\Delta(M)$ was introduced to deal with the second case. It requires a nice symmetry of the space M, that is, a group G coming with a cocompact free proper group action $G \curvearrowright M$. It measures how dense the spectrum is in intervals $(0, \lambda)$ for $\lambda \searrow 0$, and thereby whether the heat decays exponentially, in which case $\alpha_0^\Delta(M) = \infty$, or polynomially. In the latter case, $\alpha_0(M) = a$ if it decays as fast as t^{-a} for $t \to \infty$. The definition can be extended naturally to the higher Laplace operators $\Delta_k \curvearrowright L^2\Omega^k(M)$, giving rise to higher Novikov-Shubin invariants $\alpha_k^\Delta(M)$. It can be further generalised and made in terms of boundary maps d associated to M, for example the differential of the deRham cochain complex of M or the differential of the cellular chain complex (if M has a nice enough CW complex structure). The resulting invariants $\alpha_\bullet(M)$ agree for the various definitions and yield a more precise picture than the $\alpha_\bullet^\Delta(M)$. In particular, the latter can be retrieved from the former.
In general, these invariants are hard to compute. However, for $\alpha_0^\Delta = \alpha_0$, there is a complete answer known. As pointed out by J. Lott in [Lot92], a result of N. Th. Varopoulos [Var84] implies that the Novikov-Shubin invariant $\alpha_0(M)$ is finite if and only if the group G acting on M is of polynomial growth, in which case they agree with the growth rate of G. By M. Gromov [Gro81], the groups of polynomial growth are precisely the virtually nilpotent groups.

Theorem 0.2. *Let X be a free proper cocompact G-CW complex of finite type. Then the Novikov-Shubin invariant $\alpha_0(X)$ is finite if and only if G is virtually nilpotent. In this case,*

$$\alpha_0(X) = N(G),$$

where $N(G)$ denotes the growth rate G.

As the growth rate is easily computed, for example using the Bass-Guivarc'h formula, this gives not only a nice connection to geometric group theory but also a simple formula for α_0. While higher Novikov-Shubin invariants have been computed in some cases, for example for 3-dimensional manifolds by J. Lott and W. Lück [LL95], such results generally rely on explicit computations rather than general observations.

0.2 Summary of Results

Relations between Novikov-Shubin invariants and random walks. The relationship between Novikov-Shubin invariants and random walks is well-understood in degree zero for free G-CW complexes of finite type. For such a free G-CW complex X, the Novikov-Shubin invariant $\alpha_0(X)$ appears in the study of the random walk on the Cayley graph Cayley(G). Indeed, $\alpha_0(X) = 2a$ if and only if the return probability $p(n)$ of this random walk decays asymptotically like the polynomial n^{-a} for $n \to \infty$. In Chapter 3 we generalise this relation to higher Novikov-Shubin invariants $\alpha_k(X)$. To achieve this, we construct a random walk for each $0 \le k \le \dim X$. For fixed k, the state space of this random walk is given by two copies of each k-cell of X — one for each possible orientation on that k-cell — as well as one auxiliary state. We explicitly give the propagation operator P of this random walk in terms of moving probabilities of a random walker. These probabilities depend on the local structure of X. In particular the glueing maps attaching $(k+1)$-cells to the k-skeleton and their incidence numbers play a central role. For each parameter $q \in [0,1]$, there is an associated q-lazy random walk with propagation operator $P_q = q\,\mathrm{Id} + (1-q)P$. These random walks naturally induce operators B_q acting on $\ell^2 C_k^{\mathrm{cell}}(X)$, the cellular L^2-chain complex of X. We prove the following relation between B_q and the cellular upper Laplacian $\Delta_k^{\mathrm{up}} = d_{k+1} d_{k+1}^* \curvearrowright \ell^2 C_k^{\mathrm{cell}}(X)$:

Theorem (3.14)**.** *There are multiplication operators $M_{1,q}, M_{2,q} \ge 0$ on $\ell^2 C_k^{\mathrm{cell}}(X)$ such that*

$$B_q \circ M_{1,q} = \mathrm{Id} - \Delta_k^{\mathrm{up}} \circ M_{2,q}.$$

The operators $M_{1,q}$ and $M_{2,q}$ are given explicitly in terms of the local structure of X and the attaching maps of $(k+1)$-cells. If the $(k+1)$-skeleton $X^{(k+1)}$ of X is regular enough (implying that $M_{1,q}$ and $M_{2,q}$ are given by multiplication with constants), we use this equation to relate the spectrum of Δ_k^{up} to the behaviour of the random walk. The important property of the random walk here is not the return probability on its own but the difference between two probabilities: The return probability and the probability of starting at some oriented k-cell α and returning to the same cell α but with reversed orientation. If we denote the return probability for the q-lazy random walk starting at the (arbitrarily oriented) k-cell α after n steps by $p_{q,\alpha,+}(n)$ and the probability of returning to the cell with reversed orientation after n steps by $p_{q,-,\alpha}(n)$ then we show that the quantity

$$p_q(n) = \sum_{\alpha \in G \backslash X} p_{q,\alpha,+}(n) - p_{q,\alpha,-}(n),$$

defined by summing these differences over all G-types of k-cells, relates directly to the L^2-Betti number $b^{(2)}(d_{k+1}^*)$ and the Novikov-Shubin invariant $\alpha_k(X)$ in the following way:

Theorem (3.26)**.** *Let X be an upper k-regular free G-CW complex of finite type and such that $M_{1,q} \equiv C_{1,q}$ and $M_{2,q}$ are constant. Let $q \in [q_0, 1)$, with q_0 given by Lemma 3.22. Then $\alpha_k(X) = 2a$ if and only if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$C_{1,q}^{-n}\left(b^{(2)}(d_{k+1}^*) + C^{-1} n^{-a}\right) \quad \le \quad p_q(n) \quad \le \quad C_{1,q}^{-n}\left(b^{(2)}(d_{k+1}^*) + C n^{-a}\right).$$

Estimates on Novikov-Shubin invariants of nilpotent Lie groups. In Chapter 4 we review M. Rumin's approach, and a slight simplification thereof, which allows to estimate Novikov-Shubin invariants of nilpotent Lie groups in some cases. We provide a description of such an algorithm as well as an implementation in Python in Appendix A. This yields a list of estimates of some Novikov-Shubin invariants for the 34 nilpotent Lie groups of dimension less or equal to six. These estimates are listed in Table 4.1.

Refinement of Novikov-Shubin invariants for fibre bundles. In Chapter 5 we study Novikov-Shubin invariants in the setting of fibre bundles. We introduce a generalisation of Novikov-Shubin invariants. This generalisation depends on two parameters, allowing for a more detailed analysis of how the fibre and the base contribute separately to the Novikov-Shubin invariant of the total space. This uses the fact that Novikov-Shubin invariants can be defined in terms of scaling the manifold by a factor λ and studying this situation as $\lambda \searrow 0$ (see Section 2.4.4). In the setting of fibre bundles, we use two parameters μ, ν in place of λ to scale the fibre and the base independently. The definition of this two-parameter generalisation in Definition 5.6 depends a priori on a fixed Riemannian metric and a fixed connection for the fibre bundle. This definition generalises the classical Novikov-Shubin invariants as these can be recovered by scaling the fibre and base at the same speed. We explicitly compute the two-parameter Novikov-Shubin numbers for the special case of the three dimensional Heisenberg group in all relevant degrees in Section 5.2 with the following result:

Theorem (5.9)**.** *The two-parameter Novikov-Shubin invariants of the Heisenberg group $\mathbb{H}^3$ and its associated Lie algebra $\mathfrak{h}^3$ with their standard structure obtained when scaling the base at speed λ and the fibre at speed $\lambda^{1+\zeta}$ as $\lambda \searrow 0$ are given by*

$$\begin{aligned}
\alpha_0(\mathfrak{h}_3)(\lambda, \lambda^{1+\zeta}) &= 4 + 2\zeta \quad \textit{for } -1/2 \leq \zeta, \\
\alpha_1(\mathfrak{h}_3)(\lambda, \lambda^{1+\zeta}) &= 2 - 2\zeta \quad \textit{for } -1/2 < \zeta < 1, \\
\alpha_2(\mathfrak{h}_3)(\lambda, \lambda^{1+\zeta}) &= 4 + 2\zeta \quad \textit{for } -1/2 \leq \zeta.
\end{aligned}$$

Indeed, for $\zeta = 0$, this recovers the Novikov-Shubin invariants $\alpha_\bullet(\mathbb{H}^3)$.
We then study the deRham complex of such fibre bundles more closely. We give an alternative definition of this generalisation in the spirit of the near cohomological definition of Novikov-Shubin invariants. This relies on a splitting of the deRham complex that we work out explicitly:

Theorem (5.17)**.** *Let $F_\bullet \to M \to B$ be a fibre bundle, then there is an isomorphism*

$$\Omega^k(M) \xrightarrow[\cong]{\Phi} \bigoplus_{p+q=k} \Omega^p(B, \{\Omega^q(F_b)\}_{b\in B}),$$

identifying forms on M with forms on B with values in the system of forms on the fibres $\{F_b\}_{b\in B}$.

We prove several invariance properties of the two-parameter Novikov-Shubin numbers. First, we show that for a fixed connection the numbers are invariant under change of compatible metrics:

Theorem (5.20)**.** *Let $G \curvearrowright (M \to B, \nabla, g)$ be a fibre bundle with fixed connection ∇ and compatible free proper cocompact group action by a group G. Then the dilatational equivalence class of the spectral density function underlying the two-parameter Novikov-Shubin numbers*

$$\mathcal{G}_k(M \to B, \nabla) = \mathcal{G}_k(M \to B, \nabla, g)$$

does not depend on the choice of G-invariant ∇-compatible Riemannian metric g.

Then, we prove that it is further invariant under certain compatible fibre homotopy equivalences:

Theorem (5.24)**.** *If there is a G-equivariant fibre homotopy equivalence between suitable bundles $M \to B$ and $M' \to B$ such that $\nabla = f^*\nabla'$, then their spectral density functions are dilatationally equivalent,*

$$\mathcal{G}_k(M' \to B, \nabla') \sim \mathcal{G}_k(M \to B, f^*\nabla').$$

Lastly, we show that the two-parameter Novikov-Shubin numbers are invariant under change of connection as long as the fibre is shrunk at least as fast as the base:

Theorem (5.25)**.** *Let G be a group and $M \to B$ be equipped with two pairs of connection and compatible Riemannian metric such that $G \curvearrowright (M \to B, \nabla, g)$ and $G \curvearrowright (M \to B, \nabla', g')$ are Riemannian fibre bundles with connection and compatible free proper cocompact G-action. Then the two-parameter spectral density functions restricted to the subspace $\{\nu \leq \mu\}$ are dilatationally equivalent,*

$$\mathcal{G}_k(M, \nabla, g)|_{\{\nu\leq\mu\}} \sim \mathcal{G}_k(M, \nabla', g')|_{\{\nu\leq\mu\}}.$$

0.3 Structure of the Thesis

This thesis is structured as follows.

1. In the first chapter we review basic concepts to fix the notation used in the later chapters.

2. In the second chapter we give a short introduction to L^2-invariants. In particular, we discuss Novikov-Shubin invariants and different approaches to defining and studying them.

3. In the third chapter we discuss the connection between Novikov-Shubin invariants on free G-CW complexes of finite type and stochastic processes taking place on these complexes. We extend the classic connection between the Novikov-Shubin invariant $\alpha_0(X)$ of such a complex X in degree zero and the return probability of a random walk on the 1-skeleton to higher degrees. To this end, we prove in Theorem 3.26 that the L^2-Betti numbers and the Novikov-Shubin invariants describe the asymptotic behaviour of a difference of (return) probabilities of a random walk. This random walk takes place on the oriented k-cells of X and induces an operator acting on the cellular k-chains. We show in Theorem 3.14 that this induced operator can be described in terms of the Laplace operator of X and two multiplication operators capturing local glueing information of X.

4. In the forth chapter we review an approach of M. Rumin to estimating Novikov-Shubin invariants on graded nilpotent Lie groups in some cases. With a slight modification this approach can be reduced to methods of linear algebra and hence implemented as a computer program. We present a list of estimates found for low-dimensional such Lie groups in Figure 4.1, discuss some examples explicitly and give some remarks on this approach.

5. In the fifth chapter we introduce a refined version of Novikov-Shubin invariants on fibre bundles. While classical Novikov-Shubin invariants are defined in terms of one parameter λ going to zero, we use two parameters μ, ν in its place. The goal is to capture contributions of the fibre and the base of the fibre bundle to the Novikov-Shubin invariants separately. These two-parameter Novikov-Shubin numbers are defined in Definition 5.6 in terms of a fibre bundle with fixed Riemannian metric and fixed connection. We compute these two-parameter invariants in Example 5.2 for the three dimensional Heisenberg group $\mathbb{H}^3$. We show that they can be interpreted in terms of near cohomology cones. In Theorem 5.20 we show that for a fixed connection these two-parameter Novikov-Shubin numbers are independent of the (compatible) Riemannian metric. In Theorem 5.24 we show that they are invariant under certain fibre homotopies over the identity if the connections are related by pullback along the homotopy. Lastly, we show in Theorem 5.25 that the two-parameter Novikov-Shubin invariants are metric and connection invariant if we scale the fibre at least as fast as the base.

Chapter 1

Prerequisites and Notation

In this chapter we very briefly review some of the necessary prerequisites. We assume that the reader is already familiar with the topic and the main focus is on fixing the notation for later chapters. The unfamiliar reader can find introductions to these topic in many textbooks, for example, the books of W. S. Massey [Mas91] or A. Hatcher [Hat02] for CW complexes, C. Löh [Löh17] for geometric group theory and W. Woess [Woe00] for random walks.

1.1 CW complexes

1.1.1 Definition and Notation

CW complexes are a nice class of topological spaces. A CW complex X is constructed as a sequence

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \cdots$$

of topological spaces $X^{(k)}$, called the k-skeleton of X, where $X^{(k+1)}$ is obtained from $X^{(k)}$ by glueing cells $\{e_\beta^{k+1}\}_{\beta \in I_{k+1}}$ with $e_\beta^{k+1} \simeq D^{k+1}$ to $X^{(k)}$ according to continuous attaching maps

$$\chi_\beta \colon \partial e_\beta^{k+1} \simeq S^k \to X^{(k)}.$$

This glueing process can be described by the following push-out diagram:

$$\begin{tikzcd}
\bigsqcup_{\beta \in I_{k+1}} S^k \arrow[r, "(\chi_\beta)_{\beta \in I_{k+1}}"] \arrow[d, hook] & X^{(k)} \arrow[d, hook] \\
\bigsqcup_{\beta \in I_{k+1}} D^{k+1} \arrow[r] & X^{(k+1)}
\end{tikzcd}$$

Such a sequence defines the topological space X as follows:

Definition 1.1. A CW complex X is given in terms a sequence $\{X^{(i)}\}_{i \geq -1}$ as above by the space $X = \bigcup_{i \geq -1} X^{(i)}$ equipped with the corresponding weak topology.

In this thesis, we use $I_k = I_k(X)$ to denote the set of indices of k-cells. If the situation is clear from the context, we use the index $\alpha \in I_k$ to also refer to the corresponding k-cell e_α^k. Generally, we use the letter α to refer to k-cells and the letter β to refer to $(k+1)$-cells.

Note that each cell is homeomorphic to D^k and can be equipped with one out of two possible orientations. For each cell $c \in I_\bullet$ we (arbitrarily[1]) fix one of the two orientations and denote by c_+ the cell c equipped with this preferred orientation. We denote by $c_- = -c_+$ the cell c equipped with the opposite orientation.
The glueing information coming from the attaching maps can be understood by studying for each pair of $(k+1)$-cell $\beta \in I_{k+1}$ and k-cell $\alpha \in I_k$ how β is attached to α. That is, by studying the composition $\chi_{\beta,\alpha}$,

$$\begin{array}{ccccccc} \partial\beta_+ & \xrightarrow{\chi_\beta} & X^{(k)} & \twoheadrightarrow & \frac{X^{(k)}}{X^{(k)}\setminus\{\alpha\}} & \xrightarrow{\simeq} & \frac{\alpha_+}{\partial\alpha_+} \\ \simeq & & & & & & \simeq \\ S^k & & & \xrightarrow{\chi_{\beta,\alpha}} & & & S^k. \end{array}$$

Recall that maps $\chi\colon S^k \to S^k$ are classified up to homotopy by their mapping degree $\deg(\chi) \in \mathbb{Z}$. We use this for the following definition.

Definition 1.2. For $\beta \in I_{k+1}$ and $\alpha \in I_k$ we define the incidence number $[\beta : \alpha] \in \mathbb{Z}$ by

$$[\beta : \alpha] = [\beta_+ : \alpha_+] = \deg(\chi_{\beta,\alpha}) \in \mathbb{Z}.$$

Notice that the sign of the incidence number depends on the chosen preferred orientations on β and α. Changing one orientation flips the sign. The size $|[\beta : \alpha]|$ is independent of the chosen orientations.

Definition 1.3. A CW complex X is called locally finite if every point $x \in X$ is contained in finitely many cells $c \in I_\bullet$.

In the following, all CW complexes will be locally finite. This condition implies that for all $\alpha \in I_k$ there are only finitely many $\beta \in I_{k+1}$ such that $[\beta : \alpha] \neq 0$, and conversely, for all $\beta \in I_{k+1}$ there are only finitely many $\alpha \in I_k$ such that $[\beta : \alpha] \neq 0$.

1.1.2 Cellular Chain Complex

Given a locally finite CW complex X, its cellular chain complex $C_\bullet^{\mathrm{cell}}(X)$ is the chain complex given by objects

$$C_k^{\mathrm{cell}}(X) = \bigoplus_{\alpha \in I_k} \mathbb{C} \cdot \alpha = \left\{ \sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha \;\middle|\; \lambda_\alpha \in \mathbb{C},\ \lambda_\alpha = 0 \text{ for almost all } \alpha \in I_k \right\}$$

together with the boundary maps $d_\bullet \colon C_\bullet^{\mathrm{cell}}(X) \to C_{\bullet-1}^{\mathrm{cell}}(X)$ given by

$$d_{k+1}\left(\sum_{\beta \in I_{k+1}} \lambda_\beta \cdot \beta \right) = \sum_{\alpha \in I_k} \left[\sum_{\beta \in I_{k+1}} [\beta : \alpha] \cdot \lambda_\beta \right] \cdot \alpha.$$

The space $C_k^{\mathrm{cell}}(X)$ is endowed with the scalar product given by

$$\left\langle \sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha, \sum_{\alpha \in I_k} \mu_\alpha \cdot \alpha \right\rangle = \sum_{\alpha \in I_k} \lambda_\alpha \overline{\mu_\alpha},$$

[1] This might involve the Axiom of Choice if there are infinitely many cells.

defining also a norm $\|\cdot\|$ on $C_k^{\mathrm{cell}}(X)$. With respect to $\langle\cdot,\cdot\rangle$, the adjoints d^* of d are given by the maps $d_\bullet^*\colon C_{\bullet-1}^{\mathrm{cell}}(X) \to C_\bullet^{\mathrm{cell}}(X)$,

$$d_{k+1}^*\left(\sum_{\alpha\in I_k} \lambda_\alpha\cdot\alpha\right) = \sum_{\beta\in I_{k+1}}\left[\sum_{\alpha\in I_k}[\beta:\alpha]\cdot\lambda_\alpha\right]\cdot\beta.$$

For $0\le k\le \dim(X)$, we define the Laplace operators $\Delta_k = d_{k+1}d_{k+1}^* + d_k^* d_k$ acting on $C_k^{\mathrm{cell}}(X)$. We may omit the indeces if they are clear from the context. We also define the upper Laplacians $\Delta_k^{\mathrm{up}} = d_{k+1}d_{k+1}^*$ acting on the cellular chain complex, compare the following diagram:

$$\begin{tikzcd}
 & C_{k+1}^{\mathrm{cell}}(X) \arrow[dr, "d_{k+1}"] & \\
C_k^{\mathrm{cell}}(X) \arrow[ur, "d_{k+1}^*"] \arrow[rr, "\Delta"] \arrow[rr, dashed, bend left, "\Delta_k^{\mathrm{up}}"] \arrow[dr, "d_k"'] & & C_k^{\mathrm{cell}}(X) \\
 & C_{k-1}^{\mathrm{cell}}(X) \arrow[ur, "d_k^*"'] &
\end{tikzcd}$$

Computing Δ_k^{up} explicitly yields the following formula:

$$\begin{aligned}
\Delta_k^{\mathrm{up}}\left(\sum_{\alpha\in I_k}\lambda_\alpha\cdot\alpha\right) &= \sum_{\alpha\in I_k}\left[\sum_{\beta\in I_{k+1}}[\beta:\alpha]^2\lambda_\alpha + \sum_{\alpha\neq\alpha'\in I_k}\sum_{\beta\in I_{k+1}}[\beta:\alpha][\beta:\alpha']\lambda_{\alpha'}\right]\cdot\alpha \\
&= \sum_{\alpha\in I_k}\left[\sum_{\beta\in I_{k+1}}[\beta:\alpha]^2\lambda_\alpha - \sum_{\alpha\neq\alpha'\in I_k}\sum_{\beta\in I_{k+1}} -[\beta:\alpha][\beta:\alpha']\lambda_{\alpha'}\right]\cdot\alpha.
\end{aligned} \tag{1.1}$$

Example 1.4. Let us take a closer look at the Laplace operator Δ_0 acting on $C_0^{\mathrm{cell}}(X)$. Since $d_0 = 0$, we have $\Delta_0 = \Delta_0^{\mathrm{up}}$. Since $\beta\in I_1$ is an interval attached to points $\alpha_1,\alpha_2\in I_0$, the incidence numbers $[\beta:\alpha]$ vanish for all $\alpha\notin\{\alpha_1,\alpha_2\}$. Further, on the points $\alpha_1\neq\alpha_2$ in I_0, "orientations" can be chosen such that for every β as above, $[\beta:\alpha_1] = -[\beta:\alpha_2] = \pm1$. In this setup, β can be viewed as an oriented edge starting at the α_i with negative incidence number $[\beta:\alpha_i] = -1$ and ending at the α_i with positive incidence number $[\beta:\alpha_i] = 1$. This leads to the following formula for $\Delta_0 = \Delta_0^{\mathrm{up}}$:

$$\begin{aligned}
\Delta_0^{\mathrm{up}}\left(\sum_{v\in I_0}\lambda_v\cdot v\right) &= \sum_{v\in I_0}\left[\sum_{e\in I_1}[e:v]^2\lambda_v + \sum_{v\neq v'\in I_0}\sum_{e\in I_1}[e:v][e:v']\lambda_{v'}\right]\cdot v \\
&= \sum_{v\in I_0}\left[|\{e\in I_1\mid v\in\partial e\}|\cdot\lambda_v - \sum_{v\neq v'\in I_0}|\{e\in I_1\mid \partial e=\{v,v'\}\}|\cdot\lambda_{v'}\right]\cdot v.
\end{aligned} \tag{1.2}$$

Definition 1.5. We define the cellular L^2-chain complex $\ell^2C_\bullet^{\mathrm{cell}}(X)$ as the objects

$$\ell^2C_k^{\mathrm{cell}}(X) = \left\{\sum_{\alpha\in I_k}\lambda_\alpha\cdot\alpha \;\middle|\; \lambda_\alpha\in\mathbb{C},\ \sum_{\alpha\in I_k}|\lambda_\alpha|^2<\infty\right\}.$$

The differential d, the inner product together with the adjoints d^* and the Laplace operators Δ as well as Δ^{up} can all be viewed as operators acting on $\ell^2C_\bullet^{\mathrm{cell}}(X)$, formally in the same way as before, assuming that for every $0 \leq k \leq \dim(X)$, every cell $\alpha \in I_k$ only has finitely many non-zero incidence numbers $[\alpha : \gamma] \neq 0$ for $\gamma \in I_{k-1}$ and $[\beta : \alpha] \neq 0$ for $\beta \in I_{k+1}$.[2]

1.1.3 G-CW Complexes

In this thesis, we will only consider a restricted class of G-CW complexes and will therefore give an adapted definition.

Definition 1.6. Let G be a group and X a CW complex with left action $G \curvearrowright X$. Then X is called a free G-CW complex of finite type if the projection $X \twoheadrightarrow {}_G\backslash X$ is a regular covering and ${}_G\backslash X$ is a finite CW complex.

In particular, the finite CW complex ${}_G\backslash X$ comes with a CW structure such that $I_k({}_G\backslash X)$ is finite and contains exactly one k-cell of each G-type of k-cells in $G \curvearrowright I_k(X)$. In this case, we choose preferred orientations on the cells $I_k({}_G\backslash X)$ and lift them to $I_k(X)$ so that the action of G preserves the chosen orientation. In particular, the incidence numbers are invariant under the G-action, that is, $[\beta : \alpha] = [g.\beta : g.\alpha]$ for all $\alpha \in I_k(X)$, $\beta \in I_{k+1}(X)$ and $g \in G$.

1.2 Graphs

1.2.1 Definition and Notation

Definition 1.7. A graph $\mathcal{G} = (V, E, \varphi)$ consists of a set $V = \{v_i \mid i \in I_V\}$ of vertices, a set $E = \{e_j \mid j \in I_E\}$ of edges and a glueing function $\varphi\colon E \to \{U \subset V \mid |U| \in \{1, 2\}\}$ assigning to each edge $e \in E$ its set of end points $\varphi(e) = \{v_1, v_2\}$.
If $v_1 = v_2$, so $\varphi(e) = \{v_1\}$ contains only one end point, we call e a loop. If there are two edges $e_1, e_2 \in E$ with $\varphi(e_1) = \varphi(e_2)$, we call e_1 and e_2 multiedges of $\mathcal{G}$. If the meaning is clear from the context, we simplify notation by suppressing the φ in the notation and also write e for the set $\varphi(e)$.

Sometimes, it makes sense to view one of the end points v_1 as the starting point of $e = \{v_1, v_2\}$ and view e as a directed edge from v_1 to v_2. In this case, we write $e = (v_1, v_2)$. In particular, an edge $e = \{v_1, v_2\}$ may be interpreted as $e = (v_1, v_2)$ or $e = (v_2, v_1)$.

Definition 1.8. Let $\mathcal{G} = (V, E)$ be a graph. Two vertices $v_1, v_2 \in V$ are called neighbours if there is an edge $e = \{v_1, v_2\} \in E$. In this case we write $v_1 \overset{e}{\sim} v_2$ or simply $v_1 \sim v_2$.
A path w in $\mathcal{G}$ is a sequence

$$w = \left(v_0 \overset{e_1}{\sim} v_1 \overset{e_2}{\sim} \cdots \overset{e_n}{\sim} v_n \right)$$

of consecutive neighbours $v_0, \ldots, v_n \in V$ and edges $e_1, \ldots, e_n \in E$. We call w a path from v_0 to v_n. Further we call v_0 the starting point of w, v_n the end point of w and $n = |w|$ the length of w. For $0 \leq k \leq n$ we denote $w(k) = v_k$. A path is called a circle if $w(0) = w(|w|)$. To emphasize the starting point and end point of a path, for a path w from $v \in V$ to $v' \in V$ we may write $w = (v \to v')$ or $w = (v \xrightarrow{n} v')$ for such a path of length n.

Definition 1.9. A graph $\mathcal{G} = (V, E)$ is called connected if for all $v, v' \in V$ there exists a path $w = (v \to v')$ from v to v'.

[2] This will be assumed throughout this thesis and is always true in the case of free G-CW complexes of finite type, which we will see the next subsection and work with throughout this thesis. We will show that these operators are bounded later in Lemma 3.21.

Definition 1.10. Let $\mathcal{G} = (V, E)$ be a graph and $v \in V$ a vertex. We define the degree $\deg(v) \in \mathbb{N}_0$ of v to be the number of edges containing v (counting loops twice), that is

$$\deg(v) = |\{e \in E \mid \exists v \neq v' \in V\colon e = \{v, v'\}\}| + 2 \cdot |\{e \in E \mid e = \{v\}\}|.$$

A graph $\mathcal{G}$ is called d-regular if $\deg(v) = d$ holds for all $v \in V$. For $v, v' \in V$ we write

$$\deg(v, v') = \begin{cases} |\{e \in E \mid e = \{v, v'\}\}| & \text{if } v \neq v', \\ 2 \cdot |\{e \in E \mid e = \{v\}\}| & \text{if } v = v', \end{cases}$$

for the number of (multi-)edges between v and v' (counting loops twice).

Definition 1.11. Let $\mathcal{G} = (V, E)$ be a graph and let

$$\ell^2 V = \left\{ \sum_{v \in V} \lambda_v \cdot v \;\middle|\; \lambda_v \in \mathbb{C}, \sum_{v \in V} |\lambda_v|^2 < \infty \right\}.$$

We define the graph Laplacian $\Delta = \Delta_{\mathcal{G}} \curvearrowright \ell^2 V$ by

$$\begin{aligned} \Delta \left(\sum_{v \in V} \lambda_v \cdot v \right) &= \sum_{v \in V} \left[\sum_{e = \{v, v'\}} (\lambda_v - \lambda_{v'}) \right] \cdot v \\ &= \sum_{v \in V} \left[\deg(v) \lambda_v - \sum_{v \sim v'} \deg(v, v') \lambda_{v'} \right] \cdot v. \end{aligned} \tag{1.3}$$

Example 1.12. Let X be a CW complex. Then its 1-skeleton $X^{(1)}$ is a graph with vertices $V = I_0$ given by the 0-cells and edges $E = I_1$ given by the 1-cells.[3] In this case, the graph Laplacian agrees with the cellular Laplacian,

$$\Delta_{\mathcal{G}} = \Delta_0 \curvearrowright \ell^2 V = \ell^2 C_0^{\text{cell}}(X),$$

as can be seen directly by comparing Equation (1.2) and Equation (1.3).

1.2.2 Cayley Graphs and Growth Rates

A Cayley graph of a group G is a graph associated to the group G and one of its generating sets. It gives a geometric interpretation of the group G. This subsection is a very brief introduction to some concepts from geometric group theory that are needed later on in this thesis. More details and proofs to the statements can be found in C. Löh's book [Löh17].

Definition 1.13. Let G be a finitely generated group with generating set S. We call S symmetric if $S = S^{-1}$, that is, $s \in S$ implies $s^{-1} \in S$. We call S simple if $e \notin S$. We call a pair (G, S) a symmetrically simply finitely generated (in this section abbreviated to ssfg) group if G is a group with finite symmetric simple generating set S.

Definition 1.14. Let (G, S) be a ssfg group. We define the Cayley graph $\text{Cayley}(G, S)$ to be the graph $\mathcal{G} = (V, E)$ with vertices $V = G$ and exactly one edge $e = \{g, g'\}$ between $g, g' \in G$ if and only if there exists $s \in S$ such that $gs = g'$ holds.[4] If the generating set is clear from the context or if a result holds true for all ssfg generating sets we may omit it in the notation and simply write $\text{Cayley}(G)$, referring to (any) $\text{Cayley}(G, S)$.

[3] This graph possibly contains loops and multiedges.

[4] By symmetry of S this is equivalent to requiring $g's = g$ for some $s \in S$.

Remark 1.15. Note that Cayley(G,S) is a $|S|$-regular graph: Each vertex $g \in V$ has $|S|$ neighbours $\{gs \mid s \in S\}$. With $d = |S|$, the graph Laplacian $\Delta_{\mathrm{Cayley}(G,S)}$ is given by

$$\Delta_{\mathrm{Cayley}(G,S)}\left(\sum_{g\in G}\lambda_g \cdot g\right) = \sum_{g\in G}\left[d\lambda_g - \sum_{s\in S}\lambda_{gs}\right]\cdot g. \tag{1.4}$$

Definition 1.16. Let (G,S) be a ssfg group. We define the length $|g|$ of an element $g \in G$ to be the minimal number of generators $s \in S$ needed to write g as a product of these generators,

$$|g| = \min\left\{n \in \mathbb{N}_0 \mid \exists s_1,\ldots,s_n \in S\colon g = s_1\cdots s_n\right\}.$$

Using this, we define the word metric $d_{(G,S)}$ on G by

$$d_{(G,S)}(g,g') = |g^{-1}g'|.$$

We define the ball $B_{(G,S)}(n) = \{g \in G \mid |g| \leq n\}$ of radius $n \in \mathbb{N}_0$, and the growth function

$$\beta_{(G,S)}\colon \mathbb{N}_0 \to \mathbb{N}, \qquad \beta_{(G,S)}(n) = |B_{(G,S)}(n)|.$$

Definition 1.17. We call G a finitely generated group of polynomial growth if there is a finite symmetric simple generating set S such that

$$\beta_{(G,S)}(n) \sim n^{N(G,S)} \qquad \text{for } n \to \infty,$$

with some constant $N(G,S) \in \mathbb{R}_{\geq 0}$ called the growth rate of (G,S).

It turns out that the growth rate of (G,S) is independent of the finite symmetric simple generating set S, that is, for any other finite symmetric simple generating set S' we have $N(G,S) = N(G,S')$, compare [Löh17, Prop. 6.2.4]. Hence we call $N(G) = N(G,S)$ the growth rate of G if (G,S) is of polynomial growth for any finite symmetric simple generating set S.
By a result of M. Gromov [Gro81], the groups of polynomial growth are shown to be precisely the virtually nilpotent groups.

Definition 1.18. A group G is called nilpotent of step s if its lower central series $\{G_i\}_{i\in\mathbb{N}_0}$, defined iteratively by

$$G_0 = G \qquad \text{and} \qquad G_{i+1} = [G_i, G],$$

terminates in the sth step, that is, $G_s = \{e\}$ but $G_{s-1} \neq \{e\}$.
A group G is called virtually nilpotent if there is a nilpotent group G' such that G' is a subgroup of G and the index $[G : G']$ of G' in G is finite.

For nilpotent groups, H. Bass [Bas72] and Y. Guivarc'h [Gui73] have both shown independently that their growth rate can be computed by the following formula.

Theorem 1.19 (Bass-Guivarc'h formula)**.** *Let G be a nilpotent group. Then it is of polynomial growth with growth rate*

$$N(G) = \sum_{k\geq 1} k \cdot \mathrm{rk}\left(G_{k-1}/G_k\right) \quad \in \mathbb{N}.$$

Notice that this is always an integer.

Theorem 1.20 (Gromov, 1981)**.** *Let G be a finitely generated group. Then G is of polynomial growth if and only if G is virtually nilpotent.*

1.3 Random Walks

This section provides a very brief introduction to random walks in order to fix the notation used in this thesis. A detailed introduction to the topic can be found in W. Woess' book [Woe00].

1.3.1 Definition and Notation

Definition 1.21. A random walk $\mathfrak{R} = (\Omega, P)$ is a Markov chain $\{X_n \mid n \in \mathbb{N}_0\}$ of random variables taking values in the state space Ω and with transition probabilities given by the propagation operator $P = (P_{x',x})_{x,x' \in \Omega}$. That is, for every pair $x, x' \in \Omega$ there is a transition probability $P_{x',x} = \mathbb{P}(x \to x')$ such that

$$\mathbb{P}(X_{\bullet+1} = x' \mid X_\bullet = x) = \mathbb{P}(x \to x').$$

As a propagation operator, in particular $\sum_{x' \in \Omega} \mathbb{P}(x \to x') = 1$ for all $x \in \Omega$. If $X_0 = x_0$, we call x_0 the starting point. We write $\mathfrak{R}(x_0)$ for the random walk starting at x_0.

Definition 1.22. Given a graph $\mathcal{G} = (V, E)$, a (nearest neighbour) random walk $\mathfrak{R}_\mathcal{G} = (V, P)$ on $\mathcal{G}$ is given by the state space $\Omega = V$ and a propagation operator P such that

$$\mathbb{P}(v \to v') \neq 0 \quad \text{only if} \quad v \sim v',$$

that is only if there is an edge $e = (v, v') \in E$.

Definition 1.23. If $\mathcal{G}$ is a d-regular graph, then the uniform nearest neighbour random walk $\mathfrak{R}_\mathcal{G}$ is the nearest neighbour random walk on $\mathcal{G}$ satisfying $\mathbb{P}(v \to v') = {}^{d(v,v')}/_{d}$ for all $v \sim v'$ and $\mathbb{P}(v \to v') = 0$ for $v \not\sim v'$.[5]

In the following, if we talk about *the* random walk on a graph, we mean the uniform nearest neighbour random walk.
Suppose at the current step of the random walk the probability of our random walker being at $v \in V$ is given by $\lambda_v \in [0, 1]$. Then we can use the formal sum $\sum_{v \in V} \lambda_v \cdot v$ to describe the current state of the random walk. Using this, we extend the definition of the propagation operator to an operator acting on $\ell^2 V$ in the following way.

Definition 1.24. We view the propagation operator P as acting on $\ell^2 V$ by

$$P\left(\sum_{v \in V} \lambda_v \cdot v\right) = \sum_{v \in V}\left[\sum_{v' \in V} P_{v,v'} \lambda_{v'}\right] \cdot v = \sum_{v \in V}\left[\sum_{v' \in V} \mathbb{P}(v' \to v) \cdot \lambda_{v'}\right] \cdot v.$$

Note that we can model more than one step of the random walk at the same time by considering the powers of the operator P. The probability of moving from $v \in V$ to $v' \in V$ in exactly n steps is given by

$$\mathbb{P}(v \xrightarrow{n} v') = (P^n)_{v',v}.$$

We use this to define the following notion.

Definition 1.25. Let $\mathfrak{R}_\mathcal{G}(v_0)$ be a random walk on $\mathcal{G} = (V, E)$ with starting point $v_0 \in V$. We define the return probability (function) $p\colon \mathbb{N}_0 \to [0, 1]$ by

$$p(n) = \langle P^n(v_0), v_0 \rangle_{\ell^2 V} = \mathbb{P}(v_0 \xrightarrow{n} v_0).$$

[5] If $\mathcal{G}$ has no loops or multiedges, the moving probabilities are given by $\mathbb{P}(v \to v') = {}^{1}/_{d}$ for all $v \sim v'$.

We extend this to a function $p\colon \ell^2V \times \mathbb{N}_0 \to \mathbb{R}$ by

$$p(\omega, n) = p_\omega(n) = \langle P^n(\omega), \omega\rangle_{\ell^2V}$$

using the extension of P to an operator $P \curvearrowright \ell^2V$.

If $\mathcal{G}$ is a Cayley graph of a group G, the precise formula of $p(n)$ depends on the finite symmetric simple generating set S used in the construction of $\mathcal{G}$. However, the asymptotic behaviour of $p(n)$ as $n \to \infty$ is independent of the choice of such a generating set.

Example 1.26. Consider $G = (\mathbb{Z}, +)$ with the symmetric simple generating set $S = \{\pm 1\}$. The Cayley graph $\mathcal{G} = \mathrm{Cayley}(G, S)$ is given by the integers as vertices with edges between any two consecutive integers, forming the real line $\mathbb{R}$. The uniform nearest neighbour random walk $\mathfrak{R}(0)$ on $\mathcal{G}$ starting at $0 \in \mathbb{Z}$ is the random walk that moves at each step with probability $1/2$ to the left (from X_n to $X_{n+1} = X_n - 1$) and with probability $1/2$ to the right (from X_n to $X_{n+1} = X_n + 1$), compare Figure 1.1.

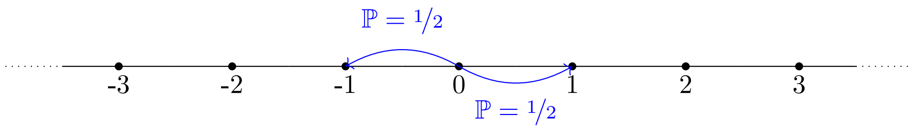


Figure 1.1: The Cayley graph of $\mathbb{Z}$

Here, the return probability $p_{\mathbb{Z}}(n) = \mathbb{P}(0 \xrightarrow{n} 0)$ can be easily computed. A random walker returning to the origin must have taken the same number of steps to the left as to the right. Hence, $p_{\mathbb{Z}}(2m+1) = 0$ for all odd natural numbers $n = 2m+1$ while for even $n = 2m$ we obtain

$$p_{\mathbb{Z}}(2m) = \binom{2m}{m}\left(\frac{1}{2}\right)^m\left(\frac{1}{2}\right)^m = \binom{2m}{m}\frac{1}{2^{2m}}.$$

Using Stirling's formula, this behaves asymptotically like

$$p_{\mathbb{Z}}(2m) \sim n^{-1/2} \qquad \text{as } m \to \infty.$$

Indeed, this asymptotic behaviour holds true for all symmetric generating sets of $\mathbb{Z}$ as we will see from the upcoming Theorem 1.27.
A simple trick can be used to see that for $G = \mathbb{Z}^2$ the asymptotic behaviour is given by

$$p_{\mathbb{Z}^2}(2m) \sim n^{-1} \qquad \text{as } m \to \infty.$$

For this, we consider the generating set $S = \{(0, \pm 1), (\pm 1, 0)\}$ of $\mathbb{Z}^2$. The Cayley graph $\mathrm{Cayley}(G, S)$ is then given by the integer points in $\mathbb{R}^2$ as vertices with edges between any vertex and the four direct neighbours. A random walker can therefore walk from each vertex to any of the four direct neighbours with probability $1/4$. If we rotate this picture by $\pi/4$ the possible moves of the random walker become diagonal moves with an up/down component and a left/right component, compare figure Figure 1.2 where the new components are drawn in red.
Computing the probabilities for the individual components, we note that they are independent random variables taking each of their two possible values with probability $1/2$ and therefore forming two random walks equivalent to the random walk on $\mathrm{Cayley}(\mathbb{Z}, \{\pm 1\})$. Since they are independent, it follows that

$$p_{\mathbb{Z}^2}(2m) = (p_{\mathbb{Z}}(2m))^2 \sim n^{-1} \qquad \text{as } m \to \infty.$$

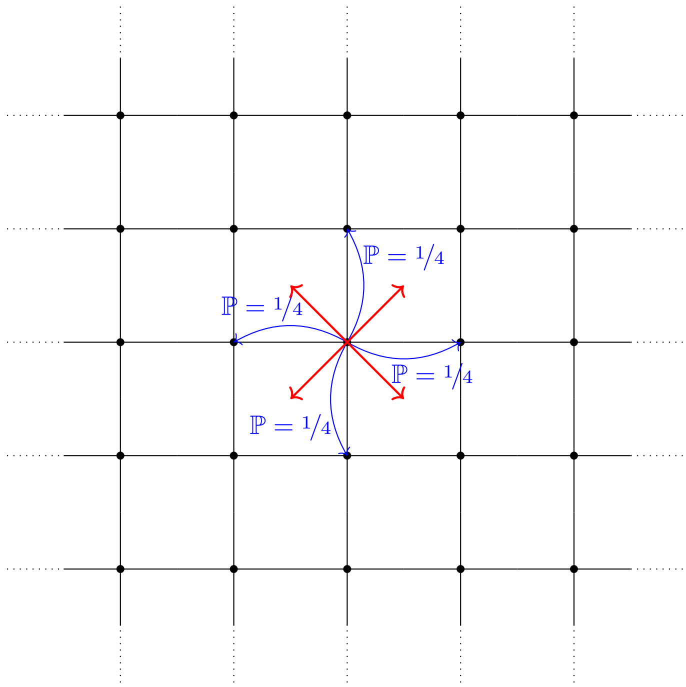


Figure 1.2: The Cayley graph of $\mathbb{Z}^2$

While this trick does not generalise to higher dimensions[6], for $N \geq 1$ one can find the generalised asymptotic behaviour of the return probability of the random walk on $\mathbb{Z}^N$ as

$$p_{\mathbb{Z}^N}(2m) \sim n^{-N/2} \qquad \text{as } m \to \infty.$$

The return probability for a group G is independent of the choice of generating set and can be given in terms of the growth rate of G, as the following theorem by N. Th. Varopoulos [Var84] shows.

Theorem 1.27 (Varopoulos, 1984)**.** *Let G be a finitely generated group. Then there is a constant $C > 0$ such that the return probability $p(n)$ of the random walk on G satisfies*

$$C^{-1} n^{-a} \leq p(n) \leq C n^{-a}$$

for all even $n \in \mathbb{N}$ if and only if the group G has polynomial growth precisely of degree $2a$.

While the constant C depends on the choice of generating set S, the exponent a does not. Notice that the restriction to even $n \in \mathbb{N}$ comes from the fact that it might be impossible to have circles of odd length in our Cayley graph, see Example 1.26. There is another way to avoid this problem, which we will consider in the following subsection.

1.3.2 Lazy Random Walks

Definition 1.28. Let $\mathfrak{R} = (\Omega, P)$ be a random walk and $q \in [0, 1]$. We define the q-lazy random walk $\mathfrak{R}_q$ associated to $\mathfrak{R}$ as the random walk $\mathfrak{R}_q = (\Omega, P_q)$ on Ω that stays put $(X_{n+1} = X_n)$ with

[6] $\mathbb{Z}^3$ with the corresponding generating set has a 6-regular Cayley graph while the graph obtained by adding one dimension to the Cayley graph of $\mathbb{Z}^2$ in the spirit of the trick would produce an 8-regular graph.

probability q and moves according to P with probability $1-q$. In other words, $P_q = q\,\mathrm{Id} + (1-q)P$ or explicitly for $v, v' \in \Omega$,

$$(P_q)_{v',v} = \mathbb{P}_q(v \to v') = q\delta_{v=v'} + (1-q)\mathbb{P}(v \to v') = \begin{cases} q + (1-q)\mathbb{P}(v \to v) & \text{if } v = v' \\ (1-q)\mathbb{P}(v \to v') & \text{if } v \neq v'. \end{cases}$$

For a random walk $\mathfrak{R} = \mathfrak{R}(\mathcal{G})$ on a graph $\mathcal{G} = (V, E)$, we denote the return probability of $\mathfrak{R}_q$ (with respect to a fixed starting vertex) in the same manner by $p_q(n)$. Indeed, for a random walk $\mathfrak{R}(v_0)$ with some starting vertex v_0 and $q \in [0, 1)$, the return probabilities $p(n)$ and $p_q(n)$ have the same asymptotic behaviour for $n \to \infty$. This can be seen since for large $n \in \mathbb{N}$ the q-lazy random walk is expected to stay put in qn moves and move $(1-q)n$ steps according to the non-lazy random walk. It follows that $p(n) \sim p_q((1-q)n)$ as $n \to \infty$. Since $1-q$ is constant, this implies the result about the asymptotic behaviour.
One can show that the return probability $p_q(n)$ of the q-lazy random walk $\mathfrak{R}_q = (\mathrm{Cayley}(G, S), P_q)$ on any Cayley graph for $q \geq 1/2$ is monotonously decreasing as n increases. This will become clearer in the following subsection. Using this, we can reformulate N. Th. Varopoulos' result by replacing $p(n)$ with $p_{1/2}(n)$ and dropping the condition that $n \in \mathbb{N}$ needs to be even.

1.3.3 Connection to the Laplacian

On Cayley graphs, the propagation operator of the random walk relates nicely to the graph Laplacian as can be easily computed. One finds the following relation:

Lemma 1.29. *Let G be finitely generated with finite symmetric simple generating set S. Then the propagation operator P of the random walk on* $\mathrm{Cayley}(G, S)$ *relates to the graph Laplacian Δ on* $\mathrm{Cayley}(G, S)$ *by the formula*

$$\mathrm{Id} - P = \frac{1}{|S|}\Delta \qquad \text{or equivalently} \qquad P = \mathrm{Id} - \frac{1}{|S|}\Delta \tag{1.5}$$

and more generally for $q \in [0, 1]$, the propagation operator P_q of the q-lazy random walk satisfies

$$\mathrm{Id} - P_q = \frac{1-q}{|S|}\Delta \qquad \text{or equivalently} \qquad P_q = \mathrm{Id} - \frac{1-q}{|S|}\Delta. \tag{1.6}$$

From this relation, we can translate properties of the graph Laplacian to properties of the random walk and vice versa. For example, we can relate the spectra of Δ and P_q for $q \in [0, 1)$.

Lemma 1.30. *Let (G, S) be finitely generated with finite symmetric simple generating set, let Δ be the graph Laplacian of $\mathcal{G} =$ $\mathrm{Cayley}(G, S)$, let $q \in [0, 1]$ and let P_q be the propagation operator of the q-lazy random walk on $\mathcal{G}$. Then the spectrum $\sigma(\Delta)$ is contained in the interval $[0, 2|S|]$ and $\sigma(P_q) \subset [-1 + 2q, 1]$. In particular, P_q is non-negative for $q \geq 1/2$ and positive for $q > 1/2$.*

Proof. It is shown, for example, by H. Kesten in [Kes59b, Lem. 2.2] that $\sigma(\Delta) \subset [0, 2|S|]$, see also [Kes59a, Lem. 1].[7] The claimed bounds of the spectra of the P_q follow directly from Equation (1.6).

[7] In the notation of the latter, $\Delta = |S| - A$, where A is the adjacency matrix, and it is shown in Lemma 1 that $\sigma(A/|S|) \subset [-1, 1]$. Alternatively, this also follows as a special case from Lemma 3.21 in this thesis.

Chapter 2

L^2-Invariants

In this chapter we review definitions and results underlying L^2-invariants. In particular, we briefly review the notions of group von Neumann algebras and modules, L^2-Betti numbers and spectral density functions. Finally, we discuss several (equivalent) definitions and approaches to Novikov-Shubin invariants. The definitions and results are mainly taken from W. Lück ([Lüc02] and [Lüc09]) and H. Kammeyer ([Kam14] and [Kam19]).
From now on, all groups are assumed to be finitely generated and all manifolds are assumed to be connected manifolds without boundary unless stated otherwise.

2.1 Group von Neumann Algebras and Modules

In this section we review the notions of group von Neumann algebras, modules over these algebras and their dimensions.
Let G be a group with neutral element $e \in G$. The set of square-summable formal complex linear combinations of G,

$$\ell^2 G = \left\{ \sum_{g\in G} \lambda_g \cdot g \;\middle|\; \lambda_g \in \mathbb{C},\ \sum_{g\in G} |\lambda_g|^2 < \infty \right\},$$

equipped with the scalar product

$$\left\langle \sum_{g\in G} \lambda_g \cdot g, \sum_{g\in G} \mu_g \cdot g \right\rangle = \sum_{g\in G} \lambda_g \overline{\mu_g}$$

is a Hilbert space. This Hilbert space contains G and in particular $e \in G \subset \ell^2 G$.

Definition 2.1. We define the group von Neumann algebra $\mathcal{N}G$ of G by

$$\mathcal{N}G = \mathcal{B}(\ell^2 G)^G,$$

the G-equivariant bounded operators on $\ell^2 G$. This forms a weakly closed $*$-subalgebra of $\mathcal{B}(\ell^2 G)$. The complex group ring $\mathbb{C}G$ can be embedded into this algebra by sending $g_0 \in G$ to right multiplication $r_{g_0^{-1}} \colon \ell^2 G \to \ell^2 G$ given on $g \in G$ by $r_{g_0^{-1}}(g) = g g_0^{-1}$ and extending linearly. On $\mathcal{N}G$, we can define the von Neumann trace $\operatorname{tr}_{\mathcal{N}G}$ by

$$\operatorname{tr}_{\mathcal{N}G} f = \langle f(e), e \rangle_{\ell^2 G}.$$

Definition 2.2. A Hilbert space V with isometric left G-action is called a Hilbert $\mathcal{N}G$-module if there exists a Hilbert space H and an isometric G-embedding $V \hookrightarrow H \otimes \ell^2 G$.

Definition 2.3. Let $\{b_i \mid i \in I\}$ be an orthonormal basis of H. For a positive endomorphism $f \curvearrowright H \otimes \ell^2 G$ we define the von Neumann trace

$$\operatorname{tr}_{\mathcal{N}G}(f) = \sum_{i \in I} \langle f(b_i \otimes e), b_i \otimes e \rangle \qquad \in [0, \infty].$$

We extend this definition to the Hilbert $\mathcal{N}G$-module V using the embedding into $H \otimes \ell^2 G$. Let $\pi\colon H \otimes \ell^2 G \to H \otimes \ell^2 G$ such that there is a G-isometric ismorphism $u\colon \operatorname{im} \pi \xrightarrow{\cong} V$. Then for any positive endomorphism $f \curvearrowright V$ we define the composition

$$\overline{f}\colon H \otimes \ell^2 G \xrightarrow{\pi} \operatorname{im} \pi \xrightarrow{u} V \xrightarrow{f} V \xrightarrow{u^{-1}} \operatorname{im} \pi \hookrightarrow H \otimes \ell^2 G$$

and define the von Neumann trace $\operatorname{tr}_{\mathcal{N}G}(f)$ of f by $\operatorname{tr}_{\mathcal{N}G}(f) = \operatorname{tr}_{\mathcal{N}G}(\overline{f})$.

This definition is independent on the choices made, compare [Lüc02, Def. 1.8], and defines a trace function satisfying several important properties listed in [Lüc02, Thm. 1.9].
Using this trace, we can define a notion of dimension of Hilbert $\mathcal{N}G$-modules as follows.

Definition 2.4. Let V be a Hilbert $\mathcal{N}G$-module. We define its von Neumann dimension by

$$\dim_{\mathcal{N}G}(V) = \operatorname{tr}_{\mathcal{N}G}(\mathrm{id}_V \colon V \to V) \qquad \in [0, \infty].$$

This defines a dimension function on Hilbert $\mathcal{N}G$-modules satisfying multiple important properties, see [Lüc02, Thm. 1.12]. Note that this dimension need not be an integer.

2.2 L^2-Betti Numbers

The L^2-Betti numbers can be defined combinatorially in terms of a CW structure or analytically in terms of a Riemannian metric. We briefly review both definitions, closely following W. Lück's book [Lüc02] but with adapted notation.

Combinatorial Definition.

Definition 2.5. Let X be a free G-CW complex of finite type and let $C_\bullet^{\mathrm{cell}}(X)$ be the cellular $\mathbb{Z}G$-chain complex of X. Then the cellular L^2-chain complex of X is given by

$$\ell^2 C_\bullet^{\mathrm{cell}}(X) = \ell^2 G \otimes_{\mathbb{Z}G} C_\bullet^{\mathrm{cell}}(X)$$

and the cellular L^2-cochain complex of X by

$$\ell^2 C^\bullet_{\mathrm{cell}}(X) = \hom_{\mathbb{Z}G}(C_\bullet^{\mathrm{cell}}(X), \ell^2(G)).$$

Here, fixing a cellular basis for $C_k^{\mathrm{cell}}(X)$ yields an explicit isomorphism

$$\ell^2 C_k^{\mathrm{cell}}(X) \cong \ell^2 C^k_{\mathrm{cell}}(X) \cong \bigoplus_{i \in I_k(G \backslash X)} \ell^2 G.$$

Definition 2.6. In the setting above we define the reduced (cellular) L^2-homology of X and its L^2-Betti numbers by

$$\begin{aligned} H_k^{(2)}(X;\mathcal{N}G) &= \ker(d_k)\big/\overline{\operatorname{im}(d_{k+1})}\ , \\ b_k^{(2)}(X;\mathcal{N}G) &= \dim_{\mathcal{N}G}(H_k^{(2)}(X;\mathcal{N}G)), \end{aligned}$$

where $d_\bullet$ denotes the differential in the L^2-chain complex. Similarly we define the reduced (cellular) L^2-cohomology of X and its L^2-Betti number by

$$\begin{aligned} H^k_{(2)}(X;\mathcal{N}G) &= \ker(\delta^k)\big/\overline{\operatorname{im}(\delta^{k-1})}\ , \\ b^k_{(2)}(X;\mathcal{N}G) &= \dim_{\mathcal{N}G}(H^k_{(2)}(X;\mathcal{N}G)), \end{aligned}$$

where $\delta^\bullet$ denotes the differential of the L^2-cochain complex.

Since $H_k^{(2)}(X;\mathcal{N}G)$ and $H^k_{(2)}(X;\mathcal{N}G)$ are isometrically G-isomorphic, the L^2-Betti numbers agree, that is,

$$b_k^{(2)}(X;\mathcal{N}G) = b^k_{(2)}(X;\mathcal{N}G),$$

compare [Lüc02, Rem. 1.31].

Remark 2.7. Notice that the definition of $H_k^{(2)}(X;\mathcal{N}G)$ differs from the classical definition of homology theories by considering $\ker(d)$ modulo the *closure* of the image $\operatorname{im}(d)$ as opposed to just the image by itself. This is important, because this ensures that $H_k^{(2)}(X;\mathcal{N}G)$ inherits the structure of a Hilbert space. However, this difference

$$\left(\ker(d_k)\big/\overline{\operatorname{im}(d_{k+1})}\right) \oplus \left(\overline{\operatorname{im}(d_{k+1})}\big/\operatorname{im}(d_{k+1})\right) \cong \ker(d_k)/\operatorname{im}(d_{k+1})$$

is very interesting and relates directly to the Novikov-Shubin invariants that we will study later.

Analytic Definition. The L^2-Betti numbers can also be viewed from an analytic point of view, giving rise to a second definition. Let M be a complete Riemannian manifold without boundary of dimension n and let $(\Omega^\bullet(M), d^\bullet)$ be its deRham cochain complex of smooth forms. Let $\Omega_c^\bullet(M)$ be the subcomplex of smooth forms with compact support. On each degree of $\Omega_c^\bullet(M)$ there is an inner product. For $\omega, \eta \in \Omega_c^k(M)$ this is given by

$$\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta,$$

where $*$ denotes the Hodge-$*$-operator $*^k\colon \Omega^k(M) \to \Omega^{n-k}(M)$. As usual, this also defines a norm by $\|\omega\|^2 = \langle \omega, \omega \rangle$. On $\Omega^\bullet(M)$ we define the adjoint $d^{k,*}\colon \Omega^{k+1}(M) \to \Omega^k(M)$ by

$$d^{k,*} = (-1)^{kn+n+1} \cdot \left(\Omega^{k+1}(M) \xrightarrow{*^{k+1}} \Omega^{n-k-1}(M) \xrightarrow{d^{n-k-1}} \Omega^{n-k}(M) \xrightarrow{*^{n-k}} \Omega^k\right).$$

Indeed, d and d^* are formally adjoint on $\Omega_c^\bullet(M)$ with respect to the inner product above. This defines Laplace operators Δ^k acting on $\Omega^k(M)$ by

$$\Delta^k = d^{k-1} \circ d^{k-1,*} + d^{k,*} \circ d^k.$$

We denote by $L^2\Omega^\bullet(M)$ the Hilbert space completion of $\Omega_c^\bullet(M)$. We call this the L^2-cochain complex of M and define the space of L^2-integrable harmonic smooth k-forms by

$$\mathcal{H}^k_{(2)}(M) = \left\{\omega \in \Omega^k(M) \,\middle|\, \Delta^k\omega = 0, \|\omega\|_{L^2} < \infty\right\}.$$

Using completeness of M, if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k+1}(M)$ and ω, $d\omega$, η and $d^*\eta$ are all square-integrable then

$$\langle d\omega, \eta\rangle = \langle \omega, d^*\eta\rangle.$$

Furthermore, we obtain a Hodge-deRham Theorem, compare [Lüc02, Thm. 1.57]:

Theorem 2.8 (L^2-Hodge-deRham Theorem)**.** *Let M be a complete Riemannian manifold without boundary. Then we obtain an orthogonal decomposition*

$$L^2\Omega^k(M) = \mathcal{H}^k_{(2)}(M) \oplus \overline{d(\Omega_c^{k-1}(M)))} \oplus \overline{d^*(\Omega_c^{k+1}(M)))}.$$

Moreover, if M is a cocompact free proper G-manifold with G-invariant Riemannian metric, it follows by a theorem of J. Dodziuk [Dod77] that $\mathcal{H}^k_{(2)}(M)$ is a finitely generated $\mathcal{N}G$-module. Hence we can define the L^2-Betti numbers

$$b^k_{(2)}(M) = \dim_{\mathcal{N}G}(\mathcal{H}^k_{(2)}(M)).$$

Remark 2.9. As W. Lück explains in more detail in [Lüc09, Sec. 2.3], originally the L^2-Betti numbers were defined by M. F. Atiyah [Ati76] in terms of heat kernels $e^{-t\Delta}$ and their asymptotic behaviour for large times $t \to \infty$. For a smooth Riemannian manifold M with cocompact free proper G-action by isometries its analytic kth L^2-Betti number is given by

$$b^k_{(2)}(M) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta^k}(x,x)) \,\mathrm{d}\operatorname{vol}_x,$$

where $\mathcal{F}$ is a fundamental domain of the G-action on M and $\operatorname{tr}_{\mathbb{C}}$ denotes the trace of an endomorphism of a finite-dimensional vector space.

Relation and Properties. As shown in W. Lück's book [Lüc02, Sec. 1.4], the combinatorial and analytic L^2-Betti numbers agree whenever both are defined. The L^2-Betti numbers satisfy many important properties, such as homotopy invariance, Poincaré duality and a Künneth formula, see [Lüc02, Thm. 1.35].

2.3 Spectral Density Functions

While L^2-Betti numbers measure the size of $\ker(d)/\overline{\operatorname{im}(d)}$, we now want to take a closer look at the difference $\overline{\operatorname{im}(d)}/\operatorname{im}(d)$ to the ordinary homology. For this purpose, we define spectral density functions. We consider the following setting:
Let U and V be Hilbert $\mathcal{N}G$-modules and let $f\colon \operatorname{dom}(f) \subset U \to V$ be a closed densely defined G-equivariant operator. Then $f^*f\colon \operatorname{dom}(f^*f) \subset U \to U$ is a self-adjoint operator and, by the spectral theorem, it defines a family $\{E_\lambda^{f^*f}\}_{\lambda\geq 0}$ of G-equivariant spectral projections.

Definition 2.10. In this setting, we define the spectral density function $F(f)$ of f by

$$F(f)\colon \mathbb{R}_{\geq 0} \to [0,\infty], \qquad F(f)(\lambda) = \dim_{\mathcal{N}G}\left(\operatorname{im} E_{\lambda^2}^{f^*f}\right) = \operatorname{tr}_{\mathcal{N}G}(E_{\lambda^2}^{f^*f}).$$

We call f Fredholm if there is $\lambda > 0$ such that $F(f)(\lambda) < \infty$.

Note that by definition, the function $F(f)$ is monotonously increasing and right-continuous.

Remark 2.11. This generalises the idea of counting eigenvalues. If f was a linear operator between finite-dimensional vector spaces, the eigenvalues of f^*f would correspond to the squares of eigenvalues of f. In this setting, the operator $E_\lambda^{f^*f}$ corresponds to the projection onto the direct sum of eigenspaces to eigenvalues less or equal to λ^2 of f^*f. In particular, taking its trace corresponds to counting (with algebraic multiplicity) eigenvalues $\leq \lambda^2$ of f^*f or equivalently eigenvalues of absolute value $\leq \lambda$ of f.

From now on, we will focus on Fredholm spectral density functions.

Definition 2.12. Let f be Fredholm. Then we define the L^2-Betti number of f by

$$b^{(2)}(f) = F(f)(0) = \dim_{\mathcal{N}G}(\ker f) \qquad \in [0, \infty),$$

measuring the dimension of the kernel of f.

This generalises the definition of L^2-Betti numbers in the sense that $b_k^{(2)}(X) = b^{(2)}(\Delta_k)$.
On spectral density functions, an important equivalence relation is given their behaviour near zero in terms of the following definition.

Definition 2.13. Let $F, G\colon \mathbb{R}_{\geq 0} \to [0, \infty]$ be monotonously increasing and right-continuous functions. They are called dilatationally equivalent if there are constants $C > 0$ and $\lambda_0 > 0$ such that

$$G(C^{-1}\lambda) \leq F(\lambda) \leq G(C\lambda) \qquad \text{for all } \lambda \in [0, \lambda_0].$$

One readily checks that this defines indeed an equivalence relation.
Lastly, we recall a part of Lemma 2.4 from W. Lück's book [Lüc02, Lem. 2.4]:

Lemma 2.14. *Let U and V be Hilbert $\mathcal{N}G$-modules. Let $f\colon \operatorname{dom}(f) \subset U \to V$ be a G-equivariant closed densely defined operator. Suppose that f is Fredholm and $b^{(2)}(f^*)$ is finite. Then f^* is Fredholm and*

$$F(f)(\lambda) - b^{(2)}(f) = F(f^*)(\lambda) - b^{(2)}(f^*).$$

We will use this throughout the thesis, and in particular in Chapter 3, to compute the Novikov-Shubin invariants (defined in the next chapter in terms of this difference) of the cellular differential d by studying the upper Laplacian $\Delta^{\mathrm{up}} = dd^*$ appearing in the definition of $F(d^*)$ instead of the lower Laplacian d^*d in the definition of $F(d)$.

2.4 Novikov-Shubin Invariants

We study these spectral density functions more closely. While the value at zero measures the size of the kernel, we can view $F(f)(\lambda)$ as measuring, in some sense, the *number of G-types of eigenvalues* or *the density of the spectrum* of f up to absolute value λ. Letting $\lambda \searrow 0$ should therefore give us a good idea of how much spectrum there is close to zero and thus of the size of the difference $\overline{\operatorname{im}(d)}/\operatorname{im}(d)$. This will give rise to our first definition of Novikov-Shubin invariants. Afterwards, we will discuss several other possible approaches to, and definitions for, these invariants.

2.4.1 via Spectral Density Functions

Given a spectral density function $F(f)$ as above, we define the Novikov-Shubin invariant $\alpha(f)$ to measure the asymptotic behaviour of $F(f)(\lambda) \to F(f)(0)$ as $\lambda \searrow 0$. The underlying idea is to attempt $F(f)(\lambda) - F(\lambda)(0) \sim \lambda^\alpha$ and solve for the exponent α.

Definition 2.15. Let $F\colon \mathbb{R}_{\geq 0} \to [0,\infty]$ be a monotonously increasing right-continuous function. We define the Novikov-Shubin invariant $\alpha(F) \in [0,\infty] \cup \{\infty^+\}$ by

$$\alpha(F) = \liminf_{\lambda \searrow 0} \frac{\log(F(\lambda) - F(0))}{\log(\lambda)}$$

if $F(\lambda) > F(0)$ for all $\lambda > 0$ and formally by $\alpha(F) = \infty^+$ otherwise.

These Novikov-Shubin invariants satisfy many interesting properties [Lüc02, Lem. 2.11], for example, they are invariant under dilatational equivalence, that is, if F and G are dilatationally equivalent then $\alpha(F) = \alpha(G)$.
We are interested, in particular, in the Novikov-Shubin invariants associated to the operators d and Δ we encountered in the cellular and analytic definitions of the L^2-Betti numbers.

Cellular Version. For a free G-CW complex X, recall the differential of the cellular L^2-chain complex $d_k\colon \ell^2 C_k^{\mathrm{cell}}(X;\mathcal{N}G) \to \ell^2 C_{k-1}^{\mathrm{cell}}(X;\mathcal{N}G)$ and the Laplace operator $\Delta_k \curvearrowright \ell^2 C_k^{\mathrm{cell}}(X;\mathcal{N}G)$.

Definition 2.16. We define the cellular spectral density functions

$$F_k(X) = F(d_k), \qquad F_k^\Delta(X) = F(\Delta_k)$$

and the corresponding Novikov-Shubin invariants

$$\alpha_k(X) = \alpha(F_{k+1}(X)) = \liminf_{\lambda \searrow 0} \frac{\log(F(d_{k+1})(\lambda) - b^{(2)}(d_{k+1}))}{\log(\lambda)},$$
$$\alpha_k^\Delta(X) = \alpha(F_k^\Delta(X)) = \liminf_{\lambda \searrow 0} \frac{\log(F(\Delta_k)(\lambda) - b_k^{(2)}(X;\mathcal{N}G))}{\log(\lambda)},$$

where we use that $b^{(2)}(\Delta_k) = b_k^{(2)}(X;\mathcal{N}G))$.

Note that we differ here from the notation used in W. Lück's book [Lüc02]: The index of the Novikov-Shubin invariant $\alpha_k(M)$ here has shifted down by one compared to W. Lück's convention and agrees with his $\alpha_{k+1}(M)$. For α^Δ the indices agree again. Both choices of indices appear in the literature, depending on the preference of homological or cohomological notation: The choice made here means that no index shift appears in the analytic definition below.

Analytic Version. Alternatively, given a complete Riemannian manifold M, we can use the analytic counterparts $d^k\colon \Omega^k(M) \to \Omega^{k+1}(M)$ and $\Delta^k \curvearrowright \Omega^k(M)$. For this, denote by $d_{\min}^k$ and $\Delta_{\min}^k$ the minimal closures of the densely defined operators

$$d^k\colon \Omega_c^k(M) \to L^2\Omega^{k+1}(M) \qquad \text{and} \qquad \Delta^k\colon \Omega_c^k(M) \to L^2\Omega^k(M),$$

respectively. The operator $d_{\min}^k$ induces an operator

$$d_{\min}^{k,\perp}\colon \operatorname{dom}(d_{\min}^k) \cap \operatorname{im}(d_{\min}^{k-1})^\perp \to \operatorname{im}(d_{\min}^{k+1,*})^\perp.$$

Definition 2.17. With the notation above, we define the analytic spectral functions of M by

$$F_k^a(M) = F(d_{\min}^{k,\perp}), \qquad F_k^{a,\Delta}(M) = F(\Delta_{\min}^k)$$

and the analytic Novikov-Shubin invariants of M by

$$\alpha_k^a(M) = \alpha(F_k^a(M)),$$
$$\alpha_k^{a,\Delta}(M) = \alpha(F_k^{a,\Delta}(M)).$$

Relation and Properties. Firstly, the invariants $\alpha_\bullet^\Delta$ can be recovered from the invariants $\alpha_\bullet$ by the following lemma [Lüc02, Lem. 2.66]:

Lemma 2.18. *Let M be a cocompact free proper G-manifold without boundary and with G-invariant Rimannian metric. Then for $0 \leq k \leq \dim(M)$,*

$$\alpha_k^\Delta(M) = \frac{1}{2}\min\{\alpha_{k-1}(M), \alpha_k(M)\}.$$

Secondly, the two different notions of Novikov-Shubin invariants agree whenever both are defined, compare [Lüc02, Thm. 2.68]:

Theorem 2.19. *Let M be a cocompact free proper G-manifold without boundary and with G-invariant Riemannian metric. Then the cellular and the analytic spectral density functions are dilatationally equivalent and in particular for all $0 \leq k \leq \dim M$,*

$$\alpha_k(M) = \alpha_k^a(M), \qquad \alpha_k^\Delta(M) = \alpha_k^{a,\Delta}(M).$$

Therefore, we omit the superscript a in the analytic version. Furthermore, we prefer to compute $\alpha_\bullet(M)$ since this yields $\alpha_\bullet^\Delta(M)$ for free.

Remark 2.20. Notice that $F(\lambda) \to F(0)$ converges more quickly if α is large, so that large Novikov-Shubin invariants indicate a very sparse spectrum close to zero or small difference $\overline{\operatorname{im}(d)}/\operatorname{im}(d)$. This is one of the reasons some authors prefer the notion of capacity $\mathfrak{c}$, defined by $\mathfrak{c} = 1/\alpha$, which describes the size of the spectrum near zero more intuitively.

2.4.2 Via Heat Kernel Asymptotics

We saw in Remark 2.9 that the L^2-Betti numbers are classically defined in terms of the heat kernel and its long-term asymptotic behaviour. This is also true for Novikov-Shubin invariants. A good account of this is given by M. Gromov and M. A. Shubin in [GS91]. Considering the same integral as in Remark 2.9,

$$\theta(t) = \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{C}}(e^{-t\Delta^k}(x,x))\,\mathrm{d}\operatorname{vol}_x,$$

where $\mathcal{F}$ is a fundamental domain for the G-action, the L^2-Betti numbers are obtained as

$$b_k^{(2)}(M) = F(\Delta^k)(0) = \lim_{t\to\infty}\theta(t).$$

We can also use this definition in terms of the heat operator to define Novikov-Shubin invariants. Writing $\lim_{t\to\infty}\theta(t) = \theta(\infty)$, the difference $\theta(t) - \theta(\infty)$ decreases as $t_0 \to \infty$ and by [GS91],

$$F(\Delta^k)(\lambda) - F(\Delta^k)(0) \;\sim\; \lambda^\alpha \quad \text{as } \lambda \searrow 0 \qquad \Longleftrightarrow \qquad \theta(t) - \theta(\infty) \;\sim\; t^{-\alpha} \quad \text{as } t \to \infty,$$

and even more precisely

$$\alpha_k^\Delta(M) = \liminf_{\lambda \searrow 0} \frac{\log(F(\Delta^k)(\lambda) - F(\Delta^k)(0))}{\log(\lambda)} = -\liminf_{t\to\infty} \frac{\log(\theta(t) - \theta(\infty))}{\log(t)},$$

giving us an equivalent description of the Novikov-Shubin invariants in terms of the long-term asymptotic behaviour of the heat operator.

Remark 2.21. In particular, the heat decays more quickly if α is large.

2.4.3 Via Near Cohomology

An approach that turned out very useful when it comes to computing Novikov-Shubin invariants is the notion of near cohomology cones introduced by M. Gromov and M. A. Shubin [GS92].

Definition 2.22. Let M be a complete Riemannian manifold without boundary and with a cocompact free proper action $G \curvearrowright M$. For $0 \leq k \leq \dim(M)$ we define the near cohomology cone of d^k of radius λ by

$$C_\lambda(d^k) = \left\{\omega \in L^2\Omega^k(M) \cap (\ker d^k)^\perp \;\middle|\; \|d^k\omega\| \leq \lambda \|\omega\|_{\operatorname{mod}\ker d^k}\right\},$$

where $\|\cdot\|_{\operatorname{mod}\ker d^k}$ denotes the quotient norm on $L^2\Omega^k(M) \cap (\ker d^k)^\perp \cong L^2\Omega^k(M)\big/\ker d^k$.

Let $\mathcal{L}_\lambda(d^k)$ denote the set of closed linear subspaces $L \subset C_\lambda(d^k)$. Then we can recover the asymptotic behaviour of the spectral density function via

$$F_k(M)(\lambda) - F_k(M)(0) = \sup_{L \in \mathcal{L}_\lambda(d^k)} \dim_{\mathcal{N}G} L,$$

so we can use these cones to compute the Novikov-Shubin invariants working concretely on the level of differential forms. For example, this was used by M. Rumin [Rum01] to find estimates on the Novikov-Shubin invariants of nilpotent Lie groups. We will review this in Chapter 4.

2.4.4 Via Scaling of the Manifolds

Let (M, g) be a Riemannian manifold with Riemannian metric g and a cocompact free proper group action $G \curvearrowright M$ acting by isometries. For $\lambda \in (0,\infty)$, we define a new metric $g_\lambda = \lambda^2 g$ by scaling the metric g with a constant factor λ^2, that is, for $x \in M$ and $v, w \in T_xM$,

$$(g_\lambda)_x(v,w) = \lambda^2 g_x(v,w) = g_x(\lambda v, \lambda w).$$

In particular, unit vectors with respect to g are now vectors of length λ with respect to g_λ. This can be interpreted as scaling our manifold by a factor of λ: If λ is small, distances shrink. The boundary operators d are metric independent, however, the scalar product on $\Omega^\bullet(M)$ and therefore the adjoint of d depends on the metric. We denote by $d^* = d^{*_1}$ the adjoint with respect to g. Then, the adjoint d^{*_λ} of d with respect to g_λ is given by $d^{*_\lambda} = \lambda^{-2} d^*$:

Lemma 2.23. *Let (M, g) be a Riemannian manifold of dimension n and $\lambda > 0$. Let $*\colon \Omega^\bullet M \to \Omega^{n-\bullet}M$ be the Hodge-star operator with respect to the metric g and $*_\lambda$ the Hodge-star operator with respect to the metric g_λ. Further let d^* respectively d^{*_λ} be the adjoint of the deRham differential d with respect to g respectively g_λ. Then in degree $0 \leq k \leq \dim(M)$,*

$$*_\lambda = \lambda^{n-2k} *\colon \Omega^k M \to \Omega^{n-k}M \quad \textit{and} \quad d^{*_\lambda} = \lambda^{-2} \cdot d^*.$$

Proof. Let $\{\vartheta_1, \dots, \vartheta_n\}$ be a basis of $\Omega^1 M$ that is orthonormal with respect to g. Because $g_\lambda(\vartheta_i, \vartheta_j) = \lambda^2 g(\vartheta_i, \vartheta_j)$, the basis $\{\vartheta_1/\lambda, \dots, \vartheta_n/\lambda\}$ is orthonormal with respect to g_λ. Then for any permutation $\sigma \in \mathcal{S}_n$ of the first n integers,

$$\begin{aligned}
*(\vartheta_{\sigma(1)} \wedge \cdots \wedge \vartheta_{\sigma(k)}) &= \pm \vartheta_{\sigma(k+1)} \wedge \cdots \wedge \vartheta_{\sigma(n)}, \\
*_\lambda(\vartheta_{\sigma(1)} \wedge \cdots \wedge \vartheta_{\sigma(k)}) &= \lambda^k *_\lambda \left(\frac{\vartheta_{\sigma(1)}}{\lambda} \wedge \cdots \wedge \frac{\vartheta_{\sigma(k)}}{\lambda} \right) \\
&= \pm \lambda^k \cdot \frac{\vartheta_{\sigma(k+1)}}{\lambda} \wedge \cdots \wedge \frac{\vartheta_{\sigma(n)}}{\lambda} \\
&= \pm \lambda^{2k-n} \cdot \vartheta_{\sigma(k+1)} \wedge \cdots \wedge \vartheta_{\sigma(n)},
\end{aligned}$$

where the $\pm$-sign depends only on σ and is the same everywhere. By linearity, it follows that $*_\lambda = \lambda^{n-2k} *\colon \Omega^k M \to \Omega^{n-k} M$. Since on k-forms $d^{k-1,*} = d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ is given by

$$(-1)^{kn+1} d^* \colon \Omega^k M \xrightarrow{*} \Omega^{n-k} M \xrightarrow{d} \Omega^{n-k+1} M \xrightarrow{*} \Omega^{k-1} M$$

and because $d^{*_\lambda} = (-1)^{kn+1} *_\lambda d *_\lambda$, the second claim follows from the first since

$$d^{*_\lambda} = (-1)^{kn+1} *_\lambda d *_\lambda = (-1)^k (\lambda^{n-2k} *) d (\lambda^{n-2n+2k-2} *) = \lambda^{-2} (-1)^{nk+1} * d * = \lambda^{-2} d^*. \quad \square$$

By definition, the Laplace operator depends on g. Denote by $\Delta = \Delta_1 = dd^* + d^* d$ the Laplace operator with respect to g and $\Delta_\lambda = dd^{*_\lambda} + d^{*_\lambda} d$ the Laplace operator with respect to g_λ. Then it follows directly that

$$\Delta_\lambda = \lambda^{-2} \Delta.$$

Thus, also the spectrum of Δ_λ is the spectrum of Δ scaled by the factor $1/\lambda^2$, such that the spectral projectors satisfy

$$E_1^{d^{*_\lambda} d} = E_{\lambda^2}^{d^* d} \quad \text{and} \quad E_1^{\Delta_\lambda} = E_{\lambda^2}^{\Delta}.$$

By definition of the spectral density functions, this implies directly that

$$\begin{aligned}
F_\bullet(M, g_\lambda)(1) = \dim_{\mathcal{N}G} \operatorname{im} E_1^{d^{*_\lambda} d} &= \dim_{\mathcal{N}G} \operatorname{im} E_{\lambda^2}^{d^* d} = F_\bullet(M, g)(\lambda), \\
F_\bullet^\Delta(M, g_{\sqrt{\lambda}})(1) = \dim_{\mathcal{N}G} \operatorname{im} E_1^{\Delta^*_{\sqrt{\lambda}} \Delta_{\sqrt{\lambda}}} &= \dim_{\mathcal{N}G} \operatorname{im} E_{\lambda^2}^{\Delta^* \Delta} = F_\bullet^\Delta(M, g)(\lambda).
\end{aligned}$$

Using that $b_\bullet^{(2)}(M)$ is metric-independent, we can write the Novikov-Shubin invariants as

$$\begin{aligned}
\alpha_k(M, g) &= \liminf_{\lambda \searrow 0} \frac{\log(F_k(M, g_\lambda)(1) - b^{(2)}(d_k))}{\log(\lambda)}, \\
\alpha_k^\Delta(M, g) &= \liminf_{\lambda \searrow 0} \frac{\log(F_k^\Delta(M, g_{\sqrt{\lambda}})(1) - b_k^{(2)}(M))}{\log(\lambda)},
\end{aligned}$$

where we consider now a fixed window of the spectrum, $\sigma(\Delta) \cap [0,1]$, however the Laplace operator varies as we scale down the manifold. This also gives a geometric interpretation of the factor $1/2$ in the relation between α^Δ and α in Lemma 2.18. While for $F(d)$ the scaling is according to the parameter λ, for $F(\Delta)$ the scaling happens with speed given by the square root $\sqrt{\lambda}$, leading to an extra factor $1/2$ in the exponent α^Δ compared to α.

2.4.5 Via Extended Cohomology

Another important view on Novikov-Shubin invariants is the extended cohomology introduced by M. Farber [Far98]. This considers the cohomology in the classical sense,

$$\mathcal{H}^{(2)}_\bullet(M) = \ker(d)/\operatorname{im}(d),$$

without taking the closure of $\operatorname{im}(d)$. M. Farber constructs a category in which this $\mathcal{H}^{(2)}_\bullet(M)$ is described nicely. In this category, objects decompose into the direct sum of a projective part and a torsion part. The projective part of $\mathcal{H}^{(2)}_\bullet(M)$ corresponds precisely to $H^{(2)}_\bullet(M)$ and the torsion part corresponds to $\overline{\operatorname{im}(d)}/\operatorname{im}(d)$ and can be used to formulate an alternative definition of the spectral density functions and Novikov-Shubin invariants.

2.4.6 Via Stochastic Methods

Lastly, given a free G-CW complex X of finite type, we show in this thesis that Novikov-Shubin invariants can also be viewed as quantities arising in certain stochastic processes related to X. Classically, this is known for $\alpha_0(X)$. Indeed, proving N. Th. Varopoulos' Theorem 0.2 one shows first that $\alpha_0 = 2a$ is equivalent to the statement that the return probability $p(n)$ of the random walk on a Cayley graph of G behaves asymptotically like $p(n) \sim n^{-a}$ as $n \to \infty$. This will be reviewed in the next chapter, Chapter 3. Then, we generalise this result to higher degrees, connecting $\alpha_k(X)$ for $0 \leq k \leq \dim(X)$ also to a random walk.

Chapter 3

Random Walks and Novikov-Shubin Invariants

In this chapter we establish the relation between the Novikov-Shubin invariants of a free G-CW complex of finite type and stochastic processes on the skeleta of this G-CW complex. While Novikov-Shubin invariants are hard to compute in general, $\alpha_0(X)$ is easy to determine using Theorem 0.2. We recall this theorem in the version given in Lück's book [Lüc02, Thm 2.46].

Theorem 3.1. *Let X be a free G-CW complex of finite type. Denote by $N(G)$ the growth rate of G. Then*

1. $\alpha_0(X) = N(G) < \infty$ *if and only if G is infinite virtually nilpotent,*
2. $\alpha_0(X) = \infty$ *if and only if G is amenable and not virtually nilpotent, and*
3. $\alpha_0(X) = \infty^+$ *if and only if G is finite or non-amenable.*

Recall that the growth rate of a (virtually[1]) nilpotent group can be easily computed using the lower central series. By the Bass-Guivarc'h formula it is given by

$$N(G) = \sum_{k \geq 1} k \cdot \operatorname{rk}\left(G_{k-1}/G_k\right),$$

where G_k is the kth term in the lower central series. This directly implies that in this case the Novikov-Shubin invariant α_0 is integer-valued. It also gives a very concrete, geometric interpretation of α_0.

It is important to note however, that this result uses a common connection between the Laplacian in degree zero and the growth rate to the return probability of a random walk taking place on the 1-skeleton $X^{(1)}$ of X. We review this connection briefly in this chapter before we construct a similar random walk taking place on the k-cells of X. We will relate a quantity related to return probabilities of this random walk to the kth Novikov-Shubin invariant $\alpha_k(X)$.

[1] The growth rate is invariant under taking finite index subgroups, so we may work with the finite index nilpotent subgroup.

3.1 Random Walks and α_0

In this section, we briefly review the connection between the growth rate of the group G, the random walks on Cayley(G) and $X^{(1)}$ and the Novikov-Shubin invariant $\alpha_0(X)$. We closely follow the outline given by W. Lück in his book [Lüc02, §2.1.4] and refer to the proof there. However, we slightly adapt the proof by using lazy random walks. This allows us to reformulate some results slightly and it will be needed when generalising this approach to higher degrees.

The first observation is that the Novikov-Shubin invariants $\alpha_0(X) = \alpha(d_1)$ of the the differential in the cellular L^2-chain complex of X and $\alpha(c_S)$ of the differential on the Cayley graph Cayley(G) agree, compare [Lüc02, Lem. 2.45].

Secondly, we have seen in Lemma 1.29 that the propagation operator P of the random walk on Cayley(G) satisfies $\operatorname{Id} - P = |S|^{-1}\Delta$ (note that the extra factor $1/2$ in Lück's book is incorrect). We also saw that we can use the q-lazy random walk to write the Laplacian as

$$\operatorname{Id} - P_q = \frac{1-q}{|S|}\Delta.$$

We know that $\sigma(P_q) \subset [-1+2q, 1]$ and therefore the operator P_q is non-negative for $q \geq 1/2$. Then, in place of Equation [Lüc02, Eq. (2.47)], in Lück's notation we obtain that

$$\operatorname{tr}_{\mathcal{N}G}(\chi_{[1-\lambda,1]}(P_q)) = F(c_S)(\sqrt{|S|(1-q)^{-1}\lambda}) - b^{(2)}(c_S). \tag{3.1}$$

Since we are interested in the exponent of the decay in λ, the constant factor $|S|(1-q)^{-1}$ does not impact the computation of the Novikov-Shubin invariant later on.

In Theorem [Lüc02, Thm. 2.48] we can replace $p(n)$ by $p_q(n)$ for $q \geq 1/2$, so for example $p_{1/2}(n)$, and require the inequality

$$C^{-1}n^{-a} \leq p_{1/2}(n) \leq Cn^{-a}$$

for all $n \geq 1$ instead of considering only even n. This works out since the requirement of even integers n comes into play precisely to make sure that P^n is a non-negative operator, which holds for $P_{1/2}$ and hence all its powers. The remaining inequalities hold true regardless of the extra factor $(1-q)^{-1}$ appearing.[2]

We can therefore reformulate Theorem [Lüc02, Thm. 2.48] in the following way.

Theorem 3.2. *The finitely generated group G has polynomial growth precisely of degree $2a$ if and only if there is a constant $C > 0$ such that*

$$C^{-1}n^{-a} \leq p_{1/2}(n) \leq Cn^{-a}$$

holds for all $n \geq 1$. If the finitely generated group G does not have polynomial growth, then for each $a > 0$ there is a constant $C(a) > 0$ such that $p_{1/2}(n) \leq C(a)n^{-a}$ holds for all $n \geq 1$.

From here, the computations work precisely the same as in Lück's book [Lüc02, p. 95f], where the extra factor $(1-q)^{-1}$ on the right hand side of Equation (3.1) disappears precisely by the same reasoning as the factor $|S|$ disappears. Thus, Theorem 3.1 follows in the same manner.

[2]Indeed, since Lück's computation includes an extra factor $1/2$, the computation he gives is precisely the one to be carried out for $q = 1/2$.

3.2 Random Walks and α_k

We now construct a random walk taking place on k-cells of a CW complex X. Then, we relate this random walk to the upper Laplacian $d_{k+1}d_{k+1}^* = \Delta_k^{\mathrm{up}}$ acting on the cellular L^2-chain complex and thereby also to the Novikov-Shubin invariant $\alpha_k(X)$ for suitable CW complexes.
The construction of the random walk generalises a construction of a random walk on the k-skeleton of finite simplicial complexes by O. Parzanchevski and R. Rosenthal in [PR17]. While the setting of CW complexes introduces some further difficulties (related to the incidence numbers of the CW complex), some of the ideas are based on the paper above as well as S. Mukherjee and J. Steenbergen's paper [MS16] and R. Rosenthal's paper [Ros14].

3.2.1 Degree k Upper Random Walks

Before we define the random walk, we introduce some quantities that will be useful later on. These quantities capture the local structure of the CW complex.

Definition 3.3. Let X be a free G-CW complex of finite type. For $\alpha \in I_k$ and $\beta \in I_{k+1}$ we define the quantities

$$\begin{aligned} d_{+,2}(\alpha) &= \sum_{\beta' \in I_{k+1}} [\beta' : \alpha]^2, \\ d_+(\alpha) &= \sum_{\beta' \in I_{k+1}} |[\beta' : \alpha]|, \\ d_-(\beta;\alpha) &= \begin{cases} \sum_{\alpha \neq \alpha' \in I_k} |[\beta : \alpha']| & \text{if } [\beta : \alpha] \neq 0, \\ 0 & \text{if } [\beta : \alpha] = 0, \end{cases} \\ d_-(\alpha) &= \max_{\beta' \in I_{k+1}} d_-(\beta;\alpha), \end{aligned} \tag{3.2}$$

where the maximum in the definition of $d_-(\alpha)$ exists since $d_-(\beta;\alpha)$ is invariant under the G-action,[3] so that it can only assume finitely many different values. Note also that these quantities are independent of the orientations chosen on α and β.[4]

These quantities generalise the idea of the degree of a k-cell, with d_+ being the incoming degree and d_- the (maximal) outgoing degree. They capture the local structure of the CW complex around the k-cell $\alpha \in I_k$.
We also introduce the notation

$$d(\alpha, \alpha', \beta) = -[\beta : \alpha][\beta : \alpha'], \tag{3.3}$$

measuring how well connected α is to α' along β.[5] Note that $d(\alpha, \alpha', \beta) = d(\alpha', \alpha, \beta)$.
From Equation (1.1), we know that Δ_k^{up} sees the incidence numbers between k- and $(k+1)$-cells in X. Hence, the incidence numbers have to appear in the definition of our random walk. Furthermore, both the sign and the size of the incidence numbers have to play a role.

[3] $d_-(\beta;\alpha) = d_-(g.\beta; g.\alpha)$ for all $g \in G$

[4] Since in the following, orientations on $(k+1)$-cells will never play a role, we will only take care of orientations on the k-cells and work with the preferred orientation on $(k+1)$-cells throughout.

[5] Here, we introduce an extra minus sign to mirror what happens in the case of graphs. There, random walkers can walk from a vertex v_1 along an (oriented) edge $e = (v_1, v_2)$ to the vertex v_2, where e is an (outgoing) edge for v_1 with $[e : v_1] = -1$ and an (incoming) edge for v_2 with $[e : v_2] = 1$, so that $[e : v_1][e : v_2] = -1$ if $e = \{v_1, v_2\}$. With this extra minus sign, the quantity $d(v_1, v_2, e) = 1$ is positive in this case.

Definition 3.4. As before, $I_k = I_k^+ = \{\alpha = \alpha_+\}$ is the set of k-cells of X, where each k-cell comes equipped with an (arbitrarily chosen) preferred orientation. For $\alpha = \alpha_+$ we denote by $-\alpha = \alpha_-$ the same k-cell but equipped with the reversed orientation and define

$$I_k^{\pm} = I_k^+ \cup \{\alpha_- \mid \alpha \in I_k\}.$$

We call two oriented k-cells $\alpha_\nu, \alpha'_{\nu'} \in I_k^{\pm}$ (upper) neighbours along $\beta \in I_{k+1}$, and write $\alpha_\nu \overset{\beta}{\sim} \alpha'_{\nu'}$, where $\nu, \nu' \in \{+,-\}$ denote orientations, if

$$\alpha_\nu \neq -\alpha'_{\nu'} \quad \text{and} \quad d(\alpha_\nu, \alpha'_{\nu'}, \beta) = \nu\nu' d(\alpha, \alpha', \beta) > 0.$$

Note that this condition is independent of the orientation chosen on $\beta \in I_{k+1}$.

Definition 3.5. The random walk $\mathfrak{R}^k = \mathfrak{R}^k(X)$ is given by the state space $I_k^* = I_k^{\pm} \cup \{\Theta\}$, where Θ is an auxiliary, absorbing state together with the following moving probabilities:

- The moving probabilities starting from the absorbing state Θ are given by

$$\mathbb{P}(\Theta \to \Theta) = 1 \quad \text{and} \quad \mathbb{P}(\Theta \to \alpha_\pm) = 0 \quad \text{for all } \alpha_\pm \in I_k^{\pm}.$$

- To define the moving probabilities starting from $\alpha_\nu \in I_k^{\pm}$, we define first for $\alpha'_{\nu'} \in I_k^{\pm}$ and $\beta \in I_{k+1}$ the quantities

$$\mathbb{P}(\alpha_\nu \nearrow \beta) = \frac{|[\beta : \alpha_\nu]|}{d_+(\alpha)} \qquad \text{and} \qquad \mathbb{P}_\alpha(\beta \searrow \alpha'_{\nu'}) = \frac{|[\beta : \alpha'_{\nu'}]|}{d_-(\alpha)}$$

These probabilities can be seen as an intermediate step of moving first from α to β and then from β to α' (keeping in mind that we started at α). In this sense, we define

$$\mathbb{P}(\alpha_\nu \xrightarrow{\beta} \alpha'_{\nu'}) = \begin{cases} \mathbb{P}(\alpha_\nu \nearrow \beta)\mathbb{P}_\alpha(\beta \searrow \alpha'_{\nu'}) = \frac{-[\beta:\alpha_\nu][\beta:\alpha'_{\nu'}]}{d_+(\alpha)d_-(\alpha)} = \frac{d(\alpha_\nu,\alpha'_{\nu'},\beta)}{d_+(\alpha)d_-(\alpha)} > 0 & \text{if } \alpha_\nu \overset{\beta}{\sim} \alpha'_{\nu'}, \\ 0 & \text{else,} \end{cases}$$

the probability of moving from α along β to α'. Recall here that $\alpha \not\sim \pm\alpha$ by definition, so that $\mathbb{P}(\alpha \xrightarrow{\beta} \pm\alpha) = 0$. Finally, we set

$$\begin{aligned}\mathbb{P}(\alpha_\nu \to \alpha'_{\nu'}) &= \sum_{\beta \in I_{k+1}} \mathbb{P}(\alpha_\nu \xrightarrow{\beta} \alpha'_{\nu'}) = \sum_{\substack{\beta \in I_{k+1} \\ \alpha \overset{\beta}{\sim} \alpha'}} \frac{d(\alpha_\nu, \alpha'_{\nu'}, \beta)}{d_+(\alpha)d_-(\alpha)} \\ &= \frac{1}{d_+(\alpha)d_-(\alpha)} \sum_{\substack{\beta \in I_{k+1} \\ \alpha_\nu \overset{\beta}{\sim} \alpha'_{\nu'}}} d(\alpha_\nu, \alpha'_{\nu'}, \beta)\end{aligned}$$

The moving probabilities would add to one if we use $d_-(\beta;\alpha)$ in place of $d_-(\alpha)$, however it will be important later on that we can pull the factor $d_-(\alpha)^{-1}$ out of the sum as it depends only on the starting cell α. Consequently, they need not add up to one.

- The complementary probability will be the probability of moving to Θ, that is

$$\mathbb{P}(\alpha_\nu \to \Theta) = 1 - \sum_{\alpha'_{\nu'} \in I_k^{\pm}} \mathbb{P}(\alpha_\nu \to \alpha'_{\nu'}).$$

As before,the propagation operator P with entries $P_{s,s'} = \mathbb{P}(s' \to s)$ for $s, s' \in I_k^*$ acts on $\ell^2(I_k^*) = \left\{ \sum_{s \in I_k^*} \lambda_s \cdot s \,\middle|\, \sum_{s \in I_k^*} |\lambda_s|^2 < \infty \right\}$ by

$$P\left(\sum_{s \in I_k^*} \lambda_s \cdot s\right) = \sum_{s \in I_k^*} \left[\sum_{s' \in I_k^*} P_{s,s'} \lambda_{s'}\right] \cdot s = \sum_{s \in I_k^*} \left[\sum_{s' \in I_k^*} \mathbb{P}(s' \to s) \lambda_{s'}\right] \cdot s.$$

This defines a random walk taking place on I_k^*.

Example 3.6. Let us consider a small example to see how we can find the moving probabilities. For this, let X be a CW complex and $\alpha \in I_k$ some k-cell. In order to find the k-cells we can move to from $\alpha = +\alpha$, we proceed as follows:

1. First we consider all $(k+1)$-cells $\beta \in I_{k+1}$ and find those, that have non-zero incidence number $[\beta : \alpha] \neq 0$. For this example, let us say there are two such $(k+1)$-cells β_1 and β_2 with $[\beta_1 : \alpha] = 1$ and $[\beta_2 : \alpha] = -2$.
2. Then we consider all other k-cells $\alpha \neq \alpha' \in I_k$ that have non-zero incidence numbers with at least one of the $(k+1)$-cells above, that is $[\beta_1 : \alpha] \neq 0$ or $[\beta_2 : \alpha] \neq 0$. For this example, let us say there are three such k-cells, α_1 with $[\beta_1 : \alpha_1] = 1$, α_2 with $[\beta_1 : \alpha_2] = 2$ and $[\beta_2 : \alpha_2] = 4$ and α_3 with $[\beta_2 : \alpha_3] = -2$.

We visualise this with the following diagram:

$$\begin{tikzcd}
 & \beta_1 \arrow[dl, "1"'] \arrow[d, "2"] & & \beta_2 \arrow[d, "4"] \arrow[dr, "-2"] & & \in I_{k+1} \\
\alpha_1 & \alpha_2 & \alpha \arrow[ul, "1"'] \arrow[ur, "-2"] & \alpha_2 & \alpha_3 & \in I_k
\end{tikzcd}$$

Next, we first change the orientations on the $(k+1)$-cells such that the incidence numbers with α are negative, so here we change the orientation on β_1.[6] Then we change the orientations on the α_i, $i \in \{1,2,3\}$ for each of the β_j independently such that the incidence numbers with the $(k+1)$-cells become positive. This changes our diagram as follows:

$$\begin{tikzcd}
 & -\beta_1 \arrow[dl, "1"'] \arrow[d, "2"] & & \beta_2 \arrow[d, "4"] \arrow[dr, "2"] & & \in I_{k+1}^{\pm} \\
-\alpha_1 & -\alpha_2 & \alpha \arrow[ul, "-1"'] \arrow[ur, "-2"] & \alpha_2 & -\alpha_3 & \in I_k^{\pm}
\end{tikzcd}$$

We now introduce the auxiliary state Θ. For each of the $(k+1)$-cells, we sum the outgoing incidence numbers. Here, we get $d_-(\beta_1; \alpha) = 1 + 2 = 3$ for β_1 and $d_-(\beta_2; \alpha) = 2 + 4 = 6$ for β_2. The maximum is therefore 6, and we add connections from each of the $(k+1)$-cells to the new state Θ until the sum of outgoing edges is equal to this maximum, i.e., $-\beta_1 \xrightarrow{3} \Theta$ in this case:

$$\begin{tikzcd}
 & & -\beta_1 \arrow[dll, "3"'] \arrow[dl, "1"'] \arrow[d, "2"] & & \beta_2 \arrow[d, "4"] \arrow[dr, "2"] & & \in I_{k+1}^{\pm} \\
\Theta & -\alpha_1 & -\alpha_2 & \alpha \arrow[ul, "-1"'] \arrow[ur, "-2"] & \alpha_2 & -\alpha_3 & \in I_k^*
\end{tikzcd}$$

[6] This orientation of β_1 has no impact on the formulas in the end and is thus suppressed in the formal definition.

The moving probabilities can now be read to be proportional to the annotations of the arrows. For the first intermediate step, $\mathbb{P}(\alpha \nearrow -\beta_1) = 1/3$ and $\mathbb{P}(\alpha \nearrow \beta_2) = 2/3$. For the second step,

$$\mathbb{P}(-\beta_1 \searrow \Theta) = 3/6, \qquad \mathbb{P}(-\beta_1 \searrow -\alpha_1) = 1/6 \quad \text{and} \quad \mathbb{P}(-\beta_1 \searrow -\alpha_2) = 2/6$$

and for β_2 we have

$$\mathbb{P}(\beta_2 \searrow \alpha_2) = 4/6 \quad \text{and} \quad \mathbb{P}(\beta_2 \searrow -\alpha_3) = 2/6.$$

The introduction of Θ guaranties that the denominator of these (unreduced) fractions is the same everywhere and the moving probabilities are proportional to the incidence numbers even if the first intermediate step leads to different β_is. Multiplying these accordingly, we find

$$\begin{aligned}
&\mathbb{P}(\alpha \xrightarrow{-\beta_1} -\alpha_1) = 1/3 \cdot 1/6, \quad \mathbb{P}(\alpha \xrightarrow{-\beta_1} -\alpha_2) = 1/3 \cdot 2/6, \quad \mathbb{P}(\alpha \xrightarrow{-\beta_1} \Theta) = 1/3 \cdot 3/6,\\
&\mathbb{P}(\alpha \xrightarrow{\beta_2} \alpha_2) = 2/3 \cdot 4/6, \qquad \mathbb{P}(\alpha \xrightarrow{-\beta_1} -\alpha_3) = 2/3 \cdot 2/6.
\end{aligned}$$

Here, every oriented k-cell can be reached only via one $(k+1)$-cell, otherwise we would have to sum over all $(k+1)$-cells, that is $\mathbb{P}(\alpha \to s) = \sum_{\pm\beta \in I^{\pm}_{k+1}} \mathbb{P}(\alpha \xrightarrow{\pm\beta} s)$ for $s \in I^*_k$. Therefore, in this example, a random walker starting at α has the following possible moves, with annotations denoting the probabilities:

$$\begin{tikzcd}
& & \alpha \arrow[dll, "1/18"'] \arrow[dl, "1/9"'] \arrow[d, "4/9"] \arrow[dr, "2/9"] \arrow[drr, "1/6"] & & \\
-\alpha_1 & -\alpha_2 & \alpha_2 & -\alpha_3 & \Theta
\end{tikzcd}$$

Note that the cell α_2 can be reached with both possible orientations. If we start at the cell $-\alpha$ with reversed orientation, we obtain the same moving probabilities as for α, but now leading to the same cells but with flipped orientations instead.

We now define an operator B acting directly on the unoriented k-skeleton I_k, which is closely related to this random walk.

Definition 3.7. We define the projection operator $T\colon \ell^2(I^*_k) \to \ell^2(I_k)$ by

$$T\left(\sum_{s \in I^*_k} \lambda_s \cdot s\right) = \sum_{\alpha \in I_k} (\lambda_{\alpha_+} - \lambda_{\alpha_-}) \cdot \alpha$$

and the inclusion operator $I\colon \ell^2(I_k) \to \ell^2(I^*_k)$, using that $I_k = I^+_k \subset I^*_k$, by

$$I\left(\sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha\right) = \sum_{\alpha_+ \in I^+_k} \lambda_{\alpha_+} \cdot \alpha_+.$$

Lastly, we define the operator $B\colon \ell^2(I_k) \to \ell^2(I_k)$ by

$$B\left(\sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha\right) = \sum_{\alpha \in I_k}\left[\sum_{\alpha \neq \alpha' \in I_k} \frac{1}{d_+(\alpha')d_-(\alpha')} \sum_{\beta \in I_{k+1}} d(\alpha, \alpha', \beta)\lambda_{\alpha'}\right] \cdot \alpha.$$

and denote $B_{\alpha,\alpha'} = \frac{1}{d_+(\alpha')d_-(\alpha')} \sum_{\beta \in I_{k+1}} d(\alpha_\nu, \alpha'_{\nu'}, \beta)$ for $\alpha \neq \alpha'$ and $B_{\alpha,\alpha} = 0$ so that

$$B\left(\sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha\right) = \sum_{\alpha \in I_k}\left[\sum_{\alpha' \in I_k} B_{\alpha,\alpha'}\lambda_{\alpha'}\right] \cdot \alpha.$$

This is captured by the diagram

$$\begin{tikzcd}
\ell^2(I_k^*) \arrow[r, "P"] \arrow[d, "T"] & \ell^2(I_k^*) \arrow[d, "T"] \\
\ell^2(I_k) \arrow[r, "B"] \arrow[u, dashed, bend left, "I"] & \ell^2(I_k). \arrow[u, dashed, bend left, "I"]
\end{tikzcd}$$

This operator B does not describe a random walk since $B_{\alpha,\alpha'}$, the "probability of moving from α' to α", may even be negative. However, this operator is closely related to the random walk described by P. Indeed, using the operators T and I we can see that B describes the process that arises from the random walk if we consider a random walker arriving at a cell α_- (equipped with the reversed orientation) as the inverse of a random walker at α_+ — that is we allow random walkers at α_+ and α_- to cancel each other out.[7]

Lemma 3.8. *The operators T, I and B defined above satisfy the equations*

$$BT = TP, \qquad B = TPI \quad \text{and} \quad B^n = TP^nI.$$

Proof. We check these equalities by direct computation. For BT we obtain

$$\begin{aligned}
BT\left(\sum_{s\in I_k^*}\lambda_s\cdot s\right) &= B\left(\sum_{\alpha\in I_k}(\lambda_{\alpha_+}-\lambda_{\alpha_-})\cdot\alpha\right)\\
&= \sum_{\alpha\in I_k}\left[\sum_{\alpha'\in I_k}\frac{1}{d_+(\alpha')d_-(\alpha')}\sum_{\beta\in I_{k+1}}d(\alpha,\alpha',\beta)\cdot(\lambda_{\alpha'_+}-\lambda_{\alpha'_-})\right]\cdot\alpha
\end{aligned}$$

and for TP we compute (omitting the coefficient of Θ as it disappears in the next step) that

$$\begin{aligned}
TP\left(\sum_{s\in I_k^*}\lambda_s s\right) &= T\left(\sum_{\alpha_\nu\in I_k^\pm}\left[\sum_{\alpha'_{\nu'}\in I_k^\pm}\frac{1}{d_+(\alpha')d_-(\alpha')}\sum_{\alpha_\nu\overset{\beta}{\sim}\alpha'_{\nu'}}-[\beta:\alpha_\nu][\beta:\alpha'_{\nu'}]\lambda_{\alpha'_{\nu'}}\right]\alpha_\nu+\cdots\Theta\right)\\
&= \sum_{\alpha\in I_k}\left[\sum_{\alpha'_{\nu'}\in I_k^\pm}\frac{1}{d_+(\alpha')d_-(\alpha')}\sum_{\alpha_+\overset{\beta}{\sim}\alpha'_{\nu'}}-[\beta:\alpha_+][\beta:\alpha'_{\nu'}]\lambda_{\alpha'_{\nu'}}\right]\cdot\alpha\\
&\quad -\sum_{\alpha\in I_k}\left[\sum_{\alpha'_{\nu'}\in I_k^\pm}\frac{1}{d_+(\alpha')d_-(\alpha')}\sum_{\alpha_-\overset{\beta}{\sim}\alpha'_{\nu'}}-[\beta:\alpha_-][\beta:\alpha'_{\nu'}]\lambda_{\alpha'_{\nu'}}\right]\cdot\alpha.
\end{aligned}$$

Now we use that $[\beta:\alpha]=[\beta:\alpha_+]=-[\beta:\alpha_-]$ and that $[\beta:\alpha][\beta:\alpha']\neq 0$ only if either $\alpha_\pm\overset{\beta}{\sim}\alpha'_\pm$

[7]While it is not clear which, if any, physical process this operator B describes, cancellation between different objects does happen in physics. For example, studying fermions via the Dirac equation suggests that for every particle there is a corresponding anti-particle, such as electrons and positrons. If they meet, they will annihilate each other. Describing this by the Dirac sea model allows also for the creation of such pairs.

or $\alpha_\pm \overset{\beta}{\sim} \alpha'_\mp$ together with $-[\beta : \alpha_\nu][\beta : \alpha'_{\nu'}] = \nu\nu' d(\alpha, \alpha', \beta)$ to find that

$$\begin{aligned} TP\left(\sum_{s \in I_k^*} \lambda_s \cdot s\right) &= \sum_{\alpha \in I_k}\left[\sum_{\alpha'_{\nu'} \in I_k^\pm} \frac{1}{d_+(\alpha')d_-(\alpha')} \sum_{\alpha_+ \overset{\beta}{\sim} \alpha'_{\nu'}} d(\alpha, \alpha', \beta) \cdot \nu' \lambda_{\alpha'_{\nu'}}\right] \cdot \alpha \\ &\quad + \sum_{\alpha \in I_k}\left[\sum_{\alpha'_{\nu'} \in I_k^\pm} \frac{1}{d_+(\alpha')d_-(\alpha')} \sum_{\alpha_- \overset{\beta}{\sim} \alpha'_{\nu'}} d(\alpha, \alpha', \beta) \cdot \nu' \lambda_{\alpha'_{\nu'}}\right] \cdot \alpha \\ &= \sum_{\alpha \in I_k}\left[\sum_{\alpha \neq \alpha' \in I_k} \frac{1}{d_+(\alpha')d_-(\alpha')} \sum_{\beta \in I_{k+1}} d(\alpha, \alpha', \beta) \cdot (\lambda_{\alpha'_+} - \lambda_{\alpha'_-})\right] \cdot \alpha \end{aligned}$$

showing the first equality. For the second equality, we have $TI = \mathrm{Id}$, hence $TPI = BTI = B$ and lastly $B^n = B^n TI = TP^n I$. □

Corollary 3.9. *For the operators B and P defined above, $n \in \mathbb{N}$ and all $\alpha, \alpha' \in I_k$,*

$$\langle B^n(\alpha), \alpha' \rangle = \langle P^n(\alpha_+), \alpha'_+ \rangle - \langle P^n(\alpha_+), \alpha'_- \rangle.$$

Proof. Using that $B^n = TP^n I$, we compute the coefficient $\langle B^n(\alpha), \alpha' \rangle$ of α' in $B(\alpha)$ by

$$\langle B^n(\alpha), \alpha' \rangle = \langle TP^n I(\alpha), \alpha' \rangle = \langle TP^n(\alpha_+), \alpha' \rangle = \langle P^n(\alpha_+), \alpha'_+ \rangle - \langle P^n(\alpha_+), \alpha'_- \rangle.$$ □

In particular, we can define the following quantities generalising the idea of return probabilities.

Definition 3.10. For the random walk described by P and $\alpha \in I_k$, we define the return probabilities $p_{\alpha,+}$ and the probabilities of returning with reversed orientation $p_{\alpha,-}$ respectively by

$$p_{\alpha,+}(n) = \langle P^n(\alpha_+), \alpha_+ \rangle \quad \text{and} \quad p_{\alpha,-}(n) = \langle P^n(\alpha_+), \alpha_- \rangle.$$

For the process described by B we define

$$p_\alpha(n) = \langle B^n(\alpha), \alpha \rangle.$$

Notice that $p_\alpha(n) = p_{\alpha,+}(n) - p_{\alpha,-}(n)$. Note also that all three quantities are independent of the choice of preferred orientations.

Corollary 3.11. *For $n \in \mathbb{N}$, the von Neumann trace of B^n is given by*

$$\mathrm{tr}_{\mathcal{N}G}(B^n) = \sum_{\alpha \in I_k(G \backslash X)} p_{\alpha,+}(n) - p_{\alpha,-}(n).$$

3.2.2 Lazy Degree k Upper Random Walks

Let X be a free G-CW complex of finite type and $0 \leq k \leq \dim(X)$ fixed. Starting with the random walk $\mathfrak{R}^k = \mathfrak{R}^k(X) = (I_k^*, P)$ above, we now introduce a lazyness parameter $q \in [0,1]$ and consider the q-lazy random walk $\mathfrak{R}^k_q = \mathfrak{R}^k_q(X)$ in the sense of Subsection 1.3.2. Recall that $\mathfrak{R}^k_q = (I_k^*, P_q)$ is the random walk on I_k^* with propagation operator

$$P_q = q\,\mathrm{Id} + (1-q)P \curvearrowright \ell^2 I_k^*.$$

In particular, the moving probabilities are given as follows:

- For the absorbing state Θ,

$$\mathbb{P}_q(\Theta \to \Theta) = 1, \qquad \mathbb{P}_q(\Theta \to \alpha_\nu) = 0.$$

- For $\alpha_\nu \in I_k^\pm$ (that is $\alpha \in I_k$ and $\nu \in \{+,-\}$),

$$\mathbb{P}_q(\alpha_\nu \to \alpha_\nu) = q, \qquad \mathbb{P}_q(\alpha_\nu \to -\alpha_\nu) = 0.$$

- For $\alpha_\nu, \alpha'_{\nu'} \in I_k^\pm$ with $\alpha_\nu \neq \pm\alpha'_{\nu'}$,

$$\mathbb{P}_q(\alpha_\nu \to \alpha'_{\nu'}) = \frac{1-q}{d_+(\alpha)d_-(\alpha)} \sum_{\substack{\beta \in I_{k+1} \\ \alpha_\nu \overset{\beta}{\sim} \alpha'_{\nu'}}} d(\alpha_\nu, \alpha_{\nu'}, \beta) = (1-q)\mathbb{P}(\alpha_\nu \to \alpha'_{\nu'}).$$

- Lastly,

$$\mathbb{P}_q(\alpha_\nu \to \Theta) = 1 - \sum_{\alpha'_{\nu'} \in I_k^\pm} \mathbb{P}_q(\alpha_\nu \to \alpha'_{\nu'}) = (1-q)\mathbb{P}(\alpha_\nu \to \Theta).$$

In the same spirit, we define $B_q \curvearrowright \ell^2 I_k$ by $B_q = q\,\mathrm{Id} + (1-q)B$. This operator is given by

$$B_q\left(\sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha\right) = \sum_{\alpha \in I_k}\left[q\lambda_\alpha + \sum_{\alpha \neq \alpha' \in I_k} \frac{1-q}{d_+(\alpha')d_-(\alpha')} \sum_{\beta \in I_{k+1}} d(\alpha, \alpha', \beta)\lambda_{\alpha'}\right] \cdot \alpha.$$

Corollary 3.12. *These operators satisfy $B_q T = TP_q$, $B_q = TP_q I$ and $B_q^n = TP_q^n I$.*

Proof. This follows directly from the previous equalities together with $TI = \mathrm{Id}$, since $B_q T = qT + (1-q)BT = qT + (1-q)TP = TP_q$ and $TP_q I = qTI + (1-q)TPI = q + (1-q)B = B_q$. $\square$

As before, we consider the probabilities of returning to the same k-cell with the same orientation or the reversed orientation respectively.

Definition 3.13. For $\alpha \in I_k$, we define the quantities

$$\begin{aligned} p_{q,\alpha,+}(n) &= \langle P_q^n(\alpha_+), \alpha_+\rangle, \\ p_{q,\alpha,-}(n) &= \langle P_q^n(\alpha_+), \alpha_-\rangle, \\ p_{q,\alpha}(n) &= \langle B_q^n(\alpha), \alpha\rangle \end{aligned}$$

Again, $p_{q,\alpha}(n) = p_{q,\alpha,+}(n) - p_{q,\alpha,-}(n)$, hence we can compute the von Neumann trace of B_q^n using the probabilities of the random walk. We define

$$p_q(n) = \mathrm{tr}_{\mathcal{N}G}(B_q^n) = \sum_{\alpha \in I_k(G\backslash X)} p_{q,\alpha,+}(n) - p_{q,\alpha,-}(n)$$

We now compare the operator B_q to the upper Laplacian Δ_k^{up}. Recall from Equation (1.1) that Δ_k^{up} acts on $\ell^2 I_k$ by

$$\begin{aligned} \Delta_k^{\mathrm{up}}\left(\sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha\right) &= \sum_{\alpha \in I_k}\left[\sum_{\beta \in I_{k+1}} [\beta : \alpha]^2 \lambda_\alpha - \sum_{\alpha' \neq \alpha \in I_k} \sum_{\beta \in I_{k+1}} -[\beta : \alpha][\beta : \alpha']\lambda_{\alpha'}\right] \cdot \alpha \\ &= \sum_{\alpha \in I_k}\left[d_{+,2}(\alpha)\lambda_\alpha - \sum_{\alpha' \neq \alpha \in I_k} \sum_{\beta \in I_{k+1}} d(\alpha, \alpha', \beta)\lambda_{\alpha'}\right] \cdot \alpha \end{aligned}$$

Theorem 3.14. *Let $B_q \curvearrowright \ell^2 I_k = \ell^2 C_k^{\mathrm{cell}}(X)$ be the operator $B_q = TP_qI$ defined as above. Then*

$$B_q \circ M_{1,q} = \mathrm{Id} - \Delta_k^{\mathrm{up}} \circ M_{2,q},$$

where $M_{1,q}, M_{2,q} \curvearrowright \ell^2 I_k$ are the non-negative multiplication operators given by

$$M_{1,q} = \frac{d_+ d_-}{q d_+ d_- + (1-q) d_{+,2}}, \quad \sum_{\alpha \in I_k} \lambda_\alpha \alpha \mapsto \sum_{\alpha \in I_k} \frac{d_+(\alpha) d_-(\alpha)}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} \cdot \lambda_\alpha \alpha,$$

$$M_{2,q} = \frac{1-q}{q d_+ d_- + (1-q) d_{+,2}}, \quad \sum_{\alpha \in I_k} \lambda_\alpha \alpha \mapsto \sum_{\alpha \in I_k} \frac{1-q}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} \cdot \lambda_\alpha \alpha.$$

Proof. For $\alpha, \alpha' \in I_k$ we compare the contributions $(B_q \circ M_{1,q})_{\alpha',\alpha}$ and $(\mathrm{Id} - \Delta_k^{\mathrm{up}} \circ M_{2,q})_{\alpha',\alpha}$ coming from the coefficient of α in the argument to the coefficient of α' in the image.[8] For $\alpha \neq \alpha'$ these contributions are given by

$$\begin{aligned}
(B_q \circ M_{1,q})_{\alpha',\alpha} &= \frac{d_+(\alpha) d_-(\alpha)}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} \cdot (B_q)_{\alpha',\alpha} \\
&= \frac{d_+(\alpha) d_-(\alpha)}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} \cdot \frac{1-q}{d_+(\alpha) d_-(\alpha)} \sum_{\beta \in I_{k+1}} d(\alpha, \alpha', \beta) \\
&= \frac{1-q}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} \sum_{\beta \in I_{k+1}} d(\alpha, \alpha', \beta) \\
&= 0 - \frac{1-q}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} \cdot \left(- \sum_{\beta \in I_{k+1}} d(\alpha, \alpha', \beta) \right) \\
&= (\mathrm{Id} - \Delta_k^{\mathrm{up}} \circ M_{2,q})_{\alpha',\alpha}
\end{aligned}$$

and for $\alpha = \alpha'$ by

$$\begin{aligned}
(B_q \circ M_{1,q})_{\alpha,\alpha} &= q \cdot \frac{d_+(\alpha) d_-(\alpha)}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} \\
&= 1 - \frac{(1-q) d_{+,2}(\alpha)}{q d_+(\alpha) d_-(\alpha) + (1-q) d_{+,2}(\alpha)} = (\mathrm{Id} - \Delta_k^{\mathrm{up}} \circ M_{2,q})_{\alpha,\alpha}.
\end{aligned}$$

Since all these coefficients agree, the claim follows. ☐

Remark 3.15. The construction here generalises the construction given by O. Parzanchevski and R. Rosenthal on simplicial complexes in [PR17] and the previous theorem generalises Proposition 2.8 of their paper. Considering the random walk of O. Parzanchevski and R. Rosenthal in degree $k = (d-1)$, the incidence numbers of a simplicial complex (viewed as a CW complex) are given as $[\beta : \alpha] \in \{0, \pm 1\}$, where ± 1 occurs if the $(d-1)$-simplex α is in the boundary of the d-simplex β, with sign depending on orientations. Therefore,

$$d_+(\alpha) = d_{+,2}(\alpha) = \deg(\alpha), \quad d_-(\alpha) = d,$$

[8] In the sense that both are operators $\Xi \curvearrowright \ell^2 C_k^{\mathrm{cell}}(X)$ that can be written as

$$\Xi \colon \sum_{\alpha \in I_k} \lambda_\alpha \alpha \mapsto \sum_{\alpha' \in I_k} \left[\sum_{\alpha \in I_k} \Xi_{\alpha',\alpha} \lambda_\alpha \right] \cdot \alpha'.$$

where $\deg(\alpha)$ denoted the number of d-simplices $\beta \in I_d$ containing α. Therefore,

$$\frac{d}{q(d-1)+1} B_q = \mathrm{Id} - \frac{1-q}{q(d-1)+1} \cdot \frac{\Delta_{d-1}^{\mathrm{up}}}{\deg(\alpha)},$$

where $\Delta_w^{\mathrm{up}} = \frac{\Delta_{d-1}^{\mathrm{up}}}{\deg(\alpha)}$ is the weighted upper Laplacian used by O. Parzanchevski and R. Rosenthal, defined by using a weighted scalar product on $\ell^2 I_k$. Note that in this case, the diagonal operator $M_{1,q}$ is given by multiplication by a constant (depending on q and d but not on $\alpha \in I_{d-1}$).

3.2.3 Connection to α_k

We now study the connection between the random walk $\mathfrak{R}_q^k = \mathfrak{R}_q^k(X)$ and the Novikov-Shubin invariant $\alpha_k(X)$ for a free G-CW complex X of finite type. In degree zero it is reasonable to consider only connected spaces (since the ℓ^2-spaces and the Laplace operator split as a direct sum with one summand for each connected component). For us, by the same reasoning, we can assume without loss of generality the following analogue in degree k.

Definition 3.16. Let X be a CW complex. We call X upper k-connected if $|I_k| \geq 2$ and for all $\alpha, \alpha' \in I_k$ there are

$$\alpha = \alpha_0, \alpha_1, \dots, \alpha_{n-1}, \alpha_n = \alpha' \in I_k \quad \text{and} \quad \beta_1, \dots, \beta_n \in I_{k+1}$$

such that $[\beta_i, \alpha_{i-1}] \neq 0$ and $[\beta_i : \alpha_i] \neq 0$ for all $1 \leq i \leq n$, that is β_i is attached non-trivially to α_{i-1} and α_i.

This condition implies for $\mathfrak{R}_q^k$ that a random walker can move from any k-cell $\alpha_\pm \in I_k^\pm$ to any other (unoriented) k-cell α' (that is, to one of the oriented k-cells α'_+ or α'_-). Furthermore, we get bounds on the quantities from Definition 3.2.

Lemma 3.17. *Let X be an upper k-connected free G-CW complex of finite type. Then there exists $D \geq 1$ such that*

$$D \geq d_{+,2}, d_+, d_- \geq 1.$$

In particular, if $q \in [0,1)$ then the operators $M_{1,q}$ and $M_{2,q}$ are positive multiplication operators bounded from below by

$$M_{1,q} \geq D^{-2} > 0 \quad \textit{and} \quad M_{2,q} = (1-q)D^{-2} > 0.$$

Proof. Let $\alpha \in I_k$ be arbitrary and let $\alpha \neq \alpha' \in I_k$ be any other k-cell. Since X is upper k-connected, by definition there is a sequence of k-cells $\alpha_i \in I_k$ and $(k+1)$-cells $\beta_i \in I_{k+1}$ connecting α to α'. In particular, there exists a $(k+1)$-cell $\beta_1 \in I_{k+1}$ such that $[\beta_1 : \alpha] \neq 0$ and a k-cell $\alpha \neq \alpha_1 \in I_k$ such that $[\beta_1 : \alpha_1] \neq 0$. Hence

$$\begin{aligned} d_{+,2}(\alpha) &\geq [\beta_1 : \alpha]^2 \geq 1, \\ d_+(\alpha) &\geq |[\beta_1 : \alpha]| \geq 1, \\ d_-(\alpha) &\geq d_-(\beta_1; \alpha) \geq |[\beta_1 : \alpha_1]| \geq 1. \end{aligned}$$

Since X is of finite type and these quantities depend only on the G-type of α, there exists

$$D = \sup_{\alpha \in I_k} \{d_{+,2}(\alpha), d_+(\alpha), d_-(\alpha)\} = \max_{\alpha \in I_k(G \backslash X)} \{d_{+,2}(\alpha), d_+(\alpha), d_-(\alpha)\} \geq 1.$$

It follows, therefore, that

$$
\begin{aligned}
M_{1,q} &= \frac{d_+d_-}{qd_+d_- + (1-q)d_{+,2}} \geq \frac{1}{qD^2 + (1-q)D} \geq \frac{1}{D^2} > 0,\\
M_{2,q} &= \frac{1-q}{qd_+d_- + (1-q)d_{+,2}} \geq \frac{1-q}{qD^2 + (1-q)D} \geq \frac{1-q}{D^2} > 0,
\end{aligned}
$$

and the claim follows. □

Generalising the notion of regular graphs, we introduce the following notion of upper k-regular free G-CW complexes.

Definition 3.18. Let X be a free G-CW complex of finite type. We call X upper k-regular if X is upper k-connected and $d_+d_- = d_+(\alpha)d_-(\alpha)$ and $d_{+,2} = d_{+,2}(\alpha)$ are independent of $\alpha \in I_k$.

In this case, also the multiplication operators $M_{1,q}$ and $M_{2,q}$ are just multiplication with a constant. Hence, the formula connecting B_q and Δ_k^{up} simplifies further.

Corollary 3.19. *Let X be an upper k-regular G-CW complex of finite type and $q \in [0,1]$. Then*

$$C_{1,q}B_q = \mathrm{Id} - C_{2,q}\Delta_k^{\mathrm{up}}$$

for the positive constants

$$C_{1,q} = \frac{d_+d_-}{qd_+d_- + (1-q)d_{+,2}} > 0 \qquad \textit{and} \qquad C_{2,q} = \frac{1-q}{qd_+d_- + (1-q)d_{+,2}} > 0.$$

Definition 3.20. Let X be an upper k-regular free G-CW complex of finite type. We define

$$\widetilde{B_q} = C_{1,q}B_q \quad \text{and} \quad \widetilde{\Delta_{q,k}^{\mathrm{up}}} = C_{2,q}\Delta_k^{\mathrm{up}}$$

so that we have the equality

$$\widetilde{B_q} = \mathrm{Id} - \widetilde{\Delta_{q,k}^{\mathrm{up}}}.$$

We now want to find bounds on the spectrum $\sigma\left(\widetilde{\Delta_{q,k}^{\mathrm{up}}}\right)$ of the operator $\widetilde{\Delta_{q,k}^{\mathrm{up}}}$.

Lemma 3.21. *Let X be a free G-CW complex of finite type, then $\Delta_k^{up} \curvearrowright \ell^2 C_k^{\mathrm{cell}}(X)$ is bounded. In particular $\sigma(\Delta_k^{up}) \subset [0, S_k]$, where*

$$S_k = \max_{\alpha \in I_k(G\backslash X)} \left\{ \sum_{\beta \in I_{k+1}} \sum_{\alpha' \in I_k} |d(\alpha, \alpha', \beta)| \right\} < \infty.$$

Proof. The differential $d_{k+1} \colon \ell^2 C_{k+1}^{\mathrm{cell}}(X) \to \ell^2 C_k^{\mathrm{cell}}(X)$ for a free G-CW complex of finite type has bounded L^2-norm. This follows easily by direct computation, for example in E. Suchla's master thesis [Suc16]. Since this is not readily available online, we quickly recall it here. However, as this is more suitable for our situation, we show boundedness of the adjoint d_{k+1}^* instead.[9]
Let $\omega = \sum_{\alpha \in I_k} \lambda_\alpha \cdot \alpha \in \ell^2 C_k^{\mathrm{cell}}(X)$, then

$$d_{k+1}^*\omega = \sum_{\beta \in I_{k+1}} \left[\sum_{\alpha \in I_k} [\beta : \alpha]\lambda_\alpha \right] \cdot \beta$$

[9] The computation for d_{k+1} is completely analogous.

and, using that $2ab \leq a^2 + b^2$ for all $a, b \geq 0$, we estimate

$$\begin{aligned}
\|d_{k+1}^*\omega\|_{L^2}^2 &= \langle d_{k+1}^*\omega, d_{k+1}^*\omega \rangle \\
&= \sum_{\beta \in I_{k+1}} \left[\sum_{\alpha \in I_k} [\beta : \alpha] \lambda_\alpha \right] \cdot \overline{\left[\sum_{\alpha' \in I_k} [\beta : \alpha'] \lambda_{\alpha'} \right]} \\
&\leq \sum_{\beta \in I_{k+1}} \sum_{\alpha \in I_k} \sum_{\alpha' \in I_k} |\lambda_\alpha| |\lambda_{\alpha'}| |[\beta : \alpha]| |[\beta : \alpha']| \\
&\leq \sum_{\beta \in I_{k+1}} \sum_{\alpha \in I_k} \sum_{\alpha' \in I_k} \frac{|\lambda_\alpha|^2 + |\lambda_{\alpha'}|^2}{2} \cdot |d(\alpha, \alpha', \beta)| \\
&= \frac{1}{2} \sum_{\beta \in I_{k+1}} \sum_{\alpha \in I_k} \sum_{\alpha' \in I_k} |\lambda_\alpha|^2 \cdot |d(\alpha, \alpha', \beta)| \\
&\quad + \frac{1}{2} \sum_{\beta \in I_{k+1}} \sum_{\alpha' \in I_k} \sum_{\alpha \in I_k} |\lambda_{\alpha'}|^2 \cdot |d(\alpha, \alpha', \beta)| \\
&= \sum_{\alpha \in I_k} |\lambda_\alpha|^2 \cdot \sum_{\beta \in I_{k+1}} \sum_{\alpha' \in I_k} |d(\alpha, \alpha', \beta)| \\
&\leq \sup_{\alpha \in I_k} \left\{ \sum_{\beta \in I_{k+1}} \sum_{\alpha' \in I_k} |d(\alpha, \alpha', \beta)| \right\} \cdot \|\omega\|_{L^2}^2.
\end{aligned}$$

Here, since X is a free G-CW complex of finite type, for every $\alpha \in I_k$ only finitely many $d(\alpha, \alpha', \beta)$ are non-zero. In particular, each sum

$$S_k(\alpha) = \sum_{\beta \in I_{k+1}} \sum_{\alpha' \in I_k} |d(\alpha, \alpha', \beta)|$$

is a finite sum. Further, the value of $S(\alpha)$ depends only on the G-type of α. Hence,

$$S_k = \sup_{\alpha \in I_k} \{S(\alpha)\} = \max_{\alpha \in I_k(G \backslash X)} \{S_k(\alpha)\} < \infty$$

and $\|d_{k+1}^*\omega\|_{L^2}^2 \leq S_k \|\omega\|_{L^2}^2$ for all $\omega \in \ell^2 C_k^{\text{cell}}(X)$. This implies that d_{k+1}^* is bounded,

$$\|d_{k+1}^*\|_{\ell^2 C_k^{\text{cell}}(X) \to \ell^2 C_{k+1}^{\text{cell}}(X)} \leq \sqrt{S_k}.$$

Since $\Delta_k^{\text{up}} = d_{k+1} d_{k+1}^*$, we obtain

$$\|\Delta_k^{\text{up}}\|_{\ell^2 C_k^{\text{cell}}(X) \to \ell^2 C_k^{\text{cell}}(X)} = \|d_{k+1}^*\|_{\ell^2 C_k^{\text{cell}}(X) \to \ell^2 C_{k+1}^{\text{cell}}(X)}^2 \leq S_k.$$

Since Δ_k^{up} is non-negative and self-adjoint, this implies $\sigma(\Delta_k^{\text{up}}) \subset [0, S_k]$. □

Note that the same argument binds the spectrum of the full Laplacian $\Delta = d^*d + dd^*$ with $\sigma(\Delta_k) \subset [0, 2\max\{S_{k-1}, S_k\}]$, though this is not needed here.
Using this, we can prove the following lemma.

Lemma 3.22. *Let X be an upper k-regular free G-CW complex of finite type. Then there exists $q_0 \in (0, 1)$ such that for all $q_0 \leq q \leq 1$ the spectrum of $\widetilde{\Delta_{q,k}^{\text{up}}}$ satisfies $\sigma(\widetilde{\Delta_{q,k}^{\text{up}}}) \subset [0, 1]$.*

Proof. By Lemma 3.21, $\sigma(\Delta_k^{\mathrm{up}}) \subset [0, S]$ for some $S > 0$, hence $\sigma(\widetilde{\Delta_{q,k}^{\mathrm{up}}}) \subset [0, C_{2,q}S]$. Note that $d_+d_- \geq 1$ and $d_{+,2} \geq 1$ so $C_{2,q} = \frac{1-q}{qd_+d_- + (1-q)d_{+,2}}$ is continuous in $q \in (0,1)$ and converges to 0 as $q \nearrow 1$. In particular, there is $q_0 \in (0,1)$ such that $C_{2,q} \leq S^{-1}$ for all $q_0 \leq q \leq 1$. $\square$

Corollary 3.23. *Let X be an upper k-regular free G-CW complex of finite type and $q \in [q_0, 1)$. Let $\widetilde{d}_{k+1} = \sqrt{C_{2,q}}d_{k+1}$. Then $\widetilde{d}_{k+1}^* = \sqrt{C_{2,q}}d_{k+1}^*$ and*

$$\widetilde{\Delta_{q,k}^{\mathrm{up}}} = \widetilde{d}_{k+1}\widetilde{d}_{k+1}^*$$

is a self-adjoint positive operator with $\sigma(\widetilde{\Delta_{q,k}^{\mathrm{up}}}) \subset [0,1]$.

Remark 3.24. Recall that for $q \in [q_0, 1)$, since d and $\widetilde{d}$ differ only by a constant factor $\sqrt{C_{2,q}}$, their spectral density functions are dilatationally equivalent and hence their Novikov-Shubin invariants agree, that is, $\alpha_k(X) = \alpha(d_{k+1}) = \alpha(\widetilde{d}_{k+1}) = \alpha(\widetilde{d}_{k+1}^*)$.

Lemma 3.25. *Let χ_I denote the indicator function of the interval I, then*

$$\mathrm{tr}_{\mathcal{N}G}(\chi_{[1-\lambda,1]}(\widetilde{B_q})) = F(\widetilde{d}_{k+1}^*)(\sqrt{\lambda}).$$

Proof. Recall that $\widetilde{B_q} = \mathrm{Id} - \widetilde{d}_{k+1}\widetilde{d}_{k+1}^*$, hence

$$\begin{aligned}\mathrm{tr}_{\mathcal{N}G}\left(\chi_{[1-\lambda,1]}\left(\widetilde{B_q}\right)\right) &= \mathrm{tr}_{\mathcal{N}G}\left(\chi_{[0,\lambda]}\left(\widetilde{d}_{k+1}\widetilde{d}_{k+1}^*\right)\right) \\ &= \mathrm{tr}_{\mathcal{N}G}\left(E_\lambda^{\widetilde{d}_{k+1}\widetilde{d}_{k+1}^*}\right) = F\left(\widetilde{d}_{k+1}^*\right)\left(\sqrt{\lambda}\right).\end{aligned}$$ $\square$

We can now proceed in the same way as in degree zero, compare Lück's book [Lüc02, §2.1.4].

Theorem 3.26. *Let X be an upper k-regular free G-CW complex of finite type and $q \in [q_0, 1)$, with q_0 given by Lemma 3.22. Then $\alpha_k(X) = 2a$ if and only if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$C_{1,q}^{-n}\left(b^{(2)}(d_{k+1}^*) + C^{-1}n^{-a}\right) \quad \leq \quad p_q(n) \quad \leq \quad C_{1,q}^{-n}\left(b^{(2)}(d_{k+1}^*) + Cn^{-a}\right).$$

Proof. Since by Lemma 3.22, $\sigma(\widetilde{\Delta_{q,k}^{\mathrm{up}}}) \subset [0,1]$ and by construction $\widetilde{\Delta_{q,k}^{\mathrm{up}}} = \mathrm{Id} - \widetilde{B_q}$, it follows that also $\sigma(\widetilde{B_q}) \subset [0,1]$. Therefore,

$$(1-\lambda)^n\chi_{[1-\lambda,1]}(\widetilde{B_q}) \quad \leq \quad \widetilde{B_q}^n \quad \leq \quad (1-\lambda)^n\chi_{[0,1-\lambda]}(\widetilde{B_q}) + \chi_{[1-\lambda,1]}(\widetilde{B_q}).$$

Taking traces with Lemma 3.25 and denoting $\widetilde{p}_q(n) = \mathrm{tr}_{\mathcal{N}G}(\widetilde{B_q}^n)$ yields

$$(1-\lambda)^n F(\widetilde{d}_{k+1}^*)(\sqrt{\lambda}) \quad \leq \quad \widetilde{p}_q(n) \quad \leq \quad (1-\lambda)^n + F(\widetilde{d}_{k+1}^*)(\sqrt{\lambda}).$$

By rearranging these terms and taking logarithms we obtain the inequalities

$$\frac{\log\left(F(\widetilde{d}_{k+1}^*)(\sqrt{\lambda}) - b^{(2)}(d_{k+1}^*)\right)}{\log\lambda} \leq \frac{\log(\widetilde{p}_q(n) - (1-\lambda)^n b^{(2)}(d_{k+1}^*))}{\log\lambda} - n \cdot \frac{\log(1-\lambda)}{\log\lambda}, \tag{3.4}$$

$$\frac{\log\left(F(\widetilde{d}_{k+1}^*)(\sqrt{\lambda}) - b^{(2)}(d_{k+1}^*)\right)}{\log\lambda} \geq \frac{\log(\widetilde{p}_q(n) - b^{(2)}(d_{k+1}^*) - (1-\lambda)^n)}{\log\lambda}. \tag{3.5}$$

Using $b^{(2)}(d_{k+1}^*) = b^{(2)}(\widetilde{d}_{k+1}^*)$ and taking the limit inferior for $\lambda \searrow 0$ on the left-hand-sides gives

$$\liminf_{\lambda\searrow 0} \frac{\log\left(F(\widetilde{d})(\sqrt{\lambda}) - b^{(2)}(d_{k+1}^*)\right)}{\log\lambda} = \frac{\alpha(\widetilde{d}_{k+1}^*)}{2} = \frac{\alpha(\widetilde{d}_{k+1})}{2} = \frac{\alpha_k(X)}{2}.$$

After substituting $p(n) = \widetilde{p}_q(n) - b^{(2)}(d_{k+1}^*)$, the term on the right-hand-side of Equation (3.5) agrees with the term in Lück's book [Lüc02, Thm. 2.48], so it follows by the same argument that

$$\alpha_k(X) \leq 2a \qquad \text{if } \widetilde{p}_q(n) \geq b^{(2)}(d_{k+1}^*) + Dn^{-a} \text{ for } n \geq 1,$$

for some constant $D > 0$.
For the right-hand-side of Equation (3.4), let $\varepsilon > 0$ be arbitrarily small and $n = n(\lambda)$ the largest integer such that $n \leq \lambda^{-\varepsilon}$, that is $n = \lfloor \lambda^{-\varepsilon} \rfloor$. If $\widetilde{p}(n) \geq Cn^{-a} + b^{(2)}(d)$ for some constant $C > 0$ and $n \geq 1$, we obtain

$$\begin{aligned} &\frac{\log(\widetilde{p}_q(n) - (1-\lambda)^n b^{(2)}(d_{k+1}^*))}{\log\lambda} - n \cdot \frac{\log(1-\lambda)}{\log\lambda} \\ &\geq \frac{\log(Cn^{-a} + [1-(1-\lambda)^n]\, b^{(2)}(d_{k+1}^*))}{\log\lambda} - \frac{\log(1-\lambda)}{\lambda^{\varepsilon}\log\lambda} \\ &\geq \frac{\log(Cn^{-a})}{\log\lambda} - \frac{\log(1-\lambda)}{\lambda^{\varepsilon}\log\lambda}, \end{aligned}$$

where we use $[1-(1-\lambda)^n]\, b^{(2)}(d_{k+1}^*) \geq 0$ (indeed, even $[1-(1-\lambda)^n]\, b^{(2)}(d_{k+1}^*) \xrightarrow{\lambda\searrow 0} 1 - e^{-\varepsilon}$). From here, we proceed precisely as in W. Lück's book [Lüc02, Thm. 2.48] and find

$$\alpha_k(X) \geq 2a \qquad \text{if } \widetilde{p}_q(n) \leq b^2(d_{k+1}^*) + Cn^{-a} \text{ for } n \geq 1,$$

concluding the proof of the theorem. □

Remark 3.27. This generalises the theorem in degree zero, since in degree zero we have $d_+ = d_{+,2} = |S|$, where $|S|$ is the size of a finite generating set of G chosen in the construction of Cayley(G), and $d_- = 1$. Thus $C_{1,q} = 1$ and the exponential decay factor $C_{1,q}^{-n} = 1$ disappears.

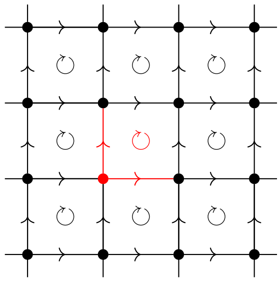
Example 3.28. Let $k \geq 2$ and let G be a finitely generated group with Cayley graph Cayley(G). Construct a G-CW complex X in the following way.

- Start with $X^{(1)} = \text{Cayley}(G)$.[10]
- For every $g \in G$ glue one k-cell α_g to $X^{(1)}$ by collapsing the boundary of α_g to the vertex v_g corresponding to $g \in G$ in the Cayley graph. This defines the k-skeleton $X^{(k)}$.
- For every edge (g, gs) in the Cayley graph, glue one $(k+1)$-cell $\beta_{g,gs}$ to $X^{(k)}$ by sending the boundary of $\beta_{g,gs}$ to $\alpha_g \cup (g, gs) \cup \alpha_{gs}$ such that $[\beta_{g,gs} : \alpha_g] = -[\beta_{g,gs} : \alpha_{gs}] \in \{\pm 1\}$. This defines $X^{(k+1)} = X$.

On X, the degree k upper random walk $\mathfrak{R}^k$ agrees with the random walk $\mathfrak{R}$ on Cayley(G) when identifying the state corresponding to $\alpha_g = (\alpha_g)_+$ in $\mathfrak{R}^k$ with the state corresponding to g in $\mathfrak{R}$. In particular, for $\mathfrak{R}^k$ we have $p_-(n) \equiv 0$ so that $p(n) = p_+(n)$ is the usual return probability. Further, the values $d_+ = d_{2,+} = |S|$ and $d_- = 1$ agree with the values on Cayley(G), so that $C_{1,q} = 1$. Therefore, the previous theorem tells us that for X we obtain $\alpha_k(X) = \alpha_0(X)$.
Indeed, we can also see this directly because $d_{k+1}d_{k+1}^*$ and $d_1 d_1^*$ are identical up to identifying α_g with g and $\beta_{g,gs}$ with (g, gs).

[10] If $k \geq 3$ and G is finitely presented, we can further glue in 2-cells according to the relations in G, so that $X^{(2)}$ is the Cayley complex of G. In that case the constructed CW complex X satisfies $\pi_1(X) = G$, see for example A. Hatcher's book [Hat02, p. 77].

3.3 Example: Degree 1 Upper Random Walk on $\mathbb{R}^2$



Consider $\mathbb{R}^2$ as a $\mathbb{Z}^2$-CW complex of finite type as shown on the right, with arrows indicating the chosen preferred orientation. For notation's sake, we will write $\mathbb{Z}^2$ as a multiplicative group with unit element $1 \in \mathbb{Z}^2$. Let x and y be two generators of $\mathbb{Z}^2 = \langle x, y \mid [x,y] = 1\rangle$ and the $\mathbb{Z}^2$-action on this CW complex be generated by x shifting to the right by one and y shifting up by one. The red cells indicate $\mathbb{Z}^2$-bases. We will denote the 0-basis $\mathcal{B}_0 = \{\gamma_\bullet\}$, the 1-basis $\mathcal{B}_1 = \{\alpha_\uparrow, \alpha_\rightarrow\}$ and the 2-basis $\mathcal{B}_2 = \{\beta_\circlearrowright\}$ in the way suggested by the indices. Given a cell c and $g = x^a y^b \in \mathbb{Z}^2$, we denote by gc the cell obtained by translating c by g, that is a units to the right and b units up.
The incidence numbers between a 2-cell β and a 1-cell α are given by $[\beta : \alpha] = 0$ if β and α do not touch and $[\beta : \alpha] = \pm 1$ if the cells touch; with sign $+1$ if the orientation β induces on α agrees with the orientation on α and -1 otherwise. This is an upper 2-regular CW complex with

$$d_+ = 2, \quad d_{+,2} = 2, \quad d_- = 3, \quad C_{1,q} = \frac{3}{2q+1}, \quad C_{2,q} = \frac{1}{2}\frac{1-q}{2q+1}, \quad C_{1,q}^{-1}C_{2,q} = \frac{1-q}{6}.$$

The upper Laplacian $\Delta = \Delta_1^{\mathrm{up}} \curvearrowright \ell^2\left((\mathbb{R}^2)^{(1)}\right)$ in degree one can be written, with respect to the basis $\mathcal{B}_1$, as the $\mathbb{C}[\mathbb{Z}^2]$-valued matrix

$$\Delta = 2 - \begin{pmatrix} x + x^{-1} & 1 - x - y^{-1} + xy^{-1} \\ 1 - x^{-1} - y + x^{-1}y & y + y^{-1} \end{pmatrix}.$$

For the non-lazy random walk on 1-cells, on $\mathcal{B}_1$ the propagation operator is given as described in Figure 3.1.

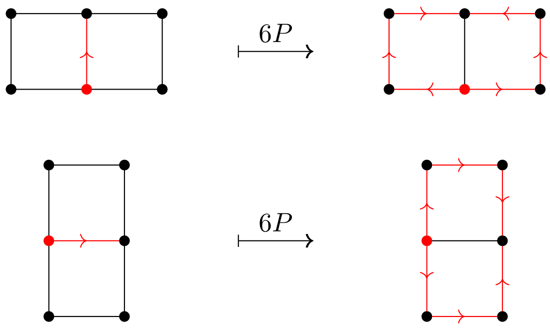


Figure 3.1: Visual representation of propagation operator P

Accounting for changing orientations with signs, this means we can write the corresponding operator $B = TPI$ with respect to $\mathcal{B}_1$ as the $\mathbb{C}[\mathbb{Z}^2]$-valued matrix

$$B = \frac{1}{6}\begin{pmatrix} x + x^{-1} & 1 - x - y^{-1} + xy^{-1} \\ 1 - x^{-1} - y + x^{-1}y & y + y^{-1} \end{pmatrix}.$$

We can readily verify that for $q \in [0,1]$ indeed

$$B_q = q \operatorname{Id} + (1-q)B = q \operatorname{Id} + \frac{1-q}{6}\left(2 \operatorname{Id} - \Delta\right) = C_{1,q}^{-1} \operatorname{Id} - C_{1,q}^{-1} C_{2,q} \Delta,$$

and thus $C_{1,q}B_q = \mathrm{Id} - C_{2,q}\Delta$. Looking at the boundary of $\beta_{\circlearrowright}$ given by

$$S = (1-x)\alpha_\uparrow + (y-1)\alpha_\rightarrow,$$

it is an eigenstate of B with eigenvalue $\frac{1}{6}\left(x + x^{-1} + y + y^{-1} - 2\right)$, compare Figure 3.2.

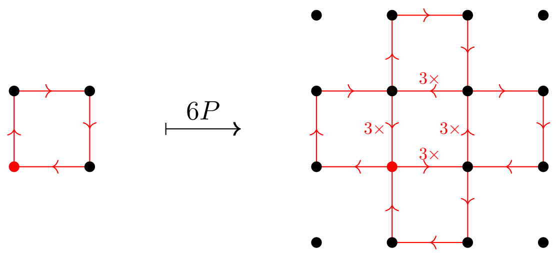


Figure 3.2: A visual representation of S and BS.

Here, $x + x^{-1} + y + y^{-1} = 4P^{\mathbb{Z}^2}$, where $P^{\mathbb{Z}^2}$ can formally also be interpreted as the propagation operator of the uniform nearest neighbour random walk on the grid Cayley($\mathbb{Z}^2$) (or, in this case, rather the 2-cells of $\mathbb{R}^2$ with this chosen CW structure). We denote $\lambda = \frac{1}{6}(4P^{\mathbb{Z}^2} - 2)$. A straight-forward computation shows that S is an eigenstate to B_q with eigenvalue

$$\lambda_q = C_{1,q}^{-1}C_{2,q}\left(4P^{\mathbb{Z}^2} + [C_{2,q}^{-1} - 4]\right).$$

Since $C_{2,q}(4 + [C_{2,q}^{-1} - 4]) = 1$, we can set $q' = 1 - 4C_{2,q} = \frac{4q-1}{2q+1}$ and can formally interpret

$$4C_{2,q}P^{\mathbb{Z}^2} + [1 - 4C_{2,q}] = P_{q'}^{\mathbb{Z}^2}$$

as the propagation operator of the corresponding q'-lazy random walk on Cayley($\mathbb{Z}^2$). Note that for $q \in [1/4, 1]$ we have $4C_{2,q} \in [0,1]$ and $q' \in [0,1]$ so this makes sense. In particular,

$$\lambda_q = C_{1,q}^{-1}P_{q'}^{\mathbb{Z}^2} \quad \text{and} \quad B_qS = C_{1,q}^{-1}P_{q'}^{\mathbb{Z}^2}S.$$

The return quantity $p_q(n)$ that we are interested in is given by $p_q(n) = p_{q,\alpha_\uparrow}(n) + p_{q,\alpha_\rightarrow}(n)$, where $p_{q,\alpha_\uparrow}(n) = \langle B_q^n\alpha_\uparrow, \alpha_\uparrow\rangle$ is the coefficient of $1\alpha_\uparrow$ in $B_q^n\alpha_\uparrow$ and similarly for $\alpha_\rightarrow$. By symmetry, $p_{q,\alpha_\uparrow}(n) = p_{q,\alpha_\rightarrow}(n)$ so that

$$p_q(n) = 2p_{q,\alpha_\uparrow}(n) = 2\langle B_q^n\alpha_\uparrow, \alpha_\uparrow\rangle.$$

We note from Figure 3.1 that

$$B_q(\alpha_\uparrow) = q\alpha_\uparrow + \frac{1-q}{6}\left(x^{-1}S - S + 2\alpha_\uparrow\right) = C_{1,q}^{-1}\alpha_\uparrow + C_{1,q}^{-1}C_{2,q}(x^{-1} - 1)S. \tag{3.6}$$

Since the random walk is $\mathbb{Z}^2$-invariant, this yields

$$B_q^n(\alpha_\uparrow) = C_{1,q}^{-1}B_q^{n-1}(\alpha_\uparrow) + C_{1,q}^{-1}C_{2,q}(x^{-1} - 1)B_q^{n-1}S$$

Resolving this recursive formula we obtain

$$\begin{aligned} B_q^n(\alpha_\uparrow) &= C_{1,q}^{-n}\alpha_\uparrow + \sum_{k=0}^{n-1} C_{1,q}^{-n+k}C_{2,q}(x^{-1} - 1)B_q^kS \\ &= C_{1,q}^{-n}\left(\alpha_\uparrow + \sum_{k=0}^{n-1} C_{2,q}(x^{-1} - 1)(P_{q'}^{\mathbb{Z}^2})^kS\right) \end{aligned}$$

In order to find the coefficient of $1\alpha_\uparrow$, we notice that

$$\langle S, 1\alpha_\uparrow\rangle = 1, \qquad \langle x^{-1}S, 1\alpha_\uparrow\rangle = -1 \quad \text{and} \quad \langle gS, 1\alpha_\uparrow\rangle = 0 \quad \text{for } g \notin \{1, x^{-1}\}$$

and therefore it follows that

$$\begin{aligned} \langle (P_{q'}^{\mathbb{Z}^2})^k S, 1\alpha_\uparrow\rangle &= \langle (P_{q'}^{\mathbb{Z}^2})^k, 1 - x^{-1}\rangle, \\ \langle x^{-1}(P_{q'}^{\mathbb{Z}^2})^k S, 1\alpha_\uparrow\rangle &= \langle (P_{q'}^{\mathbb{Z}^2})^k, x - 1\rangle. \end{aligned}$$

Using this we obtain

$$\begin{aligned} \frac{1}{2}p_q(n) = \langle B_q^n \alpha_\uparrow, \alpha_\uparrow\rangle &= C_{1,q}^{-n}\left(1 + \sum_{k=0}^{n-1} C_{2,q}\left\langle (P_{q'}^{\mathbb{Z}^2})^k, (x-1) - (1-x^{-1})\right\rangle\right) \\ &= C_{1,q}^{-n}\left(1 + \sum_{k=0}^{n-1} C_{2,q}\left\langle (P_{q'}^{\mathbb{Z}^2})^k, x + x^{-1} - 2\right\rangle\right). \end{aligned}$$

By symmetry, the coefficients of $(P_{q'}^{\mathbb{Z}^2})^k$ for x and x^{-1} agree, hence

$$\begin{aligned} \frac{1}{2}C_{1,q}^n p_q(n) &= 1 - 2C_{2,q}\sum_{k=0}^{n-1}\left\langle (P_{q'}^{\mathbb{Z}^2})^k, 1 - x\right\rangle \\ &= 1 - 2C_{2,q}\sum_{k=0}^{n-1}\left(p_{q'}^{\mathbb{Z}^2}(k) - p_{q'}^{\mathbb{Z}^2}(e \xrightarrow{k} x)\right) \end{aligned}$$

where $p_{q'}^{\mathbb{Z}^2}(k)$ is the return probability of the q'-lazy nearest neighbour random walk on $\mathbb{Z}^2$ after k steps and $p_{q'}^{\mathbb{Z}^2}(e \xrightarrow{k} x)$ the probability of the random walk to be at the vertex x after k steps. If we write $\mathbb{E}_q^g(n)$ for the expected number of visits of the vertex g in the first n steps for the q-lazy nearest neighbour random walk on $\mathbb{Z}^2$ (counting the starting position for $\mathbb{E}_q^e(n)$, if $q = 0$ we suppress it in notation), we can write this as

$$\frac{1}{2}C_{1,q}^n p_q(n) = 1 - 2C_{2,q}(\mathbb{E}_{q'}^e(n-1) - \mathbb{E}_{q'}^x(n-1)).$$

Notice that $q' = 1 - 4C_{2,q}$ implies that $2C_{2,q} = \frac{1-q'}{2}$. Hence,

$$\frac{1}{2}C_{1,q}^n p_q(n) = 1 - \frac{1-q'}{2}(\mathbb{E}_{q'}^e(n-1) - \mathbb{E}_{q'}^x(n-1)).$$

For $q < 1$ large enough, we expect[11] that

$$\mathbb{E}_{q'}^e(n-1) - \mathbb{E}_{q'}^x(n-1) \sim 1 - \Theta(n^{-1}) \qquad \text{for } n \to \infty.$$

Plugging this back into the equation above, this would imply that

$$p_q(n) \sim C_{1,q}^{-n}\left(1 + \Theta(n^{-1})\right) \qquad \text{for } n \to \infty.$$

Here, we can read off $b^{(2)}(d_2^*) = 1$, corresponding to the kernel of d_2^* of $\mathcal{N}G$-dimension one, and the Novikov-Shubin invariant $\alpha_1(\mathbb{R}^2) = \alpha(d_2^*) = 2$.

[11] To see that this asymptotic behaviour holds for the classical random walk on $\mathbb{Z}^2$, we give the following heuristic argument. We can interpret a q-lazy random walk as a non-lazy random walk by ignoring the times when the random walk stays put. When doing so, a path of length n in the q-lazy random walk is expected to give a path of length $(1-q)n$ in the non-lazy random walk. Further, the q-lazy random walk is expected to stay in one place for $(1-q)^{-1}$ steps. Therefore, it is expected that

$$\mathbb{E}_q^g(n) \sim (1-q)^{-1}\max\{\mathbb{E}^g(\lfloor(1-q)n\rfloor), \mathbb{E}^g(\lfloor(1-q)n\rfloor + 1)\} \qquad \text{for } n \to \infty.$$

For the non-lazy random walk and $e \sim x$, $\mathbb{E}^e(2n) - \mathbb{E}^x(2n) = 1$ and $\mathbb{E}^e(2n+1) - \mathbb{E}^x(2n+1) = 1 - \Theta(n^{-1})$.

3.4 Remarks and Future Directions

There are some questions coming from a stochastic point of view that can be studied further. Firstly, is it necessary to consider the asymptotic behaviour of $p_q(n)$ or is the behaviour the same for every G-type of k-cell $\alpha \in I_k(G\backslash X)$, so that we only have to consider $p_{q,\alpha}(n) \sim p_q(n)$ for some $\alpha \in I_k(G\backslash X)$? Secondly, is a lower bound on the lazy probability q necessary or does the result hold for any $q \in [0,1]$? Certainly, as in the degree zero case, such a more general result could only hold if we consider the asymptotic behaviour of $n \mapsto p(2n)$ for even arguments only, as otherwise there are examples where $p(2n+1) \leq 0$ holds. In this setup, however, it seems possible that $p(2n) > 0$ for all $n \in \mathbb{N}$ and $p_q(2n) \sim p_{q'}(n)$ for all $q \in [0,1)$ (certainly if q large enough such that $\sigma(B_q) \subset [-1,1]$) and $q' \in [q_0, 1)$ large enough (implying $\sigma(B_{q'}) \subset [0,1]$).

Going in another direction, in Theorem 3.26, we only consider complexes that are upper k-regular. This is a strong assumption on the local structure of the CW complex, so that we can assume the multiplication operators $M_{1,q}$ and $M_{2,q}$ are just multiplication by a constant.

For $M_{2,q}$ it seems like this should not be strictly necessary, as the spectral density function $F(\Delta_k^{\mathrm{up}} \circ M_{2,q})$ should be dilatationally equivalent to $F(\Delta_k^{\mathrm{up}})$. However, note that the operator $\Delta_k^{\mathrm{up}} \circ M_{2,q}$ need not be self-adjoint in general. This means that the seemingly straight-forward way to generalise Lemma 3.25 and therefore also Theorem 3.26 is not possible that easily.

Concerning $M_{1,q}$, we know that for any free G-CW complex of finite type, there are constants $0 < C_{1,q}^{\min} \leq M_{1,q} \leq C_{1,q}^{\max}$ (as $M_{1,q}$ contains only finitely many different, positive entries). However, bounds similar in nature to the theorem, like

$$(C_{1,q}^{\max})^{-n}\left(b^{(2)}(d_{k+1}^*) + C^{-1}n^{-a}\right) \quad \leq \quad p_q(n) \quad \leq \quad (C_{1,q}^{\min})^{-n}\left(b^{(2)}(d_{k+1}^*) + Cn^{-a}\right),$$

are not useful if we want to find the exponent of a secondary, polynomial decay of $p_q(n)$. If one wants to generalise the theorem, one needs a better understanding of the rate of exponential decay first. It is possible that the statement is true for some constant $C_{1,q}$ with $C_{1,q}^{\min} \leq C_{1,q} \leq C_{1,q}^{\max}$. This constant should only depend on the local structure, that is, on $G\backslash X$.
The example discussed in Section 3.3 suggests another way of defining this random walk, which may be more natural if we want to interpret it as heat spreading out through space in some higher dimensional sense. This also might lead to a better interpretation of $C_{1,q}$. In our description, a random walker moving up $\alpha \nearrow \beta$ is not allowed to move back to the cell $-\alpha$. This is a somewhat arbitrary choice made in our definition of $\mathfrak{R}_q^k$. The equation

$$B_q = C_{1,q}^{-1}\,\mathrm{Id} + C_{1,q}^{-1}C_{2,q}\Delta,$$

leading to Equation (3.6) in the example, suggests an alternative definition of the random walk. The new lazy "probability"[12] is now given by $C_{1,q}^{-1}$ and the random walker moves with "probability" $C_{1,q}^{-1}C_{2,q}$ according to any of the eight moves described by $(x^{-1}-1)S$, including the move $\alpha_\uparrow \to -\alpha_\uparrow$ twice. The increase of the lazy probability and the probability $\mathbb{P}(\alpha_\uparrow \to -\alpha_\uparrow)$ are equal, so that they cancel out when passing from the random walk to B_q. This gives some intuition for the constants $C_{1,q}^{-1}$ and $C_{2,q}$ in this example. In this new interpretation, $C_{1,q}^{-1}$ is the "lazy probability" and $C_{2,q}$ the factor such that $C_{1,q}^{-1}C_{2,q} \cdot d(\alpha, \alpha', \beta)$ is the "probability" of doing any particular move $\alpha \xrightarrow{\beta} \alpha'$, allowing now $\alpha \xrightarrow{\beta} -\alpha$ as well. Indeed, the move $\alpha \to -\alpha$ will be possible with "probability" $C_{1,q}^{-1}C_{2,q} \cdot d_{+,2}(\alpha)$.

[12] Here, the "probabilities" will not add up to one but to $1 + 2 \cdot (2C_{1,q}^{-1}C_{2,q})$. We may rescale all probabilities accordingly to obtain true probabilities if needed.

Chapter 4

Novikov-Shubin Invariants of Nilpotent Lie Groups

While higher Novikov-Shubin invariants α_k for $k \geq 1$ are hard to compute in general, a result of M. Rumin ([Rum90], [Rum99]) introduced a rather simple technique to get bounds on Novikov-Shubin invariants in some cases. By defining a suitable subcomplex of the deRham complex, now known as the Rumin complex, he proved that for any contact manifold M of dimension $2n+1$ the Novikov-Shubin invariants are given by

$$\alpha_k(M) = \begin{cases} 2n+2 & \text{for } 1 \leq k \leq 2n+1,\ k \notin \{n, n+1\}, \\ n+1 & \text{for } k \in \{n, n+1\}. \end{cases}$$

This includes the Heisenberg group $\mathbb{H}^3$ and its higher-dimensional analogues. This subcomplex can be studied in more generality and can help in the computation of Novikov-Shubin invariants for graded nilpotent Lie groups. This is explained in M. Rumin's paper [Rum01]. A slight modification allows to work with the simpler Lie algebra cohomology of the corresponding Lie algebra and a subcomplex of the underlying Chevalley-Eilenberg complex instead.

While I was working on this, this method was independently described by F. Tripaldi in [Tri20], and then in more detail by V. Fischer and F. Tripaldi in [FT22]. We will therefore only briefly recall the constructions and results and refer to these papers for more details.

4.1 The Left-Invariant Rumin Complex

Let G be a nilpotent Lie group of dimension n and with associated Lie algebra $\mathfrak{g}$. Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a basis for $\mathfrak{g}$ with structure constants $\{c_{i,j}^l\}_{1\leq i,j,l\leq n}$ defined by the equations

$$[X_i, X_j] = \sum_{l=1}^{n} c_{i,j}^l X_l$$

for $1 \leq i, j \leq n$. A weight function is a map $w \colon \mathcal{B} \to \mathbb{N}$ which satisfies that

$$w(X_i) + w(X_j) = w(X_l) \qquad \text{if } c_{i,j}^l \neq 0. \tag{4.1}$$

We write $w_i = w(X_i)$. Given a weight function, we define the subspace of elements of pure weight w_0 by

$$\mathfrak{g}(w_0) = \operatorname{span}\{X_i \mid 1 \leq i \leq n,\ w_i = w_0\}.$$

The condition given by Equation (4.1) on the weights implies that $[\mathfrak{g}(w), \mathfrak{g}(w')] \subset \mathfrak{g}(w+w')$.[1] We extend such a weight function to the dual $\mathfrak{g}^* = \Lambda^1\mathfrak{g}^*$ spanned by the canonical dual basis

$$\mathcal{B}^1 = \{\theta_i = X_i^* \mid 1 \leq i \leq n\}$$

by declaring $w(\theta_i) = w_i$ and to higher forms $\Lambda^k\mathfrak{g}^*$ spanned by

$$\mathcal{B}^k = \{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

by declaring the weights of a product as the sum of weights of the factors,

$$w(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}) = w_{i_1} + \cdots + w_{i_k}.$$

As before, we define the subspace of pure weight w_0 by $\Lambda^k\mathfrak{g}^*(w_0) = \operatorname{span}\left\{\alpha \in \mathcal{B}^k \,\middle|\, w(\alpha) = w_0\right\}$. For $k = 0$, we define $w \equiv 0$ on $\Lambda^0\mathfrak{g}^*$. For $1 \leq k \leq n$, the space $\Lambda^k\mathfrak{g}^*$ decomposes as

$$\Lambda^k\mathfrak{g}^* = \bigoplus_{w \in \mathcal{W}_k} \Lambda^k\mathfrak{g}^*(w),$$

where $\mathcal{W}_k = \left\{w(\alpha) \,\middle|\, \alpha \in \mathcal{B}^k\right\} \subset \mathbb{N}_0$ is the set of weights of k-forms in $\mathcal{B}^k$. We denote by

$$N_k^{\min} = \min \mathcal{W}_k \quad \text{respectively} \quad N_k^{\max} = \max \mathcal{W}_k$$

the minimal respectively maximal weight of k-forms. If a form $\alpha \in \Lambda^k\mathfrak{g}^*$ is not of pure weight, we denote by $w(\alpha)$ the minimal subset $W \subset \mathcal{W}_k$ such that $\alpha \in \bigoplus_{w \in W} \Lambda^k\mathfrak{g}^*(w)$.[2]
Recall that the Chevalley-Eilenberg complex $(\Lambda^k\mathfrak{g}^*, d_\mathfrak{g})$ is given by the objects $\Lambda^k\mathfrak{g}^*$ together with the differential $d_\mathfrak{g}$ defined for $\alpha \in \Lambda^k\mathfrak{g}^*$ and $X_0, \ldots, X_k \in \mathfrak{g}$ by the Cartan formula

$$d_\mathfrak{g}(\alpha)(X_0, \ldots, X_k) = \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \overset{\widehat{X_i}, \widehat{X_j}}{\ldots}, X_k),$$

where a hat indicates leaving out the corresponding argument. The Chevalley-Eilenberg complex $(\Lambda^k\mathfrak{g}^*, d_\mathfrak{g})$ can be viewed as the subcomplex of left-invariant differential forms of the deRham complex $(\Omega^k G, d)$, leading to an isomorphism

$$\Lambda^k\mathfrak{g}^* \otimes \mathcal{C}^\infty G \cong \Omega^k G.$$

Under this isomorphism the deRham differential is given on pure tensors by

$$d\colon \quad \alpha \otimes f \quad \mapsto \quad d_\mathfrak{g}\alpha \otimes f + df \wedge \alpha \otimes 1.$$

We extend the weight function w to $\Omega^k G$ by declaring subspaces of the form

$$\Omega^k G(w_0) = \Lambda^k\mathfrak{g}^*(w_0) \otimes \mathcal{C}^\infty G$$

to be of pure weight w_0 and proceeding as before. Notice that $d_\mathfrak{g}$ preserves the weight of a form since by assumption $w([X_i, X_j]) = w_i + w_j$, as can easily be seen from the Cartan formula. For the second term, the map $\alpha \otimes f \mapsto df \wedge \alpha \otimes 1$ preserves weight if and only if it vanishes since f is of weight zero but $df \neq 0$ at least of weight one.

[1] We may refer to 0 to be of any weight or ignore 0 depending on the situation, this will not be a big issue.
[2] To simplify notation, we may also write $w(\alpha) = \{w_0\}$ if α is of pure weight w_0.

The Rumin complex (E_0, d_c) is a subcomplex of, and homotopy equivalent to, the deRham complex $(\Omega^k G, d)$. The objects E_0 are given in terms of a chosen metric and the weight-preserving part of d (which we denote again by $d_\mathfrak{g}$, that is, $d_\mathfrak{g} \colon \alpha \otimes f \mapsto d_\mathfrak{g}\alpha \otimes f$), as

$$E_0^k = \ker d_\mathfrak{g} \cap (\operatorname{im} d_\mathfrak{g})^\perp \subset \Omega^k G.$$

For more details of the construction and a description of the differential d_c of the Rumin complex we refer to V. Fischer and F. Tripaldi's detailed description in [FT22]. Note that we can define a representing system of the Lie algebra cohomology after choosing a metric as

$$H^k(\mathfrak{g}) = H^k(\Lambda^\bullet \mathfrak{g}^*, d_\mathfrak{g}) \cong \ker d_\mathfrak{g} \cap (\operatorname{im} d_\mathfrak{g})^\perp \subset \Lambda^k \mathfrak{g}^*.$$

Therefore, by construction, $E_0^k \cong H^k(\mathfrak{g}) \otimes \mathcal{C}^\infty G$ and, in particular, $w(E_0^k) = w(H^k(\mathfrak{g}))$, that is, the two spaces contain forms of the same weights. We can now formulate M. Rumin's result on estimating Novikov-Shubin invariants, compare [Rum01, Thm. 3.13]:

Theorem 4.1. *Let G be a graded nilpotent Lie group with associated Lie algebra $\mathfrak{g}$ and weight function w on $\mathfrak{g}$. If $H^k(\mathfrak{g})$ is of pure weight N_k then*

$$\alpha_k(G) \in \left[\frac{N(G)}{\delta N_k^{\max}}, \frac{N(G)}{\delta N_k^{\min}}\right],$$

where $N(G) = w(d\operatorname{vol})$ is the weight of the volume form[3] *and $\delta N_{k+1}^{\min} = \max\{N_k^{\min} - N_k, 1\}$ and $\delta N_{k+1}^{\max} = N_k^{\max} - N_k$ are defined in terms of the weights gaps between $H^k(\mathfrak{g})$ and $H^{k+1}(\mathfrak{g})$.*

This theorem gives an easy way of estimating Novikov-Shubin invariants if the pure weight condition can be satisfied by some weight function on $\mathfrak{g}$. We give a brief description of an algorithm to find these estimates. A Python-implementation can be found in Appendix A.

Input: The input is a simply connected nilpotent Lie group G, given in terms of its dimension n and structure constants $c_{i,j}^l$ for $1 \leq i < j \leq n$ and $1 \leq l \leq n$ of its associated Lie algebra $\mathfrak{g}$.

Output: The output is a list of estimates $L_k \leq \alpha_k \leq U_k$ for some $0 \leq k \leq n$.

Algorithm:

1. **Computing the Lie algebra cohomology of $\mathfrak{g}$.** The groups of the Chevalley-Eilenberg complex are given by $\Lambda^k \mathfrak{g}^* \cong \mathbb{R}^{\binom{n}{k}}$, where we can identify basis k-forms $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ with subsets $\{i_1 < \cdots < i_k\} \subset \{1, \ldots, n\}$ which are a natural enumeration of $\binom{n}{k}$. The differential $d_\mathfrak{g}$ is then given by the linear operator $d_\mathfrak{g} \colon \mathbb{R}^{\binom{n}{k}} \to \mathbb{R}^{\binom{n}{k+1}}$ described by the matrix whose entries are non-zero if and only if they correspond to basis elements of the form (up to ordering) $\{l\} \cup I \to \{i, j\} \cup I$ where $I \subset \{1, \ldots n\}$ is an ordered subset with $|I| = k - 1$ and $i, j, l \notin I$. In this case, the entry is given by the structure constants,

$$(d_\mathfrak{g})_{\{l\} \cup I \to \{i,j\} \cup I} = \pm c_{i,j}^l,$$

with sign depending of the sign of permutations needed to sort the two (ordered) sets $\{l\} \cup I$ and $\{i, j\} \cup I$. Hence, computing kernel and image of $d_\mathfrak{g}$ under these isomorphisms is a problem of linear algebra and finding a system of representatives for $\ker d_\mathfrak{g} \cap (\operatorname{im} d_\mathfrak{g})^\perp \cong H^k(\mathfrak{g})$ can be done, for example, using the Gram-Schmidt algorithm.

[3] Here, the symbol $N(G)$ is used in reference to the growth rate: If G is graded, then one possible weight function is given by assigning to X_i the step in the lower central series in which X_i disappears, that is $X_i \in \mathfrak{g}_{w_i - 1}$ but $X_i \notin g_{w_i}$. In this case, $N(G)$ is equal to the growth rate. However, there may be other weight functions.

2. **Finding suitable weights.** Fix a degree k. In order to find a weight w that satisfies the pure weight condition in degree k, it needs to satisfy the linear equations coming from the weight conditions $w_i + w_j = w_l$ if $c_{i,j}^l \neq 0$ and the linear equations needed to ensure that all weights appearing in $w(H^k(\mathfrak{g}))$ are equal.[4] This produces a list of possible weight functions that lead to pure weight in degree k. The list may be empty, in which case the pure weight condition cannot be satisfied and the theorem cannot be applied.
3. **Finding the bounds.** For each weight function found in the previous step, we can now compute the weights appearing in $w(H^{k+1}(\mathfrak{g}))$.[5] The growth rate can also easily be computed from the structure constants, so that this already allows us to compute the lower bound L_k and the upper bound U_k on α_k.
4. **Apply Hodge duality.** We can improve the bounds found by this method using the known results that $\alpha_0(G) = N(G)$ and by Hodge duality $\alpha_k(G) = \alpha_{n-k-1}(G)$.

4.2 Estimates on Nilpotent Lie Groups up to Dimension 6

For nilpotent Lie groups up to dimension six (all of which are graded), we obtain the bounds given in Table 4.1 on page 57 by implementing the algorithm above as a computer program. The Python implementation can be found in Appendix A.

4.3 Examples and Remarks

In this section we go over some examples and compute the estimates on some Lie algebras.

Heisenberg groups and contact manifolds. In the table, $L_{3,2} = \mathfrak{h}^3 = \langle X, Y, Z \,|\, [X,Y] = Z \rangle$ is the (3-dimensional) Heisenberg Lie algebra associated to the Heisenberg group $\mathbb{H}^3$. For $\mathfrak{h}^3$, the Lie algebra cohomology is given by

$$H^1(\mathfrak{h}^3) = \langle \theta_X, \theta_Y \rangle, \quad H^2(\mathfrak{h}^3) = \langle \theta_X \wedge \theta_Z, \theta_Y \wedge \theta_Z \rangle, \quad H^3(\mathfrak{h}^3) = \langle \theta_X \wedge \theta_Y \wedge \theta_Z \rangle.$$

Putting weights $w(X) = 1$, $w(Y) = 1$ and $w(Z) = 2$ yields a weight function for which all homology groups are of pure weight,

$$N_0 = 0, \quad N_1 = 1, \quad N_2 = 3 \quad \text{and} \quad N_3 = 4.$$

In particular $N(G) = w(\theta_X \wedge \theta_Y \wedge \theta_Z) = 4$, hence by Theorem 4.1 we obtain

$$\alpha_0(\mathbb{H}^3) = \frac{N(G)}{N_1 - N_0} = 4, \quad \alpha_1(\mathbb{H}^3) = \frac{N(G)}{N_2 - N_1} = 2 \quad \text{and} \quad \alpha_2(\mathbb{H}^3) = \frac{N(G)}{N_3 - N_2} = 4.$$

The Lie algebra $L_{5,4} = \mathfrak{h}^5 = \langle X_1, X_2, Y_1, Y_2, Z \,|[X_1, Y_1] = Z, [X_2, Y_2] = Z \rangle$ is the 5-dimensional Heisenberg Lie algebra. For $\mathfrak{h}^5$ we obtain a similar picture, where the lower half of the homology groups, $H^0(\mathfrak{h}^5)$, $H^1(\mathfrak{h}^5)$ and $H^2(\mathfrak{h}^5)$, are spanned by all basis forms not containing θ_Z as a factor while the upper half of the homotopy groups, $H^3(\mathfrak{h}^5)$, $H^4(\mathfrak{h}^5)$ and $H^5(\mathfrak{h}^5)$ are spanned by all basis forms containing θ_Z as a factor. This is true more generally for Lie algebras associated to Lie groups that are contact manifolds. The contact form is the distinguished 1-form of weight 2, so θ_Z in case of the Heisenberg groups, while its orthogonal complement is of pure weight 1.

[4] Note that, if a non-trivial solution to this linear equation exists, it will have at least one degree of freedom. We may fix one free variable to equal 1 without changing the possible estimates.

[5] If this list still contains degrees of freedom, a bit more work may be needed to find the optimal results. Note that it is possible to restrict to integer weights and generally speaking, choosing smaller values for such free variables tends to produce the best estimates.

$\mathfrak{g}$	α_0	α_1	α_2	α_3	α_4	α_5	α_6
$L_{1,1}$	1	∞^+					
$L_{3,2}$	4	2	4	∞^+			
$L_{4,3}$	7	$[5/2, 10/3]$	$[5/2, 10/3]$	7	∞^+		
$L_{5,4}$	6	6	3	6	6	∞^+	
$L_{5,5}$	8	–	–	–	8	∞^+	
$L_{5,6}$	11	–	–	–	11	∞^+	
$L_{5,7}$	11	$[11/4, 11/2]$	–	$[11/4, 11/2]$	11	∞^+	
$L_{5,8}$	7	$[7/2, 7]$	–	$[7/2, 7]$	7	∞^+	
$L_{5,9}$	10	10/3	5	10/3	10	∞^+	
$L_{6,10}$	9	–	–	–	–	9	∞^+
$L_{6,11}$	12	–	–	–	–	12	∞^+
$L_{6,12}$	12	–	–	–	–	12	∞^+
$L_{6,13}$	12	–	–	–	–	12	∞^+
$L_{6,14}$	16	–	–	–	–	16	∞^+
$L_{6,15}$	16	–	–	–	–	16	∞^+
$L_{6,16}$	16	$[14/3, 7]$	$[16/5, 16/3]$	$[16/5, 16/3]$	$[14/3, 7]$	16	∞^+
$L_{6,17}$	16	–	–	–	–	16	∞^+
$L_{6,18}$	16	$[16/5, 8]$	–	–	$[16/5, 8]$	16	∞^+
$L_{6,19,1}$	10	$[5, 10]$	–	–	$[5, 10]$	10	∞^+
$L_{6,19,0}$	10	$[10/3, 10]$	–	–	$[10/3, 10]$	10	∞^+
$L_{6,19,-1}$	10	$[5, 10]$	–	–	$[5, 10]$	10	∞^+
$L_{6,20}$	10	$[5, 10]$	–	–	$[5, 10]$	10	∞^+
$L_{6,21,1}$	14	$[7/2, 14/3]$	–	–	$[7/2, 14/3]$	14	∞^+
$L_{6,21,0}$	14	$[7/2, 14/3]$	–	–	$[7/2, 14/3]$	14	∞^+
$L_{6,21,-1}$	14	$[7/2, 14/3]$	–	–	$[7/2, 14/3]$	14	∞^+
$L_{6,22,1}$	8	$[4, 8]$	$[4, 8]$	$[4, 8]$	$[4, 8]$	8	∞^+
$L_{6,22,0}$	8	$[4, 8]$	$[4, 8]$	$[4, 8]$	$[4, 8]$	8	∞^+
$L_{6,22,-1}$	8	$[4, 8]$	$[4, 8]$	$[4, 8]$	$[4, 8]$	8	∞^+
$L_{6,23}$	10	–	–	–	–	10	∞^+
$L_{6,24,1}$	11	–	$[4, 6]$	$[4, 6]$	–	11	∞^+
$L_{6,24,0}$	11	–	$[4, 6]$	$[4, 6]$	–	11	∞^+
$L_{6,24,-1}$	11	–	$[4, 6]$	$[4, 6]$	–	11	∞^+
$L_{6,25}$	10	$[10/3, 10]$	–	–	$[10/3, 10]$	10	∞^+
$L_{6,26}$	9	9/2	$[9/2, 9]$	$[9/2, 9]$	9/2	9	∞^+

Figure 4.1: List of estimates on Novikov-Shubin invariants for nilpotent Lie algebras of dimension up to six using M. Rumin's theorem and an implementation in Python. The list of such Lie algebras and the naming convention is taken from W. A. de Graaf's classification [Gra07]. A dash indicates that no estimate can be found, an interval indicates the interval that the α_k lies within and an explicit value gives the precise value of α_k.

Engel's group. The algebra $L_{4,3} = \langle X, Y, Z, T \mid [X, Y] = Z, [X, Z] = T \rangle$ is the Lie algebra associated to Engel's group. For this case, the computations can be found in M. Rumin's paper [Rum01]. It is an interesting case because two different weight functions can be used here to obtain better bounds than any individual weight function would yield.

The Lie algebra $L_{5,9}$. The algebra $\mathfrak{g} = L_{5,9}$ is 5-dimensional with generators $X_1, \dots, X_5$ and three non-vanishing commutator relations,

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4 \quad \text{and} \quad [X_2, X_3] = X_5$$

or equivalently, given by non-zero structure constants $c^3_{1,2} = 1$, $c^4_{1,3} = 1$ and $c^5_{2,3} = 1$. Denoting the weight of X_i by $w_i = w(X_i)$, this gives us the following initial restraints on the weights:

$$w_3 = w_1 + w_2, \quad w_4 = w_1 + w_3 = 2w_1 + w_2, \quad w_5 = w_2 + w_3 = w_1 + 2w_2.$$

Further, if we denote $\theta_i = X_i^*$ the dual basis of $\Lambda^1\mathfrak{g}^*$ and $\theta_{i_1 \dots i_k} = \theta_{i_1} \wedge \dots \wedge \theta_{i_k}$, we can read off the Lie algebra differential $d = d_{\mathfrak{g}} \colon \Lambda^1\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ by using Cartan's formula. On the basis elements of $\Lambda^1\mathfrak{g}^*$ it is given by

$$d(\theta_1) = 0, \quad d(\theta_2) = 0, \quad d(\theta_3) = -\theta_{12}, \quad d\theta_4 = -\theta_{13}, \quad d(\theta_5) = -\theta_{23}$$

and extended linearly to $\Lambda^\bullet\mathfrak{g}^*$. From this, we directly see that $d \colon \Lambda^2\mathfrak{g}^* \to \Lambda^3\mathfrak{g}^*$ vanishes on all standard basis elements $\theta_{i_1 i_2}$ except for

$$d(\theta_{15}) = \theta_{123}, \quad d(\theta_{24}) = -\theta_{123}, \quad d(\theta_{34}) = -\theta_{124}, \quad d(\theta_{35}) = -\theta_{125}, \quad d(\theta_{45}) = \theta_{234} - \theta_{135}.$$

In degree three it vanishes on standard basis elements except for

$$d(\theta_{145}) = -\theta_{1234}, \quad d(\theta_{245}) = -\theta_{1235}, \quad d(\theta_{345}) = -\theta_{1245},$$

and in degree four, the differential vanishes. We compute the Lie algebra cohomology and the weights appearing as $w(H^0(\mathfrak{g})) = \{0\}$ and

$$\begin{aligned}
H^1(\mathfrak{g}) &\cong \langle \theta_1, \theta_2 \rangle && \xrightarrow{w} \quad \{w_1, w_2\}, \\
H^2(\mathfrak{g}) &\cong \langle \theta_{14}, \theta_{25}, \theta_{15} - \theta_{24} \rangle && \xrightarrow{w} \quad \{3w_1 + w_2, w_1 + 3w_2, 2w_1 + 2w_2\}, \\
H^3(\mathfrak{g}) &\cong \langle \theta_{134}, \theta_{135} + \theta_{234}, \theta_{234} \rangle && \xrightarrow{w} \quad \{4w_1 + 2w_2, 3w_1 + 3w_2, 2w_1 + 4w_2\}, \\
H^4(\mathfrak{g}) &\cong \langle \theta_{1345}, \theta_{2345} \rangle && \xrightarrow{w} \quad \{5w_1 + 4w_2, 4w_1 + 5w_2\}, \\
H^5(\mathfrak{g}) &\cong \langle \theta_{12345} \rangle && \xrightarrow{w} \quad \{5w_1 + 5w_2\}.
\end{aligned}$$

Hence, we obtain pure weight in any/all degrees for $w_1 = w_2$, in which case

$$\begin{aligned}
w(H^0(\mathfrak{g})) &= 0, & w(H^1(\mathfrak{g})) &= 2w_1, & w(H^2(\mathfrak{g})) &= 4w_1, \\
w(H^3(\mathfrak{g})) &= 6w_1, & w(H^4(\mathfrak{g})) &= 9w_1, & w(H^5(\mathfrak{g})) &= 10w_1.
\end{aligned}$$

Since we have pure weight in all degrees, $\delta N_k^{\max} = \delta N_k^{\min} = w(H^{k+1}(\mathfrak{g})) - w(H^k(\mathfrak{g}))$ in all cases, so that we get precise answers for all Novikov-Shubin invariants. For $0 \le k \le 4$,

$$\alpha_k(L_{5,9}) = \frac{w(\theta_{12345})}{w(H^{k+1}(\mathfrak{g})) - w(H^k(\mathfrak{g}))},$$

which indeed yields:

G	α_0	α_1	α_2	α_3	α_4
$L_{5,9}$	10	10/3	5	10/3	10

Remark 4.2. Notice that M. Rumin's Theorem 4.1 has multiple assumptions that need to be satisfied in order to obtain estimates on Novikov-Shubin invariants. In particular, to estimate $\alpha_k(G)$ for some $0 \leq k \leq \dim(G)$, this includes the following two assumptions:

- The assumption of having a weight function on G.
- The assumption that the Lie algebra cohomology of $\mathfrak{g}$ is of pure weight in degree k.

Both of these assumptions are necessary and it seems that they cannot easily be lifted. A key part of M. Rumin's approach comes from studying dilatations $h_r \colon G \to G$. Given a weight function w on $\mathfrak{g}$, these dilatations can be defined using the exponential map as

$$h_r(\exp X_i) = r^{w_i} \exp X_i.$$

These dilatations induce maps on the level of differential forms, where a differential form of pure weight w_0 is scaled by a constant factor $r^{w_0 - w(d\,\mathrm{vol})/2}$, where $d\,\mathrm{vol}$ denotes a volume form. Since we can define the dilatations as functions $G \to G$ and view their pullbacks on differential forms, it is immediately clear that the pullback commutes with the deRham differential d. The requirement on w to be a weight function implies that h_r^* commutes with the commutator brackets, $h_r^*([\,\cdot\,,\cdot\,]) = [h_r^*(\,\cdot\,), h_r^*(\,\cdot\,)]$ which is needed in the proof.
Lastly, the requirement that $H^k(\mathfrak{g})$ has to be of pure weight is crucial as we study near cohomology cones, defined as subspaces of the orthogonal complements $(\ker d)^\perp$. In order for the h_r^* to define maps between near cohomology cones they have to map $(\ker d)^\perp \to (\ker d)^\perp$, which is only guaranteed if $H^k(\mathfrak{g})$ is of pure weight (so h_r^* is just scaling by a factor) but not true otherwise.

In the following chapter, Chapter 5, we will approach the problem from another angle. However, the idea of scaling different directions of our manifold with different speeds — as is done here using the dilatation h_r — will be the core feature of the main definition of the upcoming chapter.

Remark 4.3. Let us also remark that based on this data, the Novikov-Shubin invariants of an n-dimensional nilpotent Lie algebra tend to decay with the index from α_0 to $\alpha_{\lfloor n/2 \rfloor}$ (and, by Hodge duality, increase afterwards). This is not always true in the strict sense, as can be seen from the example $L_{5,9}$. However, it can be explained heuristically:
For every nilpotent Lie algebra $\mathfrak{g}$, there is a compact connected Lie group G that has $\mathfrak{g}$ as its associated Lie algebra. For such Lie groups, their deRham cohomology is isomorphic to the Lie algebra cohomology of $\mathfrak{g}$, compare the original paper of C. Chevalley and S. Eilenberg [CE48, Thm. 15.2]. Nilmanifolds, by definition quotients of nilpotent Lie groups, are precisely those manifolds, that can be described as iterated principal circle bundle, see [Bel20]. In the trivial case, the n-torus $T^n = (S^1)^n$ is a nilmanifold with Lie algebra $\mathfrak{t}^n \cong \mathbb{R}^n$. Its Lie algebra cohomology is the full $H^k(\mathfrak{t}^n) \cong \Lambda^k \mathfrak{t}^n \cong \mathbb{R}^{\binom{n}{k}}$, growing in k until $k = \lfloor n/2 \rfloor$. While generally the cohomology groups of such iterated sphere bundles will not be the full $\Lambda^k \mathfrak{t}^n$, they tend to be rather large. As Novikov-Shubin invariants measure the size of spaces related to this homology (small Novikov-Shubin invariants meaning these spaces are large), it is not surprising that they behave similarly. This also suggests a pessimistic answer to the question how often M. Rumin's approach should be expected to give estimates on Novikov-Shubin invariants. Since the Lie algebra cohomologies tend to be large (in particular, in the middle degrees $2 \leq k \leq n-2$), the pure weight assumption requires many basis forms to be of the same weight. Together with the weight restriction coming from the weight function, generally, we expect to have a overdetermined system of linear equations that will have solutions only in special cases.

Chapter 5

Two-Parameter Novikov-Shubin Invariants on Fibre Bundles

In this chapter we consider the setting of fibre bundles and define a two-parameter version of the Novikov-Shubin invariants in hopes of detecting the individual contributions from the base and from the fibre. We compute these generalised numbers explicitly for the example of the three-dimensional Heisenberg group. We then prove several invariance properties of these new, two-parameter Novikov-Shubin numbers.
This fits into the recent interest of studying fibre bundles by means of invariants, such as work based on J.-M. Bismut and J. Cheeger's study [BC89] of higher torsion invariants on fibre bundles using adiabatic limits, J.-M. Bismut's study [Bis86] of an Atiyah-Singer theorem for families of Dirac operators and characteristic classes of fibre bundles such as the Morita-Miller-Mumford classes named after D. Mumford [Mum83], E. Y. Miller [Mil86] and S. Morita [Mor87].
This new definition is different, but similar in spirit, to the study of adiabatic limits of fibre bundles. Some publications relevant to this topic include articles from L. Sanguiao [San08] and from S. Haag and J. Lampart [HL19].

5.1 Two-Parameter Novikov-Shubin Numbers

Let (M, g) be a noncompact Riemannian manifold with a cocompact free proper group action $G \curvearrowright M$ acting by isometries. As discussed in Subsection 2.4.4, the spectral density function[1] of d, defined by[2]

$$\mathcal{F}_k(M,g)(\lambda) = \dim_{\mathcal{N}G} \operatorname{im} \chi_{[0,\lambda^2]}(\Delta^k_{\mathrm{up}}(M,g)) = \operatorname{tr}_{\mathcal{N}G} \chi_{[0,\lambda^2]}(\Delta^k_{\mathrm{up}}(M,g)),$$

can instead be defined by rescaling the manifold and looking at a fixed interval of the spectrum. Since for $g_\lambda = \lambda^2 g$ the Laplace operators satisfy $\Delta^k_{\mathrm{up}}(M, g_\lambda) = \lambda^{-2}\Delta^k_{\mathrm{up}}(M, g)$, we obtain

$$\mathcal{F}_k(M,g)(\lambda) = \operatorname{tr}_{\mathcal{N}G} \chi_{[0,\lambda^2]}(\Delta^k_{\mathrm{up}}(M,g)) = \operatorname{tr}_{\mathcal{N}G} \chi_{[0,1]}(\Delta^k_{\mathrm{up}}(M,g_\lambda)) = \mathcal{F}_k(M,g_\lambda)(1).$$

[1] In this chapter, we denote spectral density functions with the calligraphic letters $\mathcal{F}$ or $\mathcal{G}$ to avoid conflicts in notation with the fibre F of the fibre bundle and the group G.

[2] As this makes the notation more suggestive in this chapter, we write $\chi_{[0,\lambda_0]}(\Delta^k_{\mathrm{up}}(M,g)) = E^{d^{k,*}d^k}_{\lambda_0}$ with the upper Laplacian $\Delta^k_{\mathrm{up}}(M,g) = d^{k,*}d^k$ depending on the Riemannian metric g.

If M is the total space of a fibre bundle and the fibre bundle structure is compatible with the G-action, then we can scale M with different speed in fibre and base directions. This way we can define a refined version of the Novikov-Shubin invariants.
More precisely, let $M \xrightarrow{\pi} B$ be a fibre bundle with fibres $\{F_b = \pi^{-1}(b)\}_{b\in B}$, where (M, g) is a Riemannian manifold with Riemannian metric g (as always, we assume the base, the fibres and thus also the total space to be connected). At every point $x \in M$ with $\pi(x) = b$, we can define the subspace

$$T_xF_b = \ker D_x\pi \subset T_xM,$$

giving rise to the vertical subbundle $VM \subset TM$ of the tangent bundle by

$$VM = T_\bullet F_\bullet = \ker(\pi_*) \subset TM.$$

Choosing a connection ∇ compatible with g on the fibre bundle is equivalent to specifying an orthogonal complement HM of VM in TM, so that the tangent space TM decomposes as

$$TM \cong_\nabla VM \perp_g HM.$$

The bundle HM is called the horizontal subbundle of TM. The Riemannian metric decomposes fibrewise into a vertical and a horizontal contribution,

$$g_x = g_{x,V} + g_{x,H},$$

where $g_{x,V}$ is supported in $V_xM \otimes V_xM$ and $g_{x,H}$ is supported in $H_xM \otimes H_xM$.
In the following, we denote the situation described here by the triple $(M \to B, \nabla, g)$ and call such a triple a Riemannian fibre bundle with connection.

Definition 5.1. We call a cocompact free proper group action $G \curvearrowright M$ compatible with this structure, and write $G \curvearrowright (M \to B, \nabla, g)$, if the Riemannian metric g is G-invariant and there is a group action $G' \curvearrowright B$ together with a surjective group homomorphism $\varphi\colon G \twoheadrightarrow G'$ such that the projection $M \xrightarrow{\pi} B$ is φ-equivariant[3].

Example 5.2. The typical example for such a Riemannian fibre bundle with connection and compatible group action is obtained by starting with a compact fibre bundle $F \to M \to B$ where F, M and B are connected. The universal covering $\widetilde{M}$ of M can be considered as a fibre bundle $\widetilde{M} \to \widetilde{B}$ over the universal covering of the base B with some fibres $F'_\bullet$ (in general, these are not the universal coverings of the fibres $F_\bullet$). On the universal coverings, we have the action of $\pi_1(M)$ on $\widetilde{M}$ and the action of $\pi_1(B)$ on $\widetilde{B}$, compare the following diagram:

$$\begin{array}{ccccc} & & \pi_1(M) & \xrightarrow{\varphi} & \pi_1(B) \\ & & \curvearrowright & & \curvearrowright \\ F' & \longrightarrow & \widetilde{M} & \longrightarrow & \widetilde{B} \\ & & \downarrow & & \downarrow \\ F & \longrightarrow & M & \longrightarrow & B \end{array}$$

The long exact sequence of homotopy groups for the fibre bundle $F \to M \to B$, given by

$$\cdots \to \pi_2(B) \to \pi_1(F) \to \pi_1(M) \overset{\varphi}{\twoheadrightarrow} \pi_1(B) \to 0,$$

[3] That is, for all $\gamma \in G$ and $x \in M$ we have $\pi(\gamma x) = \varphi(\gamma)\pi(x)$. In particular, $\ker(\varphi)$ acts on each fibre F'_b.

yields a group homomorphism $\varphi\colon \pi_1(M) \to \pi_1(B)$ that is surjective since $\pi_0(F)$ is trivial. The elements in the kernel of φ are in the image of $\pi_1(F) \to \pi_1(M)$ and act fibrewise on each fibre F'_b for $b \in \widetilde{B}$ and the projection $\widetilde{M} \to \widetilde{B}$ is φ-equivariant.

Definition 5.3. Let $(M \to B, \nabla, g)$ be a Riemannian fibre bundle with connection. For smooth positive functions $s_H, s_V \in \mathcal{C}^\infty(M, \mathbb{R}_+)$ we define the Riemannian metric g^{s_H,s_V} on M by

$$x \mapsto g_x^{s_H,s_V} = s_H(x)^2 g_{x,H} + s_V(x)^2 g_{x,V}.$$

In particular, if $s_H \equiv \overline{\mu} > 0$ and $s_V \equiv \overline{\nu} > 0$ are constant functions, this defines

$$g^{\overline{\mu},\overline{\nu}} = g^{s_H,s_V} = \overline{\mu}^2 g_V + \overline{\nu}^2 g_H.$$

This is indeed a Riemannian metric: Since g is positive definite and $s_H, s_V > 0$, also $g_x^{s_H,s_V}$ is positive definite on each fibre T_xM, and the map $x \mapsto g_x^{s_H,s_V}$ is smooth because g, s_H and s_V are smooth. We use this structure to define a refined version of the spectral density function depending on two parameters in place of the classical parameter λ.

Definition 5.4. Let $G \curvearrowright (M \to B, \nabla, g)$ be a Riemannian fibre bundle with connection and compatible G-action. Then, using the previous definition, we define the two-parameter spectral density function $\mathcal{G}_k(M \to B, \nabla, g)\colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, \infty]$ by

$$\mathcal{G}_k(M \to B, \nabla, g)(\overline{\mu}, \overline{\nu}) = \operatorname{tr}_{\mathcal{N}G} \chi_{[0,1]}(\Delta^k_{\mathrm{up}}(M, g^{\overline{\mu},\overline{\nu}})) = \mathcal{F}_k(M, g^{\overline{\mu},\overline{\nu}})(1).$$

We call two such functions $\mathcal{G}, \mathcal{G}'\colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, \infty]$ dilatationally equivalent if there exists a constant $C > 0$ such that for all $\overline{\mu}, \overline{\nu} \in \mathbb{R}_+$,

$$\mathcal{G}(C^{-1}\overline{\mu}, C^{-1}\overline{\nu}) \leq \mathcal{G}'(\overline{\mu}, \overline{\nu}) \leq \mathcal{G}(C\overline{\mu}, C\overline{\nu}).$$

In this case we write $\mathcal{G} \sim \mathcal{G}'$.

The fact that we chose the value one for the upper end of the interval is not of importance here, in the sense that the dilatational equivalence class of $\mathcal{G}$ does not depend on the upper end.

Lemma 5.5. *The dilatational equivalence class is independent of the right end chosen for the interval, that is for all $\lambda_0 > 0$,*

$$\mathcal{G}_k(M \to B, \nabla, g)(\overline{\mu}, \overline{\nu}) \sim \left((\overline{\mu}, \overline{\nu}) \mapsto \operatorname{tr}_{\mathcal{N}G} \chi_{[0,\lambda_0]}(\Delta^k_{\mathrm{up}}(M, g^{\overline{\mu},\overline{\nu}}))\right).$$

Proof. This follows directly with constant $C = \sqrt{\lambda_0}$ since

$$\begin{aligned}
\operatorname{tr}_{\mathcal{N}G} \chi_{[0,\lambda_0]}(\Delta^k_{\mathrm{up}}(M, g^{\overline{\mu},\overline{\nu}})) &= \operatorname{tr}_{\mathcal{N}G} \chi_{[0,1]}(\lambda_0^{-1}\Delta^k_{\mathrm{up}}(M, g^{\overline{\mu},\overline{\nu}})) \\
&= \operatorname{tr}_{\mathcal{N}G} \chi_{[0,1]}(\Delta^k_{\mathrm{up}}(M, \lambda_0 g^{\overline{\mu},\overline{\nu}})) \\
&= \operatorname{tr}_{\mathcal{N}G} \chi_{[0,1]}(\Delta^k_{\mathrm{up}}(M, g^{\sqrt{\lambda_0}\cdot\overline{\mu},\sqrt{\lambda_0}\cdot\overline{\nu}})) \\
&= \mathcal{G}_k(M \to B, g, \nabla)(\sqrt{\lambda_0} \cdot \overline{\mu}, \sqrt{\lambda_0} \cdot \overline{\nu}).
\end{aligned}$$ □

Instead of having two truely independent parameters $\overline{\mu}$ and $\overline{\nu}$, we would like to consider the two parameters as different speeds of scaling the manifold. Therefore, we replace these two parameters with two functions, depending on the same variable λ, governing how fast the fibre respectively the base get scaled as $\lambda \searrow 0$.

Definition 5.6. Let $G \curvearrowright (M \to B, \nabla, g)$ be a Riemannian fibre bundle with connection and compatible G-action. Let $\mu, \nu \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be monotonously increasing continuous functions with $\mu(0) = 0 = \nu(0)$. Denoting $\mathcal{G}_k = \mathcal{G}_k(M \to B, \nabla, g)$ we define the two-parameter Novikov-Shubin numbers by

$$\begin{aligned}\alpha_k(M \to B, \nabla, g)(\mu, \nu) &= \alpha\big(\ \lambda \mapsto \mathcal{G}_k(\mu(\lambda), \nu(\lambda))\ \big)\\ &= \liminf_{\lambda \searrow 0} \frac{\log(\mathcal{G}_k(\mu(\lambda), \nu(\lambda)) - b^{(2)}(d_{k+1}))}{\log(\lambda)}.\end{aligned}$$

Recall here that $b^{(2)}(d_{k+1})$ is metric invariant and measuring the size of the kernel of d_{k+1}, which we use to extend the definition of $\mathcal{G}_k$ formally by $\mathcal{G}_k(\mu(0), \nu(0)) = \mathcal{G}_k(0,0) = b^{(2)}(d_{k+1})$.

Remark 5.7. This two-parameter function generalises the usual spectral density function. Indeed, if[4] $\mu = \nu = \lambda$, then $g^{\lambda,\lambda} = \lambda^2 g = g_\lambda$ independently of the connection ∇ chosen. Hence,

$$\mathcal{G}_k(M \to B, \nabla, g)(\lambda, \lambda) = \operatorname{tr}_{\mathcal{N}G} \chi_{[0,1]}(\Delta^k_{\mathrm{up}}(M, g_\lambda)) = \mathcal{F}_k(M, g)(\lambda)$$

is the classical spectral density function of (M, g) and therefore

$$\alpha_k(M \to B, g, \nabla)(\lambda, \lambda) = \alpha_k(M)$$

recovers the Novikov-Shubin invariants.

Example 5.8. In the simplest case of a product manifold $(M, g) = (F, g_F) \times (B, g_B)$ with the canonical connection $TM \cong_\nabla TF \perp TB$, for $\mu, \nu > 0$ we have

$$\mathcal{G}_k(F \times B, \nabla, g)(\mu, \nu) = \mathcal{F}_k((F, \nu^2 g_F) \times (B, \mu^2 g_B))(1).$$

By [Lüc02, Cor. 2.44], it is therefore dilatationally equivalent to

$$\begin{aligned}\mathcal{G}_k(F \times B, \nabla, g)(\mu, \nu) &\sim \sum_{p+q=k} \mathcal{F}_p((F, \nu^2 g_F))(1) \cdot \mathcal{F}_q((B, \mu^2 g_B))(1)\\ &= \sum_{p+q=k} \mathcal{F}_p(F)(\nu) \cdot \mathcal{F}_q(B)(\mu).\end{aligned}$$

If $\mu = \lambda^r$ and $\nu = \lambda^s$, we can consider a limit as $\lambda \searrow 0$ in the spirit of the Novikov-Shubin invariants. We assume that all L^2-Betti numbers in this example vanish[5]. Following the computation in W. Lück's book [Lüc02, Thm. 2.55 (3)],

$$\begin{aligned}\alpha_k(F \times B, \nabla, g)(\mu, \nu) &= \liminf_{\lambda \searrow 0} \frac{\log(\mathcal{G}_k(F \times B, \nabla, g)(\lambda^r, \lambda^s))}{\log(\lambda)}\\ &= \liminf_{\lambda \searrow 0} \frac{\log(\mathcal{F}_k((F, \lambda^{2s} g_F) \times (B, \lambda^{2r} g_B), \nabla, g)(1))}{\log(\lambda)}\\ &= \min_{0 \leq p \leq k} \begin{Bmatrix} \alpha\left(\mathcal{F}_p(F)(\lambda^s) \cdot \mathcal{F}_{k-p}(B)(\lambda^r)\right), \\ \alpha\left(\mathcal{F}_{p+1}(F)(\lambda^s) \cdot \mathcal{F}_{k-p}(B)(\lambda^r)\right) \end{Bmatrix}\\ &= \min_{0 \leq p \leq k} \begin{Bmatrix} \alpha\left(\mathcal{F}_p(F)(\lambda^s)\right) + \alpha\left(\mathcal{F}_{k-p}(B)(\lambda^r)\right), \\ \alpha\left(\mathcal{F}_{p+1}(F)(\lambda^s)\right) + \alpha\left(\mathcal{F}_{k-p}(B)(\lambda^r)\right) \end{Bmatrix}\\ &= \min_{0 \leq p \leq k} \begin{Bmatrix} s \cdot \alpha_p(F) + r \cdot \alpha_{k-p}(B), \\ s \cdot \alpha_{p+1}(F) + r \cdot \alpha_{k-p}(B) \end{Bmatrix}.\end{aligned}$$

In this case, we see the contributions from the base and fibre scaled according to the chosen functions $\mu(\lambda) = \lambda^r$ and $\nu(\lambda) = \lambda^s$ as $\lambda \searrow 0$.

[4] By abuse of notation we denote by λ the function $\mathrm{id} \colon \lambda \mapsto \lambda$ or more generally by λ^c the function $\lambda \mapsto \lambda^c$.

[5] This is not necessary but reduces the length of notation for this example considerably. One can proceed just as in cited source by W. Lück even if the L^2-Betti numbers do not vanish.

5.2 Example: The Heisenberg Group

We consider the Heisenberg group $\mathbb{H}^3$ and its associated Lie algebra $\mathfrak{h}^3 = \langle X, Y, Z \mid [X, Y] = Z\rangle$ as a fibre bundle $\mathbb{R} \to \mathfrak{h}^3 \to \mathbb{R}^2$, where the fibre direction corresponds to the central Z-direction and the basis directions are the X- and Y-directions. A basis of left-invariant vector fields is given by the vector fields

$$\vartheta_X = \partial_X - \frac{1}{2} y \partial_Z, \qquad \vartheta_Y = \partial_Y + \frac{1}{2} x \partial_Z, \qquad \vartheta_Z = \partial_Z,$$

where x and y denote coordinates in the base $\mathbb{R}^2 = \langle X, Y\rangle$.
Requiring that ϑ_X, ϑ_Y and ϑ_Z are orthonormal yields the standard metric g and with $VM = \langle \vartheta_Z\rangle$ and $HM = \langle \vartheta_X, \vartheta_Y\rangle$. We also fix a connection ∇. The scaled metric $g^{\overline{\mu},\overline{\nu}}$ is the metric for which

$$\overline{\mu}^{-1} \cdot \vartheta_X, \quad \overline{\mu}^{-1} \cdot \vartheta_Y \quad \text{and} \quad \overline{\nu}^{-1} \cdot \vartheta_Z$$

form an orthonormal basis of $\mathfrak{h}^3$. Using results of a computation of J. Lott [Lot92, Prop. 52], we obtain the following values for the two-parameter Novikov-Shubin numbers.

Theorem 5.9. *On $\mathfrak{h}^3$, by direct computation we obtain*

$$\begin{aligned} \alpha_0(\mathfrak{h}^3)(\lambda, \lambda^{1+\zeta}) &= 4 + 2\zeta \quad \textit{for } -1/2 \leq \zeta, \\ \alpha_1(\mathfrak{h}^3)(\lambda, \lambda^{1+\zeta}) &= 2 - 2\zeta \quad \textit{for } -1/2 < \zeta < 1, \end{aligned}$$

and, by Hodge duality, also $\alpha_2(\mathfrak{h}^3)(\lambda, \lambda^{1+\zeta}) = 4 + 2\zeta$ for $-1/2 \leq \zeta$. Compare also Figure 5.1.

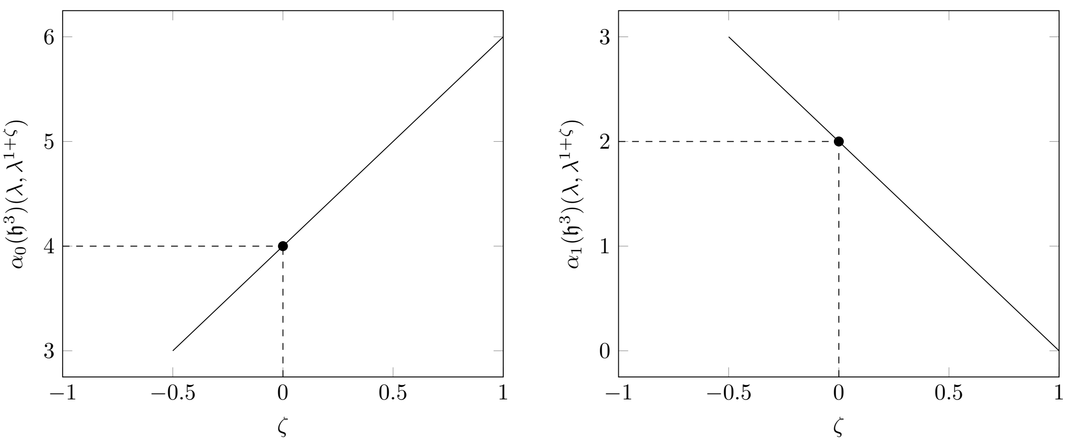


Figure 5.1: The two-parameter Novikov-Shubin numbers of $\mathfrak{h}^3$. On the left, we see a plot for $\alpha_0(\mathfrak{h}^3)(\lambda, \lambda^{1+\zeta})$ and on the right for $\alpha_1(\mathfrak{h}^3)(\lambda, \lambda^{1+\zeta})$. The marked points at $\zeta = 0$ indicate the classical Novikov-Shubin invariants $\alpha_0(\mathbb{H}^3)$ and $\alpha_1(\mathbb{H}^3)$. For α_0, the contributions of base and fibre seem to agree. In particular, as ζ increases, so does α_0. For α_1, the opposite is the case: As ζ increases, α_1 decreases. This gives an interesting insight to (classical) Novikov-Shubin invariants. Comparing the Novikov-Shubin invariants for $\mathbb{H}^3$ and $\mathbb{R}^3$, $\alpha_0(\mathbb{H}^3) = 4 > 3 = \alpha_0(\mathbb{R}^3)$ but $\alpha_1(\mathbb{H}^3) = 2 < 3 = \alpha_1(\mathbb{R}^3)$. Interestingly, for $\zeta = -1/2$, we obtain the Novikov-Shubin invariants for $\mathbb{R}^3$. This fits to the observation that in $\mathbb{H}^3$, the Z-direction scales like the product of the base directions ($[aX, bY] = abZ$), so scaling the fibre with $\lambda^{1/2}$ seems to counteract this.

Proof. It was shown by J. Lott [Lot92, Prop. 52] that in this setting of $\mathfrak{h}^3$ with metric $g^{1,c}$, the heat kernel on functions is given by

$$e^{-t\Delta_0}(0,0) = \frac{1}{4\pi^2}\frac{1}{ct^2}\int_0^\infty e^{-\frac{u^2}{c^2t}}\sinh(u)^{-1}u\,\mathrm{d}u.$$

Classically, if c is constant and we let $t\to\infty$, the normal distribution density function $e^{-\frac{u^2}{c^2t}}$ converges to the constant-1 function and therefore

$$\lim_{t\to\infty}\int_0^\infty e^{-\frac{u^2}{c^2t}}\sinh(u)^{-1}u\,\mathrm{d}u = \int_0^\infty \sinh(u)^{-1}u\,\mathrm{d}u = \frac{\pi}{4}.$$

Hence, $e^{-t\Delta_0}(0,0)$ is in $\Theta(t^{-2})$ as $t\to\infty$ and, by the observations in Subsection 2.4.2, $\alpha_0(\mathfrak{h}^3)=4$.

If we let c depend on t, the same argument remains true as long as $c(t)^2t\to\infty$ as $t\to\infty$, showing that then

$$e^{-\Delta_0}(0,0)\in\Theta(c(t)^{-1}t^{-2}).$$

Therefore, with $c=t^\zeta$ and $\zeta>-1/2$,

$$\alpha\left(\lambda\mapsto\mathcal{G}_0(\lambda,\lambda^{1+\zeta})\right)=4+2\zeta.$$

Indeed, since for $\zeta=-1/2$ the integral is a positive constant,

$$0<\int_0^\infty e^{-u^2}\sinh(u)^{-1}u\,\mathrm{d}u<\infty,$$

the argument holds also for $\zeta=-1/2$, however, the integral converges to zero for $\zeta<-1/2$, so that its asymptotic behaviour need to be taken into account. The summand 2ζ tells us that the scaling of the Z-direction contributes quadratically to the spectral density. This fits with the computation of $\alpha_0(\mathbb{H}^3)=N(\mathbb{H}^3)$ via the growth rate $N(\mathbb{H}^3)$ since by the Bass-Guivarc'h formula,

$$\begin{aligned}N(\mathbb{H}^3)&=\mathrm{rk}(\langle X,Y\rangle)+2\cdot\mathrm{rk}(\langle Z\rangle)\\&=2+2=4,\end{aligned}$$

so we also see a quadratic contribution from the central Z-direction in this picture.
On 1-forms, J. Lott computes the heat operator as

$$e^{-t\Delta_1}(0,0)=\frac{1}{2\pi^2}\frac{1}{c}\left[I_1^+ + I_1^- + I_2 + I_3\right],$$

where the summands $I_\bullet$ are the following integral expressions:

$$\begin{aligned}I_1^\pm &= \int_0^\infty\sum_{m=1}^\infty e^{-t\left[(2m+1)k+\frac{k^2}{c^2}+\frac{c^2}{2}\pm c\sqrt{(2m+1)k+\frac{k^2}{c^2}+\frac{c^2}{4}}\right]}kdk,\\ I_2 &= \int_0^\infty e^{-\frac{k^2}{c^2}t}kdk,\\ I_3 &= \int_0^\infty e^{-\left(2k+\frac{k^2}{c^2}+c^2\right)t}kdk.\end{aligned}$$

J. Lott estimates these integrals in the case where c is constant in order to compute the Novikov-Shubin invariant $\alpha_1(\mathbb{H}^3)=2$.
We compute the integrals in the case where $c=c(t)$ is a function of t.

Lemma 5.10. *The integrals I_2 and I_3 evaluate to*

$$\begin{aligned} I_2 &= \frac{1}{2}\frac{c^2}{t}, \\ I_3 &= \frac{1}{2}\frac{c^2}{t}e^{-c^2 t} + \sqrt{\pi}\frac{c^3}{\sqrt{t}} \cdot \operatorname{erfc}\left(c\sqrt{t}\right), \end{aligned}$$

where erfc *denotes the complementary Gauss error function.*

Proof. The integral I_2 can be directly evaluated by substituting $u = k^2$ as

$$I_2 = \int_0^\infty e^{-\frac{t}{c^2}\cdot k^2} k \,\mathrm{d}k = \frac{1}{2}\int_0^\infty e^{\frac{-t}{c^2}u}\,\mathrm{d}u = \frac{1}{2}\frac{c^2}{t}.$$

Substituting $u = (k/c + c)^2 t$ and $v = (k/c + c)\sqrt{t}$, we can compute

$$\begin{aligned} I_3 &= \int_0^\infty e^{-\left(2k+\frac{k^2}{c^2}+c^2\right)t} k\,\mathrm{d}k \\ &= \int_0^\infty e^{-(\frac{k}{c}+c)^2 t} k\,\mathrm{d}k \\ &= \frac{c^2}{2t}\int_0^\infty e^{-(\frac{k}{c}+c)^2 t}\left(\frac{2t}{c^2}k + 2t\right)\mathrm{d}k - c^2\int_0^\infty e^{-(\frac{k}{c}+c)^2 t}\,\mathrm{d}k \\ &= \frac{c^2}{2t}\int_{c^2 t}^\infty e^{-u}\,\mathrm{d}u - \frac{c^3}{\sqrt{t}}\int_{c\sqrt{t}}^\infty e^{-v^2}\,\mathrm{d}v \\ &= \frac{c^2}{2t}e^{-c^2 t} + \frac{\sqrt{\pi}c^3}{\sqrt{t}}\cdot\operatorname{erfc}\left(c\sqrt{t}\right). \end{aligned}$$

□

Lemma 5.11. *By substitution,*

$$I_1^\pm = c^4\int_0^\infty \left(v \mp \frac{1}{2}\right) e^{-tc^2v^2}\sum_{m=1}^\infty\left[1 - \left(\sqrt{1 + \frac{(v\mp\frac{1}{2})^2 - \frac{1}{4}}{(m+\frac{1}{2})^2}}\right)^{-1}\right]\mathrm{d}v.$$

Proof. Following J. Lott's computations, we substitute in the same way

$$\begin{aligned} u_\pm &= \sqrt{(2m+1)k + \frac{k^2}{c^2} + \frac{c^2}{4}} \pm \frac{c}{2} \\ u_\pm^2 &= (2m+1)k + \frac{k^2}{c^2} + \frac{c^2}{2} \pm c\sqrt{(2m+1)k + \frac{k^2}{c^2} + \frac{c^2}{4}} \\ k_\pm &= c\sqrt{u_\pm^2 \mp u_\pm c + c^2(m + 1/2)^2} - \left(m + \frac{1}{2}\right)c^2 \\ \frac{\mathrm{d}k_\pm}{\mathrm{d}u_\pm} &= \frac{c(u_\pm \mp c/2)}{\sqrt{u_\pm^2 \mp u_\pm c + c^2(m+1/2)^2}}. \end{aligned}$$

Omitting the index $\pm$ in notation[6], we use this with $v = {}^{u}\!/_{c}$ to obtain

$$
\begin{aligned}
I_1^{\pm} &= \int_0^\infty \sum_{m=1}^\infty e^{-t\left[(2m+1)k+\frac{k^2}{c^2}+\frac{c^2}{2}\pm c\sqrt{(2m+1)k+\frac{k^2}{c^2}+\frac{c^2}{4}}\right]} k\,\mathrm{d}k \\
&= c^2 \int_0^\infty \left(u \mp \frac{c}{2}\right) e^{-tu^2} \sum_{m=1}^\infty \frac{\sqrt{u^2 \mp cu + c^2(m+\frac{1}{2})^2} - c(m+\frac{1}{2})}{\sqrt{u^2 \mp cu + c^2(m+\frac{1}{2})^2}}\,\mathrm{d}u \\
&= c^3 \int_0^\infty \left(\frac{u}{c} \mp \frac{1}{2}\right) e^{-t\frac{u^2}{c^2}c^2} \sum_{m=1}^\infty \frac{\sqrt{\frac{u^2}{c^2} \mp \frac{u}{c} + (m+\frac{1}{2})^2} - (m+\frac{1}{2})}{\sqrt{\frac{u^2}{c^2} \mp \frac{u}{c} + (m+\frac{1}{2})^2}}\,\mathrm{d}u \\
&= c^4 \int_0^\infty \left(v \mp \frac{1}{2}\right) e^{-tc^2v^2} \sum_{m=1}^\infty \frac{\sqrt{v^2 \mp v + (m+\frac{1}{2})^2} - (m+\frac{1}{2})}{\sqrt{v^2 \mp v + (m+\frac{1}{2})^2}}\,\mathrm{d}v \\
&= c^4 \int_0^\infty \left(v \mp \frac{1}{2}\right) e^{-tc^2v^2} \sum_{m=1}^\infty \left[1 - \left(\sqrt{1 + \frac{(v \mp \frac{1}{2})^2 - \frac{1}{4}}{(m+\frac{1}{2})^2}}\right)^{-1}\right]\mathrm{d}v. \qquad \square
\end{aligned}
$$

Lemma 5.12. *We can estimate I_1^- by*

$$\frac{1}{5}\left(\frac{\sqrt{\pi}}{4}\frac{c}{\sqrt{t^3}} + \frac{1}{4}\frac{c^2}{t}\right) \le I_1^- \le \frac{\sqrt{\pi}}{4}\frac{c}{\sqrt{t^3}} + \frac{1}{4}\frac{c^2}{t}.$$

Proof. Consider the function $f\colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ describing the summands,

$$f(x) = 1 - \left(\sqrt{1 + \frac{(v \mp \frac{1}{2})^2 - \frac{1}{4}}{(x+\frac{1}{2})^2}}\right)^{-1}.$$

This function is positive, monotonously decreasing, $f(0) = 1 - (2v+1)^{-1}$ and $\lim_{x\to\infty} f(x) = 0$. We can therefore estimate the sum over the $f(n)$ by integrals,

$$\int_2^\infty f(x)\,\mathrm{d}x \le \sum_{n=1}^\infty f(n) \le \int_1^\infty f(x)\,\mathrm{d}x$$

To compute these integrals, let $w = (v+\frac{1}{2})^2 - \frac{1}{4}$, then

$$
\begin{aligned}
F(x) = \int f(x)dx &= \int 1 - \left(\sqrt{1 + \frac{w}{(x+\frac{1}{2})^2}}\right)^{-1}\mathrm{d}x \\
&= x - \left(x + \frac{1}{2}\right)\sqrt{1 + \frac{w}{(x+\frac{1}{2})^2}} + \mathrm{const}
\end{aligned}
$$

and we can compute the values

$$F(1) = 1 - \sqrt{(v + {}^1\!/_2)^2 + 2} + \mathrm{const}, \qquad F(2) = 2 - \sqrt{(v + {}^1\!/_2)^2 + 6} + \mathrm{const}$$

[6]For I_1^+, the index $+$ is to be used and for I_1^- the index $-$ is to be used.

as well as $\lim_{x\to\infty} F(x) = -1/2 + \text{const}$. Hence, we get bounds on the sum by

$$\sqrt{\left(v+\frac{1}{2}\right)^2+6}-\frac{5}{2} \le \sum_{m=1}^{\infty} f(m) \le \sqrt{\left(v+\frac{1}{2}\right)^2+2}-\frac{3}{2}.$$

For the lower bound, observe that $g\colon \mathbb{R}_{\ge 0} \to \mathbb{R}$, $v \mapsto \sqrt{(v+1/2)^2+6} - 5/2$ satisfies $g(0)=0$,

$$g'(v) = \frac{v+\frac{1}{2}}{\sqrt{\left(v+\frac{1}{2}\right)^2+6}}, \quad g''(v) = \frac{6}{((v+\frac{1}{2})^2+3)^{3/2}},$$

so $g''>0$ meaning that g' is strictly monotonously increasing and has its minimum at $g'(0)=1/5$. This implies $g(v) \ge v/5$. For the upper bound, we do the same analysis and find that for $h(v) = \sqrt{(v+1/2)^2+2} - 3/2$ we have $h(0)=0$ and $h'(v) \le \lim_{v\to\infty} h'(v) = 1$ implying that $h(v) \le v$. Hence we get new bounds

$$\frac{v}{5} \le \sum_{m=1}^{\infty} f(m) \le v.$$

Using these bounds, we get bounds on I_1^- by evaluating

$$c^4 \int_0^\infty \left(v+\frac{1}{2}\right) e^{-tc^2v^2} v \,\mathrm{d}v = c^4 \int_0^\infty v^2 e^{-tc^2v^2}\,\mathrm{d}v + \frac{c^4}{2}\int_0^\infty v e^{-tc^2v^2}\,\mathrm{d}v$$

By partial integration and with $\kappa = cv\sqrt{t}$, the first summand is given by

$$\begin{aligned} c^4 \int_0^\infty v^2 e^{-tc^2v^2}\,\mathrm{d}v &= c^2\left[-\frac{ve^{-tc^2v^2}}{2t}\right]_{v=0}^{\infty} + \frac{c^2}{2t}\int_0^\infty e^{-tc^2v^2}\,\mathrm{d}v \\ &= 0 + \frac{c}{2\sqrt{t^3}}\int_0^\infty e^{-\kappa^2}\,\mathrm{d}\kappa \\ &= \frac{\sqrt{\pi}c}{4\sqrt{t^3}} \end{aligned}$$

and with $\xi = tc^2v^2$ the second summand is

$$\frac{c^4}{2}\int_0^\infty v e^{-tc^2v^2}\,\mathrm{d}v = \frac{c^2}{4t}\int_0^\infty e^{-\xi}\,\mathrm{d}\xi = \frac{c^2}{4t}.$$

Therefore,

$$\frac{1}{5}\left(\frac{\sqrt{\pi}c}{4\sqrt{t^3}} + \frac{c^2}{4t}\right) \le I_1^- \le \frac{\sqrt{\pi}c}{4\sqrt{t^3}} + \frac{c^2}{4t}. \qquad \square$$

Lemma 5.13. *Let I_4 be the part of I_1^+ starting at 1, that is*

$$I_4 = c^4 \int_1^\infty \left(v-\frac{1}{2}\right) e^{-tc^2v^2} \sum_{m=1}^{\infty}\left[1-\left(\sqrt{1+\frac{(v-\frac{1}{2})^2-\frac{1}{4}}{(m+\frac{1}{2})^2}}\right)^{-1}\right]\mathrm{d}v.$$

Then

$$\frac{1}{5}\left[-\frac{c^2}{4t}e^{-tc^2} + \frac{\sqrt{\pi}}{4}\left(\frac{c}{\sqrt{t^3}}+\frac{c^3}{\sqrt{t}}\right)\operatorname{erfc}(c\sqrt{t})\right] \le I_4 \le \left[-\frac{c^2}{4t}e^{-tc^2} + \frac{\sqrt{\pi}}{4}\left(\frac{c}{\sqrt{t^3}}+\frac{c^3}{\sqrt{t}}\right)\operatorname{erfc}(c\sqrt{t})\right].$$

Proof. Similar as for I_-^1, in the case of I_1^+ we consider

$$f(x) = 1 - \left(\sqrt{1 + \frac{(v-\frac{1}{2})^2 - \frac{1}{4}}{(x+\frac{1}{2})^2}}\right)^{-1}.$$

If $v > 1$, the function f is again monotonously decreasing and we can estimate as before that

$$\frac{v-1}{5} \leq \sqrt{\left(v - \frac{1}{2}\right)^2 + 6} - \frac{5}{2} \leq \sum_{m=1}^{\infty} f(m) \leq \sqrt{\left(v - \frac{1}{2}\right)^2 + 2} - \frac{3}{2} \leq v - 1.$$

Therefore, we can bound the $(v > 1)$-part I_4 of I_1^+ by evaluating

$$\begin{aligned}
\widetilde{I_4} &= c^4 \int_1^\infty \left(v - \frac{1}{2}\right) e^{-tc^2v^2}(v-1)\,\mathrm{d}v \\
&= c^4 \int_1^\infty \left(v^2 - \frac{3}{2}v + \frac{1}{2}\right) e^{-tc^2v^2}\,\mathrm{d}v \\
&= c^4 \int_1^\infty v^2 e^{-tc^2v^2}\,\mathrm{d}v - \left(\frac{3}{2}c^4\right) \int_1^\infty v e^{-tc^2v^2}\,\mathrm{d}v + \frac{c^4}{2} \int_1^\infty e^{-tc^2v^2}\,\mathrm{d}v \\
&= c^2 \left[-\frac{ve^{-tc^2v^2}}{2t}\right]_{v=1}^\infty + \frac{c}{2\sqrt{t}} \int_{c\sqrt{t}}^\infty e^{-\kappa^2}\,\mathrm{d}\kappa - \left(\frac{3c^2}{4t}\right) \int_{tc^2}^\infty e^{-\xi}\,\mathrm{d}\xi + \frac{c^3}{2\sqrt{t}} \int_{c\sqrt{t}}^\infty e^{-\kappa^2}\,\mathrm{d}\kappa \\
&= \frac{c^2e^{-tc^2}}{2t} + \frac{\sqrt{\pi}c}{4\sqrt{t^3}}\mathrm{erfc}(c\sqrt{t}) - \left(\frac{3c^2}{4t}\right) e^{-tc^2} + \frac{\sqrt{\pi}c^3}{4\sqrt{t}}\mathrm{erfc}(c\sqrt{t}) \\
&= -\frac{c^2}{4t}e^{-tc^2} + \frac{\sqrt{\pi}}{4}\left(\frac{c}{\sqrt{t^3}} + \frac{c^3}{\sqrt{t}}\right)\mathrm{erfc}(c\sqrt{t})
\end{aligned}$$

with $1/5 \cdot \widetilde{I_4} \leq I_4 \leq \widetilde{I_4}$. $\square$

It remains to estimate

$$I_5 = c^4 \int_0^1 \left(v - \frac{1}{2}\right) e^{-tc^2v^2} \sum_{m=1}^\infty \left[1 - \left(\sqrt{1 + \frac{(v-\frac{1}{2})^2 - \frac{1}{4}}{(m+\frac{1}{2})^2}}\right)^{-1}\right] dv.$$

Lemma 5.14. *There is some constant $-\infty < -K < 0$ such that*

$$-K\left(\frac{1}{2}\frac{c^2}{t}e^{-tc^2} - \frac{\sqrt{\pi}}{4}\frac{c^3}{\sqrt{t}}\mathrm{erfc}(c\sqrt{t})\right) \leq I_5 \leq 0.$$

Proof. Note that the summands are non-positive and

$$\left[1 - \left(\sqrt{1 - \frac{1}{4(m+\frac{1}{2})^2}}\right)^{-1}\right] \leq \left[1 - \left(\sqrt{1 + \frac{(v-\frac{1}{2})^2 - \frac{1}{4}}{(m+\frac{1}{2})^2}}\right)^{-1}\right] \leq 0$$

so that

$$\sum_{m=1}^\infty \left[1 - \left(\sqrt{1 - \frac{1}{4(m+\frac{1}{2})^2}}\right)^{-1}\right] \cdot c^4 \int_0^1 \left(v - \frac{1}{2}\right) e^{-tc^2v^2} dv \leq I_5 \leq 0.$$

The sum converges to some constant $-\infty < -K < 0$ while

$$\begin{aligned} c^4 \int_0^1 \left(v - \frac{1}{2}\right) e^{-tc^2v^2}\,\mathrm{d}v &= \frac{c^2}{2t}\int_0^{tc^2} e^{-\xi}\,\mathrm{d}\xi - \frac{c^3}{2\sqrt{t}}\int_0^{c\sqrt{t}} e^{-\kappa^2}\,\mathrm{d}\kappa \\ &= \frac{c^2}{2t} - \frac{c^2}{2t}e^{-tc^2} - \frac{\sqrt{\pi}c}{4\sqrt{t}}\operatorname{erfc}(c\sqrt{t}). \end{aligned}$$

□

Corollary 5.15. *If $c = t^\zeta$ for $\zeta > -{}^1\!/\!{}_2$, then*

$$e^{-t\Delta_1}(0,0) \sim \frac{c}{t} = \frac{1}{t^{1-\zeta}}$$

as $t \to \infty$. In particular, for $-{}^1\!/\!{}_2 < \zeta < 1$,

$$\alpha\left(\lambda \mapsto \mathcal{G}_1(\lambda, \lambda^{1+\zeta})\right) = 2 - 2\zeta.$$

Proof. The assumption $\zeta > -{}^1\!/\!{}_2$ implies $c^2t \xrightarrow{t\to\infty} \infty$ and both e^{-tc^2} and $\operatorname{erfc}(c\sqrt{t})$ decay exponentially. By the previous computations,

$$e^{-t\Delta_1}(0,0) \sim \frac{1}{c}\left[I_1^+ + I_1^- + I_2 + I_3\right] \sim \frac{c}{t} + \frac{1}{t^{3/2}}$$

as $t \to \infty$. The assumption $\zeta > -{}^1\!/\!{}_2$ implies $t^{-3/2} \in \mathcal{O}({}^c\!/\!{}_t)$. In particular, since ${}^c\!/\!{}_t = t^{\zeta-1}$, this decays to zero as $t \to \infty$ for $\zeta < 1$. □

This concludes the computation of the asymptotics for $\alpha_\bullet(\mathfrak{h}^3)(\lambda, \lambda^{1+\zeta})$. □

5.3 Via Near Cohomology

Decomposing the tangent bundle $TM \xrightarrow{\pi} M$ as $TM \cong VM \oplus HM$ into a vertical and a horizontal subbundle gives us a diagram

$$\begin{tikzcd}
 & & VM \oplus HM \arrow[dl, bend right, "\cong"'] \arrow[dr, bend left, "\cong"] \arrow[d, phantom, "\cong"] \arrow[drr, bend left, "d\pi"] & & \\
TF_\bullet \arrow[r] \arrow[d] & T_\bullet F_\bullet \arrow[dr] & TM \arrow[d] & \pi^*TB \arrow[r, "d\pi"] \arrow[dl] & TB \arrow[d] \\
F_\bullet \arrow[rr] & & M \arrow[rr, "\pi"] & & B.
\end{tikzcd}$$

Given any vector field $X \in \Gamma(TM)$, we can decompose

$$X = Y + Z, \qquad \text{with} \qquad Y \in \Gamma(\pi^*TB), \quad Z \in \Gamma(T_\bullet F_\bullet)$$

into a horizontal component Y and a vertical component Z.
We call a vector field $Y \in \Gamma(\pi^*TB)$ basic, if there exists a vector field $\overline{Y} \in \Gamma(TB)$ such that Y is π-related to $\overline{Y}$, that is, the following diagram commutes (compare, for example, Besse [Bes08]):

$$\begin{tikzcd}
TM \arrow[r, "d\pi"] & TB \\
M \arrow[u, "Y"] \arrow[r, "\pi"'] & B \arrow[u, "\overline{Y}"']
\end{tikzcd}$$

We call Y the lift of $\overline{Y}$. For every $U \in \Gamma(TB)$ there exists a unique such lift $\widetilde{U} \in \Gamma(\pi^*TB)$. We denote by $\Gamma_b(HM) \subset \Gamma(\pi^*TB)$ the set of basic vector fields. Then $\Gamma_b(HM)$ spans $\Gamma(\pi^*TB)$ as a $\mathcal{C}^\infty(M)$-module, so every horizontal vector field $Y \in \Gamma(HM) \cong \Gamma(\pi^*(TB))$ can be written as

$$Y = \sum_{i \in I} f_i \cdot \widetilde{U_i}$$

for smooth functions $f_i \in \mathcal{C}^\infty M$ and $U_i \in \Gamma(TB)$.

Lemma 5.16. *Let $Z, Z' \in \Gamma(VM)$ be vertical vector fields and $Y \in \Gamma_b(HM)$ a basic horizontal vector field. Then*

1. $[Z, Z'] \in \Gamma(VM)$,
2. $[Y, Z] \in \Gamma(VM)$.

Proof. Recall that $VM = \ker(d\pi)$, hence $Z \sim_\pi 0$ and $Z \sim_\pi 0$ where $0 \in \Gamma(TB)$ denotes the zero section. By definition, $Y \sim_\pi \overline{Y}$ for some $\overline{Y} \in TB$. Therefore,

$$d\pi[Z, Z'] = \widetilde{[0, 0]_B} = 0, \qquad d\pi[Y, Z] = \widetilde{[\overline{Y}, 0]_B} = 0,$$

and the claim follows. $\square$

Looking at the deRham complex $\Omega^\bullet(M)$, it can be decomposed using the fibre bundle structure.

Theorem 5.17. *Let $F_\bullet \to M \to B$ be a fibre bundle, then there is an isomorphism*

$$\Omega^k(M) \xrightarrow[\cong]{\Phi} \bigoplus_{p+q=k} \Omega^p(B, \{\Omega^q(F_b)\}_{b \in B}),$$

identifying forms on M and forms on B with values in the system of forms on the fibres $\{F_b\}_{b \in B}$.[7]

Proof. Using that $TM \cong VM \oplus HM$, we decompose any $X \in \Gamma(TM)$ as a sum $X = Y + Z$ with $Y \in \Gamma(HM)$ and $Z \in \Gamma(VM)$. Given $U_1, \ldots, U_p \in \Gamma(TB)$ with basic lifts $\widetilde{U_1}, \ldots, \widetilde{U_p} \in \Gamma_b(HM)$ and $Z_{p+1}, \ldots, Z_k \in T_\bullet F_\bullet \cong VM$, for a k-form $\omega \in \Omega^k(M)$ we define

$$\Phi(\omega) = \sum_{p+q=k} (\Phi(\omega))_{p,q}$$

where the (p, q)-summand $(\Phi(\omega))_{p,q} \in \Omega^p(B, \{\Omega^q(F_\bullet)\})$ is given by

$$(\Phi(\omega))_{p,q}(U_1, \ldots, U_p)(Z_{p+1}, \ldots, Z_k) = \omega\left(\widetilde{U_1}, \ldots, \widetilde{U_p}, Z_{p+1}, \ldots, Z_k\right).$$

Decomposing $X_\bullet \in \Gamma(TM)$ as $X_\bullet = Y_\bullet + Z_\bullet$ with $Y_\bullet \in \Gamma(HM)$ and $Z_\bullet \in \Gamma(VM)$ as before, we construct the inverse

$$\Psi\colon \bigoplus_{p+q=n} \Omega^p(B, \{\Omega^q(F_\bullet)\}) \to \Omega^n(M)$$

to this map, starting with $\alpha \in \Omega^p(B, \{\Omega^q(F_\bullet)\})$ by

$$\Psi(\alpha)(X_1, \ldots, X_k) = \frac{1}{p!q!} \sum_{\sigma \in \mathcal{S}_k} \operatorname{sgn}(\sigma)\, (\pi^*\alpha)\, (Y_{\sigma(1)}, \ldots, Y_{\sigma(p)})(Z_{\sigma(p+1)}, \ldots, Z_{\sigma(k)}),$$

[7]The term on the right-hand-side is understood in the sense of A. Fomenko and D. Fuchs [FF16, Lec. 22.2].

where $\mathcal{S}_k$ is the set of permutations of the first k integers, $\{1, \dots, k\}$, and sgn the sign of the permutation. This is then extended linearly to the direct sum.
We check that Ψ and Φ are indeed inverses to each other. With the notation above, for a summand $\alpha \in \Omega^p(B, \{\Omega^q(F_\bullet)\})$,

$$\begin{aligned}
&\Phi\Psi(\alpha)(U_1, \dots, U_p)(Z_{p+1}, \dots, Z_k) \\
&\quad = \Psi(\alpha)\left(\widetilde{U_1}, \dots, \widetilde{U_p}, Z_{p+1}, \dots, Z_k\right) \\
&\quad = \frac{1}{p!q!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\,(\pi^*\alpha)\left(\widetilde{U_{\sigma(1)}}, \dots, \widetilde{U_{\sigma(p)}}\right)(Z_{\sigma(p+1)}, \dots, Z_{\sigma(k)}) \\
&\quad = (\pi^*\alpha)\left(\widetilde{U_1}, \dots, \widetilde{U_p}\right)(Z_{p+1}, \dots, Z_n) \\
&\quad = \alpha(U_1, \dots, U_p)(Z_{p+1}, \dots, Z_k),
\end{aligned}$$

where in the third equality $U_l = 0$ for $l > p$ and $Z_l = 0$ for $l \leq p$, so that after reordering the arguments, each summand appears $p!q!$ times with $+$ sign.
In the other direction, writing $X_\bullet = Y_\bullet + Z_\bullet \in \Gamma(HM) \oplus \Gamma(VM) \cong \Gamma(TM)$ as before,

$$\begin{aligned}
&\Psi\Phi(\omega)(X_1, \dots, X_k) \\
&\quad = \sum_{p+q=k} \frac{1}{p!q!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)(\pi^*\Phi(\omega))(Y_{\sigma(1)}, \dots, Y_{\sigma(p)})(Z_{\sigma(p+1)}, \dots, Z_{\sigma(k)}),
\end{aligned}$$

where pointwise for $x \in M$ with $b = \pi(x)$,

$$\begin{aligned}
&(\pi^*\Phi(\omega))_x(Y_{\sigma(1)}(x), \dots, Y_{\sigma(p)}(x))(Z_{\sigma(p+1)}(x), \dots, Z_{\sigma(k)}(x)) \\
&\quad = \Phi(\omega)_b(Y_{\sigma(1)}(x), \dots, Y_{\sigma(p)}(x))(Z_{\sigma(p+1)}(x), \dots, Z_{\sigma(k)}(x)) \\
&\quad = \omega_x(A_{\sigma(1)}(x), \dots, A_{\sigma(p)}(x), Z_{\sigma(p+1)}(x), \dots, Z_{\sigma(k)}(x)) \\
&\quad = \omega_x(Y_{\sigma(1)}(x), \dots, Y_{\sigma(p)}(x), Z_{\sigma(p+1)}(x), \dots, Z_{\sigma(k)}(x))
\end{aligned}$$

where A_i is some basic horizontal vector field with $A_i(x) = Y_i(x)$. Therefore,

$$\begin{aligned}
\Psi\Phi(\omega)(X_1, \dots, X_k) &= \sum_{p+q=k} \frac{1}{p!q!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\omega(Y_{\sigma(1)}, \dots, Y_{\sigma(p)}, Z_{\sigma(p+1)}, \dots, Z_{\sigma(k)}) \\
&= \sum_{(\Xi_1, \dots, \Xi_k) \in \{Y_1, Z_1\} \times \cdots \times \{Y_k, Z_k\}} \omega(\Xi_1, \dots, \Xi_k) \\
&= \omega(X_1, \dots, X_k),
\end{aligned}$$

where we use in the second equality that ω is antisymmetric and that after ordering each summand appears $p!q!$ times, where p is the number of $Y_\bullet$s and q the number of $Z_\bullet$s chosen. The last equality then follows by linearity of ω. □

We can now look at the deRham differential $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ under this decomposition.

Lemma 5.18. *Under the decomposition Φ of $\Omega^\bullet(M)$, the deRham differential splits into three summands, $d \cong d^{0,1} + d^{1,0} + d^{2,-1}$, where*

$$d^{i,1-i}\colon \Omega^p(B, \{\Omega^q(F_\bullet)\}) \to \Omega^{p+i}(B, \{\Omega^{q+1-i}(F_\bullet)\}).$$

Proof. By Cartan's formula, for $\omega \in \Omega^k(M)$ and $X_0, \dots, X_k \in \Gamma(TM)$, the deRham differential of ω evaluated on the $X_\bullet$s is given by

$$\begin{aligned} d(\omega)(X_1, \dots, X_{k+1}) = &\sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \overset{\widehat{X_i}}{\dots}, X_k)) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \overset{\widehat{X_i}, \widehat{X_j}}{\dots}, X_k). \end{aligned}$$

We denote by $[X_i, X_j]_H$ respectively $[X_i, X_j]_V$ the projection of $[X_i, X_j]$ to $\Gamma(HM)$ respectively $\Gamma(VM)$. Given $\alpha \in \Omega^p(B, \{\Omega^q F_\bullet\})$, we compute $\Phi d \Psi \alpha$ by looking at the (r,s)-component $(\Phi d \Psi \alpha)_{r,s} \in \Omega^r(B, \{\Omega^s F_\bullet\})$ (with $r + s = k+1$).
For this, let $U_1, \dots, U_r \in \Gamma(TB)$ and $Z_{r+1}, \dots, Z_{k+1} \in \Gamma(T_\bullet F_\bullet)$, then

$$\begin{aligned} &(\Phi d \Psi \alpha)_{r,s}(U_1, \dots, U_r)(Z_{r+1}, \dots, Z_{k+1}) \\ &\quad = (d \Psi \alpha) \left(\widetilde{U_1}, \dots, \widetilde{U_r}, Z_{r+1}, \dots, Z_{k+1} \right) \\ &\quad = \sum_{1 \le i \le r} (-1)^{i+1} \widetilde{U_i} \left(\Psi\alpha(\widetilde{U_1}, \overset{\widehat{U_i}}{\dots}, \widetilde{U_r}, Z_{r+1}, \dots, Z_{k+1}) \right) \\ &\quad\quad + \sum_{r+1 \le i \le k+1} (-1)^{i+1} Z_i \left(\Psi\alpha(\widetilde{Y_1}, \dots, \widetilde{U_r}, Z_{r+1}, \overset{\widehat{Z_i}}{\dots}, Z_{k+1}) \right) \\ &\quad\quad + \sum_{1 \le i < j \le r} (-1)^{i+j+1} \Psi\alpha \left([\widetilde{U_i}, \widetilde{U_j}], \widetilde{U_1}, \overset{\widehat{U_i}, \widehat{U_j}}{\dots}, \widetilde{U_r}, Z_{r+1}, \dots, Z_{k+1} \right) \\ &\quad\quad + \sum_{1 \le i \le r < j \le k+1} (-1)^{i+j+1} \Psi\alpha \left([\widetilde{U_i}, Z_j], \widetilde{U_1}, \overset{\widehat{U_i}}{\dots}, \widetilde{U_r}, Z_{r+1}, \overset{\widehat{Z_j}}{\dots}, Z_{k+1} \right) \\ &\quad\quad + \sum_{r+1 \le i < j \le k+1} (-1)^{i+j+1} \Psi\alpha \left([Z_i, Z_j], \widetilde{U_1}, \dots, \widetilde{U_r}, Z_{r+1}, \overset{\widehat{Z_i}, \widehat{Z_j}}{\dots}, Z_{k+1} \right). \end{aligned}$$

By definition, $\Psi(\alpha) \neq 0$ only if p of the arguments have non-zero components in $\Gamma(HM)$ and q of the arguments have non-zero components in $\Gamma(VM)$. Recall that $[Z, Z'], [\widetilde{U}, Z] \in \Gamma(VM)$ for all $Z, Z' \in \Gamma(VM)$ and $U \in \Gamma(TB)$. Therefore, the operator $\Phi d \Psi \alpha$ decomposes into the following three summands.

1. The first summand keeps the base-degree fixed and increases the fibre-degree by one. It is given for $\alpha \in \Omega^p(B, \{\Omega^q F_\bullet\})$ by

$$\begin{aligned} &(\Phi d \Psi \alpha)_{p,q+1}(U_1, \dots, U_p)(Z_{p+1}, \dots, Z_{k+1}) \\ &\quad = \sum_{p+1 \le i \le k+1} (-1)^{i+1} Z_i(\Psi\alpha(\widetilde{U_1}, \dots, \widetilde{U_p}, Z_{p+1}, \overset{\widehat{Z_i}}{\dots}, Z_{k+1})) \\ &\quad\quad + \sum_{p+1 \le i < j \le k+1} (-1)^{i+j+1-p} \Psi\alpha(\widetilde{U_1}, \dots, \widetilde{U_p}, [Z_i, Z_j], Z_{p+1}, \overset{\widehat{Z_i}, \widehat{Z_j}}{\dots}, Z_{k+1}) \\ &\quad = \sum_{p+1 \le i \le k+1} (-1)^{i+1} Z_i(\alpha(U_1, \dots, U_p))(Z_{p+1}, \overset{\widehat{Z_i}}{\dots}, Z_{k+1}) \\ &\quad\quad + \sum_{p+1 \le i < j \le k+1} (-1)^{i+j+1-p} \alpha(U_1, \dots, U_p)([Z_i, Z_j], Z_{p+1}, \overset{\widehat{Z_i}, \widehat{Z_j}}{\dots}, Z_{k+1}). \end{aligned}$$

2. The second summand increases the base-degree by one and keeps the fibre-degree fixed. It is given by

$$\begin{aligned}
&(\Phi d\Psi\alpha)_{p+1,q}(U_1,\dots,U_{p+1})(Z_{p+2},\dots,Z_{k+1})\\
&\quad= \sum_{1\le i\le p+1}(-1)^{i+1}\widetilde{U_i}(\Psi\alpha(\widetilde{U_1},\overset{\widehat{U_i}}{\dots},\widetilde{U_{p+1}},Z_{p+2},\dots,Z_{k+1}))\\
&\qquad+\sum_{1\le i<j\le p+1}(-1)^{i+j+1}\Psi\alpha([\widetilde{U_i},\widetilde{U_j}]_H,\widetilde{U_1},\overset{\widehat{U_i},\widehat{U_j}}{\dots},\widetilde{U_{p+1}},Z_{p+2},\dots,Z_{k+1})\\
&\qquad+\sum_{1\le i\le p+1<j\le k+1}(-1)^{i+j+1-p}\Psi\alpha(\widetilde{U_1},\overset{\widehat{U_i}}{\dots},\widetilde{U_{p+1}},[\widetilde{U_i},Z_j],Z_{p+2},\overset{\widehat{Z_j}}{\dots},Z_{k+1})\\
&\quad= \sum_{1\le i\le p+1}(-1)^{i+1}\widetilde{U_i}(\alpha(U_1,\overset{\widehat{U_i}}{\dots},U_{p+1})(Z_{p+2},\dots,Z_{k+1}))\\
&\qquad+\sum_{1\le i<j\le p+1}(-1)^{i+j+1}\alpha([U_i,U_j],U_1,\overset{\widehat{U_i},\widehat{U_j}}{\dots},U_{p+1})(Z_{p+2},\dots,Z_{k+1})\\
&\qquad+\sum_{1\le i\le p+1<j\le k+1}(-1)^{i+j+1-p}\alpha(U_1,\overset{\widehat{U_i}}{\dots},U_{p+1})([\widetilde{U_i},Z_j],Z_{p+2},\overset{\widehat{Z_j}}{\dots},Z_{k+1}).
\end{aligned}$$

3. The third summand increases the base-degree by two and decreases the fibre-degree by one. It is given by

$$\begin{aligned}
&(\Phi d\Psi\alpha)_{p+2,q-1}(U_1,\dots,U_{p+2})(Z_{p+3},\dots,Z_{k+1})\\
&\qquad= \sum_{1\le i<j\le p+2}(-1)^{i+j+1-p}\Psi\alpha(\widetilde{U_1},\overset{\widehat{U_i},\widehat{U_j}}{\dots},\widetilde{U_{p+2}},[\widetilde{U_i},\widetilde{U_j}]_V,Z_{p+3},\dots,Z_{k+1})\\
&\qquad= \sum_{1\le i<j\le p+2}(-1)^{i+j+1-p}\alpha(U_1,\overset{\widehat{U_i},\widehat{U_j}}{\dots},U_{p+2})([\widetilde{U_i},\widetilde{U_j}]_V,Z_{p+3},\dots,Z_{k+1}).
\end{aligned}$$

The claim follows with the differentials defined for $\alpha\in\Omega^p(B,\{\Omega^q(F_\bullet)\})$ by

$$\begin{aligned}
d^{0,1}(\alpha)&=(\Phi d\Psi\alpha)_{p,q+1},\\
d^{1,0}(\alpha)&=(\Phi d\Psi\alpha)_{p+1,q},\\
d^{2,-1}(\alpha)&=(\Phi d\Psi\alpha)_{p+2,q-1}.
\end{aligned}$$

and $d=d^{0,1}+d^{1,0}+d^{2,-1}$ extended linearly to $\bigoplus_{p+q=k}\Omega^p(B,\{\Omega^q(F_\bullet)\})$. $\square$

Denote $E_0^{p,q}=\Omega^p(B,\{\Omega^q(F_\bullet)\})$, then we can visualise this decomposition as a $\mathbb{Z}^2$-graded complex.[8] An excerpt of this is pictured below, with the maps $d^{i,1-i}$ only drawn at $E_0^{p,q}$ and as dashed arrows at their images. As usual, the parts appearing in $\Omega^k(M)$ align along the antidiagonal

[8] This is not a double complex in general as there is the diagonal $d^{2,-1}$-map. If we can choose a flat connection on M, then $d^{2,-1}$ vanishes and this is a true double complex. In terms of objects, this may be viewed as the zeroth page of the Serre spectral sequence of $F_\bullet\to M\to B$.

$p + q = k$ in the diagram.

$$
\begin{tikzcd}
E_0^{p,q+2} & E_0^{p+1,q+2} & E_0^{p+2,q+2} & E_0^{p+3,q+2} & E_0^{p+4,q+2} \\
E_0^{p,q+1} \arrow[u, dashed] \arrow[r, dashed] \arrow[rrd, dashed] & E_0^{p+1,q+1} & E_0^{p+2,q+1} & E_0^{p+3,q+1} & E_0^{p+4,q+1} \\
E_0^{p,q} \arrow[u, dashed, "d^{0,1}"] \arrow[r, "d^{1,0}"] \arrow[rrd, "d^{2,-1}"] & E_0^{p+1,q} \arrow[u, dashed] \arrow[r, dashed] \arrow[rrd, dashed] & E_0^{p+2,q} & E_0^{p+3,q} & E_0^{p+4,q} \\
E_0^{p,q-1} & E_0^{p+1,q-1} & E_0^{p+2,q-1} \arrow[u, dashed] \arrow[r, dashed] \arrow[rrd, dashed] & E_0^{p+3,q-1} & E_0^{p+4,q-1} \\
E_0^{p,q-2} & E_0^{p+1,q-2} & E_0^{p+2,q-2} & E_0^{p+3,q-2} & E_0^{p+4,q-2}
\end{tikzcd}
$$

Since $d = d^{0,1} + d^{1,0} + d^{2,-1}$ is a differential, that is, $d^2 = 0$, we obtain immediately that

$$
\begin{aligned}
&0 = (d^{0,1})^2, && 0 = d^{0,1}d^{1,0} + d^{1,0}d^{0,1}, \\
&0 = d^{0,1}d^{2,-1} + (d^{1,0})^2 + d^{2,-1}d^{0,1}, && 0 = d^{1,0}d^{2,-1} + d^{2,-1}d^{1,0}, \\
&0 = (d^{2,-1})^2.
\end{aligned}
$$

Note that $d^{1,0}$ is not a differential in general. Leaving out the terms that cancel due to the usual alternating sign[9], a direct computation shows that for $\alpha \in E_0^{p,q}$, $U_1, \ldots, U_{p+2} \in \Gamma(TB)$ and $Z_{p+3}, \ldots, Z_{k+2} \in \Gamma(T_\bullet F_\bullet)$:

$$
\begin{aligned}
&(d^{1,0})^2(\alpha)(U_1, \ldots, U_{p+3})(Z_{p+3}, \ldots, Z_{k+2}) \\
&\quad = \sum_{1 \le i < j \le p+2} (-1)^{i+j} (\widetilde{U}_j \widetilde{U}_i - \widetilde{U}_i \widetilde{U}_j)(\alpha(U_1, \overset{\widehat{U}_i, \widehat{U}_j}{\ldots}, U_{p+2})(Z_{p+3}, \ldots, Z_{k+2})) \\
&\qquad + \sum_{1 \le i < j \le p+2} (-1)^{i+j} \widetilde{[U_i, U_j]}_B (\alpha(U_1, \overset{\widehat{U}_i, \widehat{U}_j}{\ldots}, U_{p+2})(Z_{p+3}, \ldots, Z_{k+2})) \\
&\qquad + \sum_{1 \le i < j \le p+2 < l \le k+2} (-1)^{i+j+l+p} \alpha(U_1, \overset{\widehat{U}_i, \widehat{U}_j}{\ldots}, U_{p+2})([\widetilde{U}_i, [\widetilde{U}_j, Z_l]], Z_{p+3}, \ldots, Z_{k+2}).
\end{aligned}
$$

For the first two terms we have

$$
[\widetilde{U}_i, \widetilde{U}_j] - \widetilde{[U_i, U_j]}_B = [\widetilde{U}_i, \widetilde{U}_j]_V
$$

and since by the Jacobi identity $[\widetilde{U}_i, [\widetilde{U}_j, Z_l]] - [\widetilde{U}_j, [\widetilde{U}_i, Z_l]] = -[[\widetilde{U}_i, \widetilde{U}_j], Z_l]$ this precisely cancels out the terms that survive in $d^{0,1}d^{2,-1} + d^{2,-1}d^{0,1}$,

$$
\begin{aligned}
&(d^{0,1}d^{2,-1} + d^{2,-1}d^{0,1})(\alpha)(U_1, \ldots, U_{p+3})(Z_{p+3}, \ldots, Z_{k+2}) \\
&\quad = \sum_{1 \le i < j \le p+2} (-1)^{i+j} [\widetilde{U}_i, \widetilde{U}_j]_V (\alpha(U_1, \overset{\widehat{U}_i, \widehat{U}_j}{\ldots}, U_{p+2})(Z_{p+3}, \ldots, Z_{k+2})) \\
&\qquad + \sum_{1 \le i < j \le p+2 < l \le k+2} (-1)^{i+j+l+p+1} \alpha(U_1, \overset{\widehat{U}_i, \widehat{U}_j}{\ldots}, U_{p+2})([[\widetilde{U}_i, \widetilde{U}_j], Z_l], Z_{p+3}, \ldots, Z_{k+2}).
\end{aligned}
$$

[9] Coming from leaving out two arguments in two different orders.

The inner product on $\Omega^k(M)$ (coming from the Riemannian metric) induces inner products on the decomposition. For $\alpha, \alpha' \in \Omega^p(B, \{\Omega^q(F_\bullet)\})$,

$$\langle \alpha, \alpha' \rangle_{g,(p,q)} = \langle \Psi\alpha, \Psi\alpha' \rangle_{(M,g)} = \int_{(M,g)} \Psi\alpha \wedge *\Psi\alpha',$$

whereas the different direct summands are mutually orthogonal to each other since the decomposition of TM into VM and HM is orthogonal. This implies for $\omega \in \Omega^k(M)$ with $\Phi(\omega) = \alpha = \sum_{p+q=k} \alpha_{p,q} \in \bigoplus_{p+q=k} E_0^{p,q}$,

$$\|\omega\|_g^2 = \|\alpha\|_g^2 = \sum_{p+q=k} \|\alpha_{p,q}\|_g^2 = \sum_{p+q=k} \langle \alpha_{p,q}, \alpha_{p,q} \rangle_{g,(p,q)}.$$

When changing the metric from g to $g^{\mu,\nu}$ on M, the length of a vertical tangent vector $v \in V_xM$ changes by a factor ν as

$$\|v\|^2_{g^{\mu,\nu}} = \nu^2 g_x^V(v,v) = (\nu\|v\|_g)^2$$

and on horizontal tangent vectors $h \in H_xM$ by $\|h\|^2_{g^{\mu,\nu}} = (\mu\|h\|_g)^2$. Denote by ω_g the volume form on (M,g) and by $\omega_{g^{\mu,\nu}}$ the volume form on $(M, g^{\mu,\nu})$. By the observation above,

$$\omega_g = \mu^{-\dim(B)} \nu^{-\dim(F)} \cdot \omega_{g^{\mu,\nu}}.$$

The Hodge $*$-operators $*_g, *_{g^{\mu,\nu}}$ map $\Omega^k(M) \to \Omega^{n-k}(M)$ and preserve the decomposition as

$$*\colon \Omega^p(B, \{\Omega^q(F_\bullet)\}) \to \Omega^{\dim(B)-p}(B, \{\Omega^{\dim(F)-q}(F_\bullet)\}).$$

Since on $\Omega^p(B, \{\Omega^q(F_\bullet)\})$,

$$\begin{aligned}
\langle \alpha, \beta \rangle_{g^{\mu,\nu}} &= \int_{(M, g^{\mu,\nu})} \alpha \wedge *_{g^{\mu,\nu}} \beta = \mu^{-\dim(B)} \nu^{-\dim(F)} \int_{(M,g)} \alpha \wedge *_{g^{\mu,\nu}} \beta \\
&= \mu^{-\dim(B)} \nu^{-\dim(F)} \mu^{\dim(B)-2p} \nu^{\dim(F)-2q} \cdot \int_{(M,g)} \alpha \wedge *_g \beta \\
&= \mu^{-2p} \nu^{-2q} \langle \alpha, \beta \rangle_g
\end{aligned}$$

the scalar product changes by a factor $\mu^{-2p}\nu^{-2q}$.
This allows us to define the two-parameter Novikov-Shubin numbers via the near cohomology cones of the decomposed complex. Since the near cohomology cone satisfies

$$\begin{aligned}
&C^k_{\lambda_0}(M, g^{\mu,\nu}) = \left\{ \omega \in \Omega^k(M) \cap \ker(d)^\perp \;\middle|\; \|d\omega\|_{g^{\mu,\nu}} \le \lambda_0 \|\omega\|_{g^{\mu,\nu}} \right\} \\
&\cong \left\{ \alpha \in \left(\bigoplus_{p+q=k} E_0^{p,q} \right) \cap \Phi\left(\ker(d)^\perp\right) \;\middle|\; \sum_{r+s=k+1} \mu^{-r}\nu^{-s} \|(d\alpha)_{r,s}\|_g \le \lambda_0 \sum_{p+q=k} \mu^{-p}\nu^{-q} \|\alpha_{p,q}\|_g \right\},
\end{aligned}$$

we can define $\mathcal{G}_k(M \to B, \nabla, g)$ in terms of this near cohomology cone with $\lambda_0 = 1$ as follows.

Corollary 5.19. *In the notation as above,*

$$\mathcal{G}_k(M \to B, \nabla, g)(\mu, \nu) = \sup_L \dim_{\mathcal{N}G} L,$$

where the supremum runs over all closed linear subspaces L of $C_1^k(M, g^{\mu,\nu})$.

Proof. This follows immediately since $\mathcal{G}_k(M \to B, \nabla, g)(\mu,\nu) = \mathcal{F}_k(M, g^{\mu,\nu})(1)$. □

5.4 Invariance Properties

In this section we show multiple invariance properties of the two-parameter Novikov-Shubin numbers. We show that for a fibre bundle $M \to B$ and a fixed connection ∇, the dilatational equivalence class of the underlying spectral density functions is independent of the ∇-compatible Riemannian metric g on M. Then we show that the spectral density functions are dilatationally equivalent for two bundles $M \to B$ and $M' \to B$ if there exists a certain type of ∇-compatible fibre homotopy equivalence. We also show that the dilatational equivalence class of the spectral density functions is independent of the connection ∇ if we restrict them to the parameter subspace $\{\nu \leq \mu\}$, where the fibre is scaled at least as fast as the base. In particular, the two-parameter Novikov-Shubin numbers are invariant under these operations.

5.4.1 Metric Invariance for Fixed Connection

From the definition in terms of near cohomology cones, we can derive that the dilatational equivalence class of $\mathcal{G}_k(M \to B, \nabla, g)$ for a fixed connection ∇ does not depend on the metric g.

Theorem 5.20. *Let $G \curvearrowright (M \to B, \nabla, g)$ be a fibre bundle with fixed connection ∇ and compatible cocompact free proper group action by a group G. Then for $0 \leq k \leq \dim(M)$ the dilatational equivalence class of*

$$\mathcal{G}_k(M \to B, \nabla) = \mathcal{G}_k(M \to B, \nabla, g)$$

does not depend on the choice of G-invariant ∇-compatible Riemannian metric g.

Proof. On a compact manifold $\overline{M}$, any two Riemannian metrics $\overline{g}, \overline{g}'$ are quasi-equivalent, that is there exists $K \geq 1$ such that $K^{-1}\overline{g} \leq \overline{g}' \leq K\overline{g}$. By G-invariance of the Riemannian metrics and cocompactness of the action $G \curvearrowright M$, this is true for any two choices of G-invariant Riemannian metrics g, g' on M. Restricting to the subbundles V^*M and H^*M of T^*M, this inequality holds also for the vertical and horizontal parts individually. After rescaling, it follows that there is $K > 0$ such that for all $\mu, \nu > 0$,

$$K^{-1}g^{\mu,\nu} \leq (g')^{\mu,\nu} \leq Kg^{\mu,\nu}.$$

If $\omega \in C^k_\lambda(M, (g')^{\nu,\mu})$, then

$$K^{-2(k+1)}\|d\omega\|^2_{g^{\mu,\nu}} = \|d\omega\|^2_{K^{-1}g^{\mu,\nu}} \leq \|d\omega\|^2_{g'^{\mu,\nu}} \leq \lambda^2\|\omega\|^2_{g'^{\mu,\nu}} \leq \lambda^2\|\omega\|^2_{Kg^{\mu,\nu}} = K^{2k}\lambda^2\|\omega\|^2_{g^{\mu,\nu}}.$$

This implies that $\omega \in C^k_{K^{2k+1}\lambda}(M, g^{\mu,\nu}) = C^k_\lambda(M, Kg^{\mu,\nu})$. We can repeat this argument starting with $C^k_\lambda(M, g^{\mu,\nu})$ to obtain an inclusion in the other direction, so that in total

$$C^k_\lambda(M, K^{-1}g^{\mu,\nu}) \subset C^k_\lambda(M, g'^{\mu,\nu}) \subset C^k_\lambda(M, Kg^{\mu,\nu}).$$

Taking suprema over the $\mathcal{N}G$-dimensions of closed linear subspaces with $Kg^{\mu,\nu} = g^{K^{1/2}\mu, K^{1/2}\nu}$,

$$\begin{aligned}\mathcal{G}_k(M \to B, \nabla, g)(K^{-1/2}\mu, K^{-1/2}\nu) &\leq \mathcal{G}_k(M \to B, g', \nabla)(\mu, \nu) \\ &\leq \mathcal{G}_k(M \to B, \nabla, g)(K^{1/2}\mu, K^{1/2}\nu)\end{aligned}$$

and hence the spectral density functions are dilatationally equivalent,

$$G_k(M \to B, \nabla, g) \sim G_k(M \to B, g', \nabla). \qquad \square$$

5.4.2 Fibre Homotopy Invariance

Next, we want to study the behaviour of the two-parameter Novikov-Shubin numbers under fibre homotopy equivalences. Such a homotopy equivalence f, say between $M \to B$ and $M' \to B$, should respect the decomposition of $TM \cong_{\nabla} HM \oplus VM$ and $TM' \cong_{\nabla'} HM' \oplus VM'$ coming from the connections in the sense that $f^*\nabla' = \nabla$. This leads us to the following definition of geometric fibre homotopy equivalences.

Definition 5.21. Let $F_\bullet \to M \xrightarrow{\pi} B$ and $F'_\bullet \to M' \xrightarrow{\pi'} B$ be two fibre bundles over B equipped with connections

$$TM \cong_{\nabla} VM \oplus HM, \qquad TM' \cong_{\nabla'} VM' \oplus HM'.$$

A fibre homotopy equivalence $f\colon M \to M'$ is a homotopy equivalence $f\colon M \to M'$ such that f is a fibre map over the identity id_B of B, that is the diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "f"] & B \arrow[d, equal] \\ M' \arrow[r, "\pi'"] & B, \end{tikzcd}$$

commutes, and so is a homotopy equivalence inverse g of f as well as the homotopy $\Phi\colon M \times [0,1] \to M$ between gf and id_M at every time $t \in [0,1]$. We call such a fibre homotopy equivalence $f\colon M \to M'$ geometric if it satisfies $f^*\nabla' = \nabla$.

The property of being geometric implies that the pullback f^* commutes not only with the deRham differential d itself but also with each of the individual summands we identified earlier.

Lemma 5.22. *If $f\colon M \to M'$ is a geometric fibre homotopy equivalence then f^* commutes with the differential d and each of its three summands $d = d^{0,1} + d^{1,0} + d^{2,-1}$.*

Proof. Since f is geometric, the fibre homotopy equivalence f restricts fibrewise to homotopy equivalences

$$f|_{F_b}\colon F_b \xrightarrow{\simeq} F'_b.$$

and the push-forward $f_*\colon TM \to TM'$ restricts to maps

$$f_*\colon HM \to HM' \quad \text{and} \quad f_*\colon VM \to VM'.$$

Therefore, the induced chain homotopy $f^*\colon \Omega^\bullet M' \to \Omega^\bullet M$ restricts under the direct sum decompositions to maps on each (p,q)-summand, that is,

$$f^*_{p,q}\colon \Omega^p(B, \{\Omega^q(F'_\bullet)\}) \to \Omega^p(B, \{\Omega^q(F_\bullet)\})$$

given on $\alpha_{p,q} \in \Omega^p(B, \{\Omega^q(F'_\bullet)\})$ with $p + q = k$ by

$$\begin{aligned} &(f^*_{p,q}\alpha)_{p,q}(U_1, \dots, U_p)(Z_{p+1}, \dots, Z_k) \\ &\qquad = (f|_{F_\bullet})^* \left(\alpha_{p,q}(U_1, \dots, U_p)\right)(Z_{p+1}, \dots, Z_k) \\ &\qquad = \alpha_{p,q}(U_1, \dots, U_p)(df(Z_{p+1}), \dots, df(Z_k)). \end{aligned}$$

Recall that the differential d on $\Omega^p(B, \{\Omega^q(F_\bullet)\})$ splits into the following three summands:

$$\begin{aligned}
&(d^{0,1}\alpha)_{p,q+1}(U_1,\dots,U_p)(Z_{p+1},\dots,Z_{k+1})\\
&\quad= \sum_{p+1\le i\le k+1} (-1)^{i+1} Z_i(\alpha(U_1,\dots,U_p))(Z_{p+1},\overset{\widehat{Z_i}}{\dots},Z_{k+1})\\
&\quad\quad+ \sum_{p+1\le i<j\le k+1} (-1)^{i+j+1-p} \alpha(U_1,\dots,U_p)([Z_i,Z_j],Z_{p+1},\overset{\widehat{Z_i},\widehat{Z_j}}{\dots},Z_{k+1}),\\
&(d^{1,0}\alpha)_{p+1,q}(U_1,\dots,U_{p+1})(Z_{p+2},\dots,Z_{k+1})\\
&\quad= \sum_{1\le i\le p+1} (-1)^{i+1} \widetilde{U}_i(\alpha(U_1,\overset{\widehat{U_i}}{\dots},U_{p+1})(Z_{p+2},\dots,Z_{k+1}))\\
&\quad\quad+ \sum_{1\le i<j\le p+1} (-1)^{i+j+1} \alpha([U_i,U_j],U_1,\overset{\widehat{U_i},\widehat{U_j}}{\dots},U_{p+1})(Z_{p+2},\dots,Z_{k+1})\\
&\quad\quad+ \sum_{1\le i\le p+1<j\le k+1} (-1)^{i+j+1-p} \alpha(U_1,\overset{\widehat{U_i}}{\dots},U_{p+1})([\widetilde{U_i},Z_j],Z_{p+2},\overset{\widehat{Z_j}}{\dots},Z_{k+1})\\
&(d^{2,-1}\alpha)_{p+2,q-1}(U_1,\dots,U_{p+2})(Z_{p+3},\dots,Z_{k+1})\\
&\quad= \sum_{1\le i<j\le p+2} (-1)^{i+j+1-p} \alpha(U_1,\overset{\widehat{U_i},\widehat{U_j}}{\dots},U_{p+2})([\widetilde{U_i},\widetilde{U_j}]_V,Z_{p+3},\dots,Z_{k+1}).
\end{aligned}$$

Here, f^* commutes with $d^{0,1}$ as we can see directly from the formulae or from the fact that

$$(d^{0,1}\alpha)(U_1,\dots,U_p) = d_{F_\bullet}(\alpha(U_1,\dots,U_p))$$

acts as the fibre differential and therefore commutes with the pullback of f. From the formulae we see further that $d^{1,0}$ commutes with f^* since $df(\widetilde{U}) = \widetilde{U}' \circ f = \widetilde{U}$ as f preserves base points,

$$df([\widetilde{U}, Z]) = [df(\widetilde{U}), df(Z)] = [\widetilde{U}', df(Z)],$$

where $\widetilde{U}$ is the horizontal lift of U to TM and $\widetilde{U}'$ the horizontal lift to TM'. Lastly, $d^{2,-1}$ commutes with f^* since

$$\begin{aligned}
df\left([\widetilde{U_1},\widetilde{U_2}]_V\right) &= df\left([\widetilde{U_1},\widetilde{U_2}] - [\widetilde{U_1},\widetilde{U_2}]_H\right) = [df(\widetilde{U_1}), df(\widetilde{U_2})] - df(\widetilde{[U_1,U_2]})\\
&= [\widetilde{U_1}', \widetilde{U_2}'] - \widetilde{[U_1,U_2]}' = [\widetilde{U_1}', \widetilde{U_2}'] - [\widetilde{U_1}', \widetilde{U_2}']_H = [\widetilde{U}_1', \widetilde{U}_2']_V. \qquad \square
\end{aligned}$$

Lemma 5.23. *Let $G \curvearrowright (M \to B, \nabla, g)$ and $G \curvearrowright (M' \to B, \nabla', g')$ be two Riemannian fibre bundles with connection over the same base B and with compatible G-action. Let $f\colon M \to M'$ be a G-equivariant geometric fibre homotopy equivalence. If f^* and a geometric fibre homotopy inverse g^* of f^* are bounded as operators between $L^2\Omega^\bullet M'$ and $L^2\Omega^\bullet M$, then the two-parameter spectral density functions are dilatationally equivalent, that is, for $0 \le k \le \dim(M)$,*

$$\mathcal{G}_k(M' \to B, \nabla') \sim \mathcal{G}_k(M \to B, f^*\nabla').$$

Proof. By assumption, the induced map f^* is a bounded chain homotopy equivalence $L^2\Omega^\bullet M' \to L^2\Omega^\bullet M$ of Hilbert chain complexes, with bounded inverse g^*. Since

$$\mathcal{G}_k(M' \to B, \nabla', g')(\mu,\nu) = \mathcal{F}_k(M', g'^{\mu,\nu})(1)$$

and in the same way

$$\mathcal{G}_k(M \to B, f^*\nabla', g)(\mu,\nu) = \mathcal{F}_k(M, g^{\mu,\nu})(1)$$

for some[10] G-invariant Riemannian metrics compatible with the connections, the statement follows from a Proposition of M. Gromov and M. A. Shubin [GS91, Prop. 4.1]:
There exists $C(\mu,\nu)$ depending only on $\|f^*\|_{(M',g'^{\mu,\nu})\to(M,g^{\mu,\nu})}$ and $\|g^*\|_{(M',g'^{\mu,\nu})\to(M,g^{\mu,\nu})}$ with

$$\begin{aligned}\mathcal{G}_k(M\to B,f^*\nabla',g)(C(\mu,\nu)^{-1}\mu,C(\mu,\nu)^{-1}\nu) &= \mathcal{F}_k(M,C(\mu,\nu)^{-1}g^{\mu,\nu})(1)\\ &= \mathcal{F}_k(M,g^{\mu,\nu})(C(\mu,\nu)^{-1})\\ &\leq \mathcal{F}_k(M',g'^{\mu,\nu})(1)\\ &= \mathcal{G}(M'\to B,\nabla',g')(\mu,\nu)\\ &\leq \mathcal{F}_k(M,g^{\mu,\nu})(C(\mu,\nu))\\ &= \mathcal{G}_k(M\to B,f^*\nabla',g)(C(\mu,\nu)\mu,C(\mu,\nu)\nu).\end{aligned}$$

Since for $f^*\colon \Omega^p(B,\{\Omega^q F'_\bullet\})\to\Omega^p(B,\{\Omega^q F_\bullet\})$ (and in the same way for g^*),

$$\begin{aligned}\|f^*\|_{(M',g'^{\mu,\nu})\to(M,g^{\mu,\nu})} &= \sup_{0\neq\omega\in\Omega^p(B,\{\Omega^q F'_\bullet\})}\frac{\|f^*\omega\|_{g^{\mu,\nu}}}{\|\omega\|_{g'^{\mu,\nu}}}\\ &= \sup_{0\neq\omega\in\Omega^p(B,\{\Omega^q F'_\bullet\})}\frac{\mu^{-p}\nu^{-q}\cdot\|f^*\omega\|_g}{\mu^{-p}\nu^{-q}\cdot\|\omega\|_{g'}}\\ &= \sup_{0\neq\omega\in\Omega^p(B,\{\Omega^q F'_\bullet\})}\frac{\|f^*\omega\|_g}{\|\omega\|_{g'}} = \|f^*\|_{(M',g')\to(M,g)},\end{aligned}$$

the norms of f^* and g^* are independent of μ,ν and hence so is $C=C(\mu,\nu)$. Therefore, the claim follows from the inequalities above. □

Following the idea behind M. Gromov and M. A. Shubin's approach in [GS92] further, we can drop the requirement that f^* and g^* are bounded and obtain a more general invariance theorem.

Theorem 5.24. *In the notation above, if there is a G-equivariant geometric fibre homotopy equivalence between $M\to B$ and $M'\to B$, then for $0\leq k\leq\dim(M)$,*

$$\mathcal{G}_k(M'\to B,\nabla')\sim\mathcal{G}_k(M\to B,f^*\nabla').$$

Proof. In the spirit of [GS91, Thm 5.2], we show that for any geometric fibre homotopy equivalence $f\colon M\to M'$, we can construct a homotopy equivalence between the corresponding Hilbert chain complexes (which in particular is bounded). The main step here is to construct a submersive fibre homotopy equivalence $\widetilde{f}\colon M\times D^N\to M'$ from the a thickened fibre bundle $F_\bullet\times D^N\to M\times D^N\to B$ to $F'_\bullet\to M'\to B$, where D^N is a disk in $\mathbb{R}^N$.
We consider the vertical bundle $VM'\to M'$ and its pullback $f^*VM'\to M$ along f. By the smooth Serre-Swan theorem[11] there exists $N\in\mathbb{N}$ and an epimorphism p_1 of bundles over M,

$$\begin{array}{ccc}M\times\mathbb{R}^N & \xrightarrow{p_1} & f^*VM'\\ \downarrow & & \downarrow\\ M & = & M.\end{array}$$

[10] By Theorem 5.20, the dilatational equivalence classes of these spectral density functions are independent of this choice.
[11] The original Serre-Swan theorem [Swa62, Lem. 5] holds for compact topological manifolds. It since has been shown that in the smooth case it holds also for non-compact manifolds, see for example J. Nestruev's book [Nes20, Sec. 12.33] or Section 11.33 in the first edition. Here, we want the fibre bundle to be compatible with the action of G' on B, so that we may use the Serre-Swan theorem over the compact quotient $f^*V({}_G\backslash M')\to {}_G\backslash M$ and lift the bundle ${}_G\backslash M\times\mathbb{R}^N\to {}_G\backslash M$ to a bundle $M\times\mathbb{R}^N\to M$ compatible with the group action. This is possible since f is G-equivariant.

This gives us the following commutative diagram, where p_1 and p_2 are projections:

$$\begin{tikzcd}
M \times \mathbb{R}^N \arrow[r, two heads, "p_1"] \arrow[dr] & f^*VM' \arrow[r, two heads, "p_2"] \arrow[d] & VM' \arrow[d, "\pi"] \\
& M \arrow[r, "f"] \arrow[d] & M' \arrow[d] \\
& B \arrow[r, equal] & B
\end{tikzcd}$$

After fixing any ∇'-compatible Riemannian metric g' and the corresponding geodesic flows on M', on each fibre $F'_b = \pi^{-1}(b)$ of the bundle $VM' \to M'$ the exponential maps $\exp_b \colon V_bM' \to F'_b$ are defined and they glue to a map

$$\exp_V \colon VM' \to F'_\bullet.$$

For each $b \in B$, there is $\varepsilon(b) > 0$ such that the exponential map restricts to a diffeomorphism from $D^{V_bM'}_{\varepsilon(b)} = \left\{ v \in V_bM' \,\middle|\, g'_{V,b}(v,v) < \varepsilon(b)^2 \right\}$ onto its image. This radius $\varepsilon(b)$ can be chosen to depend continuously on b and be invariant under the cocompact action $G' \curvearrowright B$ and such that

$$\varepsilon = \inf_{b \in B} \{\varepsilon(b)\} = \min_{[b] \in G' \backslash B} \{\varepsilon([b])\}$$

exists and $\varepsilon > 0$. Since $g_{F,b}$ depends smoothly on $b \in B$, the set

$$U = \bigcup_{b \in B} D^{V_bM'}_{\varepsilon}$$

defines a neighbourhood of the zero section $0 \in \Gamma(VM')$. In particular, the map $\exp_V$ restricts to a diffeomorphism from U onto its image in M',

$$\exp_V \colon U \xrightarrow{\cong} \exp_V(U).$$

Further, for each $b \in B$ we can find $\delta(b) > 0$ depending continuously on b such that the subset $\{b\} \times D^N_\delta$ of the fibre over b of $M \times \mathbb{R}^N \to M$ maps into $D^{V_bM'}_\varepsilon$ via the composition $p_2 \circ p_1$. Since f preserves the base point, this can be chosen invariantly under the cocompact action $G' \curvearrowright B$ and we can define $\delta = \min_{[b] \in G' \backslash B} \{\delta([b])\} > 0$ and the image of $M \times D^N_\delta \subset M \times \mathbb{R}^N$ under $p_2 \circ p_1$ is contained in U. Hence, the composition map

$$\widetilde{f} = \exp_\bullet \circ p_2 \circ p_1$$

defines a submersion from $M \times D^N_\delta$ into M' (as a map over id_B):

$$\begin{tikzcd}
M \times \mathbb{R}^N & & VM' & & \\
\cup & & \cup & & \\
M \times D^N_\delta \arrow[r, two heads, "p_1"] \arrow[rrrr, bend right, "\widetilde{f}"] & f^*VM' \arrow[r, two heads, "p_2"] & U \arrow[r, "\exp_V", "\cong"'] & \exp_V(U) \subset & M'
\end{tikzcd}$$

Denote by $\iota \colon M \cong M \times \{0\} \hookrightarrow M \times D^N_\delta$ the inclusion as the zero section. Then the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "\iota"] \arrow[rr, bend left, "f"] \arrow[d] & M \times D^N_\delta \arrow[r, "\widetilde{f}"] \arrow[d] & M' \arrow[d] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}$$

Note that all maps are fibre maps over the identity id_B. The homotopy equivalences $L^2\Omega^k(M) \simeq L^2\Omega^k(M \times D_\delta^N)$ respect the direct sum decompositions.[12] Following [GS91, Thm. 5.2] further, the homotopy equivalence $\widetilde{f}^*$ between $L^2\Omega^\bullet M'$ and $L^2\Omega^\bullet(M \times D_\delta^N)$ induced by the submersion $\widetilde{f}$ is bounded. Since f is a bundle map over id_B, we even obtain bounded homotopy equivalences on each summand of the direct sum decomposition, $\Omega^p(B, \{\Omega^q F'_\bullet\}) \to \Omega^p(B, \{\Omega^q F_\bullet\})$. The claim now follows from the previous lemma. $\square$

5.4.3 (Partial) Metric Invariance

We have seen so far that the two-parameter Novikov-Shubin numbers behave well if the connection is fixed. If we allow the connection to vary, we can still find an invariance property if we scale the fibre at least as fast as the base, that is $\nu \le \mu$.

Theorem 5.25. *Let G be a group and $M \to B$ be equipped with two pairs of connection and Riemannian metric such that $G \curvearrowright (M \to B, \nabla, g)$ and $G \curvearrowright (M \to B, \nabla', g')$ are Riemannian fibre bundles with connection and compatible G-action. Then for all $0 \le k \le \dim(M)$ the two-parameter spectral density functions restricted to the subspace $\{\nu \le \mu\}$ are dilatationally equivalent,*

$$\mathcal{G}_k(M, \nabla, g)|_{\{\nu \le \mu\}} \sim \mathcal{G}_k(M, \nabla', g')|_{\{\nu \le \mu\}}.$$

Proof. Consider the decompositions

$$VM \oplus HM \quad \cong_\nabla \quad TM \quad \cong_{\nabla'} \quad VM \oplus H'M,$$

where the vertical bundle $VM = \ker(\pi^*)$ is independent of the connection. The identity map

$$\mathrm{id}\colon (M, g) \to (M, g')$$

induces a map $d\,\mathrm{id}\colon TM \to TM$ decomposing into maps $d\,\mathrm{id}\colon VM \to VM$ and $d\,\mathrm{id}\colon HM \to VM \oplus H'M$, so vertical tangent vectors remain vertical, but horizontal tangent vectors can obtain a vertical component. This is captured in the following diagram.

$$\begin{tikzcd}
HM \arrow[r] \arrow[ddr, "d\,\mathrm{id}"] & H'M \\
\oplus & \oplus \\
VM \arrow[r, "d\,\mathrm{id}"] & VM \\
\cong_\nabla & \cong_{\nabla'} \\
TM \arrow[r, "d\,\mathrm{id}"] \arrow[d] & TM \arrow[d] \\
(M, g) \arrow[r, "\mathrm{id}"] & (M, g')
\end{tikzcd}$$

[12] These homotopy equivalences are explicitly constructed in [GS91, Lem. 5.1]: Let $I = [0, 1]$ and $p\colon M \times I \to M$ be the natural projection and let $i_t\colon M \to M \times I$ for $t \in I$ be that map $x \mapsto (x, t)$. Then $p^*\colon L^2\Omega^k M \to L^2\Omega^k(M \times I)$ is a homotopy equivalence with inverse $J\colon L^2\Omega(M \times I) \to L^2\Omega^k M$, $J\omega = \int_0^1 i_t^*\omega dt$. Using this and the fact that $I^N \simeq D_\delta^N$ by Lipschitz maps gives the needed homotopy equivalences. Here, we consider $M \times I$ as a bundle $M \times I \to B$ with fibres $F_b \times I$ over $b \in B$.

For a form of pure degree (p,q) with respect to the direct sum decomposition coming from ∇',

$$\omega_{p,q} \in \Omega^p(B, \{\Omega^q(F_\bullet)\}) \subset_{\nabla'} \Omega^k(M, g'),$$

its pullback $\mathrm{id}^* \omega_{p,q} \in \Omega^k(M,g)$ decomposes under the direct sum decomposition coming from the connection ∇ as a sum,

$$\mathrm{id}^* \omega_{p,q} = \sum_{r+s=k} \alpha_{r,s},$$

with $\alpha_{r,s} \in \Omega^r(B, \{\Omega^s(F_\bullet)\}) \subset_\nabla \Omega^k(M,g)$. Since $\mathrm{id}^* \omega(X_1, \dots, X_k) = \omega(d\,\mathrm{id}(X_1), \dots, d\,\mathrm{id}(X_k))$, in the ∇-decomposition the (r,s)-summand $\alpha_{r,s}$ vanishes if $r < p$ or equivalently $s > q$. Hence

$$\mathrm{id}^* \omega_{p,q} = \sum_{\substack{r+s=k \\ r \geq p \,\wedge\, s \leq q}} \alpha_{r,s}.$$

Therefore, $\|\omega_{p,q}\|_{g'^{\mu,\nu}} = \mu^{-p}\nu^{-q}\|\omega_{p,q}\|_{g'}$ and

$$\begin{aligned} \|\,\mathrm{id}^* \omega_{p,q}\|_{g^{\mu,\nu}} &= \sum_{\substack{r+s=k \\ r \geq p \,\wedge\, s \leq q}} \|\alpha_{r,s}\|_{g^{\mu,\nu}} = \sum_{\substack{r+s=k \\ r \geq p \,\wedge\, s \leq q}} \mu^{-r}\nu^{-s}\|\alpha_{r,s}\|_g \\ &\overset{\nu\leq\mu}{\leq} \sum_{\substack{r+s=k \\ r \geq p \,\wedge\, s \leq q}} \mu^{-p}\nu^{-q}\|\alpha_{r,s}\|_g = \mu^{-p}\nu^{-q}\|\,\mathrm{id}^* \omega_{p,q}\|_g. \end{aligned}$$

Consequently,

$$\begin{aligned} \|\,\mathrm{id}^* |_{\Omega^p(B,\{\Omega^q(F_\bullet)\})}\|_{(M,g'^{\mu,\nu}) \to (M,g^{\mu,\nu})} &= \sup_{\omega_{p,q} \in \Omega^p(B,\{\Omega^q(F_\bullet)\})} \frac{\|\,\mathrm{id}^* \omega_{p,q}\|_{g^{\mu,\nu}}}{\|\omega_{p,q}\|_{g'^{\mu,\nu}}} \\ &\leq \sup_{\omega_{p,q} \in \Omega^p(B,\{\Omega^q(F_\bullet)\})} \frac{\mu^{-p}\nu^{-q} \cdot \|\,\mathrm{id}^* \omega_{p,q}\|_g}{\mu^{-p}\nu^{-q} \cdot \|\omega_{p,q}\|_{g'}} \\ &= \|\,\mathrm{id}^* |_{\Omega^p(B,\{\Omega^q(F_\bullet)\})}\|_{(M,g') \to (M,g)} \end{aligned}$$

Since the decomposition into the $\Omega^p(B, \{\Omega^q(F_\bullet)\})$ is orthogonal, it follows that

$$\|\,\mathrm{id}^*\,\|_{(M,g'^{\mu,\nu}) \to (M,g^{\mu,\nu})} \leq \|\,\mathrm{id}^*\,\|_{(M,g') \to (M,g)}.$$

The same argument holds if we consider the identity map as a map in the other direction, that is $\mathrm{id}\colon (M,g') \to (M,g)$. Let

$$K = \max\left\{\|\,\mathrm{id}^*\,\|_{(M,g') \to (M,g)},\ \|\,\mathrm{id}^*\,\|_{(M,g) \to (M,g')}\right\}.$$

For any $\omega \in C^k(M, g'^{\mu,\nu})(1)$ with $\nu \leq \mu$ it follows, therefore, that

$$\|d\,\mathrm{id}^* \omega\|_{g^{\mu,\nu}} = \|\,\mathrm{id}^* d\omega\|_{g^{\mu,\nu}} \leq K\|d\omega\|_{g'^{\mu,\nu}} \leq K\|\omega\|_{g'^{\mu,\nu}} = K\|\,\mathrm{id}^* \omega\|_{g'^{\mu,\nu}} \leq K^2\|\omega\|_{g^{\mu,\nu}}$$

and similarly in the other direction. These inequalities imply that

$$C^k(M, g^{\mu,\nu})(K^{-2}) \subset C^k(M, g'^{\mu,\nu})(1) \subset C^k(M, g^{\mu,\nu})(K^2).$$

Hence the spectral density functions are dilatationally equivalent and the claim follows. □

5.5 Remarks and Future Directions

In this chapter we defined the two-parameter Novikov-Shubin numbers and proved some invariance properties. The example of the Heisenberg group suggests some interesting behaviour that may prove valuable to pursue further — also in order to understand the classical Novikov-Shubin invariants on fibre bundles better.

Remark 5.26. In the way defined here, two-parameter Novikov-Shubin numbers generalise the analytic version of Novikov-Shubin invariants, as they are defined via the deRham cochain complex. This suggests the question whether there is an equivalent definition that generalises the combinatorial approach. Some progress in that direction can be made.
For example, assume that

- B is a simplicial complex,
- for every $b \in B$ the fibre F_b is a CW complex
- and the k-cells of M are precisely of the form $\sigma \times \rho \subset \sigma \times F_\sigma \cong \pi^{-1}(\sigma)$, where σ is a p-simplex in B and $\sigma \times F_\sigma \cong \pi^{-1}(\sigma)$ a suitable local trivialisation over the contractible simplex σ with fibre $F_\sigma = F_b$ for some $b \in \sigma$ (e.g., the barycenter) and ρ some q-cell in F_b.

Then, the cellular chain complex of M decomposes similarly to the deRham complex as

$$C_k^{\mathrm{cell}}(M) = \bigoplus_{p+q=k} C_p^{\mathrm{simp}}(B, \{C_q^{\mathrm{cell}}(F_\sigma)\}_{\sigma \in S_p(B)}).$$

Indeed, a chain in the (p, q)-summand on the right-hand-side is a formal sum

$$\sum_{\sigma \in S_p(B)} \lambda_\sigma \cdot \sum_{\rho \in I_q(F_\sigma)} \lambda_\rho \cdot \rho,$$

where $\sigma \in S_p(B)$ runs over the set of p-simplices of B and $\rho \in I_q(F_\sigma)$ over the set of q-cells of the corresponding fibre F_σ. Meanwhile, a chain on the left-hand-side is of the form

$$\sum_{\gamma \in I_k(M)} \lambda_\gamma \cdot \gamma.$$

By the assumed compatibility of the combinatorial structures, each γ is of the form $\gamma \cong \sigma \times \rho$ for some $\sigma \in S_p(B)$ and some $\rho \in I_q(F_\sigma)$, so we can write

$$\sum_{\gamma \in I_k(M)} \lambda_\gamma \cdot \gamma = \sum_{p+q=k} \sum_{\sigma \in S_p(B)} \sum_{\rho \in I_q(F_\sigma)} \lambda_{\sigma \times \rho} \cdot \sigma \times \rho.$$

Therefore, $\lambda_{\sigma \times \rho} \cdot \sigma \times \rho \longleftrightarrow \lambda_\sigma \lambda_\rho \cdot \rho$ yields such an isomorphism.
However, in contrast to what we saw on the deRham complex, the cellular differential does not simplify much in this picture. It will split into $(p + 1)$ differentials

$$d_i \colon C_p^{\mathrm{simp}}(B, \{C_q^{\mathrm{cell}}(F_b\}_{b \in B}) \to C_{p-i}^{\mathrm{simp}}(B, \{C_{q+i-1}^{\mathrm{cell}}(F_b)\}_{b \in B}) \qquad \text{for } 0 \leq i \leq p,$$

and, in general, more than three of the summands are non-zero — unlike in the deRham complex. This makes giving a combinatorial definition for the two-parameter Novikov-Shubin numbers, and in particular proving that it agrees with the analytic version, more difficult.

Remark 5.27. Another possible approach in the spirit of Chapter 3 of this thesis might be by stochastic methods and, in particular, random walks. Such an approach would be closely connected to a combinatorial definition from the previous remark. Nonetheless, we can make some remarks about how the scaling might interact with such a random walk.

Recall that the classical Novikov-Shubin invariants are defined in terms of the asymptotic behaviour of the spectral density function,

$$F_k(M,g)(\lambda) - b_k^{(2)}(M) \qquad \text{for } \lambda \searrow 0.$$

We saw that we can replace the argument λ by a constant and scale M by a factor λ instead,

$$F_k(M,g_\lambda)(1) - b_k^{(2)}(M) \qquad \text{for } \lambda \searrow 0.$$

We can mimic this shrinking of the manifold also on the level of random walks. Recall that the Novikov-Shubin invariants relate to the asymptotic behaviour of sums of differences of the form

$$\mathbb{P}(\alpha_+ \xrightarrow{n} \alpha_+) - \mathbb{P}(\alpha_+ \xrightarrow{n} \alpha_-) \qquad \text{for } n \to \infty.$$

We can model shrinking the manifold by a factor $1/n$ if we let the random walkers take n steps at once. Starting with a random walk $\mathfrak{R} = (\Omega, P)$, we can define $\mathfrak{R}_{1/n} = (\Omega, P^n)$ with moving probabilities $\mathbb{P}_{1/n}(s \to s') = \mathbb{P}(s \xrightarrow{n} s')$. By definition, the difference we are interested in becomes the difference of probabilities from two single steps,

$$\mathbb{P}_{1/n}(\alpha_+ \to \alpha_+) - \mathbb{P}_{1/n}(\alpha_+ \to \alpha_-) \qquad \text{for } n \to \infty,$$

and their asymptotic behaviour as we scale the manifold down.

It is less clear how scaling two directions with different speeds would translate in this picture. Scaling the base by a factor μ and the fibre by a factor ν, a random walker could possibly take roughly μ^{-1} steps at once in base directions or ν^{-1} steps in fibre directions. More generally, a random walker should be able to take a steps in base directions and b steps in fibre directions, where $a\mu^{-1} + b\nu^{-1} \approx 1$. Potentially, this could be simulated by adapting the probabilities of the underlying random walk $\mathfrak{R}$ by scaling probabilities of moves in base directions by a factor μ^{-1} and for fibre directions by ν^{-1} and then renormalising to get a random walk $\mathfrak{R}_{\mu,\nu}$. Taking

$$n \approx \mu^{-1}\mathbb{P}(\text{moving in a base direction}) + \nu^{-1}\mathbb{P}(\text{moving in a fibre direction})$$

many steps at once might capture some behaviour of the two-parameter Novikov-Shubin numbers.

Appendix A

Python Code

The code used to compute the estimates on Novikov-Shubin invariants on low dimensional Lie algebras is given below and split in the following files:

- `Rumin_Estimates_Diss.py` — Main file, to be executed.
- `Nilpotent_Lie_Algebra.py` — Given its dimension and structure constants, we save and compute relevant informations of a nilpotent Lie algebra.
- `Differential_Forms.py` — Helper classes and methods to deal with the calculus of differential forms.
- `Lie_Complex.py` — Given a nilpotent Lie algebra, compute and store relevant information about its Chevalley-Eilenberg complex and Lie algebra cohomology.
- `Weight_Handler.py` — Methods implementing the linear algebra behind finding weight functions that yield pure weight in some degree.
- `NLG_List.py` — A list of all nilpotent Lie algebras in terms of their dimension and structure constants (according to the list of W. A. de Graaf [Gra07].)
- `Compute_Rumin_Estimates.py` — Method to find estimates on Novikov-Shubin invariants based on the weights.
- `Print_Methods.py` — Wrapper method to print the result as Ascii or Latex code.

The code can be found online on GitHub in the following repository:
https://github.com/HoepfnerT/Rumin_Estimates_PhD_Thesis

Remarks on higher-dimensional Lie algebras, rounding errors and runtime estimates. The main reason that we only compute the estimates for nilpotent Lie algebras up to dimension six is that these algebras are classified nicely by W. A. de Graaf [Gra07] to be a finite list of 34 algebras. In theory, the algorithm can applied in the same manner to higher-dimensional nilpotent Lie algebras, however, it should be noted that the current implementation does not do precise computations but uses floating point approximations (for example, while applying the Gram-Schmidt algorithm via calling the `sympy`-package). This is unproblematic in lower dimensions but could lead to rounding errors impacting the computations if dimensions get large.

Further, the runtime of this algorithm does not scale well as the dimension increases. Indeed, if we consider a Lie algebra $\mathfrak{g}$ of dimension $2n$ and want to find estimates in the middle degree n, we need to study the middle degrees of the Chevalley-Eilenberg complex, where $\Lambda^n\mathfrak{g} \cong \mathbb{R}^{\binom{2n}{n}}$, and find the Lie algebra cohomology as a subspace thereof. By Stirling's formula, the dimension $\binom{2n}{n}$ grows exponentially in n, leading to similar estimates on the runtime of the estimates of the middle-degree Novikov-Shubin invariants.
Running the program for three nilpotent Lie algebras of dimension 7 (the 7-dimensional Heisenberg Lie algebra $\mathfrak{h}^7$ denoted in code as $H7$ and two arbitrarily chosen algebras from M-P. Gong's classification [Gon98]) takes roughly half as long as the computation of the full list of 34 nilpotent Lie algebras of dimension up to six. Here, running the command

```
print(Print_Methods.make_NS_invars_table([H7, L7_37A, L7_257C], MAX_DIM=7))
```

from within the `Rumin_Estimates_Diss.py`-main file yields the output after approx. 16.3 seconds (averaged over 10 runs on my local hardware) compared to 32.6 seconds for the list of 34 Lie algebras of dimension up to six.

	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7
$H7$	8	8	8	4	8	8	8	∞^+
$L7(37A)$	10	$[5,10]$	–	–	–	$[5,10]$	10	∞^+
$L7(257C)$	11	–	–	–	–	–	11	∞^+

For the 9-dimensional Heisenberg Lie algebra the program returns the correct result after approx. 307 seconds (averaged over 3 runs on my local hardware).

The time requirement grows less quickly when computing estimates on $\alpha_k(\mathfrak{g})$ for fixed $k \geq 0$ as $\dim \mathfrak{g}$ increases. An adapted algorithm could potentially run with runtime $\mathcal{O}(\binom{n}{k}) = \mathcal{O}(n^k)$.

```python
##### Start Nilpotent_Lie_Algebra.py #####
class Nilpotent_Lie_Algebra():
    """Save the dimension and the structure constants of a nilpotent Lie Algebra."""
    def __init__(self, name, dim, structure_constants):
        self.name, self.dim, self.structure_constants = name, dim, structure_constants

    # Return c_ij^k
    def get_structure_constant_ijk(self, i,j,k):
        if not k in self.structure_constants:   return 0
        for i2,j2,c in self.structure_constants[k]:
            if [i,j] == [i2, j2]:               return c
        return 0

    # Compute the lower central series
    def get_lower_central_series(self):
        lcs = []
        g = list(range(self.dim)) # g0
        while len(g) > 0:
            lcs.append(g)
            g = [ k for k in g
                  if (k in self.structure_constants)
                  and any([(i in g or j in g) and (not c == 0)
                           for i,j,c in self.structure_constants[k]]) ] # g_n+1 <-- g_n
        return lcs

    # Read off the growth rate from the lower central series
    def get_growth_rate(self):   return sum([len(g) for g in self.get_lower_central_series()])

    def __str__(self): return self.name
    def __repr__(self): return f"Nilpotent Lie Algebra {self.name} of dimension {self.dim} with structure constants {self.structure_constants}."
##### End Nilpotent_Lie_Algebra.py #####
```

```python
##### Start Differential_Forms.py #####
import sympy as sp
# Sort list, return sign of permutation needed to sort
def sort_sign(w: tuple):
    swap_counter = 0
    for i in range(len(w)-1):
        for j in range(i,len(w)):
            if w[i] > w[j]: swap_counter += 1
    if swap_counter % 2 == 0: return 1
    return -1

class Basis_Differential_Form():
    """Describe a differential form of type   coefficient * dx_I   by its coefficient and the ordered tuple I."""
    def __init__(self, coefficient, direction: tuple):
        self.coefficient = sort_sign(direction)*sp.sympify(coefficient)
        self.direction   = tuple(sorted(direction))

    def simplify(self):
        try:    self.coefficient = sp.simplify(self.coefficient)
        except: pass
        return  self

    # Compute weight based on a weight directionary for 1-forms
    def get_weight(self, weights_dict: dict):
        if self == 0: raise ValueError("The 0-form has no well-defined weight.")
        try:    return sum([weights_dict[sp.symbols("w" + str(i))] for i in self.direction])
        except KeyError:
            raise KeyError(f"{self.direction} not contained in the weights dictionary {weights}")

    ## Arithmetics
    def __add__(self, other):
        if self.direction == other.direction:
            return Basis_Differential_Form(self.coefficient + other.coefficient, self.direction)
        return NotImplemented
    def __neg__(self):          return Basis_Differential_Form(-self.coefficient, self.direction)
    def __sub__(self, other):   return self + (-other)
    def __bool__(self):         return not(self.coefficient == 0)
    def __mul__(self, other):
        if isinstance(other, (int, float)):
            return Basis_Differential_Form(other*self.coefficient, self.direction)
        if isinstance(other, Basis_Differential_Form):
            if not len(set(self.direction + other.direction)) == len(self.direction) + len(other.direction):
                return 0
```

```python
            return Basis_Differential_Form( self.coefficient * other.coefficient * sort_sign(self.direction + other.direction),
                                            tuple(sorted(self.direction + other.direction)) )
    def __str__(self):          return f"({self.coefficient}) {self.direction}"
    def __repr__(self):
        if len(self.direction) == 0:    return f"{self.coefficient}"
        if self.coefficient == 1:       return "\\vartheta_{" + "".join([str(i) for i in self.direction]) + "}"
        return f"({self.coefficient})" + "\\vartheta_{" + "".join([str(i) for i in self.direction]) + "}"

class Differential_Form():
    """Describe a differential form given as a sum of basis differential forms by the list of these basis forms"""
    def __init__(self, summands: list):
        self.summands = summands
    # Simplify a differential form by adding basis forms of same directon
    # Set flag rescale to true if rescaling is allowed
    def simplify(self, rescale = False):
        self_dict = {}
        for summand in self.summands:
            if summand.direction in self_dict:
                self_dict[summand.direction].append(summand.coefficient)
            else:
                self_dict[summand.direction] = [summand.coefficient]
        self_dict = {direction: sum( self_dict[direction] ) for direction in self_dict }
        self.summands = [ Basis_Differential_Form(self_dict[direction], direction).simplify()
                          for direction in self_dict ]
        self.summands = [ summand for summand in self if summand ] # remove 0 entries
        if rescale:
            min_coeff = min([ abs(summand.coefficient) for summand in self ])
            for summand in self.summands: summand.coefficient /= min_coeff
        self.summands.sort(key = lambda x: x.direction)
        return self

    def to_dict(self):          return {summand.direction: summand.coefficient for summand in self}

    def get_weights(self, weights_dict: dict):
        return list(set([summand.get_weight(weights_dict) for summand in self]))

    ## Arithmetics
    def __add__(self, other):   return Differential_Form(self.summands + other.summands).simplify()
    def __neg__(self):          return Differential_Form([-summand for summand in self])
    def __sub__(self, other):   return self + (-other)
    def __bool__(self):         return not(len(self.summands) == 0)

    def __mul__(self, other):
```

```
        if isinstance(other, (int, float)):
            if other == 0: return Differential_Form([Basis_Differential_Form(0, ())])
            return Differential_Form([bf1*other for bf1 in self.summands])
        elif isinstance(other, Differential_Form):
            return Differential_Form([bf1*bf2 for bf1 in self.summands for bf2 in other.summands if bf1*bf2]).simplify()

    def __iter__(self):         yield from self.summands

    def __str__(self):
        if len(self.summands) == 0: return "0"
        return "  +  ".join([str(summand) for summand in self])
    def __repr__(self):
        if len(self.summands) == 0: return "0"
        return "  +  ".join([repr(summand) for summand in self])


class Span_Differential_Forms():
    """ Describe a linear subspace of differential forms given by its basis vectors by a list of these basis vectors """
    def __init__(self, basis: list):
        self.basis = basis

    def simplify(self):
        for form in self.basis: form.simplify(rescale = True)
        self.basis = [ form for form in self.basis if form ]

    def get_weights(self, weights_dict: dict):
        return list(set().union(*[form.get_weights(weights_dict) for form in self]))

    def __iter__(self):         yield from self.basis
    def __str__(self):          return "<   " + ",\n\t".join([str(form) for form in self.basis]) + "   >"
    def __repr__(self):         return "\\langle " + ",\n\t\t".join([repr(form) for form in self.basis]) + "  \\rangle"
##### End Differential_Forms.py #####
```

```python
##### Start Lie_Complex.py #####
from Differential_Forms import Basis_Differential_Form, Differential_Form, Span_Differential_Forms, sort_sign
import itertools
import sympy as sp


class Lie_Complex():
    """
    Handle the Lie Complex and its homology for a given nilpotent Lie algebra G
    """
    def __init__(self, G):
        self.G = G
        self.chain_basis_list       = [ list(itertools.combinations(list(range(self.G.dim)), k)) for k in range(self.G.dim+1) ]
        self.differential_list      = self.setup_differential_list()
        self.homology_basis_list    = self.setup_homology_basis_list()
        self.envelope = [ direction for k in range(self.G.dim) for form in self.get_homology_group_basis(k) for direction in form.to_dict() ]


    # compute entries of matrix representing the differential
    def get_differential_entry(self, v: tuple, w: tuple):
        for k in v:
            for i, j in itertools.combinations(w, 2):
                if set(list(w) + [k] ) == set(list(v) + [i,j]):
                    sgn = sort_sign(list(w) + [k]) * sort_sign(list(w) + [i,j])
                    return -sgn*self.G.get_structure_constant_ijk(i,j,k)
        return 0

    def setup_differential_list(self):
        differential_list  = [ sp.Matrix([[0] for i in range(self.G.dim)]) ] # d^0: C^0(G) -> C^1(G)
        differential_list += [ sp.Matrix([[ self.get_differential_entry(v,w)
                                            for v in self.chain_basis_list[m]   ]
                                            for w in self.chain_basis_list[m+1] ])
                               for m in range(1, self.G.dim-1) ]            # d^m: C^m(G) -> C^(m+1)(G), n>m>0
        differential_list += [ sp.Matrix([[0 for i in range(self.G.dim)]]) ] # d^{n-1}: C^n(G) -> 0
        differential_list += [ sp.Matrix([[0]]) ]                           # d^n: C^n(G) -> 0
        return differential_list

    def setup_homology_basis_list(self):
        homology_groups = [ sp.Matrix([[1]]) ]  # H^0(G)
        for k in range(1,self.G.dim):
            rg = sp.Matrix(self.differential_list[k-1]).columnspace()    # orthogonal basis of image
            ker = sp.Matrix(self.differential_list[k]).nullspace()       # orthogonal basis of kernel
            joined = rg + ker
```

```
            H = joined[0]
            for m in joined[1:]: H = H.row_join(m)
            Q, _ = H.QRdecomposition()
            H = Q[:, len(rg):].T
            homology_groups.append(H)             # H^k(G)
        homology_groups.append(sp.Matrix([[1]]))  # H^n(G)
        return homology_groups

    def get_differential(self, k: int):
        if not k in range(self.G.dim+1): return 0
        return self.differential_list[k]

    # Return a list of span vectors of H^k(G) as a subspace in C^k(G).
    def get_homology_group_basis(self, k: int):
        if not k in range(self.G.dim+1): return 0
        H = self.homology_basis_list[k]
        basis = Span_Differential_Forms([
                    Differential_Form([
                        Basis_Differential_Form(c, bv)
                        for c,bv in zip(H.row(j), self.chain_basis_list[k]) ])
                    for j in range(H.shape[0]) ])
        basis.simplify()
        return basis

    def __str__(self): return f"Lie Complex of {self.G}."
    def __repr__(self):
        repr = f"For Lie Algebra {self.G}:\n"
        for k in range(self.G.dim+1):
            repr += f"\nDimension {k}:\nH^{k}(G) = {self.get_homology_group_basis(k)} \nd_{k} = {self.get_differential(k)}"
        return repr
##### End Lie_Complex.py #####
```

```python
##### Start Weight_Handler.py #####
import sympy as sp
from sympy.solvers.solveset import linsolve
from Differential_Forms import Span_Differential_Forms
from Lie_Complex import Lie_Complex

class Weight_Handler():
    """For a given nilpotent Lie algebra G,
    handle the computations and estimates concerning weights as used by Rumin."""

    def __init__(self, G):
        self.G = G
        self.initial_restraints = self.setup_weight_restraints()
        self.initial_homology_weights = self.setup_homology_weights()

    # Return a dict {i: w(X_i) for i = 1,..,G.dim} of possible weights on G satisfying w(X_i) + w(X_j) = w(X_k) if c_ij^k != 0.
    def setup_weight_restraints(self):
        weights     = [ sp.symbols("w" + str(i)) for i in range(self.G.dim) ]
        restraints  = [ weights[i] + weights[j] - weights[k]
                        for k in self.G.structure_constants
                        for i,j,c in self.G.structure_constants[k]
                        if not c == 0 ] # setup linear system of equations based on structure constants
        if len(restraints) == 0:  return {sp.symbols("w" + str(i)): weights[i] for i in range(self.G.dim)}
        S = linsolve(restraints, weights[::-1])
        restrained_weights = list(list(S)[0])[::-1] # solution of linear system with minimal independent variables
        return {sp.symbols("w" + str(i)): wi for i, wi in enumerate(restrained_weights)}

    # Get the subspace of a linear space of a given pure weight.
    def get_pure_weight_part( self, span: Span_Differential_Forms, pure_weight ):
        out = []
        for form in span:
            weight = form.get_weights(self.initial_restraints)
            if len(weight) > 1: raise ValueError("Basisvector of mixed weights passed to Weight_Handler.get_pure_weight_part().")
            if list(weight)[0] == pure_weight: out.append(form)
        return Span_Differential_Forms(out)

    # Compute the weights appearing in the homology algebras of G under the initial weight restrictions
    def setup_homology_weights(self):
        HG = Lie_Complex(self.G)
        return {k: HG.get_homology_group_basis(k).get_weights(self.initial_restraints) for k in range(self.G.dim+1)}

    # Find restraints on weights to make a set of weights equal
    def find_pure_weight_restraits(self, weights_list: list):
```

```
        free_vars = set().union(*[sp.sympify(w).free_symbols for w in weights_list ])
        LGS = [weights_list[i] - weights_list[i+1] for i in range(len(weights_list)-1)]
        S = linsolve(LGS, list(free_vars)[::-1])
        if len(S) == 0: return self.initial_restraints
        else:           weight_restraints = {list(free_vars)[i]: list(S)[0][::-1][i] for i in range(len(free_vars))}
        if 0 in weight_restraints.values():
            return {k: 0 for k,v in self.initial_restraints.items()}
        free_vars = set().union(*[expr.free_symbols for expr in weight_restraints.values() ])
        if len(free_vars) > 0:
            w = list(free_vars)[0]
            weight_restraints = {k: v.subs({w:1}) for k,v in weight_restraints.items()}
        return {k: v.subs(weight_restraints) for k,v in self.initial_restraints.items()}
##### End Weight_Handler.py #####
```

```
##### Start NLG_List.py #####
# Input format: (name, dimension, structure constants), where non-zero structure constants are given as a dict in the form
# {k: [(i,j, c_ij^k) for 1 <= i < j < n ] for 1 <= k <= n}
L1_1 = ("L_{1,1}", 1, {})
L3_2 = ("L_{3,2}", 3, {2: [(0, 1, 1)]})
L4_3 = ("L_{4,3}", 4, {2: [(0, 1, 1)], 3: [(0, 2, 1)]})
L5_4 = ("L_{5,4}", 5, {4: [(0, 1, 1), (2, 3, 1)]})
L5_5 = ("L_{5,5}", 5, {2: [(0, 1, 1)], 4: [(0, 2, 1), (1, 3, 1)]})
L5_6 = ("L_{5,6}", 5, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(0, 3, 1), (1, 2, 1)]})
L5_7 = ("L_{5,7}", 5, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(0, 3, 1)]})
L5_8 = ("L_{5,8}", 5, {3: [(0, 1, 1)], 4: [(0, 2, 1)]})
L5_9 = ("L_{5,9}", 5, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(1, 2, 1)]})
L6_10 = ("L_{6,10}", 6, {2: [(0, 1, 1)], 5: [(0, 2, 1), (3, 4, 1)]})
L6_11 = ("L_{6,11}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 5: [(0, 3, 1), (1, 2, 1), (1, 4, 1)], })
L6_12 = ("L_{6,12}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 5: [(0, 3, 1), (1, 4, 1)]})
L6_13 = ("L_{6,13}", 6, {2: [(0, 1, 1)], 4: [(0, 2, 1), (1, 3, 1)], 5: [(0, 4, 1), (2, 3, 1)]})
L6_14 = ("L_{6,14}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(0, 3, 1), (1, 2, 1)], 5: [(1, 4, 1), (2, 3, -1)]})
L6_15 = ("L_{6,15}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(0, 3, 1), (1, 2, 1)], 5: [(0, 4, 1), (1, 3, 1)]})
L6_16 = ("L_{6,16}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(0, 3, 1)], 5: [(1, 4, 1), (2, 3, -1)]})
L6_17 = ("L_{6,17}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(0, 3, 1)], 5: [(1, 4, 1), (1, 2, 1)]})
L6_18 = ("L_{6,18}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(0, 3, 1)], 5: [(0, 4, 1)]})
L6_19_1 = ("L_{6,19,1}", 6, {3: [(0, 1, 1)], 4: [(0, 2, 1)], 5: [(1, 3, 1), (2, 4, 1)]})
L6_19_0 = ("L_{6,19,0}", 6, {3: [(0, 1, 1)], 4: [(0, 2, 1)], 5: [(1, 3, 1), (2, 4, 0)]})
L6_19_n1 = ("L_{6,19,-1}", 6, {3: [(0, 1, 1)],4: [(0, 2, 1)], 5: [(1, 3, 1), (2, 4, -1)]})
L6_20 = ("L_{6,20}", 6, {3: [(0, 1, 1)], 4: [(0, 2, 1)], 5: [(0, 4, 1), (1, 3, 1)]})
L6_21_1 = ("L_{6,21,1}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(1, 2, 1)], 5: [(0, 3, 1), (1, 4, 1)]})
L6_21_0 = ("L_{6,21,0}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(1, 2, 1)], 5: [(0, 3, 1), (1, 4, 0)]})
L6_21_n1 = ("L_{6,21,-1}", 6, {2: [(0, 1, 1)], 3: [(0, 2, 1)], 4: [(1, 2, 1)], 5: [(0, 3, 1), (1, 4, -1)]})
L6_22_1 = ("L_{6,22,1}", 6, {4: [(0, 1, 1), (2, 3, 1)], 5: [(0, 2, 1), (2, 3, 1)]})
L6_22_0 = ("L_{6,22,0}", 6, {4: [(0, 1, 1), (2, 3, 1)], 5: [(0, 2, 1), (2, 3, 0)]})
L6_22_n1 = ("L_{6,22,-1}", 6, {4: [(0, 1, 1), (2, 3, 1)], 5: [(0, 2, 1), (2, 3, -1)]})
L6_23 = ("L_{6,23}", 6, {2: [(0, 1, 1)], 4: [(0, 2, 1), (1, 3, 1)], 5: [(0, 3, 1)]})
L6_24_1 = ("L_{6,24,1}", 6, {2: [(0, 1, 1)], 4: [(0, 2, 1), (1, 3, 1)], 5: [(0, 3, 1), (1, 2, 1)]})
L6_24_0 = ("L_{6,24,0}", 6, {2: [(0, 1, 1)], 4: [(0, 2, 1), (1, 3, 1)], 5: [(0, 3, 0), (1, 2, 1)]})
L6_24_n1 = ("L_{6,24,-1}", 6, {2: [(0, 1, 1)],4: [(0, 2, 1), (1, 3, 1)], 5: [(0, 3, -1), (1, 2, 1)]})
L6_25 = ("L_{6,25}", 6, {2: [(0, 1, 1)], 4: [(0, 2, 1)], 5: [(0, 3, 1)]})
L6_26 = ("L_{6,26}", 6, {3: [(0, 1, 1)], 4: [(0, 2, 1)], 5: [(1, 2, 1)]})
ALL_GNLG_LEQ6 = [L1_1, L3_2, L4_3, L5_4, L5_5, L5_6, L5_7, L5_8, L5_9, L6_10, L6_11, L6_12, L6_13, L6_14, L6_15,L6_16, L6_17, L6_18, L6_19_1,
                 L6_19_0, L6_19_n1, L6_20, L6_21_1, L6_21_0, L6_21_n1, L6_22_1, L6_22_0, L6_22_n1, L6_23, L6_24_1, L6_24_0, L6_24_n1, L6_25, L6_26]
H7 = ("H7", 7, {6: [(0, 1, 1), (2,3,1), (4,5,1)]}) #Some 7-dimensional Lie algebras
L7_37A = ("L7(37A)", 7, {4:[(0,1,1)], 5:[(1,2,1)], 6:[(1,3,1)]})
L7_257C = ("L7(257C)", 7, {2:[(0,1,1)], 5:[(0,2,1), (1,3,1)], 6:[(1,4,1)]})
##### End NLG_List.py #####
```

```python
##### Start Compute_Rumin_Estimates.py #####
import sympy as sp
from Lie_Complex import Lie_Complex
from Weight_Handler import Weight_Handler
# Compute the estimates on the Novikov-Shubin invariants accordingly to Rumin's method; Applying Hodge duality if flag is set True
# Returns a dict {k: (low_k, high_k)} for such k where estimate   low_k <= alpha_k <= high_k   follows from Rumin's approach
def Compute_Rumin_Estimates(G, hodge_duality = True):
    WEIGHT_HANDLER = Weight_Handler(G)
    NSinvars_estimates = {0: (G.get_growth_rate(), G.get_growth_rate()) }
    HG = Lie_Complex(G)
    for k in range(1, G.dim):
        k_weights = HG.get_homology_group_basis(k).get_weights(WEIGHT_HANDLER.initial_restraints)
        pure_weight_restraints = WEIGHT_HANDLER.find_pure_weight_restraits(k_weights)
        if 0 in pure_weight_restraints.values(): continue
        k_weights   = set([ w.subs(pure_weight_restraints) for w in WEIGHT_HANDLER.initial_homology_weights[k] ])
        k1_weights  = set([ w.subs(pure_weight_restraints) for w in WEIGHT_HANDLER.initial_homology_weights[k+1] ])
        N           = sp.sympify(WEIGHT_HANDLER.initial_homology_weights[G.dim][0].subs(pure_weight_restraints))
        diffs       = [ sp.sympify(w-v) for w in k1_weights for v in k_weights ]
        # check if still variables remain, replace by 1
        free_vars = set().union(*[list(v.free_symbols) for v in diffs])
        if len(free_vars) > 0:
            replace_dict = {fv: 1 for fv in free_vars}
            diffs = [v.subs(replace_dict) for v in diffs]
            N = N.subs(replace_dict)
            pure_weight_restraints = {k: v.subs(replace_dict) for k,v in pure_weight_restraints.items()}
        Nmax, Nmin  = max(diffs), min(diffs)
        if not Nmin > 0: Nmin = min([1/sp.sympify(w) for w in pure_weight_restraints.values()])
        NSinvars_estimates[k] = (N / Nmax, N / Nmin)
    if hodge_duality:
        for k in range(G.dim//2):
            if (k in NSinvars_estimates) and not (G.dim-k-1 in NSinvars_estimates):
                NSinvars_estimates[G.dim-k-1] = NSinvars_estimates[k]
            elif not (k in NSinvars_estimates) and (G.dim-k-1 in NSinvars_estimates):
                NSinvars_estimates[k] = NSinvars_estimates[G.dim-k-1]
            elif (k in NSinvars_estimates) and (G.dim-k-1 in NSinvars_estimates):
                n_min = max(NSinvars_estimates[k][0], NSinvars_estimates[G.dim-k-1][0])
                n_max = min(NSinvars_estimates[k][1], NSinvars_estimates[G.dim-k-1][1])
                NSinvars_estimates[k] = (n_min, n_max)
                NSinvars_estimates[G.dim-k-1] = (n_min, n_max)
    return NSinvars_estimates
##### End Compute_Rumin_Estimates.py #####
```

```python
##### Start Print_Methods.py #####
from NLG_List import *
from Nilpotent_Lie_Algebra import Nilpotent_Lie_Algebra as NLG
from Compute_Rumin_Estimates import Compute_Rumin_Estimates
# Given a list of graded nilpotent Lie algebras and the maximal dimension
# returns a table containing the NS-estimates from Rumin's method.
def make_NS_invars_table( NLG_list: list, MAX_DIM: int, COL_SEP = 14, hodge_duality = True, Latex = False ):
    outstr = ""
    if Latex:
        out_str = "Estimates on the Novikov-Shubin invariants of the deRham differentials.\n"
        out_str += "\\begin{tabular}{|c|" + "|c"*(MAX_DIM+1) + "|} \\hline \n"
        out_str += "".join([f" & $\\alpha_{k}$" for k in range(MAX_DIM+1)]) + " \\\\ \\hline \n"
    else:
        out_str = " " + "_"*((MAX_DIM+1)*(COL_SEP+3)-1) + "\n" + "|"
        out_str +=  "_"*COL_SEP + "__|_{}_|".format("_|_".join("_"*((COL_SEP-len(str(k))-2)//2) +"a_"
                                                               + str(k) + "_"*((COL_SEP-len(str(k))-1)//2) for k in range(MAX_DIM)))
        out_str += "\n"
    for LG in NLG_list:
        G = NLG(*LG)
        print(f"Starting to estimate NS invars of {G}...")
        NSinvars_estimates = Compute_Rumin_Estimates(G, hodge_duality=hodge_duality)
        NS_str = []
        if Latex:   out_str += f"${G}$"
        for k in range(G.dim):
            if k not in NSinvars_estimates:
                NS_str.append("-")
            elif NSinvars_estimates[k][0] == NSinvars_estimates[k][1]:
                if Latex:   NS_str.append(f"{NSinvars_estimates[k][0]}")
                else: NS_str.append("{}".format(NSinvars_estimates[k][0]))
            else:
                NS_str.append("[{}, {}]".format(*NSinvars_estimates[k]))
        if Latex: out_str += "".join([f" & ${a}$" for a in NS_str]) + " & $\infty^+$" + " & "*(6-G.dim) + "\\\\ \\hline \n"
        else: out_str += "| {} | {} | {}\n".format( " "*(COL_SEP-len(str(G))) + str(G),
                      " | ".join( " "*((COL_SEP-len(s)+1)//2) +s + " "*((COL_SEP-len(s))//2) for s in NS_str),
                      " | ".join(" "*(COL_SEP) for i in range(MAX_DIM-len(NS_str)+1)))
        print(f"{G} completed.")
    if Latex: out_str += "\end{tabular}"
    else:     out_str += " " + "‾"*((MAX_DIM+1)*(COL_SEP+3)-1)
    return out_str
##### End Print_Methods.py #####
```

```
##### Start Rumin_Estimates_Diss.py #####
import Print_Methods
from NLG_List import * # Here the Lie algebras are defined

if __name__=="__main__":
    RE = Print_Methods.make_NS_invars_table(ALL_GNLG_LEQ6, MAX_DIM=6, Latex = False)
    print(RE)
    #
    # For some arbitrarely chosen nilpotent Lie algebras of dimension 7:
    #print(Print_Methods.make_NS_invars_table([H7, L7_37A, L7_257C], MAX_DIM=7))
##### End Rumin_Estimates_Diss.py #####
```

Sample output for calling `Rumin_Estimates_Diss.py` (debug prints omitted, approx. runtime of 32.6 seconds averaged over 10 runs on my local hardware):

	a_0	a_1	a_2	a_3	a_4	a_5
L_{1,1}	1					
L_{3,2}	4	2	4			
L_{4,3}	7	[5/2, 10/3]	[5/2, 10/3]	7		
L_{5,4}	6	6	3	6	6	
L_{5,5}	8	-	-	-	8	
L_{5,6}	11	-	-	-	11	
L_{5,7}	11	[11/4, 11/2]	-	[11/4, 11/2]	11	
L_{5,8}	7	[7/2, 7]	-	[7/2, 7]	7	
L_{5,9}	10	10/3	5	10/3	10	
L_{6,10}	9	-	-	-	-	9
L_{6,11}	12	-	-	-	-	12
L_{6,12}	12	-	-	-	-	12
L_{6,13}	12	-	-	-	-	12
L_{6,14}	16	-	-	-	-	16
L_{6,15}	16	-	-	-	-	16
L_{6,16}	16	[14/3, 7]	[16/5, 16/3]	[16/5, 16/3]	[14/3, 7]	16
L_{6,17}	16	-	-	-	-	16
L_{6,18}	16	[16/5, 8]	-	-	[16/5, 8]	16
L_{6,19,1}	10	[5, 10]	-	-	[5, 10]	10
L_{6,19,0}	10	[10/3, 10]	-	-	[10/3, 10]	10
L_{6,19,-1}	10	[5, 10]	-	-	[5, 10]	10
L_{6,20}	10	[5, 10]	-	-	[5, 10]	10
L_{6,21,1}	14	[7/2, 14/3]	-	-	[7/2, 14/3]	14
L_{6,21,0}	14	[7/2, 14/3]	-	-	[7/2, 14/3]	14
L_{6,21,-1}	14	[7/2, 14/3]	-	-	[7/2, 14/3]	14
L_{6,22,1}	8	[4, 8]	[4, 8]	[4, 8]	[4, 8]	8
L_{6,22,0}	8	[4, 8]	[4, 8]	[4, 8]	[4, 8]	8
L_{6,22,-1}	8	[4, 8]	[4, 8]	[4, 8]	[4, 8]	8
L_{6,23}	10	-	-	-	-	10
L_{6,24,1}	11	-	[4, 6]	[4, 6]	-	11
L_{6,24,0}	11	-	[4, 6]	[4, 6]	-	11
L_{6,24,-1}	11	-	[4, 6]	[4, 6]	-	11
L_{6,25}	10	[10/3, 10]	-	-	[10/3, 10]	10
L_{6,26}	9	9/2	[9/2, 9]	[9/2, 9]	9/2	9

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