

Topological Invariants for Non-Archimedean Bornological Algebras

Dissertation
for the award of the degrees
“Doctor rerum naturalium” and “Doctor of Philosophy”
of the Georg-August-Universität Göttingen and the University of
Copenhagen

within the doctoral programs of the
“PhD School of Mathematical Sciences (SMS)” and the "PhD School
of Science"
of the Georg-August University School of Science (GAUSS) and the
University of Copenhagen

submitted by
Devarshi Mukherjee
from Calcutta, India

Göttingen, 2020

Thesis advisory committee:

Prof. Dr. Ralf Meyer
Mathematisches Institut, Universität Göttingen

Professor Ryszard Nest
Mathematical Institute, University of Copenhagen

Members of the examination committee:

Referees:

Prof. Dr. Thomas Schick
Mathematisches Institut, Universität Göttingen

Professor Yakov Kremnitzer
Mathematical Institute, University of Oxford

Dr. Dustin Clausen
Department of Mathematical Sciences, University of Copenhagen

Other members of the defense committee:

Prof. Dr. Damaris Schindler
Mathematisches Institut, Universität Göttingen

Prof. Dr. Gerlind Plonka-Hoch
Institut für Numerische und Angewandte Mathematik, Universität Göttingen

Dr. Christian Voigt
School of Mathematics and Statistics, University of Glasgow

Date of oral examination: 24.09.2020

Contents

Abstract	5
Abstrakt	7
Acknowledgments	9
Introduction	11
Background for the work in this thesis	12
Summary of papers in this thesis	13
Chapter 1. Dagger completions and bornological torsion-freeness	21
1. Introduction	21
2. Basic notions	22
3. Spectral radius and semi-dagger algebras	28
4. Bornological torsion-freeness	30
5. Dagger algebras	35
6. Dagger completions of monoid algebras	35
7. Dagger completions of crossed products	37
Chapter 2. Non-archimedean analytic cyclic homology	43
1. Introduction	43
2. Preparations	46
3. Definition of analytic cyclic homology	56
4. Analytic nilpotence and analytically quasi-free resolutions	59
5. Excision	75
6. Stability with respect to algebras of matrices	84
7. Morita functoriality	85
8. Leavitt path algebras	87
9. Filtered Noetherian rings and analytic quasi-freeness	91
Chapter 3. A cyclic homology theory in positive characteristic	99
1. Introduction	99
2. Analytic cyclic theory in mixed characteristic	101
3. The exact category of locally split extensions	107
4. Independence of the choice of lifting	117
5. Homotopy invariance and stability	121
6. Nilpotent extensions and excision	122
Chapter 4. Conclusion and outlook	133
1. What is still missing?	133
2. Some future directions	134

Bibliography

137

Abstract

In this thesis, we define a cyclic homology theory for non-archimedean bornological algebras, which we call *analytic cyclic homology*. Let V be a complete, discrete valuation ring with uniformiser π , residue field $\mathbb{F} = V/\pi V$, and quotient field F . The material we present is divided in three parts. In the first part, we interpret Monsky and Washnitzer's weak completion using the framework of bornologies, developed in [11]. Weakly complete algebras are used to define Monsky-Washnitzer cohomology, and have three characteristic features - bornological torsion-freeness, completeness and a certain spectral radius condition. We study the homological inheritance of each of these three conditions, and call a V -algebra with these properties a *dagger algebra*.

In the second part, we define analytic cyclic homology for projective systems of complete, bornologically torsion-free V -algebras. The theory we develop satisfies homotopy invariance, Morita invariance and excision. We use these properties to compute our theory for (dagger completed) Leavitt path algebras, and tensor product with such algebras. We show that our theory coincides with Berthelot's rigid cohomology for smooth commutative V -algebras of relative dimension 1.

In the third part, we define analytic cyclic homology for algebras over the residue field \mathbb{F} , by lifting them to free algebras over V and then building dagger completed tube algebras. We show that under very mild assumptions on the bornology of an \mathbb{F} -algebra A , any complete, bornologically torsion-free V -module lifting can be used to compute its analytic cyclic homology. The theory we define satisfies polynomial homotopy invariance, matricial stability, and excision for extensions of finitely generated \mathbb{F} -algebras.

Abstrakt

Hovedresultat af denne arbejde er en konstruktion af *Analytisk homologi*, en version af cyklisk homologi for ikke-arkimidiske bornologiske algebra. Lad V være en fuldstændig diskrete valueringsring, $\mathbb{F} = V/\pi V$ dens restklasselegeme og F dens kvotientlegeme. I den første del ve interpreterer Monsky og Washnitzer svag fuldstændigørelse i sprog af bornologiske algebra. Svagt fuldstændige algebra er anvendt til at definere Monsky-Washnitzer kohomologi og har tre karakteristiske egenskaber. De er torsionsfrie (som bornologiske algebra), fuldstændige og har deres spektrum har et bestemt spektral egenskab, indført i teksten nedenunder. Algebra som opfylder de egenskaber kaldes *dagger algebra* Den første del af tesen er brugt til at analysere homologisk stabilitet af dagger egenskaben.

Den anden del af arbejde introducerer analytisk cyklisk homologi af dagger algebra. Den tilsvarende konstruktion er homotopi invariant, opfylder Morita invariance og har udskærings egenskab. vi regner ud analytisk cyklisk homologi for tensor produkter med Lewitt sti-algebra. En af hovedresultater i denne del er bevis for, at for glatte V -algebra, analytisk cyklisk homologi stemmer overens med Bertholet's rigid kohomologi.

Den sidste del af arbejde giver en konstruktion af analytisk cyklisk homologi for algebraer(over \mathbb{F} ved at bruge et løft til en algebra over V . Vi viser at, under meget milde betingelser for algebra over \mathbb{F} , konstruktion er uafhængig af valg af løftet. Den teori som opnås herved opfylder polynomial homotopi invariance, Morita invariance og udskærings aksiomet for udvidelser af endelig genererede \mathbb{F} -algebra.

Acknowledgments

First of all, I would like to thank my principal co-supervisor Ralf Meyer for his immense support and patience, without which the work of this thesis would not have been conceivable. I have over the years learned vast amounts of Mathematics from him, spanning many different areas of the subject. His generosity in sharing his thoughts is something I will always be grateful for.

Secondly, I would like to thank my co-supervisor Ryszard Nest for not only providing me with valuable input for my project, but also broadening my horizons to other areas of Mathematics. The discussions with him often gave me a completely different perspective on the problems I had been working on as part of my thesis. Finally, I would also like to thank him for the hospitality during my visits to Copenhagen, which I always enjoyed.

I would also like to thank my collaborator Guillermo Cortiñas for letting me into the project of which my thesis is part. This project was originally conceived of by him and Joachim Cuntz, and their continuous input and feedback have been extremely crucial. I am also grateful for all the support I received from him surrounding my overall career prospects.

Finally, I would also like to thank my girlfriend Simran, my family in Hyderabad and my friends from Delhi and Göttingen for all the emotional support, without which these three years would have been unimaginably hard.

Introduction

The use of invariants in the understanding of geometry has a long and illustrious history. Such invariants have been applied in areas ranging from the classification of Platonic solids in Greek geometry, to the formulation of the Weil conjectures using certain *cohomology theories*, to the theory of topological insulators using *K-theory*. We dedicate this introductory section to an informal discussion on invariants for two types of ‘geometries’ - *geometry in characteristic zero*, and *geometry in positive characteristic*.

By geometry in characteristic zero, we mean associative algebras over fields of characteristic zero, that encode some geometric structure¹. For example, *affine algebraic varieties* over an algebraically closed field are equivalent to finitely generated commutative algebras over the field. Another example is the Gelfand-Naimark Theorem, which states that commutative C*-algebras are the same as *locally compact Hausdorff spaces*. Regarding varieties or schemes over the complex numbers as topological spaces, we can speak of their *singular cohomology* groups, which is a powerful invariant originating from algebraic topology. If we consider spaces that are suitably nonsingular, then a related invariant is the *de Rham cohomology* of say, a smooth manifold or a non-singular variety over \mathbb{C} . These invariants can be used to obtain important geometric information such as the *Euler characteristic* of a topological space X or the *Lefschetz number* of a continuous map on X . When we replace ‘spaces’ in the classical sense of topology or algebraic geometry by possibly non-commutative algebras, singular cohomology and de Rham cohomology are no longer meaningful, and need to be replaced with *cyclic homology*. The study of non-commutative algebras as an extension of ‘classical’ geometry is the subject matter of Alain Connes’ *non-commutative geometry*.

As the reader can probably guess, geometry in positive characteristic refers to associative algebras over fields of characteristic $p > 0$. Algebraic geometry in this context has a rich history, stemming from the pursuit reasonable invariants to prove the Weil conjectures. From the purview of non-commutative geometry, interesting geometric examples arise from finite group actions on affine schemes over finite fields, which are treated by *crossed product algebras*. Another class of examples is *graph algebras* over finite fields. It turns out, however, that the positive characteristic of the underlying ring or field poses problems that invariants such as de Rham cohomology or cyclic homology cannot easily circumvent. For instance, the Poincaré Lemma says that in characteristic zero, the de Rham complex of the (contractible) affine n -space is exact. The proof uses that the integration map on differential forms yields sections for the de Rham differential. This is however no longer true in positive characteristic.

¹I should mention here that throughout this thesis, I will only talk about geometric objects that are in some sense affine. That is, I will never refer to projective schemes, or graded algebras.

What seems desirable is a de Rham or cyclic type cohomology theory that is defined on a suitable category of ‘characteristic 0 liftings’ of an algebra over characteristic $p > 0$. Of course, the resulting lifting and invariants should be defined in a way that we can still obtain information about the original geometric object. Considerations of this nature led to the development of several related *p-adic cohomology* theories. Two such invariants which motivate this thesis are *Monsky-Washnitzer cohomology* and *rigid cohomology*. The goal of this thesis to study liftings of geometric objects in positive characteristic, to objects that are better behaved in characteristic zero. Finally, we use such liftings to construct a well-behaved homological invariant that specialises to Monsky-Washnitzer and rigid cohomology for certain classes of commutative algebras.

Background for the work in this thesis

For smooth, commutative algebras over rings containing the rationals \mathbb{Q} , the Hochschild-Kostant-Rosenberg Theorem implies that de Rham cohomology coincides with periodic cyclic homology. In other words, periodic cyclic homology extends de Rham cohomology to non-commutative algebras in characteristic zero. This thesis takes a step in the direction of extending rigid cohomology to non-commutative algebras over finite fields.

The starting point of this program is the identification of periodic cyclic homology with rigid cohomology for commutative, finite type algebras (see [11]). We describe this identification here. Denote by V a complete discrete valuation ring with uniformiser π , $\mathbb{F} = V/\pi V$ its residue field, and F its fraction field, which we assume to be of characteristic zero - we fix this notation for the rest of this thesis. Let A be a commutative \mathbb{F} -algebra, and $I \twoheadrightarrow V[S] \twoheadrightarrow A$ be a free commutative algebra presentation of A . Here S is a generating set of A , and for a natural choice, we can choose $S = A$. The next step is to construct certain bornological versions of I -adic completions $\overline{R_{J,1/n}}$ of $R = V[S]$, with $\underline{R} := R \otimes F$ and $n \in \mathbb{N}$. Concretely, the bornology is defined by specifying bounded subsets $M \subseteq R$ for which $\rho(M) < \epsilon^{1/n}$, where $\epsilon = |\pi|$ is the norm of the uniformiser of V . This provides a bornological interpretation of the *weak completions* used in Monsky-Washnitzer cohomology [32, Section 2]. In conclusion, we get a projective system of complete, bornological F -algebras $(\overline{R_{J,1/n}})_{n \in \mathbb{N}}$.

DEFINITION. Let A and R be as above. The *periodic cyclic homology* of A is defined as the homology of the homotopy inverse limit of the projective system of periodic cyclic complexes

$$(\mathbb{H}\mathbb{P}(\overline{R_{J,1/n}}), B + b)_{n \in \mathbb{N}}.$$

The main result of this paper shows that the homotopy inverse limit of $(\mathbb{H}\mathbb{P}(\overline{R_{J,1/n}}), B + b)_{n \in \mathbb{N}}$ is quasi-isomorphic to the homotopy inverse limit of the de Rham complexes $(\overline{R_{J,1/n}} \otimes_{\underline{R}} \Omega_{\underline{R}}^*, d)$, made 2-periodic. The results of Große-Klönne [19] imply that this homotopy inverse limit of de Rham complexes computes the rigid cohomology of A . In summary, we have the following:

THEOREM. [11, Theorem 6.5] *Let A be a finitely generated commutative \mathbb{F} -algebra. Then for $*$ = 0, 1, we have*

$$\mathrm{HP}_*(A) \cong \bigoplus_{j \in \mathbb{Z}} H_{\mathrm{rig}}^{2j-*}(A, F).$$

The machinery above has the important feature that it computes rigid cohomology using a chain complex that is defined naturally using free commutative algebra liftings. Furthermore, the homotopy invariance of periodic cyclic homology can be used to show that if we use a different generating set $S \subset A$ for the free commutative algebra lifting $R = V[S]$, we still get the same periodic cyclic homology. However, it is unclear how this definition can be extended to non-commutative algebras in a manner that is independent of the choice of the lifting. Invariance under different liftings is a key result about Monsky-Washnitzer cohomology, proven by Marius van der Put (see [12, Remark 8.2.8]). This motivates the need for a variant of periodic cyclic homology that is defined for non-commutative \mathbb{F} -algebras, and is independent of choices of intermediate liftings to V -algebras. Such a homology theory should of course have all the desirable formal properties that periodic cyclic homology possesses. Finally, this theory should specialise in the smooth, commutative case to periodic cyclic homology, which coincides with rigid cohomology. While we have not been able to prove the last claim in full generality, we have constructed a theory which nevertheless yields interesting computations for several classes of commutative as well as non-commutative algebras.

Summary of papers in this thesis

This thesis has been arranged in the form of a collection of three papers I have (co)-authored during my time as a PhD student.

Dagger completions and bornological torsion-freeness (joint with Ralf Meyer).²

Here we describe the *dagger completion* of a bornological V -algebra, which is a key ingredient in the construction of our homology theory. It was already observed in the seminar works of Monsky and Washnitzer, that in order to obtain a well-behaved de Rham cohomology theory, we need to ‘complete’ a torsion-free V -algebra lifting R of an \mathbb{F} -algebra. They also observed that one cannot simply work with π -adic completions and ensure the finite dimensionality of de Rham cohomology, even in simple cases such as the affine line \mathbb{F} . So the right thing to do is to take a certain subalgebra R^\dagger of the π -completion \widehat{R} , which they call the *weak completion*. If R is a polynomial algebra, then R^\dagger consists of overconvergent power series. Quotients of dagger completed polynomial algebras appear naturally in the context of Große-Klönne’s theory of rigid analytic spaces with overconvergent structure sheaf.

In this paper, we describe the dagger completion process and its various inheritance properties from the perspective of homological algebra. Our treatment uses bornologies, which is the framework we choose for our cyclic homology theories. We briefly recall some definitions from bornological analysis to keep the summaries of the papers readable:

DEFINITION.

- A *bornology* on a set S is a collection of its subsets, called *bounded sets*, such that all finite subsets are bounded, and subsets and finite unions of bounded subsets are bounded. A *bounded function* $f: S \rightarrow T$ between bornological sets is one which maps bounded sets to bounded sets.

²This paper is published in the Quarterly Journal of Mathematics (see [27]).

- A *bornological V -module* is a V -module M with a bornology \mathcal{B}_M , such that any bounded subset is contained in a bounded V -submodule. A *bornological V -algebra* R is a bornological V -module with a bounded multiplication map $R \times R \rightarrow R$.

EXAMPLE. We can always equip a V -module M with the collection of all finitely generated V -submodules. This is called the *fine bornology*.

The fine bornology is the smallest bornology on a V -module. It is in essence the ‘base’ bornology from the viewpoint of our applications, upon which we perform further analytical operations. This is because the algebras we are interested in - such as Leavitt path algebras, and coordinate rings of smooth curves - come with no further analytical structure.

We now turn to the *completion* of a bornological V -module. A bornological V -module M is said to be *complete* if every bounded subset $S \in \mathcal{B}_M$ is contained in a bounded π -adically complete V -submodule T . To define the completion of M in the most enlightening manner possible, we state a result on the structure of bornological V -modules:

PROPOSITION. [11, Proposition 2.5] *The category of bornological V -modules with bounded V -module maps is equivalent to the full subcategory of inductive systems of V -modules, with injective structure maps.*

DEFINITION. Let M be a bornological V -module, written as an inductive limit of V -submodules $M = \varinjlim M_i$. Let \widehat{M}_i denote the π -adic completion of M_i . Then the *completion* \widehat{M} of M is defined as the quotient of the inductive limit $\varinjlim \widehat{M}_i$ by the bornological closure of the trivial V -module $\{0\}$.

For the definition of bornological closure, we direct the reader to the actual paper. We only remark that taking this quotient ensures that the completion is bornologically *separated*, that is, every bounded subset is contained in a π -adically separated bounded V -submodule. Equivalently, the structure maps $M_i \rightarrow M_j$ of the inductive system defining M remain injective when we take their π -adic completions $\widehat{M}_i \rightarrow \widehat{M}_j$. The completion \widehat{M} of a bornological V -module M is complete and admits a canonical map $M \rightarrow \widehat{M}$ to it. Furthermore, it has the expected universal property: if $f: M \rightarrow N$ is a bounded V -module map into a complete bornological V -module N , then there exists a unique bounded V -module map $\widehat{M} \rightarrow N$ factorising f .

EXAMPLE. Any V -module with the fine bornology is complete. This is because finitely generated V -modules are already π -adically complete.

We now talk about torsion-freeness, which is an important aspect of our lifting constructions from the world of \mathbb{F} -algebras. Recall, a V -module is torsion-free if multiplication by π is an injective V -module map. It is *bornologically torsion-free* if this map respects bornologies.

EXAMPLE. A torsion-free V -module M with the *fine bornology* is bornologically torsion-free.

The example above implies that most algebras we are interested in studying, such as the coordinate ring of a smooth affine variety over V with the fine bornology, are automatically also bornologically torsion-free. However, in order to arrive at the weak completion of Monsky-Washnitzer, we will need to enlarge the bornology on a

given V -algebra, and complete with respect to the enlarged bornology. It is therefore natural to ask whether bornological torsion-freeness is preserved by completions. This is indeed the case:

THEOREM. [27, Theorem 4.6] *Let M be a bornologically torsion-free bornological V -module. Then \overline{M} is bornologically torsion-free.*

Finally, we come to the growth condition of bounded subsets that is inherent in the weak completion of Monsky-Washnitzer. A bornological V -algebra A is called *semi-dagger* if whenever S is a bounded V -submodule, we have $\pi \cdot S^2 \subseteq S$. Trivially, any \mathbb{F} -algebra with the fine bornology is semi-dagger. For torsion-free V -algebras, however, we need to work with a larger bornology than the fine bornology. Specifically, we equip a torsion-free bornological V -algebra A with the bornology generated by V -submodules of the form $\sum_{j=0}^{\infty} \pi^j S^{j+1}$, where S is a bounded submodule in the original bornology of A . This enlarged bornology is called the *linear growth bornology*. It is by construction the smallest semi-dagger bornology on A . We denote a bornological algebra A with the linear growth bornology by A_{lg} . The enlargement of the bornology on A to the linear growth bornology produces a canonical map $A \rightarrow A_{\text{lg}}$ that factorises any bounded algebra homomorphism into a semi-dagger algebra. The following result shows the inheritance properties of linear growth bornology and semi-dagger algebras for the notions previously introduced.

PROPOSITION. [27, Proposition 4.11, Proposition 3.8] *If A is a bornologically torsion-free V -algebra, then so is A_{lg} . If A is a semi-dagger bornological V -algebra, then so is its completion \overline{A} .*

Combining all the notions we have introduced so far, we can define a *dagger algebra*:

DEFINITION. A dagger algebra is complete, bornologically torsion-free, semi-dagger V -algebra.

We now show how to construct a dagger algebra from a given bornological V -algebra.

DEFINITION. Let A be a bornologically torsion-free V -algebra. The *dagger completion* A^\dagger of A is defined as the completion of A in its linear growth bornology, that is, $A^\dagger := \overline{A_{\text{lg}}}$.

We have only defined the dagger completion of a bornologically torsion-free V -algebra since the algebras we are interested in are already bornologically torsion-free. If A is not bornologically torsion-free, we would need to take the image A_{tf} of A inside $A \otimes F$. The dagger completion of A is then the completion of A_{tf} in its linear growth bornology. There is a universal map $A \rightarrow A^\dagger$ obtained by combining the maps $A \rightarrow A_{\text{tf}}$, $A \rightarrow A_{\text{lg}}$ and $A \rightarrow \overline{A}$, that factorises a bounded algebra homomorphism into a dagger algebra. And, the various inheritance properties of the three notions that define a dagger algebra imply that the dagger completion A^\dagger is indeed a dagger algebra.

An important feature of this bornological formalism is that these definitions make sense also for noncommutative algebras. In Section 6, we compute the dagger completion $V[S]^\dagger$ of a monoid algebra $V[S]$, which recovers Monsky-Washnitzer's weak completion if we set $S = \mathbb{N}^k$. In Section 7, we compute the dagger completion

of the crossed product algebra of a certain kinds of monoid actions. Finally, our main result that is crucial for subsequent developments of the theory is the following inheritance property for extensions of bornological algebras:

THEOREM. [27, Theorem 5.2] *Let $A \succ B \twoheadrightarrow C$ be an extension of bornological V -algebras. If A and C are dagger algebras, so is B .*

Non-archimedean analytic cyclic homology (joint with Guillermo Cortiñas and Ralf Meyer).³

In this article, we define a homology theory for projective systems of complete, bornologically torsion-free bornological algebras (briefly, *pro-algebras*) over complete discrete valuation rings V . The computations so far are directed towards two classes of algebras that are interesting from the perspective of algebraic and non-commutative geometry, namely, coordinate rings of smooth 1-dimensional affine varieties over V , and Leavitt path algebras. Furthermore, the formal properties of our theory provide computations for Laurent polynomials in n -variables and higher dimensional Toeplitz algebras and Leavitt path algebras, taking us beyond homological dimension 1.

Our construction uses dagger completions of certain generalised *tube algebras* that are motivated from rigid cohomology. The m -th tube algebra of a torsion-free V -algebra A with respect to an ideal $I \triangleleft A$ is defined as $\mathcal{U}(A, I^m) = \sum_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor} I^{mn} \subseteq A \otimes F$. Varying m , we get a projective system $\mathcal{U}(A, I^\infty) = (\mathcal{U}(A, I^m))_m$ of torsion-free bornological algebras, with the subspace bornology inherited from $A \otimes K$. In our context, we use the tube algebra of the tensor algebra presentation $JR \succ TR \twoheadrightarrow R$, of a pro-algebra R . Taking a certain relative version of dagger completion⁴ of the tube algebra $\mathcal{U}(TR, JR^\infty)$ with respect to the canonical ideal $\mathcal{U}(JR, JR^\infty)$, we obtain a pro-algebra $\mathcal{T}R$, which fits in an extension

$$\mathcal{J}R \succ \mathcal{T}R \twoheadrightarrow R,$$

with an appropriately nilpotent kernel $\mathcal{J}R$.

The definition of our homology theory uses a certain quotient of the periodic cyclic bicomplex $(\mathbb{H}\mathbb{P}, B + b)$, called the *X-complex*. We do not define this here and instead direct the reader to Section 2.7 for its definition.

DEFINITION (Analytic cyclic homology). Let R be a pro-algebra. Its *analytic cyclic complex* $\mathbb{H}\mathbb{A}(R)$ is defined as the projective system of chain complexes $X(\mathcal{T}R \otimes F)$ ⁵. The homology of the homotopy inverse limit of $\mathbb{H}\mathbb{A}$ is called the *analytic cyclic homology* of R .

Our homology theory satisfies the following formal properties:

- Homotopy invariance for dagger homotopies (see Definition 4.1.1);
- Matricial stability and more generally, Morita invariance (Section 6 and Section 7);
- Excision for semi-split extensions of pro-algebras (Section 5).

³This paper is accepted for publication in Documenta Mathematica.

⁴In light of [27, Theorem 5.2], we must use this more technical completion in place of the ‘absolute’ dagger completion of the tube-tensor algebra, because the quotient $\mathcal{U}(TR, JR^\infty)/\mathcal{U}(JR, JR^\infty) \cong R$ is not assumed to be semi-dagger.

⁵You can read the X -complex as $\mathbb{H}\mathbb{P}$ for now. In what follows, we shall explain why it makes no difference whether we use the X -complex or the periodic cyclic complex of $\mathcal{T}R$

We note here that the reason we tensor the X -complex with F in the definition of the analytic cyclic complex is to ensure homotopy invariance, on which several key results rely.

To put our machinery into context, we shed some light on the Cuntz-Quillen approach to cyclic homology theories. Their idea is to build a particular resolution $N \twoheadrightarrow E \twoheadrightarrow A$ of an algebra A , by an algebra E that has Hochschild cohomological dimension 1, and nilpotent kernel N . Nilpotence of the kernel ensures that its periodic cyclic complex is contractible. So the periodic cyclic homology of E is the same as that of A . Homological dimension 1 ensures that the periodic cyclic homology of E is computable by the X -complex. Finally, homotopy invariance of periodic cyclic homology, and the universal property of 1-dimensional algebras imply that any such extension can be used to compute the periodic cyclic homology of A . A significant part of this paper contextualises such results in the non-Archimedean setting. We recall some key definitions that appear in the paper.

DEFINITION (Analytic nilpotence). A pro-algebra $(A_n, \alpha_{m,n})_{n \in \mathbb{N}}$ is *nilpotent mod π* if, for each $m \in \mathbb{N}$, there are $n \in \mathbb{N}_{\geq m}$ and $l \in \mathbb{N}^*$ such that $\alpha_{m,n}(A_n^l) \subseteq \pi A_m$. We call a pro-algebra *analytically nilpotent* if it is isomorphic to a projective system of dagger algebras and is nilpotent mod π .

In what follows, we shall call extensions of pro-algebras $N \twoheadrightarrow E \twoheadrightarrow R$ *analytically nilpotent* if the kernel N is analytically nilpotent. We call an extension *semi-split* if it splits by a projective system of V -module maps. In other words, the splitting map for the cokernel of the extension is only V -linear.

DEFINITION (Analytic quasi-freeness). A pro-algebra R is *analytically quasi-free* if any semi-split, analytically nilpotent extension $N \twoheadrightarrow E \twoheadrightarrow R$ splits by a pro-algebra homomorphism $R \rightarrow E$.

One of our main results is the following version of Goodwillie's Theorem:

THEOREM (4.7.1). *Let $J \twoheadrightarrow E \xrightarrow{p} R$ be a semi-split, analytically nilpotent extension of pro-algebras. Then p induces a chain homotopy equivalence $\mathbb{H}\mathbb{A}(E) \simeq \mathbb{H}\mathbb{A}(R)$ and $\mathbb{H}\mathbb{A}(J)$ is contractible. So $\mathrm{HA}_*(E) \cong \mathrm{HA}_*(R)$ and $\mathrm{HA}_*(J) = 0$ for $* = 0, 1$. If E is analytically quasi-free, then $\mathbb{H}\mathbb{A}(R)$ is chain homotopy equivalent to $X(E \otimes F)$ and $\mathrm{HA}_*(R)$ is isomorphic to the homology of the homotopy projective limit of $X(E \otimes F)$.*

A particularly nice consequence of the theorem above is the case where the algebra R is itself analytically quasi-free.

COROLLARY (4.7.2). *Let R be an analytically quasi-free algebra. Then $\mathbb{H}\mathbb{A}(R)$ is chain homotopy equivalent to $X(R \otimes F)$ and $\mathrm{HA}_*(R)$ is isomorphic to the homology of $X(R \otimes F)$.*

The algebras mentioned at the beginning of the section, namely Leavitt path algebras and smooth 1-dimensional algebras are analytically quasi-free. So the result above makes our theory computable for them. The following is our main result using analytic quasi-freeness:

THEOREM (9.2.9). *Let X be a smooth affine variety over the residue field \mathbb{F} of dimension 1 and let $A = \mathcal{O}(X)$ be its algebra of polynomial functions. Let R be a smooth, commutative algebra of relative dimension 1 with $R/\pi R \cong A$. Equip R with*

the fine bornology and let R^\dagger be its dagger completion. If $\ast = 0, 1$, then $\mathrm{HA}_\ast(R^\dagger)$ is naturally isomorphic to the de Rham cohomology of R^\dagger . This is isomorphic to the Monsky–Washnitzer cohomology of A , which, if $\mathrm{char}(\mathbb{F}) > 0$, agrees with the rigid cohomology $H_{\mathrm{rig}}^\ast(A, F)$ of X .

We can in principle use analytic quasi-freeness to compute our theory also for Leavitt path algebras, but our coauthor Guillermo Cortiñas pointed out a more elegant way, that uses stability and excision. In what follows, let E be a directed graph with vertices E^0 and incidence matrix N_E . Let $C(E)$ and $L(E)$ denote the Cohn and Leavitt path algebras, respectively.

THEOREM (8.1). *Let R be a complete bornological algebra. Let E be a graph with countably many vertices. Then*

$$\begin{aligned} \mathrm{HA}(R \otimes C(E)) &\simeq \mathrm{HA}(R \otimes V^{(E^0)}), & \mathrm{HA}(C(E)) &\simeq F^{(E^0)}, \\ & & \mathrm{HA}(L(E)) &\simeq \mathrm{coker}(N_E) \oplus \ker(N_E)[1], \end{aligned}$$

If E^0 is finite, then

$$\begin{aligned} \mathrm{HA}(R \otimes C(E)) &\simeq \bigoplus_{v \in E^0} \mathrm{HA}(R), \\ \mathrm{HA}(R \otimes L(E)) &\simeq (\mathrm{coker}(N_E) \oplus \ker(N_E)[1]) \otimes \mathrm{HA}(R). \end{aligned}$$

COROLLARY (8.2). $\mathrm{HA}(R \otimes V[t, t^{-1}])$ is chain homotopy equivalent to $\mathrm{HA}(R) \oplus \mathrm{HA}(R)[1]$ and $\mathrm{HA}_\ast(R \otimes V[t, t^{-1}]) \cong \mathrm{HA}_\ast(R) \oplus \mathrm{HA}_\ast(R)[1]$.

The result above is very powerful as it provides nontrivial computations beyond homological dimension 1. This is because the tensor product of two algebras of homological dimension 1 can in general have homological dimension 2. We can therefore iterate this result to get computations for tensor products of Laurent polynomial algebras. Finally, all these results remain valid even for dagger completed Cohn and Leavitt path algebras.

Analytic cyclic homology in positive characteristic (joint with Ralf Meyer).⁶

In this paper, we define *analytic cyclic homology for an \mathbb{F} -algebra A* . The construction proceeds as follows: let W be a complete, bornologically torsion-free V -module lifting of A , which is viewed as a bornological V -algebra with the fine bornology. Here by lifting we mean *any* bornological quotient map $W \xrightarrow{q} A$. Such a lifting always exists, as we can take the free V -module over A with the fine bornology. Thereafter, we follow the Cuntz–Quillen machinery by using the tensor algebra resolution $I \twoheadrightarrow \mathrm{TW} \twoheadrightarrow A$ to build tube algebras $\mathcal{U}(\mathrm{TW}, I^\infty) = (\mathcal{U}(\mathrm{TW}, I^m))_{m \geq 1}$ of TW with respect to I .

A key difference to torsion-free theory developed previously arises here. Since the canonical lifting of A is the free V -module $W = VA$, the dagger completions of the tube-tensor algebras $\mathcal{U}(\mathrm{TW}, I^m)$ are hard to describe explicitly. So to remedy this, we equip the tube algebras with the linear growth bornology and take their X -complex. This yields chain complexes $X(\mathcal{U}(\mathrm{TW}, I^\infty)_{\mathrm{lg}}) := X(\mathcal{U}(\mathrm{TW}, I^m)_{\mathrm{lg}})_{m \geq 1}$ of incomplete bornological V -modules.

⁶This paper is currently in the form of a manuscript I have written for the purpose of this thesis. A journal version will follow shortly.

DEFINITION (Analytic cyclic homology of A). The *analytic chain complex of A with respect to W* is defined as the projective system of chain complexes of bornological F -modules $\mathbb{H}\mathbb{A}(A; W, \varrho) := (X(\mathcal{U}(\mathbb{T}W, I^\infty)_{\text{lg}} \otimes F))$. We call the analytic chain complex of A with respect to the natural lifting $W = VA$ the *analytic chain complex $\mathbb{H}\mathbb{A}(A)$ of A* .

Thereafter, we take the homotopy inverse limit $\text{holim } \mathbb{H}\mathbb{A}(A, W, \varrho)$ to obtain a chain complex of bornological F -modules. Finally, we complete this chain complex in the following weaker sense:

DEFINITION (Quasi-completion). Let M be a bornological V -module, written as an inductive limit $M = \varinjlim M_i$ of its bornological V -modules. The *quasi-completion* is defined as the inductive limit $\mathcal{Q}(M) = \varinjlim \widehat{M}_i$.

The homology of the resulting chain complex $\mathcal{Q}(\text{holim}(\mathbb{H}\mathbb{A}(A, W, \varrho)))$ is called the *analytic cyclic homology of A with respect to $\varrho: W \rightarrow A$* . The primary objective of this paper is to find the correct derived category in which all such liftings yield quasi-isomorphic analytic chain complexes. Such a derived category is constructed using a certain exact category of *locally split extensions* of torsion-free bornological V -modules (Section 3). The statement of the main result is as follows:

THEOREM (2.1.8). *Let A be an \mathbb{F} -algebra, let W be a complete, bornologically torsion-free bornological V -module, and let $\varrho: W \rightarrow A$ be a surjective V -module map, bounded in the fine bornology on A . Then ϱ induces a bounded V -algebra homomorphism $\varrho^\#: \mathbb{T}W \rightarrow A$. Then there is a canonical bornological F -module isomorphism*

$$\mathbb{H}\mathbb{A}_*(A) \xrightarrow{\cong} \mathbb{H}\mathbb{A}_*(A; W, \varrho).$$

The theory we define is homotopy invariant for polynomial homotopies and matrixially stable (see Section 5). Finally, the second key result of this paper is the following excision theorem:

THEOREM (6.3.4). *Let $K \twoheadrightarrow E \twoheadrightarrow Q$ be an extension of finitely generated \mathbb{F} -algebras. Then there exists a semi-split extension of pro-dagger algebras $\mathcal{K} \twoheadrightarrow \mathcal{E} \twoheadrightarrow \mathcal{Q}$ that lifts the original extension of \mathbb{F} -algebras. This yields the following long exact sequence in analytic cyclic homology:*

$$\begin{array}{ccccc} \mathbb{H}\mathbb{A}_0(K) & \xrightarrow{i_*} & \mathbb{H}\mathbb{A}_0(E) & \xrightarrow{p_*} & \mathbb{H}\mathbb{A}_0(Q) \\ \delta \uparrow & & & & \downarrow \delta \\ \mathbb{H}\mathbb{A}_1(Q) & \xleftarrow{p_*} & \mathbb{H}\mathbb{A}_1(E) & \xleftarrow{i_*} & \mathbb{H}\mathbb{A}_1(K). \end{array}$$

Dagger completions and bornological torsion-freeness

ABSTRACT. We define a dagger algebra as a bornological algebra over a discrete valuation ring with three properties that are typical of Monsky–Washnitzer algebras, namely, completeness, bornological torsion-freeness and a certain spectral radius condition. We study inheritance properties of the three properties that define a dagger algebra. We describe dagger completions of bornological algebras in general and compute some noncommutative examples.

1. Introduction

In [28], Monsky and Washnitzer introduce a cohomology theory for affine non-singular varieties defined over a field \mathbb{F} of nonzero characteristic. Let V be a discrete valuation ring such that the fraction field F of V has characteristic 0. Let $\pi \in V$ be a uniformiser and let $\mathbb{F} = V/\pi V$ be the residue field. Monsky and Washnitzer lift the coordinate ring of a smooth affine variety X over \mathbb{F} to a smooth commutative algebra A over V . The dagger completion A^\dagger of A is a certain subalgebra of the π -adic completion of A . If A is the polynomial algebra over V , then A^\dagger is the ring of overconvergent power series. The Monsky–Washnitzer cohomology is defined as the de Rham cohomology of the algebra $F \otimes_V A^\dagger$.

The dagger completion is interpreted in [11] in the setting of bornological algebras, based on considerations about the joint spectral radius of bounded subsets. The main achievement in [11] is the construction of a chain complex that computes the rigid cohomology of the original variety X and that is strictly functorial. In addition, this chain complex is related to periodic cyclic homology. Here we continue the study of dagger completions. We define dagger algebras by adding a bornological torsion-freeness condition to the completeness and spectral radius conditions already present in [11]. We also show that the category of dagger algebras is closed under extensions, subalgebras, and certain quotients, by showing that all three properties that define them are hereditary for these constructions.

The results in this article should help to reach the following important goal: define an analytic cyclic cohomology theory for algebras over \mathbb{F} that specialises to Monsky–Washnitzer or rigid cohomology for the coordinate rings of smooth affine varieties over \mathbb{F} . A general machinery for defining such cyclic cohomology theories is developed in [26]. It is based on a class of nilpotent algebras, which must be closed under extensions. This is why we are particularly interested in properties hereditary for extensions.

If S is a bounded subset of a F -algebra A , then its spectral radius $\varrho(S) \in [0, \infty]$ is defined in [11]. If A is a bornological V -algebra, then only the inequalities $\varrho(S) \leq s$ for $s > 1$ make sense. This suffices, however, to characterise the *linear growth bornology* on a bornological V -algebra: it is the smallest V -algebra bornology

with $\varrho(S) \leq 1$ for all its bounded subsets S . We call a bornological algebra A with this property *semi-dagger* because this is the main feature of dagger algebras. Any bornological algebra A carries a smallest bornology with linear growth. This defines a semi-dagger algebra A_{lg} . If A is a torsion-free, finitely generated, commutative V -algebra with the fine bornology, then the bornological completion $\overline{A_{\text{lg}}}$ of A_{lg} is the Monsky–Washnitzer completion of A .

Any algebra over \mathbb{F} is also an algebra over V . Equipped with the fine bornology, it is complete and semi-dagger. We prefer, however, not to call such algebras “dagger algebras.” The feature of Monsky–Washnitzer algebras that they lack is torsion-freeness. The purely algebraic notion of torsion-freeness does not work well for bornological algebras. In particular, it is unclear whether it is preserved by completions. We call a bornological V -module A *bornologically torsion-free* if multiplication by π is a bornological isomorphism onto its image. This notion has very good formal properties: it is preserved by bornological completions and linear growth bornologies and hereditary for subalgebras and extensions. So $\overline{A_{\text{lg}}}$ remains bornologically torsion-free if A is bornologically torsion-free. The bornological version of torsion-freeness coincides with the usual one for bornological V -modules with the fine bornology. Thus $\overline{A_{\text{lg}}}$ is bornologically torsion-free if A is a torsion-free V -algebra with the fine bornology.

A bornological V -module M is bornologically torsion-free if and only if the canonical map $M \rightarrow F \otimes_V M$ is a bornological embedding. This property is very important. On the one hand, we must keep working with modules over V in order to keep the original algebra over \mathbb{F} in sight and because the linear growth bornology only makes sense for algebras over V . On the other hand, we often need to pass to the F -vector space $F \otimes_V M$ – this is how de Rham cohomology is defined. Bornological vector spaces over F have been used recently to do analytic geometry in [5–7]. The spectral radius of a bounded subset of a bornological V -algebra A is defined in [11] by working in $F \otimes_V A$, which only works well if A is bornologically torsion-free. Here we define a truncated spectral radius in $[1, \infty]$ without reference to $F \otimes_V A$, in order to define semi-dagger algebras independently of torsion issues.

We prove that the properties of being complete, semi-dagger, or bornologically torsion-free are hereditary for extensions. Hence an extension of dagger algebras is again a dagger algebra.

To illustrate our theory, we describe the dagger completions of monoid algebras and crossed products. Dagger completions of monoid algebras are straightforward generalisations of Monsky–Washnitzer completions of polynomial algebras.

We thank the anonymous referee for helpful comments to improve the presentation in the paper.

2. Basic notions

In this section, we recall some basic notions on bornological modules and bounded homomorphisms. See [11] for more details. We also study the inheritance properties of separatedness and completeness for submodules, quotients and extensions.

A *bornology* on a set X is a collection \mathcal{B}_X of subsets of X , called *bounded sets*, such that all finite subsets are bounded and subsets and finite unions of bounded subsets are bounded. Let V be a complete discrete valuation ring. A *bornological V -module* is a V -module M with a bornology such that every bounded subset is

contained in a bounded V -submodule. In particular, the V -submodule generated by a bounded subset is again bounded. We always write \mathcal{B}_M for the bornology on M .

Let $M' \subseteq M$ be a V -submodule. The *subspace bornology* on M' consists of all subsets of M' that are bounded in M . The *quotient bornology* on M/M' consists of all subsets of the form $q(S)$ with $S \in \mathcal{B}_M$, where $q: M \rightarrow M/M'$ is the canonical projection. We always equip submodules and quotients with these canonical bornologies.

Let M and N be two bornological V -modules. A V -module map $f: M \rightarrow N$ is *bounded* if $f(S) \in \mathcal{B}_N$ for all $S \in \mathcal{B}_M$. Bornological V -modules and bounded V -module maps form an additive category. The isomorphisms in this category are called *bornological isomorphisms*. A bounded V -module map $f: M \rightarrow N$ is a *bornological embedding* if the induced map $M \rightarrow f(M)$ is a bornological isomorphism, where $f(M) \subseteq N$ carries the subspace bornology. It is a *bornological quotient map* if the induced map $M/\ker f \rightarrow N$ is a bornological isomorphism. Equivalently, for each $T \in \mathcal{B}_N$ there is $S \in \mathcal{B}_M$ with $f(S) = T$.

An *extension* of bornological V -modules is a diagram of V -modules

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

that is algebraically exact and such that f is a bornological embedding and g a bornological quotient map. Equivalently, g is a cokernel of f and f a kernel of g in the additive category of bornological V -modules. A *split extension* is an extension with a bounded V -linear map $s: M'' \rightarrow M$ such that $g \circ s = \text{id}_{M''}$.

Let M be a bornological V -module. A sequence $(x_n)_{n \in \mathbb{N}}$ in M *converges* towards $x \in M$ if there are $S \in \mathcal{B}_M$ and a sequence $(\delta_n)_{n \in \mathbb{N}}$ in V with $\lim |\delta_n| = 0$ and $x_n - x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. It is a *Cauchy* sequence if there are $S \in \mathcal{B}_M$ and a sequence $(\delta_n)_{n \in \mathbb{N}}$ in V with $\lim |\delta_n| = 0$ and $x_n - x_m \in \delta_j \cdot S$ for all $n, m, j \in \mathbb{N}$ with $n, m \geq j$. Since any bounded subset is contained in a bounded V -submodule, a sequence in M converges or is Cauchy if and only if it converges or is Cauchy in the π -adic topology on some bounded V -submodule of M .

We call a subset S of M *closed* if $x \in S$ for any sequence in S that converges in M to $x \in M$. These are the closed subsets of a topology on M . Bounded maps preserve convergent sequences and Cauchy sequences. Thus pre-images of closed subsets under bounded maps remain closed. That is, bounded maps are continuous for these canonical topologies.

2.1. Separated bornological modules. We call M *separated* if limits of convergent sequences in M are unique. If M is not separated, then the constant sequence 0 has a non-zero limit. Therefore, M is separated if and only if $\{0\} \subseteq M$ is closed. And M is separated if and only if any $S \in \mathcal{B}_M$ is contained in a π -adically separated bounded V -submodule.

LEMMA 2.1. *Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an extension of bornological V -modules.*

- (1) *If M is separated, so is M' .*
- (2) *The quotient M'' is separated if and only if $f(M')$ is closed in M .*
- (3) *If M' and M'' are separated and M'' is torsion-free, then M is separated.*

PROOF. Assertion (1) is trivial.

If M'' is separated, then $\{0\} \subseteq M''$ is closed. Hence $g^{-1}(\{0\}) = f(M')$ is closed in M . If M'' is not separated, then the constant sequence 0 in M'' converges to some non-zero $x'' \in M''$. That is, there are a bounded subset $S'' \subseteq M''$ and a null

sequence $(\delta_n)_{n \in \mathbb{N}}$ in V with $x'' - 0 \in \delta_n \cdot S''$ for all $n \in \mathbb{N}$. Since g is a bornological quotient map, there are $x \in M$ and $S \in \mathcal{B}_M$ with $g(x) = x''$ and $g(S) = S''$. We may choose $y_n'' \in S''$ with $x'' = \delta_n \cdot y_n''$ and $y_n \in S$ with $g(y_n) = y_n''$. So $g(x - \delta_n y_n) = 0$. Thus the sequence $(x - \delta_n y_n)$ lies in $f(M')$. It converges to x , which does not belong to $f(M')$ because $x'' \neq 0$. So $f(M')$ is not closed. This finishes the proof of (2).

We prove (3). Let $x \in M$ belong to the closure of $\{0\}$ in M . That is, there are $S \in \mathcal{B}_M$ and a null sequence $(\delta_n)_{n \in \mathbb{N}}$ in V with $x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. Then $g(x) \in \delta_n \cdot g(S)$ for all $n \in \mathbb{N}$. This implies $g(x) = 0$ because M'' is separated. So there is $y \in M'$ with $f(y) = x$. And $f(y) = x \in \delta_n \cdot S$. Choose $x_n \in S$ with $f(y) = \delta_n \cdot x_n$. We may assume $\delta_n \neq 0$ for all $n \in \mathbb{N}$ because otherwise $x \in \delta_n \cdot S$ is 0. Since M'' is torsion-free, $\delta_n \cdot x_n \in f(M')$ implies $g(x_n) = 0$. So we may write $x_n = f(y_n)$ for some $y_n \in M'$. Since f is a bornological embedding, the set $\{y_n : n \in \mathbb{N}\}$ in M' is bounded. Since M' is separated and $y = \delta_n \cdot y_n$ for all $n \in \mathbb{N}$, we get $y = 0$. Hence $x = 0$. So $\{0\}$ is closed in M . \square

The quotient $M/\overline{\{0\}}$ of a bornological V -module M by the closure of 0 is called the *separated quotient* of M . It is separated by Lemma 2.1, and it is the largest separated quotient of M . Even more, the quotient map $M \rightarrow M/\overline{\{0\}}$ is the universal arrow to a separated bornological V -module, that is, any bounded V -linear map from M to a separated bornological V -module factors uniquely through $M/\overline{\{0\}}$.

The following example shows that Lemma 2.1.(3) fails without the torsion-freeness assumption.

EXAMPLE 2.2. Let $M' = V$ and let $M = V[x]/S$, where S is the V -submodule of $V[x]$ generated by $1 - \pi^n x^n$ for all $n \in \mathbb{N}$. We embed $M' = V$ as multiples of $1 = x^0$. Then

$$M/M' = \bigoplus_{n=1}^{\infty} V/(\pi^n),$$

We endow M , M' and M/M' with the bornologies where all subsets are bounded. We get an extension of bornological V -modules $V \twoheadrightarrow M \twoheadrightarrow \bigoplus_{n=1}^{\infty} V/(\pi^n)$. Here V and $\bigoplus_{n=1}^{\infty} V/(\pi^n)$ are π -adically separated, but M is not: the constant sequence 1 in M converges to 0 because $1 = 1 - \pi^n x^n + \pi^n x^n \equiv \pi^n x^n$ in M .

2.2. Completeness. We call a bornological V -module M *complete* if it is separated and for any $S \in \mathcal{B}_M$ there is $T \in \mathcal{B}_M$ so that all S -Cauchy sequences are T -convergent. Equivalently, any $S \in \mathcal{B}_M$ is contained in a π -adically complete bounded V -submodule (see [11, Proposition 2.8]). By definition, any Cauchy sequence in a complete bornological V -module has a unique limit.

THEOREM 2.3. *Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an extension of bornological V -modules.*

- (1) *If M is complete and $f(M')$ is closed in M , then M' is complete.*
- (2) *If M' is complete, M separated, and M'' torsion-free, then $f(M')$ is closed in M .*
- (3) *Let M be complete. Then M'' is complete if and only if $f(M')$ is closed in M .*
- (4) *If M' and M'' are complete and M is separated, then M is complete. If M' and M'' are complete and M'' is torsion-free, then M is complete.*

PROOF. Statement (1) is [11, Lemma 2.13], and there is no need to repeat the proof here. It is somewhat similar to the proof of (4). Next we prove (2). Assume

that M' is complete, that M'' is torsion-free, and that $f(M')$ is not closed in M . We are going to prove that M is not separated. There is a sequence $(x_n)_{n \in \mathbb{N}}$ in M' for which $f(x_n)_{n \in \mathbb{N}}$ converges in M towards some $x \notin f(M')$. So there is a bounded set $S \subseteq M$ and a sequence $(\delta_k)_{k \in \mathbb{N}}$ in V with $\lim |\delta_k| = 0$ and $f(x_n) - x \in \delta_n \cdot S$ for all $n \in \mathbb{N}$. We may assume without loss of generality that S is a bounded V -submodule and that the sequence of norms $|\delta_n|$ is decreasing: let δ_n^* be the δ_m for $m \geq n$ with maximal norm. Then $f(x_n) - x \in \delta_n \cdot S \subseteq \delta_n^* \cdot S$ and still $\lim |\delta_n^*| = 0$. We may write $f(x_n) - x = \delta_n^* y_n$ with $y_n \in S$. Let $m < n$. Then $\delta_m^* g(y_m) = -g(x) = \delta_n^* g(y_n)$ and $\delta_n^* / \delta_m^* \in V$; this implies first $\delta_m^* \cdot (g(y_m) - g(y_n) \cdot \delta_n^* / \delta_m^*) = 0$ and then $g(y_m) = g(y_n) \cdot \delta_n^* / \delta_m^*$ because M'' is torsion-free. So there is $z_{m,n} \in M'$ with $y_m + f(z_{m,n}) = y_n \cdot \delta_n^* / \delta_m^*$. We even have $z_{m,n} \in f^{-1}(S)$ because S is a V -submodule. The subset $f^{-1}(S) \subseteq M'$ is bounded because f is a bornological embedding. We get $f(x_n) - f(x_m) = \delta_n^* y_n - \delta_m^* y_m = f(\delta_m^* z_{m,n})$ and hence $x_n - x_m = \delta_m^* z_{m,n}$ for $n > m$. This witnesses that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy in M' . Since M' is complete, it converges towards some $y \in M'$. Then $f(x_n)$ converges both towards $f(y) \in f(M')$ and towards $x \notin f(M')$. So M is not separated. This finishes the proof of (2).

Next we prove (3). If $f(M')$ is not closed, then Lemma 2.1 shows that M'' is not separated and hence not complete. Conversely, we claim that M'' is complete if $f(M')$ is closed. Lemma 2.1 shows that M'' is separated. Let $S'' \in \mathcal{B}_{M''}$. There is $S \in \mathcal{B}_M$ with $g(S) = S''$ because g is a bornological quotient map. And there is $T \in \mathcal{B}_M$ so that any S -Cauchy sequence is T -convergent. We claim that any S'' -Cauchy sequence is $g(T)$ -convergent. So let $(x''_n)_{n \in \mathbb{N}}$ be an S'' -Cauchy sequence. Thus there is a null sequence $(\delta_n)_{n \in \mathbb{N}}$ in V with $x''_n - x''_m \in \delta_j \cdot S''$ for all $n, m, j \in \mathbb{N}$ with $n, m \geq j$. As above, we may assume without loss of generality that the sequence of norms $|\delta_n|$ is decreasing. Choose any $x_0 \in M$ with $g(x_0) = x''_0$. For each $n \in \mathbb{N}$, choose $y_n \in S$ with $x''_{n+1} - x''_n = \delta_n \cdot g(y_n)$. Let

$$x_n := x_0 + \delta_0 \cdot y_0 + \cdots + \delta_{n-1} \cdot y_{n-1}.$$

Then $g(x_n) = x''_n$. And $x_{n+1} - x_n = \delta_n \cdot y_n \in \delta_n \cdot S$. Since $|\delta_n|$ is decreasing, this implies $x_m - x_n \in \delta_n \cdot S$ for all $m \geq n$. So the sequence $(x_n)_{n \in \mathbb{N}}$ is S -Cauchy. Hence it is T -convergent. Thus $g(x_n) = x''_n$ is $g(T)$ -convergent as asserted. This finishes the proof of (3).

Finally, we prove (4). So we assume M' and M'' to be complete. If M'' is torsion-free, then M is separated by Lemma 2.1. Hence the second statement in (4) is a special case of the first one. Let $S \in \mathcal{B}_M$. We must find $T \in \mathcal{B}_M$ so that every S -Cauchy sequence is T -convergent. Since M is separated, this says that it is complete. Since M'' is complete, there is a π -adically complete V -submodule $T_0 \in \mathcal{B}_{M''}$ that contains $g(S)$. Since g is a bornological quotient map, there is $T_1 \in \mathcal{B}_M$ with $g(T_1) = T_0$. Replacing it by $T_1 + S$, we may arrange, in addition, that $S \subseteq T_1$. Since f is a bornological embedding, $T_2 := f^{-1}(T_1)$ is bounded in M' . As M' is complete, there is $T_3 \in \mathcal{B}_{M'}$ so that every T_2 -Cauchy sequence is T_3 -convergent. We claim that any S -Cauchy sequence is $T_1 + f(T_3)$ -convergent. The proof of this claim will finish the proof of the theorem.

Let $(x_n)_{n \in \mathbb{N}}$ be an S -Cauchy sequence. So there are $\delta_n \in V$ and $y_n \in S$ with $\lim |\delta_n| = 0$ and $x_{n+1} - x_n = \delta_n \cdot y_n$. As above, we may assume that $|\delta_n|$ is decreasing and that $\delta_0 = 1$. Since $g(y_{n+k}) \in g(S) \subseteq T_0$ and T_0 is π -adically complete, the

following series converges in T_0 :

$$(2.4) \quad \tilde{w}_n := - \sum_{k=0}^{\infty} \frac{\delta_{n+k}}{\delta_n} g(y_{n+k}).$$

Since $\tilde{w}_n \in T_0$, there is $w_n \in T_1$ with $g(w_n) = \tilde{w}_n$. So

$$\begin{aligned} \delta_n g(w_n) &= \lim_{N \rightarrow \infty} -g \left(\sum_{k=0}^N \delta_{n+k} y_{n+k} \right) \\ &= \lim_{N \rightarrow \infty} g(x_n - x_{N+n+1}) = g(x_n) - \lim_{N \rightarrow \infty} g(x_N). \end{aligned}$$

In particular, $g(w_0) = g(x_0) - \lim_{N \rightarrow \infty} g(x_N)$. Now let

$$\tilde{x}_k := x_k - \delta_k w_k + w_0 - x_0.$$

Then

$$g(\tilde{x}_k) = g(x_k) - g(x_k) + \lim_{N \rightarrow \infty} g(x_N) + g(x_0) - \lim_{N \rightarrow \infty} g(x_N) - g(x_0) = 0.$$

So $\tilde{x}_k \in f(M')$ for all $k \in \mathbb{N}$. And

$$(2.5) \quad \begin{aligned} \tilde{x}_{n+1} - \tilde{x}_n &= x_{n+1} - x_n - \delta_{n+1} w_{n+1} + \delta_n w_n \\ &= \delta_n y_n + \delta_n w_n - \delta_{n+1} w_{n+1} = \delta_n \cdot \left(y_n + w_n - \frac{\delta_{n+1}}{\delta_n} w_{n+1} \right). \end{aligned}$$

Let $z_n := y_n + w_n - \frac{\delta_{n+1}}{\delta_n} w_{n+1}$. A telescoping sum argument shows that

$$(2.6) \quad g(z_n) = g(y_n) + \tilde{w}_n - \frac{\delta_{n+1}}{\delta_n} \tilde{w}_{n+1} = 0.$$

So $z_n \in f(M')$. And $z_n \in S + T_1 + T_1 = T_1$. Thus there is $\hat{z}_n \in f^{-1}(T_1) = T_2$ with $z_n = f(\hat{z}_n)$. Equation (2.5) means that the sequence $f^{-1}(\tilde{x}_n)$ is T_2 -Cauchy. Hence it is T_3 -convergent. So (\tilde{x}_n) is $f(T_3)$ -convergent. Then (x_n) is $T_1 + f(T_3)$ -convergent. \square

The following examples show that the technical extra assumptions in (2) and (4) in Theorem 2.3 are necessary. They only involve extensions of V -modules with the bornology where all subsets are bounded. For this bornology, bornological completeness and separatedness are the same as π -adic completeness and separatedness, respectively, and any extension of V -modules is a bornological extension.

EXAMPLE 2.7. Let $M' := \{0\}$ and $M := F$ with the bornology of all subsets. Then M' is bornologically complete, but not closed in M , and $M/M' = M$ is torsion-free. So Theorem 2.3.(2) needs the assumption that M be separated.

EXAMPLE 2.8. Let M be the V -module of all power series $\sum_{n=0}^{\infty} c_n x^n$ with $\lim |c_n| = 0$ and with the bornology where all subsets are bounded; this is the π -adic completion of the polynomial algebra $V[x]$. Let $M' = M$ and define $f: M' \rightarrow M$, $f(\sum_{n=0}^{\infty} c_n x^n) := \sum_{n=0}^{\infty} c_n \pi^n x^n$. This is a bornological embedding simply because all subsets in $M = M'$ are bounded. Let $p_n := \sum_{j=0}^n x^j$. This sequence in $M' = M$ does not converge. Nevertheless, the sequence $f(p_n) = \sum_{j=0}^n \pi^j x^j$ converges in M to $\sum_{j=0}^{\infty} \pi^j x^j$. Thus $f(M')$ is not closed in M , although M and M' are complete and f is a bornological embedding. So Theorem 2.3.(2) needs the assumption that M'' be torsion-free.

EXAMPLE 2.9. We modify Example 2.2 to produce an extension of V -modules $N' \twoheadrightarrow N \twoheadrightarrow N''$ where N' and N'' are π -adically complete, but N is not π -adically separated and hence not π -adically complete. We let $N' := V/(\pi) = \mathbb{F}$. We let N'' be the π -adic completion of the V -module M'' of Example 2.2. That is,

$$N'' := \left\{ (c_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} V/(\pi^n) : \lim |c_n| = 0 \right\}.$$

This is indeed π -adically complete. So is

$$N_1 := \left\{ (c_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} V/(\pi^{n+1}) : \lim |c_n| = 0 \right\}.$$

The kernel of the quotient map $q: N_1 \twoheadrightarrow N''$ is isomorphic to $\prod_{n=0}^{\infty} V/(\pi) = \prod_{\mathbb{N}} \mathbb{F}$. This is a \mathbb{F} -vector space, and it contains the \mathbb{F} -vector space $\sum_{n=0}^{\infty} \mathbb{F}$. Since any \mathbb{F} -vector space has a basis, we may extend the linear functional $\sum_{n=0}^{\infty} \mathbb{F} \rightarrow \mathbb{F}$, $(c_n)_{n \in \mathbb{N}} \mapsto \sum_{n=0}^{\infty} c_n$, to a \mathbb{F} -linear functional $\sigma: \prod_{\mathbb{N}} \mathbb{F} \rightarrow \mathbb{F}$. Let $L := \ker \sigma \subseteq \ker q$ and let $N := N_1/L$. The map q descends to a surjective π -linear map $N \twoheadrightarrow N''$. Its kernel is isomorphic to $\prod_{\mathbb{N}} \mathbb{F}/\ker \sigma \cong \mathbb{F} = N'$. The functional $\sigma: \prod_{\mathbb{N}} \mathbb{F} \rightarrow \mathbb{F}$ vanishes on $\delta_0 - \delta_k$ for all $k \in \mathbb{N}$, but not on δ_0 . When we identify $\prod_{\mathbb{N}} \mathbb{F} \cong \ker q$, we map δ_k to $\pi^k \delta_k \in N_1$. So δ_0 and $\pi^k \delta_k$ get identified in N , but δ_0 does not become 0: it is the generator of $N' = V/(\pi)$ inside N . Since $[\delta_0] = \pi^k [\delta_k]$ in N , the V -module N is not π -adically separated.

The *completion* \overline{M} of a bornological V -module M is a complete bornological V -module with a bounded V -linear map $M \rightarrow \overline{M}$ that is universal in the sense that any bounded V -linear map from M to a complete bornological V -module X factors uniquely through \overline{M} . Such a completion exists and is unique up to isomorphism (see [11, Proposition 2.15]). We shall describe it more concretely later when we need the details of its construction.

2.3. Vector spaces over the fraction field. Recall that F denotes the quotient field of V . Any V -linear map between two F -vector spaces is also F -linear. So F -vector spaces with F -linear maps form a full subcategory in the category of V -modules. A V -module M comes from a F -vector space if and only if the map

$$(2.10) \quad \pi_M: M \rightarrow M, \quad m \mapsto \pi \cdot m,$$

is invertible. We could define bornological F -vector spaces without reference to V . Instead, we realise them as bornological V -modules with an extra property:

DEFINITION 2.11. A bornological V -module M is a *bornological F -vector space* if the map π_M in (2.10) is a bornological isomorphism, that is, an invertible map with bounded inverse.

Given a bornological V -module M , the tensor product $F \otimes M := F \otimes_V M$ with the tensor product bornology (see [11, Lemma 2.18]) is a bornological F -vector space because multiplication by π is a bornological isomorphism on F .

LEMMA 2.12. *The canonical bounded V -linear map*

$$\iota_M: M \rightarrow F \otimes M, \quad m \mapsto 1 \otimes m,$$

is the universal arrow from M to a bornological F -vector space, that is, any bounded V -linear map $f: M \rightarrow N$ to a bornological F -vector space N factors uniquely through a bounded V -linear map $f^\#: F \otimes M \rightarrow N$, and this map is also F -linear.

PROOF. A V -linear map $f^\#: F \otimes M \rightarrow N$ must be F -linear. Hence the only possible candidate is the F -linear map defined by $f^\#(x \otimes m) := x \cdot f(m)$ for $m \in M$, $x \in F$. Any bounded submodule of $F \otimes M$ is contained in $\pi^{-k}V \otimes S$ for some bounded submodule $S \subseteq M$ and some $k \in \mathbb{N}$, and $f^\#(\pi^{-k}V \otimes S) = \pi_N^{-k}(f(S))$ is bounded in N because π_N is a bornological isomorphism. Thus $f^\#$ is bounded. \square

3. Spectral radius and semi-dagger algebras

A *bornological V -algebra* is a bornological V -module A with a bounded, V -linear, associative multiplication. We do not assume A to have a unit element. We fix a bornological V -algebra A throughout this section.

We recall some definitions from [11]. Let $\varepsilon = |\pi|$. Let $S \in \mathcal{B}_A$ and let $r \leq 1$. There is a smallest integer j with $\varepsilon^j \leq r$, namely, $\lceil \log_\varepsilon(r) \rceil$. Define

$$r \star S := \pi^{\lceil \log_\varepsilon(r) \rceil} \cdot S.$$

Let $\sum_{n=1}^{\infty} r^n \star S^n$ be the V -submodule generated by $\bigcup_{n=1}^{\infty} r^n \star S^n$. That is, its elements are finite V -linear combinations of elements in $\bigcup_{n=1}^{\infty} r^n \star S^n$.

DEFINITION 3.1. The *truncated spectral radius* $\varrho_1(S) = \varrho_1(S; \mathcal{B}_A)$ of $S \in \mathcal{B}_A$ is the infimum of all $r \geq 1$ for which $\sum_{n=1}^{\infty} r^{-n} \star S^n$ is bounded. It is ∞ if no such r exists.

By definition, $\varrho_1(S) \in [1, \infty]$. If A is an algebra over the fraction field F of V , then we may define $\sum_{n=1}^{\infty} r^{-n} \star S^n$ also for $0 < r < 1$. Then the full spectral radius $\varrho(S) \in [0, \infty]$ is defined like $\varrho_1(S)$, but without the restriction to $r \geq 1$. The arguments in the beginning of Section 3.1 in [11] assume implicitly that the V -algebra A is a bornological subalgebra in a F -algebra, so that the spectral radius is defined without truncation. This means that A is bornologically torsion-free (see Proposition 4.3). The results in [11, Section 3.1] work without this assumption if the truncated spectral radius is used throughout and the following lemma is used instead of [11, Lemma 3.1.2]:

LEMMA 3.2. *Let $S \subseteq A$ be a bounded V -submodule and $m \in \mathbb{N}_{\geq 1}$. Then $\varrho_1(S) = 1$ if and only if $\sum_{l=1}^{\infty} (\pi^m S^j)^l$ is bounded for all $j \in \mathbb{N}_{\geq 1}$.*

PROOF. Let $\varrho_1(S) = 1$ and $j \in \mathbb{N}_{\geq 1}$. Then $\sum_{l=1}^{\infty} \pi^{m \cdot l} S^{j \cdot l} \subseteq \sum_{k=1}^{\infty} (\varepsilon^{m/j})^k \star S^k$ is bounded because $\varepsilon^{m/j} < 1$. Conversely, let $\sum_{l=1}^{\infty} (\pi^m S^j)^l$ be bounded. Then $\varrho_1(\pi^m S^j) = 1$. The proof of [11, Lemma 3.1.2] shows that $\varrho_1(S) \leq \varepsilon^{-m/j}$. This inequality for all $j \in \mathbb{N}_{\geq 1}$ implies $\varrho_1(S) = 1$. \square

DEFINITION 3.3. A bornological V -algebra A is *semi-dagger* if $\varrho_1(S) = 1$ for all $S \in \mathcal{B}_A$.

PROPOSITION 3.4 ([11, Proposition 3.1.3]). *A bornological V -algebra A is semi-dagger if and only if $\sum_{i=0}^{\infty} \pi^i S^{i+1}$ is bounded for all $S \in \mathcal{B}_A$, if and only if $\sum_{i=0}^{\infty} \pi^i S^{ci+d}$ is bounded for all $S \in \mathcal{B}_A$ and $c, d \in \mathbb{N}$ with $d \geq 1$, if and only if any $S \in \mathcal{B}_A$ is contained in a bounded V -submodule $U \subseteq A$ with $\pi \cdot U \cdot U \subseteq U$.*

DEFINITION 3.5. The *linear growth bornology* on a bornological V -algebra A is the smallest semi-dagger bornology on A . That is, it is the smallest bornology \mathcal{B}'_A with $\varrho_1(S; \mathcal{B}'_A) = 1$ for all $S \in \mathcal{B}'_A$. Let A_{lg} be A with the linear growth bornology.

The existence of a smallest semi-dagger bornology is shown in [11] by describing it explicitly as follows:

LEMMA 3.6 ([11, Proposition 3.1.3 and Lemma 3.1.10]). *Let $T \subseteq A$. The following are equivalent:*

- (1) T is bounded in A_{lg} ;
- (2) $T \subseteq \sum_{i=0}^{\infty} \pi^i S^{i+1}$ for some $S \in \mathcal{B}_A$.
- (3) $T \subseteq \sum_{i=0}^{\infty} \pi^i S^{ci+d}$ for some $S \in \mathcal{B}_A$ and $c, d \in \mathbb{N}$ with $d \geq 1$.

More precisely, the proof of [11, Lemma 3.1.10] shows that the subsets in (2) form a semi-dagger bornology that contains \mathcal{B}_A . And Proposition 3.4 shows that these subsets are bounded in any semi-dagger bornology on A that contains \mathcal{B}_A . So they form the smallest semi-dagger bornology containing \mathcal{B}_A .

By definition, the algebra A_{lg} has the following universal property: if B is a semi-dagger V -algebra, then an algebra homomorphism $A \rightarrow B$ is bounded if and only if it is bounded on A_{lg} . The algebra A is semi-dagger if and only if $A = A_{\text{lg}}$.

THEOREM 3.7. *Let $A \xrightarrow{i} B \xrightarrow{q} C$ be an extension of bornological V -algebras. Then B is a semi-dagger algebra if and only if both A and C are.*

PROOF. First assume B to be semi-dagger. Let $S \in \mathcal{B}_A$. Then $\sum_{j=0}^{\infty} \pi^j i(S)^{j+1}$ is bounded in B . Since i is a bornological embedding, it follows that $\sum_{j=0}^{\infty} \pi^j S^{j+1}$ is bounded in A . That is, $\varrho_1(S; \mathcal{B}_A) = 1$. So A is semi-dagger. Now let $S \in \mathcal{B}_C$. Since q is a bornological quotient map, there is $T \in \mathcal{B}_B$ with $q(T) = S$. The subset $\sum_{j=0}^{\infty} \pi^j T^{j+1}$ is bounded in B because B is semi-dagger. Its image under q is also bounded, and this is $\sum_{j=0}^{\infty} \pi^j S^{j+1}$. So $\varrho_1(S; \mathcal{B}_C) = 1$ and C is semi-dagger.

Now assume that A and C are semi-dagger. We show that $\sum_{l=1}^{\infty} (\pi^2 S^j)^l$ is bounded in B for all $S \in \mathcal{B}_B$, $j \in \mathbb{N}_{\geq 1}$. This implies $\varrho_1(S; \mathcal{B}_B) = 1$ by Lemma 3.2.

Since C is semi-dagger, $\varrho_1(q(S); \mathcal{B}_C) = 1$. Thus $S_2 := \sum_{l=1}^{\infty} q(\pi S^j)^l$ is bounded in C by Lemma 3.2. Since q is a quotient map, there is $T \in \mathcal{B}_B$ with $q(T) = S_2$. We may choose T with $\pi S^j \subseteq T$. For each $x, y \in T$, we have $q(x \cdot y) \in S_2 \cdot S_2 \subseteq S_2 = q(T)$. Hence there is $\omega(x, y) \in T$ with $x \cdot y - \omega(x, y) \in i(A)$. Let

$$\Omega := \{x \cdot y - \omega(x, y) : x, y \in T\}.$$

This is contained in $T^2 - T$. So $\Omega \in \mathcal{B}_B$. And $T^2 \subseteq T + \Omega$. By construction, Ω is also contained in $i(A)$. Since i is a bornological embedding, $i^{-1}(\Omega)$ is bounded in A . Since A is semi-dagger, we have $\varrho_1(i^{-1}(\Omega); \mathcal{B}_A) = 1$. So $\sum_{n=1}^{\infty} (\pi \cdot \Omega)^n$ is bounded. Thus the subset

$$U := \sum_{n=1}^{\infty} (\pi \cdot \Omega)^n + \sum_{n=0}^{\infty} T \cdot (\pi \cdot \Omega)^n = \sum_{n=1}^{\infty} (\pi \cdot \Omega)^n + T + \sum_{n=1}^{\infty} T \cdot (\pi \cdot \Omega)^n$$

of B is bounded. Using $T^2 \subseteq T + \Omega$, we prove that $\pi T \cdot U \subseteq U$. Hence $(\pi T)^n \cdot U \subseteq U$ for all $n \in \mathbb{N}_{\geq 1}$ by induction. Since $T \subseteq U$, this implies $\sum_{l=1}^{\infty} \pi^l T^{l+1} \subseteq U$. Hence $\sum_{l=2}^{\infty} (\pi T)^l = \pi \cdot \sum_{l=1}^{\infty} \pi^l T^{l+1} \subseteq \pi U$. Therefore, $\sum_{l=1}^{\infty} (\pi T)^l$ is bounded. Since $\pi^2 S^j \subseteq \pi T$, it follows that $\sum_{l=1}^{\infty} (\pi^2 S^j)^l$ is bounded for all j , as desired. \square

PROPOSITION 3.8 ([11, Lemma 3.1.12]). *If A is semi-dagger, then so is its completion \overline{A} .*

Let $\overline{A_{\text{lg}}}$ be the completion of A_{lg} . This algebra is both complete and semi-dagger by Proposition 3.8. The canonical bounded homomorphism $A \rightarrow \overline{A_{\text{lg}}}$ is the universal arrow from A to a complete semi-dagger algebra, that is, any bounded homomorphism $A \rightarrow B$ for a complete semi-dagger algebra B factors uniquely

through it. This follows immediately from the universal properties of the linear growth bornology and the completion.

4. Bornological torsion-freeness

Let M be a bornological V -module. The bounded linear map $\pi_M: M \rightarrow M$, $m \mapsto \pi \cdot m$, is defined in (2.10).

DEFINITION 4.1. A bornological V -module M is *bornologically torsion-free* if π_M is a bornological embedding. Equivalently, $\pi \cdot m = 0$ for $m \in M$ only happens for $m = 0$ and any bounded subset of M that is contained in $\pi \cdot M$ is of the form $S = \pi \cdot T$ for some $T \in \mathcal{B}_M$.

Bornological F -vector spaces are bornologically torsion-free because bornological isomorphisms are bornological embeddings. We are going to show that M is bornologically torsion-free if and only if the canonical map $\iota_M: M \rightarrow F \otimes M$ defined in Lemma 2.12 is a bornological embedding. The proof uses the following easy permanence property:

LEMMA 4.2. *Let M be a bornological V -module and let $N \subseteq M$ be a V -submodule with the subspace bornology. If M is bornologically torsion-free, then so is N .*

PROOF. Let $j: N \rightarrow M$ be the inclusion map, which is a bornological embedding by assumption. Since π_M is a bornological embedding, so is $\pi_M \circ j = j \circ \pi_N$. Since j is a bornological embedding, this implies that π_N is a bornological embedding. That is, N is bornologically torsion-free. \square

PROPOSITION 4.3. *A bornological V -module M is bornologically torsion-free if and only if the canonical map $\iota_M: M \rightarrow F \otimes M$ is a bornological embedding.*

PROOF. As a bornological F -vector space, $F \otimes M$ is bornologically torsion-free. Hence M is bornologically torsion-free by Lemma 4.2 if ι_M is a bornological embedding. Conversely, assume that M is bornologically torsion-free. The map ι_M is injective because M is algebraically torsion-free. It remains to show that a subset S of M is bounded if $\iota_M(S) \subseteq F \otimes M$ is bounded. If $\iota_M(S)$ is bounded, then it is contained in $\pi^{-k} \cdot V \otimes T$ for some $k \in \mathbb{N}$ and some $T \in \mathcal{B}_M$. Equivalently, $\pi_M^k(S) = \pi^k \cdot S$ is bounded in M . Since π_M is a bornological embedding, induction shows that $\pi_M^k: M \rightarrow M$, $m \mapsto \pi^k \cdot m$, is a bornological embedding as well. So the boundedness of $\pi_M^k(S)$ implies that S is bounded. \square

PROPOSITION 4.4. *Let $M_{\text{tf}} := \iota_M(M) \subseteq F \otimes M$ equipped with the subspace bornology and the surjective bounded linear map $\iota_M: M \rightarrow M_{\text{tf}}$. This is the universal arrow from M to a bornologically torsion-free module, that is, any bounded linear map $f: M \rightarrow N$ into a bornologically torsion-free module N factors uniquely through a bounded linear map $f^\#: M_{\text{tf}} \rightarrow N$.*

PROOF. Since $F \otimes M$ is bornologically torsion-free as a bornological F -vector space, M_{tf} is bornologically torsion-free as well by Lemma 4.2. We prove the universality of the canonical map $\iota_M: M \rightarrow M_{\text{tf}}$. Let N be a bornologically torsion-free V -module and let $f: M \rightarrow N$ be a bornological V -module map. Then $N \hookrightarrow F \otimes N$ is a bornological embedding by Proposition 4.3, and we may compose to get a bounded V -linear map $M \rightarrow F \otimes N$. By Lemma 2.12, there is a unique bounded F -linear map $f': F \otimes M \rightarrow F \otimes N$ with $f'(\iota_M(m)) = f(m)$ for all $m \in M$. Since

$f'(\iota_M(M)) \subseteq N$, f' maps the submodule $M_{\text{tf}} \subseteq F \otimes M$ to the submodule $N \subseteq F \otimes N$. The restricted map $f^\#: M_{\text{tf}} \rightarrow N$ is bounded because both submodules carry the subspace bornology. This is the required factorisation of f . It is unique because $\iota_M: M \rightarrow M_{\text{tf}}$ is surjective. \square

We have seen that being bornologically torsion-free is hereditary for submodules. The obvious counterexample $\mathbb{F} = V / \pi V$ shows that it cannot be hereditary for quotients. Next we show that it is hereditary for extensions:

THEOREM 4.5. *Let $M' \xrightarrow{i} M \xrightarrow{q} M''$ be an extension of bornological V -modules. If M' and M'' are bornologically torsion-free, then so is M .*

PROOF. The exactness of the sequence $0 \rightarrow \ker \pi_{M'} \rightarrow \ker \pi_M \rightarrow \ker \pi_{M''}$ shows that π_M is injective. Let $S \in \mathcal{B}_M$ be contained in πM . We want a bounded subset $S' \in \mathcal{B}_{M'}$ with $\pi \cdot S' = S$. We have $q(S) \subseteq q(\pi \cdot M) \subseteq \pi \cdot M''$, and $q(S) \in \mathcal{B}_{M''}$ because q is bounded. Since M'' is bornologically torsion-free, there is $T'' \in \mathcal{B}_{M''}$ with $\pi \cdot T'' = q(S)$. Since q is a bornological quotient map, there is $T \in \mathcal{B}_M$ with $q(T) = T''$. Thus $q(\pi \cdot T) = q(S)$. So for any $x \in S$ there is $y \in T$ with $q(\pi \cdot y) = q(x)$. Since $i = \ker(q)$, there is a unique $z \in M'$ with $x - \pi y = i(z)$. Let T' be the set of these z . Since $x \in \pi \cdot M$ by assumption and M'' is torsion-free, we have $z \in \pi \cdot M'$. So $T' \subseteq \pi \cdot M'$. And T' is bounded because $T' \subseteq i^{-1}(S - \pi \cdot T)$ and i is a bornological embedding. Since M' is bornologically torsion-free, there is a bounded subset $U' \in \mathcal{B}_{M'}$ with $\pi \cdot U' = T'$. Then $S \subseteq \pi \cdot T + i(\pi \cdot U') = \pi \cdot (T + i(U'))$. \square

Next we prove that bornological torsion-freeness is inherited by completions:

THEOREM 4.6. *If M is bornologically torsion-free, then so is its bornological completion \overline{M} .*

The proof requires some preparation. We must look closely at the construction of completions of bornological V -modules.

PROPOSITION 4.7 ([11, Proposition 2.15]). *Let M be a bornological V -module. A completion of M exists and is constructed as follows. Write $M = \varinjlim M_i$ as an inductive limit of the directed set of its bounded V -submodules. Let \overline{M}_i denote the π -adic completion of M_i . These form an inductive system as well, and $\overline{M} \cong (\varinjlim \overline{M}_i) / \overline{\{0\}}$ is the separated quotient of their bornological inductive limit.*

The completion functor commutes with colimits, that is, the completion of a colimit of a diagram of bornological V -modules is the separated quotient of the colimit of the diagram of completions.

Since taking quotients may create torsion, the information above is not yet precise enough to show that completions inherit bornological torsion-freeness. This requires some more work. First we write M in a certain way as an inductive limit, using that it is bornologically torsion-free. For a bounded submodule S in M , let

$$\begin{aligned} \pi^{-n}S &:= \{x \in M : \pi^n \cdot x \in S\} \subseteq M \\ M_S &:= \bigcup_{n \in \mathbb{N}} \pi^{-n}S \subseteq M. \end{aligned}$$

The *gauge semi-norm* of S is defined by $\|x\|_S := \inf\{\varepsilon^n : x \in \pi^n S\}$, where $\varepsilon = |\pi|$ (see [11, Example 2.4]). A subset is bounded for this semi-norm if and only if it is contained in $\pi^{-n}S$ for some $n \in \mathbb{N}$. Since M is bornologically torsion-free,

$\pi^{-n}S \in \mathcal{B}_M$ for $n \in \mathbb{N}$. So subsets that are bounded in the gauge semi-norm on M_S are bounded in M . If $S \subseteq T$, then $M_S \subseteq M_T$ and the inclusion is contracting and hence bounded. The bornological inductive limit of this inductive system is naturally isomorphic to M because any bounded subset of M is bounded in M_S for some bounded submodule $S \subseteq M$ (compare the proof of [11, Proposition 2.5]).

The bornological completion $\overline{M_S}$ of M_S as a bornological V -module is canonically isomorphic to its Hausdorff completion as a semi-normed V -module. We call this a *Banach V -module*. Both completions are isomorphic to the increasing union of the π -adic completions $\widehat{\pi^{-n}S}$. If $S \subseteq T$, then $M_S \subseteq M_T$ and this inclusion is norm-contracting. So we get an induced contractive linear map $i_{T,S}: \overline{M_S} \rightarrow \overline{M_T}$. This map need not be injective any more (see [11, Example 2.15]). Hence the canonical maps $i_{\infty,S}: \overline{M_S} \rightarrow \overline{M}$ need not be injective. The bornological completion commutes with (separated) inductive limits by Proposition 4.7. So the completion of M is isomorphic to the separated quotient of the colimit of the inductive system formed by the Banach V -modules $\overline{M_S}$ and the norm-contracting maps $i_{T,S}$ for $S \subseteq T$.

LEMMA 4.8. *The submodules*

$$Z_S := \ker(i_{\infty,S}: \overline{M_S} \rightarrow \overline{M}) = i_{\infty,S}^{-1}(\{0\}) \subseteq \overline{M_S}$$

are norm-closed and satisfy $i_{T,S}^{-1}(Z_T) = Z_S$ if $S \subseteq T$. They are minimal with these properties in the sense that if $L_S \subseteq \overline{M_S}$ are norm-closed and satisfy $i_{T,S}^{-1}(L_T) = L_S$ for $S \subseteq T$, then $Z_S \subseteq L_S$ for all bounded V -submodules $S \subseteq M$.

PROOF. The property $i_{T,S}^{-1}(Z_T) = Z_S$ is trivial. The map $i_{\infty,S}$ is bounded and hence preserves convergence of sequences. Since \overline{M} is separated, the subset $\{0\} \subseteq \overline{M}$ is bornologically closed. Therefore, its preimage Z_S in $\overline{M_S}$ is also closed. Let (L_S) be any family of closed submodules with $i_{T,S}^{-1}(L_T) = L_S$. The quotient seminorm on $\overline{M_S}/L_S$ is again a norm because L_S is closed. And $\overline{M_S}/L_S$ inherits completeness from $\overline{M_S}$ by Theorem 2.3. If $S \subseteq T$, then $i_{T,S}$ induces an injective map $i'_{T,S}: \overline{M_S}/L_S \rightarrow \overline{M_T}/L_T$ because $L_S = i_{T,S}^{-1}(L_T)$. Hence the colimit of the inductive system $(\overline{M_S}/L_S, i'_{T,S})$ is like a directed union of subspaces, and each $\overline{M_S}/L_S$ maps faithfully into it. Thus this colimit is separated. It is even complete because each $\overline{M_S}/L_S$ is complete. Hence the map from M to this colimit induces a map on the completion \overline{M} . This implies $Z_S \subseteq L_S$. \square

Next we link \overline{M} to the π -adic completion $\widehat{M} := \varprojlim M/\pi^j M$. Equip the quotients $M/\pi^j M$ with the quotient bornology. Since $\pi^j \cdot (M/\pi^j) = 0$, any Cauchy sequence in $M/\pi^j M$ is eventually constant. So each $M/\pi^j M$ is complete. Hence the quotient map $M \rightarrow M/\pi^j M$ induces a bounded V -module homomorphism $\overline{M} \rightarrow M/\pi^j M$. Putting them all together gives a map $\overline{M} \rightarrow \widehat{M}$, which is bounded if we give \widehat{M} the projective limit bornology.

Let $S \subseteq M$ be a bounded V -submodule and let $j \in \mathbb{N}$. We have defined the submodules M_S so that $M_S \cap \pi^j M = \pi^j M_S$. That is, the map $M_S/\pi^j M_S \rightarrow M/\pi^j M$ is injective. Since M_S is dense in its norm-completion $\overline{M_S}$, we have $\overline{M_S} = M_S + \pi^j \widehat{S}$ and hence $\overline{M_S} = M_S + \pi^j \overline{M_S}$. Thus the inclusion $M_S \rightarrow \overline{M_S}$ induces an isomorphism

$$M_S/\pi^j M_S \cong \overline{M_S}/\pi^j \overline{M_S}.$$

Letting j vary, we get an injective map $\widehat{M_S} \rightarrow \widehat{M}$ and an isomorphism between the π -adic completions of M_S and $\overline{M_S}$.

PROOF OF THEOREM 4.6. For each bounded V -submodule $S \subseteq M$, define $L_S := \bigcap_{j \in \mathbb{N}} \pi^j \cdot \overline{M_S} \subseteq \overline{M_S}$. This is the kernel of the canonical map to the π -adic completion of $\overline{M_S}$. The completion $\overline{M_S}$ is torsion-free because it carries a norm. Hence L_S is also the largest F -vector space contained in $\overline{M_S}$. The subspace L_S is closed because the maps $\overline{M_S} \rightarrow \overline{M_S}/\pi^j \overline{M_S}$ for $j \in \mathbb{N}$ are bounded and their target spaces are separable, even complete.

Let $S \subseteq T$. The maps $\overline{M_S}/\pi^j \overline{M_S} \rightarrow \overline{M_T}/\pi^j \overline{M_T}$ are injective for all $j \in \mathbb{N}$, and $\overline{M_S}/\pi^j \overline{M_S} \cong \overline{M_S}/\pi^j \overline{M_S}$, $\overline{M_T}/\pi^j \overline{M_T} \cong \overline{M_T}/\pi^j \overline{M_T}$. So $i_{T,S}$ induces an injective map $\overline{M_S}/\pi^j \overline{M_S} \rightarrow \overline{M_T}/\pi^j \overline{M_T}$. This implies $i_{T,S}^{-1}(\pi^j \overline{M_T}) = \pi^j \overline{M_S}$ for all $j \in \mathbb{N}$ and then $i_{T,S}^{-1}(L_T) = L_S$.

By Lemma 4.8, the kernel $Z_S = \ker(i_{\infty,S})$ is contained in L_S for all S . Since π_{L_S} is a bornological isomorphism, the subsets $\pi \cdot Z_S \subseteq Z_S$ are also bornologically closed, and they satisfy $i_{T,S}^{-1}(\pi \cdot Z_T) = \pi \cdot i_{T,S}^{-1}(Z_T) = \pi \cdot Z_S$. Hence $Z_S \subseteq \pi \cdot Z_S$ for all S by Lemma 4.8. Thus $Z_S \subseteq L_S$ is a F -vector subspace in $\overline{M_S}$. So the quotient $\overline{M_S}/Z_S$ is still bornologically torsion-free. And any element of $\overline{M_S}/Z_S$ that is divisible by π^j lifts to an element in $\pi^j \cdot \overline{M_S}$.

Any bounded subset of \overline{M} is contained in $i_{\infty,S}(\widehat{S})$ for some bounded V -submodule $S \subseteq M$, where we view \widehat{S} as a subset of $\overline{M_S}$. Let $j \in \mathbb{N}$. To prove that \overline{M} is bornologically torsion-free, we must show that $\pi^{-j} i_{\infty,S}(\widehat{S})$ is bounded. Let $x \in \overline{M}$ satisfy $\pi^j x \in i_{\infty,S}(\widehat{S})$. We claim that $x = i_{\infty,S}(y)$ for some $y \in \overline{M_S}$ with $\pi^j y \in \widehat{S}$. This implies that $\pi^{-j} \cdot i_{\infty,S}(\widehat{S})$ is bounded in \overline{M} . It remains to prove the claim. There are a bounded V -submodule $T \subseteq M$ and $z \in \overline{M_T}$ with $x = i_{\infty,T}(z)$. We may replace T by $T + S$ to arrange that $T \supseteq S$. Let $w \in \widehat{S}$ satisfy $\pi^j x = i_{\infty,S}(w)$. This is equivalent to $\pi^j z - i_{T,S}(w) \in \ker i_{\infty,T} = Z_T$. Since Z_T is a F -vector space, there is $z_0 \in Z_T$ with $\pi^j z - i_{T,S}(w) = \pi^j z_0$. Since $x = i_{\infty,T}(z - z_0)$, we may replace z by $z - z_0$ to arrange that $\pi^j z = i_{T,S}(w)$. Since $i_{T,S}^{-1}(\pi^j \overline{M_T}) = \pi^j \overline{M_S}$, there is $y \in \overline{M_S}$ with $\pi^j \cdot y = w$. Then $\pi^j z = \pi^j i_{T,S}(y)$. This implies $z = i_{T,S}(y)$ because $\overline{M_T}$ is torsion-free. This proves the claim. \square

PROPOSITION 4.9. *Let M be a bornological V -module. Then $F \otimes \overline{M} \cong \overline{F \otimes M}$ with an isomorphism compatible with the canonical maps from M to both spaces.*

PROOF. The canonical map $M \rightarrow \overline{M}$ is the universal arrow from M to a complete V -module. The canonical map $M \rightarrow F \otimes M$ is the universal arrow from M to a bornological F -vector space by Lemma 2.12. Since $F \otimes \overline{M}$ is again complete, the canonical map $M \rightarrow F \otimes \overline{M}$ is the universal arrow from M to a complete bornological F -vector space. The completion $\overline{F \otimes M}$ is also a bornological F -vector space. The canonical map $M \rightarrow \overline{F \otimes M}$ is another universal arrow from M to a complete bornological F -vector space. Since the universal property determines its target uniquely up to canonical isomorphism, there is a unique isomorphism $F \otimes \overline{M} \cong \overline{F \otimes M}$ that makes the following diagram commute:

$$\begin{array}{ccc} F \otimes \overline{M} & \xrightarrow{\cong} & \overline{F \otimes M} \\ & \swarrow & \searrow \\ & M & \end{array} \quad \square$$

COROLLARY 4.10. *If M is bornologically torsion-free, then the canonical map $\overline{M} \rightarrow \overline{F \otimes M}$ is a bornological embedding.*

PROOF. Use the isomorphism $F \otimes \overline{M} \cong \overline{F \otimes M}$ to replace the canonical map $\overline{M} \rightarrow \overline{F \otimes M}$ by the canonical map $\overline{M} \rightarrow F \otimes \overline{M}$. This is a bornological embedding if and only if \overline{M} is bornologically torsion-free by Proposition 4.3. And this is true by Theorem 4.6. \square

Finally, we show that being bornologically torsion-free is compatible with linear growth bornologies:

PROPOSITION 4.11. *If A is a bornologically torsion-free V -algebra, then so is A_{lg} .*

PROOF. Let $S \subseteq \pi \cdot A$ be bounded in A_{lg} . Then there is $T \in \mathcal{B}_A$ with $S \subseteq T_1 := \sum_{i=0}^{\infty} \pi^i T^{i+1}$ by Lemma 3.6. The subset $T_2 := \sum_{i=0}^{\infty} \pi^i T^{i+2}$ also has linear growth. And

$$T_1 = T + \sum_{i=1}^{\infty} \pi^i T^{i+1} = T + \sum_{i=0}^{\infty} \pi^{i+1} T^{i+2} = T + \pi T_2.$$

Since T is bounded in A and A is bornologically torsion-free, $\pi^{-1} \cdot T := \{x \in A : \pi \cdot x \in T\}$ is also bounded. We have $\pi^{-1} S \subseteq \pi^{-1} T_1 \subseteq \pi^{-1} \cdot T + T_2$. This is bounded in A_{lg} . \square

The following proposition answers a question by Guillermo Cortiñas:

PROPOSITION 4.12. *Let M and N be bornological V -modules. If M and N are bornologically torsion-free, then so is $M \otimes N$ with the tensor product bornology.*

PROOF. Since M and N are torsion-free, so is $M \otimes N$, that is, multiplication by π on $M \otimes N$ is injective. Let $U \subseteq M \otimes N$ be a subset such that πU is bounded. We must show that U is bounded. By the definition of the tensor product bornology, there are bounded V -submodules $S \subseteq M$, $T \subseteq N$ such that $\pi \cdot U \subseteq S \otimes T$. Define

$$\pi^{-1} S := \{x \in M : \pi x \in S\}, \quad \pi^{-1} T := \{y \in N : \pi y \in T\}.$$

These subsets are bounded because $\pi \cdot (\pi^{-1} S) \subseteq S$ and $\pi \cdot (\pi^{-1} T) \subseteq T$ and M and N are bornologically torsion-free. We claim that $U \subseteq \pi^{-1} S \otimes \pi^{-1} T$. This shows that U is bounded.

Let $u \in U$. We may write $u = \sum_{j=1}^N x_j \otimes y_j$ with $x_j \in M$, $y_j \in N$. Since $\pi \cdot u \in S \otimes T$, we may write $\pi u = \sum_{k=1}^M z_k \otimes w_k$ with $z_k \in S$, $w_k \in T$. Let $A \subseteq M$ and $B \subseteq N$ be the V -submodules generated by the elements x_j, z_k and y_j, w_k , respectively. These submodules are finitely generated and torsion-free, hence free. And the canonical map $A \otimes B \rightarrow M \otimes N$ is injective. The submodules $A \cap S$ and $B \cap T$ are also free. Any V -module homomorphism between finitely generated free V -modules may be brought into diagonal form with entries in $\{\pi^{\mathbb{N}}\} \cup \{0\}$ along the diagonal by choosing appropriate bases in the V -modules. Therefore, there are V -module bases a_1, \dots, a_n and b_1, \dots, b_m of A and B , respectively, and $1 \leq n' \leq n$, $1 \leq m' \leq m$, $0 \geq \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{n'}$, $0 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_{m'}$, such that $\pi^{\alpha_i} \cdot a_i$ and $\pi^{\beta_j} \cdot b_j$ for $1 \leq i \leq n'$ and $1 \leq j \leq m'$ are V -module bases of $A \cap S$ and $B \cap T$, respectively. We may write $u \in A \otimes B$ uniquely in this basis as $u = \sum_{i,j} u_{i,j} a_i \otimes b_j$ with $u_{i,j} \in V$. By assumption, $\pi \cdot u = \sum_{k=1}^M y_k \otimes w_k \in (S \cap A) \otimes (T \cap B)$. By construction, the elements $\pi^{\alpha_i + \beta_j} a_i \otimes b_j$ form a V -module basis of $(S \cap A) \otimes (T \cap B)$. Since the coefficients of πu in the basis $a_i \otimes b_j$ of $A \otimes B$ are unique, it follows that $u_{i,j} = 0$ if $i > n'$ or $j > m'$, and $\pi u_{i,j} \in \pi^{\alpha_i + \beta_j} V$ for $1 \leq i \leq n'$ and $1 \leq j \leq m'$. Hence u is a V -linear combination

of $\pi^{(\alpha_i-1)+}a_i \otimes \pi^{(\beta_j-1)+}b_j$ for $1 \leq i \leq n'$, $1 \leq j \leq m'$, where $n_+ := \max\{n, 0\}$. Since $\pi^{(\alpha_i-1)+}a_i \in \pi^{-1}S$, $\pi^{(\beta_j-1)+}b_j \in \pi^{-1}T$, this implies $u \in \pi^{-1}S \otimes \pi^{-1}T$. Since $u \in U$ was arbitrary, we get $U \subseteq \pi^{-1}S \otimes \pi^{-1}T$. \square

Theorem 4.6 and Proposition 4.12 imply that bornological torsion-freeness for complete bornological V -modules is hereditary for completed tensor products.

5. Dagger algebras

DEFINITION 5.1. A *dagger algebra* is a complete, bornologically torsion-free, semi-dagger algebra.

THEOREM 5.2. Let $A \xrightarrow{i} B \xrightarrow{p} C$ be an extension of bornological V -algebras. If A and C are dagger algebras, so is B .

PROOF. All three properties defining dagger algebras are hereditary for extensions by Theorems 2.3 (because C is torsion-free), 3.7 and 4.5. \square

We have already seen that there are universal arrows $A \rightarrow A_{\text{tf}} \subseteq F \otimes A$, $A \rightarrow A_{\text{lg}}$, $A \rightarrow \overline{A}$ from a bornological algebra A to a bornologically torsion-free algebra, to a semi-dagger algebra, and to a complete bornological algebra, respectively. We now combine them to a universal arrow to a dagger algebra:

THEOREM 5.3. Let A be a bornological algebra. Then the canonical map from A to $A^\dagger := \overline{(A_{\text{tf}})_{\text{lg}}}$ is the universal arrow from A to a dagger algebra. That is, any bounded algebra homomorphism from A to a dagger algebra factors uniquely through A^\dagger . If A is already bornologically torsion-free, then $A^\dagger \cong \overline{A_{\text{lg}}}$.

PROOF. The bornological algebra A^\dagger is complete by construction. It is semi-dagger by Proposition 3.8. And it is bornologically torsion-free by Proposition 4.11 and Theorem 4.6. So it is a dagger algebra. Let B be a dagger algebra. A bounded homomorphism $A \rightarrow B$ factors uniquely through a bounded homomorphism $A_{\text{tf}} \rightarrow B$ by Proposition 4.4 because B is bornologically torsion-free. This factors uniquely through a bounded homomorphism $(A_{\text{tf}})_{\text{lg}} \rightarrow B$ because B is semi-dagger. And this factors uniquely through a bounded homomorphism $\overline{(A_{\text{tf}})_{\text{lg}}} \rightarrow B$ because B is complete. So A^\dagger has the asserted universal property. If A is bornologically torsion-free, then $A \cong A_{\text{tf}}$ and hence $A^\dagger \cong \overline{A_{\text{lg}}}$. \square

DEFINITION 5.4. We call A^\dagger the *dagger completion* of the bornological V -algebra A .

6. Dagger completions of monoid algebras

As a simple illustration, we describe the dagger completions of monoid algebras. The monoid algebra of \mathbb{N}^j is the algebra of polynomials in j variables, and its dagger completion is the Monsky–Washnitzer algebra of overconvergent power series equipped with a canonical bornology (see [11]). The case of general monoids is similar.

The monoid algebra $V[S]$ of S over V is defined by its universal property: if B is a unital V -algebra, then there is a natural bijection between algebra homomorphisms $V[S] \rightarrow B$ and monoid homomorphisms $S \rightarrow (B, \cdot)$ into the multiplicative monoid

of B . More concretely, $V[S]$ is the free V -module with basis S or, equivalently, the V -module of formal linear combinations of the form

$$\sum_{s \in S} x_s \delta_s, \quad x_s \in V, \quad s \in S,$$

with $x_s = 0$ for all but finitely many s , and equipped with the multiplication

$$\sum_{s \in S} x_s \delta_s * \sum_{t \in S} y_t \delta_t = \sum_{s, t \in S} x_s y_t \delta_{s \cdot t}.$$

We give $V[S]$ the fine bornology. Then it has an analogous universal property in the category of bornological V -algebras. So the dagger completion $V[S]^\dagger$ is a dagger algebra with the property that bounded algebra homomorphisms $V[S]^\dagger \rightarrow B$ for a dagger algebra B are in natural bijection with monoid homomorphisms $S \rightarrow (B, \cdot)$.

Assume first that S has a finite generating set F . Let $F^n \subseteq S$ be the set of all words $s_1 \cdots s_k$ with $s_1, \dots, s_k \in F$ and $k \leq n$. This gives an increasing filtration on S with $F^0 = \{1\}$ and $S = \bigcup_{n=0}^{\infty} F^n$. For $s \in S$, we define $\ell(s) \in \mathbb{N}$ as the smallest n with $s \in F^n$. This is the *word length* generated by F . Let $V[F^n] \subseteq V[S]$ be the free V -submodule of $V[S]$ spanned by F^n . Any finitely generated V -submodule of $V[S]$ is contained in $V[F^n]$ for some $n \in \mathbb{N}$. By Lemma 3.6, a subset of $V[S]$ has linear growth if and only if it is contained in $M_n := \sum_{j=0}^{\infty} \pi^j (V[F^n])^{j+1}$ for some $n \in \mathbb{N}_{\geq 1}$. That is,

$$M_{\text{lg}} = \varinjlim M_n.$$

Recall the valuation $\nu: V \rightarrow \mathbb{N} \cup \{\infty\}$ defined by

$$\nu(x) := \sup \{n \in \mathbb{N} : x \in \pi^n V\}.$$

By definition, the submodule M_n consists of all finite sums of terms $x_s \delta_s$ with $x_s \in \pi^j \cdot V$ and $\ell(s) \leq n(j+1)$ for some $j \in \mathbb{N}$ or, equivalently, $\ell(s)/n \leq j+1 \leq \nu(x_s) + 1$. That is, M_n contains a finite sum $\sum_{s \in S} x_s \delta_s$ with $x_s \in V$ and $x_s = 0$ for all but finitely many $s \in S$ if and only if $\nu(x_s) + 1 \geq \ell(s)/n$ for all $s \in S$. The π -adic completion \widehat{M}_n of M_n is the set of all formal power series $\sum_{s \in S} x_s \delta_s$ such that $x_s \in V$, $\nu(x_s) + 1 \geq \ell(s)/n$ for all $s \in S$ and $\lim_{\ell(s) \rightarrow \infty} \nu(x_s) + 1 - \ell(s)/n = \infty$. This implies $x_s \rightarrow 0$ in the π -adic norm, so that $\widehat{M}_n \subseteq \widehat{V[S]}$. So the extension $\widehat{M}_n \rightarrow \widehat{M}_{n+1}$ of the inclusion map $M_n \rightarrow M_{n+1}$ remains injective. Therefore, $\varinjlim \widehat{M}_n$ is separated, and it is contained in $\widehat{V[S]}$. Proposition 4.7 implies

$$V[S]^\dagger = \varinjlim \widehat{M}_n.$$

Elements of $\widehat{V[S]}$ are formal series $\sum_{s \in S} x_s \delta_s$ with $x_s \in V$ for all $s \in S$ and $\lim |x_s| = 0$. We have seen above that such a formal series belongs to \widehat{M}_n if and only if $\nu(x_s) + 1 \geq \ell(s)/n$ for all $s \in S$ and $\lim_{\ell(s) \rightarrow \infty} \nu(x_s) + 1 - \ell(s)/n = \infty$. If $0 < 1/n < c$, then $\nu(x_s) + 1 \geq c\ell(s)$ implies $\nu(x_s) + 1 \geq \ell(s)/n$ and $\lim_{\ell(s) \rightarrow \infty} \nu(x_s) + 1 - \ell(s)/n = \infty$. Thus all $\sum_{s \in S} x_s \delta_s \in \widehat{V[S]}$ with $\nu(x_s) + 1 \geq c\ell(s)$ belong to \widehat{M}_n . Conversely, all elements of \widehat{M}_n satisfy this for $c = 1/n$. Letting c and n vary, we see that $V[S]^\dagger$ is the set of all $\sum_{s \in S} x_s \delta_s$ in $\widehat{V[S]}$ for which there is $c > 0$ with

$$(6.1) \quad \nu(x_s) + 1 \geq c\ell(s) \quad \text{for all } s \in S,$$

and that a subset of $V[S]^\dagger$ is bounded if and only if all its elements satisfy (6.1) for the same $c > 0$. The growth condition (6.1) does not depend on the word length function ℓ because the word length functions for two different generating sets of S are related by linear inequalities $\ell \leq a\ell'$ and $\ell' \leq a\ell$ for some $a > 0$.

Now we drop the assumption that S be finitely generated. Then we may write S as the increasing union of its finitely generated submonoids. By the universal property, the monoid algebra of S with the fine bornology is a similar inductive limit in the category of bornological V -algebras, and its dagger algebra is the inductive limit in the category of dagger algebras. Since $V[S']^\dagger \subseteq \overline{V[S']} \subseteq \overline{V[S]}$ for any finitely generated $S' \subseteq S$, we may identify this inductive limit with a subalgebra of $\overline{V[S]}$ as well, namely, the union of $V[S']^\dagger$ over all finitely generated submonoids $S' \subseteq S$. That is, $V[S]^\dagger$ is the set of elements of $\overline{V[S]}$ that are supported in some finitely generated submonoid $S' \subseteq S$ and that satisfy (6.1) for some length function on S' .

We may also twist the monoid algebra. Let $V^\times = \{x \in V : |x| = 1\}$ and let $c: S \times S \rightarrow V^\times$ be a normalised 2-cocycle, that is,

$$(6.2) \quad c(r, s \cdot t) \cdot c(s, t) = c(r \cdot s, t) \cdot c(r, s), \quad c(s, 1) = c(1, s) = 1$$

for all $r, s, t \in S$. The c -twisted monoid algebra of S , $V[S, c]$, is the V -module $V[S]$ with the twisted multiplication

$$(6.3) \quad \sum_{s \in S} x_s \delta_s * \sum_{t \in S} y_t \delta_t = \sum_{s, t \in S} x_s y_t c(s, t) \cdot \delta_{s \cdot t}.$$

The condition (6.2) is exactly what is needed to make this associative and unital with unit δ_1 . Since we assume c to have values in V^\times , the twist does not change the linear growth bornology. Therefore, the dagger completion $V[S, c]^\dagger$ consists of all infinite sums $\sum_{s \in S} x_s \delta_s$ that are supported in a finitely generated submonoid of S and satisfy the growth condition (6.1), and a subset is bounded if and only if all its elements satisfy these two conditions uniformly. Only the multiplication changes and is now given by (6.3).

EXAMPLE 6.4. Let $S = (\mathbb{Z}^2, +)$ with the unit element 0. Define $c((s_1, s_2), (t_1, t_2)) := \lambda^{s_2 \cdot t_1}$ for some $\lambda \in V^\times$. This satisfies (6.2). The resulting twisted convolution algebra is an analogue of a noncommutative torus over V . Indeed, let $U_1 := \delta_{(1,0)}$ and $U_2 := \delta_{(0,1)}$ as elements of $V[\mathbb{Z}^2, c]$. Then $\delta_{(-1,0)} = U_1^{-1}$ and $\delta_{(0,-1)} = U_2^{-1}$ are inverse to them, and $\delta_{(s_1, s_2)} = U_1^{s_1} \cdot U_2^{s_2}$. So U_1, U_2 generate $V[\mathbb{Z}^2, c]$ as a V -algebra. They satisfy the commutation relation

$$(6.5) \quad U_2 \cdot U_1 = \lambda \cdot U_1 \cdot U_2.$$

And this already dictates the multiplication table in $V[\mathbb{Z}^2, c]$. The dagger completion $V[\mathbb{Z}^2, c]^\dagger$ is isomorphic as a bornological V -module to the Monsky–Washnitzer completion of the Laurent polynomial algebra $V[U_1^{\pm 1}, U_2^{\pm 1}]$, equipped with a twisted multiplication satisfying (6.5).

7. Dagger completions of crossed products

Let A be a unital, bornological V -algebra, let S be a finitely generated monoid and let $\alpha: S \rightarrow \text{End}(A)$ be an action of S on A by bounded algebra homomorphisms. The *crossed product* $A \rtimes_\alpha S$ is defined as follows. Its underlying bornological V -module is $A \rtimes_\alpha S = \bigoplus_{s \in S} A$ with the direct sum bornology. So elements of $A \rtimes_\alpha S$ are formal linear combinations $\sum_{s \in S} a_s \delta_s$ with $a_s \in A$ and $a_s = 0$ for all but finitely many $s \in S$. The multiplication on $A \rtimes_\alpha S$ is defined by

$$\left(\sum_{s \in S} a_s \delta_s \right) \cdot \left(\sum_{t \in S} b_t \delta_t \right) := \sum_{s, t \in S} a_s \alpha_s(b_t) \delta_{st}.$$

This makes $A \rtimes_\alpha S$ a bornological V -algebra. What is its dagger completion?

It follows easily from the universal property that defines $A \subseteq A \rtimes_{\alpha} S$ that

$$(A \rtimes_{\alpha} S)^{\dagger} \cong (A^{\dagger} \rtimes_{\alpha^{\dagger}} S)^{\dagger};$$

here α^{\dagger} is the canonical extension of α to the dagger completion A^{\dagger} , which exists because the latter is functorial for bounded algebra homomorphisms. Therefore, it is no loss of generality to assume that A is already a dagger algebra. It is easy to show that $(A \rtimes S)^{\dagger}$ is the inductive limit of the dagger completions $(A \rtimes S')^{\dagger}$, where S' runs through the directed set of finitely generated submonoids of S . Hence we may also assume that S is finitely generated to simplify. First we consider the following special case:

DEFINITION 7.1. The action $\alpha: S \rightarrow \text{End}(A)$ is called *uniformly bounded* if any $U \in \mathcal{B}_A$ is contained in an α -invariant $T \in \mathcal{B}_A$; α -invariance means $\alpha_s(T) = T$ for all $s \in S$.

If T is α -invariant, so is the V -module generated by T . Therefore, α is uniformly bounded if and only if any bounded subset of A is contained in a bounded, α -invariant V -submodule. If A is complete, then the image of \widehat{T} in A is also α -invariant because the maps α_s are bornological isomorphisms. Hence we may assume in this case that T in Definition 7.1 is a bounded, α -invariant π -adically complete V -submodule.

PROPOSITION 7.2. *Let A carry a uniformly bounded action α of S . Then the induced actions on \overline{A} , A_{tf} , and A_{lg} are uniformly bounded as well. Hence so is the induced action on A^{\dagger} .*

PROOF. If α is uniformly bounded, then A is the bornological inductive limit of its α -invariant bounded V -submodules. The action of α restricts to any such submodule T and then extends canonically to its π -adic completion \widehat{T} . Then the image of \widehat{T} in \overline{A} is S -invariant as well. This gives enough S -invariant bounded V -submodules in \overline{A} . So the induced action on \overline{A} is uniformly bounded.

If the action α on A is uniformly bounded, then so is the action $\text{id}_B \otimes \alpha$ on $B \otimes A$ for any bornological algebra B . In particular, the induced action on $F \otimes A$ is uniformly bounded. Since the canonical map $A \rightarrow F \otimes A$ is S -equivariant, the image A_{tf} of A in $F \otimes A$ is S -invariant. The restriction of the uniformly bounded action of S on $F \otimes A$ to this invariant subalgebra inherits uniform boundedness. So the induced action on A_{tf} is uniformly bounded.

Any subset of linear growth in A is contained in $\sum_{j=0}^{\infty} \pi^j T^{j+1}$ for a bounded V -submodule T . Since α is uniformly bounded, T is contained in an α -invariant bounded V -submodule U . Then $\sum_{j=0}^{\infty} \pi^j U^{j+1} \supseteq \sum_{j=0}^{\infty} \pi^j T^{j+1}$ is α -invariant and has linear growth. So α remains uniformly bounded for the linear growth bornology.

The uniform boundedness of the induced action on the dagger completion A^{\dagger} follows from the inheritance properties above and Theorem 5.3. \square

EXAMPLE 7.3. Let S be a finite monoid. Any bounded action of S by bornological algebra endomorphisms is uniformly bounded because we may take $T = \sum_{s \in S} \alpha_s(U)$ in Definition 7.1.

EXAMPLE 7.4. We describe a uniformly bounded action of \mathbb{Z} on the polynomial algebra $A := V[x_1, \dots, x_n]$ with the fine bornology. So a subset of A is bounded if and only if it is contained in $(V + Vx_1 + \dots + Vx_n)^k$ for some $k \in \mathbb{N}_{\geq 1}$. Let $a \in \text{GL}_n(V) \subseteq \text{End}(V^n)$ and $b \in V^n$. Then

$$\alpha_1: V[x_1, \dots, x_n] \rightarrow V[x_1, \dots, x_n], \quad (\alpha_1 f)(x) := f(ax + b),$$

is an algebra automorphism α_1 of A with inverse $(\alpha_1^{-1}f)(x) := f(a^{-1}(x-b))$. This generates an action of the group \mathbb{Z} by $\alpha_n := \alpha_1^n$ for $n \in \mathbb{Z}$. If a polynomial f has degree at most m , then the same is true for $\alpha_1 f$ and $\alpha_{-1} f$, and hence for $\alpha_n f$ for all $n \in \mathbb{Z}$. That is, the V -submodules $(V + Vx_1 + \cdots + Vx_n)^k$ in A for $k \in \mathbb{N}$ are α -invariant. So the action α on A is uniformly bounded. Proposition 7.2 implies that the induced action on $V[x_1, \dots, x_n]^\dagger$ is uniformly bounded as well.

PROPOSITION 7.5. *Let S be a finitely generated monoid with word length function ℓ . Let A be a dagger algebra and let $\alpha: S \rightarrow \text{End}(A)$ be a uniformly bounded action by algebra endomorphisms. Then $(A \rtimes S)^\dagger \subseteq \prod_{s \in S} A$. A formal series $\sum_{s \in S} a_s \delta_s$ with $a_s \in A$ for all $s \in S$ belongs to $(A \rtimes S)^\dagger$ if and only if there are $\varepsilon > 0$ and $T \in \mathcal{B}_A$ with $a_s \in \pi^{[\varepsilon \ell(s)]} T$ for all $s \in S$, and a set of formal series is bounded in $(A \rtimes S)^\dagger$ if and only if $\varepsilon > 0$ and $T \in \mathcal{B}_A$ for its elements may be chosen uniformly.*

PROOF. We first describe the linear growth bornology on $A \rtimes S$. Let \mathcal{B}' be the set of all subsets $U \subseteq A \rtimes S$ for which there are $T \in \mathcal{B}_A$ and $\varepsilon > 0$ such that any element of U is of the form $\sum_{s \in S} a_s \delta_s$ with $a_s \in \pi^{[\varepsilon \ell(s)]} T$ for all $s \in S$. We claim that \mathcal{B}' is the linear growth bornology on $A \rtimes S$. The inclusion $V[S] \subseteq A \rtimes S$ induces a bounded algebra homomorphism $V[S]_{\text{lg}} \rightarrow (A \rtimes S)_{\text{lg}}$. We have already described the linear growth bornology on $V[S]$ in Section 6. This implies easily that all subsets in \mathcal{B}' have linear growth: write $\pi^{[\varepsilon \ell(s)]} a'_s \delta_s = a'_s \cdot \pi^{[\varepsilon \ell(s)]} \delta_s$. We claim, conversely, that any subset of $A \rtimes S$ of linear growth is contained in \mathcal{B}' . All bounded subsets of $A \rtimes S$ are contained in \mathcal{B}' . It is routine to show that \mathcal{B}' is a V -algebra bornology. We only prove that the bornology \mathcal{B}' has linear growth. Since α is uniformly bounded, any $T \in \mathcal{B}_A$ is contained in a bounded, α -invariant V -submodule T_2 . Then $T_3 := \sum_{j=0}^{\infty} \pi^j T_2^{j+1}$ is a bounded, α -invariant V -submodule with $\pi \cdot T_3^2 \subseteq T_3$ and $T \subseteq T_3$ (see [11, Equation (5)]). If $a_s \in \pi^{[\varepsilon \ell(s)]} T_3$ and $a_t \in \pi^{[\varepsilon \ell(t)]} T_3$, then

$$\pi^2 \cdot a_s \cdot \alpha_t \in \pi^{2+[\varepsilon \ell(s)]+[\varepsilon \ell(t)]} T_3^2 \subseteq \pi^{[\varepsilon \ell(s+t)]} \pi T_3^2 \subseteq \pi^{[\varepsilon \ell(s+t)]} T_3$$

because $1 + [\varepsilon \ell(s)] + [\varepsilon \ell(t)] \geq [\varepsilon \ell(s+t)]$. This implies

$$\pi^2 \cdot \sum_{s \in S} \pi^{[\varepsilon \ell(s)]} T_3 \delta_s * \sum_{t \in S} \pi^{[\varepsilon \ell(t)]} T_3 \delta_t \subseteq \sum_{s, t \in S} \pi^{[\varepsilon \ell(s+t)]} T_3 \delta_{st} \subseteq \sum_{s \in S} \pi^{[\varepsilon \ell(s)]} T_3 \delta_s.$$

So any subset in \mathcal{B}' is contained in $U \in \mathcal{B}'$ with $\pi^2 \cdot U^2 \subseteq U$. By induction, this implies $(\pi^2 U)^k \cdot U \subseteq U$ for all $k \in \mathbb{N}$. Hence $\sum_{j=0}^{\infty} \pi^{2k} U^{k+1}$ is in \mathcal{B}' . Now Lemma 3.6 shows that the bornology \mathcal{B}' is semi-dagger. This proves the claim that \mathcal{B}' is the linear growth bornology on $A \rtimes S$.

Since A as a dagger algebra is bornologically torsion-free, so is $A \rtimes S$. So $(A \rtimes S)^\dagger$ is the completion of $(A \rtimes S)_{\text{lg}} = (A \rtimes S, \mathcal{B}')$. It is routine to identify this completion with the bornological V -module described in the statement. \square

Propositions 7.2 and 7.5 describe the dagger completion of $A \rtimes S$ for a uniformly bounded action of S on A even if A is not a dagger algebra. Namely, the universal properties of the crossed product and the dagger completion imply

$$(7.6) \quad (A \rtimes S)^\dagger \cong (A^\dagger \rtimes S)^\dagger.$$

EXAMPLE 7.7. Let α be the uniformly bounded action of \mathbb{Z} on $V[x_1, \dots, x_k]$ from Example 7.4. The induced action α^\dagger on $V[x_1, \dots, x_k]^\dagger$ is also uniformly bounded by Proposition 7.2. And (7.6) implies

$$(V[x_1, \dots, x_k] \rtimes_\alpha \mathbb{Z})^\dagger \cong (V[x_1, \dots, x_k]^\dagger \rtimes_{\alpha^\dagger} \mathbb{Z})^\dagger.$$

The latter is described in Proposition 7.5. Namely, $(V[x_1, \dots, x_k]^\dagger \rtimes_{\alpha^\dagger} \mathbb{Z})^\dagger$ consists of those formal series $\sum_{n \in \mathbb{Z}} a_n \delta_n$ with $a_n \in V[x_1, \dots, x_k]^\dagger$ for which there are $\varepsilon > 0$ and a bounded V -submodule T in $V[x_1, \dots, x_k]^\dagger$ such that $a_n \in \pi^{|\varepsilon||n|} T$ for all $n \in \mathbb{Z}$; notice that $|n|$ is indeed a length function on \mathbb{Z} . And a subset is bounded if some pair ε, T works for all its elements.

We combine this with the description of bounded subsets of $V[x_1, \dots, x_k]^\dagger$ in Section 6: there is some $\delta > 0$ so that a formal power series $\sum_{m \in \mathbb{N}^k} b_m x^m$ belongs to T if and only if $b_m \in \pi^{|\delta||m|} V$ for all $m \in \mathbb{N}^k$. Here we use the length function $|(m_1, \dots, m_k)| = \sum_{j=1}^k m_j$. We may merge the parameters $\varepsilon, \delta > 0$ above, taking their minimum. So $(V[x_1, \dots, x_k] \rtimes \mathbb{Z})^\dagger$ consists of the formal series $\sum_{n \in \mathbb{Z}, m \in \mathbb{N}^k} a_{n,m} x^m \delta_n$ with $a_{n,m} \in \pi^{[\varepsilon(|n|+|m|)]} V$ or, equivalently, $\nu(a_{n,m}) + 1 > \varepsilon(|n| + |m|)$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}^k$.

If the action of S on A is not uniformly bounded, then the linear growth bornology on $A \rtimes S$ becomes much more complicated. It seems unclear whether the description below helps much in practice. Let $F \subseteq S$ be a finite generating subset containing 1. Any bounded subset of $A \rtimes S$ is contained in $(\sum_{s \in F} T \cdot \delta_s)^N$ for some $N \in \mathbb{N}$ and some $T \in \mathcal{B}_A$ with $1 \in T$. Therefore, a subset of $A \rtimes S$ has linear growth if and only if it is contained in the V -submodule generated by

$$\bigcup_{n=1}^{\infty} \pi^{[\varepsilon n]} (T \cdot \{\delta_s : s \in F\})^n$$

for some $\varepsilon > 0$, $T \in \mathcal{B}_A$. Using the definition of the convolution, we may rewrite the latter set as

$$\bigcup_{n=1}^{\infty} \bigcup_{s_1, \dots, s_n \in F} \pi^{[\varepsilon n]} \cdot T \cdot \alpha_{s_1}(T) \cdot \alpha_{s_1 s_2}(T) \cdots \alpha_{s_1 \cdots s_{n-1}}(T) \delta_{s_1 \cdots s_n}.$$

The resulting V -module is the sum $\sum_{s \in S} U_s \delta_s$, where U_s is the V -submodule of A generated by finite products

$$\{\pi^{[\varepsilon n]} \cdot T \cdot \alpha_{s_1}(T) \cdots \alpha_{s_1 \cdots s_{n-1}}(T) : n \in \mathbb{N}_{\geq 1}, s_1, \dots, s_n \in F, s_1 \cdots s_n = s\}.$$

Here taking a factor $1 \in T$ is allowed. Thus we may leave out a factor $\alpha_{s_1 \cdots s_i}(T)$. This has the same effect as increasing n by 1 and putting $s_i = s_i^1 \cdot s_i^2$ with $s_i^1, s_i^2 \in F$. Since F generates S as a monoid, we may allow arbitrary $s_i \in S$ when we change the exponent of π appropriately. Namely, we must then replace n in the exponent of π by the number of factors in F that are needed to produce the desired elements s_i , which is $\ell_{\geq 1}(s_1) + \cdots + \ell_{\geq 1}(s_n)$, where $\ell_{\geq 1}(1) = 1$ and $\ell_{\geq 1}(s) = \ell(s)$ for $s \in S \setminus \{1\}$. As a result, U_s is the V -submodule of A generated by

$$\begin{aligned} & \pi^{[\varepsilon(\ell_{\geq 1}(s_1) + \cdots + \ell_{\geq 1}(s_n))]} \cdot x_0 \cdot \alpha_{s_1}(x_1) \cdots \alpha_{s_1 \cdots s_{n-1}}(x_{n-1}), \\ & n \in \mathbb{N}_{\geq 1}, x_0, \dots, x_{n-1} \in T, s_1, \dots, s_n \in S, s_1 \cdots s_n = s. \end{aligned}$$

Now assume that S is a group, not just a monoid. Then any sequence of elements $g_1, \dots, g_n \in S$ may be written as $g_i = s_1 \cdots s_i$ by putting $s_i := g_i^{-1} g_i$ with $g_0 := 1$. So U_g is the V -submodule of A generated by

$$\begin{aligned} & \pi^{[\varepsilon(\ell_{\geq 1}(g_0^{-1} g_1) + \ell_{\geq 1}(g_1^{-1} g_2) + \cdots + \ell_{\geq 1}(g_{n-1}^{-1} g_n))]} \cdot \alpha_{g_0}(x_0) \cdot \alpha_{g_1}(x_1) \cdots \alpha_{g_{n-1}}(x_{n-1}), \\ & n \in \mathbb{N}_{\geq 1}, x_0, \dots, x_{n-1} \in T, g_0, \dots, g_n \in S, g_0 = 1, g_n = g. \end{aligned}$$

These subsets U_g for fixed T and ε depend on g in a complicated way. The bornology on $A \rtimes G$ generated by these subsets is, however, also generated by the sets of the form $\sum_{g \in G} \pi^{\varepsilon \ell(g)} \cdot U \delta_g$, where U is the V -submodule of A generated by

$$\pi^{|\varepsilon(\ell_{\geq 1}(g_0^{-1}g_1) + \ell_{\geq 1}(g_1^{-1}g_2) + \dots + \ell_{\geq 1}(g_{n-2}^{-1}g_{n-1}))|} \cdot \alpha_{g_0}(x_0) \cdot \alpha_{g_1}(x_1) \cdots \alpha_{g_{n-1}}(x_{n-1}),$$

$$n \in \mathbb{N}_{\geq 1}, x_0, \dots, x_{n-1} \in T, g_0, \dots, g_{n-1} \in S, g_0 = 1,$$

for some $T \in \mathcal{B}_A$, $\varepsilon > 0$. The reason is that

$$\ell(g_n) - \ell(g_{n-1}) \leq \ell(g_{n-1}^{-1}g_n) \leq \ell(g_{n-1}) + \ell(g_n)$$

and $\ell(g_{n-1}) \leq \sum_{j=1}^{n-1} \ell(g_{j-1}^{-1}g_j)$. Therefore, replacing the exponents of π as above does not change the bornology on $A \rtimes G$ that is generated by the sets above when $\varepsilon > 0$ varies.

Non-archimedean analytic cyclic homology

ABSTRACT. Let V be a complete discrete valuation ring with fraction field F of characteristic zero and with residue field \mathbb{F} . We introduce analytic cyclic homology of complete torsion-free bornological algebras over V . We prove that it is homotopy invariant, stable, invariant under certain nilpotent extensions, and satisfies excision. We use these properties to compute it for tensor products with dagger completions of Leavitt path algebras. If R is a smooth commutative V -algebra of relative dimension 1, then we identify the analytic cyclic homology of its dagger completion with Berthelot's rigid cohomology of $R \otimes_V \mathbb{F}$.

1. Introduction

Analytic cyclic homology of complete bornological algebras over \mathbb{R} and \mathbb{C} was introduced in [24] as a bivariate generalisation from Banach to bornological algebras of the entire cyclic cohomology defined by Connes [9] and further studied by Khalkhali [22]. It was shown to be stable under tensoring with algebras of nuclear operators and invariant under differentiable homotopies and under analytically nilpotent extensions and to satisfy excision with respect to semi-split extensions [26].

Let V be a complete discrete valuation ring whose fraction field F has characteristic zero. Let π be a uniformiser and let $\mathbb{F} := V/\pi V$ be the residue field. In this article, we define and study an analytic cyclic homology theory for complete, torsion-free bornological V -algebras (see Section 2 for the definitions of these terms). For example, if R is a torsion-free, finitely generated, commutative V -algebra, then its Monsky–Washnitzer dagger completion R^\dagger introduced in [28] is such a complete bornological algebra (see [11, 27]).

We prove that analytic cyclic homology is invariant under dagger homotopies and under certain nilpotent extensions, that it is matrix stable, and that it satisfies excision with respect to semi-split extensions. We use these properties to compute the analytic cyclic homology for dagger completed Leavitt and Cohn path algebras of countable graphs. For finite graphs, we also compute the analytic cyclic homology for tensor products with such algebras. In particular, it follows that the analytic cyclic homology of the completed tensor product of R with $V[t, t^{-1}]^\dagger$ is isomorphic to the direct sum $\mathrm{HA}_*(R) \oplus \mathrm{HA}_*(R)[1]$, where HA_* denotes analytic cyclic homology. This is a variant of the fundamental theorem in algebraic K-theory.

We also compute $\mathrm{HA}_*(R^\dagger)$ for a smooth, commutative V -algebra R of relative dimension 1. Namely, it is isomorphic to the de Rham cohomology of R^\dagger . If \mathbb{F} has finite characteristic, then this agrees with Berthelot's rigid cohomology of $R \otimes \mathbb{F}$ (see [11]). Partial results that we have for smooth, commutative V -algebras of higher dimension have not been included because we have not been able to prove that analytic and periodic cyclic homology coincide in this generality.

Monsky–Washnitzer cohomology and Berthelot's rigid cohomology are defined for varieties in finite characteristic by lifting them to characteristic zero. In order to

define analogous theories for noncommutative \mathbb{F} -algebras, it is natural to replace de Rham cohomology by cyclic homology. Indeed, in [11], Berthelot’s rigid cohomology for commutative \mathbb{F} -algebras is linked to the *periodic* cyclic homology of suitable dagger completed commutative V -algebras. When we allow noncommutative algebras, however, then the dagger completion process forces us to replace periodic cyclic homology by the analytic cyclic homology that is studied here.

In work in progress, we are going to use the theory defined in this article in order to define an analytic cyclic homology theory for algebras over the residue field \mathbb{F} . We want to prove $\mathrm{HA}_*(A) \cong \mathrm{HA}_*(R^\dagger)$ whenever R is a torsion-free V -algebra and $A \cong R/\pi R$ is its reduction to an \mathbb{F} -algebra. The crucial point is that this should not depend on the choice of R – and this is where we need analytic instead of periodic cyclic homology.

All theorems in this paper require the fraction field F to have characteristic 0. This is needed for the homotopy invariance of analytic cyclic homology because the proof involves integration of polynomials. Our proofs of the excision theorem and of matrix stability use characteristic 0 indirectly because they are based on homotopy invariance. Variants of periodic cyclic homology such as (negative) cyclic homology are not homotopy invariant. This is why our methods do not apply to these theories.

Several groups of authors have recently been studying cohomology theories for varieties in finite characteristic with different approaches. We mention, in particular, the work of Petrov and Vologodsky [29] that uses topological cyclic homology.

This paper is organised as follows. Some notational conventions used throughout the article are reviewed at the end of this introduction.

In Section 2, we recall some basic notions from bornological analysis and from the Cuntz–Quillen approach to cyclic homology theories. In particular, we introduce dagger completions relative to an ideal (Section 2.2) and review the appropriate notions of extension of bornological modules, noncommutative differential forms, tensor algebra, and X -complex for bornological algebras.

Section 3 introduces the analytic cyclic pro-complex $\mathbb{H}\mathbb{A}(R)$ of a complete, torsion-free bornological algebra R . It is defined as the X -complex of the scalar extension $\mathcal{T}R \otimes_V F$ of a certain projective system $\mathcal{T}R$ of complete bornological V -algebras functorially associated to R . Hence, by definition, $\mathbb{H}\mathbb{A}(R)$ is a pro-supercomplex (that is, a projective system of $\mathbb{Z}/2$ -graded chain complexes) of complete bornological vector spaces over F . The analytic cyclic homology of R is defined as the homology of the homotopy limit of $\mathbb{H}\mathbb{A}(R)$,

$$\mathrm{HA}_*(R) := H_*(\mathrm{holim} \mathbb{H}\mathbb{A}(R)).$$

By definition, this is a $\mathbb{Z}/2$ -graded bornological vector space over F .

The results about excision, homotopy invariance and matrix stability in this article are all about $\mathbb{H}\mathbb{A}$ as a functor to the homotopy category of chain complexes of projective systems of complete bornological F -vector spaces. Here “homotopy category” means that we take chain homotopy classes of chain maps as arrows. It seems, however, that we must pass to a suitable derived category for results that compare $\mathbb{H}\mathbb{A}$ for two different liftings of an algebra over the residue field \mathbb{F} . We do not discuss here which weak equivalences must be inverted in order to get a well defined theory for algebras over \mathbb{F} .

Section 4 is concerned with analytic nilpotence. Analytically nilpotent pro-algebras and analytically nilpotent extensions of algebras and pro-algebras are introduced. A pro-algebra R is called analytically quasi-free if every semi-split, analytically nilpotent extension of R splits. In particular, the analytic tensor pro-algebra $\mathcal{T}R$ (see Definition 4.4.1) is analytically quasi-free and is part of a semi-split, analytically nilpotent extension

$$\mathcal{J}R \twoheadrightarrow \mathcal{T}R \twoheadrightarrow R.$$

We define dagger homotopy of (pro-)algebra homomorphisms using the dagger completion $V[t]^\dagger$, and we show that any semi-split analytically nilpotent extension $N \twoheadrightarrow E \twoheadrightarrow R$ with analytically quasi-free E is dagger homotopy equivalent to the extension above. We use this and the invariance of the X -complex under dagger homotopies to show that HA is invariant under dagger homotopies. This implies that HA is invariant under analytically nilpotent extensions and that $\mathbb{H}\mathbb{A}(R)$ is homotopy equivalent to $X(R \otimes F)$ if R is analytically quasi-free.

Section 5 is devoted to the proof of the Excision Theorem, which says that if

$$K \xrightarrow{i} E \xrightarrow{p} Q$$

is a semi-split pro-algebra extension, then there is a natural exact triangle

$$\mathbb{H}\mathbb{A}(K) \xrightarrow{i} \mathbb{H}\mathbb{A}(E) \xrightarrow{p} \mathbb{H}\mathbb{A}(Q) \xrightarrow{\delta} \mathbb{H}\mathbb{A}(K)[-1].$$

Applying the homotopy projective limit and taking homology, this implies a natural 6-term exact sequence

$$\begin{array}{ccccc} \mathrm{HA}_0(K) & \xrightarrow{i_*} & \mathrm{HA}_0(E) & \xrightarrow{p_*} & \mathrm{HA}_0(Q) \\ \delta \uparrow & & & & \downarrow \delta \\ \mathrm{HA}_1(Q) & \xleftarrow{p_*} & \mathrm{HA}_1(E) & \xleftarrow{i_*} & \mathrm{HA}_1(K). \end{array}$$

The proof of the excision theorem follows the structure of its archimedean version in [25, 26], and adapts it to the present case.

The stability of HA under matricial embeddings is proved in Section 6. Any pair X, Y of torsion-free bornological V -modules with a surjective bounded linear map $\langle \cdot, \cdot \rangle: Y \otimes X \rightarrow V$ gives rise to a dagger algebra $\overline{\mathcal{M}}(X, Y)$ with underlying bornological V -module $X \overline{\otimes} Y$. We show in Proposition 6.2 that HA is invariant under tensoring with $\overline{\mathcal{M}}(X, Y)$. For example, the algebra of finite matrices \mathbb{M}_n with $n \leq \infty$ and the algebra of matrices with entries going to zero at infinity are of the form $\overline{\mathcal{M}}(X, Y)$ for suitable X and Y . Thus HA is invariant under tensoring with such algebras. This implies that HA for unital algebras is functorial for certain bimodules and invariant under Morita equivalence (see Section 7).

Section 8 is concerned with Leavitt path algebras. For a directed graph E with finitely many vertices and a complete bornological algebra R , Theorems 8.1 and 8.3 compute $\mathbb{H}\mathbb{A}(R \overline{\otimes} L(E)^\dagger)$ in terms of $\mathbb{H}\mathbb{A}(R)$ and a matrix N_E related to the incidence matrix of E :

$$\mathbb{H}\mathbb{A}(R \overline{\otimes} L(E)^\dagger) \simeq (\mathrm{coker}(N_E) \oplus \mathrm{ker}(N_E)[1]) \otimes \mathbb{H}\mathbb{A}(R).$$

For trivial R , the homotopy equivalence $\mathbb{H}\mathbb{A}(L(E)^\dagger) \simeq (\mathrm{coker}(N_E) \oplus \mathrm{ker}(N_E)[1])$ is shown also for graphs with countably many vertices. If E is the graph with one

vertex and one loop, it follows that HA satisfies a version of Bass' fundamental theorem:

$$\mathbb{H}\mathbb{A}(R \overline{\otimes} V[t, t^{-1}]^\dagger) \simeq \mathbb{H}\mathbb{A}(R) \oplus \mathbb{H}\mathbb{A}(R)[-1].$$

We also compute $\mathbb{H}\mathbb{A}(R \overline{\otimes} C(E)^\dagger)$ for the Cohn path algebra of E if E has finitely many vertices, and $\mathbb{H}\mathbb{A}(C(E)^\dagger)$ if E has countably many vertices.

In Section 9 we show that if R is smooth commutative of relative dimension one, then the analytic cyclic homology of its dagger completion is the same as the rigid cohomology of its reduction modulo π (see Theorem 9.2.9). That is,

$$\mathrm{HA}_n(R^\dagger) \cong H_{\mathrm{rig}}^n(R/\pi R)$$

for $n = 0, 1$. We outline the idea of the proof. By [11], $H_{\mathrm{rig}}^n(R/\pi R)$ is isomorphic to the periodic cyclic homology of $R^\dagger \otimes F$. By Corollary 4.7.2, HA and $\mathrm{HP}(\cdot \otimes F)$ agree on analytically quasi-free bornological V -algebras. It is well known that a smooth algebra R of relative dimension 1 is quasi-free in the sense that any square-zero extension of R splits or, equivalently, that the bimodule $\Omega^1(R)$ of noncommutative differential 1-forms admits a connection. We show in Theorem 9.1.9 that if R is a torsion-free, complete bornological algebra and ∇ is a connection on $\Omega^1(R)$ that satisfies an extra condition, then R^\dagger is analytically quasi-free. We prove that a smooth commutative algebra of relative dimension 1 with the fine bornology admits such a connection (see Lemma 9.2.3).

Acknowledgement. A first draft of this manuscript was completed during a 6-month stay of the first author at the Universität Münster, supported by a Humboldt Research Award. He thanks his host, Joachim Cuntz and all the members of the Operator Algebra group, particularly its head, Wilhelm Winter, for their wonderful hospitality. He also thanks the Humboldt Foundation for its support.

1.1. Some notation. Throughout this article, we shall use the following notation. Let \mathbb{N}^* be the set of nonzero natural numbers. Let V be a complete discrete valuation ring, $\pi \in V$ a uniformiser, \mathbb{F} the residue field $V/(\pi)$ of V , and F the fraction field of V . While our definitions work in complete generality, our homotopy invariance, stability and excision theorems only work if F has characteristic zero. All tensor products \otimes are taken over V . By convention, algebras are allowed to be non-unital throughout this article. An *ideal* in a possibly non-unital V -algebra means a two-sided ideal that is also a V -submodule.

2. Preparations

2.1. Bornologies. As in [11], bornological V -algebras play a crucial role. We first recall some basic terminology about bornologies from [11, 27].

DEFINITION 2.1.1. A *bornology* on a set S is a set \mathcal{B} of subsets, called *bounded subsets*, such that finite unions and subsets of bounded subsets are bounded and finite subsets are bounded. A *bornological set* is a set with a bornology.

DEFINITION 2.1.2. A map $f: S_1 \rightarrow S_2$ between bornological sets is *bounded* if it maps bounded subsets to bounded subsets. It is a *bornological embedding* if it is injective and $T \subseteq S_1$ is bounded if and only if $f(T) \subseteq S_2$ is bounded. It is a *bornological quotient map* if it is bounded and any bounded subset $T \subseteq S_2$ is the image of a bounded subset of S_1 .

DEFINITION 2.1.3. A *bornological V -module* is a V -module R with a bornology such that any bounded subset is contained in a bounded V -submodule or, equivalently, the V -submodule generated by a bounded subset is again bounded. A *bornological V -algebra* is a bornological V -module R with a multiplication $R \times R \rightarrow R$ that is bounded in the sense that $S \cdot T$ is bounded if $S, T \subseteq R$ are bounded.

DEFINITION 2.1.4. A bornological V -module is *complete* if any bounded subset is contained in a bounded V -submodule that is π -adically complete. The *completion* \overline{M} of a bornological V -module M is a complete bornological V -module with a bounded map $M \rightarrow \overline{M}$ that is universal in the sense that any bounded map from M to a complete bornological V -module factors uniquely through it (see [11, Definition 2.14]).

EXAMPLE 2.1.5. Let M be a V -module. The *fine bornology* on M consists of those subsets of M that are contained in a finitely generated V -submodule. It is the smallest V -module bornology on M . It is the only bornology on M if M itself is finitely generated. If R is a V -algebra, then the fine bornology makes it a bornological V -algebra. The fine bornology is automatically complete.

We always equip the fraction field F with the fine bornology.

DEFINITION 2.1.6. Let M_1 and M_2 be bornological V -modules. The *tensor product bornology* on the V -module $M_1 \otimes M_2$ consists of all subsets that are contained in $S_1 \otimes S_2$ for bounded V -submodules $S_j \subseteq M_j$ for $j = 1, 2$. The *complete bornological tensor product* $M_1 \overline{\otimes} M_2$ is defined as the bornological completion of $M_1 \otimes M_2$ with the tensor product bornology.

The universal property of tensor products easily implies the following:

PROPOSITION 2.1.7. *Let M_1, M_2 and N be bornological V -modules. Bounded V -linear maps $M_1 \otimes M_2 \rightarrow N$ are in natural bijection with bounded V -bilinear maps $M_1 \times M_2 \rightarrow N$.*

COROLLARY 2.1.8. *Let M_1, M_2 and N be complete bornological V -modules. Bounded V -linear maps $M_1 \overline{\otimes} M_2 \rightarrow N$ are in natural bijection with bounded V -bilinear maps $M_1 \times M_2 \rightarrow N$.*

EXAMPLE 2.1.9. Continuing Example 2.1.5, let M_1 be a V -module with the fine bornology and let M_2 be a complete bornological V -module. Then the tensor product bornology on $M_1 \otimes M_2$ is already complete because the tensor product of a π -adically complete V -module with a finitely generated V -module is complete. Thus $M_1 \overline{\otimes} M_2 = M_1 \otimes M_2$ in this case. This applies, in particular, if $M_1 = F$. If both M_1 and M_2 carry the fine bornology, then the tensor product bornology on $M_1 \overline{\otimes} M_2 = M_1 \otimes M_2$ is the fine bornology as well.

DEFINITION 2.1.10 ([27, Definition 4.1]). A bornological V -module is (*bornologically*) *torsion-free* if multiplication by π is a bornological embedding.

REMARK 2.1.11. Let M be a bornological V -module. If $S \subseteq M$, then define

$$\pi^{-1}S := \{x \in M : \pi \cdot x \in S\}.$$

This depends on M and not just on S . By definition, M is torsion-free if and only if multiplication by π is injective and $\pi^{-1}S$ is bounded for all bounded subsets $S \subseteq M$.

PROPOSITION 2.1.12 ([27, Proposition 4.3]). *A bornological V -module M is torsion-free if and only if the canonical map $M \rightarrow M \otimes F$ is a bornological embedding.*

EXAMPLE 2.1.13. A V -module M with the fine bornology is torsion-free if and only if M is torsion-free in the usual sense.

DEFINITION 2.1.14. Let M be any bornological V -module and define $M_{\text{tf}} \subseteq M \otimes F$ as the image of the canonical map $M \rightarrow M \otimes F$, equipped with the restriction of the bornology of $M \otimes F$.

PROPOSITION 2.1.15 ([27, Proposition 4.4]). *The canonical map $M \rightarrow M_{\text{tf}}$ is the universal map from M to a torsion-free bornological V -module.*

DEFINITION 2.1.16. A bornological V -algebra R is *semi-dagger* if any bounded subset $S \subseteq R$ is contained in a bounded V -submodule $T \subseteq R$ with $\pi \cdot T \cdot T \subseteq T$ (see [27, Proposition 3.4]). Let R with the bornology \mathcal{B} be a bornological V -algebra. There is a smallest semi-dagger bornology on R that contains \mathcal{B} . It is denoted \mathcal{B}_{lg} and called the *linear growth bornology* on R ; we write R_{lg} for R with the linear growth bornology (see [27, Definition 3.5 and Lemma 3.6]).

DEFINITION 2.1.17. A *dagger algebra* is a bornological V -algebra that is complete, (bornologically) torsion-free, and semi-dagger. The *dagger completion* of a bornological V -algebra R is a dagger algebra R^\dagger with a bounded V -algebra homomorphism $R \rightarrow R^\dagger$ that is universal in the sense that any bounded homomorphism from R to a dagger algebra factors uniquely through it.

THEOREM 2.1.18 ([27, Theorem 5.3]). *If R is already torsion-free, then R^\dagger is the completion of R_{lg} . In general, it is the completion of $(R_{\text{tf}})_{\text{lg}}$.*

EXAMPLE 2.1.19. The dagger completion R^\dagger of a torsion-free, finitely generated, commutative V -algebra is usually defined as the weak completion of R by Monsky and Washnitzer [28]. This agrees with our definition of R^\dagger by [11, Theorem 3.2.1]: the dagger completion of the fine bornology on R is naturally isomorphic to the weak completion of R , equipped with a canonical bornology.

PROPOSITION 2.1.20 ([11, Proposition 3.1.25]). *Let A and B be torsion-free, complete bornological algebras. Then $(A \otimes B)_{\text{lg}} \cong A_{\text{lg}} \otimes B_{\text{lg}}$ and $(A \otimes B)^\dagger \cong A^\dagger \overline{\otimes} B^\dagger$.*

COROLLARY 2.1.21. *A completed tensor product of two dagger algebras is again a dagger algebra.*

PROOF. A completed tensor product is complete by definition. It remains semi-dagger by Proposition 2.1.20, and torsion-free by [27, Proposition 4.12]. \square

2.2. Relative dagger completions. We shall define analytic cyclic homology for torsion-free, complete bornological V -algebras R that need not be dagger algebras. This uses a variant of the linear growth bornology relative to an ideal.

Let R be a V -algebra and let M and N be V -submodules of R . Let $MN \subseteq R$ be the V -submodule generated by all products xy with $x \in M$ and $y \in N$. Let

$$(2.2.1) \quad M^\circ := \sum_{i=0}^{\infty} \pi^i M^{i+1}, \quad M^{(n)} := \sum_{i=1}^n M^i.$$

A subset of R has linear growth if and only if it is contained in M° for some bounded V -submodule M of R (with the present definitions, this is [27, Lemma 3.6]).

LEMMA 2.2.2. *Let R be a V -algebra and let $M, N \subseteq R$ be V -submodules. Then*

$$(1) \quad M^\circ + N^\circ \subseteq (M + N)^\circ;$$

- (2) $M \cdot N^\circ \subseteq ((M \cdot N + N)^{(2)})^\circ$ and $N^\circ \cdot M \subseteq ((N \cdot M + N)^{(2)})^\circ$;
- (3) $\pi \cdot M^\circ \cdot M^\circ \subseteq M^\circ$;
- (4) $M^\circ \cdot N^\circ \subseteq ((M + N)^{(2)})^\circ$;
- (5) $(M^\circ)^\circ = M^\circ$.

PROOF. The definition of M° immediately implies (1). The following computation shows the first assertion of (2):

$$\begin{aligned}
M \cdot N^\circ &= \sum_{i \geq 1} \pi^{2i-1} M N^{2i} + \sum_{i \geq 0} \pi^{2i} M N^{2i+1} \\
&= \sum_{i \geq 1} \pi^{2i-1} (M N) N^{2i-1} + \sum_{i \geq 0} \pi^{2i} (M N) (N^2)^i \\
&\subseteq (M N + N)^\circ + (M N + N^2)^\circ \subseteq ((M N + N)^{(2)})^\circ.
\end{aligned}$$

Similar calculations give the second assertion of (2) and (4). Statement (3) follows because $\pi \cdot \pi^i M^{i+1} \cdot \pi^j M^{j+1} = \pi^{i+j+1} M^{i+j+1+1}$ for all $i, j \in \mathbb{N}$. Then $\pi^i \cdot (M^\circ)^{i+1} \subseteq M^\circ$ follows by induction on i . This implies (5). \square

DEFINITION 2.2.3. Let R be a bornological V -algebra and $I \triangleleft R$ an ideal. Let $\mathcal{B}_{\text{lg}(I)}$ be the set of all subsets of R that are contained in $M + N^\circ$ for bounded V -submodules $M \subseteq R$ and $N \subseteq I$. This is a bornology on R , called the *linear growth bornology relative to I* . Let $R_{\text{lg}(I)}$ be R with this bornology.

EXAMPLE 2.2.4. By definition, $\mathcal{B}_{\text{lg}(0)} = \mathcal{B}$ and $\mathcal{B}_{\text{lg}(R)}$ is the usual linear growth bornology on R . So $R_{\text{lg}(0)} = R$ and $R_{\text{lg}(R)} = R_{\text{lg}}$.

LEMMA 2.2.5. *The bornology $\mathcal{B}_{\text{lg}(I)}$ is an algebra bornology, and its restriction to I is semi-dagger. Let S be a bornological V -algebra. A homomorphism $f: R \rightarrow S$ is bounded for the bornology $\mathcal{B}_{\text{lg}(I)}$ if and only if $f(N)$ has linear growth in S for all bounded subsets $N \subseteq I$ and $f(M)$ is bounded in S for all bounded subsets $M \subseteq R$.*

PROOF. Since I is an ideal, Lemma 2.2.2 implies that $\mathcal{B}_{\text{lg}(I)}$ makes R a bornological V -algebra. And a subset of I belongs to $\mathcal{B}_{\text{lg}(I)}$ if and only if it is contained in N° for some bounded V -submodule $N \subseteq I$. The restriction of $\mathcal{B}_{\text{lg}(I)}$ to I is semi-dagger by Lemma 2.2.2. If M and N are as in Definition 2.2.3, then $f(M + N^\circ) = f(M) + f(N)^\circ$. This is bounded in S if and only if $f(M)$ is bounded and $f(N)$ has linear growth. \square

LEMMA 2.2.6. *Let R be a bornological algebra and let I and J be ideals in R with $I \subseteq J$ and $R/I = (R/I)_{\text{lg}(J/I)}$. Then $R_{\text{lg}(J)} = R_{\text{lg}(I)}$. In particular, if R/I is semi-dagger, then $R_{\text{lg}(I)} = R_{\text{lg}}$.*

PROOF. By Lemma 2.2.5, the bornology $\mathcal{B}_{\text{lg}(J)}$ on R is the smallest one that contains the given bornology and makes J semi-dagger, and similarly for I . And the assumption $R/I = (R/I)_{\text{lg}(J/I)}$ says that $J/I \subseteq R/I$ is semi-dagger in the quotient bornology on R/I . This is the same as the quotient bornology induced by $\mathcal{B}_{\text{lg}(I)}$. [27, Theorem 3.7] says that an extension of semi-dagger algebras remains semi-dagger. This theorem applied to the extension $I \twoheadrightarrow J \twoheadrightarrow J/I$, equipped with the restrictions of the bornology $\mathcal{B}_{\text{lg}(I)}$ on I and J and the resulting quotient bornology on J/I shows that J is semi-dagger also in the bornology $\mathcal{B}_{\text{lg}(I)}$. Then $\mathcal{B}_{\text{lg}(J)} \subseteq \mathcal{B}_{\text{lg}(I)}$. And $\mathcal{B}_{\text{lg}(I)} \subseteq \mathcal{B}_{\text{lg}(J)}$ is trivial. \square

LEMMA 2.2.7. *Let R be a bornological algebra and $I \triangleleft R$ an ideal. If R is torsion-free, then so is $R_{\text{lg}(I)}$.*

PROOF. Let $S \subseteq \pi R$ be a bounded subset in $R_{\text{lg}(I)}$. By definition, there are bounded submodules $M \subseteq R$ and $N \subseteq I$ with $S \subseteq M + N^\circ$. And

$$M + N^\circ = M + N + \sum_{i=1}^{\infty} \pi^i N^{i+1} = M + N + \pi \cdot \left(\sum_{i=0}^{\infty} \pi^i N^{i+2} \right).$$

Since $\pi^i N^{i+2} \subseteq \pi^i (N^{(2)})^{i+1}$ for all $i \geq 0$, the subset $\sum_{i=0}^{\infty} \pi^i N^{i+2}$ belongs to $\mathcal{B}_{\text{lg}(I)}$. Since $M + N$ is bounded in R and R is torsion-free, $\pi^{-1} \cdot (M + N)$ is bounded. Then

$$\pi^{-1} S \subseteq \pi^{-1} (M + N^\circ) \subseteq \pi^{-1} (M + N) + \sum_{i=0}^{\infty} \pi^i N^{i+2} \in \mathcal{B}_{\text{lg}(I)}. \quad \square$$

DEFINITION 2.2.8. Let R be a torsion-free bornological algebra and $I \triangleleft R$ an ideal. The *dagger completion of R relative to I* is the completion $(R, I)^\dagger := \overline{R_{\text{lg}(I)}}$.

We shall never apply (relative) dagger completions when R is not already bornologically torsion-free. In general, the correct definition of the relative dagger completion of (R, I) would be $(R_{\text{tf}}, I_{\text{tf}})^\dagger$, where I_{tf} is identified with its image in R_{tf} (compare Theorem 2.1.18).

PROPOSITION 2.2.9. *Let R and S be torsion-free bornological V -algebras, $I \subseteq R$ an ideal, and $f: R \rightarrow S$ a bounded algebra homomorphism. Assume S to be complete. There is a bounded algebra homomorphism $(R, I)^\dagger \rightarrow S$ extending f – necessarily unique – if and only if $f(M)$ has linear growth for each bounded V -submodule M of I .*

PROOF. Use Lemma 2.2.5 and the universal property of the completion. \square

We know no analogue of Proposition 2.1.20 for relative dagger completions.

2.3. Extensions of bornological modules. An *extension* of V -modules is a diagram of V -modules

$$(2.3.1) \quad K \xrightarrow{i} E \xrightarrow{p} Q$$

that is algebraically exact and such that i is a bornological embedding and p is a bornological quotient map. Equivalently, i is a kernel of p and p is a cokernel of i in the additive category of bornological V -modules. The following elementary lemma says that this category is *quasi-abelian* (see [35]):

LEMMA 2.3.2. *Let (2.3.1) be an extension of bornological V -modules and let $f: K \rightarrow K'$ and $g: Q'' \rightarrow Q$ be bounded V -module maps. The pushout of i, f and the pullback of p, g exist and are part of morphisms of extensions*

$$\begin{array}{ccc} K \xrightarrow{i} E \xrightarrow{p} Q & & K \xrightarrow{i''} E'' \xrightarrow{p''} Q'' \\ \downarrow f & \downarrow \hat{f} & \parallel & \downarrow \hat{g} & \downarrow g \\ K' \xrightarrow{i'} E' \xrightarrow{p'} Q, & & K \xrightarrow{i} E \xrightarrow{p} Q. \end{array}$$

Here

$$E' := \frac{K' \oplus E}{\{(f(k), -i(k)) : k \in K\}}, \quad E'' := \{(e, q'') \in E \times Q'' : p(e) = g(q'')\},$$

equipped with the quotient and the subspace bornology, respectively, and $\hat{f}(e) = [(0, e)]$, $i'(k') = [(k', 0)]$, $p'[(k', e)] = p(e)$, $\hat{g}(e, q'') = e$, $p''(e, q'') = q''$, and $i''(k) = (i(k), 0)$ for $e \in E$, $k' \in K'$, $q'' \in Q''$, $k \in K$.

The following proposition is an analogue of Lemma 2.2.6 for completions, describing a situation when a partial completion relative to a submodule is equal to the completion.

PROPOSITION 2.3.3. *Assume Q in an extension (2.3.1) of bornological V -modules to be complete and bornologically torsion-free. Form the pushout diagram*

$$\begin{array}{ccccc} K & \xrightarrow{i} & E & \xrightarrow{p} & Q \\ \downarrow \text{can}_K & & \downarrow \gamma & & \parallel \\ \overline{K} & \xrightarrow{i'} & E' & \xrightarrow{p'} & Q. \end{array}$$

Then there is a unique isomorphism $\varphi: E' \xrightarrow{\cong} \overline{E}$ such that $\varphi \circ \gamma$ is the canonical map $E \rightarrow \overline{E}$.

PROOF. The bottom row is an extension by Lemma 2.3.2. Then E' is complete by [27, Theorem 2.3]. The maps $\text{can}_E: E \rightarrow \overline{E}$ and $\overline{i}: \overline{K} \rightarrow \overline{E}$ induce a bounded V -module map $\varphi: E' \rightarrow \overline{E}$ by the universal property of pushouts. Since E' is complete, the universal property of \overline{E} gives a unique map $\psi: \overline{E} \rightarrow E'$ with $\psi \circ \text{can}_E = \gamma$. Then $\varphi \circ \psi \circ \text{can}_E = \varphi \circ \gamma = \text{can}_E$. This implies $\varphi \circ \psi = \text{id}_{\overline{E}}$. Next, $\psi \circ \overline{i} \circ \text{can}_K = \gamma \circ i = i' \circ \text{can}_K$ implies $\psi \circ \overline{i} = i'$, and then $\psi \circ \varphi \circ i' = \psi \circ \overline{i} = i'$ and $\psi \circ \varphi \circ \gamma = \psi \circ \text{can}_E = \gamma$ imply $\psi \circ \varphi = \text{id}_{E'}$. So φ is an isomorphism. \square

2.4. Injective maps between completions. Unlike in the Archimedean case, all Banach spaces over F have a simple structure. This implies that they all satisfy a variant of Grothendieck's Approximation Property. This is Proposition 2.4.5, and it will be useful to describe completions of tensor products.

DEFINITION 2.4.1. Let D be a set. Let $C_0(D, V)$ be the V -module of all functions $f: D \rightarrow V$ such that for each $\delta > 0$ there is a finite subset $S \subseteq D$ with $|f(x)| < \delta$ for all $x \in D \setminus S$. Define $C_0(D, F)$ similarly. Equip $C_0(D, V)$ and $C_0(D, F)$ with the supremum norm.

THEOREM 2.4.2. *Any π -adically complete, torsion-free V -module M is isomorphic to $C_0(D, V)$ for some set D .*

PROOF. The map $M \rightarrow M \otimes F$ is an embedding because M is torsion-free. Define the *gauge norm* on $F \cdot M$ by

$$\|x\| := \inf\{|\pi|^j : \pi^{-j} \cdot x \in M\}.$$

It is a non-Archimedean norm and makes $F \cdot M$ a Banach F -vector space with unit ball M . It takes values in $\{|\pi|^n : n \in \mathbb{Z}\} \cup \{0\}$ by construction. Hence there is a set D and an isometric isomorphism $FM \cong C_0(D, F)$ (see [34, Remark 10.2]). It maps M isomorphically onto the the unit ball of $C_0(D, F)$, which is $C_0(D, V)$. \square

COROLLARY 2.4.3. *Let W be a complete, torsion-free bornological V -module. Then W is isomorphic to the colimit of an inductive system of complete V -modules of the form $(C_0(D_n, V), f_{n,m})_{n,m \in S}$ with a directed set (S, \leq) , sets D_n for $n \in S$, and injective, bounded V -linear maps $f_{n,m}: C_0(D_m, V) \hookrightarrow C_0(D_n, V)$ for $n, m \in S$, $n \geq m$.*

PROOF. The complete V -submodules of W form a directed set under inclusion. This defines an inductive system with injective structure maps and with colimit W by [11, Proposition 2.10]. Each complete V -submodule of W is π -adically complete and torsion-free. Then it is isomorphic to $C_0(D, V)$ for some set D by Theorem 4.1.2. \square

LEMMA 2.4.4. *Let $f: C_0(D_1, V) \hookrightarrow C_0(D_2, V)$ and $g: C_0(D_3, V) \hookrightarrow C_0(D_4, V)$ be injective, bounded V -linear maps. Then the induced bounded map*

$$f \widehat{\otimes} g: C_0(D_1, V) \widehat{\otimes} C_0(D_3, V) \rightarrow C_0(D_2, V) \widehat{\otimes} C_0(D_4, V)$$

is injective as well. And here $C_0(D_m, V) \widehat{\otimes} C_0(D_n, V) \cong C_0(D_m \times D_n, V)$.

PROOF. The universal property of the complete bornological tensor product implies that $C_0(D_1, V) \widehat{\otimes} C_0(D_3, V) \cong C_0(D_1 \times D_3, V)$ for all sets D_1 and D_3 . Define $C_0(D_1, C_0(D_3, V))$ to be the space of all functions $f: D_1 \rightarrow C_0(D_3, V)$ for which the gauge norm $\|f\|$ vanishes at ∞ . There is a canonical isomorphism

$$C_0(D_1 \times D_3, V) \xrightarrow{\cong} C_0(D_1, C_0(D_3, V)), \quad f \mapsto (s \mapsto f(s, \cdot)).$$

Similarly, $C_0(D_1 \times D_3, V) \cong C_0(D_3, C_0(D_1, V))$. Now we factorise the map $f \widehat{\otimes} g$ as

$$\begin{aligned} C_0(D_1, V) \widehat{\otimes} C_0(D_3, V) &\cong C_0(D_1 \times D_3, V) \cong C_0(D_1, C_0(D_3, V)) \\ &\xrightarrow{g_*} C_0(D_1, C_0(D_4, V)) \cong C_0(D_4, C_0(D_1, V)) \end{aligned}$$

$$\xrightarrow{f_*} C_0(D_4, C_0(D_2, V)) \cong C_0(D_2 \times D_4, V) \cong C_0(D_2, V) \widehat{\otimes} C_0(D_4, V);$$

here the maps f_* and g_* are injective because f and g are injective. \square

PROPOSITION 2.4.5. *Let M_1, W_1, M_2 and W_2 be complete, torsion-free bornological V -modules and let $\varphi_j: M_j \hookrightarrow W_j$ for $j = 1, 2$ be injective bounded V -module maps. Then $\varphi_1 \widehat{\otimes} \varphi_2: M_1 \widehat{\otimes} M_2 \rightarrow W_1 \widehat{\otimes} W_2$ is injective.*

PROOF. Write W_1 and W_2 as inductive limits as in Corollary 2.4.3. Then $W_1 \widehat{\otimes} W_2$ is naturally isomorphic to the inductive limit of the inductive system defined by the maps $f_{1,n_1,m_1} \otimes f_{2,n_2,m_2}: C_0(D_{n_1}, V) \otimes C_0(J_{n_2}, V) \rightarrow C_0(D_{m_1}, V) \otimes C_0(J_{m_2}, V)$, and $W_1 \widehat{\otimes} W_2$ is naturally isomorphic to the inductive limit of the inductive system defined by the maps $f_{1,n_1,m_1} \widehat{\otimes} f_{2,n_2,m_2}: C_0(D_{n_1}, V) \widehat{\otimes} C_0(J_{n_2}, V) \rightarrow C_0(D_{m_1}, V) \widehat{\otimes} C_0(J_{m_2}, V)$. All these bounded maps are injective by Lemma 2.4.4. Therefore, the tensor product is isomorphic to an ordinary union of these V -modules, equipped with the bornology cofinally generated by these V -submodules. The tensor products $M_1 \otimes M_2$ and $M_1 \widehat{\otimes} M_2$ are described similarly, and the maps φ_1 and φ_2 are described by injective maps between the entries of the appropriate inductive systems. Then Lemma 2.4.4 shows that $\varphi_1 \widehat{\otimes} \varphi_2$ is injective. \square

2.5. The bimodule of differential 1-forms. We are going to define the (complete) bimodule $\overline{\Omega}^1(A)$ of *noncommutative differential 1-forms* over a complete bornological V -algebra A .

For a unital algebra in the usual sense, $\Omega^1(A)$ is defined in [16, Section 1] as $A \otimes (A/\mathbb{C} \cdot 1)$ with a certain bimodule structure. Its elements are denoted by $a db$ with $a \in A$, $b \in A/\mathbb{C} \cdot 1$. We shall use the version for non-unital algebras, which uses A^+ instead of A . It is clear from the definition that the map $\Omega^1(A) \rightarrow A^+ \otimes A^+$, $a db \mapsto a \otimes b - ab \otimes 1$, is an isomorphism onto the kernel of the multiplication map $A^+ \otimes A^+ \rightarrow A^+$. In [26, Appendix A.3], $\Omega^1(A)$ is defined as this kernel when A is an algebra in a symmetric monoidal category. The kernel exists and is a direct

summand as a left or right module because the multiplication map $A^+ \otimes A^+ \rightarrow A^+$ splits as a left or right A -module map. This definition applies in our setting, using the tensor product $\overline{\otimes}$.

By definition, $\Omega^1(A) \subseteq A^+ \overline{\otimes} A^+$ is a complete bornological A -bimodule. The map

$$d: A \rightarrow \overline{\Omega}^1(A), \quad d(x) := 1 \otimes x - x \otimes 1,$$

is the universal bounded derivation into a complete A -bimodule, that is, any bounded derivation $\partial: A \rightarrow M$ into a complete A -bimodule factors uniquely through d . Namely, there is a unique bounded bimodule homomorphism $\overline{\Omega}^1(A) \rightarrow M$, $a_0 da_1 \mapsto a_0 \cdot \partial(a_1)$. This factorisation exists because there are bornological isomorphisms

$$\begin{aligned} A^+ \overline{\otimes} A &\rightarrow \overline{\Omega}^1(A), & x \otimes y &\mapsto x dy, \\ A \overline{\otimes} A^+ &\rightarrow \overline{\Omega}^1(A), & x \otimes y &\mapsto (dx) \cdot y = d(x \cdot y) - x dy. \end{aligned}$$

The first one is left and the second one right A -linear.

We now relate $\overline{\Omega}^1(A)$ to sections of semi-split, square-zero extensions of A (see [26, Theorem A.53] or [16, Proposition 3.3]). Let M be a complete bornological A -bimodule. Give $A \oplus M$ the multiplication

$$(a_1, m_1) \cdot (a_2, m_2) := (a_1 \cdot a_2, a_1 \cdot m_2 + m_1 \cdot a_2).$$

The inclusion $M \rightarrow A \oplus M$ and the projection $A \oplus M \rightarrow A$ form a square-zero extension that splits by the inclusion homomorphism $A \rightarrow A \oplus M$.

LEMMA 2.5.1. *Let A be a complete bornological algebra and let M be a complete bornological A -bimodule. There is a natural bijection between bounded bimodule homomorphisms $\overline{\Omega}^1(A) \rightarrow M$ and bounded V -algebra homomorphisms $A \rightarrow A \oplus M$ that split the extension $M \rightarrow A \oplus M \rightarrow A$.*

PROOF. Any bounded linear section $s: A \rightarrow A \oplus M$ has the form $a \mapsto (a, \partial(m))$ for a bounded linear map $\partial: A \rightarrow M$. And s is multiplicative if and only if ∂ is a derivation. Bounded bimodule maps $\overline{\Omega}^1(A) \rightarrow M$ are in bijection with bounded derivations. \square

We shall also apply the definition and the lemma above to incomplete bornological algebras, where we define $\Omega^1(A)$ by leaving out the completions in the construction above. And we shall use a variant of $\Omega^1(A)$ for projective systems of algebras. In general, the definition and the lemma above carry over to algebras in any additive monoidal category.

2.6. Tensor algebras and noncommutative differential forms. We describe the tensor algebra of a bornological V -module and the algebra of differential forms over a bornological algebra and relate the two. All this goes back to Cuntz and Quillen [16]. Their constructions make sense in any additive monoidal category with countable direct sums, and we specialise this generalisation of their constructions to bornological V -modules and to complete bornological V -modules. We shall mainly use the uncomplete versions below because we are going to modify tensor algebras further before completing them.

Let W be a bornological V -module. Equip $W^{\otimes n}$ for $n \geq 1$ with the tensor product bornology and $\mathbb{T}W := \bigoplus_{n \geq 1} W^{\otimes n}$ with the direct sum bornology; that is, a subset M of $\mathbb{T}W$ is bounded if and only if it is contained in the image of

$\bigoplus_{j=1}^n N^{\otimes j}$ for some $n \geq 1$ and some bounded submodule $N \subseteq W$. The multiplication $\mathbb{T}W \times \mathbb{T}W \rightarrow \mathbb{T}W$ defined by

$$(x_1 \otimes \cdots \otimes x_n) \cdot (x_{n+1} \otimes \cdots \otimes x_{n+m}) := x_1 \otimes \cdots \otimes x_{n+m}$$

makes $\mathbb{T}W$ a bornological algebra, called the *tensor algebra* of W . Let $\sigma_W: W \rightarrow \mathbb{T}W$ be the inclusion of the first summand. It is a bounded V -module homomorphism, but not an algebra homomorphism.

LEMMA 2.6.1. *The map $\sigma_W: W \rightarrow \mathbb{T}W$ is the universal bounded V -module map from W to a bornological algebra. That is, $\mathbb{T}W$ is a bornological V -algebra and if $f: W \rightarrow S$ is a bounded V -module map to a bornological V -algebra S , then there is a unique bounded algebra homomorphism $f^\#: \mathbb{T}W \rightarrow S$ with $f^\# \circ \sigma_W = f$.*

PROOF. The multiplication above is well defined and bounded by the universal property of the bornological tensor product. Let $f: W \rightarrow S$ be a bounded V -module map. Then there is a unique bounded V -module map $f^\#: \mathbb{T}W \rightarrow S$ with

$$f^\#(x_1 \otimes \cdots \otimes x_n) := f(x_1) \cdots f(x_n)$$

for all $x_1, \dots, x_n \in W$. This is a bounded algebra homomorphism. And it is the unique one with $f^\# \circ \sigma_W = f$. \square

Let W be a complete bornological V -module. The completion of $\mathbb{T}W$ is

$$\overline{\mathbb{T}W} := \bigoplus_{n \geq 1} W^{\overline{\otimes} n},$$

the direct sum of the completed tensor products, equipped with the direct sum bornology. By the universal property of completions, the canonical arrow $\overline{\sigma_W}: W \rightarrow \overline{\mathbb{T}W}$ is the universal bounded V -module map from W to a complete bornological algebra. That is, $\overline{\mathbb{T}W}$ is a complete bornological V -algebra and if $f: W \rightarrow S$ is a bounded V -module map to a complete bornological V -algebra S , then there is a unique bounded algebra homomorphism $f^\#: \overline{\mathbb{T}W} \rightarrow S$ with $f^\# \circ \overline{\sigma_W} = f$.

REMARK 2.6.2. If W is torsion-free, then so is $\mathbb{T}W$. If W is complete and torsion-free, then so is $\overline{\mathbb{T}W}$. This uses [27, Theorem 4.6 and Proposition 4.12] and that completeness and torsion-freeness are hereditary for direct sums.

Let R be a bornological V -algebra. Then so is $\mathbb{T}R$. The identity map on R induces a bounded homomorphism $p := \text{id}_R^\#: \mathbb{T}R \rightarrow R$ by Lemma 2.0.2. Let

$$(2.6.3) \quad \mathbb{J}R := \ker(p: \mathbb{T}R \rightarrow R).$$

This is a closed two-sided ideal in $\mathbb{T}R$. The inclusion $\mathbb{J}R \hookrightarrow \mathbb{T}R$ and the projection $p: \mathbb{T}R \rightarrow R$ form an extension of bornological V -algebras, which splits by the bounded V -module map $\sigma_R: R \rightarrow \mathbb{T}R$. Similarly, if R is a complete bornological V -algebra, then there is an extension of complete bornological V -algebras

$$\overline{\mathbb{J}R} \hookrightarrow \overline{\mathbb{T}R} \rightarrow R$$

that splits by the bounded V -module map $\overline{\sigma_R}$.

The *unitalisation* of R is $R^+ := R \oplus V$ with the multiplication

$$(x, \lambda) \cdot (y, \mu) := (xy + \mu x + \lambda y, \lambda \mu)$$

for $x, y \in R, \lambda, \mu \in V$. So $(0, 1)$ is the unit element in R^+ , which we denote simply by 1. The inclusion map $R \rightarrow R^+$ is the universal bounded homomorphism from R to a unital bornological algebra.

We are going to rewrite the tensor algebra using the Fedosov product on the algebra of noncommutative differential forms, following Cuntz and Quillen [16]. This alternative picture is important because it allows to describe the ideal JR and the tube algebras that we shall need. It is sketched in [26, Appendix A.3–4] why all this continues to work for algebras in symmetric monoidal categories. This observation goes back further to [15].

Let $\Omega^0 R := R$ and, for $n \geq 1$, let $\Omega^n R := R^+ \otimes R^{\otimes n}$, equipped with the tensor product bornology. That is, a submodule $N \subseteq \Omega^n R$ is bounded if and only if there is a bounded submodule $M \subseteq R$ such that N is contained in the image of $\Omega^n M = M^+ \otimes M^{\otimes n}$. Let $\Omega R := \bigoplus_{n \geq 0} \Omega^n R$, equipped with the direct sum bornology. We interpret an element $x_0 \otimes x_1 \otimes \cdots \otimes x_n \in \Omega^n R$ as a noncommutative differential form $x_0 dx_1 \dots dx_n$. There is a unique structure of differential graded algebra on ΩR whose multiplication restricts to the given multiplication on $R = \Omega^0 R$ and whose differential satisfies

$$d(x_0 dx_1 \dots dx_n) := 1 \cdot dx_0 dx_1 \dots dx_n.$$

Namely, the (graded) Leibniz rule dictates that

$$x_0 dx_1 \dots dx_n \cdot x_{n+1} dx_{n+2} \dots dx_{n+m} := \sum_{j=0}^n (-1)^{n-j} x_0 dx_1 \dots d(x_j \cdot x_{j+1}) \dots dx_{n+m}.$$

The *Fedosov product* on a differential graded algebra such as ΩR is defined by

$$(2.6.4) \quad \xi \odot \eta := \xi \eta - (-1)^{i \cdot j} d(\xi) d(\eta) \quad \text{for } \xi \in \Omega^i R, \eta \in \Omega^j R.$$

Recall the notation $M^{(n)} := \sum_{i=1}^n M^i$. If $p, q \geq 0$ and $M, N \subseteq R$ are bounded V -submodules, then

$$(2.6.5) \quad \Omega^p M \odot \Omega^q N \subseteq \Omega^{p+q}((M+N)^{(2)}) \oplus \Omega^{p+q+2}((M+N)).$$

Hence $(\Omega R, \odot)$ is a bornological algebra. Its completion $\overline{\Omega} R$ is the bornological direct sum $\bigoplus_{n \geq 0} \overline{\Omega}^n R$ of the completed differential forms. Let $\Omega^{\text{ev}} R \subseteq \Omega R$ be the bornological subalgebra of differential forms of even degree. In the following, we always equip $\Omega^{\text{ev}} R$ with the Fedosov product.

The inclusion map $R = \Omega^0 R \rightarrow \Omega^{\text{ev}} R$ induces a bounded homomorphism

$$(2.6.6) \quad \mathbb{T}R \rightarrow \Omega^{\text{ev}} R, \quad x_1 \otimes \cdots \otimes x_n \mapsto x_1 \odot \cdots \odot x_n,$$

by Lemma 2.0.2, which is, in fact, a bornological isomorphism. To understand why, let $f: R \rightarrow S$ be a V -module map. Its *curvature* is the V -module map

$$\omega_f: R \otimes R \rightarrow S, \quad \omega_f(x, y) = f(x \cdot y) - f(x) \cdot f(y).$$

It is bounded if f is. The composite of the induced homomorphism $f^\#: \mathbb{T}R \rightarrow S$ with the inverse of the map in (2.6.6) must be given by the formula

$$(2.6.7) \quad f^\#(x_0 dx_1 \dots dx_{2n}) = f(x_0) \cdot \omega_f(x_1, x_2) \cdots \omega_f(x_{2n-1}, x_{2n})$$

because the inclusion map $R \rightarrow \Omega^{\text{ev}} R$ has the curvature $(x, y) \mapsto x \cdot y - x \odot y = dx dy$. Indeed, this defines a bounded homomorphism $f^\#: \Omega^{\text{ev}} R \rightarrow S$. So $\Omega^{\text{ev}} R$ enjoys the same universal property as $\mathbb{T}R$. Then the map in (2.6.6) is a bornological isomorphism.

The map $p: \mathbb{T}R \rightarrow R$ corresponds to the map $p: \Omega^{\text{ev}} R \rightarrow R$ that vanishes on $\Omega^{2n} R$ for $n \geq 1$ and is the identity on $\Omega^0 R = R$. Therefore, the isomorphism $\mathbb{T}R \cong \Omega^{\text{ev}} R$ maps JR onto $\bigoplus_{n \geq 1} \Omega^{2n} R$. Then it follows by induction that the isomorphism maps the ideal JR^m onto $\bigoplus_{n \geq m} \Omega^{2n} R$. This simple description of all the powers JR^m is

the main point of rewriting the tensor algebra using the Fedosov product on the even-degree differential forms.

REMARK 2.6.8. The map $JR^{\otimes m} \rightarrow JR^m$ splits by the bounded V -module map given by

$$a_0 da_1 \dots da_{2(m+n)} \mapsto a_0 da_1 da_2 \otimes da_3 da_4 \otimes \dots \otimes da_{2m-3} da_{2m-2} \otimes da_{2m-1} \dots da_{2n}.$$

Thus $JR^{\otimes m} \rightarrow JR^m$ is a quotient map, and the same is true upon completion.

2.7. The X-complex. The X -complex introduced by Cuntz and Quillen in [17] is an important ingredient in their approach to cyclic homology theories. It is defined for algebras in additive monoidal categories (see also [26, Appendix A.6]). We shall specialise this definition to the additive monoidal category of complete bornological algebras over F or V .

Let $\overline{\Omega}^1(S)/[\cdot, \cdot]$ be the commutator quotient of $\overline{\Omega}^1(S)$, that is, the quotient of $\overline{\Omega}^1(S)$ by the closure of the image of

$$S \overline{\otimes} \overline{\Omega}^1(S) \rightarrow \overline{\Omega}^1(S), \quad x \otimes \omega \mapsto x \cdot \omega - \omega \cdot x.$$

With the quotient bornology, this is a complete bornological V -module (see [27, Theorem 2.3]). The closure comes in because we take a cokernel in the category of complete bornological V -modules, which forces us to make the quotient separated.

Let $q: \overline{\Omega}^1(S) \rightarrow \overline{\Omega}^1(S)/[\cdot, \cdot]$ be the quotient map. There is a unique bounded linear map $b: \overline{\Omega}^1(S) \rightarrow S$ that satisfies $b(xy) = x \cdot y - y \cdot x$. It descends to a bounded linear map $\tilde{b}: \overline{\Omega}^1(S)/[\cdot, \cdot] \rightarrow S$. The X -complex of S is the following $\mathbb{Z}/2$ -graded chain complex of complete bornological V -modules:

$$X(S) := \left(S \begin{array}{c} \xrightarrow{q \circ d} \\ \xleftarrow{\tilde{b}} \end{array} \overline{\Omega}^1(S)/[\cdot, \cdot] \right).$$

We briefly call $\mathbb{Z}/2$ -graded chain complexes *supercomplexes*. If S is a complete bornological F -algebra, then $X(S)$ is even a supercomplex of complete bornological F -vector spaces.

3. Definition of analytic cyclic homology

Let A be a torsion-free, complete bornological V -algebra. We are going to define the analytic cyclic homology of A . The idea is to make a universal ‘‘analytically nilpotent’’ extension of A and then take the X -complex of that, tensored with F to ensure its homotopy invariance. (The concept of analytic nilpotence will be introduced later in Section 4.3.) The starting point is the universal extension with a bounded linear section, which is given by the tensor algebra extension. To make the kernel of this extension nilpotent mod π , we pass to a tube algebra. Then we dagger complete this kernel to make it analytically nilpotent. The tube algebra construction produces a projective system of algebras. Tensoring with F and taking the X -complex, we thus get a projective systems of chain complexes. We could define analytic cyclic homology as an invariant in a suitable derived category of such chain complexes. Our main theorems hold in that setting. We prefer, however, to define it as an ordinary F -vector space. Therefore, we also apply the homotopy projective limit and take homology in the very end.

Now we go through the construction in small steps. In the first step, let

$$R := TA, \quad I := JA,$$

be the tensor algebra over A and the kernel of the canonical homomorphism $\mathbb{T}A \twoheadrightarrow A$.

The second step enlarges R to a projective system of tube algebras relative to powers of the ideal I :

DEFINITION 3.1. Let R be a torsion-free bornological V -algebra and I an ideal in R . Let I^j for $j \in \mathbb{N}^*$ denote the V -linear span of products $x_1 \cdots x_j$ with $x_1, \dots, x_j \in I$. The *tube algebra* of $I^l \triangleleft R$ for $l \in \mathbb{N}^*$ is

$$\mathcal{U}(R, I^l) := \sum_{j=0}^{\infty} \pi^{-j} I^{l+j} \subseteq R \otimes F$$

with the subspace bornology; this is indeed a V -subalgebra of $R \otimes F$. If $l \geq j$, then $\mathcal{U}(R, I^l) \subseteq \mathcal{U}(R, I^j)$ is a bornological subalgebra. Let $\mathcal{U}(R, I^\infty)$ be the projective system of bornological V -algebras $(\mathcal{U}(R, I^l))_{l \in \mathbb{N}^*}$.

Since $\mathcal{U}(R, I^l)$ is defined as a bornological submodule of an F -vector space, it is bornologically torsion-free. And the inclusion $R \hookrightarrow \mathcal{U}(R, I^l)$ induces a bornological isomorphism $\mathcal{U}(R, I^l) \otimes F \cong R \otimes F$.

REMARK 3.2. In [11, Definition 3.1.19], the tube algebra $\mathcal{U}(R, I^l)$ of a bornological V -algebra is equipped with a different bornology, namely, the bornology that is generated by subsets bounded in R and subsets of the form $\pi^{-1}M^l$ for bounded subsets $M \subseteq I$. This makes no difference if R carries the fine bornology. For general R , however, the two bornologies on the tube algebra need not be the same. It is easy to check that both bornologies induce the same bornology on $\mathcal{U}(R, I^l) \otimes F \cong R \otimes F$. Thus the two bornologies coincide if and only if the bornology defined in [11] is bornologically torsion-free. This concept is introduced only later in [27]. The more complicated bornology defined in [11] gives the tube algebra the expected universal property for bornological algebras that are torsion-free as algebras, but not bornologically torsion-free.

The third step equips $\mathcal{U}(R, I^l)$ for $l \in \mathbb{N}^*$ with the linear growth bornology relative to the ideal $\mathcal{U}(I, I^l)$. This gives a projective system of bornological algebras

$$\mathcal{U}(R, I^\infty)_{\text{lg}(\mathcal{U}(I, I^\infty))} = (\mathcal{U}(R, I^l)_{\text{lg}(\mathcal{U}(I, I^l))})_{l \in \mathbb{N}^*}$$

because the inclusion homomorphism $\mathcal{U}(R, I^{l+1}) \hookrightarrow \mathcal{U}(R, I^l)$ maps $\mathcal{U}(I, I^{l+1})$ to $\mathcal{U}(I, I^l)$. All these bornological algebras are torsion-free by Lemma 2.2.7.

The fourth step applies the completion functor. By [27, Theorem 4.6], this gives a projective system of complete, torsion-free bornological V -algebras

$$(\mathcal{U}(R, I^\infty), \mathcal{U}(I, I^\infty))^\dagger = ((\mathcal{U}(R, I^l), \mathcal{U}(I, I^l))^\dagger)_{l \in \mathbb{N}^*}.$$

The fifth step is to tensor with F . This gives a projective system of complete bornological F -algebras

$$(\mathcal{U}(R, I^\infty), \mathcal{U}(I, I^\infty))^\dagger \otimes F := ((\mathcal{U}(R, I^l), \mathcal{U}(I, I^l))^\dagger \otimes F)_{l \in \mathbb{N}^*}.$$

The sixth step is to take the X -complex. Being natural, it extends to a functor from projective systems of complete bornological algebras to projective systems of supercomplexes. In particular, the canonical maps $\mathcal{U}(R, I^{l+1}) \rightarrow \mathcal{U}(R, I^l)$ induce bounded chain maps

$$\sigma_l: X((\mathcal{U}(R, I^{l+1}), \mathcal{U}(I, I^{l+1}))^\dagger \otimes F) \rightarrow X((\mathcal{U}(R, I^l), \mathcal{U}(I, I^l))^\dagger \otimes F).$$

These define a projective system of supercomplexes of complete bornological F -vector spaces, which we denote by

$$\mathbb{H}\mathbb{A}(A) := X((\mathcal{U}(R, I^\infty), \mathcal{U}(I, I^\infty))^\dagger \otimes F).$$

The seventh step takes the *homotopy projective limit* $\text{holim } \mathbb{H}\mathbb{A}(A)$. More explicitly, this is the mapping cone of the chain map

$$\begin{aligned} \prod_{l \in \mathbb{N}^*} X((\mathcal{U}(R, I^l), \mathcal{U}(I, I^l))^\dagger \otimes F) &\rightarrow \prod_{l \in \mathbb{N}^*} X((\mathcal{U}(R, I^l), \mathcal{U}(I, I^l))^\dagger \otimes F), \\ (x_l) &\mapsto (x_l - \sigma_l(x_{l+1}))_{l \in \mathbb{N}^*}. \end{aligned}$$

It is a supercomplex of complete bornological F -vector spaces. The final, eighth step takes its homology:

DEFINITION 3.3. The *analytic cyclic homology* $\text{HA}_*(A)$ of a complete, torsion-free bornological \ast -algebra A for $\ast \in \mathbb{Z}/2$ is the homology of $\text{holim } \mathbb{H}\mathbb{A}(A)$, that is, the quotient of the kernel of the differential by the image of the differential. We do not take the closure of the image, so that this quotient need not be bornologically separated. For this reason, we prefer to forget the induced bornology on $\text{HA}_*(A)$.

3.1. Bivariant analytic cyclic homology. Besides the analytic cyclic homology functor HA_* , we also have the functor $\mathbb{H}\mathbb{A}$ taking values in suitable homotopy categories of chain complexes of projective systems of bornological V -modules. This functor contains more information. In particular, it yields a bivariant analytic cyclic homology theory by letting $\text{HA}_*(A_1, A_2)$ be the set of morphisms $\mathbb{H}\mathbb{A}(A_1) \rightarrow \mathbb{H}\mathbb{A}(A_2)$. Cuntz and Quillen use the same idea in [17] to extend periodic cyclic homology to a bivariant theory. The actual definition of $\text{HA}_*(A_1, A_2)$ depends on the choice of the target category, however, and this is somewhat flexible. We do not pick any choice in this article, but only point out two natural options.

The analytic cyclic homology computations in this paper often prove a chain homotopy equivalence $\mathbb{H}\mathbb{A}(A) \simeq \mathbb{H}\mathbb{A}(B)$, as supercomplexes of projective systems of bornological V -modules. These are equivalences in the homotopy category of supercomplexes, where homotopy is understood simply as chain homotopy. In all cases where we compute $\text{HA}_*(A)$ in this paper, we actually prove that $\mathbb{H}\mathbb{A}(A)$ is chain homotopy equivalent to a supercomplex with zero boundary map, so that it contains no more information than the bornological F -vector space $\text{HA}_*(A)$. Homotopy projective limits are sufficiently compatible with chain homotopies to preserve chain homotopy equivalence; and this implies an isomorphism on homology.

A larger class of weak equivalences is used in [15] to define a homotopy category of chain complexes of projective systems. A good aspect of this construction is that it clarifies the role of the homotopy projective limit: this just replaces a given complex by one that is weakly equivalent to it and fibrant in a suitable sense, so that the arrows to it in the homotopy category are the same as chain homotopy classes of chain maps. Thus $\text{HA}_*(A)$ is isomorphic to the space of arrows from the trivial supercomplex V to $\mathbb{H}\mathbb{A}(A)$ in the homotopy category of [15]. We will see later that $\mathbb{H}\mathbb{A}(V)$ is chain homotopy equivalent to the trivial supercomplex V (see Corollary 4.7.3). So the homotopy category of [15] is such that the bivariant analytic cyclic homology group $\text{HA}_*(V, A)$ simplifies to $\text{HA}_*(A)$.

4. Analytic nilpotence and analytically quasi-free resolutions

Cuntz and Quillen described the periodic cyclic homology of an algebra A as the homology of the X -complex of a certain projective system built from the tensor algebra $\mathbb{T}A$ of A . This approach to periodic cyclic homology is the key to proving that it satisfies excision. The Cuntz–Quillen approach is carried over to more analytic versions of periodic cyclic homology in [26]. Our proof of excision for HA_* in Section 5 will follow the pattern in [26]. In this section, we explain how HA_* as defined above fits into this framework.

4.1. Pro-Algebras. An important idea in [26] is that an analytic variant of periodic cyclic homology is defined by a suitable notion of “analytic nilpotence”. This leads to an analytic tensor algebra of an algebra A , which is universal among analytically nilpotent extensions of A . It also leads to the concept of analytically quasi-free algebras. The theory is set up so that any two analytically quasi-free, analytically nilpotent extensions of a given algebra are homotopy equivalent. In characteristic 0, this implies that their X -complexes are chain homotopy equivalent. Thus the X -complex of the analytic tensor algebra is chain homotopy equivalent to the X -complex of any analytically quasi-free resolution of A . In this discussion, “algebras” are always more complex objects – such as projective systems of algebras or bornological algebras – because there is no suitable concept of analytic nilpotence for mere algebras without extra structure. For the analytic cyclic homology defined above, the appropriate type of algebra is a projective system of torsion-free, complete bornological V -algebras. For brevity, we call torsion-free, complete bornological V -algebras *algebras* and projective systems of them *pro-algebras*.

A pro-algebra is given by a directed set (N, \leq) , algebras A_n for $n \in N$, and bounded algebra homomorphisms $\alpha_{m,n}: A_n \rightarrow A_m$ for $m, n \in N$ with $n \geq m$ that satisfy $\alpha_{m,m} = \mathrm{id}_{A_m}$ for all $m \in N$ and $\alpha_{m,n} \circ \alpha_{n,p} = \alpha_{m,p}$ for all $m, n, p \in N$ with $p \geq n \geq m$. The morphism set between two pro-algebras is

$$\mathrm{Hom}\left(\left(A_l\right)_{l \in L}, \left(B_n\right)_{n \in N}\right) := \lim_{\leftarrow n} \lim_{\rightarrow l} \mathrm{Hom}(A_l, B_n).$$

We shall only need pro-algebras $(A_n)_{n \in N}$ where N is countable. Restricting to a cofinal increasing sequence in N gives an isomorphic pro-algebra with $N = \mathbb{N}$. Then the maps $\alpha_{m,n}$ are uniquely determined by $\alpha_{n,n+1}: A_{n+1} \rightarrow A_n$ for $n \in \mathbb{N}$.

An algebra A is also a pro-algebra by taking $A_n = A$ and $\alpha_{n,n+1} := \mathrm{id}_A$ for all $n \in \mathbb{N}$. Such projective systems are called *constant*. For a pro-algebra $A = (A_n, \alpha_{m,n})$, there are canonical morphisms $A \rightarrow \mathrm{const}(A_n)$ for all $n \in N$.

The analytic tensor algebra of a torsion-free algebra A is the torsion-free pro-algebra $(\mathcal{U}(\mathbb{T}A, JA^\infty), \mathcal{U}(JA, JA^\infty))^\dagger$ in the above definition of analytic cyclic homology. This comes with a canonical homomorphism to A , whose kernel is the pro-algebra $(\mathcal{U}(JA, JA^\infty))^\dagger$. This projective system of complete, torsion-free bornological algebras has two important extra properties: it is semi-dagger – hence dagger – and nilpotent mod π – this concept will be defined below. A pro-algebra with these two properties is called analytically nilpotent. The tube algebra construction and the relative dagger completion in the construction of the analytic tensor algebra are the universal way to make a pro-algebra extension with an analytically nilpotent kernel.

Any functor from algebras to algebras extends canonically to an endofunctor on the category of pro-algebras by applying it entrywise. The definition of analytic cyclic homology already used this extension to pro-algebras for completions and

tensor products with F . The constructions of $\mathbb{T}A$ and $\mathbb{J}A$ for algebras are also functors and thus extend to pro-algebras. So is the tensor product bifunctor $-\overline{\otimes}-$, which extends to pro-algebras by

$$(A_n, \alpha_{m,n})_{m,n \in N} \overline{\otimes} (B_n, \beta_{m,n})_{m,n \in N'} \\ := (A_{n_1} \overline{\otimes} B_{n_2}, \alpha_{m_1, n_1} \overline{\otimes} \beta_{m_2, n_2})_{m_1, n_1 \in N, m_2, n_2 \in N'}.$$

In particular, we may tensor a pro-algebra with an algebra such as $V[t]^{\dagger}$, viewed as a constant pro-algebra.

DEFINITION 4.1.1. An *elementary dagger homotopy* between two morphisms of pro-algebras $f_0, f_1: A \rightrightarrows B$ is a morphism of pro-algebras $f: A \rightarrow B \overline{\otimes} V[t]^{\dagger}$ that satisfies $(\text{id}_A \otimes \text{ev}_t) \circ f = f_t$ for $t = 0, 1$. We call f_0, f_1 *elementary dagger homotopic* if there is such a homotopy. *Dagger homotopy* is the equivalence relation generated by elementary dagger homotopy.

4.2. The universal property of the tube algebra construction. First, we generalise the construction of tube algebras to pro-algebras. Actually, in this subsection, we drop the completeness assumption for algebras because tube algebras are usually incomplete. So ‘‘algebras’’ are torsion-free bornological algebras and pro-algebras are projective systems of such algebras until the end of this subsection.

An *ideal in a pro-algebra* $A = (A_n, \alpha_{m,n})_{i \in N}$ is a family of ideals $I_n \triangleleft A_n$ with $\alpha_{m,n}(I_n) \subseteq I_m$ for all $n, m \in N$ with $n \geq m$; then $\alpha_{m,n}$ induces homomorphisms $\mathcal{U}(A_n, I_n^l) \rightarrow \mathcal{U}(A_m, I_m^l)$ for all $l \in \mathbb{N}^*$, which intertwine the inclusion maps $\mathcal{U}(A_n, I_n^l) \hookrightarrow \mathcal{U}(A_n, I_n^j)$ for $l \geq j$. These homomorphisms form a pro-algebra

$$\mathcal{U}(A, I^\infty) := (\mathcal{U}(A_n, I_n^l))_{n \in N, l \in \mathbb{N}^*}.$$

If $l \in \mathbb{N}^*$, then $\mathcal{U}(A, I^l) := (\mathcal{U}(A_n, I_n^l))_{n \in N}$ is a pro-algebra. The pro-algebra $\mathcal{U}(A, I^l)$ for $l \in \mathbb{N}^* \cup \{\infty\}$ contains $\mathcal{U}(I, I^l)$ as an ideal. Since $A_n \subseteq \mathcal{U}(A_n, I_n^l)$ for all $n \in N$, $l \in \mathbb{N}^*$, the inclusion maps define a pro-algebra homomorphism $\iota_{A, I}: A \rightarrow \mathcal{U}(A, I^\infty)$.

REMARK 4.2.1. The notion of ideal above suffices for our purposes and is convenient to define the tube algebra quickly. It has the problem of not being invariant under isomorphism of pro-algebras. A better definition would be to define an ideal to be the kernel of a pro-algebra homomorphism. It is, however, possible to switch to isomorphic pro-algebras to make a pro-algebra homomorphism into a homomorphism of diagrams. And then the kernel becomes a family of ideals as above. This allows to extend the construction of tube algebras to ideals in the more general sense.

DEFINITION 4.2.2. A pro-algebra $(A_n, \alpha_{m,n})_{n \in N}$ is *nilpotent mod π* if, for each $m \in N$, there are $n \in N_{\geq m}$ and $l \in \mathbb{N}^*$ such that $\alpha_{m,n}(A_n^l) \subseteq \pi A_m$; here A_n^l denotes the V -submodule generated by all products $x_1 \cdots x_l$ of l factors in A_n .

REMARK 4.2.3. Let $A = (A_n, \alpha_{m,n})_{m,n \in N}$ be a pro-algebra. Let $A/(\pi)$ be the projective system of \mathbb{F} -algebras formed by the quotients $A_n/(\pi)$ with the homomorphisms induced by $\alpha_{m,n}$. By definition, A is nilpotent mod π if and only if $A/(\pi)$ has the following property: for each $n \in N$ there are $m \in N$ and $l \in \mathbb{N}^*$ such that the l -fold multiplication map $(A_m/(\pi))^{\otimes l} \rightarrow A_n/(\pi)$ is zero. This is equivalent to the definition that a projective system of \mathbb{F} -algebras is pro-nilpotent in [26, Definition 4.3].

PROPOSITION 4.2.4. *Let A and B be pro-algebras and let I and J be ideals in A and B , respectively. Let $\varphi: A \rightarrow B$ be a pro-algebra morphism that restricts to a pro-algebra morphism $I \rightarrow J$. Let $\iota_{A,I}: A \rightarrow \mathcal{U}(A, I^\infty)$ denote the canonical pro-algebra morphism.*

- (1) *The pro-algebra $\mathcal{U}(I, I^\infty)$ is nilpotent mod π .*
- (2) *If J is nilpotent mod π , then there is a unique morphism $\bar{\varphi}: \mathcal{U}(A, I^\infty) \rightarrow B$ with $\bar{\varphi} \circ \iota_{A,I} = \varphi$. It restricts to a morphism $\mathcal{U}(I, I^\infty) \rightarrow J$.*
- (3) *There is a unique morphism $\varphi_*: \mathcal{U}(A, I^\infty) \rightarrow \mathcal{U}(B, J^\infty)$ with $\varphi_* \circ \iota_{A,I} = \iota_{B,J} \circ \varphi$. It restricts to a morphism $\mathcal{U}(I, I^\infty) \rightarrow \mathcal{U}(J, J^\infty)$.*

PROOF. Write $A = (A_n, \alpha_{m,n})_{n \in \mathbb{N}}$, $I = (I_n)_{n \in \mathbb{N}}$ with ideals I_n in A_n with $\alpha_{m,n}(I_n) \subseteq I_m$ and $B = (B_n, \beta_{m,n})_{n \in \mathbb{N}'}$, $J = (J_n)_{n \in \mathbb{N}'}$ with ideals J_n in B_n with $\beta_{m,n}(J_n) \subseteq J_m$. The tube algebra $\mathcal{U}(A, I^\infty)$ is the projective limit of the tube algebras $\mathcal{U}(A_n, I_n^\infty)$ in the category of pro-algebras.

Being nilpotent mod π is hereditary for projective limits. So it suffices to prove (1) when A is a constant pro-algebra. Fix $n \in \mathbb{N}^*$ and let $m = 2n$, $l = n$. Then

$$(4.2.5) \quad \mathcal{U}(I, I^m)^l = \mathcal{U}(I, I^{2n})^n = \left(I + \sum_{j=1}^{\infty} \pi^{-j} I^{2nj} \right)^n \subseteq I^n + \sum_{j=1}^{\infty} \pi^{-j} I^{2nj}$$

because $\sum_{j=1}^{\infty} \pi^{-j} I^{2nj}$ is an ideal in $\mathcal{U}(A, I^{2n})$. Since $\pi^{-1} I^n$ and $\pi^{-2j} I^{2nj}$ are contained in $\mathcal{U}(I, I^n)$, all summands on the right hand side of (4.2.5) are contained in $\pi \cdot \mathcal{U}(I, I^n)$. Thus $\mathcal{U}(I, I^\infty)$ is nilpotent mod π .

We prove statement (2). The morphism $\varphi: A \rightarrow B$ is described by a coherent family of V -algebra homomorphisms $\varphi_n: A_{\psi(n)} \rightarrow B_n$ for all $n \in \mathbb{N}'$. Each B_n is torsion-free by our definition of ‘‘algebra’’. Then the homomorphism φ_n is determined by $\varphi_n \otimes \text{id}_F: A_{\psi(n)} \otimes F \rightarrow B_n \otimes F$. By construction, $\mathcal{U}(A_\nu, I^m) \otimes F = A_\nu \otimes F$ for all $\nu \in \mathbb{N}$, $m \in \mathbb{N}^*$. Thus a factorisation of φ through $\mathcal{U}(A, I^\infty)$ is unique if it exists.

Fix $n \in \mathbb{N}'$. Since J is nilpotent mod π , there are $m \in \mathbb{N}'_{\geq n}$ and $l \in \mathbb{N}^*$ with $\beta_{n,m}(J_m^l) \subseteq \pi \cdot J_n$. Since φ is coherent, there is $\nu \in \mathbb{N}_{\geq \psi(m)}$ with $\beta_{n,m} \circ \varphi_m \circ \alpha_{\psi(m),\nu} = \varphi_n \circ \alpha_{n,\nu}$. Since φ restricts to a morphism $I \rightarrow J$, we may also arrange that $\varphi_m \circ \alpha_{\psi(m),\nu}(I_\nu) \subseteq J_m$ by increasing ν if necessary. Hence

$$\varphi_n \circ \alpha_{n,\nu}(I_\nu^l) = \beta_{n,m} \circ \varphi_m \circ \alpha_{\psi(m),\nu}(I_\nu^l) \subseteq \beta_{n,m}(J_m^l) \subseteq \pi \cdot J_n.$$

Thus the homomorphism $(\varphi_n \circ \alpha_{n,\nu}) \otimes \text{id}_F: A_\nu \otimes F \rightarrow B_n \otimes F$ maps the tube algebra $\mathcal{U}(A_\nu, I_\nu^l) \subseteq A_\nu \otimes F$ into $B_n \subseteq B_n \otimes F$ and $\mathcal{U}(I_\nu, I_\nu^l) \subseteq A_\nu \otimes F$ into $J_n \subseteq B_n \otimes F$. This gives a homomorphism $\bar{\varphi}_n: \mathcal{U}(A_\nu, I_\nu^l) \rightarrow B_n$ with $\bar{\varphi}_n \circ \iota_{A_\nu, I_\nu^l} = \varphi_n \circ \alpha_{n,\nu}$. Since $\mathcal{U}(A_\nu, I^m) \subseteq A_\nu \otimes F$, the homomorphisms $\bar{\varphi}_n$ inherit the coherence property of a pro-algebra morphism from the maps φ_n .

We prove statement (3) of the proposition. We compose $\varphi: A \rightarrow B$ with the canonical map $B \rightarrow \mathcal{U}(B, J^\infty)$ to get a morphism $A \rightarrow \mathcal{U}(B, J^\infty)$. It restricts to a morphism $I \rightarrow J \rightarrow \mathcal{U}(J, J^\infty)$. The ideal $\mathcal{U}(J, J^\infty)$ in $\mathcal{U}(B, J^\infty)$ is nilpotent mod π by (1). So (2) shows that our morphism extends uniquely to a morphism $\mathcal{U}(A, I^\infty) \rightarrow \mathcal{U}(B, J^\infty)$ that maps $\mathcal{U}(I, I^\infty)$ to $\mathcal{U}(J, J^\infty)$. \square

We summarise the tube algebra construction in category-theoretic language. Let \mathfrak{Pro} be the category whose objects are pairs (A, I) , where A is a pro-algebra and I is an ideal in A and whose morphisms are pro-algebra morphisms that restrict to a morphism between the ideals. The pairs (A, I) where I is nilpotent mod π form a subcategory $\mathfrak{Pro}_{\text{nil}}$ in \mathfrak{Pro} . The first two statements in Proposition 4.2.4

say that the canonical arrow $(A, I) \rightarrow (\mathcal{U}(A, I^\infty), \mathcal{U}(I, I^\infty))$ is a universal arrow from (A, I) to an object in $\mathfrak{Pro}_{\text{nil}}$. Thus $\mathfrak{Pro}_{\text{nil}}$ is a reflective subcategory in \mathfrak{Pro} and the reflector acts on objects by $(A, I) \mapsto (\mathcal{U}(A, I^\infty), \mathcal{U}(I, I^\infty))$. Its functoriality is Proposition 4.2.4.(3). If I is already nilpotent mod π , then it follows that the identity map on A extends uniquely to an isomorphism of pro-algebras $\mathcal{U}(A, I^\infty) \cong A$.

The inheritance properties of nilpotence mod π proven in the following proposition are needed by the analytic cyclic homology machinery in [26].

PROPOSITION 4.2.6. *The class of nilpotent mod π pro-algebras is closed under the following operations:*

- Let $A \xrightarrow{i} B \xrightarrow{p} C$ be an extension of pro-algebras. If A and C are nilpotent mod π , then so is B , and vice versa.
- A pro-subalgebra $D \subseteq B$ is nilpotent mod π if B is so and B/D is isomorphic to a projective system of torsion-free bornological V -modules.
- Being nilpotent mod π is hereditary for projective limits.
- A tensor product $A \overline{\otimes} B$ is nilpotent mod π if A or B is nilpotent mod π .

PROOF. Remark 4.2.3 translates all these statements to statements about the class of pro-nilpotent projective systems of \mathbb{F} -algebras. In this way, the statements follow from [26, Theorem 4.4]. We briefly explain direct proofs for the first two claims. The claims about projective limits and tensor products are easy and left to the reader.

As in [26], we may write any extension of pro-algebras $A \xrightarrow{i} B \xrightarrow{p} C$ as a projective system of extensions $A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n$, with morphisms of extensions

$$\begin{array}{ccccc} A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n \\ \downarrow \alpha_{m,n} & & \downarrow \beta_{m,n} & & \downarrow \gamma_{m,n} \\ A_m & \xrightarrow{i_m} & B_m & \xrightarrow{p_m} & C_m \end{array}$$

for $n \geq m$ as structure maps (this construction is also explained during the proof of Proposition 4.3.13 below). Assume that A and C are nilpotent mod π . Pick $m \in N$. There are $n_1 \in N_{\geq m}$ and $j_1 \in \mathbb{N}^*$ so that $\alpha_{m,n_1}(A_{n_1}^{j_1}) \subseteq \pi \cdot A_m$. And there are $n_2 \in N_{\geq n_1}$ and $j_2 \in \mathbb{N}^*$ so that $\gamma_{n_1,n_2}(C_{n_2}^{j_2}) \subseteq \pi \cdot C_{n_1}$. Then $p_{n_1}(\beta_{n_1,n_2}(B_{n_2}^{j_2})) \subseteq \pi \cdot C_{n_1}$. This implies $\beta_{n_1,n_2}(B_{n_2}^{j_2}) \subseteq \pi \cdot B_{n_1} + i_{n_1}(A_{n_1})$. Then

$$\begin{aligned} \beta_{m,n_2}(B_{n_2}^{j_1 \cdot j_2}) &\subseteq \beta_{m,n_1}(\pi \cdot B_{n_1} + i_{n_1}(A_{n_1}))^{j_1} \subseteq \pi \cdot B_m + i_m(\alpha_{m,n_1}(A_{n_1}^{j_1})) \\ &\subseteq \pi \cdot B_m + i_m(\pi A_m) \subseteq \pi \cdot B_m. \end{aligned}$$

So B is nilpotent mod π . Conversely, if B is nilpotent mod π , then C is nilpotent mod π because $p_m(B_m) = C_m$ and $p_m(\pi \cdot B_m) = \pi \cdot C_m$. The claim that A is nilpotent mod π if B is follows from the claim about pro-subalgebras.

Given a pro-subalgebra $D \subseteq B$, we may write $B = (B_n, \beta_{m,n})_{n \in N}$ and $D = (D_n, \delta_{m,n})_{n \in N}$ so that $D_n \subseteq B_n$ for all $n \in N$ and $\delta_{m,n} = \beta_{m,n}|_{D_n}: D_n \rightarrow D_m$ for all $m, n \in N$ with $m \leq n$. Let $m \in N$. Since B/D is isomorphic to a projective system of torsion-free bornological V -modules, there is $n \in N_{\geq m}$ so that the structure map $B_n/D_n \rightarrow B_m/D_m$ kills all elements $x \in B_n/D_n$ with $\pi \cdot x = 0$. Equivalently, if $x \in B_n$ satisfies $\pi \cdot x \in D_n$, then $\beta_{m,n}(x) \in D_m$. Thus $\beta_{m,n}(\pi \cdot B_n \cap D_n) \subseteq \pi \cdot D_m$. If B is nilpotent mod π , then there are $l \in N_{\geq n}$ and $j \in \mathbb{N}^*$ with $\beta_{n,l}(B_l^j) \subseteq \pi \cdot B_n$. Hence

$$\delta_{m,l}(D_l^j) \subseteq \delta_{m,n}(\beta_{n,l}(B_l^j)) \subseteq \beta_{m,n}(\pi \cdot B_n \cap D_n) \subseteq \pi \cdot D_m.$$

Thus D is nilpotent mod π . \square

4.3. Analytically nilpotent pro-algebras. From now on, “algebra” means a complete, torsion-free bornological algebra.

DEFINITION 4.3.1. A pro-algebra J is *analytically nilpotent* if it is isomorphic to a pro-dagger algebra and nilpotent mod π . It is *square-zero* if its multiplication map is 0. An extension of pro-algebras $J \twoheadrightarrow E \twoheadrightarrow A$ is *analytically nilpotent* or *square-zero* if J is analytically nilpotent or square-zero, respectively.

In an analytically nilpotent pro-algebra, any power series $\sum c_n x^n$ for an “element” $x \in J$ and a bounded sequence $(c_n)_{n \in \mathbb{N}}$ in V may be evaluated (see the proof of Proposition 4.3.6 for the precise meaning of this in a pro-algebra). This uses nilpotence mod π in order to reduce to sequences whose valuation grows linearly, and being a pro-dagger algebra to ensure that such series converge.

DEFINITION 4.3.2. A *pro-linear map* between two pro-algebras is a morphism of projective systems of bornological V -modules between them; so pro-linear maps need not be multiplicative. An extension of pro-algebras $J \twoheadrightarrow E \twoheadrightarrow A$ is *semi-split* if it splits by a pro-linear map.

DEFINITION 4.3.3. A pro-algebra A is *analytically quasi-free* if any semi-split analytically nilpotent extension $J \twoheadrightarrow E \twoheadrightarrow A$ splits by a pro-algebra homomorphism $A \rightarrow E$. It is *quasi-free* if any semi-split square-zero extension $J \twoheadrightarrow E \twoheadrightarrow A$ splits by a pro-algebra homomorphism $A \rightarrow E$.

The following lemma gives an equivalent reformulation of the last definition:

LEMMA 4.3.4. *A pro-algebra A is analytically quasi-free if and only if, for any semi-split analytically nilpotent extension $J \twoheadrightarrow E \twoheadrightarrow B$, any homomorphism $f: A \rightarrow B$ lifts to a homomorphism $A \rightarrow E$. A pro-algebra A is quasi-free if and only if, for any semi-split square-zero extension $J \twoheadrightarrow E \twoheadrightarrow B$, any homomorphism $f: A \rightarrow B$ lifts to a homomorphism $A \rightarrow E$.*

PROOF. We may pull the given extension back to a semi-split extension $J \twoheadrightarrow \hat{E} \twoheadrightarrow A$, such that a section $A \rightarrow \hat{E}$ is equivalent to a lifting of f . \square

REMARK 4.3.5. A pro-algebra is square-zero if and only if it is isomorphic to a projective system of torsion-free complete bornological V -modules, each equipped with the zero map as multiplication. Then it is analytically nilpotent. As a consequence, analytically quasi-free algebras are quasi-free.

PROPOSITION 4.3.6. *The base ring V viewed as a constant pro-algebra is analytically quasi-free.*

PROOF. The proof follows [17, Section 12]. This idea is, in fact, much older, see [23, Section 3.6]. Let $J \twoheadrightarrow E \xrightarrow{p} Q$ be a semi-split, analytically nilpotent extension of pro-algebras. Analytic quasi-freeness of V is equivalent to the assertion that any idempotent in Q lifts to an idempotent in E . Here by an idempotent in a pro-algebra $A = (A_n)_n$, we mean a collection $a = (a_n)_n$ of idempotents $a_n \in A_n$. Each $a_n \in A_n$ is equivalent to a homomorphism $V \rightarrow A_n$.

Let $\hat{e} = (\hat{e}_n)_n \in Q$ be an idempotent and let $e \in E$ be the image of \hat{e} under a pro-linear section for $p: E \twoheadrightarrow Q$. Let $x := e - \hat{e}^2 \in J$. We use an Ansatz by Cuntz and Quillen to find an idempotent $\hat{e} \in E$ with $e - \hat{e} \in J$. Namely, we assume

$\hat{e} = e + (2e - 1)\varphi(x)$ for some power series $\varphi \in t\mathbb{Z}[[t]]$. As J is nilpotent mod π , for every $l \in N$, there are $m(l) \geq l$ and $j(l) \in N^*$ with $x_{m(l)}^{j(l)} = \pi y_l$. To simplify notation, we simply write this as $x^j = \pi y$ for some $y \in J$ and $j \in N^*$. Finally, since J is also a pro-dagger algebra, $\varphi(x) \in J$ for all $\varphi \in t\mathbb{Z}[[t]]$. We compute

$$\hat{e}^2 - \hat{e} = (\varphi(x)^2 + \varphi(x))(1 - 4x) - x.$$

So $\hat{e}^2 = \hat{e}$ if and only if $\varphi(x)^2 + \varphi(x) = \frac{x}{1-4x}$. This is solved by $\varphi(x) := \sum_{n=1}^{\infty} \binom{2n-1}{n} x^n$. This defines an element of J . Then \hat{e} is the desired idempotent lifting. \square

PROPOSITION 4.3.7. *An algebra A is analytically quasi-free if and only if its unitalisation A^+ is analytically quasi-free.*

PROOF. Proposition 4.3.6 implies this as in the proof of [26, Proposition 5.53]. \square

PROPOSITION 4.3.8. *Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of unital, analytically quasi-free pro-algebras. Then $\bigoplus_{n \in \mathbb{N}} A_n$ is analytically quasi-free.*

PROOF. The proof of [26, Proposition 5.53] carries over to this context. \square

COROLLARY 4.3.9. *The direct sum $\bigoplus_{n \in \mathbb{N}} V$ is analytically quasi-free.*

PROPOSITION 4.3.10. *Let $J_i \twoheadrightarrow E_i \twoheadrightarrow A_i$ for $i = 1, 2$ be semi-split, analytically nilpotent extensions of pro-algebras. Assume that E_1 is analytically quasi-free.*

- (1) *Any pro-algebra morphism $f: A_1 \rightarrow A_2$ lifts to a morphism of extensions*

$$\begin{array}{ccccc} J_1 & \twoheadrightarrow & E_1 & \xrightarrow{q_1} \twoheadrightarrow & A_1 \\ \downarrow & & \downarrow \hat{f} & & \downarrow f \\ J_2 & \twoheadrightarrow & E_2 & \xrightarrow{q_2} \twoheadrightarrow & A_2 \end{array}$$

This lifting is unique up to dagger homotopy.

- (2) *Let $\hat{f}, \hat{g}: E_1 \rightrightarrows E_2$ be pro-algebra homomorphisms that lift homomorphisms $f, g: A_1 \rightrightarrows A_2$. Then an elementary dagger homotopy $h: A_1 \rightarrow A_2 \overline{\otimes} V[t]^\dagger$ between f and g lifts to an elementary dagger homotopy $\hat{h}: E_1 \rightarrow E_2 \overline{\otimes} V[t]^\dagger$ between \hat{f} and \hat{g} .*
- (3) *Any elementary dagger homotopy $A_1 \rightarrow A_2 \overline{\otimes} V[t]^\dagger$ lifts to an elementary dagger homotopy $E_1 \rightarrow E_2 \overline{\otimes} V[t]^\dagger$.*

PROOF. Let $f: A_1 \rightarrow A_2$ be a pro-algebra homomorphism. Since E_1 is analytically quasi-free and the extension $J_2 \twoheadrightarrow E_2 \twoheadrightarrow A_2$ is semi-split and analytically nilpotent, the homomorphism $f \circ q_1$ lifts to a homomorphism $\hat{f}: E_1 \rightarrow E_2$. Since $q_2 \circ \hat{f} = f \circ q_1$ vanishes on J_1 , \hat{f} restricts to a homomorphism $J_1 \rightarrow J_2$. Thus \hat{f} gives a morphism of extensions.

The uniqueness claim in (1) follows from (2) by taking $f = g$. And (3) follows from (1) and (2). So it remains to prove (2). Assume that we are in the situation of (2). Let $\text{ev}_0, \text{ev}_1: A_2 \overline{\otimes} V[t]^\dagger \rightrightarrows A_2$ and $\text{ev}_0, \text{ev}_1: E_2 \overline{\otimes} V[t]^\dagger \rightrightarrows E_2$ denote the evaluation homomorphisms. Form the pull-back pro-algebra

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & E_2 \oplus E_2 \\ \downarrow & & \downarrow q_2 \oplus q_2 \\ A_2 \overline{\otimes} V[t]^\dagger & \xrightarrow{(\text{ev}_0 \ \text{ev}_1)} & A_2 \oplus A_2. \end{array}$$

The universal property of the pull back gives pro-algebra homomorphisms

$$\begin{aligned} q &:= (\text{ev}_0, \text{ev}_1, q_2 \overline{\otimes} \text{id}_{V[t]^\dagger})_*: E_2 \overline{\otimes} V[t]^\dagger \rightarrow \mathcal{E}, \\ (\hat{f}, \hat{g}, h \circ q_1)_* &: E_1 \rightarrow \mathcal{E}, \end{aligned}$$

because \hat{f} and \hat{g} lift $\text{ev}_t \circ h$ for $t = 0, 1$, respectively. Let

$$V[t]_0^\dagger := \{\varphi \in V[t]^\dagger : \varphi(0) = 0, \varphi(1) = 0\}.$$

We claim that q is part of a semi-split extension of pro-algebras

$$(4.3.11) \quad J_2 \overline{\otimes} V[t]_0^\dagger \twoheadrightarrow E_2 \overline{\otimes} V[t]^\dagger \twoheadrightarrow \mathcal{E}.$$

To see this, we forget multiplications and treat everything as a projective system of bornological V -modules. In this category, a pro-linear section $s: A_2 \rightarrow E_2$ for the semi-split extension $J_2 \rightarrow E_2 \rightarrow A_2$ gives a direct sum decomposition $E_2 \cong J_2 \oplus A_2$. And $V[t]^\dagger \cong V[t]_0^\dagger \oplus V \oplus V$, where the latter two summands are, say, spanned by the functions $1 - t$ and t . This induces decompositions

$$A_2 \overline{\otimes} V[t]^\dagger \cong (A_2 \overline{\otimes} V[t]_0^\dagger) \oplus A_2 \oplus A_2, \quad E_2 \overline{\otimes} V[t]^\dagger \cong (E_2 \overline{\otimes} V[t]_0^\dagger) \oplus E_2 \oplus E_2,$$

such that $(\text{ev}_0, \text{ev}_1)$ is the projection to the second and third summand both for A_2 and E_2 . These direct sum decompositions imply

$$E_2 \overline{\otimes} V[t]^\dagger \cong (J_2 \overline{\otimes} V[t]_0^\dagger) \oplus (A_2 \overline{\otimes} V[t]_0^\dagger) \oplus E_2 \oplus E_2 \cong (J_2 \overline{\otimes} V[t]_0^\dagger) \oplus \mathcal{E}.$$

And this proves the claim.

Corollary 2.1.21 and Proposition 4.2.6 imply that the tensor product $J_2 \overline{\otimes} V[t]_0^\dagger$ is analytically nilpotent. Since E_1 is analytically quasi-free, the homomorphism $(\hat{f}, \hat{g}, h \circ q_1)$ lifts to a homomorphism $\hat{h}: E_1 \rightarrow E_2 \overline{\otimes} V[t]^\dagger$ in the extension (4.3.11). This finishes the proof of (2). \square

COROLLARY 4.3.12. *Any two analytically quasi-free, analytically nilpotent extensions of a pro-algebra are dagger homotopy equivalent.*

PROOF. By Proposition 4.3.10, there are morphisms of extensions in both directions which lift the identity map on A and whose composite maps are dagger homotopic to the identity maps. \square

PROPOSITION 4.3.13. *Let $A \twoheadrightarrow E \twoheadrightarrow B$ be an extension of pro-algebras. If A and B are isomorphic to projective systems of dagger algebras, then so is E . If A and B are analytically nilpotent, then so is E .*

PROOF. Being nilpotent mod π is hereditary for pro-algebra extensions by Proposition 4.2.6. Hence the second statement follows from the first one. Its proof has several steps. First, we rewrite the given extension of pro-algebras as a projective limit of a projective system of algebra extensions. Similar ideas in a less specialised setting also appear in [4, Appendix].

Write E and B as projective systems of (torsion-free, complete bornological) algebras $(E_n, \gamma_{n,m})$ and $(B_n, \beta_{n,m})$ that are indexed by directed sets N_E and N_B , respectively. By assumption, B is isomorphic to a projective system of dagger algebras. We assume that we have picked this representative above, that is, each B_n is a dagger algebra. We describe the pro-algebra morphism $E \rightarrow B$ by a coherent family of bounded homomorphisms $\varphi_n: E_{m(n)} \rightarrow B_n$ for all $n \in N_B$. Let $N := \{(m, n) \in N_E \times N_B : m \geq m(n)\}$. Define a partial order on N by $(m_1, n_1) \geq (m_2, n_2)$ if $m_1 \geq m_2$, $n_1 \geq n_2$, $m_1 \geq m(n_2)$, and $\beta_{n_2, n_1} \circ \varphi_{n_1} \circ \gamma_{m(n_1), m_1} = \varphi_{n_2} \circ \gamma_{m(n_2), m_1}$.

This partially ordered set is directed because N_B and N_E are directed and the maps φ_n for $n \in N$ form a morphism of projective systems. The objects E_m and B_n for $(m, n) \in N$ and the maps γ_{m_1, m_2} and β_{n_1, n_2} for $m_1 \geq m_2$ and $n_1 \geq n_2$ form projective systems E' and B' of bornological algebras. They are isomorphic to E and B , respectively. The homomorphisms

$$\varphi'_{(m, n)} := \varphi_n \circ \gamma_{m(n), m}: E'_{(m, n)} = E_m \rightarrow B_n = B'_{(m, n)}$$

for $(m, n) \in N$ are coherent in the strong sense that

$$\beta'_{(m_1, n_1), (m_2, n_2)} \circ \varphi'_{(m_2, n_2)} = \varphi'_{(m_1, n_1)} \circ \gamma'_{(m_1, n_1), (m_2, n_2)}$$

for all $(m_1, n_1), (m_2, n_2) \in N$ with $(m_1, n_1) \leq (m_2, n_2)$. Here γ' and β' denote the structure maps of the projective systems E' and B' , respectively. By construction, each B'_n is a dagger algebra.

By assumption, the inclusion $A \rightarrow E$ is the kernel of the morphism $E \rightarrow B$. This is isomorphic to the kernel of $\varphi': E' \rightarrow B'$. So A is isomorphic to the projective system A' formed by the closed ideals $A'_n := \ker \varphi_n \subseteq E'_n$ for $n \in N$ with the structure maps $\alpha'_{n_1, n_2} = \gamma'_{n_1, n_2}|_{A_{n_2}}$ for $n_1, n_2 \in N$ with $n_1 \leq n_2$; and the canonical morphism $A' \rightarrow E'$ is the strongly coherent family of inclusion maps $A'_n \hookrightarrow E'_n$ for $n \in N$. Each A'_n is complete and torsion-free because E'_n and B'_n are (see [27, Theorem 2.3 and Lemma 4.2]).

The quotients E'_n/A'_n with the structure maps $\hat{\gamma}'_{n, m}$ induced by $\gamma'_{n, m}$ form a projective system of complete bornological algebras, which is the cokernel for the inclusion $A' \hookrightarrow E'$. The map φ'_n for $n \in N$ descends to an injective, bounded homomorphism $\varrho_n: E'_n/A'_n \rightarrow B'_n$. The pro-algebra morphism $\varrho = (\varrho_n)_{n \in N}$ is an isomorphism because $E \rightarrow B$ is assumed to be another cokernel for the map $A \rightarrow E$. Next, we modify our projective systems so that these become equalities; this replaces the quotients E'_n/A'_n by dagger algebras. The inverse of ϱ is given by a choice of $m(n) \in N$ for $n \in N$ and bounded homomorphisms $\psi_n: B'_{m(n)} \rightarrow E'_n/A'_n$. Increasing $m(n)$ if necessary, we may arrange that $\varrho_n \circ \psi_n = \beta'_{n, m(n)}: B'_{m(n)} \rightarrow B'_n$ and $\psi_n \circ \varrho_{m(n)} = \hat{\gamma}'_{n, m(n)}: E'_{m(n)}/A'_{m(n)} \rightarrow E'_n/A'_n$. Let $N' := \{(m, n) \in N \times N : m \geq m(n)\}$. For $(m, n) \in N'$, pull the extension $A'_n \hookrightarrow E'_n \twoheadrightarrow E'_n/A'_n$ back along ψ_n as in Lemma 2.3.2. This gives a diagram of extensions of bornological V -modules

$$\begin{array}{ccccc} A''_{(m, n)} & \hookrightarrow & E''_{(m, n)} & \twoheadrightarrow & B''_{(m, n)} \\ & & \downarrow & & \downarrow \psi_n \\ A'_n & \hookrightarrow & E'_n & \twoheadrightarrow & E'_n/A'_n \end{array}$$

with $A''_{(m, n)} = A'_n$ and $B''_{(m, n)} = B'_m$. The latter is a dagger algebra because it is equal to B_m for suitable $m \in N_B$ depending on $n \in N'$. There is a unique bornological algebra structure on $E''_{(m, n)}$ for which all maps in this diagram are homomorphisms. We claim that $E''_{(m, n)}$ is complete. First, A'_n is closed in E'_n because B'_n is separated. Then E'_n/A'_n is separated (see [27, Lemma 2.1]). Then $E''_{(m, n)}$ is closed in $B'_m \oplus E'_n$. And then $E''_{(m, n)}$ is complete by [27, Theorem 2.3]. As above, there is a partial order on N' that makes it a directed set and such that $A''_n \hookrightarrow E''_n \twoheadrightarrow B''_n$ becomes a projective system of algebra extensions. This projective system is isomorphic to $A' \hookrightarrow E' \twoheadrightarrow E'/A'$ because it is the pullback along the pro-algebra isomorphism $B' \xrightarrow{\cong} E'/A'$. Thus it is isomorphic to the original extension $A \hookrightarrow E \twoheadrightarrow B$. We

have now replaced this pro-algebra extension by a projective system of algebra extensions where the quotients B_n'' are dagger algebras.

To simplify notation, we remove the primes now and assume that our pro-algebra extension already comes to us as a projective system of algebra extensions $A_n \twoheadrightarrow E_n \twoheadrightarrow B_n$, where A_n and E_n are torsion-free, complete bornological algebras and B_n are dagger algebras for all $n \in N$. The dagger completions E_n^\dagger for $n \in N$ form a projective system of dagger algebras, and the canonical maps $E_n \rightarrow E_n^\dagger$ form a pro-algebra morphism. We claim that this pro-algebra morphism is an isomorphism. Equivalently, for each $n \in N$ there are $m \in N$ with $m \geq n$ and a bounded homomorphism $\tilde{\gamma}_{n,m}: E_m^\dagger \rightarrow E_n$ such that the composite map $E_m \rightarrow E_m^\dagger \rightarrow E_n$ is $\gamma_{n,m}$; then the other composite map $E_m^\dagger \rightarrow E_n \rightarrow E_n^\dagger$ is the map on the dagger completions induced by $\gamma_{n,m}$, and these two equalities of compositions say that we are dealing with morphisms of pro-algebras inverse to each other.

Fix $n \in N$. We are going to build the following commuting diagram, where the dashed arrow is the desired map $\tilde{\gamma}_{n,m}$:

$$\begin{array}{ccccc}
 A_m & \twoheadrightarrow & E_m & \twoheadrightarrow & B_m \\
 \downarrow f & & \swarrow & \downarrow & \parallel \\
 \tilde{A}_{n'} & \twoheadrightarrow & \tilde{E}_n & \twoheadrightarrow & B_m \\
 \downarrow g & & \swarrow \tilde{\gamma}_{n,m} & \downarrow & \downarrow \beta_{n,m} \\
 A_n & \twoheadrightarrow & E_n & \twoheadrightarrow & B_n
 \end{array}$$

$\alpha_{n,m}$ is the curved arrow from A_m to A_n . $\gamma_{n,m}$ is the curved arrow from E_m to E_n .

By assumption, A is isomorphic to a projective system of dagger algebras $(\tilde{A}_{n'})_{n' \in N'}$. Therefore, there are $m \in N$, $n' \in N'$, and maps $f: A_m \rightarrow \tilde{A}_{n'}$ and $g: \tilde{A}_{n'} \rightarrow A_n$ such that $m \geq n$ and $g \circ f = \alpha_{n,m}: A_m \rightarrow A_n$. Let \tilde{E}_n be the pushout bornological V -module of the maps $A_m \rightarrow E_m$ and $A_m \rightarrow \tilde{A}_{n'}$. This fits in an extension of bornological V -modules $\tilde{A}_{n'} \twoheadrightarrow \tilde{E}_n \twoheadrightarrow B_m$ by Lemma 2.3.2. Since $\tilde{A}_{n'}$ and B_m are torsion-free and complete, \tilde{E}_n is complete by [27, Theorem 2.3]. Since $\tilde{A}_{n'}$ is semi-dagger, the canonical map $E_m \rightarrow \tilde{E}_n$ remains bounded when we give E_m the linear growth bornology relative to the ideal A_m' . This bornology is equal to the absolute linear growth bornology on E_m by Lemma 2.2.6 because $B_m = E_m/A_m$ is a dagger algebra. Since \tilde{E}_n is complete, the map $E_m \rightarrow \tilde{E}_n$ extends to a bounded V -module homomorphism $E_m^\dagger \rightarrow \tilde{E}_n$. By construction, the map $\gamma_{n,m}: E_m \rightarrow E_n$ agrees on A_m with the composite map

$$A_m \xrightarrow{f} \tilde{A}_{n'} \xrightarrow{g} A_n \rightarrow E_n.$$

Then the universal property of pushouts gives an induced bounded V -module homomorphism $\tau: \tilde{E}_n \rightarrow E_n$. Let $\tilde{\gamma}_{n,m}: E_m^\dagger \rightarrow E_n$ be the composite of the bounded V -module homomorphisms $E_m^\dagger \rightarrow \tilde{E}_n$ and $\tilde{E}_n \rightarrow E_n$ defined above. The composite map $E_m \rightarrow E_m^\dagger \rightarrow E_n$ is $\gamma_{n,m}$ by construction. This finishes the proof that E_n is isomorphic to a projective system of dagger algebras. \square

4.4. The analytic tensor algebra. Let R be a constant pro-algebra. The definitions of $\mathbb{H}\mathbb{A}(R)$ and $\mathbb{H}\mathbb{A}_*(R)$ use a certain pro-algebra $\mathcal{T}R$ defined by completing the tensor algebra $\mathbb{T}R$. We call $\mathcal{T}R$ the analytic tensor algebra of R . We

show that there is a semi-split analytically nilpotent extension $\mathcal{J}R \twoheadrightarrow \mathcal{T}R \twoheadrightarrow R$ and that $\mathcal{T}R$ is analytically quasi-free. Since it is not more difficult, we extend the construction of the analytic tensor algebra to pro-algebras right away.

DEFINITION 4.4.1. Let $R = (R_n, \alpha_{m,n})_{m,n \in \mathbb{N}}$ be a pro-algebra. Extending the tensor algebra construction to pro-algebras gives a natural semi-split pro-algebra extension $\mathcal{J}R \twoheadrightarrow \mathcal{T}R \twoheadrightarrow R$ with $\mathcal{T}R = (\mathcal{T}R_n)_{n \in \mathbb{N}}$ and $\mathcal{J}R = (\mathcal{J}R_n)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, we form the tube algebras $\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^l)$ with the ideals $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^l)$, and their relative dagger completions $(\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^l), \mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^l))^\dagger$. These form a pro-algebra indexed by the product set $\mathbb{N} \times \mathbb{N}$, which we call the *analytic tensor algebra* of R and denote by $\mathcal{T}R$.

LEMMA 4.4.2. *The canonical homomorphism $p: \mathcal{T}R \rightarrow R$ extends uniquely to a pro-algebra homomorphism $\tilde{p}: \mathcal{T}R \rightarrow R$. The composite σ_{an} of the pro-linear map $\sigma_R: R \rightarrow \mathcal{T}R$ and the canonical homomorphism $\mathcal{T}R \rightarrow \mathcal{T}R$ is a section for \tilde{p} .*

PROOF. Fix $n \in \mathbb{N}$ and $l \in \mathbb{N}^*$. The canonical homomorphism $\mathcal{T}R_n \rightarrow R_n$ vanishes on $\mathcal{J}R_n$. Then it extends uniquely to the tube algebra $\mathcal{U}(\mathcal{T}R_n, (\mathcal{J}R_n)^l)$ by Proposition 4.2.4. This extension vanishes on $\mathcal{U}(\mathcal{J}R_n, (\mathcal{J}R_n)^l)$. Then it remains bounded for the linear growth bornology relative to this ideal and extends uniquely to a homomorphism on the relative dagger completion. These maps for all n and l form a morphism of pro-algebras $\tilde{p}: \mathcal{T}R \rightarrow R$. The canonical maps $\sigma_{R_n}: R_n \rightarrow \mathcal{T}R_n$ form a pro-linear section for $p: \mathcal{T}R \rightarrow R$. Composing with the canonical map $\mathcal{T}R \rightarrow \mathcal{T}R$ gives a section for \tilde{p} . \square

DEFINITION 4.4.3. Let $\mathcal{J}R$ be the kernel of $\tilde{p}: \mathcal{T}R \rightarrow R$.

Lemma 4.4.2 implies that there is a semi-split extension of pro-algebras

$$\mathcal{J}R \twoheadrightarrow \mathcal{T}R \begin{array}{c} \xrightarrow{\tilde{p}} \\ \xrightarrow{\sigma_R} \end{array} R.$$

PROPOSITION 4.4.4. *The pro-algebra $\mathcal{J}R$ is analytically nilpotent.*

PROOF. Let $m \in \mathbb{N}^*$. The linear growth bornology on $\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^m)$ relative to $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^m)$ restricts to the ‘‘absolute’’ linear growth bornology on $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^m)$ by Lemma 2.2.5. The tensor algebra is bornologically torsion-free by Remark 2.6.2. Then so is $\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^m)$ by the definition of the bornology on the tube algebra. Then the relative linear growth bornology on it is torsion-free by Lemma 2.2.7, and this property is preserved by completions (see [27, Theorem 4.6]). Therefore, the completion of $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^m)$ in the linear growth bornology is a dagger algebra. Then $\mathcal{J}R$ is a pro-dagger algebra. And $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^\infty)$ is nilpotent mod π by Proposition 4.2.4. This remains unaffected when we equip the tube algebras with the linear growth bornology and complete. \square

REMARK 4.4.5. Let $R = (R_n, \alpha_{m,n})_{m,n \in \mathbb{N}}$ be a projective system of dagger algebras. Since $\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^l) / \mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^l) \cong R$ is semi-dagger, the linear growth bornology on $\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^l)$ is equal to the linear growth bornology relative to $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^l)$ by Lemma 2.2.6. Hence $\mathcal{T}R$ is also equal to the ‘‘absolute’’ dagger completion,

$$\mathcal{T}R \cong \mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^\infty)^\dagger.$$

PROPOSITION 4.4.6. *The analytic tensor algebra $\mathcal{T}R$ is analytically quasi-free and quasi-free. The bimodule $\overline{\Omega}^1(\mathcal{T}R)$ is isomorphic to the free bimodule on R , that is,*

$$(4.4.7) \quad (\mathcal{T}R)^+ \overline{\otimes} R \overline{\otimes} (\mathcal{T}R)^+ \cong \overline{\Omega}^1(\mathcal{T}R);$$

the isomorphism is the map $\omega \otimes x \otimes \eta \mapsto \omega \cdot (d\sigma_R(x)) \cdot \eta$. And the following maps are isomorphisms of left or right $\mathcal{T}R$ -modules, respectively:

$$\begin{aligned} (\mathcal{T}R)^+ \overline{\otimes} R &\xrightarrow{\cong} \mathcal{T}R, & \omega \otimes x &\mapsto \omega \odot \sigma_R(x), \\ R \overline{\otimes} (\mathcal{T}R)^+ &\xrightarrow{\cong} \mathcal{T}R, & x \otimes \omega &\mapsto \sigma_R(x) \odot \omega. \end{aligned}$$

PROOF. Let $J \twoheadrightarrow E \xrightarrow{q} \mathcal{T}R$ be a semi-split, analytically nilpotent pro-algebra extension. Pull it back along the inclusion $\mathcal{J}R \hookrightarrow \mathcal{T}R$ to a pro-algebra extension $J \twoheadrightarrow K \twoheadrightarrow \mathcal{J}R$ and identify K with an ideal in E . Since J and $\mathcal{J}R$ are analytically nilpotent, so is K by Proposition 4.3.13. Let $s: \mathcal{T}R \rightarrow E$ be a pro-linear section and let $\sigma_R: R \rightarrow \mathcal{T}R$ be the canonical pro-linear section. The pro-linear map $s \circ \sigma_R$ induces a pro-algebra homomorphism $(s \circ \sigma_R)^\#: \mathcal{T}R \rightarrow E$ by Lemma 2.0.2. It satisfies $q \circ (s \circ \sigma_R)^\# = \sigma_R^\#$, and $\sigma_R^\#: \mathcal{T}R \rightarrow \mathcal{T}R$ is the canonical homomorphism because $\sigma_R^\#$ and the inclusion map agree on the image of R in $\mathcal{T}R$. In particular, $(s \circ \sigma_R)^\#$ maps $\mathcal{J}R$ into $K \triangleleft E$. Since K is nilpotent mod π , Proposition 4.2.4 shows that $(s \circ \sigma_R)^\#$ extends to the tube algebra $\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^\infty)$, in such a way that $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^\infty)$ is mapped to K . And since K is a pro-dagger algebra, the criterion in Proposition 2.2.9 shows that the morphism $\mathcal{U}(\mathcal{T}R, (\mathcal{J}R)^\infty) \rightarrow E$ extends uniquely to the dagger completion relative to $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^\infty)$. This gives a pro-algebra morphism $\mathcal{T}R \rightarrow E$ that is a section for the extension $J \twoheadrightarrow E \xrightarrow{q} \mathcal{T}R$. So $\mathcal{T}R$ is analytically quasi-free.

If $h: R \rightarrow E$ is any pro-linear map with $q \circ h = \sigma_R$, then the argument above shows that $h^\#: \mathcal{T}R \rightarrow E$ extends uniquely to a pro-algebra morphism $\mathcal{T}R \rightarrow E$ that is a section for the extension. Conversely, any multiplicative section $g: \mathcal{T}R \rightarrow E$ is of this form for $h := g \circ \sigma_R$. Thus the multiplicative sections for the extension $J \twoheadrightarrow E \xrightarrow{q} \mathcal{T}R$ are in bijection with pro-linear maps $R \rightarrow E$ with $q \circ h = \sigma_R$. Any such pro-linear map is equal to $s \circ \sigma_R + h_0$ for a unique pro-linear map $h_0: R \rightarrow J$. So multiplicative sections for our extension are in bijection with pro-linear maps $R \rightarrow J$. Combined with Lemma 2.5.1, we get a natural bijection for all $\mathcal{T}R$ -bimodules M between pro-bimodule homomorphisms $\overline{\Omega}^1(\mathcal{T}R) \rightarrow M$ and pro-linear maps $R \rightarrow M$. Thus $\overline{\Omega}^1(\mathcal{T}R)$ is isomorphic to the free bimodule on R , which is $(\mathcal{T}R)^+ \overline{\otimes} R \overline{\otimes} (\mathcal{T}R)^+$. And this isomorphism is indeed induced by the map $\omega \otimes x \otimes \eta \mapsto \omega \cdot (d\sigma_R(x)) \cdot \eta$.

Now let M be a left $\mathcal{T}R$ -module. Turn M into a $\mathcal{T}R$ -bimodule by taking the zero map as right module structure. Then a bimodule derivation $\mathcal{T}R \rightarrow M$ is just a left module map. Therefore, left module homomorphisms $\mathcal{T}R \rightarrow M$ are in bijection with pro-linear maps $R \rightarrow M$. Thus the map

$$(\mathcal{T}R)^+ \overline{\otimes} R, \quad \omega \otimes x \mapsto \omega \odot \sigma_R(x),$$

is an isomorphism of left $\mathcal{T}R$ -modules. Here we have written \odot for the multiplication in $\mathcal{T}R$ because we will later use these formulas when $\mathcal{T}R$ is identified with $\Omega^{\text{ev}}R$ with the Fedosov product. A similar argument works for right modules. \square

We now describe the analytic tensor algebra and its bornology more concretely. For this, we assume that R is a torsion-free, complete bornological algebra. A

projective system $(R_n)_{n \in \mathbb{N}}$ is treated by applying the following discussion to R_n for each $n \in \mathbb{N}$. We identify $\mathbb{T}R$ with $\Omega^{\text{ev}}R$ with the Fedosov product as in Section 2.6. Recall that the isomorphism $\mathbb{T}R \cong \Omega^{\text{ev}}R$ maps the ideal $\mathbb{J}R^m$ onto $\bigoplus_{n \geq m} \Omega^{2n}R$. Thus $\mathcal{U}(\mathbb{T}R, (\mathbb{J}R)^m)$ is spanned by $\pi^{-j} \Omega^{2n}R$ with $n \geq m \cdot j$. And $\mathcal{U}(\mathbb{J}R, (\mathbb{J}R)^m)$ is spanned by $\pi^{-j} \Omega^{2n}R$ with $n \geq m \cdot j$ and $n \geq 1$. Equivalently,

$$(4.4.8) \quad \mathcal{U}(\mathbb{T}R, (\mathbb{J}R)^m) = \sum_{n=0}^{\infty} \pi^{-\lfloor n/m \rfloor} \Omega^{2n}R, \quad \mathcal{U}(\mathbb{J}R, (\mathbb{J}R)^m) = \sum_{n=1}^{\infty} \pi^{-\lfloor n/m \rfloor} \Omega^{2n}R.$$

The following lemma estimates the growth of Fedosov products in ΩR :

LEMMA 4.4.9. *Let R be an algebra and let $M \subseteq R$ be a submodule. Let $i_0, \dots, i_n \geq 1$ and $i := i_0 + \dots + i_n$. Then*

$$\Omega^{i_0}M \odot \dots \odot \Omega^{i_n}M \subseteq \bigoplus_{j=0}^n \Omega^{i+2j}(M^{(3)}).$$

PROOF. As in the proof of [26, Theorem 5.11], we show the more precise estimate

$$(4.4.10) \quad \Omega^{i_0}M \odot \dots \odot \Omega^{i_n}M \subseteq \bigoplus_{j=0}^n (M^{(2)})^+ \text{d}(M^{(3)})^{i+2j}$$

by induction on n . This is trivial for $n = 0$. The induction step uses (2.6.5) and

$$\Omega^i M \odot (M^{(2)})^+ \subseteq (M^{(2)})^+ \text{d}(M^{(3)})^i + (\text{d}M)^{i+1} \text{d}(M^{(2)}). \quad \square$$

PROPOSITION 4.4.11. *Let R be a torsion-free bornological algebra and $m \geq 1$. If $M \subseteq R$ is bounded, $\alpha \in \mathbb{Q} \cap (0, 1/m)$, and $f \in \mathbb{N}_0$, then define*

$$(4.4.12) \quad D_m(M, \alpha, f) := \bigoplus_{n=0}^{\infty} \pi^{-\lfloor \min\{n/m, \alpha \cdot n + f\} \rfloor} \Omega^{2n}M.$$

These are V -submodules of $\mathcal{U}(\mathbb{T}R, (\mathbb{J}R)^m)$ that cofinally generate its linear growth bornology relative to the ideal $\mathcal{U}(\mathbb{J}R, (\mathbb{J}R)^m)$.

PROOF. Let $M \subseteq R$ be bounded, $\alpha \in \mathbb{Q} \cap (0, 1/m)$, and $f \in \mathbb{N}_0$. Equation (4.4.8) implies $D_m(M, \alpha, f) \subseteq \mathcal{U}(\mathbb{T}R, \mathbb{J}R^m)$. Our first goal is to show that $D_m(M, \alpha, f)$ has linear growth relative to $\mathcal{U}(\mathbb{J}R, \mathbb{J}R^m)$. Let $e \geq 1$. We claim that

$$(4.4.13) \quad M^+ \cdot \left(\sum_{n=1}^{em} \pi^{-\lfloor n/m \rfloor} (\text{d}M \text{d}M)^n \right)^{\diamond} = \bigoplus_{n=1}^{\infty} \pi^{-\lfloor n/m \rfloor + \lfloor \frac{n}{em} \rfloor - 1} \Omega^{2n}M.$$

By definition, the left hand side is spanned by Fedosov products

$$\begin{aligned} \pi^{j-1-\lfloor i_1/m \rfloor - \dots - \lfloor i_j/m \rfloor} M^+ \odot (\text{d}M, \text{d}M)^{i_1} \odot \dots \odot (\text{d}M, \text{d}M)^{i_j} \\ = \pi^{j-1-\lfloor i_0/m \rfloor - \dots - \lfloor i_j/m \rfloor} \Omega^{2(i_1 + \dots + i_j)}(M) \end{aligned}$$

for $j \geq 1$ and $1 \leq i_1, \dots, i_j \leq em$. These contribute to $\Omega^{2n}M$ if $i_1 + \dots + i_j = n$. For fixed n and j , the sum of floors $\lfloor i_1/m \rfloor + \dots + \lfloor i_j/m \rfloor$ is maximal if all but one of the i_j are divisible by m , and then it becomes $\lfloor n/m \rfloor$. For fixed n , the term $j-1-\lfloor n/m \rfloor$ becomes minimal if j is minimal. Equivalently, we choose $i_j = em$ for all but one j , and then $j = \lceil n/em \rceil$. This finishes the proof of (4.4.13).

The right hand side in (4.4.13) is one of the generators of the linear growth bornology relative to $\mathcal{U}(\mathbb{J}R, (\mathbb{J}R)^m)$. For fixed $\alpha < 1/m$ and f as above, there is

$e \in \mathbb{N}^*$ with $1/m - 1/(em) > \alpha$. Then there is $k \in \mathbb{N}$ with

$$\lfloor n/m \rfloor - \left\lfloor \frac{n}{em} \right\rfloor + 1 \geq \lfloor \min\{n/m, \alpha \cdot n + f\} \rfloor$$

for $n > k$. Then

$$D_m(M, \alpha, f) \subseteq \sum_{n=0}^k \pi^{-\lfloor \min\{n/m, \alpha \cdot n + f\} \rfloor} \Omega^{2n} M + M^+ \cdot \left(\sum_{n=1}^{em} \pi^{-\lfloor n/m \rfloor} (dM \, dM)^n \right)^\diamond.$$

The first, finite sum is already bounded in $\mathcal{U}(\mathcal{T}R, \mathcal{J}R^m)$. Therefore, $D_m(M, \alpha, f)$ has linear growth relative to $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^m)$.

Now let S be any V -submodule of $\mathcal{U}(\mathcal{T}R, \mathcal{J}R^m)$ that has linear growth relative to $\mathcal{U}(\mathcal{J}R, (\mathcal{J}R)^m)$. We claim that S is contained in $D_m(M, \alpha, f)$ for suitable M, α, f . By definition of the relative linear growth bornology, there are $k, e \in \mathbb{N}$ and a bounded submodule $M \subseteq R$ such that S is contained in the sum of $\sum_{n=0}^k \pi^{-\lfloor n/m \rfloor} \Omega^{2n} M$ and $(\sum_{i=1}^{em} \pi^{-\lfloor \frac{i}{m} \rfloor} \Omega^{2i} M)^\diamond$. The latter is spanned by Fedosov products

$$\pi^{j-1-\lfloor \frac{i_1}{m} \rfloor - \dots - \lfloor \frac{i_j}{m} \rfloor} \Omega^{2i_1} M \odot \dots \odot \Omega^{2i_j} M$$

with $j \in \mathbb{N}^*$, $1 \leq i_1, \dots, i_j \leq em$. By Lemma 4.4.9, $\Omega^{2i_1} M \odot \dots \odot \Omega^{2i_j} M$ is contained in the sum of $\Omega^{2n}(M^{(3)})$, where n lies between $i := \sum_{k=1}^j i_k$ and $i + j$. As above, the sum of the floors $\lfloor i_k/m \rfloor$ for fixed i is maximal if all but one i_k are divisible by m , and then it is $\lfloor i/m \rfloor$. The constraints $i_k \leq em$ are equivalent to the constraint $i \leq j \cdot em$. So S is contained in the sum of $\pi^{j-1-\lfloor i/m \rfloor} \Omega^{2n}(M^{(3)})$ with $i \leq n \leq i + j$ and $i \leq j \cdot em$. For fixed n, j , the exponent $j - 1 - \lfloor i/m \rfloor$ is minimal if i is maximal, so we may assume that i is the minimum of n and jem . Then the optimal choice for j is the minimal one, which is $\lfloor n/(em) \rfloor$ if $i = n$ and $j = \lfloor n/(em + 1) \rfloor$ if $i = jem$. The resulting exponents of π become $\lfloor n/(em) \rfloor - 1 - \lfloor n/m \rfloor$ in the first case and $\lfloor n/(em + 1) \rfloor - 1 - \lfloor n/(em + 1) \rfloor \cdot e$ in the second. If $\alpha > 1/m - 1/(em)$ and n is large enough, then both terms are greater or equal $-\lfloor \alpha n \rfloor$. Choosing f big enough, we may arrange that both are greater or equal $-\lfloor \min\{n/m, \alpha n + f\} \rfloor$ for all $n \in \mathbb{N}$. Then $S \subseteq D_m(M^{(3)}, \alpha, f)$. \square

COROLLARY 4.4.14. *For $m \in \mathbb{N}^*$, let \mathcal{B}_m be the bornology on $\mathcal{U}(\mathcal{T}R, \mathcal{J}R^m)$ that contains a subset if and only if it is contained in $\bigoplus_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor} \Omega^{2n} M$ for some bounded V -submodule $M \subseteq R$. This bornology makes $\mathcal{U}(\mathcal{T}R, \mathcal{J}R^m)$ a torsion-free bornological algebra. The projective system of bornological algebras $(\mathcal{U}(\mathcal{T}R, \mathcal{J}R^m), \mathcal{B}_m)_{m \in \mathbb{N}^*}$ is isomorphic to the projective system formed by $\mathcal{U}(\mathcal{T}R, \mathcal{J}R^m)$ with the linear growth bornology relative to $\mathcal{U}(\mathcal{J}R, \mathcal{J}R^m)$.*

PROOF. By Lemma 4.4.9, the Fedosov product is bounded for the bornology \mathcal{B}_m . The subsets $D_m(M, \alpha, f)$ in (4.4.12) are clearly in \mathcal{B}_m . Conversely,

$$\bigoplus_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m+1} \rfloor} \Omega^{2n} M = D_m\left(M, \frac{1}{m+1}, 0\right).$$

Thus any subset in \mathcal{B}_{m+1} is mapped to a subset of $\mathcal{U}(\mathcal{T}R, \mathcal{J}R^m)$ with linear growth relative to $\mathcal{U}(\mathcal{J}R, \mathcal{J}R^m)$. The asserted isomorphism of projective systems follows. \square

Now we can describe the completion $\mathcal{T}R$. Recall that $\overline{\Omega}^n R$ denotes the completion $R^+ \overline{\otimes} R^{\overline{\otimes} n}$ of $\Omega^n R = R^+ \otimes R^{\otimes n}$. For $m \in \mathbb{N}^*$ and a bounded V -submodule $M \subseteq R$, the canonical map $\overline{\Omega}^{2n} M \rightarrow \overline{\Omega}^{2n} R$ is injective by Proposition 2.4.5. Then we may view $\prod_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor} \overline{\Omega}^{2n} M$ as a V -submodule of $\prod_{n=0}^{\infty} \overline{\Omega}^{2n} R \otimes F$. Let $\overline{\Omega}^{\text{ev}}(R)_m$ be the

union of $\prod_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor} \overline{\Omega}^{2n} M$ for all bounded V -submodules $M \subseteq R$, with the bornology where a subset is bounded if and only if it is contained in $\prod_{n=0}^{\infty} \pi^{-\lfloor \frac{n}{m} \rfloor} \overline{\Omega}^{2n} M$ for some bounded V -submodules $M \subseteq R$. These form a decreasing sequence of subalgebras with bounded inclusion maps $\overline{\Omega}^{\text{ev}}(R)_{m+1} \hookrightarrow \overline{\Omega}^{\text{ev}}(R)_m$.

PROPOSITION 4.4.15. *If R is a torsion-free, complete bornological algebra, then $\mathcal{T}R$ is naturally isomorphic to the projective system of complete bornological algebras $(\overline{\Omega}^{\text{ev}}(R)_m)_{m \in \mathbb{N}^*}$.*

PROOF. We shall use the explicit description of the relative linear growth bornology in Proposition 4.4.11. Each $\pi^{-\lfloor n/m \rfloor} \overline{\Omega}^{2n} R$ is a direct summand of $\mathcal{U}(\mathcal{T}R, JR^m)$, and the projection is bounded in the linear growth bornology relative to $\mathcal{U}(JR, JR^m)$. This gives us maps from the completed tube to $\pi^{-\lfloor n/m \rfloor} \overline{\Omega}^{2n} R$ for all $n \in \mathbb{N}$. It is easy to see that the π -adic completion of $D_m(M, \alpha, f)$ is isomorphic to the subspace of $\prod_{n=0}^{\infty} \pi^{-\lfloor n/m \rfloor} \overline{\Omega}^{2n} M$ consisting of all $(\omega_n)_{n \in \mathbb{N}}$ for which there is a sequence $(h_j)_{j \in \mathbb{N}}$ in \mathbb{N} with $\lim h_j = \infty$ and $\omega_n \in \pi^{-\lfloor \min\{n/m, \alpha \cdot n + f\} \rfloor + h_n} \overline{\Omega}^{2n} M$ for all $n \in \mathbb{N}$. Any such subset is bounded in $\overline{\Omega}^{\text{ev}}(R)_m$. Conversely, any bounded subset in $\overline{\Omega}^{\text{ev}}(R)_{m+1}$ is contained in a subset of this form with $f = 0$ and $m < 1/\alpha < m+1$. Therefore, the projective system formed by the relative dagger completions $(\mathcal{U}(\mathcal{T}R, JR^m), \mathcal{U}(JR, JR^m))^\dagger$ is isomorphic to the projective system $(\overline{\Omega}^{\text{ev}}(R)_m)_{m \in \mathbb{N}^*}$. \square

4.5. Pro-Linear maps with nilpotent curvature. Let R and S be pro-algebras. We are going to describe pro-algebra homomorphisms $\mathcal{T}R \rightarrow S$ through a certain class of pro-linear maps $R \rightarrow S$, namely, those with analytically nilpotent curvature. This follows rather easily from the concrete description of the relative linear growth bornology above. The main issue is to define analytically nilpotent curvature. We begin with the analogue of nilpotent curvature mod π .

DEFINITION 4.5.1. Let $X = (X_{n'})_{n' \in N'}$ be a bornological pro-module and $S = (S_n)_{n \in N}$ a pro-algebra and let $\omega: X \rightarrow S$ be a pro-linear map. We call ω *nilpotent mod π* if, for every $n \in N$, there is $m \in \mathbb{N}^*$ such that the following composite is zero:

$$(4.5.2) \quad X^{\otimes m} \xrightarrow{\omega^{\otimes m}} S_m^{\otimes m} \xrightarrow{\text{mult}} S_m \rightarrow S_m/\pi S_m;$$

here mult denotes the m -fold multiplication map of S .

Let $\omega: X \rightarrow S$ be nilpotent mod π and represent ω by a coherent family of bounded V -module maps $\omega_n: X_{r(n)} \rightarrow S_n$ with $r(n) \in N'$ for $n \in N$. For $n \in N$ and $n' \in N'$ with $n' \geq r(n)$, let $\omega_{n,n'}: X_{n'} \rightarrow S_n$ be the composite map $X_{n'} \rightarrow X_{r(n)} \rightarrow S_n$. Let $n \in N$ and choose m so that the map in (4.5.2) vanishes. Then there is $n' \in N'$ with $n' \geq r(n)$ such that the composite map $X_{n'}^{\otimes m} \rightarrow S_n^{\otimes m} \rightarrow S_n \rightarrow S_n/\pi S_n$ vanishes. That is, $\omega_{n,n'}(x_1) \cdots \omega_{n,n'}(x_m) \in \pi \cdot S_n$ for all $x_1, \dots, x_m \in X_{n'}$. Let $M \subseteq X_{n'}$ be bounded. Since $\omega_{n,n'}$ is bounded and S_n is torsion-free, it follows that $\omega_{n,n'}(M)^m \subseteq \pi S_n$ and that $\pi^{-1} \cdot \omega_{n,n'}(M)^m \subseteq S_n$ is bounded. Then

$$(4.5.3) \quad \omega_{n,n'}(M)_e := \sum_{j=1}^{em} \pi^{-\lfloor j/m \rfloor} \omega_{n,n'}(M)^j$$

is bounded for every $e \geq 1$.

DEFINITION 4.5.4. Let $X = (X_m)_{m \in N'}$ be a bornological pro-module and $S = (S_n)_{n \in N}$ a pro-algebra and let $\omega: X \rightarrow S$ be a pro-linear map. Represent ω by a

coherent family of bounded V -module maps $\omega_{n,n'}: X_{n'} \rightarrow S_n$ as above. The map ω is called *analytically nilpotent* if, for every n , there are $m \in \mathbb{N}^*$ and $n' \in N'$ with $n' \geq r(n)$ such that for any bounded subset $M \subseteq X_{n'}$, the subset

$$\sum_{j=0}^{\infty} \pi^{-\lfloor j/m \rfloor} \omega_{n,n'}(M)^j \subseteq S_n \otimes F$$

is bounded in S_n .

PROPOSITION 4.5.5. *Let R and S be pro-algebras and let $f: R \rightarrow S$ be a pro-linear map. Let $\omega: R \otimes R \rightarrow S$, $x \otimes y \mapsto f(x \cdot y) - f(x) \cdot f(y)$, be its curvature. There is a pro-algebra homomorphism $f^\#: \mathcal{T}R \rightarrow S$ with $f = f^\# \sigma_R = f$ if and only if ω is analytically nilpotent.*

PROOF. Write $R = (R_{n'})_{n' \in N'}$ and $S = (S_n)_{n \in N}$ as projective systems of algebras. Then $\mathcal{T}R$ is the completion of the projective system of bornological algebras $T := (\mathcal{U}(\mathcal{T}R_{n'}, \mathcal{J}R_{n'}^m), \mathcal{B}_m)_{n' \in N', m \in \mathbb{N}^*}$ with the bornologies \mathcal{B}_m in Corollary 4.4.14. Since S is complete, any homomorphism of projective systems of bornological algebras $T \rightarrow S$ extends uniquely to $\mathcal{T}R$. Since S is torsion-free, such a homomorphism $T \rightarrow S$ is determined by its restriction to $\mathcal{T}R$. Then there is a unique pro-linear map $f: R \rightarrow S$ such that the homomorphism is $f^\#: \mathcal{T}R \rightarrow S$ as in (2.6.7). Corollary 4.4.14 shows that $f^\#$ extends to a homomorphism $T \rightarrow S$ if and only if f has analytically nilpotent curvature. \square

COROLLARY 4.5.6. *Let $f: R \rightarrow S$, $g: S \rightarrow T$ be pro-linear maps and let U be a projective system of dagger algebras. If f and g have analytically nilpotent curvature, then so do $g \circ f$ and $f \overline{\otimes} U: R \overline{\otimes} U \rightarrow S \overline{\otimes} U$.*

PROOF. The assertion about $g \circ f$ follows as in the proof of [26, Theorem 5.23], using [27, Theorems 3.7 and 4.5]. Since f has analytically nilpotent curvature, there is a homomorphism $f^\#: \mathcal{T}R \rightarrow S$ with $f^\# \circ \sigma_R = f$. The extension

$$(\mathcal{J}R) \overline{\otimes} U \twoheadrightarrow (\mathcal{T}R) \overline{\otimes} U \twoheadrightarrow R \overline{\otimes} U$$

is analytically nilpotent because $(\mathcal{J}R) \overline{\otimes} U$ is nilpotent mod π by Proposition 4.2.6 and a pro-dagger algebra by the extension of Corollary 2.1.21 to projective systems. The pro-linear section $\sigma_R \overline{\otimes} U$ induces a homomorphism $\mathcal{T}(R \overline{\otimes} U) \rightarrow (\mathcal{T}R) \overline{\otimes} U$ which, when composed with $f^\#$, gives a homomorphism $\mathcal{T}(R \overline{\otimes} U) \rightarrow S$ that extends $f \overline{\otimes} U$. Thus $f \overline{\otimes} U$ has analytically nilpotent curvature. \square

4.6. Homotopy invariance of the X -complex. In this section, we assume that the field F has characteristic 0. This is needed to prove that homotopic homomorphisms defined on a quasi-free algebra induce chain homotopic maps between the X -complexes. If we understand homotopy to mean “polynomial homotopy”, then this is already shown by Cuntz and Quillen (see [17, Sections 7–8]). In the context of complete bornological V -algebras, the proof for polynomial homotopies still works for dagger homotopies. The corresponding statement for the B, b -bicomplexes is [11, Proposition 4.3.3]. For quasi-free algebras, the canonical projection from the B, b -bicomplex to the X -complex is a chain homotopy equivalence. This implies the following:

PROPOSITION 4.6.1. *Let R and S be projective systems of complete bornological F -algebras. Let $f, g: R \rightrightarrows S$ be two homomorphisms that are dagger homotopic.*

Assume that F has characteristic 0 and that R is quasi-free. Then the induced chain maps $X(f), X(g): X(R) \rightrightarrows X(S)$ are chain homotopic.

PROOF. It suffices to treat an elementary dagger homotopy. Define

$$\eta_n: \Omega^n(S \otimes V[t]) \otimes F \rightarrow \Omega^{n-1}(S) \otimes F,$$

$$a_0 da_1 \dots da_n \mapsto \int_0^1 a_0(t) \frac{\partial a_1(t)}{\partial t} da_2(t) \dots da_n(t) dt,$$

for $n = 1, 2$. Here integration and differentiation are defined formally by rescaling the coefficients of polynomials $a_i \in S \otimes F[t]$. We claim that η_n extends to a bounded linear map $\eta_n: \Omega^n(S \otimes V[t]^\dagger) \otimes F \rightarrow \Omega^{n-1}(S) \otimes F$. To see this, let $T := S \otimes V[t]_{\text{lg}}$. Then $\Omega^n(T) \cong T^+ \otimes T^{\otimes n} \cong T^{\otimes n} \oplus T^{\otimes n+1}$. So it suffices to show that η_n is bounded on $T^{\otimes n} \otimes F \cong S^{\otimes n} \otimes V[t]_{\text{lg}}^{\otimes n} \otimes F$. This follows if the map

$$V[t]_{\text{lg}}^{\otimes n} \otimes F \rightarrow F,$$

$$a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto \int_0^1 a_0(t) \frac{\partial a_1(t)}{\partial t} \cdot a_2(t) \dots a_n(t) dt$$

is bounded. The formal differentiation on $V[t]_{\text{lg}}$ is clearly bounded. And $V[t]_{\text{lg}}$ is a bornological algebra. So this happens if and only if the integration map

$$V[t]_{\text{lg}} \otimes F \rightarrow F, \quad a(t) = \sum_{l=0}^{\infty} c_l t^l \mapsto \sum_{l=0}^{\infty} \frac{c_l}{l+1}$$

is bounded. If \mathbb{F} has characteristic 0, then $l+1$ is invertible in V for all $l \in \mathbb{N}$. If \mathbb{F} has finite characteristic p , then the valuation of $l+1$ grows at most logarithmically. In any case, this is dominated by the linear growth of the exponents of π for a subset of linear growth in $V[t]$. Thus the integration map above is bounded, and then so are the maps η_n . We still write η_n for their unique bounded extensions to the completions.

Let $\eta_0 = 0$. Then $[\eta, b] = 0$. Therefore, $\eta_2(b(\overline{\Omega}^3(S \overline{\otimes} V[t]^\dagger))) \subseteq b(\overline{\Omega}^2(S))$. So η defines a map $X^{(2)}(S \overline{\otimes} V[t]^\dagger) \rightarrow X(S)$, where $X^{(2)}$ is the truncated $B-b$ -complex defined in [26, Definition A.122].

Let $\xi_2: X^{(2)}(S \overline{\otimes} V[t]^\dagger) \rightarrow X(S \overline{\otimes} V[t]^\dagger)$ be the canonical projection. Then

$$[\eta, B + b] = (X(\text{ev}_1) - X(\text{ev}_0)) \circ \xi_2: X^{(2)}(S \overline{\otimes} V[t]^\dagger) \rightarrow X(S).$$

Now let $H: R \rightarrow S \otimes V[t]^\dagger$ be an elementary dagger homotopy between f and g . Then $\eta \circ X^{(2)}(H): X^{(2)}(R) \rightarrow X(S)$ is a chain homotopy between $X(f) \circ \xi_2$ and $X(g) \circ \xi_2$, where $\xi_2: X^{(2)}(R) \rightarrow X(R)$ is the canonical projection. Since R is analytically quasi-free, it is in particular quasi-free, so that ξ_2 is a chain homotopy equivalence. Let $\alpha: X(R) \rightarrow X^{(2)}(R)$ be the homotopy inverse of ξ_2 . Then $\eta \circ \alpha$ is the desired chain homotopy between $X(f)$ and $X(g)$. \square

THEOREM 4.6.2. *Let A and B be pro-algebras. If two homomorphisms $f_0, f_1: A \rightrightarrows B$ are dagger homotopic, then they induce homotopic chain maps $\mathbb{H}\mathbb{A}(A) \rightarrow \mathbb{H}\mathbb{A}(B)$. And then $\text{HA}_*(f_0) = \text{HA}_*(f_1)$.*

PROOF. The homomorphisms $\mathcal{T}f_0, \mathcal{T}f_1: \mathcal{T}A \rightrightarrows \mathcal{T}B$ lift f_0 and f_1 . Since $\mathcal{T}A$ is analytically quasi-free and $\mathcal{T}B$ is analytically nilpotent, Proposition 4.3.10 provides a dagger homotopy between $\mathcal{T}f_0$ and $\mathcal{T}f_1$. Then the chain maps $X(\mathcal{T}A \otimes F) \rightrightarrows X(\mathcal{T}B \otimes F)$ induced by f_0 and f_1 are chain homotopic by Proposition 4.6.1. This remains so on the homotopy projective limits. And then f_0 and f_1 induce the

same map on the homology of the homotopy projective limits. That is, $\mathrm{HA}_*(f_0) = \mathrm{HA}_*(f_1)$. \square

4.7. Invariance under analytically nilpotent extensions. We continue to assume that F has characteristic 0.

THEOREM 4.7.1. *Let $J \twoheadrightarrow E \xrightarrow{p} A$ be a semi-split, analytically nilpotent extension of pro-algebras. Then p induces a chain homotopy equivalence $\mathbb{H}\mathbb{A}(E) \simeq \mathbb{H}\mathbb{A}(A)$ and $\mathbb{H}\mathbb{A}(J)$ is contractible. So $\mathrm{HA}_*(E) \cong \mathrm{HA}_*(A)$ and $\mathrm{HA}_*(J) = 0$. If E is analytically quasi-free, then $\mathbb{H}\mathbb{A}(A)$ is chain homotopy equivalent to $X(E \otimes F)$ and $\mathrm{HA}_*(A)$ is isomorphic to the homology of the homotopy projective limit of $X(E \otimes F)$.*

PROOF. The composite map $\mathcal{T}E \twoheadrightarrow E \twoheadrightarrow A$ is pro-algebra homomorphism with a pro-linear section. Its kernel K is an extension of $\mathcal{J}E$ by J and hence analytically nilpotent by Proposition 4.3.13. Both $\mathcal{T}E$ and $\mathcal{T}A$ are analytically quasi-free by Proposition 4.4.6. Proposition 4.3.10 applied to the extensions $K \twoheadrightarrow \mathcal{T}E \twoheadrightarrow A$ and $\mathcal{J}A \twoheadrightarrow \mathcal{T}A \twoheadrightarrow A$ shows that $\mathcal{T}A$ and $\mathcal{T}E$ are dagger homotopy equivalent. This together with Proposition 4.6.1 implies that $\mathbb{H}\mathbb{A}(A) = X(\mathcal{T}A \otimes F)$ and $\mathbb{H}\mathbb{A}(E) = X(\mathcal{T}E \otimes F)$ are homotopy equivalent. This remains so for their homotopy projective limits. So $\mathrm{HA}_*(E) \cong \mathrm{HA}_*(A)$. More precisely, the isomorphism is the map induced by the quotient map $E \twoheadrightarrow A$.

Since J and $\mathcal{J}J$ are analytically nilpotent, so is $\mathcal{T}J$ by Proposition 4.3.13. Since $\mathcal{T}J$ is analytically quasi-free, Proposition 4.3.10 may be applied to the extensions $\mathcal{T}J = \mathcal{T}J \rightarrow 0$ and $0 = 0 = 0$ of 0. Thus $\mathcal{T}J$ is dagger homotopy equivalent to 0. Then $\mathbb{H}\mathbb{A}(J) \simeq 0$ and $\mathrm{HA}_*(J) \cong 0$.

Now assume E to be analytically quasi-free. Then Proposition 4.3.10 shows that the extensions of A by $\mathcal{T}A$ and E are dagger homotopy equivalent. Then $X(E) \otimes F$ is homotopy equivalent to $X(\mathcal{T}A) \otimes F$. Then $\mathbb{H}\mathbb{A}(A)$ is homotopy equivalent to the homotopy projective limit of the projective system of chain complexes $X(E) \otimes F$. \square

COROLLARY 4.7.2. *Let A be an analytically quasi-free algebra. Then $\mathbb{H}\mathbb{A}(A)$ is chain homotopy equivalent to $X(A \otimes F)$ and $\mathrm{HA}_*(A)$ is isomorphic to the homology of $X(A \otimes F)$.*

PROOF. Theorem 4.7.1 shows that $\mathbb{H}\mathbb{A}(A)$ is homotopy equivalent to $X(A \otimes F)$. Then $\mathrm{HA}_*(A)$ is isomorphic to the homology of $\mathrm{holim} X(A \otimes F)$. Since $X(A \otimes F)$ is a constant projective system, it is chain homotopy equivalent to its homotopy projective limit. So we simply get the ordinary homology of $X(A \otimes F)$. \square

COROLLARY 4.7.3. *$\mathbb{H}\mathbb{A}(V)$ is homotopy equivalent to F with zero boundary map.*

PROOF. The algebra V is analytically quasi-free by Proposition 4.3.6. Then $\mathbb{H}\mathbb{A}(V) \simeq X(F)$ by Corollary 4.7.2. A small calculation shows that any element of $\Omega^1(V)$ is a commutator. So $X(F)$ is F with zero boundary map. \square

5. Excision

The goal of this section is to prove the following excision theorem for analytic cyclic homology:

THEOREM 5.1. *Let $K \xrightarrow{i} E \xrightarrow{p} Q$ be a semi-split extension of pro-algebras with a pro-linear section $s: Q \rightarrow E$. Then there is a natural exact triangle*

$$\mathbb{H}\mathbb{A}(K) \xrightarrow{i_*} \mathbb{H}\mathbb{A}(E) \xrightarrow{p_*} \mathbb{H}\mathbb{A}(Q) \xrightarrow{\delta} \mathbb{H}\mathbb{A}(K)[-1]$$

in the homotopy category of chain complexes of projective systems of bornological V -modules. Thus there is a natural long exact sequence

$$\begin{array}{ccccc} \mathbb{H}\mathbb{A}_0(K) & \xrightarrow{i_*} & \mathbb{H}\mathbb{A}_0(E) & \xrightarrow{p_*} & \mathbb{H}\mathbb{A}_0(Q) \\ \delta \uparrow & & & & \downarrow \delta \\ \mathbb{H}\mathbb{A}_1(Q) & \xleftarrow{p_*} & \mathbb{H}\mathbb{A}_1(E) & \xleftarrow{i_*} & \mathbb{H}\mathbb{A}_1(K). \end{array}$$

Here the arrows in the ‘‘homotopy category’’ are chain homotopy classes of chain maps. This homotopy category is triangulated over any additive category, with triangles coming from mapping cones of chain maps.

The proof will take up the rest of this section. It follows [25, 26]. We use the left ideal \mathcal{L} in $\mathcal{T}E$ generated by K and prove chain homotopy equivalences $X(\mathcal{T}K) \simeq X(\mathcal{L})$ and $X(\mathcal{L}) \simeq X(\mathcal{T}E : \mathcal{T}Q)$ as chain complexes in the additive category of projective systems of bornological V -modules. First, the pro-linear section s yields two bounded maps $s_L, s_R: \Omega^{\text{ev}} Q \rightarrow \Omega^{\text{ev}} E$ defined by

$$\begin{aligned} s_L(q_0 dq_1 \dots dq_{2n}) &:= s(q_0) ds(q_1) \dots ds(q_{2n}), \\ s_R(dq_1 \dots dq_{2n} q_{2n+1}) &:= ds(q_1) \dots ds(q_{2n}) s(q_{2n+1}) \end{aligned}$$

for all $q_0, q_{2n+1} \in Q^+$ and $q_i \in Q$ for $1 \leq i \leq 2n$. Let $m \in \mathbb{N}^*$. Both s_L and s_R map $\mathbb{J}Q^{mj}$ to $\mathbb{J}E^{mj}$ for all $j \in \mathbb{N}$ by (4.4.8). Thus they induce bounded linear maps on the tubes, from $\mathcal{U}(\mathcal{T}Q, \mathbb{J}Q^m)$ to $\mathcal{U}(\mathcal{T}E, \mathbb{J}E^m)$. Both are sections for the canonical projection $\mathcal{U}(\mathcal{T}E, \mathbb{J}E^m) \rightarrow \mathcal{U}(\mathcal{T}Q, \mathbb{J}Q^m)$. These sections remain bounded for the linear growth bornologies relative to $\mathcal{U}(\mathcal{T}E, \mathbb{J}E^m)$ and $\mathcal{U}(\mathcal{T}Q, \mathbb{J}Q^m)$ by Proposition 4.4.11. Thus they extend to bounded V -module maps on the completions. These maps for all $m \in \mathbb{N}^*$ form two pro-linear sections for $\mathcal{T}p: \mathcal{T}E \rightarrow \mathcal{T}Q$. They induce two sections for the canonical chain map $X(\mathcal{T}p): X(\mathcal{T}E) \rightarrow X(\mathcal{T}Q)$. Let

$$X(\mathcal{T}E : \mathcal{T}Q) := \ker(X(\mathcal{T}p): X(\mathcal{T}E) \rightarrow X(\mathcal{T}Q)).$$

There is a semi-split extension of chain complexes

$$X(\mathcal{T}E : \mathcal{T}Q) \rightarrow \mathcal{T}E \rightarrow \mathcal{T}Q.$$

Since $X(\mathcal{T}p) \circ X(\mathcal{T}i) = X(\mathcal{T}(p \circ i)) = 0$, the chain map $X(\mathcal{T}i)$ factors through $X(\mathcal{T}E : \mathcal{T}Q)$. We are going to prove that this chain map $X(\mathcal{T}K) \rightarrow X(\mathcal{T}E : \mathcal{T}Q)$ is a chain homotopy equivalence. Then the homotopy projective limit of $X(\mathcal{T}K)$ is homotopy equivalent to that of $X(\mathcal{T}E : \mathcal{T}Q)$, and the latter fits into a semi-split extension of chain complexes with the homotopy projective limits of $X(\mathcal{T}E)$ and $X(\mathcal{T}Q)$. As a result, Theorem 5.1 follows if the inclusion map $X(\mathcal{T}K) \rightarrow X(\mathcal{T}E : \mathcal{T}Q)$ is a chain homotopy equivalence.

Our construction of the chain homotopy equivalence will, in principle, be explicit and natural, using only the multiplication maps in our pro-algebras and the pro-linear sections s_L and s_R above. Therefore, we assume for simplicity from now on that we are dealing with an extension of (complete, torsion-free bornological) algebras $K \rightarrow E \rightarrow Q$. In general, we may rewrite the semi-split extension above as a projective system of semi-split algebra extensions $K_n \rightarrow E_n \rightarrow Q_n$ with compatible

bounded linear sections; this uses arguments as in the proof of Proposition 4.3.13. To simplify notation, we write down the proof below only for a semi-split algebra extension. The chain maps and homotopies that we are going to build for the extensions $K_n \twoheadrightarrow E_n \twoheadrightarrow Q_n$ form morphisms of projective systems. So the same proof works for a semi-split extension of pro-algebras.

5.1. The pro-algebra \mathcal{L} . In the following, we identify $\mathbb{T}E$ with $\Omega^{\text{ev}}E$ and E with $\Omega^0(E) \subseteq \Omega^{\text{ev}}E$. So the map $\sigma_E: E \rightarrow \mathcal{T}E$ disappears from our notation. Proposition 4.4.6 gives an isomorphism of left $\mathcal{T}E$ -modules

$$(5.1.1) \quad (\mathcal{T}E)^+ \overline{\otimes} E \xrightarrow{\cong} \mathcal{T}E, \quad \omega \otimes x \mapsto \omega \odot x.$$

Explicitly, the inverse of this isomorphism is given by

$$(5.1.2) \quad \omega \, de_{2n-1} \, de_{2n} \mapsto \omega \otimes (e_{2n-1} \cdot e_{2n}) - (\omega \odot e_{2n-1}) \otimes e_{2n}.$$

These two maps also define an isomorphism for the purely algebraic tensor algebras:

$$(5.1.3) \quad (\mathbb{T}E)^+ \otimes E \xrightarrow{\cong} \mathbb{T}E, \quad \omega \otimes e \mapsto \omega \odot e.$$

Variants of this isomorphism and the following ones were proven already in [26, Section 4.3.2]. Let $L \subseteq \mathbb{T}E$ be the left ideal generated by K . The bounded linear section $s: Q \rightarrow E$ yields an isomorphism of bornological V -modules $E \cong K \oplus Q$. Then (5.1.3) implies an isomorphism

$$(5.1.4) \quad (\mathbb{T}E)^+ \otimes K \xrightarrow{\cong} L, \quad \omega \otimes k \mapsto \omega \odot k.$$

The explicit formula for the isomorphism in (5.1.2) and its inverse imply

$$L = K \oplus \bigoplus_{n \geq 1} \Omega^{2n-1}(E) \, dK$$

as in the proof of [26, Lemma 4.55]. Let $I := \ker(\mathbb{T}p: \mathbb{T}E \twoheadrightarrow \mathbb{T}Q)$. This is part of semi-split extensions

$$(5.1.5) \quad I \twoheadrightarrow \mathbb{T}E \begin{array}{c} \xrightarrow{\mathbb{T}p} \\ \swarrow \text{---} \\ \text{---} \\ \searrow \end{array} \mathbb{T}Q \quad I \twoheadrightarrow (\mathbb{T}E)^+ \begin{array}{c} \xrightarrow{(\mathbb{T}p)^+} \\ \swarrow \text{---} \\ \text{---} \\ \searrow \end{array} (\mathbb{T}Q)^+.$$

LEMMA 5.1.6. *The following maps are isomorphisms:*

$$(5.1.7) \quad \Psi: L^+ \otimes (\mathbb{T}Q)^+ \xrightarrow{\cong} (\mathbb{T}E)^+, \quad l \otimes \eta \mapsto l \odot s_L(\eta),$$

$$(5.1.8) \quad L \otimes (\mathbb{T}Q)^+ \xrightarrow{\cong} I, \quad l \otimes \eta \mapsto l \odot s_L(\eta),$$

$$(5.1.9) \quad (\mathbb{T}E)^+ \otimes K \otimes (\mathbb{T}Q)^+ \xrightarrow{\cong} I, \quad \omega \otimes k \otimes \eta \mapsto \omega \odot k \odot s_L(\eta),$$

$$(5.1.10) \quad (\mathbb{T}Q)^+ \otimes K \otimes (\mathbb{T}E)^+ \xrightarrow{\cong} I, \quad \eta \otimes k \otimes \omega \mapsto s_R(\eta) \odot k \odot \omega,$$

$$(5.1.11) \quad (\mathbb{T}Q)^+ \otimes K \otimes L^+ \xrightarrow{\cong} L, \quad \eta \otimes k \otimes l \mapsto s_R(\eta) \odot k \odot l.$$

PROOF. The computations in [26, Section 4.3.1] show this. We briefly sketch them. The isomorphisms (5.1.7) and (5.1.8) are equivalent because of the semi-split extension (5.1.5). And (5.1.8) and (5.1.9) are equivalent because of the isomorphism (5.1.4). The isomorphisms (5.1.9) and (5.1.10) imply each other by taking opposite algebras because this reverses the order of multiplication and exchanges s_L and s_R . And (5.1.10) implies (5.1.11) by substituting $(\mathbb{T}E)^+ \cong L^+ \otimes (\mathbb{T}Q)^+$ and $I \cong L \otimes (\mathbb{T}Q)^+$ in (5.1.11) and then cancelling the factor $(\mathbb{T}Q)^+$ on both sides.

So it suffices to prove that Ψ is an isomorphism. We describe its inverse Ψ^{-1} . Split a differential form $e_0 \, de_1 \dots de_{2n} \in \Omega^{2n}E$ so that each coefficient e_j belongs

either to K or $s(Q)$, or is 1 in case of e_0 ; this is possible because of the direct sum decomposition $E \cong K \oplus s(Q)$; write $k_i := e_i$ or $q_i := s^{-1}(e_i)$ accordingly. If no e_i belongs to K , then

$$\Psi^{-1}(s(q_0) ds(q_1) \dots ds(q_{2n})) = 1 \otimes q_0 dq_1 \dots dq_{2n}.$$

Otherwise, there is a largest $i \leq 2n$ with $e_i \in K$. If $i = 0$, then

$$\Psi^{-1}(k_0 ds(q_1) \dots ds(q_{2n})) = k_0 \otimes dq_1 \dots dq_{2n}.$$

If i is even and non-zero, then

$$\Psi^{-1}(e_0 de_1 \dots de_{i-1} dk_i ds(q_{i+1}) \dots ds(q_{2n})) = e_0 de_1 \dots de_{i-1} dk_i \otimes dq_{i+1} \dots dq_{2n}.$$

If i is odd, then

$$\begin{aligned} \Psi^{-1}(e_0 de_1 \dots de_{i-1} dk_i ds(q_{i+1}) \dots ds(q_{2n})) \\ = e_0 de_1 \dots de_{i-1} \odot (k_i \cdot s(q_{i+1})) \otimes dq_{i+2} \dots dq_{2n} \\ - e_0 de_1 \dots de_{i-1} \odot k_i \otimes q_{i+1} dq_{i+2} \dots dq_{2n}. \end{aligned}$$

A direct computation using $dk_i ds(q_{i+1}) = k_i \cdot s(q_{i+1}) - k_i \odot s(q_{i+1})$ shows that

$$\Psi \circ \Psi^{-1}(e_0 de_1 \dots de_{2n}) = e_0 de_1 \dots de_{2n}$$

for all $e_0 \in \{1\} \cup K \cup s(Q)$, $e_1, \dots, e_n \in K \cup s(Q)$. Then one shows that the map Ψ^{-1} is surjective: its image contains all elements of the form $1 \otimes \eta$ for $\eta \in (\mathbb{T}Q)^+$ and $\omega \otimes dq_1 \dots dq_{2n}$ with $\omega \in L^+$ by the first two cases with no i or even i , respectively. And modulo a term of this form, the image of Ψ^{-1} contains all $\omega \odot k \otimes q_0 dq_1 \dots dq_{2n}$ with $\omega \in (\mathbb{T}E)^+$, $k \in K$ because of the formula in the case of odd i . This exhausts $L^+ \otimes (\mathbb{T}Q)^+$ because of the isomorphism (5.1.4). \square

We are going to pass to the analytic tensor algebras and describe ‘‘analytic’’ analogues of $L, I \subseteq \mathbb{T}E$ and of the isomorphisms and semi-split extensions above. For $m \in \mathbb{N}^*$, let

$$\begin{aligned} I_{(m)} &:= \ker(\mathcal{U}(\mathbb{T}E, \mathbb{J}E^m) \rightarrow \mathcal{U}(\mathbb{T}Q, \mathbb{J}Q^m)), \\ L_{(m)} &:= K \oplus \bigoplus_{n \geq 1} \pi^{-\lfloor n/m \rfloor} \cdot \Omega^{2n-1}(E) dK. \end{aligned}$$

It is easy to see that $I_{(m)}$ is a two-sided and $L_{(m)}$ a left ideal in $\mathcal{U}(\mathbb{T}E, \mathbb{J}E^\infty)$. In particular, both are V -algebras in their own right. Inspection shows that

$$(5.1.12) \quad I_{(m)} = \mathcal{U}(\mathbb{T}E, \mathbb{J}E^m) \cap (I \otimes F), \quad L_{(m)} = \mathcal{U}(\mathbb{T}E, \mathbb{J}E^m) \cap (L \otimes F)$$

as V -submodules of $\mathbb{T}E \otimes F$. The maps in the projective system $\mathcal{U}(\mathbb{T}E, \mathbb{J}E^\infty)$ make $(I_{(m)})_{m \in \mathbb{N}^*}$ and $(L_{(m)})_{m \in \mathbb{N}^*}$ projective systems by restriction. We equip each $\mathcal{U}(\mathbb{T}E, \mathbb{J}E^m)$ with the bornology \mathcal{B}_m described in Corollary 4.4.14; using the linear growth bornology instead would slightly complicate the estimates below. We give $I_{(m)}$ and $L_{(m)}$ the subspace bornologies. So the bornology on $L_{(m)}$ is cofinally generated by

$$(5.1.13) \quad (M \cap K) \oplus \bigoplus_{n=1}^{\infty} \pi^{-\lfloor n/m \rfloor} \Omega^{2n-1} M d(M \cap K)$$

for bounded V -submodules $M \subseteq E$. Let $\mathcal{I} := (\overline{I_{(m)}})_{m \in \mathbb{N}^*}$ and $\mathcal{L} := (\overline{L_{(m)}})_{m \in \mathbb{N}^*}$ be the projective systems formed by the completions.

Since $\mathcal{U}(\mathbb{T}E, \mathbb{J}E^m)$ is a subalgebra of $\mathbb{T}E \otimes F$ and the maps in (5.1.3), (5.1.4) and (5.1.7)–(5.1.11) only involve Fedosov products and the maps s_L and s_R , (5.1.12)

implies that these maps still exist and are bounded if $\mathcal{T}E, \mathcal{T}Q, I, L$ are replaced by $\mathcal{U}(\mathcal{T}E, \mathcal{J}E^m), \mathcal{U}(\mathcal{T}Q, \mathcal{J}Q^m), I_{(m)}, L_{(m)}$, respectively, each equipped with the bornologies specified above. The inverse maps for these isomorphisms are slightly more complicated, however: they may shift the index m in the projective system:

LEMMA 5.1.14. *The inverses to the isomorphisms above extend to bounded maps*

$$\begin{aligned} \mathcal{U}(\mathcal{T}E, \mathcal{J}E^{m+1}) &\rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+ \otimes E, \\ L_{(m+1)} &\rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+ \otimes K, \\ \mathcal{U}(\mathcal{T}E, \mathcal{J}E^{2m})^+ &\rightarrow L_{(m)}^+ \otimes \mathcal{U}(\mathcal{T}Q, \mathcal{J}Q^m)^+, \\ I_{(2m)} &\rightarrow L_{(m)} \otimes \mathcal{U}(\mathcal{T}Q, \mathcal{J}Q^m)^+, \\ I_{(2m)} &\rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+ \otimes K \otimes \mathcal{U}(\mathcal{T}Q, \mathcal{J}Q^m)^+, \\ I_{(2m)} &\rightarrow \mathcal{U}(\mathcal{T}Q, \mathcal{J}Q^m)^+ \otimes K \otimes \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+, \\ L_{(2m)} &\rightarrow \mathcal{U}(\mathcal{T}Q, \mathcal{J}Q^m)^+ \otimes K \otimes L_{(m)}^+. \end{aligned}$$

PROOF. Our explicit formula for the first map shows that it reduces the total degree of a differential form by at most 2. Since $\frac{n+1}{m+1} \leq \frac{n}{m}$ for all $n \geq m$ and $\lfloor \frac{n+1}{m+1} \rfloor = \lfloor \frac{n}{m} \rfloor = 0$ if $n < m$, it follows that it defines a map $\mathcal{U}(\mathcal{T}E, \mathcal{J}E^{m+1}) \rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+ \otimes E$ that is bounded for the bornologies described in Corollary 4.4.14. The second map is a restriction of the first map, so that it is covered by the same argument.

Our explicit formula for the third map shows that it maps a differential form of degree $2n$ to a sum of tensor products involving differential forms of degree $2j$ and $2(n-j-1)$ or $2(n-j)$; in the first case, $j < n$ and the differential form in L is already explicitly written as $\omega \otimes k$, so that the isomorphism $L \rightarrow (\mathcal{T}E)^+ \otimes K$ does not reduce the degree any further. This shows that the same degree estimate applies to the fourth map in the lemma. The fifth map differs from that only by taking opposite algebras, and the sixth map is a restriction of the fifth one. This is why the following estimates cover all these maps at the same time.

That these maps are well defined between the relevant tube algebras amounts to the estimate $\lfloor n/2m \rfloor \leq \lfloor j/m \rfloor + \lfloor (n-j-1)/m \rfloor$ for all $n \in \mathbb{N}$, $0 \leq j < n$. This is trivial for $n < 2m$, so that we assume $n \geq 2m$. For fixed n , the right hand side is minimal if $j = m-1$, and then the needed estimate simplifies to $\lfloor n/2m \rfloor \leq \lfloor (n-m)/m \rfloor$. This is true for $2m \leq n < 4m$. Since adding $2m$ to n increases $\lfloor n/2m \rfloor$ by 1 and $\lfloor (n-m)/m \rfloor$ by 2, the inequality follows for all $n \in \mathbb{N}$. Now it follows that the maps in the lemma are well defined and bounded for the bornologies described in Corollary 4.4.14. \square

The composite maps

$$\begin{aligned} \mathcal{U}(\mathcal{T}E, \mathcal{J}E^{m+1})^+ \otimes E &\rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^{m+1}) \rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+ \otimes E, \\ \mathcal{U}(\mathcal{T}E, \mathcal{J}E^{m+1}) &\rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+ \otimes E \rightarrow \mathcal{U}(\mathcal{T}E, \mathcal{J}E^m)^+ \end{aligned}$$

are the structure maps in our projective systems because they extend the identity maps on $(\mathcal{T}E)^+ \otimes E$ and $\mathcal{T}E$, respectively. Thus these two families of maps for $m \in \mathbb{N}^*$ are isomorphisms of projective systems of bornological V -modules that are inverse to each other. This remains so when we complete, giving an isomorphism $(\mathcal{T}E)^+ \overline{\otimes} E \xrightarrow{\sim} \mathcal{T}E$. The same argument applies to the other isomorphisms above. Summing up, we get the following isomorphisms of projective systems of bornological

commutator quotients are computed explicitly and shown to compute the relative Hochschild homology for the quotient map $\mathcal{T}E \twoheadrightarrow \mathcal{T}Q$. And then the isomorphism on Hochschild homology implies an isomorphism in cyclic homology and thus periodic cyclic homology. \square

5.2. Analytic quasi-freeness of \mathcal{L} . The proof of the excision theorem is completed by the following theorem:

THEOREM 5.2.1. *There is a semi-split, analytically nilpotent extension $\mathcal{J}E \cap \mathcal{L} \twoheadrightarrow \mathcal{L} \twoheadrightarrow K$ and \mathcal{L} is analytically quasi-free.*

This theorem and Theorem 4.7.1 imply that $\mathbb{H}A(K)$ is chain homotopy equivalent to the X -complex of \mathcal{L} . Theorem 5.1.23 identifies this with the relative X -complex $X(\mathcal{T}E : \mathcal{T}Q)$. And this yields the excision theorem. So it only remains to prove Theorem 5.2.1.

The canonical projection $\mathcal{T}E \twoheadrightarrow E$ restricts to a semi-split projection $\mathcal{L} \twoheadrightarrow K$. Its kernel $\mathcal{J}E \cap \mathcal{L} \subseteq \mathcal{J}E$ is a projective system of closed subalgebras. These are complete and torsion-free by [27, Theorem 2.3 and Lemma 4.2]; and subalgebras also clearly inherit the property of being semi-dagger. So $\mathcal{J}E \cap \mathcal{L}$ is a projective system of dagger algebras. Proposition 4.2.6 implies that it is again nilpotent mod π because $\mathcal{J}E/(\mathcal{J}E \cap \mathcal{L})$ is torsion-free.

The proof of Theorem 5.1.23 already shows that \mathcal{L} is quasi-free. We need it to be analytically quasi-free, however. This is the main difficulty in Theorem 5.2.1. The proof of this uses the same ideas as the proof of the corresponding statement for analytic cyclic homology for bornological algebras over \mathbb{C} in [26]. First, we define a homomorphism $v: L \rightarrow \mathbb{T}L$ for the purely algebraic version L of \mathcal{L} . Then we show that this homomorphism extends uniquely to a homomorphism of pro-algebras $\mathcal{L} \rightarrow \mathcal{T}\mathcal{L}$ that is a section for the canonical projection $\mathcal{T}\mathcal{L} \twoheadrightarrow \mathcal{L}$.

We need some notation for elements of $\mathbb{T}L$ and a certain grading on $\mathbb{T}L$. Elements of $\mathbb{T}L$ are sums of differential forms $l_0 Dl_1 \dots Dl_{2n}$ with $l_0 \in L^+$, $l_1, \dots, l_{2n} \in L$. We write \odot for the Fedosov product in $\Omega^{\text{ev}}L$ to distinguish it from the Fedosov product \odot in L and the resulting usual multiplication on ΩL . Call an element of $\mathbb{T}L$ *elementary* if it is of the form $l_0 Dl_1 \dots Dl_{2n}$ with $l_j = e_{j,0} de_{j,1} \dots de_{j,2i_j}$ for $0 \leq j \leq 2n$, and $e_{j,k} \in K \cup s(Q)$ for all occurring indices j, k , except that we allow $e_{j,0} = 1$ if $i_j \geq 1$ and $l_0 = 1$ if $i_0 = 0$; here $e_{j,2i_j} \in K$ because $l_j \in L$. Any element of $\mathbb{T}L$ is a finite linear combination of such elementary elements. The *entries* of an elementary element ξ are the elements $e_{j,l} \in E$; its *internal degree* is $\deg_i(\xi) = \sum_{j=0}^{2n} 2i_j$; its *external degree* is $\deg_e(\xi) = 6n$ if $l_0 \in L$ and $\deg_e = 6n - 4$ if $l_0 = 1$, and the total degree $\deg_t(\xi)$ is the sum of these two degrees; this particular total degree already appears in the proof of [26, Lemma 5.102].

The definition of v is based on the isomorphism $L \cong (\mathcal{T}E)^+ \otimes K$ in (5.1.4). The restriction of v to $K = (\Omega^0 \mathcal{T}E \cap L) \subseteq L$ is the obvious inclusion of K into $\mathbb{T}L$. We extend this map to L using a homomorphism from $\mathcal{T}E$ to the algebra of V -module homomorphisms $\mathbb{T}L \rightarrow \mathbb{T}L$. Such a homomorphism is equivalent to a linear map $E \rightarrow \text{Hom}(\mathbb{T}L, \mathbb{T}L)$, which is, in turn, equivalent to a V -bilinear map $E \times \mathbb{T}L \rightarrow \mathbb{T}L$, which we denote as an operation $(e, \xi) \mapsto e \triangleright \xi$ for $e \in E$, $\xi \in \mathbb{T}L$. As in [26], we first define the map $\nabla: L \rightarrow \Omega^1(L)$ by $\nabla(s_R(\xi) \odot k \odot l) := s_R(\xi) \odot k Dl$ for all $\xi \in (\mathcal{T}Q)^+$, $k \in K$, $l \in L^+$, with the understanding that $D1 = 0$ if l is the unit element of L ; this

uses the inverse of the isomorphism (5.1.11). Then we let

$$\begin{aligned} e \triangleright x_0 Dx_1 \dots Dx_{2n} &= e \odot x_0 Dx_1 \dots Dx_{2n} - D\nabla(e \odot x_0) Dx_1 \dots Dx_{2n}, \\ e \triangleright Dx_1 Dx_2 \dots Dx_{2n} &= \nabla(e \odot x_1) Dx_2 \dots Dx_{2n}. \end{aligned}$$

The curvature of the corresponding map $E \rightarrow \text{Hom}(\mathbb{T}L, \mathbb{T}L)$ acts by the operation $\omega_{\triangleright}(e_1, e_2)\xi := (e_1 \cdot e_2) \triangleright \xi - e_1 \triangleright (e_2 \triangleright \xi)$. It is computed in [26, Equation (5.91)]:

$$\begin{aligned} \omega_{\triangleright}(e_1, e_2)l_0 Dl_1 \dots Dl_{2n} &= (de_1 de_2 \odot l_0) Dl_1 \dots Dl_{2n} \\ &\quad + \nabla(e_1 \odot \nabla(e_2 \odot l_0)) Dl_1 \dots Dl_{2n} \\ &\quad - D\nabla(de_1 de_2 \odot l_0) Dl_1 \dots Dl_{2n}, \\ \omega_{\triangleright}(e_1, e_2)Dl_1 \dots Dl_{2n} &= \nabla((e_1 \cdot e_2) \odot l_1) Dl_2 \dots Dl_{2n} \\ &\quad - e_1 \odot \nabla(e_2 \odot l_1) Dl_2 \dots Dl_{2n} \\ &\quad + D\nabla(e_1 \odot \nabla(e_2 \odot l_1)) Dl_2 \dots Dl_{2n}. \end{aligned}$$

Finally, we define

$$v(e_0 de_1 \dots de_{2n} \odot k) := e_0 \triangleright (\omega_{\triangleright}(e_1, e_2) \circ \dots \circ \omega_{\triangleright}(e_{2n-1}, e_{2n}))(k).$$

LEMMA 5.2.2. *The map $v: L \rightarrow \mathbb{T}L$ is an algebra homomorphism, and $p \circ v = \text{id}_L$ for the canonical projection $p: \mathbb{T}L \rightarrow L$.*

If $l \in \Omega^{2n-1}(E) dK \subseteq L$ has degree $2n$, then $v(l)$ is a sum of elementary elements of $\mathbb{T}L$ with total degree at least $2n$.

Let $M \subseteq E$ be a bounded V -submodule. There is a bounded subset $M' \subseteq E$ such that if $e_0 de_1 \dots de_{2n} \in \Omega^{2n}M \cap L$, then $v(e_0 de_1 \dots de_{2n})$ is a sum of elementary elements of $\mathbb{T}L$ with entries in M' .

PROOF. As shown in [26] or in [25], the left action \triangleright is by left multipliers, that is, $e \triangleright (\xi \odot \tau) = (e \triangleright \xi) \odot \tau$ for all $e \in E$, $\xi, \tau \in \mathbb{T}L$. And $k \triangleright \xi = k \odot \xi$ for all $k \in K$. This implies that v is a homomorphism.

A short computation shows that each summand in the formula for $\omega_{\triangleright}(e_1, e_2)$ increases the total degree defined above by at least 2; this is already shown in the proof of [26, Lemma 5.102]. By induction on n , it follows that v maps $\Omega^{2n}L$ into the subgroup spanned by elementary elements of $\mathbb{T}L$ with total degree at least $2n$.

Given a bounded subset $M \subseteq E$, the proof of [26, Lemma 5.92] provides a bounded subset $M' \subseteq E$ such that $v(e_0 de_1 \dots de_{2n} \odot k)$ is a sum of elementary elements of $\mathbb{T}L$ with entries in M' . \square

The homomorphism v induces an F -algebra homomorphism $L \otimes F \rightarrow \mathbb{T}L \otimes F$. Recall that

$$L_{(m)} := K \oplus \bigoplus_{n=1}^{\infty} \pi^{-\lfloor n/m \rfloor} \Omega^{2n-1}(E) dK$$

for $m \in \mathbb{N}^*$. These are V -subalgebras of $L \otimes F$ that satisfy $L_{(n)} \subseteq L_{(m)}$ if $n \geq m$. Each $L_{(m)}$ is equipped with the bornology cofinally generated by the submodules in (5.1.13).

Let $(\mathbb{T}L)_{(m)} \subseteq \mathbb{T}L \otimes F$ be the subgroup generated by $\pi^{-\lfloor d/m \rfloor} \xi$ for elementary elements ξ of total degree d . These are V -subalgebras of $\mathbb{T}L \otimes F$ that satisfy $(\mathbb{T}L)_{(n)} \subseteq (\mathbb{T}L)_{(m)}$ if $n \geq m$. If $M \subseteq E$ is a bounded V -submodule, then let $D_m^{\mathbb{T}}(M) \subseteq (\mathbb{T}L)_{(m)}$ be the subgroup generated by $\pi^{-\lfloor d/m \rfloor} \xi$ for elementary elements ξ of total degree d . We give $(\mathbb{T}L)_{(m)}$ the bornology that is cofinally generated by these V -submodules. This bornology is the analogue of the bornology in Corollary 4.4.14.

It is torsion-free and makes the multiplication in $(\mathbb{T}L)_{(m)}$ and the inclusion maps $(\mathbb{T}L)_{(n)} \hookrightarrow (\mathbb{T}L)_{(m)}$ for $n \geq m$ bounded. So we have turned $((\mathbb{T}L)_{(m)})_{m \in \mathbb{N}^*}$ into a projective system of torsion-free bornological algebras.

The second paragraph in Lemma 5.2.2 says that the extension $L \otimes F \rightarrow \mathbb{T}L \otimes F$ of v maps $L_{(m)}$ to $(\mathbb{T}L)_{(m)}$ for each $m \in \mathbb{N}^*$. And the third paragraph says that this homomorphism is bounded. Thus v is a homomorphism of projective systems of bornological algebras. By Corollary 4.4.14, \mathcal{L} is isomorphic to the projective system of the completions $\overline{L_{(m)}}$ for $m \in \mathbb{N}^*$, with the bornologies described above.

LEMMA 5.2.3. *The embedding $\mathbb{T}L \hookrightarrow \mathcal{T}L$ extends to an isomorphism of projective systems from the projective system of completions $\overline{(\mathbb{T}L)_{(m)}}$ for $m \in \mathbb{N}^*$ to $\mathcal{T}\mathcal{L}$.*

PROOF. For a bounded V -submodule $M \subseteq E$, let $M_K := M \cap K$ and let $\overline{\Omega}_{\mathcal{L}}^0(M) := M_K$ and $\overline{\Omega}_{\mathcal{L}}^{2k}(M) := \overline{\Omega}^{2k-1}(M) \overline{\otimes} M_K$ for $k > 1$. A proof like that for Proposition 4.4.15 shows that the completion of $(\mathbb{T}L)_{(m)}$ is the union of the products

$$(5.2.4) \quad \prod_{j \geq 0, i_0, \dots, i_{2j} \geq 0}^{\infty} \pi^{-[(6j+2i_0+\dots+2i_{2j})/m]} \overline{\Omega}_{\mathcal{L}}^{2i_0}(M) \overline{\otimes} \overline{\Omega}_{\mathcal{L}}^{2i_1}(M) \overline{\otimes} \dots \overline{\otimes} \overline{\Omega}_{\mathcal{L}}^{2i_{2j}}(M) \\ \times \prod_{j \geq 0, i_1, \dots, i_{2j} \geq 0}^{\infty} \pi^{-[(6j-4+2i_0+\dots+2i_{2j})/m]} \overline{\Omega}_{\mathcal{L}}^{2i_1}(M) \overline{\otimes} \overline{\Omega}_{\mathcal{L}}^{2i_2}(M) \overline{\otimes} \dots \overline{\otimes} \overline{\Omega}_{\mathcal{L}}^{2i_{2j}}(M)$$

taken over all bounded V -submodules $M \subseteq E$; elementary tensors in a factor of the first product correspond to differential forms $l_0 Dl_1 \dots Dl_{2j}$ with $l_0, \dots, l_{2j} \in \mathcal{L}$ and $\deg(l_j) = 2i_j$, whereas those for the second product correspond to differential forms $Dl_1 \dots Dl_{2j}$. The exponent of π is the total degree defined above.

Proposition 4.4.15 describes $\mathcal{T}E$. The pro-subalgebra \mathcal{L} is described similarly, by also asking for the last entry of all differential forms to belong to K . Then a second application of Proposition 4.4.15 describes $\mathcal{T}\mathcal{L}$. The result is very similar to the projective system above. The only difference is that the exponent of π in the bornology is replaced by $h := \lfloor j/k \rfloor + \sum_{l=0}^{2j} \lfloor i_l/m \rfloor$ for each factor in (5.2.4), for some parameters $k, m \in \mathbb{N}^*$. Here we may take $k = m$ because this gives a cofinal subset. So it remains to prove linear estimates between these two notions of ‘‘degree’’. In one direction, this is the trivial estimate

$$\left\lfloor \frac{j}{m} \right\rfloor + \sum_{l=0}^{2j} \left\lfloor \frac{i_l}{m} \right\rfloor \leq \left\lfloor \frac{j}{m} + \sum_{l=0}^{2j} \frac{i_l}{m} \right\rfloor \leq \left\lfloor \frac{1}{m} \left(6j + \sum_{l=0}^{2j} 2i_l \right) \right\rfloor$$

for $j \geq 0$ and a similar estimate with $6j - 4 = 4(j - 1) + 2j$ instead of $6j$ for $j \geq 1$. In the other direction, we distinguish two cases. Let $i := \sum i_l$. If $i < 4j \cdot m$, then $6j + 2i < j \cdot (6 + 8m)$ and we estimate

$$\left\lfloor \frac{j}{m} \right\rfloor + \sum_{l=0}^{2j} \left\lfloor \frac{i_l}{m} \right\rfloor \geq \left\lfloor \frac{j}{m} \right\rfloor \geq \left\lfloor \frac{6j + 2i}{(6 + 8m) \cdot m} \right\rfloor.$$

The other case is $i \geq 4j \cdot m$. Each floor operation changes a number by at most 1, and $6j + 2i \leq \frac{3}{2m}i + 2i \leq 4i$. So

$$\left\lfloor \frac{j}{m} \right\rfloor + \sum_{l=0}^{2j} \left\lfloor \frac{i_l}{m} \right\rfloor \geq \frac{i}{m} - 2j \geq \frac{i}{2m} \geq \left\lfloor \frac{6j + 2i}{8m} \right\rfloor. \quad \square$$

As a result, v defines a pro-algebra homomorphism $\mathcal{L} \rightarrow \mathcal{T}\mathcal{L}$. Then \mathcal{L} is analytically quasi-free. This ends the proof of the excision theorem.

6. Stability with respect to algebras of matrices

A *matricial pair* (X, Y) consists of two torsion-free bornological modules X and Y and a surjective linear map $\langle \cdot, \cdot \rangle: Y \otimes X \rightarrow V$. Any such map is bounded. A *homomorphism* from (X, Y) to another matricial pair (W, Z) is a pair $f = (f_1, f_2)$ of bounded linear homomorphisms $f_1: X \rightarrow W$, $f_2: Y \rightarrow Z$ such that $\langle f_2(y), f_1(x) \rangle = \langle y, x \rangle$ for all $x \in X$ and $y \in Y$. An *elementary homotopy* is a pair $H = (H_1, H_2)$ of bounded linear maps, where $H_1: X \rightarrow W[t]$ and $H_2: Y \rightarrow Z$ or $H_1: X \rightarrow W$ and $H_2: Y \rightarrow Z[t]$, such that the following diagram commutes:

$$\begin{array}{ccc} Y \otimes X & \xrightarrow{H_2 \otimes H_1} & Z \otimes W[t] \\ \downarrow \langle \cdot, \cdot \rangle & & \downarrow \langle \cdot, \cdot \rangle \otimes \text{id} \\ V & \xrightarrow{\text{inc}} & V[t] \end{array}$$

Let (X, Y) be a matricial pair. Let $\mathcal{M} = \mathcal{M}(X, Y)$ be $X \otimes Y$ with the product

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle y_1, x_2 \rangle x_1 \otimes y_2.$$

This product is associative and bounded, and it even makes \mathcal{M} a semi-dagger algebra. The bornological algebra \mathcal{M} is also torsion-free by [27, Proposition 4.12]. Thus the completion $\overline{\mathcal{M}}$ is a dagger algebra and $\overline{\mathcal{M}} = \mathcal{M}^\dagger$.

Homomorphisms and homotopies of matricial pairs induce homomorphisms and homotopies of the corresponding algebras. Any pair $(\xi, \eta) \in X \times Y$ with $\langle \eta, \xi \rangle = 1$ yields a bounded algebra homomorphism

$$\iota = \iota_{\xi, \eta}: V \rightarrow \mathcal{M}, \quad \iota(1) = \xi \otimes \eta.$$

We shall also write ι for the composite of the map above with the completion map $\mathcal{M} \rightarrow \overline{\mathcal{M}} = \mathcal{M}^\dagger$. If R is a torsion-free bornological algebra, then $R \overline{\otimes} \mathcal{M}^\dagger$ is torsion-free by [27, Theorem 4.6 and Propositions 14.11 and 14.12]. Define

$$(6.1) \quad \iota_R := \text{id}_R \otimes \iota: R \rightarrow R \overline{\otimes} \mathcal{M}^\dagger.$$

PROPOSITION 6.2. *Let R be a complete, torsion-free bornological algebra. Then the map ι_R induces a chain homotopy equivalence $\mathbb{H}\mathbb{A}(R) \simeq \mathbb{H}\mathbb{A}(R \overline{\otimes} \mathcal{M}^\dagger)$ and an isomorphism $\mathbb{H}\mathbb{A}_*(R) \simeq \mathbb{H}\mathbb{A}_*(R \overline{\otimes} \mathcal{M}^\dagger)$.*

PROOF. Corollary 4.5.6 yields a natural pro-algebra homomorphism $\mathcal{T}(R \overline{\otimes} \mathcal{M}^\dagger) \rightarrow \mathcal{T}(R) \overline{\otimes} \mathcal{M}^\dagger$ covering the identity of $R \overline{\otimes} \mathcal{M}^\dagger$. And any elementary homotopy between matricial pairs (X, Y) and (W, Z) yields an elementary dagger homotopy $\mathcal{M}(X, Y)^\dagger \rightarrow \mathcal{M}(Z, W)^\dagger \overline{\otimes} V[t]^\dagger$. The X -complex is invariant under dagger homotopies by Proposition 4.6.1. Taking all this into account, the argument of the proof of [26, Theorem 5.65] now applies verbatim and proves the proposition. \square

Let Λ be a set. We now describe increasingly complicated algebras of matrices indexed by the set Λ .

EXAMPLE 6.3. Let Λ be a set and let $V^{(\Lambda)}$ be the V -module of finitely supported functions $\Lambda \rightarrow V$. This is the free module with basis $\{\chi_\lambda: \lambda \in \Lambda\}$ formed by the characteristic functions of the singletons. The algebra $\mathcal{M}(V^{(\Lambda)}, V^{(\Lambda)})$ associated to the bilinear form $\langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda, \mu}$ is just the algebra M_Λ of finitely supported matrices indexed by $\Lambda \times \Lambda$, equipped with the fine bornology. The latter algebra is already a dagger algebra. Proposition 6.2 implies $\mathbb{H}\mathbb{A}(R) \simeq \mathbb{H}\mathbb{A}(M_\Lambda \otimes R)$ for all R .

EXAMPLE 6.4. Define $V^{(\Lambda)}$ as in Example 6.3. Its π -adic completion is the Banach module $c_0(\Lambda) := c_0(\Lambda, V)$ with the supremum norm. The bilinear form in Example 6.3 extends to $c_0(\Lambda)$. The π -adic completion of $\mathcal{M}(c_0(\Lambda), c_0(\Lambda))$ is isomorphic to the Banach V -algebra $M_\Lambda^0 \cong c_0(\Lambda \times \Lambda)$ of matrices indexed by $\Lambda \times \Lambda$ with entries in V that go to zero at infinity. The Banach V -modules above become bornological by declaring all subsets to be bounded. Then the completions and tensor products as Banach V -modules and as bornological V -modules are the same. Therefore, Proposition 6.2 implies $\mathbb{H}\mathbb{A}(R) \cong \mathbb{H}\mathbb{A}(M_\Lambda^0 \overline{\otimes} R)$ for all R .

EXAMPLE 6.5. Let $\ell: \Lambda \rightarrow \mathbb{N}$ be a proper function, that is, for each $n \in \mathbb{N}$ the set of $x \in \Lambda$ with $\ell(x) \leq n$ is finite. Define $V^{(\Lambda)}$ as in Example 6.3 and give it the bornology that is cofinally generated by the V -submodules

$$S_m := \sum_{\lambda \in \Lambda} \pi^{\lfloor \ell(\lambda)/m \rfloor} \chi_\lambda$$

for $m \in \mathbb{N}^*$. The bilinear form in Example 6.3 remains bounded for this bornology on $V^{(\Lambda)}$. So $\mathcal{M}(V^{(\Lambda)}, V^{(\Lambda)})$ with the tensor product bornology from the above bornology is a bornological algebra as well. It is torsion-free and semi-dagger. So its dagger completion is the same as its completion. We denote it by M_Λ^ℓ . It is isomorphic to the algebra of infinite matrices $(c_{x,y})_{x,y \in \Lambda}$ for which there is $m \in \mathbb{N}^*$ such that $c_{x,y} \in \pi^{\lfloor (\ell(x)+\ell(y))/m \rfloor}$ for all $x, y \in \Lambda$; this is the same as asking for $\lim |c_{x,y} \pi^{-\lfloor (\ell(x)+\ell(y))/m \rfloor}| = 0$ because ℓ is proper. It makes no difference to replace the exponent of π by $\lfloor \ell(x)/m \rfloor + \lfloor \ell(y)/m \rfloor$ or $\lfloor \max\{\ell(x), \ell(y)\}/m \rfloor$ because we may vary m . Proposition 6.2 implies $\mathbb{H}\mathbb{A}(R) \cong \mathbb{H}\mathbb{A}(M_\Lambda^\ell \overline{\otimes} R)$ for all R .

The following completed matrix algebras will be needed in Section 8.

EXAMPLE 6.6. Let Λ be a set with a filtration by a directed set I . That is, there are subsets $\Lambda_S \subseteq \Lambda$ for $S \in I$ with $\Lambda_S \subseteq \Lambda_T$ for $S \leq T$ and $\Lambda = \bigcup_{S \in I} \Lambda_S$. Let $\ell: \Lambda \rightarrow \mathbb{N}$ be a function whose restriction to Λ_S is proper for each $S \in I$. For $S \in I$, form the matrix algebra $M_{\Lambda_S}^\ell$ as in Example 6.5. These algebras for $S \in I$ form an inductive system. Let $\varinjlim M_{\Lambda_S}^\ell$ be its bornological inductive limit. This bornological algebra is also associated to a matricial pair, namely, the pair based on $\varinjlim V^{(\Lambda_S)}$, where each $V^{(\Lambda_S)}$ carries the bornology described in Example 6.5. Thus Proposition 6.2 implies $\mathbb{H}\mathbb{A}(R) \cong \mathbb{H}\mathbb{A}(\varinjlim M_{\Lambda_S}^\ell \overline{\otimes} R)$ for all R .

7. Morita functoriality

In this section, we show that analytic cyclic homology is functorial for certain bimodules. Let A and B be unital, torsion-free, complete bornological V -algebras and let P be an A - B -bimodule. Assume P to be finitely generated and projective as a right B -module. Then there are $n \in \mathbb{N}$ and an idempotent matrix $e \in \mathbb{M}_n(B)$ such that $P \cong eB^n$. The left action of A on P induces a V -algebra homomorphism

$$\iota_A: A \rightarrow \text{End}_B(P) \cong e\mathbb{M}_n(B)e \subseteq \mathbb{M}_n(B).$$

Proposition 6.2 describes a chain homotopy equivalence $\mathbb{H}\mathbb{A}(B) \cong \mathbb{H}\mathbb{A}(\mathbb{M}_n(B))$ for any $n \in \mathbb{N}_{\geq 1}$. Composing this with the map induced by ι_A gives a chain map

$$\mathbb{H}\mathbb{A}(P): \mathbb{H}\mathbb{A}(A) \rightarrow \mathbb{H}\mathbb{A}(B).$$

This induces maps $\mathbb{H}\mathbb{A}_*(P): \mathbb{H}\mathbb{A}_*(A) \rightarrow \mathbb{H}\mathbb{A}_*(B)$ for $* = 0, 1$.

LEMMA 7.1. *The homotopy class of $\mathbb{H}\mathbb{A}(P)$ only depends on the isomorphism class of P . That is, if $P \cong e \cdot B^n \cong f \cdot B^m$ for $n, m \in \mathbb{N}$ and idempotent $e \in \mathbb{M}_n(B)$, $f \in \mathbb{M}_m(B)$, then the resulting chain maps $\mathbb{H}\mathbb{A}(A) \rightarrow \mathbb{H}\mathbb{A}(B)$ are chain homotopic. If $A = B = P$, then $\mathbb{H}\mathbb{A}(P)$ is homotopic to the identity chain map.*

PROOF. The chain homotopy equivalence $\mathbb{H}\mathbb{A}(B) \cong \mathbb{H}\mathbb{A}(\mathbb{M}_n(B))$ in Proposition 6.2 is induced by the corner embedding $\iota_n: B \rightarrow \mathbb{M}_n(B)$, $b \mapsto b \cdot E_{11}$. If $k \geq n$, then the inclusion $j_{kn}: \mathbb{M}_n(B) \rightarrow \mathbb{M}_k(B)$ that appends zeros on the right and at the bottom satisfies $j_{kn} \circ \iota_n = \iota_k$. Hence there is a commuting diagram of chain homotopy equivalences

$$\begin{array}{ccc} \mathbb{H}\mathbb{A}(B) & \xrightarrow{\mathbb{H}\mathbb{A}(\iota_n)} & \mathbb{H}\mathbb{A}(\mathbb{M}_n(B)) \\ & \searrow \mathbb{H}\mathbb{A}(\iota_k) & \downarrow \mathbb{H}\mathbb{A}(j_{kn}) \\ & & \mathbb{H}\mathbb{A}(\mathbb{M}_k(B)). \end{array}$$

Therefore, the maps $\mathbb{H}\mathbb{A}(A) \rightarrow \mathbb{H}\mathbb{A}(B)$ remain unchanged when we replace e and f by $j_{kn}(e) \in \mathbb{M}_k(B)$ and $j_{km}(f) \in \mathbb{M}_k(B)$ for $k \geq n, m$. This allows us to reduce to the case $n = m$. And then we may still choose $k = 2n = 2m$ to create extra room.

Since $fB^m \cong eB^m$, the idempotent matrices e and f are *Murray-von-Neumann equivalent*. That is, there are matrices $v, w \in \mathbb{M}_m(B)$ with

$$e = vw, \quad f = wv, \quad v w v = v, \quad w v w = w.$$

Let $\iota_A^e, \iota_A^f: A \rightarrow \mathbb{M}_m(B)$ be the two homomorphisms defined above using the idempotent elements e and f , respectively. By construction, $\iota_A^e(a) = v \iota_A^f(a) w$ and $\iota_A^f(a) = w \iota_A^e(a) v$ for all $a \in A$. It is well known that $j_{2m,m}(e)$ and $j_{2m,m}(f)$ are homotopic. We recall the elementary proof. Let

$$v_t := tv + (1-t), \quad w_t := tw + (1-t)$$

in $\mathbb{M}_m(B[t])$ and let

$$u_t := \begin{pmatrix} v_t & v_t w_t - 1 \\ 1 - w_t v_t & 2w_t - w_t v_t w_t \end{pmatrix}, \quad u_t^{-1} := \begin{pmatrix} 2w_t - w_t v_t w_t & 1 - w_t v_t \\ v_t w_t - 1 & v_t \end{pmatrix}.$$

Easy computations show that the latter two elements of $\mathbb{M}_{2m}(B[t])$ satisfy $u_0 = 1$ and $u_t u_t^{-1} = 1 = u_t^{-1} u_t$. And

$$u_1 \begin{pmatrix} \iota_A^f(a) & 0 \\ 0 & 0 \end{pmatrix} u_1^{-1} = \begin{pmatrix} \iota_A^e(a) & 0 \\ 0 & 0 \end{pmatrix}$$

for all $a \in A$. Therefore, conjugation by u_t defines a polynomial homotopy between the homomorphisms ι_A^e and ι_A^f . Since $\mathbb{H}\mathbb{A}$ is homotopy invariant by Theorem 4.6.2, it follows that the chain maps $\mathbb{H}\mathbb{A}(A) \rightarrow \mathbb{H}\mathbb{A}(\mathbb{M}_{2m}(B))$ induced by ι_A^e and ι_A^f are chain homotopic.

To prove the last claim about $\mathbb{H}\mathbb{A}(A)$ for the identity bimodule A , use $m = 1$ and $e = 1$. Then $\iota_A: A \rightarrow \mathbb{M}_m(A)$ is the identity map. \square

LEMMA 7.2. *Let A, B, C be unital, torsion-free, complete bornological V -algebras and let P be an A - B -bimodule and Q a B - C -bimodule. Assume P and Q to be finitely generated and projective as right modules. Then $P \otimes_B Q$ is finitely generated and projective as a right module, and $\mathbb{H}\mathbb{A}(P \otimes_B Q)$ is chain homotopic to $\mathbb{H}\mathbb{A}(P) \circ \mathbb{H}\mathbb{A}(Q)$.*

PROOF. By assumption, there are $m, n \in \mathbb{N}$ and idempotent matrices $e \in \mathbb{M}_m(B)$ and $f \in \mathbb{M}_n(C)$ with $P \cong eB^m$ and $Q \cong fC^n$. Then

$$P \otimes_B Q \cong (e \cdot B^m) \otimes_B (f \cdot C^n) \cong (e \otimes_B 1) \cdot (B^m \otimes_B (f \cdot C^n)) = (e \otimes_B 1) \cdot (f \cdot C^n)^m.$$

More precisely, this identifies $P \otimes_B Q$ with the image of $\mathbb{M}_m(\iota_B)(e) \in \mathbb{M}_{m \cdot n}(C)$, where $\iota_B: B \rightarrow \mathbb{M}_n(C)$ is the homomorphism associated to f and $\mathbb{M}_m(\iota_B): \mathbb{M}_m(B) \rightarrow \mathbb{M}_{m \cdot n}(C)$ is the induced homomorphism on matrices. Inspection shows that the homomorphism $A \rightarrow \mathbb{M}_{m \cdot n}(C)$ defined by realising $P \otimes_B Q$ in this way is equal to the composite homomorphism $\mathbb{M}_m(\iota_B) \circ \iota_A: A \rightarrow \mathbb{M}_m(B) \rightarrow \mathbb{M}_{m \cdot n}(C)$. This implies the claim because the chain homotopy equivalences $\mathbb{H}\mathbb{A}(B) \cong \mathbb{H}\mathbb{A}(\mathbb{M}_m(B))$ are natural. \square

THEOREM 7.3. *Let A and B be unital, torsion-free, complete bornological V -algebras. A Morita equivalence between them induces a chain homotopy equivalence $\mathbb{H}\mathbb{A}(A) \simeq \mathbb{H}\mathbb{A}(B)$ and isomorphisms $\mathbb{H}\mathbb{A}_*(A) \cong \mathbb{H}\mathbb{A}_*(B)$ for $* = 0, 1$.*

PROOF. The Morita equivalence is given by bimodules P and Q over A, B and B, A with $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$. It is well known that the equivalence bimodules P and Q are finitely generated and projective as right modules. Hence they induce well defined chain maps $\mathbb{H}\mathbb{A}(A) \leftrightarrow \mathbb{H}\mathbb{A}(B)$ by the construction above. These are inverse up to chain homotopy by Lemma 7.2. This homotopy equivalence implies isomorphisms $\mathbb{H}\mathbb{A}_*(A) \cong \mathbb{H}\mathbb{A}_*(B)$ for $* = 0, 1$ on analytic cyclic homology. \square

When dealing with non-unital algebras, Morita theory gets more difficult. In particular, we know less about the bimodules involved in a Morita equivalence. The issue is to impose the right assumptions on an A, B -bimodule so that there are a matricial pair as in Section 6, an idempotent double centraliser e of $B \overline{\otimes} \overline{\mathcal{M}}$, and an algebra homomorphism $A \rightarrow e(B \overline{\otimes} \overline{\mathcal{M}})e$. We do not discuss sufficient conditions on bimodules that allow to associate such data to them.

8. Leavitt path algebras

Our next goal is to compute the analytic cyclic homology for tensor products with Leavitt and Cohn path algebras of directed graphs and their dagger completions. A *directed graph* E consists of a set E^0 of vertices and a set E^1 of edges together with source and range maps $s, r: E^1 \rightarrow E^0$. A vertex $v \in E^0$ is *regular* if $0 < |s^{-1}(\{v\})| < \infty$. Let $\text{reg}(E) \subseteq E^0$ be the subset of regular vertices. Define

$$N_E: E^0 \times \text{reg}(E) \rightarrow \mathbb{Z}, \quad (v, w) \mapsto \delta_{v,w} - |s^{-1}(\{w\}) \cap r^{-1}(\{v\})|.$$

Let $L(E)$ and $C(E)$ be the Leavitt and Cohn path algebras over V , as defined in [1, Definitions 1.2.3 and 1.2.5]. We consider them as bornological algebras with the fine bornology. The following theorem follows easily from the results in [13] and the formal properties of analytic cyclic homology:

THEOREM 8.1. *Assume $\text{char } F = 0$. Let R be a complete bornological algebra. Let E be a graph with countably many vertices. Then*

$$\begin{aligned} \mathbb{H}\mathbb{A}(R \otimes C(E)) &\simeq \mathbb{H}\mathbb{A}(R \otimes V^{(E^0)}), & \mathbb{H}\mathbb{A}(C(E)) &\simeq F^{(E^0)}, \\ \mathbb{H}\mathbb{A}(L(E)) &\simeq \text{coker}(N_E) \oplus \ker(N_E)[1], \end{aligned}$$

If E^0 is finite, then

$$\begin{aligned}\mathbb{H}\mathbb{A}(R \otimes C(E)) &\simeq \bigoplus_{v \in E^0} \mathbb{H}\mathbb{A}(R), \\ \mathbb{H}\mathbb{A}(R \otimes L(E)) &\simeq (\text{coker}(N_E) \oplus \ker(N_E)[1]) \otimes \mathbb{H}\mathbb{A}(R).\end{aligned}$$

PROOF. We define a functor H from the category of V -algebras to the triangulated category of pro-supercomplexes by giving A the fine bornology and taking $\mathbb{H}\mathbb{A}(R \otimes A)$. The functor H is homotopy invariant for polynomial (and even dagger) homotopies by Theorem 4.6.2, stable for algebras of finite matrices over any set Λ by Proposition 6.2 applied to Example 6.3, and exact on semi-split extensions by Theorem 5.1. Theorem 5.1 also implies that $\mathbb{H}\mathbb{A}$ is finitely additive. It is not countably additive in general, but Corollary 4.3.9 shows that it is countably additive on the ground ring V . Now [13, Theorem 4.2] proves a homotopy equivalence

$$\mathbb{H}\mathbb{A}(R \otimes C(E)) \simeq \mathbb{H}\mathbb{A}(R \otimes V^{(E^0)}).$$

If E^0 is finite, then this is homotopy equivalent to $\mathbb{H}\mathbb{A}(R) \otimes V^{(E^0)} = \bigoplus_{v \in E^0} \mathbb{H}\mathbb{A}(R)$ by finite additivity. And if $R = V$, then Corollary 4.3.9 identifies $\mathbb{H}\mathbb{A}(V^{(E^0)}) \simeq F^{(E^0)}$.

[13, Proposition 5.2] yields a distinguished triangle of pro-supercomplexes

$$\mathbb{H}\mathbb{A}(R \otimes V^{\text{reg}(E)}) \xrightarrow{f} \mathbb{H}\mathbb{A}(R \otimes V^{(E^0)}) \rightarrow \mathbb{H}\mathbb{A}(R \otimes L(E)) \rightarrow \mathbb{H}\mathbb{A}(R \otimes V^{\text{reg}(E)})$$

and partly describes the map f . If $R = V$ and E^0 is countable, then Corollary 4.3.9 identifies $\mathbb{H}\mathbb{A}(V^{(E^0)}) \simeq F^{(E^0)}$ and $\mathbb{H}\mathbb{A}(V^{\text{reg}(E)}) \simeq F^{\text{reg}(E)}$, and the information about the map f in [13, Proposition 5.2] shows that it multiplies vectors with the matrix N_E . If E^0 is finite, then $\mathbb{H}\mathbb{A}$ is E^0 -additive and [13, Theorem 5.4] gives a distinguished triangle

$$\mathbb{H}\mathbb{A}(R) \otimes F^{\text{reg}(E)} \xrightarrow{\text{id} \otimes N_E} \mathbb{H}\mathbb{A}(R) \otimes F^{E^0} \rightarrow \mathbb{H}\mathbb{A}(R \otimes L(E)) \rightarrow \dots$$

Since $\text{char}(F) = 0$, there are invertible matrices x, y with entries in F such that xN_Ey is a diagonal matrix with only zeros and ones in the diagonal. We may replace the map N_E or $\text{id} \otimes N_E$ above by $\text{id} \otimes (xN_Ey)$. Then the formulas for $\mathbb{H}\mathbb{A}(L(E))$ in general and for $\mathbb{H}\mathbb{A}(R \otimes L(E))$ for finite E^0 follow. \square

COROLLARY 8.2. $\mathbb{H}\mathbb{A}(R \otimes V[t, t^{-1}])$ is chain homotopy equivalent to $\mathbb{H}\mathbb{A}(R) \oplus \mathbb{H}\mathbb{A}(R)[1]$ and $\text{HA}_*(R \otimes V[t, t^{-1}]) \cong \text{HA}_*(R) \oplus \text{HA}_*(R)[1]$.

PROOF. Apply Theorem 8.1 to the graph consisting of one vertex and one loop. \square

The following theorem says that Theorem 8.1 remains true for the dagger completions $C(E)^\dagger$ and $L(E)^\dagger$ of $C(E)$ and $L(E)$:

THEOREM 8.3. Let R be a complete bornological algebra and let E be a graph. Then

$$\mathbb{H}\mathbb{A}(R \otimes C(E)) \simeq \mathbb{H}\mathbb{A}(R \overline{\otimes} C(E)^\dagger), \quad \mathbb{H}\mathbb{A}(R \otimes L(E)) \simeq \mathbb{H}\mathbb{A}(R \overline{\otimes} L(E)^\dagger).$$

So the formulas in Theorem 8.1 also compute $\mathbb{H}\mathbb{A}(R \overline{\otimes} C(E)^\dagger)$ and $\mathbb{H}\mathbb{A}(R \overline{\otimes} L(E)^\dagger)$ – assuming E^0 to be countable or finite or $R = V$ for the different cases.

COROLLARY 8.4 (Fundamental Theorem). $\mathbb{H}\mathbb{A}(R \overline{\otimes} V[t, t^{-1}]^\dagger)$ is chain homotopy equivalent to $\mathbb{H}\mathbb{A}(R) \oplus \mathbb{H}\mathbb{A}(R)[1]$ and $\text{HA}_*(R \overline{\otimes} V[t, t^{-1}]^\dagger) \cong \text{HA}_*(R) \oplus \text{HA}_*(R)[1]$.

PROOF. Combine Theorem 8.3 and Corollary 8.2. \square

We are going to prove Theorem 8.3 by showing that the proofs in [13] continue to work when we suitably complete all algebras that occur there. We must be careful, however, because the dagger completion is *not* an exact functor. We first recall some basic facts that are used in [13]. These will be used to describe the dagger completions $C(E)^\dagger$ and $L(E)^\dagger$.

By definition, $L(E)$ has the same generators as $C(E)$ and more relations. This provides a quotient map $p: C(E) \twoheadrightarrow L(E)$. Let $K(E) \subseteq C(E)$ be its kernel.

LEMMA 8.5. *There is a semi-split extension of V -algebras*

$$K(E) \twoheadrightarrow C(E) \twoheadrightarrow L(E).$$

PROOF. Let \mathcal{P} be the set of finite paths in E . For $v \in \text{reg}(E)$, choose $e_v \in s^{-1}(\{v\})$. Let

$$\begin{aligned} \mathcal{B} &:= \{\alpha\beta^* : \alpha, \beta \in \mathcal{P}, r(\alpha) = r(\beta)\}, \\ \mathcal{B}' &:= \mathcal{B} \setminus \{\alpha e_v e_v^* \beta^* : v \in \text{reg}(E), \alpha, \beta \in \mathcal{P}, r(\alpha) = r(\beta) = v\}. \end{aligned}$$

By [1, Propositions 1.5.6 and 1.5.11], \mathcal{B} is a basis of $C(E)$ and \mathcal{B}' is a basis of $L(E)$. Let $\sigma: L(E) \rightarrow C(E)$ be the linear map that sends each element of \mathcal{B}' to itself. This is a section for the quotient map $p: C(E) \rightarrow L(E)$. \square

Next we describe $K(E)$ as in [1, Proposition 1.5.11]. Let $v \in \text{reg}(E)$. Define

$$q_v := v - \sum_{s(e)=v} ee^*.$$

Let $\mathcal{P}_v \subseteq \mathcal{P}$ be the set of all paths with $r(\alpha) = v$. Let $V^{(\mathcal{P}_v)}$ be the free V -module on the set \mathcal{P}_v and let $\mathcal{M}_{\mathcal{P}_v}$ be the algebra of finite matrices indexed by \mathcal{P}_v as in Example 6.3. The map

$$\bigoplus_{v \in \text{reg}(E)} \mathcal{M}_{\mathcal{P}_v} \rightarrow K(E), \quad \alpha \otimes \beta \mapsto \alpha q_v \beta^*,$$

is a V -algebra isomorphism by [1, Proposition 1.5.11]. Each $\mathcal{M}_{\mathcal{P}_v}$ with the fine bornology is a dagger algebra because it is a union of finite-dimensional subalgebras. Thus $K(E)$ is a dagger algebra as well. In contrast, $C(E)$ and $L(E)$ with the fine bornology are not semi-dagger algebras. And the restriction to $K(E)$ of the linear growth bornology of $C(E)$ is *not* just the fine bornology: this is already visible in the special case where $C(E)$ is the Toeplitz algebra and $L(E) = V[t, t^{-1}]$.

We are going to describe the linear growth bornology on $C(E)$. Let \mathcal{F} be the set of all finite subsets $S \subseteq E^0 \cup E^1$ such that

$$e \in S \cap E^1 \text{ and } s(e) \in \text{reg}(E) \Rightarrow \{s(e)\} \cup s^{-1}(s(e)) \subseteq S.$$

Let $S^{(\infty)}$ for $S \in \mathcal{F}$ be the set of all paths that consist only of edges in S . Let $|\alpha|$ be the length of a path $\alpha \in \mathcal{P}$. For $n \in \mathbb{N}$, let

$$S_n := \{\alpha\beta^* : \alpha, \beta \in S^{(\infty)}, |\alpha| + |\beta| \leq n\} \subseteq \mathcal{B}.$$

This is an increasing filtration on the basis \mathcal{B} of $C(E)$.

LEMMA 8.6. *A subset of $C(E)$ has linear growth if and only if there are $S \in \mathcal{F}$ and $m \in \mathbb{N}^*$ such that it is contained in the V -linear span of $\bigcup_{n \in \mathbb{N}} \pi^{\lfloor n/m \rfloor} S_n$.*

PROOF. It is easy to see that the V -linear span of $\bigcup_{n \in \mathbb{N}} \pi^{\lfloor n/m \rfloor} S_n$ in $C(E)$ has linear growth. Conversely, we claim that any subset of linear growth is contained in one of this form. Every finite subset of $E^0 \cup E^1$ is contained in an element of \mathcal{F} . It follows that, for every finitely generated submodule $M \subseteq C(E)$, there are $S \in \mathcal{F}$ and $m \geq 1$ such that M is contained in the V -submodule generated by S_m . Then M^j is contained in the V -submodule generated by S_{mj} for all $j \in \mathbb{N}^*$. Thus M° is contained in the V -submodule generated by $\pi^{j-1} S_{mj}$ for all $j \in \mathbb{N}^*$. This is the V -linear span of $\bigcup_{n \in \mathbb{N}^*} \pi^{\lfloor n/m \rfloor - 1} S_n$. Letting m vary, we may replace $\lfloor n/m \rfloor - 1$ by $\lfloor n/m \rfloor$. \square

Constructing linear growth bornologies commutes with taking quotients. So a subset of $L(E)$ has linear growth if and only if it is the image of a subset of linear growth in $C(E)$. Next we show that the section $\sigma: L(E) \rightarrow C(E)$ is bounded for the linear growth bornologies, and we describe the restriction to $K(E)$ of the linear growth bornology on $C(E)$:

LEMMA 8.7. *Give $V^{\mathcal{P}_v} \subseteq V^{\mathcal{P}}$ the bornology where a subset is bounded if and only if it is contained in the linear span of $\{\pi^{\lfloor |\alpha|/m \rfloor} \alpha : \alpha \in S^{(\infty)}\}$ for some $S \in \mathcal{F}$ and some $m \in \mathbb{N}^*$. Equip the matrix algebra $\mathcal{M}_{\mathcal{P}_v} = V^{\mathcal{P}_v \times \mathcal{P}_v}$ with the resulting tensor product bornology and the multiplication defined by the obvious bilinear pairing as in Section 6, and give $\bigoplus_{v \in \text{reg}(E)} \mathcal{M}_{\mathcal{P}_v}$ the direct sum bornology. There is a semi-split extension of bornological algebras*

$$\bigoplus_{v \in \text{reg}(E)} \mathcal{M}_{\mathcal{P}_v} \xrightarrow{i} C(E)_{\text{lg}} \xrightarrow{p} L(E)_{\text{lg}}.$$

σ

PROOF. Let $S \in \mathcal{F}$. We claim that $\sigma \circ p$ maps the linear span of S_n into itself. If $\alpha\beta^* \in \mathcal{B}'$, then $\sigma \circ p(\alpha\beta^*) = \alpha\beta^*$. If $\alpha\beta^* \notin \mathcal{B}'$, then $\alpha = \alpha_0 e_v$, $\beta = \beta_0 e_v$ for some $v \in \text{reg}(E)$, $\alpha_0, \beta_0 \in \mathcal{P}_v$. And then

$$p(\alpha\beta^*) = p(\alpha_0\beta_0^*) - \sum_{s(e)=v, e \neq e_v} p(\alpha_0 e e^* \beta_0).$$

Since $\alpha_0\beta_0^*$ is shorter than $\alpha\beta^*$ and $\alpha_0 e e^* \beta_0 \in \mathcal{B}'$ for $e \in E^1$ with $s(e) = v$ and $e \neq e_v$, an induction over $|\alpha| + |\beta|$ shows that $\sigma \circ p(\alpha\beta^*)$ is always a V -linear combination of shorter words; in addition, all edges in these words are again contained in S because $S \in \mathcal{F}$. This proves the claim. Now Lemma 8.6 implies that $\sigma \circ p$ preserves linear growth of subsets. Equivalently, σ is a bounded map $L(E)_{\text{lg}} \rightarrow C(E)_{\text{lg}}$. Then a subset of $K(E)$ has linear growth in $C(E)$ if and only if it is of the form $(\text{id} - \sigma \circ p)(M)$ for a V -submodule $M \subseteq C(E)$ that has linear growth. The projection $\text{id} - \sigma \circ p$ kills $\alpha\beta^* \in \mathcal{B}'$. Thus we may disregard these generators when we describe the restriction to $K(E)$ of the linear growth bornology on $C(E)$. Instead of applying $\text{id} - \sigma \circ p$ to the remaining basis vectors $\alpha e_v e_v^* \beta^*$ for $r(\alpha) = r(\beta) = v \in \text{reg}(E)$, we may also apply it to $\alpha e_v e_v^* \beta^* - \alpha\beta^*$ because $\alpha\beta^*$ is a shorter basis vector that involves the same edges. And

$$\begin{aligned} (\text{id} - \sigma \circ p)(\alpha e_v e_v^* \beta^* - \alpha\beta^*) &= \alpha e_v e_v^* \beta^* - \alpha\beta^* + \sigma \left(\sum_{s(e)=v, e \neq e_v} p(\alpha e e^* \beta^*) \right) \\ &= -\alpha\beta^* + \sum_{s(e)=v} \alpha e e^* \beta^* = -\alpha q_v \beta^*. \end{aligned}$$

Now Lemma 8.6 implies that a subset of $K(E)$ has linear growth in $C(E)$ if and only if there are $S \in \mathcal{F}$ and $m \in \mathbb{N}^*$ so that it belongs to the V -linear span of $\pi^{\lfloor n/m \rfloor} \alpha q_v \beta^*$

with $v \in \text{reg}(E)$, $\alpha, \beta \in \mathcal{P}_v \cap S^{(\infty)}$, and $|\alpha| + |\beta| + 2 \leq n$. Under the isomorphism $\bigoplus_{v \in \text{reg}(E)} \mathcal{M}_{\mathcal{P}_v} \cong K(E)$, this becomes equal to the bornology on $\bigoplus_{v \in \text{reg}(E)} \mathcal{M}_{\mathcal{P}_v}$ specified in the statement of the lemma. \square

The semi-split extension in Lemma 8.7 implies a similar semi-split extension involving the dagger completions $C(E)^\dagger$, $L(E)^\dagger$ and the completion of $\bigoplus_{v \in \text{reg}(E)} \mathcal{M}_{\mathcal{P}_v}$ for the bornology specified in Lemma 8.7.

Now Theorem 8.3 is proven by showing that all homomorphisms and quasi-homomorphisms that are used in [13] remain bounded and all homotopies among them remain dagger homotopies when we give all algebras that occur the suitable “linear growth” bornology, defined using the lengths of paths to define linear growth. This is because all maps in [13] are described by explicit formulas in terms of paths, which change the length only by finite amounts. We have put linear growth in quotation marks because the correct bornologies on the ideals $K(E)$ and $\hat{K}(E)$ in [13] are restrictions of linear growth bornologies on larger algebras as in Lemma 8.7. These bornological algebras are special cases of Example 6.6, and so $\mathbb{H}\mathbb{A}$ is stable for such matrix algebras. The bornology on $K(E)$ in Lemma 8.7 actually deserves to be called a “linear growth bornology”. But the relevant length function is specified by hand and not by the length of products as for the official linear growth bornology in Definition 2.1.16.

9. Filtered Noetherian rings and analytic quasi-freeness

In Section 9.1, we develop a criterion for a quasi-free algebra to be analytically quasi-free. It uses a connection with a growth condition, called finite-degree. In Section 9.2, we show that the criterion from Section 9.1 applies to dagger completions of smooth, commutative V -algebras of relative dimension 1. And we show that any smooth curve over \mathbb{F} lifts to such a V -algebra.

9.1. Finite-degree connections. Recall that a complete bornological V -algebra R is called quasi-free if all its square-zero extensions split. This is equivalent to the existence of a *connection* on $\overline{\Omega}^1(R)$, that is, a linear map $\nabla: \overline{\Omega}^1(R) \rightarrow \overline{\Omega}^2(R)$ satisfying

$$\nabla(a\omega) = a\nabla(\omega) \quad \text{and} \quad \nabla(\omega a) = \nabla(\omega)a + \omega da,$$

for all $a \in R$ and $\omega \in \overline{\Omega}^1(R)$. And this is, in turn, equivalent to $\overline{\Omega}^1(R)$ being projective for extensions of complete bornological R -bimodules with a bounded V -linear section. The above claims go back to Cuntz and Quillen [16] for algebras without extra structure. They also hold for algebras in additive symmetric monoidal categories and hence for complete bornological V -algebras (see, for instance, [26]). A related result is Proposition 4.4.6.

We are going to prove that a quasi-free algebra R is analytically quasi-free if $\overline{\Omega}^1(R)$ has a connection whose growth is controlled in a certain way. This uses increasing filtrations. An (increasing) *filtration* on a V -module M is an increasing sequence of V -submodules $(\mathcal{F}_n M)_{n \in \mathbb{N}}$ with $\bigcup \mathcal{F}_n M = M$. For a V -algebra R , we require, in addition, that $\mathcal{F}_n R \cdot \mathcal{F}_m R \subseteq \mathcal{F}_{n+m} R$ for all $n, m \in \mathbb{N}$. And for a module M over a V -algebra R with a fixed filtration $(\mathcal{F}_n R)_{n \in \mathbb{N}}$, we require, in addition, that $\mathcal{F}_n R \cdot \mathcal{F}_m M \subseteq \mathcal{F}_{n+m} M$ for all $n, m \in \mathbb{N}$. Then we speak of a *filtered algebra* and a *filtered module*, respectively.

DEFINITION 9.1.1. A map $f: M \rightarrow N$ between filtered V -modules has *finite degree* if there is $a \in \mathbb{N}$ – the degree – such that $f(\mathcal{F}_n M) \subseteq \mathcal{F}_{n+a}(N)$ for all $n \in \mathbb{N}$. Two filtrations $(\mathcal{F}_n M)_n$ and $(\mathcal{F}'_n M)_n$ on a filtered V -module M are called *shift equivalent* if there is $a \in \mathbb{N}$ such that $\mathcal{F}_n M \subseteq \mathcal{F}'_{n+a} M$ and $\mathcal{F}'_n M \subseteq \mathcal{F}_{n+a} M$ for all $n \in \mathbb{N}$.

EXAMPLE 9.1.2. Let R be a torsion-free bornological V -algebra. Define $M^{(j)}$ for a complete bounded submodule $M \subseteq R$ and $j \geq 0$ as in (2.2.1). Put

$$(9.1.3) \quad \mathcal{F}_r^M \overline{\Omega}^j R := \sum_{i_0 + \dots + i_j \leq r} M^{(i_0)} dM^{(i_1)} \dots dM^{(i_j)} \oplus \sum_{i_1 + \dots + i_j \leq r} dM^{(i_1)} \dots dM^{(i_j)}$$

for $r \in \mathbb{N}$. This is an increasing filtration on the differential j -forms of the subalgebra $M^{(\infty)} \subseteq R$ generated by M .

The following lemma relates such filtrations to the linear growth bornology:

LEMMA 9.1.4. *Let R be a torsion-free bornological algebra, $M \subseteq R$ a bounded V -submodule and $n \geq 0$. Then*

$$\sum_{i \geq 0} \pi^i \mathcal{F}_{i+n}^M \overline{\Omega}^n R \subseteq \overline{\Omega}^n(M^\circ) \subseteq \sum_{i \geq 0} \pi^i \mathcal{F}_{i+n+1}^M \overline{\Omega}^n R.$$

PROOF. We compute

$$\begin{aligned} \overline{\Omega}^n(M^\circ) &= M^\circ d(M^\circ)^n \oplus d(M^\circ)^n \\ &= \sum_{i \geq 0} \pi^i \left(\sum_{i_0 + \dots + i_n = i} M^{(i_0+1)} dM^{(i_1+1)} \dots dM^{(i_n+1)} \right. \\ &\quad \left. \oplus \sum_{i_1 + \dots + i_n = i} dM^{(i_1+1)} \dots dM^{(i_n+1)} \right). \quad \square \end{aligned}$$

LEMMA 9.1.5. *Let $M \subseteq R$ be a bounded submodule, $r, b \geq 1$ and $s \geq 0$. Then*

$$\mathcal{F}_r^M \overline{\Omega}^s R \subseteq \mathcal{F}_{\lfloor r/b \rfloor + s}^{M^{(b)}} \overline{\Omega}^s R \subseteq \mathcal{F}_{r+b(s+1)}^M \overline{\Omega}^s R.$$

PROOF. Straightforward. \square

LEMMA 9.1.6. *Let X and Y be torsion-free bornological modules. Let (f_n) be a sequence of bounded linear maps $X \rightarrow Y$. Assume that for each bounded submodule $M \subseteq X$ there is a bounded submodule $N \subseteq Y$ and a sequence of nonnegative integers (a_n) with $\lim a_n = \infty$ and $f_n(M) \subseteq \pi^{a_n} N$ for all $n \in \mathbb{N}$. Then the series $s(x) := \sum_n f_n(x)$ converges in \overline{Y} for every $x \in X$, and the assignment $x \mapsto s(x)$ is bounded and linear. So it extends to a bounded linear map $s: \overline{X} \rightarrow \overline{Y}$.*

PROOF. Straightforward. \square

DEFINITION 9.1.7. Let R be a torsion-free bornological V -algebra. A connection $\nabla: \overline{\Omega}^1(R) \rightarrow \overline{\Omega}^2(R)$ has *finite degree* on a bounded submodule $M \subseteq R$ if it has finite degree as a V -module map with respect to the filtrations on $\overline{\Omega}^1(M^{(\infty)})$ and $\overline{\Omega}^2(M^{(\infty)})$ from Example 9.1.2. A connection ∇ has *finite degree* on R if any bounded subset is contained in a bounded submodule of R on which ∇ has finite degree.

REMARK 9.1.8. Lemma 9.1.5 implies that if ∇ has finite degree on M , then it also has finite degree on $M^{(b)}$ for all b . Then ∇ is a finite degree connection on $M^{(\infty)}$ with the bornology that is cofinally generated by $M^{(n)}$ for $n \in \mathbb{N}$.

The following theorem is an analytic version of the formal tubular neighbourhood theorem by Cuntz and Quillen in [16].

THEOREM 9.1.9. *Let R be a complete, torsion-free bornological algebra. If $\overline{\Omega}^1(R)$ has a connection of finite degree, then R^\dagger is analytically quasi-free.*

PROOF. We introduce some notation on Hochschild cochains. If X is a complete, bornological R -bimodule and $\psi: R^{\overline{\otimes}^n} \rightarrow X$ is an n -cochain, write $\delta(\psi)$ for its Hochschild coboundary. If $\xi: R^{\overline{\otimes}^m} \rightarrow Y$ is another cochain, write $\psi \cup \xi: R^{\overline{\otimes}^{n+m}} \rightarrow X \overline{\otimes}_R Y$ for the cup product. Let $\nabla: \overline{\Omega}^1 R \rightarrow \overline{\Omega}^2 R$ be a connection of finite degree, and let $M \subseteq R$ be a bounded submodule and $a \geq 0$ an integer such that ∇ has degree a on M . The connection ∇ is equivalent to a 1-cochain $\varphi_2: R \rightarrow \overline{\Omega}^2 R$ satisfying $\delta(\varphi_2) = d \cup d$, via $\nabla(x_0 dx_1) = x_0 \varphi_2(x_1)$ for $x_0 \in R^+$, $x_1 \in R$. Then φ_2 raises the M -filtration degree by at most a . If X is a filtered R -bimodule and $\psi: R \overline{\otimes} R \rightarrow X$ is a 2-cocycle of degree at most b , then

$$\bar{\psi}: \overline{\Omega}^2 R \rightarrow X, \quad \bar{\psi}(x_0 dx_1 dx_2) = x_0 \psi(x_1, x_2)$$

is a bimodule homomorphism. And the 1-cochain

$$\psi' = \bar{\psi} \circ \varphi_2$$

raises filtration degree by at most $a+b$ and satisfies $\delta(\psi') = \psi$. For $n \geq 1$, inductively define a 2-cocycle and a 1-cochain with values in $\overline{\Omega}^{2(n+1)} R$ as follows:

$$\begin{aligned} \psi_{2(n+1)} &:= \sum_{j=0}^n d\varphi_{2j} \cup d\varphi_{2(n-j)} - \sum_{j=1}^n \varphi_{2j} \cup \varphi_{2(n+1-j)}, \\ \varphi_{2(n+1)} &:= \psi'_{2(n+1)}. \end{aligned}$$

Put $\varphi_0 = \text{id}: R \rightarrow R$. To see that the maps ψ_{2n} are cocycles, one proves first that

$$\delta(d\varphi_{2n}) = - \sum_{j=0}^n d(\varphi_{2j} \cup \varphi_{2(n-j)}).$$

Then a long but straightforward calculation using the Leibniz rule for both d and δ shows by induction that $\delta(\psi_{2n}) = 0$ (see [10, Theorem 2.1]). By construction, the bounded linear map $\varphi_{\leq 2n} := \sum_{i=0}^n \varphi_{2i}$ is a section of the canonical projection $\text{TR} \rightarrow R$, and its curvature vanishes modulo JR^{n+1} . So it defines a bounded algebra homomorphism $R \rightarrow \text{TR}/JR^{n+1}$. Hence the infinite series $\sum_{i=0}^{\infty} \varphi_{2i}$ is an algebra homomorphism into the projective limit. It suffices to show that, for each m , the series $\sum_{i=0}^{\infty} \varphi_{2i}$ defines a bounded linear homomorphism $R_{\text{lg}} \rightarrow (\mathcal{U}(\text{TR}_{\text{lg}}, JR_{\text{lg}}^m), \mathcal{U}(JR_{\text{lg}}, JR_{\text{lg}}^m))^\dagger$.

One checks by induction on n that $\varphi_{2n}(M^{(i)}) \subseteq \mathcal{F}_{i+(2n-1)a}^M \overline{\Omega}^{2n} R$. Hence

$$(9.1.10) \quad \varphi_{2n}(M^\circ) \subseteq \sum_{i=0}^{\infty} \pi^i \mathcal{F}_{i+(2n-1)a+1}^M \overline{\Omega}^{2n} R.$$

Next let $m \geq 1$ and choose an integer $c > \max\{1, 2am\}$. Then

$$(9.1.11) \quad i + \left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{i + (2n-1)a + 1}{c} \right\rfloor \geq (1 - 1/c)i \geq 0$$

for all $i \geq 0$ and sufficiently large n . Then $i \geq \left\lceil \frac{i+(2n-1)a+1}{c} \right\rceil - \lfloor \frac{n}{m} \rfloor$. Set $D(i, n, c) := \left\lceil \frac{i+(2n-1)a+1}{c} \right\rceil$. Equations (9.1.10) and (9.1.11) and Lemmas 9.1.4 and 9.1.5 imply

$$\begin{aligned} \varphi_{2n}(M^\circ) &\subseteq \sum_{i \geq 0} \pi^i \mathcal{F}_{D(i, n, c)+2n}^{M^{(c)}} \overline{\Omega}^{2n}(R) \\ &\subseteq \pi^{-\lfloor \frac{n}{m} \rfloor} \sum_{i \geq 0} \pi^{D(i, n, c)} \mathcal{F}_{D(i, n, c)+2n}^{M^{(c)}} \overline{\Omega}^{2n}(R) \subseteq \pi^{-\lfloor \frac{n}{m} \rfloor} \overline{\Omega}^{2n}((M^{(c)})^\circ) \end{aligned}$$

By Proposition 4.4.15, the subset of infinite series $\sum_{n=0}^\infty \varphi_{2n}(M^\circ)$ is bounded in $(\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, \mathrm{JR}_{\mathrm{lg}}^m), \mathcal{U}(\mathrm{JR}_{\mathrm{lg}}, \mathrm{JR}_{\mathrm{lg}}^m))^\dagger$. So $\sum_{n=0}^\infty \varphi_{2n}$ defines a bounded homomorphism

$$R \rightarrow (\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, \mathrm{JR}_{\mathrm{lg}}^m), \mathcal{U}(\mathrm{JR}_{\mathrm{lg}}, \mathrm{JR}_{\mathrm{lg}}^m))^\dagger$$

for each $m \geq 1$; this completes the proof. \square

COROLLARY 9.1.12. *Let R be as in Theorem 9.1.9. Then the natural map $\mathrm{HA}(R^\dagger) \rightarrow X(R^\dagger \otimes F)$ is a chain homotopy equivalence and $\mathrm{HA}_*(R)$ is isomorphic to the homology of $X(R^\dagger \otimes F)$.*

PROOF. Immediate from Theorem 9.1.9 and Corollary 4.7.2. \square

9.2. Filtered Noetherian rings and smooth algebras. We now show that some quasi-free algebras have a connection of finite degree. In particular, this includes smooth, commutative finitely generated V -algebras of relative dimension 1. For the remainder of this section, let R be a finitely generated V -algebra, equipped with the fine bornology. Let $S \subseteq R$ be a finite generating subset and let $S^{\leq n}$ be the set of all products of elements of S of length at most n . As above, let $\mathcal{F}_n R \subseteq R$ be the V -submodule generated by $S^{\leq n}$. By convention, $S^{\leq 0} = \{1\}$ and $\mathcal{F}_0 R = V \cdot 1$. This is an increasing filtration on R . It induces filtrations on the bimodules $\Omega^l(R)$ as in Example 9.1.2. More concretely, $\mathcal{F}_n(\Omega^l(R))$ is the V -submodule of $\Omega^l(R)$ generated by $x_0 dx_1 \dots dx_l$ with $x_0 \in \mathcal{F}_{n_0}(R)$ or $x_0 = 1$ and $n_0 = 0$, and $x_i \in \mathcal{F}_{n_i}(R)$ for $i = 1, \dots, l$, and $n_0 + \dots + n_l \leq n$. By construction, the V -module $\mathcal{F}_n R \cdot \mathcal{F}_m R$ that is generated by products $x \cdot y$ with $x \in \mathcal{F}_n R$, $y \in \mathcal{F}_m R$ is equal to $\mathcal{F}_{n+m} R$ for all $n, m \in \mathbb{N}$. This is more than what is required for a filtered algebra, and the extra information is crucial for the filtration to generate the linear growth bornology.

Let M be an R -module with a finite generating set $S_M \subseteq M$. Then we define a filtration on M , called the *canonical filtration*, by letting $\mathcal{F}_n M$ be the V -submodule generated by $a \cdot x$ with $a \in \mathcal{F}_n R$ and $x \in S_M$. This satisfies $\mathcal{F}_m R \cdot \mathcal{F}_n M \subseteq \mathcal{F}_{n+m} M$ because $\mathcal{F}_m R \cdot \mathcal{F}_n R \subseteq \mathcal{F}_{n+m} R$. The following proposition characterises canonical filtrations by a universal property:

PROPOSITION 9.2.1. *Let R be a filtered V -algebra and let M be a finitely generated R -module. Equip M with the filtration described above. Then any R -module map from M to a filtered R -module Y is of finite degree. The canonical filtrations for two different finite generating sets of M are shift equivalent.*

PROOF. Let $\{m_1, \dots, m_n\}$ be a finite generating set for M as an R -module. Let $h: M \rightarrow Y$ be an R -module homomorphism into a filtered R -module Y . Since $Y = \bigcup \mathcal{F}_l Y$, there is an $l \in \mathbb{N}$ with $h(m_i) \in \mathcal{F}_l Y$ for all $i = 1, \dots, m$. Then $h(a \cdot m_i) \in \mathcal{F}_{n+l} Y$ for $a \in \mathcal{F}_n R$. Hence $h(\mathcal{F}_n M) \subseteq \mathcal{F}_{n+l} Y$ for all $n \in \mathbb{N}$. That is, h has finite degree. In particular, if we equip M with another filtration $(\mathcal{F}'_n M)_{n \in \mathbb{N}}$, then the identity map has finite degree, that is, there is $l \in \mathbb{N}$ with $\mathcal{F}_n M \subseteq \mathcal{F}'_{n+l} M$ for all

$n \in \mathbb{N}$. If the other filtration comes from another finite generating set, then we may reverse the roles and also get $l' \in \mathbb{N}$ with inclusions $\mathcal{F}'_n M \subseteq \mathcal{F}_{n+l'} M$ for all $n \in \mathbb{N}$. \square

DEFINITION 9.2.2. A filtered V -algebra R is called (left) *filtered Noetherian* if every left ideal I is finitely generated and the filtration $(\mathcal{F}_n R \cap I)_{n \in \mathbb{N}}$ is shift equivalent to the canonical filtration of Proposition 9.2.1 from a finite generating set. In other words, there are finitely many $x_1, \dots, x_n \in I$ and $l \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and $y \in \mathcal{F}_m R \cap I$, there are $a_i \in \mathcal{F}_{m+l} R$ with $y = \sum_{i=1}^n a_i x_i$.

LEMMA 9.2.3. *Let R be a finitely generated, quasi-free V -algebra. Assume that $R^+ \otimes (R^+)^{\text{op}}$ is filtered Noetherian. Then $\Omega^1(R)$ has a connection of finite degree.*

PROOF. Since R is quasi-free, the left multiplication map $R^+ \otimes \Omega^1(R) \rightarrow \Omega^1(R)$ splits by an R -bimodule homomorphism $s: \Omega^1(R) \rightarrow R^+ \otimes \Omega^1(R)$. By definition, $\Omega^1(R)$ is a left ideal in $R^+ \otimes (R^+)^{\text{op}}$. By assumption, it is finitely generated as such, and the filtration on $R^+ \otimes (R^+)^{\text{op}}$ restricted to $\Omega^1(R)$ is the canonical filtration on $\Omega^1(R)$ as a module over $R^+ \otimes (R^+)^{\text{op}}$. Now Proposition 9.2.1 shows that the section s above has finite degree. The section s yields a connection $\nabla: \Omega^1(R) \rightarrow \Omega^2(R)$, which is defined by $\nabla(\omega) = 1 \otimes \omega - s(\omega)$. It follows that ∇ has finite degree. \square

Our next goal is to show that a commutative, finitely generated V -algebra with the filtration coming from a finite generating set is filtered Noetherian. First consider the polynomial ring in n variables. The filtration defined by the obvious generating set is the total degree filtration, where $\mathcal{F}_m(V[x_1, \dots, x_n])$ is the V -submodule generated by the monomials of total degree at most m , that is, terms of the form $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leq m$.

THEOREM 9.2.4. *The polynomial ring $R = V[x_1, \dots, x_n]$ with the total degree filtration is filtered Noetherian.*

PROOF. Let I be any ideal in R . Since R is Noetherian, I is finitely generated. Since V is a principal ideal domain, I has a finite, strong Gröbner basis with respect to any term order on the monomials x^α (see [2, Theorem 4.5.9]). We use the degree lexicographic order (see [2, Definition 1.4.3]); the only property we need is that $|\alpha| < |\beta|$ implies $x^\alpha < x^\beta$. The chosen order on monomials defines the *leading term* $\text{lt}(f)$ of a polynomial f . Let $G = \{f_1, \dots, f_N\}$ be a strong Gröbner basis for I . By [2, Theorem 4.1.12], any $g \in I$ can be written as $g = \sum_{j=1}^M c_j t_j f_{i_j}$, where $M \in \mathbb{N}$, $c_j \in V$, t_j is a monomial in R , $i_j \in \{1, \dots, N\}$, and $\text{lt}(t_j f_{i_j}) < \text{lt}(g)$ for each j . So the total degree of $t_j f_{i_j}$ is at most the total degree of g for each $j = 1, \dots, M$, and this remains so for the total degree of t_j . Combining the monomials t_j with the same i_j , we write any element $g \in I$ of total degree at most m in the form $\sum_{i=1}^N p_i f_i$ with $p_i \in \mathcal{F}_m R$. \square

PROPOSITION 9.2.5. *A quotient of a filtered Noetherian V -algebra with the induced filtration is again filtered Noetherian.*

PROOF. Let R be a filtered Noetherian V -algebra and let I be an ideal. Any ideal in the quotient ring R/I is of the form J/I for a unique ideal J in R containing I . Let $x_1, \dots, x_n \in J$ and $l \in \mathbb{N}$ be such that for all $m \in \mathbb{N}$ and $y \in \mathcal{F}_m R \cap I$, there are $a_i \in \mathcal{F}_{m+l} R$ with $y = \sum_{i=1}^n a_i x_i$. Then the images of x_1, \dots, x_n in J/I and the same l will clearly work for the ideal J/I in the quotient R/I . \square

COROLLARY 9.2.6. *Any finitely generated, commutative V -algebra is filtered Noetherian.*

PROOF. Let A be a finitely generated, commutative V -algebra. Let S be any finite generating set. Turn it into a surjective homomorphism from the polynomial algebra $R = V[x_1, \dots, x_n]$ onto A . This identifies $A \cong R/I$ for an ideal I in R . The filtration on A defined by S is equal to the filtration on the quotient R/I defined by the degree filtration on R . Now the claim follows from Theorem 9.2.4 and Proposition 9.2.5. \square

PROPOSITION 9.2.7. *Let R be a smooth, finitely generated commutative V -algebra of relative dimension 1. Then R admits a connection of finite degree.*

PROOF. The assumptions on R imply that $\Omega^1(R)$ is a projective, finitely generated R -bimodule. Furthermore, by Corollary 9.2.6, R is filtered Noetherian. The result now follows from Lemma 9.2.3. \square

REMARK 9.2.8. In their seminal article [28], Paul Monsky and Gerard Washnitzer introduced the so-called *Monsky–Washnitzer cohomology* $H_{\text{MW}}^*(A)$ for a smooth unital \mathbb{F} -algebra A that has a “very smooth” lift. This is a presentation $A = S/\pi S$ where S is dagger complete and very smooth ([28, Definition 2.5]); by definition, $H_{\text{MW}}^*(A) = H_{\text{dR}}(S \otimes F)$ is the de Rham cohomology of $S \otimes F$. As in the current article, Monsky and Washnitzer assumed that $\text{char}(F) = 0$ but made no assumption about the characteristic of \mathbb{F} . The very smooth liftability assumption in [28] was crucial for their proof of the functoriality of H_{MW}^* . Later on, Marius van der Put [32] managed to remove that assumption; for any smooth commutative unital \mathbb{F} -algebra A of finite type, he defines $H_{\text{MW}}^*(A)$ as the de Rham cohomology of the dagger completion of any smooth V -algebra R with $R/\pi R = A$. The existence of such a lift follows from a theorem of Renée Elkik [18]; van der Put proves functoriality of H_{MW}^* using Artin approximation. However, in his paper he assumes that \mathbb{F} is finite. More recently, under very general assumptions (in particular, for \mathbb{F} of arbitrary characteristic) Alberto Arabia [3] proved that every smooth \mathbb{F} -algebra admits a very smooth lift, and extended the original definition of Monsky and Washnitzer. In a parallel development, Pierre Berthelot introduced rigid cohomology $H_{\text{rig}}^*(X)$ of general schemes X over a field \mathbb{F} with $\text{char}(\mathbb{F}) > 0$, which for smooth affine $X = \text{sp}(A)$ agrees with $H_{\text{MW}}^*(A)$. With no assumptions on $\text{char}(\mathbb{F})$, Große-Klönne [19] introduced the de Rham cohomology of dagger spaces over V , and he related it to rigid cohomology in the case when $\text{char}(\mathbb{F}) > 0$.

The following is one of the main applications of our theory:

THEOREM 9.2.9. *Let X be a smooth affine variety over the residue field \mathbb{F} of dimension 1 and let $A = \mathcal{O}(X)$ be its algebra of polynomial functions. Let R be a smooth, commutative algebra of relative dimension 1 with $R/\pi R \cong A$. Equip R with the fine bornology and let R^\dagger be its dagger completion. If $* = 0, 1$, then $\text{HA}_*(R^\dagger)$ is naturally isomorphic to the de Rham cohomology of R^\dagger . This is isomorphic to the Monsky–Washnitzer cohomology of A , which, if $\text{char}(\mathbb{F}) > 0$, agrees with the rigid cohomology $H_{\text{rig}}^*(A, F)$ of X .*

PROOF. By our hypothesis and Proposition 9.2.7, R is quasi-free. Equipping R with the fine bornology, we are in the situation of Theorem 9.1.9. Then Corollary 9.1.12 and [11, Theorem 5.5] imply that $\text{HA}_*(R^\dagger)$ is isomorphic to the de

Rham cohomology of R^\dagger . Remark 9.2.8 discusses the generality in which the latter is known to be isomorphic to different cohomology theories over \mathbb{F} . \square

Elkik [18] has shown that any smooth curve over \mathbb{F} has a smooth lift over V . The following lemma shows that we may also arrange this lift to have relative dimension 1 as required in Theorem 9.2.9:

LEMMA 9.2.10. *Let R be a smooth algebra, $A = R/\pi R$, and $d = \dim A$. Then $R = R_1 \times R_2$, where R_1 is smooth of relative dimension d and $R_1/\pi R_1 = A$.*

PROOF. Since R is smooth, $\Omega_{R/V}^1$ is projective. So its rank r is a continuous function on $\text{Spec}(R)$. Thus the set of primes \mathfrak{P} where $r(\mathfrak{P}) = d$ is clopen. This clopen subset induces a product decomposition $R \cong R_1 \times R_2$. Since $\dim A = d$, the relative dimension is d at all primes containing π . Now the lemma follows. \square

CHAPTER 3

A cyclic homology theory in positive characteristic

ABSTRACT. Let V be a complete discrete valuation ring with uniformiser π and residue field $\mathbb{F} = V/\pi V$. We define a cyclic homology theory for algebras over \mathbb{F} , which we call *analytic cyclic homology*, by lifting them to free algebras over V and then building a tube algebra and completing to a dagger algebra. We show that this theory may be computed using any complete, torsion-free V -module lifting of an \mathbb{F} -algebra. We show that our theory is polynomially homotopy invariant, matricially stable, and satisfies excision.

1. Introduction

Cyclic homology and its variants have become fundamental invariants in non-commutative geometry. Developed by Connes and Tsygan in the 1980s, these invariants vastly generalise de Rham cohomology for smooth manifolds and schemes to non-commutative algebras. The approach of Cuntz and Quillen provided a new perspective in cyclic homology by introducing a non-commutative analogue of non-singular replacements of arbitrary algebraic varieties. Specifically, they build a certain universal nilpotent extension of an associative algebra A over \mathbb{C} , by a quasi-free algebra $\mathcal{T}(A)$. From the viewpoint of cyclic homology, the consequence of this quasi-free replacement is that the periodic cyclic homology of A can be computed by a much simpler chain complex – the X -complex of $\mathcal{T}(A)$. So a quasi-free replacement plays the same role in cyclic homology that a smooth embedding does in de Rham cohomology. The use of suitably defined nilpotent extensions has thereafter become a standard approach in the definition of local, analytic and equivariant cyclic homology theories.

The robustness of the Cuntz-Quillen approach leads us to the natural question of extending periodic cyclic homology to the setting of non-Archimedean algebras. The ultimate aim of the program – which is far from complete – is that our homology theory specialises to rigid cohomology in the commutative setting. Our motivation comes from recent work in [11], which links Berthelot’s rigid cohomology for commutative \mathbb{F}_p -algebras to the periodic cyclic homology of a certain dagger completed \mathbb{Z}_p -algebra lifting.

As a first step towards a theory for noncommutative \mathbb{F}_p -algebras, we introduced *analytic cyclic homology* in [12] for torsion-free non-Archimedean bornological algebras. We briefly recall its definition here. Let V be a complete discrete valuation ring with uniformiser π , residue field \mathbb{F} and fraction field F of characteristic 0. Given a complete, bornologically torsion-free V -algebra R , we use the Cuntz-Quillen machinery to build an *analytically nilpotent* extension

$$\mathcal{J}R \twoheadrightarrow \mathcal{T}R \twoheadrightarrow R$$

with an *analytically quasi-free* algebra $\mathcal{T}R$. We define the analytic cyclic complex $\mathbb{H}\mathbb{A}(R)$ of R as the projective system of X -complexes of $\mathcal{T}R \overline{\otimes} F$. These are projective systems of \mathbb{Z}_2 -graded bornological F -vector spaces. We show that our homology theory is homotopy invariant for dagger homotopies, matricially stable and satisfies excision for semi-split extensions. In one of our main applications, we show that if R is a smooth algebra of relative dimension 1, then its analytic cyclic homology can be identified with the rigid cohomology of $R/\pi R$ with coefficients in F , when $\text{char } F = 0$.

We assume throughout this paper that $\text{char } F = 0$.

In this article, we define *analytic cyclic homology* for \mathbb{F} -algebras by lifting them to projective systems of torsion-free V -algebras (briefly, *pro-algebras*). We show that our theory is homotopy invariant for polynomial homotopies. Furthermore, we use the analytic cyclic theory in [12] to show that our theory for \mathbb{F} -algebras is matricially stable and satisfies excision for extensions of finitely generated algebras. These properties are desirable from the perspective of algebraic bivariant K -theory ([14]), which is the universal excisive, homotopy invariant, matricially stable functor from a category of algebras into a triangulated category. In our most important result, we show that different choices of complete, bornological torsion-free V -module liftings all compute the same theory.

The paper is organised as follows:

In Section 2, we define the analytic cyclic homology complex $\mathbb{H}\mathbb{A}(A, W, \varrho)$ of an \mathbb{F} -algebra A with respect to a complete, bornologically torsion-free V -module lift $W \xrightarrow{\varrho} A$. It is defined as the X -complex of a pro-algebra, built from the tensor algebra TW and powers of the ideal $I := \ker(\text{TW} \rightarrow A)$. We refer to the analytic cyclic complex complex of A with respect to the free V -module lift $VA \twoheadrightarrow A$ as ‘the’ *analytic cyclic homology complex* of A and denote it by $\mathbb{H}\mathbb{A}(A)$. This is a projective system of \mathbb{Z}_2 -graded chain complexes of bornological F -vector spaces. The homology $\text{HA}_*(A)$ of a certain bornological completion of the homotopy inverse limit of $\mathbb{H}\mathbb{A}(A)$ is called the *analytic cyclic homology* of A .

In Section 3, we recall the simplified axioms of an exact category, following [21]. We need this generality to define the correct *derived category* in which different choices of torsion-free liftings of an algebra over \mathbb{F} yield quasi-isomorphic chain complexes. We call an extension

$$K \twoheadrightarrow E \xrightarrow{p} Q$$

in the category of bornological V -modules \mathfrak{B} *locally split* if for every bounded submodule $M \subseteq Q$, there is a bounded V -linear section for p . We also define locally split extensions in the category $\overleftarrow{\mathfrak{B}}$ of projective systems of bornological V -modules. The class of locally split extensions are exact category structures on \mathfrak{B} and $\overleftarrow{\mathfrak{B}}$. We use these exact category structures to define *quasi-isomorphisms* in the homotopy category of chain complexes $\text{HoKom}(\overleftarrow{\mathfrak{B}})$ over $\overleftarrow{\mathfrak{B}}$. The *derived category* of the exact category $\overleftarrow{\mathfrak{B}}$ is defined as the localisation of $\text{HoKom}(\overleftarrow{\mathfrak{B}})$ at the quasi-isomorphisms.

In Section 4, we use the exact category framework defined in Section 3 to show that for any complete, bornologically torsion-free V -module lifting $W \rightarrow A$, the complexes $\mathbb{H}\mathbb{A}(A, W, \varrho)$ and $\mathbb{H}\mathbb{A}(A)$ are quasi-isomorphic.

Sections 5 and 6 are about formal properties of our theory. We show that analytic cyclic homology is homotopy invariant for polynomial homotopies. We also show that if Λ is a set and M_Λ is the algebra of finitely supported matrices over \mathbb{F} ,

then the canonical map

$$A \rightarrow A \otimes_{\mathbb{F}} M_{\Lambda}$$

induces a quasi-isomorphism $\mathbb{H}\mathbb{A}(A) \cong \mathbb{H}\mathbb{A}(A \otimes_{\mathbb{F}} M_{\Lambda})$. Finally, we show that there is a diagram of extensions

$$\begin{array}{ccccc} \mathcal{K} & \hookrightarrow & \mathcal{E} & \xrightarrow{\rho} \twoheadrightarrow & \mathcal{Q} \\ \downarrow & & \downarrow & & \downarrow \\ K & \hookrightarrow & E & \xrightarrow{p} \twoheadrightarrow & Q, \end{array}$$

where the top row is a semi-split extension of pro-dagger algebras that surjects onto the bottom row of finitely generated \mathbb{F} -algebras with the fine bornology. This is used to show that analytic cyclic homology satisfies excision in the sense that if $K \twoheadrightarrow E \twoheadrightarrow Q$ is an extension of finitely generated \mathbb{F} -algebras, then there is a natural 6-term long exact sequence in homology

$$\begin{array}{ccccc} \mathrm{HA}_0(K) & \xrightarrow{i_*} & \mathrm{HA}_0(E) & \xrightarrow{p_*} & \mathrm{HA}_0(Q) \\ \delta \uparrow & & & & \downarrow \delta \\ \mathrm{HA}_1(Q) & \xleftarrow{p_*} & \mathrm{HA}_1(E) & \xleftarrow{i_*} & \mathrm{HA}_1(K). \end{array}$$

2. Analytic cyclic theory in mixed characteristic

In this section, we define analytic cyclic homology for \mathbb{F} -algebras. The definition follows an approach similar to that introduced in [12]. Namely, we lift an \mathbb{F} -algebra to a suitably defined tensor algebra, which has desirable homological properties in the sense of Cuntz and Quillen. For an \mathbb{F} -algebra A that admits a dagger algebra lifting D – which is the situation considered in Monsky-Washnitzer cohomology – the analytic cyclic homology of A coincides with the analytic cyclic homology of D . Therefore, the definition we introduce is the right passage from torsion-free V -algebras to \mathbb{F} -algebras.

DEFINITION 2.0.1. Let W be a complete bornological V -module. Its (incomplete) *tensor algebra* TW is the direct sum $\bigoplus_{n=1}^{\infty} W^{\otimes n}$ with its canonical bornology and with the multiplication defined by

$$(x_1 \otimes \cdots \otimes x_n) \cdot (x_{n+1} \otimes \cdots \otimes x_{n+m}) := x_1 \otimes \cdots \otimes x_{n+m}.$$

Let $\sigma_W: W \rightarrow TW$ be the inclusion of the first summand. It is a bounded V -module homomorphism, but not an algebra homomorphism.

A subset $S \subseteq TW$ is bounded if and only if there are $a \in \mathbb{N}$ and a bounded, π -adically complete V -submodule $B \subseteq W$ such that S is contained in the image in TW of $\sum_{n=1}^a B^{\otimes n}$. The tensor algebra has the following universal property:

LEMMA 2.0.2. *Let S be a bornological V -algebra. Composition with σ_W is a bijection from bounded homomorphisms $TW \rightarrow S$ to bounded V -module maps $W \rightarrow S$.*

PROOF. Let $f: W \rightarrow S$ be a bounded V -module homomorphism. Then there is a unique bounded homomorphism $f^{\#}: TW \rightarrow S$ with $f^{\#} \circ \sigma_W = f$, which is defined by

$$f^{\#}(x_1 \otimes \cdots \otimes x_n) := f(x_1) \cdots f(x_n)$$

for all $x_1, \dots, x_n \in W$. \square

In what follows, we go through a sequence of steps as in [12, Section 3], which leads to the definition of our homology theory.

The first step. Let A be an \mathbb{F} -algebra and let W be a torsion-free, complete bornological V -module with a surjective map $\varrho: W \twoheadrightarrow A$. We assume that $\varrho(M) \subseteq A$ is finite-dimensional for each bounded V -submodule $M \subseteq W$. Equivalently, ϱ is bounded when A carries the fine bornology. Let $R := \mathrm{TW}$; this is a torsion-free bornological V -algebra. It is torsion-free because W is torsion-free, and torsion-freeness is inherited by n -fold tensor products and direct sums. Since A with the fine bornology is a bornological V -algebra, Lemma 2.0.2 also shows that ϱ induces a bounded V -algebra homomorphism $\varrho^\#: R = \mathrm{TW} \twoheadrightarrow A$. Let $I := \ker \varrho^\#$. Then we have a torsion-free V -algebra resolution

$$I \twoheadrightarrow R \twoheadrightarrow A$$

of A .

The second step. We have built a torsion-free bornological V -algebra R with an ideal I . As in the case of our homology theory for complete, bornologically torsion-free algebras, we enlarge R to tube algebras for powers of I in R . We recall their definition:

DEFINITION 2.0.3. Let R be a torsion-free bornological V -algebra and I an ideal in R . Let I^j for $j \in \mathbb{N}^*$ denote the V -linear span of products $x_1 \cdots x_j$ with $x_1, \dots, x_j \in I$. The *tube algebra* of $I^l \triangleleft R$ for $l \in \mathbb{N}^*$ is defined as

$$\mathcal{U}(R, I^l) := \sum_{j=0}^{\infty} \pi^{-j} I^{l+j} \subseteq R \otimes F,$$

equipped with the subspace bornology; this is indeed a V -subalgebra of $R \otimes F$. If $l \geq j$, then $\mathcal{U}(R, I^l) \subseteq \mathcal{U}(R, I^j)$ is a bornological subalgebra. Let $\mathcal{U}(R, I^\infty)$ denote the resulting projective system of bornological V -algebras $(\mathcal{U}(R, I^l))_{l \in \mathbb{N}^*}$.

Since $\mathcal{U}(R, I^l)$ is a bornological submodule of an F -vector space, it is bornologically torsion-free. Furthermore, the inclusion $R \hookrightarrow \mathcal{U}(R, I^l)$ induces a bornological isomorphism $\mathcal{U}(R, I^l) \otimes F \cong R \otimes F$.

The third step. In this step, we equip each tube-tensor algebra $\mathcal{U}(\mathrm{TW}, I^l)$ for $l \in \mathbb{N}^*$, with the relative linear growth bornology with respect to the ideal $\mathcal{U}(I, I^l)$ (see [12, Definition 2.2.3]). Since $\pi \cdot \mathcal{U}(R, I^l) \subseteq \mathcal{U}(I, I^l)$, the quotient $\mathcal{U}(R, I^l)/\mathcal{U}(I, I^l)$ is semi-dagger. So by [12, Lemma 2.26], there is no difference between the relative and the absolute linear growth bornologies on $\mathcal{U}(R, I^l)$.

The fourth step. Tensor with F . This gives a projective system of bornological F -algebras

$$\mathcal{U}(R, I^\infty)_{\mathrm{lg}} \otimes F := (\mathcal{U}(R, I^l)_{\mathrm{lg}} \otimes F)_{l \in \mathbb{N}^*}.$$

The fifth step. In this step, we take the X -complex, defined for incomplete algebras as follows:

DEFINITION 2.0.4. Let S be a bornological V -algebra. The X -complex of S is defined as

$$X(S) := \left(S \begin{array}{c} \xrightarrow{q \circ d} \\ \xleftarrow{\bar{b}} \end{array} \Omega^1(S)/[\cdot, \cdot] \right),$$

where $\Omega^1(S)/[\cdot, \cdot]$ denotes the quotient of the noncommutative 1-forms by the cokernel of the map

$$S \otimes \Omega^1(S) \rightarrow \Omega^1(S), \quad x \otimes \omega \mapsto x \cdot \omega - \omega \cdot x.$$

The maps q , d and \tilde{b} are the same maps already defined in [12, Section 2.7].

In our case, we take the X -complex of the projective system of incomplete bornological algebras $\mathcal{U}(R, I^\infty)_{\text{ig}} \otimes F$. These define a projective system of $\mathbb{Z}/2$ -graded chain complexes of bornological F -vector spaces, which we call the *analytic cyclic complex* $\mathbb{H}\mathbb{A}(A; W, \varrho)$ of A with respect to the lifting $\varrho: W \rightarrow A$.

The sixth step. The next step takes the homotopy inverse limit $\text{holim } \mathbb{H}\mathbb{A}(A, W, \varrho)$ of the analytic chain complex. More explicitly, this is the mapping cone of the chain map

$$\begin{aligned} \prod_{l \in \mathbb{N}^*} X(\mathcal{U}(R, I^l)) \otimes F &\rightarrow \prod_{l \in \mathbb{N}^*} X(\mathcal{U}(R, I^l) \otimes F), \\ (x_l) &\mapsto (x_l - \sigma_l(x_{l+1}))_{l \in \mathbb{N}^*}. \end{aligned}$$

We will discuss the homotopy inverse limit construction in greater depth in Section 3. This gives us a $\mathbb{Z}/2$ -graded chain complex of bornological F -vector spaces.

The seventh step. Finally, we impart a notion of *completion* to our bornological structures that has better exactness properties. Recall that we can explicitly describe the completion of a bornological V -module as follows: given a bornological V -module $M = \varinjlim M_i$, its completion is defined as $\overline{M} = \varinjlim \overline{M_i}/\{\overline{0}\}$. The functor that takes a bornological V -module to the quotient by the bornological closure of the trivial module $\{0\}$ is called the *separated quotient functor*. In general, this functor is not exact and the separated quotient of a bornological V -module is hard to describe explicitly. This is also why we delay the process of completion; instead of directly taking the dagger completion of the tube-tensor algebra in step 2 as in [11, 12]. More specifically, the definition of analytic cyclic homology involves a certain analytic tensor algebra lifting $\mathcal{T}R$ of a complete, bornologically torsion-free V -algebra R . We described this tensor algebra explicitly in [12, Section 4.4], using the bornology of R and subcompleteness of the tensor product (see [12, Proposition 2.4.5]). However, to associate a natural chain complex to an \mathbb{F} -algebra A , we need to use *free algebra* liftings in place of tensor algebras, for which the separated quotient functor is hard to control. Therefore, we replace the completion functor with the *quasi-completion* of a bornological V -module:

DEFINITION 2.0.5. Let M be a bornological V -module written as an inductive limit $M \cong \varinjlim M_i$ of the directed set of its bounded V -submodules M_i . The *quasi-completion* of M is defined as $\mathcal{Q}(M) := \varinjlim \overline{M_i}$. It is a bornological V -module with the inductive limit bornology.

We now take the *quasi-completion* of the homotopy inverse limit of $\mathbb{H}\mathbb{A}(A, W, \varrho)$ at each degree. This yields a \mathbb{Z}_2 -graded bornological chain complex.

The eighth and the last step. In what follows, let A and $\varrho: W \rightarrow A$ be as in Step 1.

DEFINITION 2.0.6. The *analytic cyclic homology of an \mathbb{F} -algebra A with respect to W* is the homology $\text{HA}_*(A, W, \varrho)$ of the chain complex $\mathcal{Q}(\text{holim}(\mathbb{H}\mathbb{A}(A, W, \varrho)))$, that is, the quotient of the kernel of the differential by the image of the differential, equipped with the induced bornology.

REMARK 2.0.7. The definitions above also work if the lifting W of an \mathbb{F} -algebra is a projective system of V -modules. We then need a tube algebra that is a projective system over the powers of the ideal as well as over the indexing set of the pro-tensor algebra $\mathbb{T}W$ (see [12, Section 4.2]). We still denote the analytic cyclic complex of a pro- V -module lifting $W \twoheadrightarrow A$ by $\mathbb{H}\mathbb{A}(A, W, \varrho)$.

The following result formulates the functoriality of our homology theory:

LEMMA 2.0.8. *For $j = 1, 2$, let A_j be \mathbb{F} -algebras with the fine bornology, let W_j be torsion-free, complete bornological V -modules with bounded V -module maps $\varrho_j: W_j \twoheadrightarrow A_j$. Let $g: W_1 \twoheadrightarrow W_2$ be a bounded V -module homomorphism that satisfies $g(\ker \varrho_1) \subseteq \ker \varrho_2$. Then g induces a bounded F -linear map $\mathbb{H}\mathbb{A}_*(A_1; W_1, \varrho_1) \rightarrow \mathbb{H}\mathbb{A}_*(A_2; W_2, \varrho_2)$.*

PROOF. The map $g: W_1 \twoheadrightarrow W_2$ induces a bounded V -algebra homomorphism

$$\mathbb{T}(g): \mathbb{T}W_1 \rightarrow \mathbb{T}W_2.$$

Furthermore, the condition $g(\ker \varrho_1) \subseteq \ker \varrho_2$ means that the \mathbb{F} -linear map $A_1 \rightarrow A_2$, $\tilde{g}: x + \ker(\varrho_1) \mapsto g(x) + \ker(\varrho_2)$ is well-defined. Finally, the maps $\varrho_j: W_j \twoheadrightarrow A_j$ induce bounded V -algebra homomorphisms $\varrho_j^*: \mathbb{T}W_j \rightarrow A_j$ such that $\varrho_2^* \circ \mathbb{T}(g) = \tilde{g} \circ \varrho_1^*$. So $\mathbb{T}(g)$ maps $I_1 := \ker(\mathbb{T}W_1 \twoheadrightarrow A_1)$ to $I_2 := \ker(\mathbb{T}W_2 \twoheadrightarrow A_2)$. Hence $\mathbb{T}(g)$ induces a map $\mathcal{U}(\mathbb{T}W_1, I_1^l) \rightarrow \mathcal{U}(\mathbb{T}W_2, I_2^l)$ for each l . Since tensor product with F , the X -complex, homotopy projective limits, and quasi-completion are functors, we obtain a chain map $\mathcal{Q}(\operatorname{holim}(\mathbb{H}\mathbb{A}(A_1, W_1, \varrho_1))) \rightarrow \mathcal{Q}(\operatorname{holim}(\mathbb{H}\mathbb{A}(A_2, W_2, \varrho_2)))$. It induces a map in homology $\mathbb{H}\mathbb{A}_*(A_1, W_1, \varrho_1) \rightarrow \mathbb{H}\mathbb{A}_*(A_2, W_2, \varrho_2)$ for each $* = 0, 1$. \square

2.1. Special lifts of an algebra over the residue field. We now consider two special types of liftings that arise in practice.

Monsky-Washnitzer type lifting. The situation in Monsky-Washnitzer cohomology is as follows: consider a smooth, unital \mathbb{F} -algebra A with a weakly complete, “very smooth” V -algebra D that satisfies $D/\pi D \cong A$. In our setup, this means that if we equip A with the fine bornology, then there is a resolution by a dagger algebra $\pi D \twoheadrightarrow D \twoheadrightarrow A$. We can now apply our machinery and define the analytic cyclic homology of A with respect to its dagger algebra lifting D . This turns out to be the same as the analytic cyclic homology of D as defined in [12, Section 3]. Importantly, there is no difference between the quasi-completion and the completion of the tube-tensor algebra $\mathcal{U}(\mathbb{T}D, JD^m)$ in the linear growth bornology.

PROPOSITION 2.1.1. *Let D be a dagger algebra with the property that the quotient bornology on $D/\pi D$ is the fine bornology. Then $\mathbb{H}\mathbb{A}(D/\pi D, D, q)$ is quasi-isomorphic to the analytic cyclic homology complex $\mathbb{H}\mathbb{A}(D)$ of the dagger algebra D .*

PROOF. Let $A := D/\pi D$. We claim that the ideals JD and

$$\ker(\mathbb{T}D \twoheadrightarrow A) = JD \oplus \pi \cdot \sigma_D(D) = JD + \pi \cdot \mathbb{T}D$$

in $\mathbb{T}D$ generate the same tube algebras. To check the claim, let $l \in \mathbb{N}_{\geq 1}$ and compute

$$\sum_{a=0}^{\infty} \pi^{-a} (JD + \pi \cdot \mathbb{T}D)^{al} \subseteq \sum_{a=0}^{\infty} \sum_{i=0}^{al} \pi^{-a+i} (JD)^{al-i} = \sum_{a=0}^{\infty} \pi^{-a} (JD)^{al}.$$

So the tube algebras defining $\mathbb{H}\mathbb{A}(D)$ and $\mathbb{H}\mathbb{A}(D/\pi D, D, q)$ are the same. We now show that quasi-completion commutes with homotopy inverse limits. To show this, we will need the following:

LEMMA 2.1.2. *Quasi-completions commute with countable products.*

PROOF. Let $(M_n)_{n \in \mathbb{N}}$ be a collection of bornological V -modules and let $M = \prod_{n \in \mathbb{N}} M_n$. Writing each M_n as an inductive limit $M_n = \varinjlim_{i_n \in I_n} M_{n, i_n}$, we can describe the bornology on M explicitly as follows: a subset in M is bounded if and only if it is contained in some bounded submodule of the form $\prod_{n \in \mathbb{N}} M_{n, i_n}$, for $i_n \in I_n$. So the quasi-completion of M is the inductive limit of $\prod_{n \in \mathbb{N}} \overline{M_{n, i_n}}$. This is also precisely the product of the quasi-completions $\mathcal{Q}(M_n)$. \square

It remains to show that the quasi-completion of the complex $X(\mathcal{U}(\mathbb{T}D, JD^\infty)_{\text{lg}} \otimes F)$ is isomorphic to the completed analytic chain complex $\overline{X}(\mathbb{T}D \overline{\otimes} F)$ that defines the analytic cyclic homology of the dagger algebra D . Let $l \in \mathbb{N}^*$ be fixed. In degree 0, the X -complex is $\mathcal{U}(\mathbb{T}D, JD^l)_{\text{lg}} \otimes F$, and in degree 1 it is the commutator quotient of $\Omega^1(\mathcal{U}(\mathbb{T}D, JD^l)_{\text{lg}} \otimes F)$. Here by commutator quotient, we mean the quotient by the image of the map $\mathcal{U}(\mathbb{T}D, JD^l)_{\text{lg}} \otimes \Omega^1(\mathcal{U}(\mathbb{T}D, JD^l)_{\text{lg}}) \rightarrow \Omega^1(\mathcal{U}(\mathbb{T}D, JD^l)_{\text{lg}})$, $x \otimes \omega \mapsto x \cdot \omega - \omega \cdot x$.

We first observe that submodules of the form $\bigoplus_{j=0}^{\infty} \pi^{-l \lfloor \frac{\cdot}{l} \rfloor} \Omega^{2j}(M)$, for bounded π -adically complete submodules of D , cofinally generate a bornology that is isomorphic to the linear growth bornology of the tube algebra $\mathcal{U}(\mathbb{T}D, JD^l)_{\text{lg}}$ (see [12, Corollary 4.1.4]). Now let $M \hookrightarrow N$ be a bornological embedding between bounded, π -adically complete submodules of D . Then by [12, Proposition 2.4.5], we have an inclusion $\overline{\Omega}^1(M) \subseteq \overline{\Omega}^1(N)$. So we have $\prod_{n=0}^{\infty} \pi^{-l \lfloor \frac{\cdot}{l} \rfloor} \overline{\Omega}^{2n}(M) \hookrightarrow \prod_{n=0}^{\infty} \pi^{-l \lfloor \frac{\cdot}{l} \rfloor} \overline{\Omega}^{2n}(N)$, which implies the separatedness of the inductive limit. Varying l , we can identify $\Omega^1(\mathcal{U}(\mathbb{T}D, JD^\infty)_{\text{lg}})/[\cdot, \cdot]$ with the pro-bimodule differential forms of odd degree. So we can repeat the same argument as above to deduce that the quasi-completion of this pro-module is isomorphic to $\overline{\Omega}^1(\mathcal{U}(\mathbb{T}D, JD^\infty)^\dagger)/[\cdot, \cdot]$. \square

COROLLARY 2.1.3. *Let A be the coordinate ring of a smooth, affine variety X over \mathbb{F} of relative dimension 1. Suppose R is a smooth, commutative V -algebra with the fine bornology, satisfying $R/\pi R \cong A$. Then for $\ast = 0, 1$, the following homology groups are isomorphic at each degree:*

- (1) $\text{HA}_\ast(A, R^\dagger, R^\dagger \rightarrow A)$;
- (2) *the de Rham cohomology of $R^\dagger \otimes F$;*
- (3) *the Monsky-Washnitzer cohomology of X ;*
- (4) *if $\text{char}(\mathbb{F}) > 0$, then the rigid cohomology $H_{\text{rig}}^\ast(A, F)$ of X .*

PROOF. The homology groups in (2), (3) and (4) are all isomorphic to $\text{HA}_\ast(R^\dagger)$ by [12, Theorem 8.2.9]. \square

We would like to show that the homology theories above are isomorphic to $\text{HA}_\ast(A)$, which is our definition for the analytic cyclic homology for an \mathbb{F} -algebra. This is true by Proposition 2.1.8, which we prove in the Section 4.

Pro-dagger algebra lift with pro-nilpotent kernel. We end this section with a generalisation of Proposition 2.1.1. This generalisation will be needed to prove excision in Section 6. Recall that a pro-algebra N is called *analytically nilpotent* if it is isomorphic to a projective system of dagger algebras, and is nilpotent mod π (see [12, Definition 4.3.1]).

PROPOSITION 2.1.4. *Let A be an \mathbb{F} -algebra with the fine bornology, viewed as a constant projective system. Let $I_n \twoheadrightarrow D_n \xrightarrow{\varrho_n} A$ be an extension of pro-algebras, where*

$D = (D_n)_{n \in \mathbb{N}}$ is a pro-dagger algebra and $I = (I_n)_{n \in \mathbb{N}}$ is analytically nilpotent. Denote the surjections $(\varrho_n: D_n \rightarrow A)_{n \in \mathbb{N}}$ by $\varrho: D \rightarrow A$. Then $\mathbb{H}\mathbb{A}(A, D, \varrho)$ is quasi-isomorphic to $\mathbb{H}\mathbb{A}(D)$.

PROOF. Let $J = \ker(\mathbb{T}D \twoheadrightarrow A)$ be the pro-algebra $J_n = \ker(\mathbb{T}D_n \twoheadrightarrow A)$. By hypothesis, for each n , there are $m \geq n$ and $k \geq 1$ such that $I_m^k \subseteq \pi \cdot I_n$. So the image of J_m^k is contained in $\mathbb{J}D_n \oplus \pi I_n$. By a computation as in the proof of Proposition 2.1.1, it follows that for each n , there are $m \geq n$ and $l = jk \geq j$, such that the image of $\mathcal{U}(\mathbb{T}D_m, J_m^l)$ is contained in $\mathcal{U}(\mathbb{T}D_n, \mathbb{J}D_n^j)$. And the image of $\mathcal{U}(\mathbb{T}D_n, \mathbb{J}D_n^j)$ is contained in $\mathcal{U}(\mathbb{T}D_n, J_n^j)$ for each $n \in \mathbb{N}$ and $j \in \mathbb{N}^*$. Therefore, the pro-algebras $\mathcal{U}(\mathbb{T}D, I^\infty)$ and $\mathcal{U}(\mathbb{T}D, \mathbb{J}D^\infty)$ are isomorphic as projective systems. Applying the X -complex gives an isomorphism of projective systems of chain complexes $X(\mathcal{U}(\mathbb{T}D, I^\infty)) \rightarrow X(\mathcal{U}(\mathbb{T}D, \mathbb{J}D^\infty))$. Now the functoriality of homotopy inverse limits, quasi-completions and Proposition 2.1.1 imply that $\mathbb{H}\mathbb{A}_*(D) \cong \mathbb{H}\mathbb{A}_*(A, D, \varrho)$ for $* = 0, 1$. \square

REMARK 2.1.5. Let A be an \mathbb{F} -algebra and let $N \twoheadrightarrow D \twoheadrightarrow A$ be an extension by a pro-algebra D and a nilpotent mod π kernel N . Then D is automatically isomorphic to a projective system of semi-dagger algebras. This is because by [12, Remark 4.22], the reduction mod $N/\pi N$ is nilpotent, and hence isomorphic to a projective system of semi-dagger algebras. Since πN is also a projective system of semi-dagger algebras, so is N by [12, Proposition 4.3.13]. Of course, since $\pi \cdot A = 0$, A is also semi-dagger. Viewing A as a constant pro-algebra, we can again use [12, Proposition 4.3.13] to conclude that D is isomorphic to a projective system of semi-dagger algebras.

2.1.1. *The free V -algebra lifting of an \mathbb{F} -algebra.* The approach in [11] uses the free commutative V -algebra lifting $V[A] \twoheadrightarrow A$ of a commutative \mathbb{F} -algebra A , and associates to it a functorial chain complex. If the algebra A is finitely generated, their construction yields rigid cohomology. In our setting, the algebra A is non-commutative. So we must replace the free commutative algebra by the free V -algebra $V\langle A \rangle$, defined as follows:

DEFINITION 2.1.6. Let S be a set. Let $\langle S \rangle$ denote the free semigroup generated by S ; this is the set of all non-empty words with letters in S , with concatenation of words as multiplication. Let $V\langle S \rangle$ be the free V -module generated by $\langle S \rangle$, equipped with the multiplication induced by the product in $\langle S \rangle$. This is the *free algebra generated by S* . Namely, it has the following universal property: for any V -algebra B , there is a natural bijection between algebra homomorphisms $V\langle S \rangle \rightarrow B$ and maps $S \rightarrow B$. We write $f^\#: V\langle S \rangle \rightarrow B$ for the algebra homomorphism induced by a map $f: S \rightarrow B$.

If S is a subset of A , then the inclusion map $i: S \rightarrow A$ induces a V -algebra homomorphism $i^\#: V\langle S \rangle \rightarrow A$. It is surjective if and only if S generates A as an \mathbb{F} -algebra. A functorial choice for the generating set is $S = A$.

LEMMA 2.1.7. *Let S be a set and let $V\langle S \rangle$ be the free V -module over S . Then $V\langle S \rangle \cong \mathbb{T}(V\langle S \rangle)$.*

PROOF. Let B be a V -algebra and let $S \rightarrow B$ be a map. Since B is in particular a V -module, the universal property of free modules, gives a unique V -linear map $V\langle S \rangle \rightarrow B$ extending $S \rightarrow B$. Since B is a V -algebra, the universal property of the

tensor algebra gives a unique extension $\mathbb{T}(VS) \rightarrow B$. The claim now follows from the Yoneda Lemma. \square

We equip the free V -module VA and the tensor algebra $R := V\langle A \rangle \cong \mathbb{T}(VA)$ with the fine bornology. Then VA and R are both complete as V -modules. Since VA is free, the algebra R is torsion-free and since it has the fine bornology, it is also bornologically torsion-free. So we obtain a bornologically torsion-free resolution

$$I \twoheadrightarrow R \xrightarrow{p} A,$$

which we feed into our machinery. We denote the resulting analytic chain complex by $\mathbb{H}\mathbb{A}(A) := \mathbb{H}\mathbb{A}(A, VA, VA \twoheadrightarrow A)$ and call the resulting homology $\mathrm{HA}_*(A)$ ‘the analytic cyclic homology of A ’. This name is justified in our main theorem, which we now formulate:

PROPOSITION 2.1.8. *Let A be an \mathbb{F} -algebra, let W be a complete, bornologically torsion-free bornological V -module, and let $\varrho: W \rightarrow A$ be a surjective V -module map that is bounded in the fine bornology on A . Assume that ϱ has a bounded section. Then there is a canonical bornological F -module isomorphism*

$$\mathrm{HA}_*(A) \cong \mathrm{HA}_*(A; W, \varrho).$$

Let A , W and $\varrho: W \rightarrow A$ be as in Proposition 2.1.8. Let $I := \ker \varrho^\#$. The homology $\mathrm{HA}_*(A; W, \varrho)$ is based on the projective system of bornological V -algebras $\mathcal{U}(\mathbb{T}W, I^l)$ for $l \in \mathbb{N}_{\geq 1}$. The homology $\mathrm{HA}_*(A)$ is based on the projective system of bornological V -algebras $\mathcal{U}(\mathbb{T}(VA), J^l)$ for $l \in \mathbb{N}_{\geq 1}$, where $J := \ker(p: \mathbb{T}(VA) \twoheadrightarrow A)$. We first build a natural map

$$\mathrm{HA}_*(A) \rightarrow \mathrm{HA}_*(A; W, \varrho).$$

The starting point is a map $s: A \rightarrow W$ with $\varrho \circ s = \mathrm{id}_A$. This induces a V -linear map $VA \rightarrow W$, which we compose with the canonical linear map $\sigma_W: W \rightarrow \mathbb{T}W$. The composite induces a homomorphism s_* from $V\langle A \rangle = \mathbb{T}(VA) \rightarrow \mathbb{T}W$. It satisfies $\varrho^\# \circ s_* = p$ and hence maps J to I . Then it extends uniquely to compatible homomorphisms $s_*^{(l)}: \mathcal{U}(\mathbb{T}(VA), J^l) \rightarrow \mathcal{U}(\mathbb{T}W, I^l)$ for all $l \in \mathbb{N}$. The bornology on $\mathcal{U}(\mathbb{T}(VA), J^l)$ is the fine one, so that these homomorphisms are bounded. We get $\mathrm{HA}_*(A)$ and $\mathrm{HA}_*(A; W, \varrho)$ by applying the steps 3–8 in the definition of analytic cyclic homology to the projective systems of bornological algebras $\mathcal{U}(\mathbb{T}(VA), J^l)$ and $\mathcal{U}(\mathbb{T}W, I^l)$. Since each of these steps is functorial, the family of homomorphisms $(s_*^{(l)})_{l \in \mathbb{N}_{\geq 1}}$ induces a map $\mathrm{HA}_*(A) \rightarrow \mathrm{HA}_*(A; W, \varrho)$. To show that this map is invertible, we first need to develop some machinery that is motivated by local cyclic homology [26, Section 2.3]. Using this machinery, we will construct ‘local’ algebra homomorphisms in the opposite direction that are inverse to $s_*^{(l)}$ in a suitable sense. We will ultimately show that the resulting chain complexes $\mathbb{H}\mathbb{A}(A)$ and $\mathbb{H}\mathbb{A}(A, W, \varrho)$ are ‘locally’ chain homotopy equivalent.

3. The exact category of locally split extensions

In this section, we define exact category structures on the categories of bornological V -modules and projective systems of bornological V -modules (over countable directed sets). Let \mathcal{C} be an additive category \mathcal{C} with kernels and cokernels. A diagram of the form

$$(3.0.1) \quad K \xrightarrow{i} E \xrightarrow{p} Q$$

is called an *extension* if $i = \ker(p)$ and $p = \operatorname{coker}(i)$. An *exact category* is an additive category \mathcal{C} with a distinguished class of extensions \mathcal{E} called *conflations*, satisfying certain properties. An arrow in \mathcal{C} is called an *inflation* (respectively, *deflation*) if it is the arrow i (respectively, p) in a conflation. The conflations must satisfy the following axioms:

- the identity map on the zero object is a deflation;
- if $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are deflations, so is $A \xrightarrow{g \circ f} C$;
- the pullback of a deflation along an arbitrary map exists and is again a deflation;
- the pushout of an inflation along an arbitrary map exists and is again an inflation.

3.1. Locally split extensions of bornological modules. We now define the relevant exact category structures on the categories \mathfrak{B} and $\overleftarrow{\mathfrak{B}}$ of torsion-free bornological V -modules and projective systems of torsion-free bornological V -modules, respectively. Briefly, we require extensions of bornological V -modules that split *locally*, that is, on each bounded V -submodule of the quotient.

DEFINITION 3.1.1. An extension as in (3.0.1) of torsion-free bornological V -modules is called *locally split* if for any bounded V -submodule $M \subseteq Q$, there is a bounded V -module map $s: M \rightarrow E$ such that $p \circ s: M \rightarrow Q$ is the inclusion map. We call a bounded V -module map $X \xrightarrow{f} Q$ *locally liftable* into the extension (3.0.1) if for any bounded V -submodule $M \subseteq X$, there is a bounded V -module map $\hat{f}: M \rightarrow E$ such that $p \circ \hat{f} = f \circ \iota_M$, where ι_M is the inclusion $M \subseteq X$;

$$\begin{array}{ccc} & M \subseteq X & \\ & \swarrow \hat{f} & \downarrow f \\ E & \xrightarrow{p} & Q. \end{array}$$

REMARK 3.1.2. An extension is locally split if and only if id_Q is locally liftable, if and only if any bounded V -module map $X \rightarrow Q$ is locally liftable.

In what follows, and throughout this article, we shall only consider projective systems indexed by countable directed sets N . We can actually assume $N = \mathbb{N}$. This is because restriction to cofinal subsets yields isomorphic projective systems.

DEFINITION 3.1.3. An extension of projective systems of torsion-free bornological V -modules is called *locally split* if it is isomorphic to a projective system of extensions $(K_n \twoheadrightarrow E_n \twoheadrightarrow Q_n)_{n \in \mathbb{N}}$, where for each $n \in \mathbb{N}$, there exists $m \geq n$ such that the canonical map $Q_m \rightarrow Q_n$ is locally liftable with respect to the extension $K_n \twoheadrightarrow E_n \twoheadrightarrow Q_n$.

A few remarks are in order here. The proof of [12, Proposition 4.3.13] shows that an extension $K \twoheadrightarrow E \twoheadrightarrow Q$ of projective systems of bornological V -modules is isomorphic to a projective system of extensions of bornological V -modules $(K'_n \twoheadrightarrow E'_n \twoheadrightarrow Q'_n)_{n \in \mathbb{N}}$, such that $(K'_n)_n$, $(E'_n)_n$ and $(Q'_n)_n$ are isomorphic as projective systems to K , E and Q , respectively. We do not assume that the local sections are compatible with the structure maps of E and Q . Finally, it is easy to see that the definition of locally split extensions is independent of the choice of representation as a projective system of extensions.

Denote by \mathfrak{E} and $\overleftarrow{\mathfrak{E}}$ the classes of locally split extensions in the categories \mathfrak{B} and $\overleftarrow{\mathfrak{B}}$, respectively.

REMARK 3.1.4. The class of locally split extensions defined above is inspired by the *locally split extensions* defined in [26, Section 2.3.6] for categories of projective and inductive systems over an additive category.

LEMMA 3.1.5. *The pair $(\mathfrak{B}, \mathfrak{E})$ is an exact category.*

PROOF. Clearly, the zero map on the zero module is a deflation. To see that the composition of two deflations $p: A \rightarrow B$ and $q: B \rightarrow C$ is a deflation, let M be a bounded V -submodule of C . Then there exists a bounded V -module map $s_M: M \rightarrow B$ such that $q \circ s_M$ is the inclusion of M into C . Let N_M be a bounded submodule of B containing the image of s_M . Then there is a bounded V -module map $t_M: N_M \rightarrow A$ such that $p \circ t_M$ is the inclusion of N_M into B . So $t_M \circ s_M$ is the required local section for the map $A \rightarrow C$. It remains to verify the pushout and pullback axioms. By [12, Lemma 2.3.1], the pullback of a bornological quotient map $A \xrightarrow{p} B$ along an arbitrary bounded V -module map $C \xrightarrow{f} B$ exists, and is a bornological quotient map $A \times_{p,f} C \xrightarrow{p'} C$, where $A \times_{p,f} C = \{(x, y) \in A \oplus C : p(x) = f(y)\}$ and p' is the canonical projection onto C . This has an obvious local section: let $M \subseteq C$ be a bounded V -submodule. Then there exists a bounded V -module map $s_{f(M)}: f(M) \rightarrow A$ that locally splits p . So $(s_{f(M)} \circ f, \iota_M): M \rightarrow A \oplus E$, $m \mapsto (s_{f(M)}f(m), m)$, is the required local section of p' .

Finally, let $i: A \rightarrow B$ be an inflation, and $f: A \rightarrow A'$ an arbitrary bounded V -module map. Then by Lemma [12, Lemma 2.3.1], the pushout exists, and is given by $B' := \frac{A' \oplus B}{\{(f(a), -i(a)) : a \in A\}}$ with the canonical maps $i': A' \rightarrow B'$, $i'(a') = [(a', 0)]$ and $\hat{f}: B \rightarrow B'$, $\hat{f}(b) = [(0, b)]$. Furthermore, i' is a bornological embedding. It remains to show that the cokernel $B' \xrightarrow{q'} \text{coker}(i')$ is locally split. Since i is an inflation, we know that for every bounded V -submodule M of $\text{coker}(i)$, there is a local section $s_M: M \rightarrow B$ for $B \rightarrow \text{coker}(i)$. Then $\hat{f} \circ s$ is a local splitting of q' . \square

PROPOSITION 3.1.6. *The pair $(\overleftarrow{\mathfrak{B}}, \overleftarrow{\mathfrak{E}})$ is an exact category.*

PROOF. Let $p: A \rightarrow B$ and $q: B \rightarrow C$ be two deflations. By definition, we can represent p and q as diagrams of locally split cokernels $(p'_n: A'_n \rightarrow B'_n)_{n \in \mathbb{N}}$ and $(q''_n: B''_n \rightarrow C''_n)_{n \in \mathbb{N}}$, where $A \cong (A'_n)_{n \in \mathbb{N}}$, $(B'_n)_{n \in \mathbb{N}} \cong B \cong (B''_n)_{n \in \mathbb{N}}$ and $C \cong (C''_n)_{n \in \mathbb{N}}$. Taking the pullback of the projective system of cokernels $(p'_n: A'_n \rightarrow B'_n)_{n \in \mathbb{N}}$ along the isomorphism $(B''_n)_{n \in \mathbb{N}} \rightarrow (B'_n)_{n \in \mathbb{N}}$, we obtain a diagram of cokernels $p'': (A''_n)_{n \in \mathbb{N}} \rightarrow (B''_n)_{n \in \mathbb{N}}$ such that $(A''_n)_{n \in \mathbb{N}} \cong (A'_n)_{n \in \mathbb{N}} \cong A$. It is easy to see that p'' is locally split, so that it represents the deflation p . So for each $n \in \mathbb{N}$, there exists $k \geq n$, such that the structure map $B''_k \rightarrow B''_n$ is locally liftable with respect to $p''_n: A''_n \rightarrow B''_n$. Similarly, there exists $l \geq k$ such that the structure map $C''_l \rightarrow C''_k$ is locally liftable with respect to $q''_k: B''_k \rightarrow C''_k$. Composition of these local liftings $C''_l \rightarrow B''_k \rightarrow A''_n$ yields the required local lifting of the structure map $C''_l \rightarrow C''_n$ into the quotient $A''_n \rightarrow C''_n$.

To verify the other axioms, it is well known (see [30, Proposition 7.1.5]) that if the pullback (respectively, pushout) of a cokernel (respectively, kernel) along an arbitrary morphism exists and is a cokernel (respectively, a kernel), then the same holds in the projective category. We sketch the proof here as follows: let $q: B \rightarrow C$

be a cokernel of projective systems and let $f: X \rightarrow C$ be an arbitrary pro-linear map. Then q can be represented as a cokernel of diagrams $(q_n: B_n \twoheadrightarrow C_n)_{n \in \mathbb{N}}$. Modifying the terms B_n and C_n if necessary (up to an isomorphism of projective systems), we get a morphism of diagrams $f_n: X_n \rightarrow C_n$ into $B_n \twoheadrightarrow C_n$. Of course, the modified morphism of diagrams $B_n \twoheadrightarrow C_n$ still represents the deflation $q: B \twoheadrightarrow C$, so we continue to denote it by q_n . By Lemma 3.1.5, we can take the pullback of $q_n: B_n \twoheadrightarrow C_n$ along $f_n: X_n \rightarrow C_n$ and again obtain a cokernel $(B_n \times_{q,f} X_n) \twoheadrightarrow (X_n)$ of diagrams. Fix $n \in \mathbb{N}$. Since q is a deflation, there is an $m \geq n$ such that the structure map $C_m \twoheadrightarrow C_n$ lifts locally with respect to the cokernel $B_n \twoheadrightarrow C_n$. Precomposing with $f_m: X_m \rightarrow C_m$ yields the required local lifting $X_m \rightarrow B_n \times_{q,f} X_n$ of the structure map $X_m \rightarrow X_n$.

The pushout axiom can be dealt with similarly: let $i: A \twoheadrightarrow B$ be an inflation and $f: A \rightarrow A'$ a pro-linear map. By definition, we can represent the extension $i: A \twoheadrightarrow B \xrightarrow{q} \text{coker}(i)$ as a diagram of extensions $(A_n \xrightarrow{i_n} B_n \twoheadrightarrow \text{coker}(i_n))_{n \in \mathbb{N}}$ that satisfies: for each $n \in \mathbb{N}$, there is an $m \geq n$ such that the structure map $\text{coker}(i_m) \rightarrow \text{coker}(i_n)$ is locally liftable. By modifying A and B up to an isomorphism of projective systems, we can represent f as a morphism of diagrams $f_n: A_n \rightarrow A'_n$ out of the kernels $(i_n: A_n \twoheadrightarrow B_n)_{n \in \mathbb{N}}$. We continue to denote the kernel $A_n \twoheadrightarrow B_n$ by i_n as both these morphisms represent $i: A \twoheadrightarrow B$. By Lemma 3.1.5, we can take the pushout $A'_n \xrightarrow{i'_n} B'_n$, whose cokernel isomorphic as a projective system to $\text{coker}(i)$. \square

3.2. Definition of the derived category. We define the derived category of the exact category $(\overleftarrow{\mathfrak{B}}, \overleftarrow{\mathfrak{E}})$. Throughout this section, we will refer to \mathbb{Z}_2 -graded chain complexes, also known as *supercomplexes*.

LEMMA 3.2.1. *Let R be a ring that is finitely generated as a \mathbb{Z} -module. Let (X, σ_X) and (Y, σ_Y) be projective systems of R -modules, and $f: X \rightarrow Y$ a morphism of projective system of \mathbb{Z} -modules that is R -linear in the sense that*

$$\begin{array}{ccc} R \otimes_{\mathbb{Z}} X & \xrightarrow{m_X} & X \\ \text{id} \otimes f \downarrow & & \downarrow f \\ R \otimes_{\mathbb{Z}} Y & \xrightarrow{m_Y} & Y \end{array}$$

commutes, where m_X and m_Y are the multiplication maps of X and Y , respectively. Then f can be represented as a morphism of projective system of R -modules.

PROOF. We can represent f by \mathbb{Z} -linear maps $(f_n: X_{m(n)} \rightarrow Y_n)_{n \in \mathbb{N}}$. The condition of R -linearity says that for each generator $r \in R$, there are indices $k_r(n) \geq l_r(n) \geq m(n)$ to have $f_n(\sigma^X(r \cdot x)) = r \cdot \sigma^Y(f_{l_r}(x))$ for $x \in X_{k_r(n)}$. Now since R is finitely generated, we can arrange that this equality holds simultaneously on all the generators of R , by taking the maximum of all such indices k_r . So $f: X \rightarrow Y$ is a morphism of projective systems of R -modules. \square

THEOREM 3.2.2. *Let \mathcal{C} be an additive category with cokernels. Then any supercomplex in $\overleftarrow{\mathcal{C}}$ is a projective limit of a projective system of supercomplexes in \mathcal{C} and, this yields an equivalence of categories*

$$\overleftarrow{\text{Kom}}(\mathcal{C}) \cong \text{Kom}(\overleftarrow{\mathcal{C}}).$$

PROOF. Let R be the ring with the presentation $\{g, d: g^2 = 1, gd + dc = 0, d^2 = 0\}$. Then a \mathbb{Z}_2 -graded chain complex over \mathcal{C} is equivalent to an object $X \in \mathcal{C}$, together with a ring homomorphism $R \rightarrow \text{End}(X)$. Since \mathcal{C} is additive and has cokernels, so does $\overleftarrow{\mathcal{C}}$. Therefore any chain complex in $\text{Kom}(\overleftarrow{\mathcal{C}})$ is a projective system $X = (X_n)_{n \in \mathbb{N}}$ in $\overleftarrow{\mathcal{C}}$, together with a ring homomorphism $f: R \rightarrow \text{End}(X)$. Viewing R as an R -module in an obvious way, we obtain a projective system $R \otimes_{\mathbb{Z}} X$ of R -modules. Since R is finitely generated and free as an abelian group and, since $\overleftarrow{\mathcal{C}}$ is an additive category, $R \otimes_{\mathbb{Z}} X$ is well-defined. It is concretely given by a direct sum of finitely many copies of X . The map f induces a morphism $R \otimes_{\mathbb{Z}} X \rightarrow X$ of projective systems in $\overleftarrow{\mathcal{C}}$. Tensoring on the left with R , we obtain a morphism $R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} X \rightarrow R \otimes_{\mathbb{Z}} X$ of projective systems in $\overleftarrow{\mathcal{C}}$, which is R -linear. Since R is finitely generated, Lemma 3.2.1 implies that we can represent this map as a projective system of R -module maps $R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} X \rightarrow R \otimes_{\mathbb{Z}} X$. These can be further represented as a diagram of R -modules $(R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} X_n \rightarrow R \otimes_{\mathbb{Z}} X_n)_{n \in M}$, after suitably reindexing by some directed set M , with $(X_n)_{n \in M} \cong X$ in $\overleftarrow{\mathcal{C}}$. For each n , the cokernel of $R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} X_n \rightarrow R \otimes_{\mathbb{Z}} X_n$ is X_n , so that each X_n is an R -module. Therefore, X is a projective system of R -modules. By naturality of the bar resolution, this assignment is indeed a functor $\text{Kom}(\overleftarrow{\mathcal{C}}) \rightarrow \overleftarrow{\text{Kom}}(\mathcal{C})$, which is inverse to the functor $\overleftarrow{\text{Kom}}(\mathcal{C}) \rightarrow \text{Kom}(\overleftarrow{\mathcal{C}})$ that forgets the R -action on a diagram in \mathcal{C} . \square

DEFINITION 3.2.3. A chain complex (C, δ) in an exact category is called *exact* if $\ker(\delta)$ exists and

$$\ker(\delta) \rightarrow C \rightarrow \ker(\delta)$$

is a conflation, where $\ker(\delta) \rightarrow C$ is the identical inclusion and $C \rightarrow \ker(\delta)$ is the map δ at each level of the chain complex. A chain map $f: C \rightarrow D$ is called a *quasi-isomorphism* if its mapping cone is exact.

Consider the exact category $(\overleftarrow{\mathfrak{B}}, \overleftarrow{\mathfrak{E}})$ of projective systems of torsion-free bornological V -modules with locally split extensions as conflations as in Definition 3.1.3. Then $\text{Kom}(\overleftarrow{\mathfrak{B}})$ is an exact category, whose conflations are given degreewise. We use the equivalence of categories $\overleftarrow{\text{Kom}}(\overleftarrow{\mathfrak{B}}) \cong \text{Kom}(\overleftarrow{\mathfrak{B}})$ to describe conflations in the category $\overleftarrow{\text{Kom}}(\overleftarrow{\mathfrak{B}})$, which is easier to work with. The induced exact category structure on $\overleftarrow{\text{Kom}}(\overleftarrow{\mathfrak{B}})$ is the set of all kernel-cokernel pairs that are mapped by the forgetful functor to conflations in $\text{Kom}(\overleftarrow{\mathfrak{B}})$. By [8, Section 5], this is indeed an exact structure on $\overleftarrow{\text{Kom}}(\overleftarrow{\mathfrak{B}})$. We describe these conflations more explicitly.

Let $K \rightarrow E \rightarrow Q$ be a locally split extension of chain complexes in each degree. This means that at each chain complex degree, this extension is isomorphic to a projective system of extensions of bornological V -modules

$$(K_n \rightarrow E_n \rightarrow Q_n)_{n \in \mathbb{N}}$$

that is locally split. Since the forgetful functor $\overleftarrow{\text{Kom}}(\overleftarrow{\mathfrak{B}}) \rightarrow \text{Kom}(\overleftarrow{\mathfrak{B}})$ is part of an equivalence of categories, it is essentially surjective and fully faithful. So the extension above lifts to the extension of projective systems of chain complexes

$$(K', d^{K'}) \rightarrow (E', d^{E'}) \rightarrow (Q', d^{Q'})$$

that is isomorphic to the original extension and satisfies the following: for each $n \in \mathbb{N}$, there exists $m \geq n$ such that the canonical chain map $Q'_m \rightarrow Q'_n$ is locally liftable with respect to the extension $K'_n \twoheadrightarrow E'_n \twoheadrightarrow Q'_n$ taken degreewise.

In what follows, we also refer to conflations in $\overleftarrow{\text{Kom}}(\mathfrak{B})$ as locally split extensions and not explicitly distinguish between conflations in $\overleftarrow{\text{Kom}}(\mathfrak{B})$ and $\text{Kom}(\overleftarrow{\mathfrak{B}})$.

DEFINITION 3.2.4. The *derived category* of the exact category $\overleftarrow{\mathfrak{B}}$ is the localisation of the homotopy category of chain complexes $\text{HoKom}(\overleftarrow{\mathfrak{B}})$ at the quasi-isomorphisms.

3.3. Locally contractible chain complexes and local chain homotopy equivalences. In this section, we describe quasi-isomorphisms and exact chain complexes as local chain homotopy equivalences and locally contractible complexes. This description will be useful in the next section, when we compare the analytic cyclic homology of an \mathbb{F} -algebra A and the analytic cyclic homology of a torsion-free lifting. We will make use of the local structure of complete, torsion-free bornological V -modules to construct an explicit local chain homotopy equivalence between them.

Let $C = (C_n, d_n^C)$ be a chain complex of bornological V -modules. For each n , we write the bornological V -module C_n as an inductive limit of its bounded submodules as follows $C_n = \varinjlim_{i \in I_n} C_n^i$.

DEFINITION 3.3.1. (1) A *local chain map* $f: (C, d^C) \rightarrow (D, d^D)$ between chain complexes in $\text{Kom}(\mathfrak{B})$ is a collection of V -linear maps $(f_n^i: C_n^i \rightarrow D_n)_{(n \in \mathbb{Z}_2, i \in I_n)}$ satisfying $d_n^D \circ f_n^i(x) = f_{n-1}^j \circ d_n^C(x)$ for all $x \in C_n^i$, and some $j \in I_{n-1}$ so that $d_n^C(C_n^i) \subseteq C_{n-1}^j$.

EXAMPLE 3.3.2. The collection of canonical maps $(C_n^i \hookrightarrow C_n)_{(n \in \mathbb{Z}, i \in I_n)}$ is a local chain map $C \rightarrow C$.

(2) Two local chain maps $f, g: (C, d^C) \rightarrow (D, d^D)$ in $\text{Kom}(\mathfrak{B})$ are said to be *locally chain homotopic* if there is a collection of bounded V -module maps

$$(h_n^{i_n}: C_n^{i_n} + d_n^C(C_n^{i_n}) \rightarrow D_{n+1})_{(n \in \mathbb{Z}_2, i_n \in I_n)}$$

that satisfies

$$f_n^k(x) - g_n^k(x) = (h_{n-1}^{i_{n-1}} \circ d_n^C + d_{n+1}^D \circ h_n^{i_n})(x)$$

for all $x \in C_n^{i_n} + d_n^C(C_n^{i_n})$ and all $(n, i_n) \in (\mathbb{Z}_2, I_n)$. We call the collection above a local chain homotopy and denote it by $h: C \rightarrow D[1]$.

REMARK 3.3.3. Before moving on, we simplify our notation. Since we rarely need the chain complex degree n , we omit it altogether. A bounded submodule at each chain complex degree will simply be denoted as $M \subseteq C$. Accordingly, a local chain homotopy is a collection of bounded V -module maps $h: M + d^C(M) \rightarrow D$ that satisfies $f(x) - g(x) = (h \circ d^C + d^D \circ h)(x)$ for all $x \in M + d^C(M)$ and every bounded submodule M .

DEFINITION 3.3.4. A chain complex (C, d^C) in $\text{Kom}(\mathfrak{B})$ is *locally contractible* if for any bounded submodule $M \subseteq C$, there is a bounded V -module map $h^M: M + d^C(M) \rightarrow C$ such that $(h^M \circ d^C + d^D \circ h^M)(x) = x$ for all $x \in M + d^C(M)$. In other words, the identical inclusions of each bounded subcomplex of C is null homotopic.

We now define local contractibility and local chain homotopy equivalences for projective systems of chain complexes and chain maps in $\overleftarrow{\text{Kom}}(\mathfrak{B})$.

DEFINITION 3.3.5. A projective system of chain complexes $C = (C_n, \gamma_{m,n}: C_m \rightarrow C_n)_{n \in \mathbb{N}}$ in $\overleftarrow{\text{Kom}}(\mathfrak{B})$ is locally contractible if and only if for every n , there are an $m \geq n$ such that the structure map $\gamma_{m,n}: C_m \rightarrow C_n$ is locally null-homotopic.

DEFINITION 3.3.6. A projective system of chain maps $f: C \rightarrow D$ in $\overleftarrow{\text{Kom}}(\mathfrak{B})$ represented by a projective system of chain maps $(f_n: C_n \rightarrow D_n)_{n \in \mathbb{N}}$ is called a *local chain homotopy equivalence* if for each $n \in \mathbb{N}$, there are an $m \geq n$, and a local chain map and local chain homotopies

$$g_m: D_m \rightarrow C_n, \quad h: D_m \rightarrow D_n[1], \quad h: C_m \rightarrow C_n[1]$$

between $f_n g_m$ and $g_m f_m$, and the canonical structure maps $D_m \rightarrow D_n$ and $C_m \rightarrow C_n$, respectively.

LEMMA 3.3.7. A projective system $f: C \rightarrow D$ of chain maps in $\overleftarrow{\text{Kom}}(\mathfrak{B})$ is a local chain homotopy equivalence if and only if $\text{cone}(f)$ is locally contractible. A projective system of chain complexes C is locally contractible if and only if the zero map $0 \rightarrow C$ is a local chain homotopy equivalence.

PROOF. Let $\text{cone}(f) = C[-1] \oplus D$ be a locally contractible complex in $\overleftarrow{\text{Kom}}(\mathfrak{B})$. Then for each n , there exists $m(n) \geq n$ such that the structure map

$$\text{cone}(f)_m = C[-1]_m \oplus D_m \xrightarrow{\gamma_{m,n}^{C[-1]} \oplus \gamma_{m,n}^D} C[-1]_n \oplus D_n = \text{cone}(f)_n$$

is locally null homotopic. So for every bounded subset $M \subseteq \text{cone}(f)_m$, there exists a bounded contracting homotopy $h: M + d^{\text{cone}(f)}(M) \rightarrow \text{cone}(f)_n$. Since $\text{cone}(f)_m$ has the direct sum bornology, M is contained in $T_C \oplus S_D$, with bounded subsets $T_C \subseteq C[-1]$ and $S_D \subseteq D$. More concretely, these bounded subsets are of the form $T_C = M^{C[-1]} + d^{C[-1]}(M^{C[-1]})$ and $S_D = M^D + f_m(M^{C[-1]}) + d^D(M^D)$, for bounded subsets $M^{C[-1]} \subseteq C[-1]_m$ and $M^D \subseteq D_m$. So the local homotopy $h: T_C \oplus S_D \rightarrow \text{cone}(f)_n$ is given by a block matrix of the form

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}: T_C \oplus S_D \rightarrow C[-1]_n \oplus D_n.$$

For two linear maps, denote $[f, g] := fg + gf$. Then the local null homotopy of the structure map $\text{cone}(f)_m \rightarrow \text{cone}(f)_n$ says that

$$(3.3.8) \quad \begin{pmatrix} \gamma_{m,n}^{C[-1]} & 0 \\ 0 & \gamma_{m,n}^D \end{pmatrix} = \begin{pmatrix} [d^{C[-1]}, h_{11}] + h_{12} \circ f & d^{C[-1]} \circ h_{12} - h_{12} d^D \\ h_{21} \circ d^{C[-1]} - d^D \circ h_{21} + h_{22} \circ f + f \circ h_{11} & f \circ h_{12} - [d^D, h_{22}] \end{pmatrix}$$

on the bounded submodule $T_C \oplus S_D$. Then $g_m = h_{12}: S_D \rightarrow D_n$ is the required local chain homotopy inverse of $f_n: C_n \rightarrow D_n$.

Conversely, if f is a local chain homotopy equivalence, then the local chain homotopy inverse and the local chain homotopies can be used to satisfy (3.3.8), which shows that $\text{cone}(f)$ is locally contractible. \square

THEOREM 3.3.9. *A chain complex C in $\overleftarrow{\text{Kom}}(\mathfrak{B}) \cong \text{Kom}(\overleftarrow{\mathfrak{B}})$ is locally contractible if and only if it is locally split exact. A chain map $f: C \rightarrow D$ in $\overleftarrow{\text{Kom}}(\mathfrak{B})$ is a local chain homotopy equivalence if and only if it is a quasi-isomorphism.*

PROOF. Let $C = (C_n, d_n)_{n \in \mathbb{N}}$ be a locally contractible projective system of chain complexes. Then for each $n \in \mathbb{N}$, there exists $m \geq n$ such that the structure map $C_m \xrightarrow{\gamma_{m,n}} C_n$ is locally null-homotopic. So the restriction of $\gamma_{m,n}$ to $\ker(d_m) \subseteq C_m$ is locally null-homotopic. So for any bounded submodule $X \subseteq \ker(d_m)$, there exists a bounded V -module map $h: X \rightarrow C_n$ witnessing the null-homotopy, that is, $d_n \circ h = \gamma_{n,m} \circ \iota_X$. This means precisely that the canonical map $\ker(d_m) \rightarrow \ker(d_n)$ is locally liftable with respect to the map $C_n \rightarrow \ker(d_n)$. Finally, since C has zero homology as a projective system, the map $d: C \rightarrow \ker(d)$ is surjective as a morphism of projective systems. Therefore, we can arrange that $\ker(d) \rightarrow C \rightarrow \ker(d)$ is isomorphic to a projective system of extensions in $\overleftarrow{\mathfrak{B}}$ that is locally split exact. So C is locally split exact.

Conversely, suppose C is locally split exact. Then $(\text{im}(d_n))_{n \in \mathbb{N}} \cong (\ker(d_n))_{n \in \mathbb{N}}$ as projective systems. And local split exactness implies that for each n , there is an $m \geq n$, such that for every bounded submodule $X \subseteq \ker(d_m)$, there is a bounded V -module map $h_1: X \rightarrow C_m$ satisfying $d_n \circ h_1 = \gamma_{n,m} \circ \iota_X$. Similarly, for this m , there is an $l \geq m$ and a bounded V -module map $h_2: d_l(Y) \rightarrow C_m$ satisfying $d_m \circ h_2 = \gamma_{m,l} \circ \iota_{d_l(Y)}$ for any bounded submodule $Y \subseteq C_l$. Define $\varphi_Y := \gamma_{l,m} \circ \iota_Y - h_2 \circ d_l: Y \rightarrow C_m$. Then φ_Y is a bounded V -module map for the subspace bornology on $Y \subseteq C_l$, whose image is contained in $\ker(d_m)$. By hypothesis, h_1 exists on this bounded submodule to yield a local contracting homotopy for the structure map $\gamma_{n,l}: C_l \rightarrow C_n$.

The claim for local homotopy equivalences follows from Lemma 3.3.7 and the definition of quasi-isomorphisms. \square

3.4. Homotopy inverse limit. Let $C = (C_n, \gamma_{m,n})_{n \in \mathbb{N}} \in \overleftarrow{\text{Kom}}(\mathfrak{B})$ be a projective system of chain complexes of torsion-free bornological V -modules. Then for each $n \in \mathbb{N}$, there is a chain map

$$\prod_{k=0}^n C_k \xrightarrow{\text{id-Shift}_n} \prod_{k=0}^{n-1} C_k,$$

given explicitly as

$$(\text{id} - \text{Shift}_n)(c_0, \dots, c_n) \mapsto (c_0 - \gamma_{0,1}(c_1), c_1 - \gamma_{1,2}(c_2), \dots, c_{n-1} - \gamma_{n-1,n}(c_n))$$

for $c_k \in C_k$, $k = 0, \dots, n$. Furthermore, we have a canonical inclusion $C_n \xrightarrow{i_n} \prod_{k=0}^n C_k$, $i_n(x_n) = (\gamma_{0,n}(x_n), \dots, \gamma_{n-1,n}(x_n))$ whose composition with $\text{id} - \text{Shift}_n$ vanishes.

LEMMA 3.4.1. *Let $C \in \overleftarrow{\text{Kom}}(\mathfrak{B})$ be a projective system of chain complexes indexed by \mathbb{N} . Then we have a split exact sequence*

$$C_n \xrightarrow{i_n} \prod_{k=0}^n C_k \xrightarrow{\text{id-Shift}_n} \prod_{k=0}^{n-1} C_k.$$

PROOF. The splitting of i is given by the obvious projection

$$\prod_{k=0}^n C_k \rightarrow C_n \quad (x_0, \dots, x_n) \mapsto x_n.$$

Now use the splitting lemma. \square

Using the lemma above, we deduce that for each $n \in \mathbb{N}$, there is a chain homotopy equivalence

$$C_n \rightarrow \text{cone}(1 - \text{Shift}_n).$$

Denote by RC the projective system of chain complexes $(\text{cone}(1 - \text{Shift}_n))_{n \in \mathbb{N}}$. We then have a local chain homotopy equivalence between projective systems of chain complexes

$$\alpha: C \rightarrow RC.$$

REMARK 3.4.2. We note that the assignment $C \mapsto RC$ is only functorial for diagrams $\mathbb{N}^{\text{op}} \rightarrow \text{Kom}(\mathfrak{B})$. To show that it is functorial for projective systems of chain maps, we must pass to the derived category $\text{Der}(\overleftarrow{\mathfrak{B}})$. Let $f: C \rightarrow D$ be a projective system of chain maps. Since $C \rightarrow RC$ and $D \rightarrow RD$ are both isomorphisms in the derived category $\text{Der}(\overleftarrow{\mathfrak{B}})$, the chain map f lifts to a morphism $R(f): RC \rightarrow RD$ in $\text{Der}(\overleftarrow{\mathfrak{B}})$, which is unique up to chain homotopy. So defining the action of R on morphisms this way turns it into a functor $\overleftarrow{\text{Kom}}(\mathfrak{B}) \rightarrow \text{Der}(\overleftarrow{\mathfrak{B}})$. The functoriality of homotopy inverse limits on a quasi-abelian category of pro-objects over arbitrary indexing categories is treated in detail in [30, Definition 7.3.5].

DEFINITION 3.4.3. A functor \mathcal{F} from $\overleftarrow{\text{Kom}}(\mathfrak{B})$ or $\text{HoKom}(\overleftarrow{\mathfrak{B}})$ to another category is called *local* if it maps local chain homotopy equivalences to isomorphisms.

To see how such functors can arise, let $F: \overleftarrow{\mathfrak{B}} \rightarrow \mathfrak{X}$ be an additive, exact functor for the exact category structure on $\overleftarrow{\mathfrak{B}}$ given by locally split extensions. Here \mathfrak{X} is any exact category. Then F induces a functor $F_*: \overleftarrow{\text{Kom}}(\mathfrak{B}) \rightarrow \text{Kom}(\mathfrak{X})$ that descends to a functor $\text{HoKom}(\overleftarrow{\mathfrak{B}}) \rightarrow \text{Der}(\mathfrak{X})$. By Theorem 3.3.9, the functor F is exact if and only if the functor $F_*: \text{HoKom}(\overleftarrow{\mathfrak{B}}) \rightarrow \text{Der}(\mathfrak{X})$ is local.

A local functor extends to a functor on the derived category $\text{Der}(\overleftarrow{\mathfrak{B}})$, by the universal property of localisations. We show that the functor $\overleftarrow{\lim} \circ R: \overleftarrow{\text{Kom}}(\mathfrak{B}) \rightarrow \text{Der}(\mathfrak{B})$ is a local functor.

PROPOSITION 3.4.4. *The functor $\overleftarrow{\lim} \circ R: \overleftarrow{\text{Kom}}(\mathfrak{B}) \rightarrow \text{Der}(\mathfrak{B})$ maps local chain homotopy equivalences to isomorphisms. So it descends to a functor*

$$\text{holim} := \overleftarrow{\lim} \circ R: \text{Der}(\overleftarrow{\mathfrak{B}}) \rightarrow \text{Der}(\mathfrak{B}),$$

which is called the homotopy inverse limit. The functor holim is the total right derived functor of the inverse limit functor.

PROOF. Let $f: C \rightarrow D$ be a local chain homotopy equivalence. By Lemma 3.3.7, the mapping cone $\text{cone}(f)$ is a locally contractible projective system of chain complexes. We claim that $\overleftarrow{\lim} \circ R(f)$ is a local chain homotopy equivalence. There is a representative of f of the form $(f_n: C_n \rightarrow D_n)_{n \in \mathbb{N}}$. By naturality of the mapping cone, we get $\text{cone}(\overleftarrow{\lim} \circ R(f_n)) \cong \overleftarrow{\lim} \circ R(\text{cone}(f))$. Using Remark 3.4.2, we see that $\overleftarrow{\lim} \circ R(\text{cone}(f))$ is locally contractible, so that $\text{cone}(\overleftarrow{\lim} \circ R(f_n))$ is locally contractible. Hence $\overleftarrow{\lim} \circ R(f)$ is a local chain homotopy equivalence.

Finally, to see that $\overleftarrow{\lim} \circ R$ is the total right derived functor of $\overleftarrow{\lim}$, one checks that if F is any local functor on $\text{HoKom}(\overleftarrow{\mathfrak{B}})$, then any natural transformation

$F \Rightarrow \varprojlim$ factors uniquely through $\varprojlim \circ R$. This is a standard result in category theory (see [33, Proposition 6.4.11] for details in a more general context). \square

REMARK 3.4.5. We end this section with a remark on the notion of homology that we need for our purposes. We define the *homology of a chain complex* $C \in \overleftarrow{\text{Kom}}(\mathfrak{B})$ as the homology of the homotopy inverse limit of C . By Proposition 3.4.4, the homology functor

$$H_*: \overleftarrow{\text{Kom}}(\mathfrak{B}) \rightarrow \mathfrak{B}$$

is already local.

3.5. Interaction of quasi-isomorphisms with quasi-completion. We now discuss the exactness of the quasi-completion functor introduced in Step 7 of Section 2.0.5. To do this, we first describe the internal structure of torsion-free bornological V -modules.

PROPOSITION 3.5.1. *The category of torsion-free bornological V -modules \mathfrak{B} is equivalent to the full subcategory of the category of strict inductive systems of torsion-free V -modules $\overrightarrow{\mathfrak{B}}$.*

PROOF. By [11, Proposition 2.5], the result is true if we do not restrict to torsion-free bornological V -modules. We work out the same proof in the subcategory of torsion-free V -modules. In one direction, we have the *dissection functor*

$$\text{diss}: \mathfrak{B} \rightarrow \overrightarrow{\mathfrak{B}},$$

which maps a torsion-free bornological V -module to the strict inductive system of its bounded submodules. Of course, these submodules are in particular torsion-free. The functor in the other direction is the *inductive limit functor*, that assigns to an inductive system $(M_i)_{i \in I}$ of torsion-free V -modules, its inductive limit $M = \varinjlim M_i$ with the inductive limit bornology. This is a torsion-free bornological V -module. \square

The assignment $M \rightarrow \mathcal{Q}(M)$ is functorial: it is the composition of the dissection functor $M \mapsto (M_i)_{i \in I}$, the π -adic completion functor, and the inductive limit functor. Now suppose $A \twoheadrightarrow B \twoheadrightarrow C$ is a locally split extension of torsion-free bornological V -modules. Then the exactness of each of the component functors of \mathcal{Q} implies that we have an extension of torsion-free bornological V -modules

$$\mathcal{Q}(A) \twoheadrightarrow \mathcal{Q}(B) \twoheadrightarrow \mathcal{Q}(C).$$

Note that π -adic completion is indeed an exact functor since in particular, the quotient C is torsion-free. Denoting by $\mathfrak{B}_{\text{triv}}$ the exact structure whose conflations are all extensions in \mathfrak{B} , what we have shown is the following:

PROPOSITION 3.5.2. *The quasi-completion functor $\mathcal{Q}: \mathfrak{B} \rightarrow \mathfrak{B}_{\text{triv}}$ is an exact functor.*

The exactness of \mathcal{Q} implies the following easy consequence:

COROLLARY 3.5.3. *The functor \mathcal{Q} induces a local functor*

$$\mathcal{Q}: \text{HoKom}(\mathfrak{B}) \rightarrow \text{Der}(\mathfrak{B}_{\text{triv}}).$$

That is, if $f: C \rightarrow D$ is a local chain homotopy equivalence, then $\mathcal{Q}(f): \mathcal{Q}(C) \rightarrow \mathcal{Q}(D)$ is a quasi-isomorphism.

PROOF. Since \mathcal{Q} is an additive functor, it extends to the homotopy category of chain complexes over \mathfrak{B} , by termwise application. Now if $f: C \rightarrow D$ is a local chain homotopy equivalence, then its cone $\text{cone}(f)$ is locally split-exact by Theorem 3.3.9. By naturality of the mapping cone, we have $\text{cone}(\mathcal{Q}(f)) \cong \mathcal{Q}(\text{cone}(f))$. By Proposition 3.5.2, $\mathcal{Q}(\text{cone}(f))$ is exact, and hence $\text{cone}(\mathcal{Q}(f))$ is exact. So $\mathcal{Q}(f)$ is a quasi-isomorphism. \square

4. Independence of the choice of lifting

In this section, we prove Proposition 2.1.8, which is our main result.

4.1. Some non-Archimedean analysis. The proof of Proposition 2.1.8 requires a lot of preparation for which we first recall some results on the structure of non-Archimedean Banach spaces.

DEFINITION 4.1.1. Let D be a set. Let $C_0(D, V)$ be the set of all functions $f: D \rightarrow V$ such that for each $\delta > 0$ there is a finite subset $F \subseteq D$ with $|f(x)| < \delta$ for all $x \in D \setminus F$. Define $C_0(D, F)$ similarly. Equip both with the supremum norm.

THEOREM 4.1.2. [12, Theorem 2.4.2] *Let W be a complete, bornologically torsion-free bornological V -module. Any π -adically complete bounded V -submodule M of W is isomorphic to $C_0(D, V)$ for some set D .*

LEMMA 4.1.3. *Let W be a bornologically torsion-free, complete bornological V -module. Assume that the quotient bornology on $W/\pi W$ is the fine one. Then the quotient bornology on $W/\pi^m W$ is the fine one for all $m \in \mathbb{N}$.*

PROOF. The claim is proven by induction on m , the case $m = 0$ being trivial. Assume that the claim is true for m . Let $S \subseteq W$ be a bounded V -module. We must show that its image in $W/\pi^{m+1}W$ is finitely generated. By assumption, its image in $W/\pi W$ is finitely generated. So we may pick a finite set x_1, \dots, x_n with $S \subseteq \sum_{j=1}^n Vx_j + \pi W$. Let $S_1 := (S + \sum_{j=1}^n Vx_j) \cap \pi W$. Then $S_1 \subseteq \pi W$ is bounded and $S \subseteq \sum_{j=1}^n Vx_j + S_1$. Since W is bornologically torsion-free, $\pi^{-1}S_1$ is bounded as well. And $S \subseteq \sum_{j=1}^n Vx_j + \pi \cdot (\pi^{-1}S_1)$. The induction assumption applied to the image of $\pi^{-1}S_1$ in $W/\pi^m W$ gives finitely many elements x_{n+1}, \dots, x_m such that $\pi^{-1}S_1 \subseteq \sum_{j=n+1}^m Vx_j + \pi^m W$. Then $S \subseteq \sum_{j=1}^m Vx_j + \pi^{m+1}W$ as desired. \square

LEMMA 4.1.4. *Let W_1, W_2 be torsion-free, complete bornological V -modules. If the quotient bornologies on $W_j/\pi W_j$ for $j = 1, 2$ are fine, then the same is true for $W_1 \otimes W_2/\pi W_1 \otimes W_2$ and also for $\text{TW}_1/\pi \cdot \text{TW}_1$.*

PROOF. Any bounded subset of $W_1 \otimes W_2$ is contained in the image of $M_1 \otimes M_2$ for π -adically complete bounded V -submodules $M_j \subseteq W_j$ for $j = 1, 2$. By assumption, for $j = 1, 2$ there are finite subsets $S_j \subseteq W_j$ with $M_j = \sum_{x \in S_j} V \cdot x + \pi W_j$. Hence the image of $M_1 \otimes M_2$ in $W_1 \otimes W_2$ is contained in $\sum_{x \in S_1, y \in S_2} x \otimes y + \pi \cdot W_1 \otimes W_2$. Then the quotient bornology on $W_1 \otimes W_2/\pi W_1 \otimes W_2$ is the fine one. By induction, it follows that the quotient bornology on $W_1^{\otimes n}/\pi W_1^{\otimes n}$ is the fine one for all $n \in \mathbb{N}$. And this is inherited by the direct sum $\text{TW}_1 = \bigoplus W_1^{\otimes n}$ because any bounded subset of TW_1 is already contained in a finite subsum. \square

4.2. Proof of Proposition 2.1.8. We now return to the proof of Proposition 2.1.8. Recall that W is a complete, bornologically torsion-free V -module with a V -module surjection $\varrho: W \rightarrow A$, that we assume is bounded in the fine bornology on A . Let $s: A \rightarrow W$ be a set-theoretic section of ϱ . The map s induces a V -algebra homomorphism $s_*: V\langle A \rangle \rightarrow \mathbb{T}W$. The following lemma gives us the necessary control on bounded subsets of $\mathcal{U}(\mathbb{T}W, I^l)$:

LEMMA 4.2.1. *In the situation of Proposition 2.1.8, let N be a bounded V -submodule of $\mathcal{U}(\mathbb{T}W, I^l)$.*

- *There are $a \in \mathbb{N}$, a finite subset $S \subseteq I$, and a bounded subset $B \subseteq \mathbb{T}W$ such that N is contained in $\sum_{j=1}^a V\pi^{-j}S^{lj} + B$.*
- *There is a bounded subset $M \subseteq W$ such that N is the image of a bounded subset in $\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)$, and the natural bounded map from $\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)$ to $\mathcal{U}(\mathbb{T}W, I^l)$ is injective.*
- *There is a bounded homomorphism $f_M: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l) \rightarrow \mathcal{U}(V\langle A \rangle, J^l)$ with $p \circ f_M = \varrho^\#$.*
- *There is a bounded homomorphism*

$$H_M: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l) \rightarrow \mathcal{U}(\mathbb{T}W, I^l) \otimes V[t]$$

such that $\text{ev}_0 \circ H_M$ is the homomorphism $\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l) \hookrightarrow \mathcal{U}(\mathbb{T}W, I^l)$ induced by the inclusion map, $\text{ev}_1 \circ H_M = s_ \circ f_M$, and $(\varrho^\# \otimes \text{id}_{V[t]}) \circ H_M$ is the constant homotopy $\varrho^\#$.*

- *If $N_A \subseteq VA$ is a bounded V -submodule with $s_*(N_A) \subseteq M$, then s_* induces a bounded homomorphism $s_*: \mathcal{U}(\mathbb{T}N_A, (J \cap \mathbb{T}N_A)^l) \rightarrow \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)$. And there is a bounded homomorphism*

$$H_{VA}: \mathcal{U}(\mathbb{T}N_A, (J \cap \mathbb{T}N_A)^l) \rightarrow \mathcal{U}(V\langle A \rangle, J^l) \otimes V[t]$$

such that $\text{ev}_1 \circ H_{VA} = f_M \circ s_$, the map $\text{ev}_0 \circ H_{VA}$ is equal to the inclusion map $\mathcal{U}(\mathbb{T}N_A, (J \cap \mathbb{T}N_A)^l) \hookrightarrow \mathcal{U}(V\langle A \rangle, J^l)$, and $(p \otimes \text{id}_{V[t]}) \circ H_{VA}$ is the constant homotopy associated to $p: \mathcal{U}(\mathbb{T}N_A, (J \cap \mathbb{T}N_A)^l) \rightarrow A$.*

PROOF. By construction, $\mathcal{U}(\mathbb{T}W, I^l)$ carries the subspace bornology from $\mathbb{T}W \otimes F$. Hence N is bounded in $\mathbb{T}W \otimes F$. Then it is contained in $\pi^{-b}N'$ for a bounded subset $N' \subseteq \mathbb{T}W$. Lemmas 4.1.3 and 4.1.4 imply that the quotient bornology on $\mathbb{T}W/\pi^b\mathbb{T}W$ is fine. Thus there are $x_1, \dots, x_n \in N'$ that generate the image of N' in $\mathbb{T}W/\pi^b\mathbb{T}W$ as a V -module. Since $N' \subseteq \mathcal{U}(\mathbb{T}W, I^l) = \sum_{j=0}^\infty \pi^{-j}I^{jl}$, each $x_j \in N'$ may be written as a finite sum of products $\pi^{-1}y_1 \cdots y_l$ with $y_1, \dots, y_l \in I$, plus a term in $\mathbb{T}W$. Let S be the set of all factors y_m that appear in these products. Then each element of N may be written as a finite V -linear combination of elements of $\bigcup_{j=1}^a \pi^{-j}S^{jl}$, plus an element of $\mathbb{T}W$. Let $B := \left(N + \sum_{j=1}^a V\pi^{-j}S^{jl}\right) \cap \mathbb{T}W$. This is bounded because $\mathbb{T}W$ is bornologically torsion-free, and B and S verify the first claim.

The subset $B \cup S$ of $\mathbb{T}W$ is bounded and hence contained in the image of $\bigoplus_{j=1}^b M^{\otimes n}$ for some π -adically complete, bounded V -submodule $M \subseteq W$ and some $b \in \mathbb{N}$. The inclusion $M \hookrightarrow W$ induces an injective bounded homomorphism $\mathbb{T}M \hookrightarrow \mathbb{T}W$ and this remains so after tensoring with F . Hence we get an injective bounded homomorphism $\mathcal{U}(\mathbb{T}M, (I \cap \mathbb{T}M)^l) \hookrightarrow \mathcal{U}(\mathbb{T}W, I^l)$. By construction of B and S , the subset N is the image of a bounded subset of $\mathcal{U}(\mathbb{T}M, (I \cap \mathbb{T}M)^l)$.

The image of the map $\varrho|_M: M \rightarrow A$ has finite dimension. Let $a_1, \dots, a_j \in A$ be a basis. Identify M with $C_0(D, V)$. For $i \in D$, let $\delta_i \in C_0(D, V)$ be the characteristic function of $\{i\}$. Write $\varrho(\delta_i)$ as a linear combination of the basis elements a_1, \dots, a_j with coefficients in \mathbb{F} and lift the coefficients to V in any way. This defines an element of VA supported in the finite set $\{a_1, \dots, a_j\}$, which we call $f(\delta_i)$. Doing this for all $i \in D$, we get a unique V -linear map $M \rightarrow VA$ with the specified values on characteristic functions. It has finite rank and thus it is bounded for the fine bornology on VA . It is compatible with the projection to A . We lift further to a linear map $M \rightarrow \mathbb{T}(VA) = V\langle A \rangle$ in the canonical way and let f_M be the induced homomorphism $\mathbb{T}M \rightarrow V\langle A \rangle$. Since this is compatible with the projections to A , it maps $\mathbb{T}M \cap I$ to J . Being a homomorphism, it then maps $(\mathbb{T}M \cap I)^l$ to J^l and extends to a bounded homomorphism $f_M: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l) \rightarrow \mathcal{U}(V\langle A \rangle, J^l)$.

To build H_M , we use the linear homotopy from the canonical inclusion $M \hookrightarrow W \hookrightarrow \mathbb{T}W$ to the map $s_* \circ f_M \circ \sigma_W: M \hookrightarrow W \rightarrow \mathbb{T}W$. This bounded V -module map induces a bounded homomorphism $H_M: \mathbb{T}M \rightarrow \mathbb{T}W \otimes V[t]$. When we project to $A \otimes V[t]$, this map becomes the constant homotopy on $\varrho^\#|_{\mathbb{T}M}$. Therefore, it maps the ideal $I \cap \mathbb{T}M$ into $I \otimes V[t]$. So its extension to $H_M \otimes \text{id}_F: \mathbb{T}M \otimes F \rightarrow \mathbb{T}W \otimes V[t] \otimes F$ maps the tube algebra $\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)$ into the tube algebra $\mathcal{U}(\mathbb{T}W \otimes V[t], (I \otimes V[t])^l)$. The latter is easily seen to be equal to $\mathcal{U}(\mathbb{T}W, I^l) \otimes V[t]$. This provides the homotopy H_M with the desired properties. The homotopy H_{VA} is constructed in exactly the same way. \square

The statement of Lemma 4.2.1 is somewhat analogous to [12, Corollary 4.3.12], which says that all semi-split, analytically quasi-free, analytically nilpotent extensions of the same algebra are homotopy equivalent. Since the extensions we are dealing with are no longer semi-split, and then we only get locally defined homomorphisms.

The tube algebras $(\mathcal{U}(\mathbb{T}W, I^l))_{l \in \mathbb{N}^*}$ are only quasi-free as a pro-algebra, and not for each individual l . Therefore, we cannot apply the homotopy invariance of the X -complex to the local homomorphism in Lemma 4.2.1 directly. We could in principle describe a version of Lemma 4.2.1 which would say that the different tube algebra resolutions of A are locally dagger homotopic as projective systems of bornological V -algebras. This would then have to be followed up with a version of homotopy invariance for the X -complex that works for locally dagger-homotopy equivalent algebras. Instead, we use the B - b -bicomplex $\mathbb{H}\mathbb{P}$, which is homotopy invariant even for individual tube algebras. Of course, the result will not change when we vary the tube algebra parameter l , since for quasi-free algebras, $\mathbb{H}\mathbb{P}$ is chain homotopy equivalent to the X -complex. The following lemma shows that the tube algebras that we require are indeed quasi-free.

LEMMA 4.2.2. *The pro-algebras $\mathcal{U}(V\langle A \rangle, J^\infty)_{\text{lg}}$ and $\mathcal{U}(\mathbb{T}W, I^\infty)_{\text{lg}}$ are quasi-free.*

PROOF. The proof is similar to the proof of [12, Proposition 4.4.6]. We only need to observe that the properties of being nilpotent mod π , and semi-dagger are both hereditary for extensions of pro-algebras, by [12, Proposition 4.2.5] and by adapting the proof of [12, Proposition 4.3.13]. \square

PROOF OF PROPOSITION 2.1.8. The map $f_M^l: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l) \rightarrow \mathcal{U}(V\langle A \rangle, J^l)$ from Lemma 4.2.1 is also a bounded algebra homomorphism

$$f_M^l: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)_{\text{lg}} \rightarrow \mathcal{U}(V\langle A \rangle, J^l)_{\text{lg}}$$

because application of the linear growth bornology is functorial. Furthermore, Lemma 4.2.1 implies that the bornological algebra $\mathcal{U}(\mathbb{T}W, I^l)_{\text{lg}}$ is isomorphic to the inductive limit of the strict inductive system $\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)_{\text{lg}}$ over the directed set of bounded subsets $M \subseteq W$.

The polynomial homotopy $H_M: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l) \rightarrow \mathcal{U}(\mathbb{T}W, I^l) \otimes V[t]$ between $s_* \circ f_M^l$ and the inclusion map $\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l) \xrightarrow{i_M} \mathcal{U}(\mathbb{T}W, I^l)$ is also a bounded algebra homomorphism

$$H_M: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)_{\text{lg}} \rightarrow \mathcal{U}(\mathbb{T}W, I^l)_{\text{lg}} \otimes V[t]_{\text{lg}}$$

such that $\text{ev}_0 \circ H_M$ is the inclusion $\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)_{\text{lg}} \rightarrow \mathcal{U}(\mathbb{T}W, I^l)_{\text{lg}}$ and $\text{ev}_1 \circ H_M$ is the composition $s_* \circ f_M^l$. Here we have used that the linear growth bornology commutes with tensor products (see [11, Proposition 3.1.25]). Similarly, the homomorphism $H_{V\langle A \rangle}$ is a polynomial homotopy between $f_M^l \circ s_*$ and the inclusion $\iota: \mathcal{U}(\mathbb{T}N_A, (J \cap \mathbb{T}N_A)^l)_{\text{lg}} \rightarrow \mathcal{U}(V\langle A \rangle, J^l)_{\text{lg}}$. All this remains true after tensoring with F .

Fix $l \geq 1$. The bounded homomorphisms

$$s_*^l \otimes F: \mathcal{U}(V\langle A \rangle, J^l)_{\text{lg}} \otimes F \rightarrow \mathcal{U}(\mathbb{T}W, I^l)_{\text{lg}} \otimes F$$

and

$$f_M^l \otimes F: \mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)_{\text{lg}} \otimes F \rightarrow \mathcal{U}(V\langle A \rangle, J^l)_{\text{lg}} \otimes F$$

induce bounded chain maps between the bornological chain complexes

$$\mathbb{H}\mathbb{P}(\mathcal{U}(V\langle A \rangle, J^l)_{\text{lg}} \otimes F) \text{ and } \mathbb{H}\mathbb{P}(\mathcal{U}(\mathbb{T}M, (\mathbb{T}M \cap I)^l)_{\text{lg}} \otimes F).$$

By homotopy invariance of periodic cyclic homology, the polynomial homotopies H_M and $H_{V\langle A \rangle}$ induce chain homotopies between $\mathbb{H}\mathbb{P}(s_*^l) \circ \mathbb{H}\mathbb{P}(f_M^l)$ and $\mathbb{H}\mathbb{P}(i_M)$ and between $\mathbb{H}\mathbb{P}(f_M^l) \circ \mathbb{H}\mathbb{P}(s_*^l)$ and $\mathbb{H}\mathbb{P}(i)$, respectively. Since M is an arbitrary bounded submodule of W , we conclude that the pro-chain map

$$\mathbb{H}\mathbb{P}(s_*^l): \mathbb{H}\mathbb{P}(\mathcal{U}(V\langle A \rangle, J^l)_{\text{lg}} \otimes F) \rightarrow \mathbb{H}\mathbb{P}(\mathcal{U}(\mathbb{T}W, I^l)_{\text{lg}} \otimes F)$$

is a local chain homotopy equivalence.

By Lemma 4.2.2, the b - B -bicomplexes are homotopy equivalent to the X -complexes of these quasi-free pro- F -algebras. We then get a local chain homotopy equivalence

$$\mathbb{H}\mathbb{A}(A) = X(\mathcal{U}(V\langle A \rangle, J^\infty)_{\text{lg}} \otimes F) \rightarrow X(\mathcal{U}(\mathbb{T}W, I^\infty)_{\text{lg}} \otimes F) = \mathbb{H}\mathbb{A}(A, W, \varrho)$$

between projective systems of chain complexes of bornological V -modules. This is a quasi-isomorphism by Theorem 3.4.4 and Corollary 3.5.3. \square

We end this section with a version of Proposition 2.1.8 for pro- V -module liftings. This version will be used to prove excision for our theory. Let A be an \mathbb{F} -algebra viewed as a constant projective system. Let $W = (W_n)_{n \in \mathbb{N}}$ be a projective system of complete, bornologically torsion-free bornological V -modules with a compatible family of surjective V -module maps $\varrho_n: W_n \twoheadrightarrow A$ denoted by $\varrho: W \twoheadrightarrow A$. Let $s: A \rightarrow W$ be a bounded pro-set section. Assume that each ϱ_n is bounded in the fine bornology on A .

PROPOSITION 4.2.3. *With A , $\varrho: W \twoheadrightarrow A$ and $s: A \rightarrow W$ as above, there is a canonical bornological F -module isomorphism*

$$\text{HA}_*(A) \cong \text{HA}_*(A; W, \varrho).$$

PROOF. By construction, there are maps $s_n: A \rightarrow W_n$ that split $\varrho_n: W_n \twoheadrightarrow A$ for each $n \in \mathbb{N}$. These maps induce V -algebra homomorphisms $V\langle A \rangle \rightarrow \mathrm{TW}_n$ for each n , which in turn induce pro-chain maps $\mathbb{H}\mathbb{A}(A) \rightarrow \mathbb{H}\mathbb{A}(A, W_n, \varrho_n)$. Since the local algebra homomorphisms of Lemma 4.2.1 are defined entrywise on each tensor algebra TW_n , the pro-chain map above is a quasi-isomorphism for each n . \square

5. Homotopy invariance and stability

In this section, we prove certain formal properties of our theory. We start with a homotopy lifting result that is analogous to [12, Proposition 4.3.10]. Using this, we prove that our theory is homotopy invariant for polynomial homotopies. We then use the stability result [12, Proposition 6.2] of analytic cyclic homology for torsion-free algebras to show that the same result holds for stability over finitely supported matrices.

LEMMA 5.0.1. *Let $f_0, f_1: A \rightarrow B$ be homomorphisms between \mathbb{F} -algebras, and let $H: A \rightarrow B \otimes_{\mathbb{F}} \mathbb{F}[t]$ be a polynomial homotopy between them.*

- (1) *there are V -algebra homomorphisms $f_0^*, f_1^*: V\langle A \rangle \rightarrow V\langle B \rangle$ lifting f_0 and f_1 ;*
- (2) *there is a polynomial homotopy $H^*: V\langle A \rangle \rightarrow V\langle B \rangle \otimes V[t]$ between f_0^* and f_1^* that lifts H .*

PROOF. The liftings f_0^* and f_1^* exist because of the universal property of free algebras. The proof of the second statement is similar to the proof of [12, Proposition 4.3.10] and the universal property of free algebras. Concretely, we take the pull-back \mathcal{E} of the evaluation maps $(\mathrm{ev}_0, \mathrm{ev}_1): B \otimes \mathbb{F}[t] \rightarrow B \oplus B$ along the canonical projection $V\langle B \rangle \oplus V\langle B \rangle \xrightarrow{p_B \oplus p_B} B \oplus B$. Then the maps

$$\rho: V\langle B \rangle \otimes V[t] \rightarrow V\langle B \rangle \oplus V\langle B \rangle, \quad b \mapsto (b(0), b(1)),$$

and $V\langle B \rangle \otimes V[t] \xrightarrow{\eta} B \otimes_{\mathbb{F}} \mathbb{F}[t]$ induce a unique map $\phi: V\langle B \rangle \otimes V[t] \rightarrow \mathcal{E}$ by the universal property of pullbacks. Here $\eta = p_B \otimes \mathbb{F}$. We have the following commuting diagram:

$$\begin{array}{ccccc} & & V\langle B \rangle \otimes V[t] & & \\ & & \downarrow \phi & \nearrow \rho & \\ V\langle A \rangle & \xrightarrow{\bar{H}} & \mathcal{E} & \longrightarrow & V\langle B \rangle \oplus V\langle B \rangle \\ \downarrow p_A & & \downarrow & \searrow & \downarrow p_B \oplus p_B \\ A & \xrightarrow{H} & B \otimes_{\mathbb{F}} \mathbb{F}[t] & \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} & B \oplus B \end{array}$$

where \bar{H} is defined on A by $a \mapsto (H(a), f_0^*(a), f_1^*(a))$. Composing with a section of ϕ , we obtain a map of sets $A \rightarrow V\langle B \rangle \otimes V[t]$. By the universal property of free algebras, this extends to a V -algebra homomorphism $H^*: V\langle A \rangle \rightarrow V\langle B \rangle \otimes V[t]$, finishing the proof. \square

PROPOSITION 5.0.2. *Let A be an \mathbb{F} -algebra and let $A[t] := A \otimes_{\mathbb{F}} \mathbb{F}[t]$.*

- (1) *the inclusion $\iota_A: A \rightarrow A \otimes_{\mathbb{F}} \mathbb{F}[t]$ induces a quasi-isomorphism $\mathbb{H}\mathbb{A}(A) \cong \mathbb{H}\mathbb{A}(A[t])$;*

- (2) let Λ be a set and let $M_\Lambda(A)$ be the algebra of finitely supported matrices indexed by $\Lambda \times \Lambda$, with values in A . Then for each $\lambda \in \Lambda$, the canonical inclusion $\iota_\lambda: A \rightarrow M_\Lambda(A)$, $a \mapsto e_{\lambda,\lambda} \otimes a$ induces a quasi-isomorphism

$$\mathbb{H}\mathbb{A}(A) \cong \mathbb{H}\mathbb{A}(M_\Lambda(A)).$$

PROOF. Let $\text{ev}_0: A[t] \rightarrow A$ be the evaluation map at $t = 0$. Then $H: A[t] \rightarrow A[t, s]$, $f \mapsto sf + (1-s)(\iota_A \circ \text{ev}_0)(f)$ is an affine homotopy between $\iota \circ \text{ev}_0$ and $\text{id}_{A[t]}$. By Lemma 5.0.1, this extends to a polynomial homotopy $H^*: V\langle A[t] \rangle \rightarrow V\langle A[t] \rangle \otimes V[s]$ between the induced maps $\iota^* \circ \text{ev}_0^*$ and the identity map. Let $I_{V\langle A[t] \rangle} = \ker(V\langle A[t] \rangle \twoheadrightarrow A[t])$. Then H^* maps $I_{V\langle A[t] \rangle}$ to $I_{V\langle A[t] \rangle} \otimes V[s]$. Hence it extends to an algebra homomorphism

$$\mathcal{U}(V\langle A[t] \rangle, I_{V\langle A[t] \rangle}^m) \rightarrow \mathcal{U}(V\langle A \rangle, I_{V\langle A \rangle}^m) \otimes V[s],$$

for each $m \geq 1$. This further induces bounded algebra homomorphisms

$$\mathcal{U}(V\langle A[t] \rangle, I_{V\langle A[t] \rangle}^m)_{\text{lg}} \rightarrow \mathcal{U}(V\langle A \rangle, I_{V\langle A \rangle}^m)_{\text{lg}} \otimes V[s]_{\text{lg}}$$

for each $m \geq 1$, which are polynomial homotopies between the corresponding maps of semi-dagger algebras induced by $\iota^* \circ \text{ev}_0^*$ and the identity map on $V\langle A[t] \rangle$. By Lemma 4.2.2, the pro-algebra $\mathcal{U}(V\langle A[t] \rangle, I_{V\langle A[t] \rangle}^m)_{\text{lg}}$ is quasi-free. Then the homotopy invariance of the X -complex and homotopy inverse limits implies that $\mathbb{H}\mathbb{A}(\iota_A) \circ \mathbb{H}\mathbb{A}(\text{ev}_0)$ is chain homotopic to $\mathbb{H}\mathbb{A}(\text{id})$.

Next we prove (2). There is a canonical V -algebra lifting $V\langle M_\Lambda(A) \rangle \rightarrow M_\Lambda(V\langle A \rangle)$ of the identity map on $M_\Lambda A$, induced by the obvious maps $V M_\Lambda(A) \rightarrow M_\Lambda(VA) \rightarrow M_\Lambda(V\langle A \rangle)$. Let $I = \ker(V\langle M_\Lambda(A) \rangle \twoheadrightarrow M_\Lambda(A))$ and $J = \ker(V\langle A \rangle \twoheadrightarrow A)$. Then we get a pro-algebra homomorphism between tube algebras

$$\mathcal{U}(V\langle M_\Lambda(A) \rangle, I^\infty) \rightarrow M_\Lambda(\mathcal{U}(V\langle A \rangle, J^\infty)).$$

The functoriality of the X -complex gives a pro-chain map

$$X(\mathcal{U}(V\langle M_\Lambda(A) \rangle, I^\infty)) \rightarrow X(M_\Lambda(\mathcal{U}(V\langle A \rangle, J^\infty))).$$

We can now proceed as in the proof of [26, Theorem 5.65], adapted to the situation of [12, Example 6.3]. \square

6. Nilpotent extensions and excision

6.1. Nilpotent resolutions of \mathbb{F} -algebras. In this section, we discuss analytically nilpotent resolutions that can be naturally associated to \mathbb{F} -algebras. Our motivation here is to prove that our theory satisfies excision for finitely generated \mathbb{F} -algebras. Specifically, we will show that an extension of finitely generated \mathbb{F} -algebras extends to an extension of pro-dagger algebras with analytically nilpotent kernels, and then appeal to Proposition 2.1.4. An important feature of our framework is that we do not require our \mathbb{F} -algebras to admit V -algebra lifts R that satisfy $R/\pi R \cong A$. Instead, we resolve A by an extension of tube algebras

$$\mathcal{U}(I, I^\infty) \twoheadrightarrow \mathcal{U}(\text{TR}, I^\infty) \twoheadrightarrow A,$$

where $I = \ker(\text{TR} \twoheadrightarrow A)$, and R is a V -module with a non-associative multiplication, that reduces mod π to the multiplication on A . Such a lift R always exists.

In what follows, let A be an \mathbb{F} -algebra and let R be a V -module. Let $\Omega^l(R) := R^+ \otimes R^{\otimes l}$ for $l > 0$ and $\Omega^0(R) = R$ as V -modules, where $R^+ := R \oplus V$ is the V -algebra

obtained by adjoining a unit to R . Let $\Omega(R) = \bigoplus_{l=0}^{\infty} \Omega^l(R)$, $\Omega^{\text{ev}}(R) = \bigoplus_{l=0}^{\infty} \Omega^{2l}(R)$, and $\Omega^{\text{odd}}(R) = \bigoplus_{l=0}^{\infty} \Omega^{2l+1}(R)$. We now define a V -module isomorphism

$$\iota: \Omega^{\text{ev}}(R) \xrightarrow{\cong} \mathbb{T}R,$$

using a possibly non-associative V -bilinear multiplication $\mu: R \otimes R \rightarrow R$. Let \otimes denote the multiplication in $\mathbb{T}R$. For $x_1, x_2 \in R$, we define

$$\iota(dx_1 dx_2) := \mu(x_1, x_2) - x_1 \otimes x_2 \in \mathbb{T}R.$$

More generally,

$$\begin{aligned} \iota(dx_1 dx_2 \dots dx_{2l-1} dx_{2l}) &:= \iota(dx_1 dx_2) \otimes \dots \otimes \iota(dx_{2l-1} dx_{2l}), \\ \iota(x_0 dx_1 dx_2 \dots dx_{2l-1} dx_{2l}) &:= x_0 \otimes \iota(dx_1 dx_2) \otimes \dots \otimes \iota(dx_{2l-1} dx_{2l}) \end{aligned}$$

for $x_0, \dots, x_{2l} \in R$. These maps $\Omega^{2l}(R) \rightarrow \mathbb{T}R$ combine to a map $\iota: \Omega^{\text{ev}}(R) \rightarrow \mathbb{T}R$.

LEMMA 6.1.1. *The map $\iota: \Omega^{\text{ev}}(R) \rightarrow \mathbb{T}R$ is an isomorphism of V -modules.*

PROOF. We define filtrations on $\mathbb{T}R$ and $\Omega^{\text{ev}}R$. Let $(\mathbb{T}R)_j$ be the V -submodule generated by $x_1 \otimes \dots \otimes x_m$ with $x_i \in R$, $m \leq j$. Let $(\Omega^{\text{ev}}R)_j$ be the V -submodule generated by $\Omega^{2k}R$ for all k with $2k < j$ and by the ‘‘closed’’ $2k$ -forms $dx_1 \dots dx_{2k}$ if $2k = j$. We prove by induction on j that ι restricts to an isomorphism from $(\Omega^{\text{ev}}R)_j$ onto $(\mathbb{T}R)_j$. The assertion is empty for $j = 0$. Assuming it for $j - 1 \geq 0$, we prove it for j . We have $(\mathbb{T}R)_j / (\mathbb{T}R)_{j-1} \cong R^{\otimes j}$. By definition, ι maps $dx_1 \dots dx_{2k}$ and $x_0 dx_1 \dots dx_{2k}$ to $(-1)^k x_1 \otimes \dots \otimes x_{2l}$ and $(-1)^k x_0 \otimes x_1 \otimes \dots \otimes x_{2l}$ modulo shorter terms. So ι maps $(\Omega^{\text{ev}}R)_j$ into $(\mathbb{T}R)_j$ and induces an isomorphism from $(\Omega^{\text{ev}}R)_j / (\Omega^{\text{ev}}R)_{j-1}$ onto $(\mathbb{T}R)_j / (\mathbb{T}R)_{j-1}$. By the induction assumption and the Five Lemma, it induces an isomorphism from $(\Omega^{\text{ev}}R)_j$ onto $(\mathbb{T}R)_j$. \square

In the following, we identify $\mathbb{T}R$ and $\Omega^{\text{ev}}R$ using this isomorphism. Let \odot be the associative multiplication on $\Omega^{\text{ev}}R$ that corresponds to the multiplication in $\mathbb{T}R$. If μ is associative, then \odot is the well known *Fedosov product*, $\omega \odot \eta = \omega\eta - d\omega d\eta$, where we use the usual multiplication of differential forms dictated by the Leibniz rule (see [16]). When μ is non-associative, this has to be corrected by terms involving the associator of μ . Writing μ multiplicatively, the basic rule is

$$(6.1.2) \quad (dx_1 dx_2) \odot x_3 - dx_1 d(x_2 x_3) + d(x_1 x_2) dx_3 - x_1 dx_2 dx_3 = (x_1 x_2) x_3 - x_1 (x_2 x_3)$$

for all $x_1, x_2, x_3 \in R$. The homomorphism $p: \mathbb{T}R \rightarrow A$ satisfies $p(dx_1 dx_2) = 0$ because μ lifts the multiplication in A . Hence it corresponds to the map $\Omega^{\text{ev}}R \rightarrow A$, $\sum \omega_{2n} \mapsto \omega_0 \bmod \pi$ for $\omega_{2n} \in \Omega^{2n}R$. So

$$(6.1.3) \quad I := \ker(\mathbb{T}R \rightarrow A) = \pi R \oplus \bigoplus_{j=1}^{\infty} \Omega^{2j}R.$$

LEMMA 6.1.4. *If $m \in \mathbb{N}^*$, then*

$$\mathcal{U}(\mathbb{T}R, I^m) = \sum_{j=0}^{\infty} \pi^{-\lfloor j/m \rfloor} \Omega^{2j}R, \quad \mathcal{U}(I, I^m) = I \oplus \sum_{j=1}^{\infty} \pi^{-\lfloor j/m \rfloor} \Omega^{2j}R,$$

PROOF. To simplify the writing, we adjoin a unit element to $\mathbb{T}R$ and allow $x_0 \in R^+ := R \oplus V \cdot 1$ to treat forms with and without x_0 on an equal footing. Let $I^{(+)} := I \oplus \pi \cdot V \subseteq R^+$, so that $\pi \in I^{(+)}$. We claim that this does not affect the tube algebra $\sum_{j=0}^{\infty} \pi^{-j} I^m j$. Let $I^{(m)} := \sum_{j=1}^m \pi^{m-j} I^j$. Then induction shows that

$$(I^{(+)})^m := (I \oplus \pi \cdot V)^m = I^{(m)} \oplus \pi^m \cdot V.$$

And

$$\sum_{j=1}^{\infty} \pi^{-j} I^{(mj)} = \sum_{j=1}^{\infty} \sum_{k=1}^{mj} \pi^{mj-k-j} I^k = \sum_{j=1}^{\infty} \pi^{-j} I^{mj}.$$

Let $m \geq 1$ and $j \geq 0$. Let $(m-j)_+$ be $m-j$ if $m-j \geq 0$ and 0 otherwise. Let $x_0 \in R^+$, $x_1, \dots, x_{2j} \in R$. Then

$$(6.1.5) \quad \pi^{(m-j)_+} x_0 dx_1 \dots dx_{2j} = (\pi^{(m-j)_+} x_0) \odot dx_1 dx_2 \odot \dots \odot dx_{2j-1} dx_{2j}$$

is a product of j terms $dx_{2i-1} dx_{2i} \in I$ and the term $\pi^{(m-j)_+} x_0$, which involves $(m-j)_+ \geq m-j$ factors $\pi \cdot 1 \in I^{(+)}$. Thus $\pi^{(m-j)_+} x_0 dx_1 \dots dx_{2j} \in I^{(m)}$. We prove by induction on m that these elements generate $I^{(m)}$.

Let $m \geq 1$ and assume the assertion is shown for $I^{(m)}$. We must prove it for $I^{(m+1)} = I \odot I^{(m)} + \pi \cdot I^{(m)}$. The second summand is easy to handle. The first is generated by products $\pi x_0 \odot \pi^{(m-j)_+} y_0 dy_1 \dots dy_{2j}$ and $x_0 dx_1 \dots dx_{2i} \odot \pi^{(m-j)_+} y_0 dy_1 \dots dy_{2j}$ with $i > 0$. The first type of product gives

$$\begin{aligned} \pi x_0 \odot \pi^{(m-j)_+} y_0 dy_1 \dots dy_{2j} \\ = \pi^{1+(m-j)_+} (x_0 y_0) dy_1 \dots dy_{2j} - \pi^{1+(m-j)_+} dx_0 dy_0 dy_1 \dots dy_{2j}, \end{aligned}$$

which has the desired form. In the second type of product, we rewrite $x_0 dx_1 \dots dx_{2i} \odot \pi^{(m-j)_+} y_0$ using (6.1.2). All associators that appear here belong to πR because $A = R/\pi R$ is associative. Hence we get a sum of forms that have non-zero degree or belong to πR . So each summand in the product has the desired form. This completes our description of $I^{(m)}$. The assertions on tube algebras follow immediately. \square

Lemma 6.1.4 implies $\mathcal{U}(I, I^m) \cap \mathbb{T}R = I$ for $m \geq 2$. Hence there is a natural extension of V -algebras

$$\mathcal{U}(I, I^m) \twoheadrightarrow \mathcal{U}(\mathbb{T}R, I^m) \twoheadrightarrow A$$

for all $m \geq 2$. Letting m vary, this becomes an extension of pro-algebras

$$(6.1.6) \quad \mathcal{U}(I, I^\infty) \twoheadrightarrow \mathcal{U}(\mathbb{T}R, I^\infty) \twoheadrightarrow A,$$

where A is viewed as a constant pro-algebra. Its kernel is nilpotent mod π by [12, Proposition 4.2.3]. Lemma 6.1.4 also shows that $\bigcap_{m \in \mathbb{N}^*} \mathcal{U}(\mathbb{T}R, I^m) = \mathbb{T}R$ or, equivalently,

$$\varprojlim \mathcal{U}(\mathbb{T}R, I^m) = \mathbb{T}R.$$

We may choose a representative in R and then in $\mathbb{T}R$ for each element of A . So the extension (6.1.6) splits by a morphism of pro-sets. We cannot expect much more because A is an \mathbb{F} -algebra, whereas $\mathcal{U}(\mathbb{T}R, I^m)$ is torsion-free for all $m \in \mathbb{N}^*$.

Equation (6.1.6) gives us a pro-algebra lifting of an \mathbb{F} -algebra with a nilpotent kernel. In order to be in the situation of Proposition 2.1.4, however, we need an extension of A by a pro-dagger algebra and an analytically nilpotent kernel. To obtain such a resolution in a natural way, we take $R = VA$ as the free V -module over A , and equip A and R with the fine bornology. Equipping the tube algebra $\mathcal{U}(\mathbb{T}R, I^\infty) = \mathcal{U}(V\langle A \rangle, I^\infty)$ with the fine bornology, we now show that we can enlarge (6.1.6) to an *analytically* nilpotent extension

$$\mathcal{U}(I, I^\infty)^\dagger \twoheadrightarrow \mathcal{U}(\mathbb{T}R, I^\infty)^\dagger \twoheadrightarrow A.$$

We first describe the quotient bornology of dagger complete tube algebras $\mathcal{U}(R, I^\infty)^\dagger$ by their canonical ideals $\mathcal{U}(I, I^\infty)^\dagger$. Here R is any V -algebra presentation

of an \mathbb{F} -algebra A with $I = \ker(R \twoheadrightarrow A)$, and the tube algebras $\mathcal{U}(R, I^m)$ are equipped with the fine bornology for each $m \in \mathbb{N}^*$.

PROPOSITION 6.1.7. *The canonical map from $\mathcal{U}(I, I^m)^\dagger$ to $\mathcal{U}(R, I^m)^\dagger$ is a bornological embedding. Its image is a closed ideal. It induces an isomorphism*

$$\mathcal{U}(I, I^m)^\dagger \otimes F \cong \mathcal{U}(R, I^m)^\dagger \otimes F.$$

The inclusions $\mathcal{U}(I, I^m) \rightarrow \mathcal{U}(I, I^m)^\dagger$ and $\mathcal{U}(R, I^m) \rightarrow \mathcal{U}(R, I^m)^\dagger$ induce an isomorphism

$$\mathcal{U}(R, I^m) / \mathcal{U}(I, I^m) \cong \mathcal{U}(R, I^m)^\dagger / \mathcal{U}(I, I^m)^\dagger.$$

The quotient $\mathcal{U}(R, I^m) / \mathcal{U}(I, I^m)$ is an \mathbb{F} -algebra with the fine bornology.

PROOF. We claim that the linear growth bornology of $\mathcal{U}(R, I^m)$ restricted to $\mathcal{U}(I, I^m)$ is the linear growth bornology of $\mathcal{U}(I, I^m)$. To prove this, we equip $\mathcal{U}(R, I^m)$ with the bornology generated by the inclusions of $\mathcal{U}(I, I^m)$ and R . That is, the bounded subsets of $\mathcal{U}(R, I^m)$ are those of the form $S + T$ where S is a finitely generated V -submodule in R and T has linear growth in $\mathcal{U}(I, I^m)$. We claim that $\mathcal{U}(R, I^m)$ with this bornology is a semi-dagger algebra.

The restriction of our bornology on $\mathcal{U}(R, I^m)$ to $\mathcal{U}(I, I^m)$ is the linear growth bornology because $S \cap \mathcal{U}(I, I^m)$ for finitely generated $S \subseteq R$ is again finitely generated and hence bounded in $\mathcal{U}(I, I^m)$. The induced bornology on $A_1 := \mathcal{U}(R, I^m) / \mathcal{U}(I, I^m)$ is the fine bornology. This quotient algebra is a quotient of the \mathbb{F} -algebra A because $R/I \cong A$. This is semi-dagger because $\pi \cdot A = 0$. Hence $\mathcal{U}(R, I^m)$ with our tailor-made bornology is an extension of semi-dagger algebras, hence semi-dagger. Thus our bornology contains all subsets of linear growth in $\mathcal{U}(R, I^m)$. Conversely, all subsets of the form $S + T$ as above have linear growth in $\mathcal{U}(R, I^m)$. So our new bornology on $\mathcal{U}(R, I^m)$ is just the linear growth bornology. Hence the linear growth bornology on $\mathcal{U}(R, I^m)$ restricts to the linear growth bornology on $\mathcal{U}(I, I^m)$ and induces the fine bornology on the quotient A_1 . We also see that the inclusion $\mathcal{U}(I, I^m)_{\text{lg}} \rightarrow \mathcal{U}(R, I^m)_{\text{lg}}$ induces bornological isomorphisms $\mathcal{U}(I, I^m)_{\text{lg}} \otimes F \cong \mathcal{U}(R, I^m)_{\text{lg}} \otimes F$ and $\mathcal{U}(R, I^m)_{\text{lg}} / \mathcal{U}(I, I^m)_{\text{lg}} \cong A_1$.

Both $\mathcal{U}(I, I^m)$ and $\mathcal{U}(R, I^m)$ are bornologically torsion-free. This remains so for the linear growth bornologies by [12, Lemma 2.2.7]. Hence

$$\mathcal{U}(I, I^m)^\dagger \otimes F = \overline{\mathcal{U}(I, I^m)_{\text{lg}} \otimes F} = \overline{\mathcal{U}(R, I^m)_{\text{lg}} \otimes F} = \mathcal{U}(R, I^m)^\dagger \otimes F.$$

Since both $\mathcal{U}(I, I^m)^\dagger$ and $\mathcal{U}(R, I^m)^\dagger$ are bornologically torsion-free, they bornologically embed into $\overline{\mathcal{U}(I, I^m)_{\text{lg}} \otimes F} = \overline{\mathcal{U}(R, I^m)_{\text{lg}} \otimes F}$. Hence the canonical map $\mathcal{U}(I, I^m)^\dagger \rightarrow \mathcal{U}(R, I^m)^\dagger$ is a bornological embedding. Since $\mathcal{U}(I, I^m)^\dagger$ is complete, its image is closed. It is a two-sided ideal because $\mathcal{U}(I, I^m)$ is a two-sided ideal in $\mathcal{U}(R, I^m)$. Since the quotient $\mathcal{U}(R, I^m)_{\text{lg}} / \mathcal{U}(I, I^m)_{\text{lg}} \cong A_1$ is an \mathbb{F} -algebra with the fine bornology, it is already complete. The completion functor preserves cokernels because it is a left adjoint functor. Hence the isomorphism above implies $\overline{\mathcal{U}(R, I^m)_{\text{lg}} / \mathcal{U}(I, I^m)_{\text{lg}}} \cong A_1$. That is, $\mathcal{U}(R, I^m)^\dagger / \mathcal{U}(I, I^m)^\dagger \cong \mathcal{U}(R, I^m) / \mathcal{U}(I, I^m)$. \square

COROLLARY 6.1.8. *Let A be an \mathbb{F} -algebra with the fine bornology. Then we have an analytically nilpotent extension*

$$\mathcal{U}(I, I^\infty)^\dagger \twoheadrightarrow \mathcal{U}(V\langle A \rangle, I^\infty)^\dagger \twoheadrightarrow A$$

of pro-bornological algebras, that splits by a bounded pro-map of sets.

PROOF. Set $R = V\langle A \rangle$ with the fine bornology in Proposition 6.1.7 and then use Equation (6.1.6). The kernel $\mathcal{U}(I, I^\infty)^\dagger$ is analytically nilpotent as nilpotence mod π is unaffected by dagger completion. \square

REMARK 6.1.9. In Corollary 6.1.8, we can replace the free algebra $V\langle A \rangle$ by the free algebra over a smaller set. Namely, let S be a basis for A and $R := VS$. Then $V\langle A \rangle \cong \mathrm{TR}$. So Proposition 6.1.7 and Equation (6.1.6) yield an extension

$$\mathcal{U}(I, I^\infty)^\dagger \twoheadrightarrow \mathcal{U}(\mathrm{TR}, I^\infty)^\dagger \twoheadrightarrow A.$$

6.2. An explicit analytic tensor algebra lifting. The analytic tensor algebras constructed from free algebras in Corollary 6.1.8 and Remark 6.1.9 are too large. Consequently, they do not give us any information about their structure, by way of explicit descriptions of their bornologies. In this section, we start with a finitely generated \mathbb{F} -algebra A , whose generating set we use to build a specific basis $S \subseteq A$. We then take the free V -module R over S , and replace the fine bornology by a larger bornology R_{lg} . It turns out that the linear growth bornology on the resulting tube algebra $(\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m))_{m \in \mathbb{N}^*}$ is easier to describe. The completion of the tube algebra in this bornology is dagger homotopy equivalent to $\mathcal{U}(\mathrm{TR}, I^\infty)^\dagger$.

Let A be a finitely generated \mathbb{F} -algebra and let S_1 be a generating subset of A . We assume S_1 to be linearly independent. We shall describe a filtration on A by \mathbb{F} -vector subspaces. Let $\mathcal{F}_j A$ be the \mathbb{F} -vector subspace of A generated by elements of the form $s_1 \dots s_l$, where $s_i \in S$ for $1 \leq i \leq l \leq j$. Set $\mathcal{F}_0 A = \{0\}$. We have

$$\mathcal{F}_j A \cdot \mathcal{F}_k A \subseteq \mathcal{F}_{j+k} A, \quad \mathcal{F}_n A \subseteq \mathcal{F}_{n+1} A.$$

So $(\mathcal{F}_n A)_{n \in \mathbb{N}}$ is an increasing filtration of A . Furthermore, S_1 is a basis for $\mathcal{F}_1 A$. The set S_1 together with products of the form $s_1 \cdot s_2$, with $s_1, s_2 \in S_1$ generates $\mathcal{F}_2 A$. Omitting some products if necessary, we obtain a basis S_2 for $\mathcal{F}_2 A$ containing S_1 . Continuing like this, for each $j \geq 1$, we obtain bases S_j for $\mathcal{F}_j A$ containing S_{j-1} . Then $S = \bigcup_{j=1}^\infty S_j$ is a basis for A .

Let R be the free V -module generated by the set S . By construction, $R/\pi R \cong A$, and R inherits an increasing filtration $\mathcal{F}_j R := VS_j$. Let $q: R \rightarrow A$ be the quotient map. Then $q(\mathcal{F}_j R) = \mathcal{F}_j A$ by construction. We shall now construct a map $\mu: R \otimes R \rightarrow R$ that reduces mod π to the (associative) multiplication map $\mu_0: A \otimes_{\mathbb{F}} A \rightarrow A$ for the algebra A in the sense that the following diagram commutes:

$$\begin{array}{ccc} R \otimes R & \xrightarrow{\mu} & R \\ q \otimes q \downarrow & & \downarrow q \\ A \otimes_{\mathbb{F}} A & \xrightarrow{\mu_0} & A. \end{array}$$

Let $x \in S_j$ and $y \in S_k$ for some j and k . Then there is $z \in \mathcal{F}_{j+k} R$ such that $q(z) = \mu_0(q \otimes q(x \otimes y))$. We then set $\mu(x \otimes y) := z$. This map is well-defined as different representations of $\mu_0(x \otimes y)$ all map to z for a specific choice of section $A \rightarrow R$. By construction, we have the following:

LEMMA 6.2.1. *The map $\mu: R \otimes R \rightarrow R$ satisfies $\mu(\mathcal{F}_j R \otimes \mathcal{F}_k R) = \mathcal{F}_{j+k} R$ for each j and $k \in \mathbb{N}$.*

PROOF. By construction, we have

$$\mu(\mathcal{F}_j(R) \otimes \mathcal{F}_k(R)) \subseteq \mathcal{F}_{j+k}(R).$$

When we divide out by π , the map μ is simply μ_0 , which restricts to a surjective map $\mathcal{F}_j(A) \otimes_{\mathbb{F}} \mathcal{F}_k(A) \twoheadrightarrow \mathcal{F}_{j+k}(A)$. We have

$$\mu_0(\mathcal{F}_j(A) \otimes_{\mathbb{F}} \mathcal{F}_k(A)) \cong \mu(\mathcal{F}_j(R) \otimes \mathcal{F}_k(R)) / \pi \cdot \mu(\mathcal{F}_j(R) \otimes \mathcal{F}_k(R)).$$

Here $\mathcal{F}_j(R) \otimes \mathcal{F}_k(R)$ and $\mathcal{F}_{j+k}(R)$ are finitely generated, free V -modules. So a V -module map between them is surjective once it is surjective mod π . \square

The multiplication map μ is in general non-associative and this means that we must modify the definition of the linear growth bornology for (associative) algebras. By Lemma 6.2.1, any element of R may be written as a finite sum of products of elements in $\mathcal{F}_1(R)$. Conversely, an element belongs to $\mathcal{F}_j(R)$ if and only if we can get it using only products of length at most j . Here the way we put parentheses does not matter as they all give the same V -submodule in R . Hence the submodules of linear growth should be those of the form $\sum_{j=0}^{\infty} \pi^j (\mathcal{F}_1(R))^{l(j+1)} = \sum_{j=0}^{\infty} \pi^j \mathcal{F}_{l(j+1)}(R)$ for some $l \in \mathbb{N}^*$. This leads to the following definition:

DEFINITION 6.2.2. A subset of R has *linear growth* if it is contained in a submodule of the form

$$\sum_{j=0}^{\infty} \pi^j \mathcal{F}_{l(j+1)}(R)$$

for some $l \in \mathbb{N}^*$. We denote R with this bornology by R_{lg} .

LEMMA 6.2.3. Let $\lambda: \mathbb{N} \rightarrow \mathbb{N}$ be a function of linear growth, that is, $\lambda(n) \leq an + b$ for some $a, b \in \mathbb{N}$. Then $\sum_{j=0}^{\infty} \pi^j \mathcal{F}_{\lambda(j)}(R)$ has linear growth in R .

PROOF. Let $l > \max\{a, b\}$. Then $\lambda(j) \leq l(j+1)$, so that we have

$$\sum_{j=0}^{\infty} \pi^j \mathcal{F}_{\lambda(j)}(R) \subseteq \sum_{j=0}^{\infty} \pi^j \mathcal{F}_{l(j+1)}(R).$$

\square

If R is an associative algebra with the fine bornology, then this bornology is analogous to the usual definition of the linear growth bornology for algebras. By construction, R_{lg} is bornologically torsion-free and its reduction mod π is $R/\pi R = A$ with the fine bornology. So $q: R \rightarrow A$ is a bornological quotient map. Completing R with respect to this bornology yields a complete, bornologically torsion-free V -module $\overline{R_{\text{lg}}}$. Since the completion functor is a left adjoint functor and since an \mathbb{F} -algebra is already complete with respect to the fine bornology, we have

$$\overline{R_{\text{lg}}} / \pi \overline{R_{\text{lg}}} \cong R/\pi R \cong A.$$

So $\overline{R_{\text{lg}}} \twoheadrightarrow A$ is a bornological quotient map.

The construction above defines a bornologically torsion-free, bornological V -module R_{lg} , whose bornology is described by a filtration $(\mathcal{F}_j R)_{j \in \mathbb{N}}$. We can use the filtration on R to induce a filtration on differential forms $\Omega^l(R)$ as follows: let $\mathcal{F}_j(\Omega^l(R))$ be the V -module generated by differential forms $x_0 dx_1 \cdots dx_l$, where $x_0 \in (\mathcal{F}_{j_0}(R))^+$, $x_i \in \mathcal{F}_{j_i}(R)$ such that $\sum_{i=0}^l j_i = j$. This is an increasing filtration on the module $\Omega^l(R)$. We use the description of the tube algebra from Lemma 6.1.4, $\mathcal{U}(\text{TR}, I^m) = \bigoplus_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor} \Omega^{2j}(R)$ to equip it with the bornology \mathcal{B}_m generated by V -modules of the form

$$(6.2.4) \quad \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lfloor \frac{j}{l} \rfloor + k} \mathcal{F}_{l(k+1)}(\Omega^{2j}(R)),$$

for some $l \in \mathbb{N}^*$.

PROPOSITION 6.2.5. *For each $m \in \mathbb{N}^*$, the bornological algebra $(\mathcal{U}(\mathrm{TR}, I^m), \mathcal{B}_m)$ is bornologically torsion-free and semi-dagger. So its completion is a dagger algebra. The ideal $\mathcal{U}(I, I^m)$ is bornologically closed, and the bornology \mathcal{B}_m still induces the fine bornology on the quotient $\mathcal{U}(\mathrm{TR}, I^m) / \mathcal{U}(I, I^m) \cong A$.*

PROOF. We first show that $(\mathcal{U}(\mathrm{TR}, I^m), \mathcal{B}_m)$ is bornologically torsion-free. Let S be a bounded subset in \mathcal{B}_m . Then there is an α such that

$$\begin{aligned} S \subseteq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lceil \frac{j}{\alpha} \rceil + k} \mathcal{F}_{\alpha(k+1)}(\Omega^{2j}(R)) &= \sum_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lceil \frac{j}{\alpha} \rceil} \mathcal{F}_{\alpha}(\Omega^{2j}(R)) \\ &+ \pi \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lceil \frac{j}{\alpha} \rceil + k} \mathcal{F}_{\alpha(k+2)}(\Omega^{2j}(R)). \end{aligned}$$

Then

$$\pi^{-1}S \subseteq \sum_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lceil \frac{j}{\alpha} \rceil - 1} \mathcal{F}_{\alpha}(\Omega^{2j}(R)) + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lceil \frac{j}{\alpha} \rceil + k} \mathcal{F}_{\alpha(k+2)}(\Omega^{2j}(R)).$$

Denote the first term and the second term in the equation above by T and U , respectively. The term T is contained in

$$\sum_{j=0}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lceil \frac{j-1}{\alpha} \rceil} \mathcal{F}_{\alpha}(\Omega^{2j}(R)) \subseteq \pi^{-1} \mathcal{F}_{\alpha}(R) + \sum_{j=1}^{\infty} \pi^{-\lfloor \frac{j}{m} \rfloor + \lceil \frac{j}{2\alpha} \rceil} \mathcal{F}_{2\alpha}(\Omega^{2j}(R)),$$

which is bounded as R_{lg} is bornologically torsion-free and the latter term is a generator for the bornology \mathcal{B}_m . The second term U is bounded in \mathcal{B}_m by Lemma 6.2.3. Therefore, $\pi^{-1}S$ is bounded in \mathcal{B}_m .

To show that the bornology \mathcal{B}_m is semi-dagger, we need the following estimate on the growth of the Fedosov product on $\Omega(R)$:

LEMMA 6.2.6. *Let R be a V -module with a filtration $(\mathcal{F}_l(R))_l$ by V -submodules. Then with the induced filtration on differential forms, we have*

$$\mathcal{F}_{l_1} \Omega^{2i_1}(R) \odot \mathcal{F}_{l_2} \Omega^{2i_2}(R) \subseteq \bigoplus_{j=0}^1 \mathcal{F}_l(\Omega^{2i_1+2j}(R)) \oplus \bigoplus_{k=1}^{i_1} \pi^k \mathcal{F}_l \Omega^{2(i-k)}(R),$$

where $i = i_1 + i_2$ and $l = l_1 + l_2$.

PROOF. Let $\omega_1 := x_0 dx_1 \cdots dx_{2i_1}$ and $\omega_2 := y_0 dy_1 \cdots dy_{2i_2}$ be differential forms in $\mathcal{F}_{l_1} \Omega^{2i_1}(R)$ and $\mathcal{F}_{l_2} \Omega^{2i_2}(R)$. Since the Fedosov product is only determined by the action of ω_1 on the 0-th entry y_0 of ω_2 , it suffices to prove the case $i_2 = 0$. So we have $i = i_1$ and we rewrite $y_0 = x_{2i_1+1}$. We write ω_1 as i blocks of 2-forms $x_0 \odot dx_1 dx_2 \odot \cdots \odot dx_{2i-1} dx_{2i}$.

Let $a(x, y, z) = x(yz) - (xy)z$ denote the associator for the multiplication μ of R . Since the multiplication on R is associative mod π , $a(x, y, z) \in \pi R$. Using the rule in (6.1.2), when we take the product $\omega_1 \odot x_{2i_1+1}$, we get a $2i + 2$ -form $d\omega_1 dx_{2i_1+1}$ and sums of $2i$ -forms

$$x_0 dx_1 dx_2 \cdots d(x_l x_{l+1}) \cdots dx_{2i+1}$$

that do not simplify further. We also get a decomposable $2i$ -form

$$x_0 \odot \cdots \odot dx_{2i-3} dx_{2i-2} \odot x_{2i-1} \odot dx_{2i} dx_{2i+1},$$

and a $2i - 2$ form

$$x_0 \odot dx_1 dx_2 \odot \cdots \odot a(x_{2i-1}, x_{2i}, x_{2i+1}).$$

The $2i$ and $2i+2$ -forms that do not decompose further are contained in $\mathcal{F}_l(\Omega^{2i}(R))$ and $\mathcal{F}_l(\Omega^{2i+2}(R))$, respectively.

Applying the product rule of the Fedosov product to the term $x_0 \odot dx_1 dx_2 \odot \cdots \odot a(x_{2i-1}, x_{2i}, x_{2i+1})$ yields an indecomposable $2i$ -form in $\mathcal{F}_l(\Omega^{2i}(R))$, indecomposable sums of $2i-2$ -forms in $\pi\mathcal{F}_l(\Omega^{2i-2}(R))$, and a $2i-4$ -form

$$x_0 \odot dx_1 dx_2 \odot \cdots \odot a(x_{2i-3}, x_{2i-2}, a(x_{2i-1}, x_{2i}, x_{2i+1})) \in \pi^2 \mathcal{F}_l(\Omega^{2i-4}(R)).$$

The same argument for the term $x_0 \odot \cdots \odot dx_{2i-3} dx_{2i-2} \odot x_{2i-1} \odot dx_{2i} dx_{2i+1}$ yields sums of $2i$ -forms that do not decompose further, a $2i$ -form

$$x_0 \odot dx_1 dx_2 \odot \cdots \odot x_{2i-3} \odot \cdots \odot dx_{2i} dx_{2i+1},$$

and a $2i-2$ -form

$$x_0 \odot dx_1 dx_2 \odot \cdots \odot dx_{2i-5} dx_{2i-4} \odot a(x_{2i-3}, x_{2i-2}, x_{2i-1}) \odot dx_{2i} dx_{2i+1} \in \pi \mathcal{F}_l(\Omega^{2i-2}(R)).$$

So we see that each $2i-2$ -form arises precisely as an indecomposable term in the Fedosov expansion of a term of the form

$$x_0 \odot \cdots \odot a(x_{2i-(2j+1)}, x_{2i-2j}, x_{2i-(2j-1)}) \odot \cdots \odot dx_{2i} dx_{2i+1} \in \pi \mathcal{F}_l(\Omega^{2i-2}(R)),$$

with $0 \leq j \leq i-1$. Iterating the same argument, we see that any $2i-2k$ -form arises as an indecomposable term in the Fedosov expansion of a term of the form

$$x_0 \odot \cdots \odot a(x_{2i-(2j+1)}, x_{2i-2j}, a(x_{2i-(2j-1)}, x_{2i-(2j+3)}, a(\dots))) \odot \cdots,$$

where the number of times the associator term occurs is k . And such a term must lie in $\pi^k \mathcal{F}_l(\Omega^{2i-2k}(R))$. \square

Now let $T_\alpha = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \pi^{-[j/m]+[j/\alpha]+k} \mathcal{F}_{\alpha(k+1)} \Omega^{2j}(R)$ be a subset in \mathcal{B}_m for some $\alpha \in \mathbb{N}^*$. We claim that $\pi T_\alpha^2 \subseteq T_\alpha$. The V -module πT_α^2 is generated by elements of the form

$$\pi^{-[j_1/m]-[j_2/m]+[j_2/\alpha]+[j_2/\alpha]+k_1+k_2+1} \mathcal{F}_{\alpha(k_1+1)} \Omega^{2j_1}(R) \odot \mathcal{F}_{\alpha(k_2+1)} \Omega^{2j_2}(R),$$

for $j_1, j_2, k_1, k_2 \geq 0$. By Lemma 6.2.6,

$$\begin{aligned} & \mathcal{F}_{\alpha(k_1+1)} \Omega^{2j_1}(R) \odot \mathcal{F}_{\alpha(k_2+1)} \Omega^{2j_2}(R) \\ & \subseteq \mathcal{F}_{\alpha(k_1+k_2+2)} \Omega^{2(j_1+j_2)}(R) \oplus \mathcal{F}_{\alpha(k_1+k_2+2)} \Omega^{2(j_1+j_2+1)}(R) \\ & \quad \oplus \bigoplus_{l=0}^{j_1} \pi^l \mathcal{F}_{\alpha(k_1+k_2+2)} \Omega^{2(j_1+j_2-l)}(R). \end{aligned}$$

So πT_α^2 is contained in sums of V -modules of the form

$$T_{\alpha,1} = \sum_{j_1, j_2=0}^{\infty} \sum_{k_1, k_2=0}^{\infty} \pi^{-[j_1/m]-[j_2/m]+[j_2/\alpha]+[j_2/\alpha]+k_1+k_2+1} \mathcal{F}_{\alpha(k_1+k_2+2)} \Omega^{2n}(R),$$

and

$$T_{\alpha,2} = \pi^{-[j_1/m]-[j_2/m]+[j_2/\alpha]+[j_2/\alpha]+l+k_1+k_2+1} \mathcal{F}_{\alpha(k_1+k_2+2)} \Omega^{2t}(R),$$

where $n = j_1 + j_2$ or $j_1 + j_2 + 1$, $t = j_1 + j_2 - l$, and l varies between 1 and j_1 .

If $n = j_1 + j_2$, then the estimates $-[j_1/m] - [j_2/m] \geq -[n/m]$ and $[j_1/\alpha] + [j_2/\alpha] \geq [n/\alpha]$ imply that $T_{\alpha,1} \subseteq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \pi^{-[n/m]+[n/\alpha]+k} \mathcal{F}_{\alpha(k+1)} \Omega^{2n}(R) = T_\alpha$. Similarly, if $n = j_1 + j_2 + 1$, the same estimates on the exponent of π show that

$$T_{\alpha,1} \subseteq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \pi^{-[\frac{n-1}{m}] + [\frac{n-1}{\alpha}] + k} \mathcal{F}_{\alpha(k+1)} \Omega^{2n}(R) \subseteq T_\alpha.$$

Now let $1 \leq l \leq j_1$ be fixed. Then

$$-\lfloor j_1/m \rfloor - \lfloor j_2/m \rfloor + \lceil j_1/\alpha \rceil + \lceil j_2/\alpha \rceil + l \geq -\lfloor \frac{t+l}{m} \rfloor + \lceil \frac{t+l}{\alpha} \rceil + l \geq -\lfloor \frac{t}{m} \rfloor + \lceil \frac{t}{\alpha} \rceil,$$

so $T_{\alpha,2} \subseteq \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \pi^{-\lfloor \frac{t}{m} \rfloor + \lceil \frac{t}{\alpha} \rceil + k} \mathcal{F}_{\alpha(k+1)} \Omega^{2t}(R) = T_{\alpha}$. Hence $\pi T_{\alpha}^2 \subseteq T_{\alpha}$, so \mathcal{B}_m is semi-dagger.

The ideal $\mathcal{U}(I, I^m)$ with the subspace bornology inherits bornological torsion-freeness from $(\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m)$. And, $(\mathcal{U}(I, I^m), \mathcal{B}_m) \hookrightarrow (\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m)$ is a bornological embedding. Since their quotient is an \mathbb{F} -algebra, tensoring with F yields a bornological isomorphism

$$\mathcal{U}(I, I^m), \mathcal{B}_m \otimes F \cong \mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m \otimes F.$$

By [27], the completions $\overline{\mathcal{U}(I, I^m), \mathcal{B}_m}$ and $\overline{\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m}$ are bornologically torsion-free. Hence, $\overline{\mathcal{U}(I, I^m), \mathcal{B}_m}$ and $\overline{\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m}$ both embed into

$$\overline{\mathcal{U}(I, I^m), \mathcal{B}_m} \otimes F \cong \overline{\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m} \otimes F.$$

So the canonical map $\overline{\mathcal{U}(I, I^m), \mathcal{B}_m} \rightarrow \overline{\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m}$ is a bornological embedding. Since $\overline{\mathcal{U}(I, I^m), \mathcal{B}_m}$ is complete, its image is closed, as required.

Finally, we show that the quotient bornology on $\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m)/\mathcal{U}(I, I^m)$ is the fine bornology. Any finite-dimensional subspace of A is contained in some \mathbb{F} -vector subspace of the form $\mathcal{F}_k(A)$ for some $k \in \mathbb{N}$. This lifts to the bounded V -submodule $\mathcal{F}_k(R)$ by construction. We have $\mathcal{F}_k(R) \subseteq \sum_{i=0}^{\infty} \pi^i \mathcal{F}_{k+i}(R)$, which is bounded in the ‘‘linear growth bornology’’ on R and hence in \mathcal{B}_m for each $m \geq 1$. So the quotient map $\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m) \twoheadrightarrow A$ is bounded in the fine bornology on A . Since an \mathbb{F} -algebra with the fine bornology is complete, and since the completion functor commutes with quotients, we obtain bornological isomorphisms

$$\overline{\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m} / \overline{\mathcal{U}(I, I^m), \mathcal{B}_m} \cong A,$$

completing the proof. \square

In summary, the above construction tells us that given a finitely generated \mathbb{F} -algebra A , we can construct a pro-dagger algebra $\mathcal{T}_R A := \overline{\mathcal{U}(\mathrm{TR}_{\mathrm{lg}}, I^m), \mathcal{B}_m}$ that lifts A . Furthermore, the dagger algebra lifting respects bornologies in the sense that the quotient map $\mathcal{T}_R A \twoheadrightarrow A$ is bounded with respect to the fine bornology on A .

6.3. The excision theorem. In this section, we use our theory of nilpotent extensions to show that our homology theory satisfies excision. Let

$$K \xrightarrow{i} E \xrightarrow{p} Q$$

be an extension of finitely generated \mathbb{F} -algebras. We view them as bornological V -algebras with the fine bornology. We will prove that there is a long exact sequence in analytic cyclic homology:

$$(6.3.1) \quad \begin{array}{ccccc} \mathrm{HA}_0(K) & \xrightarrow{i_*} & \mathrm{HA}_0(E) & \xrightarrow{p_*} & \mathrm{HA}_0(Q) \\ \delta \uparrow & & & & \downarrow \delta \\ \mathrm{HA}_1(Q) & \xleftarrow{p_*} & \mathrm{HA}_1(E) & \xleftarrow{i_*} & \mathrm{HA}_1(K). \end{array}$$

LEMMA 6.3.2. *Suppose we have a morphism of extensions of the form*

$$\begin{array}{ccccc} \mathcal{K} & \twoheadrightarrow & \mathcal{E} & \xrightarrow{p} & \mathcal{Q} \\ \downarrow & & \downarrow & & \downarrow \\ K & \twoheadrightarrow & E & \xrightarrow{p} & Q, \end{array}$$

where the top row is a projective system of extensions of dagger algebras. Suppose further that the algebras in the bottom row carry the fine bornology, and that the kernels of the morphisms from the top row to the bottom row are analytically nilpotent. Then there exists a long exact sequence as in Equation (6.3.1).

PROOF. By Proposition 4.2.3, the algebras in the top row compute the homology of the \mathbb{F} -algebras in the bottom row, since by hypothesis, the reduction mod π has the fine bornology. We then use Proposition 2.1.4 and the Excision Theorem ([12, Theorem 5.1]) for pro- V -algebras. \square

Now given an extension of finitely generated \mathbb{F} -algebras (6.3), we pick bases S, T and non-associative liftings $M = VS$ and $N = VT$ of E and Q in the manner described in Section 6.2. By the universal property of free modules, an \mathbb{F} -linear map $p: E \rightarrow Q$ extends uniquely to a V -module map $\hat{p}: M \rightarrow N$. If $p: E \rightarrow Q$ is bounded in the fine bornology, then since the free lifts M and N are filtration preserving, the map \hat{p} is bounded in the fine bornology on M and N . This remains true for the linear-growth bornologies on M and N . So we obtain a bounded V -module map $M \rightarrow N$ with a bounded V -module section $s: N \rightarrow M$. By functoriality of the tensor algebra, we obtain a bounded V -algebra homomorphism $\mathbb{T}M \rightarrow \mathbb{T}N$ that splits by the following V -module section

$$\sigma(x_0 dx_1 \dots dx_{2n}) := s(x_0) ds(x_1) \dots ds(x_{2n})$$

for all $x_0, x_{2n+1} \in N^+$ and $x_i \in N$ for $1 \leq i \leq 2n$. Furthermore, s maps even differential forms to even differential forms. Lemma 6.1.4 shows that they define linear maps on tube algebras $\sum_{j=0}^{\infty} \pi^{-\lfloor j/m \rfloor} \Omega^{2j}(N) \rightarrow \sum_{j=0}^{\infty} \pi^{-\lfloor j/m \rfloor} \Omega^{2j}(M)$.

LEMMA 6.3.3. *The V -module section $\sigma: \mathcal{U}(\mathbb{T}N, I_N^m) \rightarrow \mathcal{U}(\mathbb{T}M, I_M^m)$ is bounded in the bornology defined in (6.2.4).*

PROOF. Since the section $s: N \rightarrow M$ is bounded, for each k , there is an $l \in \mathbb{N}$ with $s(\mathcal{F}_k(N)) \subseteq \mathcal{F}_l(M)$. Let $\omega = x_0 dx_1 \dots dx_{2j} \in \mathcal{F}_k(\Omega^{2j}(N))$. Then $\sigma(\omega) = s(x_0) d(s(x_1)) \dots d(s(x_{2j})) \in \mathcal{F}_{l_i} d(\mathcal{F}_{l_1}(M)) \dots d(\mathcal{F}_{l_{2j}}(M)) \subseteq \mathcal{F}_l(\Omega^{2j}(M))$, for $l = l_0 + \dots + l_{2j}$. So

$$\sigma\left(\sum_j \sum_k \pi^{-\lfloor j/m \rfloor + k} \mathcal{F}_k \Omega^{2j}(N)\right) \subseteq \sum_l \pi^{-\lfloor j/m \rfloor + l} \mathcal{F}_l \Omega^{2j}(M).$$

\square

By Lemma 6.3.3, the section σ extends to completions and yields a section for the canonical bounded algebra homomorphism $\rho: \mathcal{T}_M E \twoheadrightarrow \mathcal{T}_N Q$.

What we have therefore proved is the following:

PROPOSITION 6.3.4. *An extension $K \twoheadrightarrow E \twoheadrightarrow Q$ of finitely generated \mathbb{F} -algebras with the fine bornology extends to a semi-split extension of pro-dagger algebras*

$$\begin{array}{ccccc}
\ker(\mathcal{T}_M E \twoheadrightarrow \mathcal{T}_N Q) & \twoheadrightarrow & \mathcal{T}_M E & \xrightarrow{\rho} & \mathcal{T}_N Q \\
\downarrow & & \downarrow & & \downarrow \\
K & \twoheadrightarrow & E & \xrightarrow{p} & Q.
\end{array}$$

Furthermore, the quotient maps from the dagger algebras $\ker(\mathcal{T}_M E \twoheadrightarrow \mathcal{T}_N Q)$, $\mathcal{T}_M E \twoheadrightarrow E$ and $\mathcal{T}_N Q \twoheadrightarrow Q$ are bounded with respect to the fine bornologies on K , E and Q .

PROOF. By Proposition 6.2.5, for any \mathbb{F} -algebra A with the fine bornology, there is a bornological quotient map from the dagger algebra $\mathcal{T}_R A$ to A , where R is a filtered non-associative lifting. This, coupled with the discussion preceding this theorem, shows that any bounded \mathbb{F} -algebra homomorphism $E \rightarrow Q$ lifts to a bounded V -algebra homomorphism $\mathcal{T}_M E \twoheadrightarrow \mathcal{T}_N Q$. Finally, the kernel $\mathcal{T}_M E \twoheadrightarrow \mathcal{T}_N Q$ is a pro-subalgebra of a pro-dagger algebra and is hence a dagger algebra. The induced map to K is bounded in the fine bornology. Finally, it splits since the pro-set section $E \rightarrow \mathcal{T}_M E$ comes from a set-theoretic section $E \rightarrow M$, which in turn restricts to a section $K \rightarrow \ker(M \rightarrow N) \rightarrow \ker(\mathcal{T}_M E \twoheadrightarrow \mathcal{T}_N Q)$. \square

By Lemma 6.3.2, the lifted pro-algebras $\ker(\mathcal{T}_M E \twoheadrightarrow \mathcal{T}_N Q)$, $\mathcal{T}_M E$ and $\mathcal{T}_N Q$ are all admissible liftings of K , E and Q , that is, they compute their analytic cyclic homologies. This completes the proof of excision for analytic cyclic homology for finitely generated \mathbb{F} -algebras.

REMARK 6.3.5. The methods in this section should also help us prove excision for extensions of countably generated \mathbb{F} -algebras. This would entail building a non-associative lifting R from a countable generating set for the \mathbb{F} -algebra A we start with. The lifting R should then be an inductive limit of finitely generated non-associative algebras, for which we can again build tube-tensor algebras with explicit semi-dagger bornologies. However, we do not treat this more general version of excision here.

Conclusion and outlook

The results in this thesis achieve three main things. First, they formalise Monsky and Washnitzer’s weak completion using bornologies, which is the framework for our homology theories. Second, we construct a homology theory that satisfies certain desirable formal properties. Third, we provide computations of our homology theory for certain classes of algebras that are interesting in non-commutative geometry and algebraic geometry. In particular, for algebras corresponding to smooth 1-dimensional varieties over a finite field, our homology theory agrees with rigid cohomology.

1. What is still missing?

The main result of [11] shows that for finitely generated, commutative \mathbb{F} -algebras, periodic cyclic homology coincides with rigid cohomology. In [12], we have shown that for smooth curves (that is, one-dimensional affine varieties), our theory coincides with periodic cyclic homology and hence, rigid cohomology. This is because in Hochschild cohomological dimension 1, smooth algebras are the same as quasi-free algebras, for which the X -complex computes the analytic and the periodic cyclic theories.

In general, if R is a smooth commutative V -algebra, it has Hochschild cohomological dimension n , for some $n \in \mathbb{N}$. Using standard results from homological perturbation theory, one can then show that a certain projection

$$\mathbb{H}\mathbb{P}(R^\dagger \overline{\otimes} F) \twoheadrightarrow X^n(R^\dagger \overline{\otimes} F)$$

of the periodic cyclic complex is a chain homotopy equivalence. We refer the reader to [26, Definition A.122, Theorem A.123] for the definition of the X^n -complex and a proof of this result. Viewing the analytic chain complex as the $b - B$ -bicomplex $(\mathbb{H}\mathbb{P}, b + B)$, with a certain *analytic bornology*, we can show that there is a bounded projection map

$$\mathbb{H}\mathbb{A}(R^\dagger) \rightarrow X^n(R^\dagger \overline{\otimes} F) \sim \mathbb{H}\mathbb{P}(R^\dagger \overline{\otimes} F).$$

For Banach algebras over \mathbb{C} with the von Neumann bornology, results by Masoud Khalkhali (see [22, Theorem 2.50]) can be used to show that if A has finite cohomological dimension n , then $\mathbb{H}\mathbb{A}(A) \twoheadrightarrow X^n(A)$ is a chain homotopy equivalence. The proof of this result uses an explicit formula for the contracting homotopy of the kernel of the projection map $\mathcal{F}_n(\mathbb{H}\mathbb{P}(A)) = \ker(\mathbb{H}\mathbb{P}(A) \twoheadrightarrow X^n(A))$. However, in our case, it is not obvious why such contracting homotopies should be bounded in the analytic bornology on the complex $\mathbb{H}\mathbb{A}(R^\dagger)$. The comparison of the analytic and periodic cyclic theories in the non-Archimedean setting is therefore an important issue that this thesis has not resolved in full generality.

2. Some future directions

We now discuss some future projects that I plan to pursue, using the tools developed in this thesis.

2.1. More on liftings mod π . The main result in Chapter 3 says that if D is any dagger algebra lifting, then $\mathrm{HA}_*(D) \cong \mathrm{HA}_*(D/\pi D)$, provided $D/\pi D$ has the fine bornology. An important question now is how such liftings arise and are related to π -adic completions, which seem more ‘canonical’¹.

Let R be a π -adically complete V -algebra and let $A := R/\pi R$. We give R the *compactoid bornology*: a subset $S \subseteq R$ is bounded if and only if for every $n \in \mathbb{N}$, there is a finite set $F \subseteq R$ such that $S \subseteq F + \pi^n R$. This bornology appears in connection with the work of Bambozzi, Ben Bassat and Kremnitzer on derived analytic geometry (see [7]). It can then be shown that R with the compactoid bornology is a dagger algebra. Furthermore, the analysis results from Section 4.1 yield the following:

LEMMA. *Let M be a complete, torsion-free bornological V -module. Then $M/\pi M$ has the fine bornology if and only if the bornology on M is compactoid, that is, for any bounded submodule $N \subseteq M$, there is a bounded compactoid submodule N' that contains N .*

The lemma above can be used to feed a π -adically complete bornological algebra R with the compactoid bornology into the statement of Proposition 2.1.8 to yield $\mathrm{HA}_*(R) \cong \mathrm{HA}_*(A)$. So if R is any reasonable V -algebra lifting mod π of A , and A has the fine bornology, then we expect the following:

THEOREM. *With R and A as above, we have*

$$\mathrm{HA}_*(R^\dagger) \cong \mathrm{HA}_*(A) \cong \mathrm{HA}_*(\widehat{R}).$$

Another result we expect due to Proposition 2.1.8 is the invariance of analytic cyclic homology under homotopies $A \rightarrow B \overline{\otimes} \widehat{V[t]}$, where $\widehat{V[t]}$ has the compactoid bornology and A and B are dagger algebras whose reductions mod π have the fine bornology. This should hold because of the following:

THEOREM. *Let B be a dagger algebra whose reduction mod π has the fine bornology. Then $\mathrm{HA}_*(B) \cong \mathrm{HA}_*(B \overline{\otimes} \widehat{V[t]})$.*

PROOF. The proof follows from a string of isomorphisms:

$$\begin{aligned} \mathrm{HA}_*(B \overline{\otimes} \widehat{V[t]}) &\cong \mathrm{HA}_*(B \overline{\otimes} \widehat{V[t]}/\pi(B \overline{\otimes} \widehat{V[t]})) \\ &\cong \mathrm{HA}_*(B/\pi B \otimes \mathbb{F}[t]) \cong \mathrm{HA}_*(B/\pi B) \cong \mathrm{HA}_*(B). \end{aligned}$$

□

These results will be added to the manuscript of Chapter 3.

¹I am thankful to the organisers for inviting me to the Oberwolfach Workshop *Non-commutative geometry and cyclic homology*, where these ideas were first discussed.

2.2. Product structures in non-Archimedean cyclic homology. Here we consider the behaviour of analytic cyclic homology with tensor products. In periodic cyclic homology, and archimedean versions of analytic and local cyclic homology, such results are due to Michael Puschnigg. Briefly, if \mathbb{H} denotes any of the chain complexes of any of these theories, we have under certain conditions chain homotopy equivalences

$$\mathbb{H}(A \otimes B) \sim \mathbb{H}(A) \otimes \mathbb{H}(B),$$

that are associative, symmetric and monoidal in a suitable sense. If $\mathbb{H} = \mathbb{H}\mathbb{P}$, then the above holds for any pro-algebra A and B . But for $\mathbb{H}\mathbb{A}$, we need that the algebras A and B are locally multiplicative in the sense that they are direct limits of semi-normed subalgebras. Such results are used to construct an *exterior product* operation on bivariate versions of these theories.

In the non-archimedean theory for torsion-free V -algebras, I expect a similar result to hold, stated as follows:

THEOREM. *Let A and B be complete, bornologically torsion-free unital V -algebras. Then there is a chain homotopy equivalence $\mathbb{H}\mathbb{A}(A \overline{\otimes} B) \sim \mathbb{H}\mathbb{A}(A) \overline{\otimes} \mathbb{H}\mathbb{A}(B)$.*

To see why this could be true, we first observe that the identity map on the algebra $A \overline{\otimes} B$ extends to a morphism $\mathcal{T}(A \overline{\otimes} B) \rightarrow \mathcal{T}(A) \overline{\otimes} \mathcal{T}(B)$ of analytically nilpotent extensions. The proof that the kernel of $\mathcal{T}A \overline{\otimes} \mathcal{T}B \rightarrow A \overline{\otimes} B$ is analytically nilpotent uses [12, Proposition 4.2.5]. It is precisely here that the assumption of local multiplicativity is used in analytic cyclic homology for \mathbb{C} -algebras. So invariance under analytically nilpotent extensions gives

$$\mathbb{H}\mathbb{A}(A \overline{\otimes} B) \sim \mathbb{H}\mathbb{A}(\mathcal{T}A \overline{\otimes} \mathcal{T}B).$$

Now a typical issue arises involving homological dimensions. In the entire theory developed in this thesis, analytic cyclic homology is mostly only easily computable in cohomological dimension 1. However, $\mathcal{T}A \overline{\otimes} \mathcal{T}B$ in general has cohomological dimension 2. In dimension 2, periodic cyclic homology is chain homotopy equivalent to a certain quotient of it, namely the X^2 -complex, which also came up in the proof of homotopy invariance of our theory (see Proposition 4.6.1). So the task is to show that there is a chain homotopy equivalence

$$\mathbb{H}\mathbb{A}(\mathcal{T}A \overline{\otimes} \mathcal{T}B) \sim X^2(\mathcal{T}A \otimes \mathcal{T}B) \overline{\otimes} F.$$

In the Archimedean case, Puschnigg (see [31]) constructs certain explicit maps at the level of the tensor algebras, that implement the chain homotopy equivalence above. We hope that similar formulas can also be used in our setting. Suppose we can adapt Puschnigg's methods, the rest of the proof goes as expected:

$$X^2(\mathcal{T}A \overline{\otimes} \mathcal{T}B) \overline{\otimes} F \sim X(\mathcal{T}A \overline{\otimes} F) \overline{\otimes} X(\mathcal{T}B \overline{\otimes} F) = \mathbb{H}\mathbb{A}(A) \overline{\otimes} \mathbb{H}\mathbb{A}(B).$$

2.3. Equivariant non-Archimedean periodic cyclic homology. An important motivation for the introduction of our homology theory is to study of group(oid) actions on affine varieties over a finite field. The orbit spaces of such actions are studied through crossed product algebras, which are noncommutative, and fit within the scope of application of our homology theory. In this realm, we wish to develop a non-Archimedean version of equivariant periodic cyclic homology $\mathbb{H}\mathbb{P}^G$ in the sense of Christian Voigt [36]. We propose to investigate under what conditions, the following version of the Green-Julg Theorem holds:

THEOREM. *Let G be a discrete group acting on a dagger algebra A . Then*

$$\mathrm{HP}_*^G(V, A) \cong \mathrm{HA}_*(G \rtimes A).$$

In more geometric situations, A is the dagger completion of the coordinate ring of say a smooth, affine variety X . The idea would then be to relate these equivariant invariants to the more geometric, de Rham type invariants such as the rigid cohomology of the orbit space X/G .

2.4. Topological cyclic homology and final insights. An important development alongside ‘classical’ cyclic homology is *topological Hochschild homology* and its variants. These invariants are constructed by working over the sphere spectrum, instead of the derived category of \mathbb{Z} -modules. While topological versions of cyclic homology coincide with the classical versions in the Archimedean setting, their effect is most remarkable in the non-Archimedean setting. The pioneering work of Hesselholt and more recent work by Scholze-Nikolaus ([20]) and Petrov-Vologodsky ([29]) establishes connections between topological periodic cyclic homology and crystalline cohomology. However the methods used in the topological theories seem drastically different from our more concrete, analytical methods, which are better suited to the examples we find important. It is therefore very desirable to find connections between the two worlds. One possible approach could be the development of a bornological version of topological cyclic homology - a project I plan to take up in the future with Kobi Kremnitzer. Furthermore, I would also be interested in studying *condensed mathematics* developed by Scholze and Clausen, as a framework in which cyclic homology can be defined for topological algebras.

Bibliography

- [1] Gene Abrams, Pere Ara, and Mercedes Siles Molina, *Leavitt path algebras*, Lecture Notes in Mathematics, vol. 2191, Springer, London, 2017. doi: 10.1007/978-1-4471-7344-1MR **3729290**
- [2] William W. Adams and Philippe Loustaunau, *An introduction to Gröbner bases*, Graduate Studies in Mathematics, vol. 3, Amer. Math. Soc., Providence, RI, 1994. doi: 10.1090/gsm/003MR **1287608**
- [3] Alberto Arabia, *Relèvements des algèbres lisses et de leurs morphismes*, Comment. Math. Helv. **76** (2001), no. 4, 607–639, doi: 10.1007/s00014-001-8322-y. MR **1881700**
- [4] Michael Artin and Barry Mazur, *Etale homotopy*, Lecture Notes in Mathematics, vol. 100, Springer, 1969. MR **0245577**
- [5] Federico Bambozzi, *On a generalization of affinoid varieties*, Ph.D. Thesis, Università degli Studi di Padova, 2014. arXiv: 1401.5702.
- [6] Federico Bambozzi and Oren Ben-Bassat, *Dagger geometry as Banach algebraic geometry*, J. Number Theory **162** (2016), 391–462, doi: 10.1016/j.jnt.2015.10.023. MR **3448274**
- [7] Federico Bambozzi, Oren Ben-Bassat, and Kobi Kremnizer, *Stein domains in Banach algebraic geometry*, J. Funct. Anal. **274** (2018), no. 7, 1865–1927, doi: 10.1016/j.jfa.2018.01.003. MR **3762090**
- [8] Theo Bühler, *Exact categories*, Expo. Math. **28** (2010), no. 1, 1–69, doi: 10.1016/j.exmath.2009.04.004. MR **2606234**
- [9] Alain Connes, *Entire cyclic cohomology of Banach algebras and characters of θ -summable Fredholm modules*, K-Theory **1** (1988), no. 6, 519–548, doi: 10.1007/BF00533785. MR **953915**
- [10] Guillermo Cortiñas, *On the derived functor analogy in the Cuntz–Quillen framework for cyclic homology*, Algebra Colloq. **5** (1998), no. 3, 305–328. MR **1679567**
- [11] Guillermo Cortiñas, Joachim Cuntz, Ralf Meyer, and Georg Tamme, *Nonarchimedean bornologies, cyclic homology and rigid cohomology*, Doc. Math. **23** (2018), 1197–1245, doi: 10.25537/dm.2018v23.1197-1245. MR **3874948**
- [12] Guillermo Cortiñas, Ralf Meyer, and Devarshi Mukherjee, *Non-Archimedean analytic cyclic homology* (2019), eprint. arXiv: 1912.09366.
- [13] Guillermo Cortiñas and Diego Montero, *Algebraic bivarient K-theory and Leavitt path algebras*, J. Noncommut. Geom., accepted. arXiv: 1806.09204.
- [14] Guillermo Cortiñas and Andreas Thom, *Bivarient algebraic K-theory*, J. Reine Angew. Math. **610** (2007), 71–123, doi: 10.1515/CRELLE.2007.068. MR **2359851**
- [15] Guillermo Cortiñas and Christian Valqui, *Excision in bivarient periodic cyclic cohomology: a categorical approach*, K-Theory **30** (2003), no. 2, 167–201, doi: 10.1023/B:KTHE.0000018383.93721.dd. MR **2064238**
- [16] Joachim Cuntz and Daniel Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289, doi: 10.1090/S0894-0347-1995-1303029-0. MR **1303029**
- [17] ———, *Cyclic homology and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 373–442, doi: 10.1090/S0894-0347-1995-1303030-7. MR **1303030**
- [18] Renée Elkik, *Solutions d’équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 553–603 (1974), available at http://www.numdam.org/item=ASENS_1973_4_6_4_553_0. MR **0345966**
- [19] Elmar Große-Klönne, *De Rham cohomology of rigid spaces*, Math. Z. **247** (2004), no. 2, 223–240, doi: 10.1007/s00209-003-0544-9. MR **2064051**
- [20] Lars Hesselholt and Peter Scholze, *Arbeitsgemeinschaft: Topological Cyclic Homology*, Oberwolfach Reports **15** (2019), no. 2, 805–940.
- [21] Bernhard Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), no. 4, 379–417, doi: 10.1007/BF02568439. MR **1052551**

- [22] Masoud Khalkhali, *Algebraic connections, universal bimodules and entire cyclic cohomology*, Comm. Math. Phys. **161** (1994), no. 3, 433–446, available at <https://projecteuclid.org/euclid.cmp/1104270005>. MR **1269386**
- [23] Joachim Lambek, *Lectures on rings and modules*, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1966. With an appendix by Ian G. Connell. MR **0206032**
- [24] Ralf Meyer, *Analytic cyclic cohomology*, Ph.D. Thesis, Westf. Wilhelms-Universität Münster, 1999. arXiv: math.KT/9906205.
- [25] ———, *Excision in entire cyclic cohomology*, J. Eur. Math. Soc. (JEMS) **3** (2001), no. 3, 269–286, doi: 10.1007/s100970100035. MR **1848947**
- [26] ———, *Local and analytic cyclic homology*, EMS Tracts in Mathematics, vol. 3, European Mathematical Society (EMS), Zürich, 2007. doi: 10.4171/039 MR **2337277**
- [27] Ralf Meyer and Devarshi Mukherjee, *Dagger completions and bornological torsion-freeness*, Q. J. Math. **70** (2019), no. 3, 1135–1156, doi: 10.1093/qmath/haz012. MR **4009486**
- [28] Paul Monsky and G. Washnitzer, *Formal cohomology. I*, Ann. of Math. (2) **88** (1968), 181–217, doi: 10.2307/1970571. MR **0248141**
- [29] Alexander Petrov and Vadim Vologodsky, *On the periodic topological cyclic homology of DG categories in characteristic p* (2019), eprint. arXiv: 1912.03246.
- [30] Fabienne Prosmans, *Derived limits in quasi-abelian categories*, Bull. Soc. Roy. Sci. Liège **68** (1999), no. 5-6, 335–401, available at <http://popups.ulg.ac.be/0037-9565/index.php?id=2128>. MR **1743618**
- [31] Michael Puschnigg, *Explicit product structures in cyclic homology theories*, K-Theory **15** (1998), no. 4, 323–345, doi: 10.1023/A:1007748727044. MR **1656222**
- [32] Marius van der Put, *The cohomology of Monsky and Washnitzer*, Mém. Soc. Math. France (N.S.) **23** (1986), 4, 33–59, available at http://www.numdam.org/item?id=MSMF_1986_2_23_33_0. Introductions aux cohomologies p -adiques (Luminy, 1984). MR **865811**
- [33] Emily Riehl, *Categories in context*, Dover, 2017.
- [34] Peter Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. doi: 10.1007/978-3-662-04728-6 MR **1869547**
- [35] Jean-Pierre Schneiders, *Quasi-abelian categories and sheaves*, Mém. Soc. Math. Fr. (N.S.) **76** (1999), vi+134, doi: 10.24033/msmf.389. MR **1779315**
- [36] Christian Voigt, *Equivariant periodic cyclic homology*, J. Inst. Math. Jussieu **6** (2007), no. 4, 689–763, doi: 10.1017/S1474748007000102. MR **2337312**