The Barban-Davenport-Halberstam Theorem for tuples of k-free numbers

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vorgelegt von **Tomos Parry** aus Llanberis

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Betreuungsausschuss

Prof. Jörg Brüdern, Mathematisches Institut Göttingen

Prof. Valentin Blomer, Mathematisches Institut Bonn

Mitglieder der Prüfungskommission

Referent: Prof. Jörg Brüdern, Mathematisches Institut Göttingen

Koreferent: Prof. Valentin Blomer, Mathematisches Institut Bonn

Weitere Mitglieder der Prüfungskommission

Prof. Preda Mihăilescu, Mathematisches Institut Göttingen

Prof. Thomas Schick, Mathematisches Institut Göttingen

Prof. Gerlind Plonka-Hoch, Institut für Numerische und Angewandte Mathematik

Prof. Tatyana Krivobokova, Institut für Mathematische Stochastik

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Contents

Thesis Overview

Denote by $\pi(x)$ the number of primes up to x and by $\pi_x(q,a)$ the number of those lying in a given arithmetic progression $(a+nq)_{n\geq 0}$. Denote by $\phi(n)$ Euler's phi function.

Counting prime numbers has long been a large part of number theory. Euclid already documented (around 300BC) that they were infinite in number, since if not then one may be added to the product of the finitely many primes $p_1,...,p_R$, a list of all the primes, to obtain a new number

$$N = p_1 \cdots p_R + 1$$

which is divisible by none of the $p_1,...,p_R$, a contradiction. Quite some time later on, Gauss conjectured (1792/3, so in his teenage years) on the basis of his per hand calculations that up to a given x there are approximately $x/\log x$ primes, and this was confirmed a century later independently by Hadamard and Vallée Poussin, the method laid out by Riemann in 1859, when he connected the distribution of the primes with the complex analytic properties of his zeta function $\zeta(s)$; by Perron's formula

$$\sum_{\substack{n\leq x \\ n \text{ is a power} \\ \text{of a prime } p}} \log p = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta'(s)x^s ds}{\zeta(s)s}.$$

Today the approximation is known to be of strength

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}\left(xe^{-c\sqrt{\log x}}\right)$$

where

$$\mathrm{li}(x) = \int_2^x \frac{du}{\log u}$$

but there is no $\delta > 0$ for which we know

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}\left(x^{1-\delta}\right). \tag{1}$$

In expectation that the primes be randomly distributed there should be nothing better than "square root cancellation" and indeed Littlewood showed (1914)

$$\pi(x) = \mathrm{li}(x) + \Omega\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right).$$

Establishing (1) for some $\delta \in (0,1/2]$ is in the number theory community accepted as out of reach (the validity of $\delta = 1/2$, up to logarithms, amounts to the Riemann Hypothesis), and therefore we settle for "average type" results, where this averaging is done over arithmetic progressions. The first result in this context is a famous theorem by Dirichlet which asserts, for given coprime

q and a, that there are infinitely many primes congruent to a modulo q; this he established with the introduction of his characters

$$\sum_{\substack{n\leq x\\ n\equiv a(q)\\ n \text{ is a power}\\ \text{of a prime } p}} \log p = \frac{1}{\phi(q)}\sum_{\chi}\overline{\chi(a)} \sum_{\substack{n\leq x\\ n \text{ is a power}\\ \text{of a prime } p}} \chi(n)\log p$$

and the analytic properties of their corresponding L-functions

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

In fact for any $A > 0$, some $c = c(A) > 0$ and any $q \leq (\log x)^A$ we have the Siegel-Walfisz Theorem

$$\pi_x(q,a) = \frac{\text{li}(x)}{\phi(q)} + \mathcal{O}_A\left(xe^{-c\sqrt{\log x}}\right).$$

The problem of counting primes restricted to a given arithmetic progression no easier than counting all of them, the above discussion about the size of the error term still applies, however now we can ask what happens when we vary q and a. Define

$$E_x(q,a) = \sum_{\substack{n\leq x\\ n \text{ is a power}\\ \text{of a prime } p}} \log p - \frac{x}{\phi(q)}.$$

One of the major theorems in 20th century number theory is the Bombieri-Vinogradov Theorem (1965):

$$\sum_{q\leq Q}\max_{\substack{1\leq a\leq q\\ (a,q)=1}}\max_{y\leq x}|E_y(q,a)| \ll \sqrt{x}Q(\log x)^5,$$

for any $A > 0$ and $\sqrt{x}/(\log x)^A \leq Q \leq \sqrt{x}$. Shortly after (1966) came the Barban-Davenport-Halberstam Theorem:

$$\sum_{q\leq Q}\sum_{\substack{a=1\\ (a,q)=1}}^{q}|E_x(q,a)|^2 \ll xQ\log x + \frac{x^2}{(\log x)^A}$$

for $Q \leq x$. This was improved to an asymptotic formula by Montgomery and Hooley (1970s): for any $A > 0$ and some $c \in \mathbb{R}$

$$\sum_{q\leq Q}\sum_{\substack{a=1\\ (a,q)=1}}^{q}|E_x(q,a)|^2 = xQ\log Q + cxQ + \mathcal{O}\left(x^{3/4}Q^{5/4} + \frac{x^2}{(\log x)^A}\right). \qquad (2)$$

These results are important because they are suggested by the truth of

$$E_x(q,a) \approx \sqrt{\frac{x}{q}}.$$

Yet more insight into the primes might be gained by looking at their arithmetical patterns, say tuples $n+h_1, ..., n+h_r$, for some fixed non-negative integers $0 \leq h_1 < h_2 < \cdots < h_r$, such that the $n+h_i$ are all prime. But unfortunately this strategy already fails at Euclid's Theorem, since it isn't (yet) known that there are infinitely many pairs of primes $n, n+2$, never mind the Hardy-Littlewood Conjecture - this states that for some $\mathfrak{S}_{\mathbf{h}} \in \mathbb{R}$

$$\sum_{\substack{n \leq x \\ n+h_i \text{ all prime}}} 1 \sim \frac{\mathfrak{S}_{\mathbf{h}} x}{(\log x)^r}.$$

It is perhaps a natural question whether such motives are present when we discuss sequences other than the primes. Let $k \geq 2$. If for a given n there is no prime p for which $p^k | n$ then n is said to be k-free. An asymptotic formula of similar shape to that in (2) is indeed known for the k-free numbers, the current state of knowledge attained and summarised by Vaughan in [17]. Crucially, the corresponding questions on r-tuples of k-free numbers *are* accessible; for given non-negative integers $0 \leq h_1 < h_2 < \cdots < h_r$ we call $n+h_1, ..., , n+h_r$ a k-free r-tuple if the $n+h_i$ are all k-free. The asymptotic count

$$\sum_{n \leq x} \mu_k(n+h_1) \cdots \mu_k(n+h_r) = \mathfrak{S}_{\mathbf{h}} x + \mathcal{O}\left(x^{2/(k+1)+\epsilon}\right),$$

for some $\mathfrak{S}_{\mathbf{h}} \in \mathbb{R}$, is easily established (see [10]) and we look at the distribution in arithmetic progressions. In [2] twins in arithmetic progressions were investigated and it was shown easily that

$$\sum_{\substack{n \leq x \\ n \equiv a(q)}} \mu_k(n)\mu_k(n+1) = \eta(q,a)x + \mathcal{O}\left(x^{2/(k+1)+\epsilon}\right).$$

for some $\eta(q,a) \in \mathbb{R}$, and that

$$V(x,Q) := \sum_{q \leq Q} \sum_{a=1}^{q} \left| \sum_{\substack{n \leq x \\ n \equiv a(q)}} \mu_k(n)\mu_k(n+1) - \eta(q,a)x \right|^2 \ll Q^2 \left(\frac{x}{Q}\right)^{2/k}.$$

In [9] the method of Vaughan (that in [17]) is followed to show that

$$V(x,Q) \ll Q^2 \left(\frac{x}{Q}\right)^{1/k+\epsilon} + x^{1+2/k} \log Q + x^{3/2+1/2k+\epsilon}.$$

These results are important because the same results for primes are out of reach.

As far as we can see, however, there is no recorded asymptotic formula for the variance of twins of k-free numbers. The main result of this thesis is an asymptotic formula for the variance

$$\sum_{q \leq Q} \sum_{a=1}^{q} \left| \sum_{\substack{n \leq x \\ n+h_i \ k\text{-free} \\ n \equiv a(q)}} 1 - \eta_{\mathbf{h}}(q,a)x \right|^2$$

of k-free r-tuples; here $\eta_{\mathbf{h}}(q,a)$ is a suitable main term. We will state the precise result shortly and it is the content of Chapter 2. It seems to be the first instance of an asymptotic formula for this variance. We follow closely Vaughan's argument in [17].

The above variance is somtimes referred to as the "improper" variance. We will also look at the "proper" variance

$$\sum_{a=1}^{q}\left|\sum_{\substack{n\leq x\\ n\ k\text{-free}\\ n\equiv a(q)}} 1-\eta(q,a)x\right|^2$$

of k-free numbers; that is, averaging just over the residues. An asymptotic formula was found (when averaging over the reduced classes) for squarefrees by Ramon Nunes [12], and we improve one of the error terms (when averaging over all the classes). This result is the content of Chapter 3.

We now state these results precisely.

Theorem 1. *Fix non-negative integers $0 \leq h_1 < h_2 < \cdots < h_r$ and let*

$$\mathcal{S} = \{n \in \mathbb{N} | n + h_i \in \{\ k\text{-frees}\ \},\ \ i = 1, ..., r\}. \tag{3}$$

Let for $q, a \in \mathbb{N}$ and $Q, x > 0$

$$\eta(q,a) = \sum_{\substack{d_1,\dots,d_r=1\\ (d_i^k,d_j^k)|h_i-h_j\ (1\leq i,j\leq r)\\ (q,d_i^k)|a+h_i\ (1\leq i\leq r)}}^{\infty} \frac{\mu(d_1)\cdots\mu(d_r)}{[q,d_1^k,...,d_r^k]}, \qquad E_x(q,a) = \sum_{\substack{n\leq x\\ n\in\mathcal{S}\\ n\equiv a(q)}} 1 - x\eta(q,a) \tag{4}$$

and

$$V(x,Q) = \sum_{q\leq Q}\sum_{a=1}^{q} E_x(q,a)^2. \tag{5}$$

Take $\mathfrak{c} = \mathfrak{c}(r)$ to be any number in $[1/2, 1)$ for which we know

$$\int_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^r dt}{(1+|t|)^{3/2}}$$

converges absolutely for all $\sigma \geq \mathfrak{c}$. For each prime p write R_p for the number of different residues represented by the $h_1, ..., h_r$ modulo p^k. If always $R_p < p^k$ then for $Q \leq x$ and $\epsilon > 0$

$$V(x,Q) = Q^2\left(\frac{x}{Q}\right)^{1/k} P\Big(\log(x/Q)\Big) + \mathcal{O}_{k,r,\mathbf{h},\epsilon}\left(Q^2\left(\frac{x}{Q}\right)^{\mathfrak{c}(r)/k} + x^{1+2/(k+1)+\epsilon}\right)$$

where $P = P(r,k,\mathbf{h})$ is a polynomial of degree $r-1$.

As already mentioned the only theorems in this direction, that we know of, are upper bound result for twins of squarefree numbers. In this case we can take $\mathfrak{c} = 1/2$ since it is contained in classical results that for $\sigma \geq 1/2$

$$\int_T^{2T} |\zeta(\sigma+it)|^2 dt \ll_\sigma T\log T$$

and so our theorem then implies

$$V(x,Q) = x^{1/2}Q^{3/2}P\Big(\log(x/Q)\Big) + \mathcal{O}_\epsilon\left(x^{1/4}Q^{7/4} + x^{5/3+\epsilon}\right)$$

for some linear function P. Of course if the h_i cover a complete residue system modulo some p^k then there are no k-free r-tuples.

Theorem 2. *Let $k \geq 2$ and denote by $\mathcal{S}$ the set of k-free numbers. For $q, a \in \mathbb{N}$ define*

$$\eta(q,a) = \sum_{\substack{d=1 \\ (q,d^k)|a}}^{\infty} \frac{\mu(d)}{[q,d^k]}, \qquad E_x(q,a) = \sum_{\substack{n \leq x \\ n \in \mathcal{S} \\ n \equiv a(q)}} 1 - x\eta(q,a) \tag{6}$$

and for $x > 0$

$$V_x(q) = \sum_{a=1}^{q} |E_x(q,a)|^2. \tag{7}$$

Define

$$C_k = \frac{2k}{(1/k-1)\zeta(2)} \prod_p \frac{1 - 2/(p^k + p^{k-1})}{1 - p^{1-1/k}}$$

and

$$f_k(q) = C_k \prod_{p|q} \frac{1 - 2/p^k + (q,p^k)^{1/k-1}/p}{1 - 2/p^k + 1/p}.$$

For large x and for $q \leq x$ we have for every $\epsilon > 0$

$$V_x(q) = q\left(\frac{x}{q}\right)^{1/k} f_k(q) + \mathcal{O}_{k,\epsilon}\left(x^{\epsilon}\left(q\left(\frac{x}{q}\right)^{2/(9-2/k)} + \frac{x^{1+2/(k+1)}}{q}\right)\right).$$

This is an asymptotic formula for $k = 2, 3, 4$. Averaging just over the reduced classes an asymptotic formula for $V_x(q)$, in the squarefree case, is already established in [12] with error essentially

$$\ll q\left(\frac{x}{q}\right)^{1/3} + \left(\frac{x}{q}\right)^{23/15}.$$

Before this only upper bound results are recorded (see [8] and the references therein), although these are stronger in the range where the above asymptotic formulas don't hold and are concerned with more general sequences than the squarefrees. The relevance of our result is the improvement in the first error term, which for $k = 2$ seems decently small. This is obtained by a careful analysis of the integrals arising from an application of Perron's formula. (Our second error term is weaker than in [12] but most likely can be made to be just as small for the squarefress by arguing, as in that paper, with the square sieve.)

Chapter 1

Number theoretical sequences in arithmetic progressions

By a "number theoretical sequence" we are being rather vague - we mean simply a sequence that would be of interest to a number theorist. For clarity, let's just say a number theoretical sequence is a subset of $\mathbb{N}$.

Let $\mathcal{N}$ denote a number theoretical sequence and suppose $\mathcal{N}$ satisfies for some $\eta : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$

$$\sum_{\substack{n \le x \\ n \in \mathcal{N} \\ n \equiv a(q)}} 1 = \eta(q,a)x + o(x) \tag{8}$$

with $x \to \infty$. Of course, the question of uniformity in q and a is an important one but, as a minimum, we should require that the above is true for each fixed q and a if we are to study the average behaviour over arithmetic progressions. Define

$$E_x(q,a) = \sum_{\substack{n \le x \\ n \in \mathcal{N} \\ n \equiv a(q)}} 1 - \eta(q,a)x \qquad \text{and} \qquad V(x,Q) = \sum_{q \le Q} \sum_{a=1}^{q} |E_x(q,a)|^2.$$

We have already alluded to the relevance of the Barban-Davenport-Halberstam (BDH) Theorem and to the question of when $\mathcal{N}$ obeys a BDH type law. Hooley asks this question in [5] and answers

$$V(x,Q) \ll_A xQ + \frac{x^2}{(\log x)^A}$$

for each $A > 0$, whenever

$$E_x(q,a) \ll_A \frac{x}{(\log x)^A}$$

and η satisfies

$$\eta(q,a) = \eta\left(q,(q,a)\right); \tag{9}$$

in [7] he removes this condition on η. Of course, should a better approximation in arithmetic progressions hold, this would be reflected in the second error term. As with (the shorter proofs of) the Bombieri-Vinogradov Theorem, Hooley's result was made possible with the large sieve: if we define

$$W(\alpha) = \sum_{\substack{n \le x \\ n \in \mathcal{N}}} e(\alpha n)$$

then the Large Sieve Inequality says

$$\sum_{q \le Q} {\sum_{a=1}^{q}}' |W(a/q)|^2 \ll x\Big(x + Q^2\Big); \tag{10}$$

here and in what follows we write Σ'^q to mean the sum is taken over $a = 1, ..., q$ coprime to q. For illustrative purposes we present a weak form of his argument in the stronger setting where $\mathcal{N}$ satisfies

$$E_x(q,a) \ll x^{\Delta}. \tag{11}$$

Assume $Q \le x$ and Q is large. By orthogonality

$$\begin{aligned} E_x(q,a) &= \frac{1}{q}\sum_{b=1}^{q} e\left(-\frac{ab}{q}\right)\left(W(b/q) - x\sum_{N=1}^{q}\eta(q,N)e\left(\frac{Nb}{q}\right)\right) \\ &=: \frac{1}{q}\sum_{b=1}^{q} e\left(-\frac{ab}{q}\right) X_{q,b} \end{aligned}$$

so that

$$\sum_{a=1}^{q} |E_x(q,a)|^2 = \frac{1}{q}\sum_{b=1}^{q}|X_{q,b}|^2$$

and therefore

$$\begin{aligned} V(x,Q) &= \sum_{q\le Q}\frac{1}{q}\sum_{d|q}\sum_{\substack{b=1\\(b,q)=d}}^{q}|X_{q,b}|^2 \\ &= \sum_{q\le Q}\frac{1}{q}\sum_{d|q}{\sum_{b=1}^{d}}'|X_{d,b}|^2 \\ &\ll \log Q\sum_{d\le Q}\frac{1}{d}{\sum_{b=1}^{d}}'|X_{d,b}|^2 \\ &\ll (\log Q)^2\max_{R\le Q}\left(\frac{1}{R}\sum_{R\le q<2R}{\sum_{b=1}^{q}}'|X_{q,b}|^2\right) \\ &\ll (\log Q)^2\max_{R\le Q}\frac{1}{R}\sum_{R\le q<2R}{\sum_{b=1}^{q}}'\left(|W(b/q)|^2 + \left|x\sum_{N=1}^{q}\eta(q,N)e\left(\frac{Nb}{q}\right)\right|^2\right) \\ &=: (\log Q)^2\max_{R\le Q}\frac{\Upsilon_R}{R}. \end{aligned} \tag{12}$$

By (9)

$$\begin{aligned} {\sum_{b=1}^{q}}'\left|\sum_{N=1}^{q}\eta(q,N)e\left(\frac{Nb}{q}\right)\right|^2 &= \sum_{N,N'=1}^{q}\eta(q,N)\overline{\eta(q,N')}c_q(N-N') \\ &= \sum_{d,d'|q}\eta(q,d)\overline{\eta(q,d')}{\sum_{N=1}^{q/d}}'{\sum_{N'=1}^{q/d'}}'c_q(Nd-N'd') \end{aligned}$$

and here the N, N' sum is

$${\sum_{a=1}^{q}}' c_{q/d}(a)c_{q/d'}(-a) = \mu(q/d)\mu(q/d')\phi(q) \ll q$$

so that

$${\sum_{b=1}^{q}}'\left|\sum_{N=1}^{q}\eta(q,N)e\left(\frac{Nb}{q}\right)\right|^2 \ll q\sum_{d,d'|q}\left|\eta(q,d)\eta(q,d')\right|. \tag{13}$$

Clearly

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}\\ n\equiv a(q)}} 1 \ll \frac{x}{q}$$

so

$$\eta(q,a)=\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\le x\\ n\in\mathcal{N}\\ n\equiv a(q)}} 1 \ll \frac{1}{q}$$

therefore (13) becomes

$$\sum_{b=1}^{q}{}' \left|\sum_{N=1}^{q}\eta(q,N)e\left(\frac{Nb}{q}\right)\right|^2 \quad\ll\quad q^{\epsilon-1}. \tag{14}$$

This with (10) says

$$\Upsilon_R \ll x(x+R^2)+x^2\sum_{q\le R} q^{\epsilon-1} \ll xR^2+x^{2+\epsilon};$$

on the other hand (11) implies

$$\Upsilon_R \ll \sum_{R\le q<2R}\sum_{a=1}^{q}|E_x(q,a)|^2 \ll x^{2\Delta}R^2$$

and therefore (12) says

$$V(x,Q)\ll x^{\epsilon}\max_{R\le Q}\left(\min\left\{xR+\frac{x^2}{R},x^{2\Delta}R\right\}\right).$$

For $R\le x^{1-\Delta}$ the minimum is $\le x^{1+\Delta}$ and for $x^{1-\Delta}<R\le Q$ the minimum is $\le xQ+x^{1+\Delta}$, therefore

$$V(x,Q)\ll x^{\epsilon}\Big(xQ+x^{1+\Delta}\Big)$$

which is a weak result of BDH type.

In [6] Hooley refines his BDH bound to an asymptotic formula on the further assumption that $\eta(q,q)$ is essentially multiplicative, that is there is $\rho\in\mathbb{R}^{\ge}$ and multiplicative $f:\mathbb{N}\to\mathbb{R}^{\ge}$ for which

$$\eta(q,q)=\rho f(q)$$

and we make a few comments as to the proof. Opening the square in the definition of $V(x,Q)$ we see that the difficulty lies in evaluating sums of type

$$S(x,Q):=\sum_{Q<q\le x}\sum_{\substack{n'<n\le x\\ n\equiv n'(q)\\ n,n'\in\mathcal{N}}} 1$$

which with a "modulus swap" becomes

$$\sum_{l\le x/Q}\sum_{\substack{Q<(n-n')/l\le x\\ n\equiv n'(l)\\ n,n'\in\mathcal{N}}} 1=\sum_{l\le x/Q}\sum_{\delta|l}\sum_{\substack{b=1\\ (b,l)=\delta}}^{l}\sum_{\substack{n'\le x-Ql\\ n'\in\mathcal{N}\\ n'\equiv b(l)}}\sum_{\substack{n'+Ql<n\le x\\ n\in\mathcal{N}\\ n\equiv b(l)}} 1$$

the point being the modulus is now small. In view of (8) and (9) we expect

$$\sum_{\substack{X<n\le Y\\ n\in\mathcal{N}\\ n\equiv a(q)}} 1 \approx \eta\left(q,(q,a)\right)(Y-X)$$

so that the sums over n' and n above are

$$\approx \eta(l,\delta)\sum_{\substack{n'\le x-Ql\\ n'\in\mathcal{N}\\ n'\equiv b(l)}}\Big(x-Ql-n'\Big) \approx \eta(l,\delta)\int_0^{x-Ql}\left(\sum_{\substack{n'\le t\\ n'\in\mathcal{N}\\ n'\equiv b(l)}} 1\right)dt \approx \frac{\eta(l,\delta)^2(x-Ql)^2}{2}$$

and therefore

$$\begin{aligned} S(x,Q) &\approx \frac{1}{2}\sum_{l\le x/Q}\Big(x-Ql\Big)^2\sum_{\delta|l}\eta(l,\delta)^2\sum_{\substack{b=1\\ (b,l)=\delta}}^{l} 1 \\ &=: \frac{1}{2}\sum_{l\le x/Q}\Big(x-Ql\Big)^2 M(l). \end{aligned} \tag{15}$$

An important consequence of (9) is

$$\eta(l,\delta)=\frac{1}{\phi(l/\delta)}\sum_{r|l/\delta}\mu(r)\eta(r,r)$$

(this can be shown in a few lines) which with the multiplicativity assumption means that the Dirichlet series of $M(l)$ is associated to an Euler product and so we can apply Perron's formula to (15). Hooley obtains with this argument

$$V(x,Q)=\Big(c+o(1)\Big)xQ+\mathcal{O}_A\left(\frac{x^2}{(\log x)^A}\right) \tag{16}$$

for some constant c and where $o(1)\to 0$ with $x/Q\to\infty$. An alternative approach to Hooley's was developed by Goldston and Vaughan (see [3]) through which the variance is expressed as a binary additive problem tractable by the circle method. In this case the approach to deal with sums of type

$$\sum_{q\le Q}\sum_{\substack{n'<n\le x\\ n,n'\in\mathcal{N}\\ n\equiv n'(q)}} 1$$

is to write the double sum as

$$\sum_{q\le Q}\sum_{l\le x/q}\sum_{\substack{n,n'\le x\\ n-n'=ql\\ n,n'\in\mathcal{N}}} 1$$

and then to apply the circle method to the inner sum. In [16] a general result is obtained in this way, again under (9) of course.

A subtle point is that the constant (16) may be zero. As seen from the BDH Theorem this is not the case for the primes, but as seen from [17] this is the case for the k-free numbers.

Brüdern has a theory for this principle (see [1]) which characterises those sequences with smaller variances as those *limit periodic* sequences. On the other hand, since we expect for the k-free numbers

$$E_x(q,a) \ll \left(\frac{x}{q}\right)^{1/2k},$$

we should also expect that in this case $c = 0$ and moreover that

$$V(x,Q) \ll Q^2 \left(\frac{x}{Q}\right)^{1/k}.$$

In fact an asymptotic formula for the k-free variance is long known and a strong asymptotic formula is given in [17], as already mentioned. As far as strong asymptotics in arithmetic progressions are concerned, the k-free numbers are indeed easier than the primes, the k-frees in a vague sense perhaps approximating the primes with decreasing k.

In summary, the variance in arithmetic progressions of both the primes and the k-frees are particularly well understood, and there is a theory for sequences in general. Once we look at r-tuples, however, we no longer have (9). In this case the situation for primes is hopeless, but we can at least address the k-frees in this thesis.

Chapter 2

Proof of Theorem 1

In [17] the evaluation of the variance is translated into a binary additive problem in k-free numbers which can be tackled with the circle method. We use the method laid out there for our proof except at the very end, worried that the r-tuples not being multiplicative would cause some problems otherwise.

Write μ_k for the indicator function of the k-frees and write $g = g_k$ for the unique function satisfying

$$\mu_k = g \star 1; \tag{17}$$

specifically

$$g(d) = \begin{cases} \mu(d^{1/k}) & \text{if } d \text{ is a } k\text{th power} \\ 0 & \text{if not} \end{cases} \tag{18}$$

since

$$\mu_k(n) = \sum_{d^k|n} \mu(d) \tag{19}$$

is a well known identity. Then from (4)

$$\eta(q,a) = \sum_{\substack{d_1,\dots,d_r=1 \\ (d_i,d_j)|h_i-h_j \ (1\le i,j\le r) \\ (q,d_i)|a+h_i \ (1\le i\le r)}}^{\infty} \frac{g(d_1)\cdots g(d_r)}{[q,d_1,...,d_r]}. \tag{20}$$

Throughout this chapter the implied constants in the $\mathcal{O}$ symbol will always be understood to be dependent on $k, r, \mathbf{h}$ and ϵ, and ϵ may be taken to be arbitrarily small at each of its occrences. Whenever s, σ and t appear in the same context we will always mean a complex number s with real and imaginary parts σ and t. For real positive numbers c and T we often write the shorthand

$$\int_{\pm T} \text{ or } \int_{c\pm iT} \text{ or } \int_{c\pm i\infty}$$

for

$$\int_{-T}^{T} \text{ or } \int_{c-iT}^{c+iT} \text{ or } \int_{c-i\infty}^{c+i\infty}.$$

We will write statements that involve r-tuples using vectors and mean that that statement is to hold for each vector component. For example, $\nu \equiv \mathbf{d}\ (\mathbf{q})$ would mean $\nu \equiv d_i\ (q_i)$ for each $i = 1, ..., R$, where the R, d_i, q_i would be clear from context. A sum Σ'^q will mean that the summation variables are restricted to numbers coprime to q. The R-fold divisor estimate $d_R(n) \ll n^\epsilon$ is well known, as is the (General) Chinese Remainder Theorem which says

$$n \equiv \mathbf{a} \bmod (\mathbf{q})$$

has exactly one solution modulo $[q_1, ..., q_R]$ if $(q_i, q_j)|a_i - a_j$ and has no solutions otherwise. We will use both these facts frequently but often forget to mention where they come from. A coprimality condition may often disappear from one line to the next with the introduction of the Möbius function; here we are using

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

2.1 - Lemmas

Our theorem rests on having a concrete expression for the Gauss sums associated with the k-frees. The difficulties in the transition to r-tuples is contained in the following lemma.

Lemma 1. *For $a, q \in \mathbb{N}$ define*

$$G(q, a) = \sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right) \eta(q, \nu).$$

For each prime p and each q a non-negative power of p define

$$\mathfrak{g}_p(q) = \begin{cases} 1/p^k & \text{if } q|p^k \\ 0 & \text{if } p^{k+1}|q, \end{cases}$$

For each prime p write R_p for the number of different residue classes modulo p^k represented by the $h_1, ..., h_r$. If for all primes p we have $R_p < p^k$ define for $q \in \mathbb{N}$

$$G(q) = \prod_{p^\beta || q} \frac{\mathfrak{g}_p(p^\beta)}{1 - R_p/p^k}.$$

For each prime p write $H_1, ..., H_{R_p}$ for the different residue classes modulo p^k represented by the $h_1, ..., h_r$. For $A \in \mathbb{N}$ and P a power of p define for $P|p^k$

$$\Omega_A(P) = -\sum_{n=1}^{R_p} e\left(-\frac{AH_n}{P}\right)$$

and for $p^{k+1}|P$ define the LHS to be zero. For $Q, A \in \mathbb{N}$ define

$$H(Q, A) = \prod_{i=1}^{\omega} \Omega_{A\overline{Q/Q_i}}(Q_i),$$

where $Q = Q_1 \cdots Q_\omega$ with Q_i the distinct prime powers of Q and each $\overline{Q/Q_i}$ the inverse of Q/Q_i mod Q_i; if $Q = 1$ the product is to be read as 1. Define

$$\rho = \prod_p \left(1 - \frac{R_p}{p^k}\right).$$

If $R_p < p^k$ for all p then for any $a, q \in \mathbb{N}$

$$G(q, a) = \rho G\left(\frac{q}{(q, a)}\right) H\left(\frac{q}{(q, a)}, \frac{a}{(q, a)}\right).$$

Proof. We first set quite a bit of notation. For given $d_1, ..., d_r$ write $d^* = [d_1, ..., d_r]$ and $g(\mathbf{d}) = g(d_1)\cdots g(d_r)$, where g is as in (17). By a sum Σ^* over variables $d_1, ..., d_r$ we will mean always that for all i, j we have $(d_i, d_j)|h_i - h_j$. For given $Q, d_1, ..., d_r \in \mathbb{N}$ with $Q|d^*$ and $(d_i, d_j)|h_i - h_j$ write $\nu_{Q,\mathbf{d}}$ for the unique solution modulo $[(Q, d_1), ..., (Q, d_r)] = (Q, d^*) = Q$ to the system $\nu \equiv -\mathbf{h} \mod ((Q, \mathbf{d}))$. For $Q, A, N \in \mathbb{N}$ with $Q|N$ define

$$g^*_{A,Q}(N) = \sideset{}{^*}\sum_{\substack{d_1,...,d_r \\ d^*=N}} g(\mathbf{d}) e\left(\frac{A\nu_{Q,\mathbf{d}}}{Q}\right)$$

so that

$$\sum_{\substack{N=1 \\ Q|N}}^{\infty} \frac{g^*_{A,Q}(N)}{N} = \sideset{}{^*}\sum_{\substack{d_1,...,d_r \\ Q|d^*}} \frac{g(\mathbf{d})}{d^*} e\left(\frac{A\nu_{Q,\mathbf{d}}}{Q}\right), \tag{21}$$

and for P a non-negative power of a prime p define

$$\mathfrak{g}^*_{A,p}(P) := \sum_{\substack{t\geq 0 \\ P|p^t}} \frac{g^*_{A,P}(p^t)}{p^t} = \sideset{}{^*}\sum_{\substack{d_1,...,d_r \\ P|d^* \\ \mathbf{d}\text{ non-negative} \\ \text{powers of } p}} \frac{g(\mathbf{d})}{d^*} e\left(\frac{A\nu_{P,\mathbf{d}}}{P}\right). \tag{22}$$

For $\omega, D^0_1, ..., D^0_r, D^1_1, ..., D^1_r, ..., D^\omega_1, ..., D^\omega_r \in \mathbb{N}$ write $\mathbf{D}_j = (D^j_1, ..., D^j_r) \in \mathbb{N}^r$, write $\mathbf{D}_0 \cdots \mathbf{D}_r = (D^0_1 \cdots D^\omega_1, ..., D^0_r \cdots D^\omega_r) \in \mathbb{N}^r$, and write $D^*_j = [D^j_1, ..., D^j_r]$. Now we can start the proof proper.

Take $Q, D^0_1, ..., D^\omega_r \in \mathbb{N}$, where $Q = Q_1 \cdots Q_\omega$ as a product of prime powers, where for $1 \leq j \leq \omega$ each D^j_i contains only primes of Q_j, and where each D^0_i is coprime to Q. Suppose $Q_j|D^*_j$. Then we have modulo Q

$$\begin{aligned} \nu_{Q,\mathbf{D}_0\cdots\mathbf{D}_r} &\equiv \nu_{Q,\mathbf{D}_1\cdots\mathbf{D}_r} \\ &\equiv \overline{Q/Q_1}\,(Q/Q_1)\,\nu_{Q_1,\mathbf{D}_1} + \cdots + \overline{Q/Q_\omega}\,(Q/Q_\omega)\,\nu_{Q_\omega,\mathbf{D}_\omega} \end{aligned}$$

so that for $N, A \in \mathbb{N}$, with $N = N_0 N_1 \cdots N_\omega$ where the N_i are the prime powers of N and where $(Q, N_0) = 1$ and $Q_i|N_i$, we have

$$\begin{aligned} g^*_{A,Q}(N) &= \sideset{}{^*}\sum_{\substack{d_1,...,d_r \\ d^*=N}} g(d_1)\cdots g(d_r) e\left(\frac{A\nu_{Q,\mathbf{d}}}{Q}\right) \\ &= \sum_{j=0}^{\omega} \sideset{}{^*}\sum_{\substack{D^j_0,...,D^j_r \\ D^*_j=N_j}} g\left(D^0_1\cdots D^\omega_1\right)\cdots g\left(D^0_r\cdots D^\omega_r\right) e\left(\frac{A\nu_{Q,\mathbf{D}_0\cdots\mathbf{D}_\omega}}{Q}\right) \\ &= \left(\sideset{}{^*}\sum_{\substack{D^0_1,...,D^0_r \\ D^*_0=N_0}} g(D^0_1)\cdots g(D^0_r)\right) \prod_{j=1}^{\omega} \left\{\sideset{}{^*}\sum_{\substack{D^j_1,...,D^j_r \\ D^*_j=N_j}} g(D^j_1)\cdots g(D^j_r) e\left(\frac{A\overline{Q/Q_j}\nu_{Q_j,\mathbf{D}_j}}{Q_j}\right)\right\} \\ &= g^*_{1,1}(N_0) \prod_{j=1}^{\omega} g^*_{A\overline{Q/Q_j},Q_j}(N_j) \end{aligned}$$

and therefore

$$\sum_{\substack{N=1\\ Q|N}}^{\infty} \frac{g^*_{A,Q}(N)}{N} = \prod_{p\not| Q}\left(\sum_{t\geq 0}\frac{g^*_{1,1}(p^t)}{p^t}\right)\prod_{j=1}^{\omega}\left(\sum_{\substack{t\geq 0\\ Q_j|p_j^t}} \frac{g^*_{\overline{AQ/Q_j},Q_j}(p_j^t)}{p^t}\right)$$

$$= \prod_{p\not| Q}\mathfrak{g}^*_{1,p}(1)\prod_{j=1}^{\omega}\mathfrak{g}^*_{\overline{AQ/Q_j},p_j}(Q_j), \tag{23}$$

where obviously the p_j are the primes of Q_j; if $Q=1$ this all goes through easier and gives the same result as long as we read the finite product as 1. We now evaluate $\mathfrak{g}^*_{A,p}(P)$ on non-negative powers P of a prime p and on any $A\in\mathbb{N}$ coprime to p. For d a non-negative power of p we have from (18)

$$g(d) = \begin{cases} 1 & \text{if } d=1 \\ -1 & \text{if } d=p^k \\ 0 & \text{otherwise}\end{cases}$$

so from (22)

$$\mathfrak{g}^*_{A,p}(P) = \sum_{X\subseteq\{1,\ldots,r\}}(-1)^{|X|}\sideset{}{^*}\sum_{\substack{d_1,\ldots,d_r\\ P|d^*\\ \mathbf{d}\text{ non-negative}\\ \text{powers of p}\\ d_i=p^k \text{ if } i\in X\\ d_i=1 \text{ if } i\notin X}}\frac{1}{d^*}e\left(\frac{A\nu_{P,\mathbf{d}}}{P}\right)$$

$$= \sum_{\substack{\mathbf{d}\\ P|d^*\\ \mathbf{d}\text{ non-negative}\\ \text{powers of p}\\ \text{all } d_i=1}}\frac{1}{d^*}e\left(\frac{A\nu_{P,\mathbf{d}}}{P}\right)$$

$$+ \sum_{n=1}^{R_p}e\left(-\frac{AH_n}{P}\right)\sum_{\substack{X\subseteq\{1,\ldots,r\}\\ h_i\equiv H_n(p^k)\ \forall i\in X\\ X\neq\emptyset}}(-1)^{|X|}\sum_{\substack{\mathbf{d}\\ P|d^*\\ \mathbf{d}\text{ non-negative}\\ \text{powers of p}\\ d_i=p^k \text{ if } i\in X\\ d_i=1 \text{ if } i\notin X}}\frac{1}{d^*}. \tag{24}$$

Note that

$$\sum_{\mathcal{N}\subseteq\{1,\ldots,n\}}(-1)^{|\mathcal{N}|} = \sum_{i=0}^{n}(-1)^i\binom{n}{i} = (1-1)^n = 0$$

so that

$$\sum_{\substack{\mathcal{N}\subseteq\{1,\ldots,n\}\\ \mathcal{N}\neq\emptyset}}(-1)^{|\mathcal{N}|} = -1$$

and therefore

$$\sum_{\substack{\mathcal{N}\subseteq\{1,\ldots,r\}\\ h_i\equiv H_n(p^k)\ \forall i\in\mathcal{N}\\ \mathcal{N}\neq\emptyset}}(-1)^{|\mathcal{N}|} = -1. \tag{25}$$

If $p^{k+1}|P$ there are no terms in (24). If $P|p^k$ the non-empty X (i.e. those with $d^*=p^k$) contribute

$$\frac{1}{p^k}\sum_{n=1}^{R_p}e\left(-\frac{AH_n}{P}\right)\sum_{\substack{X\subseteq\{1,\ldots,r\}\\ h_i\equiv H_n(p^k)\ \forall i\in X\\ X\neq\emptyset}}(-1)^{|X|} = p^{-k}\begin{cases}\Omega_A(P) & \text{if } P\neq 1\\ -R_p & \text{if } P=1\end{cases}$$

from (25), and the empty set term (i.e. $d^*=1$) only gives a contribution if $P=1$, in which case it contributes 1. Consequently for $(A,P)=1$

$$\begin{aligned}\mathfrak{g}^*_{A,p}(P) &= \begin{cases} 1-p^{-k}R_p & \text{if } P=1 \\ p^{-k}\Omega_A(P) & \text{if } 1\neq P|p^k \\ 0 & \text{if } p^{k+1}|P \end{cases} \\ &= \begin{cases} 1-p^{-k}R_p & \text{if } P=1 \\ \mathfrak{g}_p(P)\Omega_A(P) & \text{if } P\neq 1 \end{cases}\end{aligned}$$

and therefore from (23) we have for $(A,Q)=1$

$$\sum_{\substack{N=1\\ Q|N}}^{\infty} \frac{g^*_{A,Q}(N)}{N} \;=\; \prod_{p\nmid Q}\left(1-p^{-k}R_p\right)\prod_{j=1}^{\omega}\left(\mathfrak{g}_{p_j}(Q_j)\Omega_{\overline{AQ/Q_j}}(Q_j)\right).$$

so from (21)

$$\begin{aligned}\sum_{\substack{d_1,\ldots,d_r\\ Q|d^*}}{}^{*} \frac{g(\mathbf{d})}{d^*}e\left(\frac{A\nu_{Q,\mathbf{d}}}{Q}\right) &= \prod_{p}\left(1-\frac{R_p}{p^k}\right)\prod_{j=1}^{\omega}\frac{\mathfrak{g}_{p_j}(Q_j)\Omega_{\overline{AQ/Q_j}}(Q_j)}{1-R_{p_j}/p_j^k} \\ &= \rho G(Q)H(Q,A) \qquad (26)\end{aligned}$$

for any $Q,A\in\mathbb{N}$ with $(A,Q)=1$. We now take $Q=q/(q,a)$ and $A=a/(q,a)$. Since for $(d_i,d_j)|h_i-h_j$ we have

$$\begin{aligned}\sum_{\substack{\nu=1\\ (q,\mathbf{d})|\nu+\mathbf{h}}}^{q} e\left(\frac{a\nu}{q}\right) &= \frac{q}{[(q,d_1),...,(q,d_r)]}\begin{cases} e\left(a\nu_{q,\mathbf{d}}/q\right) & \text{if } q/[(q,d_1),...,(q,d_r)] \mid a \\ 0 & \text{if not}\end{cases} \\ &= \frac{q}{(q,d^*)}\begin{cases} e\left(A\nu_{Q,\mathbf{d}}/Q\right) & \text{if } Q|d^* \\ 0 & \text{if not}\end{cases}\end{aligned}$$

we see from (20) and (26)

$$G(q,a)=q\sum_{\substack{d_1,\ldots,d_r\\ (d_i,d_j)|h_i-h_j\\ Q|d^*}}\frac{g(\mathbf{d})}{[q,d^*](q,d^*)}e\left(\frac{A\nu_{Q,\mathbf{d}}}{Q}\right)=\rho G(Q)H(Q,A).$$

□

This $H(q,a)$ function of the previous lemma is special to the r-tuple case. We deal now with its main properties. When comparing with [17] it is perhaps useful to think of $\Phi_q(n)$ as Ramanujan's sum.

Lemma 2. *Define H as in Lemma 1. Define for $q,n\in\mathbb{N}$*

$$\Phi_q(n)=\sum_{a=1}^{q}{}' |H(q,a)|^2 e\left(\frac{an}{q}\right),\qquad \Phi^*_q(n)=\sum_{a=1}^{q}{}' \overline{H(q,a)}e\left(\frac{an}{q}\right)\qquad \textit{and}\qquad \Phi(q)=\Phi_q(0).$$

(i) Both $\Phi_q(n)$ and $\Phi^*_q(n)$ are, for each n, multiplicative in q.

(ii) Suppose we have a function $F(q,d)$ defined for $q \in \mathbb{N}$ and $d|q$. If for all $(q,q')=1$ and $d|q, d'|q'$ we have $F(qq',dd')=F(q,d)F(q',d')$ then the sum

$$\sum_{A=1}^{q} F\Big(q,(q,A)\Big)\Phi_q(A)$$

is multiplcative in q.

(iii) For q a power of a prime and for $d|q$

$$\sum_{A=1}^{q/d}{}' \Phi_q(-Ad) = \Phi(q)\mu(q/d).$$

(iv) For any $q \in \mathbb{N}$

$$\sum_{A=1}^{q} |\Phi_q(A)| \ll q^{1+\epsilon}$$

and the same claim holds with $\Phi_q(A)$ replaced by $\Phi_q^*(A)$.

(v) Let $G(\cdot)$ and ρ be as in Lemma 1. For any $q \in \mathbb{N}$

$$\sum_{a=1}^{q} \eta(q,a)^2 = \frac{\rho^2}{q}\sum_{d|q}\Phi(d)G(d)^2.$$

(vi) For any $q,a \in \mathbb{N}$

$$H(q,a) \ll q^{\epsilon}.$$

Proof. For given $d_1,...,d_r$ write always $d^* = [d_1,...,d_r]$ and $g(\mathbf{d}) = g(d_1)\cdots g(d_r)$, where g is as in (17). By a sum Σ^* over variables $d_1,...,d_r$ we will mean always that for all i,j we have $(d_i,d_j)|h_i-h_j$. For P a power of a prime p and $A \in \mathbb{N}$ we have

$$H(P,A) = -\sum_{n=1}^{R_p} e\left(-\frac{AH_n}{P}\right)\begin{cases} 1 & \text{for } P|p^k \\ 0 & \text{for } p^{k+1}|P \end{cases} \tag{27}$$

where $H_1,...,H_{R_p}$ are the distinct residues represented by $h_1,...,h_r$ modulo p^k.

(i) Take $q,q' \in \mathbb{N}$ with $(q,q')=1$ and, for any prime p, define $\beta,\beta' \geq 0$ through $p^\beta||q$ and $p^{\beta'}||q'$. For a prime p denote by $\overline{R}$ the inverse of any R modulo $p^{\beta+\beta'}$. Then $q'\overline{qq'/p^{\beta+\beta'}} \equiv \overline{q/p^\beta} \mod (p^\beta)$ for any $p|q$. A similar comment applies when considering $p|q'$ so that for any $a,a' \in \mathbb{N}$

$$\begin{aligned} H(qq',aq'+a'q) &= \prod_{p|qq'}\left(-\sum_{n=1}^{R_p} e\left(\frac{(aq'+a'q)\overline{qq'/p^{\beta+\beta'}}H_n}{p^{\beta+\beta'}}\right)\right) \\ &= \prod_{p|q}\left(-\sum_{n=1}^{R_p} e\left(\frac{a\overline{q/p^\beta}H_n}{p^\beta}\right)\right)\prod_{p|q'}\left(-\sum_{n=1}^{R_p} e\left(\frac{a'\overline{q'/p^{\beta'}}H_n}{p^{\beta'}}\right)\right) \\ &= H(q,a)H(q',a'); \end{aligned} \tag{28}$$

here we are assuming $P|p^k$ but if not the equality is trivial. As a, a' respectively run over reduced residue systems modulo q, q' the quantity $aq' + a'q$ runs over a reduced residue sytem modulo qq', therefore (28) implies

$$\begin{aligned}
\Phi_{qq'}(n) &= \sum_{A=1}^{qq'}{}' |H(qq', A)|^2 e\left(\frac{An}{qq'}\right) \\
&= \sum_{a=1}^{q}{}' \sum_{a'=1}^{q'}{}' \left|H\left(qq', aq' + a'q\right)\right|^2 e\left(\frac{(aq' + a'q)n}{qq'}\right) \\
&= \left(\sum_{a=1}^{q}{}' \left|H(q, a)\right|^2 e\left(\frac{an}{q}\right)\right)\left(\sum_{a'=1}^{q'}{}' \left|H(q', a')\right|^2 e\left(\frac{a'n}{q'}\right)\right) \\
&= \Phi_q\left(n\right)\Phi_{q'}\left(n\right)
\end{aligned}$$

and a similar proof obviously works for $\Phi_q^*(n)$.

(ii) Suppose $M, M' \in \mathbb{N}$ with $(M, M') = 1$ so that

$$F\Big(MM', (MM', aM' + a'M)\Big) = F\Big(M, (M, a)\Big)F\Big(M', (M', a')\Big). \tag{29}$$

As a and a' run over complete residue system modulo M and M' respectively then so do $a\overline{M}'$ and $a'\overline{M}$ and the quantity $aM' + a'M$ runs over a complete residue system modulo MM'. Therefore by (29) and part (i)

$$\begin{aligned}
&\sum_{A=1}^{MM'} F\Big(MM', (MM', A)\Big)\Phi_{MM'}(A) \\
&= \sum_{a=1}^{M}\sum_{a'=1}^{M'} F\Big(MM', (MM', aM' + a'M)\Big)\Phi_{MM'}\left(aM' + a'M\right) \\
&= \sum_{a=1}^{M}\sum_{a'=1}^{M'} F\Big(M, (M, a)\Big)F\Big(M', (M', a')\Big)\Phi_M\left(aM' + a'M\right)\Phi_{M'}\left(a'M + aM'\right) \\
&= \sum_{a=1}^{M}\sum_{a'=1}^{M'} F\Big(M, (M, a\overline{M'})\Big)F\Big(M', (M', a'\overline{M})\Big)\Phi_M\left(a\right)\Phi_{M'}\left(a'\right)
\end{aligned}$$

and the claim follows.

(iii) The equality is trivial if $p^{k+1}|q$, otherwise for $q|p^k$ we have from (27)

$$\Phi_q(n) = \sum_{n,n'=1}^{R_p} \sum_{a=1}^{q}{}' e\left(\frac{a(H_n - H_{n'} + n)}{q}\right) \tag{30}$$

so that

$$\begin{aligned}\sum_{A=1}^{q/d}{}' \Phi_q(-Ad) &= \sum_{n,n'=1}^{R_p} \sum_{a=1}^{q}{}' e\left(\frac{a(H_n - H_{n'})}{q}\right) \sum_{A=1}^{q/d}{}' e\left(\frac{-aAd}{q}\right) \\ &= \mu(q/d) \sum_{n,n'=1}^{R_p} \sum_{a=1}^{q}{}' e\left(\frac{a(H_n - H_{n'})}{q}\right) \\ &= \mu(q/d)\Phi_q(0)\end{aligned}$$

from (30).

(iv) The same argument as in part (ii) shows the sum in qustion to be multiplicative so it is enough to prove the bound for q a prime power. In that case (30) implies

$$\Phi_q(n) = \sum_{n,n'=1}^{R_p} c_q(H_n - H_{n'} + n)$$

so that

$$\sum_{n=1}^{q} |\Phi_q(n)| \ll \sum_{n=1}^{q} |c_q(n)| \ll q^{1+\epsilon}$$

by a standard bound for Ramanujan's sum and the proof is obviously the same for $\Phi_q^*(n)$.

(v) From part (i)

$$\sum_{b=1}^{d}{}' H(d,b)\, e\left(\frac{-ab}{d}\right)$$

is, for each $a \in \mathbb{N}$, multiplicative in d and therefore so is

$$\begin{aligned}\eta^*(q,a) &:= \sum_{b=1}^{q} G\left(\frac{q}{(q,b)}\right) H\left(\frac{q}{(q,b)}, \frac{b}{(q,b)}\right) e\left(\frac{-ab}{q}\right) \\ &= \sum_{d|q} G(d) \sum_{b=1}^{d}{}' H(d,b)\, e\left(\frac{-ab}{d}\right).\end{aligned}$$

As in (ii) it can then be shown that

$$\sum_{a=1}^{q} |\eta^*(q,a)|^2$$

is mutiplicative.

Suppose q is a prime power so that for $d|q, p^k$ we have from (27)

$$\sum_{b=1}^{d}{}' H(d,b)\, e\left(\frac{-ab}{d}\right) = -\sum_{n=1}^{R_p} c_d(H_n - a).$$

and therefore (since G vanishes for $p^{k+1}|d$)

$$\eta^*(q,a) = -\sum_{n=1}^{R_p} \sum_{d|q,p^k} G(d) c_d(H_n - a)$$

so that

$$\sum_{a=1}^{q} |\eta^*(q,a)|^2 = \sum_{n,n'=1}^{R_p} \sum_{d,d'|q} G(d)\overline{G(d')} \sum_{a=1}^{q} c_d(H_n - a) c_{d'}(a - H_{n'})$$
$$= q \sum_{n,n'=1}^{R_p} \sum_{d,d'|q} \frac{G(d)\overline{G(d')}}{[d,d']} \sum_{a=1}^{[d,d']} c_d(H_n - a) c_{d'}(a - H_{n'}). \qquad (31)$$

Then

$$\sum_{a=1}^{[d,d']} c_d(H_n - a) c_{d'}(a - H_{n'})$$
$$= \sum_{A=1}^{d}{}' \sum_{A'=1}^{d'}{}' e\left(\frac{AH_n}{d} - \frac{A'H_{n'}}{d'}\right) \sum_{a=1}^{[d,d']} e\left(\frac{a(-A[d,d']/d + A'[d,d']/d')}{[d,d']}\right)$$
$$= [d,d'] \sum_{A=1}^{d}{}' \sum_{\substack{A'=1 \\ [d,d']|-A[d,d']/d + A'[d,d']/d'}}^{d'}{}' e\left(\frac{AH_n}{d} - \frac{A'H_{n'}}{d'}\right)$$

but the only (prime power) d, d' which can satisfy these summation conditions are those with $d = d'$, in which case the A, A' sum becomes

$$\sum_{A=1}^{d}{}' e\left(\frac{A(H_n - H_{n'})}{d}\right) = c_d(H_n - H_{n'})$$

and so

$$\frac{1}{[d,d']} \sum_{a=1}^{[d,d']} c_d(H_n - a) c_{d'}(H_{n'} - a) = \begin{cases} c_d(H_n - H_{n'}) & \text{if } d = d' \\ 0 & \text{if not.} \end{cases}$$

Therefore (31) says

$$\sum_{a=1}^{q} |\eta^*(q,a)|^2 = q \sum_{n,n'=1}^{R_p} \sum_{d|q} |G(d)|^2 c_d(H_n - H_{n'})$$
$$= q \sum_{d|q} |G(d)|^2 \sum_{a=1}^{q}{}' \left| - \sum_{n=1}^{R_p} e\left(\frac{aH_n}{d}\right) \right|^2$$
$$= q \sum_{d|q} \Phi(d) |G(d)|^2. \qquad (32)$$

from (28). This holds initially only for q a prime power, but we've already said that the LHS is multiplicative and the RHS is multiplicative from part (i), so (32) holds for general q. Let $G(\cdot,\cdot)$ be as in Lemma 1. From orthogonality and then Lemma 1

$$\eta(q,a) = \frac{1}{q} \sum_{b=1}^{q} G(q,b) e\left(\frac{-ab}{q}\right)$$
$$= \frac{\rho \eta^*(q,a)}{q}$$

so

$$\sum_{a=1}^{q} |\eta(q,a)|^2 = \frac{\rho^2}{q^2} \sum_{a=1}^{q} |\eta^*(q,a)|^2$$

and the result follows from (32).

(vi) We have

$$H(q,a) \ll \prod_{p|q} R_p.$$

□

Lemma 3. *Suppose $R_p < p^k$ holds for all primes p, where R_p is the number of different residues represented by the h_i modulo p^k and let $G(\cdot)$ be as in Lemma 1. Then for any $q \in \mathbb{N}$*

$$G(q) \ll q^{\epsilon-1}$$

and for and any $Z > 0$

$$\sum_{Z<q\leq 2Z} |G(q)| \ll Z^{1/k-1+\epsilon}.$$

This implies in particular

$$\sum_{q\leq Z} |G(q)| \ll Z^{\theta+\epsilon} \qquad \text{and} \qquad \sum_{q>Z} |G(q)| \ll Z^{\theta-1+\epsilon}.$$

Proof. For any prime p and any $t \geq 0$ we have

$$G(p^t) = \frac{1}{1-R_p/p^k} \begin{cases} 1/p^k & \text{if } t \leq k \\ 0 & \text{if } t > k. \end{cases}$$

For large p we have $R_p = r$ so $1 - R_p/p^k \geq 1/2$ and so

$$|G(p^t)| \leq \begin{cases} 2/p^k & \text{for large } p \\ 0 & \text{for all } p \text{ and } t > k \end{cases} \tag{33}$$

therefore for all p

$$|G(p^t)| \ll \begin{cases} 1/p^k & \text{always, in particular for } t \leq k, \\ 0 & \text{for } t > k \end{cases}$$

so certainly the first claim holds for prime powers, and by multiplicativity also for general q. By (33) we have

$$p^{t(1-1/k)}|G(p^t)| \leq \begin{cases} 2/p & \text{for large } p \text{ and } 1 \leq t \leq k \\ 0 & \text{for all } p \text{ and } t > k \end{cases}$$

therefore by multiplicativity

$$\begin{aligned} \sum_{q\leq 2Z} q^{1-1/k}|G(q)| &\leq \prod_{p\ll 1} \left(1 + \sum_{t\leq k} p^{t(1-1/k)}|G(p^t)|\right) \prod_{p\leq 2Z} \left(1 + \sum_{t\leq k} p^{t(1-1/k)}|G(p^t)|\right) \\ &\ll \prod_{p\leq 2Z} (1+2k/p) \\ &\leq \prod_{p\leq 2Z} (1+1/p)^{2k} \\ &\ll (\log Z)^{2k} \end{aligned}$$

by one of Merten's formulas. Consequently

$$\sum_{Z<q\le 2Z} |G(q)| \ll Z^{1/k-1} \sum_{q\le 2Z} q^{1-1/k} |G(q)| \ll Z^{1/k-1} (\log Z)^{2k}.$$

□

Counting k-free numbers amounts to counting solutions of congruences modulo k-th powers. The precision we need for r-tuples is contained in [10], but we reproduce the proof since we need a slightly different result to the one stated there.

Lemma 4. *(i) For all $R, D \in \mathbb{N}$ with D being k-free we have for $Z > 0$*

$$\sum_{\substack{d_1,\ldots,d_R \\ [d_1^k,\ldots,d_R^k,D]\le Z}} 1 \ll_R Z^\epsilon \left(\frac{Z}{D}\right)^{1/k}.$$

(ii) For all $R, a_1, ..., a_R \in \mathbb{N}$ and $Y > 0$

$$\sum_{\substack{d_1\cdots d_R > Y \\ (d_i^k, d_j^k)|a_i-a_j}} [d_1, ..., d_R]^{k(\epsilon-1)} \ll_{R,\mathbf{a}} Y^{1-k+\epsilon}.$$

(iii) For $t > 0$, $R, d_1, ..., d_R \in \mathbb{N}$ and $a_1, ..., a_R \in \mathbb{N}_0$ denote by $\mathcal{N}_{\mathbf{d};\mathbf{a}}(t)$ the number of solutions $n \le t$ to the system $n \equiv -\mathbf{a}\ (\mathbf{d}^k)$. Then for $Y > 0$ we have

$$\sum_{d_1\cdots d_R > Y} \mathcal{N}_{\mathbf{d};\mathbf{a}}(t) \ll_{R,\mathbf{a}} t^\epsilon \Big(tY^{1-k+\epsilon} + t^{2/(k+1)}\Big) + 1. \tag{34}$$

Proof. (i) We have

$$\sum_{[d_1^k,\ldots,d_R^k,D]\le Z} 1 = \sum_{[n^k,D]\le Z} \sum_{[d_1,\ldots,d_R]=n} 1 \ll_R Z^\epsilon \sum_{[n^k,D]\le Z} 1 = \sum_{l|D} \sum_{\substack{n^k \le Zl/D \\ (n^k,D)=l}} 1.$$

Write $l_0 = \prod_{p|l} p$. Then the inner sum above is

$$\sum_{\substack{n^k \le Zl/Dl_0^k \\ (n^k l_0^k, D)=l}} 1 \le \left(\frac{Z}{D}\right)^{1/k} \frac{l^{1/k}}{l_0} \le \left(\frac{Z}{D}\right)^{1/k}$$

since D and therefore l is k-free, and the claim follows for $D \le Z$. If $D > Z$ the LHS of the sum in question is zero.

(ii) For any $\mathbf{d}, \mathbf{a}$ with $(d_i^k, d_j^k)|a_i - a_j$

$$\frac{1}{[d_1^k, ..., d_R^k]} \ll_{R,\mathbf{a}} \frac{1}{d_1^k \cdots d_R^k}$$

and so

$$\sum_{\substack{d_1\cdots d_R > Y \\ (d_i^k, d_j^k)|a_i-a_j}} [d_1, ..., d_R]^{k(\epsilon-1)} \ll_{R,\mathbf{a}} \sum_{d_1\cdots d_R > Y} (d_1 \cdots d_R)^{k(\epsilon-1)} \ll \sum_{n>Y} n^{k(\epsilon-1)+\epsilon} \ll Y^{1+k(\epsilon-1)+\epsilon}.$$

(iii) We prove the claim by induction on R. Suppose t is large since otherwise the LHS of the sum in question is

$$\leq \sum_{d_1,\ldots,d_R} \sum_{\substack{n \ll 1 \\ d_i^k | n+a_i}} 1 \ll_{R,\mathbf{a}} 1$$

anyway. We have

$$\sum_{d>Y} \mathcal{N}_{d;a}(t) = \sum_{d>Y} \sum_{\substack{n \leq t \\ n \equiv -a (d^k)}} 1 \leq \sum_{Y<d \leq (t+a)^{1/k}} \left(\frac{t}{d^k} + 1 \right) \ll_a tY^{1-k} + t^{1/k} \tag{35}$$

which is (stronger than) the result for $R = 1$ so suppose now the result holds for some R and let $Z > 0$ be a parameter. We have

$$\begin{aligned} &\sum_{\substack{d_1 \cdots d_{R+1} > Y \\ d_1 \cdots d_R > Z}} \mathcal{N}_{d_1,\ldots,d_{R+1};a_1,\ldots,a_{R+1}}(t) \\ &\leq \sum_{d_1 \cdots d_R > Z} \sum_{\substack{n \leq t \\ n \equiv -a_i (d_i^k) \\ i=1,\ldots,R}} \sum_{\substack{d_{R+1} \\ n \equiv -a_{R+1} (d_{R+1}^k)}} 1 \\ &\ll (t + a_{R+1})^{\epsilon} \sum_{d_1 \cdots d_R > Z} \mathcal{N}_{d_1,\ldots,d_R;a_1,\ldots,a_R}(t) \\ &\ll_{R,\mathbf{a}} t^{\epsilon} \Big(tZ^{1-k+\epsilon} + t^{2/(k+1)} \Big) \end{aligned}$$

by assumption, and since the argument would obviously be the same if we had the summation condition $d_1 \cdots d_{R+1}/d_i > Z$ for some $1 \leq i \leq R$ instead of $i = R+1$ we deduce

$$\sum_{\substack{d_1,\ldots,d_{R+1} > Y \\ d_1 \cdots d_{R+1}/d_i > Z \\ \text{for some } i}} \mathcal{N}_{d_1,\ldots,d_{R+1},a_1,\ldots,a_{R+1}}(t) \ll t^{\epsilon} \Big(tZ^{1-k+\epsilon} + t^{2/(k+1)} \Big). \tag{36}$$

On the other hand if always $d_1 \cdots d_{R+1}/d_i \leq Z$ then we must have $d_1 \cdots d_{R+1} \leq Z^{1+1/R}$ so that

$$\begin{aligned} &\sum_{\substack{d_1 \cdots d_{R+1} > Y \\ d_1 \cdots d_{R+1}/d_i \leq Z \\ \text{for all } i}} \mathcal{N}_{d_1,\ldots,d_{R+1};a_1,\ldots,a_{R+1}}(t) \\ &\leq \sum_{Y < d_1 \cdots d_{R+1} \leq Z^{1+1/R}} \mathcal{N}_{d_1,\ldots,d_{R+1};a_1,\ldots,a_{R+1}}(t) \\ &\leq \sum_{\substack{Y < d_1 \cdots d_{R+1} \leq Z^{1+1/R} \\ (d_i^k, d_j^k) | a_i - a_j}} \left(\frac{t}{[d_1^k, \ldots, d_{R+1}^k]} + 1 \right) \\ &\ll_{R,\mathbf{a}} \Big(tY^{1-k+\epsilon} + Z^{1+1/R+\epsilon} \Big) \end{aligned} \tag{37}$$

from part (ii). Together (36) and (37) imply, assuming $Z \leq t$,

$$\begin{aligned}\sum_{d_1\cdots d_{R+1}>Y} \mathcal{N}_{d_1,\ldots,d_{R+1};a_1,\ldots,a_{R+1}}(t) &\ll_{R,\mathbf{a}} t^{\epsilon}\left(tY^{1-k+\epsilon}+Z^{1+1/R}+tZ^{1-k}+t^{2/(k+1)}\right)\\ &\ll t^{\epsilon}\left(tY^{1-k+\epsilon}+t^{(R+1)/(Rk+1)}+t^{2/(k+1)}\right)\end{aligned}$$

having chosen $Z = t^{R/(Rk+1)}$. The second term being less than the third, this is the result for $R+1$. □

We use the last lemma to handle k-free r-tuples in arithmetic progressions.

Lemma 5. *Let $E_t(\cdot,\cdot)$ be as in (4). Define for $t>0$ and $q,a\in\mathbb{N}$*

$$\Delta_t(q,a)=\sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right)E_t(q,\nu)$$

and write $\theta=1/k$ and $\Delta=2/(k+1)$.

(i) For $t>0$ and $q,a\in\mathbb{N}$

$$\sum_{\substack{n\leq t\\ n\in\mathcal{S}\\ n\equiv a(q)}} 1=\eta(q,a)t+\mathcal{O}\left(t^{\Delta+\epsilon}\right).$$

(ii) For $t,\gamma>0$

$$\begin{aligned}&\sum_{q\leq\gamma}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\ll\gamma^{2-2\theta}t^{2\theta}+t^{2\Delta},\\ &\sum_{q\leq\gamma}\frac{1}{q}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\ll\gamma^{1-2\theta}t^{2\theta}+t^{2\Delta},\\ &\sum_{q\leq\gamma}\frac{1}{q^{1-\theta}}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\ll\gamma^{1-\theta}t^{2\theta}+t^{2\Delta}\\ \textit{and}\quad &\sum_{q\leq\gamma}\frac{1}{q^{2-\theta}}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\ll t^{2\Delta};\end{aligned}$$

here the $\ll$ symbol may include terms of size $t^{\epsilon},\gamma^{\epsilon}$.

Proof. A sum Σ^* over variables $d_1,\ldots,d_r$ will mean that for all i,j we have $(d_i^k,d_j^k)|h_i-h_j$. For natural numbers $d_1,\ldots,d_r$ write $d^*=[d_1,\ldots,d_r]$. For given $q,a,d_1,\ldots,d_r\in\mathbb{N}$ and $t>0$ write $\mathcal{N}_{\mathbf{d};\mathbf{h}}^{q,a}(t)$ for the number of solutions $n\leq t$ to the system of congruences $n\equiv-\mathbf{h}$ mod $(\mathbf{d}^k)$ and $n\equiv a$ mod (q) and write $\mathcal{N}_{\mathbf{d};\mathbf{h}}(t)=\mathcal{N}_{\mathbf{d};\mathbf{h}}^{1,1}(t)$, as in Lemma 4. From (3) and (19)

$$\sum_{\substack{n\leq t\\ n\in\mathcal{S}\\ n\equiv a(q)}} 1={\sum_{d_1^k,\ldots,d_r^k\leq t}}^{*}\mu(\mathbf{d})\mathcal{N}_{\mathbf{d};\mathbf{h}}^{q,a}(t).\tag{38}$$

(i) From (38) we have for a parameter $Y \le t^{r/k}$ to be chosen

$$\begin{aligned}
\sum_{\substack{n\le t\\ n\in\mathcal{S}\\ n\equiv a(q)}} 1 &= \sideset{}{^*}\sum_{d_1\cdots d_r\le Y} \mu(\mathbf{d})\mathcal{N}^{q,a}_{\mathbf{d};\mathbf{h}}(t) + \mathcal{O}\left(\sideset{}{^*}\sum_{d_1\cdots d_r> Y} \mathcal{N}_{\mathbf{d};\mathbf{h}}(t)\right)\\
&= \sideset{}{^*}\sum_{d_1\cdots d_r\le Y} \mu(\mathbf{d})\mathcal{N}^{q,a}_{\mathbf{d};\mathbf{h}}(t) + \mathcal{O}\Big(t^{\epsilon}\Big(tY^{1-k+\epsilon}+t^{\Delta}\Big)\Big) \qquad (39)
\end{aligned}$$

from Lemma 4 (iii). The main term here is

$$\begin{aligned}
&\sideset{}{^*}\sum_{\substack{d_1\cdots d_r\le Y\\ (q,\mathbf{d}^k)|a+\mathbf{h}}} \mu(\mathbf{d})\left(\frac{t}{[q,d^{*k}]}+\mathcal{O}(1)\right)\\
&\quad= t\sideset{}{^*}\sum_{\substack{d_1,\ldots,d_r\\ (q,\mathbf{d}^k)|a+\mathbf{h}}} \frac{\mu(\mathbf{d})}{[q,d^{*k}]} + \mathcal{O}\left(t\sideset{}{^*}\sum_{d_1\cdots d_r>Y}\frac{1}{d^{*k}} + \sum_{d_1\cdots d_r\le Y} 1\right)\\
&\quad= t\eta(q,a) + \mathcal{O}\left(tY^{1-k+\epsilon}+Y^{1+\epsilon}\right)
\end{aligned}$$

from (4) and Lemma 4 (ii), so (39) becomes

$$\sum_{\substack{n\le t\\ n\in\mathcal{S}\\ n\equiv a(q)}} 1 = t\eta(q,a) + \mathcal{O}\Big(tY^{1-k+\epsilon}+Y^{1+\epsilon}+t^{\epsilon}\Big(tY^{1-k+\epsilon}+t^{\Delta}\Big)\Big)$$

which gives (i) on choosing $Y = t^{1/k}$.

(ii) From (38) we have for a parameter $Y \le t^{r/k}$ to be chosen

$$\sum_{\substack{n\le t\\ n\in\mathcal{S}\\ n\equiv \nu(q)}} 1 = \sideset{}{^*}\sum_{d_1\cdots d_r\le Y} \mu(\mathbf{d})\mathcal{N}^{q,\nu}_{\mathbf{d};\mathbf{h}}(t) + \mathcal{O}\left(\sideset{}{^*}\sum_{\substack{d_1\cdots d_r>Y\\ \nu\equiv-\mathbf{h}(q,\mathbf{d}^k)}} \sum_{\substack{n\le t\\ n\equiv-\mathbf{h}(\mathbf{d}^k)\\ n\equiv\nu(q)}} 1\right). \qquad (40)$$

The main term here is

$$\begin{aligned}
\sideset{}{^*}\sum_{\substack{d_1\cdots d_r\le Y\\ (q,\mathbf{d}^k)|\nu+\mathbf{h}}} \mu(\mathbf{d})\left(\frac{t}{[q,d^{*k}]}+\mathcal{O}(1)\right) &= t\sideset{}{^*}\sum_{\substack{d_1,\ldots,d_r\\ (q,\mathbf{d}^k)|\nu+\mathbf{h}}} \frac{\mu(\mathbf{d})}{[q,d^{*k}]} + \mathcal{O}\left(t\sideset{}{^*}\sum_{\substack{d_1\cdots d_r>Y\\ (q,\mathbf{d}^k)|\nu+\mathbf{h}}}\frac{1}{[q,d^{*k}]} + \sum_{d_1\cdots d_r\le Y} 1\right)\\
&= t\eta(q,\nu) + \mathcal{O}\left(t\sideset{}{^*}\sum_{\substack{d_1\cdots d_r>Y\\ \nu\equiv-\mathbf{h}(q,\mathbf{d}^k)}}\frac{1}{[q,d^{*k}]} + Y^{1+\epsilon}\right)
\end{aligned}$$

from (4), therefore (40) implies

$$\begin{aligned}
\sum_{\substack{n\le t\\ n\in\mathcal{S}\\ n\equiv \nu(q)}} 1 - t\eta(q,\nu) &\ll \sideset{}{^*}\sum_{\substack{d_1\cdots d_r>Y\\ \nu\equiv-\mathbf{h}(q,\mathbf{d}^k)}} \left(\sum_{\substack{n\le t\\ n\equiv-\mathbf{h}(\mathbf{d}^k)\\ n\equiv\nu(q)}} 1 + \frac{t}{[q,d^{*k}]}\right) + Y^{1+\epsilon}\\
&=: \mathcal{T}_Y(q,\nu) + Y^{1+\epsilon}. \qquad (41)
\end{aligned}$$

Now

$$\sum_{\nu=1}^{q} |\mathcal{T}_Y(q,\nu)|^2 \quad \leq \quad \sum_{\substack{d_1\cdots d_r>Y \\ d_1'\cdots d_r'>Y}}^{*} \; \sum_{\substack{\nu=1 \\ \nu\equiv-\mathbf{h}(q,\mathbf{d}^k) \\ \nu\equiv-\mathbf{h}(q,\mathbf{d'}^k)}}^{q} \left(\sum_{\substack{n,n'\leq t \\ n\equiv-\mathbf{h}(\mathbf{d}^k) \\ n'\equiv-\mathbf{h}(\mathbf{d'}^k) \\ n\equiv n'\equiv\nu(q)}} 1 + \frac{t^2}{[q,d^{*k}][q,d'^{*k}]} \right).$$

and the congruence conditions in the ν sum amount to one conrgruence modulo

$$[(q,d_1^k),...,(q,d_r^k),(q,{d_1'}^k),...,(q,{d_r'}^k)] = \frac{(q,d^{*k})(q,d'^{*k})}{(q,d^{*k},d'^{*k})}$$

so that the whole ν sum is

$$\leq \sum_{\nu=1}^{q} \sum_{\substack{n,n'\leq t \\ n\equiv-\mathbf{h}(\mathbf{d}^k) \\ n'\equiv-\mathbf{h}(\mathbf{d'}^k) \\ n\equiv n'\equiv\nu(q)}} 1 + \frac{t^2 q(q,d^{*k},d'^{*k})}{[q,d^{*k}][q,d'^{*k}](q,d^{*k})(q,d'^{*k})} = \sum_{\substack{n,n'\leq t \\ n\equiv-\mathbf{h}(\mathbf{d}^k) \\ n'\equiv-\mathbf{h}(\mathbf{d'}^k) \\ n\equiv n'(q)}} 1 + \frac{t^2(q,d^{*k},d'^{*k})}{qd^{*k}d'^{*k}}$$

and therefore

$$\sum_{\nu=1}^{q} |\mathcal{T}_Y(q,\nu)|^2 \quad \leq \quad \sum_{\substack{d_1\cdots d_r>Y \\ d_1'\cdots d_r'>Y}}^{*} \left(\sum_{\substack{n,n'\leq t \\ n\equiv-\mathbf{h}(\mathbf{d}^k) \\ n'\equiv-\mathbf{h}(\mathbf{d'}^k) \\ n\equiv n'(q)}} 1 + \frac{t^2(q,d^{*k},d'^{*k})}{qd^{*k}d'^{*k}} \right).$$

Since for any $N \in \mathbb{N}$

$$\sum_{q\leq\gamma} \frac{(q,N)}{q} \ll N^{\epsilon}(\log\gamma + 1) \ll N^{\epsilon}\gamma^{\epsilon}$$

we see that

$$\begin{aligned}
\sum_{q\leq\gamma}\sum_{\nu=1}^{q} |\mathcal{T}_Y(q,\nu)|^2 \quad &\ll \quad \sum_{\substack{d_1\cdots d_r>Y \\ d_1'\cdots d_r'>Y}}^{*} \left(t^{\epsilon} \sum_{\substack{n,n'\leq t \\ n\equiv-\mathbf{h}(\mathbf{d}^k) \\ n'\equiv-\mathbf{h}(\mathbf{d'}^k)}} 1 + \frac{t^2(d^{*k},d'^{*k})^{\epsilon}\gamma^{\epsilon}}{d^{*k}d'^{*k}} \right) \\
&\leq \quad t^{\epsilon}\left(\sum_{d_1\cdots d_r>Y}^{*} \mathcal{N}_{\mathbf{d},\mathbf{h}}(t) \right)^2 + t^2\gamma^{\epsilon}\left(\sum_{d_1\cdots d_r>Y}^{*} d^{*k(\epsilon-1)} \right)^2 \\
&\ll \quad t^{\epsilon}\left(t^2 Y^{2-2k+2\epsilon} + t^{2\Delta} \right) + t^2 Y^{2-2k+2\epsilon}\gamma^{\epsilon}
\end{aligned}$$

from Lemma 4 (iii) and (ii). Putting this in (41) we get, assuming $Y \le t$,

$$\begin{aligned}
&\sum_{q\le\gamma}\sum_{\nu=1}^{q}\left|\sum_{\substack{n\le t\\ n\in\mathcal{S}\\ n\equiv\nu(q)}} 1 - t\eta(q,\nu)\right|^2 \\
&\ll \quad t^{\epsilon}\left(t^2Y^{2-2k+2\epsilon}+t^{2\Delta}\right)+t^2Y^{2-2k+2\epsilon}\gamma^{\epsilon}+Y^{2+2\epsilon}\sum_{q\le\gamma}\sum_{\nu=1}^{q}1 \\
&\ll \quad t^2Y^{2-2k}+t^{2\Delta}+Y^2\gamma^2
\end{aligned}$$

the $t^{\epsilon}, \gamma^{\epsilon}$ terms going into the $\ll$ symbol. Choosing $Y = (t/\gamma)^{1/k}$ gives the first claim and the others follow from partial summation. $\square$

In the last stage of the proof we will be left with a quantity which we have chosen to analyse with Perron's formula. The main difficulty will be evaluating

$$\sum_{n\le X}\frac{\chi(n)}{n^s}$$

where χ is a Dirichlet character.

We make the convention that a primitive character may be principal, as in Chapter 9 of [11].

Lemma 6. *For any $M \in \mathbb{N}$, $Q > 0$, $t_0 \in \mathbb{R}$, $T \ge 1$, and any primitive character χ modulo M,*

$$\int_1^T \frac{L(iv+it_0,\chi)Q^{iv}dv}{v} \ll \sqrt{M}\left(1+\sqrt{|t_0|}\right).$$

Here the $\ll$ may contain terms up to $M^{\epsilon}, T^{\epsilon}, |t_0|^{\epsilon}$.

Proof. Throughout we allow the $M^{\epsilon}, T^{\epsilon}, |t_0|^{\epsilon}$ terms to go into the $\ll, \mathcal{O}$ symbols - we are basically telling the reader to ignore logs and epsilons.

We first suppose χ is non-principal. Take parameters $Z_0 > Z \gg 1$. Summing by parts and applying the Polya-Vinogradov Inequality (Theorem 9.18 of [11]) we have for any $t \in \mathbb{R}$

$$\begin{aligned}
\sum_{Z<n\le Z_0}\frac{\overline{\chi}(n)}{n^{1-it}} &= \frac{1}{Z^{1-it}}\sum_{Z<n\le Z_0}\overline{\chi}(n)+(1-it)\int_Z^{Z_0}\frac{1}{y^{2-it}}\left(\sum_{Z<n\le y}\overline{\chi}(n)\right)dy \\
&\ll \frac{(1+|t|)\sqrt{M}}{Z}
\end{aligned}$$

so that letting $Z_0 \to \infty$

$$L(1-it,\overline{\chi}) = \sum_{n\le Z}\frac{\overline{\chi}(n)}{n^{1-it}}+\mathcal{O}\left(\frac{1}{1+|t|}\right) \tag{42}$$

so long as

$$Z > (1+|t|)^2\sqrt{M}; \tag{43}$$

on the other hand Theorem 4.11 of [15] says that if $Z > 1+|t|$ then

$$\zeta(1-it) = \sum_{n\le Z}\frac{1}{n^{1-it}}+\mathcal{O}\left(\frac{1}{1+|t|}\right)$$

so that (42) subject to (43) remains true also in the case of principal χ (that is, $M = 1$). For any κ there is some A_κ for which

$$\sin\left(\frac{\pi(it+\kappa)}{2}\right) = A_\kappa e^{\pi|t|/2}\left(1+\mathcal{O}\left(\frac{1}{1+|t|}\right)\right)$$

and by standard formulas for the Gamma function there is some B for which

$$\Gamma(1-it) = B|t|^{1/2-it}e^{-\pi|t|/2+it}\left(1+\mathcal{O}\left(\frac{1}{1+|t|}\right)\right)$$

so that with (42) we have for any $t \in \mathbb{R}$

$$\begin{aligned} L(1-it,\overline{\chi})\Gamma(1-it)\sin\left(\frac{\pi(it+\kappa)}{2}\right) & \\ = \left(BA_\kappa|t|^{1/2-it}e^{it} + \mathcal{O}\left(\frac{1}{(1+|t|)^{1/2}}\right)\right)&\left(\sum_{n\le Z}\frac{\overline{\chi}(n)}{n^{1-it}} + \mathcal{O}\left(\frac{1}{1+|t|}\right)\right) \\ = BA_\kappa|t|^{1/2-it}e^{it}\sum_{n\le Z}\frac{\overline{\chi}(n)}{n^{1-it}} + \mathcal{O}(\log Z) & \end{aligned}$$

so long as (43) holds. Let κ and $\epsilon(\chi)$ be given respectively as in (10.15) and (10.17) of [11]; from the comments immediately following (10.17) the quantity $\epsilon(\chi)$ is bounded. Therefore Corollary 10.9 of [11] and the last equality say that for some $C_\chi \ll 1$ we have for any $t \in \mathbb{R}$

$$\begin{aligned} L(it,\chi) &= \pi^{-1}\epsilon(\chi)\sqrt{M}\left(\frac{2\pi}{M}\right)^{it} L(1-it,\overline{\chi})\Gamma(1-it)\sin\left(\frac{\pi(it+\kappa)}{2}\right) \\ &= C_\chi\sqrt{M}\left(\frac{2\pi}{M}\right)^{it}|t|^{1/2-it}e^{it}\sum_{n\le Z}\frac{\overline{\chi}(n)}{n^{1-it}} + \mathcal{O}\left(\sqrt{M}\log Z\right) \\ &= C_\chi\sqrt{M|t|}\sum_{n\le Z}\frac{\overline{\chi}(n)e^{it(\log(2\pi/M)-\log|t|+1+\log n)}}{n} + \mathcal{O}\left(\sqrt{M}\log Z\right) \end{aligned}$$

so long as (43) holds. Therefore for any $1 \le v \le T$ and so long as

$$Z > (1+T+|t_0|)^2\sqrt{M} \tag{44}$$

we have

$$\frac{L\left(i(v+t_0),\chi\right)Q^{iv}}{v} = \frac{C_\chi\sqrt{M|v+t_0|}}{v}\sum_{n\le Z}\frac{\overline{\chi}(n)e\left(f(v)\right)}{n} + \mathcal{O}\left(\frac{\sqrt{M}\log Z}{v}\right),$$

where

$$f(v) = f_{n,M,Q,t_0}(v) = \frac{(v+t_0)(\log(2\pi/M)-\log|v+t_0|+1+\log n)+v\log Q}{2\pi};$$

note that f is twice differentiable for $v + t_0 \neq 0$ and there we have

$$f''(v) = \pm\frac{1}{2\pi|v+t_0|}. \tag{45}$$

Therefore

$$\begin{aligned}
&\int_1^T \frac{L(iv+it_0,\chi)Q^{iv}dv}{v} \\
&= \quad C_\chi\sqrt{M}\int_1^T \frac{\sqrt{|v+t_0|}}{v}\left(\sum_{n\le Z}\frac{\overline{\chi}(n)e\left(f(v)\right)}{n}\right)dv + \mathcal{O}\left(\sqrt{M}\log Z\int_1^T\frac{dv}{v}\right) \\
&\ll \quad \sqrt{M}\sum_{n\le Z}\frac{1}{n}\left|\int_1^T G(v)e\left(f(v)\right)dv\right| + \sqrt{M}\log Z \qquad (46)
\end{aligned}$$

subject to (44), where

$$G(v) = \frac{\sqrt{|v+t_0|}}{v}.$$

We now bound the integral in (46). Take $R \ge 1$. For $v \in (R, 2R)$ we have $|v+t_0| \ll R + |t_0|$ so from (45)

$$v \in (R,2R)\backslash\{t_0\} \quad\Longrightarrow\quad |f''(v)| \gg \frac{1}{R+|t_0|} \quad\text{and}\quad G(v) \ll \frac{\sqrt{R+|t_0|}}{R} \qquad (47)$$

and we now consider the various scenarios for the sizes of R and t_0. Suppose first that R is large and $-t_0 \in (R+1, 2R-1)$. Then the above bounds become

$$v \in (R,2R)\backslash\{t_0\} \quad\Longrightarrow\quad |f''(v)| \gg \frac{1}{R} \quad\text{and}\quad G(v) \ll \frac{1}{\sqrt{R}}$$

so that from Lemma 4.5 of [15]

$$\begin{aligned}
\int_R^{2R} G(v)e\left(f(v)\right)dv &= \left(\int_R^{-t_0-1} + \int_{-t_0-1}^{-t_0+1} + \int_{-t_0+1}^{2R}\right) G(v)e\left(f(v)\right)dv \\
&\ll 1
\end{aligned}$$

having bounded the second integral crudely with (47). If $-t_0 \notin (R+1, 2R-1)$ then $v + t_0 \neq 0$ for $v \in (R+1, 2R-1)$ so the bounds in (47) and the same lemma imply

$$\begin{aligned}
\int_R^{2R} G(v)e\left(f(v)\right)dv &= \left(\int_R^{R+1} + \int_{R+1}^{2R-1} + \int_{2R-1}^{2R}\right) G(v)e\left(f(v)\right)dv \\
&\ll \frac{\sqrt{R+|t_0|}}{R} + \frac{\sqrt{R+|t_0|}}{R}\cdot\frac{1}{\sqrt{R+|t_0|}} + \frac{\sqrt{R+|t_0|}}{R} \\
&\ll 1 + \sqrt{|t_0|}
\end{aligned}$$

having bounded the first and third integrals crudely with (47). If R is not large then

$$\int_R^{2R} G(v)e\left(f(v)\right)dv \ll 1 + \sqrt{|t_0|} \qquad (48)$$

is clear from (47) so we conclude that (48) holds for all $R \ge 1$ and subject to no constraints on t_0. Consequently

$$\int_1^T G(v)e\left(f(v)\right)dv \ll 1 + \sqrt{|t_0|}$$

so (46) implies

$$\begin{aligned}\int_1^T \frac{L(iv+it_0,\chi)Q^{iv}dv}{v} &\ll \sqrt{M}\Big(1+\sqrt{|t_0|}\Big)\sum_{n\le Z}\frac{1}{n}+\sqrt{M}\log Z \\ &\ll \sqrt{M}\log Z\Big(1+\sqrt{|t_0|}\Big)\end{aligned}$$

which proves the lemma if we set for example $Z = 1+(1+T+|t_0|)^2\sqrt{M}$ in accordance with (44). $\square$

Suppose $x>0$ and $m\in\mathbb{N}$ with $m\le x^{\mathcal{O}(1)}$. For $w\in\mathbb{C}$ with $\mathfrak{Im}(w)\ll x^{\mathcal{O}(1)}$, and $\mathfrak{Re}(w)\ge 1/2$ and for a primitive character χ^* modulo m, we have from [13]

$$L(w,\chi^*)\ll x^{\epsilon}\sqrt{m(1+|\mathfrak{Im}(w)|)}$$

which with the functional equation (Corollary 10.10 of [11]) implies

$$L(w,\chi^*)\ll x^{\epsilon}\sqrt{m(1+|\mathfrak{Im}(w)|)} \tag{49}$$

for $\mathfrak{Re}(w)\in[0,1]$.

Lemma 7. *For $q,d\in\mathbb{N}$ with $d|q$ and $s\in\mathbb{C}$ define*

$$U_s(q,d)=\frac{1}{q}\sum_{D|d}D^s\phi(q/D).$$

Then for any $d,M\in\mathbb{N}$, $x,t\in\mathbb{R}$, $Q>0$, and any Dirichlet character χ mod M, we have for $d,M,|t|,Q\le x^{\mathcal{O}(1)}$

$$\sum_{u\le Q}\frac{\chi\left(u/(u,d)\right)}{\left(u/(u,d)\right)^s}=\frac{U_s(dM,d)Q^{1-s}}{1-s}+\mathcal{O}\left(x^{\epsilon}\sqrt{M(1+|t|)}\right)$$

where $s=it$, and where the main term is present if and only if χ is principal.

Proof. Throughout we write $s=it$ and for $w\in\mathbb{C}$ always $w=u+iv$, for real u,v. As in the last proof we allow the $\ll,\mathcal{O}$ symbols to contain terms up to x^{ϵ}.

Let χ^* be a primitive character of modulus m say, with $m\le x^{\mathcal{O}(1)}$. Since χ^* is principal if and only if $m=1$ we may define for any $X>0$

$$R_{\chi^*}(X)=\frac{X^{1-s}}{1-s}\begin{cases}1 & \text{if } m=1\\ 0 & \text{if } \chi^* \text{ is not principal.}\end{cases}$$

Write A for the implied constant in the hypothesis and take parameters $2\le X,T\le x^{A+2}$ with T so large that

$$T>|t| \tag{50}$$

and

$$T>X^2. \tag{51}$$

Perron's formula (Theorem 2 in Part II, Section 2 of [14]) implies for $\kappa>1$

$$\begin{aligned}\sum_{n\le X}\frac{\chi^*(n)}{n^s} &= \frac{1}{2\pi i}\int_{\kappa\pm iT}\frac{L(w+s,\chi^*)X^w dw}{w}+\mathcal{O}\left(X^{\kappa}\sum_{n=1}^{\infty}\frac{1}{n^{\kappa}\big(1+T|\log(X/n)|\big)}\right)\\ &=: I(X,T)+\mathcal{O}\Big(E(X,T)\Big).\end{aligned} \tag{52}$$

If $m = 1$ then $L(w+s,\chi^*) = \zeta(w+s)$ and if χ^* is non-principal then $L(w+s,\chi^*)$ is holomorphic for $u > 0$, so by the Residue Theorem and (50)

$$I(X,T) = R_{\chi^*}(X) \quad - \quad \frac{1}{2\pi i}\left(\int_{\kappa+iT}^{iT} + \int_{\mathcal{L}} + \int_{-iT}^{\kappa-iT}\right) \frac{L(w+s,\chi^*)X^w dw}{w}, \tag{53}$$

where $\mathcal{L}$ is the vertical line from iT to $-iT$ except for a half circle $\mathcal{C}$ from δi to $-\delta i$ to the right of 0, where $\delta = 1/\log X$. From (49) we have

$$\begin{aligned} \int_{\kappa+iT}^{iT} \frac{L(w+s,\chi^*)X^w dw}{w} &\ll X^\kappa \int_0^\kappa \frac{|L(u+iT+it)|du}{|u+iT|} \\ &\ll \frac{x^\epsilon X^\kappa \sqrt{m(1+T+|t|)}}{T} \end{aligned}$$

and similarly for the other horizontal integral in (53). For the vertical integral Lemma 6 and (49) imply

$$\begin{aligned} &\int_{\mathcal{L}} \frac{L(w+s,\chi^*)X^w dw}{w} \\ &\ll \left|\int_1^T \frac{L\left(iv+it,\chi^*\right)X^{iv}dv}{v}\right| + \left(\int_{\mathcal{C}} + \int_{\delta\le|v|\le1}\right) \frac{|L\left(w+s,\chi^*\right)|\cdot|X^w|\cdot dw}{|w|} \\ &\ll x^\epsilon\sqrt{m(1+|t|)} + X^\delta\sqrt{m(1+|t|)}\left(\int_{\mathcal{C}} + \int_{\delta\le|v|\le1}\right)\frac{dw}{|w|} \\ &\ll x^\epsilon\sqrt{m(1+|t|)} + X^{1/\log X}\sqrt{m(1+|t|)}\cdot|\log\delta| \\ &\ll x^\epsilon\sqrt{m(1+|t|)}. \end{aligned}$$

Using these bounds for the integrals in (53) and inserting the result into (52) we get

$$\sum_{n\le X}\frac{\chi^*(n)}{n^s} \quad = \quad R_{\chi^*}(X) + \mathcal{O}\left(x^\epsilon\left(\frac{X^\kappa\sqrt{m(1+T+|t|)}}{T} + \sqrt{m(1+|t|)} + E(X,T)\right)\right). \tag{54}$$

In general for $Z > -1$

$$|\log(1+Z)| \ge \frac{|Z|}{1+Z}.$$

For $X/2 \le n \le 3X/2$ we have $(n-X)/X > -1$ so that

$$|\log(X/n)| = \left|\log\left(1+\frac{n-X}{X}\right)\right| \ge \frac{|n-X|}{n} \ge \left\lfloor|n-X|\right\rfloor \Big/ n$$

and therefore

$$\sum_{X/2\le n\le 3X/2}\frac{1}{n^\kappa|\log(X/n)|} \le X^{1-\kappa}\left(1 + 2\sum_{h\le X}\frac{1}{h}\right) \ll X^{1-\kappa}. \tag{55}$$

If n is not in this range then $|\log(X/n)| \gg 1$ so from (55)

$$\begin{aligned} X^{\kappa}\sum_{n=1}^{\infty}\frac{1}{n^{\kappa}\big(1+T|\log(X/n)|\big)} &\ll X^{\kappa}\left(\frac{\zeta(\kappa)}{T}+\frac{X^{1-\kappa}}{T}\right) \\ &\ll \frac{1}{T}\left(\frac{X^{\kappa}}{\kappa-1}+X\right) \\ &\ll \frac{X}{T} \end{aligned}$$

if we set $\kappa = 1 + 1/\log X$. Therefore $E(X,T) \ll X/T$ which we put in (54) to get

$$\begin{aligned} \sum_{n\le X}\frac{\chi^*(n)}{n^s} &= R_{\chi^*}(X)+\mathcal{O}\left(x^{\epsilon}\left(\frac{X\sqrt{m(1+T+|t|)}}{T}+\sqrt{m(1+|t|)}+\frac{X}{T}\right)\right) \\ &= R_{\chi^*}(X)+\mathcal{O}\left(x^{\epsilon}\sqrt{m(1+|t|)}\right) \end{aligned}$$

from (51). The equality obviously still valid if $0 \le X \le 2$ we conclude that for any $0 < X \le x^{A+2}$

$$\sum_{n\le X}\frac{1}{n^s}=\frac{X^{1-s}}{1-s}+\mathcal{O}\left(x^{\epsilon}\sqrt{(1+|t|)}\right) \tag{56}$$

and

$$\sum_{n\le X}\frac{\chi^*(n)}{n^s} \ll x^{\epsilon}\sqrt{m(1+|t|)} \tag{57}$$

if χ^* is non-principal.

If χ is non-principal then there is an m with $m|M$ and non-principal primitive character χ^* mod m for which

$$\chi(n)=\begin{cases}\chi^*(n) & \text{if } (n,M)=1\\ 0 & \text{if not}\end{cases}$$

so that

$$\begin{aligned} \sum_{u\le Q}\frac{\chi\left(u/(u,d)\right)}{\left(u/(u,d)\right)^s} &= \sum_{D|d}\sum_{\substack{u\le Q/D\\(u,d/D)=1}}\frac{\chi(u)}{u^s} \\ &= \sum_{D|d}\sum_{\substack{u\le Q/D\\(u,dM/D)=1}}\frac{\chi^*(u)}{u^s} \\ &= \sum_{\substack{D|d\\ \Delta|dM/D}}\frac{\mu(\Delta)\chi^*(\Delta)}{\Delta^s}\sum_{u\le Q/D\Delta}\frac{\chi^*(u)}{u^s} \\ &\ll \sum_{D,\Delta|dM}\left|\sum_{u\le Q/D\Delta}\frac{\chi^*(u)}{u^s}\right| \\ &\ll x^{\epsilon}\sqrt{m(1+|t|)} \end{aligned}$$

from (57), which proves the lemma for χ non-principal. If χ is principal then we use (56) to deduce

$$\begin{aligned}
\sum_{u\leq Q}\frac{\chi\left(u/(u,d)\right)}{\left(u/(u,d)\right)^s} &= \sum_{\substack{u\leq Q\\(u/(u,d),M)=1}}\frac{1}{\left(u/(u,d)\right)^s}\\
&= \sum_{D|d}\sum_{\Delta|dM/D}\frac{\mu(\Delta)}{\Delta^s}\sum_{u\leq Q/D\Delta}\frac{1}{u^s}\\
&= \frac{Q^{1-s}}{1-s}\sum_{D|d}\frac{1}{D^{1-s}}\sum_{\Delta|dM/D}\frac{\mu(\Delta)}{\Delta}+\mathcal{O}\left(x^{\epsilon}\sqrt{1+|t|}\right)\\
&= \frac{Q^{1-s}}{1-s}\sum_{D|d}\frac{1}{D^{1-s}}\cdot\frac{\phi(dM/D)}{dM/D}+\mathcal{O}\left(x^{\epsilon}\sqrt{1+|t|}\right)
\end{aligned}$$

which proves the lemma for χ principal. □

2.2 - The circle method argument

Let $x, Q > 0$ be given. If $Q \leq \sqrt{x}$ then the first claim of Lemma 5 (ii) implies

$$V(x,Q) \ll x^{1+\theta+\epsilon}$$

which is stronger than our theorem, so we assume

$$Q > \sqrt{x}. \tag{58}$$

Since for $(d_i^k, d_j^k)|h_i - h_j$

$$\frac{1}{[d_1^k,...,d_r^k]} \ll \frac{1}{d_1^k\cdots d_r^k}$$

we have from (18)

$$\sum_{\substack{d_1,...,d_r\\d_i-d_j|h_i-h_j}}\frac{|g(\mathbf{d})|}{[q,[d_1,...,d_r]]} \ll \frac{1}{q}\sum_{d_1,...,d_r}\frac{(q,d_1^k\cdots d_r^k)}{d_1^k\cdots d_r^k} \ll \frac{1}{q}\sum_{n=1}^{\infty}(q,n^k)n^{\epsilon-k} \ll q^{\epsilon-1}. \tag{59}$$

From (5) and (4)

$$\begin{aligned}
V(x,Q) &= \sum_{q\leq Q}\sum_{a=1}^{q}\sum_{\substack{n,m\leq x\\n,m\in\mathcal{S}\\n\equiv m\equiv a(q)}}1-2x\sum_{q\leq Q}\sum_{a=1}^{q}\eta(q,a)\sum_{\substack{n\leq x\\n\in\mathcal{S}\\n\equiv a(q)}}1+x^2\sum_{q\leq Q}\sum_{a=1}^{q}\eta(q,a)^2\\
&=: S_1(x,Q)-2xS_2(x,Q)+x^2\sum_{q\leq Q}W(q).
\end{aligned}\tag{60}$$

For $d_1, ..., d_r \in \mathbb{N}$ write $d^* = [d_1, ..., d_r]$. Denote by V the unique solution modulo $[(q, d_1), ..., (q, d_r)] = (q, d^*)$ to $x \equiv -\mathbf{h}\left((q, \mathbf{d})\right)$. From (20)

$$\sum_{a=1}^{q}\eta(q,a)\sum_{\substack{n\leq x\\n\in\mathcal{S}\\n\equiv a(q)}}1=\sum_{\substack{n\leq x\\n\in\mathcal{S}}}\eta(q,n)=\sum_{\substack{d_1,...,d_r\\d_i-d_j|h_i-h_j}}\frac{g(\mathbf{d})}{[q,d^*]}\sum_{\substack{n\leq x\\n\in\mathcal{S}\\n\equiv V((q,d^*))}}1.$$

From Lemma 5 (i) the inner sum here is

$$A_{q,d^*}x + \mathcal{O}\left(x^{\Delta+\epsilon}\right)$$

for some A_{q,d^*} and therefore from (59)

$$\begin{aligned}\sum_{a=1}^{q}\eta(q,a)\sum_{\substack{n\leq x\\ n\in\mathcal{S}\\ n\equiv a(q)}}1 &= x\sum_{\substack{d_1,\ldots,d_r\\ d_i-d_j\mid h_i-h_j}}\frac{A_{q,d^*}g(\mathbf{d})}{[q,d^*]}+\mathcal{O}\left(x^{\Delta}\sum_{\substack{d_1,\ldots,d_r\\ d_i-d_j\mid h_i-h_j}}\frac{|g(\mathbf{d})|}{[q,d^*]}\right)\\ &=: xB_q+\mathcal{O}\left(x^{\Delta+\epsilon}q^{\epsilon-1}\right). \qquad (61)\end{aligned}$$

On the other hand Lemma 5 (i) says

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}\\ n\equiv a(q)}}1 = x\eta(q,a)+o(x)$$

so from (60)

$$\sum_{a=1}^{q}\eta(q,a)\sum_{\substack{n\leq x\\ n\in\mathcal{S}\\ n\equiv a(q)}}1 = x\sum_{a=1}^{q}\eta(q,a)^2+o(x)=xW(q)+o(x).$$

Therefore (61) implies

$$B_q = W(q)$$

and becomes

$$\sum_{a=1}^{q}\eta(q,a)\sum_{\substack{n\leq x\\ n\in\mathcal{S}\\ n\equiv a(q)}}1 = xW(q)+\mathcal{O}\left(x^{\Delta+\epsilon}q^{\epsilon-1}\right)$$

so that

$$S_2(x,Q)=\sum_{q\leq Q}\left(xW(q)+\mathcal{O}\left(x^{\Delta}q^{\epsilon-1}\right)\right)=x\sum_{q\leq Q}W(q)+\mathcal{O}\left(x^{\Delta+\epsilon}\right). \qquad (62)$$

Let ρ be as in Lemma 1. That lemma says $\eta(1,0)=\rho$ so from Lemma 5 (i)

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}}}1=\rho x+\mathcal{O}\left(x^{\Delta+\epsilon}\right)$$

so that

$$\sum_{q\leq Q}\sum_{\substack{n\leq x\\ n\in\mathcal{S}}}1=\rho xQ+\mathcal{O}\left(x^{1+\Delta+\epsilon}\right)$$

and therefore

$$\begin{aligned}S_1(x,Q) &= \sum_{q\leq Q}\sum_{\substack{n,m\leq x\\ n,m\in\mathcal{S}\\ n\equiv m(q)}}1\\ &= 2\sum_{q\leq Q}\sum_{\substack{m<n\leq x\\ n,m\in\mathcal{S}\\ n\equiv m(q)}}1+\sum_{q\leq Q}\sum_{\substack{n\leq x\\ n\in\mathcal{S}}}1\\ &=: 2S_4(x,Q)+\rho xQ+\mathcal{O}\left(x^{1+\Delta+\epsilon}\right).\end{aligned}$$

Putting this and (62) in (60) gives

$$V(x,Q) = 2S_4(x,Q) + \rho xQ - x^2 \sum_{q\le Q} W(q) + \mathcal{O}\left(x^{1+\Delta+\epsilon}\right) \tag{63}$$

and now our task is to study $S_4(x,Q)$ using the circle method. This argument is due to Vaughan (see [16] and, for the particular case of the k-free numbers, see [17]).

Let $\gamma > 0$ be a parameter. Consider the set of all irreducible fractions in $[0,1]$ with denominator not exceeding γ; the Farey fractions. If a/q is a Farey fraction, denote by $\mathfrak{F}(a/q)$ the interval from the median of a_1/q_1 and a/q to the median of a_2/q_2, where a_1/q_1 is the Farey fraction immediately preceeding a/q and a_2/q_2 that immediately proceeding a/q so that

$$\bigcup_{\substack{1\le a\le q\le\gamma\\(a,q)=1}} \mathfrak{F}(a/q)$$

is a partition of a unit interval; we are being a bit lazy concerning the end-points of the intervals but it will be clear soon enough that this is not important since we will only be integrating (functions continuous on $\mathbb{R}$) over these intervals and of course end-points then play no role. Denote by $\mathfrak{U}(a/q)$ the interval of unit length centered at a/q. It can be shown that

$$\left(\frac{a}{q} - \frac{1}{2q\gamma}, \frac{a}{q} + \frac{1}{2q\gamma}\right) \subseteq \mathfrak{F}(a/q) \subseteq \left(\frac{a}{q} - \frac{1}{q\gamma}, \frac{a}{q} + \frac{1}{q\gamma}\right) \subseteq \mathfrak{U}(a/q); \tag{64}$$

for a discussion of these matters, see Sections 3.1 and 3.8 of [4].

Circle method lemma. *Let $\theta, \Delta, \rho, G(\cdot), H,$ and Δ_t be as in Lemma 1 and Lemma 5, and denote by $\mathfrak{F}(a/q)$ the Farey arc at a/q in the Farey dissection of order γ. For $\alpha \in \mathbb{R}$ define*

$$f(\alpha) = \sum_{\substack{n\le x\\n\in\mathcal{S}}} e(n\alpha), \qquad F(\alpha) = \sum_{\substack{uv\le x\\u\le Q}} e(\alpha uv) \qquad \textit{and} \qquad I(\alpha) = \int_1^x e(\alpha t)dt.$$

For $\alpha \in \mathfrak{F}(a/q)$ write $\beta = \alpha - a/q$ and define for $\alpha \in \mathfrak{F}(a/q)$

$$J(\alpha) = -2\pi i\beta \int_1^x e(\beta t)\Delta_t(q,a)dt \qquad \textit{and} \qquad \hat{J}(\alpha) = e(x\beta)\Delta_x(q,a) + J(\alpha).$$

Then for $2\sqrt{x} \le \gamma \le x^{3/4}$

(A) *for $\alpha \in \mathfrak{F}(a/q)$,* $\quad f(\alpha) = \rho G(q)H(q,a)I(\beta) + \hat{J}(\alpha)$

(B) *for any $\beta \in \mathbb{R}$ and $q \le x$,* $\quad \sum_{a=1}^{q}{}' |H(q,a)|^2 F(-a/q-\beta) \ll x$

(C) $$\sum_{q\le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |\hat{J}(\alpha)|^2 d\alpha \ll x^{1+\theta} + \frac{x^{1+2\Delta}}{\gamma}$$

(D) $$\sum_{q\le 2\sqrt{x}} \overline{G(q)} \sum_{a=1}^{q}{}' \overline{H(q,a)} \int_{\mathfrak{F}(a/q)} F(-\alpha)\overline{I(\beta)}\hat{J}(\alpha)d\alpha \ll x^{1+\Delta} + \gamma x^{1/2+\theta}$$

(E) $$\sum_{2\sqrt{x}<q\le\gamma} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |f(\alpha)|^2 d\alpha \ll x\gamma$$

(F) $$\sum_{a=1}^{q}{}' \int_{\mathfrak{U}(a/q)\setminus\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |I(\alpha - a/q)|^2 d\alpha \ll q^2\gamma^2.$$

Here the $\ll$ symbol is allowed to contain terms of size x^ϵ.

Proof. This is essentially all contained in [17]. We will use the notaton $\prec\!\!\prec$ whenever the letter x is in context - we will write $f(x) \prec\!\!\prec g(x)$ to mean $f(x) \ll x^\epsilon g(x)$. We are basically telling the reader once again to ignore logs and epsilons, but the proof being longer we want to use an extra notation. Write $\lambda = 1/q\gamma$ so that (64) reads

$$\left(\frac{a}{q} - \lambda/2, \frac{a}{q} + \lambda/2\right) \subseteq \mathfrak{F}(a/q) \subseteq \left(\frac{a}{q} - \lambda, \frac{a}{q} + \lambda\right) \subseteq \mathfrak{U}(a/q). \tag{65}$$

For $\alpha \in \mathfrak{F}(a/q)$ we have from (65) that $|\beta| \leq \lambda \leq 1/2\sqrt{x}$ so we have, firstly, from display (2.7), Lemma 2.9, (the second part of) Lemma 2.11 and Lemma 2.12 of [17]

$$F(\alpha) \prec\!\!\prec \frac{x}{q\,(1 + x|\beta|)} + \sqrt{x} + q \ll \frac{x}{q} + q, \qquad \text{for } \alpha \in \mathfrak{F}(a/q), \tag{66}$$

so that from (65)

$$\int_{\mathfrak{F}(a/q)} |F(\alpha)| d\alpha \prec\!\!\prec \frac{1}{q} \int_{\pm\lambda} \frac{x d\beta}{1 + x|\beta|} + \lambda(\sqrt{x} + q) \prec\!\!\prec \frac{1}{q}, \tag{67}$$

and secondly

$$|F(\alpha)| \cdot |\beta| \prec\!\!\prec \frac{1}{q} + \left(\sqrt{x} + q\right)|\beta| \ll \frac{1}{q}, \qquad \text{for } \alpha \in \mathfrak{F}(a/q). \tag{68}$$

From displays (2.7), (2.9), (2.11), Lemma 2.9 and (taking $q = 1$ in the first part of) Lemma 2.11 of [17]

$$F(\alpha) \ll \sum_{u \leq \sqrt{x}} \frac{x}{u + x||u\alpha||} \qquad \text{for } \alpha \in \mathbb{R}. \tag{69}$$

Simply integrating shows

$$I(\beta) \ll \frac{x}{1 + x|\beta|}. \tag{70}$$

Now we prove the claims of the lemma.

(A) Write $f_t(\alpha) = \Sigma_{n \leq t, n \in \mathcal{S}} e(n\alpha)$ and recall from Lemma 5 the definition of $\Delta_t(q, a)$. Sorting the n according to the residue modulo q and using Lemma 1 we have for $t \leq x$

$$\begin{aligned} f_t(a/q) &= \sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right) \sum_{\substack{n \leq t \\ n \in \mathcal{S} \\ n \equiv \nu(q)}} 1 \\ &= t \sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right) \eta(q, \nu) + \Delta_t(q, a) \\ &= \rho G(q) H(q, a) t + \Delta_t(q, a) \end{aligned}$$

so that partial summation to

$$f(\alpha) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}}} e\left(\frac{an}{q} + \beta n\right)$$

gives

$$\begin{aligned} f(\alpha) &= e(x\beta)f_x(a/q) - 2\pi i\beta \int_1^x e(\beta t)f_t(\alpha)dt \\ &= \rho G(q)H(q,a)\left(xe(x\beta) - 2\pi i\beta\int_1^x te(\beta t)dt\right) \\ &+ \quad e(x\beta)\Delta_x(q,a) - 2\pi i\beta\int_1^x e(\beta t)\Delta_t(q,a)dt \end{aligned}$$

which gives the result after an integration by parts.

(B) Take $\Phi_q(n)$ as in Lemma 2. By part (iv) of that lemma

$$\begin{aligned} \sum_{a=1}^{q}{}' |H(q,a)|^2 F(-a/q-\beta) &= \sum_{\substack{uv\le x\\ u\le Q}} \Phi_q(-uv)e\left(-uv\beta\right) \\ &\ll x^\epsilon\left(x/q+1\right)\sum_{n=1}^{q}|\Phi_q(n)| \\ &\ll x^\epsilon\left(x/q+1\right)q^{1+\epsilon}. \end{aligned}$$

(C) From (68) and (65) we have for $\alpha \in \mathfrak{F}(a/q)$

$$|F(\alpha)|\cdot|J(\alpha)|^2 \ll |F(\alpha)|\cdot|\beta|^2\left|\int_1^x \Delta_t(q,a)e(\beta t)dt\right|^2 \lll \frac{1}{q^2\gamma}\left|\int_1^x \Delta_t(q,a)e(\beta t)dt\right|^2$$

and therefore from (65)

$$\begin{aligned} &\sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(\alpha)|\cdot|J(\alpha)|^2 d\alpha \\ &\quad\lll \frac{1}{q^2\gamma}\sum_{a=1}^{q}\int_{\pm\lambda}\left|\int_1^x \Delta_t(q,a)e(\beta t)dt\right|^2 d\beta \\ &\quad= \frac{1}{q^2\gamma}\int_1^x\int_1^x\left(\sum_{a=1}^{q}\Delta_t(q,a)\overline{\Delta_{t'}}(q,a)\right)\left(\int_{\pm\lambda} e\Big(\beta(t-t')\Big)d\beta\right)dt'dt. \end{aligned}$$

We have (Δ_t and E_t are defined in Lemma 5)

$$\sum_{a=1}^{q}\Delta_t(q,a)\overline{\Delta_{t'}}(q,a) = q\sum_{\nu=1}^{q}E_t(q,\nu)\overline{E_{t'}(q,\nu)} \tag{71}$$

and the second factor in the double integral above is

$$\ll \min\left(\frac{1}{|t-t'|},\lambda\right)$$

so

$$\begin{aligned}
&\sum_{q\le\gamma}\sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(\alpha)|\cdot|J(\alpha)|^2 d\alpha \\
&\quad \lll \quad \frac{1}{\gamma}\int_1^x\int_1^x \left(\sum_{q\le\gamma}\frac{1}{q}\sum_{\nu=1}^{q}\left|E_t(q,\nu)\overline{E_{t'}(q,\nu)}\right|\min\left(\frac{1}{|t-t'|},\lambda\right)\right)dt'dt \\
&\quad =: \quad \frac{\mathcal{V}(x,\gamma)}{\gamma}. \qquad (72)
\end{aligned}$$

Applying twice the Cauchy-Schwarz inequality we see that

$$\begin{aligned}
\mathcal{V}(x,\gamma) \quad &\le \quad \int_1^x\left(\sum_{q\le\gamma}\frac{1}{q}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\int_1^x\min\left(\frac{1}{|t-t'|},\lambda\right)dt'\right)dt \\
&\lll \quad x\cdot\max_{t\le x}\left(\sum_{q\le\gamma}\frac{1}{q}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\right) \qquad (73)
\end{aligned}$$

therefore from (72)

$$\sum_{q\le\gamma}\sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(\alpha)|\cdot|J(\alpha)|^2 d\alpha \lll \frac{x}{\gamma}\cdot\max_{t\le x}\left(\sum_{q\le\gamma}\frac{1}{q}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\right). \qquad (74)$$

From (67) and (71)

$$\sum_{a=1}^{q}{}' |\Delta_x(q,a)|^2\int_{\mathfrak{F}(a/q)}|F(\alpha)|d\alpha \lll \sum_{\nu=1}^{q}|E_x(q,\nu)|^2$$

therefore

$$\sum_{q\le 2\sqrt{x}}\sum_{a=1}^{q}{}' |\Delta_x(q,a)|^2\int_{\mathfrak{F}(a/q)}|F(\alpha)|d\alpha \lll \max_{t\le x}\left(\sum_{q\le 2\sqrt{x}}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\right)$$

which with (74) says

$$\begin{aligned}
&\sum_{q\le 2\sqrt{x}}\sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(\alpha)|\cdot|\hat{J}(\alpha)|^2 d\alpha \\
&\quad \lll \quad \max_{t\le x}\left(\sum_{q\le 2\sqrt{x}}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2+\frac{x}{\gamma}\sum_{q\le\gamma}\frac{1}{q}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\right) \\
&\quad \lll \quad x^{1+\theta}+x^{2\Delta}+\frac{x}{\gamma}\left(\gamma^{1-2\theta}x^{2\theta}+x^{2\Delta}\right)
\end{aligned}$$

from Lemma 5 (ii). The second term is bounded by the fourth term and the third is bounded by $x^{1+\theta}$.

(D) For $\alpha \in \mathfrak{F}(a/q)$ write

$$C(\alpha) = G(q)H(q,a)I(\beta).$$

For $\alpha \in \mathfrak{F}(a/q)\backslash(a/q - \lambda/2, a/q + \lambda/2)$ we have $|\beta| \gg \lambda$ so from (66) and (70)

$$F(-\alpha)|I(\beta)|^2 \lll \left(\frac{x}{q(1+x|\beta|)} + \gamma\right)\left(\frac{x}{1+x|\beta|}\right)^2 \ll \left(\frac{1}{q\lambda} + \gamma\right)\frac{1}{\lambda^2}$$

so that from (65)

$$\int_{\mathfrak{F}(a/q)\backslash(a/q-\lambda/2,a/q+\lambda/2)} |F(-\alpha)| \cdot |I(\beta)|^2 d\alpha \lll \frac{1}{\lambda}\left(\frac{1}{q\lambda} + \gamma\right) = q\gamma^2$$

and therefore by Lemma 2 (vi)

$$\begin{aligned}\sum_{q \le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)\backslash(a/q-\lambda/2,a/q+\lambda/2)} |F(-\alpha)| \cdot |C(\alpha)|^2 d\alpha \quad &\lll \quad \gamma^2 \sum_{q \le 2\sqrt{x}} |G(q)|^2 q^2 \\ &\lll \quad \gamma^2 x^{\theta/2}\end{aligned}$$

from Lemma 3. Therefore the Cauchy-Schwarz Inequality and part (C) imply

$$\begin{aligned}&\left(\sum_{q \le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)\backslash(a/q-\lambda/2,a/q+\lambda/2)} |F(-\alpha)| \cdot |C(\alpha)| \cdot |\hat{J}(\alpha)| d\alpha\right)^2 \\ &\quad \le \quad \left(\sum_{q \le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)\backslash(a/q-\lambda/2,a/q+\lambda/2)} |F(-\alpha)| \cdot |C(\alpha)|^2 d\alpha\right) \\ &\quad\quad \times \quad \left(\sum_{q \le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |\hat{J}(\alpha)|^2 d\alpha\right) \\ &\quad \lll \quad \gamma^2 x^{\theta/2}\left(x^{1+\theta} + \frac{x^{1+2\Delta}}{\gamma}\right)\end{aligned}$$

so that

$$\begin{aligned}&\sum_{q \le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)\backslash(a/q-\lambda/2,a/q+\lambda/2)} |F(-\alpha)| \cdot |C(\alpha)| \cdot |\hat{J}(\alpha)| d\alpha \\ &\quad\quad \lll \quad \gamma x^{1/2+3\theta/4} + \gamma^{1/2} x^{1/2+\Delta+\theta/4} \\ &\quad\quad \ll \quad \gamma x^{1/2+\theta} + x^{1+\Delta}. \qquad (75)\end{aligned}$$

For $\alpha \in \mathfrak{F}(a/q)$ and $u \le \sqrt{x}$ we have from (65)

$$|u\beta| \le \frac{1}{2q}$$

so if $q \not| u$ then

$$||u\alpha|| \ge \left|\left|\frac{ua}{q}\right|\right| - ||u\beta|| \gg \left|\left|\frac{ua}{q}\right|\right|$$

and therefore using the standard bound for a linear exponential sum

$$\begin{aligned}
H_q(\alpha) &:= \sum_{\substack{u\le\sqrt{x}\\ q\nmid u}}\left(\sum_{v\le x/u}+\sum_{\sqrt{x}<u\le Q,x/v}\right)e(\alpha uv)\\
&\ll \sum_{\substack{u\le\sqrt{x}\\ q\nmid u}}\frac{1}{||ua/q||}\\
&\ll (\sqrt{x}/q+1)\sum_{u=1}^{q}\frac{1}{||ua/q||}\\
&\ll q^{\epsilon}\Big(\sqrt{x}+q\Big) \qquad (76)
\end{aligned}$$

so that, breaking the u summation in the definition of F at $\sqrt{x}$ in view of (58),

$$\begin{aligned}
F(\alpha) &= \sum_{u\le\sqrt{x}}\sum_{v\le x/u}e(\alpha uv)+\sum_{v\le\sqrt{x}}\sum_{\sqrt{x}<u\le Q,x/v}e(\alpha uv)\\
&= \sum_{\substack{u\le\sqrt{x}\\ q|u}}\sum_{v\le x/u}e(\alpha uv)+\sum_{\substack{u\le\sqrt{x}\\ q|u}}\sum_{\sqrt{x}<v\le Q,x/u}e(\alpha uv)+H_q(\alpha)\\
&= \sum_{u\le\sqrt{x}/q}\sum_{v\le x/uq}e(\beta quv)+\sum_{u\le\sqrt{x}/q}\sum_{\sqrt{x}<v\le Q,x/uq}e(\beta quv)+\mathcal{O}\left(q^{\epsilon}\gamma\right)\\
&=: K_q(\beta)+\mathcal{O}\left(q^{\epsilon}\gamma\right) \qquad (77)
\end{aligned}$$

whenever $\alpha\in\mathfrak{F}(a/q)$. Therefore by Lemma 2 (vi)

$$\begin{aligned}
&\sum_{a=1}^{q}{}'\,\overline{H(q,a)}\int_{a/q\pm\lambda/2}F(-\alpha)\overline{I(\beta)}\hat{J}(\alpha)d\alpha\\
&\quad= \int_{\pm\lambda/2}K_q(-\beta)I(-\beta)\left(\sum_{a=1}^{q}{}'\,\overline{H(q,a)}\hat{J}(a/q+\beta)\right)d\beta+\mathcal{O}\left(q^{\epsilon}\gamma\sum_{a=1}^{q}{}'\int_{\mathfrak{F}(a/q)}\Big|I(\beta)\hat{J}(\alpha)\Big|\,d\alpha\right)\\
&\quad=: \int_{\pm\lambda/2}K_q(-\beta)A_q(\beta)d\beta+\mathcal{O}\left(q^{\epsilon}\gamma\sum_{a=1}^{q}{}'\int_{\mathfrak{F}(a/q)}|B(\alpha)|d\alpha\right)
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_{q\le2\sqrt{x}}\sum_{a=1}^{q}{}'\int_{a/q\pm\lambda/2}F(-\alpha)\overline{C(\alpha)}\hat{J}(\alpha)d\alpha\\
&\quad\ll_{\epsilon} \sum_{q\le2\sqrt{x}}|G(q)|\left|\int_{\pm\lambda/2}K_q(-\beta)A_q(\beta)d\beta\right|+\gamma\sum_{q\le2\sqrt{x}}|G(q)|\sum_{a=1}^{q}{}'\int_{\mathfrak{F}(a/q)}|B(\alpha)|d\alpha
\end{aligned}$$

and therefore from (75)

$$\begin{aligned}
&\sum_{q\le 2\sqrt{x}} \overline{G(q)} \sum_{a=1}^{q}{}' \overline{H(q,a)} \int_{\mathfrak{F}(a/q)} F(-\alpha)\overline{I(\beta)}\hat{J}(\alpha)d\alpha \\
&= \sum_{q\le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{a/q\pm\lambda/2} F(-\alpha)\overline{C(\alpha)}\hat{J}(\alpha)d\alpha \\
&+ \mathcal{O}\left(\sum_{q\le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)\setminus(a/q-\lambda/2,a/q+\lambda/2)} |F(-\alpha)|\cdot|C(\alpha)|\cdot|\hat{J}(\alpha)|d\alpha \right) \\
&\lll \sum_{q\le\gamma} |G(q)| \left| \int_{\pm\lambda/2} K_q(-\beta)A_q(\beta)d\beta \right| \\
&+ \gamma \sum_{q\le\gamma} |G(q)| \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |B(\alpha)|d\alpha + \gamma x^{1/2+\theta} + x^{1+\Delta}. \qquad (78)
\end{aligned}$$

Recall the definition of $\Delta_t(q,a)$ from Lemma 5 and of $H(q,a)$ from Lemma 2. Let $\Phi^*(n)$ be as in Lemma 2. From Lemma 5 (A) and then Lemma 2 (iv) we have for $q,t\le x$

$$\begin{aligned}
\sum_{a=1}^{q}{}' \Delta_t(q,a)\overline{H(q,a)} &= \sum_{\nu=1}^{q} \Phi_q^*(\nu) \left(\sum_{\substack{n\le t \\ n\in\mathcal{S} \\ n\equiv\nu(q)}} 1 - t\eta(q,v) \right) \\
&\lll t^{\Delta} \sum_{\nu=1}^{q} |\Phi_q^*(\nu)| \\
&\lll qt^{\Delta}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{a=1}^{q}{}' \overline{H(q,a)}\hat{J}(a/q+\beta) &\ll \left| \sum_{a=1}^{q}{}' \Delta_x(q,a)\overline{H(q,a)} \right| + |\beta| \int_1^x \left| \sum_{a=1}^{q}{}' \Delta_t(q,a)\overline{H(q,a)} \right| dt \\
&\lll qx^{\Delta}(1+|\beta|x)
\end{aligned}$$

and therefore from (70)

$$A_q(\alpha) \lll qx^{1+\Delta}. \qquad (79)$$

Therefore from (77), (65) and (67)

$$\int_{\pm\lambda/2} K_q(-\beta)A_q(\beta)d\beta \lll qx^{1+\Delta} \int_{\pm 1/2q\gamma} \Big(|F(-a/q-\beta)| + \gamma \Big) d\beta \lll qx^{1+\Delta} \left(\frac{1}{q} + \gamma\lambda \right) \ll x^{1+\Delta}$$

and so from Lemma 3

$$\sum_{q\le\gamma} |G(q)| \left| \int_{\pm\lambda/2} K_q(-\beta)A_q(\beta)d\beta \right| \lll x^{1+\Delta}. \qquad (80)$$

We have from (65)

$$\begin{aligned}\sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |J(\alpha)|^2 d\alpha \\ \leq \quad \int_1^x \int_1^x \left(\sum_{a=1}^{q} \Delta_t(q,a)\overline{\Delta_{t'}(q,a)} \right) \cdot \left| \int_{\pm\lambda} |\beta|^2 e\Big(\beta(t-t')\Big) d\beta \right| dt' dt.\end{aligned}$$

The first factor is from (71)

$$q \sum_{\nu=1}^{q} E_t(q,\nu)\overline{E_{t'}(q,\nu)}$$

and the second factor is

$$\ll \min\left(\frac{\lambda^2}{t-t'}, \lambda^3\right)$$

so

$$\begin{aligned}&\sum_{q\leq\gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |J(\alpha)|^2 d\alpha \\ &\ll \quad \int_1^x \int_1^x \left(\sum_{q\leq\gamma} q^\theta \sum_{\nu=1}^{q} E_t(q,\nu)\overline{E_{t'}(q,\nu)} \min\left(\frac{\lambda^2}{t-t'}, \lambda^3\right) \right) dt' dt \\ &=: \quad \mathcal{U}(x,\gamma). \qquad (81)\end{aligned}$$

As in (73) we have

$$\begin{aligned}\mathcal{U}(x,\gamma) \quad &\leq \quad \int_1^x \left(\sum_{q\leq\gamma} q^\theta \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \int_1^x \min\left(\frac{\lambda^2}{t-t'}, \lambda^3\right) dt' \right) dt \\ &\lll \quad \frac{x}{\gamma^2} \cdot \max_{t\leq x} \left(\sum_{q\leq\gamma} q^{\theta-2} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \right)\end{aligned}$$

so that (81) says

$$\sum_{q\leq\gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |J(\alpha)|^2 d\alpha \lll \frac{x}{\gamma^2} \cdot \max_{t\leq x} \left(\sum_{q\leq\gamma} \frac{1}{q^{2-\theta}} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \right). \qquad (82)$$

From (65) and (71)

$$\sum_{a=1}^{q}{}' |\Delta(a/q)|^2 \int_{\mathfrak{F}(a/q)} d\alpha \lll q\lambda \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 = \frac{1}{\gamma} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2$$

so that

$$\sum_{q\leq\gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |\Delta(a/q)|^2 d\alpha \lll \frac{1}{\gamma} \cdot \max_{t\leq x} \left(\sum_{q\leq\gamma} \frac{1}{q^{1-\theta}} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \right)$$

and therefore from (82) and Lemma 5 (ii)

$$\begin{aligned}
&\sum_{q\le\gamma}\frac{1}{q^{1-\theta}}\sideset{}{'}\sum_{a=1}^{q}\int_{\mathfrak{F}(a/q)}|\hat{J}(\alpha)|^2 d\alpha \\
&\quad\lll \frac{1}{\gamma}\cdot\max_{t\le x}\left(\sum_{q\le\gamma}\frac{1}{q^{1-\theta}}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2+\frac{x}{\gamma}\sum_{q\le\gamma}\frac{1}{q^{2-\theta}}\sum_{\nu=1}^{q}|E_t(q,\nu)|^2\right) \\
&\quad\lll \frac{1}{\gamma}\left(\gamma^{1-\theta}x^{2\theta}+x^{2\Delta}+\frac{x^{1+2\Delta}}{\gamma}\right) \\
&\quad\ll x^{2\theta}+\frac{x^{1+2\Delta}}{\gamma^2}. \qquad (83)
\end{aligned}$$

From orthogonality

$$\int_{\pm\lambda}|I(\beta)|^2\ll x$$

so from Lemma 3

$$\sum_{q\le\gamma}q^{1-\theta}|G(q)|^2\sum_{a=1}^{q}\int_{\pm\lambda}|I(\beta)|^2 d\beta\lll x. \qquad (84)$$

From the Cauchy-Schwarz inequality and then (83) and (84)

$$\begin{aligned}
&\sum_{q\le\gamma}|G(q)|\sideset{}{'}\sum_{a=1}^{q}\int_{\mathfrak{F}(a/q)}|B_q(\alpha)|d\alpha \\
&\quad\le\left(\sum_{q\le\gamma}q^{1-\theta}|G(q)|^2\sum_{a=1}^{q}\int_{\pm\lambda}|I(\beta)|^2 d\beta\right)^{1/2}\left(\sum_{q\le\gamma}\frac{1}{q^{1-\theta}}\sideset{}{'}\sum_{a=1}^{q}\int_{\mathfrak{F}(a/q)}|\hat{J}(\alpha)|^2 d\alpha\right)^{1/2} \\
&\quad\lll x^{1/2+\theta}+\frac{x^{1+\Delta}}{\gamma}. \qquad (85)
\end{aligned}$$

From this, (78) and (80) we deduce

$$\sum_{q\le 2\sqrt{x}}\overline{G(q)}\sideset{}{'}\sum_{a=1}^{q}\overline{H(q,a)}\int_{\mathfrak{F}(a/q)}F(-\alpha)\overline{I(\beta)}\hat{J}(\alpha)d\alpha\lll\gamma\left(x^{1/2+\theta}+\frac{x^{1+\Delta}}{\gamma}\right).$$

(E) For $q\gg\sqrt{x}$ and $\alpha\in\mathfrak{F}(a/q)$ we have from (66)

$$F(\alpha)\ll\gamma$$

therefore

$$\sum_{2\sqrt{x}<q\le\gamma}\sideset{}{'}\sum_{a=1}^{q}\int_{\mathfrak{F}(a/q)}|F(-\alpha)|\cdot|f(\alpha)|^2 d\alpha\ll\gamma\int_0^1|f(\alpha)|^2\ll x\gamma \qquad (86)$$

by orthogonality.

(F) Write

$$\mathcal{V}=\sideset{}{'}\sum_{a=1}^{q}\sum_{u\le\sqrt{x}}\int_{u/2q\gamma<|\beta|\le 1/2}\frac{x d\beta}{|\beta|^2(u+x||ua/q+u\beta||)}$$

so that (69) and the bound $I(\beta) \ll 1/|\beta|$ (from (70)) says

$$\sum_{a=1}^{q}{}' \int_{1/2q\gamma<|\beta|\leq 1/2} |F(-a/q-\beta)| \cdot |I(\beta)|^2 d\beta \lll \mathcal{V}. \tag{87}$$

We have

$$\int_{1/2q\gamma<|\beta|\leq 1/2} \frac{xd\beta}{\beta^2 \left(u + x||ua/q + u\beta||\right)} \quad = \quad u \int_{u/2q\gamma<|t|\leq u/2} \frac{xdt}{t^2 \left(u + x||ua/q + t||\right)}$$

and the part of the integral with $t \geq 1/2$ is

$$\leq \sum_{0<|j|\leq u/2} \int_{-1/2}^{1/2} \frac{xdt}{(j+t)^2 \left(u + x||ua/q + j + t||\right)} \ll 1$$

so that the whole integral in the definition of $\mathcal{V}$ is

$$u \int_{u/2q\gamma<|t|\leq 1/2} \frac{xdt}{t^2 \left(u + x||ua/q + t||\right)} + \mathcal{O}(u)$$

and therefore

$$\begin{aligned}
\mathcal{V} \quad &= \quad \sum_{u\leq\sqrt{x}} u \int_{u/2q\gamma<|t|\leq 1/2} \frac{1}{t^2} \left(\sum_{a=1}^{q}{}' \frac{x}{u + x||ua/q + t||} \right) dt + \mathcal{O}\left(\sum_{a=1}^{q}{}' \sum_{u\leq\sqrt{x}} u \right) \\
&= \quad \sum_{u\leq\sqrt{x}} u \int_{u/2q\gamma<|t|\leq 1/2} \frac{1}{t^2} \left(\sum_{a=1}^{q}{}' \frac{x}{u + x||ua/q + t||} \right) dt + \mathcal{O}\left(xq\right).
\end{aligned}$$

Write $q' = q/(q,u)$ and $u' = u/(q,u)$. The inner sum is

$$\begin{aligned}
&\leq \quad (q,u) \sum_{a=1}^{q'} \frac{x}{u + x||u'a/q' + t||} \\
&= \quad (q,u) \sum_{a=1}^{q'} \frac{x}{u + x||a/q' + t||} \\
&\ll \quad (q,u) \sum_{\substack{a=1 \\ ||a/q'+t||\leq 1/3q'}}^{q'/2} \frac{x}{u + x||a/q' + t||} + q'(q,u)
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{V} \quad &\ll \quad \sum_{u\leq\sqrt{x}} u(q,u) \int_{u/2q\gamma<|t|\leq 1/2} \frac{1}{t^2} \left(\sum_{\substack{a=1 \\ ||a/q'+t||\leq 1/3q'}}^{q'/2} \frac{x}{u + x||a/q' + t||} \right) dt \\
&\qquad + \quad xq \quad + \quad q \sum_{u\leq\sqrt{x}} u \int_{u/q\gamma<|t|\leq 1/2} \frac{dt}{t^2} \\
&\ll \quad \sum_{u\leq\sqrt{x}} u(q,u) \int_{u/2q\gamma<|t|\leq 1/2} \left(\sum_{\substack{a=1 \\ ||a/q'+t||\leq 1/3q'}}^{q'/2} \frac{x}{t^2(u + x||a/q' + t||)} \right) dt + q^2\gamma\sqrt{x}.
\end{aligned}$$

In the sum we have $t \gg a/q'$ so that the whole integral is for $q \le x$

$$\ll \quad {q'}^2 \int_{u/2q\gamma<|t|\le 1/2} \left(\sum_{a=1}^{q'/2} \frac{x}{a^2(u+x||a/q'+t||)} \right) dt \lll {q'}^2$$

and therefore

$$\mathcal{V} \lll \sum_{u\le\sqrt{x}} u(q,u){q'}^2 + q^2\gamma\sqrt{x} \ll q^2\gamma\sqrt{x}$$

so that from (87)

$$\sum_{a=1}^{q}{}' \int_{1/2q\gamma<|\beta|\le 1/2} |F(-a/q-\beta)| \cdot |I(\beta)|^2 d\beta \lll q^2\gamma^2$$

or in other words

$$\sum_{a=1}^{q}{}' \int_{\mathfrak{U}(a/q)\setminus X} |F(-\alpha)| \cdot |I(\alpha-a/q)|^2 d\alpha \ll q^2\gamma^2$$

for any subset

$$\left(\frac{a}{q}-\frac{1}{2q\gamma}, \frac{a}{q}+\frac{1}{2q\gamma}\right) \subseteq X \subseteq \mathfrak{U}(a/q).$$

The result now follows from (65). □

Let F, f be as in the Circle Method Lemma. Writing the congruence condition in $S_4(x,Q)$ out explicitly and using orthogonality we have

$$S_4(x,Q) \quad = \quad \sum_{q\le Q} \sum_{l\le x/q} \sum_{\substack{n,m\le x \\ n,m\in\mathcal{S} \\ n-m=ql}} 1 = \int_0^1 F(-\alpha)|f(\alpha)|^2 d\alpha. \tag{88}$$

As in the comments preceeding the Circle Method Lemma, denote by $\mathfrak{F}(a/q)$ the Farey arc at a/q in the Farey dissection of order γ, where $(a,q)=1$ and $2\sqrt{x} \le \gamma \le x^{3/4}$. Then (88) implies

$$\begin{aligned} S_4(x,Q) \quad &= \quad \sum_{q\le 2\sqrt{x}} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} F(-\alpha)|f(\alpha)|^2 d\alpha \\ &\qquad + \quad \mathcal{O}\left(1 + \sum_{2\sqrt{x}<q\le\gamma} \sum_{a=1}^{q}{}' \int_{\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |f(\alpha)|^2 d\alpha\right) \\ &=: \quad M(\gamma) + \mathcal{O}\Big(1 + E(\gamma)\Big). \end{aligned} \tag{89}$$

Let $\theta, \Delta, \rho, G, H, I$ and $\hat{J}$ be as in the Circle Method Lemma and as in that lemma write $\alpha = a/q + \beta$ whenever $\alpha \in \mathfrak{F}(a/q)$. From the Circle Method Lemma (A)

$$|f(\alpha)|^2 = |\rho G(q)H(q,a)I(\beta)|^2 + 2\mathfrak{Re}\left(\overline{\rho G(q)H(q,a)I(\beta)}\hat{J}(\alpha)\right) + |\hat{J}(\alpha)|^2$$

so that from the Circle Method Lemma (D) and (C) we have

$$\begin{aligned} M(\gamma) &= \rho^2 \sum_{q\le 2\sqrt{x}} |G(q)|^2 \sideset{}{'}\sum_{a=1}^{q} |H(q,a)|^2 \int_{\mathfrak{F}(a/q)} F(-\alpha)|I(\beta)|^2 d\alpha \\ &+ 2\mathfrak{Re}\left(\overline{\rho} \sum_{q\le 2\sqrt{x}} \overline{G(q)} \sideset{}{'}\sum_{a=1}^{q} \overline{H(q,a)} \int_{\mathfrak{F}(a/q)} F(-\alpha)\overline{I(\beta)}\hat{J}(\alpha)d\alpha\right) \\ &+ \sum_{q\le 2\sqrt{x}} \sideset{}{'}\sum_{a=1}^{q} \int_{\mathfrak{F}(a/q)} F(-\alpha)|\hat{J}(\alpha)|^2 d\alpha \\ &= \rho^2 \sum_{q\le 2\sqrt{x}} |G(q)|^2 \sideset{}{'}\sum_{a=1}^{q} |H(q,a)|^2 \int_{\mathfrak{F}(a/q)} F(-\alpha)|I(\beta)|^2 d\alpha + \mathcal{O}\left(x^\epsilon\left(x^{1+\Delta} + \gamma x^{1/2+\theta} + \frac{x^{1+2\Delta}}{\gamma}\right)\right) \\ &=: M^*(\gamma) + \mathcal{O}\left(x^\epsilon\left(x^{1+\Delta} + \gamma x^{1/2+\theta} + \frac{x^{1+2\Delta}}{\gamma}\right)\right) \end{aligned}$$

so that (89) says

$$S_4(x,Q) = M^*(\gamma) + \mathcal{O}\left(x^\epsilon\left(x^{1+\Delta} + \gamma x^{1/2+\theta} + \frac{x^{1+2\Delta}}{\gamma} + E(\gamma)\right)\right). \tag{90}$$

Take a parameter $Z \le 2\sqrt{x}$. From Lemma 3

$$\sum_{q\le Z} q^2 |G(q)|^2 \ll Z^{\theta+\epsilon}$$

so from Lemma 2 (vi) and the Circle Method Lemma (F)

$$\sum_{q\le Z} |G(q)|^2 \sideset{}{'}\sum_{a=1}^{q} |H(q,a)|^2 \int_{\mathfrak{U}(a/q)\setminus\mathfrak{F}(a/q)} |F(-\alpha)|\cdot|I(\alpha - a/q)|^2 d\alpha \ll x^\epsilon \gamma^2 Z^\theta, \tag{91}$$

where $\mathfrak{U}(a/q)$ denotes the unit interval centered at a/q. On the other hand the Circle Method Lemma (B) and then orthogonality gives

$$\sideset{}{'}\sum_{a=1}^{q} |H(q,a)|^2 \int_X F(-a/q-\beta)|I(\beta)|^2 d\beta \ll x^{1+\epsilon} \int_{-1/2}^{1/2} |I(\beta)|^2 d\beta \le x^{2+\epsilon}$$

for any $X \subseteq [-1/2, 1/2]$, and from Lemma 3

$$\sum_{q>Z} |G(q)|^2 \ll Z^{\theta-2+\epsilon},$$

therefore

$$\sum_{q>Z} |G(q)|^2 \left| \sideset{}{'}\sum_{a=1}^{q} |H(q,a)|^2 \int_X F(-a/q-\beta)|I(\beta)|^2 d\beta \right| \ll x^{2+\epsilon} Z^{\theta-2}. \tag{92}$$

From (91) and (92)

$$
\begin{aligned}
M^*(\gamma) = \rho^2 \sum_{q\le x} |G(q)|^2 \sum_{a=1}^{q}{}' |H(q,a)|^2 \int_{\mathfrak{U}(a/q)} F(-\alpha)|I(\alpha - a/q)|^2 d\alpha \\
+ \quad \mathcal{O}\Bigg(\sum_{q\le Z} |G(q)|^2 \sum_{a=1}^{q}{}' |H(q,a)|^2 \int_{\mathfrak{U}(a/q)\setminus\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |I(\alpha - a/q)|^2 d\beta \\
+ \quad \sum_{q>Z} |G(q)|^2 \left| \sum_{a=1}^{q}{}' |H(q,a)|^2 \int_{\mathfrak{U}(a/q)\setminus\mathfrak{F}(a/q)} F(-\alpha)|I(\alpha - a/q)|^2 d\alpha \right| \\
+ \quad \sum_{q>Z} |G(q)|^2 \left| \sum_{a=1}^{q}{}' |H(q,a)|^2 \int_{\mathfrak{U}(a/q)} F(-\alpha)|I(\alpha - a/q)|^2 d\alpha \right| \Bigg) \\
= \quad \rho^2 \sum_{q\le x} |G(q)|^2 \sum_{a=1}^{q}{}' |H(q,a)|^2 \int_{\mathfrak{U}(a/q)} F(-\alpha)|I(\alpha - a/q)|^2 d\alpha + \mathcal{O}\left(x^{\epsilon}\left(\gamma^2 Z^{\theta} + x^2 Z^{\theta-2}\right)\right) \\
= \quad \rho^2 \sum_{q\le x} |G(q)|^2 \sum_{a=1}^{q}{}' |H(q,a)|^2 \int_{-1/2}^{1/2} F(-a/q-\beta)|I(\beta)|^2 d\beta + \mathcal{O}\left(\gamma^{2-\theta} x^{\theta+\epsilon}\right) \qquad (93)
\end{aligned}
$$

on choosing $Z = x/\gamma$. Since

$$
\int_{\mathfrak{U}(a/q)} |I(\alpha - a/q)|^2 e\left(-(\alpha - a/q)n\right) d\beta = \sum_{\substack{N,N'\le x \\ N-N'=n}} 1 = x - n + \mathcal{O}(1)
$$

we have (F is defined in the Circle method lemma)

$$
\int_{\mathfrak{U}(a/q)} F(-\alpha)|I(\alpha - a/q)|^2 d\alpha = \sum_{\substack{uv\le x \\ u\le Q}} e(-auv/q)(x - uv) + \mathcal{O}(x)
$$

so we deduce from (93), Lemma 2 (vi) and Lemma 3

$$
\begin{aligned}
M^*(\gamma) \quad &= \quad \rho^2 \sum_{q\le x} |G(q)|^2 \sum_{a=1}^{q}{}' |H(q,a)|^2 \left(\sum_{\substack{uv\le x \\ u\le Q}} e(-auv/q)\Big(x - uv\Big) + \mathcal{O}(x) \right) \\
&+ \quad \mathcal{O}\left(x \sum_{q\le x} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \right) + \mathcal{O}\left(\gamma^{2-\theta} x^{\theta+\epsilon}\right) \\
&= \quad \rho^2 \sum_{q\le x} |G(q)|^2 \sum_{\substack{uv\le x \\ u\le Q}} \Big(x - uv\Big)\Phi_q(-uv) + \mathcal{O}\left(x^{\epsilon}\Big(x + \gamma^{2-\theta}x^{\theta}\Big)\right),
\end{aligned}
$$

where $\Phi_q(n)$ is as in Lemma 2, so from (90)

$$
\begin{aligned}
S_4(x,Q) \quad &= \quad \rho^2 \sum_{q\le x} |G(q)|^2 \sum_{\substack{uv\le x \\ u\le Q}} \Big(x - uv\Big)\Phi_q(-uv) \\
&+ \quad \mathcal{O}\left(x^{\epsilon} \left(x^{1+\Delta} + \gamma x^{1/2+\theta} + \frac{x^{1+2\Delta}}{\gamma} + E(\gamma) + \gamma^{2-\theta} x^{\theta} \right) \right). \qquad (94)
\end{aligned}
$$

Recall that $\theta = 1/k$ and $\Delta = 2/(k+1)$. If $k > 2$ we set $\gamma = 2\sqrt{x}$ so that $E(\gamma) = 0$ and $\gamma \geq x^{\Delta}$ to deduce that

$$\begin{aligned} S_4(x, Q) &= \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{\substack{uv \leq x \\ u \leq Q}} \Big(x - uv\Big) \Phi_q(-uv) + \mathcal{O}\left(x^{1+\Delta+\epsilon}\right) \\ &=: \rho^2 \mathcal{J}(x, Q) + \mathcal{O}\left(x^{1+\Delta+\epsilon}\right). \end{aligned} \tag{95}$$

If $k = 2$ we set $\gamma = x^{2/3}$ and deduce from the Circle method lemma (E) that the error term in (94) is up to an x^{ϵ} bound

$$\ll x^{5/3} + \gamma x + \frac{x^{7/3}}{\gamma} + x\gamma + \gamma^{3/2} x^{1/2} \ll x^{5/3} = x^{1+\Delta}$$

to conclude that (95) holds for all $k \geq 2$.

2.3 - Completion of proof

The circle method work being done it remains to evaluate $\mathcal{J}(x, Q)$. We use Perron's formula.

We make the convention that whenever we have the letter $\mathcal{D}$ appearing in a context involving natural numbers q, a we mean $\mathcal{D} = (q, a)$. For any $u \in \mathbb{N}$ we then write $u' = u/(u, \mathcal{D})$. Sorting the uv according to the residue a mod q we have

$$\begin{aligned} \mathcal{J}(x, Q) &= \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} \Phi_q(-a) \sum_{\substack{uv \leq x \\ u \leq Q \\ uv \equiv a(q)}} \Big(x - uv\Big) \\ &= \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} \Phi_q(-a) \sum_{\substack{uv \leq x \\ u \leq Q \\ \mathcal{D} | uv \\ uv/\mathcal{D} \equiv a'(q')}} \Big(x - uv\Big) \\ &=: \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} \Phi_q(-a) \mathcal{U}(q, a). \end{aligned} \tag{96}$$

For $n \in \mathbb{N}$ with $(n, q') = 1$ denote by $\overline{n}$ the inverse of n modulo q'. We have

$$\begin{aligned} \mathcal{U}(q, a) &= \sum_{u \leq Q} \sum_{\substack{v \leq x/u \\ \mathcal{D}/(\mathcal{D}, u) | v \\ uv/\mathcal{D} \equiv a'(q')}} \Big(x - uv\Big) \\ &= \sum_{\substack{u \leq Q \\ (u', q') = 1}} \mathcal{D}u' \sum_{\substack{v \leq x/\mathcal{D}u' \\ v \equiv \overline{u'} a'(q')}} \Big(x/\mathcal{D}u' - v\Big) \\ &=: \sum_{\substack{u \leq Q \\ (u', q') = 1}} \mathcal{D}u' \mathcal{V}_{q,a}(u). \end{aligned} \tag{97}$$

Through the orthogonality of Dirichlet characters and a Perron formula (taking $w = 1$ in (11) of Section 2, Part II in [14], the relevant quantities being defined at the start of that section) we have

$$\begin{aligned}\mathcal{V}_{q,a}(u) &= \frac{1}{\phi(q')}\sum_{\chi}\overline{\chi}(\overline{u'}a')\sum_{v\le x/\mathcal{D}u'}\chi(v)\Big(x/\mathcal{D}u'-v\Big)\\ &= \frac{1}{\phi(q')}\sum_{\chi}\overline{\chi}(\overline{u'}a')\int_{2-i\infty}^{2+i\infty}\frac{L(s,\chi)}{s(s+1)}\left(\frac{x}{\mathcal{D}u'}\right)^{s+1}ds; \qquad (98)\end{aligned}$$

here and in what follows the sum Σ_χ runs over the Dirichlet characters modulo q' and for $s\in\mathbb{C}$ we always write $s=\sigma+it$ for real numbers σ,t. Denote by $\mathcal{L}$ the contour from $-ix^6$ to ix^6 which is a vertical line except for a small detour to the right of 0. Define

$$\delta(\chi)=\begin{cases}1 & \text{if } \chi=\chi_0\\ 0 & \text{if } \chi\neq\chi_0.\end{cases}$$

The parts of the above integral with $|t|\ge x^6$ contribute to the integral

$$\ll x^3\int_{|t|\ge x^6}\frac{|L(2+it,\chi)|dt}{t^2}\ll x^3\int_{|t|\ge x^6}\frac{dt}{t^2}\ll\frac{1}{x^3}$$

so pulling the remaining part of the integral to the left, and so picking up a simple pole at $s=1$ if $\chi=\chi_0$, we see that the integral in (98) is

$$\begin{aligned}&\int_{\mathcal{L}}\frac{L(s,\chi)}{s(s+1)}\left(\frac{x}{\mathcal{D}u'}\right)^{s+1}ds+\frac{Res_{s=1}L(s,\chi_0)}{2}\left(\frac{x}{\mathcal{D}u'}\right)^{2}\delta(\chi)+\mathcal{O}\left(\frac{1}{x^3}\right)\\ =\ &\int_{\mathcal{L}}\frac{L(s,\chi)}{s(s+1)}\left(\frac{x}{\mathcal{D}u'}\right)^{s+1}ds+\frac{\phi(q')}{2q'}\left(\frac{x}{\mathcal{D}u'}\right)^{2}\delta(\chi)+\mathcal{O}\left(\frac{1}{x^3}\right)\end{aligned}$$

so that for $(u',q')=1$

$$\begin{aligned}\mathcal{V}_{q,a}(u) &= \frac{1}{\phi(q')}\sum_{\chi}\chi(u')\overline{\chi}(a')\int_{\mathcal{L}}\frac{L(s,\chi)}{s(s+1)}\left(\frac{x}{\mathcal{D}u'}\right)^{s+1}ds\\ &\quad+\frac{1}{2q'}\left(\frac{x}{\mathcal{D}u'}\right)^{2}+\mathcal{O}\left(\frac{1}{x^3\phi(q')}\sum_{\chi}1\right)\end{aligned}$$

and so from (97) for $q\le x$

$$\begin{aligned}\mathcal{U}(q,a) &= \frac{1}{\phi(q')}\int_{\mathcal{L}}\frac{x^{s+1}}{s(s+1)\mathcal{D}^s}\left(\sum_{\chi}\overline{\chi}(a')L(s,\chi)\sum_{u\le Q}\frac{\chi(u')}{u'^s}\right)ds\\ &\quad+\frac{x^2}{2q'}\sum_{\substack{u\le Q\\(u',q')=1}}\frac{1}{\mathcal{D}u'}+\mathcal{O}\left(\frac{1}{x^3}\sum_{u\le Q}\mathcal{D}u'\right)\\ &=: \mathcal{I}(q,a)+\frac{x^2}{2q}\sum_{\substack{u\le Q\\(u/(u,\mathcal{D}),q/\mathcal{D})=1}}\frac{(u,\mathcal{D})}{u}+\mathcal{O}\left(1\right). \qquad (99)\end{aligned}$$

We have

$$L(s,\chi_0)=\zeta(s)\prod_{p|q'}(1-p^{-s})=:\zeta(s)\omega_s(q'); \qquad (100)$$

define for $d|q$

$$\theta_s(q,d) = \frac{\omega_s(q/d)U_s(q,d)}{d^s\phi(q/d)} \tag{101}$$

where $U_s(q,d)$ is as in Lemma 7. For $q', |t| \leq x$ we have the standard estimate

$$L(s,\chi) \ll x^\epsilon \sqrt{q'(1+|t|)}, \qquad 0 \leq \sigma \leq 1,$$

so that with Lemma 7 we see that the term in the brackets in $\mathcal{I}(q,a)$ is for $q \leq x$

$$\begin{aligned} &= \frac{\overline{\chi}_0(a')L(s,\chi_0)U_s(\mathcal{D}q',\mathcal{D})Q^{1-s}}{1-s} + \mathcal{O}\left(x^\epsilon\sqrt{q'(1+|t|)}\sum_\chi |\overline{\chi}(a')L(s,\chi)|\right) \\ &= \frac{Q^2\zeta(s)\mathcal{D}^s\phi(q')\theta_s(q,\mathcal{D})}{(1-s)Q^{s+1}} + \mathcal{O}\Big(x^\epsilon q'(1+|t|)\phi(q')\Big) \end{aligned} \tag{102}$$

and so

$$\begin{aligned} \mathcal{I}(q,a) &= Q^2\int_{\mathcal{L}} \frac{\zeta(s)\theta_s(q,\mathcal{D})}{s(s+1)(1-s)}\left(\frac{x}{Q}\right)^{s+1} ds + \mathcal{O}\left(x^\epsilon q'\int_{\mathcal{L}}\frac{\big|(1+|t|)x^{s+1}\big|ds}{|s(s+1)\mathcal{D}^s|}\right) \\ &= -Q^2\int_{\mathcal{L}} \frac{\zeta(s)\theta_s(q,\mathcal{D})}{(s-1)s(s+1)}\left(\frac{x}{Q}\right)^{s+1} ds + \mathcal{O}\left(x^{1+\epsilon}q\right) \end{aligned}$$

so from Lemma 2 (iv) we have for $q \leq x$

$$\sum_{a=1}^q \Phi_q(-a)\mathcal{I}(q,a) = -Q^2\int_{\mathcal{L}}\frac{\zeta(s)(x/Q)^{s+1}}{(s-1)s(s+1)}\left(\sum_{a=1}^q \theta_s(q,\mathcal{D})\Phi_q(-a)\right) ds + \mathcal{O}\left(x^{1+\epsilon}q^2\right)$$

and therefore from Lemma 3

$$\begin{aligned} &\sum_{q\leq x}|G(q)|^2\sum_{a=1}^q \Phi_q(-a)\mathcal{I}(q,a) \\ &\qquad = -Q^2\int_{\mathcal{L}}\frac{\zeta(s)(x/Q)^{s+1}}{(s-1)s(s+1)}\left(\sum_{q\leq x}|G(q)|^2\sum_{a=1}^q\theta_s(q,\mathcal{D})\Phi_q(-a)\right) ds \\ &\qquad\quad + \mathcal{O}\left(x^{1+\epsilon}\sum_{q\leq x}q^2|G(q)|^2\right) \\ &\qquad =: -Q^2\int_{\mathcal{L}}\frac{\zeta(s)(x/Q)^{s+1}}{(s-1)s(s+1)}\left(\sum_{q\leq x}|G(q)|^2\Delta_s(q)\right) ds + \mathcal{O}\left(x^{1+\theta+\epsilon}\right). \end{aligned} \tag{103}$$

For $\sigma \geq 0$ and $d|q$ it is clear that $U_s(q,d) \ll |d^s|q^\epsilon$ and so from (101) that $\theta_s(q,d) \ll q^\epsilon$, so from Lemma 2 (iv) we have

$$\Delta_s(q) \ll q^{1+\epsilon}$$

and therefore from Lemma 3

$$\sum_{q>x}|G(q)|^2\Delta_s(q) \ll x^{\theta-1+\epsilon}$$

so we can add in these terms to (103) at the cost of an error of size

$$\ll x^{\theta-1+\epsilon}Q^2\int_{\mathcal{L}}\left|\frac{\zeta(s)(x/Q)^{s+1}}{(s-1)s(s+1)}\right|ds \ll x^{\theta+\epsilon}Q\int_{\pm\infty}\frac{|t|^{1/2}dt}{1+|t|^3}\ll x^{1+\theta+\epsilon}$$

to get

$$\begin{aligned}\sum_{q\leq x}|G(q)|^2\sum_{a=1}^{q}\Phi_q(-a)\mathcal{I}(q,a) &= -Q^2\int_{\mathcal{L}}\frac{\zeta(s)(x/Q)^{s+1}}{(s-1)s(s+1)}\left(\sum_{q=1}^{\infty}|G(q)|^2\Delta_s(q)\right)ds+\mathcal{O}\left(x^{1+\theta+\epsilon}\right)\\ &=: -Q^2\int_{\mathcal{L}}\frac{\zeta(s)(x/Q)^{s+1}\mathcal{G}(s)ds}{(s-1)s(s+1)}+\mathcal{O}\left(x^{1+\theta+\epsilon}\right)\\ &=: -Q^2\mathcal{V}(x/Q)+\mathcal{O}\left(x^{1+\theta+\epsilon}\right),\end{aligned}$$

where $\mathcal{G}(s)$ converges absolutely (at least) for $\sigma \geq 0$. The last equality with (96) and (99) implies

$$\begin{aligned}\mathcal{J}(x,Q) &= -Q^2\mathcal{V}(x/Q)+\frac{x^2}{2}\sum_{q\leq x}\frac{|G(q)|^2}{q}\sum_{a=1}^{q}\Phi_q(-a)\sum_{\substack{u\leq Q\\ (u/(u,\mathcal{D}),q/\mathcal{D})=1}}\frac{(u,\mathcal{D})}{u}\\ &+ \mathcal{O}\left(x^{1+\theta+\epsilon}+x^{1+\theta+\epsilon}\sum_{q\leq x}|G(q)|^2\sum_{a=1}^{q}|\Phi_q(-a)|\right)\\ &=: -Q^2\mathcal{V}(x/Q)+\frac{x^2\mathcal{W}(Q)}{2}+\mathcal{O}\left(x^{1+\theta+\epsilon}\right) \qquad (104)\end{aligned}$$

with Lemma 2 (iv) and Lemma 3. Our application of Perron's formula is complete and now our task is now to evaluate $\mathcal{G}(s)$. The main point is we can write down an analytic continuation for this thanks to the explicit expressions for the Gauss sum in Lemma 1. Recall the assumption of our theorem: we always have $R_p < p^k$, where R_p is the number of distinct residue classes represented by the $h_1, ..., h_r$. Denote these different residues by $H_1, ..., H_{R_p}$.

It is straightforward to establish that $U_s(q,d)$ satisfies $U_s(qq',dd') = U_s(q,d)U_s(q',d')$ for $(q,q') = 1$ and $\mathbf{d}|\mathbf{q}$, so the same must be true of $\theta_s(q,d)$ and therefore Lemma 2 (ii) says that $\Delta_s(q)$ is multiplicative, so from Lemma 1 we have for $\sigma \geq 0$

$$\mathcal{G}(s) = \prod_p\left(\sum_{t\geq 0}|G(q)|^2\Delta_s(q)\right) = \prod_p\left(1+\frac{1}{p^{2k}(1-R_p/p^k)^2}\sum_{1\neq q|p^k}\Delta_s(q)\right). \qquad (105)$$

Define ρ as in Lemma 1, namely

$$\rho = \prod_p\left(1-\frac{R_p}{p^k}\right);$$

then from (105) for $\sigma \geq 0$

$$\rho^2\mathcal{G}(s) = \prod_p\left(\left(1-\frac{R_p}{p^k}\right)^2+\frac{1}{p^{2k}}\sum_{1\neq q|p^k}\Delta_s(q)\right). \qquad (106)$$

We have

$$U_s(q,d) = \frac{1}{q}\sum_{D|d}D^s\phi(q/D) =: \frac{F_s^*(q,d)}{q}$$

so that from (101)

$$\theta_s\left(q,(q,a)\right)=\frac{\omega_s(q/d)F_s^*(q,d)}{qd^s\phi(q/d)}$$

so from (103)

$$\Delta_s(q) \quad = \quad \frac{1}{q}\sum_{d|q}\frac{\omega_s(q/d)F_s^*(q,d)}{d^s\phi(q/d)}{\sum_{a=1}^{q/d}}' \Phi_q(-ad). \tag{107}$$

Define $H(q,a)$ and $\Phi(q)$ as in Lemma 2, and take a prime p. For q a power of p

$$H(q,a)=-\sum_{n=1}^{R_p}e\left(\frac{aH_n}{q}\right)$$

so that

$$\Phi(q)=\sum_{n,n'=1}^{R_p}c_q(H_n-H_{n'})$$

so from Lemma 2 (iii)

$${\sum_{a=1}^{q/d}}'\Phi_q(-ad)=\mu(q/d)\sum_{n,n'=1}^{R_p}c_q(H_n-H_{n'}).$$

and therefore from (107)

$$\begin{aligned}\Delta_s(q) \quad &= \quad \frac{1}{q}\left(\sum_{d|q}\frac{\mu(q/d)\omega_s(q/d)F^*(q,d)}{d^s\phi(q/d)}\right)\left(\sum_{n,n'=1}^{R_p}c_q(H_n-H_{n'})\right)\\ &=: \quad \frac{P_s(q)}{q}\sum_{n,n'=1}^{R_p}c_q(H_n-H_{n'}).\end{aligned} \tag{108}$$

Simple calculations show

$$F_s^*(q,q)=q^s+\frac{\phi(q)\left(q^{s-1}-1\right)}{p^{s-1}-1}$$

and

$$F_s^*(q,q/p)=F_s^*(q,q)-q^s$$

so that

$$\begin{aligned}P_s(q) \quad &= \quad \frac{F_s^*(q,q)}{q^s}-\frac{(1-p^{-s})F_s^*(q,q/p)}{(q/p)^s\phi(p)}\\ &= \quad \left(q^s+\frac{\phi(q)\left(q^{s-1}-1\right)}{p^{s-1}-1}\right)\left(\frac{1}{q^s}-\frac{1-p^{-s}}{(q/p)^s\phi(p)}\right)+\frac{q^s(1-p^{-s})}{(q/p)^s\phi(p)}\\ &= \quad q^{1-s}\end{aligned}$$

so from (108)

$$\Delta_s(q)=q^{-s}\sum_{n,n'=1}^{R_p}c_q(H_n-H_{n'}).$$

The sum here is for $q|p^k$

$$\sum_{\substack{n,n'=1 \\ n\neq n'}}^{R_p} c_q(H_n - H_{n'}) + R_p\phi(q) =: \Delta^*(q) + R_p\phi(q)$$

so

$$\Delta_s(q) = q^{-s}\Big(\Delta^*(q) + R_p\phi(q)\Big)$$

and therefore, writing $X = p$ and $Y = p^{-s}$,

$$\begin{aligned}
\sum_{1\neq q|p^k} \Delta_s(q) &= \sum_{1\neq q|p^k} q^{-s}\Delta^*(q) + R_p\left(\sum_{1\neq q|p^{k-1}} q^{-s}\phi(q) + p^{-sk}\phi(p^k)\right) \\
&= \sum_{1\neq q|p^k} q^{-s}\Delta^*(q) + R_p\left(\frac{(1-1/p)(p^{k(1-s)} - p^{1-s})}{p^{1-s}-1} - p^{k(1-s)-1}\right) + R_p p^{k(1-s)} \\
&= \sum_{1\leq t\leq k} Y^t\Delta^*(X^t) + R_p\left(\frac{(1-1/X)((XY)^k - XY)}{XY-1} - \frac{(XY)^k}{X}\right) + R_p(XY)^k \\
&=: P_1(X,Y) + R_p(XY)^k
\end{aligned}$$

so from (106) for $\sigma \geq 0$

$$\begin{aligned}
\rho^2\mathcal{G}(s) &= \prod_p \left(1 - \frac{2R_p}{X^k} + \frac{R_p^2}{X^{2k}} + \frac{P_1(X,Y)}{X^{2k}} + R_p\left(\frac{Y}{X}\right)^k\right) \\
&= \prod_p \left(\big(1+(Y/X)^k\big)^{R_p} + 1 + R_p\left(\frac{Y}{X}\right)^k - \big(1+(Y/X)^k\big)^{R_p} - \frac{2R_p}{X^k} + \frac{R_p^2}{X^{2k}} + \frac{P_1(X,Y)}{X^{2k}}\right) \\
&=: \prod_p \left(\big(1+(Y/X)^k\big)^{R_p} + P_2(X,Y) + \frac{P_1(X,Y)}{X^{2k}}\right); \qquad (109)
\end{aligned}$$

in particular

$$\rho^2\mathcal{G}(0) = \prod_p \left(1 - \frac{2R_p}{X^k} + \frac{R_p^2}{X^{2k}} + \frac{P_1(X,0)}{X^{2k}} + \frac{R_p}{X^k}\right). \qquad (110)$$

For q a power of p and $n \in \mathbb{N}$ with $q \nmid n$ we have $c_q(n) \ll q/p$, so for $q|p^k$ all the summands in $\Delta^*(q)$ are $\ll p^{k-1}$. Therefore

$$\sum_{1\leq t\leq k} Y^t\Delta^*(X^t) \ll R_p^2 X^{k-1} \sum_{1\leq t\leq k} |Y|^t \ll X^{k-1}\Big(|Y| + |Y|^k\Big). \qquad (111)$$

Take a small $\delta > 0$. If $\sigma \in [-1, 1/2]$ then $|Y|/X \leq 1$. Moreover $5X|Y|/7 \geq 5\sqrt{2}/7 > 1$ so with (111)

$$\begin{aligned}
P_1(X,Y) &\ll X^{k-1}\Big(|Y| + |Y|^k\Big) + \underbrace{\frac{X^k|Y|^k + X|Y|}{X|Y|}}_{\ll X^{k-1}(|Y|^k+|Y|)+1} + \frac{X^k|Y|^k}{X} \\
&\ll X^k + X^{2k-1}\left(\frac{|Y|}{X}\right)^k \\
&\ll \frac{X^{2k}}{X^{1+\delta k}}, \qquad \text{for } \sigma \geq -1+\delta. \qquad (112)
\end{aligned}$$

From the Binomial Theorem

$$P_2(X,Y) \ll \left(\frac{Y}{X}\right)^{2k} + ... + \left(\frac{Y}{X}\right)^{rk} + \frac{1}{X^k} + \frac{1}{X^{2k}} \ll \frac{1}{p^{1+2k\delta}}, \qquad \text{for } \sigma \geq -1 + \frac{1}{2k} + \delta$$

so that with (112) we have

$$P_2(X,Y) + \frac{P_1(X,Y)}{X^{2k}} \ll \frac{1}{p^{1+\delta}}, \qquad \text{for } \sigma \geq -1 + \frac{1}{2k} + \delta. \tag{113}$$

For $\sigma \geq -1 + 1/2k + \delta$ we have $|(Y/X)^k| \leq 1/p^{1/2+\delta k}$ so

$$1 + (Y/X)^k \gg 1, \qquad \text{for } \sigma \geq -1 + 1/2k + \delta. \tag{114}$$

Since $(Y/X)^k \ll 1/p^{1+\delta k}$ if $\sigma \geq -1 + 1/k + \delta$ we deduce from (113) that the product in (109) is absolutely convergent for $\sigma \geq -1 + 1/k + \delta$ and so from (114) that

$$\begin{aligned} \rho^2 \mathcal{G}(s) &= \prod_p \left(1 + (Y/X)^k\right)^{R_p} \prod_p \left(1 + \frac{P_2(X,Y) + P_1(X,Y)/X^{2k}}{(1+(Y/X)^k)^{R_p}}\right) \\ &= \prod_p \left(1 + (Y/X)^k\right)^r \prod_{p^k \leq h_r} \frac{\left(1 + (Y/X)^k\right)^{R_p}}{\left(1 + (Y/X)^k\right)^r} \prod_p \left(1 + \frac{P_2(X,Y) + P_1(X,Y)/X^{2k}}{(1+(Y/X)^k)^{R_p}}\right) \\ &=: \frac{\zeta(sk+k)^r \mathcal{F}(s)}{\zeta(2s+2k)^r} \end{aligned}$$

where the third product in the second line is absolutely convergent and uniformly bounded for $\sigma \geq -1 + 1/2k + \delta$, so that $\mathcal{F}(s)$ is holomorphic and uniformly bounded for $\sigma \geq -1 + 1/2k + \delta$.

We now have our analytic extension for $\mathcal{G}(s)$ in place. With this extension the integral in $\mathcal{V}(y)$ is certainly absolutely convergent for $\sigma \geq -1 + 1/k + \delta$ and so we may pull it to the left, picking up a simple pole at $s = 0$, to deduce

$$\begin{aligned} \rho^2 \mathcal{V}(y) &= -\rho^2 \zeta(0) \mathcal{G}(0) y + \int_{(-1+1/k+\delta)} \frac{\zeta(s)\zeta(sk+k)^r \mathcal{F}(s) y^{s+1} ds}{(s-1)s(s+1)\zeta(2sk+2k)^r} \\ &= \frac{\rho^2 \mathcal{G}(0) y}{2} + \frac{1}{k} \int_{(1+k\delta)} \frac{\zeta(-1+s/k)\zeta(s)^r \mathcal{F}(-1+s/k) y^{s/k} ds}{(-2+s/k)(-1+s/k)(s/k)\zeta(2s)^r} \\ &=: \frac{\rho^2 \mathcal{G}(0) y}{2} + \int_{(1+k\delta)} f(s) y^{s/k} ds. \end{aligned}$$

Since for $1/2 \leq \sigma \leq 3/2$ and $|t| \geq 1$ we have the standard bounds $\zeta(-1+s/k) \ll |t|^{3/2-\sigma/k+\epsilon}$ and $\zeta(2s) \gg 1/(\log t)^7$ we see that

$$f(s) \ll \frac{(1+|t|^{3/2-\sigma/k+\epsilon})|\zeta(s)|^r}{(1+|t|)^3} \ll \frac{|\zeta(s)|^r}{(1+|t|)^{3/2}}, \qquad \text{for} \quad \frac{1}{2} \leq \sigma \leq \frac{3}{2}.$$

By the definition of $\mathfrak{c}$ the integral above therefore converges absolutely for $\sigma \geq \mathfrak{c}$, and we may move the line of integration to $\sigma = \mathfrak{c}$, picking up a pole at $s = 1$, to deduce

$$\begin{aligned} \rho^2 \mathcal{V}(y) &= \frac{\rho^2 \mathcal{G}(0) y}{2} + Res_{s=1}\left(f(s) y^{s/k}\right) + \int_{(\mathfrak{c})} f(s) y^{s/k} ds \\ &= \frac{\rho^2 \mathcal{G}(0) y}{2} + Res_{s=1}\left(f(s) y^{s/k}\right) + \mathcal{O}\left(y^{\mathfrak{c}/k}\right). \end{aligned} \tag{115}$$

Since the pole of f is of order r a standard formula from complex analysis tells us that

$$\begin{aligned} Res_{s=1}\Big(f(s)y^{s/k}\Big) &= \frac{1}{(r-1)!}\sum_{i+j=r-1}\left(\left(\frac{d}{ds}\right)^{(j)}(y^{s/k})\Big|_{s=1}\right)\left(Res_{s=1}\Big((s-1)^{r-i-1}f(s)\Big)\right) \\ &= \frac{y^{1/k}}{(r-1)!}\sum_{i+j=r-1}\left(\frac{\log y}{k}\right)^{j}\left(Res_{s=1}\Big((s-1)^{r-i-1}f(s)\Big)\right) \\ &= -\frac{y^{1/k}P(\log y)}{2}, \end{aligned}$$

where $P = P_r$ is a polynomial of degree at most $r-1$, so from (115)

$$-2\rho^2\mathcal{V}(y) = -\rho^2\mathcal{G}(0)y + y^{1/k}P(\log y) + \mathcal{O}\left(y^{\mathfrak{c}/k}\right). \tag{116}$$

It is straightforward to establish that for $q \not| n$

$$\sum_{d|q} c_d(n) = 0$$

so

$$\sum_{1\le t\le k}\Delta^*(p^t) = \sum_{\substack{n,n'=1\\ n\ne n'}}^{R_p}\sum_{\substack{d|p^k\\ d\ne 1}} c_d(H_n - H_{n'}) = -R_p(R_p-1)$$

so

$$P_1(X,0) = \sum_{1\le t\le k}\Delta^*(p^t) + R_p\left(\frac{(1-1/X)(X^k-X)}{X-1} - X^{k-1}\right) = -R_p^2$$

and so from (110)

$$\rho^2\mathcal{G}(0) = \prod_p\left(1-\frac{R_p}{X^k}\right) = \rho. \tag{117}$$

From (104)

$$\begin{aligned} \mathcal{W}(Q) &= \sum_{u\le Q}\frac{1}{u}\sum_{q=1}^{\infty}\frac{|G(q)|^2}{q}\sum_{\substack{a=1\\ (u/(u,\mathcal{D}),q/\mathcal{D})=1}}^{q}(u,\mathcal{D})\Phi_q(-a) \\ &=: \sum_{u\le Q}\frac{1}{u}\sum_{q=1}^{\infty}\frac{|G(q)|^2 f_u(q)}{q}. \end{aligned} \tag{118}$$

From Lemma 2 (ii) we see that $f_u(q)$ is multiplicative and for prime powers q Lemma 2 (iii) implies

$$\begin{aligned} f_u(q) &= \sum_{\substack{d|q\\ (u/(u,d),q/d)=1}}(u,d)\sum_{a=1}^{q/d}{}'\,\Phi_q(-a) \\ &= \Phi(q)\sum_{\substack{d|q\\ (u/(u,d),q/d)=1}}(u,d)\mu(q/d) \\ &= (u,q)\Phi(q)\sum_{\substack{d|q\\ (u,q)=(u,d)}}\mu(q/d) \\ &=: (u,q)\Phi(q)F_u(q) \end{aligned}$$

so that for general q

$$f_u(q) \quad = \quad (u,q)\Phi(q)\left(\prod_{p^\beta || q} F_u(p^\beta)\right).$$

If $p^\beta | u$ then $F_u(p^\beta)$ has only the term $d = q$ and so $F_u(p^\beta) = 1$. If $p^\beta \not| u$ then $F_u(p^\beta)$ has the terms $d = q$ and $d = q/p$ so $F_u(p^\beta) = 0$. Therefore

$$f_u(q) = \begin{cases} q\Phi(q) & \text{if } q|u \\ 0 & \text{if not} \end{cases}$$

and so from (118)

$$\mathcal{W}(Q) = \sum_{u \le Q} \frac{1}{u} \sum_{q|u} \Phi(q)|G(q)|^2 = \frac{1}{\rho^2} \sum_{u \le Q} W(u) \tag{119}$$

from Lemma 2 (v) and (60). Theorem 1 now follows from (63), (95), (104), (116), (117) and (119).

Chapter 3

Proof of Theorem 2

We consider k and q as fixed throughout. Each time ϵ appears it is to be understood that it may be taken arbitrarily small at each occurence. Fix some $0 < \delta < 1/2k$. All $\ll, \mathcal{O}$ constants depend on ϵ, k and δ. For real positive numbers c and T we often write the shorthand

$$\int_{\pm T} \quad \text{or} \quad \int_{c \pm iT} \quad \text{or} \quad \int_{c \pm i\infty}$$

for

$$\int_{-T}^{T} \quad \text{or} \quad \int_{c-iT}^{c+iT} \quad \text{or} \quad \int_{c-i\infty}^{c+i\infty}.$$

As said in the introduction, an asymptotic formula for a similar variance to the one in question was already found in [12], and our interest is in improving one of the error terms.

3.1 - Lemmas

For $\mathfrak{Re}(s) > 1$ define

$$\mathcal{F}(s) = \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{[d^k, d'^k][q, (d^k, d'^k)]^s}$$

and for $\mathfrak{Re}(s) \geq -1 + \delta$ define

$$\mathcal{F}^*(s) = \prod_{p|q} \frac{1 + (q, p^k)^s / p^{k(1+s)}}{1 + 1/p^{k(1+s)}} \prod_{p} \left(1 - \frac{2}{p^k \left(1 + (q, p^k)^s / p^{k(1+s)} \right)} \right).$$

The first series converges since the summands are bounded by

$$\frac{1}{[d^k, d'^k](d^k, d'^k)};$$

for $\mathfrak{Re}(s) \geq -1 + \delta$

$$\begin{aligned} \left| (q, p^k)^s / p^{k(1+s)} \right| &\leq \begin{cases} 1/p^k & \text{for } \mathfrak{Re}(s) \geq 0 \\ 1/p^{k\delta} & \text{for } \mathfrak{Re}(s) < 0 \end{cases} \\ &\ll 1 \end{aligned} \tag{120}$$

and therefore

$$1 + (q, p^k)^s / p^{k(1+s)} \geq 1 - 1/2^{k\delta} \gg 1$$

so that each Euler factor of the infinite product in $\mathcal{F}^*(s)$ is of the form

$$1 + \mathcal{O}\left(1/p^k\right)$$

and therefore this product converges; for $\mathfrak{Re}(s) \geq -1+\delta$ we have

$$\frac{1}{p^{k(1+s)}} \geq \frac{1}{p^{k\delta}}$$

and therefore

$$1 + 1/p^{k(1+s)} \geq 1 - 1/2^{k\delta} \gg 1$$

so from (120) the finite product in $\mathcal{F}^*(s)$ is $\ll q^\epsilon$ for $\mathfrak{Re}(s) \geq -1+\delta$.

Lemma 1. *If* $\mathfrak{Re}(s) > 1$ *then*

$$\mathcal{F}(s) = \frac{\zeta\left(k(s+1)\right)\mathcal{F}^*(s)}{q^s \zeta\left(2k(s+1)\right)}.$$

Proof. We have

$$\begin{aligned}\sum_{d,d'} \frac{\mu(d)\mu(d')(d^k, d'^k)^{1-s}(q, d^k, d'^k)^s}{d^k d'^k} &= \sum_{N=1}^{\infty} \frac{1}{N^k} \sum_{dd'=N} \mu(d)\mu(d')(d^k, d'^k)^{1-s}(q, d^k, d'^k)^s \\ &=: \sum_{N=1}^{\infty} \frac{a_q(N)}{N^k}. \qquad (121)\end{aligned}$$

Clearly $a_q(N)$ is multiplcative and simple calculations show

$$a_q(p) = -2,$$

$$a_q(p^2) = p^{k(1-s)}(q, p^k)^s$$

and $a_q(p^t) = 0$ for $t \geq 3$. Consequently

$$\begin{aligned}\sum_{N=1}^{\infty} \frac{a_q(N)}{N^k} &= \prod_p \left(1 - \frac{2}{p^k} + \frac{(q,p^k)^s}{p^{k(1+s)}}\right) \\ &= \prod_p \left(1 + \frac{(q,p^k)^s}{p^{k(1+s)}}\right) \prod_p \left(1 - \frac{2}{p^k\left(1 + (q,p^k)^s/p^{k(1+s)}\right)}\right) \\ &= \prod_p \left(1 + \frac{1}{p^{k(1+s)}}\right) \prod_{p|q} \frac{1 + (q,p^k)^s/p^{k(1+s)}}{1 + 1/p^{k(1+s)}} \prod_p \left(1 - \frac{2}{p^k\left(1 + (q,p^k)^s/p^{k(1+s)}\right)}\right) \\ &= \frac{\zeta\left(k(1+s)\right)\mathcal{F}^*(s)}{\zeta\left(2k(1+s)\right)} \qquad (122)\end{aligned}$$

so that (121) becomes

$$\sum_{d,d'} \frac{\mu(d)\mu(d')(d^k, d'^k)^{1-s}(q, d^k, d'^k)^s}{d^k d'^k} = \frac{\zeta\left(k(1+s)\right)\mathcal{F}^*(s)}{\zeta\left(2k(1+s)\right)}$$

and the claim follows. □

Lemma 2. *Suppose* q *has* ω *distinct prime factors* $p_1, ..., p_\omega$. *For each* $n \in \mathbb{N}$ *and each* $l_1, .., l_\omega, l'_1, ..., l'_\omega \geq 0$ *there are* $\lambda_n, W_n, C_{\mathbf{l},\mathbf{l}'}, Z_{\mathbf{l},\mathbf{l}'} \in \mathbb{R}$ *such that*

$$\mathcal{F}^*(s) = \sum_{\substack{l_1,\ldots,l_\omega \geq 0 \\ l'_1,\ldots,l'_\omega \geq 0}} \sum_{n=1}^{\infty} C_{\mathbf{l},\mathbf{l}'} Z_{\mathbf{l},\mathbf{l}'}^s \lambda_n W_n^{1+s}$$

for $\mathfrak{Re}(s) \geq -1+\delta$. *Moreover for* $-1+\delta \leq \mathfrak{Re}(s) \leq 0$

$$\sum_{\substack{l_1,\ldots,l_\omega \geq 0 \\ l'_1,\ldots,l'_\omega \geq 0}} \sum_{n=1}^{\infty} \left| C_{\mathbf{l},\mathbf{l}'} Z^s_{\mathbf{l},\mathbf{l}'} \lambda_n W_n^{1+s} \right| \ll 1.$$

Proof. From (120) we have $(q,p^k)^s/p^{k(1+s)} < 1$ and therefore

$$\begin{aligned} \prod_p \left(1 - \frac{2}{p^k\left(1+(q,p^k)^s/p^{k(1+s)}\right)}\right) &= \prod_p \left(1 - \frac{2}{p^k} \sum_{t\geq 1} \left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1}\right) \\ &= \sum_{n=1}^{\infty} f_s^*(n) \end{aligned} \tag{123}$$

where $f_s^*(n)$ is the multiplicative function given on prime powers by

$$f_s^*(p^t) = -\frac{2}{p^k}\left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1}.$$

Then for any $n = \prod_{p^t|n} p^t$

$$\begin{aligned} f_s^*(n) &= \prod_{p^t|n}\left(-\frac{2}{p^k}\right)\left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1} \\ &= \left(\prod_{p^t|n}(-1)^{t-1}\right)\left(\prod_{p^t|n}\frac{-2}{p^k}\right)\left(\prod_{p^t|n}(q,p^k)^{-(t-1)}\right)\left(\prod_{p^t|n}\frac{(q,p^k)^{(t-1)(1+s)}}{p^{(t-1)k(1+s)}}\right). \end{aligned} \tag{124}$$

If we now define

$$\lambda_n = \left(\prod_{p^t|n}(-1)^{t-1}\right)\left(\prod_{p^t|n}\frac{-2}{p^k}\right)\left(\prod_{p^t|n}(q,p^k)^{1-t}\right)$$

and

$$W_n = \prod_{p^t|n}\frac{(q,p^k)^{t-1}}{p^{(t-1)k}}$$

then (124) becomes

$$f^*(n) = \lambda_n W_n^{1+s}$$

so (123) becomes

$$\prod_p \left(1 - \frac{2}{p^k\left(1+(q,p^k)^s/p^{k(1+s)}\right)}\right) = \sum_{n=1}^{\infty} \lambda_n W_n^{1+s}. \tag{125}$$

Just as (123) is true so is

$$\sum_{n=1}^{\infty} |f_s^*(n)| = \prod_p \left(1 - \frac{2}{p^k}\sum_{t\geq 1}\left|\left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1}\right|\right). \tag{126}$$

For $-1+\delta \le \mathfrak{Re}(s) \le 0$ the t sum here is from (120)

$$\ll \sum_{t\geq 1}\left(\frac{1}{p^{k\delta}}\right)^{t-1} = \frac{1}{1-p^{k\delta}} \ll 1$$

so the Euler product in (126) is uniformly bounded in this range and therefore

$$\sum_{n=1}^{\infty} |f^*(n)| \ll 1, \qquad \text{for } -1+\delta \le \mathfrak{Re}(s) \le 0. \tag{127}$$

We have for $\mathfrak{Re}(s) \ge -1+\delta$

$$\frac{1}{1+1/p^{k(1+s)}} = \sum_{l\geq 0}\left(\frac{-1}{p^{k(1+s)}}\right)^{l} = \sum_{l\geq 0}\frac{C_p(l)}{p^{k(1+s)l}} \tag{128}$$

for some $C_p(l)$ with

$$\sum_{l\geq 0}\left|\frac{C_p(l)}{p^{k(1+s)l}}\right| \ll \sum_{l\geq 0}\left(\frac{1}{p^{k\delta}}\right)^{l} \ll 1. \tag{129}$$

as well as

$$1+\frac{(q,p^k)^s}{p^{k(1+s)}} = \sum_{l'\geq 0}\frac{C_p'(l')(q,p^k)^{sl'}}{p^{k(1+s)l'}} \tag{130}$$

for some $C_p'(l')$ with

$$\sum_{l'\geq 0}\left|\frac{C_p'(l')(q,p^k)^{sl'}}{p^{k(1+s)l'}}\right| \le 1+1. \tag{131}$$

from (120). From (128), (129), (130) and (131) there are for each prime p and $l,l' \in \mathbb{N}$ some $C_p(l), C_p'(l')$ for which

$$\frac{1+(q,p^k)^s/p^{k(1+s)}}{1+1/p^{k(1+s)}} = \sum_{l,l'\geq 0}\frac{C_p(l)C_p'(l')(q,p^k)^{sl'}}{p^{k(1+s)(l+l')}}$$

and

$$\sum_{l,l'\geq 0}\left|\frac{C_p(l)C_p'(l')(q,p^k)^{sl'}}{p^{k(1+s)(l+l')}}\right| \ll 1.$$

Consequently

$$\prod_{p|q}\frac{1+(q,p^k)^s/p^{k(1+s)}}{1+1/p^{k(1+s)}} = \sum_{\substack{l_1,\dots,l_\omega\geq 0\\ l_1',\dots,l_\omega'\geq 0}}\frac{C_{p_1}(l_1)C_{p_1}'(l_1')\cdots C_{p_\omega}(l_\omega)C_{p_\omega}'(l_\omega')(q,p_1^k)^{sl_1'}\cdots(q,p_\omega^k)^{sl_\omega'}}{p_1^{k(1+s)(l_1+l_1')}\cdots p_\omega^{k(1+s)(l_\omega+l_\omega')}}$$

and

$$\sum_{\substack{l_1,\dots,l_\omega\geq 0\\ l_1',\dots,l_\omega'\geq 0}}\left|\frac{C_{p_1}(l_1)C_{p_1}'(l_1')\cdots C_{p_\omega}(l_\omega)C_{p_\omega}'(l_\omega')(q,p_1^k)^{sl_1'}\cdots(q,p_\omega^k)^{sl_\omega'}}{p_1^{k(1+s)(l_1+l_1')}\cdots p_\omega^{k(1+s)(l_\omega+l_\omega')}}\right| \ll 1$$

for $\mathfrak{Re}(s) \geq -1+\delta$. If we now define

$$C^*_{\mathbf{l},\mathbf{l}'} = \prod_{i=1}^{\omega} C_{p_i}(l_i) C'_{p_i}(l'_i), \qquad W_{\mathbf{l},\mathbf{l}'} = \left(\prod_{i=1}^{\omega} p_i^{l_i+l'_i}\right)^k, \qquad C_{\mathbf{l},\mathbf{l}'} = \frac{C^*_{\mathbf{l},\mathbf{l}'}}{W_{\mathbf{l},\mathbf{l}'}},$$

$$D_{\mathbf{l}'} = \prod_{i=1}^{\omega} (q, p_i^k)^{l'_i}, \quad \text{and} \quad Z_{\mathbf{l},\mathbf{l}'} = \frac{D_{\mathbf{l}'}}{W_{\mathbf{l},\mathbf{l}'}}$$

then

$$\prod_{p|q} \frac{1+(q,p^k)^s/p^{k(1+s)}}{1+1/p^{k(1+s)}} = \sum_{\substack{l_1,\ldots,l_\omega \geq 0 \\ l'_1,\ldots,l'_\omega \geq 0}} C_{\mathbf{l},\mathbf{l}'} Z^s_{\mathbf{l},\mathbf{l}'}$$

with

$$\sum_{\substack{l_1,\ldots,l_\omega \geq 0 \\ l'_1,\ldots,l'_\omega \geq 0}} |C_{\mathbf{l},\mathbf{l}'} Z^s_{\mathbf{l},\mathbf{l}'}| \ll 1 \tag{132}$$

for $\mathfrak{Re}(s) \geq -1+\delta$. The first claim now follows from (125) and the boundedness claim from (127) and (132). □

Lemma 3. *(A) Define*

$$\alpha = \prod_{p|q} \frac{1+(q,p^k)/p^{2k} - 2/p^k}{1+1/p^{2k} - 2/p^k} \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{2k}}\right),$$

$$\beta = \prod_p \left(1 - \frac{1}{p^k}\right)$$

and

$$\gamma = \prod_{p|q} \frac{1+(q,p^k)^{-1+1/k}/p - 2/p^k}{1+1/p-2/p^k} \prod_p \frac{1-2/(p^k+p^{k-1})}{1-p^{1-1/k}}.$$

Then

$$\frac{\zeta(2k)\mathcal{F}^*(1)}{\zeta(4k)} = \alpha, \qquad \frac{\zeta(k)\mathcal{F}^*(0)}{\zeta(2k)} = \beta \quad \textit{and} \quad \zeta(-1+1/k)\mathcal{F}^*(-1+1/k) = \gamma.$$

(B) Define $\eta(q,a)$ as in (6). For any $q, n \in \mathbb{N}$

$$\eta(q,n) = \eta\left(q,(q,n)\right) \ll q^{\epsilon-1}$$

and

$$\sum_{a=1}^{q} \eta(q,a)^2 = \frac{\alpha}{q}.$$

Proof. (A) So long as there are no problems with zeros of denominators we have

$$\begin{aligned}\mathcal{F}^*(s) &= \prod_{p|q} \frac{1+(q,p^k)^s/p^{k(1+s)} - 2/p^k}{1+1/p^{k(1+s)}} \prod_{p\nmid q} \left(1 - \frac{2}{p^k(1+1/p^{k(1+s)})}\right) \\ &= \prod_{p|q} \frac{1+(q,p^k)^s/p^{k(1+s)} - 2/p^k}{1+1/p^{k(1+s)} - 2/p^k} \prod_p \left(1 - \frac{2}{p^k(1+1/p^{k(1+s)})}\right).\end{aligned}$$

For $s = 1, 0, -1+1/k$ this is clearly not the case and therefore from the Euler product expressions for the Riemann zeta function

$$\begin{aligned}\frac{\zeta(2k)\mathcal{F}^*(1)}{\zeta(4k)} &= \prod_{p|q} \frac{1+(q,p^k)/p^{2k}-2/p^k}{1+1/p^{2k}-2/p^k} \prod_p \left(\frac{1-p^{-4k}}{1-p^{-2k}}\right)\left(1-\frac{2}{p^k(1+1/p^{2k})}\right) \\ &= \prod_{p|q} \frac{1+(q,p^k)/p^{2k}-2/p^k}{1+1/p^{2k}-2/p^k} \prod_p \left(1+\frac{1}{p^{2k}}-\frac{2}{p^k}\right),\end{aligned}$$

$$\begin{aligned}\frac{\zeta(k)\mathcal{F}^*(0)}{\zeta(2k)} &= \prod_{p|q} \frac{1+1/p^k-2/p^k}{1+1/p^k-2/p^k} \prod_p \left(\frac{1-p^{-2k}}{1-p^{-k}}\right)\left(1-\frac{2}{p^k(1+1/p^k)}\right) \\ &= \prod_p \left(1+\frac{1}{p^k}-\frac{2}{p^k}\right)\end{aligned}$$

and

$$\begin{aligned}&\zeta(-1+1/k)\mathcal{F}^*(-1+1/k) \\ &= \prod_{p|q} \frac{1+(q,p^k)^{-1+1/k}/p-2/p^k}{1+1/p-2/p^k} \prod_p \left(1-p^{1-1/k}\right)^{-1}\left(1-\frac{2}{p^k(1+1/p)}\right).\end{aligned}$$

(B) From (6)

$$\eta(q,a) = \sum_{\substack{D|q \\ D \text{ is } (k+1)\text{-free}}} \sum_{\substack{d=1 \\ (q,d^k)|a \\ (q,d^k)=D}}^{\infty} \frac{\mu(d)}{[q,d^k]}.$$

Writing l_0 for the squarefree part of D the d sum must be

$$\frac{D}{q} \sum_{\substack{d=1 \\ (q,d^k)|a \\ (q,d^k)=D}}^{\infty} \frac{\mu(d)}{d^k} = \frac{D}{q} \sum_{\substack{d=1 \\ (q,(dl_0)^k)|a \\ (q,(dl_0)^k)=D}}^{\infty} \frac{\mu(dl_0)}{(dl_0)^k} \ll \frac{D}{ql_0^k}$$

so that

$$\eta(q,a) \ll \sum_{\substack{D|q \\ D \text{ is } (k+1)\text{-free}}} \frac{D}{ql_0^k} \ll q^{\epsilon-1}$$

which is the second claim and the first is trivial. We have

$$\begin{aligned}
\sum_{a=1}^{q} \eta(q,a)^2 &= \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{[q,d^k][q,d'^k]} \sum_{\substack{a=1 \\ (q,d^k),(q,d'^k)|a}}^{q} 1 \\
&= q \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{[q,d^k][q,d'^k][(q,d^k),(q,d'^k)]} \\
&= \frac{1}{q} \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')(q,d^k,d'^k)}{d^k d'^k} \\
&= \frac{1}{q} \sum_{N=1}^{\infty} \frac{1}{N^k} \sum_{dd'=N} \mu(d)\mu(d')(q,d^k,d'^k) \\
&=: \frac{1}{q} \sum_{N=1}^{\infty} \frac{b_q(N)}{N^k}. \qquad (133)
\end{aligned}$$

Clearly $b_q(N)$ is multiplicative and simple calculations show

$$b_q(p) = -2,$$

$$b_q(p^2) = (q,p^k)$$

and $b_q(p^t) = 0$ for $t \geq 3$. Consequently

$$\begin{aligned}
\sum_{N=1}^{\infty} \frac{b_q(N)}{N^k} &= \prod_p \left(1 - \frac{2}{p^k} + \frac{(q,p^k)}{p^{2k}}\right) \\
&= \prod_{p|q} \frac{1 - 2/p^k + (q,p^k)/p^{2k}}{1 - 2/p^k + 1/p^{2k}} \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{2k}}\right) \\
&= \alpha
\end{aligned}$$

which with (133) is the third claim. □

Lemma 4. *Let $c > 1$, let*

$$\mathcal{A}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\mathfrak{Re}(s) > c$, and let

$$A(Q) = \max_{Q/2 \leq n \leq 3Q/2} |a_n|.$$

Then for $T \geq 1$ and non-integer $Q > 0$

$$\sum_{n \leq Q} a_n \Big(Q - n\Big) = \frac{1}{2\pi i} \int_{c \pm iT} \frac{\mathcal{A}(s) Q^{s+1} ds}{s(s+1)} + \mathcal{O}\left(\frac{QA(Q)2^c}{T}\left(1 + \frac{Q\left(\log Q + 1\right)}{T}\right) + \frac{Q^{c+1}}{T^2} \sum_{n=1}^{\infty} \frac{|a_n|}{n^c}\right).$$

In particular if $c = 1 + \mathcal{O}\left(1/\log Q\right)$ then

$$\sum_{n \leq Q} \Big(Q - n\Big) = \frac{1}{2\pi i} \int_{c \pm iT} \frac{\zeta(s) Q^{s+1} ds}{s(s+1)} + \mathcal{O}\left(Q^{\epsilon}\left(\frac{Q}{T} + \frac{Q^2}{T^2}\right)\right).$$

Proof. Take $X > 0$ and define

$$\delta(X) = \begin{cases} 0 & \text{if } 0 < X < 1 \\ X - 1 & \text{if } X > 1 \end{cases}$$

and

$$I_X(T) = \frac{1}{2\pi i} \int_{c \pm iT} \frac{X^{s+1} ds}{s(s+1)}.$$

We first prove

$$|I_X(T) - \delta(X)| \ll \frac{X^{c+1}}{T} \min\left\{1, \frac{1}{T|\log X|}\right\}. \tag{134}$$

Suppose first $0 < X < 1$ so that for $R > c$ we have $X^R \ll X^c \ll 1$. The integrand is holomorphic to the right of 0 so for $R > c$

$$\begin{aligned} 2\pi i I_X(T) &= -\left(\int_{c+iT}^{R+iT} + \int_{R+iT}^{R-iT} + \int_{R-iT}^{c-iT}\right) \frac{X^{s+1} ds}{s(s+1)} \\ &\ll \frac{1}{T^2} \int_c^R X^{\sigma+1} d\sigma + \frac{X^{R+1}}{R^2} \int_{\pm T} dt \\ &\ll \frac{X^{c+1} + X^{R+1}}{T^2 |\log X|} + \frac{X^{R+1} T}{R^2} \ll \frac{X^{c+1}}{T^2 |\log X|} \end{aligned}$$

with $R \to \infty$. Suppose now that $X > 1$ so that for $R < -1$ we have $X^R \ll 1$ we have $X^{s+1} \ll 1$. The integrand is holomorphic except for at two places, so for $R < -1$

$$\begin{aligned} 2\pi i I_X(T) &= Res_{s=0}\left(\frac{X^{s+1}}{s(s+1)}\right) + Res_{s=-1}\left(\frac{X^{s+1}}{s(s+1)}\right) - \left(\int_{c+iT}^{R+iT} + \int_{R+iT}^{R-iT} + \int_{R-iT}^{c-iT}\right) \frac{X^{s+1} ds}{s(s+1)} \\ &= X - 1 + \mathcal{O}\left(\frac{1}{T^2} \int_c^R X^{\sigma+1} d\sigma + \frac{X^{R+1}}{R^2} \int_{\pm T} dt\right) \\ &= X - 1 + \mathcal{O}\left(\frac{1}{T^2} \int_c^\infty X^{\sigma+1} d\sigma + \frac{X^{R+1} T}{|R|^2}\right) \\ &= X - 1 + \mathcal{O}\left(\frac{X^{c+1} + X^{R+1}}{T^2 |\log X|} + \frac{X^{R+1} T}{|R|^2}\right) \\ &= X - 1 + \mathcal{O}\left(\frac{X^{c+1}}{T^2 |\log X|}\right) \end{aligned}$$

so we can conclude that for all $X > 0$ the second bound in (134) is clear; now for the first bound. If $0 < X < 1$ and if $\mathcal{C}$ is the arc of the circle going clockwise from $c + iT$ to $c - iT$ (so a circle of radius $\sqrt{T^2 + c^2} > T$) then on $\mathcal{C}$ we have $X^s \ll X^c$ on $\mathcal{C}$). Noting again that the integrand is holomorphic to the right of 0 we have

$$\begin{aligned} 2\pi i I_X(T) &= -\int_{\mathcal{C}} \frac{X^{s+1} ds}{s(s+1)} \\ &\ll X^{c+1} \int_{\mathcal{C}} \frac{ds}{|s| \cdot |s+1|} \ll \frac{X^{c+1}}{T}. \end{aligned}$$

If $X > 1$ the remaining part of the circle should be taken as the contour so that $X^s \ll X^c$ holds on the contour, and this gives a similar result. We conclude that the first bound in (134) also holds for any value of $X > 0$ and so the proof of (134) is complete.

By (134) (and absolute convergence)

$$\begin{aligned}\int_{c\pm iT} \frac{\mathcal{A}(s)Q^{s+1}ds}{s(s+1)} &= \sum_{n=1}^{\infty} a_n n \int_{c\pm iT} \frac{1}{s(s+1)} \left(\frac{Q}{n}\right)^{s+1} ds \\ &= \sum_{n=1}^{\infty} a_n n \delta(Q/n) + \mathcal{O}\left(\frac{Q^{c+1}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^c} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\}\right).\end{aligned}$$

For $Q/2 \leq n \leq 3Q/2$

$$|\log(Q/n)| = \left|\log\left(1 + \frac{n-Q}{Q}\right)\right| \gg \frac{|n-Q|}{n} \geq \Big\lfloor |n-Q| \Big\rfloor \Big/ n.$$

Therefore

$$\begin{aligned}\sum_{Q/2 \leq n \leq 3Q/2} \frac{|a_n|}{n^c} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\} &\leq A(Q)\left(\frac{1}{(Q/2)^c} + \frac{2\,(Q/2)^{1-c}}{T} \sum_{h \leq Q/2+1} \frac{1}{h}\right) \\ &\ll Q^{-c}A(Q)2^c\left(1 + \frac{Q\,(\log Q + 1)}{T}\right)\end{aligned}$$

and if n is not in this range then $|\log(Q/n)| \gg 1$ so we deduce

$$\frac{Q^{c+1}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^c} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\} \ll \frac{QA(Q)2^c}{T}\left(1 + \frac{Q\,(\log Q + 1)}{T}\right) + \frac{Q^{c+1}}{T^2} \sum_{n=1}^{\infty} \frac{|a_n|}{n^c}.$$

Therefore the error term in (135) is of the right order of magnitude and of course the main term is

$$\sum_{n \leq Q} a_n \Big(Q - n\Big)$$

so the main claim is proven. For the "in particular claim" the main claim implies an error term

$$Q^{\epsilon}\left(\frac{Q}{T} + \frac{Q^2}{T^2} + \frac{Q^{c+1}\zeta(c)}{T^2}\right);$$

now use $\zeta(c) \ll 1/(c-1) \ll \log Q$ and $Q^c \ll Q$. □

Lemma 5. *Take $Q > 0$, $L \geq 2$ and $\Delta \in [1/2k, 1/k]$. Let*

$$R_1 = -1 + \Delta \quad \textit{and} \quad R_2 = \Delta k.$$

Then

$$\int_1^L \frac{\zeta(R_1 + it)\zeta(R_2 + it)Q^{it}dt}{t^2} \ll L^{1/4 - 1/2k} \log L.$$

Proof. Write $s = \sigma + it$, suppose always $t \geq 1$ and take two parameters $N, M \gg 1$ with $NM = t/2\pi$. Let

$$\chi(s) = \frac{2^{s-1}\pi^s \sec(s\pi/2)}{\Gamma(s)}.$$

By formula (4.12.3) of [15] (the definition of $\chi(s)$ comes just before) we have for $-1 \leq \sigma \leq 1$

$$\begin{aligned} \chi(s) &= \left(\frac{t}{2\pi}\right)^{1/2-\sigma-it} e^{i(t+\pi/4)} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right) \\ &= \left(\frac{t}{2\pi}\right)^{1/2-\sigma-it} e^{i(t+\pi/4)} + \mathcal{O}\left(\frac{1}{t^{1/2+\sigma}}\right) \end{aligned} \tag{135}$$

so that

$$\chi(R_2+it) \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} = \left(\frac{t}{2\pi}\right)^{1/2-R_2-it} e^{i(t+\pi/4)} \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} + \mathcal{O}\left(\frac{M^{R_2}}{t^{1/2+R_2}}\right)$$

so by the approximate functional equation (formula (4.12.4) of [15])

$$\begin{aligned} \zeta(R_2+it) &= \sum_{n \leq N} \frac{1}{n^{R_2+it}} + \chi(R_2+it) \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} \\ &+ \mathcal{O}\left(N^{-R_2} + t^{1/2-R_2} M^{R_2-1}\right) \\ &= \sum_{n \leq N} \frac{1}{n^{R_2+it}} + \left(\frac{t}{2\pi}\right)^{1/2-R_2-it} e^{i(t+\pi/4)} \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} \\ &+ \mathcal{O}\left(\left(\frac{M}{t}\right)^{R_2} \left(1 + \frac{t^{1/2}}{M}\right)\right). \end{aligned} \tag{136}$$

From the functional equation (this just preceeds formula (4.12.1) of [15]) and from (135) we have

$$\begin{aligned} \zeta(R_1+it) &= \left(\left(\frac{t}{2\pi}\right)^{1/2-R_1-it} e^{i(t+\pi/4)} + \mathcal{O}\left(\frac{1}{t^{1/2+R_1}}\right)\right) \zeta(1-R_1-it) \\ &= \left(\frac{t}{2\pi}\right)^{1/2-R_1-it} e^{i(t+\pi/4)} \zeta(1-R_1-it) + \mathcal{O}\left(\frac{1}{t^{1/2+R_1}}\right) \end{aligned}$$

so that with (136) we get

$$\begin{aligned} &\zeta(R_1+it)\zeta(R_2+it) \\ &= \left(\frac{t}{2\pi}\right)^{1/2-R_1-it} e^{i(t+\pi/4)} \zeta(1-R_1-it) \sum_{n \leq N} \frac{1}{n^{R_2+it}} \\ &+ \left(\frac{t}{2\pi}\right)^{1-R_1-R_2-2it} e^{2i(t+\pi/4)} \zeta(1-R_1-it) \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} \\ &+ \mathcal{O}\Bigg(t^{1/2-R_1} |\zeta(1-R_1-it)| \left(\frac{M}{t}\right)^{R_2} \left(1 + \frac{t^{1/2}}{M}\right) \\ &+ \frac{(t/M)^{1-R_2} + t^{1/2-R_2} M^{R_2}}{t^{1/2+R_1}} + \frac{1}{t^{1/2+R_1}} \left(\frac{M}{t}\right)^{R_2} \left(1 + \frac{t^{1/2}}{M}\right)\Bigg) \\ &=: M_1(t) + M_2(t) + \mathcal{O}\left(t^{1/2-R_1} \left(\frac{M}{t}\right)^{R_2} \left(1 + \frac{t^{1/2}}{M}\right)\right). \end{aligned}$$

Write $N = t^{1/A}$ and $M = t^{1/B}$ so the above reads

$$\zeta(R_1 + it)\zeta(R_2 + it) = M_1(t) + M_2(t) + \mathcal{O}\left(t^{1/2-R_1+R_2/B-R_2}\left(1 + t^{1/2-1/B}\right)\right). \tag{137}$$

For some constant C

$$\begin{aligned} M_1(t)Q^{it} &= Ct^{1/2-R_1}\sum_{n\le N}\sum_{m=1}^{\infty}\frac{e^{it(-\log t+1-\log n+\log m+\log Q)}}{n^{R_2}m^{1-R_1}} \\ &= Ct^{1/2-R_1}\sum_{\substack{n^A\le t \\ 2\pi nM\le t}}\sum_{m=1}^{\infty}\frac{e\left(f_{mQ/n}(t)\right)}{n^{R_2}m^{1-R_1}} \end{aligned}$$

where

$$f_X(t) = \frac{t(-\log t + 1 + \log X)}{2\pi}$$

and the two summation conditions on n are equivalent. So for any $T \ge 1$

$$\int_T^{2T}\frac{M_1(t)Q^{it}dt}{t^2} = C\sum_{m=1}^{\infty}\frac{1}{m^{1-R_1}}\sum_{n^A\le T}\frac{1}{n^{R_2}}\int_{\max(2\pi nM,T)}^{2T}\frac{e\left(f_{mQ/n}(t)\right)dt}{t^{3/2+R_1}}. \tag{138}$$

We now bound this oscillatory integral. We have

$$2\pi f_X'(t) = -\log t + \log X. \tag{139}$$

Suppose first that T is large and $0 < X \ll 1$. For $\max(2\pi nM, T) < t < 2T$ we have from (139)

$$f_X'(t) \gg 1$$

and

$$t^{3/2+R_1} \gg T^{3/2+R_1}$$

so from Lemma 4.3 of [15]

$$\int_{\max(2\pi nM,T)}^{2T}\frac{e\left(f_X(t)\right)dt}{t^{3/2+R_1}} \ll \frac{1}{T^{3/2+R_1}}, \qquad \text{if } 0 < X \ll 1. \tag{140}$$

Suppose now X is large. Since from (139)

$$\begin{aligned} f_X'(t) &\gg |\log(t/X)| \\ &= |\log(1 + (t-X)/X)| \\ &\gg \begin{cases} |t-X|/X & \text{if } t \in (X/2, 3X/2) \\ 1 & \text{if not} \end{cases} \\ &\gg \begin{cases} 1/\sqrt{X} & \text{if } t \in (X/2, X-\sqrt{X}) \cup (X+\sqrt{X}, 3X/2) \\ 1 & \text{if } t \notin (X/2, 3X/2) \end{cases} \end{aligned}$$

and since for $t > T$

$$t^{3/2+R_1} \gg T^{3/2+R_1} \tag{141}$$

we have from Lemma 4.3 of [15]

$$\int_{\max(2\pi nM,T)}^{2T} \frac{e\left(f_X(t)\right)dt}{t^{3/2+R_1}}$$

$$\ll \begin{cases} \sqrt{X}/T^{3/2+R_1} & \text{if } (T,2T) \subseteq (X/2, X-\sqrt{X}) \cup (X+\sqrt{X}, 3X/2) \\ 1/T^{3/2+R_1} & \text{if } (T,2T) \subseteq (1,\infty)\backslash(X/2,3X/2) \end{cases}$$

$$\ll \frac{1}{T^{1+R_1}}$$

and this bound is trivial if $(T,2T) \subseteq (X-\sqrt{X}, X+\sqrt{X})$. Therefore from (140)

$$\int_{\max(2\pi nM,T)}^{2T} \frac{e\left(f_X(t)\right)dt}{t^{3/2+R_1}} \ll \frac{1}{T^{1+R_1}}$$

holds in fact for all $X > 0$, so we deduce from (138)

$$\begin{aligned} \int_T^{2T} \frac{M_1(t)Q^{it}dt}{t^2} &\ll \frac{1}{T^{1+R_1}} \sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n^A \le T} \frac{1}{n^{R_2}} \\ &\ll \frac{1}{T^{1+R_1}} \left(T^{1/A}\right)^{1-R_2} \\ &\ll T^{1/A-1-R_1-R_2/A}. \end{aligned} \tag{142}$$

Similarly we have

$$\int_T^{2T} \frac{M_2(t)Q^{it}dt}{t^2} = C\sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n^B \le T} \frac{1}{n^{1-R_2}} \int_{\max(nN,T)}^{2T} \frac{e\left(f_{mQ/n}(t)\right)dt}{t^{1+R_1+R_2}}$$

where the oscillatroy integral is

$$\ll \frac{1}{T^{1/2+R_1+R_2}}$$

so that

$$\begin{aligned} \int_T^{2T} \frac{M_2(t)Q^{it}dt}{t^2} &\ll \frac{1}{T^{1/2+R_1+R_2}} \sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n^B \le T} \frac{1}{n^{1-R_2}} \\ &\ll \frac{1}{T^{1/2+R_1+R_2}} \left(T^{1/B}\right)^{R_2} \\ &\ll T^{R_2/B-1/2-R_1-R_2}. \end{aligned} \tag{143}$$

Note that

$$-1/2 - R_1 - R_2/2 = 1/2 - \Delta - \Delta k/2 \le 1/2 - \Delta - 1/4 \tag{144}$$

so taking $A = B = 2$ we see from (142) and (143)

$$\int_T^{2T} \frac{\left(M_1(t)+M_2(t)\right)Q^{it}dt}{t^2} \ll T^{-1/2-R_1-R_2/2} \ll T^{1/4-\Delta}.$$

We assumed that T is large but the bound is trivial for T not large so we conclude

$$\int_1^L \frac{\left(M_1(t)+M_2(t)\right)Q^{it}dt}{t^2} \ll L^{1/4-\Delta}\log L$$

and so from (137) and (144)

$$\begin{aligned}\int_1^L \frac{\zeta(R_1+it)\zeta(R_2+it)Q^{it}dt}{t^2} &\ll L^{1/4-\Delta}\log L + \int_1^L t^{-3/2-R_1-R_2/2}dt \\ &\ll L^{1/4-\Delta}\log L.\end{aligned}$$

□

Lemma 6. *Let α, β, and γ be as in Lemma 3. For $X > 0$ and $T, c > 1$*

$$\begin{aligned}&\int_{c\pm iT} \frac{\zeta(s)\zeta\left(k(s+1)\right)\mathcal{F}^*(s)X^{s+1}ds}{s(s+1)\zeta\left(2k(s+1)\right)} \\ &\quad= \frac{\alpha X^2}{2} - \frac{\beta X}{2} + \frac{k\gamma X^{1/k}}{(-1+1/k)\zeta(2)} \\ &\quad+ \mathcal{O}\left(\log(2+X)\log(2+T)\left(T^{1/4}\left(\frac{X}{T}\right)^{1/2k} + \frac{X^{c+1}}{T^2}\right)\right).\end{aligned}$$

Proof. Suppose X and T are large since otherwise the claim is clear. For $s \in \mathbb{C}$ write always $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$ and let

$$\mathcal{I}(s) = \frac{\zeta(s)\zeta\left(k(s+1)\right)\mathcal{F}^*(s)}{\zeta\left(2k(s+1)\right)}. \tag{145}$$

Let $R_1 = -1 + 1/2k + \tau$ for some $0 < \tau < 1/k$. From Lemma 2 we have $\mathcal{F}^*(s) \ll 1$ for $\sigma \geq R_1$, therefore

$$\mathcal{I}(s) \ll \frac{|\zeta(s)\zeta\left(k(s+1)\right)|}{|\zeta\left(2k(s+1)\right)|}, \qquad \text{for } \sigma \geq R_1. \tag{146}$$

By Lemma 1 we see that $\mathcal{I}(s)$ is holomorphic except for simple poles at $s = 1$ and $s = -1 + 1/k$ so by the Residue Theorem

$$\begin{aligned}2\pi i \int_{c\pm iT} \frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)} &= \frac{X^2 Res_{s=1}\mathcal{I}(s)}{2} + \mathcal{I}(0)X + \frac{kX^{1/k}Res_{s=-1+1/k}\mathcal{I}(s)}{-1+1/k} \\ &- 2\pi i\left(\int_{c+iT}^{R_1+iT} + \int_{R_1+iT}^{R_1-iT} + \int_{R_1-iT}^{c-iT}\right)\frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)}.\end{aligned} \tag{147}$$

It is standard that for $t \geq 1$

$$\zeta(s) \ll \begin{cases} t^{1/2-\sigma+\epsilon} & \text{for } \sigma \leq 0 \\ t^{1/2-\sigma/2+\epsilon} & \text{for } \sigma \geq 0 \end{cases}$$

and

$$\zeta(\sigma) \ll \begin{cases} 1 & \text{for } \sigma \geq 2k \\ 1/|\sigma - 1| & \text{for } 1 \leq \sigma \leq 2; \end{cases}$$

we will now use these bounds freely without comment. If $0 \leq \sigma \leq 2$ and $t \geq 1$ we have

$$\zeta(s) \ll \max\{1, t^{1/2-\sigma/2+\epsilon}\},$$

$$\zeta\left(k(s+1)\right) \ll 1$$

and

$$\frac{1}{\zeta\left(2k(s+1)\right)} \ll \zeta\left(2k(\sigma+1)\right) \ll 1,$$

so from (146)

$$\mathcal{I}(s) \ll \max\{1, t^{1/2-\sigma/2+\epsilon}\}$$

and therefore

$$\int_{iT}^{c+iT} \frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)} \ll T^{\epsilon-3/2}X + \frac{X^{c+1}}{T^2} \ll \frac{X}{T} + \frac{X^{c+1}}{T^2}. \tag{148}$$

If $R_1 \le \sigma \le 0$ then for $t \ge 1$

$$\zeta(s) \ll t^{1/2-\sigma},$$
$$\zeta\left(k(s+1)\right) \ll t^{1/2}$$

and

$$\frac{1}{\zeta\left(2k(s+1)\right)} \ll \zeta\left(2k(\sigma+1)\right) \ll \frac{1}{|2k(\sigma+1)-1|} \ll \frac{1}{\tau} + 1,$$

so from (146)

$$\mathcal{I}(s) \ll \frac{t^{1-\sigma}}{\tau} \tag{149}$$

and therefore

$$\int_{R_1+iT}^{iT} \frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)} \ll \frac{1}{\tau}\left(\frac{X^{R_1+1}}{T^{1+R_1}} + \frac{X}{T}\right) \ll \frac{1}{\tau}\left(1 + \frac{X}{T}\right). \tag{150}$$

From (148) and (150) we have

$$\left(\int_{c+iT}^{R_1+iT} + \int_{R_1-iT}^{c-iT}\right) \frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)} \ll \frac{1}{\tau}\left(1 + \frac{X}{T} + \frac{X^{c+1}}{T^2}\right) \ll \frac{1}{\tau}\left(1 + \frac{X^{c+1}}{T^2}\right) \tag{151}$$

a similar argument for the second integral obviously valid. We now turn to the vertical contribution in (147). Denote by ω the number of prime factors of q. For given integers $n, l_1, ..., l_\omega, l'_1, ..., l'_\omega \ge 0$ write $\mathbf{n} = (n, l_1, ..., l_\omega, l'_1, ..., l'_\omega)$. Let $W_n, Z_{\mathbf{l},\mathbf{l}'}$ be as in Lemma 2. Then that lemma says that for given $\mathbf{n}$ there are $a_{\mathbf{n}} = a_{\mathbf{n}}(\sigma) \in \mathbb{R}$ such that for $\sigma \ge -1 + 1/2k$

$$\mathcal{F}^*(s) = \sum_{\mathbf{n}} a_{\mathbf{n}} \left(W_n Z_{\mathbf{l},\mathbf{l}'}\right)^{it}$$

and

$$\sum_{\mathbf{n}} a_{\mathbf{n}} \ll 1. \tag{152}$$

Therefore

$$\frac{\mathcal{F}^*(R_1+it)X^{it}}{\zeta\left(2k(R_1+it+1)\right)} = \sum_{m,\mathbf{n}} \frac{\mu(m)a_{\mathbf{n}}}{m^{2k(R_1+1)}} \left(\frac{XW_nZ_{\mathbf{l},\mathbf{l}'}}{m^{2k}}\right)^{it},$$

so from (145), Lemma 5 and (152)

$$\begin{aligned}\int_1^T \frac{\mathcal{I}(R_1+it)X^{it}dt}{t^2} &= \sum_{m,\mathbf{n}} \frac{\mu(n)a_{\mathbf{n}}}{m^{2k(R_1+1)}} \int_1^T \frac{\zeta(R_1+it)\zeta\left(k(R_1+it+1)\right)}{t^2} \left(\frac{XW_nZ_{\mathbf{l},\mathbf{l}'}}{m^{2k}}\right)^{it} dt \\ &\ll T^{1/4-1/2k}\log T \sum_{m,\mathbf{n}} \left|\frac{\mu(m)a_{\mathbf{n}}}{m^{2k(R_1+1)}}\right| \\ &\ll T^{1/4-1/2k}(\log T)\zeta(1+2k\tau) \ll \frac{T^{1/4-1/2k}\log T}{\tau}. \end{aligned} \tag{153}$$

For $|t| \geq 1$ we have

$$\frac{1}{s(s+1)} = \frac{1}{t^2} + \mathcal{O}\left(\frac{1}{t^3}\right)$$

therefore from (153)

$$\begin{aligned}
\int_{R_1}^{R_1+iT} \frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)} &= X^{R_1+1}\int_1^T \frac{\mathcal{I}(R_1+it)X^{it}ds}{t^2} \\
&+ \mathcal{O}\left(X^{R_1+1}\int_{\substack{R_1+i\infty \\ |t|\geq 1}} \frac{|\mathcal{I}(s)|ds}{t^3} + X^{R_1+1}\int_{R_1}^{R_1+i} \frac{|\mathcal{I}(s)|ds}{|s(s+1)|}\right) \\
&\ll \frac{X^{R_1+1}T^{1/4-1/2k}\log T}{\tau} \\
&= \frac{X^\tau T^{1/4}\log T}{\tau}\left(\frac{X}{T}\right)^{1/2k}
\end{aligned}$$

since for $0 \leq t \leq 1$ we obviously have $\mathcal{I}(s) \ll 1$ and since for $\sigma \geq -1$ and $|t| \geq 1$ we have clearly $\mathcal{I}(s) \ll |t|^{7/4}$. A similar bound obviously holding for t negative we conclude

$$\int_{R_1+iT}^{R_1-iT} \frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)} \ll \frac{X^\tau T^{1/4}\log T}{\tau}\left(\frac{X}{T}\right)^{1/2k}. \tag{154}$$

From Lemma 3 (A) we have

$$Res_1\mathcal{I}(s) = \frac{\zeta(2k)\mathcal{F}^*(1)}{\zeta(4k)} = \alpha,$$

$$\mathcal{I}(0) = \frac{\zeta(0)\zeta(k)\mathcal{F}^*(0)}{\zeta(2k)} = -\frac{\beta}{2}$$

and

$$Res_{-1+1/k}\mathcal{I}(s) = \frac{\zeta(-1+1/k)\mathcal{F}^*(-1+1/k)}{\zeta(2)} = \frac{\gamma}{\zeta(2)}$$

so the main terms in (147) are

$$\frac{\alpha X^2}{2} - \frac{\beta X}{2} + \frac{k\gamma X^{1/k}}{(-1+1/k)\zeta(2)} =: M(X).$$

This with (151) and (154) means (147) becomes

$$\begin{aligned}
\int_{c\pm iT} \frac{\mathcal{I}(s)X^{s+1}ds}{s(s+1)} &= M(X) + \mathcal{O}\left(\frac{1}{\tau}\left(X^\tau T^{1/4}\log T\left(\frac{X}{T}\right)^{1/2k} + \frac{X^{c+1}}{T^2}\right)\right) \\
&= M(X) + \mathcal{O}\left(\log T\log X\left(T^{1/4}\left(\frac{X}{T}\right)^{1/2k} + \frac{X^{c+1}}{T^2}\right)\right)
\end{aligned}$$

on taking $\tau = 1/\log X$. □

Lemma 7. *For any $x, y > 0$*

$$\sum_{[d,d']\leq y} 1 \leq y^{1+\epsilon},$$

$$\sum_{[d,d']>y} \frac{1}{[d^k, {d'}^k]} \ll y^{1-k+\epsilon}$$

and, for $ql \leq x$,

$$\sum_{\substack{d,d \\ [d,d']>y}} \sum_{\substack{n \leq x \\ n \equiv 0(d^k) \\ n \equiv -ql({d'}^k)}} 1 \ll xy^{1-k+\epsilon} + x^{2/(k+1)+\epsilon}.$$

Proof. Since

$$\sum_{[d,d']=n} 1 \ll n^\epsilon$$

we have

$$\sum_{[d,d']\leq y} 1 = \sum_{n \leq y} \sum_{[d,d']=n} 1 \ll y^{1+\epsilon}$$

and

$$\sum_{[d,d']>y} \frac{1}{[d^k, {d'}^k]} = \sum_{n>y} \frac{1}{n^k} \sum_{[d,d']=n} 1 \ll y^{1-k+\epsilon}$$

which are the first two claims. Let Z be a parameter. We have with a divisor estimate

$$\sum_{\substack{d,d' \\ d>Z}} \sum_{\substack{n \leq x \\ n \equiv 0(d^k) \\ n \equiv -ql({d'}^k)}} 1 \ll x^\epsilon \sum_{\substack{d^k \leq x \\ d>Z}} \sum_{\substack{n \leq x \\ n \equiv 0(d^k)}} 1 \ll x^{1+\epsilon} \sum_{d>Z} \frac{1}{d^k} \ll x^{1+\epsilon} Z^{1-k}$$

and similarly for the terms with $d' > Z$. On the other hand the second claim implies

$$\begin{aligned} \sum_{\substack{d,d' \leq Z \\ [d,d']>y}} \sum_{\substack{n \leq x \\ n \equiv 0(d^k) \\ n \equiv -ql({d'}^k)}} 1 &\ll \sum_{\substack{d,d' \leq Z \\ [d,d']>y}} \left(\frac{x}{[d^k, {d'}^k]} + 1 \right) \\ &\ll xy^{1-k+\epsilon} + Z^2 \end{aligned}$$

and therefore

$$\sum_{[d,d']>y} \sum_{\substack{n \leq x \\ n \equiv 0(d^k) \\ n \equiv -ql({d'}^k)}} 1 \ll xy^{1-k+\epsilon} + Z^2 + x^{1+\epsilon} Z^{1-k}$$

which gives the claim on choosing $Z = x^{1/(k+1)}$. □

3.2 - Proof of theorem

Let $x > 0$ large be given and define $\eta(q,a)$ and $V_x(q)$ as in (6) and (7). Opening the square we have

$$\begin{aligned} V_x(q) &= \sum_{a=1}^{q} \sum_{\substack{n,n' \le x \\ n,n' \in \mathcal{S} \\ n \equiv n' \equiv a(q)}} 1 - 2x \sum_{a=1}^{q} \eta(q,a) \sum_{\substack{n \le x \\ n \in \mathcal{S} \\ n \equiv a(q)}} 1 + x^2 \sum_{a=1}^{q} \eta(q,a)^2 \\ &= \sum_{\substack{n,n' \le x \\ n,n' \in \mathcal{S} \\ n \equiv n'(q)}} 1 - 2x \sum_{\substack{n \le x \\ n \in \mathcal{S}}} \eta(q,n) + x^2 \sum_{a=1}^{q} \eta(q,a)^2 \\ &=: A_x(q) - 2x B_x(q) + x^2 \sum_{a=1}^{q} \eta(q,a)^2. \end{aligned} \tag{155}$$

From Lemma 3 (B) we have $\eta(q,n) = \eta\left(q,(q,n)\right)$ and $\eta(q,d) \ll 1$. Therefore from Lemma 2.2 (ii) of [17] we have for some constants c_{dh}, c_q

$$\begin{aligned} B_x(q) &= \sum_{d|q} \eta(q,d) \sum_{\substack{n \le x \\ n \in \mathcal{S} \\ (n,q)=d}} 1 \\ &= \sum_{d|q} \eta(q,d) \sum_{h|q/d} \mu(h) \sum_{\substack{n \le x \\ n \in \mathcal{S} \\ dh|n}} 1 \\ &= x \sum_{d|q} \eta(q,d) \sum_{h|q/d} \mu(h) c_{dh} + \mathcal{O}\left(x^{1/k+\epsilon} \sum_{d|q} |\eta(q,d)| \sum_{h|q/d} |\mu(h)| \right) \\ &= x c_q + \mathcal{O}\left(x^{1/k+\epsilon} \right). \end{aligned} \tag{156}$$

But evidently with $x \to \infty$

$$B_x(q) = \sum_{a=1}^{q} \eta(q,a) \sum_{\substack{n \le x \\ n \in \mathcal{S} \\ n \equiv a(q)}} 1 \sim x \sum_{a=1}^{q} \eta(q,a)^2$$

so (156) must read

$$B_x(q) = x \sum_{a=1}^{q} \eta(q,a)^2 + \mathcal{O}\left(x^{1/k+\epsilon} \right). \tag{157}$$

It is well known that

$$\sum_{\substack{n \le x \\ n \in \mathcal{S}}} 1 = \frac{x}{\zeta(k)} + \mathcal{O}\left(x^{1/k} \right)$$

therefore

$$\begin{aligned} A_x(q) &= 2\sum_{\substack{n<n'\le x\\ n,n'\in\mathcal{S}\\ n\equiv n'(q)}} 1+\sum_{\substack{n\le x\\ n\in\mathcal{S}}}1 \\ &= 2\sum_{l\le x/q}\sum_{\substack{n,n'\le x\\ n,n'\in\mathcal{S}\\ n'-n=ql}} 1+\frac{x}{\zeta(k)}+\mathcal{O}\left(x^{1/k}\right) \\ &=: 2C_x(q)+\frac{x}{\zeta(k)}+\mathcal{O}\left(x^{1/k}\right) \qquad (158)\end{aligned}$$

so we deduce from (155) and (157)

$$V_x(q) = 2C_x(q)+\frac{x}{\zeta(k)}-x^2\sum_{a=1}^{q}\eta(q,a)^2+\mathcal{O}\left(x^{1/k+\epsilon}\right). \qquad (159)$$

Take a parameter $y\le x^{1/k}$ so that $[d,d']\le y$ is a stronger condition than $d^k, d'^k\le x$. Using

$$\sum_{d^k|n}\mu(d)=\begin{cases}1 & \text{if } n \text{ is } k\text{-free}\\ 0 & \text{if not}\end{cases}$$

we see that

$$\begin{aligned} \sum_{\substack{n,n'\le x\\ n,n'\in\mathcal{S}\\ n'-n=ql}} 1 &= \sum_{d,d'}\mu(d)\mu(d')\sum_{\substack{n,n'\le x\\ n\equiv 0(d^k)\\ n'\equiv 0(d'^k)\\ n'-n=ql}} 1 \\ &= \sum_{d,d'}\mu(d)\mu(d')\sum_{\substack{n\le x-ql\\ n\equiv 0(d^k)\\ n\equiv -ql(d'^k)}} 1 \\ &= \sum_{\substack{[d,d']\le y\\ (d^k,d'^k)|ql}}\mu(d)\mu(d')\left(\frac{x-ql}{[d^k,d'^k]}+\mathcal{O}(1)\right)+\mathcal{O}\left(\sum_{[d,d']>y}\sum_{\substack{n\le x\\ n\equiv 0(d^k)\\ n\equiv -ql(d'^k)}}1\right) \\ &= (x-ql)\sum_{\substack{d,d'=1\\ (d^k,d'^k)|ql}}^{\infty}\frac{\mu(d)\mu(d')}{[d^k,d'^k]}+\mathcal{O}\left(\sum_{[d,d']\le y}1\right) \\ &+ \mathcal{O}\left((x-ql)\sum_{[d,d']>y}\frac{1}{[d^k,d'^k]}\right)+\mathcal{O}\left(\sum_{[d,d']>y}\sum_{\substack{n\le x\\ n\equiv 0(d^k)\\ n\equiv -ql(d'^k)}}1\right).\end{aligned}$$

From Lemma 7 the error terms here are for $ql\le x$

$$y^{1+\epsilon}+xy^{1-k+\epsilon}+x^{2/(k+1)+\epsilon}\ll x^{2/(k+1)+\epsilon}$$

after setting $y=x^{1/k}$, so that

$$\sum_{\substack{n,n'\le x\\ n,n'\in\mathcal{S}\\ n'-n=ql}} 1=(x-ql)\sum_{\substack{d,d'=1\\ (d^k,d'^k)|ql}}^{\infty}\frac{\mu(d)\mu(d')}{[d^k,d'^k]}+\mathcal{O}\left(x^{2/(k+1)+\epsilon}\right)$$

so from (158)

$$\begin{aligned} C_x(q) &= \sum_{d,d'} \frac{\mu(d)\mu(d')}{[d^k, d'^k]} \sum_{\substack{l \le x/q \\ (d^k, d'^k) | ql}} \Big(x - ql \Big) + \mathcal{O}\left(\left(x^{2/(k+1)+\epsilon} \right) \sum_{l \le x/q} 1 \right) \\ &= \sum_{d,d'} \frac{\mu(d)\mu(d')[q,(d^k,d'^k)]}{[d^k, d'^k]} \sum_{l \le x/[q,(d^k,d'^k)]} \left(\frac{x}{[q,(d^k,d'^k)]} - l \right) + \mathcal{O}\left(\frac{x^{1+2/(k+1)+\epsilon}}{q} \right) \\ &=: \mathcal{J}(x) + \mathcal{O}\left(\frac{x^{1+2/(k+1)+\epsilon}}{q} \right). \end{aligned} \tag{160}$$

From now on all $\ll$ symbols will denote bounds up to x^ϵ bounds so that (159) and (160) read

$$V_q(x) = 2\mathcal{J}(x) + \frac{x}{\zeta(k)} - x^2 \sum_{a=1}^{q} \eta(q,a)^2 + \mathcal{O}\left(\frac{x^{1+2/(k+1)}}{q} \right). \tag{161}$$

Let $c = 1 + 1/\log x$ so that $x^{c+1} \ll x^2$ and $\zeta(c) \ll \log x \ll 1$. From Lemma 4 the inner sum in $\mathcal{J}(x,q)$ is for any $T > 0$

$$\int_{c \pm iT} \frac{\zeta(s)}{s(s+1)} \left(\frac{x}{[q,(d^k,d'^k)]} \right)^{s+1} ds + \mathcal{O}\left(\frac{x^2}{T^2 [q,(d^k,d'^k)]^2} \right)$$

and therefore from Lemma 1 and Lemma 6

$$\begin{aligned} \mathcal{J}(x) &= \int_{c \pm iT} \frac{\zeta(s) x^{s+1}}{s(s+1)} \left(\sum_{d,d'} \frac{\mu(d)\mu(d')}{[d^k,d'^k][q,(d^k,d'^k)]^s} \right) ds + \mathcal{O}\left(\frac{x^2}{T^2} \sum_{d,d'} \frac{|\mu(d)\mu(d')|}{[d^k,d'^k][q,(d^k,d'^k)]} \right) \\ &= q \int_{c \pm iT} \frac{\zeta(s)\zeta\left(k(s+1)\right)\mathcal{F}^*(s)}{s(s+1)\zeta\left(2k(s+1)\right)} \left(\frac{x}{q} \right)^{s+1} ds + \mathcal{O}\left(\frac{x^2}{qT^2} \right) \\ &= \frac{\alpha x^2}{2q} - \frac{\beta x}{2} + \frac{k\gamma q^{1-1/k} x^{1/k}}{(-1+1/k)\zeta(2)} + \mathcal{O}\left(q \left(T^{1/4} \left(\frac{x}{qT} \right)^{1/2k} + \left(\frac{x}{qT} \right)^2 \right) \right) \end{aligned}$$

where α, β, γ are as in Lemma 3. Setting

$$T = \left(\frac{x}{q} \right)^V$$

where

$$V = \frac{2 - 1/2k}{9/4 - 1/2k}$$

the error term becomes

$$\ll q \left(\frac{x}{q} \right)^{2/(9-2/k)}$$

and so from (161)

$$\begin{aligned} V_x(q) = \left(\frac{\alpha}{q} - \sum_{a=1}^{q} \eta(q,a)^2 \right) x^2 + \left(\frac{1}{\zeta(k)} - \beta \right) x + \frac{2k\gamma q^{1-1/k} x^{1/k}}{(-1+1/k)\zeta(2)} \\ + \quad \mathcal{O}\left(q \left(\frac{x}{q} \right)^{2/(9-2/k)} \right) + \mathcal{O}\left(\frac{x^{1+2/(k+1)}}{q} \right). \end{aligned}$$

From Lemma 3 (B) the x^2 coefficient vanishes. Directly from the definitions (Lemma 3) we see that $\beta = \zeta(k)^{-1}$ so the x coefficient also vanishes. Again from the definitions the $x^{1/k}$ coefficient is

$$\frac{2kq^{1-1/k}}{(-1+1/k)\zeta(2)} \prod_p \left(\frac{1 - 2/(p^k + p^{k-1})}{1 - p^{1-1/k}} \right) \prod_{p|q} \left(\frac{1 + (q, p^k)^{-1+1/k}/p - 2/p^k}{1 + 1/p - 2/p^k} \right)$$

and we have Theorem 2.

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