L^2-Invariants for Self-Similar CW-Complexes

Dissertation
for the award of the degree
"Doctor rerum naturalium" (Dr. rer. nat.)
of the Georg-August-Universität Göttingen

within the doctoral program Mathematical Sciences
of the Georg-August University School of Science
(GAUSS)

submitted by
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from Braunschweig

Göttingen, 2020

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Date of the oral examination

7 October 2020

Acknowledgements

I would like to thank my advisor, Prof. Dr. Thomas Schick, for sending me on this mathematical journey. His enthusiastically shared knowledge and neverending support made this thesis possible.

I would like to thank my second advisor, Prof. Dr. Dorothea Bahns, for fruitful discussions, insightful advice, and always having an open ear for me.

I would like to thank Prof. Dr. Gábor Elek und Dr. Łukasz Grabowski for the invitation to Lancaster and for showing me their mathematical point of view.

I would like to thank my family, who are always there for me when I need them most.

Finally, my thanks go to the German Research Foundation (DFG), who financially supported this thesis through the Research Training Group 1493 "Mathematical structures in modern quantum physics" at the University of Göttingen.

Contents

1 Introduction

Algebraic topology means to use algebraic tools to answer topological questions. We take some description of a topological space, often in combinatorial or geometrical terms, and turn it into an algebraic structure. That structure tends to be large and unsightly at first, but the algebraic machinery will eventually distill it down to succinct statements about the topology of our space. And hopefully, the result will be independent of the choice of the description we gave in the beginning or the algebraic detours we took in between.

Homology theory is one of the two most important such machines.[1] Most topological spaces can be considered as cell complexes: they can be built up from vertices (0-cells), edges (1-cells), faces (2-cells), etc. Let $\mathcal{E}_j X$ be the set of j-cells of a space X, and $\mathbb{C}[\mathcal{E}_j X]$ be the abstract vector space they generate. Then, the geometric description of X translates into a series of *boundary maps*

$$\ldots \to \mathbb{C}[\mathcal{E}_3 X] \xrightarrow{\partial_3} \mathbb{C}[\mathcal{E}_2 X] \xrightarrow{\partial_2} \mathbb{C}[\mathcal{E}_1 X] \xrightarrow{\partial_1} \mathbb{C}[\mathcal{E}_0 X]$$

where ∂_j sends each j-cell to the sum (or, depending on the orientations, the difference) of the $(j-1)$-cells that make up its boundary.

Let us combine the boundary maps into *Laplacian operators*:

$$\Delta_j = \partial_j^* \partial_j + \partial_{j+1} \partial_{j+1}^* \colon \mathbb{C}[\mathcal{E}_j X] \to \mathbb{C}[\mathcal{E}_j X].$$

The kernels of these operators are the *homology groups* of X:

$$H_j(X; \mathbb{C}) = \ker \Delta_j.$$

These are not only much smaller than the vector spaces $\mathbb{C}[\mathcal{E}_j X]$, but also independent of the precise geometric description of the space – they only measure topological properties. Their dimensions are the *Betti numbers* of X:

$$\beta_j(X) = \dim_{\mathbb{C}} H_j(X; \mathbb{C}) = \dim_{\mathbb{C}}(\ker \Delta_j).$$

L^2-invariants are an approach to homology for spaces with infinitely many cells. Completing the vector spaces $\mathbb{C}[\mathcal{E}_j X]$ yields Hilbert spaces $\ell^2(\mathcal{E}_j X)$, and the Laplacians extend (under certain conditions) to bounded operators on these spaces. Unlike in the finite case, these new Laplacians usually have a continuous spectrum, and it turns out that the entire spectrum – not just the size of the kernel – can carry topological information. To measure this, we require a *spectral density function*[2], which, for any $\lambda \geq 0$, quantifies the size of the largest subspace on which the operator's norm is bounded by λ.

Defining such a function poses one main challenge: to describe the size of infinite-dimensional spaces with finite numbers.

[1] The other being homotopy theory.

[2] Often also called the *integrated density of states*.

Periodic spaces

Let us call a complex X *periodic* if there are a finite subcomplex K and a group G acting freely and cellularly on X such that $G \cdot K = X$. Then the infinitely many cells of X form finitely many G-orbits, and each $\ell^2(\mathcal{E}_j X)$ can be identified with a space $(\ell^2 G)^n$ for some $n \in \mathbb{N}$.

For any G-equivariant operator $T \in \mathcal{B}(\ell^2(\mathcal{E}_j X))$, the value of $\langle \sigma, T\sigma \rangle$ (for $\sigma \in \mathcal{E}_j X$) is constant along any G-orbit! Taking the trace over only one representative per orbit yields the *von Neumann trace*

$$\operatorname{tr}_{\mathcal{N}(G)}(T) = \sum_{[\sigma] \in (\mathcal{E}_j X)/G} \langle \sigma, T\sigma \rangle \,.$$

The Laplacian is G-equivariant, since it only depends on the geometric structure of the space that is preserved by the G-action. Furthermore, the G-equivariant operators form a *von Neumann algebra* (that is, a weakly closed C^*-algebra), so any spectral projections $\chi_{[0,\lambda]}(\Delta_j)$ are G-equivariant as well, and we can define the desired spectral density function as

$$F(\Delta_j)(\lambda) = \operatorname{tr}_{\mathcal{N}(G)}\big(\chi_{[0,\lambda]}(\Delta_j)\big).$$

Especially, its value at zero measures the size of the kernel of Δ_j, and constitutes the j-th L^2*-Betti number* of X:

$$b_j^{(2)}(X) = F(\Delta_j)(0) = \operatorname{tr}_{\mathcal{N}(G)}(\operatorname{proj}_{\ker \Delta_j}).$$

This is the starting point of the theory of L^2-invariants, invented by Atiyah [Ati76].

Novikov and Shubin [NS86] found a topological invariant that quantifies the "almost-kernel" (the part of the spectrum very *close* to zero):

$$\alpha_j(X) = \lim_{\lambda \to 0} \frac{\log(F(\Delta_j)(\lambda) - F(\Delta_j)(0))}{\log(\lambda)}.$$

Finally, the spectral density function allows to define a determinant in the sense of Fuglede and Kadison [FK52] for such operators.

L^2-invariants have been studied in great detail (see [Lüc02] for an extensive treatment, and [Kam19] for an overview). However, their construction relied heavily on the existence of a suitable group action on the space – in other words, on periodicity.

However, there is a completely different approach to these invariants, in which the group structure fades into the background: approximation.

Let us again write $X = G \cdot K$ with a compact subcomplex K. At first, the L^2-Betti numbers of X have little to nothing in common with the Betti numbers of K or the quotient space X/G: Evaluating the Laplacian on a cell near the boundary of K will produce drastically different results depending on whether crossing that boundary will lead into another copy of K (when we are working on X), or back into K itself (when we are working on X/G), or into nothingness (when we are working just on K). Thus, if we want to "approximate" X by a finite subcomplex K, the boundary of K will be where the similarities end. Consequently, K will only show similar properties to X if its boundary is insignificant compared to its interior!

This is one of the many definitions of *amenability*: A space X is amenable if there is a *Følner sequence* of finite subspaces

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq X, \quad \bigcup_m K_m = X,$$

such that, in some suitable measure, the share of points in K_m that are close to its boundary converges to zero.

In such an amenable periodic space, Dodziuk and Mathai [DM98] proved that the L^2-Betti numbers can indeed be obtained from ordinary Betti numbers of larger and larger subspaces: If n_m counts how many representatives of each G-orbit lie in K_m, then

$$b_j^{(2)}(X;G) = \lim_{m\to\infty} \frac{\beta_j(K_m)}{n_m}, \tag{$*$}$$

and their proof can be extended to approximate not just L^2-Betti numbers, but whole spectral density functions.

In this final formula, the group structure barely appears any more (only in the normalization factor n_m, which could be replaced by e. g. the number of cells in K_m). Thus, we can begin to ask the question: Can this limit exist if there is no group action on X?

Aperiodic spaces

The existence of the limit ($*$) depends mainly on two factors. On the one hand, it needs amenability: For example, any d-regular tree with $d \geq 3$ has a positive first L^2-Betti number, while each finite subtree of it has $\beta_1 = 0$. On the other hand, it requires that finite subcomplexes are in some sense "representative" for the whole space: any structure that can be found in the space must be found at a similar frequency in every sufficiently large finite subspace. Periodic spaces certainly satisfy this condition – but they are not the only ones.

A first such observation was made by Geerse and Hof [GH91], who studied self-similar tilings of $\mathbb{R}^n$ (such as the decidedly non-periodic Penrose tilings) in an effort to model the physical properties of quasicrystals, and proved the existence of various thermodynamic means.

Kellendonk [Kel95] studied the same tilings from a mathematical point of view. He used the geometry of the tiling itself to define a C^*-algebra of operators, and established the existence of a spectral density function for such operators.

Cipriani, Guido and Isola [CGI09] constructed *self-similar complexes*:

Beginning with a finite CW-complex K_0, define a sequence of complexes K_m, where each K_m is the union of several copies of K_{m-1}, glued together along a small number of overlapping cells. Identifying each K_m with one part of K_{m+1}, one then obtains the self-similar space as the union

$$X = \bigcup_{m=0}^{\infty} K_m.$$

Under the condition that (K_m) is a Følner sequence in X, Cipriani, Guido and Isola were able to define traces for "geometric" operators on such spaces. However, geometric operators do not form a von Neumann algebra, and their spectral projections are not geometric. Thus, with no access to spectral density functions, Cipriani, Guido and Isola defined Betti numbers as

$$\beta(\Delta) = \lim_{t\to\infty} \operatorname{tr}(e^{-t\Delta})$$

and Novikov–Shubin invariants as

$$\alpha(\Delta) = 2 \lim_{t\to\infty} \frac{\log(\operatorname{tr}(e^{-t\Delta}) - \beta(\Delta))}{-\log(t)}.$$

They proved that the Euler characteristics of K_m converged to that of X, and calculated Novikov–Shubin invariants for certain complexes.

Meanwhile, Elek [Ele06] gave a precise definition for *aperiodic order* on general graphs (that is, one-dimensional complexes): In a graph, let the *r-pattern* of a vertex v be the isomorphism class of the (rooted) graph spanned by all the vertices that are at most r steps away from v. Then a graph has *aperiodic order* if every such pattern appears at a well-defined frequency: in any Følner sequence, the share of vertices with this pattern converges to the same number.

Elek then defined the algebra of *pattern-invariant operators* on the space $\ell^2(\text{vertices})$, whose values on a vertex only depend on the pattern of the vertex, and proved that their spectral density functions can be obtained as a uniform limit over finite subgraphs – provided that the graph has aperiodic order. (The pattern-invariant operators do not form a von Neumann algebra either; Elek avoided this problem by passing to the Gelfand–Naimark–Segal construction – an abstract algebra based on the representation of an algebra on "itself".)

In a second paper [Ele08], Elek found a large class of graphs that actually satisfy this condition by relating it to Benjamini–Schramm convergence of the graphs themselves.

Content and results of this thesis

In this thesis, we combine and expand the ideas of Elek and Cipriani–Guido–Isola to define and study L^2-invariants for self-similar complexes.

In Chapter 2, we extend Elek's framework of aperiodic order to higher-dimensional complexes. This includes the existence of a trace for geometric operators on such complexes, and the extension of the trace to a suitable von

Neumann algebra, which allows us to define spectral density functions for any such operators.

In Chapter 3, we show that Cipriana–Guido–Isola's self-similar complexes always have aperiodic order, and prove the approximation theorem for spectral density functions:

Theorem (3.5 and 3.11)**.** *Let X be a self-similar complex with Følner sequence (K_m), and let $T \in \mathcal{B}(\ell^2(\mathcal{E}_j X))$ be any geometric operator. Then the renormalized spectral density functions of $T|_{K_m}$ converge uniformly to the spectral density function of T.*

In Chapter 4, we define L^2-Betti numbers and Novikov–Shubin invariants for self-similar complexes, and we study their properties. Especially, we show that the L^2-Betti numbers of a self-similar complex are approximated by those of its subcomplexes and discuss this possibility for Novikov–Shubin invariants, and we prove that both of these are indeed invariant under self-similar homotopies:

Theorem (4.13 and 4.14)**.** *Let X and Y be self-similar complexes that are self-similarly homotopy equivalent. Then we have*

$$b_j^{(2)}(X) = c \cdot b_j^{(2)}(Y) \quad \text{and} \quad \alpha_j(X) = \alpha_j(Y) \quad \text{for all } j,$$

where the constant

$$c = \lim_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{E}_j K_m|}$$

adjusts for the number of cells used in the specific cell structure of each complex. It is independent of the choice of self-similar Følner sequences (K_m) for X and (L_m) for Y, as long as they fulfill $K_m \simeq L_m$ for all $m \in \mathbb{N}$.

In Chapter 5, we discuss Fuglede–Kadison determinants of geometric operators. We can prove that these determinants in general share many of the properties of their classical equivalents, especially multiplicativity, and that the Laplacians of self-similar complexes are of determinant class; this lets us also define L^2-torsion for self-similar complexes. Whether the determinants or the torsion can be approximated in general remains an open question, but we can show convergence for the Laplacians of some self-similar CW-complexes.

In Chapter 6, we show that the cartesian product of self-similar complexes is again such a complex, and we prove product formulas for all three L^2-invariants:

Theorem (6.3, 6.5 and 6.6)**.** *Let X and Y be self-similar complexes, and normalize every trace by the numbers of vertices. Then we have:*

(a) *L^2-Betti numbers fulfill the Künneth formula:*

$$b_\ell^{(2)}(X \times Y) = \sum_{j+k=\ell} b_j^{(2)}(X) \cdot b_k^{(2)}(Y).$$

(b) *If X and Y have the limit property, then so does $X \times Y$, and in this case, the Novikov–Shubin invariants fulfill*

$$\alpha_\ell(X \times Y) = \min\Big(\{\alpha_j(X) + \alpha_k(Y) \,|\, j + k = \ell\} \\ \cup \Big\{\alpha_j(X) \,\Big|\, b^{(2)}_{\ell-j}(Y) > 0\Big\} \cup \Big\{\alpha_k(Y) \,\Big|\, b^{(2)}_{\ell-k}(X) > 0\Big\}\Big)$$

(c) *Let $\rho^{(2)}$ denote L^2-torsion and $\chi^{(2)}$ denote the L^2-Euler characteristic. Then*

$$\rho^{(2)}(X \times Y) = \chi^{(2)}(X)\,\rho^{(2)}(Y) + \chi^{(2)}(Y)\,\rho^{(2)}(X).$$

Finally, a short appendix summarizes the most important facts about the Borel functional calculus that is necessary to define and work with spectral density functions.

2 Pattern-invariant operators and traces

Throughout this thesis, we aim to use geometrical (or topological) properties of spaces to ensure the analytical convergence of algebraic properties. In this chapter, we will lay the groundwork for all of that.

First, we will look at the geometric structure of *regular CW-complexes* and how it translates into algebra. Then, we will define the concept of *geometric operators*, that is, operators on the L^2-chain groups whose values only depend on the geometric *patterns* of the space.

The most important geometric operators are the Laplacians of the space, and every L^2-invariant will later be derived from their spectra. We therefore turn to functional analysis to construct a tool that measures these spectra, namely, the *spectral density function* (or *integrated density of states*). This function is usually defined as the trace of the spectral projections of the operators – which poses two challenges: There is a priori no trace on the set of operators on an infinite-dimensional space, and spectral projections of geometric operators are in general not geometric.

Constructing a trace for the geometric operators themselves requires to take a mean over the infinite set of cells. To ensure such a mean is well-defined, we will make use of the concept of *aperiodic order*: We will consider only spaces where every pattern appears with a well-defined frequency. (We will show in the next chapter that self-similar complexes do indeed have this property.) In that situation, the defining property of geometric operators ensures the existence of the trace.

The trace is unfortunately not weakly continuous, and it therefore does not simply extend to the weak closure of the algebra of geometric operators (which would contain the spectral projections we are interested in). Instead, we will construct a different von Neumann algebra containing all geometric operators to which the trace can be extended. This will finally allow us to define the desired spectral density functions.

2.1 Preliminaries

As a compromise between the algebraically simple, but rigid structure of simplicial complexes and the flexible, but algebraically complicated structure of CW-complexes, we will be using *regular CW-complexes*. Let us briefly look at their definition and most important properties.

Unless otherwise noted, every map of topological spaces will be assumed to be continuous.

2.1 Definition. Let X be a CW-complex, and denote by $\mathcal{E}_j X$ the set of j-cells of X. As a special case, if X is one-dimensional, it is a graph with vertex set $\mathcal{E}_0 X$ and edge set $\mathcal{E}_1 X$.

X is the disjoint union of its cells. Denote by $X^{(j)}$ the j-skeleton of X, that is, the union of all cells of dimension $\leq j$.

For any cell $\sigma \in \mathcal{E}_j X$, let $f_\sigma \colon S^{j-1} \to X^{(j-1)}$ be the attaching map. It extends to a map $F_\sigma \colon \overline{D^j} \to X$ such that $F_\sigma(D^j) = \sigma$. Denote by $\partial\sigma = f_\sigma(S^{j-1})$ the topological boundary of σ in X.

A subcomplex $K \subseteq X$ is called *full* if, whenever K contains the boundary of a cell σ of X, it also contains σ.

The complex X is *regular* if for each cell σ, the extended attaching map $F_\sigma \colon \overline{D^j} \to \overline{\sigma} \subseteq X$ is a homeomorphism onto its image.

The complex X is *bounded* if there is a constant $C > 0$ such that each cell $\sigma \in \mathcal{E}_j X$ (for arbitrary j) fulfills

$$|\{\rho \in \mathcal{E}_{j-1} X \mid \rho \subseteq \partial\sigma\}| \leq C$$

and

$$|\{\tau \in \mathcal{E}_{j+1} X \mid \sigma \subseteq \partial\tau\}| \leq C.$$

Regularity is a rather strong restriction for CW-complexes. On the one hand, it can necessitate much more complicated cell structures: For example, the n-sphere has a CW-structure with only two cells (a 0- and an n-cell) but its smallest regular CW-structure consists of $2n+1$ cells (two of each dimension between 0 and n).

On the other hand, regularity allows to treat the cells in a much more intuitive way: For example, it allows us to say that the boundary of a cell σ *consists of* certain other cells, and it ensures that the closure of every cell is a subcomplex:

2.2 Lemma. *Let X be a regular CW-complex. Let $\rho \in \mathcal{E}_{j-1} X$ and $\sigma \in \mathcal{E}_j X$. Then either $\rho \subseteq \partial\sigma$ or $\rho \cap \partial\sigma = \emptyset$.*

Proof. Assume the contrary and choose a point

$$x \in \overline{\rho \cap \partial\sigma} \cap \overline{\rho \setminus \partial\sigma}.$$

(The intersection is nonempty because ρ is connected.) Since $\partial\sigma$ is closed in X, we have $x \in \partial\sigma$.

Using the attaching map $f_\sigma \colon S^{j-1} \to \partial\sigma \subseteq X$, define $U_r = f_\sigma(B_r(f_\sigma^{-1}(x)))$, where $B_r(\xi)$ means the open r-ball around ξ in $S^{j-1} \subseteq \mathbb{R}^j$. Each of the U_r is by definition homeomorphic to D^{j-1} and contained in $\partial\sigma$.

If there were an $r > 0$ such that $U_r \subseteq \rho$, then this U_r would also be an open neighborhood of x in ρ (since ρ itself is homeomorphic to a disc D^{j-1}). But then x could not be contained in $\overline{\rho \setminus \partial\sigma}$ – contradiction.

Thus, there is a sequence of points $y_r \in U_r$ that are not contained in ρ. Since it is compact, $\partial\sigma$ intersects only finitely many cells, so we can assume that all y_r are contained in the same k-cell ρ' (for some $k \leq j-1$), and therefore $x \in \overline{\rho'}$. However, by construction of the CW-complex, the open cell ρ must be disjoint from the closure of any other cell of dimension $\leq j-1$, so this, too, is a contradiction. $\square$

2.3 Corollary. *If $S \subseteq X$ is a union of cells of X, then its closure $\overline{S}$ is a subcomplex of X.*

Note that both this lemma and its corollary are false for general CW-complexes: For example, given a one-dimensional CW-complex X, one could attach a 2-cell by mapping its entire boundary to a single point of X that is not a 0-cell. Then the boundary of this cell contains one point of a 1-cell, but not the rest of that cell, and its closure in X is not a subcomplex.

2.4 Remark. In fact, regular CW-complexes are relatively close to simplicial complexes. Allen Hatcher describes their relations as follows ([Hat02], p. 534):

"A CW complex is called regular if its characteristic maps can be chosen to be embeddings. The closures of the cells are then homeomorphic to closed balls, and so it makes sense to speak of closed cells in a regular CW complex. The closed cells can be regarded as cones on their boundary spheres, and these cone structures can be used to subdivide a regular CW complex into a regular Δ-complex, by induction over skeleta. [...] The barycentric subdivision of a regular unordered Δ-complex is a simplicial complex."

Therefore, working in a category of regular CW-complexes is very close to working in the simplicial category. Compared to simplicial complexes, the main advantage of regular CW-complexes is their compatibility with product spaces, as the product of two regular cells is again a regular cell, while the product of two simplices is almost never a simplex.

For regular CW-complexes, the cellular chain complex takes a particularly simple form: Write the chain groups as $\mathbb{C}[\mathcal{E}_j X]$ and the differential as

$$\partial_j \colon \mathbb{C}[\mathcal{E}_j X] \to \mathbb{C}[\mathcal{E}_{j-1} X],\ \sigma \mapsto \sum_{\rho \in \mathcal{E}_{j-1} X} [\sigma : \rho]\, \rho$$

with *incidence numbers* $[\sigma : \rho] \in \mathbb{Z}$. Then we have:

2.5 Lemma. *Let X be a regular CW-complex, $\sigma \in \mathcal{E}_j X$ and $\rho \in \mathcal{E}_{j-1} X$. Then*

$$[\sigma : \rho] = \begin{cases} \pm 1, & \text{if } \rho \subseteq \partial\sigma, \\ 0, & \text{otherwise} \end{cases}$$

Proof. See [Suc16], Lemma 1.5. □

As our goal is to consider L^2-invariants, we will soon pass to the Hilbert space completion of the chain groups, namely, $\ell^2(\mathcal{E}_j X)$. The properties of boundedness and regularity together will ensure that the differentials extend to bounded operators on these spaces.

2.6 Definition. Let X be a regular CW-complex.

Define the *combinatorial distance* of two j-cells $\sigma, \sigma' \in \mathcal{E}_j X$ as follows:

- $d_{\mathrm{comb}}(\sigma, \sigma') = 0$ if and only if $\sigma = \sigma'$.
- $d_{\mathrm{comb}}(\sigma, \sigma') = 1$ if $\sigma \neq \sigma'$ and there is a $(j-1)$-cell ρ such that $\rho \subseteq \partial\sigma \cap \partial\sigma'$ or if there is a $(j+1)$-cell τ such that $\sigma \cup \sigma' \subseteq \partial\tau$.

- $d_{\mathrm{comb}}(\sigma, \sigma') = n$ if n is the smallest integer for which there are $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ such that $d_{\mathrm{comb}}(\sigma_i, \sigma_{i+1}) = 1$ for all i.

For $\sigma \in \mathcal{E}_j X$ define

$$B_r(\sigma) = \{\sigma' \in \mathcal{E}_j X \mid d_{\mathrm{comb}}(\sigma, \sigma') \leq r\}.$$

This turns $\mathcal{E}_j X$ into a discrete metric space, except that the distance of two cells might be infinite if there is no "path" of adjacent cells connecting them.

If X is connected, then d_{comb} is a metric on $\mathcal{E}_0 X$ and $\mathcal{E}_1 X$, but it might not be a metric on $\mathcal{E}_j X$ for $j \geq 2$. See Figure 1 for an example.

2.7 Definition. In analogy to simplicial complexes, and to simplify language, a k-cell ρ contained in the boundary of a j-cell σ will sometimes be referred to as a *(k-)face* of σ. Then, two (distinct) j-cells are *adjacent* to each other if they share a $(j-1)$-face or if both of them are faces of the same $(j+1)$-cell.

2.8 Lemma. *If X is bounded with constant C (compare Def. 2.1), then*

$$|B_r(\sigma)| \leq \big(2C(C-1)\big)^r.$$

Especially, there is a bound on the size of r-balls around cells of X depending only on r.

Proof. By induction on r: For $r = 1$, note that $\partial\sigma$ contains at most C cells of dimension $j-1$, and each of those is contained in the boundaries of at most $C-1$ other j-cells; and analogously, σ is contained in the boundaries of at most C cells of dimension $j+1$, each of which contains at most $C-1$ other j-cells. For $r > 1$, simply use $B_{r+1}(\sigma) \subseteq \bigcup_{\sigma' \in B_r(\sigma)} B_1(\sigma')$. □

Local isomorphisms and patterns

To find some kind of order in infinite complexes, we require a way to compare small parts of the complex to each other, that is, a notion of *local isomorphism*. However, in order to translate these topological similarities into algebraical ones, we are interested in something significantly stronger than an isomorphism of CW-complexes:

2.9 Definition. An (*orientation-preserving*) *regular isomorphism* between two regular CW-complexes K and L is a map $\gamma\colon K \to L$ such that for each j-cell σ of K, the image $\gamma(\sigma) \subseteq L$ is a j-cell of L and $\gamma\colon \sigma \to \gamma(\sigma)$ is an orientation-preserving homeomorphism.

A *local isomorphism* of a regular CW-complex X is a regular isomorphism $\gamma\colon K \to L$ between two (finite) subcomplexes $K, L \subseteq X$.

This definition of a local isomorphism is explicitly about preserving a particular cell structure, not just a topological shape. Nonetheless, it appears very often when we build cell structures for infinite CW-complexes – simply put, local isomorphisms describe a copy-and-paste approach to putting cell structures on larger spaces by some kind of "tiling", periodic or not.

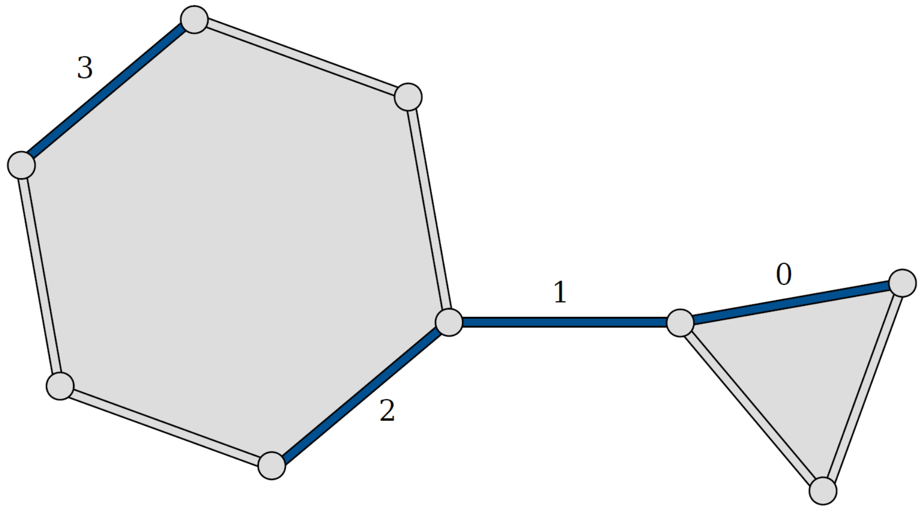

Figure 1: *The combinatorial distance.* In this complex, edge 0 is adjacent to edge 1, as they share a vertex, and edge 1 is adjacent to edge 2 for the same reason. Edge 2 is adjacent to edge 3 since they are both contained in the same 2-cell (the hexagon). Thus, the edges 0 and 3 have combinatorial distance three. Meanwhile, the triangle and the hexagon have combinatorial distance ∞, since they are neither adjacent to each other nor to any other 2-cell.

2.10 Definition. Let σ be a j-cell of X. Let $\widehat{B}_r(\sigma)$ be the smallest full subcomplex of X that contains $B_r(\sigma) = \{\sigma' \in \mathcal{E}_j X \mid d_{\text{comb}}(\sigma, \sigma') \leq r\}$, and $\widehat{\sigma}$ be the subcomplex given by the closure of σ in X. Then the *r-pattern* of σ is the regular isomorphism type of the pair $\big(\widehat{B}_r(\sigma), \widehat{\sigma}\big)$.

Denote by $\operatorname{Pat}_{j,r}(X)$ the set of all r-patterns of j-cells in X.

2.11 Lemma. *If X is a bounded regular CW-complex, the set $\operatorname{Pat}_{j,r}(X)$ is finite.*

Proof. Since X is bounded, Lemma 2.8 ensures that there is an upper bound for the number of cells in any subcomplex $\widehat{B}_r(\sigma)$.

Using Hatcher's argument (see Remark 2.4), we can turn every finite regular CW-complex K into a finite simplicial complex K^{simp}, and two complexes K_1, K_2 are regularly isomorphic if and only if there is a simplicial isomorphism $K_1^{\text{simp}} \to K_2^{\text{simp}}$. Furthermore, we obtain a new bound for the maximal number of simplices in such a complex.

In this process, a cell $\sigma \in \mathcal{E}_j X$ turns into one or several simplices; its closure will be a simplicial subcomplex.

For obvious combinatorial reasons, there are only finitely many simplicial pairs $\big(\widehat{B}_r(\sigma)^{\text{simp}}, \widehat{\sigma}^{\text{simp}}\big)$, and the claim follows. $\square$

Frontiers

Local isomorphisms show the similarity between two parts of a complex, but this similarity inevitably ends somewhere – presumably at the boundary of

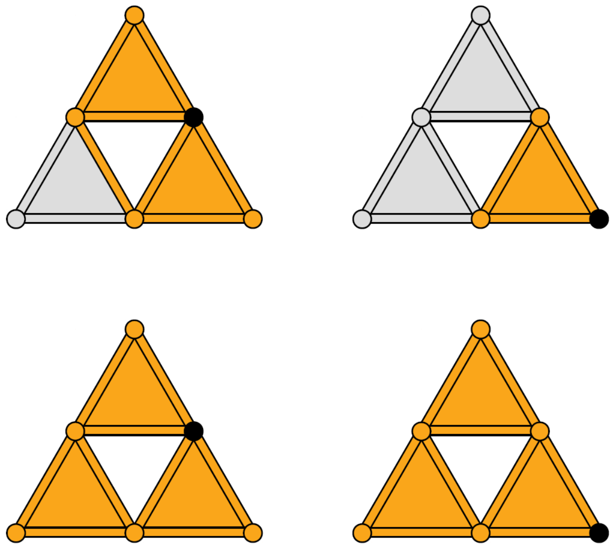

Figure 2: *Patterns.* The two vertices marked black have clearly different 1-patterns (top row). In their 2-patterns (bottom row), the complexes $\widehat{B}_2(\sigma)$ are isomorphic, but the pairs $\big(\widehat{B}_2(\sigma), \widehat{\sigma}\big)$ are not, so the 2-patterns are also different. (For any other vertex in this complex, the patterns are identical to one of these two.)

these subcomplexes. When we look at the algebraic side of things, it turns out that this affects not just the cells that form the topological boundary of such a subcomplex; instead, we need to consider every cell "adjacent" to the outside of the subcomplex with regard to the combinatorial distance. To distinguish these cells from those in the actual boundary, we will call them *frontiers*:

2.12 Definition. The *original j-frontiers* of a subcomplex $K \subseteq X$ are the j-cells adjacent to $X \setminus K$. The set of original j-frontiers is denoted $\mathcal{F}_j^{\mathrm{orig}} K$, so

$$\mathcal{F}_j^{\mathrm{orig}} K = \big\{\sigma \in \mathcal{E}_j K \,\big|\, d_{\mathrm{comb}}\big(\sigma, (\mathcal{E}_j X \setminus \mathcal{E}_j K)\big) = 1\big\}\,.$$

It is desirable that local isomorphisms preserve frontiers, that is, $\gamma(\mathcal{F}_j K) = \mathcal{F}_j(\gamma K)$. Unfortunately, this definition does not deliver that property: If a cell $\sigma \in \mathcal{E}_j K$ lies "at the margin" of X itself, then it will often not be a frontier of K, but many local isomorphisms $\gamma\colon K \to L$ will map σ to a frontier of L. For example, consider the simplicial complex $X = [0, \infty)$, with $\mathcal{E}_0 X = \mathbb{N}_0$ and $\mathcal{E}_1 X = \{(n, n+1) \,|\, n \in \mathbb{N}_0\}$, and the subcomplex $K = [0, 5]$. By definition, $\mathcal{F}_0^{\mathrm{orig}} K = \{5\}$; but for any $n > 0$, the local isomorphism $\gamma\colon [0,5] \to [n, n+5]$, $x \mapsto x + n$ will also map 0 to a frontier.

To remedy this problem, let us extend the definition of frontiers:

2.13 Definition. The (*generalized*) *j-frontiers* of a subcomplex $K \subseteq X$ are given by

$$\mathcal{F}_j K = \bigcup_{\gamma \in \Gamma(K, ?)} \gamma^{-1}\big(\mathcal{F}_j^{\mathrm{orig}}(\gamma K)\big)$$

where $\Gamma(K, ?)$ is the set of all local isomorphisms $\gamma\colon K \to \gamma K \subseteq X$.

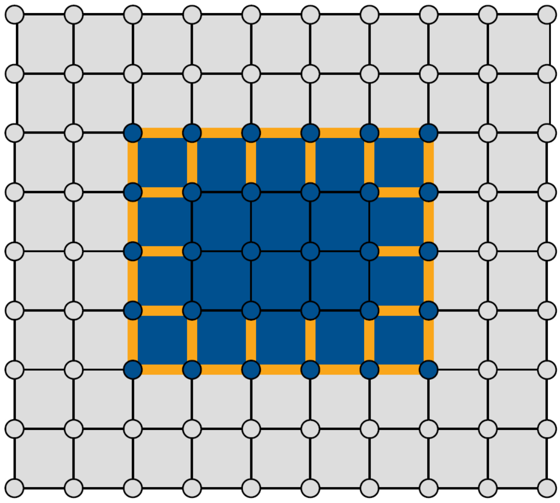

Figure 3: *Frontiers.* All 1-frontiers of the dark blue subcomplex are marked in orange. Note that not all frontiers are part of the topological boundary of the subcomplex.

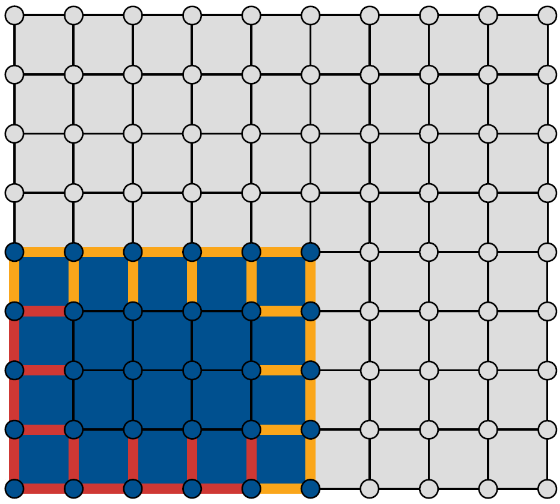

Figure 4: *Generalized frontiers.* The original 1-frontiers of this subcomplex are only those marked in orange. However, as there is a local isomorphism mapping this complex to the one from Figure 3, the 1-cells marked in red are generalized frontiers.

2.14 Lemma. *If $\gamma\colon K \to \gamma K$ is a local isomorphism, then $\gamma(\mathcal{F}_j K) = \mathcal{F}_j(\gamma K)$.*

Proof. Let $\sigma \in \mathcal{F}_j K$. Then there is a local isomorphism $\gamma'\colon K \to \gamma' K$ such that $\gamma'\sigma \in \mathcal{F}^{\mathrm{orig}}(\gamma' K)$. As $\gamma' \circ \gamma^{-1}\colon \gamma K \to \gamma' K$ is also a local isomorphism and $(\gamma' \circ \gamma^{-1})(\gamma\sigma) = \gamma'\sigma \in \mathcal{F}^{\mathrm{orig}}(\gamma' K)$, one obtains $\gamma\sigma \in \mathcal{F}_j(\gamma K)$. This proves $\gamma(\mathcal{F}_j K) \subseteq \mathcal{F}_j(\gamma K)$.

Applying the same argument to the local isomorphism $\gamma^{-1}\colon \gamma K \to K$ shows $\gamma^{-1}\big(\mathcal{F}_j(\gamma K)\big) \subseteq \mathcal{F}_j K$. Since $\gamma\colon \mathcal{E}_j K \to \mathcal{E}_j(\gamma K)$ is a bijection, this implies $\mathcal{F}_j(\gamma K) \subseteq \gamma(\mathcal{F}_j K)$. $\square$

From now on, generalized frontiers will simply be called "frontiers".[3]

2.15 Remark. The set of generalized frontiers can be rather large: For any cell in K, there could be *some* local isomorphism mapping it to a frontier. The easiest way to prove that a cell is *not* a frontier is to use the boundedness of the complex: Any cell $\sigma \in \mathcal{E}_j K$ whose combinatorial neighborhood $B_1(\sigma) \subseteq K$ already has the maximal possible size cannot be a generalized frontier of K. Namely, for any local isomorphism $\gamma\colon K \to \gamma K$, the cell $\gamma\sigma$ already has the maximal number of neighbors in $\gamma(B_1(\sigma)) \subseteq \gamma K$, so it cannot also be adjacent to a cell outside of γK.

2.16 Lemma. *Let $K \subseteq X$ be a full subcomplex and $\sigma \in \mathcal{E}_j K$. Let $\gamma\colon K \to \gamma K$ be a local isomorphism.*

(a) *$d_{\mathrm{comb}}(\sigma, \mathcal{F}_j K) = d_{\mathrm{comb}}(\gamma\sigma, \mathcal{F}_j(\gamma K))$.*

(b) *If $d_{\mathrm{comb}}(\sigma, \mathcal{F}_j K) \geq r$, then σ and $\gamma\sigma$ have the same r-pattern.*

Proof. (a) Let $d_{\mathrm{comb}}(\sigma, \mathcal{F}_j K) = r$. Write $\sigma = \sigma_0$ and choose cells $\sigma_1, \ldots, \sigma_r \in \mathcal{E}_j X$ such that $d_{\mathrm{comb}}(\sigma_i, \sigma_{i+1}) = 1$ for all $0 \leq i \leq r-1$ and $\sigma_r \in \mathcal{F}_j K$. Note that all σ_i actually lie in $\mathcal{E}_j K$ since otherwise $d_{\mathrm{comb}}(\sigma, \mathcal{F}_j K)$ would be smaller than r. Since $\gamma(\mathcal{F}_j K) = \mathcal{F}_j(\gamma K)$ by Lemma 2.14, we have $\gamma\sigma_r \in \mathcal{F}_j(\gamma K)$.

Furthermore, $d_{\mathrm{comb}}(\gamma\sigma_i, \gamma\sigma_{i+1}) = 1$ for all i: If σ_i and σ_{i+1} share a face $\rho \in \mathcal{E}_{j-1} X$, then ρ lies in K and $\gamma\sigma_i$ and $\gamma\sigma_{i+1}$ share the face $\gamma\rho$. If σ_i and σ_{i+1} are both faces of a cell $\tau \in \mathcal{E}_{j+1} X$, then τ must lie in K: If any other j-face of τ were not contained in K, then σ_i would already be a frontier of K, which it is not; so all j-faces of τ lie in K; as K is full, this implies that τ lies in K. Consequently, $\gamma\tau$ exists and has both $\gamma\sigma_i$ and $\gamma\sigma_{i+1}$ as faces.

Thus, we obtain

$$d_{\mathrm{comb}}\big(\gamma\sigma, \mathcal{F}_j(\gamma K)\big) \leq d_{\mathrm{comb}}(\gamma\sigma, \gamma\sigma_r) \leq r = d_{\mathrm{comb}}(\sigma, \mathcal{F}_j K).$$

Applying the same argument to γ^{-1} yields

$$d_{\mathrm{comb}}\big(\gamma\sigma, \mathcal{F}_j(\gamma K)\big) \geq d_{\mathrm{comb}}(\sigma, \mathcal{F}_j K).$$

(b) By part (a), $d_{\mathrm{comb}}(\sigma, \mathcal{F}_j K) \geq r$ implies $d_{\mathrm{comb}}(\gamma\sigma, \mathcal{F}_j(\gamma K)) \geq r$, and thus $B_r(\gamma\sigma) \subseteq \mathcal{E}_j(\gamma K)$, which implies $\widehat{B_r}(\gamma\sigma) \subseteq \gamma K$. Thus, $\gamma\colon K \to \gamma K$ restricts to an isomorphism

$$\gamma\colon \left(\widehat{B_r}(\sigma), \widehat{\sigma}\right) \to \left(\widehat{B_r}(\gamma\sigma), \widehat{\gamma\sigma}\right),$$

so the patterns are the same. □

[3] In [Suc16], the original frontiers were denoted $\mathcal{F}_j K$ and the generalized frontiers $\mathcal{F}_j^{\mathcal{G}} K$.

2.2 Aperiodic order and the trace

With the stage set, we can now begin to tame infinite complexes.

2.17 Definition. An *amenable exhaustion* or *Følner sequence* of a regular CW-complex X is a sequence of finite full subcomplexes $K_m \subseteq X$ such that

- $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq X$ and $\bigcup_{m\in\mathbb{N}} K_m = X$ (*exhaustion*),
- $\lim_{m\to\infty} \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} = 0$ for all j with $\mathcal{E}_j X \neq \emptyset$ (*amenability*).

Note that if X is finite, then there must be an $m_0 \in \mathbb{N}$ such that $K_m = X$ for all $m \geq m_0$.

The following definitions are a generalization of those given in [Ele06] (where they were only used for graphs).

2.18 Definition. An (regular and bounded) CW-complex X has *aperiodic order* if for every $j, r \in \mathbb{N}$ there is a function

$$\mathbb{P}_{j,r} \colon \operatorname{Pat}_{j,r}(X) \to [0,1]$$

such that every amenable exhaustion $(K_m)_{m\in\mathbb{N}}$ satisfies

$$\lim_{m\to\infty} \frac{\left|\mathcal{E}_j^\alpha K_m\right|}{|\mathcal{E}_j K_m|} = \mathbb{P}_{j,r}(\alpha),$$

where $\mathcal{E}_j^\alpha K_m$ is the set of cells $\sigma \in \mathcal{E}_j K_m$ whose r-patterns are equal to $\alpha \in \operatorname{Pat}_{j,r}(X)$.

$\mathbb{P}_{j,r}(\alpha)$ is called the *frequency* of the pattern α. The definition immediately implies

$$\sum_{\alpha\in\operatorname{Pat}_{j,r}(X)} \mathbb{P}_{j,r}(\alpha) = 1.$$

Note that if X is finite, then every amenable exhaustion is eventually constant, and the complex automatically has aperiodic order.

2.19 Example. The property that any amenable exhaustion produces the same pattern frequencies is far from automatic. As a simple counterexample, define a CW-complex X as follows: Let $\mathcal{E}_0 X \cong \mathbb{Z}$ with 0-cells σ_n for $n \in \mathbb{Z}$. Connect σ_n to σ_{n+1} by one edge if $n < 0$, and by two edges if $n \geq 0$:

$$\cdots \longrightarrow \sigma_{-2} \longrightarrow \sigma_{-1} \longrightarrow \sigma_0 \rightrightarrows \sigma_1 \rightrightarrows \sigma_2 \rightrightarrows \sigma_3 \rightrightarrows \cdots$$

The 0-cells of this complex have three different 1-patterns: For σ_n with $n < 0$, the pattern is $\circ \longrightarrow \bullet \longrightarrow \circ$, for σ_0 it is $\circ \longrightarrow \bullet \rightrightarrows \circ$, and for σ_n with $n > 0$ it is $\circ \rightrightarrows \bullet \rightrightarrows \circ$.

For any positive integers $a, b \in \mathbb{N}$, the full subcomplexes $\big(K_m^{[a,b]}\big)_{m\in\mathbb{N}}$ spanned by $\mathcal{E}_0 K_m^{[a,b]} = \{\sigma_n \,|\, -am \leq n \leq bm\}$ form an amenable exhaustion, and in that exhaustion, we find the pattern frequencies

$$\mathbb{P}_{1,1}^{[a,b]}(\,\circ \longrightarrow \bullet \longrightarrow \circ\,) = \lim_{m\to\infty} \frac{am-1}{am+bm+1} = \frac{a}{a+b},$$
$$\mathbb{P}_{1,1}^{[a,b]}(\,\circ \longrightarrow \bullet \rightrightarrows \circ\,) = \lim_{m\to\infty} \frac{1}{am+bm+1} = 0,$$
$$\mathbb{P}_{1,1}^{[a,b]}(\,\circ \rightrightarrows \bullet \rightrightarrows \circ\,) = \lim_{m\to\infty} \frac{bm-1}{am+bm+1} = \frac{b}{a+b},$$

which clearly depend on the choice of the exhaustion. Thus, this complex does not have aperiodic order.

2.20 Definition. The *propagation* of an operator $A \in \mathcal{B}\big(\ell^2(\mathcal{E}_j X)\big)$ is given by

$$\operatorname{prop}(A) = \max\left\{d_{\mathrm{comb}}(\sigma,\sigma') \,|\, \sigma,\sigma' \in \mathcal{E}_j X \text{ and } \langle \sigma, A\sigma'\rangle \neq 0\right\}.$$

An operator $A \in \mathcal{B}\big(\ell^2(\mathcal{E}_j X)\big)$ is called *r-pattern-invariant* if $\operatorname{prop}(A) \leq r$ and the following commutativity condition holds: If $\gamma\colon K \to L$ is a local isomorphism and $\sigma \in \mathcal{E}_j K$ such that $B_r(\sigma) \subseteq K$ and $B_r(\gamma\sigma) \subseteq L$, then $A\gamma\sigma = \gamma A\sigma$ and $A^*\gamma\sigma = \gamma A^*\sigma$.

An operator is called *geometric* if it is r-pattern-invariant for some $r \in \mathbb{N}$.

Denote by $\mathcal{A}_j^{\mathrm{geo}}(X)$ the set of all geometric operators in $\mathcal{B}\big(\ell^2(\mathcal{E}_j X)\big)$.

2.21 Definition and Lemma. Let X be a regular and bounded CW-complex.

(a) For each $j \in \mathbb{N}_0$, let $\partial_j\colon \ell^2(\mathcal{E}_j X) \to \ell^2(\mathcal{E}_{j-1} X)$ be the operator induced by the differential of the cellular chain complex of X.

That is, for any cells $\sigma \in \mathcal{E}_j X$ and $\rho \in \mathcal{E}_{j-1} X$, the value of $\langle \rho, \partial_j \sigma\rangle$ is given by the degree of the map

$$S^{j-1} \xrightarrow{f_\sigma} X^{(j-1)} \xrightarrow{\mathrm{proj}} X^{(j-1)} \big/ \big(X^{(j-1)} \setminus \rho\big) \xrightarrow{\approx} \overline{\rho}/\partial\rho \xrightarrow{g_\rho} S^{j-1},$$

where f_σ is the attaching map of σ and g_ρ is induced by the inverse of the attaching map of ρ.

Each ∂_j is a bounded operator.

(b) Define the *j-th combinatorial Laplacian* of X by

$$\Delta_j = \partial_{j+1}\partial_{j+1}^* + \partial_j^*\partial_j.$$

Each Δ_j is a positive 1-pattern-invariant operator on $\ell^2(\mathcal{E}_j X)$, and thus geometric.

Proof. By definition of the combinatorial distance and Lemma 2.5, each Δ_j has propagation ≤ 1 and is indeed 1-pattern invariant. For a proof that ∂_j and Δ_j are bounded, see [Suc16], Lemma 2.2 / Def. 2.5 / Remark 2.6. $\square$

2.22 Lemma. *$\mathcal{A}_j^{geo}(X)$ is a $*$-algebra.*

Proof. If A is r_1-pattern-invariant and B is r_2-pattern-invariant, then clearly $A + cB$ is $\max(r_1, r_2)$-pattern-invariant for every $c \in \mathbb{C}$.

The composition AB is $(r_1 + r_2)$-pattern-invariant: Given $\gamma\colon K \to L$ and $\sigma \in \mathcal{E}_j K$ such that $B_{r_1+r_2}(\sigma) \subseteq K$ and $B_{r_1+r_2}(\gamma\sigma) \subseteq L$, we can write $B\sigma = \sum_{\sigma' \in B_{r_2}(\sigma)} b_{\sigma'}\sigma'$ (since $\operatorname{prop}(B) \le r_2$) and thus obtain

$$AB\gamma\sigma = A\gamma B\sigma = \sum_{\sigma' \in B_{r_2}(\sigma)} b_{\sigma'} A\gamma\sigma' = \sum_{\sigma' \in B_{r_2}(\sigma)} b_{\sigma'}\gamma A\sigma' = \gamma AB\sigma$$

using that for all $\sigma' \in B_{r_2}(\sigma)$ we have $B_{r_1}(\sigma') \subseteq B_{r_1+r_2}(\sigma) \subseteq K$ and $B_{r_1}(\gamma\sigma') \subseteq B_{r_1+r_2}(\gamma\sigma) \subseteq L$. (The latter follows from Lemma 2.16.)

Finally, if A is r-pattern-invariant, then so is A^*; this follows directly from the definition. □

2.23 Definition and Lemma. Let X be a complex with aperiodic order and (K_m) an amenable exhaustion of X. Then the following defines a tracial state on $\mathcal{A}_j^{\mathrm{geo}}(X)$:

$$\operatorname{tr}_{\mathcal{A}}(T) = \lim_{m\to\infty} \frac{1}{|\mathcal{E}_j K_m|} \sum_{\sigma \in \mathcal{E}_j K_m} \langle \sigma, T\sigma \rangle \tag{1}$$

This is independent of the choice of (K_m), and if $T \in \mathcal{A}_j^{\mathrm{geo}}(X)$ is r-pattern-invariant, then

$$\operatorname{tr}_{\mathcal{A}}(T) = \sum_{\alpha \in \mathrm{Pat}_{j,r}(X)} \mathbb{P}_r(\alpha) \langle \sigma_\alpha, T\sigma_\alpha \rangle\,, \tag{2}$$

where $\sigma_\alpha \in \mathcal{E}_j X$ is any j-cell with r-pattern α.

Proof. Well-definedness: Let $T \in \mathcal{A}_j^{\mathrm{geo}}(X)$ be r-pattern-invariant. If two j-cells $\rho, \sigma \in \mathcal{E}_j X$ have the same r-pattern, then there is a local isomorphism $\gamma\colon \widehat{B}_r(\rho) \to \widehat{B}_r(\sigma)$ such that $\gamma\rho = \sigma$. Thus,

$$\langle \sigma, T\sigma \rangle = \langle \gamma\rho, T\gamma\rho \rangle = \langle \gamma\rho, \gamma T\rho \rangle = \langle \rho, T\rho \rangle$$

because $\operatorname{supp}(T\rho) \subseteq B_r(\rho)$. Therefore, $\langle \sigma, T\sigma \rangle$ only depends on the r-pattern of σ, and we obtain

$$\frac{1}{|\mathcal{E}_j K_m|} \sum_{\sigma \in \mathcal{E}_j K_m} \langle \sigma, T\sigma \rangle = \sum_{\alpha \in \mathrm{Pat}_{j,r}(X)} \frac{|\mathcal{E}_j^\alpha K_m|}{|\mathcal{E}_j K_m|} \langle \sigma_\alpha, T\sigma_\alpha \rangle \xrightarrow{m\to\infty} \sum_{\alpha \in \mathrm{Pat}_{j,r}(X)} \mathbb{P}_r(\alpha) \langle \sigma_\alpha, T\sigma_\alpha \rangle\,.$$

This proves that the limit in Equation (1) exists and does not depend on the choice of amenable exhaustion, and it proves Equation (2).

Linearity is clear from the definition.

State: The Cauchy–Schwarz inequality and the convention $\|\sigma\| = 1$ yield $|\langle \sigma, T\sigma \rangle| \le \|T\|$ for all $\sigma \in \mathcal{E}_j X$, and thus $|\operatorname{tr}_{\mathcal{A}} T| \le \|T\|$ for all $T \in \mathcal{A}_j^{\mathrm{geo}}(X)$. Conversely, $\operatorname{tr}_{\mathcal{A}}(\mathrm{id}) = 1 = \|\mathrm{id}\|$.

Trace property: Let $S, T \in \mathcal{A}_j^{\text{geo}}(X)$ be r-pattern-invariant. (If S is r_1-pattern-invariant and T is r_2-pattern-invariant, simply let $r = \max(r_1, r_2)$.) Define the the set of "r-frontiers" of K_m

$$\mathcal{F}_j^r K_m = \mathcal{E}_j K_m \cap B_{r-1}(\mathcal{F}_j K_m) = \left\{\sigma \in \mathcal{E}_j K_m \,\middle|\, d_{\text{comb}}\big(\sigma, (\mathcal{E}_j X \setminus \mathcal{E}_j K_m)\big) \le r\right\}.$$

Note that by boundedness of X there is $C > 0$ (depending on X and r, but not on m) such that $\left|\mathcal{F}_j^r K_m\right| \le C\,|\mathcal{F}_j K_m|$ for all m.

$$\begin{aligned}
\sum_{\sigma \in \mathcal{E}_j K_m} \langle \sigma, ST\sigma\rangle &= \sum_{\sigma \in \mathcal{E}_j K_m \setminus \mathcal{F}_j^r K_m} \langle \sigma, ST\sigma\rangle + \mathcal{O}(|\mathcal{F}_j K_m|) \\
&= \sum_{\sigma \in \mathcal{E}_j K_m \setminus \mathcal{F}_j^r K_m} \sum_{\rho \in \mathcal{E}_j K_m} \langle \sigma, S\rho\rangle\langle \rho, T\sigma\rangle + \mathcal{O}(|\mathcal{F}_j K_m|) \\
&= \sum_{\sigma \in \mathcal{E}_j K_m \setminus \mathcal{F}_j^r K_m} \sum_{\rho \in \mathcal{E}_j K_m \setminus \mathcal{F}_j^r K_m} \langle \sigma, S\rho\rangle\langle \rho, T\sigma\rangle + \mathcal{O}(|\mathcal{F}_j K_m|)
\end{aligned}$$

In the first line, at most $\left|\mathcal{F}_j^r K_m\right|$ terms are left out; in the third line, at most $C\left|\mathcal{F}_j^r K_m\right|$ terms are left out: For each ρ in $\mathcal{F}_j^r K_m$, there are at most C cells σ for which $\langle \rho, T\sigma\rangle \neq 0$. Each of the dropped terms is bounded by $\|S\|\,\|T\|$. Thus, the $\mathcal{O}$-constants depend on S and T, but not on m.

The same computation yields

$$\sum_{\rho \in \mathcal{E}_j K_m} \langle \rho, TS\rho\rangle = \sum_{\rho \in \mathcal{E}_j K_m \setminus \mathcal{F}_j^r K_m} \sum_{\sigma \in \mathcal{E}_j K_m \setminus \mathcal{F}_j^r K_m} \langle \rho, T\sigma\rangle\langle \sigma, S\rho\rangle + \mathcal{O}(|\mathcal{F}_j K_m|).$$

Thus,

$$\frac{1}{|\mathcal{E}_j K_m|} \sum_{\sigma \in \mathcal{E}_j K_m} \langle \sigma, (ST - TS)\sigma\rangle = \mathcal{O}\left(\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|}\right) \xrightarrow{m\to\infty} 0. \qquad \square$$

2.3 The algebra of pattern-invariant operators

The $*$-algebra $\mathcal{A}_j^{\text{geo}}(X)$ can easily be extended to a C^*-algebra:

2.24 Definition. Let $\mathcal{A}_j(X)$ be the operator-norm closure of $\mathcal{A}_j^{\text{geo}}(X)$ in $\mathcal{B}\big(\ell^2(\mathcal{E}_j X)\big)$. As the norm $\operatorname{tr}_{\mathcal{A}}$ is norm-continuous, it immediately extends to a trace on $\mathcal{A}_j(X)$.

This allows us to define a functional calculus $f(T)$ for every geometric operator $T \in \mathcal{A}_j^{\text{geo}}(X)$ and every continuous function f, and to take the trace $\operatorname{tr}_{\mathcal{A}}(f(T))$. However, we are aiming to define spectral projections for these operators, that is, $\chi_{[0,\lambda]}(T)$ with the clearly discontinuous characteristic functions $\chi_{[0,\lambda]}$. This requires a von Neumann algebra!

The obvious next step would be to take the weak closure of $\mathcal{A}_j^{\text{geo}}(X)$ in $\mathcal{B}\big(\ell^2(\mathcal{E}_j X)\big)$, and extend the trace by weak continuity. Unfortunately, the trace fails to be weakly continuous:

2.25 Example. Consider $X = [0, \infty)$ with the standard CW-structure given by $\mathcal{E}_0 X = \mathbb{N}_0$ and $\mathcal{E}_1 X = \{(n, n+1) \mid n \in \mathbb{N}_0\}$. Here, every vertex has degree two, except for $\{0\}$, which has degree one.

For each $r \in \mathbb{N}$, define an operator $P_r \in \mathcal{B}(\ell^2(\mathcal{E}_0 X))$ by

$$P_r \sigma = \begin{cases} \sigma, & \text{if every vertex in } B_r(\sigma) \text{ has degree two,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, P_r is r-pattern-invariant and thus contained in $\mathcal{A}_j^{\mathrm{geo}}(X)$, and for every r we have $\operatorname{tr}_{\mathcal{A}}(P_r) = 1$ because $P_r \sigma = \sigma$ for almost all $\sigma \in \mathcal{E}_0 X$.

But on the other hand, $(P_r)_{r \in \mathbb{N}}$ is a decreasing sequence of projections that weakly (even strongly) converge to zero! As $\operatorname{tr}_{\mathcal{A}}(P_r) \xrightarrow{r \to \infty} 1 \neq 0 = \operatorname{tr}_{\mathcal{A}}(0)$, the trace is not weakly continuous.

To obtain a more suitable algebra, we employ the Gelfand–Naimark–Segal construction.

First of all, the trace on $\mathcal{A}_j(X)$ defines a scalar product on the algebra itself:

2.26 Definition. Define a hermitian form and the corresponding seminorm on $\mathcal{A}_j(X)$ by

$$\langle S, T \rangle_{\mathcal{H}} = \operatorname{tr}_{\mathcal{A}}(S^* T), \qquad \|T\|_{\mathcal{H}} = \sqrt{\operatorname{tr}_{\mathcal{A}}(T^* T)}.$$

2.27 Lemma. *Let $S, T \in \mathcal{A}_j(X)$. Then we have:*

(a) $\|T\|_{\mathcal{H}} \leq \|T\|$

(b) $\|T\|_{\mathcal{H}} = \|T^*\|_{\mathcal{H}}$

(c) $\|ST\|_{\mathcal{H}} \leq \|S\| \cdot \|T\|_{\mathcal{H}}$

(d) $\|ST\|_{\mathcal{H}} \leq \|S\|_{\mathcal{H}} \cdot \|T\|$

(e) *The set $\mathcal{K}_j(X) = \{T \in \mathcal{A}_j(X) \mid \|T\|_{\mathcal{H}} = 0\}$ is a closed ideal of $\mathcal{A}_j(X)$.*

(f) *$\mathcal{K}_j(X) = \{0\}$ if and only if for every $r \in \mathbb{N}$ and every $\sigma \in \mathcal{E}_j X$, the r-pattern of σ has positive frequency. Then, $\| \ \|_{\mathcal{H}}$ is a norm on $\mathcal{A}_j(X)$.*

Proof. (a) This holds since $\operatorname{tr}_{\mathcal{A}}$ is a state (and by the C^*-property):

$$\|T\|_{\mathcal{H}}^2 = \operatorname{tr}_{\mathcal{A}}(T^* T) \leq \|T^* T\| = \|T\|^2 .$$

(b) This follows directly from the trace property:

$$\|T\|_{\mathcal{H}}^2 = \operatorname{tr}_{\mathcal{A}}(T^* T) = \operatorname{tr}_{\mathcal{A}}(T T^*) = \|T^*\|_{\mathcal{H}}^2 .$$

(c) $\|ST\|_{\mathcal{H}}^2 = \lim\limits_{m \to \infty} \dfrac{1}{|\mathcal{E}_j K_m|} \sum\limits_{\sigma \in \mathcal{E}_j K_m} \|ST\sigma\|^2 \leq \lim\limits_{m \to \infty} \dfrac{1}{|\mathcal{E}_j K_m|} \sum\limits_{\sigma \in \mathcal{E}_j K_m} \|S\|^2 \|T\sigma\|^2$
$= \|S\|^2 \cdot \|T\|_{\mathcal{H}}^2.$

(d) This follows from (b) and (c) combined.

(e) The triangle inequality for seminorms gives $\|S + \lambda T\|_{\mathcal{H}} \le \|S\|_{\mathcal{H}} + |\lambda| \, \|T\|_{\mathcal{H}}$ for all $\lambda \in \mathbb{C}$, so $\mathcal{K}_j(X)$ is a linear subspace. By (c), it is a left ideal, and by (d) it is a right ideal. Finally, it is closed (in the original norm topology) because $\mathrm{tr}_{\mathcal{A}}$ and thus $\| \ \|_{\mathcal{H}}$ are norm-continuous.

(f) Assume there are a j-cell $\sigma \in \mathcal{E}_j X$ and an $r \in \mathbb{N}$ such that the r-pattern α of σ has $\mathbb{P}_{j,r}(\alpha) = 0$. Then the operator given by $T\rho = \rho$ if ρ has the same r-pattern as σ and $T\rho = 0$ otherwise is clearly r-pattern-invariant and non-zero, but its $\mathcal{H}$-norm vanishes. Thus, $\mathcal{K}_j(X)$ is nontrivial.

Conversely, assume that every pattern of every j-cell in the complex has positive frequency. Let $T \in \mathcal{A}_j(X)$ and $\sigma \in \mathcal{E}_j X$ such that $T\sigma \neq 0$. By definition of $\mathcal{A}_j(X)$, there is $S \in \mathcal{A}_j^{\mathrm{geo}}(X)$ such that $\|T - S\| \le \frac{1}{3} \|T\sigma\|$, and S is s-pattern-invariant for some $s \in \mathbb{N}$. Let α_σ be the s-pattern of σ. By assumption, $\mathbb{P}_{j,s}(\alpha_\sigma) > 0$, and every $\rho \in \mathcal{E}_j X$ with this pattern fulfills

$$\begin{aligned} \|S\rho\| = \|S\sigma\| &\ge \|T\sigma\| - \|T\sigma - S\sigma\| \ge \frac{2}{3} \|T\sigma\| \\ \implies \|T\rho\| &\ge \|S\rho\| - \|S\rho - T\rho\| \ge \frac{1}{3} \|T\sigma\| \end{aligned}$$

This implies

$$\|T\|_{\mathcal{H}}^2 = \mathrm{tr}_{\mathcal{N}}(T^*T) \ge \frac{\mathbb{P}_{j,s}(\alpha_\sigma) \, \|T\sigma\|^2}{9} > 0. \qquad \square$$

2.28 Remark. It should be noted that the $\mathcal{H}$-norm is not submultiplicative: Consider a complex with just three cells, and let

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

On $\mathrm{Mat}_3(\mathbb{C})$, we have $\mathrm{tr}_{\mathcal{A}} = \frac{1}{3}\,\mathrm{tr}$, and we obtain $\|T\|_{\mathcal{H}}^2 = 3 < 3\sqrt{3} = \|T^2\|_{\mathcal{H}}$.

With the newly constructed scalar product, we can complete $\mathcal{A}_j(X)$ into a Hilbert space and have it act on this extended version of itself:

2.29 Definition and Lemma. Define a Hilbert space $\mathcal{H}_j(X)$ as the completion of the pre-Hilbert space $\big(\mathcal{A}_j(X)/\mathcal{K}_j(X), \langle -, - \rangle_{\mathcal{H}}\big)$.

Then the action of $\mathcal{A}_j(X)$ on $\mathcal{H}_j(X)$ by left multiplication yields a $*$-homomorphism $\mathcal{A}_j(X) \to \mathcal{B}(\mathcal{H}_j(X))$. If $\mathcal{K}_j(X) = 0$, this map is isometric (with respect to the operator norms on each side).

Define the von Neumann algebra $\mathcal{N}_j(X)$ as the weak closure of $\mathcal{A}_j(X)$ in $\mathcal{B}(\mathcal{H}_j(X))$.

When the space X is clear, simply write $\mathcal{A}_j, \mathcal{H}_j$ and $\mathcal{N}_j$ instead of $\mathcal{A}_j(X)$, $\mathcal{H}_j(X)$ and $\mathcal{N}_j(X)$.

Proof. Note first that the statements of Lemma 2.27 (b), (c) and (d) still hold if T (in (b) and (c)) respectively S (in (d)) are replaced by elements of $\mathcal{H}_j$. This shows that for every $T \in \mathcal{A}_j$, the map $\mathcal{H}_j \to \mathcal{H}_j, \Xi \mapsto T \cdot \Xi$ is well-defined and has $\mathcal{B}(\mathcal{H}_j)$-operator norm less than or equal to $\|T\|$. (In particular, if we change the representative of Ξ by something of $\mathcal{H}$-norm 0, then the result will also change by something of $\mathcal{H}$-norm 0.)

To see that $\mathcal{A}_j \to \mathcal{B}(\mathcal{H}_j)$ is a $*$-homomorphism, note that for $A, B, T \in \mathcal{A}_j$,

$$\langle A, TB \rangle_{\mathcal{H}} = \operatorname{tr}_{\mathcal{A}}(A^*TB) = \operatorname{tr}_{\mathcal{A}}\big((T^*A)^*B\big) = \langle T^*A, B \rangle_{\mathcal{H}},$$

where T^* denotes the adjoint of T in $\mathcal{A}_j$. Since $\mathcal{A}_j/\mathcal{K}_j$ is dense in $\mathcal{H}_j$ (w.r.t. the $\mathcal{H}$-norm), this proves $\langle \Xi, T\Upsilon \rangle_{\mathcal{H}} = \langle T^*\Xi, \Upsilon \rangle_{\mathcal{H}}$ for all $\Xi, \Upsilon \in \mathcal{H}_j$, as desired.

Finally, if $\mathcal{K}_j = 0$, then the map $\mathcal{A}_j \to \mathcal{B}(\mathcal{H}_j)$ is injective (because $\mathrm{id} \in \mathcal{H}_j$, and $T \neq 0 \implies T \cdot \mathrm{id} \neq 0$), and every injective $*$-homomorphism between C^*-algebras is isometric. $\square$

2.30 Example. Let X be a finite complex, fix some $j \in \{0, \ldots, \dim X\}$, and let $n = |\mathcal{E}_j X|$. Then $\mathcal{B}(\ell^2(\mathcal{E}_j X)) \cong \mathrm{Mat}_n(\mathbb{C})$, and the trace on $\mathcal{A}_j^{\mathrm{geo}}(X) \subseteq \mathrm{Mat}_n(\mathbb{C})$ is given by the normalized matrix trace. The $\mathcal{H}$-norm is given by the normalized Frobenius norm

$$\|T\|_{\mathcal{H}} = \sqrt{\frac{1}{n}\sum_{i,j=1}^{n} |t_{ij}|^2},$$

and obviously $\mathcal{K}_j(X) = \{0\}$. As the spaces are all finite-dimensional, all norms are equivalent, and we obtain $\mathcal{H}_j(X) = \mathcal{A}_j(X) = \mathcal{A}_j^{\mathrm{geo}}(X)$. Furthermore, $\mathcal{B}(\mathcal{H}_j)$ is finite-dimensional, and thus $\mathcal{A}_j(X) \subseteq \mathcal{B}(\mathcal{H}_j)$ is closed, so we also obtain $\mathcal{N}_j(X) = \mathcal{A}_j(X) = \mathcal{A}_j^{\mathrm{geo}}(X)$.

2.31 Example. Let $X = \mathbb{R}$ with the standard CW-structure, so $\mathcal{E}_0\mathbb{R} \cong \mathbb{Z} \cong \mathcal{E}_1\mathbb{R}$. In this case, every local isomorphism extends to a global isomorphism, and the group of global isomorphisms is generated by $\mathbb{Z}$-translations and the reflection at zero.

Let us determine the geometric operators on $\mathcal{E}_0\mathbb{R}$. Since they must be $\mathbb{Z}$-equivariant, we can use the standard Fourier isomorphisms $\ell^2\mathbb{Z} \cong L^2(S^1)$ and $\mathcal{B}(\ell^2\mathbb{Z})^{\mathbb{Z}} \cong L^\infty(S^1)$. Here, the reflection at zero corresponds to

$$R\colon L^2(S^1) \to L^2(S^1), f(z) \mapsto f(z^{-1}).$$

Thus, if a geometric operator T on $\ell^2(\mathcal{E}_0\mathbb{R})$ is given by a function $t \in L^\infty(S^1)$, that function must fulfill

$$t(z) \cdot f(z^{-1}) = TRf(z) = RTf(z) = (t \cdot f)(z^{-1}) = t(z^{-1}) \cdot f(z^{-1})$$

for any $f \in L^2(S^1)$, and thus $t(z) = t(z^{-1})$.

On the other hand, a $\mathbb{Z}$-equivariant operator of propagation r must be a linear combination of shifts by distances $\leq r$, so its corresponding function in $L^\infty(S^1)$ is a Laurent polynomial of degrees between $-r$ and r.

Consequently, $\mathcal{A}_0^{\mathrm{geo}}(\mathbb{R})$ corresponds to symmetric polynomials in z and z^{-1}, or equivalently, to polynomials in $\mathrm{Re}(z) = \frac{1}{2}(z + z^{-1})$. By the Weierstrass approximation theorem, the norm closure is given by

$$\mathcal{A}_0(\mathbb{R}) \cong C([-1, 1]).$$

The $\mathcal{H}$-norm on $\mathcal{A}_0$ is clearly equivalent to the L^2-norm on $[-1, 1]$, and thus

$$\mathcal{H}_0(\mathbb{R}) \cong L^2([-1, 1]),$$

which immediately implies

$$\mathcal{N}_0(\mathbb{R}) \cong L^\infty([-1, 1]).$$

In both examples, $\mathcal{N}_j$ can be identified with a linear subspace of $\mathcal{H}_j$. This holds in general:

2.32 Lemma. *The map $\mathcal{N}_j(X) \to \mathcal{H}_j(X), T \mapsto T[\mathrm{id}]$ is injective and has dense image. Thus, $\mathcal{N}_j(X)$ can be identified with a dense subspace of $\mathcal{H}_j(X)$.*

Proof. By Lemma 2.27 (d), right multiplication by an element of $\mathcal{A}_j$ is also a bounded operator on $\mathcal{H}_j$, and it certainly commutes with any operator given by left multiplication with an element of $\mathcal{A}_j$.

By the double commutant theorem, that means that right multiplication by an element of $\mathcal{A}_j$ also commutes with every operator $T \in \mathcal{N}_j$. Therefore, if $[A]$ is the element of $\mathcal{H}_j$ represented by $A \in \mathcal{A}_j$, we have

$$T[A] = T([\mathrm{id}] \cdot A) = T[\mathrm{id}] \cdot A,$$

so the restriction of T to $\mathcal{A}_j/\mathcal{K}_j \subseteq \mathcal{H}_j$ is uniquely determined by the value of $T[\mathrm{id}]$. As $\mathcal{A}_j/\mathcal{K}_j$ is dense in $\mathcal{H}_j$, this implies that $\mathcal{N}_j \to \mathcal{H}_j, T \mapsto T[\mathrm{id}]$ is injective. Finally, the image of this map certainly contains $\mathcal{A}_j/\mathcal{K}_j$, which is dense in $\mathcal{H}_j$. $\square$

2.33 Corollary. *The trace on $\mathcal{A}_j(X)$ extends to a weakly continuous faithful trace on $\mathcal{N}_j$, namely by*

$$\mathrm{tr}_{\mathcal{N}} \colon \mathcal{N}_j(X) \to \mathbb{C},\ T \mapsto \langle [\mathrm{id}], T[\mathrm{id}] \rangle_{\mathcal{H}}.$$

Proof. The functional $\mathrm{tr}_{\mathcal{N}}$ is by definition weakly continuous on $\mathcal{B}(\mathcal{H}_j)$.

For $A \in \mathcal{A}_j$, we have

$$\mathrm{tr}_{\mathcal{N}}(A) = \langle [\mathrm{id}], A[\mathrm{id}] \rangle_{\mathcal{H}} = \mathrm{tr}_{\mathcal{H}}(\mathrm{id}^* A\, \mathrm{id}) = \mathrm{tr}_{\mathcal{H}}(A) = \mathrm{tr}_{\mathcal{A}}(A),$$

so this indeed coincides with the original trace when applied to $\mathcal{A}_j$.

If $P \in \mathcal{N}_j$ is a non-zero projection, then

$$\mathrm{tr}_{\mathcal{N}}(P) = \mathrm{tr}_{\mathcal{N}}(P^* P) = \left\| P[\mathrm{id}] \right\|_{\mathcal{H}}^2 \neq 0$$

by Lemma 2.32. Thus, $\mathrm{tr}_{\mathcal{N}}$ is faithful.

It remains to prove the trace property on $\mathcal{N}_j$. Given $S, T \in \mathcal{N}_j$, find nets $(A_i)_{i\in I}, (B_k)_{k\in K} \subseteq \mathcal{A}_j$ such that $S = \lim_{i\in I} A_i$ and $T = \lim_{k\in K} B_k$ in the weak operator topology. As $\operatorname{tr}_{\mathcal{N}}$ is weakly continuous, multiplication is weakly continuous in each factor, and the trace property holds on $\mathcal{A}_j$, we obtain

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}}(ST) &= \lim_{i\in I} \operatorname{tr}_{\mathcal{N}}(A_i T) = \lim_{i\in I}\lim_{k\in K} \operatorname{tr}_{\mathcal{N}}(A_i B_k) \\ &= \lim_{i\in I}\lim_{k\in K} \operatorname{tr}_{\mathcal{N}}(B_k A_i) = \lim_{i\in I} \operatorname{tr}_{\mathcal{N}}(T A_i) = \operatorname{tr}_{\mathcal{N}}(TS). \qquad \square\end{aligned}$$

2.34 Remark. For completeness, let us show that the map $\mathcal{N}_j \to \mathcal{H}_j$, $T \mapsto T[\mathrm{id}]$ is in general not surjective. One such example is given in 2.31, where we show $\mathcal{N}_j \cong L^\infty([-1,1]) \subsetneq L^2([-1,1]) \cong \mathcal{H}_j$.

Here is a second example: Assume that in the complex X there are patterns $\alpha_n \in \mathrm{Pat}_{r_n,j}(X)$, with $0 = r_0 < r_1 < r_2 < \ldots$, such that each $\sigma \in \mathcal{E}_j X$ with r_n-pattern α_n also has r_m-pattern α_m for all $m \le n$, but only half of these cells also have r_{n+1}-pattern α_{n+1}. Then α_n has frequency 2^{-n}. Now define $A_n \in \mathcal{A}_j$ by

$$A_n \sigma = 2^{m/3}\sigma, \text{ where } m = \min\left(n, \max\{k \mid \sigma \text{ has } r_k\text{-pattern } \alpha_k\}\right).$$

This is a Cauchy sequence in $\mathcal{H}$:

$$\begin{aligned}\|A_n - A_m\|_{\mathcal{H}}^2 &= \operatorname{tr}_{\mathcal{H}}\big((A_n - A_m)^2\big) \\ &= \sum_{k=m+1}^{n} 2^{-(k+1)} \cdot \big(2^{k/3} - 2^{m/3}\big)^2 + \sum_{k=n+1}^{\infty} 2^{-(k+1)} \cdot \big(2^{n/3} - 2^{m/3}\big)^2 \\ &\le \sum_{k=m+1}^{n} 2^{-(k+1)} \cdot 2^{2k/3} + \sum_{k=n+1}^{\infty} 2^{-(k+1)} \cdot 2^{2n/3} \\ &= \sum_{k=m+1}^{n} 2^{-k/3-1} + 2^{-(n+1)} \cdot 2^{2n/3} \\ &\le \frac{1}{2}\sum_{k=m+1}^{\infty} 2^{-k/3} + 2^{-n/3-1} \xrightarrow{m,n\to\infty} 0.\end{aligned}$$

Thus, $\Xi = \lim_{n\to\infty} A_n$ exists in $\mathcal{H}_j$. Assume that there were $T \in \mathcal{N}_j$ such that $T[\mathrm{id}] = \Xi$. Then, by the argument from the proof of Lemma 2.32, we would have $T[A_m] = T[\mathrm{id}] \cdot A_m = \Xi \cdot A_m$ for every $m \in \mathbb{N}$. Since T is by assumption continuous, this gives $T\Xi = \lim_{m\to\infty} \Xi \cdot A_m$. Conversely, as right multiplication by A_m is continuous, $\Xi \cdot A_m = \lim_{n\to\infty} A_n \cdot A_m$.

For all $m \le n$, we have

$$\begin{aligned}\|A_m A_n\|_{\mathcal{H}}^2 &= \operatorname{tr}_{\mathcal{H}}(A_n A_m^2 A_n) \\ &\ge \operatorname{tr}_{\mathcal{H}}(A_m^4) = \sum_{k=0}^{m} 2^{-(k+1)} \cdot 2^{4k/3} + 2^{-(m+1)} \cdot 2^{4m/3} \\ &= \sum_{k=0}^{m} 2^{k/3-1} + 2^{m/3-1} \xrightarrow{m\to\infty} \infty.\end{aligned}$$

Thus, $T\Xi$ cannot be an element of $\mathcal{H}_j$. Contradiction!

2.4 Dimensions

In the finite-dimensional world, every dimension of a vector space can be expressed as the trace of the projection to that space. With the trace on $\mathcal{N}_j(X)$ developed above, we can apply the same concept to define finite dimensions for certain subspaces of $\mathcal{H}_j(X)$:

2.35 Definition. A closed subspace $V \subseteq \mathcal{H}_j(X)$ is called *geometric*, if the orthogonal projection to V lies in $\mathcal{N}_j(X)$. For every such subspace, define $\dim_{\mathcal{N}}(V) = \mathrm{tr}_{\mathcal{N}}(\mathrm{proj}_V)$.

Let us collect some basic properties of this dimension:

2.36 Lemma. (a) *If $V, W \subseteq \mathcal{H}_j(X)$ are geometric subspaces, then so are $V^\perp$, $V \cap W$ and $V + W$.*

(b) *If $V \perp W$, then* $\dim_{\mathcal{N}}(V \oplus W) = \dim_{\mathcal{N}}(V) + \dim_{\mathcal{N}}(W)$.

(c) *If $T \in \mathcal{N}_j(X)$, then* $\ker(T)$ *and* $\overline{\mathrm{im}(T)}$ *are geometric subspaces, and*

$$\dim_{\mathcal{N}}\big(\overline{\mathrm{im}(T)}\big) = \dim_{\mathcal{N}}\big(\mathcal{H}_j(X)\big) - \dim_{\mathcal{N}}\big(\ker(T)\big).$$

(d) *If $V, W \subseteq \mathcal{H}_j(X)$ are geometric subspaces, then*

$$\dim_{\mathcal{N}}(V + W) = \dim_{\mathcal{N}}(V) + \dim_{\mathcal{N}}(W) - \dim_{\mathcal{N}}(V \cap W).$$

Proof. (a) If P_V is the orthogonal projection to $V \in \mathcal{N}_j(X)$, then $\mathrm{id} - P_V \in \mathcal{N}_j(X)$ projects to $V^\perp$, so $V^\perp$ is geometric.

By a theorem of von Neumann [von50], the projection to $V \cap W$ is given by

$$P_{V\cap W} = \lim_{n\to\infty} (P_V P_W)^n$$

in strong operator topology. This is clearly contained in $\mathcal{N}_j(X)$, so $V \cap W$ is geometric.

Since $(V + W)^\perp = V^\perp \cap W^\perp$, the third statement follows from the first two.

(b) This follows directly from $P_{V\oplus W} = P_V + P_W$.

(c) Since $\chi_{\{0\}}(T) \in \mathcal{N}_j(X)$ projects to $\ker(T)$, the kernel is geometric, and as $\overline{\mathrm{im}(T)} = \ker(T^*)^\perp$, it follows from (a) that $\overline{\mathrm{im}(T)}$ is also geometric.

Write $T = U\,|T|$ with $|T| = \sqrt{T^*T}$ and U unitary. Clearly, $U, |T| \in \mathcal{N}_j(X)$ and $\ker(T) = \ker|T|$. If Q projects to $\overline{\mathrm{im}\,|T|}$, then UQU^* projects to $\overline{\mathrm{im}(T)}$, so we get

$$\dim_{\mathcal{N}}\big(\ker(T)\big) = \dim_{\mathcal{N}}\big(\ker|T|\big), \qquad \dim_{\mathcal{N}}\big(\overline{\mathrm{im}(T)}\big) = \dim_{\mathcal{N}}\big(\overline{\mathrm{im}\,|T|}\big).$$

Finally, as $|T|$ is self-adjoint, we have

$$\mathcal{H}_j(X) = \ker|T| \oplus \overline{\mathrm{im}\,|T|},$$

so the statement follows from (b).

(d) Let $\widetilde{W} = (\mathrm{id} - P_V)W$ be the projection of W to $V^\perp$. This gives the orthogonal decomposition $V + W = V \oplus \widetilde{W}$, and by (b), $\dim_{\mathcal{N}}(V+W) = \dim_{\mathcal{N}}(V) + \dim_{\mathcal{N}}(\widetilde{W})$. Note that $\widetilde{W}$ is geometric by (c), because it is the image of $(\mathrm{id} - P_V)P_W$. Furthermore, $\ker\big((\mathrm{id} - P_V)P_W\big) = W^\perp \oplus (V \cap W)$, and so (b) and (c) yield

$$\begin{aligned}\dim \widetilde{W} &= \dim_{\mathcal{N}}\big(\overline{\mathrm{im}((\mathrm{id} - P_V)P_W)}\big) \\ &= \dim_{\mathcal{N}}\big(\mathcal{H}_j(X)\big) - \dim_{\mathcal{N}}(W^\perp) - \dim_{\mathcal{N}}(V \cap W) \\ &= \dim_{\mathcal{N}}(W) - \dim_{\mathcal{N}}(V \cap W)\end{aligned}$$

This completes the proof. □

2.37 Corollary. *If $V, W \subseteq \mathcal{H}_j(X)$ are two geometric subspaces such that $\dim_{\mathcal{N}}(V) < \dim_{\mathcal{N}}(W)$, then $W \cap V^\perp \neq \{0\}$.*

Proof. Apply Lemma 2.36 to $P_V P_W$. We have $\ker(P_V P_W) = W^\perp \oplus (W \cap V^\perp)$, and $\overline{\mathrm{im}(P_V P_W)} \subseteq V$. Therefore,

$$\begin{aligned}\dim_{\mathcal{N}}(V) &\geq \dim_{\mathcal{N}}\big(\overline{\mathrm{im}(P_V P_W)}\big) \\ &= \dim_{\mathcal{N}}\big(\mathcal{H}_j(X)\big) - \dim_{\mathcal{N}}\big(W^\perp\big) - \dim_{\mathcal{N}}\big(W \cap V^\perp\big) \\ &= \dim_{\mathcal{N}}(W) - \dim_{\mathcal{N}}\big(W \cap V^\perp\big) \\ \implies \dim_{\mathcal{N}}\big(W \cap V^\perp\big) &\geq \dim_{\mathcal{N}}(W) - \dim_{\mathcal{N}}(V) > 0.\end{aligned}$$

As a zero space would have dimension zero, this completes the proof. □

2.5 Spectral density functions

We are now, finally, ready to define and discuss the spectral density functions (or "integrated densities of states") of geometric operators:

2.38 Definition. Given a positive operator $T \in \mathcal{N}_j(X)$ and $\lambda \in [0, \infty)$, define the *spectral projections* of T by

$$E^T(\lambda) = \chi_{(-\infty,\lambda]}(T) = \chi_{[0,\lambda]}(T) \in \mathcal{N}_j(X)$$

and the *spectral density function* of T by

$$F^T \colon [0, \infty) \to [0, 1], \lambda \mapsto \mathrm{tr}_{\mathcal{N}}\big(E_T(\lambda)\big).$$

In general, for any operator $T \in \mathcal{N}_j(X)$, define

$$E^T(\lambda) = E^{T^*T}(\lambda^2) \quad \text{and} \quad F^T(\lambda) = F^{T^*T}(\lambda^2).$$

(This is well-defined, as Lemma A.8 proves the equality $E^T(\lambda) = E^{T^*T}(\lambda^2)$ for self-adjoint T.)

2.39 Lemma. *Let $T \in \mathcal{N}_j$. Then its spectral density function F^T fulfills the following:*

(a) $F^T(0) = \dim_{\mathcal{N}}(\ker T)$.

(b) F^T *is increasing.*

(c) F^T *is right-continuous.*

(d) $F^T(\lambda) = \dim_{\mathcal{N}}(\mathcal{H}_j) = 1$ *for every* $\lambda \geq \|T\|$.

Proof. All of this follows from the definition and Theorem A.6. $\square$

2.40 Lemma. *For any $T \in \mathcal{N}_j(X)$ and all $\lambda \geq 0$, we have $F^T(\lambda) = F^{|T|}(\lambda) = F^{T^*}(\lambda)$.*

Proof. By definition,

$$F^T(\lambda) = \operatorname{tr}_{\mathcal{N}} E^{T^*T}(\lambda) = \operatorname{tr}_{\mathcal{N}} E^{|T|^2}(\lambda) = F^{|T|}(\lambda)$$

and $F^{T^*}(\lambda) = \operatorname{tr}_{\mathcal{N}} E^{TT^*}(\lambda)$. Since $\operatorname{spec}(T^*T) \subseteq [0, \infty)$, we have $E^{T^*T}(\lambda) = \chi_{[0,\lambda]}(T^*T)$ and the same for TT^*. Choose a sequence of polynomials $(p_m)_{m\in\mathbb{N}}$ that converge pointwise to $\chi_{[0,\lambda]}$. Then Theorem A.3 implies that $p_m(T^*T)$ converges weakly to $E^{T^*T}(\lambda)$, and as the trace is weakly continuous, we get $F^T(\lambda) = \lim_{m\to\infty} \operatorname{tr}_{\mathcal{N}} p_m(T^*T)$ and $F^{T^*}(\lambda) = \lim_{m\to\infty} \operatorname{tr}_{\mathcal{N}} p_m(TT^*)$.

On the other hand, the trace property implies $\operatorname{tr}_{\mathcal{N}}\big((T^*T)^k\big) = \operatorname{tr}_{\mathcal{N}}\big((TT^*)^k\big)$ for all k, so (by linearity of the trace) those two limits are equal. $\square$

2.41 Remark. If $\mathcal{E}_jX$ is finite, every self-adjoint operator $T \in \mathcal{N}_j(X)$ corresponds to a hermitian $(n \times n)$-matrix, and there is an orthonormal basis of $\mathbb{C}^n$ with respect to which T has the form $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$. Then the spectral projection $E^T(\mu)$ is given by the projection to the first k basis vectors, where k is given by $\lambda_k \leq \mu < \lambda_{k+1}$, and $F^T(\mu) = \operatorname{tr}_{\mathcal{N}}(E^T(\mu)) = k/n$. Especially, the spectral density function of an operator on a finite-dimensional space is always a right-continuous step function.

The idea of spectral density functions is that $F(\lambda)$ measures the size of the maximal subspace on which T is bounded by λ:

2.42 Lemma. *Let $T \in \mathcal{N}_j(X)$ be self-adjoint and $\mu \geq 0$. Then*

$$F^T(\mu) = \max\left\{\dim_{\mathcal{N}} V \mid V \subseteq \mathcal{H}_j(X) \textit{ geometric}, \|T|_V\| \leq \mu\right\}.$$

(Here, $T|_V$ is considered as an operator $T\colon V \to \mathcal{H}_j(X)$.)

Proof. By definition, $F^T(\mu) = \operatorname{tr}_{\mathcal{N}}(E^T(\mu)) = \dim_{\mathcal{N}}\big(\operatorname{im} E^T(\mu)\big)$, and the space $\operatorname{im} E^T(\mu)$ is geometric. Lemma A.9 gives $\|Tv\| \leq \mu\,\|v\|$ for all $v \in \operatorname{im} E^T(\mu)$, and thus

$$F^T(\mu) = \dim_{\mathcal{N}}(\operatorname{im} E^T(\mu)) \leq \max\left\{\dim_{\mathcal{N}} V \mid V \text{ geometric}, \|T|_V\| \leq \mu\right\}.$$

Conversely, let $V \subseteq \ell^2(\mathcal{E}_jX)$ be a geometric subspace such that $\dim_{\mathcal{N}} V > F^T(\mu)$. By Corollary 2.37, this implies that there is a nonzero vector $x \in V \cap \big(\operatorname{im} E^T(\mu)\big)^\perp$. Then, Lemma A.9 yields $\|Tx\| > \mu\,\|x\|$, and therefore $\|T|_V\| > \lambda$. $\square$

Finally, it should be noted that the spectral density function of an operator T contains all necessary information to determine the trace for any operator that can be obtained from T through functional calculus:

2.43 Lemma. *For any self-adjoint $T \in \mathcal{N}(X)$ and any function $f \in L^\infty(\mathbb{R})$, we have*

$$\operatorname{tr}_{\mathcal{N}}\big(f(T)\big) = \int_{\mathbb{R}} f(\lambda)\, dF^T(\lambda),$$

where the measure $dF^T(\lambda)$ is given by $dF^T\big((a,b]\big) = F^T(b) - F^T(a)$.

Proof. Using the definition of F and Theorem A.6, we obtain:

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}}\big(f(T)\big) &= \langle [\mathrm{id}_X], f(T)[\mathrm{id}_X]\rangle_{\mathcal{H}} = \int_{\mathbb{R}} f(\lambda)\, d\big\langle [\mathrm{id}_X], E^T(\lambda)[\mathrm{id}_X]\big\rangle_{\mathcal{H}} \\ &= \int_{\mathbb{R}} f(\lambda)\, d\big(\operatorname{tr}_{\mathcal{N}}\big(E^T(\lambda)\big)\big) = \int_{\mathbb{R}} f(\lambda)\, dF^T(\lambda).\end{aligned}$$ □

2.44 Remark. If one is interested solely in the spectral density functions of geometric operators, but not in their von Neumann algebra, Lemma 2.43 can serve as an alternative definition:

For any bounded continuous function $f\colon \operatorname{spec}(T) \to \mathbb{C}$, we have $f(T) \in \mathcal{A}_j$, so $\operatorname{tr}_{\mathcal{A}}(f(T))$ is immediately defined. Especially, this defines a positive linear functional

$$C_c(\operatorname{spec}(T)) \to \mathbb{C},\ f \mapsto \operatorname{tr}_{\mathcal{A}}(f(T)).$$

By the *Riesz–Markov–Kakutani representation theorem* (see [Els11], p. 358), this implies the existence of a unique locally finite inner regular measure μ^T on $\operatorname{spec}(T)$ such that

$$\operatorname{tr}_{\mathcal{N}}\big(f(T)\big) = \int_{\operatorname{spec}(T)} f\, d\mu^T$$

holds for every $f \in C_c(\operatorname{spec}(T))$. One can then define the spectral density function by

$$F^T(\lambda) = \int_{\operatorname{spec}(T)} \chi_{[0,\lambda]}\, d\mu^T,$$

obtaining the same function as in our Definition 2.38 (and, of course, $dF^T = \mu^T$). The author would like to thank Ralf Meyer for pointing out this approach.

3 Self-similar complexes and uniform convergence

In the previous chapter, we have been using spaces with *aperiodic order*, that is, spaces where every pattern of cells appears at a frequency that becomes approximately constant on a large scale. Now, we shall discuss a way to actually construct such spaces using self-similarity.

In short, a self-similar space is obtained through an iterative process: We start with a finite cell complex and glue several copies of it together. We can then use the resulting (still finite) complex and repeat the process ad infinitum, eventually obtaining the self-similar complex as the union of all iteration steps.

It is intuitively clear that patterns that are present in the finite subcomplexes will repeat infinitely often in the final complex. On the other hand, whenever two subcomplexes are glued together, new patterns can be created. This lets the whole complex be more than the sum of its parts, but it also holds potential for instability and divergence. To keep this variation in check, we need *amenability*: The number of cells at which different subcomplexes meet each other must be small compared to the total number of cells.

Under these conditions, we will show first that the self-similar structure indeed implies the aperiodic order required in the previous chapter. Then, we will come to the centerpiece of this thesis: We will prove that, on a self-similar complex, the spectral density function for any geometric operator can be approximated uniformly by the spectral density functions of the finite subcomplexes that form the self-similar structure.

3.1 Definition. A *self-similar complex* is a bounded regular CW-complex X for which there is a *self-similar exhaustion*, that is, an amenable exhaustion $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq X$ by connected[4] subcomplexes as in Def. 2.17, such that for each $m \in \mathbb{N}$ there is a finite set $\mathcal{G}(m, m+1)$ of local isomorphisms $\gamma \colon K_m \to \gamma K_m \subseteq X$ that fulfills

$$K_{m+1} = \bigcup_{\gamma \in \mathcal{G}(m,m+1)} \gamma K_m,$$
$$\mathcal{E}_j(\gamma_1 K_m) \cap \mathcal{E}_j(\gamma_2 K_m) = \mathcal{F}_j(\gamma_1 K_m) \cap \mathcal{F}_j(\gamma_2 K_m) \text{ for all } \gamma_1 \neq \gamma_2.$$

Thus, each subcomplex K_{m+1} consists of “copies” of the next-smaller subcomplex K_m that overlap only at their frontiers. Write

$$\mathcal{G}(m, m+k) = \{\gamma_{m+k-1} \circ \ldots \circ \gamma_{m+1} \circ \gamma_m \,|\, \gamma_j \in \mathcal{G}(j, j+1)\},$$
$$\mathcal{G}(m) = \bigcup_{k=0}^{\infty} \mathcal{G}(m, m+k).$$

[4] Connectedness in the topological sense implies that any two vertices in K_m have a finite combinatorial distance; this is relevant for the proof of Theorem 3.4. On the other hand, it does not imply that any two j-cells for $j \geq 2$ have finite combinatorial distance, and that is not needed for any proofs.

Then we obtain

$$K_{m+k} = \bigcup_{\gamma \in \mathcal{G}(m,m+k)} \gamma K_m \quad \text{and} \quad X = \bigcup_{\gamma \in \mathcal{G}(m)} \gamma K_m,$$

where the various copies of K_m still only overlap at their frontiers.

Note that the self-similar exhaustion (K_m) and its local isomorphism sets $\mathcal{G}(m, m+1)$ are not a fixed part of the structure – they only need to exist.

3.2 Example. $\mathbb{R}^d$ with the standard CW-structure (that is, $\mathcal{E}_0\mathbb{R}^d = \mathbb{Z}^d$, etc.) is self-similar. For example, let $K_m = [-3^m, 3^m]^d$ for all $m \in \mathbb{N}_0$, where the 3^d local isomorphisms in $\mathcal{G}(m, m+1)$ are given by shifts by $2 \cdot 3^m \cdot \sum_{i=1}^d \epsilon_i e_i$, where $\epsilon_i \in \{-1, 0, 1\}$ and e_i are the standard basis vectors in $\mathbb{R}^d$.

Check that these K_m are indeed a Følner sequence: for the top-dimensional cells, we have

$$\frac{|\mathcal{F}_d K_m|}{|\mathcal{E}_d K_m|} = \frac{2d \cdot (2 \cdot 3^m)^{d-1}}{(2 \cdot 3^m)^d} = \frac{d}{3^m} \xrightarrow{m\to\infty} 0,$$

in other dimensions, we get terms of the same order of magnitude.

An entirely different self-similar exhaustion of $\mathbb{R}^2$ is shown in Figure 5. Here, every K_{m+1} consists of only four copies of K_m (instead of nine as in the first structure).

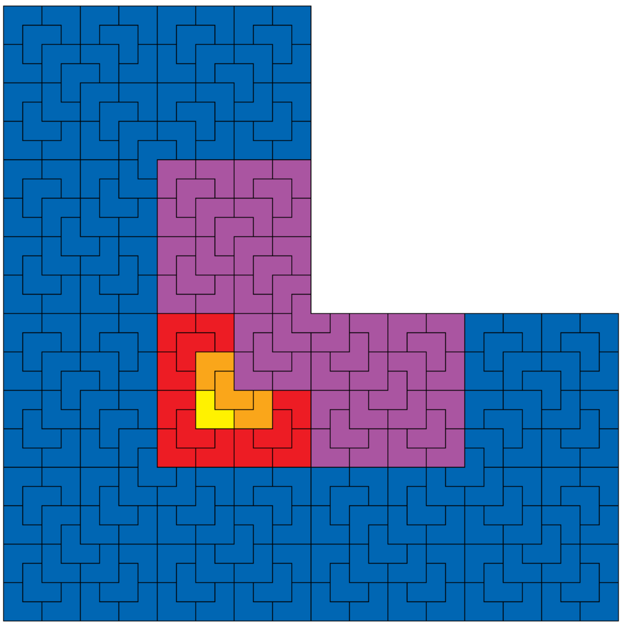

Figure 5: *Another self-similar structure on* $\mathbb{R}^2$. In order to cover the entirety of $\mathbb{R}^2$, new "corners" have to be attached on alternating sides: K_1 (orange) and K_3 (purple) extend to the top right of K_0 (yellow), while K_2 (red) and K_4 (blue) extend to the bottom left. Note that each subcomplex includes its predecessors, so K_4 is actually the entire picture (and not just the blue parts).

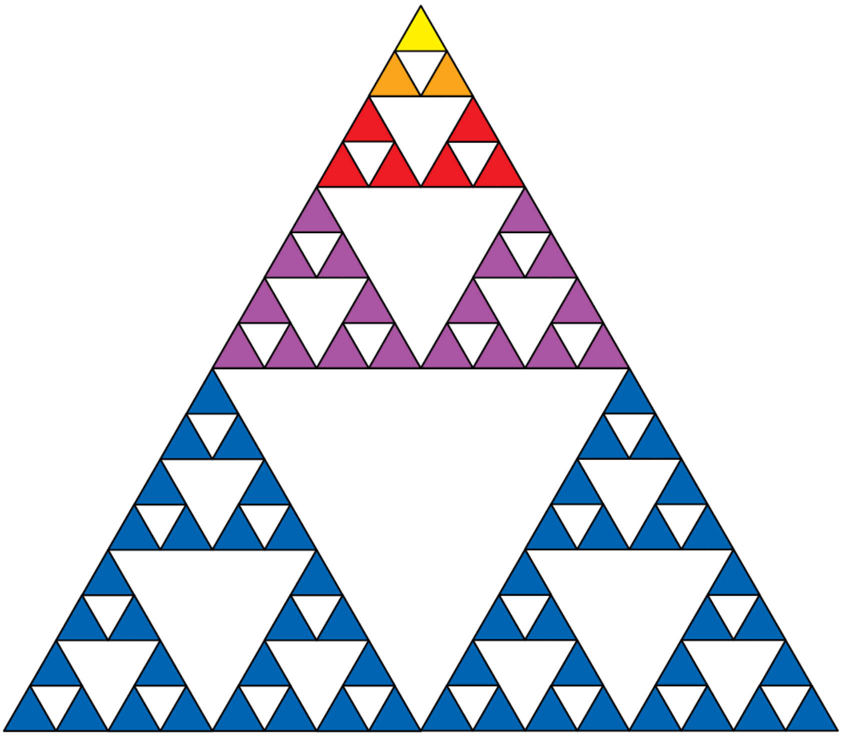

Figure 6: *Subcomplexes K_0 to K_4 of Sierpiński's triangle.* K_0 is the yellow triangle on top, K_1 consists of the yellow and orange parts, K_2 adds the red parts, etc. The whole space would extend infinitely towards the bottom of the page.

3.3 Example (*Sierpiński's triangle*)**.** Let K_0 be a triangle and form K_{m+1} from three copies of K_m each connected at their "corners" (see Figure 6).

In this case, amenability is trivial: each K_m has $|\mathcal{F}_0 K_m| = 3$, $|\mathcal{F}_1 K_m| = 6$, $|\mathcal{F}_2 K_m| = 0$, regardless of m. (There are two slightly counterintuitive aspects to this: First, the top corner is not an original frontier, but it is a generalized frontier, as it gets mapped to a frontier by many local isomorphisms. Second, any two 2-cells have combinatorial distance ∞, as they can only border each other at a vertex, not along an edge.)

The top corner has a unique 1-pattern: It is the only vertex of degree two, all other vertices have degree four. Thus, this is an example of a space where a pattern has frequency zero. Consequently, an operator $P \in \mathcal{B}(\ell^2(\mathcal{E}_0 X))$ that projects to the space spanned by vertices of degree two would of course be geometric, but have $\mathrm{tr}_{\mathcal{A}}(P) = 0$.

To avoid this, we can instead use the "twin triangle" (see Figure 7). Here, every pattern has positive frequency. (It will be shown in the next section that the pattern frequencies indeed exist.)

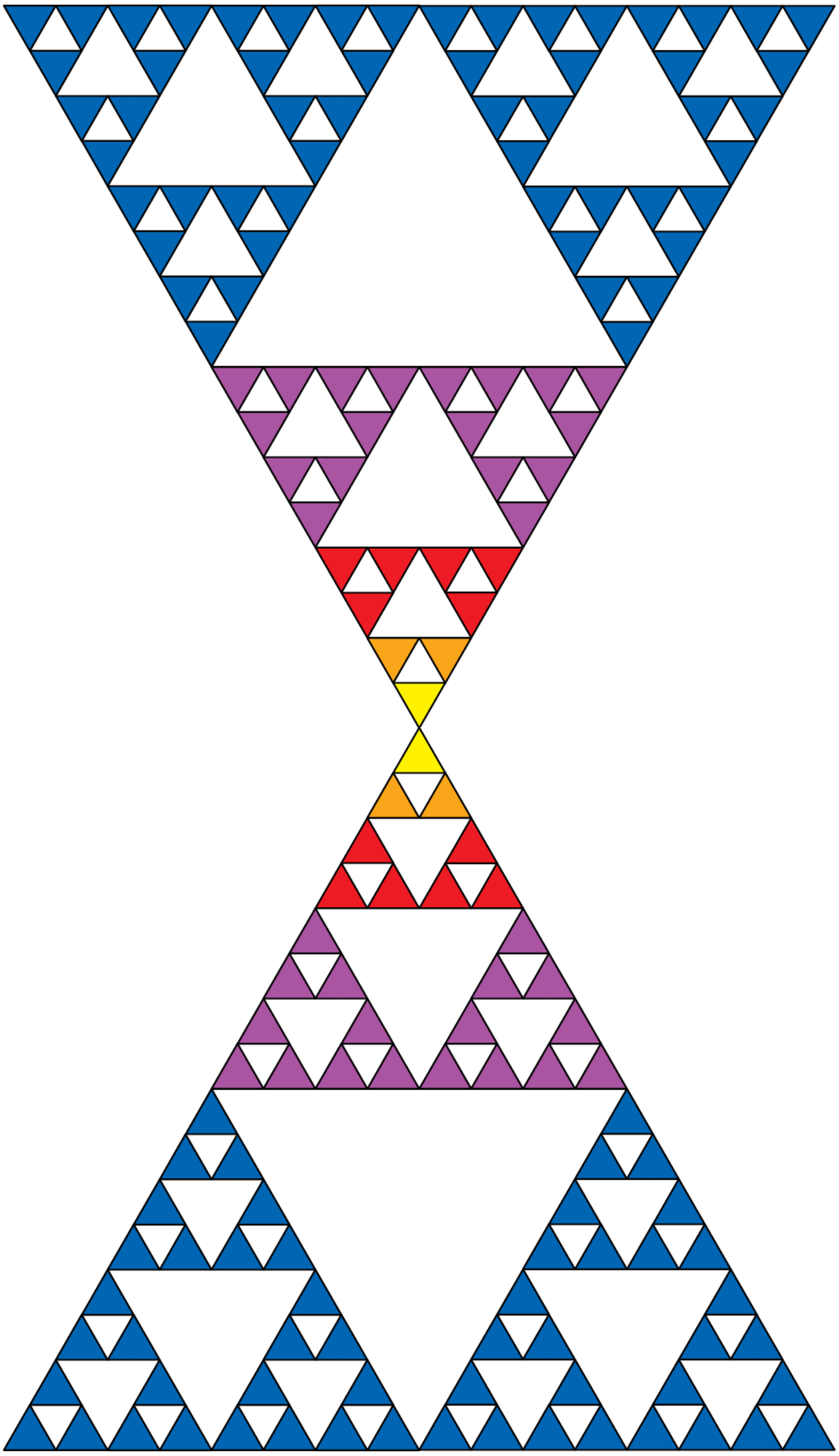

Figure 7: *The "twin triangle".* K_0 now consists of two triangles (yellow), but every K_m for $m > 0$ still consists of three copies of K_{m-1}. Unlike the original triangle, the twin triangle has no "special vertex": For any r, the r-neighborhood of the "central vertex" (the one between the two yellow triangles) looks no different than the r-neighborhood of infinitely many other points.

3.1 Self-similarity implies aperiodic order

Let us now prove that self-similar complexes do indeed have aperiodic order. This is not surprising, as their construction has clearly a "repetitive" nature, but it is not completely obvious either, since we need to check that the frequency of patterns converges not just along a given self-similar exhaustion, but along *any* Følner sequence, self-similar or not.

3.4 Theorem. *Every self-similar complex has aperiodic order.*

Proof. Let (K_m) be a self-similar exhaustion of X. Fix $j, r \in \mathbb{N}_0$ and a pattern $\alpha \in \mathrm{Pat}_{j,r}(X)$.

For any subcomplex $K \subseteq X$, consider the "r-interior":

$$\mathcal{I}_j^r K = \{\sigma \in \mathcal{E}_j K \mid d_{\mathrm{comb}}(\sigma, X \setminus K) > r\},$$
$$\mathcal{I}_j^{r,\alpha} K = \left\{\sigma \in \mathcal{I}_j^r K \,\middle|\, \sigma \text{ has } r\text{-pattern } \alpha\right\}.$$

By Lemma 2.16, we have

- $\left|\mathcal{I}_j^{r,\alpha}(\gamma K_m)\right| = \left|\mathcal{I}_j^{r,\alpha} K_m\right|$ for every $\gamma \in \mathcal{G}(m,n)$,
- $\mathcal{I}_j^r(\gamma_1 K_m) \cap \mathcal{I}_j^r(\gamma_2 K_m) = \emptyset$ whenever $\gamma_1, \gamma_2 \in \mathcal{G}(m,n)$ with $\gamma_1 \neq \gamma_2$,
- $\left|\mathcal{E}_j K_n \setminus \bigsqcup_{\gamma \in \mathcal{G}(m,n)} \mathcal{I}_j^r(\gamma K_m)\right| \leq C_r \, |\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|,$

  where $C_r = \max_{\sigma \in \mathcal{E}_j X} |B_r(\sigma)|$. ($C_r$ is finite since X is bounded.)

Therefore, the number of times the pattern α appears in K_n is given by

$$\begin{aligned}
\left|\mathcal{E}_j^\alpha K_n\right| &= \sum_{\gamma \in \mathcal{G}(m,n)} \left|\mathcal{I}_j^{r,\alpha}(\gamma K_m)\right| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|) \\
&= |\mathcal{G}(m,n)| \left|\mathcal{I}_j^{r,\alpha} K_m\right| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|) \\
&= |\mathcal{G}(m,n)| \left|\mathcal{E}_j^\alpha K_m\right| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|)
\end{aligned}$$

where the last line follows from $|\mathcal{E}_j K_m| - \left|\mathcal{I}_j^r K_m\right| \leq C_r \, |\mathcal{F}_j K_m|$.

On the other hand, the total number of j-cells in K_n is

$$\begin{aligned}
|\mathcal{E}_j K_n| &= \sum_{\gamma \in \mathcal{G}(m,n)} \left|\mathcal{I}_j^1(\gamma K_m)\right| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|) \\
&= |\mathcal{G}(m,n)| \left|\mathcal{I}_j^1 K_m\right| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|) \\
&= |\mathcal{G}(m,n)| \, |\mathcal{E}_j K_m| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|)
\end{aligned}$$

One obtains the pattern frequency:

$$\begin{aligned}
\frac{\left|\mathcal{E}_j^\alpha K_n\right|}{|\mathcal{E}_j K_n|} &= \frac{|\mathcal{G}(m,n)| \left|\mathcal{E}_j^\alpha K_m\right| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|)}{|\mathcal{G}(m,n)| \, |\mathcal{E}_j K_m| + \mathcal{O}(|\mathcal{G}(m,n)| \, |\mathcal{F}_j K_m|)} \\
&= \frac{\left|\mathcal{E}_j^\alpha K_m\right| + \mathcal{O}(|\mathcal{F}_j K_m|)}{|\mathcal{E}_j K_m| + \mathcal{O}(|\mathcal{F}_j K_m|)}
\end{aligned}$$

Now fix any $\varepsilon > 0$. Choose m large enough that for all $n \geq m$, the $\mathcal{O}$-terms are less than $\varepsilon\, |\mathcal{E}_j K_m|$, which gives:

$$\frac{1-\varepsilon}{1+\varepsilon}\,\frac{\left|\mathcal{E}_j^\alpha K_m\right|}{|\mathcal{E}_j K_m|} \leq \frac{\left|\mathcal{E}_j^\alpha K_n\right|}{|\mathcal{E}_j K_n|} \leq \frac{1+\varepsilon}{1-\varepsilon}\,\frac{\left|\mathcal{E}_j^\alpha K_m\right|}{|\mathcal{E}_j K_m|}$$

Thus, the sequence of frequencies $\left(\frac{\left|\mathcal{E}_j^\alpha K_m\right|}{|\mathcal{E}_j K_m|}\right)$ is convergent, and it has a limit

$$\mathbb{P}_{j,r}(\alpha) = \lim_{m\to\infty} \frac{\left|\mathcal{E}_j^\alpha K_m\right|}{|\mathcal{E}_j K_m|}.$$

It remains to show that if (L_k) is a different amenable exhaustion of X, then $\left(\frac{\left|\mathcal{E}_j^\alpha L_k\right|}{|\mathcal{E}_j L_k|}\right)$ converges to the same limit.

Again, fix $\varepsilon > 0$. Let

$$C_r = \max_{\sigma\,\in\,\mathcal{E}_j X} |B_r(\sigma)|\,.$$

Choose m large enough that

$$\left|\mathbb{P}_{j,r}(\alpha) - \frac{\left|\mathcal{E}_j^\alpha K_m\right|}{|\mathcal{E}_j K_m|}\right|, \quad \left|\mathbb{P}_{j,r}(\alpha) - \frac{\left|\mathcal{I}_j^{1,\alpha} K_m\right|}{|\mathcal{E}_j K_m|}\right| \quad \text{and} \quad \frac{C_r\,|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|}$$

are all smaller than ε.

Let[5]

$$b_m = \max_{\rho,\rho'\,\in\,\mathcal{E}_0 K_m} d_{\mathrm{comb}}(\rho,\rho')$$

(note that this is always finite) and

$$D_m = \max_{\rho\,\in\,\mathcal{E}_0 X} |B_{b_m}(\rho)|\,.$$

Then choose k_0 large enough such that for all $k \geq k_0$,

$$\frac{|\mathcal{F}_j L_k|}{|\mathcal{E}_j L_k|} < \frac{\varepsilon}{D_m}.$$

For $m \in \mathbb{N}$, let

$$\begin{aligned}
\mathcal{G}_{\mathrm{in}}(m,k) &= \{\gamma \in \mathcal{G}(m) \mid \gamma K_m \subseteq L_k\}\,,\\
\mathcal{G}_{\mathrm{out}}(m,k) &= \{\gamma \in \mathcal{G}(m) \mid \gamma K_m \cap L_k \neq \emptyset\}\,,\\
\mathcal{G}_{\mathrm{front}}(m,k) &= \mathcal{G}_{\mathrm{out}}(m,k) \setminus \mathcal{G}_{\mathrm{in}}(m,k).
\end{aligned}$$

Then the frequency of the pattern α in L_k can be estimated by

$$\frac{|\mathcal{G}_{\mathrm{in}}(m,k)|\,\left|\mathcal{I}_j^{1,\alpha} K_m\right|}{|\mathcal{G}_{\mathrm{out}}(m,k)|\,|\mathcal{E}_j K_m|} \leq \frac{\left|\mathcal{E}_j^\alpha L_k\right|}{|\mathcal{E}_j L_k|} \leq \frac{|\mathcal{G}_{\mathrm{out}}(m,k)|\left(\left|\mathcal{E}_j^\alpha K_m\right| + C_r\,|\mathcal{F}_j K_m|\right)}{|\mathcal{G}_{\mathrm{in}}(m,k)|\,\left|\mathcal{I}_j^{1} K_m\right|}$$

[5] Unlike C_r, which counts the j-cells in r-patterns, the constant D_m always counts 0-cells. This is necessary because we will soon use that any two 0-cells in the complex are connected by a "path" of adjacent cells, which does not hold for general j-cells.

where the term $|\mathcal{G}_{\text{out}}(m,k)|\, C_r\, |\mathcal{F}_j K_m|$ estimates the number of cells whose r-pattern in L_k may stretch across multiple copies of K_m.

By choice of m, we already know

$$(1-\varepsilon)\, \mathbb{P}_{j,r}(\alpha) \le \frac{\left|\mathcal{I}_j^{1,\alpha} K_m\right|}{|\mathcal{E}_j K_m|} \quad \text{and} \quad \frac{\left|\mathcal{E}_j^{\alpha} K_m\right| + C_r\, |\mathcal{F}_j K_m|}{\left|\mathcal{I}_j^{1} K_m\right|} \le \frac{1+2\varepsilon}{1-\varepsilon}\, \mathbb{P}_{j,r}(\alpha).$$

It remains to bound the ratio of $|\mathcal{G}_{\text{in}}(m,k)|$ and $|\mathcal{G}_{\text{out}}(m,k)|$.

If $\gamma \in \mathcal{G}_{\text{front}}(m,k)$, then γK_m contains a vertex in L_k and a vertex in $X \setminus L_k$; since K_m (and thus γK_m) is connected, it must also contain a vertex $\rho \in \mathcal{F}_0 L_k$. Therefore,

$$\bigcup_{\gamma \in \mathcal{G}_{\text{front}}(m,k)} \mathcal{E}_0(\gamma K_m) \subseteq B_{b_m}(\mathcal{F}_0 L_k)$$

and thus

$$\left| \bigcup_{\gamma \in \mathcal{G}_{\text{front}}(m,k)} \mathcal{E}_0(\gamma K_m) \right| \le D_m\, |\mathcal{F}_0 L_k|\, .$$

Conversely, as the different copies of K_m only overlap at their frontiers, we have

$$|\mathcal{G}_{\text{front}}(m,k)| \cdot (|\mathcal{E}_0 K_m| - |\mathcal{F}_0 K_m|) \le \left| \bigcup_{\gamma \in \mathcal{G}_{\text{front}}(m,k)} \mathcal{E}_0(\gamma K_m) \right|.$$

Combining the previous two equations yields the estimate

$$|\mathcal{G}_{\text{front}}(m,k)| \cdot (|\mathcal{E}_0 K_m| - |\mathcal{F}_0 K_m|) \le D_m\, |\mathcal{F}_0 L_k|\, ,$$

and if m is large enough to ensure $|\mathcal{F}_0 K_m| \le \frac{1}{2}\, |\mathcal{E}_0 K_m|$, we obtain

$$|\mathcal{G}_{\text{front}}(m,k)| \le \frac{D_m\, |\mathcal{F}_0 L_k|}{|\mathcal{E}_0 K_m| - |\mathcal{F}_0 K_m|} \le 2 D_m\, \frac{|\mathcal{F}_0 L_k|}{|\mathcal{E}_0 K_m|}$$

On the other hand,

$$L_k \subseteq \bigcup_{\gamma \in \mathcal{G}_{\text{out}}(m,k)} \gamma K_m \quad \Longrightarrow \quad |G_{\text{out}}(m,k)| \ge \frac{|\mathcal{E}_0 L_k|}{|\mathcal{E}_0 K_m|},$$

which implies

$$\frac{|\mathcal{G}_{\text{front}}(m,k)|}{|G_{\text{out}}(m,k)|} \le 2 D_m\, \frac{|\mathcal{F}_0 L_k|}{|\mathcal{E}_0 L_k|} < 2\varepsilon$$

and therefore

$$\frac{|\mathcal{G}_{\text{in}}(m,k)|}{|G_{\text{out}}(m,k)|} \ge 1 - 2\varepsilon.$$

Thus, we finally end up with

$$(1-2\varepsilon)(1-\varepsilon)\, \mathbb{P}_{j,r}(\alpha) \le \frac{\left|\mathcal{E}_j^{\alpha} L_k\right|}{|\mathcal{E}_j L_k|} \le \frac{1+\varepsilon}{(1-2\varepsilon)(1-\varepsilon)}\, \mathbb{P}_{j,r}(\alpha).$$

As ε was arbitrary and this holds for all $k \ge k_0$, the limits indeed coincide. $\square$

3.2 Approximating spectral density functions

We now come to the main theorem about spectral density functions of geometric operators on self-similar complexes.

Given any geometric operator T on a self-similar complex X, we form a sequence of "restricted" operators T_m on the finite subcomplexes K_m.[6]

These operators T_m can always be obtained by $T_m := P_m T I_m$, where $I_m \colon \ell^2(\mathcal{E}_j K_m) \to \ell^2(\mathcal{E}_j X)$ is the inclusion and $P_m \colon \ell^2(\mathcal{E}_j X) \to \ell^2(\mathcal{E}_j K_m)$ is the orthogonal projection.

However, as the proof of convergence extensively uses that the frontiers of K_m are negligible compared to its interior, the behavior of T_m near these frontiers is equally negligible. Thus, our choice of T_m is rather flexible. In the most interesting case, when $T = \Delta_j^{(X)}$ is the Laplacian on X, we can choose $T_m = \Delta_j^{(K_m)}$ to be the Laplacian on K_m (which ignores the existence of all cells in $X \setminus K_m$), even if that operator takes significantly different values on frontiers of K_m.

Firstly, in Theorem 3.5, we will show that the spectral density functions of the operators T_m are uniformly convergent (without yet specifying the limit). Then, in Theorem 3.11, we will show that their limit is actually the spectral density function of T as defined in Def. 2.38.

This is remarkable in so far as the function from 2.38 measured the sizes of subspaces of $\mathcal{H}_j(X)$ (the abstract Hilbert space formed from the geometric operators themselves), while the spectral density functions of the T_m actually measure subspaces spanned by cells in K_m. Furthermore, it proves that the choice of (K_m) does not matter at all – the spectral density functions will always converge to the same limit!

3.5 Theorem. *Let X be a self-similar CW-complex with self-similar exhaustion $(K_m)_{m\in\mathbb{N}}$, and let $T \in \mathcal{A}_j^{geo}(X)$ be a positive r-pattern-invariant operator.*

Choose a sequence of positive operators $T_m \in \mathcal{B}(\ell^2(\mathcal{E}_j K_m))$ such that, for all m, $\|T_m\| \leq C$ for some constant C, $\operatorname{prop}(T_m) \leq r$, and $T\sigma = T_m\sigma$ for every $\sigma \in \mathcal{I}_j^{r+1} K_m$.

Let F_m be the spectral density functions of T_m, that is,

$$F_m(\lambda) = \frac{1}{|\mathcal{E}_j K_m|} \max \left\{ \dim_{\mathbb{C}} W \;\middle|\; \begin{array}{c} W \subseteq \ell^2(\mathcal{E}_j K_m) \text{ linear subspace such that} \\ \|T_m v\| \leq \lambda \cdot \|v\| \text{ for all } w \in W \end{array} \right\}.$$

Then the sequence of functions F_m converges uniformly, and the limit does not depend on the choice of the sequence (T_m).

Before we begin the actual proof, some elementary lemmas:

3.6 Lemma. *Let A, B be finite-dimensional vector spaces and $C \subseteq A \oplus B$ be a subspace. Then*

$$\dim(A \cap C) \geq \dim(C) - \dim(B).$$

[6] The exhaustion (K_m) is not fixed, and it will become clear later that its choice does not matter.

Proof. We have $A \oplus B \supseteq A + C$ and thus

$$\begin{aligned}
& & \dim(A \oplus B) &\geq \dim(A + C) \\
&\implies & \dim(A) + \dim(B) &\geq \dim(A) + \dim(C) - \dim(A \cap C) \\
&\implies & \dim(B) &\geq \dim(C) - \dim(A \cap C) \\
&\implies & \dim(A \cap C) &\geq \dim(C) - \dim(B).
\end{aligned}$$

□

3.7 Lemma. *Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be a self-adjoint operator and $\mu \geq 0$. Assume that exactly k of the eigenvalues of T (counted with multiplicity) have absolute value $\leq \mu$. Let $V \subset \mathbb{C}^n$ be a subspace such that $\|Tv\| \leq \mu \|v\|$ for all $v \in V$. Then:*

(a) $\dim V \leq k$.

(b) *If $\dim V < k$, there is a subspace $W \supset V$ such that $\|Tw\| \leq \mu \|w\|$ for all $w \in W$ and $\dim W = k$.*

Proof. Let $(e_i)_{i=1}^n$ be an ONB of eigenvectors of T, and $(\lambda_i)_{i=1}^n$ be the corresponding eigenvalues. Without loss of generality we have

$$|\lambda_1| \leq \ldots \leq |\lambda_k| \leq \mu < |\lambda_{k+1}| \leq \ldots \leq |\lambda_n|.$$

(a) Assume $\dim V > k$. Then the spaces V and $\operatorname{span}(e_{k+1}, \ldots, e_n)$ must have a nontrivial intersection (since their dimensions add up to more than n). But any nonzero vector $x = \sum_{i=k+1}^n x_i e_i$ fulfills

$$\|Tx\|^2 = \sum_{i=k+1}^{n} \lambda_i^2 |x_i|^2 > \sum_{i=k+1}^{n} \mu^2 |x_i|^2 = \mu^2 \|x\|^2,$$

so it cannot lie in V. Contradiction!

(b) Assume $\dim V < k$. It suffices to construct a space $W \supset V$ such that $\|Tw\| \leq \mu \|w\|$ for all $w \in W$ and $\dim W = \dim V + 1$. If that dimension is not yet equal to k, simply repeat the process a finite number of times.

Let $\operatorname{Eig}(\mu) = \{x \in \mathbb{C}^n \mid Tx = \mu x\}$ be the eigenspace of μ. (This will often, but not always, be $\{0\}$.)

Case 1: $\operatorname{Eig}(\mu) \not\subseteq V$. Define $W = V + \operatorname{Eig}(\mu)$. Then any vector $v + x$, where $v \in V$ and $x \in \operatorname{Eig}(\mu)$, fulfills

$$\begin{aligned}
\|T(v + x)\|^2 &= \|Tv\|^2 + 2\operatorname{Re}\langle Tv, Tx\rangle + \|Tx\|^2 \\
&= \|Tv\|^2 + 2\operatorname{Re}\langle v, T^2 x\rangle + \|Tx\|^2 \\
&= \|Tv\|^2 + 2\mu^2 \operatorname{Re}\langle v, x\rangle + \mu^2 \|x\|^2 \\
&\leq \mu^2 \|v\|^2 + 2\mu^2 \operatorname{Re}\langle v, x\rangle + \mu^2 \|x\|^2 \\
&= \mu^2 \|v + x\|^2.
\end{aligned}$$

Case 2: $\operatorname{Eig}(\mu) \subseteq V$. (This includes the case $|\lambda_k| < \mu$, when $\operatorname{Eig}(\mu) = \{0\}$.) Define

$$B = \sqrt{|T^2 - \mu^2 \mathrm{id}|}\colon \mathbb{C}^n \to \mathbb{C}^n.$$

(That is, $Be_i = \sqrt{|\lambda_i^2 - \mu^2|} \cdot e_i$ for all $1 \leq i \leq n$.) Let $P\colon \mathbb{C}^n \to \mathbb{C}^n$ be the orthogonal projection to $\operatorname{span}(e_1, \dots, e_k)$. Note that P and B commute and that $\ker(B) = \operatorname{Eig}(\mu) = \operatorname{span}\{e_i \mid |\lambda_i| = \mu\} \subseteq \operatorname{im}(P)$. Then

$$\dim(PBV) \leq \dim(BV) = \dim(V) - \dim \operatorname{Eig}(\mu) < \operatorname{rank}(P) - \dim \operatorname{Eig}(\mu),$$

and therefore, there is a nonzero vector

$$y_0 \in \operatorname{im}(P) \cap (PBV)^{\perp} \cap \operatorname{Eig}(\mu)^{\perp}.$$

Since $B = B^*$ and everything is finite-dimensional, we have $\operatorname{Eig}(\mu)^{\perp} = \ker(B)^{\perp} = \operatorname{im}(B)$, so there is a pre-image $x_0 = B^{-1}y_0$. Now define $W = V \oplus \mathbb{C}x_0$.

We have

$$T^2 - \mu^2 \mathrm{id} = -\left|T^2 - \mu^2 \mathrm{id}\right| P + \left|T^2 - \mu^2 \mathrm{id}\right| (\mathrm{id} - P) = -B^2 P + B^2(\mathrm{id} - P)$$

and so, for any vector $z \in \mathbb{C}^n$,

$$\begin{aligned} \|Tz\|^2 - \mu^2 \|z\|^2 &= \left\langle T^2 z, z \right\rangle - \left\langle \mu^2 z, z \right\rangle \\ &= \left\langle (T^2 - \mu^2 \mathrm{id}) z, z \right\rangle \\ &= -\left\langle B^2 P z, z \right\rangle + \left\langle B^2 (\mathrm{id} - P) z, z \right\rangle \\ &= -\|PBz\|^2 + \|(\mathrm{id} - P)Bz\|^2 . \end{aligned}$$

Therefore, for $v \in V$ and $x \in \mathbb{C}x_0$, we obtain

$$\begin{aligned} \|T(v+x)\|^2 - \mu^2 \|v+x\|^2 &= -\|PBv + PBx\|^2 + \|(\mathrm{id} - P)Bv + (\mathrm{id} - P)Bx\|^2 \\ &= -\|PBv\|^2 - \|Bx\|^2 + \|(\mathrm{id} - P)Bv\|^2 \\ &\leq -\|PBv\|^2 + \|(\mathrm{id} - P)Bv\|^2 \\ &= \|Tv\|^2 - \mu^2 \|v\|^2 \\ &\leq 0, \end{aligned}$$

where we used that $PBx = Bx \perp PBv$ and $(\mathrm{id} - P)Bx = 0$. Thus, we obtain indeed $\|T(v+x)\| \leq \mu \|v + x\|$. $\square$

3.8 Definition. Let $\mathcal{H}$ be an (at most countably infinite-dimensional) Hilbert space, $\mathcal{J} \subseteq \mathcal{H}$ a closed subspace, $T \in \mathcal{B}(\mathcal{H})$ and $\lambda \geq 0$. Then define

$$\mathcal{L}(T, \lambda, \mathcal{J}) = \{V \subseteq \mathcal{J} \text{ closed subspace} \mid \|Tv\| \leq \lambda \|v\| \text{ for all } v \in V\}.$$

A subspace $W' \in \mathcal{L}(T, \lambda, \mathcal{J})$ is called *of maximal dimension* if

$$\dim_{\mathbb{C}} W' = \max \{\dim_{\mathbb{C}} W \mid W \in \mathcal{L}(T, \lambda, \mathcal{J})\}.$$

Note: With this notation, the spectral density functions from Theorem 3.5 can be written as $F_m(\lambda) = \frac{1}{|\mathcal{E}_j K_m|} \cdot \max\{\dim_{\mathbb{C}} W \mid W \in \mathcal{L}(T_m, \lambda, \ell^2(\mathcal{E}_j K_m))\}$.

3.9 Lemma. *Let $\mathcal{H}$ be an (at most countably infinite-dimensional) Hilbert space, $\mathcal{J} \subseteq \mathcal{H}$ a finite-dimensional subspace, $T \in \mathcal{B}(\mathcal{H})$ self-adjoint and $\lambda \geq 0$. Let $V \in \mathcal{L}(T, \lambda, \mathcal{J})$. Then there is a subspace $W \in \mathcal{L}(T, \lambda, \mathcal{J})$ of maximal dimension such that $V \subseteq W$.*

Proof. Let $S\colon \mathcal{H} \to \mathcal{H}$ be a unitary operator such that $ST\mathcal{J} \subseteq \mathcal{J}$. (This exists because $\dim(T\mathcal{J}) \leq \dim(\mathcal{J})$.) Then consider the restriction $ST|_{\mathcal{J}}$ as an operator $\mathcal{J} \to \mathcal{J}$ and let $ST|_{\mathcal{J}} = UA$ be its polar decomposition, that is, $U\colon \mathcal{J} \to \mathcal{J}$ is unitary and $A\colon \mathcal{J} \to \mathcal{J}$ is positive. By construction, every $x \in \mathcal{J}$ fulfills

$$\|Ax\| = \|U^* STx\| = \|STx\| = \|Tx\| .$$

Therefore, $\mathcal{L}(T, \lambda, \mathcal{J}) = \mathcal{L}(A, \lambda, \mathcal{J})$, and A is self-adjoint. Now the claim follows from Lemma 3.7. □

3.10 Lemma. *Let $T \in \mathcal{B}(\mathbb{C}^k)$ be positive and $\lambda \geq 0$. Let $W \subseteq \mathbb{C}^k$ and $V \in \mathcal{L}(T, \lambda, W)$ be maximal. For $n \in \mathbb{N}$ define*

$$\begin{aligned} T^{\oplus n} &= T \oplus T \oplus \ldots \oplus T\colon \mathbb{C}^{kn} \to \mathbb{C}^{kn}, \\ V^{\oplus n} &= V \oplus V \oplus \ldots \oplus V, \\ W^{\oplus n} &= W \oplus W \oplus \ldots \oplus W. \end{aligned}$$

Then $V^{\oplus n}$ is a maximal element of $\mathcal{L}\big(T^{\oplus n}, \lambda, W^{\oplus n}\big)$.

Proof. To see that $V^{\oplus n} \in \mathcal{L}\big(T^{\oplus n}, \lambda, W^{\oplus n}\big)$, write $v \in V^{\oplus n}$ as $v = \bigoplus_{i=1}^n v_i$ with $v_i \in V$. Then

$$\left\|T^{\oplus n} v\right\|^2 = \left\|\bigoplus_{i=1}^n T v_i\right\|^2 = \sum_{i=1}^n \|T v_i\|^2 \leq \sum_{i=1}^n \lambda^2 \|v_i\|^2 = \lambda^2 \left\|\bigoplus_{i=1}^n v_i\right\|^2 = \lambda^2 \|v\|^2 .$$

As in the proof of Lemma 3.9, there is a self-adjoint operator $A\colon W \to W$ such that $\|Aw\| = \|Tw\|$ for every $w \in W$, and thus $A^{\oplus n}\colon W^{\oplus n} \to W^{\oplus n}$ fulfills $\|A^{\oplus n} w\| = \|T^{\oplus n} w\|$ for every $w \in W^{\oplus n}$.

Let k be the number of eigenvalues of A (counted with multiplicities) that have absolute value $\leq \lambda$. Clearly, $A^{\oplus n}$ has nk such eigenvalues. Thus, by Lemma 3.7, every maximal element of $\mathcal{L}(T, \lambda, W)$ is k-dimensional and every maximal element of $\mathcal{L}(T, \lambda, W^{\oplus n})$ is nk-dimensional.

Finally, V is maximal in $\mathcal{L}(T, \lambda, W)$, so $\dim V = k$, thus $\dim V^{\oplus n} = nk$, and thus $V^{\oplus n}$ is maximal in $\mathcal{L}(T, \lambda, W^{\oplus n})$. □

Proof of Theorem 3.5. Fix $m \in \mathbb{N}$ large enough that $\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} \leq \frac{1}{2}$. Decompose $\mathcal{E}_j K_m$ into an interior part and a part close to the frontier:

$$\begin{aligned} \mathcal{I}_j^{r+1} K_m &= \{\sigma \in \mathcal{E}_j K_m \,|\, d_{\mathrm{comb}}(\sigma, X \setminus K_m) > r+1\} , \\ \mathcal{F}_j^{r+1} K_m &= \{\sigma \in \mathcal{E}_j K_m \,|\, d_{\mathrm{comb}}(\sigma, X \setminus K_m) \leq r+1\} . \end{aligned}$$

(Especially, $\mathcal{F}_j^1 K_m = \mathcal{F}_j K_m$.) We obviously obtain

$$\ell^2(\mathcal{E}_j K_m) = \ell^2(\mathcal{I}_j^{r+1} K_m) \oplus \ell^2(\mathcal{F}_j^{r+1} K_m)$$

and

$$T_m|_{\ell^2(\mathcal{I}_j^{r+1}K_m)} = T|_{\ell^2(\mathcal{I}_j^{r+1}K_m)}.$$

Choose a subspace

$$V_m(\lambda) \in \mathcal{L}(T_m, \lambda, \ell^2(\mathcal{I}_j^{r+1}K_m)) = \mathcal{L}(T, \lambda, \ell^2(\mathcal{I}_j^{r+1}K_m))$$

of maximal dimension. Clearly, $\mathcal{L}(T_m, \lambda, \ell^2(\mathcal{I}_j^{r+1}K_m)) \subseteq \mathcal{L}\big(T_m, \lambda, \ell^2(\mathcal{E}_j K_m)\big)$. Thus, by Lemma 3.9, there is a subspace

$$W_m(\lambda) \in \mathcal{L}\big(T_m, \lambda, \ell^2(\mathcal{E}_j K_m)\big)$$

of maximal dimension such that

$$V_m(\lambda) \subseteq W_m(\lambda).$$

By definition, we get

$$F_m(\lambda) = \frac{\dim(W_m(\lambda))}{|\mathcal{E}_j K_m|}.$$

Since $W_m(\lambda) \cap \ell^2(\mathcal{I}_j^{r+1}K_m)$ is an element of $\mathcal{L}(T_m, \lambda, \ell^2(\mathcal{I}_j^{r+1}K_m))$ and contains $V_m(\lambda)$, we must have $W_m(\lambda) \cap \ell^2(\mathcal{I}_j^{r+1}K_m) = V_m(\lambda)$ (otherwise $V_m(\lambda)$ wouldn't be of maximal dimension). By Lemma 3.6, this implies

$$\dim(V_m(\lambda)) \geq \dim(W_m(\lambda)) - \dim\big(\ell^2(\mathcal{F}_j^{r+1}K_m)\big)\,,$$

and therefore

$$\left| F_m(\lambda) - \frac{\dim(V_m(\lambda))}{|\mathcal{E}_j K_m|} \right| \leq \frac{\dim(\ell^2(\mathcal{F}_j^{r+1}K_m))}{|\mathcal{E}_j K_m|} \leq C_r \, \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|},$$

where again $C_r = \max_{\sigma \in \mathcal{E}_j X} |B_{r+1}(\sigma)|$. Since the exhaustion is amenable, this error term will go to zero for large m; note that it depends neither on λ nor on the choice of T_m. On the other hand, on the "interior part" $\ell^2(\mathcal{I}_j^{r+1}K_m)$, T and T_m coincide, so $\dim(V_m(\lambda))$ does not depend on the choice of T_m.

Now let $n \geq m$. Then there is a set $\mathcal{G}(m,n)$ of local isomorphisms such that

$$K_n \;=\; \bigcup_{\gamma \in \mathcal{G}(m,n)} \gamma K_m.$$

Decompose $\mathcal{E}_j K_n$ into "images of interiors" and the rest:

$$\mathcal{I}(m,n)_j^{r+1} K_n := \bigsqcup_{\gamma \in \mathcal{G}(m,n)} \mathcal{I}_j^{r+1}(\gamma K_m),$$
$$\mathcal{F}(m,n)_j^{r+1} K_n := \bigcup_{\gamma \in \mathcal{G}(m,n)} \mathcal{F}_j^{r+1}(\gamma K_m)$$

Note that the first union is disjoint (while the second one is not). We get

$$\ell^2(\mathcal{E}_j K_n) = \ell^2\big(\mathcal{I}(m,n)_j^{r+1} K_n\big) \oplus \ell^2\big(\mathcal{F}(m,n)_j^{r+1} K_n\big)$$

and

$$\begin{aligned}\dim(\ell^2(\mathcal{I}(m,n)_j^{r+1}K_n)) &= |\mathcal{G}(m,n)| \cdot \dim(\ell^2(\mathcal{I}_j^{r+1}K_m)),\\ \dim(\ell^2(\mathcal{F}(m,n)_j^{r+1}K_n)) &\le |\mathcal{G}(m,n)| \cdot \dim(\ell^2(\mathcal{F}_j^{r+1}K_m))\\ &\le |\mathcal{G}(m,n)| \cdot C_{r+1} \cdot |\mathcal{F}_j K_m|\,.\end{aligned}$$

We also have

$$\ell^2\big(\mathcal{I}(m,n)_j^{r+1}K_n\big) \subseteq \ell^2\big(\mathcal{I}_j^{r+1}K_n\big)$$

and thus

$$T_n|_{\ell^2(\mathcal{I}(m,n)_j^{r+1}K_n)} = T|_{\ell^2(\mathcal{I}(m,n)_j^{r+1}K_n)}.$$

(Proof of this fact: It suffices to show that $\mathcal{F}_j K_n \subseteq \bigcup_\gamma \gamma(\mathcal{F}_j K_m)$. So assume $\sigma \in \mathcal{F}_j K_n$. Then there is $\gamma \in \mathcal{G}(n)$ such that $d(\gamma(\sigma), X \setminus \gamma K_n) = 1$. On the other hand, there must be $\sigma' \in \mathcal{E}_j K_m$ and $\gamma' \in \mathcal{G}(m,n)$ such that $\sigma = \gamma'(\sigma')$. Thus, $\gamma \circ \gamma' \in \mathcal{G}(m)$ and $d(\gamma\circ\gamma'(\sigma'), X \setminus (\gamma\circ\gamma')K_m) = 1$ (because $\gamma' K_m \subseteq K_n$), so $\sigma' \in \mathcal{F}_j K_m$, so $\sigma \in \gamma'(\mathcal{F}_j K_m)$.)

Define

$$V_{m,n}(\lambda) = \bigoplus_{\gamma \in \mathcal{G}(m,n)} \gamma(V_m(\lambda))\,.$$

By Lemma 3.10, $V_{m,n}(\lambda)$ is indeed a maximal element of $\mathcal{L}\big(T, \lambda, \ell^2\big(\mathcal{I}(m,n)_j^{r+1}K_n\big)\big)$.

We have

$$\mathcal{L}(T,\lambda,\ell^2\big(\mathcal{I}(m,n)_j^{r+1}K_n\big)) = \mathcal{L}(T_n,\lambda,\ell^2\big(\mathcal{I}(m,n)_j^{r+1}K_n\big)) \subseteq \mathcal{L}(T_n,\lambda,\ell^2(\mathcal{E}_j K_n)),$$

so by Lemma 3.9 there is $W_n(\lambda) \in \mathcal{L}(T_n, \lambda, \ell^2(\mathcal{E}_j K_n))$ of maximal dimension such that $W_n(\lambda) \supseteq V_{m,n}(\lambda)$, and

$$F_n(\lambda) = \frac{\dim(W_n(\lambda))}{|\mathcal{E}_j K_n|}.$$

As above, we have $W_n(\lambda) \cap \ell^2\big(\mathcal{I}(m,n)_j^{r+1}K_n\big) = V_{m,n}(\lambda)$ and thus, by Lemma 3.6,

$$\dim(W_n(\lambda)) - \dim(V_{m,n}(\lambda)) \le \dim(\ell^2\big(\mathcal{F}(m,n)_j^{r+1}K_n\big)).$$

Since $K_n = \bigcup_{\gamma \in \mathcal{G}(m,n)} \gamma K_m$ and the different copies of K_m overlap only at their frontiers, we have

$$|\mathcal{E}_j K_n| \ge |\mathcal{G}(m,n)| \cdot (|\mathcal{E}_j K_m| - |\mathcal{F}_j K_m|) \ge \frac{1}{2}\,|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|$$

(since m is large enough to fulfill $\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} \le \frac{1}{2}$). This gives

$$\begin{aligned}\left| F_n(\lambda) - \frac{\dim(V_{m,n}(\lambda))}{|\mathcal{E}_j K_n|}\right| &\le \frac{\dim(\ell^2\big(\mathcal{F}(m,n)_j^{r+1}K_n\big))}{|\mathcal{E}_j K_n|}\\ &\le \frac{|\mathcal{G}(m,n)| \cdot C_{r+1} \cdot |\mathcal{F}_j K_m|}{\frac{1}{2}\,|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|}\\ &\le 2\,C_{r+1}\,\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|}.\end{aligned}$$

Again, the term $\dim(V_{m,n}(\lambda))$ does not depend on the choice of T_n, and the error term does not depend on λ.

We can also express the denominator in terms of $|\mathcal{E}_j K_m|$:

$$\left|\frac{1}{|\mathcal{E}_j K_n|} - \frac{1}{|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|}\right| = \frac{|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m| - |\mathcal{E}_j K_n|}{|\mathcal{E}_j K_n| \cdot |\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|}$$

$$\leq \frac{|\mathcal{G}(m,n)| \cdot |\mathcal{F}_j K_m|}{|\mathcal{E}_j K_n| \cdot |\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|} = \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_n| \cdot |\mathcal{E}_j K_m|}$$

We certainly have $\dim(V_{m,n}(\lambda)) \leq |\mathcal{E}_j K_n|$, so this gives

$$\left|\frac{\dim(V_{m,n}(\lambda))}{|\mathcal{E}_j K_n|} - \frac{\dim(V_{m,n}(\lambda))}{|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|}\right| \leq \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|}.$$

It remains to compute

$$\left|\frac{\dim(V_m(\lambda))}{|\mathcal{E}_j K_m|} - \frac{\dim(V_{m,n}(\lambda))}{|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|}\right|.$$

But this term equals zero, since $V_{m,n}(\lambda) = \bigoplus_\gamma \gamma(V_m(\lambda))$!
Finally, we can combine all the estimates to obtain

$$\begin{aligned}
|F_m(\lambda) - F_n(\lambda)| &\leq \left|F_m(\lambda) - \frac{\dim(V_m(\lambda))}{|\mathcal{E}_j K_m|}\right| \\
&\quad + \left|\frac{\dim(V_m(\lambda))}{|\mathcal{E}_j K_m|} - \frac{\dim(V_{m,n}(\lambda))}{|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|}\right| \\
&\quad + \left|\frac{\dim(V_{m,n}(\lambda))}{|\mathcal{G}(m,n)| \cdot |\mathcal{E}_j K_m|} - \frac{\dim(V_{m,n}(\lambda))}{|\mathcal{E}_j K_n|}\right| \\
&\quad + \left|\frac{\dim(V_{m,n}(\lambda))}{|\mathcal{E}_j K_n|} - F_n(\lambda)\right| \\
&\leq C_{r+1} \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} + 0 + \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} + 2\, C_{r+1} \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} \\
&\leq (3\, C_{r+1} + 1) \cdot \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} \xrightarrow{m \to \infty} 0.
\end{aligned}$$

The error in the end depends neither on λ nor on n, so the sequence $(F_m)_{m \in \mathbb{N}}$ is uniformly Cauchy, and thus uniformly convergent. □

Now, we come to the second part of the main result: The limit of the spectral density functions F_m of the "restricted" operators T_m indeed coincides with the spectral density function of T defined in 2.38.

3.11 Theorem. *Let $X, (K_m)$ and $T \in \mathcal{A}_j^{geo}(X)$, $T_m \in \mathcal{B}(\ell^2 K_m)$ be as in Theorem 3.5, and denote again by F and F_m the spectral density functions of T respectively T_m. Then the sequence (F_m) converges uniformly to F.*

3.12 Definition. To simplify notation, write

$$\operatorname{tr}_m(T) := \frac{1}{|\mathcal{E}_j K_m|} \sum_{\sigma \in \mathcal{E}_j K_m} \langle \sigma, T\sigma \rangle .$$

as long as the exhaustion (K_m) is fixed. This makes sense both for "global" operators $T \in \mathcal{B}(\ell^2 X)$ and for "local" operators $T_m \in \mathcal{B}(\ell^2 K_m)$.

3.13 Lemma. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and T, T_m as in 3.5. Then*

$$\operatorname{tr}_{\mathcal{N}} f(T) = \lim_{m\to\infty} \operatorname{tr}_m \left(f(T_m) \right).$$

(Here, $f(T)$ is intended as functional calculus in $\mathcal{A}_j(X)$, and $f(T_m)$ as functional calculus in $\mathcal{B}(\ell^2(\mathcal{E}_j K_m))$.)

Proof. As f is continuous, we have indeed $f(T) \in \mathcal{A}_j(X)$, so the trace can be computed by $\operatorname{tr}_{\mathcal{N}} f(T) = \lim_{m\to\infty} \operatorname{tr}_m\bigl(f(T)\bigr)$. If f is a polynomial, then $f(T_m)\sigma = f(T)\sigma$ for every $\sigma \in \mathcal{I}_j^{nr} K_m$, where r is the propagation of T and n is the degree of f. As the nr-neighborhood of the frontier is negligible for the trace, the claim follows.

For general f, note that (by assumption) the norms $\{\|T_m\|_{\mathrm{op}} \mid m \in \mathbb{N}\}$ are bounded, and T and all T_m are positive. Thus, there is $D > 0$ such that $[0, D]$ contains all spectra of T and T_m, and by the Weierstrass approximation theorem there is a polynomial p with $|f(\lambda) - p(\lambda)| < \varepsilon$ for all $\lambda \in [0, D]$.

Then, using that both $\operatorname{tr}_{\mathcal{N}}$ and tr_m have norm one, we obtain

$$\begin{aligned}
\left|\operatorname{tr}_{\mathcal{N}} f(T) - \lim_{m\to\infty} \operatorname{tr}_m \left(f(T_m)\right)\right| &\le \left|\operatorname{tr}_{\mathcal{N}} f(T) - \operatorname{tr}_{\mathcal{N}} p(T)\right| \\
&\quad + \left|\operatorname{tr}_{\mathcal{N}} p(T) - \lim_{m\to\infty} \operatorname{tr}_m \left(p(T_m)\right)\right| \\
&\quad + \left|\lim_{m\to\infty} \operatorname{tr}_m \left(p(T_m)\right) - \lim_{m\to\infty} \operatorname{tr}_m \left(f(T_m)\right)\right| \\
&\le \varepsilon + 0 + \varepsilon.
\end{aligned}$$

As ε was arbitrary, the claim follows. ∎

3.14 Lemma. *For all $\delta > 0$ and $\lambda \in \mathbb{R}$ define the continuous function*

$$f_{\lambda,\delta}(x) = \begin{cases} 1, & x \in (-\infty, \lambda], \\ 1 - \frac{x-\lambda}{\delta}, & x \in [\lambda, \lambda+\delta], \\ 0, & x \in [\lambda+\delta, \infty). \end{cases}$$

Then for every $\varepsilon > 0$ there is $\delta > 0$ (independent of m) such that

$$|F(\lambda) - \operatorname{tr}_{\mathcal{N}} f_{\lambda,\delta}(T)| < \varepsilon \quad \text{and} \quad |F_m(\lambda) - \operatorname{tr}_m f_{\lambda,\delta}(T_m)| < \varepsilon$$

for all $m \in \mathbb{N}$.

Proof. First, note that

$$\operatorname{tr}_{\mathcal{N}} f_{\lambda,\delta}(T) = \int_{\mathbb{R}} f_{\lambda,\delta}(x)\, dF(x),$$
$$\operatorname{tr}_m f_{\lambda,\delta}(T_m) = \int_{\mathbb{R}} f_{\lambda,\delta}(x)\, dF_m(x).$$

Note that it suffices to let the integrals run over $\left[0, \|T\|_{\mathrm{op}}\right]$.

From now on, focus on the first integral, the second one actually works the same.

The Lebesgue–Stieltjes integral fulfills the following version of integration by parts (due to Hewitt, [Hew60]): If in each point either f or g is continuous, then

$$\int_a^b f\, dg + \int_a^b g\, df = f(b+)g(b+) - f(a-)g(a-),$$

where $f(b+) = \lim_{x\to b+} f(x)$, etc. In our case, $f_{\lambda,\delta}$ is everywhere continuous, so the condition is fulfilled. With $a = 0$ and $b = \|T\|_{\mathrm{op}}$ we have

$$f_{\lambda,\delta}(b+)F(b+) - f_{\lambda,\delta}(a-)F(a-) = 0 \cdot 1 - 1 \cdot 0 = 0.$$

Thus,

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}} f_{\lambda,\delta}(T) &= \int_0^b f_{\lambda,\delta}(x)\, dF(x) = -\int_0^b F(x)\, df_{\lambda,\delta}(x) \\ &= -\int_0^b F(x)\, \frac{df_{\lambda,\delta}}{dx}\, dx = \frac{1}{\delta}\int_\lambda^{\lambda+\delta} F(x)\, dx.\end{aligned}$$

(Note: While $f_{\lambda,\delta}$ is only almost everywhere differentiable, it is everywhere continuous, and thus, no single point has positive measure in $df_{\lambda,\delta}$, and we can leave out the two non-differentiable points λ and $\lambda + \delta$. On the rest of the interval, $df = \frac{df}{dx}\, dx$ holds.) The same argument applied to F_m yields

$$\operatorname{tr}_m f_{\lambda,\delta}(T_m) = \frac{1}{\delta}\int_\lambda^{\lambda+\delta} F_m(x)\, dx.$$

Every spectral density function is right-continuous and non-decreasing. Thus, there is $\delta_\infty > 0$ such that $|F(\lambda) - F(x)| < \varepsilon$ for all $x \in [\lambda, \lambda + \delta_\infty]$.

By Theorem 3.5, the functions F_m are uniformly convergent to some function F_{lim}, so there is M such that $|F_m(x) - F_{\mathrm{lim}}(x)| < \varepsilon/3$ for all $m \geq M$. As a limit of right-continuous functions, F_{lim} is right-continuous. Thus, there is δ_{lim} such that $|F_{\mathrm{lim}}(\lambda) - F_{\mathrm{lim}}(x)| < \varepsilon/3$ for all $x \in [\lambda, \lambda + \delta_{\mathrm{lim}}]$, and that implies

$$\begin{aligned}&|F_m(\lambda) - F_m(x)| \\ &\qquad < |F_m(\lambda) - F_{\mathrm{lim}}(\lambda)| + |F_{\mathrm{lim}}(\lambda) - F_{\mathrm{lim}}(x)| + |F_{\mathrm{lim}}(x) - F_m(x)| < \varepsilon\end{aligned}$$

for all $m \geq M$ and all $x \in [\lambda, \lambda + \delta_{\mathrm{lim}}]$.

Finally, for each of the functions $F_1, F_2, \ldots, F_{M-1}$, there is a $\delta_m > 0$ such that $|F_m(\lambda) - F_m(x)| < \varepsilon$ for all $x \in [\lambda, \lambda + \delta_m]$.

Setting $\delta = \min(\delta_1, \ldots, \delta_{M-1}, \delta_{\lim}, \delta_\infty)$, we get indeed

$$|F(\lambda) - \operatorname{tr}_{\mathcal{N}} f_{\lambda,\delta}(T)| < \varepsilon \quad \text{and} \quad |F_m(\lambda) - \operatorname{tr}_m f_{\lambda,\delta}(T_m)| < \varepsilon$$

for all $m \in \mathbb{N}$ and all $x \in [\lambda, \lambda + \delta]$. □

Proof of Theorem 3.11. First show pointwise convergence.

Fix $\varepsilon > 0$. By Lemma 3.14, there are $\delta > 0$ and a continuous function $f_{\lambda,\delta}$ such that

$$|F(\lambda) - \operatorname{tr}_{\mathcal{N}} f_{\lambda,\delta}(T)| < \frac{\varepsilon}{3} \quad \text{and} \quad |F_m(\lambda) - \operatorname{tr}_m f_{\lambda,\delta}(T_m)| < \frac{\varepsilon}{3}$$

for all $m \in \mathbb{N}$.

By Lemma 3.13, the function $f_{\lambda,\delta}$ satisfies

$$\operatorname{tr}_{\mathcal{N}} f_{\lambda,\delta}(T) = \lim_{m\to\infty} \operatorname{tr}_m \left(f_{\lambda,\delta}(T_m)\right),$$

so there is $M \in \mathbb{N}$ such that for every $m \geq M$,

$$\left|\operatorname{tr}_{\mathcal{N}} f_{\lambda,\delta}(T) - \operatorname{tr}_m \left(f_{\lambda,\delta}(T_m)\right)\right| < \frac{\varepsilon}{3}.$$

Thus, for every $m \geq M$,

$$|F(\lambda) - F_m(\lambda)| < \varepsilon.$$

This proves pointwise convergence, and thus, $F(\lambda) = F_{\lim}(\lambda)$ for all $\lambda \in \mathbb{R}$. As we already know that $F_m \to F_{\lim}$ uniformly, uniform convergence follows. □

3.3 Different normalizations

From an algebraical point of view, it is natural to normalize the trace on $\mathcal{B}\big(\ell^2(\mathcal{E}_j K_m)\big)$ by the dimension of that space, that is, by $|\mathcal{E}_j K_m|$. (This turns the normalized trace into a state, that is, a linear functional of norm one. In particular, it leads to $\dim_{\mathcal{N}}(\mathcal{H}_j) = \operatorname{tr}_{\mathcal{N}}(\mathrm{id}) = 1$.) Topologically, however, this is not the most useful normalization:

First, it causes problems when we want to compare L^2-invariants of different dimensions, e. g. when computing Euler characteristics. For this, it is desirable to divide all traces on $\mathcal{B}\big(\ell^2(\mathcal{E}_j K_m)\big)$ (for the various values of j) by the same number.

Second, it is unsuitable for comparing different self-similar CW-complexes. For example, different CW-structures on the same complex will often result in different numbers of cells (in any dimension), but they should not change the topological invariants.

Third, it proves to be unfortunate for product spaces, where we need the normalizations on X, Y and $X \times Y$ to be compatible, but $|\mathcal{E}_j K_m| \cdot |\mathcal{E}_j L_m| \neq |\mathcal{E}_j(K_m \times L_m)|$ for any $j \geq 1$.

There are several possibilities for this normalization factor – one might choose the number of vertices, or of top-dimensional cells, or even the number of local isomorphisms in $\mathcal{G}(1,m)$ (that is, how many copies of K_1 are needed to "build" K_m). To show that all of these normalizations are indeed equivalent (in so far as they only produce a constant prefactor to the trace), let us show that their ratios converge:

3.15 Lemma. (a) *For every $j \in \mathbb{N}_0$, the limit* $\lim_{m\to\infty} \dfrac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|}$ *exists.*

(b) *If $\mathcal{E}_j X \neq \emptyset$, then* $\lim_{m\to\infty} \dfrac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|} > 0$.

Proof. (a) We have $K_{m+1} = \bigcup_{\gamma\in\mathcal{G}(m,m+1)} \gamma K_m$ and therefore $|\mathcal{E}_j K_{m+1}| \leq |\mathcal{G}(m,m+1)| \cdot |\mathcal{E}_j K_m|$, while $|\mathcal{G}(1,m+1)| = |\mathcal{G}(1,m)| \cdot |\mathcal{G}(m,m+1)|$. Thus,

$$\frac{|\mathcal{E}_j K_{m+1}|}{|\mathcal{G}(1,m+1)|} \leq \frac{|\mathcal{G}(m,m+1)| \cdot |\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)| \cdot |G(m,m+1)|} \leq \frac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|}$$

for all m. Hence, the sequence $\left(\frac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|}\right)_m$ is non-increasing (and bounded from below by 0), so it converges.

(b) If X contains any j-cell, then amenability demands that there is $m_0 \in \mathbb{N}$ such that $|\mathcal{E}_j K_m| > |\mathcal{F}_j K_m|$ for all $m \geq m_0$. Then, we have for all $m \geq m_0$ and thus also for the limit $m \to \infty$,

$$\frac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|} \geq \frac{|\mathcal{G}(m_0,m)| \cdot (|\mathcal{E}_j K_{m_0}| - |\mathcal{F}_j K_{m_0}|)}{|\mathcal{G}(1,m_0)| \cdot |\mathcal{G}(m_0,m)|} \geq \frac{1}{|\mathcal{G}(1,m_0)|} > 0. \qquad \square$$

3.16 Remark. We can define the shorthand

$$\frac{|\mathcal{E}_j X|}{|\mathcal{G}(1,\infty)|} = \lim_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|}.$$

Note, however, that this depends on the choice of (K_m): For example, if a second exhaustion were given by $L_m = K_{m+1}$ with the local isomorphisms $\mathcal{G}'(m,n) = \mathcal{G}(m+1,n+1)$, then we would obtain

$$\lim_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{G}'(1,m)|} = \lim_{m\to\infty} \frac{|\mathcal{E}_j K_{m+1}|}{|\mathcal{G}(2,m+1)|} = |\mathcal{G}(1,2)| \cdot \lim_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|}.$$

In contrast, the ratio of cell numbers is an intrinsic property of X itself:

3.17 Definition and Lemma. For any $j, k \in \mathbb{N}_0$ such that $\mathcal{E}_k X \neq \emptyset$, the limit

$$\frac{|\mathcal{E}_j X|}{|\mathcal{E}_k X|} := \lim_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_k K_m|}$$

exists and is independent of the exhaustion (K_m). If $\mathcal{E}_j X \neq \emptyset$, the limit is positive.

Proof. The existence follows directly from Lemma 3.15; it remains to show the independence of (K_m). A general proof is given in [Suc16] (Lemma 4.12); as an illustration, let us give the (much shorter) proof for the simplicial case here:

Consider specifically $k = 0$ and define $T \in \mathcal{A}_0^{\mathrm{geo}}(X)$ as

$$T\rho = \frac{|\{\sigma \in \mathcal{E}_j X \mid \rho \in \sigma\}|}{j+1}\,\rho.$$

As every j-simplex contains exactly $j+1$ vertices, this immediately yields

$$\begin{aligned}
\operatorname{tr}_m T \;&=\; \frac{1}{|\mathcal{E}_0 K_m|} \sum_{\rho \in \mathcal{E}_0 K_m} \frac{1}{j+1} \sum_{\substack{\sigma \in \mathcal{E}_j K_m \\ \rho \in \sigma}} 1 \;=\; \frac{1}{|\mathcal{E}_0 K_m|} \sum_{\sigma \in \mathcal{E}_j K_m} \frac{1}{j+1} \sum_{\substack{\rho \in \mathcal{E}_0 K_m \\ \rho \in \sigma}} 1 \\
&=\; \frac{1}{|\mathcal{E}_0 K_m|} \sum_{\sigma \in \mathcal{E}_j K_m} 1 \;=\; \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_0 K_m|}.
\end{aligned}$$

Now the claim follows from Theorem 3.4, as $\operatorname{tr}_{\mathcal{A}}(T) = \lim_{m\to\infty} \operatorname{tr}_{K_m}(T)$ is independent of the choice of (K_m). Finally, for $k > 0$, simply use

$$\lim_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_k K_m|} = \frac{\lim\limits_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_0 K_m|}}{\lim\limits_{m\to\infty} \frac{|\mathcal{E}_k K_m|}{|\mathcal{E}_0 K_m|}}.$$

For the positivity of the limit, combine this result with Lemma 3.15 (b). $\square$

4 L^2-Betti numbers and Novikov–Shubin invariants

Let us now turn towards the actual L^2-invariants that can be derived from the spectral density functions of Laplacians. First among these are the L^2-Betti numbers, which simply measure the sizes of the kernels of the Laplacians, or the amount of "harmonic chains" on the complex. Second, we will consider Novikov–Shubin invariants, which look at the spectrum *near* (but not at) zero; one might say that they measure the amount of "almost harmonic chains" on the complex.[7]

For both of these invariants, we will look at two main questions:

(a) We will justify the name "invariants" by proving that they are indeed invariant under suitable homotopies. This, of course, requires to first construct the notion of a self-similar homotopy compatible with the structure of the spaces.

(b) We will attempt to approximate the invariants by their equivalents on finite subcomplexes. For L^2-Betti numbers, this turns out to be a simple corollary; for Novikov–Shubin invariants the situation is much more complicated.

4.1 Approximation of L^2-Betti numbers

The L^2-Betti numbers of an operator measure the size of its kernel – although, strictly speaking, the "$\ker T$" in the following definition is a subspace of $\mathcal{H}_j$, not of $\ell^2(\mathcal{E}_j X)$, so we do not directly measure how many chains in $\ell^2(\mathcal{E}_j X)$ get mapped to zero!

Nonetheless, L^2-Betti numbers measure a "failure to be injective", and the approximation result below shows that their values are very much correlated with the behavior of the operator on $\ell^2(\mathcal{E}_j X)$.

4.1 Definition. Given a positive operator $T \in \mathcal{N}_j$, the L^2-*Betti number* of T is

$$b^{(2)}(T) = F^T(0) = \dim_{\mathcal{N}}(\ker T).$$

Given a self-similar complex X, define its j-*th* L^2-*Betti number* as

$$b_j^{(2)}(X) = b^{(2)}\big(\Delta_j^{(X)}\big).$$

In this chapter, $\dim_{\mathcal{N}}$ is normalized such that the total space has dimension one. As mentioned in 3.3, it is occasionally useful to use other normalizations; we will use a different one in Chapter 6.

[7] The third main invariant (L^2-torsion and the Fuglede–Kadison determinants necessary to construct it) will be considered in the next chapter.

As L^2-Betti numbers are simply the values of the spectral density functions at zero, their approximation follows directly from the work done in the previous chapter:

4.2 Corollary. *Let X be a self-similar CW-complex with self-similar exhaustion $(K_m)_{m\in\mathbb{N}}$. Denote by*

$$\beta_j(K_m) = \dim_{\mathbb{C}} H_j(K_m;\mathbb{C})$$

the classical Betti numbers of K_m, and by

$$b_j^{(2)}(X) = \dim_{\mathcal{N}}\left(\ker \Delta_j^{(X)}\right)$$

the L^2-Betti numbers of X. Then

$$b_j^{(2)}(X) = \lim_{m\to\infty} \frac{\beta_j(K_m)}{|\mathcal{E}_j K_m|}.$$

Proof. In Theorem 3.11, put $T = \Delta_j^{(X)}$ and $T_m = \Delta_j^{(K_m)}$. Then $b_j^{(2)}(X) = F(0)$ and $\frac{\beta_j(K_m)}{|\mathcal{E}_j K_m|} = F_m(0)$, and convergence follows. $\square$

In the classical case, L^2-Betti numbers over (elementary) amenable groups can often only take certain rational values.[8] For our version of self-similar complexes, we cannot expect such a restriction:

4.3 Example. Given any number $y \in [0,1]$, write $y = \sum_{j=1}^{\infty} y_j 2^{-j}$ with $y_j \in \{0,1\}$, and define complexes $(K_m)_{m=0}^{\infty}$ as follows:

Let $K_0 = [0,1]^2$ be a square with the standard CW-structure. If $y_m = 0$, define K_m as the union of two copies of K_{m-1} that are glued together along an edge. If $y_m = 1$, instead define K_m as the union of two copies of K_{m-1} that are glued together only at the two endpoints of an edge. (See Figure 8.)

With $\beta_1(K_0) = 0$ and $|\mathcal{E}_1 K_0| = 4$, we obtain

$$\beta_1(K_m) = 2\cdot\beta_1(K_{m-1}) + y_m = \sum_{j=1}^{m} 2^{m-j} y_j,$$

$$\begin{aligned}|\mathcal{E}_1 K_m| &= 2\cdot|\mathcal{E}_0 K_{m-1}| + y_m - 1 = 2^m\cdot 4 + \sum_{j=1}^{m} 2^{m-j}(y_j - 1)\\ &= 3\cdot 2^m - 1 + \sum_{j=1}^{m} 2^{m-j} y_j.\end{aligned}$$

Thus, if we set $X = \bigcup_{m=0}^{\infty} K_m$,

$$b_1^{(2)}(X) = \lim_{m\to\infty} \frac{\beta_1(K_m)}{|\mathcal{E}_1 K_m|} = \frac{y}{y+3},$$

and therefore any number between 0 and 1/4 appears as the first L^2-Betti number of such a complex.

[8] This is a special case of *Atiyah's conjecture*, compare [DLM+03].

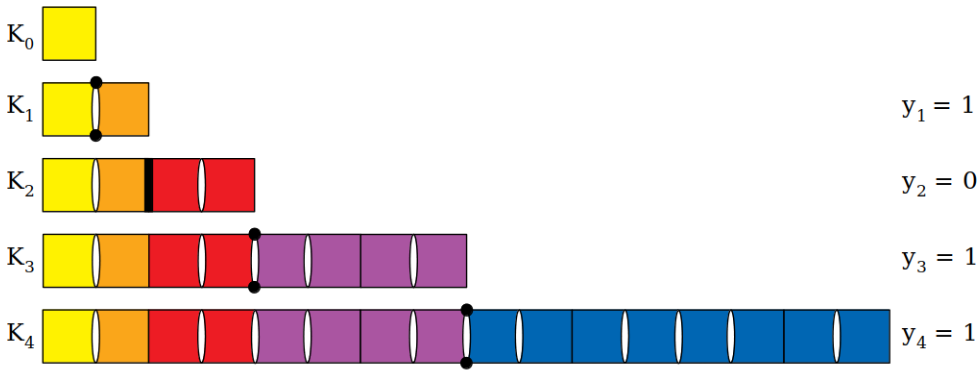

Figure 8: *K_0 to K_4 for the complex of Example 4.3.* In the picture for K_m, the places where the two copies of K_{m-1} are glued together are marked in black: two vertices in K_1, K_3 and K_4, and an entire edge in K_2. Thus, this complex encodes the number $y = (0.1011\ldots)_2$.

4.2 Novikov–Shubin invariants

The Betti numbers of a space measure the size of the kernel of its Laplacian; in a way, they count how many "harmonic" chains of cells exist on that space. For finite complexes, the Laplacian is invertible whenever there are no such harmonic chains, and if that happens in all dimensions greater zero, the chain complex is contractible. (Note that the space itself need not be contractible, as the homology with complex coefficients does not "see" all topological properties. The contractibility of the chain complex is thus more of an algebraic statement than a topological one.)

In infinite complexes, this simple statement is no longer true, as an injective Laplacian is not always invertible. Thus, it becomes necessary to consider "almost harmonic" chains: elements $\Xi \in \mathcal{H}_j(X)$ for which $\Delta_j\Xi$ is not zero, but arbitrarily small. This "almost-kernel" is measured by the Novikov–Shubin invariant, which is defined as the doubly-logarithmic slope of the spectral density function at zero:

4.4 Definition. Given a non-decreasing function $F\colon [0,\infty) \to [0,\infty)$, define its *Novikov–Shubin invariant* by

$$\alpha(F) = \begin{cases} \liminf\limits_{\lambda\to 0^+} \dfrac{\log(F(\lambda) - F(0))}{\log(\lambda)}, & \text{if } F(\lambda) > F(0) \text{ for all } \lambda > 0, \\ \infty^+, & \text{otherwise,} \end{cases}$$

where ∞^+ is understood as a new symbol meaning "more than ∞".

For an operator $T \in \mathcal{N}_j(X)$, define $\alpha(T) = \alpha(F^T)$, where F^T is the spectral density function of T.

For a self-similar complex X, define $\alpha_j(X) = \alpha\big(\Delta_j^{(X)}\big)$. (This is a deviation from the literature standard, where $\alpha_j(X)$ is defined as $\alpha_j(\partial_j^{(X)})$ instead; we prefer using the Laplacian as $\partial_j^{(X)}$ is not a self-adjoint operator.)

Novikov–Shubin invariants can take any value in $[0,\infty]$ as well as the additional value ∞^+, which indicates a *spectral gap* around zero: $\alpha(F)$ is set to ∞^+ whenever F is constant on an interval $[0,\varepsilon)$ for some $\varepsilon > 0$; if F is the spectral density function of some operator T, this implies that $\operatorname{spec}(T) \cap (0,\varepsilon) = \emptyset$. If T is injective, this implies that it is bounded from below by ε, and conversely, every operator that is bounded from below will exhibit this spectral gap. As being bounded from below is equivalent to invertibility, this shows:

4.5 Lemma. *An operator $T \in \mathcal{N}_j(X)$ is invertible if and only if* $\ker(T) = \{0\}$ *and $\alpha(T) = \infty^+$.*

4.6 Remark. A space X is said to have the *limit property* if the limit inferior in 4.4 is a true limit. Lück notes that every G-CW-complex known to him has the limit property ([Lüc02], page 92, Remark 2.42).

It is not hard to write down a *function* that does not fulfill the limit property; for example, let

$$F(0) = 0 \quad \text{and} \quad F(\lambda) = 2^{-k} \text{ if } \lambda \in [2^{-2k}, 2^{-k}) \text{ for } k \in \mathbb{N}.$$

Then clearly

$$\alpha(F) = \liminf_{\lambda \to 0^+} \frac{\log(F(\lambda) - F(0))}{\log(\lambda)} = \frac{1}{2} \quad \text{but} \quad \limsup_{\lambda \to 0^+} \frac{\log(F(\lambda) - F(0))}{\log(\lambda)} = 1.$$

However, it is unclear whether such a counterexample could occur as the spectral density function of a Laplacian of a self-similar complex.

Let us quickly note some obvious properties of Novikov–Shubin invariants of functions. Most of these follow trivially from the definition; compare [Lüc02], p. 77 ff.

4.7 Lemma. *Let $F, G\colon [0,\infty) \to [0,\infty)$ be non-decreasing and $r > 0$. Then (using $r \cdot \infty = \infty$ and $r \cdot \infty^+ = \infty^+$)*

$$\begin{aligned}
\alpha(r \cdot F(\lambda)) &= \alpha(F(\lambda)), \\
\alpha(F(r \cdot \lambda)) &= \alpha(F(\lambda)), \\
\alpha(F(\lambda^r)) &= r \cdot \alpha(F(\lambda)), \\
\alpha(F + G) &= \min\bigl(\alpha(F), \alpha(G)\bigr),
\end{aligned}$$

and, if $F(0) = G(0)$, then

$$F \le G \implies \alpha(F) \ge \alpha(G).$$

4.8 Definition and Lemma. Two non-decreasing functions $F, G\colon [0,\infty) \to [0,\infty)$ are *dilatationally equivalent*, written $F \simeq G$, if there are $\varepsilon, C > 0$ such that

$$F(\lambda) \le G(C\lambda) \quad \text{and} \quad G(\lambda) \le F(C\lambda) \quad \text{for all } \lambda \in [0,\varepsilon).$$

In that case,

$$\alpha(F) = \alpha(G).$$

Proof. This is a direct consequence of Lemma 4.7. $\square$

4.9 Corollary. *Let X be a self-similar complex. For any two operators $S, T \in \mathcal{N}_j(X)$ we have the following properties:*

(a) $\alpha(T^*T) = \alpha(TT^*) = \frac{1}{2}\,\alpha(T^*) = \frac{1}{2}\,\alpha(T)$.

(b) *If S is invertible, then $\alpha(ST) = \alpha(TS) = \alpha(T)$.*

Proof. (a) This follows from Lemmas 2.40 and 4.7.

(b) If S is invertible, there is a constant $C > 0$ such that $C^{-1}\,\|\Xi\| \leq \|S\Xi\| \leq C\,\|\Xi\|$ for all $\Xi \in \mathcal{H}_j$. Therefore, if $V \subseteq \mathcal{H}_j$ is a subspace fulfilling $\|T|_V\| \leq \lambda$, then $\|ST|_V\| \leq C\lambda$, so Lemma 2.42 yields $F(ST)(C\lambda) \geq F(T)(\lambda)$.

Conversely, if $W \subseteq \mathcal{H}_j$ is a subspace fulfilling $\|ST|_W\| \leq C^{-1}\lambda$, then $\|T|_V\| \leq \lambda$, so Lemma 2.42 yields $F(T)(\lambda) \geq F(ST)(C^{-1}\lambda)$.

Now $\alpha(ST) = \alpha(T)$ follows from Lemma 4.8, and since S^* is also invertible, part (a) gives $\alpha(TS) = \alpha((TS)^*) = \alpha(S^*T^*) = \alpha(T^*) = \alpha(T)$. $\square$

Even though we mostly work with the Laplacians Δ_j, we will sometimes need to refer back to the boundary operators ∂_j. To this end, we use the following relation:

4.10 Corollary. *Let X be a self-similar complex, and $\partial_j \colon \ell^2(\mathcal{E}_j X) \to \ell^2(\mathcal{E}_{j-1} X)$ the differential in its L^2-chain complex. Then*

$$\alpha_j(\Delta_j) = \min\left(\frac{\alpha(\partial_j)}{2}, \frac{\alpha(\partial_{j+1})}{2}\right).$$

Proof. Define $\Delta_{j-} = \partial_j^*\partial_j$ and $\Delta_{j+} = \partial_{j+1}\partial_{j+1}^*$, so the Laplacian takes the form $\Delta_j = \Delta_{j-} + \Delta_{j+}$. (Of course, both of the operators $\Delta_{j\pm}$ are geometric.) By definition, $F(\partial_j)(\lambda) = F(\Delta_{j-})(\lambda^2)$, and by Lemma 2.40 we also have $F(\partial_{j+1})(\lambda) = F(\partial_{j+1}^*)(\lambda) = F(\Delta_{j+})(\lambda^2)$. Therefore, Lemma 4.7 gives

$$\alpha(\Delta_{j-}) = \frac{\alpha(\partial_j)}{2} \quad \text{and} \quad \alpha(\Delta_{j+}) = \frac{\alpha(\partial_{j+1})}{2}.$$

It is a well-known fact that on every finite CW-complex, the chain complex can be decomposed into the orthogonal sum

$$\mathbb{C}[\mathcal{E}_j X] = \ker(\Delta_j) \oplus \operatorname{im}(\partial_j^*) \oplus \operatorname{im}(\partial_{j+1}),$$

and that w. r. t. this decomposition, the Laplacian takes the form

$$\Delta_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{j-} & 0 \\ 0 & 0 & \Delta_{j+} \end{pmatrix}.$$

Therefore, the (renormalized) spectral density function on every finite subcomplex $K_m \subseteq X$ fulfills

$$F\big(\Delta_j^{(K_m)}\big) = F\big(\Delta_{j-}^{(K_m)}\big) + F\big(\Delta_{j+}^{(K_m)}\big).$$

By Theorem 3.11, the same holds on X:

$$F\big(\Delta_j^{(X)}\big) = F\big(\Delta_{j-}^{(X)}\big) + F\big(\Delta_{j+}^{(X)}\big).$$

Now the claim follows from Lemma 4.7. $\square$

4.3 Homotopy invariance of L^2-Betti numbers

The name *L^2-invariant* refers to homotopy invariance. However, this cannot mean invariance under *any* homotopy: For example, all of the complexes defined in Example 4.3 (except for the one corresponding to $y = 0$) are homotopy equivalent, yet their L^2-Betti numbers take many different values. Namely, each of these complexes has countably many holes, but their density is different – something that a conventional homotopy cannot account for.

Thus, a homotopy of self-similar complexes must in some way preserve the self-similar structure to have any chance of preserving our version of L^2-invariants. It is important to note that the self-similar structure on a complex is not part of the space itself – rather, a complex is self-similar as soon as there is at least one such structure, and the invariants are independent of its choice. Therefore, it should suffice that a self-similar map is compatible with *some* self-similar structure on each space:

4.11 Definition. Let X and Y be self-similar complexes. A map $f\colon X \to Y$ is a *self-similar map* if there are self-similar structures $\big(K_m, \mathcal{G}(m, m+1)\big)$ on X and $\big(L_m, \mathcal{G}'(m, m+1)\big)$ on Y such that for all m:

(a) $f(K_m) \subseteq L_m$,

(b) there is a bijection $\varphi_m\colon \mathcal{G}(m, m+1) \to \mathcal{G}'(m, m+1)$,

(c) and the map is "equivariant" with respect to these local isomorphisms; that is, the following diagram commutes for each $\gamma \in \mathcal{G}(m, m+1)$:

$$\begin{tikzcd} K_m \arrow[r, "\gamma"] \arrow[d, "f"] & \gamma K_m \arrow[d, "f"] \\ L_m \arrow[r, "\varphi_m(\gamma)"] & \varphi_m(\gamma) L_m \end{tikzcd}$$

A homotopy equivalence requires *four* such maps (one in each direction and a homotopy in each space), and it appears necessary that all of them use the same structure:

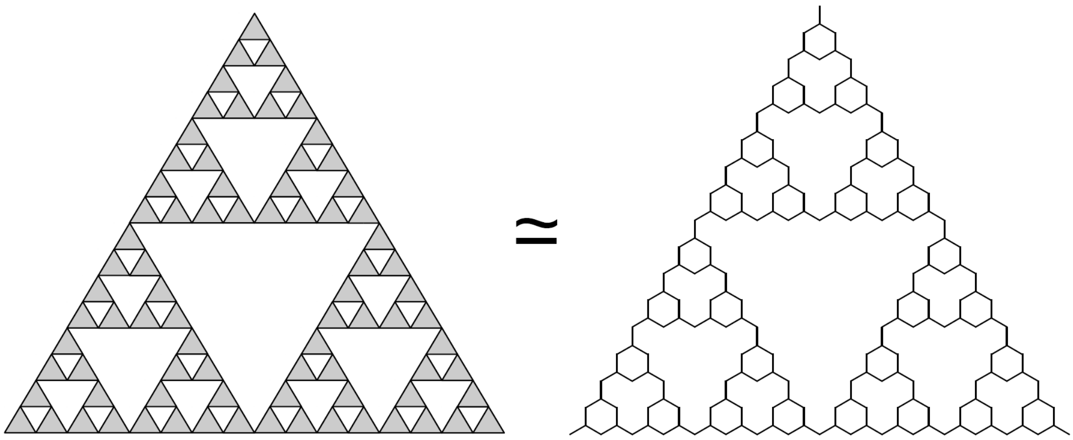

Figure 9: *A self-similar homotopy.* The Sierpiński triangle is self-similarly homotopy equivalent to a 1-dimensional complex.

4.12 Definition. Let X and Y be self-similar complexes. A pair of maps $f\colon X \to Y$ and $g\colon Y \to X$ is a *self-similar homotopy equivalence* if there are self-similar structures $\big(K_m, \mathcal{G}(m, m+1)\big)$ on X and $\big(L_m, \mathcal{G}'(m, m+1)\big)$ on Y such that:

(a) $f(K_m) \subseteq L_m$ and $g(L_m) \subseteq K_m$,

(b) the homotopy $h_1\colon gf \simeq \mathrm{id}_X$ restricts to $h_1|_{K_m}\colon (g|_{L_m})(f|_{K_m}) \simeq \mathrm{id}_{K_m}$,

(c) the homotopy $h_2\colon fg \simeq \mathrm{id}_Y$ restricts to $h_2|_{L_m}\colon (f|_{K_m})(g|_{L_m}) \simeq \mathrm{id}_{L_m}$,

(d) there is a bijection $\varphi_m\colon \mathcal{G}(m, m+1) \to \mathcal{G}'(m, m+1)$,

(e) and both maps and homotopies are "equivariant" with respect to these local isomorphisms; that is, all of the following diagrams commute for each m, each $\gamma \in \mathcal{G}(m, m+1)$ and each $\gamma' \in \mathcal{G}'(m, m+1)$:

$$\begin{tikzcd} K_m \arrow[r, "\gamma"] \arrow[d, "f"] & \gamma K_m \arrow[d, "f"] \\ L_m \arrow[r, "\varphi_m(\gamma)"] & \varphi_m(\gamma) L_m \end{tikzcd} \qquad \begin{tikzcd} K_m \times [0,1] \arrow[r, "\gamma\times\mathrm{id}"] \arrow[d, "h_1"] & \gamma K_m \times [0,1] \arrow[d, "h_1"] \\ K_m \arrow[r, "\gamma"] & \gamma K_m \end{tikzcd}$$

$$\begin{tikzcd} L_m \arrow[r, "\gamma'"] \arrow[d, "g"] & \gamma' L_m \arrow[d, "g"] \\ K_m \arrow[r, "\varphi_m^{-1}(\gamma')"] & \varphi_m^{-1}(\gamma') K_m \end{tikzcd} \qquad \begin{tikzcd} L_m \times [0,1] \arrow[r, "\gamma'\times\mathrm{id}"] \arrow[d, "h_2"] & \gamma' L_m \times [0,1] \arrow[d, "h_2"] \\ L_m \arrow[r, "\gamma'"] & \gamma' L_m \end{tikzcd}$$

Let us now turn to L^2-Betti numbers. By definition, a homotopy equivalence between two self-similar complexes X and Y implies that there are self-similar structures (K_m) on X and (L_m) on Y such that $K_m \simeq L_m$ for all m, and thus the classical Betti numbers of these subcomplexes are equal: $\beta_j(K_m) = \beta_j(L_m)$.

By Corollary 4.2, the L^2-Betti numbers of X and Y can be approximated by $\beta_j(K_m)$ and $\beta_j(L_m)$, but only after normalization – and our usual normalization factor is the number of cells (of some dimension) in each subcomplex, which is certainly not homotopy invariant.

By Lemma 3.15, the limits $\lim\limits_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|G(1,m)|}$ and $\lim\limits_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|G'(1,m)|}$ exist, and by property 4.12 (d) we have $|\mathcal{G}(1,m)| = |\mathcal{G}'(1,m)|$. Thus, the limit $\lim\limits_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_j L_m|}$ exists as well, and we can write

$$\begin{aligned} b_j^{(2)}(X) &= \lim_{m\to\infty} \frac{\beta_j(K_m)}{|\mathcal{E}_j K_m|} = \lim_{m\to\infty} \frac{\beta_j(L_m)}{|\mathcal{E}_j K_m|} = \lim_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{E}_j K_m|} \cdot \lim_{m\to\infty} \frac{\beta_j(L_m)}{|\mathcal{E}_j L_m|} \\ &= \lim_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{E}_j K_m|} \cdot b_j^{(2)}(Y). \end{aligned}$$

This already proves that if $b_j^{(2)}(X) = 0$, then also $b_j^{(2)}(Y) = 0$.

If the j-th Betti numbers are non-zero, it remains to show that the normalization factor $\lim\limits_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{E}_j K_m|}$ does not depend on the choices of (K_m) and (L_m):

Assume (K'_m) and (L'_m) are other self-similar exhaustions of X and Y such that $K'_m \simeq L'_m$ for every m. Again, we obtain $\beta_j(K'_m) = \beta_j(L'_m)$. From Corollary 4.2 we see that

$$\begin{aligned} \lim_{m\to\infty} \frac{\beta_j(K_m)}{|\mathcal{E}_j K_m|} &= b_j^{(2)}(X) = \lim_{m\to\infty} \frac{|\beta_j(K'_m)|}{|\mathcal{E}_j K'_m|}, \\ \lim_{m\to\infty} \frac{\beta_j(L_m)}{|\mathcal{E}_j L_m|} &= b_j^{(2)}(Y) = \lim_{m\to\infty} \frac{|\beta_j(L'_m)|}{|\mathcal{E}_j L'_m|}. \end{aligned}$$

Therefore, using $K_m \simeq L_m$ and $K'_m \simeq L'_m$,

$$\lim_{m\to\infty} \frac{|\mathcal{E}_j K'_m|}{|\mathcal{E}_j K_m|} = \lim_{m\to\infty} \frac{\beta_j(K'_m)}{\beta(K_m)} = \lim_{m\to\infty} \frac{\beta_j(L'_m)}{\beta(L_m)} = \lim_{m\to\infty} \frac{|\mathcal{E}_j L'_m|}{|\mathcal{E}_j L_m|},$$

and thus indeed

$$\lim_{m\to\infty} \frac{|\mathcal{E}_j K'_m|}{|\mathcal{E}_j L'_m|} = \lim_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{E}_j L_m|}.$$

This yields the following statement:

4.13 Theorem. *If two self-similar complexes X and Y are homotopy equivalent in the sense of Definition 4.12, then their L^2-Betti numbers are related by the formula*

$$b_j^{(2)}(X) = \lim_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{E}_j K_m|} \cdot b_j^{(2)}(Y)$$

where the factor $\lim\limits_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{E}_j K_m|}$ is the same for any two self-similar exhaustions $(K_m), (L_m)$ that fulfill $K_m \simeq L_m$ for every m.

4.4 Homotopy invariance of Novikov–Shubin invariants

Unlike L^2-Betti numbers, Novikov–Shubin invariants are not affected by normalization factors: Since they only use the logarithm of the spectral density function, any constant factor becomes negligible. That allows us to obtain “true” invariance under self-similar homotopy equivalences:

4.14 Theorem. *If two self-similar complexes X and Y are homotopy equivalent in the sense of Definition 4.12, then their Novikov–Shubin invariants are equal. More precisely, we have both $\alpha\big(\partial_j^{(X)}\big) = \alpha\big(\partial_j^{(Y)}\big)$ and $\alpha\big(\Delta_j^{(X)}\big) = \alpha\big(\Delta_j^{(Y)}\big)$ for all j.*

In order to work with self-similar maps, we first need to prove that they indeed induce bounded operators on the L^2-chain complex:

4.15 Lemma. *Let $f\colon X \to Y$ be a self-similar map. Then the induced chain maps $f_j\colon \mathbb{C}[\mathcal{E}_j X] \to \mathbb{C}[\mathcal{E}_j Y]$ are bounded, and thus extend to bounded operators $f_j\colon \ell^2(\mathcal{E}_j X) \to \ell^2(\mathcal{E}_j Y)$.*

Proof. Use the same notation as before: (K_m) is a self-similar exhaustion of X with local isomorphisms $\mathcal{G}(m, m+1)$, and (L_m) is the corresponding self-similar exhaustion of Y with local isomorphisms $\mathcal{G}'(m, m+1)$

Choose m large enough that L_m has a non-frontier cell. Then there is a constant D such that every γL_m (for $\gamma \in \mathcal{G}'(m)$) overlaps at most D other such copies of L_m. Namely, let r be the maximal combinatorial distance between vertices in L_m (this is finite, as we assume L_m to be connected). Then, for any vertex $\rho \in \mathcal{E}_0(\gamma L_m)$, the $2r$-ball around ρ contains an interior vertex of every other copy $\gamma' L_m$ that intersects γL_m. As different translates only overlap at their frontiers, all these interior vertices are distinct, and as the complex Y is bounded, $B_{2r}(\rho)$ is finite. Thus, we can set $D = |B_{2r}(\rho)|$.

Write $x \in \mathbb{C}[\mathcal{E}_j X]$ as a sum $\sum_{\gamma \in \mathcal{G}(m)} x_\gamma$, where the $x_\gamma \in \mathbb{C}[\mathcal{E}_j(\gamma K_m)]$ are mutually orthogonal and only finitely many x_γ are nonzero. By assumption, $f_j(x_\gamma) \in \mathbb{C}[\varphi(\gamma) L_m]$ for some $\varphi(\gamma) \in \mathcal{G}'(m)$, but since the translates of L_m can overlap at their frontiers, the $f_j(x_\gamma)$ are not necessarily mutually orthogonal.

Enumerate the $\gamma \in \mathcal{G}(m)$ with $x_\gamma \neq 0$ as $\gamma_1, \ldots, \gamma_N$ such that

$$\|f_j(x_{\gamma_1})\| \geq \|f_j(x_{\gamma_2})\| \geq \ldots \geq \|f_j(x_{\gamma_N})\| .$$

Then, in the sum

$$\|f_j(x)\|^2 = \sum_{i,k=1}^{N} \langle f_j(x_{\gamma_i}), f_j(x_{\gamma_k}) \rangle$$

there are at most $2D$ nonzero terms containing $f(x_{\gamma_1})$, because $\operatorname{supp}(f(x_{\gamma_1})) \subseteq \varphi(\gamma_1)(L_m)$ and $\varphi(\gamma_1) L_m$ intersects at most D other translates $\varphi(\gamma_k) L_m$. Each of those terms fulfills

$$\big|\langle f_j(x_{\gamma_1}), f_j(x_{\gamma_k}) \rangle\big| \leq \|f_j(x_{\gamma_1})\| \cdot \|f_j(x_{\gamma_k})\| \leq \|f_j(x_{\gamma_1})\|^2 .$$

Thus,

$$\|f_j(x)\|^2 \leq 2D \|f_j(x_{\gamma_1})\|^2 + \sum_{i,k=2}^{N} \langle f_j(x_{\gamma_i}), f_j(x_{\gamma_k})\rangle \,,$$

and by induction we obtain

$$\|f_j(x)\|^2 \leq 2D \sum_{i=1}^{N} \|f_j(x_{\gamma_i})\|^2 \,.$$

Finally, since f commutes with local isomorphisms, so does f_j, and thus

$$\|f_j(x_{\gamma_i})\| \leq \left\| f_j|_{\mathbb{C}[\mathcal{E}_j K_m]} \right\| \cdot \|x_\gamma\|$$

for all γ. We conclude

$$\|f_j\| \leq \sqrt{2D} \left\| f_j|_{\mathbb{C}[\mathcal{E}_j K_m]} \right\| \,,$$

and $\left\| f_j|_{\mathbb{C}[\mathcal{E}_j K_m]} \right\|$ is finite because $\mathcal{E}_j K_m$ is finite. □

4.16 Remark. Given a self-similar homotopy $h\colon X \times [0,1] \to Y$ between two maps $X \to Y$, the induced chain homotopy $\eta_j\colon \mathbb{C}[\mathcal{E}_j X] \to \mathbb{C}[\mathcal{E}_{j+1} Y]$ can be obtained as $(-1)^j$ times the restriction of

$$h_j\colon \mathbb{C}\big[\mathcal{E}_{j+1}\big(X \times [0,1]\big)\big] \to \mathbb{C}[\mathcal{E}_{j+1} Y],$$

using the Künneth isomorphism

$$\mathcal{E}_{j+1}\big(X \times [0,1]\big) \cong \mathcal{E}_{j+1} X \sqcup \mathcal{E}_{j+1} X \sqcup \mathcal{E}_j X.$$

(Compare [Lüc05], page 61.) Applying Lemma 4.15 to h_j therefore also proves that η_j is bounded.

As a preparation for the proof of homotopy invariance, let us show that if two *finite* CW-complexes are homotopy equivalent, then the spectral density functions of their differentials are dilatationally equivalent in a precisely controlled way:

4.17 Lemma. *Let* $K \underset{g}{\overset{f}{\rightleftarrows}} L$ *be a homotopy equivalence between two finite CW-complexes with homotopies* $h_1\colon gf \simeq \mathrm{id}_K$ *and* $h_2\colon fg \simeq \mathrm{id}_L$, *and let* $\eta_{1,j}\colon \mathbb{C}[\mathcal{E}_j K] \to \mathbb{C}[\mathcal{E}_{j+1} K]$ *and* $\eta_{2,j}\colon \mathbb{C}[\mathcal{E}_j L] \to \mathbb{C}[\mathcal{E}_{j+1} L]$ *be the chain homotopies induced by* h_1 *and* h_2.

Denote by $F_{\mathrm{nn}}(\partial_j)$ *the non-normalized spectral density functions of the differentials, and let* $F_{\mathrm{nn}}^{\perp}(\partial_j)(\lambda) = F_{\mathrm{nn}}(\partial_j)(\lambda) - F_{\mathrm{nn}}(\partial_j)(0)$. *Then*

$$F_{\mathrm{nn}}(\partial_j^K)^{\perp}(\lambda) \leq F_{\mathrm{nn}}(\partial_j^L)^{\perp}(2\,\|f_{j-1}\|\,\|g_j\|\,\lambda)$$
$$F_{\mathrm{nn}}(\partial_j^L)^{\perp}(\lambda) \leq F_{\mathrm{nn}}(\partial_j^K)^{\perp}(2\,\|g_{j-1}\|\,\|f_j\|\,\lambda)$$

for all

$$0 < \lambda < \min\left(\frac{1}{2\,\|\eta_{1,j-1}\|}, \frac{1}{2\,\|\eta_{2,j-1}\|}\right).$$

Proof. Fix j and λ and let $V \subseteq \mathbb{C}[\mathcal{E}_j K]$ be the vector space spanned by all eigenvectors of $(\partial_j^K)^* \partial_j^K$ with eigenvalues $\leq \lambda^2$. (In other words, this is the image of the spectral projection $E(\lambda)$ for the operator ∂_j^K.)

Furthermore, let

$$P_j \colon \mathbb{C}[\mathcal{E}_j K] \to \ker(\partial_j^K)^\perp, \qquad Q_j \colon \mathbb{C}[\mathcal{E}_j L] \to \ker(\partial_j^L)^\perp$$

be the orthogonal projections. Thus, the non-normalized spectral density function of ∂_j^K takes the values

$$\begin{aligned} F_{\mathrm{nn}}(\partial_j^K)(\lambda) &= \dim(V), \\ F_{\mathrm{nn}}(\partial_j^K)^\perp(\lambda) &= F_{\mathrm{nn}}(\partial_j^K)(\lambda) - F_{\mathrm{nn}}(\partial_j^K)(0) = \dim(P_j V). \end{aligned}$$

Define

$$W = Q_j f_j P_j V.$$

Let $x \in P_j V$ and write $f_j(x) = y + z$ with $y = Q_j f_j(x) \in W$ and $z \in \ker(\partial_j^L)$. By definition of the chain homotopy, we have

$$x = g_j(y) + g_j(z) + \eta_{1,j-1} \partial_j^K(x) + \partial_{j+1}^K \eta_{1,j}(x).$$

However, g_* commutes with the differentials, so $\partial_j^K g_j(z) = g_{j-1} \partial_j^L(z) = 0$. Thus, both the second and the fourth summand lie in $\ker(\partial_j^K) \perp x$, and we obtain

$$x = P_j x = P_j g_j(y) + P_j \eta_{1,j-1} \partial_j^K(x).$$

By choice of x and λ, we have

$$\begin{aligned} \|x\| &\leq \|P_j g_j(y)\| + \left\|P_j \eta_{1,j-1} \partial_j^K(x)\right\| \\ &\leq \|g_j(y)\| + \|\eta_{1,j-1}\| \left\|\partial_j^K(x)\right\| \\ &\leq \|g_j\| \, \|y\| + \|\eta_{1,j-1}\| \, \lambda \, \|x\| \\ &\leq \|g_j\| \, \|y\| + \frac{1}{2} \|x\| \\ \implies \|x\| &\leq 2 \|g_j\| \, \|y\| . \end{aligned}$$

This proves that $Q_j f_j \colon P_j V \to W, x \mapsto y$ is bounded from below and therefore injective, so

$$\dim(P_j V) = \dim(W).$$

Elements of W are trivially orthogonal to the kernel of ∂_j^L, and furthermore,

$$\begin{aligned} \left\|\partial_j^L(y)\right\| &= \left\|\partial_j^L f_j(x)\right\| = \left\|f_{j-1} \partial_j^K(x)\right\| \\ &\leq \|f_{j-1}\| \left\|\partial_j^K x\right\| \leq \|f_{j-1}\| \, \lambda \, \|x\| \\ &\leq 2\lambda \|f_{j-1}\| \, \|g_j\| \, \|y\| , \end{aligned}$$

so the finite-dimensional analogue of Lemma 2.42 shows that

$$F_{\mathrm{nn}}(\partial_j^L)^\perp(2 \|f_{j-1}\| \, \|g_j\| \, \lambda) \geq \dim(W) = \dim(P_j V) = F_{\mathrm{nn}}(\partial_j^K)^\perp(\lambda).$$

This proves the first estimate. To prove the second estimate, simply swap K with L, f with g, and η_1 with η_2. $\square$

Now we can use approximation of spectral density functions to prove 4.14:

Proof of Thm. 4.14. Use all notation from Def. 4.12.

The maps f, g and the homotopies h_1, h_2 induce chain maps and chain homotopies

$$\begin{aligned} f_j\colon \mathbb{C}[\mathcal{E}_j X] &\to \mathbb{C}[\mathcal{E}_j Y] & g_j\colon \mathbb{C}[\mathcal{E}_j Y] &\to \mathbb{C}[\mathcal{E}_j X] \\ \eta_{1,j}\colon \mathbb{C}[\mathcal{E}_j X] &\to \mathbb{C}[\mathcal{E}_{j+1} X] & \eta_{2,j}\colon \mathbb{C}[\mathcal{E}_j Y] &\to \mathbb{C}[\mathcal{E}_{j+1} Y] \end{aligned}$$

on the cellular (non-L^2) chain complexes of X and Y. By assumption, the maps can be restricted to K_m and L_m, and thus the same applies to their induced maps on the chain complexes, yielding

$$\begin{aligned} f_j\colon \mathbb{C}[\mathcal{E}_j K_m] &\to \mathbb{C}[\mathcal{E}_j L_m] & g_j\colon \mathbb{C}[\mathcal{E}_j L_m] &\to \mathbb{C}[\mathcal{E}_j K_m] \\ \eta_{1,j}\colon \mathbb{C}[\mathcal{E}_j K_m] &\to \mathbb{C}[\mathcal{E}_{j+1} K_m] & \eta_{2,j}\colon \mathbb{C}[\mathcal{E}_j L_m] &\to \mathbb{C}[\mathcal{E}_{j+1} L_m] \end{aligned}$$

On the other hand, Lemma 4.15 and Remark 4.16 show that all these chain maps are indeed bounded operators. Thus, they extend to the L^2-chain complexes $\ell^2(\mathcal{E}_j X)$ resp. $\ell^2(\mathcal{E}_j Y)$, and we can estimate $\left\| f_j|_{\mathbb{C}[\mathcal{E}_j K_m]} \right\| \leq \|f_j\|$, and the same for $g_j, \eta_{1,j}$ and $\eta_{2,j}$. Especially, these norms are bounded uniformly in m.

Apply Lemma 4.17 to K_m and L_m to get

$$\begin{aligned} F_{\mathrm{nn}}(\partial_j^{K_m})^\perp(\lambda) &\leq F_{\mathrm{nn}}(\partial_j^{L_m})^\perp\big(\sqrt{C}\cdot\lambda\big) \\ F_{\mathrm{nn}}(\partial_j^{L_m})^\perp(\lambda) &\leq F_{\mathrm{nn}}(\partial_j^{K_m})^\perp\big(\sqrt{C}\cdot\lambda\big) \end{aligned}$$

for all $\lambda \in (0, \varepsilon)$, where

$$\begin{aligned} \sqrt{C} &= \max\Big(2\,\|f_{j-1}\|\,\|g_j\|\,, 2\,\|g_{j-1}\|\,\|f_j\|\Big) \\ \varepsilon &= \min\left(\frac{1}{2\,\|\eta_{1,j-1}\|}, \frac{1}{2\,\|\eta_{2,j-1}\|}\right) \end{aligned}$$

and neither ε nor C depend on m.

Remember that $F_{\mathrm{nn}}(\partial_j)^\perp(\lambda) = F_{\mathrm{nn}}(\Delta_{j-})^\perp(\lambda^2)$. Therefore, we have

$$\begin{aligned} F_{\mathrm{nn}}(\Delta_{j-}^{K_m})^\perp(\lambda) &\leq F_{\mathrm{nn}}(\Delta_{j-}^{L_m})^\perp\big(C\cdot\lambda\big) \\ F_{\mathrm{nn}}(\Delta_{j-}^{L_m})^\perp(\lambda) &\leq F_{\mathrm{nn}}(\Delta_{j-}^{K_m})^\perp\big(C\cdot\lambda\big) \end{aligned}$$

and furthermore, Δ_{j-}^X and Δ_{j-}^Y are geometric operators, they are approximated by $\Delta_{j-}^{K_m}$ and $\Delta_{j-}^{L_m}$, and Theorem 3.11 is applicable. However, in order to apply it, we have to renormalize:

Recall from Lemma 3.15 that the limits

$$\lim_{m\to\infty} \frac{|\mathcal{E}_j K_m|}{|\mathcal{G}(1,m)|} \quad \text{and} \quad \lim_{m\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{G}'(1,m)|}$$

exist and are nonzero whenever $\mathcal{E}_j X \neq \emptyset \neq \mathcal{E}_j Y$. We are assuming that

$$|\mathcal{G}(1,m)| = |\mathcal{G}'(1,m)|$$

for all m. Thus, the limit

$$r = \lim_{n\to\infty} \frac{|\mathcal{E}_j L_m|}{|\mathcal{E}_j K_m|}$$

exists, and we get from 3.11 that for every $\lambda \in (0, \varepsilon)$,

$$\begin{aligned} F(\Delta_{j-}^X)(\lambda) &= \lim_{m\to\infty} \frac{F_{\mathrm{nn}}(\Delta_{j-}^{K_m})(\lambda)}{|\mathcal{E}_j K_m|} \leq \lim_{m\to\infty} \frac{F_{\mathrm{nn}}(\Delta_{j-}^{L_m})^{\perp}(C\lambda)}{|\mathcal{E}_j K_m|} \\ &= r \cdot \lim_{m\to\infty} \frac{F_{\mathrm{nn}}(\Delta_{j-}^{L_m})^{\perp}(C\lambda)}{|\mathcal{E}_j L_m|} = r \cdot F(\Delta_{j-}^Y)(C\lambda) \end{aligned}$$

and, by the same argument,

$$r \cdot F(\Delta_{j-}^Y)(\lambda) \leq F(\Delta_{j-}^X)(C\lambda).$$

Now, Lemmas 4.7 and 4.8 yield

$$\alpha\big(F(\Delta_{j-}^X)\big) = \alpha\big(r \cdot F(\Delta_{j-}^Y)\big) = \alpha\big(F(\Delta_{j-}^Y)\big)$$

and thus

$$\alpha(\partial_j^{(X)}) = 2\alpha(\Delta_{j-}^X) = 2\alpha(\Delta_{j-}^Y) = \alpha(\partial_j^{(Y)}).$$

Finally, the statement for Δ follows from Corollary 4.10. $\square$

4.5 Novikov–Shubin invariants, random walks and growth

Betti numbers have a clear topological meaning ("counting holes in a space"), and their approximation carries that meaning over to L^2-Betti numbers (counting the "frequency of holes"). It is significantly harder to ascribe such meaning to Novikov–Shubin invariants, especially since they are completely trivial for finite subspaces. (The spectral density function of any operator on a finite complex is always a step function, giving it $\alpha = \infty^+$.)

For the classical Novikov–Shubin invariants of (Laplacians on) spaces with a suitable group action, there is indeed a geometrical (though not truly topological) meaning of at least the zeroth Novikov–Shubin number:

4.18 Definition. Let G be a finitely generated infinite group and C its Cayley graph with regard to some finite generating set S. Let d be the combinatorial distance of vertices of C (or, equivalently, the word metric of G with respect to S), and

$$B_r(e) = \{g \in G \mid d(e,g) \leq r\}.$$

G has *polynomial growth*, if there is a number $d > 0$ such that

$$|B_r(e)| \sim r^d,$$

and in that case, d is called the *degree of growth* of G. (It turns out that d does not depend on the choice of S.)

All infinite virtually nilpotent groups have polynomial growth ([Gro81]), and the degree of growth is closely linked to the first Novikov–Shubin invariant. This theorem is originally due to Lott ([Lot92]); we cite the fully generalized version from Lück ([Lüc02], Lemmas 2.45 and 2.46):

4.19 Theorem. *Let G be a finitely generated group and X a connected free G-CW-complex of finite type.*

(a) *If G is finite or not amenable, $\alpha_1(X) = \infty^+$.*

(b) *If G is infinite and amenable, but not virtually nilpotent, $\alpha_1(X) = \infty$.*

(c) *If G is infinite and virtually nilpotent, $\alpha_1(X)$ equals the degree of growth of G.*

However, the proof of this theorem, as given by Lück, makes it clear that α and the degree of growth are not directly related to each other. Rather, they are both connected to a third quantity: the return probability of random walks.

For any vertex $\rho \in \mathcal{E}_0 X$, let $S_1(\rho) = \{\sigma \in \mathcal{E}_0 X \mid d_{\mathrm{comb}}(\rho, \sigma) = 1\}$ (this is the set of all neighboring vertices of ρ), and define the *transition operator*

$$P\colon \ell^2(\mathcal{E}_0 X) \to \ell^2(\mathcal{E}_0 X),\ \rho \mapsto \frac{1}{|S_1(\rho)|} \sum_{\sigma \in S_1(\rho)} \sigma.$$

Then $\langle \sigma, P^n \rho \rangle$ is the probability that a simple random walk starting at ρ will be at σ after exactly n steps. In particular, the probability that the random walk will return to ρ after exactly n steps is given by

$$p_\rho(n) = \langle \rho, P^n \rho \rangle\,.$$

As vertices of X can have very different neighborhoods, $p_\rho(n)$ does indeed depend on ρ, or, more precisely, on the n-pattern of ρ. However, as the operator P^n is clearly pattern-invariant, we can take the average value

$$p(n) = \lim_{m \to \infty} \frac{1}{|\mathcal{E}_0 K_m|} \sum_{\sigma \in \mathcal{E}_0 K_m} \langle \sigma, P^n \sigma \rangle = \operatorname{tr}_{\mathcal{A}}(P^n).$$

(On a homogenous space – a space with a transitive group action – there is only one pattern, and $p_\rho(n)$ does *not* depend on ρ.)

These return probabilities are closely linked to $\alpha_0(X)$, and this holds for G-CW-complexes as well as for spaces with aperiodic order:

4.20 Lemma. *If there are constants $C > c > 0$ and $a > 0$ such that $c\,n^a \leq p(n) \leq C\,n^a$, then $\alpha_0(X) = a$.*

Proof. Let $\Theta\rho = \frac{1}{\deg(\rho)}\rho$. As X is a bounded complex, Θ is bounded from below and thus invertible, and it fulfills

$$\Delta_0 \Theta = (\mathrm{id} - P).$$

Thus, Lemma 4.9 yields

$$\alpha_0(X) = \alpha(\Delta_0) = \alpha(\mathrm{id} - P).$$

The rest of the proof carries over verbatim from [Lüc02], p. 95f. $\square$

The second half of the proof of Theorem 4.19 follows from a theorem of Varopoulos:

4.21 Theorem ([Var87])**.** *Let G be a finitely generated group.*

(a) *If G does not have polynomial growth, then for every $a > 0$ there is $C > 0$ such that $p(n) \leq C\, n^{-a}$.*

(b) *G has polynomial growth of degree $2a$ if and only if there are $C > c > 0$ such that $c\, n^{-a} \leq p(n) \leq C\, n^{-a}$ for every even $n \in \mathbb{N}$.*

This, however, does *not* hold for spaces with aperiodic order, and not even for self-similar spaces. The Sierpiński triangle is actually a counterexample, as proven by Woess ([Woe00], p. 171):

"... we have seen that under certain conditions (quasi-homogeneity), polynomial growth with degree r and decay of order $n^{-r'/2}$ for transition probabilities occur with the same exponents $r' = r$. In this section we shall study a class of graphs with polynomial growth, where r' is strictly smaller than r. These are the simplest "fractal" graphs, strongly related to the Sierpiński fractals in $d \geq 2$ dimensions."

Thus, for self-similar spaces like the Sierpiński triangle, the zeroth Novikov–Shubin invariant is indeed related to return probabilities, but it is *not* tied to growth.

4.6 Approximation of Novikov–Shubin invariants

As mentioned above, the Novikov–Shubin invariants of finite subcomplexes always take the value ∞^+, since the spectral density function of an operator on a finite-dimensional space is always constant around zero. Thus, $\alpha_j(X)$ will usually not be the limit of $\alpha_j(K_m)$. However, there is still a possibility that $\alpha_j(X)$ might be computable as the limit of *some* number derived from K_m.

For G-CW-complexes, Kammeyer [Kam17] explored the possibility of approximating $\alpha_j(X)$ by the "alpha numbers" of approximating step functions:

4.22 Definition. Let $F\colon [0,\infty) \to [0,\infty)$ be a non-decreasing right-continuous step function with finitely many steps. Then let $\lambda^+ = \min\{\lambda \in \mathbb{R} \mid F(\lambda) > F(0)\}$ and define the *alpha number* of F by

$$\alpha_{\text{discrete}}(F) = \frac{\log(F(\lambda^+) - F(0))}{\log(\lambda^+)}.$$

Kammeyer finds that for a G-CW-complex with virtually cyclic group G, the $\limsup$ of the alpha numbers[9] indeed converges to the Novikov–Shubin invariants of X, but the $\liminf$ does not. However, his methods make extensive use of the specific group structure and cannot carry over to the self-similar case.

It would be appealing to assume that if a sequence of step functions converges uniformly to a continuous function, then their alpha numbers converge to its Novikov–Shubin invariant. Unfortunately, this is false in general: Let[10]

$$G_m \colon [0,\infty) \to [0,\infty),\ \lambda \mapsto \begin{cases} 0, & \text{if } x \in [0, e^{-m}), \\ \frac{1}{m}, & \text{if } x \in [e^{-m}, 1), \\ 1, & \text{if } x \in [1, \infty). \end{cases}$$

Each G_m is non-decreasing, and the sequence (G_m) converges uniformly to

$$G \colon [0,\infty) \to [0,\infty),\ \lambda \mapsto \begin{cases} 0, & \text{if } x \in [0, 1), \\ 1, & \text{if } x \in [1, \infty). \end{cases}$$

Then the alpha numbers of the G_m are

$$\alpha_{\text{discrete}}(G_m) = \frac{\log(1/m)}{\log(e^{-m})} = \frac{\log(m)}{m} \xrightarrow{m\to\infty} 0,$$

while clearly

$$\alpha(G) = \infty^+,$$

yielding the maximal possible difference between the Novikov–Shubin invariant and its "approximation".

Thus, the alpha numbers could only be guaranteed to converge if we could reliably control the smallest eigenvalue of the Laplacian on K_m. This is clearly not possible, as we know very little about the frontiers (and have consistently ignored their contributions in the previous approximations). Consequently, it appears unlikely that the alpha numbers would yield a reliable approximation for Novikov–Shubin invariants on self-similar complexes.

[9] Kammeyer considers finite subcovers of the covering $X \to X/G$, instead of amenable subcomplexes of X itself.

[10] This counterexample will appear again in Remark 5.10.

5 Fuglede–Kadison determinants and torsion

The Fuglede–Kadison determinant is, for the most part, a generalization of the usual determinant to operators on infinite-dimensional spaces, and it preserves several important properties of the determinant.

However, it deviates from the classical determinant in two regards: First, as its definition relies on the spectral density function of an operator, it only depends on the "absolute value" of the operator, and will never carry a sign.

Second, and most importantly, the Fuglede–Kadison determinant of an operator ignores the operator's kernel – especially, it does not become zero whenever the operator is not injective! The determinant *can* become zero, but this instead requires a large amount of very small spectral values. Thus, a zero determinant indicates a large "almost-kernel" instead of a non-trivial kernel.[11]

In the first part of this chapter, we will show that the Fuglede–Kadison determinants in our setting show most of the properties of their classical counterparts, especially multiplicativity. In the second part, we discuss whether these determinants can be approximated by finite-dimensional analogues. The third and last part of the chapter defines and briefly discusses the third L^2-invariant, L^2-torsion, that is constructed from the Fuglede–Kadison determinants of Laplacians.

5.1 Definition and properties

5.1 Definition. Given an operator T with spectral density function F^T, define the *Fuglede–Kadison determinant* of T by

$$\det_{\mathrm{FK}}(T) = \exp \int_{(0,\infty)} \log(\lambda)\, dF^T(\lambda),$$

where we set $\exp(-\infty) = 0$ and the measure dF^T is given by

$$dF^T\big((a,b]\big) = F^T(b) - F^T(a).$$

T is called *of determinant class* if $\det_{\mathrm{FK}}(T) > 0$, that is, if the integral in the definition is finite.

5.2 Remark. (a) As the definition of $\det_{\mathrm{FK}}(T)$ only depends on the spectral density function of T, one can also speak of "the determinant of F" for any non-decreasing right-continuous function $F\colon [0,\infty) \to [0,\infty)$.

(b) If T has eigenvalues, F^T can have jump discontinuities, and thus the measure dF^T can have atoms. Especially, if T is not injective, it can happen that $dF^T(\{0\}) > 0$, so it is important to note that the domain of integration does not contain the point 0.

[11] Large "almost-kernels" correspond to small Novikov–Shubin numbers, and indeed $\alpha(T) > 0$ implies $\det_{\mathrm{FK}}(T) > 0$. See [Lüc02], Theorem 3.14 (4).

(c) On the other hand, the upper bound of the domain of integration is irrelevant: For any $\lambda > \|T\|_{\mathrm{op}}$, we have $F^T(\lambda) = \mathrm{tr}_{\mathcal{N}}(E^T(\lambda)) = 1$, so $dF^T\big((\|T\|_{\mathrm{op}}, \infty)\big) = 0$. Thus, the Fuglede–Kadison determinant can be computed as

$$\det_{\mathrm{FK}}(T) = \exp \int_{(0,\|T\|_{\mathrm{op}}]} \log(\lambda)\, dF^T(\lambda).$$

For the practical computation of the Fuglede–Kadison determinant, the following lemma is most useful:

5.3 Lemma ([Lüc02], Lemma 3.15)**.** *If $\int_{(0,b]} \log(\lambda)\, dF(\lambda) > -\infty$, then*

$$\int_{(0,b]} \log(\lambda)\, dF(\lambda) \;=\; \log(b)\big(F(b) - F(0)\big) - \int_0^b \frac{F(\lambda) - F(0)}{\lambda}\, d\lambda.$$

To simplify notation, let

$$F^{\perp}(\lambda) = F(\lambda) - F(0).$$

Then we obtain

$$\int_{(0,b]} \log(\lambda)\, dF(\lambda) \;=\; \log(b)\, F^{\perp}(b) - \int_0^b \frac{F^{\perp}(\lambda)}{\lambda}\, d\lambda.$$

5.4 Remark. If $T\colon \mathbb{C}^n \to \mathbb{C}^n$ is a positive self-adjoint operator, then

$$\det_{\mathrm{FK}}(T) = \sqrt[n]{\prod \{\lambda \text{ eigenvalue of } T \mid \lambda > 0\}}.$$

Especially, the Fuglede–Kadison determinant of an operator on a finite-dimensional space is never zero. (The determinant of the zero operator is given by an empty product, and thus it equals 1.)

Many properties of the Fuglede–Kadison determinant, most importantly their multiplicativity, carry over from the classical case.

In the following theorem, most statements and proofs follow Lück ([Lüc02], Theorem 3.14 and Lemma 3.15), except for 5.5 (b):

5.5 Theorem. *Always let $S, T \in \mathcal{N}_j(X)$.*

(a) $\det_{\mathrm{FK}}(T) = \det_{\mathrm{FK}}(T^*) = \sqrt{\det_{\mathrm{FK}}(T^*T)} = \sqrt{\det_{\mathrm{FK}}(TT^*)}$.

(b) *If T is self-adjoint, then $T + P_{\ker T}$ is injective and*

$$\det_{\mathrm{FK}}(T + P_{\ker T}) = \det_{\mathrm{FK}}(T).$$

(c) *If T is positive and injective, then*

$$\det_{\mathrm{FK}}(T) = \lim_{\varepsilon \to 0+} \det_{\mathrm{FK}}(T + \varepsilon\, \mathrm{id}).$$

(d) *If T is invertible, then*

$$\det_{\mathrm{FK}}(T) = \exp\left(\frac{1}{2}\,\mathrm{tr}_{\mathcal{N}}\big(\log(T^*T)\big)\right).$$

(e) *If S, T are injective and positive, then*

$$S \leq T \implies \det_{\mathrm{FK}}(S) \leq \det_{\mathrm{FK}}(T).$$

(f) *If S is injective and T has dense image, then*

$$\det_{\mathrm{FK}}(ST) = \det_{\mathrm{FK}}(S)\det_{\mathrm{FK}}(T).$$

Proof. (a) By Lemma A.8, every self-adjoint operator fulfills $F^T(\lambda) = F^{T^2}(\lambda^2)$, and for every other operator T, the spectral density function is defined by $F^T(\lambda) = F^{T^*T}(\lambda^2)$. Therefore, we get

$$dF^{T^*T}((a,b]) = F^T(\sqrt{b}) - F^T(\sqrt{a}) = dF^T((\sqrt{a},\sqrt{b}])$$

and thus

$$\begin{aligned}\det_{\mathrm{FK}}(T^*T) &= \exp\int_{(0,\infty)} \log(\lambda)\,dF^{T^*T}(\lambda)\\ &= \exp\int_{(0,\infty)} \log(\lambda^2)\,dF^T(\lambda) = \det_{\mathrm{FK}}(T)^2\end{aligned}$$

The same argument shows $\det_{\mathrm{FK}}(T^*) = \sqrt{\det_{\mathrm{FK}}(TT^*)}$, and applying Lemma 2.40 yields $\det_{\mathrm{FK}}(T) = \det_{\mathrm{FK}}(T^*)$.

(b) Write $\widetilde{T} = T + P_{\ker T}$. Given a vector $z = x + y \in \ker(T) \oplus \ker(T)^\perp$, we have $\widetilde{T}z = x + Ty$ and $\langle x, Ty\rangle = \langle Tx, y\rangle = 0$. Therefore,

$$\left\|\widetilde{T}z\right\|^2 = \|x\|^2 + \|Ty\|^2,$$

and this implies that the spectral density function of $\widetilde{T}$ is given by

$$F^{\widetilde{T}}(\lambda) = \begin{cases} F^T(\lambda) - F^T(0) & \text{if } \lambda \in [0,1),\\ F^T(\lambda), & \text{if } \lambda \in [1,\infty).\end{cases}$$

Thus, the resulting measure on $(0,\infty)$ is given by

$$dF^{\widetilde{T}} = dF^T + F^T(0)\cdot\delta_1,$$

and as $\log(1) = 0$, this implies

$$\det_{\mathrm{FK}}(T) = \exp\int_{(0,\infty)} \log(\lambda)\,dF^T(\lambda) = \exp\int_{(0,\infty)} \log(\lambda)\,dF^{\widetilde{T}}(\lambda) = \det_{\mathrm{FK}}(\widetilde{T}).$$

(c) This follows from $F^{T+\varepsilon\mathrm{id}}(\lambda) = F^T(\lambda - \varepsilon)$ and Beppo Levi's monotone convergence theorem.

(d) If T and thus T^*T are invertible, they are bounded from below, so $\operatorname{spec}(T^*T) \subseteq (0,\infty)$ and $\log(T^*T) \in \mathcal{N}_j$. Applying part (a) and Theorem A.6 gives

$$\begin{aligned}
\log \det_{\mathrm{FK}}(T) &= \frac{1}{2} \log \det_{\mathrm{FK}}(T^*T) \\
&= \frac{1}{2} \int_{(0,\infty)} \log(\lambda)\, dF^{T^*T}(\lambda) \\
&= \frac{1}{2} \operatorname{tr}_{\mathcal{N}} \int_{(0,\infty)} \log(\lambda)\, dE^{T^*T}(\lambda) \\
&= \frac{1}{2} \operatorname{tr}_{\mathcal{N}}\bigl(\log(T^*T)\bigr).
\end{aligned}$$

(e) From

$$\left\|\sqrt{S}x\right\|^2 = \langle x, Sx\rangle \leq \langle x, Tx\rangle = \left\|\sqrt{T}x\right\|^2$$

and Lemma 2.42, we get $F^{\sqrt{T}}(\lambda) \leq F^{\sqrt{S}}(\lambda)$ for all $\lambda \in [0,\infty)$, and with part (a), this yields $F^T(\lambda) \leq F^S(\lambda)$ for all $\lambda \in [0,\infty)$. Using Lemma 5.3 with $b = \max\bigl(\|S\|, \|T\|\bigr)$, we have

$$\log \det_{\mathrm{FK}}(S) = \log(b)\, F^S(b) - \int_0^b \frac{F^S(\lambda)}{\lambda}\, d\lambda$$

and the same for T. (Note that S, T are injective, so $F^S(0) = 0 = F^T(0)$.) As $F^S(b) = \dim_{\mathcal{N}}(\ell^2(\mathcal{E}_j X)) = F^T(b)$, this implies the result.

(f) Show first: If A, B are both positive and invertible, then

$$\det_{\mathrm{FK}}(ABBA) = \det_{\mathrm{FK}}(A)^2 \det_{\mathrm{FK}}(B)^2.$$

Use Lemma 3.18 from [Lüc02]: If the operator-valued function $X\colon [0,1] \to \mathcal{B}(\mathcal{H})$ is differentiable (in the sense that there is $X'(t) \in \mathcal{B}(\mathcal{H})$ such that $\frac{1}{\varepsilon}\|X(t+\varepsilon) - X(t) - \varepsilon X'(t)\|_{\mathrm{op}} \xrightarrow{\varepsilon\to 0} 0$), and if $\bigcup_t \operatorname{spec}(X(t))$ lies in the interior of the domain of a holomorphic function f, then

$$\operatorname{tr}_{\mathcal{N}}\left(\frac{d}{dt} f\bigl(X(t)\bigr)\right) = \operatorname{tr}_{\mathcal{N}}\Bigl(f'\bigl(X(t)\bigr) \circ X'(t)\Bigr).$$

Lück proves this for the von Neumann trace on $\mathcal{N}(G)$, but the only property of the trace his proof requires is that it commutes with integration, which holds for $\operatorname{tr}_{\mathcal{N}}$, too, as every integral $\int_0^1 X(t)\, dt$ fulfills by definition

$$\left\langle \xi_1, \int_0^1 X(t)\, dt\, \xi_2 \right\rangle = \int_0^1 \langle \xi_1, X(t)\xi_2\rangle\, dt \quad \text{for all } \xi_1, \xi_2 \in \mathcal{H}.$$

As $A, B > 0$ are invertible, their spectra are contained in $[\varepsilon, \infty)$ for some $\varepsilon > 0$, so $\log(A)$, $\log(B)$ and $\log(ABBA)$ are defined and bounded. Applying the previous lemma gives

$$\frac{d}{dt}\operatorname{tr}_{\mathcal{N}}\left(\log\left(A \circ \left(tB^2 + (1-t)\mathrm{id}\right) \circ A\right)\right) = \frac{d}{dt}\operatorname{tr}_{\mathcal{N}}\left(\log\left(tB^2 + (1-t)\mathrm{id}\right)\right).$$

As for $t = 0$ we have

$$\begin{aligned}\operatorname{tr}_{\mathcal{N}}\left(\log\left(A \circ \left(0 \cdot B^2 + 1 \cdot \mathrm{id}\right) \circ A\right)\right) &= \operatorname{tr}_{\mathcal{N}}\left(\log(A^2)\right), \\ \operatorname{tr}_{\mathcal{N}}\left(\log\left(0 \cdot B^2 + 1 \cdot \mathrm{id}\right)\right) &= 0,\end{aligned}$$

this yields

$$\begin{aligned}&\operatorname{tr}_{\mathcal{N}}\left(\log(A \circ B^2 \circ A)\right) - \operatorname{tr}_{\mathcal{N}}\left(\log(A^2)\right) \\ &\qquad = \int_0^1 \frac{d}{dt}\operatorname{tr}_{\mathcal{N}}\left(\log\left(A \circ \left(tB^2 + (1-t)\mathrm{id}\right) \circ A\right)\right) dt \\ &\qquad = \int_0^1 \frac{d}{dt}\operatorname{tr}_{\mathcal{N}}\left(\log\left(tB^2 + (1-t)\mathrm{id}\right)\right) dt \\ &\qquad = \operatorname{tr}_{\mathcal{N}}\left(\log(B^2)\right) - \operatorname{tr}_{\mathcal{N}}\left(\log(\mathrm{id})\right) \\ &\qquad = \operatorname{tr}_{\mathcal{N}}\left(\log(B^2)\right).\end{aligned}$$

Together with part (d), this implies $\det_{\mathrm{FK}}(ABBA) = \det_{\mathrm{FK}}(A)^2 \det_{\mathrm{FK}}(B)^2$ for invertible A, B.

Now extend this statement to any injective $A, B \geq 0$. Choose $C > 0$ such that for all $\varepsilon \in [0,1]$,

$$ABBA \leq A\left(B + \varepsilon\,\mathrm{id}\right)^2 A \leq ABBA + C\,\varepsilon\,\mathrm{id}.$$

Since $ABBA \geq 0$ is injective, we can then use part (e) to get

$$\lim_{\varepsilon \to 0+} \det_{\mathrm{FK}}\left(A\left(B + \varepsilon\mathrm{id}\right)^2 A\right) = \det_{\mathrm{FK}}(ABBA).$$

Setting $X = BA$ in $\det_{\mathrm{FK}}(X^*X) = \det_{\mathrm{FK}}(XX^*)$, we get $\det(ABBA) = \det(BAAB)$, and so,

$$\begin{aligned}\det_{\mathrm{FK}}(ABBA) &= \lim_{\varepsilon \to 0+} \det_{\mathrm{FK}}\left(A\left(B + \varepsilon\,\mathrm{id}\right)^2 A\right) \\ &= \lim_{\varepsilon \to 0+} \det_{\mathrm{FK}}\left(\left(B + \varepsilon\,\mathrm{id}\right) A^2 \left(B + \varepsilon\,\mathrm{id}\right)\right) \\ &= \lim_{\varepsilon \to 0+} \lim_{\delta \to 0+} \det_{\mathrm{FK}}\left(\left(B + \varepsilon\,\mathrm{id}\right)\left(A + \delta\,\mathrm{id}\right)^2 \left(B + \varepsilon\,\mathrm{id}\right)\right).\end{aligned}$$

Now apply the previous result, as $B + \varepsilon\,\mathrm{id}$ and $A + \delta\,\mathrm{id}$ are positive and invertible. This gives, for any injective $A, B > 0$,

$$\begin{aligned}\det_{\mathrm{FK}}(ABBA) &= \lim_{\varepsilon \to 0+} \det_{\mathrm{FK}}\left(B + \varepsilon\,\mathrm{id}\right)^2 \lim_{\delta \to 0+} \det_{\mathrm{FK}}\left(A + \delta\,\mathrm{id}\right)^2 \\ &= \det_{\mathrm{FK}}(B)^2 \det_{\mathrm{FK}}(A)^2.\end{aligned}$$

Finally, return to the original statement. Let S be injective and T have dense image. Write $S = VB$ and $T = AU$ with U, V unitary and $B = \sqrt{S^*S}$, $A = \sqrt{TT^*}$ positive. Note that since T^* is injective, the operators A and B are both injective. Thus, we can conclude

$$\begin{aligned}
\det_{\mathrm{FK}}(ST) &= \sqrt{\det_{\mathrm{FK}}(T^*S^*ST)} \\
&= \sqrt{\det_{\mathrm{FK}}(U^*ABV^*VBAU)} \\
&= \sqrt{\det_{\mathrm{FK}}(U^*ABBAU)} \\
&= \sqrt{\det_{\mathrm{FK}}(ABBA)} \\
&= \sqrt{\det_{\mathrm{FK}}(A)^2 \det_{\mathrm{FK}}(B)^2} \\
&= \det_{\mathrm{FK}}(A) \det_{\mathrm{FK}}(B) \\
&= \det_{\mathrm{FK}}(S) \det_{\mathrm{FK}}(T).
\end{aligned}$$

□

5.6 Remark. Even in the finite-dimensional case, properties (c), (e) and (f) fail without injectivity: Simply consider $S, T \in \mathrm{Mat}_2(\mathbb{C})$, where the trace is given by

$$\mathrm{tr}_{\mathcal{N}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2}$$

and thus the Fuglede–Kadison determinant is the square root of the product of the absolute values of the non-zero singular values of a matrix.

(c) Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This is positive but not injective, and $\det_{\mathrm{FK}}(T) = 1$. On the other hand, $\det_{\mathrm{FK}}(T + \varepsilon\,\mathrm{id}) = \sqrt{(1+\varepsilon)\cdot\varepsilon} \xrightarrow{\varepsilon\to 0} 0$.

(e) Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 0 \\ 0 & 1/4 \end{pmatrix}$. Then we have $S \leq T$, but $\det_{\mathrm{FK}}(S) = 1 \geq 1/\sqrt{2} = \det_{\mathrm{FK}}(T)$.

(f) Using the same example matrices as for (e), we have

$$\det_{\mathrm{FK}}(ST) = \sqrt{2} \neq 1 \cdot \frac{1}{\sqrt{2}} = \det_{\mathrm{FK}}(S) \cdot \det_{\mathrm{FK}}(T).$$

To show that the “essential surjectivity” of the second operator (i. e. dense image) is also necessary, consider

$$\det_{\mathrm{FK}}(TS) = \sqrt{2} \neq \frac{1}{\sqrt{2}} \cdot 1 = \det_{\mathrm{FK}}(T) \cdot \det_{\mathrm{FK}}(S).$$

If $T \in \mathcal{N}_j(X)$ is positive but not injective, 5.5 (c) can be generalized as follows:

5.7 Corollary. *Assume $T \in \mathcal{N}_j(X)$ and $T \geq 0$, and let $\beta := \dim_{\mathcal{N}}(\ker T)$. Then*

$$\det_{\mathrm{FK}}(T) = \lim_{\varepsilon\to 0+} \varepsilon^{-\beta} \det_{\mathrm{FK}}(T + \varepsilon\,\mathrm{id}).$$

Proof. Write $P = P_{\ker T}$ for the orthogonal projection to $\ker(T) \subseteq \mathcal{H}$. Abbreviate $\beta = \dim_{\mathcal{N}}(\ker T)$ and $\eta = \dim_{\mathcal{N}}(\mathcal{H}_j)$. (Under the standard normalization, $\eta = 1$, but the statement still holds for other normalizations.)

Combining 5.5 (b) and (c), we have

$$\det_{\mathrm{FK}}(T) = \lim_{\varepsilon \to 0+} \det_{\mathrm{FK}}(T + P + \varepsilon\,\mathrm{id}).$$

Since $TP = 0$, we have $(T + \varepsilon\,\mathrm{id})(\varepsilon^{-1}P + \mathrm{id}) = T + P + \varepsilon\,\mathrm{id}$, and as both $T + \varepsilon\,\mathrm{id}$ and $\varepsilon^{-1}P + \mathrm{id}$ are invertible for every $\varepsilon > 0$, we get from 5.5 (f)

$$\det_{\mathrm{FK}}(T) = \lim_{\varepsilon \to 0+} \Big(\det_{\mathrm{FK}}(T + \varepsilon\,\mathrm{id}) \cdot \det_{\mathrm{FK}}(\varepsilon^{-1}P + \mathrm{id}) \Big).$$

Finally, note that the spectral density function of $\varepsilon^{-1}P + \mathrm{id}$ is given by

$$F^{\varepsilon^{-1}P+\mathrm{id}}(\lambda) = \begin{cases} 0 & \text{for } \lambda \in [0,1), \\ \dim_{\mathcal{N}}(\ker(T)^{\perp}) = \eta - \beta & \text{for } \lambda \in [1, 1+\varepsilon^{-1}), \\ \dim_{\mathcal{N}}(\mathcal{H}_j) = \eta & \text{for } \lambda \in [1+\varepsilon^{-1}, \infty), \end{cases}$$

which yields

$$\begin{aligned} \det_{\mathrm{FK}}(\varepsilon^{-1}P + \mathrm{id}) &= \exp \int_{(0,\infty)} \log(\lambda)\, dF^{\varepsilon^{-1}P+\mathrm{id}}(\lambda) \\ &= \exp\big((\eta - \beta)\log(1) + \beta \log(1+\varepsilon^{-1})\big) \\ &= (1+\varepsilon^{-1})^{\beta}. \end{aligned}$$

For $\varepsilon \to 0+$, this is asymptotically equal to $\varepsilon^{-\beta}$, and therefore,

$$\det_{\mathrm{FK}}(T) = \lim_{\varepsilon \to 0+} \Big(\det_{\mathrm{FK}}(T+\varepsilon\,\mathrm{id})\ \det_{\mathrm{FK}}(\varepsilon^{-1}P+\mathrm{id}) \Big) = \lim_{\varepsilon \to 0+} \varepsilon^{-\beta}\ \det_{\mathrm{FK}}(T+\varepsilon\,\mathrm{id}).$$

□

The most interesting determinants are those of Laplacians. In the classical case, Dodziuk and Mathai ([DM98]) showed that Laplacians on G-CW-complexes are always of determinant class, and their proof can be adapted to the self-similar case as well:

5.8 Theorem. *The Laplacians $\Delta_j^{(X)}$ of a self-similar CW-complex X are of determinant class.*

Proof. Show first that $\det_{\mathrm{FK}}\big(\Delta_j^{(K_m)}\big) \geq 1$ for every m.

Let $k = \dim_{\mathbb{C}} \ker\big(\Delta_j^{(K_m)}\big)$ and let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of $\Delta_j^{(K_m)}$, where $\lambda_i = 0$ for $i \leq k$ and $\lambda_i > 0$ for $i > k$. As mentioned in Remark 5.4,

$$\det_{\mathrm{FK}}\big(\Delta_j^{(K_m)}\big) = \sqrt[r]{\prod_{i=k+1}^{r} \lambda_i} > 0.$$

The characteristic polynomial of $\Delta_j^{(K_m)}$ is

$$\chi(x) = \prod_{i=1}^{r}(\lambda_i - x) = x^k \cdot \prod_{i=k+1}^{r}(\lambda_i - x) =: (-x)^k \cdot \psi(x).$$

Note that

$$\psi(0) = \det_{\mathrm{FK}}\big(\Delta_j^{(K_m)}\big)^r > 0.$$

Since this can also be computed as

$$\chi(x) = \det\Big(\Delta_j^{(K_m)} - x \cdot \mathrm{id}\Big)$$

and $\Delta_j^{(K_m)}$ is an integer matrix, we know that $\chi(x)$ has integer coefficients, and so the same must hold for $\psi(x)$! Therefore, $\psi(0) > 0$ implies $\psi(0) \geq 1$, and this proves the claim.

Now let F_m be the normalized spectral density function of $\Delta_j^{(K_m)}$. Choose $b > 0$ large enough that $\left\|\Delta_j^{(X)}\right\| \leq b$ and $\left\|\Delta_j^{(K_m)}\right\| \leq b$ for all m. (This is possible since the bound on the norm of the Laplacian only depends on the number of "neighbors" a cell can have, which is bounded throughout X.) We get for all m

$$\log\det_{\mathrm{FK}}(F_m) = \log\det_{\mathrm{FK}}\big(\Delta_j^{(K_m)}\big) \geq 0.$$

On the other hand,

$$\log\det_{\mathrm{FK}}(F_m) = \log(b)\, F_m^{\perp}(b) - \int_{(0,b]} \frac{F_m^{\perp}(\lambda)}{\lambda}\, \mathrm{d}\lambda,$$

so for all m

$$\log(b)\, F_m^{\perp}(b) \geq \int_{(0,b]} \frac{F_m^{\perp}(\lambda)}{\lambda}\, \mathrm{d}\lambda.$$

Since F_m converges pointwise[12] to $F = F\big(\Delta_j^{(X)}\big)$, there is $M \in \mathbb{N}$ such that for all $m \geq M$,

$$\log(b)\, F^{\perp}(b) \geq \int_{(0,b]} \frac{F_m^{\perp}(\lambda)}{\lambda}\, \mathrm{d}\lambda.$$

Now, Fatou's lemma gives

$$\log(b)\, F^{\perp}(b) \geq \int_{(0,b]} \frac{F^{\perp}(\lambda)}{\lambda}\, \mathrm{d}\lambda,$$

and therefore

$$\log\det_{\mathrm{FK}}(F) = \log(b)\, F^{\perp}(b) - \int_{(0,b]} \frac{F^{\perp}(\lambda)}{\lambda}\, \mathrm{d}\lambda \geq 0. \qquad \square$$

[12] Even uniformly, but we don't need that here.

5.2 Approximation

Let us explore under which circumstances the Fuglede–Kadison determinants of geometric operators can be approximated by finite-dimensional restrictions.[13]

By Theorem 3.11, the spectral density function of an operator T on a self-similar complex is approximated uniformly by those of restrictions of T to finite subcomplexes K_m. Recalling Lemma 5.3, we can express the determinants as follows (with $F^{\perp}(\lambda) = F(\lambda) - F(0)$ and $b \geq \|T\|$):

$$\log \det_{\mathrm{FK}}(F) = \log(b)\, F^{\perp}(b) - \int_{(0,b]} \frac{F^{\perp}(\lambda)}{\lambda}\, d\lambda.$$

However, as $1/\lambda$ is unbounded on $(0, b]$, even uniform convergence $F_m \to F$ will not ensure convergence of these integrals.

This problem can of course be avoided if the domain of integration is bounded away from zero. This is satisfied when T has a *spectral gap* at zero:

5.9 Theorem. *Let X be a self-similar complex with Følner sequence (K_m). Assume that $T \in \mathcal{A}_j(X)$ is positive and has a spectral gap at zero, that is,* $\operatorname{spec}(T) \cap (0, \varepsilon) = \emptyset$ *for some $\varepsilon > 0$. (This is equivalent to T being invertible when restricted to* $\ker(T)^{\perp}$*.)*

Define $T_m := P_m T I_m$, where $I_m \colon \ell^2(\mathcal{E}_j K_m) \to \ell^2(\mathcal{E}_j X)$ is the inclusion and $P_m \colon \ell^2(\mathcal{E}_j X) \to \ell^2(\mathcal{E}_j K_m)$ is the orthogonal projection. Then

$$\det_{\mathrm{FK}}(T) = \lim_{m\to\infty} \det_{\mathrm{FK}} T_m.$$

Proof. Note first that the spectrum of T does not depend on whether we consider it as an element of $\mathcal{B}(\ell^2(\mathcal{E}_j X))$ or an element of $\mathcal{B}(\mathcal{H}_j(X))$.

By assumption, there is $\varepsilon > 0$ such that

$$\langle v, Tv\rangle \geq \varepsilon \|v\|^2 \text{ for all } v \in \ker(T)^{\perp} \subseteq \ell^2(\mathcal{E}_j X).$$

Using $P_m^* = I_m$, we have

$$\langle w, T_m w\rangle = \langle w, P_m T I_m w\rangle = \langle w, Tw\rangle \text{ for all } w \in \ell^2(\mathcal{E}_j K_m),$$

so $\ker(T_m) = \ker(T) \cap \ell^2(\mathcal{E}_j K_m)$, and we obtain

$$\langle w, T_m w\rangle = \langle w, Tw\rangle \geq \varepsilon \|w\|^2 \text{ for all } w \in \ker(T_m)^{\perp} \subseteq \ell^2(\mathcal{E}_j K_m).$$

Consequently, both F^T and every F^{T_m} are constant on $[0, \varepsilon)$. Furthermore, note that $\|T_m\| \leq \|T\|$ for all m. As the function $\lambda \mapsto 1/\lambda$ is bounded on $[\varepsilon, \infty)$, uniform convergence $F^{T_m} \to F^T$ implies convergence of the integrals

$$\int_{[\varepsilon, \|T\|]} \frac{F^{T_m}(\lambda)}{\lambda}\, d\lambda \xrightarrow{m\to\infty} \int_{[\varepsilon, \|T\|]} \frac{F^{T}(\lambda)}{\lambda}\, d\lambda.$$

By Lemma 5.3, the claim follows. ∎

[13] In the classical case, the approximation of Fuglede–Kadison determinants of any operator $T \in \mathrm{Mat}_{r,s}(\mathbb{Q}G)$ is known if G is an infinite virtually cyclic group. For $G = \mathbb{Z}^n$ it is known that $\det_{\mathrm{FK}}(T) = \limsup_m \det_{\mathrm{FK}}(T_m)$. See [Lüc16], Remark 6.5.

In this lemma, we have used that a spectral gap of T directly implies that the restrictions of T to smaller subcomplexes have the exact same spectral gap. This would not necessarily work with the more general "approximating operators" in Theorem 3.11, as a single "rogue eigenvalue" getting too close to zero can destroy the convergence:

5.10 Example. Define a sequence of functions

$$G_m\colon [0,\infty) \to [0,\infty),\ \lambda \mapsto \begin{cases} 0, & \text{if } x \in [0, e^{-m}), \\ \frac{1}{m}, & \text{if } x \in [e^{-m}, 1), \\ 1, & \text{if } x \in [1, \infty). \end{cases}$$

Each G_m is non-decreasing, and the sequence (G_m) converges uniformly to

$$G\colon [0,\infty) \to [0,\infty),\ \lambda \mapsto \begin{cases} 0, & \text{if } x \in [0, 1), \\ 1, & \text{if } x \in [1, \infty). \end{cases}$$

Yet, one clearly has for all $m \in \mathbb{N}$

$$\begin{aligned} \log \det_{\mathrm{FK}}(G_m) &= \log(1)\, G_m(1) - \int_0^1 \frac{G_m(\lambda)}{\lambda}\, \mathrm{d}\lambda \\ &= 0 - \int_{e^{-m}}^1 \frac{1}{m\lambda}\, d\lambda \\ &= -\frac{\log(1) - \log(e^{-m})}{m} \\ &= -1, \\ \log \det_{\mathrm{FK}}(G) &= \log(1)\, G(1) - \int_0^1 \frac{G(\lambda)}{\lambda}\, \mathrm{d}\lambda \\ &= 0, \end{aligned}$$

and thus

$$\det_{\mathrm{FK}}(G_m) = \frac{1}{e} \not\to 1 = \det_{\mathrm{FK}}(G).$$

More generally, for any bounded non-decreasing function $F\colon [0,\infty) \to [0,\infty)$, define a sequence of functions $\widetilde{F}_m = F + G_m$. Then each $\widetilde{F}_m$ is non-decreasing, the sequence $(\widetilde{F}_m)$ converges uniformly to the function $\widetilde{F} = F + G$, and, since $\log \det_{\mathrm{FK}}(F)$ is linear in the function F,

$$\lim_{m\to\infty} \log \det_{\mathrm{FK}}(\widetilde{F}_m) = \log \det_{\mathrm{FK}}(F) - 1.$$

If the functions in counterexample 5.10 were indeed the spectral density functions of a sequence of operators, the lowest "eigenvalue" of those operators would have to decay exponentially in m, while their normalized multiplicity decayed only polynomially (indicating that the number of cells in K_m increased polynomially in m). This indicates that the convergence of determinants might be achievable in cases where the smallest positive eigenvalues of the restrictions T_m can be controlled:

Assume that $T \in \mathcal{A}_j(X)$ is of determinant class but without spectral gap, and let T_m, F_m and F be as above. Abbreviate $\varepsilon_m = \|F_m - F\|_\infty$ and let μ_m be the smallest positive eigenvalue of T_m. Since T has no spectral gap and F_m approximates F, we know that

$$\lim_{m\to\infty} \varepsilon_m = 0 = \lim_{m\to\infty} \mu_m.$$

Since $F_m^\perp(\lambda) = 0$ for $\lambda < \mu_m$, we obtain the estimate

$$\begin{aligned}\left|\int_0^b \frac{F^\perp(\lambda) - F_m^\perp(\lambda)}{\lambda}\,\mathrm{d}\lambda\right| &\leq \int_0^{\mu_m} \frac{F^\perp(\lambda)}{\lambda}\,\mathrm{d}\lambda + \int_{\mu_m}^b \frac{\varepsilon_m}{\lambda}\,\mathrm{d}\lambda \\ &\leq \int_0^{\mu_m} \frac{F^\perp(\lambda)}{\lambda}\,\mathrm{d}\lambda + \varepsilon_m \cdot (\log(b) - \log(\mu_m))\,.\end{aligned}$$

Since F is of determinant class, we know that

$$\int_0^{\mu_m} \frac{F^\perp(\lambda)}{\lambda}\,\mathrm{d}\lambda \xrightarrow{\mu_m\to 0} 0,$$

and obviously

$$\varepsilon_m \log(b) \xrightarrow{m\to\infty} 0.$$

Therefore, $\det_{\mathrm{FK}}(T_m)$ will converge to $\det_{\mathrm{FK}}(T)$ if

$$\varepsilon_m \log(\mu_m) \xrightarrow{m\to\infty} 0.$$

This would be satisfied if μ_m were bounded from below by any power of ε_m.

The proof of 3.11 shows that

$$\varepsilon_m := \|F_m - F\|_\infty = \mathcal{O}\left(\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|}\right).$$

Thus, ε_m falls faster if the complex is "more amenable".

The problem of bounding this smallest positive eigenvalue has been studied extensively for the Laplacians of graphs. However, the answer points in the wrong direction for our purpose: The smallest positive eigenvalue becomes "large", when the graph is a *magnifier*, that is, all subsets of the graph have "large" neighborhoods. In other words, we obtain a better bound on the smallest eigenvalue if the complex is *less* amenable! More concretely, consider the following result by Alon:

5.11 Lemma ([Alo86], Lemma 2.4)**.** *If K is a finite graph such that every vertex subset $A \subseteq \mathcal{E}_0 K$ with $|A| \leq \frac{1}{2}|\mathcal{E}_0 K|$ fulfills*

$$|\{\sigma \in \mathcal{E}_0 K \mid d_{\mathrm{comb}}(\sigma, A) = 1\}| \geq c\,|A|$$

for some constant $c > 0$, then the smallest positive eigenvalue of $\Delta_0^{(K)}$ is at least $c^2/(2c^2+4)$.

For well-chosen Følner sequences (K_m), these two properties can sometimes be reconciled. Essentially, this requires an exhaustion (K_m) such that no K_m has a proper subset that is "much more amenable" than K_m itself:

5.12 Theorem. *Let X be a self-similar complex and assume there is a self-similar Følner sequence (K_m) for which there are constants $c, e > 0$ such that for every vertex subset $A_m \subseteq \mathcal{E}_0 K_m$ with $|A| \leq \frac{1}{2} |\mathcal{E}_0 K_m|$,*

$$\frac{|\{\sigma \in \mathcal{E}_0 K_m \mid d_{\mathrm{comb}}(\sigma, A) = 1\}|}{|A|} \geq c \cdot \left(\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} \right)^e .$$

Then

$$\lim_{m \to \infty} \det_{\mathrm{FK}}\big(\Delta_0^{(K_m)}\big) = \det_{\mathrm{FK}}\big(\Delta_0^{(X)}\big).$$

Proof. Use the same notation as above. (Note, however, that unlike in Theorem 5.9, $\Delta_0^{(K_m)}$ is not the restriction of $\Delta_0^{(X)}$ to K_m.) By Alon's lemma and Theorem 3.11, we have $\mu_m \geq d\, c^2 \varepsilon_m^{2e}$, where $d > 0$ combines the factor 4 from 5.11 with the $\mathcal{O}$-constant of $\varepsilon_m = \mathcal{O}\Big(\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|}\Big)$, and that gives

$$|\varepsilon_m \log \mu_m| \leq \big|\varepsilon_m \cdot \big(\log(dc^2) + 2e \log(\varepsilon_m)\big)\big| \xrightarrow{m \to \infty} 0. \qquad \square$$

5.13 Example. (a) Consider the Laplacian on one-cells of $\mathbb{R}^d$ with the standard CW-structure (compare Example 3.2), and choose a Følner sequence consisting of cubes, $K_m = [-3^m, 3^m]^d$. We obtain

$$\varepsilon_m \sim \frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} \sim \frac{1}{3^m}$$

On the other hand, among subsets $A \subseteq \mathcal{E}_0 K_m$ with $|A| \leq \frac{1}{2} |\mathcal{E}_0 K_m|$, the one with the fewest frontiers in K_m will simply be a half-cube, containing $\frac{1}{2} \cdot (2 \cdot 3^m)^d$ vertices, of which $(2 \cdot 3^m)^{d-1}$ are frontiers, so 5.11 yields a lower bound

$$\mu_m \geq \left(\frac{c}{3^m}\right)^2$$

for some constant $c > 0$. This easily suffices to ensure convergence of determinants.

(b) For Sierpiński's triangle (see Example 3.3), there are numerous subcomplexes of any size with exactly three frontiers (namely, large subtriangles). Thus, the constant c in 5.11 frequently takes values proportional to $1/|A|$, but one can also choose (K_m) in such a way that $\varepsilon_m \sim 1/|\mathcal{E}_0 K_m|$. Again, we obtain $\mu_m \sim \varepsilon_m^2$, which suffices for convergence.

5.3 L^2-torsion

The Fuglede–Kadison determinants of Laplacians of a space are not homotopy invariants by themselves.[14] However, they give a rise to another L^2-invariant, the L^2-torsion.

Classically, L^2-torsion is defined as the logarithm of the alternating product of Fuglede–Kadison determinants of the differentials $\partial_j^{(X)}$. This is easily shown to be equal to the following expression in terms of the Laplacians, which shall serve as our definition:

5.14 Definition. The L^2-*torsion* of an aperiodically ordered CW-complex X is

$$\rho^{(2)}(X) = -\frac{1}{2} \sum_{j=0}^{\dim X} (-1)^j \cdot j \cdot \log \det_{\mathrm{FK}}\big(\Delta_j^{(X)}\big),$$

provided every $\Delta_j^{(X)}$ is of determinant class.

This definition makes sense for any self-similar complex, as their Laplacians are always of determinant class (see 5.8).

Whether the L^2-torsion is in fact a homotopy invariant remains a difficult question. In the classical case, this can be proven for G-CW-complexes that are "det-L^2-acyclic", that is, where every Laplacian is of determinant class and all L^2-Betti numbers vanish:

5.15 Theorem ([Lüc02], Theorem 3.93 and Lemma 13.6)**.** *Let X and Y be finite free G-CW-complexes, where G is an amenable group. If X and Y are G-homotopy equivalent and one of the two is det-L^2-acyclic, then both of them are det-L^2-acyclic and $\rho^{(2)}(X) = \rho^{(2)}(Y)$.*

Unfortunately, proving an analogous statement for the self-similar case is beyond the scope of this thesis: In the classical case, the L^2-torsion of X and Y will a priori differ by a term related to the Whitehead torsion of the homotopy equivalence between the two; the classical proof then relies on properties of the group G to ensure that this term (always) vanishes. In the aperiodical case, no such Whitehead torsion exists; it remains to be determined if there is a suitable analogue.

As an additional complication, even if the L^2-Betti numbers vanish, the L^2-chain complex of a self-similar complex may not be truly L^2-acyclic, since some Laplacians may have (small) kernels in $\ell^2(\mathcal{E}_j X)$ even if the induced operators on $\mathcal{H}_j(X)$ are injective. Finally, there can be no finite-dimensional analogue that carries over to the whole space by approximation, as the usual (non-L^2) chain complex of a finite CW-complex cannot be acyclic.

[14]Finite example: In a circle formed from three 0-cells and three 1-cells, we have $\det_{\mathrm{FK}}(\Delta_1) = \sqrt[3]{9} \approx 2.08$, while in a circle with four 0-cells and four 1-cells, $\det_{\mathrm{FK}}(\Delta_1) = \sqrt[4]{16} = 2$.

6 Product spaces

Intuitively, the cartesian product of two self-similar spaces should again be a self-similar space. That is indeed the case (although it takes a nonzero amount of work to prove amenability), and thus, it makes sense to try and express the L^2-invariants of such a product space through those of its factors. In this final chapter, we will derive such formulas for L^2-Betti numbers, Novikov–Shubin invariants and L^2-torsion.

In doing so, we hope to demonstrate once more how the approximation theorem for spectral density functions allows us to gain information about L^2-invariants, even when those invariants themselves cannot be approximated by finite-dimensional analogues.

6.1 Products of self-similar complexes are self-similar

6.1 Theorem. *Let X and Y be self-similar CW-complexes with self-similar exhaustions (K_m) respectively (L_m). Then $X \times Y$ is a self-similar CW-complex with self-similar exhaustion $(K_m \times L_m)$.*

Proof. Note first that the product of regular CW-complexes is again regular: For any two cells $\sigma \in \mathcal{E}_j X$ and $\tau \in \mathcal{E}_k Y$, the extended attaching maps $f_\sigma \colon \overline{D^j} \to \overline{\sigma} \subseteq X$ and $f_\tau \colon \overline{D^k} \to \overline{\tau} \subseteq Y$ are by assumption homeomorphisms, so the product attaching map $f_{\sigma\times\tau} \colon \overline{D^{j+k}} \approx \overline{D^j} \times \overline{D^k} \xrightarrow{f_\sigma \times f_\tau} \overline{\sigma} \times \overline{\tau} \subseteq X \times Y$ is again a homeomorphism.

Second, the product of bounded complexes is bounded: The topological boundary of a product cell is $\partial(\sigma \times \tau) = \partial\sigma \times \tau \cup \sigma \times \partial\tau$, and thus we get

$$\begin{aligned}
&|\{\kappa \in \mathcal{E}_{j+k-1}(X \times Y) \mid \kappa \subseteq \partial(\sigma \times \tau)\}| \\
&\qquad = \sum_{a+b=j+k-1} |\{\lambda \times \mu \in \mathcal{E}_a X \times \mathcal{E}_b Y \mid \lambda \subseteq \overline{\sigma},\, \mu \subseteq \overline{\tau})\}| \\
&\qquad \le |\{\lambda \in \mathcal{E}_* X \mid \lambda \subseteq \overline{\sigma}\}| \cdot |\{\mu \in \mathcal{E}_* Y \mid \mu \subseteq \overline{\tau})\}|
\end{aligned}$$

and analogously

$$\begin{aligned}
&|\{\kappa \in \mathcal{E}_{j+k+1}(X \times Y) \mid (\sigma \times \tau) \subseteq \partial\kappa\}| \\
&\qquad \le \left|\left\{\lambda \in \mathcal{E}_* X \,\middle|\, \sigma \subseteq \overline{\lambda}\right\}\right| \cdot |\{\mu \in \mathcal{E}_* Y \mid \tau \subseteq \overline{\mu})\}|
\end{aligned}$$

where $\mathcal{E}_* X = \bigcup_j \mathcal{E}_j X$ is the set of all cells of X. As X and Y are bounded, these numbers are bounded, showing that $X \times Y$ is bounded.

To show that $(K_m \times L_m)$ is again a Følner sequence, we need to understand the frontiers in a product space:

6.2 Lemma. *Let X, Y be regular CW-complexes. Let $\sigma_1 \in \mathcal{E}_{j_1} X$, $\sigma_2 \in \mathcal{E}_{j_2} X$, $\tau_1 \in \mathcal{E}_{k_1} Y$ and $\tau_2 \in \mathcal{E}_{k_2} Y$ such that $j_1 + k_1 = d = j_2 + k_2$. Then the cells $\sigma_1 \times \tau_1$ and $\sigma_2 \times \tau_2$ are adjacent in $\mathcal{E}_d(X \times Y)$ if and only if one of the following holds:*

(a) $(j_1, k_1) = (j_2, k_2)$, *σ_1 is adjacent to σ_2 in X and $\tau_1 = \tau_2$.*

(b) $(j_1, k_1) = (j_2, k_2)$, *$\sigma_1 = \sigma_2$ and τ_1 is adjacent to τ_2 in Y.*

(c) $(j_1, k_1) = (j_2 + 1, k_2 - 1)$, *$\partial\sigma_1 \supseteq \sigma_2$ and $\tau_1 \subseteq \partial\tau_2$.*

(d) $(j_1, k_1) = (j_2 - 1, k_2 + 1)$, *$\sigma_1 \subseteq \partial\sigma_2$ and $\partial\tau_1 \supseteq \tau_2$.*

Proof. [$\Longleftarrow$] follows immediately from $\partial(\sigma \times \tau) = (\partial\sigma \times \tau) \cup (\sigma \times \partial\tau)$.
[$\Longrightarrow$]: Assume that $\sigma_1 \times \tau_1$ and $\sigma_2 \times \tau_2$ share a $(j_1 + k_1 - 1)$-face $\sigma_3 \times \tau_3$:

$$\begin{aligned}
\sigma_3 \times \tau_3 &\subseteq \partial(\sigma_1 \times \tau_1) \cap \partial(\sigma_2 \times \tau_2) \\
&= ((\partial\sigma_1 \times \tau_1) \cup (\sigma_1 \times \partial\tau_1)) \cap ((\partial\sigma_2 \times \tau_2) \cup (\sigma_2 \times \partial\tau_2)) \\
&= ((\partial\sigma_1 \times \tau_1) \cap (\partial\sigma_2 \times \tau_2)) \cup ((\partial\sigma_1 \times \tau_1) \cap (\sigma_2 \times \partial\tau_2)) \\
&\quad \cup ((\sigma_1 \times \partial\tau_1) \cap (\partial\sigma_2 \times \tau_2)) \cup ((\sigma_1 \times \partial\tau_1) \cap (\sigma_2 \times \partial\tau_2)) \\
&= ((\partial\sigma_1 \cap \partial\sigma_2) \times (\tau_1 \cap \tau_2)) \cup ((\partial\sigma_1 \cap \sigma_2) \times (\tau_1 \cap \partial\tau_2)) \\
&\quad \cup ((\sigma_1 \cap \partial\sigma_2) \times (\partial\tau_1 \cap \tau_2)) \cup ((\sigma_1 \cap \sigma_2) \times (\partial\tau_1 \cap \partial\tau_2))
\end{aligned}$$

By Lemma 2.2, $\sigma_3 \times \tau_3$ must be fully contained in one of the four "$\cup$-summands". For purely dimensional reasons (and because any two open cells are either identical or disjoint),
$\sigma_3 \times \tau_3 \subseteq (\partial\sigma_1 \cap \partial\sigma_2) \times (\tau_1 \cap \tau_2)$ implies case (a),
$\sigma_3 \times \tau_3 \subseteq (\partial\sigma_1 \cap \sigma_2) \times (\tau_1 \cap \partial\tau_2)$ implies case (c),
$\sigma_3 \times \tau_3 \subseteq (\sigma_1 \cap \partial\sigma_2) \times (\partial\tau_1 \cap \tau_2)$ implies case (d),
$\sigma_3 \times \tau_3 \subseteq (\sigma_1 \cap \sigma_2) \times (\partial\tau_1 \cap \partial\tau_2)$ implies case (b).

Conversely, assume there is a $(j_1 + k_1 + 1)$-cell $\sigma_4 \times \tau_4$ whose boundary contains both $\sigma_1 \times \tau_1$ and $\sigma_2 \times \tau_2$:

$$(\sigma_1 \times \tau_1) \cup (\sigma_2 \times \tau_2) \subseteq (\partial\sigma_4 \times \tau_4) \cup (\sigma_4 \times \partial\tau_4)$$

Using Lemma 2.2 again, each of the two $\sigma_i \times \tau_i$ must be contained in $\partial\sigma_4 \times \tau_4$ or in $\sigma_4 \times \partial\tau_4$. We obtain the following cases:
$(\sigma_1 \times \tau_1) \cup (\sigma_2 \times \tau_2) \subseteq \partial\sigma_4 \times \tau_4$ implies case (a),
$(\sigma_1 \times \tau_1) \cup (\sigma_2 \times \tau_2) \subseteq \sigma_4 \times \partial\tau_4$ implies case (b),
$\sigma_1 \times \tau_1 \subseteq \partial\sigma_4 \times \tau_4$ and $\sigma_2 \times \tau_2 \subseteq \sigma_4 \times \partial\tau_4$ implies case (d),
$\sigma_1 \times \tau_1 \subseteq \sigma_4 \times \partial\tau_4$ and $\sigma_2 \times \tau_2 \subseteq \partial\sigma_4 \times \tau_4$ implies case (c). □

Proof of Theorem 6.1, continued. Let $\mathcal{G}(m, m+1)$ and $\mathcal{G}'(m, m+1)$ be the sets of local isomorphisms for X resp. Y associated to the Følner sequences (K_m) and (L_m). It is clear that

$$K_{m+1} \times L_{m+1} = \bigcup_{\substack{\gamma \in \mathcal{G}(m,m+1) \\ \delta \in \mathcal{G}'(m,m+1)}} (\gamma \times \delta)(K_m \times L_m).$$

It remains to show that the exhaustion $(K_m \times L_m)$ is amenable (i. e. a Følner sequence).

Let $\sigma_1 \in \mathcal{E}_j X$ and $\tau_1 \in \mathcal{E}_k Y$ such that $\sigma_1 \times \tau_1$ is a frontier of $K_m \times L_m$. Let $\sigma_2 \times \tau_2$ be a cell of $(X \times Y) \setminus (K_m \times L_m)$ adjacent to it and apply Lemma 6.2. Case (a) implies $\sigma_1 \in \mathcal{F}_j K_m$ and case (b) implies $\tau_1 \in \mathcal{F}_k L_m$. In case (c), $\sigma_2 \subseteq \partial\sigma_1 \subseteq K_m$, so $\tau_2 \nsubseteq L_m$. As L_m is a full subcomplex, τ_2 must have a face outside L_m, and that face is adjacent (via τ_2) to τ_1, so $\tau_1 \in \mathcal{F}_k L_m$. Analogously, we get $\sigma_1 \in \mathcal{F}_j K_m$ in case (d). Thus, we obtain indeed

$$\begin{aligned}
\frac{|\mathcal{F}_d(K_m \times L_m)|}{|\mathcal{E}_d(K_m \times L_m)|} &\leq \frac{\sum_{j+k=d} |(\mathcal{F}_j K_m \times \mathcal{E}_k L_m) \cup (\mathcal{E}_j K_m \times \mathcal{F}_k L_m)|}{\sum_{j+k=d} |\mathcal{E}_j K_m \times \mathcal{E}_k L_m|} \\
&\leq \sum_{j+k=d} \frac{|(\mathcal{F}_j K_m \times \mathcal{E}_k L_m) \cup (\mathcal{E}_j K_m \times \mathcal{F}_k L_m)|}{|\mathcal{E}_j K_m \times \mathcal{E}_k L_m|} \\
&\leq \sum_{j+k=d} \left(\frac{|\mathcal{F}_j K_m \times \mathcal{E}_k L_m|}{|\mathcal{E}_j K_m \times \mathcal{E}_k L_m|} + \frac{|\mathcal{E}_j K_m \times \mathcal{F}_k L_m|}{|\mathcal{E}_j K_m \times \mathcal{E}_k L_m|} \right) \\
&= \sum_{j+k=d} \left(\frac{|\mathcal{F}_j K_m|}{|\mathcal{E}_j K_m|} + \frac{|\mathcal{F}_k L_m|}{|\mathcal{E}_k L_m|} \right) \xrightarrow{m\to\infty} 0. \qquad \square
\end{aligned}$$

6.2 L^2-Betti numbers of product spaces

If K and L are finite CW-complexes, then $K \times L$ is a finite CW-complex whose ℓ-cells are given by

$$\mathcal{E}_\ell(K \times L) \cong \bigcup_{j+k=\ell} \mathcal{E}_j K \times \mathcal{E}_k L.$$

And in perfect analogy, their (non-normalized) Betti numbers are given by the *Künneth formula*

$$\beta_\ell(K \times L; \mathbb{C}) = \sum_{j+k=\ell} \beta_j(K; \mathbb{C}) \cdot \beta_k(L; \mathbb{C}).$$

Until now, we have usually normalized the trace on $\ell^2(\mathcal{E}_j X) = \mathbb{C}[\mathcal{E}_j X]$ by the number of j-cells themselves (such that the trace becomes a state). Unfortunately, this normalization is incompatible with the Künneth formula:

$$\frac{\beta_\ell(K \times L; \mathbb{C})}{|\mathcal{E}_\ell(K \times L)|} = \frac{\sum_{j+k=\ell} \beta_j(K; \mathbb{C}) \cdot \beta_k(L; \mathbb{C})}{\sum_{j+k=\ell} |\mathcal{E}_j K \times \mathcal{E}_j L|} \neq \sum_{j+k=\ell} \frac{\beta_j(K; \mathbb{C})}{|\mathcal{E}_j K|} \cdot \frac{\beta_j(K; \mathbb{C})}{|\mathcal{E}_k L|}$$

Getting the Künneth formula to work with our approximation therefore requires a different normalization. The easiest solution is to always normalize by the number of vertices instead: As a zero-cell in $K \times L$ must be the product of two zero-cells of K and L, we have

$$|\mathcal{E}_0(K \times L)| = |\mathcal{E}_0 K| \cdot |\mathcal{E}_0 L|\,,$$

and thus the renormalized Künneth formula holds:

$$\frac{\beta_\ell(K \times L; \mathbb{C})}{|\mathcal{E}_0(K \times L)|} = \sum_{j+k=\ell} \frac{\beta_j(K; \mathbb{C})}{|\mathcal{E}_0 K|} \cdot \frac{\beta_j(K; \mathbb{C})}{|\mathcal{E}_0 L|}$$

Now, the approximation theorem for L^2-Betti numbers yields a Künneth formula for self-similar complexes:

6.3 Theorem. *Let X and Y be self-similar complexes, and normalize every trace by the numbers of vertices. Then L^2-Betti numbers fulfill the Künneth formula:*

$$b_\ell^{(2)}(X \times Y) = \sum_{j+k=\ell} b_j^{(2)}(X) \cdot b_k^{(2)}(Y).$$

Proof. Let (K_m) and (L_m) be Følner sequences for X resp. Y. By Theorem 6.1, $(K_m \times L_m)$ is a Følner sequence for $X \times Y$, and Corollary 4.2 together with Lemma 3.17 gives us

$$\begin{aligned} b_\ell^{(2)}(X \times Y) &= \lim_{m\to\infty} \frac{\beta_\ell(K_m \times L_m)}{|\mathcal{E}_0(K_m \times L_m)|} = \sum_{j+k=\ell} \lim_{m\to\infty} \frac{\beta_j(K_m)}{|\mathcal{E}_0 K_m|} \cdot \lim_{m\to\infty} \frac{\beta_k(L_m)}{|\mathcal{E}_0 L_m|} \\ &= \sum_{j+k=\ell} b_j^{(2)}(X) \cdot b_k^{(2)}(Y). \end{aligned}$$ □

6.3 Novikov–Shubin invariants of product spaces

While there is no approximation theorem for Novikov–Shubin invariants, we can make use of the approximation theorem for the spectral density functions themselves to obtain a formula for the Novikov–Shubin invariants of a product space.

For any two finite subcomplexes K, L, the Laplacian of the product is given by

$$\Delta_\ell^{(K\times L)} = \bigoplus_{j+k=\ell} \left(\Delta_j^{(K)} \otimes \mathrm{id}_{\ell^2(\mathcal{E}_k L)} + \mathrm{id}_{\ell^2(\mathcal{E}_j K)} \otimes \Delta_k^{(L)}\right),$$

and for the non-normalized spectral density functions of operators on finite-dimensional spaces, we have

$$F_{\mathrm{nn}}(f \oplus g) = F_{\mathrm{nn}}(f) + F_{\mathrm{nn}}(g)$$

and (compare [Lüc02], Lemma 2.31)

$$F_{\mathrm{nn}}(f)(\lambda/2) \cdot F_{\mathrm{nn}}(g)(\lambda/2) \;\le\; F_{\mathrm{nn}}(f \otimes \mathrm{id} + \mathrm{id} \otimes g)(\lambda) \;\le\; F_{\mathrm{nn}}(f)(\lambda) \cdot F_{\mathrm{nn}}(g)(\lambda).$$

Therefore, the non-normalized spectral density function of $\Delta_\ell^{(K\times L)}$ fulfills

$$F_{\mathrm{nn}}\big(\Delta_\ell^{(K\times L)}\big) \simeq \sum_{j+k=\ell} F_{\mathrm{nn}}\big(\Delta_j^{(K)}\big) \cdot F_{\mathrm{nn}}\big(\Delta_k^{(L)}\big),$$

where the dilational equivalence only dilates by a factor of 2.

This still holds if we normalize with the number of vertices, as $|\mathcal{E}_0(K \times L)| = |\mathcal{E}_0 K| \cdot |\mathcal{E}_0 L|$, and as the dilatational equivalence uses the same constant (namely, 2) for every subcomplex, the approximation theorem 3.11 yields

$$F\big(\Delta_\ell^{(X\times Y)}\big) \simeq \sum_{j+k=\ell} F\big(\Delta_j^{(X)}\big) \cdot F\big(\Delta_k^{(Y)}\big).$$

Finally, Lemma 4.7 gives $\alpha(F+G) = \min\bigl(\alpha(F), \alpha(G)\bigr)$, and therefore we arrive at the following statement, analogous to [Lüc02], Lemma 2.35 (1):

6.4 Lemma. *Let X and Y be self-similar complexes. Then*

$$\alpha_\ell(X \times Y) = \min_{j+k=\ell} \alpha\Bigl(F\bigl(\Delta_j^{(X)}\bigr) \cdot F\bigl(\Delta_k^{(Y)}\bigr)\Bigr)\,.$$

It remains to express $\alpha(F \cdot G)$ in terms of $\alpha(F)$ and $\alpha(G)$. Recalling the notation $F^\perp(\lambda) = F(\lambda) - F(0)$, we have

$$\begin{aligned}\alpha(F \cdot G) &= \alpha\bigl(F^\perp \cdot G^\perp + F^\perp \cdot G(0) + F(0) \cdot G^\perp + F(0) \cdot G(0)\bigr)\\ &= \min\bigl\{\alpha(F^\perp \cdot G^\perp), \alpha(F^\perp \cdot G(0)), \alpha(F(0) \cdot G^\perp), \alpha(F(0) \cdot G(0))\bigr\}\end{aligned}$$

The constant function $F(0) \cdot G(0)$ has $\alpha = \infty^+$, so the last term is irrelevant. The middle terms can be relevant, depending on $F(0)$ and $G(0)$:

$$\alpha(F^\perp \cdot G(0)) = \begin{cases}\alpha(F^\perp) = \alpha(F) & \text{if } G(0) > 0,\\ \infty^+ & \text{if } G(0) = 0.\end{cases}$$

$$\alpha(F(0) \cdot G^\perp) = \begin{cases}\alpha(G^\perp) = \alpha(G) & \text{if } F(0) > 0,\\ \infty^+ & \text{if } F(0) = 0.\end{cases}$$

The first term is always relevant, but not always obvious: We have

$$\alpha(F^\perp \cdot G^\perp) = \liminf_{\lambda \to 0} \frac{\log(F^\perp(\lambda) \cdot G^\perp(\lambda))}{\log(\lambda)} = \liminf_{\lambda \to 0} \frac{\log(F^\perp(\lambda)) + \log(G^\perp(\lambda))}{\log(\lambda)},$$

but a limit inferior, unlike a true limit, does not necessarily commute with addition!

Thus, the final theorem about Novikov–Shubin invariants of product spaces is only guaranteed to be true if the complexes fulfill the limit property (see Remark 4.6). Fortunately, the arguments outlined above also show that if X and Y have the limit property, then so does $X \times Y$.

6.5 Theorem. *Let X and Y be self-similar complexes satisfying the limit property. Then*

$$\begin{aligned}\alpha_\ell(X \times Y) = \min\Bigl(&\{\alpha_j(X) + \alpha_k(Y) \,|\, j + k = \ell\}\\ &\cup \Bigl\{\alpha_j(X) \,\Big|\, b^{(2)}_{\ell-j}(Y) > 0\Bigr\} \cup \Bigl\{\alpha_k(Y) \,\Big|\, b^{(2)}_{\ell-k}(X) > 0\Bigr\}\Bigr)\end{aligned}$$

Unlike in Theorems 6.3 and 6.6, the choice of normalization does not actually matter in this theorem, since replacing a spectral density function F by $r \cdot F$ for any constant $r > 0$ does not change the Novikov–Shubin invariant of F (compare Lemma 4.7).

6.4 L^2-torsion of product spaces

Finally, we aim to prove the following formula for the L^2-torsion of a product space:

6.6 Theorem. *Let X and Y be self-similar complexes, and normalize every trace by the numbers of vertices. Then*

$$\rho^{(2)}(X \times Y) = \chi^{(2)}(X)\,\rho^{(2)}(Y) + \chi^{(2)}(Y)\,\rho^{(2)}(X),$$

where $\chi^{(2)}$ denotes the L^2-Euler characteristic.

Although the torsion of $X \times Y$ is not necessarily equal to the limit of the torsions of $K_m \times L_m$, it is worth it to prove this formula for finite complexes first – every lemma in the following subsection will be needed for the proof of Theorem 6.6.

Torsion in the finite-dimensional case

6.7 Definition. Let (C_*, c_*) be a finite-dimensional chain complex of $\mathbb{C}$-vector spaces (i. e. all but finitely many C_p are 0, and the non-zero ones are finite-dimensional). Define the Laplacians of C_* by $\Delta_p^{(C)} = c_{p+1}c_{p+1}^* + c_p^*c_p$, and let $F\big(\Delta_p^{(C)}\big)$ be their (non-normalized) spectral density functions.[15] Then define the *torsion* of C_* by

$$\rho(C_*) = -\frac{1}{2}\sum_p (-1)^p \cdot p \cdot \log \det\nolimits_{\mathrm{FK}} F\big(\Delta_p^{(C)}\big).$$

6.8 Theorem. *Let C_* and D_* be chain complexes as above. Then*

$$\rho(C_* \otimes D_*) = \chi(C_*)\,\rho(D_*) + \chi(D_*)\,\rho(C_*).$$

The proof of this builds on two main lemmas. The first one explains why the Euler characteristic appears in the formula:

6.9 Lemma. *Let C_* be a chain complex as above. Then*

$$\sum_p (-1)^p F\big(\Delta_p^{(C)}\big) = \chi(C_*) \cdot \chi_{[0,\infty)}.$$

Proof. We have an orthogonal decomposition

$$C_p = \ker \Delta_p^C \oplus \operatorname{im}(c_{p+1}) \oplus \ker(c_p)^\perp.$$

Define $c_p^\perp \colon \ker(c_p)^\perp \to \operatorname{im}(c_p)$ as the restriction of c_p. Then the Laplacian decomposes as

$$\Delta_p^{(C)} = 0 \oplus c_{p+1}^\perp (c_{p+1}^\perp)^* \oplus (c_p^\perp)^* c_p^\perp$$

[15] While we are dealing purely with the finite-dimensional case, the non-normalized spectral density functions will simply be denoted F instead of F_{nn}.

Therefore, and because $F(f^*f) = F(ff^*)$,

$$\begin{aligned} F\big(\Delta_p^{(C)}\big) &= \dim\big(\ker \Delta_p^C\big) \cdot \chi_{[0,\infty)} + F\big(c_{p+1}^\perp (c_{p+1}^\perp)^*\big) + F\big((c_p^\perp)^* c_p^\perp\big) \\ &= \beta_p(C_*) \cdot \chi_{[0,\infty)} + F\big((c_{p+1}^\perp)^* c_{p+1}^\perp\big) + F\big((c_p^\perp)^* c_p^\perp\big)\,. \end{aligned}$$

This gives

$$\begin{aligned} &\sum_p (-1)^p F\big(\Delta_p^{(C)}\big) \\ &= \sum_p (-1)^p \beta_p(C_*) \cdot \chi_{[0,\infty)} + \sum_p (-1)^p F\big((c_{p+1}^\perp)^* c_{p+1}^\perp\big) + \sum_p (-1)^p F\big((c_p^\perp)^* c_p^\perp\big) \\ &= \chi(C_*) \cdot \chi_{[0,\infty)} - \sum_p (-1)^p F\big((c_p^\perp)^* c_p^\perp\big) + \sum_p (-1)^p F\big((c_p^\perp)^* c_p^\perp\big) \\ &= \chi(C_*) \cdot \chi_{[0,\infty)}. \end{aligned}$$

□

The second main lemma expresses spectral density functions of the product complex via spectral density functions of the two “factors”:

6.10 Lemma. *Let C_* and D_* be chain complexes as above. Then*

$$F\big(\Delta_n^{(C\otimes D)}\big) = \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} F\big(\Delta_p^{(C)}\big) * F\big(\Delta_q^{(D)}\big),$$

where $*$ denotes convolution.

Proof. Note first:

$$\begin{aligned} \chi_{[\mu,\infty)} * \chi_{[\nu,\infty)}(\lambda) &= \int_{-\infty}^{+\infty} \chi_{[\mu,\infty)}(\xi) \cdot \chi_{[\nu,\infty)}(\lambda - \xi)\,\mathrm{d}\xi \\ &= \int_{-\infty}^{+\infty} \chi_{[\mu,\infty)}(\xi) \cdot \chi_{(-\infty,\lambda-\nu]}(\xi)\,\mathrm{d}\xi \\ &= \begin{cases} \lambda - \mu - \nu, & \text{if } \lambda \geq \mu + \nu, \\ 0, & \text{if } \lambda \leq \mu + \nu, \end{cases} \end{aligned}$$

and therefore, for almost all λ,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \chi_{[\mu,\infty)} * \chi_{[\nu,\infty)}(\lambda) = \left.\begin{cases} 1, & \text{if } \lambda > \mu + \nu, \\ 0, & \text{if } \lambda < \mu + \nu, \end{cases}\right\} = \chi_{[\mu+\nu,\infty)}(\lambda).$$

By definition, the tensor product of chain complexes has the form

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and it is known that

$$\Delta_n^{(C\otimes D)} = \bigoplus_{p+q=n} \left(\Delta_p^{(C)} \otimes \mathrm{id}_{D_q} + \mathrm{id}_{C_p} \otimes \Delta_q^{(D)}\right),$$

which immediately implies

$$F\big(\Delta_n^{(C\otimes D)}\big) = \sum_{p+q=n} F\big(\Delta_p^{(C)} \otimes \mathrm{id}_{D_q} + \mathrm{id}_{C_p} \otimes \Delta_q^{(D)}\big)\,.$$

Define $m_p = \dim(C_p)$ and $n_q = \dim(D_q)$. Let $(\mu_{p,i})_{i=1}^{m_p}$ be the eigenvalues of $\Delta_p^{(C)}$ and $(\nu_{q,j})_{j=1}^{n_q}$ be the eigenvalues of $\Delta_q^{(D)}$. Then the eigenvalues of $\Delta_p^{(C)} \otimes \mathrm{id}_{D_q} + \mathrm{id}_{C_p} \otimes \Delta_q^{(D)}$ are exactly $(\mu_{p,i} + \nu_{q,j})_{i,j}$, and we obtain

$$\begin{aligned}
\sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} F\big(\Delta_p^{(C)}\big) * F\big(\Delta_q^{(D)}\big) &= \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\sum_{i=1}^{m_p} \chi_{[\mu_{p,i},\infty)} * \sum_{j=1}^{n_q} \chi_{[\nu_{q,j},\infty)} \right) \\
&= \sum_{p+q=n} \sum_{i=1}^{m_p} \sum_{j=1}^{n_q} \frac{\mathrm{d}}{\mathrm{d}\lambda} \big(\chi_{[\mu_{p,i},\infty)} * \chi_{[\nu_{q,j},\infty)}\big) \\
&= \sum_{p+q=n} \sum_{i=1}^{m_p} \sum_{j=1}^{n_q} \chi_{[\mu_{p,i}+\nu_{q,j},\infty)} \\
&= \sum_{p+q=n} F\big(\Delta_p^{(C)} \otimes \mathrm{id}_{D_q} + \mathrm{id}_{C_p} \otimes \Delta_q^{(D)}\big) \\
&= F\big(\Delta_n^{(C\otimes D)}\big)\,. \qquad \square
\end{aligned}$$

Now we are ready for the proof of Theorem 6.8. Unlike the lemmas, this proof will not be needed to prove Theorem 6.6; but it uses the same methods and can largely serve as a blueprint for the proof of that theorem.

Proof of Theorem 6.8. Choose $b > a > 0$ such that $b > \mu_{p,i} + \nu_{q,j}$ hold for all p, q, i, j and $\mu_{p,i} > a$, $\nu_{q,j} > a$ holds whenever $\mu_{p,i} > 0$, $\nu_{q,j} > 0$. (This is possible since C_* and D_* are finite-dimensional.)

Start by using the linearity of the determinant:

$$\begin{aligned}
\rho(C_* \otimes D_*) &= -\frac{1}{2} \sum_n (-1)^n \cdot n \cdot \log \det\nolimits_{\mathrm{FK}}\big(\Delta_n^{(C\otimes D)}\big) \\
&= -\frac{1}{2} \log \det\nolimits_{\mathrm{FK}} \left(\sum_n (-1)^n \cdot n \cdot F\big(\Delta_n^{(C\otimes D)}\big) \right)
\end{aligned}$$

(Note that only finitely many terms in the sum are non-zero.)

Now compute this sum of spectral density functions using Lemma 6.9:

$$\begin{aligned}
\sum_n (-1)^n \cdot n \cdot F\big(\Delta_n^{(C\otimes D)}\big) &= \sum_n (-1)^n \cdot n \cdot \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} F\big(\Delta_p^{(C)}\big) * F\big(\Delta_q^{(D)}\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda} \sum_p (-1)^p F\big(\Delta_p^{(C)}\big) * \sum_q (-1)^q \cdot q \cdot F\big(\Delta_q^{(D)}\big) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}\lambda} \sum_p (-1)^p \cdot p \cdot F\big(\Delta_p^{(C)}\big) * \sum_q (-1)^q F\big(\Delta_q^{(D)}\big)
\end{aligned}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}\,\big(\chi(C_*)\cdot\chi_{[0,\infty)}\big) * \sum_q (-1)^q\cdot q\cdot F\big(\Delta_q^{(D)}\big)$$
$$+ \frac{\mathrm{d}}{\mathrm{d}\lambda}\,\sum_p (-1)^p\cdot p\cdot F\big(\Delta_p^{(C)}\big) * \big(\chi(D_*)\cdot\chi_{[0,\infty)}\big)$$

As in the proof of Lemma 6.10, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\chi_{[\mu,\,\infty)} * \chi_{[0,\,\infty)}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\,\chi_{[0,\,\infty)} * \chi_{[\mu,\,\infty)}(\lambda) = \chi_{[\mu,\,\infty)}(\lambda)$$

for all μ and almost all λ (namely, $\lambda \neq \mu$). Since $F\big(\Delta_p^{(C)}\big)$ and $F\big(\Delta_q^{(D)}\big)$ are linear combinations of such step functions, we get for almost all λ

$$\sum_n (-1)^n\cdot n\cdot F\big(\Delta_n^{(C\otimes D)}\big) = \chi(C_*)\cdot\sum_q (-1)^q\cdot q\cdot F\big(\Delta_q^{(D)}\big)$$
$$+ \chi(D_*)\cdot\sum_p (-1)^p\cdot p\cdot F\big(\Delta_p^{(C)}\big)\,.$$

Again by linearity of $\log\det_{\mathrm{FK}}$, we obtain the claim:

$$\begin{aligned}
\rho(C_*\otimes D_*) &= -\frac{1}{2}\log\det\nolimits_{\mathrm{FK}}\left(\sum_n (-1)^n\cdot n\cdot F\big(\Delta_n^{(C\otimes D)}\big)\right)\\
&= \chi(C_*)\cdot\Big(-\frac{1}{2}\Big)\sum_q (-1)^q\cdot q\cdot\log\det\nolimits_{\mathrm{FK}} F\big(\Delta_q^{(D)}\big)\\
&\quad + \chi(D_*)\cdot\Big(-\frac{1}{2}\Big)\sum_p (-1)^p\cdot p\cdot\log\det\nolimits_{\mathrm{FK}}\big(F\big(\Delta_p^{(C)}\big)\big)\\
&= \chi(C_*)\,\rho(D_*) + \chi(D_*)\,\rho(C_*).
\end{aligned}$$

□

Torsion in the infinite case

We are now moving towards the proof of Theorem 6.6. To carry over as much as possible from the previous subsection, we need some analytical preparations:

6.11 Lemma. *Let $f_m, g_m\colon \mathbb{R}\to\mathbb{R}$ be two sequences of functions such that*

$$0\le f_m\le 1,\quad 0\le g_m\le 1\quad\text{and}\quad f_m(\lambda) = 0 = g_m(\lambda)\ \text{for all}\ \lambda\le 0$$

hold for all m. Assume that $f_m \xrightarrow{m\to\infty} f$ and $g_m \xrightarrow{m\to\infty} g$ uniformly.

Then $f_m * g_m \xrightarrow{m\to\infty} f*g$, and the convergence is uniform on any compact interval.

Proof.

$$\begin{aligned}
|f_m * g_m(\lambda) - f*g(\lambda)| &\le |(f_m - f)*g_m(\lambda)| + |f*(g_m-g)(\lambda)|\\
&\le \int_0^\lambda |f_m(\xi)-f(\xi)|\,|g_m(\lambda-\xi)|\;\mathrm{d}\xi\\
&\quad + \int_0^\lambda |f(\xi)|\,|g_m(\lambda-\xi)-g(\lambda-\xi)|\;\mathrm{d}\xi\\
&\le |\lambda|\,\|f_m-f\|_\infty\,\|g\|_\infty + |\lambda|\,\|f\|_\infty\,\|g_m-g\|_\infty\,.
\end{aligned}$$

By assumption, the last line converges to 0 for $m \to \infty$. On any compact interval, $|\lambda|$ is bounded and the convergence is uniform. $\square$

We also need to make sure that even in a sequence of only almost everywhere differentiable functions, limit and derivative can be exchanged:

6.12 Theorem ([Heu09], Theorem 104.3, slightly generalized)**.** *Let $f_m \colon [a,b] \to \mathbb{R}$ be a sequence of continuous almost everywhere differentiable functions, such that $\lim_{m\to\infty} f_m(\lambda_0)$ exists for at least one $\lambda_0 \in [a,b]$ and the sequence of derivatives f'_m is almost everywhere uniformly convergent (that is, there is a set $N \subset [a,b]$ of measure zero such that f'_m converges uniformly on $[a,b] \setminus N$).*

Then the sequence f_m is uniformly convergent, its limit is almost everywhere differentiable, and $\lim_{m\to\infty} \frac{df_m}{d\lambda} = \frac{d}{d\lambda} \lim_{m\to\infty} f_m$ almost everywhere.

Proof. Let f_m be differentiable on $[a,b] \setminus N_m$, where N_m has measure zero. Define $N = \bigcup_{m=1}^{\infty} N_m$; this is again a set of measure zero.

Fix $\varepsilon > 0$. By assumption there is $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have

$$\begin{aligned} |f_m(\lambda_0) - f_n(\lambda_0)| &< \frac{\varepsilon}{2}, \\ |f'_m(\lambda) - f'_n(\lambda)| &< \frac{\varepsilon}{2(b-a)} \quad \text{for all } \lambda \in [a,b] \setminus N. \end{aligned}$$

Since all f_m are continuous, the mean value theorem still applies: For all $a \leq \alpha < \beta \leq b$,

$$\begin{aligned} && f_m(\beta) - f_m(\alpha) &= \int_{[\alpha,\beta]\setminus N} f'_m(\lambda) \,\mathrm{d}\lambda \\ \Longrightarrow && |f_m(\beta) - f_m(\alpha)| &\leq (\beta - \alpha) \cdot \sup_{\lambda \in [\alpha,\beta]\setminus N} |f'_m(\lambda)| \,. \end{aligned}$$

Therefore, for all $m, n \geq n_0$ and all $\lambda, \mu \in [a,b]$,

$$\big|\big(f_m(\lambda) - f_n(\lambda)\big) - \big(f_m(\mu) - f_n(\mu)\big)\big| < |\lambda - \mu| \cdot \frac{\varepsilon}{2(b-a)}.$$

On the one hand, putting $\mu = \lambda_0$, this implies

$$|f_m(\lambda) - f_n(\lambda)| < |f_m(\lambda_0) - f_n(\lambda_0)| + \frac{\varepsilon\,|\lambda - \mu|}{2(b-a)} < \varepsilon,$$

so the sequence f_m is uniformly convergent.

On the other hand, we can re-order this to get

$$\big|\big(f_m(\lambda) - f_m(\mu)\big) - \big(f_n(\lambda) - f_n(\mu)\big)\big| < |\lambda - \mu| \cdot \frac{\varepsilon}{2(b-a)}$$

and, after division by $|\lambda - \mu|$,

$$\left|\frac{f_m(\lambda) - f_m(\mu)}{\lambda - \mu} - \frac{f_n(\lambda) - f_n(\mu)}{\lambda - \mu}\right| < \frac{\varepsilon}{2(b-a)}.$$

Therefore, the sequence of functions $(\lambda, \mu) \mapsto \frac{f_m(\lambda)-f_m(\mu)}{\lambda-\mu}$ converges uniformly.

Now let $\lambda \in [a,b] \setminus N$. Then $f'_m(\lambda) = \lim_{\mu\to\lambda} \frac{f_m(\lambda)-f_m(\mu)}{\lambda-\mu}$ exists for all m, and therefore the limits

$$\lim_{m\to\infty} \frac{df_m}{d\lambda} = \lim_{m\to\infty} \lim_{\mu\to\lambda} \frac{f_m(\lambda) - f_m(\mu)}{\lambda - \mu}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \lim_{m\to\infty} f_m = \lim_{\mu\to\lambda} \lim_{m\to\infty} \frac{f_m(\lambda) - f_m(\mu)}{\lambda - \mu}$$

both exist and are equal (compare the same book by Heuser, Theorem 104.1). □

Now we can prove an analogue of Lemma 6.10 for self-similar complexes.

6.13 Lemma. *Let X and Y be self-similar CW-complexes as above. Then*

$$F\big(\Delta_n^{(X\times Y)}\big) = \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} F\big(\Delta_p^{(X)}\big) * F\big(\Delta_q^{(Y)}\big),$$

where the sum runs over all $(p,q) \in \mathbb{Z}^2$ such that $p + q = n$.

Proof. By Lemma 6.10, we get for each m

$$F_{\mathrm{nn}}\big(\Delta_n^{(K_m\times L_m)}\big) = \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} F_{\mathrm{nn}}\big(\Delta_p^{(K_m)}\big) * F_{\mathrm{nn}}\big(\Delta_q^{(L_m)}\big),$$

where $F_{\mathrm{nn}}\big(\Delta^{(\dots)}\big)$ are the non-normalized spectral density functions. Normalizing by the number of vertices, we obtain

$$F\big(\Delta_n^{(K_m\times L_m)}\big) = \frac{F_{\mathrm{nn}}\big(\Delta_n^{(K_m\times L_m)}\big)}{|\mathcal{E}_0(K_m \times L_m)|} = \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{F_{\mathrm{nn}}\big(\Delta_p^{(K_m)}\big)}{|\mathcal{E}_0 K_m|} * \frac{F_{\mathrm{nn}}\big(\Delta_q^{(L_m)}\big)}{|\mathcal{E}_0 L_m|}$$

for all m.

By Theorem 3.11, the normalized spectral density functions converge uniformly, so we know

$$F\big(\Delta_n^{(X\times Y)}\big) = \lim_{m\to\infty} \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{F_{\mathrm{nn}}\big(\Delta_p^{(K_m)}\big)}{|\mathcal{E}_0 K_m|} * \frac{F_{\mathrm{nn}}\big(\Delta_q^{(L_m)}\big)}{|\mathcal{E}_0 L_m|}$$

and, using Lemma 6.11,

$$\sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} \lim_{m\to\infty} \frac{F_{\mathrm{nn}}\big(\Delta_p^{(K_m)}\big)}{|\mathcal{E}_0 K_m|} * \frac{F_{\mathrm{nn}}\big(\Delta_q^{(L_m)}\big)}{|\mathcal{E}_0 L_m|} = \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} F\big(\Delta_p^{(X)}\big) * F\big(\Delta_q^{(Y)}\big).$$

It remains to show that the limit commutes with the derivative. That follows from Theorem 6.12 applied to the function

$$f_m = \sum_{p+q=n} \frac{F_{\mathrm{nn}}\big(\Delta_p^{(K_m)}\big)}{|\mathcal{E}_0 K_m|} * \frac{F_{\mathrm{nn}}\big(\Delta_q^{(L_m)}\big)}{|\mathcal{E}_0 L_m|}$$

Namely, we know that this function is almost everywhere differentiable, we know that its derivative is uniformly convergent to $F\big(\Delta_n^{(K_m\times L_m)}\big)$, and we know that $f_m(\lambda)=0$ for all $\lambda\le 0$. $\square$

Next, we need an analogue of Lemma 6.9 for X, and a last technical result:

6.14 Lemma. *Let X be a self-similar chain complex as always. Then*

$$\sum_p(-1)^pF\big(\Delta_p^{(X)}\big)=\chi^{(2)}(X)\cdot\chi_{[0,\infty)}.$$

Proof. Applying Lemma 6.9 to $C_*(K_m)$ yields

$$\sum_p(-1)^pF_{\mathrm{nn}}\big(\Delta_p^{(K_m)}\big)=\chi(K_m)\cdot\chi_{[0,\infty)}.$$

Divide by $|\mathcal{E}_0K_m|$ and take $\lim_{m\to\infty}$. By previous results, the claim follows. $\square$

6.15 Lemma. *Let $F\colon\mathbb{R}\to\mathbb{R}$ be continuous at $\lambda\in\mathbb{R}$. Then $\frac{\mathrm{d}}{\mathrm{d}\lambda}(F*\chi_{[0,\infty)})(\lambda)=\frac{\mathrm{d}}{\mathrm{d}\lambda}(\chi_{[0,\infty)}*F)(\lambda)=F(\lambda)$.*

Proof. For any $\lambda,\mu\in\mathbb{R}$, we have

$$\begin{aligned}
&F*\chi_{[0,\infty)}(\lambda)-F*\chi_{[0,\infty)}(\mu)\\
&\qquad=\int_{\mathbb{R}}F(\xi)\,\chi_{[0,\infty)}(\lambda-\xi)\,\mathrm{d}\xi-\int_{\mathbb{R}}F(\xi)\,\chi_{[0,\infty)}(\mu-\xi)\,\mathrm{d}\xi\\
&\qquad=\int_{-\infty}^{\lambda}F(\xi)\,\mathrm{d}\xi-\int_{-\infty}^{\mu}F(\xi)\,\mathrm{d}\xi\\
&\qquad=\int_{\mu}^{\lambda}F(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Since F is continuous at λ, the last line tends to $(\lambda-\mu)\cdot F(\lambda)$ as μ tends to λ, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(F*\chi_{[0,\infty)})(\lambda)=\lim_{\mu\to\lambda}\frac{F*\chi_{[0,\infty)}(\lambda)-F*\chi_{[0,\infty)}(\mu)}{\lambda-\mu}=F(\lambda).$$

By commutativity of convolution, the same holds for $\chi_{[0,\infty)}*F$. $\square$

Now we can prove the main result:

Proof of Theorem 6.6. Start by using the linearity of the determinant:

$$\begin{aligned}
\rho^{(2)}(X\times Y)&=-\frac{1}{2}\sum_n(-1)^n\cdot n\cdot\log\det\nolimits_{\mathrm{FK}}\big(\Delta_n^{(X\times Y)}\big)\\
&=-\frac{1}{2}\log\det\nolimits_{\mathrm{FK}}\Big(\sum_n(-1)^n\cdot n\cdot F\big(\Delta_n^{(X\times Y)}\big)\Big)
\end{aligned}$$

(Note that only finitely many terms in the sum are non-zero.)

Insert the result of Lemma 6.13, then use Lemma 6.14:

$$
\begin{aligned}
\sum_n (-1)^n \cdot n \cdot F\big(\Delta_n^{(X\times Y)}\big) &= \sum_n (-1)^n \cdot n \cdot \sum_{p+q=n} \frac{\mathrm{d}}{\mathrm{d}\lambda} F\big(\Delta_p^{(X)}\big) * F\big(\Delta_q^{(Y)}\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda} \sum_p (-1)^p F\big(\Delta_p^{(X)}\big) * \sum_q (-1)^q \cdot q \cdot F\big(\Delta_q^{(Y)}\big) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}\lambda} \sum_p (-1)^p \cdot p \cdot F\big(\Delta_p^{(X)}\big) * \sum_q (-1)^q F\big(\Delta_q^{(Y)}\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda} \big(\chi^{(2)}(X) \cdot \chi_{[0,\infty)}\big) * \sum_q (-1)^q \cdot q \cdot F\big(\Delta_q^{(Y)}\big) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}\lambda} \sum_p (-1)^p \cdot p \cdot F\big(\Delta_p^{(X)}\big) * \big(\chi^{(2)}(Y) \cdot \chi_{[0,\infty)}\big)
\end{aligned}
$$

From Lemma 6.15, we have

$$
\frac{\mathrm{d}}{\mathrm{d}\lambda} \big(\chi_{[0,\infty)} * F\big(\Delta_q^{(Y)}\big)\big) = F\big(\Delta_q^{(Y)}\big), \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} \big(F\big(\Delta_p^{(X)}\big) * \chi_{[0,\infty)}\big) = F\big(\Delta_p^{(X)}\big)
$$

at each point where those spectral density functions are continuous. $F\big(\Delta_q^{(Y)}\big)$ and $F\big(\Delta_p^{(X)}\big)$ can have at most countably many discontinuities[16], so these equalities hold almost everywhere. Inserting this into the previous calculation gives

$$
\begin{aligned}
\sum_n (-1)^n \cdot n \cdot F\big(\Delta_n^{(X\times Y)}\big) = \chi^{(2)}(X) \cdot \sum_q (-1)^q \cdot q \cdot F\big(\Delta_q^{(Y)}\big) \\
+ \chi^{(2)}(Y) \cdot \sum_p (-1)^p \cdot p \cdot F\big(\Delta_p^{(X)}\big),
\end{aligned}
$$

again almost everywhere. Finally, $\log \det_{\mathrm{FK}}(F)$ is defined via an integral over $F(\lambda)/\lambda$ and the value of F at a sufficiently large $b \in \mathbb{R}$. Choosing b large enough, each $F\big(\Delta_q^{(Y)}\big)$ and $F\big(\Delta_p^{(X)}\big)$ will be constant around b, while the almost-everywhere equality yields an equality of the integrals. Thus,

$$
\begin{aligned}
\rho^{(2)}(X \times Y) &= -\frac{1}{2} \log \det_{\mathrm{FK}} \Bigg(\sum_n (-1)^n \cdot n \cdot F\big(\Delta_n^{(X\times Y)}\big) \Bigg) \\
&= \chi^{(2)}(X) \cdot \Big(-\frac{1}{2}\Big) \sum_q (-1)^q \cdot q \cdot \log \det_{\mathrm{FK}} F\big(\Delta_q^{(Y)}\big) \\
&\quad + \chi^{(2)}(Y) \cdot \Big(-\frac{1}{2}\Big) \sum_p (-1)^p \cdot p \cdot \log \det_{\mathrm{FK}} F\big(\Delta_p^{(X)}\big) \\
&= \chi^{(2)}(X)\, \rho^{(2)}(Y) + \chi^{(2)}(Y)\, \rho^{(2)}(X).
\end{aligned}
$$

□

[16] Proof: Every $F(\Delta^{(K_m)})$ has at most finitely many discontinuities. If all $F(\Delta^{(K_m)})$ are continuous at λ, then so is $F(\Delta^{(X)})$ (because of uniform convergence). Thus, the set of discontinuities of $F(\Delta^{(X)})$ is contained in the union of the sets of discontinuities of $F(\Delta^{(K_m)})$, and thus countable.

A Borel functional calculus

For the reader's convenience, this appendix summarizes the definition and some properties of the Borel functional calculus for self-adjoint operators. It mostly follows [RS72] and [Lüc02].

Most of these results do *not* require the operator to be bounded, and they will be stated here in full generality, even though this thesis in general only deals with bounded operators.

In the following, let $T\colon \operatorname{dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ a densely defined (possibly unbounded) self-adjoint operator on a separable Hilbert space $\mathcal{H}$. In particular, the word "self-adjoint" implies that $\operatorname{dom}(T) = \operatorname{dom}(T^*)$ and that T is closed.

A.1 Theorem. *There are a measure space (X, μ), with μ a finite measure, a real-valued function $t \in L^2(X, \mu)$ and a unitary operator $U\colon \mathcal{H} \to L^2(X, \mu)$ such that*

$$v \in \operatorname{dom}(T) \qquad \Longleftrightarrow \qquad Uv \in \left\{ f \in L^2(X,\mu) \,\middle|\, t \cdot f \in L^2(X,\mu) \right\}$$

and the following diagram commutes:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "T"] \arrow[d, "U"'] & \mathcal{H} \arrow[d, "U"] \\ L^2(X,\mu) \arrow[r, "M_t"] & L^2(X,\mu) \end{tikzcd}$$

where M_t is given by multiplication by t, that is,

$$(M_t f)(x) = t(x) \cdot f(x).$$

Proof. See [RS72], Theorem VIII.4. □

A.2 Lemma. *Under the notation and conditions of Theorem A.1, the spectrum of T is equal to the essential range of t.*

Proof. Since $\|Uv\|_2 = \|v\|_{\mathcal{H}}$ for al $v \in \mathcal{H}$ and $(T - \lambda \mathrm{id}_{\mathcal{H}}) = U^* M_{t-\lambda} U$, it is clear that $\operatorname{spec} T = \operatorname{spec} M_t$.

Let $t(X)_{\mathrm{ess}}$ be the essential range of t; that is, the set of all $r \in \mathbb{R}$ such that $\mu\big(\{x \in X \mid |r - t(x)| < \varepsilon\}\big) > 0$ for all $\varepsilon > 0$.

Assume $\lambda \in t(X)_{\mathrm{ess}}$, and for any $n \in \mathbb{N}$ pick a measurable set $A_n \subseteq X$ such that $|t(x) - \lambda| < \frac{1}{n}$ for all $x \in A_n$ and $\mu(A_n) > 0$. Let χ_{A_n} be the indicator function of A_n. Then $\chi_{A_n} \in L^2(X, \mu)$ (since the measure is finite), and

$$\|(t-\lambda)\cdot\chi_{A_n}\|_2^2 = \int_X |(t-\lambda)\cdot\chi_{A_n}|^2 \, d\mu < \frac{1}{n^2}\int_X |\chi_{A_n}|^2 \, d\mu = \frac{1}{n^2}\|\chi_{A_n}\|_2^2 .$$

As n was arbitrary, $M_{t-\lambda} = M_t - \lambda \operatorname{id}$ is not bounded from below, and thus cannot be invertible. It follows that $\lambda \in \operatorname{spec}(M_t) = \operatorname{spec}(T)$.

Conversely, assume $\lambda \notin t(X)_{\text{ess}}$. Then there is an $\varepsilon > 0$ and a set $B \subseteq X$ such that $|t(x) - \lambda| \geq \varepsilon$ for all $x \in B$ and $\mu(B) = \mu(X)$. Hence, every function $f \in L^2(X, \mu)$ satisfies

$$\|(t-\lambda) \cdot f\|_2^2 = \int_B |(t-\lambda) \cdot f|^2 \, d\mu \geq \varepsilon^2 \int_B |f|^2 \, d\mu = \varepsilon^2 \|f\|_2^2 ,$$

so $M_{t-\lambda} = M_t - \lambda \mathrm{id}$ is bounded from below, and thus invertible. It follows that $\lambda \notin \operatorname{spec}(M_t) = \operatorname{spec}(T)$. $\square$

A.3 Theorem (Borel functional calculus)**.** *Under the notation and conditions of Theorem A.1, the map*

$$L^\infty(\mathbb{R}) \to \mathcal{B}(\mathcal{H}), \; h \mapsto h(T) := U^* M_{h \circ t} U$$

does not depend on the choice of (X, μ) and has the following properties:

(a) *It is a $*$-homomorphism of algebras.*

(b) *It is norm-continuous. More precisely, $\|h(T)\|_{op} \leq \|h\|_\infty$.*

(c) *If the sequence $(h_n) \subseteq L^\infty(\mathbb{R})$ converges pointwise to $h \in L^\infty(\mathbb{R})$ and the sequence $\big(\|h_n\|_\infty\big)$ is bounded, then $\big(h_n(T)\big)$ converges to $h(T)$ in strong operator topology.*

(d) *If $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$ and $v \in \mathcal{H}$, then $h(T)v = h(\lambda)v$.*

(e) *If $h \geq 0$, then $h(T) \geq 0$.*

Proof. See [RS72], Theorem VIII.5. $\square$

A.4 Corollary. *The operator $h(T)$ only depends on the values of h on $\operatorname{spec}(T)$, so functional calculus can be considered as a map $L^\infty(\operatorname{spec}(T)) \to \mathcal{B}(\mathcal{H})$.*

Proof. By definition, $h(T)$ only depends on the function $h \circ t \in L^\infty(X, \mu)$. If two functions h, h' agree on $t(X)_{\text{ess}}$, then $h \circ t$ and $h' \circ t$ agree μ-almost everywhere on X, and thus represent the same element of $L^\infty(X, \mu)$. $\square$

A.5 Corollary. (a) *If $h(\lambda) \in \{0, 1\}$ for all $\lambda \in \mathbb{R}$, then $h(T)$ is a projection.*

(b) *If $|h(\lambda)| = 1$ for all $\lambda \in \mathbb{R}$, then $h(T)$ is unitary.*

Proof. Since $h \mapsto h(T)$ is an algebra homomorphism, we have $\big(h(T)\big)^* = h^*(T)$, where $*\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is given by pointwise complex conjugation. This gives:

(a) If $h(\lambda) \in \{0, 1\}$ for all $\lambda \in \mathbb{R}$, then $|h(\lambda)|^2 = h(\lambda)$ for all $\lambda \in \mathbb{R}$, and thus

$$h(T)^* h(T) = (h^* h)(T) = |h|^2 (T) = h(T).$$

(b) If $|h(\lambda)| = 1$ for all $\lambda \in \mathbb{R}$, then $1/h(\lambda) = \overline{h(\lambda)} = h^*(\lambda)$ for all $\lambda \in \mathbb{R}$, and thus

$$h(T)^{-1} = (1/h)(T) = h^*(T) = h(T)^*. \qquad \square$$

A.6 Theorem. *For any measurable subset $\Omega \subseteq \mathbb{R}$, define a projection $E^T_\Omega := \chi_\Omega(T)$ (where χ_Ω is the characteristic function of Ω). This defines a* projection-valued measure *on $\mathbb{R}$:*

(a) $E^T_\emptyset = 0$

(b) $E^T_\mathbb{R} = \mathrm{id}$

(c) $\Omega = \bigsqcup_{n=1}^\infty \Omega_n \implies E^T_\Omega = \sum_{n=1}^\infty E^T_{\Omega_n}$
(*The sum converges in strong operator topology.*)

(d) $E^T_{\Omega \cap \tilde{\Omega}} = E^T_\Omega E^T_{\tilde{\Omega}}$

This implies that for any $v, w \in \mathcal{H}$, the map $\Omega \mapsto \langle v, E^T_\Omega w \rangle$ defines a real-valued measure on $\mathbb{R}$, and the functional calculus of Theorem A.3 can be computed as

$$\langle v, h(T)w \rangle = \int_\mathbb{R} h(\lambda)\, d\langle v, E^T(\lambda)w \rangle,$$

where $E^T(\lambda) := E^T_{(-\infty,\lambda]}$. In short, write

$$h(T) = \int_\mathbb{R} h(\lambda)\, dE^T(\lambda).$$

Proof. See [RS72], pp. 262f. $\square$

Finally, if the operator in question lies in a particular sub-algebra, it is highly desirable that the same holds for the results of its functional calculus. That is indeed the case:

A.7 Corollary. *Assume that T is bounded. Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ a von Neumann algebra such that $T \in \mathcal{A} \subseteq \mathcal{N}$.*

(a) *If $h \in L^\infty(\mathrm{spec}(T))$ is continuous, we have $h(T) \in \mathcal{A}$.*

(b) *For any $h \in L^\infty(\mathrm{spec}(T))$, we have $h(T) \in \mathcal{N}$.*

Proof. (a) As T is bounded, there is a compact interval I containing $\mathrm{spec}(T)$. By the Weierstrass approximation theorem, there are polynomials $(p_n)_{n\in\mathbb{N}}$ such that $\|h - p_n\|_\infty \xrightarrow{n\to\infty} 0$. Clearly, $p_n(T) \in \mathcal{A}$ for all n (because $\mathcal{A}$ is an algebra), and $\|h(T) - p_n(T)\|_{\mathrm{op}} \xrightarrow{n\to\infty} 0$ (by A.3 (b)). As a $\mathcal{A}$ is norm-closed, this proves the claim.

(b) Show first that for all $\lambda \in \mathbb{R}$ the spectral projection $E^T(\lambda) = \chi_{(-\infty,\lambda]}(T)$ lies in $\mathcal{N}$: Setting

$$f_n(r) = \begin{cases} 1, & \text{if } r \in (-\infty, \lambda], \\ 1 - n(r-\lambda), & \text{if } r \in [\lambda, \lambda + \frac{1}{n}], \\ 0, & \text{if } r \in [\lambda + \frac{1}{n}, \infty), \end{cases}$$

we obtain $f_n(T) \in \mathcal{A}$ for all n (because the f_n are continuous) and $f_n \xrightarrow{n\to\infty} \chi_{(-\infty,\lambda]}$ pointwise. Clearly, $\|f_n\|_\infty = 1$ for all n.

Thus, $f_n(T) \xrightarrow{n\to\infty} E^T(\lambda)$ in strong (and thus weak) operator topology by A.3 (c). As $\mathcal{N}$ is weakly closed, this proves $E^T(\lambda) \in \mathcal{N}$.

Now let $h \in L^\infty(\operatorname{spec}(T))$ be any measurable function. By A.6, the integral $h(T) = \int_{\operatorname{spec}(T)} h(\lambda)\, d\lambda$ is the weak limit of finite sums of spectral projections $E^T(\lambda)$, and therefore lies in $\mathcal{N}$. $\square$

In the definition of spectral density functions (2.38), operators that are not positive or map a Hilbert space to a different Hilbert space are treated by considering T^*T instead of T. To ensure consistency with the usual definition, one needs to check what this does to an operator that is already positive:

A.8 Lemma. *Let $T \geq 0$ as above. Then $E^{T^2}(\lambda^2) = E^T(\lambda)$.*

Proof. The commutative diagram of Theorem A.1 can be expanded to

$$\begin{tikzcd} \mathcal{H} \arrow[r, "T"] \arrow[d, "U"'] \arrow[rr, bend left, "T^2"] & \mathcal{H} \arrow[r, "T"] \arrow[d, "U"] & \mathcal{H} \arrow[d, "U"] \\ L^2(X,\mu) \arrow[r, "M_t"] \arrow[rr, bend right, "M_{t^2}"'] & L^2(X,\mu) \arrow[r, "M_t"] & L^2(X,\mu) \end{tikzcd}$$

Thus, using Theorem A.3, we obtain

$$E^{T^2}(\lambda^2) = \chi_{(-\infty,\lambda^2]}(T^2) = U^* M_{\chi_{(-\infty,\lambda^2]} \circ t^2} U,$$

and for all $r \in \mathbb{R}$ we have (note $t(r) \geq 0$):

$$\left(\chi_{(-\infty,\lambda^2]} \circ t^2\right)(r) = \begin{Bmatrix} 1 & \text{if } t(r)^2 \leq \lambda^2 \\ 0 & \text{otherwise} \end{Bmatrix} = \begin{Bmatrix} 1 & \text{if } t(r) \leq \lambda \\ 0 & \text{otherwise} \end{Bmatrix} = \left(\chi_{(-\infty,\lambda]} \circ t\right)(r)$$

Thus, indeed,

$$E^{T^2}(\lambda^2) = U^* M_{\chi_{(-\infty,\lambda^2]} \circ t^2} U = U^* M_{\chi_{(-\infty,\lambda]} \circ t} U = E^T(\lambda). \qquad \square$$

(It therefore makes sense to define $F^T(\lambda) := F^{T^*T}(\lambda^2)$ for any not self-adjoint operator $T \in \mathcal{N}_j(X)$. It would *not* make sense to define $E^T(\lambda)$ that way, because that would yield $\int \lambda\, dE^T(\lambda) = |T|$ in contradiction to A.6.)

Finally, let us note the following "interpretation" of the spectral projections: Simply speaking, the image of $E^T(\lambda)$ is a maximal subspace on which T is bounded by λ.

A.9 Lemma. *Assume the conditions and notation of Theorem A.1, and let $v \in \mathcal{H}$ be nonzero. Then*

$$\begin{aligned} E^T(\lambda)\, v = v &\implies \|Tv\| \leq \lambda \, \|v\| \,, \\ E^T(\lambda)\, v = 0 &\implies \|Tv\| > \lambda \, \|v\| \,. \end{aligned}$$

Proof. Either see [Lüc02], Lemma 2.2, p. 73, or consider the following:

Using Theorems A.1, A.3 and A.6, we have the commutative diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "E^T(\lambda)"] \arrow[d, "U"'] & \mathcal{H} \arrow[d, "U"] \\ L^2(X,\mu) \arrow[r, "M_\chi"] & L^2(X,\mu) \end{tikzcd}$$

where $\chi = \chi_{(-\infty,\lambda]} \circ t \in L^\infty(X)$ is the characteristic function of the set $\{x \in X \mid |t(x)| \leq \lambda\}$. (Thus, $1 - \chi$ is the characteristic function of the set $\{x \in X \mid |t(x)| > \lambda\}$.) Fix $v \in \mathcal{H}$ and let Uv be given by the function $f \in L^2(X, \mu)$. Then we have:

$$\begin{aligned} \|Tv\|_{\mathcal{H}}^2 = \|UTv\|_2^2 &= \int_X |t \cdot f|^2 \, \mathrm{d}\mu \\ &= \int_X |t \cdot f|^2 \cdot \chi \, d\mu + \int_X |t \cdot f|^2 \cdot (1-\chi)\, d\mu \end{aligned}$$

If $E^T(\lambda)v = v$, then $\chi \cdot f = f$ (and thus $(1-\chi) \cdot f = 0$), so

$$\begin{aligned} \|Tv\|_{\mathcal{H}}^2 &= \int_X |t \cdot f|^2 \cdot \chi \, d\mu = \int_{\{x \mid |t(x)| \leq \lambda\}} |t \cdot f|^2 \; d\mu \\ &\leq \lambda^2 \int_X |f|^2 \; d\mu = \lambda^2 \, \|Uv\|_2^2 = \lambda^2 \, \|v\|_{\mathcal{H}}^2 \,. \end{aligned}$$

Finally, if $E^T(\lambda)v = 0$, then $\chi \cdot f = 0$, so

$$\begin{aligned} \|Tv\|_{\mathcal{H}}^2 &= \int_X |t \cdot f|^2 \cdot (1-\chi) \, d\mu = \int_{\{x \mid |t(x)| > \lambda\}} |t \cdot f|^2 \; d\mu \\ &> \lambda^2 \int_X |f|^2 \; d\mu = \lambda^2 \, \|Uv\|_2^2 = \lambda^2 \, \|v\|_{\mathcal{H}}^2 \,. \end{aligned}$$

(Of course, the "greater than" requires that the integral is nonzero; that is the case since $\|f\|_2 = \|v\|_{\mathcal{H}} > 0$.) ☐

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