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Contents

Chapter 1

Introduction and main results

The field of epidemiological models has been an active field of research for a long time. An epidemiological model describes and simulates a possible course of an infection, which spreads through a given population. The question of how to model the spread of different infections in an realistic manner has brought forth dozens of models. But the spread of an infection is a highly complex problem. Thus, there are still “real-world phenomena”, which cannot be adequately explained.

The contact process is a particularly simple example of an epidemiological model. This process models the spread of an infection over time in a spatially structured population, where this structure is given through a graph $G = (V, E)$. The vertex set V labels the individuals and two individuals $x, y \in V$ are considered neighbours, i.e. they have physical contact, if there exists an edge $\{x, y\} \in E$. If an individual is infected, it can pass on its infection to its neighbours. The contact process has been around for almost half a century. Thus, it is not surprising that there exist many variations of this model, which try to incorporate more realistic assumptions and try to shed light on different aspects. Nevertheless, certain aspects are still not well understood. This is something which the current global pandemic, caused by Covid-19, has made us aware of. For example it has become apparent that with the implementation of preventive measures such as social distancing the spread of Covid-19 has slowed down significantly. Strong evidence for these effectiveness of this measures in Germany has, for example, been provide by Dehning et al. [Deh+20]. Of course the situation vastly differs between different countries. These phenomena indicate that the spatial structure of the population has an huge impact on the course of the pandemic.

Of course there have been variations of the contact process, which incorporate random spatial structures, in order to take into account that one does not exactly know this structure. This mostly was done in a static setting. By this we mean that even through these variations considered for example a random population structures, it is was still fixed at the beginning and could not change halfway through. This does not appear to be realistic, since we are not always in contact with the same people. Only recently have people started to consider models which model infections in dynamical spatial structures. This means that the spatial structure can change on the same time scale as the spread of the infection happens. Since theory and knowledge regarding this type of models is still limited we were motivated to further study the impact of such dynamical structures on the course of an infection.

Therefore, in this thesis we study a contact process in an evolving random environment. The model we consider is a variation of the contact process that allows for the neighbourhood relations to change over time by introducing a time evolving random environment.

We will assume that our evolving environment will always converge to a unique equilibrium regardless of its initial state. If we additionally assume convergence to be fast enough, then the initial state of the environment is inconsequential to the fact if the infection can persist in the population for all time or eventually dies out. We will also study the equilibrium states of the system or to be precise the invariant laws. If we further assume that the environment evolves according to a reversible dynamic we can determine conditions under which we can fully characterize all invariant laws.

As an application we consider a contact process on top of a dynamical percolation as random environment and we assume that the underlying graph is a d-dimensional integer lattice. A dynamical percolation is a stochastic process which assigns to every edge independently a state of being open or closed, where the infection can only use open edges. Furthermore, the state of every edges is independently of the other edges updated with a certain rate. This infection model was first proposed by Linker and Remenik in [LR20]. We can augment some of their result, and therefore contribute to a more complete picture of the behaviour of this particular model.

The class of models we consider is defined on a graphs with bounded degrees. This means that the number of neighbours of a individual is bound uniformly. Of course in reality nobody can have infinitely many acquaintance or friends, with whom they interact. But a uniform bound also seems somewhat unnatural. Thus, in the last part of this thesis we consider an extension of the model proposed in [LR20]. To be precise we consider a contact process on a dynamical long range percolation and extend some of the results known in the finite range case to this setting.

1.1 The contact process in an evolving random environment

In this section we will formally introduce a *contact process in an evolving random environment*, which we abbreviate with CPERE. As already mentioned the spatially structure of the population will be given through a graph $G = (V, E)$, where V is a countable set and denotes the vertex set and E the edge set. We will assume throughout this thesis that G is transitive, connected and has bounded degree. Furthermore, we assume that G is an infinite graph since otherwise the answer to the question, if a infection can persist for all time, is always no.

The CPERE $(\mathbf{C}, \mathbf{B}) = (\mathbf{C}_t, \mathbf{B}_t)_{t\geq 0}$ is a Feller process on $\mathcal{P}(V) \times \mathcal{P}(E)$, where $\mathcal{P}(V)$ and $\mathcal{P}(E)$ are the power sets of V and E. We call the process $\mathbf{B}$ the background process, since it describes the evolving random environment and assume that it is an autonomous Feller process with values in $\mathcal{P}(E)$. On top of this space-time random environment we define an infection process $\mathbf{C}$ with values in $\mathcal{P}(V)$ and transitions

$$\begin{aligned} \mathbf{C}_{t-} = C &\to C \cup \{x\} \quad \text{at rate } \lambda \cdot \#\{y \in C : \{x, y\} \in \mathbf{B}_{t-}\} \text{ and} \\ \mathbf{C}_{t-} = C &\to C \backslash \{x\} \quad \text{at rate } r, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ denotes the *infection rate* and $r > 0$ the *recovery rate*. If $x \in \mathbf{C}_t$, then we call x *infected* at time t. If $e \in \mathbf{B}_t$ we call e *open* at time t and *closed* otherwise.

We equip $\mathcal{P}(V) \times \mathcal{P}(E)$ with the topology which induces the point wise convergence. This means if $\big((C_n, B_n)\big)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{P}(V) \times \mathcal{P}(E)$, then $(C_n, B_n) \to (C, B)$ as $n \to \infty$ if and only if $\mathbb{1}_{\{(x,e)\in(C_n,B_n)\}} \to \mathbb{1}_{\{(x,e)\in(C,B)\}}$ as $n \to \infty$ for every $(x, e) \in V \times E$. Furthermore, we denote by " $\Rightarrow$ " the weak convergence of probability measures on $\mathcal{P}(V) \times \mathcal{P}(E)$.

Remark 1.1.1. Besides $\mathcal{P}(V) \times \mathcal{P}(E)$ we could also choose $\{0, 1\}^V \times \{0, 1\}^E$ as the state space of the CPERE, since we can identify every element (C, B) with the function $\mathbb{1}_{\{\,\cdot\,\in(C,B)\}}$ and vice versa. Note that on $\{0, 1\}^V \times \{0, 1\}^E$ the product topology induces the point wise convergence. In the literature both choices of states spaces are common. We decided to use $\mathcal{P}(V) \times \mathcal{P}(E)$ out of preference and notational convenience.

It is common to add the initial configuration (C, B) as a superscript to the process, i.e. $(\mathbf{C}^{C,B}, \mathbf{B}^B)$. Sometimes it is more convenient to use the usual notation to indicate the initial configuration of a Markov process by adding it to the law as a superscript, i.e.

$$\mathbb{P}^{(C,B)}_{\lambda,r}\big((\mathbf{C}_t,\mathbf{B}_t)\in\,\cdot\,\big)=\mathbb{P}_{\lambda,r}\big((\mathbf{C}_t,\mathbf{B}_t)\in\,\cdot\,\big|(\mathbf{C}_0,\mathbf{B}_0)=(C,B)\big).$$

We do not only consider deterministic initial configurations. Thus, if we want to consider an initial distribution μ of $(\mathbf{C},\mathbf{B})$ we write

$$\mathbb{P}^{\mu}_{\lambda,r}\big((\mathbf{C}_t,\mathbf{B}_t)\in\,\cdot\,\big)=\int\mathbb{P}^{(C,B)}_{\lambda,r}\big((\mathbf{C}_t,\mathbf{B}_t)\in\,\cdot\,\big)\mu\big(\mathsf{d}(C,B)\big).$$

Note that if $\mu=\delta_C\otimes\mu_2$, where μ_2 is a probability measure on $\mathcal{P}(E)$, then we abuse the notation slightly and write $\mathbb{P}^{(C,\mu_2)}_{\lambda,r}$.

The CPERE can be defined for a fairly general class of interacting particles systems, acting as the background process $\mathbf{B}$. In this thesis we focus on the case where the background is a *spin system* on $\mathcal{P}(E)$. An interacting particle systems is called a spin system if it has a generator of the form

$$\mathcal{A}_{Spin}f(B)=\sum_{e\in E}q(e,B)\big(f(B\triangle\{e\})-f(B)\big),$$

where $q(e,B)$ is the flip rate of e with respect to the "present" configuration $B\subset E$ and $\triangle$ is the symmetric difference of sets, i.e. $B_1\triangle B_2=(B_1\backslash B_2)\cup(B_2\backslash B_1)$.
We additionally equip the edge set E with a spatial structure by considering the line graph $L(G)$ (see Definition 2.4.9). In the line graph the original edge set E is considered to be the vertex set and edges $e_1,e_2\in E$ are defined to be adjacent if they have a vertex in common, i.e. it exists $x\in V$ such that $x\in e_1,e_2$. Let $\mathbb{B}^L_n(e)$ denote the ball with centre $e\in E$ of radius n with respect to the graph distance of $L(G)$. We assume that the spin system satisfies the following three properties.

1. It is *attractive*, i.e. the spin rate $q(\cdot,\cdot)$ satisfies that if $B_1\subset B_2$, then

$$q(e,B_1)\leq q(e,B_2)\text{ if }e\notin B_2\quad\text{and}\quad q(e,B_1)\geq q(e,B_2)\text{ if }e\in B_1.$$

2. It is *translation invariant*, i.e. if σ is a graph automorphism (see Definition 2.4.3)then

$$q(e,B)=q(\sigma(e),\sigma(B))\qquad\text{for all }B\subset E.$$

3. The spin system is of *finite range*, i.e. there exists a constant $R \in \mathbb{N}$ such that

$$q(e,B) = q(e, B \cap \mathbb{B}_R^L(e))$$

for all $e \in E$ and $B \subset E$. We call such a spin system of range R.

We will now list some examples of spin systems we consider for the background dynamic.

Example 1.1.2. Let $\mathcal{N}_e^L$ denote the neighbourhood of e in the line graph $L(G)$.

(i) The probably easiest possible non-trivial choice is the *dynamical percolation*. This system will be our main example. The dynamical percolation updates every edge independently from all other edges. Hence, the background $\mathbf{B}$ is a Feller process with transition

$$\begin{aligned} \mathbf{B}_{t-} = B \to B \cup \{e\} \quad &\text{at rate } \alpha \text{ and} \\ \mathbf{B}_{t-} = B \to B \backslash \{e\} \quad &\text{at rate } \beta, \end{aligned}$$

where $\alpha, \beta > 0$.

(ii) Next we consider a *noisy voter model* on $G = (V, E)$ with

$$V = \mathbb{Z} \quad \text{and} \quad E = \{\{x,y\} \subset \mathbb{Z} : |x-y| = 1\}.$$

In this case $L(G)$ is again a 1-dimensional nearest neighbour integer lattice just like G. The background $\mathbf{B}$ has transitions

$$\begin{aligned} \mathbf{B}_{t-} = B \to B \cup \{e\} \quad &\text{at rate } \frac{\alpha}{2} + \beta |B \cap \mathcal{N}_e^L| \text{ and} \\ \mathbf{B}_{t-} = B \to B \backslash \{e\} \quad &\text{at rate } \frac{\alpha}{2} + \beta |B^c \cap \mathcal{N}_e^L|, \end{aligned}$$

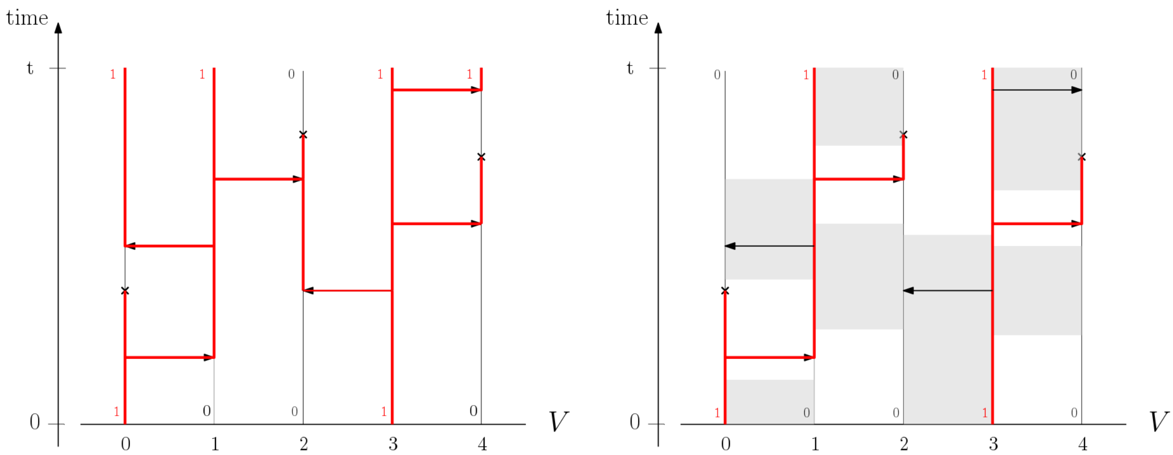
where $\alpha, \beta > 0$.

(iii) The last example is the *ferromagnetic stochastic Ising model* with inverse temperature $\beta > 0$. Here, the transitions of $\mathbf{B}$ are

$$\begin{aligned} \mathbf{B}_{t-} = B \to B \cup \{e\} \quad &\text{at rate } 1 - \tanh\big(\beta\big(|\mathcal{N}_e^L| - 2|B \cap \mathcal{N}_e^L|\big)\big) \text{ and} \\ \mathbf{B}_{t-} = B \to B \backslash \{e\} \quad &\text{at rate } 1 - \tanh\big(\beta\big(|\mathcal{N}_e^L| - 2|B^c \cap \mathcal{N}_e^L|\big)\big). \end{aligned}$$

(iv) A trivial example is $\mathbf{B}_t \equiv E$ for all $t \geq 0$. With this choice we recover the classical contact process, since all edges are open at all times, and thus the infection process $\mathbf{C}$ is not affect by $\mathbf{B}$ at all. We will abbreviate the classical contact process with CP.

The CP can be constructed via the so called graphical representation, which is a general concept to construct an interacting particle system via a graphical approach. In the case of the CP one draws infection and recovery events according to a Poisson point process, which are respectively depicted by arrows pointing from an individual x to a neighbour y and by crosses at a site x. Now if x is infected the arrow causes the infection of y. On the other hand a cross at x leads to the recovery of x. See Figure 1.1(a) for a visualization. The CPERE is essentially constructed in the same way as the CP with the difference that we incorporate the background into the graphical representation as visualized in Figure 1.1(b).Basically an infection arrow from x to y can only transmit an infection at a time t if the edge is open, i.e. $\{x, y\} \in \mathbf{B}_t$.



(a) The arrows form sites x to y correspond to a possible transmission of an infection from x to y and the crosses correspond to a possible recovery of the respective site. The red lines indicate the infection paths.

(b) Grey areas indicate that an edge is closed with respect to the background. Infection arrows in a grey area are ignored. The red lines again indicate the infection paths.

Figure 1.1: Visualization of a graphical representation of the classical contact process and the contact process in a evolving random environment.

One of the key quantities in infinite systems which model the spread of infections is the so called survival probability of the infection $\mathbf{C}$, which is defined as follows:

Definition 1.1.3 (Survival probability θ)**.** Let $C \subset V$, $B \subset E$ and $\lambda, r > 0$. Then

$$\theta(\lambda, r, C, B) := \mathbb{P}^{(C,B)}_{\lambda,r}\big(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0\big)$$

is called the *survival probability* (of $\mathbf{C}$).

We often omit some or even all arguments of the survival probability θ to aid the readability of texts and formulas. We will see that CPERE exhibits a so called phase transition. This means that the process drastically changes its behaviour with regards to θ if the parameter of interest (λ, r) crosses a certain critical threshold and thus, one can divide the parameter set in different phases. This means that if we increase the infection rate or respectively decrease the recovery rate, the drastic change which occurs is the possibility for the infection to survive, i.e. $\theta(\lambda, r, C, B) > 0$.

Definition 1.1.4 (Critical infection rate for survival)**.** Let $C \subset V$ be finite, $B \subset E$ and $r > 0$. We define the *critical infection rate for survival* by

$$\lambda_c(r, C, B) := \inf\{\lambda > 0 : \theta(\lambda, r, C, B) > 0\}.$$

We will show that the survival probability θ is monotone in λ and r, and thus the infimum attains a unique value. Note that we can analogously define a critical recovery rate $r_c(\lambda, C, B)$. In this case the infection rate is a variable instead of the recovery rate.

Remark 1.1.5. As already mentioned the dynamical percolation introduced in Example 1.1.2 (i) can be considered as our main example for a background process. In this special case we will call the process $(\mathbf{C}, \mathbf{B})$ a *contact process on a dynamical percolation*, which we abbreviate with CPDP. In this model we have two additional parameters α and β corresponding to the rates at which edges open or close. Thus, we denote by

$$\theta_{DP}(\lambda, r, \alpha, \beta, C, B) = \mathbb{P}^{(C,B)}_{\lambda,r,\alpha,\beta}\big(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0\big)$$

the survival probability of CPDP and by $\lambda_c^{\mathrm{DP}}(r, \alpha, \beta, C, B)$ the critical value of the contact process on a dynamical percolation, where $C \subset V$ non-empty and finite and $B \subset E$. If it is clear from the context that we consider the dynamical percolation as background, then we will drop the super/subscript DP.

1.2 History

To the best of our knowledge the CP was first introduced by Harris [Har74]. It is a Markov process which models the spread of an infection in a structured population via a contact interaction. This means we consider a collection of individuals and we know which of these individuals have physical contact. So if we assume that a number of these individuals are infected, then with at a certain rate a "sick" individual can infect a "healthy" one with whom they are in contact. On the other hand, if one individual is "sick", it recovers at a certain rate, i.e. after a random amount of time. Now the dynamics of the spread of the infection is modelled via the CP $\mathbf{X}$. Again, this is a Markov process with state space $\mathcal{P}(V)$ and transitions

$$\begin{aligned} \mathbf{X}_{t-} &= A \to A \cup \{x\} \quad \text{at rate } \lambda \cdot |\{y \in A : \{x, y\} \in E\}| \text{ and} \\ \mathbf{X}_{t-} &= A \to A \backslash \{x\} \quad \text{at rate } r, \end{aligned}$$

where $\lambda > 0$ is the infection rate and with rate $r > 0$ a infected person recovers.

Remark 1.2.1. For the CP the survival probability only depends on the fraction λ/r, since by rescaling the time the problem reduces to the case $r = 1$. This is not the case for the CPERE since rescaling time also affects the background $\mathbf{B}$.

Thus, in this section we assume that $r = 1$ and in the context of the CP we denote the survival probability by $\theta(\lambda, C) = \mathbb{P}_\lambda^C(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0)$, where $C \subset V$ denotes the set of initially infected individuals. Since one is mainly interested in whether the survival probability is non-zero the quantity of interest is again $\lambda_c = \inf\{\lambda \geq 0 : \theta(\lambda, \{x\}) > 0\}$, which is called the critical infection rate of survival. For the CP one can show that for any two finite and non-empty sets $C, C' \subset V$, $\theta(C) > 0 \Leftrightarrow \theta(C') > 0$. This means that the critical value λ_c does not depend on the choice of the initial configuration as long as at least one and only finitely many individuals are initially infected. Till this day Liggetts books [Lig12] and [Lig13] are the standard reference for interacting particles systems and in particular the CP. Thus, for a detailed introduction and description of the CP and interacting particle in general we refer the reader to these two books.

The above mentioned graphical representation was introduced by Harris [Har78] for a certain class of Markov processes. Besides its obvious use to construct the CP this representation turned out to be one of the most powerful tools for studying the CP, since it enables us to use a wide range of coupling methods. For example, it is immediately clear by this construction that the survival probability is monotone with respect to an increase in the infection rate or the initial infections. In a lot of situation it enables one to couple the CP to a different model which is much easier to study in the particular situation. For example, Durrett [Dur91] construct a coupling between the CP on the 1-dimensional integer lattice and an oriented percolation on $\mathbb{Z}_+ \times \mathbb{Z}$, which allows them to conclude that $\lambda_c < \infty$. This shows in particular that the critical value is finite if the graph G is infinite and connected. On the other hand, for graphs with bounded degree one can show that $\lambda_c > 0$ by a comparison with branching random walk.

A different aspect which was intensively studied were the corresponding invariant laws. In Markov process theory the invariant distributions are often investigate, since they determine the asymptotic behaviour as $t \to \infty$. A typical question is if there exists a unique invariant law and if not, if it is possible to classify the infinitely many invariant laws. Note that $\delta_\emptyset$ is obviously an invariant law of the CP. Thus, we can excluded that no invariant law exists in case of the CP. There has been considerable effort to study these questions for the CP. Since the CP is a monotone Feller process one can show quite easily that a so called upper invariant law $\overline{\nu}$ exists, i.e. $\mathbf{X}_t^V \Rightarrow \overline{\nu}$ as $t \to \infty$, where the superscript denotes that $\mathbf{X}_0^V = V$. In the subcritical phase, i.e for $\lambda < \lambda_c$, clearly $\mathbf{X}_t \Rightarrow \delta_\emptyset$ as $t \to \infty$, and therefore the process is ergodic, which means $\overline{\nu} = \delta_\emptyset$. In the supercritical phase, i.e. $\lambda > \lambda_c$, this is not so clear and in fact with the concept of duality one can show that in this phase $\overline{\nu} \neq \delta_\emptyset$, and thus the contact process is not ergodic.

This motivates the question: How do the invariant laws look in the supercritical phase? Is it possible to characterize them? On the d-dimensional integer lattices with nearest neighbour structure, i.e. $V = \mathbb{Z}^d$ and $E = \big\{\{x, y\} \subset V : ||x - y||_1 = 1\big\}$, where $||\cdot||_1$ denotes the 1-norm, Durrett and Griffeath [DG82] managed to formulate equivalent conditions such that the so called *complete convergence* holds for the CP, i.e.

$$\mathbf{X}_t^C \Rightarrow [1 - \theta(\lambda, C)]\delta_\emptyset + \theta(\lambda, C)\overline{\nu}$$

as $t \to \infty$. This means that there exist only two *extremal* measures, which are $\delta_\emptyset$ and $\overline{\nu}$ and that every other invariant law is only a convex combination of these two.

These conditions are also equivalent to a result which is closely related to complete convergence, the so called asymptotic shape theorem. Let $\mathbf{0}$ denote the zero vector in $\mathbb{Z}^d$ and set

$$\mathbf{H}_t := \bigcup_{s \leq t} \mathbf{X}_s^{\{\mathbf{0}\}} \quad \text{and} \quad \mathbf{K}_t := \mathbf{X}_t^V \triangle \mathbf{X}_t^{\{\mathbf{0}\}}.$$

The set $\mathbf{H}_t$ contains all sites which were at least once infected until time t and $\mathbf{K}_t$ is the coupled region at time t. A very lose interpretation of the coupled region $\mathbf{K}_t$, given in [DG82], is that $\mathbf{X}_t^{\{\mathbf{0}\}}$ is in its "equilibrium" in this region. Let us set $\mathbf{H}'_t := \mathbf{H}_t + \big[-\frac{1}{2}, \frac{1}{2}\big]^d$

and $\mathbf{K}'_t := \mathbf{K}_t + \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ for all $t \geq 0$. Formally the asymptotic shape theorem states that there exists a compact and convex set $U \subset \mathbb{R}^d$ such that for every $\varepsilon > 0$

$$\mathbb{P}\big(\exists s > 0 : (1-\varepsilon)tU \subset (\mathbf{K}'_t \cap \mathbf{H}'_t) \subset \mathbf{H}'_t \subset (1+\varepsilon)tU \ \forall t \geq s \,\big|\, \mathbf{X}_t^{\{\mathbf{0}\}} \neq \emptyset \ \forall t \geq 0\big) = 1.$$

In words this result states that the CP $\mathbf{X}$ is a linear growth model. This means that $\mathbf{H}_t$ expands asymptotically linear in time with respect to the spatial distance. The supplement that $(\mathbf{K}'_t \cap \mathbf{H}'_t)$ grows asymptotically linear, means in broad terms that the area where $\mathbf{X}_t^{\{\mathbf{0}\}}$ is already in its equilibrium expands asymptomatically linear in time. Such shape theorems seem to be prominent for models defined on integer lattices with nearest neighbour structure since the graph distance is given through the 1-norm. Usually $\mathbb{Z}^d$ is endowed with $||\cdot||_1$, and thus the graph distance not only describes a "social" distance but can also seen as a "geographical" distance.

Furthermore, in [DG82] was shown that for λ large enough these equivalent condition are satisfied. But they and many others believed that these results hold for every $\lambda > \lambda_c$. Only years later Bezuidenhout and Grimmett [BG90] developed a technique which could be used to show that the equivalent conditions stated in [DG82] are satisfied for $\lambda > \lambda_c$. At this point we should mention that the techniques developed in [BG90] have more applications. For example they can be used to show that the CP goes extinct almost surely at criticality, i.e. $\theta(\lambda_c, C) = 0$ for all $C \subset V$ finite. We will later come back to these techniques since we will modify them and apply them to our model.

On other graphs the complete convergence does not always hold true. For example the CP on the d-regular tree $\mathbb{T}^d$ exhibits a intermediate phase, where survival is possible, but there exist infinitely many extremal invariant measures. This can be found in [Lig12, Chapter I.4]. Salzano and Schonmann have studied in [SS97] and [SS99] the so-called second lowest extremal invariant measure and with it partial convergence and complete convergence results of the CP. In [Sal99] Salzano actually provided examples of trees on which, as the infection parameter increases, complete convergence alternates between holding and failing infinitely many times. Thus, on general graphs it is near to impossible to make an exact statement whether complete convergence holds or not.

The results we described until now assume that the graph G is known. This is equivalent to the assumption that we know the complete spatial structure of the population. In fact this is a rather unrealistic assumption, since determining the exact structure of a population is extremely difficult if not impossible. This is one motivation to study the contact process in a random environment. One of the first works to consider a contact process on $\mathbb{Z}$ in a random environment was from Bramson, Durrett and Schonmann [BDS91], where they considered the recovery rates $(r_i)_{i\in V}$ to be distributed identical and independent across sites. There have been others, which additionally choose the infection rate according to some probability distribution independently for all sites, see for example [Lig92] and [Kle94]. But, they assume that the infection rates are strictly positive. Therefore, the underlying graph structure has not changed.

If the random environment is allowed to prevent the transmission of an infection between two adjacent sites, then this would really change the underlying graph, since this corresponds to erasing an edge. From a geometric perspective, one could speak of a contact process on a random graph. One of the first examples for such a model was considered by Pemantle and Stacey [PS01]. They studied among other things a contact process on a Galton-Watson tree. There has been a considerable amount of effort to study such variations of the CP. Maybe one of the most natural choices is to consider an infection rate randomly chosen between 0 and some constant λ independently for each edge. This can be seen as a contact process on top of a bond percolation model. This infection model was for example considered by Xue [Xue14], who investigated survival of the infection and proposed an upper bound on the critical infection rate. Another related work was done by Chen and Yao [YC12]. They studied complete convergence of a contact processes on a percolation clusters of $\mathbb{Z}^d \times \mathbb{Z}_+$. Note that they needed to introduce one oriented spatial direction for their techniques to work. Certainly closely related to the complete convergence is the asymptotic shape theorem. Garet and Marchand [GM12] proved such a result for the contact process on $\mathbb{Z}^d$ in a rather general random environments. Van Hao Can [Can15] studied the contact process on a long range percolation cluster. In comparison to the other models we listed here the resulting underlying graph has no longer bounded degrees. It is only locally finite.

These works all consider contact processes in a *static* random environment, i.e. the random environment is random but fixed for the whole time horizon. But in reality, connections between individuals obviously change over time. Therefore, with the aim in mind to formulate an infection model closer to reality, people tried to incorporate this effect. Such models can be called a contact process in a dynamical random environment. To the best of our knowledge the first to explicitly consider a contact process with dynamical rates was Broman [Bro07]. In this work they considered a contact process on top of a vertex dynamical percolation, which affects the recovery rate in such a way that the recovery rate of a individual alters between two values. Thus, they study a contact process with varying recovery rates. In [Bro07] they considered general graphs and assumed that the dynamical percolation is started stationary. They studied mainly comparison methods of the critical value with respect to the classical contact process. [SW08] can be considered as a follow up, since they studied the same model on $\mathbb{Z}^d$ and studied the influence of the initial configuration of the dynamical percolation on the critical value, i.e. it is no longer started stationary, and proved that this variation of the contact process dies out at criticality. [SW08] considered a multi-type contact process, where a state of temporary immunization of an individual was introduced. Hence, individuals in this state cannot be infected, and thus one could say this is closely related to the asymptotic behaviour of the model introduced by [Bro07], where the recovery rates alter between r and ∞. In [Rem08] they even managed to show complete convergence of their model. There is a rich literature on multi-type contact process see for example [DS91], [DM91] and [Kuo16].

The three works [Bro07], [SW08] and [SW08], all studied a contact process with varying recovery rates. Only recently have people started to study what we would consider contact processes on dynamical random graphs. For example [JM17] and [JLM19] studied the contact process on finite and scale free graphs with vertex updates. This means, that when a vertex x is updated all edges connected to x are removed and afterwards new edges are randomly added. The first work to consider a dynamical random environment affecting the infections on an infinite graph was [LR20].

1.3 The contact process on a dynamical percolation

In this section we recapitulate the results of [LR20] in more detail since they can be considered the starting point for our work. The process considered in [LR20] is a contact process on a dynamical percolation. This model is a special case of the CPERE as seen in Example 1.1.2 (i). They considered a particular choice of the rates, namely $r = 1$, $\alpha = vp$ and $\beta = v(1-p)$ for $v > 0$ and $p \in (0,1)$. The parameter v can be understood as the *update speed* of an edge and p is the probability for an edge to be open afterwards. Additionally they consider $\mathbf{B}$ to be started stationary, i.e its initial distribution is its unique invariant law which we denote by π.

The main object of [LR20] was to study the existence of a phase transition, i.e if the critical infection rate

$$\lambda_c(v,p) := \inf\Big\{\lambda > 0 : \int \theta_{DP}(\lambda, 1, vp, v(1-p), \{x\}, B)\pi(\mathsf{d}B) > 0\Big\}$$

is non-trivial, where $x \in V$ is arbitrary. Note that since the background is started in its invariant law via the graphical representation, one can easily see that the CPDP is translation invariant with respect to spatial shifts, which is the reason why $\lambda_c(v,p)$ does not depend on the choice of x. First they showed a weaker version of monotonicity.

Proposition 1.3.1. *For every $p \in [0,1]$ the function $v \mapsto \frac{1}{v}\lambda_c(v,p)$ is non-increasing.*

This result corresponds to [LR20, Proposition 2.1]. Now they were able to show existence of a phase which they called the *immunization* phase, which basically states that if the background parameters are chosen favourable enough the infection cannot survive regardless of the infection rate. This phenomenon is not present in the classical case. The following theorem is a combination of Theorem 2.5 and Theorem 2.6 in [LR20].

Theorem 1.3.2 (Immunization)**.** *Let $G = (V,E)$ be a connected and vertex transitive graph with bounded degrees. Then*

(*i*) *For every $v > 0$ there exists a $p_0(v) > 0$ such that $\lambda_c(v,p) = \infty$ for every $p < p_0(v)$.*

(*ii*) *There exists a $p_1 \in (0,1)$ such that for every $p > p_1$, $\lambda_c(v,p) < \infty$ for every $v > 0$.*

Theorem 1.3.2 shows the existence of a critical curve $v \mapsto p_0(v)$ such that for every (v,p) which lies above the curve we have $\lambda_c(v,p) < \infty$, i.e. there exists a infection rate such that survival is possible. On the other hand for every pair (v,p) which lies below it holds that $\lambda_c(v,p) = \infty$, i.e. regardless of the infection rate extinction happens almost surely. Note that Proposition 1.3.1 states that the critical value can at most grow linear with respect to v. This yields that the curve $v \mapsto p_0(v)$ is non-increasing, since if $\lambda(v,p) < \infty$ for a $v > 0$, then Proposition 1.3.1 implies that $\lambda(v',p) < \infty$ for all $v' > v$. See Figure 1.2 for a visualization. The next result is a combination of Theorem 2.3 and Theorem 2.4 in [LR20] and is about the extreme case $v \to 0$ and $v \to \infty$.

Theorem 1.3.3 (Asymptotic behaviour)**.** *Let $G = (V,E)$ be a connected and vertex transitive graph with bounded degrees.*

(*i*) *For every $p \in (0,1)$, $\lambda_c(v,p) \to \frac{\lambda_c(G)}{p}$ as $v \to \infty$, where λ_c^G denotes the critical value of the classical contact process on G.*

(*ii*) *For the $V = \mathbb{Z}$ and $E = \{\{x,y\} \subset \mathbb{Z} : |x-y| = 1\}$ it holds for every $p \in (0,1)$ that the critical value $\lambda_c(v,p) \to \infty$ as $v \to 0$.*

This theorem characterizes the asymptotic behaviour. As one expects for $v \to \infty$, i.e. fast speed, the critical behaviour is that of CP on G with rescaled infection rate, i.e. λp. A non-rigorous argument for this is that by letting v tend to ∞ the update events happen so frequently that it is no different from throwing a coin with success probability p after encountering a infection event. In particular the immunization phase shrinks as $v \to \infty$ and ceases to exist for $v = \infty$, which means that $v \to p_0(v)$ decreases to 0 (see Figure 1.2).

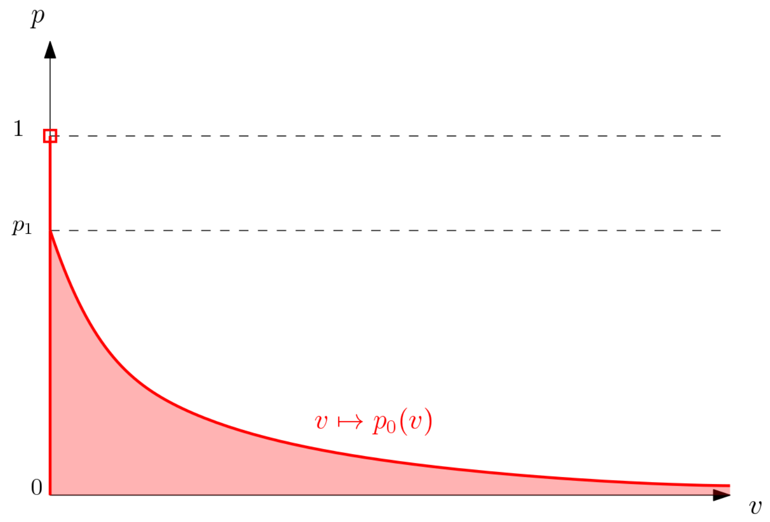


Figure 1.2: This is a sketch of the phase diagram of a CPDP on the 1-dimensional integer lattice $\mathbb{Z}$. The red curve denotes the critical parameter configuration and the red area is the immunization phase, i.e. certain extinction regardless of the infection rate. For parameter in the white area above the red curve there exists a infection rate λ such that the infection has a positive survival probability. For $v = 0$ extinction happens almost surely with exception of $p = 1$.

On the other hand the asymptotic for slow speed are only fully characterized for the 1-dimensional integer lattice. Theorem 1.3.3 (ii) states that in this case $\lambda_c(v, p) \to \infty$ for $p < 1$ as $v \to 0$ (see Figure 1.2), which agrees with the our intuition. One would expect that the critical behaviour for $v \to 0$ is the same as the contact process on a percolation cluster. Since on $\mathbb{Z}$ no infinite cluster occurs for $p < 1$, survival is not possible.

On more general graphs one would expect that the asymptotic behaviour for slow speed depends on the parameter p, since a percolation cluster of infinite size becomes possible. Proposition 1.3.1 and Theorem 1.3.2 (i) show that for p small enough $\lambda_c(v, p) \to \infty$ as $v \to 0$. But recently Hilário et al. [Hil+21] have studied a robust renormalization approach for generalized contact process. They call any process that is obtained from a percolative structure of recovery and transmission marks in same way as the contact process, but the distribution of these marks is given through some other Poisson point. This renormalization approach allows them to study survival or extinction of processes in this class. In fact the CPDP is part of this class and also one of the two examples they treat in [Hil+21]. Thus, they managed to provide some further results on the asymptotic behaviour of the critical infection rate for slow speed on the d-dimensional integer lattice.

Theorem 1.3.4. *Let* $V = \mathbb{Z}^d$ *and* $E = \{\{x, y\} \subset \mathbb{Z}^d : ||x-y||_1 = 1\}$ *and* $p_c(d)$ *denotes the critical probability of an independent percolation model on* $(\mathbb{Z}^d, E)$*.*

(i) *For all* $p < p_c(d)$ *and* $\lambda > 0$ *there exists* $v_0(p, \lambda, d) > 0$ *such that for any* $v \in (0, v_0)$ *the infection dies out almost surely.*

(ii) *For any* $p > p_c(d)$ *we have* $\sup\{\lambda_c(v, p') : v \geq 0, p' \in [p, 1]\} < \infty$*.*

We illustrated the phase diagram of CPDP on $\mathbb{Z}^d$, where $d \geq 0$, in Figure 1.3. If we compare this setting to the behaviour on 1-dimensional lattice we see that there exist an additional phase where survival is always possible and the critical infection rate $\lambda_c(v, p)$ is uniformly bounded if $p > p_c(d)$.

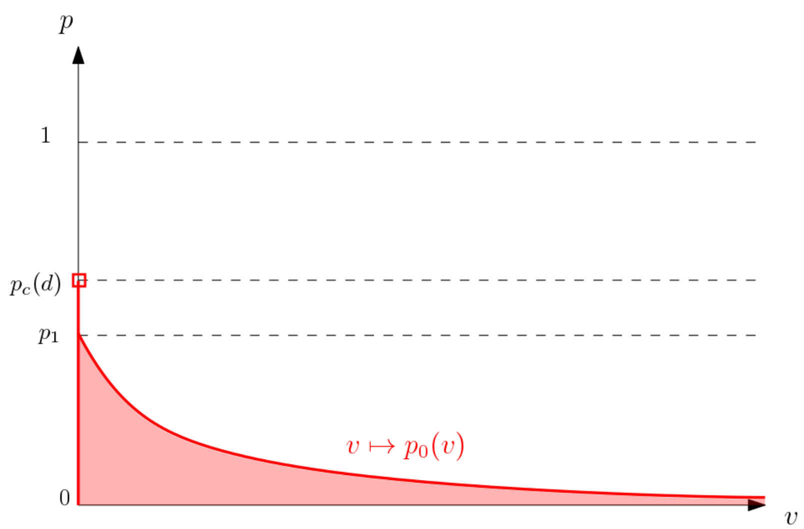


Figure 1.3: This is an illustration of the phase diagram of a CPDP on the d-dimensional integer lattice $\mathbb{Z}^d$. Again the red curve denotes the critical parameter configuration and the red area is the immunization phase.

1.4 Summary of the main results

In this section we give an overview of this thesis and at the same time a detailed summary of the main results. The results are proven in the corresponding chapters later on.

First we will study the CPERE on graphs with bounded degree. We study the influence of the initial configuration of the background on the survival probability. Next we focus on the invariant laws and therefore on the question whether the CPERE is ergodic or not. The goal here is to derive two conditions which imply complete convergence of the CPERE. We finish this part by considering the special case of the CPDP. Here, we formulate a block construction of the CPDP, which enables us to couple this process with an oriented percolation in the spirit of [BG90]. Among other things this enables us to show that the two conditions which imply complete convergences are satisfied in this special case. In the last part we will study a contact process on a dynamical long range percolation.

First of all in **Chapter 2** we introduce some basic notions. We start with a short introduction of Feller processes, and clarify some notation and definitions which we need in this thesis. Then we introduce the Poisson point process and with this process we formulate the graphical representation of a interacting particle system, which is one of the most essential tools in this thesis. We finish this chapter with the introduction of some notation and useful results on graphs.

In the first part of this thesis we start to study the CPERE. Thus, we need to clarify the setting we work in for the next chapters. Let $G = (V, E)$ be a connected and transitive graph with bounded degree. We denote by ρ the exponential growth of the graph G, (see Definition 2.4.6), i.e. $\rho = \lim_{n\to\infty} \frac{1}{n}\log(|\mathbb{B}_n(x)|)$, where $\mathbb{B}_n(x)$ denotes the ball of radius n with $x \in V$ as centre with respect to the graph distance. Note that since G is transitive ρ does not depend on the choice of x. If $\rho = 0$ we call G of subexponential growth. Next we define the coupled region of the background at time t by

$$\Psi_t = \Psi_t(\mathbf{B}) := \{e \in E : e \notin \mathbf{B}_t^{B_1} \triangle \mathbf{B}_t^{B_2} \;\; \forall B_1, B_2 \subset E\} \tag{1.2}$$

and the permanently coupled region at time t through

$$\Psi'_t = \Psi'_t(\mathbf{B}) := \{e \in E : e \in \Psi_s \forall s \geq t\}, \tag{1.3}$$

where $t \geq 0$. Recall that $\mathbf{B}$ is an attractive, translation invariant and finite range spin system. But we need some further assumption on the background process $\mathbf{B}$.

Assumption 1.4.1. *The background* $\mathbf{B}$ *satisfies the following assumptions:*

(*i*) $\mathbf{B}$ *is ergodic, i.e there exists a unique invariant law* π *such that* $\mathbf{B}_t^B \Rightarrow \pi$ *as* $t \to \infty$ *for all* $B \subset E$.

(*ii*) *There exist constants* $T, K, \kappa > 0$ *such that* $\mathbb{P}(e \notin \Psi'_t) < K \exp(-\kappa t)$ *for every* $e \in E$ *and for all* $t \geq T$.

(*iii*) $\mathbf{B}$ *is a reversible Feller process (see Definition 2.1.7).*

Loosely speaking if we assume that $\mathbf{B}$ is ergodic, i.e. that (i) is satisfied, then (ii) refers to the expansion speed of the permanently coupled region. This gives us a rough insight on how fast the background process convergences to the invariant law π.

Chapter 3 is basically divided in two parts. The first part is dedicated to the construction of a finite range spin system and the expansion behaviour of its permanently coupled region. In Section 3.1 we explicitly state one possible graphical representation for a general finite range spin system. Thus, we show that all spin systems we consider can be constructed via a graphical representation, which is a useful and important fact since we heavily rely on coupling methods which use such a representation. In Section 3.2 we study the expansion speed of the permanently coupled region Ψ'_t, $t \geq 0$. As readers familiar with interacting particle systems might know, the question if a spin systems satisfies (i) or not, is in general not trivial to determine. Hence, it may not be even harder to additionally show (ii). Thus, the main goal of Section 3.2 is to show the following result.

Proposition 1.4.2. *Suppose that Assumption 1.4.1* (i) *is satisfied and there exist constants* $S, K' > 0$ *and* $\gamma > \rho$ *such that* $\mathbb{P}(e \notin \Psi_s) \leq K' e^{-\gamma s}$ *for every* $e \in E$ *and* $s \geq S$. *Then there exist* $T, K > 0$ *and* $\kappa > 0$ *such that*

$$\mathbb{P}(e \notin \Psi'_t) \leq K e^{-\kappa t}$$

for all $t > T$ *and* $e \in E$.

With this result we are able to state a sufficient condition such that a spin system satisfies (i) and (ii) of Assumption 1.4.1, which is based on the so-called $M < \varepsilon$ condition, see [Lig12, Theorem I.4.1]. Recall that $\mathcal{N}_e^L$ denotes the neighbourhood of e with respect to the line graph $L(G)$.

Corollary 1.4.3. *Suppose $\varepsilon - M > \rho$, where*

$$M := \sum_{a \in \mathcal{N}_e^L} \sup_{B \subset E} |q(e,B) - q(e, B \triangle \{a\})| \quad \textit{and} \quad \varepsilon := \inf_{B \subset E} |q(e,B) + q(e, B \triangle \{e\})|,$$

then Assumption 1.4.1 (i) and (ii) are satisfied.

The proof of Corollary 1.4.3 can be found in Section 3.2. The definition of both M and ε do not depend on the choice of e since the background is translation invariant. M is a measure for the maximal dependence of the transition rates on the state of a single edge, while ε is a measure for the minimal rate at which the state of a single edge changes. Note that we simplified the definitions of M and ε in comparison to [Lig12, Chapter I], since we only consider finite range spin systems.

Remark 1.4.4. The constants M and ε can be explicitly calculated for the three systems defined in Example 1.1.2. The calculation can be found in Appendix A. Thus, with Corollary 1.4.3 we can state sufficient conditions on the rates such that these spin systems satisfy Assumption 1.4.1 (i) and (ii)

1. For the dynamical percolation the two quantities are $M = 0$ and $\varepsilon = \alpha + \beta$ and hence $\alpha + \beta > \rho$ is sufficient.
2. In case of the noisy voter model, $M = \beta|\mathcal{N}_e^L|$ and $\varepsilon = \alpha + \beta|\mathcal{N}_e^L|$. This implies that $\alpha > \rho$ suffices.
3. For the ferromagnetic stochastic Ising model the calculation is more lengthy but can still be carried out in a straightforward manner and the result is that

$$\varepsilon = 2 \quad \text{and} \quad M = \begin{cases} |\mathcal{N}_e^L| \frac{2(e^{2\beta} - e^{-2\beta})}{e^{2\beta} + e^{-2\beta} + 2} & \text{if } |\mathcal{N}_e^L| \text{ odd} \\ |\mathcal{N}_e^L| \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}} & \text{if } |\mathcal{N}_e^L| \text{ even,} \end{cases}$$

and therefore $0 \leq \beta < \frac{1}{4} \log\left(\frac{|\mathcal{N}_e^L| + 2 - \rho}{|\mathcal{N}_e^L| - 2 + \rho}\right)$ is sufficient, where the right hand side is only positive if $\rho < 2$. Note that if $|\mathcal{N}_e^L|$ is odd we are able to obtain a slightly better bound on β, which can be found at the end of Appendix A.

The second part of Chapter 3 is dedicated to obtaining some basic knowledge about the CPERE. In Section 3.3 we rigorously formulate the graphical representation of the CPERE. A direct consequence of this construction is the existence of a Feller process $(\mathbf{C}, \mathbf{B})$ with rates as in (1.1). In Section 3.4 we state some basic properties of the CPERE, such as some monotonicity properties of the CPERE, additivity of the infection process **C** and more. These properties follow with relatively small effort via different couplings derived from the graphical representation. We end Chapter 3 with a comparison result between the CPDP and CPERE. Let $(\mathbf{C}, \mathbf{B})$ be a CPERE such that **B** is a spin system with rate $q(\cdot,\cdot)$ and set

$$\begin{aligned} \alpha_{\min} &:= \min_{F \subset \mathcal{N}_e^L} q(e,F), \qquad & \beta_{\min} &:= \min_{F \subset \mathcal{N}_e^L} q(e, F \cup \{e\}) \\ \alpha_{\max} &:= \max_{F \subset \mathcal{N}_e^L} q(e,F) \ \text{ and } & \beta_{\max} &:= \max_{F \subset \mathcal{N}_e^L} q(e, F \cup \{e\}). \end{aligned} \tag{1.4}$$

Recall that θ denotes the survival probability of a CPERE and θ_{DP} the survival probability of a CPDP as mentioned in Remark 1.1.5.

Corollary 1.4.5. *Let $\lambda, r > 0$ and $\alpha_{\max}$,$\alpha_{\min}$, $\beta_{\max}, \beta_{\min} \geq 0$ as in* (1.4)*. Then*

$$\theta_{DP}(\lambda, r, \alpha_{\max}, \beta_{\min}, C, B) \geq \theta(\lambda, r, C, B) \geq \theta_{DP}(\lambda, r, \alpha_{\min}, \beta_{\max}, C, B)$$

where $C \subset V$ and $B \subset E$.

This result is a direct consequence of Proposition 3.4.5.

Example 1.4.6. If we consider the background process **B** to be a noisy voter model on $\mathbb{Z}$ as defined in Example 1.1.2 (ii) with rates $\alpha, \beta > 0$ we obtain the following bounds on the survival probability of **C**:

$$\theta_{\mathrm{DP}}(\lambda, r, \alpha+\beta, \beta, C, B) \geq \theta(\lambda, r, C, B) \geq \theta_{\mathrm{DP}}(\lambda, r, \beta, \alpha+\beta, C, B).$$

In **Chapter 4** we study the influence of the initial configuration of the background process on the chances of survival. In this chapter we will only use Assumption 1.4.1 (i) and (ii). Let us fix the following notation:

Definition 1.4.7 (Survival probability for stationary background)**.** Let $C \subset V$, $B \subset E$ and $\lambda, r > 0$. Then

$$\theta^{\pi}(\lambda, r, C) := \mathbb{P}_{\lambda,r}^{(C,\pi)}\big(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0\big)$$

is the survival probability of **C** with $\mathbf{B}_0 \sim \pi$, i.e. the background being stationary, and we define the critical infection rate as

$$\lambda_c^{\pi}(r) := \inf\{\lambda > 0 : \theta^{\pi}(\lambda, r, \{x\}) > 0\}.$$

Note that we will show later, in Corollary 4.0.2, that for a stationary background the definition of the critical infection rate does not depend on the choice of $x \in V$. We denote by $\mathcal{N}_x$ the neighbourhood of x in G. Let $c_1(\lambda, \rho)$ be the unique solution of

$$c\lambda - 1 - \log(c\lambda|\mathcal{N}_x|)) = \rho \tag{1.5}$$

which satisfies $0 < c_1(\lambda, \rho) \leq \frac{1}{\lambda}$, where $\lambda > 0$ and $x \in V$. We will later see, in Lemma 4.1.2, that $c_1(\lambda, \rho)$ is an upper bound for the maximal growth rate of the set of all infections. The main goal in this chapter is to show that under an additional condition, the initial configuration of the background $\mathbf{B}$ has no influence on whether survival of the infection is possible or not.

Theorem 1.4.8. *Let $\lambda, r > 0$ and $C \subset V$ be finite and non-empty. Suppose (i) and (ii) of Assumption 1.4.1 are satisfied and $c_1(\lambda, \rho) > \kappa^{-1}\rho$, then $\theta(\lambda, r, C, B_1) > 0$ if and only if $\theta(\lambda, r, C, B_2) > 0$ for all $B_1, B_2 \subset E$.*

Note that the statement is obviously true if $|C| \in \{0, \infty\}$, since then the survival probability θ is either 0 or 1. We will see that if the inequality $c_1(\lambda, \rho) > \kappa^{-1}\rho$ holds, then asymptotically the growth speed of the infection $\mathbf{C}$ is slower than the expansion of the permanently coupled region Ψ', with respect to time. Furthermore, by Proposition 4.0.1 (iii) if follows that $\theta^{\pi}(\lambda, r, C_1) > 0$ if and only if $\theta^{\pi}(\lambda, r, C_2) > 0$ for any two non-empty and finite $C_1, C_2 \subset V$. Thus, as a direct consequence of Theorem 1.4.8 we get the following result regarding the critical infection rate.

Corollary 1.4.9. *Let $r > 0$ and suppose Assumption 1.4.1 (i) and (ii) are satisfied. If there exists a non-empty and finite set $C' \subset V$ and a $B' \subset E$ such that $c_1(\lambda_c(r, C', B'), \rho) > \kappa^{-1}\rho$, then it follows that $\lambda_c(r, C, B) = \lambda_c^{\pi}(r)$ for all non-empty and finite $C \subset V$ and $B \subset E$. Then we denote the critical infection rate simply by $\lambda_c(r)$.*

Note that if we consider graphs with subexponential growth, i.e. $\rho = 0$, the inequality $c_1(\lambda, \rho) > \kappa^{-1}\rho$ is obviously satisfied for all $\lambda > 0$. Thus, on graphs with subexponential growth Theorem 1.4.8 and Corollary 1.4.9 are true as long as Assumption 1.4.1 (i) and (ii) are satisfied.

Since Corollary 1.4.9 provides us with sufficient conditions to determine if the critical infection rate $\lambda_c(r)$ is independent of the initial conditions, we can naturally extend Theorem 1.3.2(i) and Theorem 1.3.3, which were proven in [LR20], in the sense that we drop the assumption of stationarity, i.e. $\mathbf{B}_0 \sim \pi$. Recall from Remark 1.1.5 that we denote the survival probability and the critical infection rate by θ_{DP} and λ_c^{DP} for the contact process on a dynamical percolation.

Corollary 1.4.10. *Let $\rho \geq 0$ be the exponential growth of G.*

(i) For every $p \in (0,1)$, $\lambda_c^{DP}\big(1, vp, v(1-p), C, B\big) \to \frac{\lambda_c^G}{p}$ as $v \to \infty$, for all $C \subset V$ non-empty and finite and all $B \subset E$, where λ_c^G denotes the critical infection rate of the classical contact process with recovery rate 1 on the graph G.

(ii) If G is of subexponential growth, i.e $\rho = 0$, then for every $r > 0$ and $v > 0$ there exists a $p_0 = p_0(r, v) > 0$ such that for every $p < p_0$, $\lambda_c^{DP}\big(r, vp, v(1-p), C, B\big) = \infty$ for all $C \subset V$ non-empty and finite and all $B \subset E$.

(iii) If $V = \mathbb{Z}$ and $E = \big\{\{x, y\} \subset \mathbb{Z} : |x - y| = 1\big\}$, i.e. G is the 1-dimensional integer lattice, then for every $r > 0$ and $p \in (0,1)$, $\lambda_c^{DP}\big(r, vp, v(1-p), C, B\big) \to \infty$ as $v \to 0$, for all $C \subset V$ non-empty and finite and all $B \subset E$

In **Chapter 5** we study a quite different aspect of the CPERE. In this chapter we will focus on the connection between survival and non-ergodicity, i.e. that there exists more than one invariant law. Note that Assumption 1.4.1 *(iii)* will be pivotal and therefore, we briefly discuss when this assumption is satisfied.

For a given spin systems it is by no means trivial to see if it is reversible or not. But the class of *stochastic Ising models* satisfies reversibility by definition. Therefore, it seems to be natural to choose our background from this class of spins systems. By the definition given in [Lig12, Section IV.2] a stochastic Ising model is a spin system which is reversible with respect to the probability measure

$$\nu(B) \sim \exp\Big(\sum_{D \subset E} (-1)^{|B^c \cap D|} J_D\Big),$$

where $(J_D)_{D \subset E} \subset \mathbb{R}$ such that $\sum_{D \subset E} |J_D| < \infty$. Note that the sequence $(J_D)_{D \subset E}$ is called a *potential* of an Ising model. Hence, Assumption 1.4.1 *(iii)* is already naturally satisfied. On the other hand by [Lig12, Theorem IV.2.13] we know that every reversible, finite range spin system with strictly positive spin rates must already be a stochastic Ising model. Given a *potential* $(J_D)_{D \subset E}$ there are obviously infinitely many ways to choose the spin rates $q(\cdot, \cdot)$. One common choice of the spin rate is

$$q(e, B) := 1 - \tanh\Big(\sum_{e \ni D} (-1)^{|B^c \cap D|} J_D\Big) = 2\Big(1 + \exp\Big(2 \sum_{e \ni D} (-1)^{|B^c \cap D|} J_D\Big)\Big)^{-1}. \quad (1.6)$$

Of course, if Assumption 1.4.1 (i) and (ii) are satisfied is a different question and also not always the case. For example in case of the ferromagnetic Ising model stated in Example 1.1.2 (iii) if the parameter β is small enough, it satisfy these two assumptions. But depending on the underlying graph G, this system can exhibit a non-trivial phase transition between ergodicity and non-ergodicity, i.e. for β large enough there exist more than one invariant law. See for example [Lig12, Theorem IV.3.14].

Remark 1.4.11. In fact with the choice (1.6) of the spin rates we can show that all three systems in Example 1.1.2 are part of the class of stochastic Ising models.

1. Let $p \in (0,1)$. We choose $J_D = \frac{1}{2}\log\left(\frac{p}{1-p}\right)$ for $|D| = 1$ and $J_D = 0$ otherwise. Next plugging this choice of a potential into the spin rates (1.6) we get that $q(e,B) = p\mathbb{1}_{\{e\notin B\}} + (1-p)\mathbb{1}_{\{e\in B\}}$. Now rescaling time with a constant $v > 0$ and setting $\alpha := vp$ and $\beta := v(1-p)$ yields that the dynamical percolation is a stochastic Ising model for all $\alpha, \beta > 0$.
2. To show this for the noisy voter model on the 1-dimensional nearest neighbour lattice, let $\gamma > 0$ and choose $J_D = \frac{1}{4}\log(1+\gamma^{-1})$, for $|D| = 2$ and $J_D = 0$ otherwise. Inserting this into (1.6) yields that
$$q(e,B) = \frac{1}{2\gamma+1}\Big(\frac{1}{2}\big(|B\cap\mathcal{N}_e|\mathbb{1}_{\{e\notin B\}} + |B^c\cap\mathcal{N}_e|\mathbb{1}_{\{e\in B\}}\big) + \gamma\Big).$$
Again rescaling time with the factor $\frac{\alpha(2\gamma+1)}{2\gamma}$, where $\alpha > 0$ and setting $\beta := \frac{\alpha}{2\gamma}$ we see that the spin rate corresponds to the spin rate of a noisy voter model as given in Example 1.1.2 (ii).
3. That the ferromagnetic stochastic Ising model introduced in Example 1.1.2 (iii) is part of this class is quite obvious, but for the sake of completeness we also state the concrete potential $(J_D)_{D\subset E}$. For $\beta > 0$ we choose $J_D = \beta$ if $|D| = 2$ and $D \subset \mathcal{N}_e^L$ and $J_D = 0$ otherwise. This choice yields
$$q(e,B) = 1 - \tanh\Big(2\beta\big(|\mathcal{N}_e^L| - 2(\mathbb{1}_{\{e\in B\}}|B\cap\mathcal{N}_e^L| + \mathbb{1}_{\{e\notin B\}}|B^c\cap\mathcal{N}_e^L|)\big)\Big).$$

Remark 1.4.12 (General noisy voter model)**.** One might question why we do not consider a more general noisy voter model, as for example a process with transitions

$$\begin{aligned}\mathbf{B}_{t-} = B \to B\cup\{e\} &\quad \text{at rate } \alpha_1 + \beta|B\cap\mathcal{N}_e^L| \text{ and}\\ \mathbf{B}_{t-} = B \to B\backslash\{e\} &\quad \text{at rate } \alpha_2 + \beta|B^c\cap\mathcal{N}_e^L|,\end{aligned}$$

where $\alpha_1, \alpha_2, \beta > 0$. It is not difficult to show that this process satisfies (i) and (ii) of Assumptions 1.4.1. But we do not know if this process always satisfy (iii) in this general setting. For example it is not clear if this process is part of the class of stochastic Ising models, which would imply that (iii) is satisfied.

In Chapter 5, Section 5.1, we start with proving the existence of a so-called *upper invariant law* $\overline{\nu} = \overline{\nu}_{\lambda,r}$. This law has the property that if ν is a invariant law of the CPERE, then this implies that $\nu \preceq \overline{\nu}$, where $\preceq$ denotes the stochastic order. This explains why its called upper invariant law. At this point it is not clear if $\overline{\nu}$ differs from the trivial invariant law $\delta_\emptyset \otimes \pi$. The question if $\overline{\nu} = \delta_\emptyset \otimes \pi$ is equivalent to asking if this system is ergodic, i.e. if there exists a unique invariant law which is the weak limit of the process. By monotonicity, we know that if $\lambda_1 \leq \lambda_2$, then $\overline{\nu}_{\lambda_1,r} \preceq \overline{\nu}_{\lambda_2,r}$ and the reversed order holds for the recovery rate. Thus, we define the following critical value.

Definition 1.4.13 (Critical infection rate for non-triviality of $\overline{\nu}$)**.** For $r > 0$ we define

$$\lambda_c'(r) := \inf\{\lambda > 0 : \overline{\nu}_{\lambda,r} \neq \delta_\emptyset \otimes \pi\}.$$

The first aim is to show that this phase transition corresponds to the already known phase transition between certain extinction and persistence of the infection in the population with positive probability. Here we will again need the growth assumption $c_1(\lambda, \rho) > \kappa^{-1}\rho$. Recall that $c_1(\lambda, \rho)$ is the unique solution of (1.5), κ is given through Assumption 1.4.1 (ii) and ρ is the exponential growth of the graph G.

Proposition 1.4.14. *Let $r > 0$ and suppose Assumptions 1.4.1 (i)-(iii) are satisfied. Then $\lambda_c'(r) = \lambda_c^\pi(r)$. If additionally $c_1\big(\lambda_c^\pi(r), \rho\big) > \kappa^{-1}\rho$, then $\lambda_c'(r) = \lambda_c(r)$.*

In Section 5.2 we derive the main result of this chapter. We state two conditions which are equivalent to the so-called complete convergence of the CPERE, i.e. for every initial configuration $C \subset V$ and $B \subset E$

$$(\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \Rightarrow \theta(C,B)\overline{\nu} + [1 - \theta(C,B)](\delta_\emptyset \otimes \pi). \tag{1.7}$$

Note that if we know that complete convergence holds true, then we have already characterized all invariant laws of $(\mathbf{C}, \mathbf{B})$. We abuse notation somewhat by writing

$$\{x \in \mathbf{C}_t \text{ i.o.}\} = \{x \in \mathbf{C}_t \text{ for a sequence of times } t \uparrow \infty\},$$

where i.o. is short for "infinitely often".

Theorem 1.4.15. *Let $\lambda, r > 0$ such that $c_1(\lambda, \rho) > \kappa^{-1}\rho$. Furthermore, let Assumptions 1.4.1 (i)-(iii) be satisfied. Suppose*

$$\mathbb{P}^{(C,B)}_{\lambda,r}(x \in \mathbf{C}_t \text{ i.o.}) = \theta(\lambda, r, C, B) \tag{1.8}$$

for all $x \in V$, $C \subset V$ and $B \subset E$ and

$$\lim_{n\to\infty} \limsup_{t\to\infty} \mathbb{P}_{\lambda,r}(\mathbf{C}_t^{\mathbb{B}_n(x),\emptyset} \cap \mathbb{B}_n(x) \neq \emptyset) = 1 \tag{1.9}$$

for any $x \in V$. Then (1.7) is satisfied. Conversely if (1.7) holds and additionally $\overline{\nu} \neq \delta_\emptyset \otimes \pi$, then (1.8) and (1.9) are satisfied.

We finish this chapter with Section 5.3, where we discuss continuity of the survival probability θ. If $|C| \in \{0, \infty\}$, then $\theta(\cdot, C, B)$ is constant, and thus obviously continuous. Therefore, we will only consider the case where C is non-empty and finite. We define for such initial configurations (C, B) the region of survival by

$$\mathcal{S}(C, B) := \{(\lambda, r) \in (0, \infty)^2 : \theta(\lambda, r, C, B) > 0\}. \tag{1.10}$$

On the complement $\big(\mathcal{S}(C, B)\big)^c$ we see that the survival probability is again 0, and thus obviously again continuous. So the only interesting question is if $\theta(\cdot, C, B)$ is continuous on $\mathcal{S}(C, B)$. Unfortunately, on general graphs we are not able to determine if the survival probability is continuous on the whole survival region. Thus, for technical reasons, we need to restrict ourselves to the parameter set

$$\mathcal{S}_{c_1} := \{(\lambda, r) : \exists \lambda' \leq \lambda \text{ s.t. } (\lambda', r) \in \mathcal{S}(\{x\}, \emptyset) \text{ and } c_1(\lambda', \rho) > \kappa^{-1}\rho\}, \tag{1.11}$$

which contains all parameter (λ, r) such that a $\lambda' \leq \lambda$ exists fo which survival is still possible and the already known growth condition is satisfied. This is actually equivalent to assuming that for $r > 0$, $c_1\big(\lambda_c^\pi(r), \rho\big) > \kappa^{-1}\rho$. Note that by Theorem 1.4.8, the set $\mathcal{S}_{c_1}$ does not depend on the choice of the initial configurations (C, B) of the CPERE with C being non-empty and finite. We denote by $\mathring{U}$ the interior of a set $U \subset \mathbb{R}^d$, i.e. the largest open set which is contained in U.

Theorem 1.4.16. *Let $C \subset V$ be finite and non-empty and $B \subset E$. Then the survival probability $\theta(\cdot, C, B)$ is continuous on $\mathring{\mathcal{S}}_{c_1}$.*

Note that on subexponential graphs, i.e. $\rho = 0$, we know that $c_1(\lambda, \rho) > \kappa^{-1}\rho$ is always satisfied and thus $\mathcal{S}_{c_1} = \mathcal{S}(C, B)$ for all (C, B) with C being non-empty and finite.

This means in particular that on subexponential graphs Theorem 1.4.16 shows that the survival provability is everywhere continuous, except at criticality, i.e. the boundary of the survival region. With the techniques used in this chapter we are neither able to prove or disprove continuity at criticality. Such a result is much more involved and not even know for the CP on every graph G. An exception is for example the CP on the d-dimensional integer lattice. For this model [BG90] showed that the process goes almost surely extinct at criticality, which implies continuity on the whole parameter set. We can use their techniques to show, among other things, continuity of the CPDP in this setting.

Thus, in **Chapter 6** we focus on our main example introduced in Example 1.1.2 (i). The CPDP on the d-dimensional lattice, i.e.

$$V = \mathbb{Z}^d \quad \text{and} \quad E = \{\{x,y\} \subset \mathbb{Z}^d : ||x-y||_1 = 1\},$$

where $||\cdot||_1$ denotes the 1-norm. Note that we denote by $\mathbf{0} \in \mathbb{Z}^d$ the d-dimensional vector of zeros. Since we consider the concrete case of a dynamical percolation as background process we have two additional parameters α and β to consider. First of all the d-dimensional lattice is obviously of subexponential growth, and thus by Remark 1.4.4 the background process $\mathbf{B}$ satisfies Assumption 1.4.1 for all $\alpha, \beta > 0$. Furthermore, recall from Remark 1.1.5 that we denote the survival probability by $\theta(\lambda, r, \alpha, \beta, C, B)$. Since we only consider the dynamical percolation as background we drop the subscript DP. As mentioned in the same remark. Since $\rho = 0$, by Corollary 1.4.9 the critical infection rate is given through

$$\lambda_c(r, \alpha, \beta) = \inf\{\lambda > 0 : \theta(\lambda, r, \alpha, \beta, \{\mathbf{0}\}, \emptyset) > 0\}.$$

Another property of dynamical percolation is that every edge is independent of the other edges, i.e. if $e \neq e'$, then $\{e \in \mathbf{B}_t\}$ and $\{e' \in \mathbf{B}_t\}$ are independent for every $t \geq 0$. Thus, we can explicitly state the invariant law $\pi = \pi_{\alpha,\beta}$ of the background process. According to this measure the state of every edge is independently distributed with respect to a Bernoulli distribution with parameter $\frac{\alpha}{\alpha+\beta}$, i.e. for every $e \in E$

$$\pi(\{B \subset E : e \in B\}) = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \pi(\{B \subset E : e \notin B\}) = \frac{\beta}{\alpha+\beta}.$$

The main topic of Chapter 6 is to adapt the techniques developed by [BG90] to the CPDP. The revolutionary aspect of this work was that they managed to formulate conditions equivalent to the survival of the CP, which only depend on an arbitrarily large but finite space-time box. On the other hand the survival of the CP is obviously a event which depends on the whole space-time path of the CP. Essentially these conditions state that if we consider a large enough space-time box $[-L, L]^d \times [0, T]$ and start with a smaller fully infected space box $[-n, n]^d$, with high probability we find a spatially shifted box $[-n, n]^d + x$ at the sides or the top of the large box $[-L, L]^d \times [0, T]$, which is again fully infected. With these conditions they managed to formulate a coupling between an oriented percolation and the CP such that if the percolation model survives, then the CP survives and vice versa.

In Section 6.1 we start with formulating appropriate finite space-time events for the CPDP and eventually we prove that if we are in the supercritical phase, i.e that $\theta(\lambda, r, \alpha, \beta, \{\mathbf{0}\}, \emptyset) > 0$, then these events occur with high probability, which means that the finite space-time conditions are satisfied. In Section 6.2 we construct the previously mentioned coupling with an oriented percolation such that if this model percolates it implies survival of the CPDP and vice versa. This is a powerful tool and has far reaching consequences, since it enables us to show the following results.

In this case we can again denote the survival by

$$\mathcal{S} := \{(\lambda, r, \alpha, \beta) \in (0, \infty)^2 : \theta\big(\lambda, r, \alpha, \beta, \{\mathbf{0}\}, \emptyset\big) > 0\}, \tag{1.12}$$

where we know by Theorem 1.4.8 that this set does not depend on the intial configuration of the CPDP as long as the set of initially infected sites C is non-empty and finite. We also include the two additional parameter.

Theorem 1.4.17. *The CPDP goes almost surely extinct at criticality, i.e.*

$$\theta\big(\lambda, r, \alpha, \beta, \{\mathbf{0}\}, \emptyset\big) = 0$$

for all $(\lambda, r, \alpha, \beta) \in (0, \infty)^4 \backslash \mathring{\mathcal{S}}$.

Furthermore, for the parameters α and β we can obtain the same monotonicity and continuity properties as for the infection and recovery rates λ and r, which we showed in Section 3.4 and Section 5.3. Therefore, a direct consequence of Theorem 1.4.17 is the following result:

Corollary 1.4.18. *Let* $C \subset V$ *and* $B \subset E$. *The survival probability is continuous, i.e.*

$$(\lambda, r, \alpha, \beta) \mapsto \theta(\lambda, r, \alpha, \beta, C, B)$$

is continuous seen as function from $(0,\infty)^4$ *to* $[0,1]$.

At last we are able to show that complete convergence holds for the CPDP for every choice of parameters.

Theorem 1.4.19. *The CPDP* $(\mathbf{C},\mathbf{B})$ *satisfies complete convergence, i.e. for every* $C \subset V$ *and* $B \subset E$

$$(\mathbf{C}_t^{C,B},\mathbf{B}_t^B) \Rightarrow [1-\theta(C,B)](\delta_\emptyset \otimes \pi) + \theta(C,B)\overline{\nu} \quad \text{as } t \to \infty.$$

We will end this chapter by showing that for a general CPERE on the d-dimensional integer lattice, complete convergence holds on a subset of its survival region. To be precise this subset will be the interior of the survival region of a suitable CPDP, which lies "below" the CPERE. This CPDP is obtain by Proposition 3.4.5. Here we will again use the subscript DP since we need to distinguish between a CPERE and a CPDP, i.e. θ denotes the survival probability of the CPERE and θ_{DP} of the CPDP (see Remark 1.1.5).

Theorem 1.4.20. *Let* $(\mathbf{C},\mathbf{B})$ *be a CPERE on the* d*-dimensional integers lattice* $(\mathbb{Z}^d, E)$ *with infection rates* $\lambda > 0$*, recovery rate* $r > 0$ *and spin rate of the background* $q(\cdot,\cdot)$ *and suppose that* (*i*)-(*iii*) *of Assumption 1.4.1 are satisfied. Let* $\alpha_{\min}$ *and* $\beta_{\max}$ *be defined as in* (1.4)*. If* $\theta_{DP}(\lambda, r, \alpha_{\min}, \beta_{\max}, \{\mathbf{0}\}, \emptyset) > 0$ *then complete convergence holds, i.e. for every* $C \subset V$ *and* $B \subset E$

$$(\mathbf{C}_t^{C,B},\mathbf{B}_t^B) \Rightarrow [1-\theta(\lambda,r,C,B)](\delta_\emptyset \otimes \pi) + \theta(\lambda,r,C,B)\overline{\nu} \quad \text{as } t \to \infty.$$

In Chapter 7 we will consider a *contact process on a dynamical long range percolation*, which we abbreviate with CPLDP. The term "long range" refers to the fact that connections of any length are possible, and therefore we are no longer in the setting of bounded degrees. To be precise we consider the set

$$\mathcal{E} := \{e = \{x,y\} : x,y \in V, x \neq y\},$$

which contains edges between all vertices and not only neighbours, i.e. vertices $x, y \in V$ such that $d(x,y) = 1$. Recall $d(\cdot,\cdot)$ is the graph distance induced by the graph $G = (V,E)$. We define the CPLDP $(\mathbf{C},\mathbf{B})$ on the state space $\mathcal{P}(V) \times \mathcal{P}(\mathcal{E})$. In this chapter we adapt the methods developed in [LR20] and extend some of their result, which we summarize in Section 1.3, to the long range or rather infinite range setting.

But, let us first state the concrete setting and process. We consider two sequences $(p_e)_{e\in\mathcal{E}} \subset [0,1]$ and $(v_e)_{e\in\mathcal{E}} \subset (0,\infty)$ of real numbers. Here p_e will be the probability that after an update the edge e is open and v_e is the update speed of this particular edge. Additionally we assume that if $\{x_i, y_i\} \in \mathcal{E}$ for $i \in \{1,2\}$ with $d(x_1,y_1) = d(x_2,y_2)$, then $p_{\{x_1,y_1\}} = p_{\{x_2,y_2\}}$ and $v_{\{x_1,y_1\}} = v_{\{x_2,y_2\}}$. Thus, edges which are of the same length according to the graph distance $d(\cdot,\cdot)$ have the same probability to being open after an update and the same update speed. We want to remain in a similar setting as the CPDP, where the behaviour of the background is governed by two parameters. Thus, let $\gamma > 0$ and $q \in (0,1)$ and set

$$\hat{p}_e = \hat{p}_e(q) := qp_e \quad \text{and} \quad \hat{v}_e = \hat{v}_e(\gamma) := \gamma v_e$$

for all $e \in \mathcal{E}$. Note that q and γ have similar interpretations as the parameters p and v considered in Section 1.3. Now we are ready to define the dynamical long range percolation process which will be our background process. Thus, $\mathbf{B}$ is again a Feller process on $\mathcal{P}(\mathcal{E})$ with transitions

$$\begin{aligned} \mathbf{B}_{t-} = B &\to B \cup \{e\} \quad \text{at rate } \hat{v}_e\hat{p}_e \text{ and} \\ \mathbf{B}_{t-} = B &\to B\backslash\{e\} \quad \text{at rate } \hat{v}_e(1-\hat{p}_e). \end{aligned} \tag{1.13}$$

Note that we choose $\mathbf{B}_0 \sim \pi$, where π is the invariant law of $\mathbf{B}$ which means that the events $(\{e \in \mathbf{B}_0\})_{e\in\mathcal{E}}$ are independent and $\mathbb{P}(e \in \mathbf{B}_0) = \hat{p}_e$ for all $e \in \mathcal{E}$.

As usual in a long range setting we need some assumptions regarding the decay of the flip rates of the background process.

Assumption 1.4.21. *Assume that the sequences $(p_e)_{e\in\mathcal{E}}$ and $(v_e)_{e\in\mathcal{E}}$ satisfy*

(i) $\sum_{y\in V} v_{\{x,y\}}p_{\{x,y\}} < \infty$ *for all* $x \in V$ *and*

(ii) $\sum_{y\in V} v_{\{x,y\}}^{-1} < \infty$ *for all* $x \in V$.

In Section 7.1 we will discuss the construction of this process via a graphical representation and prove that is well defined. We need to adjust the construction use for the CPERE specifically for this case, since $(V,\mathcal{E})$ is no longer a graph with bounded degree. This is possible since Assumption 1.4.21 implies that $v_{\{x,y\}}p_{\{x,y\}} \to 0$ and $v_{\{x,y\}} \to \infty$ as $d(x,y) \to \infty$, this indicates that the probability that a long edge is open, i.e. an edge connecting two vertices over a long distance, becomes exceedingly unlikely. Therefore, heuristically speaking a successful infection over a long distance is getting more unlikely as the distance increases. Since the probability that this particular edge is closed in the moment a infection event takes places increases. Simply put, at any time the percolation cluster will be locally finite graph, and thus the construction still works out. With Assumption 1.4.21 we are able to show in Lemma 7.1.3 that $(\mathbf{C}, \mathbf{B})$ is a well-defined Markov process, in the sense that $|\mathbf{C}_t^C| < \infty$ almost surely for all $t \geq 0$, if $C \subset V$ is finite. Although we do not show that this process has the Feller property. As before we again focus on the survival of the CPLDP, and thus for $\lambda, r, \gamma > 0$ and $q \in (0, 1)$ we again denote by

$$\theta(\lambda, r, \gamma, q, C) := \mathbb{P}_{\lambda,r,\gamma,q}(\mathbf{C}_t^C \neq \emptyset \ \forall t \geq 0)$$

the survival probability and the critical infection rate for survival by

$$\lambda_c(r, \gamma, q) := \inf\{\lambda \geq 0 : \theta(\lambda, r, \gamma, q, \{x\}) > 0\}.$$

It is not hard to see that the CPLDP is again monotone regarding changes in the infection rate λ, and thus the infimum takes a unique value. Also note that again the definition does not depend on the choice of $x \in V$, since we started the background in its stationary state, and therefore this follows again by translation invariance. Actually, monotonicity in the rates λ, r and q can be easily concluded by a coupling argument via the graphical representation, similarly to Lemma 3.4.2 and Lemma 6.3.1 for the CPDP. Thus, we will not show this again. But as in the setting in [LR20] it is not clear at all if the survival probability is monotone in γ. Hence, we show at least the following result.

Proposition 1.4.22. *The function $\gamma \mapsto \gamma^{-1}\lambda_c(r, \gamma, q)$ is monotone decreasing.*

Another application of the graphical representation enables us to compare the CPLDP to a long range version of the contact process. Let us now define this long range version. Let $r > 0$ and $(a_e)_{e \in \mathcal{E}}$ be a sequence of positive real numbers such that $a_{\{x,y\}} = a_{\{x',y'\}}$ if $d(x, y) = d(x', y')$ and

$$\sum_{y \in V} a_{\{x,y\}} < \infty$$

for all $x \in V$, where we again used the convention $a_{\{x,x\}} = 0$. Then a Feller process $\mathbf{X}$ on the state space $\mathcal{P}(V)$ with transitions

$$\mathbf{X}_{t-} = C \to C \cup \{x\} \quad \text{at rate } \sum_{y \in C} a_{\{x,y\}} \text{ and}$$

$$\mathbf{X}_{t-} = C \to C \backslash \{x\} \quad \text{at rate } r,$$

is called a long range contact process. For more details on this type of process one may consult [Swa09].

Proposition 1.4.23 (Comparison with a long range contact process)**.** *Let $C \subset V$ and $(\mathbf{C}_t^C, \mathbf{B}_t)_{t\geq 0}$ be a CPLDP with parameter $\lambda, r, \gamma > 0$ and $q \in (0,1)$. Then there exists a long range contact process $(\overline{\mathbf{X}}_t^C)_{t\geq 0}$ with $\overline{\mathbf{X}}_0^C = C$, infection rates*

$$\overline{a}_e(\lambda, \gamma, q) = \frac{1}{2}\Big(\lambda + \gamma v_e - \sqrt{(\lambda + \gamma v_e)^2 - 4 v_e p_e \lambda \gamma q}\Big).$$

for all $e \in \mathcal{E}$ and recovery rate r such that $\overline{\mathbf{X}}_t^C \subset \mathbf{C}_t^C$ for all $t \geq 0$.

This result in particular yields that if $\overline{\mathbf{X}}^C$ survives with positive probability so does $\mathbf{C}^C$. Furthermore, it is not difficult to see that for every $e \in \mathcal{E}$

$$\lim_{\gamma \to \infty} \overline{a}_e(\lambda, \gamma, q) = \lambda q p_e.$$

In Section 7.2 we will provide some preliminary ground work for Section 7.3, where we prove existence of a immunization phase. The techniques applied in these two sections use among other things a comparison argument between a long range percolation model and the background process on a finite time interval $[nT, (n+1)T)$, where $n \in \mathbb{N}_0$ and $T > 0$. Thus, we first state a bound on the probability that an edge e is closed throughout such a time interval of length T and then, introduce a long range percolation model and show some results which guarantee absence of infinite connected component in such a model.

In Section 7.3 we study the critical infection rate $\lambda_c(r, q, \gamma)$ with respect to small q. For the arguments in this section Assumption 1.4.21 (ii) will be crucial, i.e. that

$$\sum_{y \in V} v_{\{x,y\}}^{-1} < \infty$$

for all $x \in V$. This assumption implies that $v_{\{x,y\}} \to \infty$ as $d(x,y) \to \infty$. Heuristically speaking, this assumption might be interpreted in the following way. Since the updates of long edges happen very frequently one can assume that before every infection event an update already took place, and thus an successful transmission of an infection via a long edges e occurs approximately with rate $\lambda \hat{p}_e$. We show with a strategy similar to the proof of [LR20, Theorem 2.5] that for the CPLDP there exists a immunization phase.

Theorem 1.4.24. *Suppose Assumption 1.4.21 is satisfied. Then, for a given* $r > 0$ *and* $\gamma > 0$*, there exists* $q^* = q^*(r, \gamma) \in (0, 1)$ *such that* $\mathbf{C}$ *dies out almost surely for all* $q < q^*$*, regardless of the choice of* $\lambda > 0$*, i.e.* $\lambda_c(r, \gamma, q) = \infty$ *for all* $q < q^*$*.*

In Section 7.4 we will study the asymptotic behaviour of the critical infection rate $\lambda_c(r, q, \gamma)$ as $\gamma \to 0$. For general countable vertex sets V a direct consequence of Proposition 1.4.22 and Theorem 1.4.24 is the following result.

Corollary 1.4.25. *Let* $r > 0$*. There exists a* $q^* = q^*(r) \in (0, 1)$ *such that for every* $q < q^*$*, there exists a* $\gamma_0 = \gamma_0(q) > 0$ *such that* $\lambda_c(r, \gamma, q) = \infty$ *for all* $\gamma < \gamma_0$*. This implies in particular that* $\lim_{\gamma \to 0} \lambda_c(r, \gamma, q) = \infty$ *for every* $q < q^*$*.*

But if we choose $V = \mathbb{Z}$ and $E = \{\{x, y\} \subset \mathbb{Z} : |x - y| = 1\}$, i.e. $G = (V, E)$ is the 1-dimensional lattice, then we can fully describe the asymptotic behaviour for slow speed of the CPDLP, i.e. $\gamma \to 0$, under suitable assumption.

Assumption 1.4.26. *Assume that the sequences* $(p_e)_{e \in \mathcal{E}}$ *and* $(v_e)_{e \in \mathcal{E}}$ *satisfy*

$$\sum_{y \in \mathbb{N}} y v_{\{0,y\}}^{-1} < \infty \qquad \textit{and} \qquad \sum_{y \in \mathbb{N}} y v_{\{0,y\}} p_{\{0,y\}} < \infty$$

This is basically a stronger version of Assumption 1.4.21.

Theorem 1.4.27. *Suppose Assumption 1.4.26 is satisfied. Let* $r > 0$*,* $q \in (0, 1)$ *and* $C \subset V$ *be non-empty and finite. Then, for every* $\lambda > 0$ *there exists* $\gamma^*(\lambda) = \gamma^* > 0$ *such that* $\mathbf{C}^C$ *dies out almost surely for all* $\gamma \le \gamma^*$*, i.e.* $\theta(\lambda, r, \gamma, q, C) = 0$ *for all* $\gamma \le \gamma^*$*. Thus, in particular* $\lim_{\gamma \to 0} \lambda_c(r, \gamma, q) = \infty$*.*

With this result we have proven that in the regime for which Assumption 1.4.26 is fulfilled we have a similar overall behaviour as for the contact process on a finite range dynamical percolation, and thus in this case the phase diagram with respect to the background parameters should also look as the visualization in Figure 1.2.

Chapter 2

Basic notions and graphical representation

2.1 Markov process theory

Here we give a short recap of some results and notation used in (homogenous) Markov process theory. For a self-contained and detailed introduction to this topic we refer the reader to [EK09] or [Lig12]. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space and $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ be a measure space, where we assume that $\mathbb{S}$ is a compact Polish space and $\mathcal{B}(\mathbb{S})$ is the Borel σ-algebra.

Definition 2.1.1 (Markov property)**.** We call an ($\mathcal{F}_t$–adapted) stochastic process $(\mathbf{X}_t)_{t\geq 0}$ a *Markov process* if for $t \geq s$, $\mathcal{F}_s$ is independent of $\mathbf{X}_t$ given $\mathbf{X}_s$, i.e.

$$\mathbb{E}[f(\mathbf{X}_t)|\mathcal{F}_s] = \mathbb{E}[f(\mathbf{X}_t)|\mathbf{X}_s].$$

for all measurable and bounded functions $f : \mathbb{S} \to \mathbb{R}$.

Recall that a Markov process is called (time)-homogeneous if the conditional distribution of $\mathbf{X}_t$ given $\mathbf{X}_s$ only depends on the difference $t - s$, i.e. $\mathbb{E}[f(\mathbf{X}_t)|\mathbf{X}_s] = \mathbb{E}[f(\mathbf{X}_{t-s})|\mathbf{X}_0]$ for all measurable and bounded functions $f : \mathbb{S} \to \mathbb{R}$. Furthermore, let us denote by $B(\mathbb{S})$ the set of all bounded and measurable functions and by $C(\mathbb{S})$ the space of all continuous functions. We equip $C(\mathbb{S})$ with the supremum norm $||f|| = \sup_{x\in\mathbb{S}} |f(x)|$.

Definition 2.1.2 (Transition semigroup)**.** For $t \geq 0$ we call $T(t) : C(\mathbb{S}) \to B(\mathbb{S})$ which maps $f \mapsto \big(x \mapsto \mathbb{E}[f(\mathbf{X}_t)|\mathbf{X}_0 = x]\big)$, the *transition operator* of $\mathbf{X}$ and $(T(t))_{t\geq 0}$ its *transition semigroup.*

It is easy to see that $(T(t))_{t\geq 0}$ has semigroup structure since obviously $T(0)f = f$ holds for every $f \in C(\mathbb{S})$ and by the Markov property we can show that the so called Chapman-Kolmogorov equation holds, i.e. $T(t+s)f = T(t)T(s)f$ for every $f \in C(\mathbb{S})$. Note that we denote by $D_{\mathbb{S}}([0,\infty))$ the Skorokhod space. This is the function space which contains all cadlag functions $f : [0,\infty) \to \mathbb{S}$, i.e. f is right continuous and has left limits everywhere. Often we need stronger assumptions on the process $\mathbf{X}$, which leads us to the notion of a Feller process.

Definition 2.1.3 (Feller process)**.** We call a Markov process $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ a *Feller process* if $\mathbf{X}$ has almost surely paths in $D_{\mathbb{S}}([0,\infty))$ and $T(t)f \in C(\mathbb{S})$ for all $f \in C(\mathbb{S})$.

Note that the transition semigroup of a Markov process characterizes the finite dimensional marginals and thus, characterizes the distribution of $\mathbf{X}$ completely.

Definition 2.1.4 (Markov semigroup)**.** We call a collection $(T(t))_{t\geq 0}$ of operators on $C(\mathbb{S})$, i.e. $T(t) : C(\mathbb{S}) \to C(\mathbb{S})$ for all $t \geq 0$, a *Markov semigroup* if the following is satisfied:

1. $T(0)f = f$ for all $f \in C(\mathbb{S})$
2. The mapping $t \mapsto T(t)f$ from $[0,\infty)$ to $C(\mathbb{S})$ is right continuous for all $f \in C(\mathbb{S})$.
3. $T(t)T(s) = T(t+s)$ for all $f \in C(\mathbb{S})$ and all $t, s \geq 0$
4. For all $t \geq 0$ it holds $T(t)\mathbb{1}_{\mathbb{S}} = \mathbb{1}_{\mathbb{S}}$.
5. $T(t)f \geq 0$ if $f \geq 0$ for all $t \geq 0$.

It is not difficult show that a transition semigroup of a Markov process is a Markov semigroup as seen in [Lig12, Proposition 1.3]. More importantly the reverse is also true, i.e. if $(T(t))_{t\geq 0}$ is a Markov semigroup, then there exists a unique Feller process $\mathbf{X}$ such that

$$T(t)f(x) = \mathbb{E}[f(\mathbf{X}_t)|\mathbf{X}_0 = x]$$

for all $x \in \mathbb{S}$, $f \in C(\mathbb{S})$ and $t \geq 0$. See [Lig12, Theorem 1.5]. The correspondence between Feller process and Markov semigroup is without doubt of great importance in the Markov process theory. But in most cases it can be exceedingly difficult or impossible to explicitly determine a Markov semigroup. Thus, it is more convenient to work with the so-called generator $\mathcal{A}$.

Definition 2.1.5 (Generator)**.** Let $\mathbf{X}$ be a Markov process and $(T(t))_{t\geq 0}$ its corresponding transition semigroup, then we define

$$\mathcal{A}f(x) := \lim_{t\to 0} t^{-1}(T(t)f(x) - f(x)).$$

for all $f \in \mathbf{D}(\mathcal{A}) := \{f \in B(\mathbb{S}) : \lim_{t\to 0} t^{-1}||T(t)f - f||_\infty \text{ exists}\}$. We call $\mathcal{A}$ the *generator* of the semigroup $(T(t))_{t\geq 0}$.

The generator is the time derivative of the Markov semigroup at the time point 0, i.e.

$$\frac{\mathsf{d}}{\mathsf{d}t}T(t) = \mathcal{A}.$$

Since $(T(t))_{t\geq 0}$ also satisfies the semigroup structure we have two defining properties of a operator-valued exponential function, and thus $T(t) = e^{t\mathcal{A}}$ for all $t \geq 0$ such that in turn $\mathcal{A}$ determines the Markov semigroup. Of course if $\mathbb{S}$ is not a finite set it is by no means trivial if these objects are well defined. To provide sufficient conditions for $\mathcal{A}$ such that these objects are properly defined, one would need to use the theory for operator semigroups developed by Hille and Yosida. We will not go further into detail here and again refer the interested reader to [Lig12, Section 1.2] or to [EK09, Chapter 1], where this is described in a more general setting.

Let us proceed with introducing some further notation and useful results. First of all, since we don't always start the Feller process $\mathbf{X}$ in a deterministic value $x \in \mathbb{S}$ but rather with a initial distribution ν we use the notation

$$\nu T(t)f = \int T(t)f(x)\nu(\mathsf{d}x)$$

for all $f \in C(\mathbb{S})$ and the shorthand $\nu T(t)$.

Definition 2.1.6 (Stationary distribution)**.** Let $\mathbf{X}$ be a Feller process with state space $\mathbb{S}$ and $(T(t))_{t\geq 0}$ its Markov semigroup. Then a probability measure ν on Ω is called *stationary* or *invariant* if $\nu T(t) = \nu$ for all $t \geq 0$.

Note that the definition obviously implies that if $\mathbf{X}_0 \sim \nu$, then $(\mathbf{X}_{t+s})_{t\geq 0} \stackrel{d}{=} (\mathbf{X}_t)_{t\geq 0}$ for all $s > 0$. But this means that if the Feller process is stationary we can easily extend the definition to the whole negative real line such that $(\mathbf{X}_t)_{t\in\mathbb{R}}$ is stationary process. Later in Chapter 5 we need the concept of (time)-reversibility.

Definition 2.1.7 (Reversible)**.** A Feller process $\mathbf{X}$ is said to be reversible with respect to the probability measure ν if

$$\int fT(t)g\mathsf{d}\nu = \int gT(t)f\mathsf{d}\nu$$

for all $f, g \in C(\mathbb{S})$, where $(T(t))_{t\geq 0}$ is the corresponding Markov-semigroup.

Obviously if $\mathbf{X}$ is reversible with respect to ν, then ν must be an invariant distribution of $\mathbf{X}$. This can be obtained by setting $g \equiv \mathbb{1}_{\mathbb{S}}$. An equivalent and maybe somewhat more intuitive interpretation of reversibility is the following:

Proposition 2.1.8. *Let $\mathbf{X}$ be Feller process. Then $\mathbf{X}$ is reversible with respect to the probability measure ν if and only if $(\mathbf{X}_t)_{t\in\mathbb{R}}$ and $(\mathbf{X}_{-t})_{t\in\mathbb{R}}$ have the same joint distributions, where $(\mathbf{X}_t)_{t\in\mathbb{R}}$ is the stationary process obtained by using the initial distribution ν and the transition mechanism corresponding to $T(t)$.*

Proof. See [Lig12, Proposition II.5.3] □

Next we introduce a partial order on the space of all probability measures on $\mathbb{S}$, which is called the stochastic order. We assume that $\mathbb{S}$ is equipped with a partial order "$\leq$". Furthermore with respect to this partial order we call a function $f : \mathbb{S} \to \mathbb{R}$ increasing if $x \leq y$ implies $f(x) \leq f(y)$, where $x, y \in \mathbb{S}$.

Definition 2.1.9 (Stochastic order)**.** Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be probability measures on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$. Then we say $\mathbb{P}_2$ dominates $\mathbb{P}_1$ stochastically, which we denote by $\mathbb{P}_1 \preceq \mathbb{P}_2$ if and only if

$$\int f\mathsf{d}\mathbb{P}_1 \leq \int f\mathsf{d}\mathbb{P}_2$$

for all measurable, increasing and bounded functions $f : \mathbb{S} \to \mathbb{R}$. Let X_1 and X_2 be $\mathbb{S}$-valued random variables. We write $X_1 \preceq X_2$ if $\mathbb{P}^{X_1} \preceq \mathbb{P}^{X_2}$.

Now we are able to introduce the notion of monotonicity for Feller processes.

Definition 2.1.10 (Monotone Feller process)**.** Let μ_1, μ_2 be two probability measures on $\mathbb{S}$. We call a Feller process $\mathbf{X}$ monotone if $\mu_1 \preceq \mu_2$ implies $\mu_1 T(t) \preceq \mu_2 T(t)$ for all $t \geq 0$.

Definition 2.1.11 (Coupling of probability measures)**.** A coupling of two probability measure $\mathbb{P}_1$ and $\mathbb{P}_2$ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is any probability measure $\hat{\mathbb{P}}$ on $(\mathbb{S}^2, \mathcal{B}(\mathbb{S}^2))$ such that $\hat{\mathbb{P}}(A \times \mathbb{S}) = \mathbb{P}_1(A)$ and $\hat{\mathbb{P}}(\mathbb{S} \times A) = \mathbb{P}_2(A)$ for all $A \in \mathcal{B}(\mathbb{S})$.

Another useful result for ordered probability measures is Strassen's Theorem.

Theorem 2.1.12 (Strassen)**.** *Let* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *be probability measures on* $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$*. If* $\mathbb{P}_1 \preceq \mathbb{P}_2$*, then there exists a coupling* $\hat{\mathbb{P}}$ *such that* $\hat{\mathbb{P}}(\{(x, y) \in \mathbb{S}^2 : x \leq y\}) = 1$*.*

Proof. See [Hol12, Theorem 7.9] □

Remark 2.1.13. This result can again be formulated for random variables. Assume that X_1 and X_2 are two random variables with values in $\mathbb{S}$ and are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $X_1 \preceq X_2$. Then Theorem 2.1.12 implies that there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and on it copies $\hat{X}_1 \overset{d}{=} X_2$ and $\hat{X}_2 \overset{d}{=} X_1$ such that $\hat{X}_1 \leq \hat{X}_2$ holds $\hat{\mathbb{P}}$-almost surely.

2.2 Poisson point processes

In this section we briefly introduce Poisson point processes, since we need them to formulate the graphical representation of an interacting particle system in the next section. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(\mathbb{S}, \mathcal{T})$ a topological space which is second countable, Hausdorff and locally compact, i.e. for every $x \in \mathbb{S}$ there exists a set $U \in \mathcal{T}$ with $x \in U$ and a compact set $C \subset \mathbb{S}$ such that $U \subset C$. Recall that by definition $\sigma(\mathcal{T}) = \mathcal{B}(\mathbb{S})$. First we need to state some general definitions and results concerning random measures and point processes.

Definition 2.2.1 (Locally finite measures)**.** Let μ be a measure on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$. We call μ *locally finite* if for every $x \in \mathbb{S}$ there exists $U \in \mathcal{B}(\mathbb{S})$ with $x \in U$ such that $\mu(U) < \infty$. We denote by $\mathfrak{M} = \mathfrak{M}(\mathbb{S})$ the set of all locally finite measures. Furthermore we define the set of all locally finite counting measures by

$$\mathfrak{N} := \{\mu \in \mathfrak{M} : \mu(A) \in \mathbb{N}_0 \cup \{\infty\} \text{ for all } A \in \mathcal{B}(\mathbb{S})\}.$$

Definition 2.2.2. A random measure $\Xi : \Omega \to \mathfrak{N}$ is called a *point process*. We call a point process *simple* if $\mathbb{P}(\Xi(\{x\}) \leq 1 \text{ for all } x \in \mathbb{S}) = 1$.

The next result justifies that a point process is seen as a random point cloud in $\mathbb{S}$.

Proposition 2.2.3. *Let* Ξ *be a simple point process, then there exists a* $\mathbb{N}_0 \cup \{\infty\}$*-valued random variable* N *and a sequence of* $\mathbb{S}$*-valued random variables* $X_0, X_1, \ldots$ *such that*

$$\Xi = \sum_{i=0}^{N} \delta_{X_i}.$$

Proof. See [Kal17, Lemma 1.6] □

Now we finally introduce the Poisson point process.

Definition 2.2.4 (Poisson point process)**.** $\Xi : \Omega \to \mathfrak{N}$ is called a *Poisson point process* on $\mathbb{S}$ with intensity measure $\xi : \mathcal{P}(\mathbb{S}) \to \mathbb{R}_+$ if

1. $\Xi(B) \sim \text{Poi}(\xi(B))$ for every bounded $B \in \mathcal{B}(\mathbb{S})$ and
2. $\Xi(B_1), \ldots, \Xi(B_n)$ are independent for every $n \in \mathbb{N}$ and every collection of bounded disjoint sets $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{S})$.

The standard example is $\mathbb{S} = \mathbb{R}^d$ and ξ chosen as the Lebesgue measure. In this case the Poisson point process is often described as an ideal gas. We end this section with a characterization when a Poisson point process is simple, which justifies when the physical picture of an ideal gas is appropriate.

Proposition 2.2.5. *Let Ξ be a Poisson point process with intensity measure ξ. The Poisson point process Ξ is simple if and only if $\xi(\{x\}) = 0$ for all $x \in \mathbb{S}$*

Proof. See [Kal06, Propsition 10.4.] □

2.3 Interacting particle systems and their graphical representation

In this section we introduce a graphical Poisson construction for interacting particle systems. Interacting particle system are a particular class of Feller processes. In the literature this term is not really standardized so we will briefly explain what we mean by it.
Let Λ be a finite or countably infinite set, where the interpretation of the elements contained in Λ are locations and we assume that on each location sits exactly one particle. Thus, we can identify each particle with its location $x \in \Lambda$. Now we want to assign to each particle a state, which may change over time. We denote by S the set of all possible states and we assume it to be finite. Now $f \in S^\Lambda$ is a configuration of the states of all particles, i.e. $f(x) \in S$ denotes the state of the particle $x \in \Lambda$. An interacting particle system is a Feller process with state space S^Λ and is specified via *local interaction* between particles. With a local interaction we mean that this particular interaction only depends on the states of a finite number of particles and can

only affect a finite number of particles, i.e. change their states. A common example for such a system is an opinion model, i.e. the particles correspond to people which can have a variety of opinions and S denotes all possible opinions. The local interactions describe how the opinion of a particular person is affect by another person's opinion. For a detailed introduction of interacting particle systems we refer the reader to [Lig12].

We only consider the special case where a particle can assume one of two distinct states, i.e. $S = \{0, 1\}$. This actually allows an alternative interpretation, where the state 1 or 0 describe whether a particle is present at the location $x \in \Lambda$ or not. Since we consider a infection model this interpretation seems more apt. In this context Λ is the population of individual and the particles are the infection. Thus, if a particle is present at $x \in \Lambda$ it means individual x is sick if it is not present the individual is healthy. Out of notational convenience we work with the power set $\mathcal{P}(\Lambda)$ as state space instead of $\{0, 1\}^\Lambda$. Note that it is not difficult to see that $\{0, 1\}^\Lambda$ only contains indicator functions, i.e. for every $f \in \{0, 1\}^\Lambda$ there exist a set $A \subseteq \Lambda$ such that $f \equiv \mathbb{1}_A$. Hence, it is easy to see that there exists a one-to-one correspondence between these two sets, i.e. $\{0, 1\}^\Lambda \backsimeq \mathcal{P}(\Lambda)$.

The graphical Poisson construction we are about to introduce is often called the *graphical representation*. As the name suggests this construction of particle systems is done with the help of an underlying Poisson point process. The standard reference for interacting particle system [Lig12] describes this for the case of additive systems. Besides this standard reference, [Swa17] explains in detail a graphical construction for a broader class of interacting particle systems. Since we intend to use this approach here we recapitulate some of the notation and results. Again, for a detailed description we refer to [Swa17].

For a map $m : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ we define the set

$$\mathcal{D}(m) := \{x \in \Lambda : \exists A \in \mathcal{P}(\Lambda) \text{ s.t. } x \in m(A) \triangle A\}.$$

This set is the collection of all $x \in \Lambda$, which can possibly be changed by m. Next for a given $x \in \Lambda$ we call $y \in \Lambda$ m-relevant if there exist $A, B \in \mathcal{P}(\Lambda)$ such that $x \in m(A) \triangle m(B)$ and $A \triangle B = \{y\}$, in words this means that the state of y, i.e. y being contained in the configuration or not, may affect which state x is in after the application of m. We define

$$\mathcal{R}_x(m) := \{y \in \Lambda : y \text{ is } m\text{-relevant w.r.t } x\}.$$

Definition 2.3.1 (local map). A map $m : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ is called a *local* map if the following three conditions are fulfilled.

1. $\mathcal{D}(m)$ is finite.
2. $\mathcal{R}_x(m)$ is finite for all $x \in \Lambda$.
3. For each $x \in \Lambda$, if $y \notin A \triangle B$ for all $y \in \mathcal{R}_x(m)$, then $x \notin m(A) \triangle m(B)$.

See [Swa17, Exercise 4.9] for a map which satisfies the first two properties, but not the last. This map is in fact discontinuous. Just before [Swa17, Exercise 4.9] it is mentioned that one can show that a map $m : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ is continuous if and only if the second and the third property are satisfied.

Let $\mathcal{M}$ be a countable set of local maps and let $(r_m)_{m\in\mathcal{M}}$ be non-negative constants, where r_m will be the rate corresponding to a map $m \in \mathcal{M}$. Let Ξ be a Poisson point process on $(\mathcal{M} \times \mathbb{R}, \mathcal{B}(\mathcal{M} \times \mathbb{R}))$ with intensity measure ξ such that

$$\xi(M \times [s,t)) = \sum_{m\in M} r_m(t-s),$$

where $M \subset \mathcal{M}$ is finite. Note that we fully characterized ξ since $\{M \times [s,t) : s < t, M \subset \mathcal{M} \text{ finite}\}$ is a π-system which generates $\mathcal{B}(\mathcal{M} \times \mathbb{R})$. We use a short hand notation and write $\mathsf{d}\xi = r_m \mathsf{d}t$ as the intensity measure of Ξ. By Proposition 2.2.5 the Poisson point process is simple and therefore by Proposition 2.2.3 there exist random variables $(m_1, t_1), (m_2, t_2), \dots$ with state space $\mathcal{M} \times \mathbb{R}$ such that $\Xi = \sum_{k=0}^{\infty} \delta_{(m_k,t_k)}$. Since Ξ is *supported* by these random variables and we denote

$$\omega := \operatorname{supp}(\Xi) := \{(m_k, t_k) : k \in \mathbb{N}\}.$$

Furthermore, we set $\omega_{s,u} := \omega \cap \mathcal{M} \times (s,u]$ with $s < u$. Now for every random set $\tilde{\omega}_n := \{(m_1,t_1), \dots, (m_n,t_n)\} \subset \omega_{s,u}$, where we assume that $t_1 < \cdots < t_n$ and $n \in \mathbb{N}$ we can define the map $\mathbf{X}_{s,u}^{\tilde{\omega}_n}(A) := m_n \circ \cdots \circ m_1(A)$ pointwise for $A \in \mathcal{P}(\Lambda)$. By [Swa17, Theorem 4.14] if the rates satisfy

$$\sup_{x\in\Lambda} \sum_{m\in\mathcal{M}, \mathcal{D}(m)\ni x} r_m(|\mathcal{R}_x(m)| + 1) < \infty, \tag{2.1}$$

then, for every $A \in \mathcal{P}(\Lambda)$ and $s \le u$, the pointwise limit $\mathbf{X}_{s,u}(A) := \lim_{\tilde{\omega}_n \uparrow \omega_{s,u}} \mathbf{X}_{s,u}^{\tilde{\omega}_n}(A)$ exists almost surely and does not depend on the choice of the finite sets $\tilde{\omega}_n \uparrow \omega_{s,u}$. Furthermore [Swa17, Theorem 4.14] states, that if X_0 is a $\mathcal{P}(\Lambda)$-valued random variable, independent of ω, then $\mathbf{X}_t := \mathbf{X}_{0,t}(X_0)$, where $t \geq 0$, defines a Feller process with generator

$$\mathcal{A}f(A) = \sum_{m \in \mathcal{M}} r_m(f(m(A)) - f(A)).$$

for all $f \in C(\mathcal{P}(\Lambda))$ and initial state $\mathbf{X}_0 = X_0$. Recall that this also means that $\mathbf{X}$ has almost surely paths in $D_{\mathcal{P}(V)}([0,\infty))$. In case that the initial configuration is deterministic, i.e. $\mathbf{X}_0 = A \in \mathcal{P}(\Lambda)$, we sometimes add it as a superscript $\mathbf{X}^A = (\mathbf{X}_t^A)_{t \geq 0}$.

Example 2.3.2 (The classical contact process)**.** As an example for an interacting particle system constructed via this graphical representation we consider the classical contact process. For $x, y \in V$ such that $\{x, y\} \in E$ one considers the maps

$$\mathbf{inf}_{x,y}(A) := \begin{cases} A \cup \{x\} & \text{if } y \in A \\ A & \text{otherwise,} \end{cases}$$
$$\mathbf{rec}_x(A) := A \backslash \{x\}$$

with rates $r_{\mathbf{inf}_{x,y}} = \lambda > 0$ and $r_{\mathbf{rec}_x} = r > 0$ such that by construction the process $\mathbf{X}$ with transitions

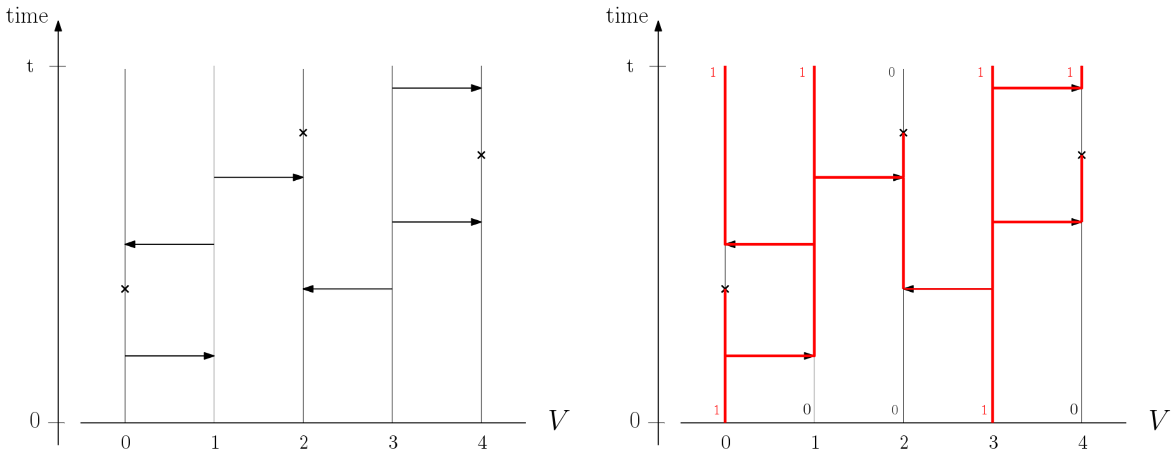
$$\mathbf{X}_{t-} = A \to A \cup \{x\} \quad \text{at rate } \lambda \cdot |\{y \in A : \{x,y\} \in E\}| \text{ and}$$
$$\mathbf{X}_{t-} = A \to A \backslash \{x\} \quad \text{at rate } r.$$

See Figure 2.1 for a visualization. The $\mathbf{inf}_{x,y}$ map refers to an infection event, which means that a potential infection is transmitted from individual x to its neighbour y. On the other hand the $\mathbf{rec}_x$ map refers to a recovery event, which means that individual x recovers from a potential infection and is healthy afterwards.

2.4 Basic notions of graphs

Let V be a countable set and $E \subset \{e = \{x, y\} : x, y \in V, x \neq y\}$. We call V the set of all vertices and E the set of all *unoriented* edges. We call the tuple $G = (V, E)$ a graph. Note that by assumption E contains no loops and if there exists an edge between x and y it is unique. In the literature such graphs are often called simple or strict.

Definition 2.4.1. Let $G = (V, E)$ be a graph. Let $x \in V$ and we denote the *neighbourhood* of x by $\mathcal{N}_x := \{y \in V : \{x, y\} \in E\}$ and $|\mathcal{N}_x|$ is the *degree* of x. If $\sup_{x \in V} |\mathcal{N}_x| < \infty$, we say G is of bounded degree.



(a) The arrows from x to y correspond to the $\mathbf{inf}_{x,y}$ map, which transmits an infection from x to y if x is infected. The crosses signify a map $\mathbf{rec}_x$, which causes x to recover.

(b) The red lines indicates an infection path, i.e. if a site is red at time t it is infected.

Figure 2.1: Visualization of a graphical representation of a classical contact process.

Definition 2.4.2. We call $x, y \in V$ *adjacent* if $\{x, y\} \in E$. We call $x, y \in V$ *connected*, if there exists a finite sequence $(v_i)_{0 \leq i \leq n} \subset V$ such that $x = v_0$, $y = v_n$ and $\{v_i, v_{i+1}\} \in E$ for all $0 \leq i \leq n-1$. If all $x, y \in V$ are connected, then we call the graph G *connected*.

With the notion of connectedness we can introduce the so called *graph distance* d as follows. Let $(v_i)_{0 \leq i \leq n}$ be a sequences with the smallest number of vertices needed to connect x to y, then set $d(x, y) = n$. We call the set $\mathbb{B}_k(x) := \{y \in V : d(x, y) \leq k\}$, the ball of radius k around $x \in V$. See Figure 2.2(a) for a visualization.

Definition 2.4.3. (Graph automorphism) Let $\sigma : V \to V$ be a permutation such that $\{x, y\} \in E$ if and only if $\{\sigma(x), \sigma(y)\} \in E$. We call such a σ a *graph automorphism* and $\mathrm{Aut}(G)$ the set of all graph automorphisms.

Remark 2.4.4. Let $G = (V, E)$ be a graph

1. Note that the set $\mathrm{Aut}(G)$ of all graph automorphism endowed with the concatenation $\circ$ as operation is a group.
2. Since by assumption $\{x, y\} \in E \Leftrightarrow \{\sigma(x), \sigma(y)\} \in E$ for any $\sigma \in \mathrm{Aut}(G)$, we slightly abuse notation and for and write $\sigma(e) = \{\sigma(x), \sigma(y)\}$ for a given $e = \{x, y\}$.

Now we will introduce the notion of transitivity, which basically describes that a graph looks locally the same everywhere.

Definition 2.4.5. (Transitivity) We call the graph $G = (V, E)$

1. *vertex transitive* if for every $x, y \in V$ a $\sigma \in \mathrm{Aut}(G)$ exists such that $\sigma(x) = y$.
2. *edge transitive* if for every $e_1, e_2 \in E$ a $\sigma \in \mathrm{Aut}(G)$ exists such that $\sigma(e_1) = e_2$.
3. *transitive* if the graph is vertex and edge transitive

Note that all vertices of a vertex transitive graph $G = (V, E)$ have the same degree, i.e. $|\mathcal{N}_x| = |\mathcal{N}_y|$ for all $x, y \in V$. Note that we can describe the *growth* of a connected and vertex transitive graph $G = (V, E)$ through the following notion:

Definition 2.4.6 ((Sub-)exponential growth)**.** Let G be a vertex transitive and connected graph. We say G has *exponential growth* ρ if

$$\lim_{n\to\infty} \frac{1}{n} \log(|\mathbb{B}_n(x)|) := \rho.$$

If $\rho = 0$ we call G of *subexponential growth.* The limit does not depend on the choice of $x \in V$.

Lemma 2.4.7. *Let G be a vertex transitive and connected graph with bounded degree. Then $\rho \leq \log(|\mathcal{N}_x| - 1)$, where $x \in V$ and hence in particular $\rho < \infty$.*

Proof. Let us fix some $x \in V$ and let $n \geq 1$. Now let $y \in \partial\mathbb{B}_n(x)$, i.e. $d(x, y) = n$. Also there must exist at least one $z \in \mathcal{N}_y$ such that $d(x, z) = n - 1$, otherwise y could not be connected to x which would be a contradiction. Now we see that

$$|\partial\mathbb{B}_{n+1}(x)| \leq \sum_{y\in\partial\mathbb{B}_n(x)} (|\mathcal{N}_y| - 1) = (|\mathcal{N}_x| - 1)|\partial\mathbb{B}_n(x)|,$$

where we used that $|\mathcal{N}_x| = |\mathcal{N}_y|$ for every $y \in V$. Recursive application implies that $|\partial\mathbb{B}_n(x)| \leq (|\mathcal{N}_x| - 1)^n$. Thus, we can conclude that

$$|\mathbb{B}_n(x)| = \sum_{k=1}^{n} |\partial\mathbb{B}_k(x)| + 1 \leq \sum_{k=0}^{n} (|\mathcal{N}_x| - 1)^n = \frac{(|\mathcal{N}_x| - 1)^{n+1} - 1}{|\mathcal{N}_x|}.$$

But with this inequality we see that

$$\frac{1}{n} \log(|\mathbb{B}_n(x)|) \leq \log(|\mathcal{N}_x| - 1) + \frac{1}{n} \log\left(|\mathcal{N}_x|^{-1}(|\mathcal{N}_x| - 1)\right).$$

Hence if we let $n \to \infty$ it follows that $\rho \leq \log(|\mathcal{N}_x| - 1)$. □

Remark 2.4.8. Note that $\lim_{n\to\infty} \frac{1}{n}\log(|\mathbb{B}_n(x)|) = \rho$ and $\lim_{n\to\infty} |\mathbb{B}_n(x)|e^{-\rho n} = 1$ are equivalent statements. Thus, the term exponential growth refers to the growth of the cardinality of a ball $\mathbb{B}_n(x)$ as n tends to infinity. It also makes sense to call the graph of subexponential growth if $\rho = 0$, since $\rho = 0$ implies that $\lim_{n\to\infty} |\mathbb{B}_n(x)|e^{-Cn} = 0$ for every $C > 0$. Also Lemma 2.4.7 implies that connected and vertex transitive graphs G with bounded degree can not have a *superexponential* growth. As a by-product we get a upper bound on the constant ρ.

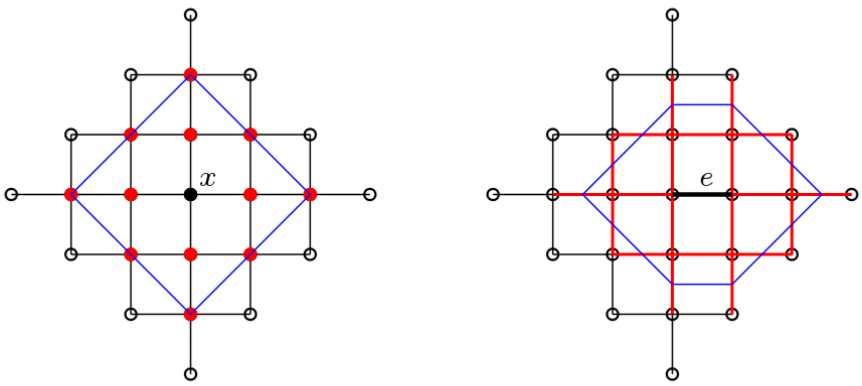
Next we introduce the line graph.

Definition 2.4.9 (Line graph)**.** Let $G = (V, E)$ be a graph, then we call $L(G)$ the *line* graph of G. The vertex set of the line graph is the edge set E and two elements in E are defined to be adjacent if they share a vertex, e.g. e_1 and e_2 are adjacent if $|e_1 \cap e_2| = 1$.

We will denote the neighbourhood and ball of radius n around an element e in the line graph by $\mathcal{N}_e^L$ and $\mathbb{B}_n^L(e)$. See Figure 2.2(b) for a visualization.

Remark 2.4.10. Note that we can express the neighbourhood of an element $\{x, y\}$ in the line graph as $\mathcal{N}_e^L = \{a \in E : |e \cap a| = 1\}$ and the balls as

$$\mathbb{B}_n^L(\{x, y\}) = \{\{z, z'\} \in E : z \in \mathbb{B}_n(x), z' \in \mathbb{B}_n(y)\}.$$



(a) Visualization of $\mathbb{B}_2(x)$ (b) Visualization of $\mathbb{B}_2^L(e)$

Figure 2.2: Here we illustrate the two different types of balls on the lattice $\mathbb{Z}^2$. The blue line indicates the "boundary" of the balls of distance two and the red vertices/edges are the elements contained in the balls.

Lemma 2.4.11. *Let G be a graph and $L(G)$ its corresponding line graph.*

(i) If G is connected, then $L(G)$ is connected.

(ii) If G has bounded degree, then also $L(G)$ has bounded degree.

(iii) If G is edge transitive, then $L(G)$ is vertex transitive.

(iv) Let G be connected and transitive. If G is of exponential growth ρ, then $L(G)$ is also of exponential growth ρ.

Proof. (i) This is clear by definition

(ii) For any $e \in E$ there exist $x, y \in V$ such that $e = \{x, y\}$. Now we can identify each edge contained $\mathcal{N}_{\{x,y\}}$ by the vertex which is not equal to x or y and see that

$$|\mathcal{N}_{\{x,y\}}| = |(\mathcal{N}_x \cup \mathcal{N}_y) \backslash \{x, y\}| \leq |\mathcal{N}_x| + |\mathcal{N}_y| < \infty$$

where we used that G is of bounded degree.

(iii) First we prove that $\sigma(\mathcal{N}_x) = \mathcal{N}_{\sigma(x)}$ for every $x \in V$. Let us assume that $\sigma(\mathcal{N}_x) \neq \mathcal{N}_{\sigma(x)}$ then either there exists a $y \in \mathcal{N}_x$ such that $\sigma(y) \notin \mathcal{N}_{\sigma(x)}$ and thus $\{\sigma(x), \sigma(y)\} \notin E$ which is a contradiction to $\{x, y\} \in E$, or there exists a $z \in \mathcal{N}_{\sigma(x)}$ with $\sigma^{-1}(z) \notin \mathcal{N}_x$, but $\{\sigma(x), z\} \in E$ implies $\{\sigma^{-1}(\sigma(x)), \sigma^{-1}(z)\} = \{x, \sigma^{-1}(z)\} \in E$ since σ^{-1} is again a graph automorphism, which is a contradiction. Now let $\sigma \in \mathrm{Aut}(G)$ and recall that for $e = \{x, y\}$, $\sigma(e) = \{\sigma(x), \sigma(y)\}$. Let $e_1, e_2 \in E$ if e_1 and e_2 are adjacent, i.e. they have a vertex x in common, then $\sigma(e_1)$ and $\sigma(e_2)$ are adjacent as well, since $\sigma(\mathcal{N}_x) = \mathcal{N}_{\sigma(x)}$. Thus, every $\sigma \in \mathrm{Aut}(G)$ induces a graph automorphism on $L(G)$. In the line graph E has the role of the set of all vertices. Now it is clear that if G is edge transitive, then $L(G)$ is vertex transitive.

(iv) Let $x, y \in V$ with $e = \{x, y\} \in E$, then analogously to (ii) we can again uniquely identify each edge $e' \neq e$ with a vertex $z \notin \{x, y\}$, which is contained in the union $\mathbb{B}_n(x) \cup \mathbb{B}_n(y)$. Note that since G is transitive and connected each vertex has the same number of neighbours, i.e. $|\mathbb{B}_n(x)| = |\mathbb{B}_n(y)|$ for all $x, y \in V$, and at least two. Thus, it follows that

$$|\mathbb{B}_n(x)| \leq |\mathbb{B}_n(x) \cup \mathbb{B}_n(y)| \leq |\mathbb{B}^L_n(e)| \leq 2|\mathbb{B}_n(x)|,$$

Now the claim follows by Remark 2.4.8.

□

Chapter 3

Graphical representation and consequences

3.1 Graphical representation of finite range spin systems

In this section we show that every finite range spin system can be constructed via the graphical representation discussed in Section 2.3. We explicitly state a set of maps $\mathcal{M}$ and corresponding rates $(r_m)_{m\in\mathcal{M}}$ such that the generator of the resulting Feller process agrees with the generator of a previously specified finite range spin system. One reason for this effort is that the techniques used in the next section heavily rely on this representation.

Recall that we assumed that the graph $G = (V, E)$ is transitive, connected and has bounded degree. Now by Lemma 2.4.11 we know that the line graph $L(G)$ is vertex transitive, connected and has bounded degree. Therefore, we consider in the current and next section a slightly more general setting. Let $\mathbf{X}$ be an attractive and translation invariant spin system of range R on some connected and vertex transitive graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with bounded degree. This is notationally more convenient, than using the line graph $L(G)$. Recall that the generator of a spin system is

$$\mathcal{A}_{Spin} f(A) = \sum_{x\in\mathcal{V}} q(x, A)(f(A \triangle \{x\}) - f(A)), \tag{3.1}$$

where $f \in C(\mathcal{P}(\mathcal{V}))$ and $A \subset \mathcal{V}$. The interpretation of a spin system is that at a site $x \in \mathcal{V}$ a spin flip takes places with a spin rate $q(x, A)$, which depends on the configuration A. Such a spin flip can be seen as the action of a map $m : A \mapsto A \triangle \{x\}$.

As already mentioned we intend to construct these systems by using the representation introduced in Section 2.3. Therefore, one issue is that we can not use the set of all spin flip maps m, since the rates of these maps would need to be $q(x,A)$ and thus depend on the configuration A. This is not in line with the setting in Section 2.3, where every rate r_m is constant with respect to the configuration A. Hence, we need to choose different maps.

Since we only consider finite range systems we know that there exists a range $R \in \mathbb{N}$ such that $q(x,A) = q(x, A \cap \mathbb{B}_R(x))$ for any $A \subset \mathcal{V}$. This means in particular that $q(x,\cdot)$ only depends on finitely many elements and thus we can work around this by just defining separate maps for every relevant configuration in $\mathbb{B}_R(x)$. We distinguish between an up or down flip, i.e. if $x \in A$ or $x \notin A$. Then we consider every possible configuration of the R-neighbourhood of x, which we denote by $\mathcal{N}_x(R) := \mathbb{B}_R(x) \backslash \{x\}$. This leads to the following maps and rates. For every $x \in \mathcal{V}$ and $F \subset \mathcal{N}_x(R)$ we set

$$\mathbf{up}_{x,F}(A) := \begin{cases} A \cup \{x\} & \text{if } x \notin A \text{ and } A \cap \mathcal{N}_x(R) = F \\ A & \text{otherwise,} \end{cases}$$

$$\mathbf{down}_{x,F}(A) := \begin{cases} A \backslash \{x\} & \text{if } x \in A \text{ and } A \cap \mathcal{N}_x(R) = F \\ A & \text{otherwise,} \end{cases}$$

for $A \subset \mathcal{V}$ and choose the rates to be

$$r_{\mathbf{up}_{x,F}} = q(x,F) \quad \text{and} \quad r_{\mathbf{down}_{x,F}} = q(x, F \cup \{x\}). \tag{3.2}$$

Note that $x \notin F$, since $F \subset \mathcal{N}_x(R)$. We denote the sets of the two types of maps by

$$\mathcal{M}_{\text{up}} = \{\mathbf{up}_{x,F} : x \in \mathcal{V}, F \subset \mathcal{N}_x(R)\} \text{ and } \mathcal{M}_{\text{down}} = \{\mathbf{down}_{x,F} : x \in \mathcal{V}, F \subset \mathcal{N}_x(R)\}$$

and define the set of all maps as $\mathcal{M} := \mathcal{M}_{\text{up}} \cup \mathcal{M}_{\text{down}}$. Now let Ξ^q be a Poisson point process on $\mathcal{M} \times \mathbb{R}$ with rates $(r_m)_{m \in \mathcal{M}}$. Obviously (2.1) is satisfied, and thus via the construction in Section 2.3, we get a Feller process $\mathbf{X}$ with the generator

$$\mathcal{A}f(A) = \sum_{m \in \mathcal{M}} r_m(f(m(A)) - f(A)).$$

Now it suffices to show that this generator is the same as the generator stated in (3.1), i.e. that $\mathcal{A}f(A) = \mathcal{A}_{\text{Spin}}f(A)$ for all $f \in C(\mathcal{P}(\mathcal{V}))$ and $A \in \mathcal{P}(\mathcal{V})$. By plugging in the maps and rates we get that

$$\begin{aligned}\mathcal{A}f(A) = \sum_{x\in\mathcal{V}} \sum_{F\subset\mathcal{N}_x} & q(x,F)(f(\mathbf{up}_{x,F}(A)) - f(A)) \\ & + q(x, F\cup\{x\})(f(\mathbf{down}_{x,F}(A)) - f(A)).\end{aligned}$$

By using that $q(x,F) = q(x, F\cup D)$ for any $D \subset \mathcal{V}$ with $D \cap \mathbb{B}_R(x) = \emptyset$, we see that

$$\begin{aligned}&\sum_{F\subset\mathcal{N}_x(R)} q(x,F)(f(\mathbf{up}_{x,F}(A)) - f(A)) \\ =&\mathbb{1}_{\{x\notin A\}} \sum_{F\subset\mathcal{N}_x(R)} \mathbb{1}_{\{A\cap\mathcal{N}_x(R)=F\}} q(x,A)(f(A\cup\{x\}) - f(A)) \\ =&\mathbb{1}_{\{x\notin A\}} q(x,A)(f(A\cup\{x\}) - f(A)).\end{aligned}$$

An analogous calculation to the one just performed for the maps $\mathbf{up}_{x,F}$ can also be formulated for the maps $\mathbf{down}_{x,F}$ and therefore,

$$\begin{aligned}\mathcal{A}f(A) =& \sum_{x\in\mathcal{V}} \mathbb{1}_{\{x\notin A\}} q(x,A)(f(A\cup\{x\}) - f(A)) + \mathbb{1}_{\{x\in A\}} q(x,A)(f(A\backslash\{x\}) - f(A)) \\ =& \sum_{x\in\mathcal{V}} q(x,A)(f(A \triangle \{x\}) - f(A)) = \mathcal{A}_{\text{Spin}}f(A).\end{aligned}$$

Thus, we constructed a spin system with spin rate $q(\cdot,\cdot)$.

The first consequence of this representation is that we are able to couple a general finite range spin system with a dynamical percolation (see Example 1.1.2 (i)). Set

$$\begin{aligned}\alpha_{\min} &:= \min_{F\subset\mathcal{N}_x(R)} r_{\mathbf{up}_{x,F}}, & \beta_{\min} &:= \min_{F\subset\mathcal{N}_x(R)} r_{\mathbf{down}_{x,F}} \\ \alpha_{\max} &:= \max_{F\subset\mathcal{N}_x(R)} r_{\mathbf{up}_{x,F}} \quad \text{and} & \beta_{\max} &:= \max_{F\subset\mathcal{N}_x(R)} r_{\mathbf{down}_{x,F}},\end{aligned} \tag{3.3}$$

as already seen in (1.4). Note that $\alpha_{\min}$, $\beta_{\min}$, $\alpha_{\max}$ and $\beta_{\max}$ do not depend on x since the spin system is translation invariant.

Proposition 3.1.1. *Let $\mathbf{X}$ be a spin system with spin rate $q(\cdot,\cdot)$. Furthermore let $\alpha_{\min}$, $\beta_{\min}$, $\alpha_{\max}$ and $\beta_{\max}$ be defined as in (3.3). There exists a dynamical percolation $\overline{\mathbf{Y}}$ with rates $\alpha_{\max}$ and $\beta_{\min}$ and $\underline{\mathbf{Y}}$ with rates $\alpha_{\min}$ and $\beta_{\max}$ such that if $\underline{\mathbf{Y}}_0 = \mathbf{X}_0 = \overline{\mathbf{Y}}_0$ then $\underline{\mathbf{Y}}_t \subset \mathbf{X}_t \subset \overline{\mathbf{Y}}_t$ almost surely for all $t > 0$.*

Proof. Let $\mathbf{X}$ be the spin system obtained via the graphical representation described above, which uses the maps $\mathbf{up}_{x,F}$ and $\mathbf{down}_{x,F}$ and the rates defined as in (3.3), i.e. $r_{\mathbf{up}_{x,F}} = q(x, F)$ and $r_{\mathbf{down}_{x,F}} = q(x, F \cup \{x\})$, where $x \in \mathcal{V}$ and $F \subset \mathcal{N}_x(R)$. By construction we see that $\mathbf{X}$ has the spin rate $q(\cdot, \cdot)$. Now we adjust the construction in the following way. We use the same maps but choose the rates to be $\underline{r}_{\mathbf{up}_{x,F}} = \alpha_{\min}$ and $\underline{r}_{\mathbf{down}_{x,F}} = \beta_{\max}$ for any $x \in \mathcal{V}$ and any $F \subset \mathcal{N}_x(R)$. This yields a spin system $\underline{\mathbf{Y}}$ with spin rate

$$\underline{q}(x, A) = \alpha_{\min} \mathbb{1}_{\{x \in A\}} + \beta_{\max} \mathbb{1}_{\{x \notin A\}},$$

where $x \in \mathcal{V}$ and $A \subset \mathcal{V}$. Thus, $\underline{\mathbf{Y}}$ is a dynamical percolation with rates $\alpha_{\min}$ and $\beta_{\max}$. Analogously by choosing the rates to be $\overline{r}_{\mathbf{up}_{x,F}} = \alpha_{\min}$ and $\overline{r}_{\mathbf{down}_{x,F}} = \beta_{\max}$ for any $x \in \mathcal{V}$ and any $F \subset \mathcal{N}_x(R)$ we obtain a dynamical percolation $\overline{\mathbf{Y}}$ with rates $\alpha_{\max}$ and $\beta_{\min}$. Let $\overline{q}(\cdot, \cdot)$ denote the spin rate of $\overline{\mathbf{Y}}$, then by definition of the rates in (3.3) it follows that

$$\begin{aligned} \underline{q}(x, A) \leq q(x, A) \leq \overline{q}(x, A) \quad &\text{if} \quad x \notin A \quad \text{and} \\ \underline{q}(x, A) \geq q(x, A) \geq \overline{q}(x, A) \quad &\text{if} \quad x \in A. \end{aligned}$$

for any $x \in \mathcal{V}$ and $A \subset \mathcal{P}(\mathcal{V})$. Now by [Lig12, Theorem III.1.5] it follows that there exist a coupling such that if $\underline{\mathbf{Y}}_0 = \mathbf{X}_0 = \overline{\mathbf{Y}}_0$ then $\underline{\mathbf{Y}}_t \subset \mathbf{X}_t \subset \overline{\mathbf{Y}}_t$ almost surely for all $t > 0$. □

3.2 Expansion speed of the permanently coupled region

Recall the definition of the coupled and permanently coupled region from (1.2) and (1.3). On a general graph $\mathcal{G}$ for the spin system $\mathbf{X}$ the coupled region at time t is

$$\Psi_t = \Psi_t(\mathbf{X}) = \{x \in \mathcal{V} : x \notin \mathbf{X}_t^{A_1} \triangle \mathbf{X}_t^{A_2} \ \forall A_1, A_2 \subset \mathcal{V}\}$$

and the permanently coupled region at time t is

$$\Psi'_t = \{x \in \mathcal{V} : x \in \Psi_s \forall s \geq t\},$$

for $t \geq 0$. The main goal of this section is to show Proposition 1.4.2. To be precise we show that if there exist constants $S, K' > 0$ and $\gamma > \rho$ such that

$$\mathbb{P}(x \notin \Psi_s) \leq K' e^{-\gamma s} \tag{3.4}$$

for every $x \in \mathcal{V}$ and $s \geq S$. Then there exist constants $T, K > 0$ and $\kappa > 0$ such that $\mathbb{P}(x \notin \Psi'_t) \leq K e^{-\kappa t}$ for all $t > T$ and $x \in \mathcal{V}$. In particular Proposition 1.4.2 follows for $\mathcal{G} = L(G)$. The strategy is to use the Borel-Cantelli Lemma and the fact that $t \mapsto \Psi'_t$ is non-decreasing. We see that

$$\begin{aligned} \mathbb{P}(\exists t \geq s : x \notin \Psi_t) \leq & \sum_{k=\lfloor s \rfloor}^{\infty} \mathbb{P}(\mathbb{B}_k(x) \not\subset \Psi_k) \\ & + \sum_{k=\lfloor s \rfloor}^{\infty} \mathbb{P}(\mathbb{B}_k(x) \subset \Psi_k, \exists t \in [k, k+1) \text{ s.t. } x \notin \Psi_t). \end{aligned} \tag{3.5}$$

The idea is that with (3.4) we are able to show that for discrete time points $k \geq \lfloor s \rfloor$ with a high probability $\mathbb{B}_k(x)$ is already contained in the coupled region Ψ_k if s is large enough. Then we show that on the event that $\mathbb{B}_k(x) \subset \Psi_k$ it is unlikely that the site x is affected by some $y \in \mathbb{B}^c_k(x)$ within one unit of time, which is necessary for a $t \in [k, k+1)$ to exist such that $x \notin \Psi_t$.

We briefly explain why this is the case. We know that $\mathcal{R}_x(m) \subset \mathbb{B}_R(x)$ for all $x \in \mathcal{D}(m)$ and only finitely many $m \in \mathcal{M}$ exist with $x \in \mathcal{D}(m)$. Thus, we define the set

$$\mathcal{R}_x := \bigcup_{m: x \in \mathcal{D}(m)} \mathcal{R}_x(m) \subset \mathbb{B}_R(x)$$

and see that this set is finite. In line with the notion of m-relevance we call $\mathcal{R}_x$ the set of all relevant elements with respect to x, i.e. if $y \in \mathcal{R}_x$ there exists an m with $x \in \mathcal{D}(m)$ such that y is m-relevant with respect to x. Now if $\mathcal{R}_x \subset \Psi_{t-}$ it is impossible that $x \notin \Psi_t$ since all relevant elements with respect to x are contained in the coupled region. Therefore, x can only "decouple" in the time interval $[k, k+1)$ if it is affected by some $y \notin \Psi_k$.

To formalize this, we use so-called paths of potential influence. Recall some notation from Section 2.3, which are $\omega = \text{supp}(\Xi^q)$ and $\omega_{s,u} = \omega \cap \mathcal{M} \times (s, u]$, where Ξ^q is the Poisson point process used in the graphical representation of $\mathbf{X}$. We took the following definition from [Swa17].

Definition 3.2.1 (Path of potential influence)**.** Let $x, y \in \mathcal{V}$ and $s < u$. A path of *potential influence* from (x, s) to (y, u) is a cadlag function $\gamma : [s, u] \to \mathcal{V}$ such that $\gamma(s) = \gamma(s-) = x$ and $\gamma(u) = y$, and

1. if $\gamma(t-) \neq \gamma(t)$ for some $t \in (s, u]$, then there exists some $m \in \mathcal{M}$ such that $(m, t) \in \omega$, $\gamma(t) \in \mathcal{D}(m)$ and $\gamma(t-) \in \mathcal{R}_{\gamma(t)}(m)$,

2. for each $(m, t) \in \omega$ with $t \in (s, u]$ and $\gamma(t) \in \mathcal{D}(m)$, one has $\gamma(t-) \in \mathcal{R}_{\gamma(t)}(m)$.

We write $(x, s) \rightsquigarrow (y, u)$ if there exists a path of potential influence between (x, s) and (y, u).

The first property ensures that every jump of a path of potential influence γ corresponds to a point $(t, m) \in \omega$, such that m can actually affect the site $\gamma(t)$. The second property guarantees that for any point $(m, t) \in \omega_{s,u}$, which could have caused the position $\gamma(t)$, i.e. $\gamma(t) \in \mathcal{D}(m)$, the "previous" site $\gamma(t-)$ must have been m-relevant with respect to the current state, i.e. $\gamma(t-) \in \mathcal{R}_{\gamma(t)}(m)$. This implies in particular that maps m with $\mathcal{R}_{\gamma(t)}(m) = \emptyset$ cannot play a role for such a path γ, an example for such a map is the map $\mathbf{rec}_x$ since obviously $\mathcal{R}_x(\mathbf{rec}_x) = \emptyset$.

Now let us repeat and reformulate what we described before the definition. Let $x \in \Psi_k$, then if there exists an $t \in [k, k+1)$ such that $x \notin \Psi_t$, then there exists a $y \notin \Psi_k$ such that $(y, k) \rightsquigarrow (x, t)$.

Before we continue, we first need to derive a bound on the probability of the sum of n exponentially distributed random variables with parameter λ. This sum is gamma distributed with paramter n and λ, which we will denote by $\Gamma(n, \lambda)$. Now we show the following result:

Lemma 3.2.2. *Let $T_n \sim \Gamma(n, \lambda)$ with $\lambda > 0$ and $n \in \mathbb{N}$ and let θ be a constant such that $0 < \theta < \frac{1}{\lambda}$ and $\theta\lambda - \log(\theta\lambda) - 1 > 0$. Then*

$$\mathbb{P}(T_n < \theta n) \leq \exp\big(-n\big(\theta\lambda - \log(\theta\lambda) - 1\big)\big).$$

Proof. Let $c > 0$, then the generalized Markov inequality yields

$$e^{-c\theta n}\mathbb{P}(T_n < \theta n) = e^{-c\theta n}\mathbb{P}(e^{-c\theta n} < e^{-cT_n}) \leq \mathbb{E}[e^{-cT_n}] = \lambda^n(\lambda + c)^{-n}.$$

Rearranging and renaming yields $\mathbb{P}(T_n < \theta n) \leq e^{nf_\theta(c)}$, where

$$f_\theta(c) = c\theta + \log(\lambda) - \log(\lambda + c).$$

For a fixed θ, the function f_θ has its minimum at $c_\theta = \frac{1}{\theta} - \lambda$, which has the function value $f_\theta(c_\theta) = 1 - \theta\lambda + \log(\theta\lambda)$. Note that it is necessary that $\theta \in (0, \frac{1}{\lambda})$ since otherwise $c_\theta \leq 0$. This proves the claim. □

Now we define $C_{\max} := \sup_{x \in \mathcal{V}} \sum_{m \in \mathcal{M}, \mathcal{D}(m) \ni x} r_m |\mathcal{R}_x(m)|$. The constant $C_{\max}$ is an upper bound on the rate at which a map m is drawn, which could affect the state of an arbitrary $x \in \mathcal{V}$ with $|\mathcal{R}_x(m)| \neq \emptyset$. Note that $0 < C_{\max} < \infty$ by (2.1). Now we are able to derive the necessary bound. Recall that R denotes the range of the spin system.

Lemma 3.2.3. *Let $L \geq 0$ and $s \geq 0$. Then there exists $K' > 0$, $\kappa' > L$ and $L' > C_{\max}$ such that for all $x, y \in \mathcal{V}$ with $d(x,y) > L'$,*

$$\mathbb{P}(\exists u \in [s, s+1) : (x,s) \rightsquigarrow (y,u)) \leq K' e^{-\kappa' \lceil R^{-1} d(x,y) \rceil}.$$

Proof. Let us assume $(x,s) \rightsquigarrow (y,u)$. Thus, there must exist a path of potential influence γ from (x,s) to (y,u). The first thing we observe is that $\mathcal{R}_z(m) \subset \mathbb{B}_R(z)$ for all $z \in \mathcal{V}$ and all $m \in \mathcal{M}$, and therefore we conclude that the path γ must at least jump $\lceil R^{-1} d(x,y) \rceil$ times. Hence, for every path γ there must exist a sequence $\{(m_1, s_1), \ldots, (m_n, s_n)\} \subset \omega_{s,u}$ with $s := s_0 < s_1 < \cdots < s_n \leq u$ and $n \geq \lceil R^{-1} d(x,y) \rceil$ such that the s_k correspond to the jump times of γ, $\gamma(s_k) \in \mathcal{D}(m_k)$ and $\gamma(s_k -) \in \mathcal{R}_{\gamma(s_k)}(m_k)$ for all $k \leq n$. Therefore, for every γ there exists a sequence $(x_k)_{0 \leq k \leq n} \subset \mathcal{V}$ such that $\gamma(t) = x_k$ for $t \in [s_{k-1}, s_k)$ for all k and $x_0 = x$ and $x_n = y$. Note that $1 \leq d(x_k, x_{k-1}) \leq R$. For a given sequence $(x_k)_{0 \leq k \leq n} \subset \mathcal{V}$ we can define the times

$$T_k := \inf\{t > T_{k-1} : (m,t) \in \Xi \text{ with } x_k \in \mathcal{D}(m) \text{ and } |\mathcal{R}_{x_k}(m)| \neq \emptyset\},$$

where $T_0 := 0$. Now define $\gamma_{\max} : (s,u] \to \mathcal{V}$ such that $\gamma_{\max}(t) = x_k$ for all $t \in [T_{k-1}, T_k)$. By definition it is clear that out of all paths which pass through the points $(x_k)_{0 \leq k \leq n}$ the path $\gamma_{\max}$ is the first to reach y, i.e. $T_n \leq s_n$ for any path of potential influence γ which passes through $(x_k)_{0 \leq k \leq n}$. Note that by translation invariance the distribution of $T_n \sim \Gamma(n, C_{\max})$ is in particular independent of the exact sequence $(x_k)_{0 \leq k \leq n}$. Furthermore the number of all possible sequences $(x_k)_{0 \leq k \leq n}$ which connect x to y and satisfy $1 \leq d(x_k, x_{k-1}) \leq R$ for all $k \leq n$ is bounded by the number M^n, where $M := |\mathcal{N}_x(R)|$. This implies that

$$\mathbb{P}(\exists u \in [s, s+1) : (x,s) \rightsquigarrow (y,u) \text{ with } n \text{ jumps}) \leq M^n \mathbb{P}(T_n < 1).$$

Now we observe that

$$\begin{aligned} &\{\exists u \in [s, s+1) : (x,s) \rightsquigarrow (y,u)\} \\ = &\bigcup_{n = \lceil R^{-1} d(x,y) \rceil}^{\infty} \Big\{\exists u \in [s, s+1) : (x,s) \rightsquigarrow (y,u) \text{ with } n \text{ jumps}\Big\}, \end{aligned}$$

and therefore via σ-additivity of $\mathbb{P}$

$$\mathbb{P}(\exists u \in [s, s+1) : (x,s) \rightsquigarrow (y,u)) \leq \sum_{n=\lceil R^{-1}d(x,y)\rceil}^{\infty} M^n \mathbb{P}(T_n < n^{-1}n).$$

Note that $\theta \mapsto \theta C_{\max} - \log(\theta C_{\max} M) - 1$ is a continuous function and converges to ∞ as $\theta \to 0$. Thus, there exists $\theta < (RC_{\max} + 1)^{-1}$ such that

$$\kappa' := \theta C_{\max} - \log(\theta C_{\max} M) - 1 > L. \tag{3.6}$$

Now we set $L' := \theta^{-1}$. Note that $d(x,y) > L'$ implies that $\lceil R^{-1}d(x,y)\rceil > L' > C_{\max}$, where $x, y \in \mathcal{V}$. Since $M \geq 1$, (3.6) implies in particular that

$$\theta C_{\max} - \log(\theta C_{\max}) - 1 > 0.$$

Thus, by Lemma 3.2.2 and the fact that we consider n with $n^{-1} \leq \lceil R^{-1}d(x,y)\rceil^{-1} < \theta$ such that $\mathbb{P}(T_n < n^{-1}n) \leq \mathbb{P}(T_n < \theta n)$ we get that

$$\begin{aligned}\mathbb{P}(\exists u \in [s, s+1) : (x,s) \rightsquigarrow (y,u)) &\leq \sum_{n=\lceil R^{-1}d(x,y)\rceil} M^n \exp\big(-n(\theta C_{\max} - \log(\theta C_{\max}) - 1)\big) \\ &\leq \frac{\exp\big(-\lceil R^{-1}d(x,y)\rceil(\theta C_{\max} - \log(\theta C_{\max} M) - 1)\big)}{1 - \exp(1 - \theta C_{\max} + \log(\theta C_{\max} M))}.\end{aligned}$$

Now we set $K' := \big(1 - \exp(1 - \theta C_{\max} + \log(\theta C_{\max} M))\big)^{-1}$ and by (3.6) we know that $K' > 0$. Therefore, we conclude that

$$\mathbb{P}(\exists t \in [k, k+1) : (x,k) \rightsquigarrow (y,t)) \leq K' e^{-\kappa' \lceil R^{-1}d(x,y)\rceil}. \qquad \square$$

Now we can finally prove Proposition 1.4.2. Note that we show this result on arbitrary connected, vertex transitive graphs $\mathcal{G}$ with bounded degree. In Section 1.4 these results are formulated on the line graph $L(G)$ which is only a special case by setting $\mathcal{G} = L(G)$.

Proof of Proposition 1.4.2. Recall from (3.4) that we assume that there exist constants $S, K' > 0$ and $\gamma > \rho$ such that $\mathbb{P}(x \notin \Psi_s) \leq K' e^{-\gamma s}$ for every $x \in \mathcal{V}$ and $s \geq S$. Furthermore, in (3.5) we saw that

$$\mathbb{P}(x \notin \Psi'_s) \leq \sum_{k=\lfloor s\rfloor}^{\infty} \mathbb{P}(\mathbb{B}_k(x) \not\subset \Psi_k) + \sum_{k=\lfloor s\rfloor}^{\infty} \mathbb{P}(\mathbb{B}_k(x) \subset \Psi_k, \exists t \in [k, k+1) \text{ s.t. } x \notin \Psi_t).$$

We begin with considering the first sum. With (3.4) we can conclude for all $D \subset \mathcal{V}$ that

$$\mathbb{P}(D \not\subset \Psi_t) = \mathbb{P}(\exists y \in D : y \notin \Psi_t) \leq \sum_{y \in D} \mathbb{P}(y \notin \Psi_t) \leq |D| K' e^{-\gamma t}.$$

Thus, by setting $t = k$ and $D = \mathbb{B}_k(x)$, we get $\mathbb{P}(\mathbb{B}_k(x) \not\subset \Psi_k) \leq |\mathbb{B}_k(x)| K' e^{-\gamma k}$. We know that $|\mathbb{B}_k(x)| e^{-\rho k} \to 1$, since $\mathcal{G}$ is of exponential growth ρ. We also assumed that $\gamma - \rho > 0$. Hence, there exists a $0 < \kappa_1 < \gamma - \rho$, such that for s large enough $|\mathbb{B}_k(x)| e^{(\gamma - \kappa_1)k} \leq 1$ for all $k \geq \lfloor s \rfloor$, and thus

$$\sum_{k=\lfloor s \rfloor}^{\infty} |\mathbb{B}_k(x)| K' e^{-\gamma k} = K' \sum_{k=\lfloor s \rfloor}^{\infty} \underbrace{|\mathbb{B}_k(x)| e^{-(\gamma - \kappa_1)k}}_{\leq 1} e^{-\kappa_1 k} \leq K' \sum_{k=\lfloor s \rfloor}^{\infty} e^{-\kappa_1 k} = K_1 e^{-\kappa_1 \lfloor s \rfloor},$$

where $K_1 = K'(1 - e^{-\kappa_1})^{-1}$. Therefore, the sum convergences to 0 as $s \to \infty$ and in particular we also get an exponential bound. Now it suffices to find a similar bound for the second sum. Recall that $\partial \mathbb{B}_k(x) = \mathbb{B}_k(x) \backslash \mathbb{B}_{k-1}(x)$. We see that

$$\begin{aligned} &\sum_{k=\lfloor s \rfloor}^{\infty} \mathbb{P}(\mathbb{B}_k(x) \subset \Psi_k, \exists t \in [k, k+1) \text{ s.t. } x \notin \Psi_t) \\ &\leq \sum_{k=\lfloor s \rfloor}^{\infty} \sum_{m=k}^{\infty} \mathbb{P}(\exists y \in \partial \mathbb{B}_{m+1}(x) \text{ and } \exists t \in [k, k+1) \text{ s.t. } (y, k) \rightsquigarrow (x, t)) \\ &\leq \sum_{k=\lfloor s \rfloor}^{\infty} \sum_{m=k}^{\infty} \sum_{y \in \partial \mathbb{B}_{m+1}(x)} \mathbb{P}(\exists t \in [k, k+1) \text{ s.t. } (y, k) \rightsquigarrow (x, t)). \end{aligned}$$

Note that $d(x, y) > \lfloor s \rfloor$. Hence, by choosing s large enough such that the conditions of Lemma 3.2.3 are satisfied, we can conclude that there exists $K^* > 0$ and $\kappa^* > \rho(R+1)$ such that

$$\sum_{k=\lfloor s \rfloor}^{\infty} \mathbb{P}(\mathbb{B}_k(x) \subset \Psi_k, \exists t \in [k, k+1) \text{ s.t. } x \notin \Psi_t) \leq \sum_{k=\lfloor s \rfloor}^{\infty} \sum_{m=k}^{\infty} |\partial \mathbb{B}_{m+1}(x)| K^* e^{-\kappa^* \lceil R^{-1} m \rceil}.$$

By using again that the graph is of exponential growth ρ and a comparison with the geometric sum we get that there exists $K_2, \kappa_2 > 0$ such that

$$\sum_{k=\lfloor s \rfloor}^{\infty} \mathbb{P}(\mathbb{B}_k(x) \subset \Psi_k, \exists t \in [k, k+1) \text{ s.t. } x \notin \Psi_t) \leq K_2 e^{-\kappa_2 \lfloor s \rfloor}.$$

Thus, we can conclude that $\mathbb{P}(x \in \Psi'_s) \leq K_1 e^{-\kappa_1 \lfloor s \rfloor} + K_2 e^{-\kappa_2 \lfloor s \rfloor}$, which proves the claim that there exists $\kappa, K > 0$ such that $\mathbb{P}(x \in \Psi'_s) \leq K e^{-\kappa \lfloor s \rfloor}$. □

Remark 3.2.4. Taking a close look at the proof of Proposition 1.4.2 we see that κ is chosen such that $\gamma - \kappa > \rho$, i.e. the exponent is smaller by a value of ρ. Here we want to emphasize that depending on the concrete spin system, this might not be the best possible choice. For example in case of the dynamical percolation (see Example 1.1.2 (i)) we know that $\Psi'_t = \Psi_t$ for all $t \geq 0$, and therefore one can easily calculate that for all $t \geq 0$,

$$\mathbb{P}(x \in \Psi_t) = \mathbb{P}(x \in \Psi'_t) \leq e^{-(\alpha+\beta)t}.$$

Next we prove Corollary 1.4.3. Note that we again prove this on $\mathcal{G}$, which is the more general case and the statement follows by considering $\mathcal{G} = L(G)$. Let us briefly recall the statement of Corollary 1.4.3 on $\mathcal{G}$. We consider

$$M = \sum_{y \in \mathcal{N}_x(R)} \sup_{A \subset \mathcal{V}} |q(x, A) - q(x, B \triangle \{y\})| \quad \text{and} \quad \varepsilon = \inf_{A \subset \mathcal{V}} |q(x, A) + q(x, A \triangle \{x\})|,$$

and show that if $\varepsilon - M > \rho$, then it follows that the process $\mathbf{X}$ is ergodic and there exists a $T > 0$ such that, there exist $\kappa, K > 0$ with $\mathbb{P}(x \notin \Psi'_t) \leq K e^{-\kappa t}$ for all $t > T$ and $x \in \mathcal{V}$.

Proof of Corollary 1.4.3. Since we assumed that $\varepsilon - M > \rho$ by [Lig12, Theorem I.4.1] it follows that the process $\mathbf{X}$ is ergodic, i.e. there exists an unique invariant measure π, and there exists a $K > 0$ such that

$$\sup_{A \subset \mathcal{V}} |\mathbb{P}(\mathbf{X}_t^A \in D) - \pi(D)| \leq K \frac{e^{-(\varepsilon - M)t}}{\varepsilon - M}. \tag{3.7}$$

for any $D \subset \mathcal{V}$. Since $\mathbf{X}$ is a monotone Feller process, by Theorem 2.1.12 we find a version $\widetilde{\mathbf{X}}$ such that monotonicity holds almost surely, i.e. that $\widetilde{\mathbf{X}}^{A_1} \subset \widetilde{\mathbf{X}}^{A_2}$ if $A_1 \subset A_2$ almost surely, where the superscript indicates the initial condition. Therefore, the coupled region simplifies to $\Psi_t = \widetilde{\mathbf{X}}_t^{\mathcal{V}} \triangle \widetilde{\mathbf{X}}_t^{\emptyset}$, and thus by using monotonicity we get that

$$\mathbb{P}(x \in \Psi_t) = \mathbb{P}(x \in \widetilde{\mathbf{X}}_t^{\mathcal{V}}, x \notin \widetilde{\mathbf{X}}_t^{\emptyset}) = \mathbb{P}(x \in \widetilde{\mathbf{X}}_t^{\mathcal{V}}) - \mathbb{P}(x \in \widetilde{\mathbf{X}}_t^{\emptyset}) \leq 2K \frac{e^{-(\varepsilon - M)t}}{\varepsilon - M},$$

where we have used $\{x \in \widetilde{\mathbf{X}}_t^{\emptyset}\} \subset \{x \in \widetilde{\mathbf{X}}_t^{\mathcal{V}}\}$ and the triangle inequlity as well as (3.7). Finally an application of Proposition 1.4.2 proves the claim. □

We end this section with a useful lemma, which we will need in Chapter 5.

Lemma 3.2.5. *Let* $\mathbf{X}$ *be a spin system with spin rate* $q(\cdot,\cdot)$, $A \subset \mathcal{V}$ *and* $x \in A$. *Furthermore, let* $u > 0$ *and* $n \in \mathbb{N}$, *then for every* $\varepsilon > 0$ *there exists a* $k > n$ *such that for all sets* $D \subset \mathcal{V}$ *with* $A \cap \mathbb{B}_k(x) = D \cap \mathbb{B}_k(x)$,

$$\mathbb{P}(\mathbf{X}_t^A \cap \mathbb{B}_n(x) = \mathbf{X}_t^D \cap \mathbb{B}_n(x) \ \forall t < u) > 1 - \varepsilon.$$

Proof. Without loss of generality we can assume that $u = 1$. Otherwise we rescale time in an appropriate manner and consider $(\mathbf{X}_{ut})_{t\geq 0}$ instead of $(\mathbf{X}_t)_{t\geq 0}$. Similarly to the proof of Proposition 1.4.2 we see that, if there exists $t \in [0,1)$ such that $\mathbf{X}_t^A \neq \mathbf{X}_t^D$ on $\mathbb{B}_n(x)$, then there must exist $y \in E\backslash\mathbb{B}_k(x)$, $z \in \mathbb{B}_n(x)$ and $t \in [0,1)$ such that $(y,0) \rightsquigarrow (z,t)$. Therefore,

$$\begin{aligned}
&\mathbb{P}\big(\exists t < 1 : \mathbf{X}_t^A \cap \mathbb{B}_n(x) \neq \mathbf{X}_t^D \cap \mathbb{B}_n(x)\big) \\
&\leq \sum_{z\in\mathbb{B}_n(x)} \sum_{m=k}^{\infty} \mathbb{P}\big(\exists y \in \partial\mathbb{B}_{m+1}(x) \text{ and } \exists t \in [0,1) \text{ s.t. } (y,0) \rightsquigarrow (z,t)\big) \\
&\leq \sum_{z\in\mathbb{B}_n(x)} \sum_{m=k}^{\infty} \sum_{y\in\partial\mathbb{B}_{m+1}(x)} \mathbb{P}\big(\exists t \in [0,1) \text{ s.t. } (y,0) \rightsquigarrow (z,t)\big).
\end{aligned}$$

Note that $d(z,y) > k - n$. Now choose k large enough such that the assumptions of Lemma 3.2.3 are satisfied and thus, there exists a $K' > 0$ and $\kappa' > \rho(R+1)$ such that

$$\mathbb{P}\big(\exists t \in [0,1) \text{ s.t. } (y,k) \rightsquigarrow (z,t)\big) \leq K' e^{-\kappa' \lceil R^{-1} d(z,y) \rceil},$$

where ρ was the exponential growth of $\mathcal{G}$. We get that

$$\mathbb{P}\big(\exists t < 1 : \mathbf{X}_t^A \cap \mathbb{B}_n(x) \neq \mathbf{X}_t^D \cap \mathbb{B}_n(x)\big) \leq |\mathbb{B}_n(x)| K' \sum_{m=k}^{\infty} |\partial\mathbb{B}_{m+1}(x)| e^{-\kappa' \lceil R^{-1}(m-n) \rceil},$$

since $d(z,y) > m - n$ for $y \in \partial\mathbb{B}_{m+1}(x)$ and $z \in \mathbb{B}_n(x)$. Note that

$$(R+1)\lceil R^{-1}(m-n) \rceil \geq m - n \quad \text{ and } \quad \sup_{m\geq 0} |\partial\mathbb{B}_{m+1}(x)| e^{-\rho m} < \infty$$

where we used for the second term that $\mathcal{G}$ is of exponential growth ρ. Hence, the sum on the right hand side convergences, since $\kappa' > \rho(R+1)$. This implies in particular that the right hand side tends to 0 as $k \to \infty$. Hence, for every $\varepsilon > 0$ there exists $k > n$ large enough such that the right hand side is smaller than ε, which provides the claim. □

3.3 Construction of the CPERE via graphical representation

In this section we explicitly construct the CPERE via the graphical representation introduced in Section 2.3 on a connected and transitive graph $G = (V, E)$ with bounded degree. This provides of course existence of the Feller process $(\mathbf{C}, \mathbf{B})$ and the graphical representation is an important tool in a lot of proofs in the subsequent chapters.

We assume that the maps and rates used to construct the (autonomous) background $\mathbf{B}$ via the graphical representation are known, i.e. $\mathcal{M}_{\text{Back}}$ is a countable set which contains local maps $m : \mathcal{P}(E) \to \mathcal{P}(E)$ with corresponding rates $(r_m)_{m \in \mathcal{M}_{\text{Back}}}$ such that the bound on the rates given in (2.1) is satisfied. For examples see Section 3.1 on the graph $L(G)$. Then $\mathbf{B}$ is a Feller process with generator

$$\mathcal{A}_{\text{Back}} f(B) = \sum_{m \in \mathcal{M}_{\text{Back}}} r_m(f(m(B)) - f(B)).$$

For the construction we are about to formulate we will use $\mathcal{P}(V \cup E)$ as a state space of the process. The reason for that is that with this choice we fit into the setting of Section 2.3. This is of course no issue, since $\mathcal{P}(V \cup E)$ and $\mathcal{P}(V) \times \mathcal{P}(E)$ can be easily identified with each other since for every set $A \subset V \cup E$ there exists a $C \subset V$ and $B \subset E$ such that $A = C \cup B$, and thus A corresponds to (C, B) and vice versa. Therefore, we first extend the maps $m \in \mathcal{M}_{\text{Back}}$ to maps $m^* : \mathcal{P}(V \cup E) \to \mathcal{P}(V \cup E)$. As we already mentioned for every set $A \subset V \cup E$ exist $C \subset V$ and $B \subset E$ such that $A = C \cup B$. Then we set $m^*(A) := C \cup m(B)$ for every $m \in \mathcal{M}_{\text{Back}}$. Let $\mathcal{M}^*_{\text{Back}}$ denote the set of all maps m^* and we use the same rates as before, i.e. $r_{m^*} = r_m$. Next for $A \subset V \cup E$ and $x, y \in V$ such that $\{x, y\} \in E$ we define

$$\mathbf{coop}_{x,y}(A) := \begin{cases} A \cup \{y\} & x \in A \text{ and } \{x, y\} \in A \\ A & \text{otherwise,} \end{cases}$$

$$\mathbf{rec}_x(A) := A \backslash \{x\}.$$

and set the rates to be $r_{\mathbf{coop}_{x,y}} = \lambda$ and $r_{\mathbf{rec}_x} = r$. The map $\mathbf{coop}_{x,y}$ is called the cooperative infection map. The name comes from the fact that for x to successfully infect y it needs the edge $\{x, y\}$ to be open. In this sense x and $\{x, y\}$ must cooperate such that the infection spreads to y.

Now define the set of all maps relevant for the infection process to be

$$\mathcal{M}_{\mathrm{CP}} := \underbrace{\{\mathbf{coop}_{x,y} : x, y \in V \text{ s.t. } \{x,y\} \in E\}}_{=\mathcal{M}_{\mathrm{inf}}} \cup \underbrace{\{\mathbf{rec}_x : x \in V\}}_{=\mathcal{M}_{\mathrm{rec}}}.$$

Let us denote by $\Xi = \Xi_{\lambda,r}$ the Poisson point process with respect to $\mathcal{M} := \mathcal{M}_{\mathrm{CP}} \cup \mathcal{M}^*_{\mathrm{Back}}$ and the corresponding rates $(r_m)_{m\in\mathcal{M}}$. Obviously (2.1) is satisfied, and thus there exists a Feller process $\mathbf{X}$ on $\mathcal{P}(V \cup E)$ with generator

$$\begin{aligned}\mathcal{A}f(A) &= \sum_{m\in\mathcal{M}} r_m(f(m(A)) - f(A)) \\ &= \sum_{x\in V} \lambda \sum_{y\in V:\{x,y\},x\in A} (f(A \cup \{y\}) - f(A)) + \sum_{x\in V} r(f(A\backslash\{x\}) - f(A)) \\ &+ \sum_{m\in\mathcal{M}_{\mathrm{Back}}} r_m(f((A\backslash E) \cup m(A\backslash V)) - f(A)),\end{aligned}$$

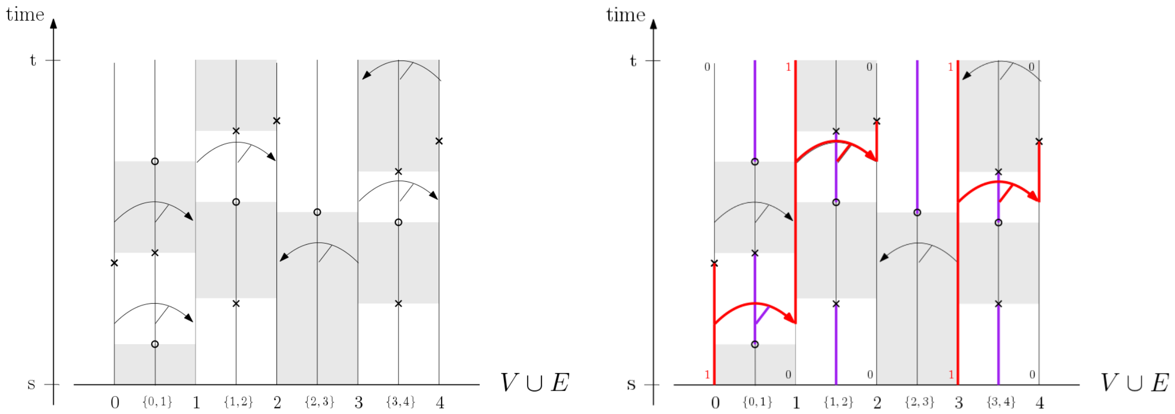
where $A \subset V \cup E$. The process $\mathbf{X}$ is a combination of infection process and the background in one. But, it is far more convenient to treat these two parts as separate object. Therefore, we switch back to the state space $\mathcal{P}(V) \times \mathcal{P}(E)$, which we achieve by setting $\mathbf{C}_t := \mathbf{X}_t\backslash E$ and $\mathbf{B}_t := \mathbf{X}_t\backslash V$ for all $t \geq 0$. With this we obtained the CPERE as described in Section 1.1, i.e. $(\mathbf{C}, \mathbf{B})$ is a Feller process on the state space $\mathcal{P}(V) \times \mathcal{P}(E)$ and $\mathbf{C}$ has jump rates (1.1).

We visualized this construction in Figure 3.1 for the contact process on a dynamical percolation, i.e. $\mathbf{B}$ is a dynamical percolation (see Example 1.1.2 (i)). In this case $\mathbf{B}$ can be constructed via the maps

$$\mathbf{birth}_e(B) := B \cup \{e\} \qquad \mathbf{death}_e(B) := B\backslash\{e\}$$

for $B \subset E$ and rates $r_{\mathbf{birth}_e} = \alpha$ and $r_{\mathbf{death}_e} = \beta$ for all $e \in E$.

Remark 3.3.1. The Poisson point process Ξ used in the graphical representation can be represented as the sum of three independent Poisson point process. These are Ξ^{inf} on $\mathcal{M}_{\mathrm{inf}} \times \mathbb{R}$, which are in the graphical representation (see Figure 3.1), the infection arrows, Ξ^{rec} on $\mathcal{M}_{\mathrm{rec}} \times \mathbb{R}$ corresponding to the recovery symbols and Ξ^{Back} on $\mathcal{M}^*_{\mathrm{Back}} \times \mathbb{R}$ which are the maps used to construct the background process. The sum of these three processes is again Ξ, i.e. $\Xi = \Xi^{\mathrm{inf}} + \Xi^{\mathrm{rec}} + \Xi^{\mathrm{Back}}$. It is useful to distinguish the three parts since we will often use couplings based on one or more of these three point process, while the remaining maps stay the same.



(a) The two tailed arrows correspond to the **coop** maps and the crosses on the sites refer to a **rec** maps. These are the infection and recovery events. The circles and crosses on the edges correspond to **birth** or **death** maps, which respectively cause an edge to open or to close. The grey area indicates that the edge is closed.

(b) The vertical red lines indicate when a site is infected and the vertical purple lines when a edge is open. An infection path is visualized by a path of red vertical lines and red arrows, which lead from a site at time s to a site at time t. Note that arrows are only red, i.e. transmit the infection, if the edge is open.

Figure 3.1: Visualization of a graphical representation of a contact process on a dynamical percolation defined on finite subgraph of the 1-dimensional integer lattice. In this image we consider $V \cup E$ as state space and therefore added time lines to the edges.

3.4 Basic properties of the CPERE

We denote by $\mathbb{P}_{\lambda,r}$ the probability law associated with the Poisson point process $\Xi = \Xi_{\lambda,r}$. Note that we defined the CPERE $(\mathbf{C},\mathbf{B})$ on the same probability space as Ξ. Furthermore, by construction via the graphical representation it is clear that $(\mathbf{C},\mathbf{B})$ is a strong Markov process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t$ is generated by the Poisson point process Ξ up until time t.

Recall that we add superscripts either to the process $\mathbf{C}^{C,B}, \mathbf{B}^{B}$ or the law $\mathbb{P}^{(C,B)}_{\lambda,r}$ to indicate the initial configuration (C,B). Also recall that we introduced the short notation $\mu T_{\lambda,r}(t)$ in Section 2.1, where $(T_{\lambda,r}(t))_{t\geq 0}$ is the corresponding semigroup and μ the initial distribution of the CPERE. We equip $\mathcal{P}(V)$ and $\mathcal{P}(E)$ with the inclusion as a partial order. Furthermore, we equip $\mathcal{P}(V) \times \mathcal{P}(E)$ with the following partial order. Let $C, C' \subset V$ and $B, B' \subset E$, then $(C,B) \subset (C',B')$ if $C \subset C'$ and $B \subset B'$.

Lemma 3.4.1 (Monotone Feller process)**.** *Let* $(\mathbf{C},\mathbf{B})$ *be a CPERE and* $(T_{\lambda,r}(t))_{t\geq 0}$ *its corresponding Feller semigroup. Then* $(\mathbf{C},\mathbf{B})$ *is a* monotone *Feller process, i.e. let* μ_1 *and* μ_2 *be probability measures on* $\mathcal{P}(V) \times \mathcal{P}(E)$*, if* $\mu_1 \preceq \mu_2$ *then this implies* $\mu_1 T_{\lambda,r}(t) \preceq \mu_2 T_{\lambda,r}(t)$ *for all* $t \geq 0$.

Proof. Since by assuming that $\mathbf{B}$ is an attractive spin system it follows due to construction in Section 3.3 that $(\mathbf{C}, \mathbf{B})$ is also an attractive spin system. Thus, $(\mathbf{C}, \mathbf{B})$ is a monotone Feller process by [Lig12, Theorem III.2.2]. □

Not only is the CPERE monotone with respect to its initial condition it is also monotone with respect to the infection and recovery rate.

Lemma 3.4.2 (Monotonicity of CPERE). *Let* $(\mathbf{C}, \mathbf{B})$ *be a CPERE with parameter* $\lambda, r > 0$. *Let* $\widehat{\lambda} \geq \lambda$ *then there exists an CPERE* $(\widehat{\mathbf{C}}, \mathbf{B})$ *with infection rate* $\widehat{\lambda}$, *the same initial configuration and recovery rate* r *such that* $\mathbf{C}_t \subseteq \widehat{\mathbf{C}}_t$ *for all* $t \geq 0$. *In words* $\mathbf{C}$ *is monotone increasing in* λ. *On the other hand* $\mathbf{C}$ *is monotone decreasing in* r.

Proof. The two properties follow from a coupling via the graphical representation, which does not depend on the initial configuration. We only prove monotonicity in λ. Recall that $\mathcal{M}_{\text{inf}} = \{\mathbf{coop}_{x,y} : x, y \in V \text{ with } \{x, y\} \in E\}$. Let $\widehat{\lambda} \geq \lambda$ and consider a Poisson point process $\widehat{\Xi}^{\text{inf}}$ on $\mathbb{R}_+ \times \mathcal{M}_{\text{inf}}$ with intensity measure $(\widehat{\lambda} - \lambda)\mathsf{d}t$, i.e. all maps $m \in \mathcal{M}_{\text{inf}}$ occur with rate $(\widehat{\lambda} - \lambda)$. Also let $\widehat{\Xi}^{\text{inf}}$ be independent of Ξ, which is the process used in the graphical representation of $(\mathbf{C}, \mathbf{B})$ (see Section 3.3). Next we define $\widehat{\Xi} := \Xi + \widehat{\Xi}^{\text{inf}}$. This is again a Poisson point process on $\mathbb{R} \times \mathcal{M}$, with the only difference compared to Ξ, that the rates $r_m = \widehat{\lambda}$ for all $m \in \mathcal{M}_{\text{inf}}$. Now let $(\widehat{\mathbf{C}}, \mathbf{B})$ be the process constructed by the graphical representation where we use $\widehat{\Xi}$ instead of Ξ and we use the same initial configuration. Since only more infection events can happen it is obvious that $\mathbf{C}_t \subseteq \widehat{\mathbf{C}}_t$ for all $t \geq 0$. Since $\mathbf{B}$ is exactly the same process for both constructions the claim follows. The proof of monotonicity in r follows analogously. □

Let us also add that the process $\mathbf{C}^{C,B}$ is additive in the following sense:

Lemma 3.4.3 (Additivity). *Let* $t \geq 0$, *then* $\mathbf{C}_t^{C,B} \cup \mathbf{C}_t^{C',B} = \mathbf{C}_t^{C \cup C',B}$ *for all* $B \subseteq E$ *and* $C, C' \subseteq V$.

Proof. This follows immediately via the graphical representation in Section 3.3. □

Furthermore the probabilities of events where $\mathbf{C}$ depends only on a finite time horizon are continuous with respect to the infection and recovery rate, if we consider finitely many initially infected sites.

Lemma 3.4.4 (Continuity for finite times and finite initial infections)**.** *Let* $(\mathbf{C},\mathbf{B})$ *be a CPERE with initial configuration* $C \subset V$ *and* $B \subset E$ *such that* $|C| < \infty$*. Furthermore for* $t \geq 0$ *let* $\mathcal{A} \subset D_{\mathcal{P}(V)}([0,t])$*, then*

$$\lambda \mapsto \mathbb{P}^{(C,B)}_{\lambda,r}((\mathbf{C}_s)_{s\leq t} \in \mathcal{A}) \quad \textit{and} \quad r \mapsto \mathbb{P}^{(C,B)}_{\lambda,r}((\mathbf{C}_s)_{s\leq t} \in \mathcal{A})$$

are continuous.

Proof. We will only prove that $\lambda \mapsto \mathbb{P}_{\lambda,r}((\mathbf{C}_s)_{s\leq t} \in \mathcal{A})$ is continuous the statement for r follows again analogously. Let $(\widehat{\mathbf{C}},\mathbf{B})$ be a CPERE with infection rate $\widehat{\lambda} > \lambda$ and the same recovery rate and initial configuration as $(\mathbf{C},\mathbf{B})$. Let $(\widehat{\mathbf{C}},\mathbf{B})$ be coupled via the graphical representation to $(\mathbf{C},\mathbf{B})$ such that $\mathbf{C}_t \subset \widehat{\mathbf{C}}_t$ for all $t > 0$ and $\mathbf{C}_0 = \widehat{\mathbf{C}}_0$. We already used this coupling in the proof of Lemma 3.4.2. We just introduce more infection arrows via a Poisson point process $\widehat{\Xi}^{\text{inf}}$ with intensity measure $\big(\widehat{\lambda} - \lambda\big)\mathsf{d}t$ and use $\widehat{\Xi} = \Xi + \widehat{\Xi}^{\text{inf}}$ to construct $(\widehat{\mathbf{C}},\mathbf{B})$. Now it suffices to show that

$$\mathbb{P}(\mathbf{C}_s \neq \widehat{\mathbf{C}}_s \text{ for some } s \leq t) \to 0$$

as $|\widehat{\lambda} - \lambda| \to 0$. Now set $X_s(x) := |\{y \in \mathbf{C}_s : \{x,y\} \in \mathbf{B}_s\}|$, which is the number of infected neighbours of x, which share an open edge with x at time s. Note that any additional infection paths of $\widehat{\mathbf{C}}$ up until time t must have started through an infection event $(s, \mathbf{coop}_{x,y}) \in \operatorname{supp}\big(\widehat{\Xi}^{\text{inf}}\big)$ with $\{x,y\} \in \mathbf{B}_s$. This means that these events happen with the random intensity

$$(\widehat{\lambda} - \lambda)\int_0^t \sum_{x\in V} X_s(x)ds.$$

It follows immediately that

$$\mathbb{E}_{\lambda,r}\Big[(\widehat{\lambda} - \lambda)\int_0^t \sum_{x\in V} X_s(x)ds\Big] \leq |\mathcal{N}_x|(\widehat{\lambda} - \lambda)\mathbb{E}_{\lambda,r}\Big[\int_0^t |\mathbf{C}_s|ds\Big].$$

Now let $\overline{\mathbf{C}}$ be a classical contact process with infection rate λ constructed via Ξ^{inf} and Ξ^{rec} (see Remark 2.3.2), thus $\mathbf{C}_t \subseteq \overline{\mathbf{C}}_t$ for all $t \geq 0$. We know that

$$\mathbb{E}_{\lambda,r}\Big[\int_0^t |\mathbf{C}_s|ds\Big] \leq \mathbb{E}_{\lambda,r}\Big[\int_0^t |\overline{\mathbf{C}}_s|ds\Big] < \infty$$

where the second inequality follows by [Lig13, Chapter I, (1.19)].

Now we see by conditioning and using independence of $\widehat{\Xi}_{\text{inf}}$

$$\begin{aligned}\mathbb{P}(\mathbf{C}_s \neq \widehat{\mathbf{C}}_s \text{ for some } s \leq t) =& 1 - \mathbb{E}_{\lambda,r}\Big[\exp\Big(-(\widehat{\lambda}-\lambda)\int_0^t \sum_{x\in V} X_s(x)ds\Big)\Big]\\ \leq& |\mathcal{N}_x|(\widehat{\lambda}-\lambda)\mathbb{E}_{\lambda,r}\Big[\int_0^t |\overline{\mathbf{C}}_s|ds\Big],\end{aligned}$$

where we used $1-e^{-x} \leq x$ and that $|\mathcal{N}_x| = |\mathcal{N}_y|$ for all $y \in V$. By letting $|\widehat{\lambda}-\lambda| \to 0$, the right hand side convergences to zero, which proves the claim. □

We end this chapter with a comparison result between CPERE and CPDP.

Proposition 3.4.5. *Let* $(\mathbf{C}, \mathbf{B})$ *be a CPERE with infection and recovery rate* $\lambda, r > 0$. *Furthermore let* α_{max},α_{min}, β_{max} *and* β_{min} *be chosen as in* (3.3). *Then there exists two CPDP* $(\overline{\mathbf{C}}, \overline{\mathbf{B}})$ *and* $(\underline{\mathbf{C}}, \underline{\mathbf{B}})$ *with the same infection and recovery rates and the dynamical percolations* $\overline{\mathbf{B}}$ *and* $\underline{\mathbf{B}}$ *with respectively the rates* $\alpha_{\text{max}}, \beta_{\text{min}}$ *and* $\alpha_{\text{min}}, \beta_{\text{max}}$. *These processes have the property that if* $(\underline{\mathbf{C}}_0, \underline{\mathbf{B}}_0) = (\mathbf{C}_0, \mathbf{B}_0) = (\overline{\mathbf{C}}_0, \overline{\mathbf{B}}_0)$ *then* $\underline{\mathbf{C}}_t \subset \mathbf{C}_t \subset \overline{\mathbf{C}}_t$ *and* $\underline{\mathbf{B}}_t \subset \mathbf{B}_t \subset \overline{\mathbf{B}}_t$ *for all* $t > 0$ *almost surely.*

Proof. First of all we can construct analogously as in the proof of Proposition 3.1.1 a spin system $\mathbf{B}$ with spin rate $q(\cdot,\cdot)$ and two dynamical percolations $\underline{\mathbf{B}}$ and $\overline{\mathbf{B}}$ which have respectively the rates $\alpha_{\text{max}}, \beta_{\text{min}}$ and $\alpha_{\text{min}}, \beta_{\text{max}}$, where any of the three processes has values in $\mathcal{P}(E)$. Furthermore let $\underline{q}(\cdot,\cdot)$ and $\overline{q}(\cdot,\cdot)$ denote the spin rates of $\underline{\mathbf{B}}$ and $\overline{\mathbf{B}}$, then again by choice of the rates it follows that

$$\begin{aligned}\underline{q}(e,B) \leq q(e,B) \leq \overline{q}(e,B) \quad &\text{if} \quad e \notin B \quad \text{and}\\ \underline{q}(e,B) \geq q(e,B) \geq \overline{q}(e,B) \quad &\text{if} \quad e \in B\end{aligned}$$

for any $e \in E$ and any $B \subset E$. Let $\lambda, r > 0$, then we define the function

$$f(x,A) := \lambda|\{y \in V : \{x,y\} \in A, y \in A\}|\mathbb{1}_{\{x\notin A\}} + r\mathbb{1}_{\{x\in A\}},$$

where $x \in V$ and $A \subset V \cup E$. Next we can construct three process $\underline{\mathbf{X}}$, $\mathbf{X}$ and $\overline{\mathbf{X}}$ via the graphical representation as described in Section 3.3, which have respectively the spin rates

$$\begin{aligned}q_{\underline{\mathbf{X}}}(z,A) &= f(z,A)\mathbb{1}_{\{z\in V\}} + \underline{q}(z,A\backslash V)\mathbb{1}_{\{z\in E\}},\\ q_{\mathbf{X}}(z,A) &= f(z,A)\mathbb{1}_{\{z\in V\}} + q(z,A\backslash V)\mathbb{1}_{\{z\in E\}} \quad \text{and}\\ q_{\overline{\mathbf{X}}}(z,A) &= f(z,A)\mathbb{1}_{\{z\in V\}} + \overline{q}(z,A\backslash V)\mathbb{1}_{\{z\in E\}},\end{aligned}$$

where $z \in V \cup E$ and $A \subset V \cup E$. Now it follows immediately that

$$\begin{aligned} q_{\underline{\mathbf{X}}}(z,A) \leq q_{\mathbf{X}}(z,A) \leq q_{\overline{\mathbf{X}}}(z,A) \quad &\text{if} \quad z \notin A \quad \text{and} \\ q_{\underline{\mathbf{X}}}(z,A) \geq q_{\mathbf{X}}(z,A) \geq q_{\overline{\mathbf{X}}}(z,A) \quad &\text{if} \quad z \in A \end{aligned}$$

holds for all $z \in V \cup E$ and $A \subset V \cup E$. Thus, we can again use [Lig12, Theorem III.1.5], which implies that there exist a coupling such that if $\underline{\mathbf{X}}_0 = \mathbf{X}_0 = \overline{\mathbf{X}}_0$ then $\underline{\mathbf{X}}_t \subset \mathbf{X}_t \subset \overline{\mathbf{X}}_t$ for all $t > 0$ almost surely. Now we can again use the one to one correspondence between $\mathcal{P}(V \cup E)$ and $\mathcal{P}(V) \times \mathcal{P}(E)$ as we did in the end of the Section 3.3 to obtain $(\underline{\mathbf{C}}, \underline{\mathbf{B}})$,$(\mathbf{C}, \mathbf{B})$ and $(\overline{\mathbf{C}}, \overline{\mathbf{B}})$. This proves the claim. □

Chapter 4

Influence of the initial state of the background on survival

The main objective in this chapter is to prove Theorem 1.4.8, i.e. that the chance of survival does not depend on the initial configuration of the background if a certain growth condition is satisfied. Recall G is a connected and transitive graph with bounded degree and is of exponential growth $\rho \geq 0$. Additionally in this section we assume that the background $\mathbf{B}$ satisfies Assumption 1.4.1 (i) and (ii).

Before we start we briefly state some properties of the survival probability which follow from the monotonicity and additivity results shown in Section 3.4.

Proposition 4.0.1 (Monotonicity)**.** *Let $C \subset V$ be a finite subset $B \subset E$ also let $\lambda, r > 0$. The following properties hold.*

(i) *The survival probabilities θ and θ^{π} are monotone in all arguments separately.*

(ii) *Assume that $x \in C$ then*

$$\theta(\lambda, r, \{x\}, B) > 0 \Rightarrow \theta(\lambda, r, C, B) > 0,$$
$$\theta(\lambda, r, C, B) > 0 \Rightarrow \exists y \in C : \theta(\lambda, r, \{y\}, B) > 0.$$

(iii) *$\theta^{\pi}(\lambda, r, \{x\}) = \theta^{\pi}(\lambda, r, \{y\})$ and $\theta(\lambda, r, \{x\}, B) = \theta(\lambda, r, \{y\}, B)$ for all $x, y \in V$ for $B \in \{\emptyset, E\}$.*

Proof. (i) This is a direct consequence of Lemma 3.4.1 and Lemma 3.4.2.

(ii) The first implication is a direct consequence of Lemma 3.4.1. The second implication follows from additivity (see Lemma 3.4.3).

(*iii*) This is a direct consequence of the assumption that $\mathbf{B}$ is translation invariant, i.e. Assumption 1.4.1 (*ii*). This implies that the CPERE is translation invariant. Therefore, let $\sigma \in \mathrm{Aut}(G)$ such that it maps x to y, which exist since G is transitive. Thus, we can conclude that $\theta(\lambda, r, \{x\}, B) = \theta(\lambda, r, \{y\}, \sigma(B))$. Now if $\mathbf{B}_0 \sim \pi$, by translation invariance it follows that $\sigma(\mathbf{B}_0) \sim \pi$ and obviously $\sigma(B) = B$ for $B \in \{\emptyset, E\}$ for any $\sigma \in \mathrm{Aut}(G)$. This yields the claim. □

Corollary 4.0.2. *Let $\lambda, r > 0$, $C \subset V$ finite and non-empty. Then, $\theta^\pi(\lambda, r, C) > 0$ if and only if $\theta^\pi(\lambda, r, C') > 0$ for all $C' \subset V$ finite and non-empty. This shows in particular that the Definition 1.4.7 of the critical infection rate $\lambda_c^\pi(r)$ does not depend on the initial condition $C \subset V$ as long as the set is non-empty and finite.*

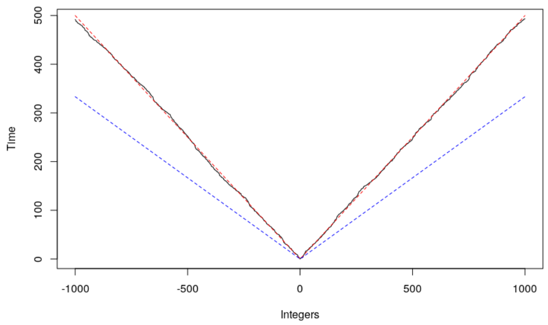
Proof. Suppose $\theta^\pi(\lambda, r, C) > 0$. Then, by Proposition 4.0.1 (*ii*) we get that there exists a $y \in C'$ such that the survival probability $\theta^\pi(\lambda, r, \{y\}) > 0$. Furthermore, by Proposition 4.0.1 (*iii*) it follows that $\theta^\pi(\lambda, r, \{y\}) = \theta^\pi(\lambda, r, \{x\})$ for all $x \in V$. Thus, by monotonicity we get that $\theta^\pi(\lambda, r, C') > 0$ for all $C' \subset V$ non-empty and finite. On the other hand if $\theta^\pi(\lambda, r, C) = 0$, then obviously $\theta^\pi(\lambda, r, \{y\}) = 0$ for all $y \in C$. But with Proposition 4.0.1 (*iii*) it follows that $\theta^\pi(\lambda, r, \{x\}) = 0$ for all $x \in V$. Now suppose that there exists a finite and non-empty $C' \subset V$ such that $\theta^\pi(\lambda, r, C') > 0$. But then Proposition 4.0.1 (*ii*) would imply that there exists a $y \in C'$ such $\theta^\pi(\lambda, r, \{y\}) > 0$. This is a contradiction, since we already showed that $\theta^\pi(\lambda, r, \{x\}) = 0$ for all $x \in V$.

That $\lambda_c^\pi(r)$ does not depend on the choice of C as long as it is non-empty and finite is a obvious consequence. □

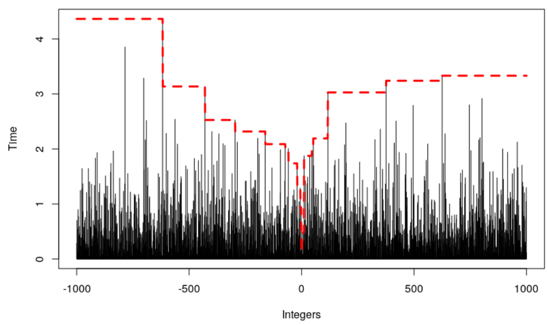
4.1 Comparison between the expansion speed of the infection and the permanently coupled region

In this section we study the asymptotic growth speed of the infection process $\mathbf{C}$ and the asymptotic speed at which the state of the edges in the background couple if started in different initial conditions, i.e. the asymptotic growth speed of the permanently coupled region Ψ'. At last we will compare these two objects in terms of expansion speed.

The maximal number of infected sites can be represented by a classical contact process $\widetilde{\mathbf{C}}^C = (\widetilde{\mathbf{C}}_t^C)_{t \geq 0}$ with infection rate $\lambda > 0$, recovery rate $r = 0$ and $\widetilde{\mathbf{C}}_0^C = C \subset V$, which is coupled with the CPERE $(\mathbf{C}^{C,B}, \mathbf{B}^B)$ such that $\mathbf{C}_t^{C,B} \subset \widetilde{\mathbf{C}}_t^C$ for all $t \geq 0$ for any $B \subset E$. This can be achieved via the graphical representation (see Remark 2.3.2) by exchanging the maps $\mathbf{coop}_{x,y}$ with the maps $\mathbf{inf}_{x,y}$. In words, this means that we ignore the background $\mathbf{B}$ and consider every infection arrow to be valid regardless of the state of the edge at the time of the transmission. In Figure 4.1(a) we visualized the spread of the infection $\widetilde{\mathbf{C}}^{\{0\}}$ on $\mathbb{Z}$ and in Figure 4.1(b) the expansion speed of Ψ'_t for the case where $\mathbf{B}$ is a dynamical percolation. The comparison suggest that Ψ' expands much faster than $\widetilde{\mathbf{C}}^{\{0\}}$.



(a) The black lines represent the right and left most particle of $\widetilde{\mathbf{C}}^{\{0\}}$ on $\mathbb{Z}$ with $\lambda = 2$. The red line has a slope of $\frac{1}{2}$ and the blue line $\frac{1}{3}$.



(b) A simulation of the first update times of a dynamical percolation on $\mathbb{Z}$ with speed $v = \alpha + \beta = 2$. The black bars are the waiting times until the first update. The red dashed line is the right and left most edge of the connected component of Ψ' containing the edge $\{0, 1\}$.

Figure 4.1: Simulations on the lattice $\mathbb{Z}$ of the infected area $\widetilde{\mathbf{C}}^{\{0\}}$ on the left and the first update times of a dynamical percolation and thus Ψ' on the right.

We start with the set of all infections, i.e. the process $\widetilde{\mathbf{C}}^{C}$, which is also often called the simplest growth model or Richardson model. See [Dur88] for a more detailed description. It is well known that asymptotically the infected area can grow at most at some linear speed in time. This is also visible in Figure 4.1(a). Next we provide an explicit upper bound for this linear speed. To be precise, for given infection paramter $\lambda > 0$ this upper bound will be $(c_1(\lambda, \rho))^{-1}$, where $c_1(\lambda, \rho)$ is a solution of

$$c\lambda - 1 - \log(c\lambda|\mathcal{N}_x|) = \rho. \tag{4.1}$$

Lemma 4.1.1. *Let $\lambda > 0$ and $x \in V$. There exists a unique solution $0 < c_1(\lambda, \rho) < \lambda^{-1}$ of (4.1). Furthermore, $\lambda \mapsto c_1(\lambda, \rho)$ is continuous, strictly decreasing, $c_1(\lambda, \rho) \to \infty$ as $\lambda \to 0$ and $c_1(\lambda, \rho) \to 0$ as $\lambda \to \infty$.*

Proof. The solution $c_1(\lambda,\rho)$ can actually be stated explicitly with the help of the Lambert W-function. It is also called the product logarithm since it is the inverse function of $t \mapsto te^t$. As domain of the function we consider $(-e^{-1},\infty)$ such that $W:(-e^{-1},\infty)\to(-1,\infty)$. This means that $W(s)e^{W(s)}=s$ for all $s\in(-e^{-1},\infty)$. Let us state some properties of W. The function W is continuous and strictly increasing. Furthermore, $W(s)\to -1$ as $s\to -e^{-1}$, $W(s)\to\infty$ as $s\to\infty$ and $W(0)=0$. Then one can show that $c_1(\lambda,\rho)=-\frac{1}{\lambda}W\big(-|\mathcal{N}_x|^{-1}e^{-(1+\rho)}\big)$, which can be verified by inserting our guess into (4.1). First wee see that $c\lambda-1-\log(c\lambda|\mathcal{N}_x|)=\rho$ if and only if $|\mathcal{N}_x|^{-1}\exp(-(1+\rho))=c\lambda\exp(-c\lambda)$. Therefore, inserting our guess in the right-hand side and using that W is the inverse function of $t\mapsto te^t$ verifies that this is a solution of (4.1). Note that

$$-e^{-1}<|\mathcal{N}_x|^{-1}\exp(-(1+\rho))<0$$

and $-1<W(s)<0$ for $-e^{-1}<s<0$, and thus it follows that $c_1(\lambda,\rho)<\frac{1}{\lambda}$. Now for $\lambda>0$ fixed set $g_\rho(c):=c\lambda-1-\log(c\lambda|\mathcal{N}_x|))-\rho$ for all $c>0$. Obviously the function g_ρ is smooth on $(0,\infty)$ and its derivative is $g_\rho'(c)=\lambda-\frac{1}{c}>0$ for all $c<\frac{1}{\lambda}$ which implies that g_ρ is strictly decreasing on $(0,\frac{1}{\lambda})$, and thus $c_1(\lambda,\rho)$ must be the unique solution of (4.1) on $(0,\frac{1}{\lambda})$. At last the two properties follow immediately. □

Let us define the first hitting time of $y\in V$ for $\widetilde{\mathbf{C}}$ with initial infections $C\subset V$ as $\tau_y(C):=\inf\{t\geq 0: y\in\widetilde{\mathbf{C}}_t^C\}$. The special case of following lemma, where $V=\mathbb{Z}^d$ with nearest neighbour structure and $\lambda=1$ can be found in [Dur88, Lemma 1.9].

Lemma 4.1.2. *Let $\lambda>0$ and set $g_\rho(c):=c\lambda-1-\log(c\lambda|\mathcal{N}_x|))-\rho$ for all $c>0$. Then for every $0<c<c_1(\lambda,0)$ we have $g_0(c)>0$ and*

$$\mathbb{P}\big(\tau_y(\{x\})<cd(x,y)\big)\leq\frac{\exp(-g_0(c)d(x,y))}{1-\exp(-g_0(c))},$$

where $x\neq y$. This implies in particular for all $c<c_1(\lambda,\rho)$ that for any $x\in V$

$$\mathbb{P}(\exists s\geq 0:\widetilde{\mathbf{C}}_{ct}^{\{x\}}\subset\mathbb{B}_{\lfloor t\rfloor}(x)\ \forall t\geq s)=1$$

To understand this result more clearly let us consider Figure 4.1(a). In this figure we visualized that the set of all infection expands asymptotically linear in time with some slope $c'>0$. What Lemma 4.1.2 basically states is that for every slope $c<c_1(\lambda,\rho)$ from some time point $s\geq 0$ onwards the boundary of the set of all infected individuals will expand with a steeper slope than c, and thus $c<c'$.

Proof. Let $0 < c < c_1(\lambda, 0)$. If $\tau_y(\{x\}) < cd(x,y)$, then the site x must have been infected before the time point $cd(x,y)$. This means in particular that there exists a sequence of distinct points $x = x_0, x_1 \dots, x_m = y$ such that $(x_{n-1}, x_n) \in E$ for $n \in \{1, \dots, m\}$, along which the infection travels. Note that obviously $m \geq d(x,y)$. Now we wet $T_0 := 0$ and define $T_n := \inf\{t > T_{n-1} | (\mathbf{inf}_{x_{n-1},x_n}, t) \in \Xi^{\text{inf}}\}$ for $1 \leq n \leq m$. It is clear from the construction that $\tau_y(\{x\}) \leq T_m$. The memorylessness property implies that $T_m \sim \Gamma(\lambda, m)$. Therefore, the event $\{\tau_y(\{x\}) < cd(x,y)\}$ is equivalent to the statement that there exists a sequence $(x_n, T_n)_{0 \leq n \leq m}$ with $m \geq d(x,y)$, $x_0 = x$ and $x_m = y$ such that $T_m < cd(x,y)$.

It is easy to see that the number of paths of length m is bounded by K^m, where $K := |\mathcal{N}_z|$ for an arbitrary $z \in V$. The number K^m is obviously also a bound on the number of paths of length m connecting x to y. This implies the inequality

$$\mathbb{P}\big(\tau_y(\{x\}) < cd(x,y)\big) \leq \sum_{m=d(x,y)}^{\infty} K^m \mathbb{P}\big(T_m < cd(x,y)\big) \leq \sum_{m=d(x,y)}^{\infty} K^m \mathbb{P}(T_m < cm),$$

where we used that $d(x,y) \leq m$. Furthermore by Lemma 3.2.2 we see that

$$\begin{aligned}
\mathbb{P}\big(\tau_y(\{x\}) < cd(x,y)\big) &\leq \sum_{m=d(x,y)}^{\infty} K^m \exp(-m(c\lambda - 1 - \log(c\lambda))) \\
&= \frac{\exp\big(-d(x,y)(c\lambda - 1 - \log(c\lambda K)\big)}{1 - \exp(1 - c\lambda + \log(c\lambda K)\big)}.
\end{aligned}$$

By Lemma 4.1.1 we have that $c\lambda - 1 - \log(c\lambda K) > 0$ for $c < c_1(\lambda, 0)$, and thus the first claim follows. For the second claim we conclude that

$$\begin{aligned}
\mathbb{P}\big(\widetilde{\mathbf{C}}^{\{x\}}_{c(n+1)} \not\subseteq \mathbb{B}_n(x)\big) &\leq \mathbb{P}\big(\exists y \in V : d(x,y) = n+1, \tau_y(\{x\}) < c(n+1)\big) \\
&\leq \sum_{y \in V:\, d(x,y) = n+1} \mathbb{P}\big(\tau_y(\{x\}) < cd(x,y)\big) \\
&\leq |\partial \mathbb{B}_{n+1}(x)| \frac{\exp(-g_0(c)(n+1))}{1 - \exp(-g_0(c))}.
\end{aligned}$$

Note that if $c < c_1(\lambda, \rho)$, then $g_0(c) > \rho$. Thus,

$$\mathbb{P}\big(\widetilde{\mathbf{C}}^{\{x\}}_{c(n+1)} \not\subseteq \mathbb{B}_n(x)\big) \leq \Big(\frac{\sup_{k \geq 0} |\partial \mathbb{B}_{k+1}(x)| e^{-\rho(k+1)}}{1 - \exp(-g_0(c))}\Big) \exp(-\underbrace{(g_0(c) - \rho)}_{= g_\rho(c) > 0}(n+1))$$

and since G is of exponential growth ρ the first factor is finite. Since $g_\rho(c) > 0$ by a comparison with the geometric sum we see that the right hand side is summable. Thus, applying the Borel-Cantelli Lemma we get that

$$\mathbb{P}\big(\exists N \geq 1 : \widetilde{\mathbf{C}}^{\{x\}}_{c(n+1)} \subseteq \mathbb{B}_n(x), \forall n \geq N\big) = 1.$$

Since $\widetilde{\mathbf{C}}^{\{x\}}_{ct} \subset \widetilde{\mathbf{C}}^{\{x\}}_{c(n+1)}$ for all $t \in (n, n+1]$ it follows that

$$\mathbb{P}\big(\exists s \geq 0 : \widetilde{\mathbf{C}}^{\{x\}}_{ct} \subseteq \mathbb{B}_{\lfloor t \rfloor}(x), \forall t \geq s\big) = 1.$$

□

Next we consider the speed of expansion of the permanently coupled region Ψ' defined in (1.3). Recall that $\mathbb{B}^L_k(e)$ denotes the ball of radius $k \in \mathbb{N}$ around an edge $e \in E$ in the line graph $L(G)$ (see Section 2.4).

Proposition 4.1.3. *Let $e \in E$ and κ as in Assumption 1.4.1 (ii). If $c > \kappa^{-1}\rho$, then*

$$\mathbb{P}(\exists s \geq 0 : \mathbb{B}^L_{\lfloor t \rfloor + 1}(e) \subset \Psi'_{ct} \ \forall t \geq s) = 1.$$

Proof. Fix an arbitrary $e \in E$ and recall that by Assumption 1.4.1 (ii) there exist $T, K, \kappa > 0$ such that $\mathbb{P}(e \notin \Psi'_t) \leq K e^{-\kappa t}$ for all $t > T$. Thus, it follows that

$$\sum_{n=\lceil T \rceil}^{\infty} \mathbb{P}(\mathbb{B}^L_{n+1}(e) \not\subseteq \Psi'_{cn}) \leq K \sum_{n=\lceil T \rceil}^{\infty} |\mathbb{B}^L_{n+1}(e)| e^{-\kappa c n}. \tag{4.2}$$

By Remark 2.4.8 we see that $|\mathbb{B}^L_{n+1}(e)| e^{-\rho n} \to 1$ as $n \to \infty$, if G has exponential growth $\rho > 0$. Therefore, $K_1 := K \cdot \sup_{n \in \mathbb{N}} |\mathbb{B}^L_{n+1}(e)| \exp(-\rho n) < \infty$ and

$$\sum_{n=\lceil T \rceil}^{\infty} \mathbb{P}(\mathbb{B}^L_{n+1}(e) \not\subseteq \Psi'_{cn}) \leq K \sum_{n=\lceil T \rceil}^{\infty} |\mathbb{B}^L_{n+1}(e)| e^{-\rho n} e^{(\rho - \kappa c) n} \leq K_1 \sum_{n=\lceil T \rceil}^{\infty} e^{(\rho - \kappa c) n} < \infty,$$

where we used $\kappa c > \rho$. If $\rho = 0$, we know by Remark 2.4.8 that $|\mathbb{B}^L_{n+1}(e)| e^{-Cn} \to 0$ as $n \to \infty$ for all $C > 0$, and thus the right hand side of (4.2) is finite. Since we know that the left hand side of (4.2) is summable, the Borel-Cantelli Lemma yields that

$$\mathbb{P}(\exists N \geq 1 : \mathbb{B}^L_{n+1}(e) \subseteq \Psi'_{cn}, \forall n \geq N) = 1.$$

Note that $\Psi'_{cn} \subset \Psi'_{ct}$ for all $t \geq n$, which proves the claim. □

At the end of this section we use these two results such that we can compare the asymptotic expansion speed of the infection and the coupled region. Since one process has values in $\mathcal{P}(V)$ and the other in $\mathcal{P}(E)$ we need to introduce the following notation. We denote by

$$\Phi_t := \{x \in V : \{x, y\} \in \Psi'_t \ \forall y \in \mathcal{N}_x\}.$$

the set of all vertices whose attached edges are already permanently coupled at time t.

Theorem 4.1.4. *Let $\lambda > 0$, $C \subset V$ be non-empty and finite, κ as in Assumption 1.4.1 (ii) and $c_1(\lambda, \rho)$ chosen as in Lemma 4.1.1. If $c_1(\lambda, \rho) > \kappa^{-1}\rho$, then*

$$\mathbb{P}(\exists s \geq 0 : \widetilde{\mathbf{C}}_t^C \subseteq \Phi_t \ \forall t \geq s) = 1.$$

Proof. Let $x \in V$ and $y \in \mathcal{N}_x$. First we consider $C = \{x\}$. Note that we assumed $c_1(\lambda, \rho) > \kappa^{-1}\rho$, and thus there exists a $c < c_1(\lambda, \rho)$ such that $c\kappa > \rho$. Since $c < c_1(\lambda, \rho)$ by Lemma 4.1.2 we get that

$$\mathbb{P}(\exists s > 0 : \widetilde{\mathbf{C}}_{ct}^{\{x\}} \subset \mathbb{B}_{\lfloor t \rfloor}(x) \ \forall t \geq s) = 1. \tag{4.3}$$

On the other hand we know that $c\kappa > \rho$, and hence Proposition 4.1.3 implies that

$$\mathbb{P}(\exists s > 0 : \mathbb{B}^L_{\lfloor t \rfloor + 1}(\{x, y\}) \subset \Psi'_{ct} \ \forall t > s) = 1.$$

Since $\mathbb{B}^L_{\lfloor t \rfloor + 1}(\{x, y\})$ contains all edges attached to any vertex in $\mathbb{B}_{\lfloor t \rfloor}(x)$, we see by definition of the random set Φ_{ct} that

$$\mathbb{P}(\exists s > 0 : \mathbb{B}_{\lfloor t \rfloor}(x) \subset \Phi_{ct} \ \forall t > s) = 1. \tag{4.4}$$

By combining (4.3) and (4.4) we get that

$$\mathbb{P}(\exists s \geq 0 : \widetilde{\mathbf{C}}_t^{\{x\}} \subseteq \Phi_t \ \forall t \geq s) = 1.$$

Now let $C \subset V$ be an arbitrary non-empty and finite subset. Then we see with Lemma 3.4.3 that

$$\mathbb{P}(\nexists s \geq 0 : \widetilde{\mathbf{C}}_t^C \subseteq \Phi_t \ \forall t \geq s) \leq \sum_{x \in C} \mathbb{P}(\nexists s \geq 0 : \widetilde{\mathbf{C}}_t^{\{x\}} \subseteq \Phi_t \ \forall t \geq s).$$

But we already showed that $\mathbb{P}(\nexists s \geq 0 : \widetilde{\mathbf{C}}_t^{\{x\}} \subseteq \Phi_t \ \forall t \geq s) = 0$ for all $x \in V$ and thus, the right hand side is already equal to 0. This proves the claim. $\square$

4.2 Proofs of Theorem 1.4.8, Corollary 1.4.9 and Corollary 1.4.10

We are finally ready to prove the main results of this chapter. We begin with the proof of Theorem 1.4.8. Let us briefly recapitulate its content. Let $\lambda, r > 0$ and $C \subset V$ be finite and non-empty. Suppose that $c_1(\lambda, \rho) > \kappa^{-1}\rho$ is satisfied, then we show that $\theta(\lambda, r, C, B_1) > 0$ if and only if $\theta(\lambda, r, C, B_2) > 0$ for all $B_1, B_2 \subset E$.

Proof of Theorem 1.4.8. Let $\lambda, r > 0$. As mentioned at the beginning of this chapter we assume that Assumptions 1.4.1 (i)-(ii) are satisfied. Additionally we suppose that $c_1(\lambda, \rho) > \kappa^{-1}\rho$ holds, where $c_1(\lambda, \rho)$ is the solution of (4.1) and κ as in Assumption 1.4.1 (ii). Furthermore let $x \in V$ be fixed. The proof strategy is to use $\theta^{\pi}(\{x\})$ as a reference, i.e. $\mathbf{B}_0 \sim \pi$. Note that we omit the infection and recovery rate as variables since they are considered constant throughout the whole proof. By Proposition 4.0.1 (i) it suffices to show that $\theta(C, \emptyset) > 0$ if and only if $\theta(C, E) > 0$.

Let $A \subset V$ be an arbitrary finite non-empty set. Then by Corollary 4.0.2 it follows that $\theta^{\pi}(C) > 0$ if and only if $\theta^{\pi}(A) > 0$. Since also $\theta^{\pi}(\{x\}) = \theta^{\pi}(\{y\})$ for all $y \in V$ it is enough to show:

a) If $\theta^{\pi}(\{x\}) > 0$, then $\theta(\{x\}, \emptyset) > 0$.

b) If $\theta^{\pi}(\{x\}) = 0$, then $\theta(\{x\}, E) = 0$.

The key idea is that we prove this by coupling the CPERE $(\mathbf{C}, \mathbf{B})$ to processes $\underline{\mathbf{C}}$ and $\overline{\mathbf{C}}$, which act as a upper and lower bound, i.e. $\underline{\mathbf{C}}_0 = \mathbf{C}_0 = \overline{\mathbf{C}}_0$ and $\underline{\mathbf{C}}_t \subset \mathbf{C}_t \subset \overline{\mathbf{C}}_t$ for all $t > 0$. Note that all three infection processes will depend on the same background process $\mathbf{B}$. Let $s > 0$, then we define $\underline{\mathbf{C}}^{C,B,s}$ as follows.

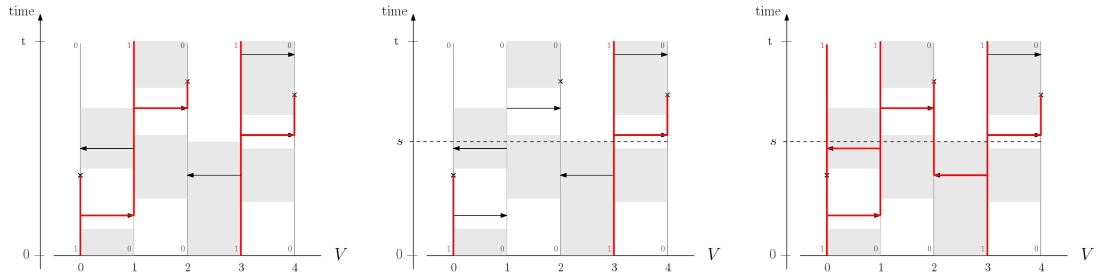
1. We set $\underline{\mathbf{C}}_0^{C,B,s} = C$. On $[0, s]$ we only consider the recovery symbols caused by Ξ^{rec} and ignore all infection arrows, i.e. $\mathbf{coop}_{x,y}$ maps.
2. On (s, ∞) we use the same graphical representation as for the $\mathbf{C}^{C,B}$, i.e. the same infection arrows and recovery symbols generated by Ξ^{inf} and Ξ^{rec} and the same background $\mathbf{B}^B$.

Next we define $\overline{\mathbf{C}}^{C,B,s}$ as follows.

1. We set $\overline{\mathbf{C}}_0^{C,B,s} = C$. On $[0, s]$ we only consider the infection events caused by Ξ^{inf}. This means we ignore all recovery symbols caused by Ξ^{rec} and also the background $\mathbf{B}^B$ in the sense that we treat all edges as open. Hence, instead of the maps $\mathbf{coop}_{x,y}$ we apply the maps $\mathbf{inf}_{x,y}$ (see Example 2.3.2).

2. On (s, ∞) we again use the same graphical representation as for $\mathbf{C}^{C,B}$ and we use the same background $\mathbf{B}^B$.

See Figure 4.2 for a visualization of $\overline{\mathbf{C}}^s$, $\mathbf{C}$ and $\underline{\mathbf{C}}^s$ on the same realization of $\mathbf{B}$. Recall



(a) Construction of $\mathbf{C}$ via the Graphical representation. (b) Construction of $\underline{\mathbf{C}}^s$ via the Graphical representation. (c) Construction of $\overline{\mathbf{C}}^s$ via the Graphical representation.

Figure 4.2: Here we visualize, how the three process $\underline{\mathbf{C}}^s$, $\mathbf{C}$ and $\overline{\mathbf{C}}^s$ a constructed by using the same Poisson point process Ξ.

that $\widetilde{\mathbf{C}}^C$ is the classical contact process without recoveries which is coupled to the CPERE $(\mathbf{C}^{C,B}, \mathbf{B}^B)$ such that $\mathbf{C}_0^{C,B} = \widetilde{\mathbf{C}}_0^C = C$ and $\mathbf{C}_t^{C,B} \subset \widetilde{\mathbf{C}}_t^C$ for all $t \geq 0$. By construction $\overline{\mathbf{C}}_t^{C,B,s} = \widetilde{\mathbf{C}}_t^C$ for all $t \leq s$.

We set $A_s(C) := \{\widetilde{\mathbf{C}}_t^C \subseteq \Phi_t \; \forall t \geq s\}$. Another reason why we consider these two processes is that by the construction of $\underline{\mathbf{C}}^s$ and $\overline{\mathbf{C}}^s$ it is clear that

$$\mathbb{P}(A_s(C) \cap \{\underline{\mathbf{C}}_t^{C,\emptyset,s} \neq \emptyset \; \forall t \geq 0\}) = \mathbb{P}(A_s(C) \cap \{\underline{\mathbf{C}}_t^{C,E,s} \neq \emptyset \; \forall t \geq 0\}), \tag{4.5}$$

$$\mathbb{P}(A_s(C) \cap \{\overline{\mathbf{C}}_t^{C,\emptyset,s} \neq \emptyset \; \forall t \geq 0\}) = \mathbb{P}(A_s(C) \cap \{\overline{\mathbf{C}}_t^{C,E,s} \neq \emptyset \; \forall t \geq 0\}), \tag{4.6}$$

since both processes are independent of the background $\mathbf{B}$ on $[0, s]$ and in the time interval (s, ∞) all infection paths stay in the coupled region, i.e. the initial configuration of the background process has no influence.

We start by proving a). To avoid clutter we set $A_s := A_s(\{x\})$. We see that

$$\theta(\{x\}, \emptyset) \geq \mathbb{P}(A_s \cap \{\underline{\mathbf{C}}_t^{\{x\},\emptyset,s} \neq \emptyset \; \forall t \geq 0\})$$

for every $s > 0$ and by (4.5) we get that

$$\theta(\{x\}, \emptyset) \geq \int \mathbb{P}(A_s(\{x\}) \cap \{\underline{\mathbf{C}}_t^{\{x\},B,s} \neq \emptyset \; \forall t \geq 0\}) \pi(\mathrm{d}B). \tag{4.7}$$

The state $\emptyset$ is obviously an absorbing state for the infection. Hence,

$$\begin{aligned}&\int \mathbb{P}(A_s \cap \{\underline{\mathbf{C}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq 0\})\pi(\mathrm{d}B)\\ =&\int \mathbb{P}(A_s \cap \{\underline{\mathbf{C}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq s\})\pi(\mathrm{d}B).\end{aligned} \tag{4.8}$$

Let $\underline{\underline{\mathbf{C}}}^s$ be a process which is constructed analogously as $\underline{\mathbf{C}}^s$ with the difference that on $[0, s]$ also no recovery symbols have an effect. Therefore, $\underline{\underline{\mathbf{C}}}$ is just a delayed CPERE. By construction it is clear that it is only possible for $\underline{\mathbf{C}}^{s,\{x\},B}$ to survive if until time s the site x is not hit by a recovery symbol, i.e. let $T := \inf\{t > 0 : (\mathbf{rec}_x, t) \in \Xi^{\mathrm{rec}}\}$, then $\underline{\mathbf{C}}^{s,\{x\},B}$ goes extinct almost surely on the event $\{T \leq s\}$. Note that $\underline{\underline{\mathbf{C}}}^{\{x\},B,s} = \underline{\mathbf{C}}^{\{x\},B,s}$ on $\{T > s\}$ and thus,

$$\begin{aligned}&\int \mathbb{P}\big(A_s \cap \{\underline{\mathbf{C}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq s\}\big)\pi(\mathrm{d}B)\\ =&\int \mathbb{P}\big(A_s \cap \{\underline{\underline{\mathbf{C}}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq s\} \cap \{T > s\}\big)\pi(\mathrm{d}B).\end{aligned} \tag{4.9}$$

Furthermore we know that the event $\{T > s\}$ only depends on Ξ^{rec} in the time interval $[0, s]$. Since A_s only depends on Ξ^{inf} and the point processes Ξ^{Back} and Ξ^{rec} have no impact on the survival of $\underline{\underline{\mathbf{C}}}$ on $[0, s]$, we get that

$$\begin{aligned}&\int \mathbb{P}\big(A_s \cap \{\underline{\underline{\mathbf{C}}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq s\} \cap \{T > s\}\big)\pi(\mathrm{d}B)\\ =&\mathbb{P}(T > s) \int \mathbb{P}\big(A_s \cap \{\underline{\underline{\mathbf{C}}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq s\}\big)\pi(\mathrm{d}B).\end{aligned} \tag{4.10}$$

By construction it follows that $(\underline{\underline{\mathbf{C}}}_t^s)_{t\leq s}$ and $(\mathbf{B}_t)_{t\leq s}$ are independent. Also since π is the unique invariant law of the background process we see that

$$\int \mathbb{P}\big(\underline{\underline{\mathbf{C}}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq 0\big)\pi(\mathrm{d}B) = \int \mathbb{P}\big(\mathbf{C}_t^{\{x\},B} \neq \emptyset \ \forall t \geq 0\big)\pi(\mathrm{d}B) = \theta^\pi(\{x\}) > 0,$$

for every $s \geq 0$, where the last inequality follows by assumption. As already mentioned $\underline{\underline{\mathbf{C}}}$ is just a delayed CPERE and if it is started stationary the survival probability is constant in s. By Theorem 4.1.4 for every $\theta^\pi(\{x\}) > \varepsilon > 0$ there exists a $S > 0$ such that $\mathbb{P}(A_s) > 1-\varepsilon$ for all $s > S$, where we used that $A_s \subset A_{s'}$ if $s \leq s'$. We can use this to conclude that

$$\left| \int \mathbb{P}\big(\underline{\underline{\mathbf{C}}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq 0\big)\pi(\mathsf{d}B) - \int \mathbb{P}\big(A_s \cap \{\underline{\underline{\mathbf{C}}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq s\}\big)\pi(\mathsf{d}B) \right| < \varepsilon. \tag{4.11}$$

Now using (4.7)-(4.11) successively yields that $\theta(\{x\},\emptyset) \geq \mathbb{P}(T > s)(\theta^{\pi}(\{x\}) - \varepsilon) > 0$, where we used that $\mathbb{P}(T > s) > 0$ for all $s \geq 0$. This proves a).

It remains to show b). Here, it suffices to show that

$$\mathbb{P}(A_s \cap \{\mathbf{C}_t^{\{x\},E} \neq \emptyset \ \forall t \geq 0\}) = 0 \tag{4.12}$$

for all $s > 0$. This is because Theorem 4.1.4 yields that

$$\mathbb{P}\Big(\bigcup_{n\in\mathbb{N}_0} A_n\Big) = \mathbb{P}(\exists s \geq 0 : \widetilde{\mathbf{C}}_t^{C} \subseteq \Phi_t \ \forall t \geq s) = 1,$$

where we used in the first equality that $A_s \subset A_{s'}$ if $s \leq s'$. Hence,

$$\begin{aligned}\mathbb{P}(\mathbf{C}_t^{\{x\},E} \neq \emptyset \ \forall t \geq 0) &= \mathbb{P}(\{\exists s \geq 0 : \widetilde{\mathbf{C}}_t^{\{x\}} \subseteq \Phi_t \, \forall t \geq s\} \cap \{\mathbf{C}_t^{\{x\},E} \neq \emptyset \ \forall t \geq 0\}) \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}(A_n \cap \{\mathbf{C}_t^{\{x\},E} \neq \emptyset \ \forall t \geq 0\}),\end{aligned}$$

and therefore (4.12) implies that the right hand side is 0. By constructions of $\overline{\mathbf{C}}$ we see that

$$\mathbb{P}(A_s \cap \{\mathbf{C}_t^{\{x\},E} \neq \emptyset \ \forall t \geq 0\}) \leq \mathbb{P}(A_s \cap \{\overline{\mathbf{C}}_t^{\{x\},E,s} \neq \emptyset \ \forall t \geq 0\}).$$

Furthermore by (4.6) it follows that

$$\mathbb{P}(A_s \cap \{\overline{\mathbf{C}}_t^{\{x\},E,s} \neq \emptyset \ \forall t \geq 0\}) = \int \mathbb{P}(A_s \cap \{\overline{\mathbf{C}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq 0\})\pi(\mathsf{d}B)$$

and since $\overline{\mathbf{C}}_s^{\{x\},B,s} = \widetilde{\mathbf{C}}_s^{\{x\}}$ for all $B \subset E$ we get

$$\int \mathbb{P}(A_s \cap \{\overline{\mathbf{C}}_t^{\{x\},B,s} \neq \emptyset \ \forall t \geq 0\})\pi(\mathsf{d}B) \leq \mathbb{E}^{\{x\}}[\mathbb{P}^{(\widetilde{\mathbf{C}}_s,\pi)}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0)] = 0,$$

where we used that by assumption $\theta^\pi(C) = 0$ for all finite C and $|\widetilde{\mathbf{C}}_s^{\{x\}}| < \infty$ almost surely. Therefore,

$$\mathbb{P}^{\{x\},E}(A_s \cap \{\mathbf{C}_t^{\{x\},E} \neq \emptyset \ \forall t \geq 0\}) = 0$$

for all $s \geq 0$, which implies $\theta(\{x\}, E) = 0$. □

Now we have shown that if $c_1(\lambda, \rho) > \kappa^{-1}\rho$ holds, then the chance to survive is independent of the initial configuration of the background. Next we will show as a corollary that if for a $r > 0$ there exists a non-empty and finite set $C \subset V$ and $B \subset E$ such that $c_1(\lambda_c(r, C, B), \rho) > \kappa^{-1}\rho$, then it follows that $\lambda_c(r, C, B) = \lambda_c^\pi(r)$ for all non-empty and finite $C \subset V$ and $B \subset E$. This basically means that if for $r > 0$ there exists an λ such that survival is possible and additionally $c_1(\lambda, \rho) > \kappa^{-1}\rho$ then the critical infection rate is independent of the choice of the initial configuration (C, B) as long as $C \subset V$ is non-empty and finite.

Proof of Proposition 1.4.9. Let $r > 0$ and suppose there exists a non-empty and finite $C' \subset V$ and set $B' \subset E$ such that $c_1\big(\lambda_c(r, C', B'), \rho\big) > \kappa^{-1}\rho$. We know by Lemma 4.1.1 that $\lambda \mapsto c_1(\lambda, \rho)$ is continuous and strictly decreasing. Hence, there exists an $\varepsilon > 0$ such that all $\lambda < \lambda_c(r, C', B') + \varepsilon$ satisfy $c_1(\lambda, \rho) > \kappa^{-1}\rho$. Now we consider $\lambda < \lambda_c(r, C', B') + \varepsilon$. Theorem 1.4.8 implies in particular that

$$\theta(\lambda, r, C', B') > 0 \Leftrightarrow \theta^\pi(\lambda, r, C') > 0. \tag{4.13}$$

Furthermore, in Corollary 4.0.2 we already showed that

$$\theta^\pi(\lambda, r, C') > 0 \Leftrightarrow \theta^\pi(\lambda, r, C) > 0, \tag{4.14}$$

for every non-empty and finite $C \subset V$. This, in particular implies that

$$\lambda_c(r, C', B') = \lambda^\pi(r).$$

Next we use again that $c_1(\lambda, \rho) > \kappa^{-1}\rho$ such that Theorem 1.4.8 together with (4.13) and (4.14) yield that $\theta(\lambda, r, C', B') > 0$ if and only if $\theta(\lambda, r, C, B) > 0$ for all non-empty and finite $C \subset V$ and all $B \subset E$. This obviously implies that

$$\lambda_c(r, C', B') = \lambda_c^\pi(r) = \lambda_c(r, C, B)$$

for all finite and non-empty $C \subset V$ and $B \subset E$. □

We end this chapter by showing an extension of the results concerning the asymptotic behaviour and the immunization region shown by [LR20].

Proof of Corollary 1.4.10. Recall from Remark 1.1.5 that for a CPDP with parameters $\lambda, r, \alpha, \beta$ and initial configuration (C, B) we denoted the survival probability by $\theta_{\mathrm{DP}}(\lambda, r, \alpha, \beta, C, B)$ and the critical infection rate by $\lambda_c^{\mathrm{DP}}(r, \alpha, \beta, C, B)$. Let $v > 0$ and $p \in (0, 1)$, then we set $\alpha = vp$ and $\beta = v(1-p)$. Since $\mathbf{B}$ is a dynamical percolation we know from Remark 3.2.4 that for the constant κ in Assumption 1.4.1 (ii) it holds that $\kappa \geq \alpha + \beta = v$. From here on throughout the proof we again drop the sub- and superscript DP out of notational convenience.

Fix some $x \in V$ and recall that λ_c^G denotes the critical infection rate of the classical contact process with recovery rate 1 on the graph G. We first show (i), which states that for every $p \in (0, 1]$, $\lambda_c\big(1, vp, v(1-p), C, B\big) \to \frac{\lambda_c^G}{p}$ as $v \to \infty$, for all $C \subset V$ non-empty and finite and all $B \subset E$.

Theorem 1.3.3 (i) implies in particular that for every $p \in (0, 1)$, we can choose for every $\varepsilon > 0$ a $v_0 > 0$ large enough such that

$$\lambda_c^{\pi}\big(1, vp, v(1-p)\big) < \frac{\lambda_c^G}{p} + \varepsilon$$

for all $v > v_0$. Thus, next we choose $v_1 > v_0$ such that $c_1(p^{-1}\lambda_c^G + \varepsilon, \rho) > v^{-1}\rho$. Because c_1 is monotone deceasing in the first coordinate we see that

$$c_1(\lambda_c^{\pi}\big(1, vp, v(1-p)\big), \rho) > v^{-1}\rho$$

for all $v > v_1$. Since we know that $\kappa \geq v$ by Corollary 1.4.9 it follows that for all $v > v_1$ the critical infection rate $\lambda_c\big(1, vp, v(1-p)\big)$ is independent of the initial configuration, i.e.

$$\lambda_c\big(1, vp, v(1-p)\big) = \lambda_c\big(1, vp, v(1-p), C, B\big)$$

for all $C \subset V$ non-empty and finite and all $B \subset E$. So finally, Theorem 1.3.3 (i) yields that

$$\lim_{v \to \infty} \lambda_c\big(1, vp, v(1-p)\big) = \frac{\lambda_c^G}{p}.$$

Next we show (ii) and (iii). In both cases we consider graphs of subexponential growth, i.e. $\rho = 0$. Therefore, the inequality $c_1(\lambda, \rho) > \kappa^{-1}\rho$ is obviously satisfied, and thus by Proposition 1.4.9 it follows that the critical infection rate $\lambda_c(r, vp, v(1-p))$ is independent of the initial configuration for any choice of the parameter.

Now we first show (ii), which states that for every $r > 0$ and $v > 0$ there exists a $p_0 = p_0(r, v) > 0$ such that for every $p < p_0$, $\lambda_c\big(r, vp, v(1-p)\big) = \infty$. Now Theorem 1.3.2 (i) yields that for every $v > 0$ there exists a $p_0 = p_0(v) > 0$ such that for every $p < p_0$, $\lambda_c^{\pi}\big(1, vp, v(1-p)\big) = \infty$ Since we showed that the critical value does not depend on the initial conditions a direct consequence is that $\lambda_c\big(1, vp, v(1-p)\big) = \infty$, i.e. for every $\lambda > 0$,

$$\theta\big(\lambda, 1, vp, v(1-p), C, B\big) = 0.$$

for every finite $C \subset V$ and every $B \subset E$. But, by rescaling time with the factor r we see that

$$\theta(\lambda, 1, vp, v(1-p), C, B) = \theta\big(\lambda r, r, vrp, vr(1-p), C, B\big),$$

and therefore by setting $v' := vr$ we see that for every $\lambda > 0$ the survival probability $\theta\big(\lambda, r, v'p, v'(1-p), C, B\big) = 0$ for all finite $C \subset V$ and all $B \subset E$. This proves the claim.

Claim (iii) follows via a similar argument. Hence, we will now show that for every $p \in [0, 1)$, $\lambda_c\big(r, vp, v(1-p)\big) \to \infty$ as $v \to 0$. By Theorem 1.3.3 (ii) we know that $\lambda_c(1, vp, v(1-p)) \to \infty$ as $v \to 0$, i.e. the special case $r = 1$. Thus, for every $\lambda > 0$ there exists a $v_0 > 0$ such that

$$\theta(\lambda, 1, vp, v(1-p), C, B) = 0$$

for every $C \subset V$ finite, $B \subset E$ and for every $v < v_0$. Now again rescaling time by the fixed factor r and setting $\lambda' := \lambda r$ and $v_0' := v_0 r$ yields that for every $\lambda' > 0$ there exists a $v_0' > 0$ such that

$$\theta(\lambda', r, vp, v(1-p), C, B) = 0$$

for every $v < v_0'$, and thus $\lambda_c^{\mathrm{DP}}\big(r, vp, v(1-p)\big) \to \infty$ as $v \to 0$. □

Chapter 5

The CPERE and its invariant laws

In this chapter we mainly study the invariant laws of the CPERE. We assume through out this whole chapter that the background $\mathbf{B}$ satisfies the Assumption 1.4.1 (i)-(iii).

5.1 Upper invariant law and the dual process of $\mathbf{C}$

First we introduce the notion of duality. Let $\mathbf{X}$ and $\mathbf{Y}$ be two processes on the same probability space and let the Polish spaces $\mathbb{S}_X$ and $\mathbb{S}_Y$ denote their respective state spaces.

Definition 5.1.1 (Duality)**.** Let $t \geq 0$. We call $(\mathbf{X}_u)_{0\leq u\leq t}$ and $(\mathbf{Y}_u)_{0\leq u\leq t}$ dual with respect to a function $H : \mathbb{S}_X \times \mathbb{S}_Y \to \mathbb{R}$ if $s \mapsto \mathbb{E}[H(\mathbf{X}_{t-s}, \mathbf{Y}_s)]$ is a constant function for $0 \leq s \leq t$.

For the classical contact process $\mathbf{X}$ (see Example 2.3.2) one can use the graphical representation to construct a dual process $\widehat{\mathbf{X}}$ such that $s \mapsto \mathbb{P}(\mathbf{X}_s \cap \widehat{\mathbf{X}}_{t-s} \neq \emptyset)$ is a constant function for $s \leq t$ and $\widehat{\mathbf{X}}$ is again a classical contact process. The process $\widehat{\mathbf{X}}$ which satisfies this "self" duality with respect to function $H(A, B) := \mathbb{1}_{\{A\cap B\neq\emptyset\}}$ is obtained by the following construction: Consider the graphical representation backwards in time and reverse the infection arrows. The recovery symbols stay as they are. See Figure 5.1 for a visualization. In case of the classical contact process $\mathbf{X}$ (see Remark 2.3.2), duality is a powerful tool to analyse its invariant laws. It can in particular be used to provide a connection between the survival probability and the upper invariant law.

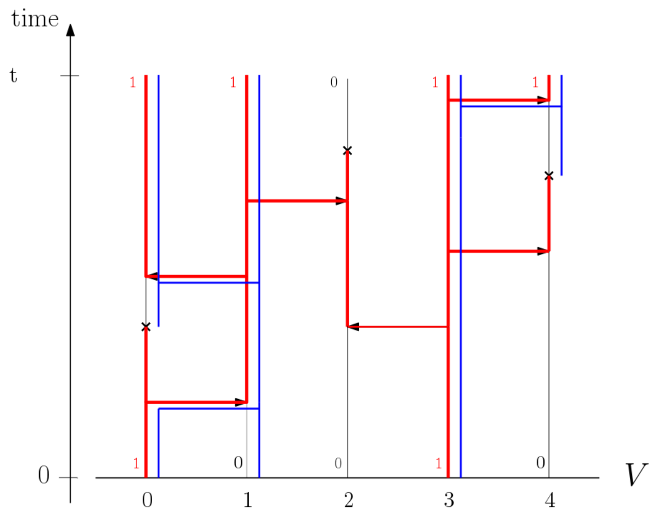


Figure 5.1: The red lines indicate the original infection paths in the construction of $\mathbf{X}$. The blue lines indicate the infection paths which result from considering the graphical representation backward in time. The blue paths define the dual process $\widehat{\mathbf{X}}$, which also runs backwards in time.

We are not able to construct a dual process for $(\mathbf{C}, \mathbf{B})$ in this manner. But if we first fix the background $\mathbf{B}$ in the time interval $[0, t]$, we can construct a process $\widehat{\mathbf{C}}$ which satisfies a conditional duality relation with respect to $\mathbf{C}$, i.e.

$$\mathbb{P}(\mathbf{C}_t^{C,B} \cap A \neq \emptyset | \mathcal{G}) = \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_{t-s}^{A,B,t} \neq \emptyset | \mathcal{G}) = \mathbb{P}(C \cap \widehat{\mathbf{C}}_t^{A,B,t} \neq \emptyset | \mathcal{G}) \tag{5.1}$$

holds almost surely for all $s \leq t$, where $\mathcal{G} := \sigma(\mathbf{B}_s : 0 \leq s \leq t)$ is the σ-algebra generated from the background process until time t. Obviously $\widehat{\mathbf{C}}$ will in general not be CPERE, but this process will nevertheless prove useful.

Define $\widehat{\mathbf{B}}_s^{B,t} := \mathbf{B}_{(t-s)_-}^B$, i.e. fix the background, reverse the time flow and start at some fixed time $t > 0$. Now we define the dual process $(\widehat{\mathbf{C}}_s^{A,B,t})_{0 \leq s \leq t}$ with $\widehat{\mathbf{C}}_0^{A,B,t} = A$ as follows: We define this process analogously to $\mathbf{C}$ with the help of the graphical representation using the same infection and recovery events just backwards in time and the direction of the infection is reversed, i.e.

$$(u, \mathbf{coop}_{x,y}) \to (t-u, \mathbf{coop}_{y,x}) \text{ and } (u, \mathbf{rec}_x) \to (t-u, \mathbf{rec}_x),$$

where $x, y \in V$ such that $\{x, y\} \in E$. Note that the superscript B does not denote the initial configuration of the time reversed background $\widehat{\mathbf{B}}$ but of the original $\mathbf{B}$. Now we just let the infection run backwards in time, starting at time t till time 0. See

Figure 5.2 for a visualization of the construction. We see that we coupled $\widehat{\mathbf{C}}$ to $\mathbf{C}$ in such a way that (5.1) holds.

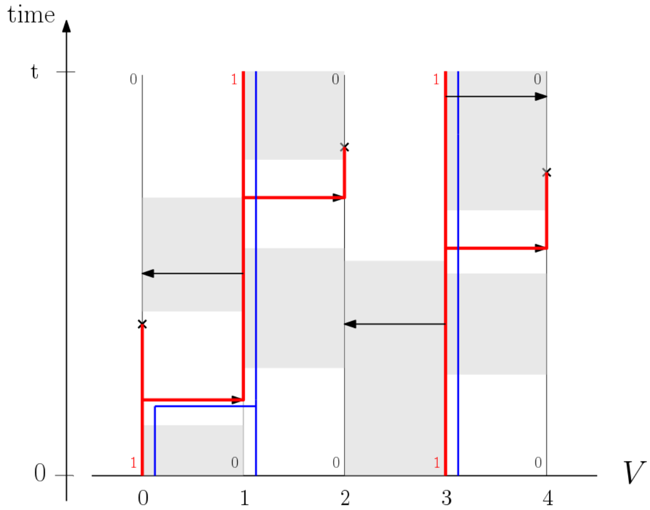


Figure 5.2: The arrows and crosses are respectively the infection and recovery events. The grey areas are the blocked edges. Thus, if an arrow is contained in a grey area it is not considered. The red lines are the infection paths of the forward-time process $\mathbf{C}$. The blue lines are the infections backwards in time with respect to the mirrored arrows, which define the process $\widehat{\mathbf{C}}$.

Next we show amongst other things that we can recover a self duality in the case where we assume stationarity of $\mathbf{B}$, i.e. $\mathbf{B}_0 \sim \pi$.

Proposition 5.1.2 (Distributional duality)**.** *Let $t \geq 0$, $A, C \subseteq V$ and $B, H \subset E$ then*

$$s \mapsto \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_{t-s}^{A,B,t} \neq \emptyset, \mathbf{B}_t^B \cap H \neq \emptyset) \quad \textit{and} \quad s \mapsto \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_{t-s}^{A,B,t} \neq \emptyset)$$

are constant functions. If $\mathbf{B}$ is reversible this implies in particular that for all $t \geq 0$

$$\mathbb{P}^{(C,\pi)}(\mathbf{C}_t \cap A \neq \emptyset) = \mathbb{P}^{(A,\pi)}(\mathbf{C}_t \cap C \neq \emptyset).$$

Proof. Let $t \geq 0$. By using (5.1) we see that

$$\begin{aligned}
\mathbb{P}(\mathbf{C}_t^{C,B} \cap A \neq \emptyset, \mathbf{B}_t^B \cap H \neq \emptyset) &= \mathbb{E}[\mathbb{P}(\mathbf{C}_t^{C,B} \cap A \neq \emptyset | \mathcal{G}) \mathbb{1}_{\{\mathbf{B}_t^B \cap H \neq \emptyset\}}] \\
&= \mathbb{E}[\mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_{t-s}^{A,B,t} \neq \emptyset | \mathcal{G}) \mathbb{1}_{\{\mathbf{B}_t^B \cap H \neq \emptyset\}}] \\
&= \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_{t-s}^{A,B,t} \neq \emptyset, \mathbf{B}_t^B \cap H \neq \emptyset)
\end{aligned}$$

for all $s \leq t$. The equality $\mathbb{P}(\mathbf{C}_t^{C,B} \cap A \neq \emptyset) = \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_{t-s}^{A,B,t} \neq \emptyset)$ for all $s \leq t$ follows by taking expectations in (5.1), which proves the first claim. For the second claim choose $s = 0$ and integrate both sides with respect to π, and thus

$$\int \mathbb{P}^{(C,B)}(\mathbf{C}_t \cap A \neq \emptyset)\pi(\mathsf{d}B) = \int \mathbb{P}(\widehat{\mathbf{C}}_t^{A,B,t} \cap C \neq \emptyset)\pi(\mathsf{d}B). \tag{5.2}$$

We assumed that $\mathbf{B}$ is reversible with respect to its invariant law π. Let us consider $(\mathbf{B}_s)_{s\leq t}$ with $\mathbf{B}_0 \sim \pi$ and as before set $\widehat{\mathbf{B}}_s^{\pi,t} := \mathbf{B}_{(t-s)-}$ for $0 \leq s \leq t$, then by Proposition 2.1.8 it follows that $(\mathbf{B}_s)_{s\leq t} \stackrel{d}{=} (\widehat{\mathbf{B}}_s^{\pi,t})_{s\leq t}$. Again define by the reversed graphical representation $(\widehat{\mathbf{C}}_s^{A,\pi,t})_{s\leq t}$ with respect to the background $(\widehat{\mathbf{B}}_s^{\pi,t})_{s\leq t}$. Now the process $(\widehat{\mathbf{C}}_s^{A,\pi,t}, \widehat{\mathbf{B}}_s^{\pi,t})_{s\leq t}$ is again a CPERE with initial distribution $\delta_A \otimes \pi$. Hence, this fact together with (5.2) yields that

$$\mathbb{P}^{(C,\pi)}(\mathbf{C}_t \cap A \neq \emptyset) = \mathbb{P}^{(A,\pi)}(\mathbf{C}_t \cap C \neq \emptyset). \qquad \square$$

Now we study the upper invariant law $\overline{\nu}$ of $(\mathbf{C}, \mathbf{B})$. We start with the existence of such a law. Recall that we denoted by $T(t) = T_{\lambda,r}(t)$ the Feller semigroup corresponding to the CPERE $(\mathbf{C}, \mathbf{B})$ with parameters λ and r.

Proposition 5.1.3 (Upper invariant law)**.** *There exists a probability measure $\overline{\nu}$ such that $(\delta_V \otimes \delta_E)T(t) \Rightarrow \overline{\nu}$ as $t \to \infty$.*

Proof. Obviously it holds that $\mu := \delta_V \otimes \delta_E \succeq (\delta_V \otimes \delta_E)T(t) = \mu T(t)$ for all $t > 0$ and thus by Lemma 3.4.1, $\mu T(s) \succeq \mu T(t)T(s) = \mu T(t+s)$ for all $t, s \geq 0$, where we used the semigroup property. Next let f be an arbitrary bounded, measurable and monotone increasing function. Then by definition of the stochastic order it holds that

$$T(s)f(V,E) = \int f \mathsf{d}\mu T(s) \geq \int f \mathsf{d}\mu T(t+s) = T(t+s)f(V,E)$$

and thus, $s \mapsto T(s)f(V,E)$ is non-increasing, real-valued function and obviously bounded from below. This implies that $T(s)f(V,E)$ convergences as $s \to \infty$. Since this is the case for any measurable, increasing and bounded function and the set of these functions is dense in the set of all measurable and bounded functions we get weak convergence of $\mu T(s)$, which yields the claim. $\square$

Next we show two properties of the upper invariant law $\overline{\nu}$. The measure $\overline{\nu}$ derives its name from the first property.

Lemma 5.1.4. *Let $\overline{\nu} = \overline{\nu}_{\lambda,r}$ be the upper invariant law of the CPERE $(\mathbf{C}, \mathbf{B})$ with infection rate $\lambda > 0$ and recovery rate $r > 0$. Then we have:*

(*i*) *If ν is an invariant law of $(\mathbf{C}, \mathbf{B})$, then $\nu \preceq \overline{\nu}$.*

(*ii*) *If $\lambda_1 \leq \lambda_2$, then $\overline{\nu}_{\lambda_1,r} \preceq \overline{\nu}_{\lambda_2,r}$ and if $r_1 \geq r_2$ then $\overline{\nu}_{\lambda,r_1} \preceq \overline{\nu}_{\lambda,r_2}$.*

Proof. (*i*) Lemma 3.4.1 states that $(\mathbf{C}, \mathbf{B})$ is a monotone Feller process, this implies that for any invariant law ν holds that $\nu = \nu T(t) \preceq (\delta_V \otimes \delta_E)T(t) \Rightarrow \overline{\nu}$ as $t \to \infty$, where Proposition 5.1.3 provides the weak convergence.

(*ii*) Let μ be a probability distribution on $\mathcal{P}(V) \times \mathcal{P}(E)$. By Lemma 3.4.2 it follows that if $\lambda_1 \leq \lambda_2$ then $\mu T_{\lambda_1,r}(t) \preceq \mu T_{\lambda_2,r}(t)$ for all $t \geq 0$ and if $r_1 \geq r_2$ then $\mu T_{\lambda,r_1}(t) \preceq \mu T_{\lambda,r_2}(t)$. Thus, the claim follows by setting $\mu = \delta_V \otimes \delta_E$ and letting $t \to \infty$ by Proposition 5.1.3. □

We do not need to start the background with every edge in the open state, i.e. $\mathbf{B}_0 = E$, to have convergence towards the upper invariant law. As long as the initial distribution of the background dominates π stochastically, this is enough to ensure convergence towards $\overline{\nu}$.

Lemma 5.1.5. *Let μ be a probability measure with $\pi \preceq \mu$ then $(\delta_V \otimes \mu)T(t) \Rightarrow \overline{\nu}$ as $t \to \infty$.*

Proof. First of all it is clear that $\delta_V \otimes \pi \preceq \delta_V \otimes \mu$, and therefore

$$\lim_{t\to\infty} (\delta_V \otimes \pi)T(t) \preceq \lim_{t\to\infty} (\delta_V \otimes \mu)T(t).$$

if the limit exists. So its enough to prove convergence for $\pi = \mu$. Since π is the invariant law of the background and the infection process can only occupy fewer sites than all of V it follows that $(\delta_V \otimes \pi)T(s) \preceq (\delta_V \otimes \pi)$ for all $s \geq 0$ and by Lemma 3.4.1 we get that

$$(\delta_V \otimes \pi)T(t+s) \preceq (\delta_V \otimes \pi)T(t) \quad \text{for all } t, s \geq 0.$$

Again using the same procedure as in Proposition 5.1.3 we see that a measure ν' exists such that $(\delta_V \otimes \pi)T(t) \Rightarrow \nu'$ as $t \to \infty$. By Lemma 5.1.4 (*i*) we know that $\nu' \preceq \overline{\nu}$. This means that if we can show that $\overline{\nu} \preceq \nu'$ we are finished. By Assumption 1.4.1 (*i*) we know that π is the unique invariant law of $\mathbf{B}$. Thus, the second marginal of any invariant law of $(\mathbf{C},\mathbf{B})$ must be π. Therefore it is clear that for every invariant law ν, $\nu \preceq \delta_V \otimes \pi$ must hold. Therefore by monotonicity and stationarity we know that

$$\nu = \nu T(t) \preceq (\delta_V \otimes \pi)T(t) \Rightarrow \nu' \quad \text{as } t \to \infty.$$

Since this holds for any invariant law $\overline{\nu}$ it also holds for the upper invariant law $\nu = \overline{\nu}$. □

This enables us to uncover a connection between the survival probability θ^π of the infection process $\mathbf{C}$ started with stationary background and the upper invariant law $\overline{\nu}$ in the next result.

Proposition 5.1.6. *Let $C \subset V$ be finite, then*

$$\theta^\pi(C) = \mathbb{P}^{(C,\pi)}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0) = \overline{\nu}(\{A \subset \mathcal{P}(V) : C \cap A \neq \emptyset\} \times \mathcal{P}(E)),$$

and thus in particular $\theta^\pi(\lambda, r, \{x\}) > 0$ if and only if $\overline{\nu}_{\lambda,r} \neq \delta_\emptyset \otimes \pi$, where $x \in V$ is arbitrary.

Proof. By the self duality relation from Proposition 5.1.2 we get for $C \subset V$

$$\mathbb{P}^{(V,\pi)}(\mathbf{C}_t \cap C \neq \emptyset) = \mathbb{P}^{(C,\pi)}(\mathbf{C}_t \neq \emptyset) \to \mathbb{P}^{(C,\pi)}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0) \quad \text{as } t \to \infty,$$

where we used continuity of the probability measure. On the other hand, since C is finite we get

$$\mathbb{P}^{(V,\pi)}(\mathbf{C}_t \cap C \neq \emptyset) = \int \mathbb{1}_{\{A \cap C \neq \emptyset\}}(\delta_V \otimes \pi)T(t)(\mathsf{d}(A,B)) \to \int \mathbb{1}_{\{A \cap C \neq \emptyset\}}\overline{\nu}(\mathsf{d}(A,B))$$

as $t \to \infty$, where we used Lemma 5.1.5. Now we can conclude that

$$\overline{\nu}(\{A \subset V : A \cap C \neq \emptyset\} \times \mathcal{P}(E)) = \mathbb{P}^{(C,\pi)}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0). \tag{5.3}$$

which yields the first claim.

Next by Proposition 4.0.1 (iii) we know that $\theta^\pi(\{x\}) = \theta^\pi(\{y\})$ for all $x, y \in V$. This yields in particular that the second claim does not depend on the choice of x. Now choose $C = \{x\}$ for some $x \in V$. Suppose that $\theta^\pi(\{x\}) > 0$, then we see by (5.3) that

$$\overline{\nu}(\{(A,B) \in \mathcal{P}(V) \times \mathcal{P}(E) : x \in A\}) > 0. \tag{5.4}$$

This implies that $\overline{\nu} \neq \delta_{\emptyset} \otimes \pi$. For the converse direction we assume that $\theta^{\pi}(\{x\}) = 0$, and hence $\theta^{\pi}(\{y\}) = 0$ for all $y \in V$. Now we see by (5.3) that

$$\overline{\nu}(\{(A,B) \in \mathcal{P}(V) \times \mathcal{P}(E) : y \in A\}) = 0.$$

for all $y \in V$. Now let us consider the set $\mathcal{D} := \{(A,B) \in \mathcal{P}(V) \times \mathcal{P}(E) : A \neq \emptyset\}$. By using σ-subadditivity and (5.4) we see that

$$\overline{\nu}(\mathcal{D}) \leq \sum_{y \in V} \overline{\nu}(\{(A,B) \in \mathcal{P}(V) \times \mathcal{P}(E) : y \in A\}) = 0,$$

and thus it follows that $\overline{\nu} = \delta_{\emptyset} \otimes \pi$. This provides the second claim. □

This connection between the survival probability θ^{π} and the upper invariant law $\overline{\nu}$ already suggests that the parameter regime where the upper invariant law agrees with $\delta_{\emptyset} \otimes \pi$ is the same as the regime of almost certain extinction. Note that if $\overline{\nu} = \delta_{\emptyset} \otimes \pi$, then by Lemma 5.1.3 and Lemma 5.1.4(i) follows that the CPERE convergences weakly towards the measure $\delta_{\emptyset} \otimes \pi$ and if $\overline{\nu} \neq \delta_{\emptyset} \otimes \pi$ we already know that at least two distinct invariant laws exist, and therefore there are obviously infinitely many invariant laws. Thus, if $\overline{\nu}$ is trivial or non-trivival also determines if the system is ergodic or non-ergodic.

Now we show that the critical value $\lambda_c'(r)$ of the phase transition between triviality and non-triviality of the upper invariant law indeed agrees with the critical value for survival $\lambda_c^{\pi}(r)$, where the background is assumed to be stationary. If we additionally assume that $c_1\big(\lambda_c^{\pi}(r), \rho\big) > \kappa^{-1}\rho$, then we know that the critical infection rate of survival does not depend on the initial configuration.

Proof of Corollary 1.4.14. Let $r > 0$, then as a direct consequence of Proposition 5.1.6 follows that $\lambda_c'(r) = \lambda_c^{\pi}(r)$. If we assume additionally $c_1\big(\lambda_c^{\pi}(r), \rho\big) > \kappa^{-1}\rho$ by Corollary 1.4.9 follows that there exists a $\lambda_c(r)$ such that $\lambda_c(r) = \lambda_c(r, C, B)$ for every $C \subset V$ non-empty and finite and every $B \subset E$, and thus in particular $\lambda_c'(r) = \lambda_c(r)$. □

For the remainder of this section we provide some ground work for the subsequent sections which consider complete convergence and continuity properties of the survival probability.

Proposition 5.1.7. *The measure $\overline{\nu}$ has the property that $\overline{\nu}(\{\emptyset\} \times \mathcal{P}(E)) \in \{0,1\}$.*

Proof. Let $D := \{\emptyset\} \times \mathcal{P}(E)$. If $\overline{\nu}(D) = 1$ then it follows that $\overline{\nu} = \delta_\emptyset \otimes \pi$. Thus, we assume that $\overline{\nu}(D^c) = q \in (0,1]$. Recall that the second marginal of $\overline{\nu}$ is π. Now define $\nu(\cdot) := \overline{\nu}(\cdot | D^c)$ and write

$$\overline{\nu} = q\nu + (1-q)(\delta_\emptyset \otimes \pi).$$

This equality together with the fact that $\delta_\emptyset \otimes \pi$ and $\overline{\nu}$ are invariant measures implies that the measure ν is again invariant. Let $f : \mathcal{P}(V) \times \mathcal{P}(E) \to \mathbb{R}$ be a bounded, measurable and monotone increasing function. Then $\delta_\emptyset \otimes \pi \preceq \overline{\nu}$ implies that

$$\int f \mathsf{d}\overline{\nu} = q \int f \mathsf{d}\nu + (1-q) \int f \mathsf{d}(\delta_\emptyset \otimes \pi) \leq q \int f \mathsf{d}\nu + (1-q) \int f \mathsf{d}\overline{\nu},$$

and therefore $q \int f \mathsf{d}\overline{\nu} \leq q \int f \mathsf{d}\nu$. Since $q > 0$, this implies that for all such functions $\int f \mathsf{d}\overline{\nu} \leq \int f \mathsf{d}\nu$ which yields that $\overline{\nu} \preceq \nu$. On the other hand since we know that $\overline{\nu}$ is the upper invariant law by Lemma 5.1.4 (ii) it follows that $\nu \preceq \overline{\nu}$, and thus $\nu = \overline{\nu}$. But this implies that $\overline{\nu}(D^c) = 1$, and therefore $\overline{\nu}(D) = 0$. □

A consequence of this proposition is that if $\overline{\nu} \neq \delta_\emptyset \otimes \pi$, then

$$\begin{aligned} \lim_{n\to\infty} \theta^\pi(\mathbb{B}_n(x)) &= \lim_{n\to\infty} \overline{\nu}(\{A \subset V : A \cap \mathbb{B}_n(x) \neq \emptyset\} \times \mathcal{P}(E)) \\ &= \overline{\nu}(\{A \subset V : A \neq \emptyset\} \times \mathcal{P}(E)) = 1, \end{aligned} \tag{5.5}$$

where we used Proposition 5.1.6 in the first equality and $x \in V$. We want to extend this result to

$$\lim_{n\to\infty} \theta(\mathbb{B}_n(x), \emptyset) = 1.$$

Recall that $\mathbf{B}$ is an autonomous Feller process. Thus, we denote by $(S(t))_{t\geq 0}$ the Feller semigroup associated with the background process. Let $s > 0$, then we set $\pi_s := \delta_\emptyset S(s)$ and

$$\theta^{\pi_s}(C) := \int \mathbb{P}(\mathbf{C}_t^{C,B} \neq \emptyset \ \forall t \geq 0) \pi_s(\mathsf{d}B).$$

By Assumption 1.4.1 (i) there exists a unique invariant law π of the background process $\mathbf{B}$ such that $\pi_s \Rightarrow \pi$ as $s \to \infty$. Recall that $\widetilde{\mathbf{C}}$ denotes a classical contact process with infection rate $\lambda > 0$ without recovery, i.e. only infection arrows are taken into account and the background as well as recovery symbols are completely ignored.

Lemma 5.1.8. *Let $t > 0$, $\varepsilon > 0$ and $A \subset V$ finite. Then there exists a finite $D = D(t, \varepsilon, A) \subset V$ such that*

$$\mathbb{P}(\widetilde{\mathbf{C}}_t^A \subset D) > 1 - \varepsilon.$$

Proof. Let $t > 0$ and $A \subset V$ finite and fixed. We know that for every finite initial configuration A the random set $|\widetilde{\mathbf{C}}_t^A| < \infty$ almost surely. This implies that for some $x \in A$,

$$\mathbb{P}\big(\exists n \geq 1 : \widetilde{\mathbf{C}}_t^A \subset \mathbb{B}_n(x)\big) = 1.$$

Thus, since $\{\widetilde{\mathbf{C}}_t^A \subset \mathbb{B}_n(x)\} \subset \{\widetilde{\mathbf{C}}_t^A \subset \mathbb{B}_m(x)\}$ if $m \geq n$ and because of continuity of $\mathbb{P}$, it follows that for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\mathbb{P}\big(\widetilde{\mathbf{C}}_t^A \subset \mathbb{B}_n(x)\big) > 1 - \varepsilon$$

for all $n > N$, which proves the claim. □

Recall that $\mathbb{B}_n^L(e)$ denotes the ball in the line graph $L(G)$ of radius $n \in \mathbb{N}$ with $e \in E$ as centre.

Lemma 5.1.9. *Let $e \in E$ and $k \in \mathbb{N}$. There exists a probability law μ_s on $\mathcal{P}(E^2)$ with marginals π and π_s such that for every $\varepsilon > 0$ there exists a $s > 0$ such that*

$$\mu_s\big(\{(B, D) \in E^2 : B \cap \mathbb{B}_k^L(e) = D \cap \mathbb{B}_k^L(e)\}\big) > 1 - \varepsilon.$$

Proof. Let $\mathbf{B}^\pi$ be the background process such that $\mathbf{B}_0^\pi \sim \pi$. Now let $\mathbf{B}^\pi$ be coupled to $\mathbf{B}^\emptyset$ via the graphical representation. Recall that the coupled region was defined as follows:

$$\Psi_t = \{e \in E : e \notin \mathbf{B}_t^{B_1} \triangle \mathbf{B}_t^{B_2} \ \forall B_1, B_2 \subset E\}.$$

Choose $c > 0$ such that $c\kappa > \rho$. By Theorem 4.1.3 we know that

$$\mathbb{P}(\exists s \geq 0 : \mathbb{B}_{t+1}^L(e) \subset \Psi_{ct} \ \forall t \geq s) = 1.$$

By continuity of the law $\mathbb{P}$ and monotonicity of the event, there exists an $s > k$ such that $\mathbb{P}(\mathbb{B}_{t+1}^L(e) \subset \Psi_{ct} \ \forall t \geq s) > 1 - \varepsilon$, which in particular implies that

$$\mathbb{P}(\mathbf{B}_{cs}^\pi \cap \mathbb{B}_k^L(e) = \mathbf{B}_{cs}^\emptyset \cap \mathbb{B}_k^L(e)) > 1 - \varepsilon.$$

Now set $s' = cs$ and let $\mu_{s'}$ be the joint probability distribution of $(\mathbf{B}_{s'}^\pi, \mathbf{B}_{s'}^\emptyset)$. This distribution satisfies the claim. □

With these two lemmas we are able to show the following useful approximation result of the survival probability. Recall that $c_1(\lambda, \rho)$ is the solution of (1.5), κ is the constant from Assumption 1.4.1 *(ii)* and ρ denotes the exponential growth of the graph G.

Lemma 5.1.10. *Let $\lambda, r > 0$ and suppose that $c_1(\lambda, \rho) > \kappa^{-1}\rho$. Then for any $C \subset V$,*

$$\lim_{s\to\infty} \theta^{\pi_s}(\lambda, r, C) = \theta^{\pi}(\lambda, r, C).$$

Proof. Note that if $|C| = \infty$ or $C = \emptyset$ the statement is trivial, since either both sides are 1 or 0. Thus, we assume that C is a finite non-empty subset of V. Fix $x \in C$ and $y \in \mathcal{N}_x$. Since $c_1(\lambda, \rho) > \kappa^{-1}\rho$ by Proposition 4.1.4 we know that

$$\mathbb{P}(\exists u \geq 0 : \widetilde{\mathbf{C}}_t^C \subset \Phi_t \ \forall t \geq u) = 1.$$

Set $A_u^1(C) := \{\widetilde{\mathbf{C}}_t^C \subseteq \Phi_t \ \forall t \geq u\}$. We see that for every $\varepsilon > 0$ there exists a $T > 0$ such that $\mathbb{P}(A_u^1(C)) \geq 1 - \varepsilon$ for all $u \geq T$, where we used that $A_u^1(C) \subset A_{u'}^1(C)$ for $u \leq u'$ and continuity of the law $\mathbb{P}$. Next we fix $u \geq T$ and define $A_{u,m}^2(C) := \{\widetilde{\mathbf{C}}_t^C \subset \mathbb{B}_m(x) \ \forall t \leq u\}$ for $m \in \mathbb{N}$. By Lemma 5.1.8 we can choose a $m = m(u)$ large enough such that $\mathbb{P}(A_{u,m}^2(C)) > 1 - \varepsilon$. Together this yields

$$\theta(C, B) \leq \mathbb{P}\big(A_u^1(C) \cap A_{u,m}^2(C) \cap \{\mathbf{C}_t^{C,B} \neq \emptyset \ \forall t \geq 0\}\big) + 2\varepsilon \tag{5.6}$$

for any $B \subset E$. By Lemma 3.2.5 we can choose a $k = k(m) > m + 1$ large enough such that

$$\mathbb{P}(\mathbf{B}_t^B \cap \mathbb{B}_{m+1}^L(\{x, y\}) = \mathbf{B}_t^D \cap \mathbb{B}_{m+1}^L(\{x, y\}) \ \forall t \leq u) > 1 - \varepsilon, \tag{5.7}$$

for any $D \subset E$ with $B \cap \mathbb{B}_k^L(\{x, y\}) = D \cap \mathbb{B}_k^L(\{x, y\})$. Note that $\mathbb{B}_{m+1}^L(\{x, y\})$ contains in particular all edges which are attached to all vertices in $\mathbb{B}_m(x)$. Now for notational convenience define $A_{u,m}^3(C) := A_u^1(C) \cap A_{u,m}^2(C)$. Furthermore set

$$\begin{aligned} A_{m,u}(C, (B, D)) :=& \{\mathbf{B}_t^B \cap \mathbb{B}_{m+1}^L(\{x, y\}) = \mathbf{B}_t^D \cap \mathbb{B}_{m+1}^L(\{x, y\}) \ \forall t \leq u\} \cap A_{u,m}^3(C), \\ E_k(B, D) :=& \{(B, D) \in E^2 : B \cap \mathbb{B}_k^L(\{x, y\}) = D \cap \mathbb{B}_k^L(\{x, y\})\}. \end{aligned}$$

By Lemma 5.1.9 there exists a distribution μ_s on $\mathcal{P}(E^2)$ with marginals π and π_s, such that for $s > 0$ large enough

$$\mu_s\big(E_k(B, D)\big) > 1 - \varepsilon. \tag{5.8}$$

Note that by choice of these events

$$\begin{aligned} &A_{m,u}(C, (B, D)) \cap \{\mathbf{C}_t^{C,B} \neq \emptyset \ \forall t \geq 0\} \\ =&A_{m,u}(C, (B, D)) \cap \{\mathbf{C}_t^{C,D} \neq \emptyset \ \forall t \geq 0\} \subset \{\mathbf{C}_t^{C,D} \neq \emptyset \ \forall t \geq 0\}, \end{aligned} \tag{5.9}$$

since on the event $A_{m,u}(C,(B,D))$ the infection stays in $\mathbb{B}_m(x)$ until time u and afterwards only travels along edges already contained in the permanently coupled region. But, for any of the initial configuration B or D the background does not differ in the ball $\mathbb{B}^L_{m+1}(\{x,y\})$ at any time $t \in [0,u]$ and thus, we can interchange B and D on $A_{m,u}(C,(B,D))$. Finally we can conclude that

$$\begin{aligned}
&\int \mathbb{P}\big(A^1_u(C) \cap A^2_{u,m}(C) \cap \{\mathbf{C}^{C,B}_t \neq \emptyset\ \forall t \geq 0\}\big)\pi(\mathsf{d}B) \\
\leq &\int \mathbb{P}\big(A_{m,u}(C,(B,D)) \cap \{\mathbf{C}^{C,B}_t \neq \emptyset\ \forall t \geq 0\}\big)\mathbb{1}_{E_k(B,D)}\mu_s(\mathsf{d}(B,D)) + 2\varepsilon \qquad (5.10)\\
\leq &\int \mathbb{P}\big(\mathbf{C}^{C,D}_t \neq \emptyset\ \forall t \geq 0\}\big)\pi_s(\mathsf{d}D) + 2\varepsilon,
\end{aligned}$$

where we used (5.7) and (5.8) in the first inequality and in the second the definition of $E_k(B,D)$ together with (5.9). Hence, by combining (5.6) and (5.10) we obtain

$$\theta^{\pi}(C) \leq \theta^{\pi_s}(C) + 4\varepsilon.$$

On the other hand we have that $\pi_s = \delta_{\emptyset}S(s)$. Since **B** is by assumption a monotone Feller process we get that $\pi_s \preceq \pi$ for all $s \geq 0$, and thus by monotonicity of the survival probability it follows that

$$\theta^{\pi_s}(C) \leq \theta^{\pi}(C) \leq \theta^{\pi_s}(C) + 4\varepsilon,$$

which proves the claim. □

With this approximation result we are able to show the desired result.

Lemma 5.1.11. *Let $x \in V$ and $r > 0$. Suppose that $c_1(\lambda^{\pi}_c(r),\rho) > \kappa^{-1}\rho$, then for all $\lambda > \lambda_c(r) = \lambda^{\pi}_c(r)$*

$$\lim_{n\to\infty} \theta(\lambda, r, \mathbb{B}_n(x), \emptyset) = 1.$$

Proof. Let us fix $x \in V$. By Lemma 4.1.1 we know that $\lambda \mapsto c_1(\lambda,\rho)$ is continuous and strictly decreasing. Thus, if $c_1(\lambda^{\pi}(r),\rho) > \kappa^{-1}\rho$, then there exists an $\varepsilon' > 0$ such that $c_1(\lambda,\rho) > \kappa^{-1}\rho$ for all $\lambda \in (\lambda^{\pi}_c(r), \lambda^{\pi}_c(r) + \varepsilon')$. Note that by Proposition 1.4.14 $\lambda^{\pi}_c(r) = \lambda_c(r)$. Let $n \geq 0$ and fix $\lambda \in (\lambda^{\pi}_c(r), \lambda^{\pi}_c(r) + \varepsilon')$ by (5.5) we know that for every $\varepsilon > 0$ there exists n large enough such that $\theta^{\pi}(\mathbb{B}_n(x)) > 1 - \varepsilon$ and by Lemma 5.1.10 we know that for given n and ε there exist $s > 0$ large enough such that

$$\theta^{\pi_s}(\mathbb{B}_n(x)) > 1 - \varepsilon. \qquad (5.11)$$

Choose a set $\{x_i : i \in \mathbb{N}\} \subset V$ such that $d(x_i, x_j) > 2n$ for $i \neq j$. Note that by this choice the sets $(\mathbb{B}_n(x_i))_{i\in\mathbb{N}}$ are disjoint. Let us consider the event

$$A^s_{m,n} := \{\exists i \le m : (t, \mathbf{rec}_x) \notin \operatorname{supp}(\Xi^{\text{rec}}) \ \forall (t,x) \in [0,s] \times \mathbb{B}_n(x_i)\},$$

i.e. in words for some $i \le m$ no recovery symbols occurs up to time s in $\mathbb{B}_n(x_i)$. For given s and n choose m large enough such that

$$\mathbb{P}(A^s_{m,n}) > 1 - \varepsilon. \tag{5.12}$$

Let $k = k(m,n)$ be large enough such that $\bigcup_{i=1}^m \mathbb{B}_n(x_i) \subset \mathbb{B}_k(x)$. Now by the choice of s it follows that

$$\mathbb{P}(\mathbf{C}_t^{\mathbb{B}_k(x),\emptyset} \neq \emptyset \ \forall t \ge 0 | A^s_{m,n}) \ge \mathbb{P}^{(\mathbb{B}_n(x),\pi_s)}(\mathbf{C}_t \neq \emptyset \ \forall t \ge 0) = \theta^{\pi_s}(\mathbb{B}_n(x)) \tag{5.13}$$

where we used the translation invariance of $(\mathbf{C}, \mathbf{B})$ and that $A^s_{m,n}$ is independent of the background. Now by (5.11), (5.12) and (5.13) we get that

$$\theta(\mathbb{B}_k(x), \emptyset) \ge \mathbb{P}(\mathbf{C}_t^{\mathbb{B}_k(x),\emptyset} \neq \emptyset \ \forall t \ge 0 | A^s_{m,n})\mathbb{P}(A^s_{m,n}) \ge \theta^{\pi_s}(\mathbb{B}_n(x))(1-\varepsilon) \ge (1-\varepsilon)^2,$$

which yields that $\lim_{n\to\infty} \theta(\lambda, r, \mathbb{B}_n(x), \emptyset) = 1$, for all $\lambda \in (\lambda_c(r), \lambda_c(r) + \varepsilon)$. Since $\lambda \mapsto c_1(\lambda, \rho)$ is strictly decreasing it is possible that there exists $\lambda' > \lambda$ such that $c_1(\lambda', \rho) > \kappa^{-1}\rho$ is no longer satisfied. In this case we can use monotonicity and see that

$$\lim_{n\to\infty} \theta(\lambda', r, \mathbb{B}_n(x), \emptyset) \ge \lim_{n\to\infty} \theta(\lambda, r, \mathbb{B}_n(x), \emptyset) = 1. \qquad \square$$

This result actually plays a key role for some of the continuity properties concerning the survival probability with respect to the infection and recovery rate. It seems appropriate to mention here that there is a different way to prove Lemma 5.1.11 without relying on the duality and hence it would be possible to drop Assumption 1.4.1 (*iii*), i.e. reversibility of the background, in this particular case. Therefore this might be relevant for further analysis of the CPERE with a non-reversible background. Of course this comes with the price of posing some different assumptions on the graph G.

This proof strategy uses ergodicity theory. Hence, we will clarify some notions and objects. For details we refer the interested reader to [Kal06, Chapter 9]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $S : \Omega \mapsto \Omega$ be a measure-preserving map, i.e. $\mathbb{P}^S = \mathbb{P}$. We denote by $\mathcal{I} = \{A \in \mathcal{F} : A = S^{-1}(A)\}$ the invariant σ-algebra. We call the 4-tupel

$(\Omega, \mathcal{F}, \mathbb{P}, S)$ an ergodic system if $\mathcal{I}$ is $\mathbb{P}$-trivial, i.e. if $A \in \mathcal{I}$, then $\mathbb{P}(A) \in \{0,1\}$. Let X be the identity on Ω, i.e. $X(\omega) = \omega$, if $(\Omega, \mathcal{F}, \mathbb{P}, S)$ is ergodic then we call (X, S) ergodic. Let $f : \Omega \to \mathbb{R}$ be a measurable function. The mean ergodic theorem of Birkhoff, see [Kal06, Theorem 9.6], in particular states that if (X, S) is ergodic, then

$$\frac{1}{2n} \sum_{k=-n}^{n} f(S^{-k}X) \to \mathbb{E}[f(X)] \quad \text{as } n \to \infty.$$

Now we will briefly summarize the proof strategy.

Remark 5.1.12. Let us consider the special setting of the 1-dimensional integer lattice, i.e. $V = \mathbb{Z}$ and $E = \{\{x,y\} \subset V : |x-y| = 1\}$. Now define $Y_x := \mathbb{1}_{\{\mathbf{C}_t^{\{x\},\emptyset} \neq \emptyset \ \forall t \geq 0\}}$ to be the indicator variable for the event of survival in case the process starts only with site x being infected and all edges closed.

Recall that we constructed $(\mathbf{C}, \mathbf{B})$ via a graphical construction with respect to a Poisson point proces Ξ. From a different perspective Ξ can be seen as a family of independent Poisson process $(\Xi_z)_{z \in V \cup E}$ on $\mathbb{R}$, where $\Xi_x \stackrel{d}{=} \Xi_y$ for all $x, y \in V$ and $\Xi_e \stackrel{d}{=} \Xi_{e'}$ for all $e, e' \in E$. Let S be a shift operator which maps $\xi_x \to \xi_{x+1}$ and $\xi_{\{x-1,x\}} \to \xi_{\{x,x+1\}}$ for all $x \in V$, where $\xi = (\xi_z)_{z \in V \cup E}$ is a realization of Ξ. Now it is clear that the shift S is a measure preserving map with respect to the distribution of this family of Poisson processes, since it maps vertices to vertices and edges to edges. Furthermore, since the processes are all independent it follows immediately that (Ξ, S) is ergodic.

Now since $(\mathbf{C}, \mathbf{B})$ is constructed via the graphical construction we see that there must exist a measurable function f from the state space of Ξ to $\{0,1\}$ such that

$$f(S^{-k}(\Xi)) = \mathbb{1}_{\{\mathbf{C}_t^{\{k\},\emptyset} \neq \emptyset \ \forall t \geq 0\}} = Y_k$$

for every $k \in \mathbb{Z}$. Note that by translation invariance, $\mathbb{P}(Y_0 = 1) = \mathbb{P}(Y_x = 1)$ for all $x \in \mathbb{Z}$. Now if we assume that $\theta^{\pi}(\lambda, r, \{0\}) > 0$ we see that $\mathbb{P}(Y_0 = 1) > 0$ by Theorem 1.4.8. Then by Birkhoff's mean ergodic theorem [Kal06, Theorem 9.6], it follows that

$$\frac{1}{2n} \sum_{x=-n}^{n} Y_x = \frac{1}{2n} \sum_{x=-n}^{n} f(S^{-k}(\Xi)) \to \mathbb{E}[f(\Xi)] = \mathbb{P}^{(\{0\},\emptyset)}(\mathbf{C}_t \neq \emptyset \, \forall \, t \geq 0) > 0$$

almost surely. But this implies that almost surely there must exist a y for which $Y_y = 1$. Moreover, by additivity it follows that the event $\{\mathbf{C}_t^{\mathbb{B}_n(x),\emptyset} \neq \emptyset \ \forall t \geq 0\}$ occurs as soon as the event $\{Y_y = 1\}$ occurs for some site in $y \in \mathbb{B}_n(x)$ which proves the statement.

This approach can be adapted for more general graphs $G = (V, E)$. For example if we assume that V is a finitely generate group and $G = (V, E)$ the Cayley graph of V. Since V is equipped with a group action we can again define a shift operator S_x, which maps $y \mapsto y + x$, and thus adjust the proof with a multivariate version of the mean ergodic theorem. See for example [Kal06, Theorem 9.9]. Note that Cayley graphs are always vertex transitive, but not necessarily edge transitive.

5.2 Equivalent conditions for complete convergence

This section is dedicated to proving Theorem 1.4.15. Recall that $c_1(\lambda, \rho)$ is the solution of (1.5), κ is the constant from Assumption 1.4.1 (ii) and ρ denotes the exponential growth of the graph G. In this section we assume that $\lambda, r > 0$ and that the already familiar growth condition $c_1(\lambda, \rho) > \kappa^{-1}\rho$ is satisfied.

Therefore, the main goal is to show that the two conditions (1.8) and (1.9), which are

$$\mathbb{P}^{(C,B)}_{\lambda,r}(x \in \mathbf{C}_t \text{ i.o.}) = \theta(\lambda, r, C, B) \tag{5.14}$$

for all $x \in V$, $C \subset V$ and $B \subset E$ and

$$\lim_{n\to\infty} \limsup_{t\to\infty} \mathbb{P}_{\lambda,r}(\mathbf{C}_t^{\mathbb{B}_n(x),\emptyset} \cap \mathbb{B}_n(x) \neq \emptyset) = 1 \tag{5.15}$$

for any $x \in V$, are equivalent to complete convergence of the CPERE, i.e.

$$(\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \Rightarrow \theta(C, B)\overline{\nu} + [1 - \theta(C, B)](\delta_\emptyset \otimes \pi) \quad \text{ as } t \to \infty \tag{5.16}$$

for all $C \subset V$ and $B \subset E$. We first show convergence of the marginals $\mathbf{C}$ and $\mathbf{B}$ and then conclude that this already implies that the CPERE $(\mathbf{C}, \mathbf{B})$ convergences.

By Assumption 1.4.1 (i) we already know that $\mathbf{B}_t^B \Rightarrow \pi$ as $t \to \infty$ for all $B \subset E$. Hence it remains to show that the two conditions (5.14) and (5.15) imply that the infection process $\mathbf{C}$ convergences weakly as $t \to \infty$. We show that for any $C \subset V$ and $B \subset E$

$$\mathbb{P}_{\lambda,r}(\mathbf{C}_t^{C,B} \cap C' \neq \emptyset) \to \theta(\lambda, r, C, B)\theta^\pi(\lambda, r, C'), \tag{5.17}$$

as $t \to \infty$ for every $C' \subset V$ finite, which suffices to conclude weak convergence of the infection process $\mathbf{C}$ since the function class $\{\mathbb{1}_{\{\cdot \cap C' \neq \emptyset\}} : C' \subset V \text{ finite}\}$ is convergence determining. This actually turns out to be the major share of the workload. At last we show that the converse holds true as well, which provides that (5.16) implies (5.14) and (5.15). Once we know that the marginals converge we show that this already implies the convergence of the joint distribution, i.e. $(\mathbf{C}_t, \mathbf{B}_t)$ converges weakly as $t \to \infty$.

As the readers familiar with the classical contact process might know a similar result holds in the classical case as well. In fact the proof strategy to derive the equivalence of the two conditions and convergence of the first marginal is inspired by the proof for the classical contact process. The idea is basically the same, but since we introduced a background we lose some important properties for which we need to formulate a work around. Therefore, we briefly summarize the important points of this approach to give the reader more intuition before we start with the actual proofs.

As in Remark 2.3.2 we denote by $\mathbf{X}$ a classical contact process. In the beginning of Section 5.1 we already explained how to construct a dual process $\widehat{\mathbf{X}}^{t+s} = (\widehat{\mathbf{X}}^{t+s}_u)_{u \le t+s}$ for $\mathbf{X}$ such that $s \mapsto \mathbb{P}\big(\mathbf{X}_s \cap \widehat{\mathbf{X}}^t_{t-s} = \emptyset\big)$ is a constant function on $[0, t]$. It is not difficult to see that $(\mathbf{X}^C_u)_{u \le s}$ and the dual process $(\widehat{\mathbf{X}}^{C',t+s}_u)_{u \le t}$ are independent, since they are defined on disjoint sections of the graphical representation. Furthermore, it is also know that the dual process $\widehat{\mathbf{X}}$ has again the dynamics of a classical contact process. These facts can be used to conclude that

$$
\begin{aligned}
&\mathbb{P}\big(\mathbf{X}^C_{t+s} \cap C' \neq \emptyset\big) = \mathbb{P}(\mathbf{X}^C_s \cap \widehat{\mathbf{X}}^{C',t+s}_t \neq \emptyset\big) \\
=&\mathbb{P}\big(\mathbf{X}^C_s \neq \emptyset, \widehat{\mathbf{X}}^{C',t+s}_t \neq \emptyset\big) - \mathbb{P}\big(\mathbf{X}^C_s \neq \emptyset, \widehat{\mathbf{X}}^{C',t+s}_t \neq \emptyset, \mathbf{X}^C_s \cap \widehat{\mathbf{X}}^{C',t+s}_t = \emptyset\big) \\
=&\mathbb{P}\big(\mathbf{X}^C_s \neq \emptyset)\mathbb{P}(\mathbf{X}^{C'}_t \neq \emptyset\big) - \mathbb{P}\big(\mathbf{X}^C_s \neq \emptyset, \widehat{\mathbf{X}}^{C',t+s}_t \neq \emptyset, \mathbf{X}^C_s \cap \widehat{\mathbf{X}}^{C',t+s}_t = \emptyset\big).
\end{aligned}
$$

Now obviously $\mathbb{P}\big(\mathbf{X}^C_s \neq \emptyset\big)\mathbb{P}\big(\mathbf{X}^{C'}_t \neq \emptyset\big) \to \theta(C)\theta(C')$ as $s, t \to 0$, where $\theta(\cdot)$ denotes the survival probability of $\mathbf{X}$. Graphically, the event in the last term means that two independent contact processes which will not go extinct share no infected site after a long time. If the graph is "nice" enough, it seems reasonable to assume that this gets more unlikely as s, t grow larger such that

$$\mathbb{P}\big(\mathbf{X}^C_s \neq \emptyset, \widehat{\mathbf{X}}^{C',t+s}_t \neq \emptyset, \mathbf{X}^C_s \cap \widehat{\mathbf{X}}^{C',t+s}_t = \emptyset\big) \to 0.$$

as $s, t \to \infty$. Note that this property is somewhat similar to the second condition (5.15) and hence indicates its necessity.

The two major issues, or rather the two properties we do not have in our setting are:

1. For the CPERE the process $(\mathbf{C}^{C,B}_u)_{u \le s}$ and the dual process $\big(\widehat{\mathbf{C}}^{C',B,t+s}_u\big)_{u \le t}$ are not independent, since both processes depend on the background $(\mathbf{B}^B_u)_{u \le t+s}$.

2. For an arbitrary $B \subset E$ the process $\big(\widehat{\mathbf{C}}_u^{C',B,t+s}, \widehat{\mathbf{B}}_u^{B,t+s}\big)_{u \leq t+s}$ is not necessarily a CPERE again.

The approach to solve these two problems is to construct a process $\big(\check{\mathbf{C}}, \check{\mathbf{B}}\big)$ which satisfies these two properties and does not differ on a finite time horizon from $\big(\widehat{\mathbf{C}}, \widehat{\mathbf{B}}\big)$ with an arbitrarily high probability. This is possible since we know by Proposition 5.1.2 that if $\mathbf{B}_0 \sim \pi$, i.e. if we start in its invariant law, then the dual process $\big(\widehat{\mathbf{C}}, \widehat{\mathbf{B}}\big)$ is again a CPERE. So in case we do not start stationary, the idea is that we use the fact that the background couples itself faster than the infection can spread through the population, i.e.

$$\mathbb{P}\big(\exists s \geq 0 : \widetilde{\mathbf{C}}_t^C \subseteq \Phi_t \ \forall t \geq s\big) = 1,$$

where Φ_t denotes the set of all vertices whose attached edges are already permanently coupled at time t. This holds by Proposition 4.1.4 since we assumed $c_1(\lambda, \rho) > \kappa^{-1}\rho$. Thus, we can basically wait long enough for $\mathbf{B}$ to forget its initial configuration in the relevant area and restart the process in its invariant law.

Now we start by formulating this in a rigorous manner. For that we first introduce some shorthand notation to keep the formulas somewhat cleaner. For $A \subset V$ we set

$$\begin{aligned} A_E &:= \big\{\{x,y\} \in E : x \in A\big\}, \\ A^N &:= \bigcup_{x \in A} \mathbb{B}_N(x), \\ A_E^N &:= \big\{\{x,y\} \in E : x \in A^N\big\}, \end{aligned}$$

where $\mathbb{B}_N(x)$ is the ball with centre x and radius N with respect to the graph distance of G (see Section 2.4).

Let $(\check{\mathbf{B}}_r^{s/2})_{r \geq s/2}$ denote a process with same dynamics as the background process $\mathbf{B}$, which is coupled with the original background in such a way that it starts at time $s/2$ with an initial distribution π and is assumed to be independent from $(\mathbf{B}_r^B)_{r<s/2}$, but from $s/2$ onwards it uses the same graphical representation as $\mathbf{B}^B$. For a visualization see Figure 5.3.

Lemma 5.2.1. *Let $D \subset V$, $B \subset E$ be finite and fixed. Then for every $\varepsilon > 0$ there exists an $S > 0$ such that for all $s \geq S$*

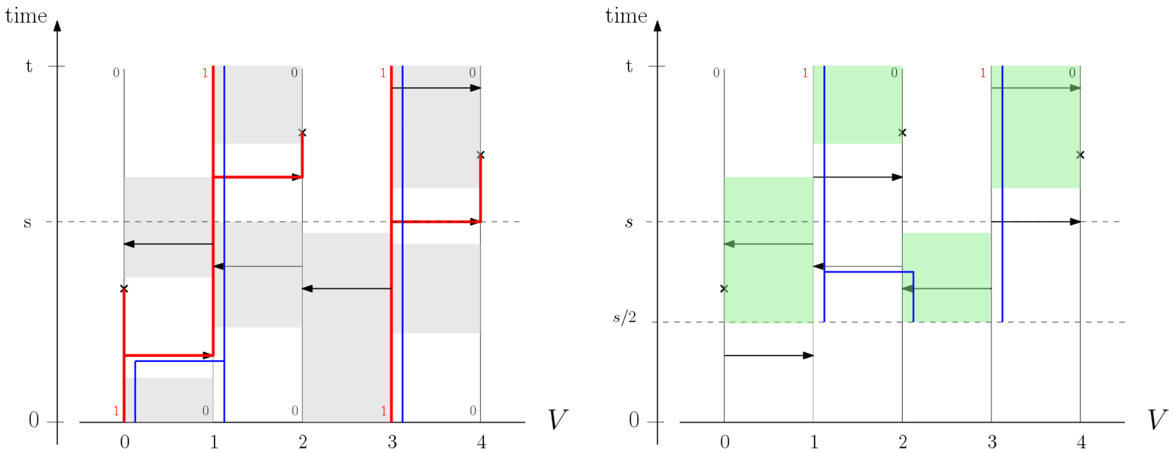
$$\mathbb{P}\big(\check{\mathbf{B}}_u^{s/2} \cap D_E = \mathbf{B}_u^B \cap D_E \ \forall u \geq s\big) > 1 - \varepsilon.$$

Proof. Let $x \in D$. Let $c > 0$ be chosen such that $c\kappa > \rho$, then by Proposition 4.1.3 we know that $\mathbb{P}(\exists s \geq 0 : \mathbb{B}_{\lfloor c^{-1}t \rfloor}(x) \subseteq \Phi_t \ \forall t \geq s) = 1$. Let $S' > 0$ be chosen such that $D \subset \mathbb{B}_{\lfloor c^{-1}t \rfloor}$ for all $t \geq S'/2$. By continuity of the measure $\mathbb{P}$, for every $\varepsilon > 0$ there exists a $S > S' > 0$ such that $\mathbb{P}\big(\mathbb{B}_{\lfloor c^{-1}t \rfloor}(x) \subseteq \Phi_t, \forall t \geq \frac{S}{2}\big) > 1 - \varepsilon$ then this already implies that for all $s \geq S$

$$\mathbb{P}(\check{\mathbf{B}}_u^{s/2} \cap D_E = \mathbf{B}_u^B \cap D_E \ \forall u \geq s) > 1 - \varepsilon. \qquad \square$$

Let $t, s > 0$ and recall the dual process $(\widehat{\mathbf{C}}_r^{A,B,t+s})_{r \leq t+s}$ of $(\mathbf{C}_r^{C,B})_{r \leq t+s}$. In the definition of the dual process we fixed the background $(\mathbf{B}_r^B)_{r \leq t+s}$, reversed the graphical representation with respect to the time axis at the time point $t + s$ and fixed A as the initial set of infected sites for the dual process.

Now let $(\check{\mathbf{C}}_u^{A,s/2,t+s})_{u \leq t+s/2}$ be a process coupled to $\widehat{\mathbf{C}}^{A,B,t+s}$ by using the same time-reversed infection arrows and recovery symbols, but the background at time $s/2$ (foward in time) is reset and independently drawn according to the law π, i.e. we use $(\check{\mathbf{B}}_r^{s/2})_{r \geq s/2}$ instead of $(\mathbf{B}_r^B)_{r \geq s/2}$. Again see Figure 5.3 for a illustration.



(a) Visualization of the graphical representation of $(\mathbf{C}_u, \mathbf{B}_u)_{u \leq t+s}$ and the dual $(\widehat{\mathbf{C}}_u^{t+s})_{u \leq t+s}$.

(b) Visualization of the graphical representation of $(\check{\mathbf{B}}_u^{s/2})_{s/2 \leq u \leq t}$ and $(\check{\mathbf{C}}_u^{s/2,t+s})_{u \leq t+s/2}$.

Figure 5.3: As usual the arrows and crosses denote the infection and recovery symbols. The grey area visualize the closed edges according to $\mathbf{B}$ (left picture) and the light green areas the closed edges according to $\check{\mathbf{B}}$ (right picture). The red line visualizes the infection path forward in time, i.e. $\mathbf{C}$, and the blue line the infection path backward in time, i.e. $\widehat{\mathbf{C}}$ in the left and $\check{\mathbf{C}}$ on the right.

Lemma 5.2.2. *Let $t > 0$, $A \subset V$ be finite and $B \subset E$. Then for every $\varepsilon > 0$ there exists an $S > 0$ such that for all $s > S$,*

$$\mathbb{P}(\widehat{\mathbf{C}}_u^{A,B,t+s} = \check{\mathbf{C}}_u^{A,s/2,t+s}\ \forall u \leq t) > 1 - \varepsilon.$$

Proof. First, by Lemma 5.1.8 we know that for every $\varepsilon_1 > 0$ there exists a finite $D = D(t, \varepsilon_1, A) \subset V$ such that

$$\mathbb{P}(\widehat{\mathbf{C}}_u^{A,B,t+s}, \check{\mathbf{C}}_u^{A,s/2,t+s} \subset D\ \forall u \leq t) > 1 - \varepsilon_1.$$

Now for D given via Lemma 5.2.1 we obtain that for every $\varepsilon_2 > 0$ there exists an $S > 0$ such that for every $s > S$

$$\mathbb{P}(\check{\mathbf{B}}_u^{s/2} \cap D_E = \mathbf{B}_u^B \cap D_E\ \forall u \geq s) > 1 - \varepsilon_2.$$

Recall that $D_E \subset E$ was the set which contains every edge attached to D. But now we see that

$$\begin{aligned}\{\widehat{\mathbf{C}}_u^{A,B,t+s}, \check{\mathbf{C}}_u^{A,s/2,t+s} \subset D\ \forall u \leq t\} \cap \{\check{\mathbf{B}}_u^{s/2} \cap D_E = \mathbf{B}_u^B \cap D_E\ \forall u \geq s\}\\ \subseteq \{\widehat{\mathbf{C}}_u^{A,B,t+s} = \check{\mathbf{C}}_u^{A,s/2,t+s}\ \forall u \leq t\}.\end{aligned}$$

But by choosing $\varepsilon_1, \varepsilon_2 \leq \frac{\varepsilon}{2}$ we see that $\mathbb{P}(\widehat{\mathbf{C}}_u^{A,B,t+s} \neq \check{\mathbf{C}}_u^{A,s/2,t+s}\ \forall u \leq t) \leq \varepsilon$, which yields the claim. □

With Lemma 5.2.1 and Lemma 5.2.2 we formalized what we before described loosely as $(\check{\mathbf{C}}, \check{\mathbf{B}})$ not differing from $(\widehat{\mathbf{C}}, \widehat{\mathbf{B}})$ with an arbitrarily high probability. Now we can begin to show the convergence of the first marginal. We will split this in two steps by first proving an upper bound and in the second step we use (5.14) and (5.15) to show that this upper bound also acts as a lower bound which provides the desired result.

Proposition 5.2.3. *Let $t, s > 0$, $C, C' \subset V$ with C' being finite and $B \subset E$, then for every $\varepsilon > 0$ there exist $S, T > 0$ such that*

$$\mathbb{P}_{\lambda,r}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) \leq \theta(\lambda, r, C, B)\theta^{\pi}(\lambda, r, C') + \varepsilon$$

for all $s > S$, $t > T$. This implies in particular for any finite $C' \subset V$

$$\limsup_{t \to \infty} \mathbb{P}_{\lambda,r}(\mathbf{C}_t^{C,B} \cap C' \neq \emptyset) \leq \theta(\lambda, r, C, B)\theta^{\pi}(\lambda, r, C').$$

Proof. By Proposition 5.1.2 it follows that

$$\mathbb{P}(\mathbf{C}_{t+s}^{C,B} \cap C' \neq \emptyset) = \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) \leq \mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset).$$

Thus, it suffices to show that for every $\varepsilon > 0$ there exist $S, T > 0$ such that

$$\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) \leq \theta(C,B)\theta^{\pi}(C') + \varepsilon$$

for all $s > S$ and $t > T$. We denote the extinction time of the infection process $\mathbf{C}$ by

$$\tau_{ex} = \tau_{ex}(C,B) := \inf\{t > 0 : \mathbf{C}_t^{C,B} = \emptyset\}.$$

First we observe that for $C' \subset V$ finite that $\mathbb{P}^{(C',\pi)}(\tau_{ex} > t) \to \theta^{\pi}(C')$ as $t \to \infty$. Thus, for every $\varepsilon > 0$ there exists a $T > 0$ such that $|\mathbb{P}^{(C',\pi)}(\tau_{ex} > t) - \theta^{\pi}(C')| < \varepsilon$ for all $t > T$. So we fix t such that this is satisfied. Note that

$$\mathbb{P}^{(C,B)}(u < \tau_{ex} < \infty) = 1 - \mathbb{P}^{(C,B)}(\tau_{ex} \leq u) - \mathbb{P}^{(C,B)}(\tau_{ex} = \infty),$$

and thus it follows that $\lim_{u\to\infty} \mathbb{P}^{(C,B)}(u < \tau_{ex} < \infty) = 0$. Now we can use that $\{\mathbf{C}_s^{C,B} \neq \emptyset\} = \{\tau_{ex} > s\}$ to see that for every $\varepsilon > 0$ there exists an $S_1 > 0$ such that

$$|\mathbb{P}(\tau_{ex} > s/2, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) - \mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset)| \leq \mathbb{P}(s/2 < \tau_{ex} < \infty) < \varepsilon, \tag{5.18}$$

for all $s > S_1$, which implies that

$$\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) \leq \mathbb{P}(\tau_{ex} > s/2, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) + \varepsilon.$$

Applying Lemma 5.2.2 yields that for given $\varepsilon > 0$ there exists $S_2 > 0$ such that

$$\mathbb{P}(\widehat{\mathbf{C}}_u^{C',B,t+s} = \check{\mathbf{C}}_u^{C',s/2,t+s} \ \forall u \leq t) > 1 - \varepsilon$$

for all $s > S_2$, and thus for $s > \max(S_1, S_2)$

$$\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) \leq \mathbb{P}(\tau_{ex} > s/2, \check{\mathbf{C}}_t^{C',s/2,t+s} \neq \emptyset) + 2\varepsilon.$$

Furthermore, we know that for every $\varepsilon > 0$ there exists an $S_3 > 0$ such that

$$|\mathbb{P}^{(C,B)}(\tau_{ex} > s/2) - \theta(C,B)| < \varepsilon$$

for all $s > S_3$. Note that by construction $(\mathbf{C}_r^{C,B})_{r\leq s/2}$ and $(\check{\mathbf{C}}_r^{C',s/2,t+s})_{r\leq t+s/2}$ are independent, and thus

$$\begin{aligned}\mathbb{P}(\tau_{ex} > s/2, \check{\mathbf{C}}_t^{C',s/2,t+s} \neq \emptyset) &= \mathbb{P}^{(C,B)}(\tau_{ex} > s/2)\mathbb{P}(\check{\mathbf{C}}_t^{C',s/2,t+s} \neq \emptyset) \\ &= \mathbb{P}^{(C,B)}(\tau_{ex} > s/2)\mathbb{P}^{(C',\pi)}(\tau_{ex} > t),\end{aligned}$$

where we used in the second equality that $\big(\check{\mathbf{C}}_u^{C',s/2,t+s}, \check{\mathbf{B}}_{t+s-u}^{s/2}\big)_{u\leq t+s/2}$ is again a CPERE with intial distribution $\delta_{C'} \otimes \pi$. Set $\kappa := 4\varepsilon + \varepsilon^2$. We obtain at last that for any $t > T$ and $s > S := \max(S_1, S_2, S_3)$ (note that S depends on T) we have

$$\begin{aligned}\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{C',B,t+s} \neq \emptyset) &\leq \mathbb{P}^{(C,B)}(\tau_{ex} > s/2)\mathbb{P}^{(C',\pi)}(\tau_{ex} > t) + 2\varepsilon \\ &\leq \theta(C,B)\theta^{\pi}(C') + \kappa,\end{aligned}$$

which proves the claim. □

The next step is to prove a lower bound. For that we need the following stopping time

$$\tau_{A,H}(C,B) := \inf\{t \geq 0 : (\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \supset (A,H)\}, \tag{5.19}$$

which is the first time that at least all sites in A are infected and all edges in H are open.

Lemma 5.2.4. *Let $A, C \subset V$ and $H, B \subset E$ be non-empty and A and H finite. Let $x \in C$ then*

$$\mathbb{P}^{(C,B)}(\tau_{A,H} < \infty) \geq \mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.})$$

Proof. Suppose that $\pi \neq \delta_\emptyset$. Otherwise $\mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}) = 0$, and thus the inequality is trivially true. First of all note that

$$\{\tau_{A,H}(C,B) < \infty\} = \{(\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \supset (A,H) \text{ for some } t \geq 0\}. \tag{5.20}$$

Next we define the stopping times $T_k = \inf\{t > T_{k-1} + 1 : x \in \mathbf{C}_t\}$, where $T_0 = 0$. Recall that $\mathcal{F}_t$ is the σ-algebra generated from all Poisson point processes used in the graphical representation until time t. Let us assume that $x \in C$, since $\pi \neq \delta_\emptyset$ and we know that the background process is translation invariance, we can guarantee that $\varepsilon = \mathbb{P}^{(\{x\},\emptyset)}(\mathbf{C}_1 \supseteq A, \mathbf{B}_1 \supseteq H) > 0$. This implies by monotonicity

$$\mathbb{P}(\mathbf{C}_{T_k+1} \supseteq A, \mathbf{B}_{T_k+1} \supseteq H | \mathcal{F}_{T_k}) \geq \varepsilon \text{ almost surely on } \{T_k < \infty\}. \tag{5.21}$$

Set $A_k := \{\mathbf{C}_{T_k+1} \supseteq A, \mathbf{B}_{T_k+1} \supseteq H\} \in \mathcal{F}_{T_k+1} \subset \mathcal{F}_{T_{k+1}} := \mathcal{G}_{k+1}$ and then we see that

$$\sum_{k=0}^{\infty} \mathbb{P}(A_k|\mathcal{G}_k) = \infty \quad \text{almost surely on} \quad \bigcap_{k=0}^{\infty}\{T_k < \infty\}.$$

Since $A_k \in \mathcal{G}_{k+1}$ for all $k \in \mathbb{N}$ we can apply an extension of the Borel-Cantelli Lemma [Dur19, Theorem 4.3.4] and we get that

$$\Big\{\sum_{k=0}^{\infty} \mathbb{P}(A_k|\mathcal{G}_k) = \infty\Big\} = \{A_k \text{ i.o.}\}.$$

Note that $\bigcap_{k=1}^{\infty}\{T_k < \infty\} = \{x \in \mathbf{C}_t \text{ i.o.}\}$. Hence, by (5.20) and (5.21) we get that

$$\mathbb{P}^{(C,B)}(\tau_{A,H} < \infty) \geq \mathbb{P}^{(C,B)}(\{A_k \text{ i.o.}\} \cap \{x \in \mathbf{C}_t \text{ i.o.}\}) = \mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}). \qquad \square$$

Proposition 5.2.5. *Let $C \subset V$ and $B \subset E$. Suppose* (5.14) *and* (5.15) *are satisfied, then*

$$\liminf_{t\to\infty} \mathbb{P}_{\lambda,r}(\mathbf{C}_t^{C,B} \cap C' \neq \emptyset) \geq \theta(\lambda, r, C, B)\theta^{\pi}(\lambda, r, C').$$

for every $C' \subset V$ finite.

Proof. Let $A \subset V$ and $H \subset E$ with A and H being finite sets. We can assume that $\pi \neq \delta_\emptyset$, since if $\pi = \delta_\emptyset$, then $\theta^\pi(C') = 0$ for all $C' \subset V$ finite, and thus the right hand side is zero. Recall from (5.19) that the first time that at least all sites in A are infected and all edges in H are open is denoted by $\tau_{A,H}(C, B)$. Furthermore, set $\sigma_A^N := \tau_{A,A_E^N}$ and $\tau_A := \tau_{A,\emptyset}$. Now we see that

$$\begin{aligned}
\mathbb{P}^{(C,B)}(\mathbf{C}_{t+s+u} \cap C' \neq \emptyset) &\geq \mathbb{P}^{(C,B)}(\sigma_A^N < s, \mathbf{C}_{t+u+s} \cap C' \neq \emptyset) \\
&= \mathbb{E}^{(C,B)}[\mathbb{1}_{\{\sigma_A^N < s\}} \mathbb{P}(\mathbf{C}_{t+u+s} \cap C' \neq \emptyset | \mathcal{F}_{\sigma_A^N})] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{\sigma_A^N < s\}} \mathbb{P}^{(\mathbf{C}_{\sigma_A^N}, \mathbf{B}_{\sigma_A^N})}(\mathbf{C}_{t+u+(s-\sigma_A^N)} \cap C' \neq \emptyset\Big] \\
&\geq \mathbb{P}^{(C,B)}(\sigma_A^N < s) \inf_{r > t+u} \mathbb{P}^{(A,A_E^N)}(\mathbf{C}_r \cap C' \neq \emptyset),
\end{aligned} \tag{5.22}$$

where we used that $(\mathbf{C}, \mathbf{B})$ is a strong Markov process. As already mentioned before one major issue is that in comparison to the classical case our duality is weaker in the sense that $\widehat{\mathbf{C}}_{t+u+r}^{C',A_E^N,t+u+r}$ is not again a CPERE, and therefore our process is not self dual. But now we show that the difference is not big if we choose $t + u$ large enough. Recall that Φ_t was the set of all vertices x such that all edges attached to x are contained in the coupled region at time t. By Proposition 4.1.4 we know that

$$\mathbb{P}(\exists s > 0 : \widetilde{\mathbf{C}}_t^A \subset \Phi_t \ \forall t \geq s) = 1,$$

and thus for every $\varepsilon' > 0$ there exists an $S > 0$ such that

$$\mathbb{P}(\widetilde{\mathbf{C}}_t^A \subset \Phi_t \ \forall t \geq S) > 1 - \varepsilon'.$$

As an application of Lemma 5.1.8 we find an $N = N(S) \in \mathbb{N}$ such that

$$\mathbb{P}(\widetilde{\mathbf{C}}_t^A \subset A^N \ \forall t \leq S) > 1 - \varepsilon'.$$

Furthermore, by Lemma 3.2.5 there exists an $M > N$ such that

$$\mathbb{P}(\underbrace{\mathbf{B}_t^{A_E^M} = \mathbf{B}_t^{A_E^M \cup B} \text{ on } A_E^N \text{ for all } t \leq S}_{:=E_S(B,N,M)}) > 1 - \varepsilon'$$

where $B \subset E$ is chosen arbitrarily and ε' is independent of the choice of B. Thus, we can conclude for a given $A \subset V$ that for every $\varepsilon > 0$ there exists an $S = S(\varepsilon) > 0$, $N = N(S) \in \mathbb{N}$ and $M > N$ such that

$$\mathbb{P}(\{\widetilde{\mathbf{C}}_t^A \subset \Phi_t \ \forall t \geq S, \widetilde{\mathbf{C}}_t^A \subset A^N \ \forall t \leq S\} \cap E_S(B, N, M)) > 1 - \varepsilon$$

for all $B \subset E$. Note that ε depends on A. On this event the process $\mathbf{C}^{A,A_E^M}$ does not differ from $\mathbf{C}^{A,A_E^M \cup B}$ for any $B \subset E$, since on this event the infection paths have either not yet left A^N and the edges in A_E^N will have the same state open or closed with the two chosen initial configuration or the infection paths stay in Φ, the area where every edge attached to an infected site has already been coupled. Thus, we get

$$\left| \mathbb{P}^{(A,A_E^M)}(\mathbf{C}_r \cap C' \neq \emptyset) - \int \mathbb{P}^{(A,A_E^M \cup B)}(\mathbf{C}_r \cap C' \neq \emptyset) \pi(\mathsf{d}B) \right| < \varepsilon. \tag{5.23}$$

Furthermore, by monotonicity (see Lemma 3.4.1) it follows that

$$\mathbb{P}^{(A,A_E^M)}(\mathbf{C}_r \cap C' \neq \emptyset) > \mathbb{P}^{(A,\pi)}(\mathbf{C}_r \cap C' \neq \emptyset) - \varepsilon.$$

Using this and the fact that if the background is started stationary the CPERE is self dual by Proposition 5.1.2, and therefore we get with (5.22) that

$$\mathbb{P}^{(C,B)}(\mathbf{C}_{t+s+u} \cap C' \neq \emptyset) \geq \mathbb{P}^{(C,B)}(\sigma_A^N < s) \inf_{r>t+u} \big(\mathbb{P}^{(C',\pi)}(A \cap \mathbf{C}_r \neq \emptyset) - \varepsilon\big).$$

Then, analogously to (5.22) by considering τ_D with $D \subset V$ finite instead of σ_A^N we can find a similar lower bound for the last probability such that

$$\mathbb{P}^{(C,B)}(\mathbf{C}_{t+s+u} \cap C' \neq \emptyset) \geq \mathbb{P}^{(C,B)}(\sigma_A^N < s)\mathbb{P}^{(C',\pi)}(\tau_D < t) \inf_{r>u} \mathbb{P}^{(D,\emptyset)}(A \cap \mathbf{C}_r \neq \emptyset) - \varepsilon.$$

For $A \subset V$ and $B \subset E$ finite we know by Lemma 5.2.4 that

$$\mathbb{P}^{(C,B)}(\tau_{A,H} < \infty) \geq \theta(C, B),$$

and thus by letting $s, t, u \to \infty$ we see that

$$\liminf_{t\to\infty} \mathbb{P}^{(C,B)}(\mathbf{C}_t \cap C' \neq \emptyset) \geq \theta(C, B)\theta^\pi(C') \liminf_{t\to\infty} \mathbb{P}^{(D,\emptyset)}(A \cap \mathbf{C}_t \neq \emptyset) - \varepsilon.$$

Now for an arbitrary $x \in V$ we choose $A = D = \mathbb{B}_n(x)$ and use (5.15) which means that for all $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\liminf_{t\to\infty} \mathbb{P}^{(\mathbb{B}_n(x),\emptyset)}(\mathbb{B}_n(x) \cap \mathbf{C}_t \neq \emptyset) > 1-\delta$ for all $n > n_0$. Note that ε depends on $\mathbb{B}_n(x)$, which means we first need to choose n_0 and then the parameter accordingly such that (5.23) holds for $\varepsilon = \delta$ and such that

$$\liminf_{t\to\infty} \mathbb{P}^{(C,B)}(\mathbf{C}_t \cap C' \neq \emptyset) \geq \theta(C, B)\theta^\pi(C') - 2\delta.$$

Since this holds for all $\delta > 0$, the claim follows. □

We showed one direction of the equivalence. Next we show the converse direction.

Proposition 5.2.6. *Suppose* (5.17) *holds and assume that* $\overline{\nu}_{\lambda,r} \neq \delta_\emptyset \otimes \pi$, *then* (5.14) *and* (5.15) *are satisfied.*

Proof. Note that $\overline{\nu}_{\lambda,r} \neq \delta_\emptyset \otimes \pi$ can only occur if $\pi \neq \delta_\emptyset$. Choose $C = C' = \mathbb{B}_n$ and $B = \emptyset$, then by (5.17) follows that $\lim_{t\to\infty} \mathbb{P}^{(\mathbb{B}_n,\emptyset)}(\mathbb{B}_n \cap \mathbf{C}_t \neq \emptyset) = \theta(\mathbb{B}_n, \emptyset)\theta^\pi(\mathbb{B}_n)$. Using Lemma 5.1.11 yields that the right hand side converges to 1 as $n \to \infty$. This proves (5.15). Now all what is left to show is (5.14). We see that

$$\{\mathbf{C}_t \cap C' \neq \emptyset \text{ i.o.}\} = \bigcap_{n\in\mathbb{N}} \{\mathbf{C}_s \cap C' \neq \emptyset \text{ for some } s \geq n\},$$

and thus by continuity of the law $\mathbb{P}$ we get that

$$\mathbb{P}^{(C,B)}(\mathbf{C}_t \cap C' \neq \emptyset \text{ i.o.}) = \lim_{n\to\infty} \mathbb{P}^{(C,B)}(\mathbf{C}_s \cap C' \neq \emptyset \text{ for some } s \geq n) \geq \theta(C,B)\theta^{\pi}(C'),$$

where we again used (5.17). Now using the fact that $\mathbb{P}^{(\{y\},\emptyset)}(x \in \mathbf{C}_1) > 0$ for all $x, y \in V$ it follows analogously as in the proof of Lemma 5.3.3, that the event $\{x \in \mathbf{C}_t^{C,B} \text{ i.o.}\}$ almost surely happens on $\{\mathbf{C}_t^{C,B} \cap C' \neq \emptyset \text{ i.o.}\}$, and thus

$$\mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}) \geq \theta(C,B)\theta^{\pi}(C').$$

Furthermore, if we choose $C' = \mathbb{B}_n$ and let $n \to \infty$, then Lemma 5.1.11 yields that

$$\mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}) \geq \theta(C,B)$$

for all $x \in V$ and $C \subset V$. Since the reversed inequality "$\leq$" obviously holds as well, this provides (5.14). □

Since we have shown that the conditions (5.14) and (5.15) are equivalent to the fact that the two marginal processes converge, the only thing left to show is that convergence of the marginals already implies convergences of the joint distribution.

Proposition 5.2.7. *Suppose that* (5.17) *holds, then for all $C \subset V$ and $B \subset E$ it follows that*

$$\begin{aligned} &\mathbb{P}_{\lambda,r}^{(C,B)}(\mathbf{C}_t \cap A \neq \emptyset, \mathbf{B}_t \cap H \neq \emptyset) \\ &\qquad \to \theta(C,B)\overline{\nu}(\{(C',B') : C' \cap A \neq \emptyset, B' \cap H \neq \emptyset\}) \end{aligned} \tag{5.24}$$

as $t \to \infty$, for every $A \subset V$ and $H \subset E$ finite.

Proof. Let $A, C \subset V$ and $B, H \subset E$ be chosen arbitrary with $A \subset V$ and $H \subset E$ finite. We consider these sets as fixed. We again exploit the duality relation we derived in Proposition 5.1.2, which states that

$$\mathbb{P}(\mathbf{C}_{t+s}^{C,B} \cap A \neq \emptyset, \mathbf{B}_{t+s}^{B} \cap H \neq \emptyset) = \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{B}_{t+s}^{B} \cap H \neq \emptyset) \tag{5.25}$$

where $t, s > 0$. Let $\tau = \tau_{ex}(C,B)$ denote the extinction time with initial configuration $C \subset V$ and $B \subset E$.

Some simple calculations yield that

$$\begin{aligned}
0 \leq &\mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset) - \mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset) \\
= &\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_t^{A,B,t+s} = \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset) \\
\leq &\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_t^{A,B,t+s} = \emptyset) \\
= &\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset) - \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset).
\end{aligned} \tag{5.26}$$

Now we fix an arbitrary $\varepsilon > 0$. Then by a combination of Proposition 5.2.3 and (5.17) we get that there exists a $S_1 > 0$ and $T > 0$ such that

$$\left|\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset) - \mathbb{P}(\mathbf{C}_s^{C,B} \cap \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset)\right| < \frac{\varepsilon}{3}$$

for all $s > S_1$ and $t > T$. By using the duality relation (5.25) together with (5.26) we can conclude that

$$\left|\mathbb{P}(\mathbf{C}_{t+s}^{C,B} \cap A \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset) - \mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset)\right| < \frac{\varepsilon}{3}$$

for all $s > S_1$ and $t > T$. Furthermore, there exists an $S_2 = S_2(C, B, \varepsilon) > 0$ such that

$$\begin{aligned}
|\mathbb{P}(\mathbf{C}_s^{C,B} \neq \emptyset, &\widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset) \\
&- \mathbb{P}(\tau > s/2, \widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset)| < \frac{\varepsilon}{3}
\end{aligned}$$

for $s \geq S_2$, which can be shown analogously to (5.18). In the last step we conclude that there exists an $S_3 = S_3(t, A, H, \varepsilon) > 0$ such that for $s \geq S_3$

$$\begin{aligned}
\big|\mathbb{P}(\tau > s/2, &\widehat{\mathbf{C}}_t^{A,B,t+s} \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset) \\
&- \mathbb{P}(\tau > s/2, \check{\mathbf{C}}_t^{A,s/2,t+s} \neq \emptyset, \check{\mathbf{B}}_{t+s/2}^{s/2} \cap H \neq \emptyset)\big| < \frac{\varepsilon}{3},
\end{aligned}$$

which follows as a combination of Lemma 5.2.1 and Lemma 5.2.2. Finally by putting everything together and using the triangle inequality we get that for every $t > T$ there exists an $S > 0$ such that

$$\left|\mathbb{P}(\mathbf{C}_{t+s}^{C,B} \cap A \neq \emptyset, \mathbf{B}_{t+s}^B \cap H \neq \emptyset) - \mathbb{P}(\tau > s/2, \check{\mathbf{C}}_t^{A,s/2,t+s} \neq \emptyset, \check{\mathbf{B}}_{t+s/2}^{s/2} \cap H \neq \emptyset)\right| < \varepsilon$$

for every $s > S$. To be precise one can choose $S = \max\{S_1, S_2, S_3\}$.

This means that if we first let $s \to \infty$ and then $t \to \infty$, the two probabilities converge to the same limit. So it suffices to show that

$$\begin{aligned}\mathbb{P}(\tau > s/2, &\check{\mathbf{C}}_t^{A,s/2,t+s} \neq \emptyset, \check{\mathbf{B}}_{t+s/2}^{s/2} \cap H \neq \emptyset) \\ &\to \theta(C,B)\overline{\nu}(\{(C,B) : C \cap A \neq \emptyset, B \cap H \neq \emptyset\})\end{aligned}$$

as $s, t \to \infty$. Recall that we already concluded above that $(\mathbf{C}_r^{C,B})_{r<s/2}$ is independent of $(\check{\mathbf{C}}_r^{A,s/2,t+s})_{r \leq t+s/2}$ and it is also independent of $(\check{\mathbf{B}}_r^{s/2})_{r \geq s/2}$. Thus, we get that

$$\begin{aligned}\mathbb{P}(\tau > s/2, &\check{\mathbf{C}}_t^{A,s/2,t+s} \neq \emptyset, \check{\mathbf{B}}_{t+s/2}^{s/2} \cap H \neq \emptyset) \\ &= \mathbb{P}(\tau > s/2)\mathbb{P}(\check{\mathbf{C}}_t^{A,s/2,t+s} \neq \emptyset, \check{\mathbf{B}}_{t+s/2}^{s/2} \cap H \neq \emptyset).\end{aligned}$$

Next we use that $(\check{\mathbf{C}}_r^{A,s/2,t+s}, \check{\mathbf{B}}_{t+s-r}^{s/2})_{r \leq t}$ is again a CPERE with initial distribution $\delta_A \otimes \pi$, and thus by duality

$$\mathbb{P}(\check{\mathbf{C}}_t^{A,s/2,t+s} \neq \emptyset, \check{\mathbf{B}}_{t+s/2}^{s/2} \cap H \neq \emptyset) = \mathbb{P}^{(V,\pi)}(\mathbf{C}_t \cap A \neq \emptyset, \mathbf{B}_t \cap H \neq \emptyset),$$

which converges to the desired limit since we have already shown that $(\delta_V \otimes \pi)T(t) \Rightarrow \overline{\nu}$ as $t \to \infty$ by Lemma 5.1.5. The claim follows, since $\mathbb{P}^{(C,B)}(\tau > s/2) \to \theta(C,B)$ as $s \to \infty$. □

Note that analogously as before (5.24) is equivalent to complete convergence, i.e for every initial configuration $C \subset V$ and $B \subset E$

$$(\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \Rightarrow \theta(C,B)\overline{\nu} + [1 - \theta(C,B)](\delta_\emptyset \otimes \pi),$$

since the function class $\{\mathbb{1}_{\{\cdot \cap A \neq \emptyset, \cdot \cap H \neq \emptyset\}} : C' \subset V, H \subset E \text{ finite}\}$ is convergence determining. Now we can conclude the main result of this chapter.

Proof of Theorem 1.4.15. The theorem follows as a combination of the four Propositions 5.2.3, 5.2.5, 5.2.6 and 5.2.7. To be precise Propositions 5.2.3 and 5.2.5 yield that (5.14) and (5.15) imply the convergence of the first marginal, i.e. (5.17). But in Proposition 5.2.7 we already concluded that (5.17) suffices to conclude weak convergence of the CPERE, i.e. (5.24). At last Propostion 5.2.6 provides equivalence of the conditions and complete convergence. □

5.3 Continuity of the survival probability

In this section we study continuity of the survival probability with respect to the infection rate λ and recovery rate r. We start with determining on which regions of the parameter space the functions

$$\lambda \mapsto \theta(\lambda, r, C, B) \quad \text{and} \quad r \mapsto \theta(\lambda, r, C, B)$$

are left or right continuous. Before we proceed we need the following result concerning the limit of a sequence of monotone and continuous functions.

Lemma 5.3.1. *Let $f : \mathbb{R}_+ \to [0,1]$ and $f_n : \mathbb{R}_+ \to [0,1]$ for every $n \geq 1$ with $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}_+$. Let f_n be a continuous and monotone function for all $n \in \mathbb{N}$, and furthermore $f_n(x) \geq f_{n+1}(x)$ for all $x \in \mathbb{R}_+$. Then if f_n is increasing for all $n \in \mathbb{N}$, it follows that f is right continuous and if f_n is decreasing, then f is left continuous.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of increasing and continuous functions and let $x_n \downarrow x$. We show that $\lim_{n\to\infty} f(x_n) = f(x)$. By our assumptions it is clear that $(f_n(x_n))_{n\in\mathbb{N}}$ is a decreasing sequence which is bounded from below by $f(x)$, and thus the sequence converge. Hence, it holds that $\lim_{n\to\infty} f_n(x_n) \geq f(x)$.

Suppose $\lim_{n\to\infty} f_n(x_n) > f(x)$. Since $(f_n(x_n))_{n\in\mathbb{N}}$ converges there must exist $y > f(x)$ such that $\lim_{n\to\infty} f_n(x_n) = y$. Since f is the pointwise limit of $(f_n)_n$ there must exist an $m \in \mathbb{N}$ such that $f_m(x) < y$. Also f_m is continuous and $x_n \downarrow x$. Thus, there must exist $k \in \mathbb{N}$ such that $f_m(x_k) < y$. Now let $l := \max(k, m)$. Because of monotonicity it follows $f_l(x_l) < y$, which is a contradictions, since $(f_n(x_n))_{n\in\mathbb{N}}$ is strictly decreasing to y, and therefore $\lim_{n\to\infty} f_n(x_n) = f(x)$ but since $f_n(x_n) \geq f(x_n) \geq f(x)$ for all $n \in \mathbb{N}$ it follows $\lim_{n\to\infty} f(x_n) = f(x)$. Note that we used here that f is monotone increasing, which follows by the fact that $(f_n)_{n\in\mathbb{N}}$ is a sequence of monotone increasing functions, and therefore the limit function f must also be monotone increasing.

If we assume f_n is decreasing instead of increasing with a similar line of arguments it follows that f must be left continuous, since for every sequence $z_n \uparrow z$ we see that $(f_n(z_n))_{n\in\mathbb{N}}$ is a decreasing sequence. □

As a direct consequence of this lemma we can conclude right continuity in the following proposition.

Proposition 5.3.2. *Let $C \subset V$ and $B \subset E$. Then, for $r > 0$ the function*

$$\lambda \mapsto \theta(\lambda, r, C, B),$$

is right continuous on $(0, \infty)$ and for $\lambda > 0$ the function $r \mapsto \theta(\lambda, r, C, B)$ is left continuous on $(0, \infty)$.

Proof. By Lemma 3.4.4 we know that the function $\lambda \mapsto \mathbb{P}_{\lambda,r}^{(C,B)}(\mathbf{C}_t \neq \emptyset)$ is continuous for any $t \geq 0$ and also $\mathbb{P}_{\lambda,r}^{(C,B)}(\mathbf{C}_s \neq \emptyset) \geq \mathbb{P}_{\lambda,r}^{(C,B)}(\mathbf{C}_t \neq \emptyset)$ if $s \leq t$. Thus, we can conclude that

$$\mathbb{P}_{\lambda,r}^{(C,B)}(\mathbf{C}_t \neq \emptyset) \downarrow \theta(\lambda, r, C, B) \quad \text{as} \quad t \to \infty,$$

by continuity of $\mathbb{P}$. Since $\mathbb{P}_{\lambda,r}^{(C,B)}(\mathbf{C}_t \neq \emptyset)$ is increasing with respect to the infection rate λ, we can use Lemma 5.3.1 to conclude that $\lambda \mapsto \theta(\lambda, r, C, B)$ is right continuous.

Analogously it follows that $r \mapsto \theta(\lambda, r, C, B)$ is left continuous since $\mathbb{P}_{\lambda,r}^{(C,B)}(\mathbf{C}_t \neq \emptyset)$ is decreasing with respect to the recovery rate r. □

The continuity from the respective other side is more difficult to prove. Before we proceed with this we need the following somewhat technical result.

Lemma 5.3.3. *Let $(\mathbf{C}, \mathbf{B})$ be a CPERE, $\emptyset \neq C \subset V$ be finite and $B \subset E$. Set*

$$D_{n,t}(C, B) := \{\exists x \in V \text{ such that } \mathbb{B}_n(x) \subseteq \mathbf{C}_s^{C,B} \text{ for some } s \leq t\}$$

for $n \in \mathbb{N}$ and $t \geq 0$. In words $D_{n,t}(C, B)$ is the event that for some $s \leq t$ there exists a site x such that all sites in the ball $\mathbb{B}_n(x)$ with centre x are infected at time s. Then

$$\lim_{t \to \infty} \mathbb{P}\big(D_{n,t}(C, B)\big) \geq \theta(C, B) \quad \text{for all } n \in \mathbb{N}.$$

Proof. We can assume that $\pi \neq \delta_\emptyset$ since otherwise the survival probability is 0 which makes the statement trivial. We omit for most parts of the proof the initial configuration (C, B) since it remains unchanged throughout this proof. Note that since $D_{n,t}$ is increasing in t, it follows that $\lim_{t\to\infty} \mathbb{P}(D_{n,t}) = \mathbb{P}(D_{n,\infty})$. The idea of this proof is that if a site x is infected at time $k \in \mathbb{N}$, i.e. $x \in \mathbf{C}_k$, the probability that all sites in a radius of n get infected by time $k+1$, i.e. $\mathbf{C}_{k+1} \supseteq \mathbb{B}_n(x)$, is positive for every fixed $n \in \mathbb{N}$. But if we assume that $\mathbf{C}$ survives we know that for every $t \geq 0$ there exists an $x \in V$ such that $x \in \mathbf{C}_t$ and this will imply $\mathbb{P}_{\lambda,r}(D_{n,\infty}) \geq \theta(\lambda, r)$ for every $n \in \mathbb{N}$. In fact

$$\{\mathbf{C}_t \neq \emptyset \ \forall t \geq 0\} = \{\forall k \in \mathbb{N}_0, \exists x \in V \text{ such that } x \in \mathbf{C}_k\}$$

since $\emptyset$ is an absorbing state.

Recall that $\mathcal{F}_k$ is the σ-algebra generated from the Poisson point processes Ξ used in the graphical representation until time k. Then we set

$$\varepsilon = \varepsilon(n) = \mathbb{P}^{(\{x\},\emptyset)}(\mathbf{C}_1 \supseteq \mathbb{B}_n(x)) > 0.$$

We see that $\mathbb{P}(\mathbf{C}_{k+1} \supseteq \mathbb{B}_n(x)|\mathcal{F}_k) \geq \varepsilon$ almost surely on $\{x \in \mathbf{C}_k\}$, where we used monotonicity with respect to the initial configurations (see Lemma 3.4.1). This yields that for any $x^* \in V$

$$\mathbb{P}\Big(\bigcup_{x\in V}\{\mathbf{C}_{k+1} \supseteq \mathbb{B}_n(x)\}\Big|\mathcal{F}_k\Big) \geq \mathbb{P}(\mathbf{C}_{k+1} \supseteq \mathbb{B}_n(x^*)|\mathcal{F}_k) \geq \varepsilon.$$

almost surely on $\{x^* \in \mathbf{C}_k\}$. We set $A^n_{k+1} := \bigcup_{x\in V}\{\mathbf{C}_{k+1} \supseteq \mathbb{B}_n(x)\} \in \mathcal{F}_{k+1}$ for $k \in \mathbb{N}_0$. We see that

$$\sum_{k=0}^{\infty} \mathbb{P}(A^n_{k+1}|\mathcal{F}_k) = \infty \quad \text{a.s. on} \quad \{\forall k \in \mathbb{N}_0,\ \exists x \in V \text{ such that } x \in \mathbf{C}_k\}.$$

Now analogous to Lemma 5.2.4 we can use the extension of the Borel-Cantelli Lemma, found in [Dur19, Theorem 4.3.4] and get that

$$\Big\{\sum_{k=0}^{\infty} \mathbb{P}(A^n_{k+1}|\mathcal{F}_k) = \infty\Big\} = \{A^n_k \text{ i.o.}\}.$$

This implies $\{\mathbf{C}_t = \emptyset\ \forall t \geq 0\} \subset \{A_k \text{ i.o.}\}$. Obviously $\mathbb{P}^{(C,B)}(\{A^n_k \text{ i.o.}\}) \leq \theta(C,B)$, and thus with what we just shown it follows that actually $\mathbb{P}^{(C,B)}(\{A^n_k \text{ i.o.}\}) = \theta(C,B)$ holds. This yields for all $n > 0$

$$\mathbb{P}_{\lambda,r}(D_{n,\infty}(C,B)) \geq \mathbb{P}^{(C,B)}_{\lambda,r}(A^n_k \text{ i.o.}) = \theta(\lambda,r,C,B). \qquad \square$$

Finally, we are prepared to prove the second continuity property. Recall from (1.11) that

$$\mathcal{S}_{c_1} = \{(\lambda,r) : \exists\lambda' \leq \lambda \text{ s.t. } (\lambda',r) \in \mathcal{S}(\{x\},\emptyset) \text{ and } c_1(\lambda',\rho) > \kappa^{-1}\rho\},$$

where $\mathcal{S}(\{x\},\emptyset)$ denotes the survival region for the initial configuration $(\{x\},\emptyset)$ defined in (1.10), i.e. $(\lambda,r) \in \mathcal{S}(\{x\},\emptyset)$ if and only if $\theta(\lambda,r,\{x\},\emptyset) > 0$.

Proposition 5.3.4. *Let $C \subset V$, $B \subset E$ and $x \in V$.*

(*i*) *Let $r > 0$. Then the function $\lambda \mapsto \theta(\lambda, r, C, B)$ is left continuous, and thus continuous, on $\{\lambda : (\lambda, r) \in \mathring{\mathcal{S}}_{c_1}\}$.*

(*ii*) *Let $\lambda > 0$. Then the function $r \mapsto \theta(\lambda, r, C, B)$ is right continuous, and thus continuous, on $\{r : (\lambda, r) \in \mathring{\mathcal{S}}_{c_1}\}$.*

Proof. We assume that $C \subset V$ is finite and non-empty. Otherwise the surival probability is 0 or 1 and a constant function is obviously continuous. We only show (i) since (ii) follows analogously, i.e only some minor changes are needed in the proof. We fix $r > 0$ and assume that $\{\lambda : (\lambda, r) \in \mathring{\mathcal{S}}_{c_1}\} \neq \emptyset$. Thus, let $(\lambda, r) \in \mathring{\mathcal{S}}_{c_1}$, fix some $x \in V$ and define $\tau = \tau_n := \inf\{t \geq 0 : \exists x \in V \text{ s.t. } \mathbf{C}_t \supseteq \mathbb{B}_n(x)\}$, where $n \in \mathbb{N}$. We see that

$$\begin{aligned}\theta(\lambda) &= \mathbb{P}(\mathbf{C}_s \neq \emptyset \ \forall s \geq 0) \geq \mathbb{P}(\{\tau < t\} \cap \{\mathbf{C}_s \neq \emptyset \ \forall s \geq \tau\}) \\ &= \mathbb{E}[\mathbb{1}_{\{\tau < t\}} \mathbb{P}(\mathbf{C}_s \neq \emptyset \ \forall s \geq \tau | \mathcal{F}_\tau)]\end{aligned}$$

for any $t \geq 0$, where we used again that if $\mathbf{C}_t \neq \emptyset$ for $t \geq \tau$, then this must also be true for all $t \leq \tau$. Now we use the fact that $(\mathbf{C}, \mathbf{B})$ is a Feller process, and see that

$$\mathbb{P}(\mathbf{C}_s \neq \emptyset \ \forall s \geq \tau | \mathcal{F}_\tau) = \mathbb{P}\big(\mathbf{C}_{\tau+s} \neq \emptyset \ \forall s \geq 0 \,\big|\, (\mathbf{C}_\tau, \mathbf{B}_\tau)\big),$$

where we used the strong Markov property. From the definition of τ it is clear that there exists an $x \in V$ such that $\mathbf{C}_\tau \supseteq \mathbb{B}_n(x)$. Now we know that

$$\mathbb{P}\big(\mathbf{C}_{\tau+s} \neq \emptyset \ \forall s \geq 0 \,\big|\, (\mathbf{C}_\tau, \mathbf{B}_\tau)\big) \geq \mathbb{P}^{(\mathbb{B}_n(x), \emptyset)}\big(\mathbf{C}_s \neq \emptyset \ \forall s \geq 0\big),$$

and by translation invariance the right-hand side is independent of x. Thus we can omit the site x and write $\mathbb{B}_n$. So we get that

$$\theta(\lambda) \geq \mathbb{P}_\lambda(D_{n,t}) \mathbb{P}_\lambda^{(\mathbb{B}_n, \emptyset)}\big(\mathbf{C}_s \neq \emptyset \ \forall s \geq 0\big) = \mathbb{P}_\lambda(D_{n,t}) \theta(\lambda, \mathbb{B}_n, \emptyset),$$

where we used that $\{\tau < t\} = D_{n,t}$. The set $D_{n,t}$ is defined as in Lemma 5.3.3. Now let $\lambda_c(r) < \lambda'' < \lambda' < \lambda$, and thus $\lambda', \lambda'' \in \{\lambda : (\lambda, r) \in \mathring{\mathcal{S}}_{c_1}\}$. Then we see that

$$\theta(\lambda') \geq \mathbb{P}_{\lambda'}(D_{n,t}) \theta(\lambda', \mathbb{B}_n, \emptyset) \geq \mathbb{P}_{\lambda'}(D_{n,t}) \theta(\lambda'', \mathbb{B}_n, \emptyset),$$

where we used monotonicity which was shown in Lemma 3.4.2. Letting $\lambda' \uparrow \lambda$ yields

$$\theta(\lambda-) \geq \mathbb{P}_\lambda(D_{n,t}) \theta(\lambda'', \mathbb{B}_n, \emptyset), \tag{5.27}$$

where we used continuity of $\lambda \mapsto \mathbb{P}_\lambda(D_{n,t})$ which follows by Lemma 3.4.4. Recall that (C, B) was the initial value of the CPERE, using Lemma 5.3.3 we get thet

$$\lim_{t\to\infty} \mathbb{P}_\lambda\big(D_{n,t}(C,B)\big) \geq \theta(\lambda, C, B),$$

and thus letting $t \to \infty$ in (5.27) yields

$$\theta(\lambda-, C, B) \geq \lim_{t\to\infty} \mathbb{P}_\lambda(D_{n,t}(C,B))\theta(\lambda'', \mathbb{B}_n, \emptyset) \geq \theta(\lambda, C, B)\theta(\lambda'', \mathbb{B}_n, \emptyset).$$

Since we know that $\lambda'' \in \{\lambda : (\lambda, r) \in \mathring{\mathcal{S}}_{c_1}\}$, by Lemma 5.1.11 it follows that

$$\theta(\lambda'', \mathbb{B}_n, \emptyset) \to 1 \text{ as } n \to \infty.$$

Putting everything together yields $\theta(\lambda-, C, B) \geq \theta(\lambda, C, B)$. But since we know that the function is monotone increasing in λ, this yields left continuity on the parameter set $\{\lambda : (\lambda, r) \in \mathring{\mathcal{S}}_{c_1}\}$. Right continuity, and therefore continuity follows by Proposition 5.3.2. □

We end this section with the following proof:

Proof of Theorem 1.4.16. By Proposition 5.3.2 and Proposition 5.3.4 it follows that

$$(\lambda, r) \mapsto \theta(\lambda, r, C, B)$$

is separately continuous on the open set $\mathring{\mathcal{S}}_{c_1} \subset \mathbb{R}^2$, which means that the function is continuous in all variable separately, i.e. $\lambda \mapsto \theta(\lambda, r, C, B)$ and $r \mapsto \theta(\lambda, r, C, B)$ are continuous on $\{\lambda : (\lambda, r) \in \mathcal{S}_{c_1}\}$ and $\{r : (\lambda, r) \in \mathcal{S}_{c_1}\}$ respectively. Since the survival probability θ is monotone in the infection rate λ and the recovery rate r it follows that the function is jointly continuous on $\mathring{\mathcal{S}}_{c_1}$, see [KD69, Proposition 2]. □

Chapter 6

CPDP on the d-dimenstional integer lattice $\mathbb{Z}^d$

In the previous sections we considered the CPERE in a fairly general setting. In this section we focus on the main example introduced in Example 1.1.2 (i). The CPDP on the d-dimensional lattice with nearest neighbour structure. Therefore, $V = \mathbb{Z}^d$ and $E = \{\{x, y\} \subset \mathbb{Z}^d : ||x - y||_1 = 1\}$, where $|| \cdot ||_1$ denotes the 1-norm. Thus, in this chapter the background $\mathbf{B}$ is assumed to be the dynamical percolation. Let us recall that $\mathbf{B}$ is a Feller process with transitions

$$\begin{aligned} \mathbf{B}_{t-} = B &\to B \cup \{e\} \quad \text{ at rate } \alpha \text{ and} \\ \mathbf{B}_{t-} = B &\to B \backslash \{e\} \quad \text{ at rate } \beta, \end{aligned}$$

where $\alpha, \beta > 0$. In words this means that with rate α an edge is updated to the state open and with rate β it is closed.

In Section 3.1 we discussed the graphical representation of spin systems. In case of the dynamical percolation one can give a simpler choice of maps which yield the same dynamics, which are

$$\mathbf{birth}_e(B) := B \cup \{x\} \quad \text{and} \quad \mathbf{death}_e(B) := B \backslash \{x\}$$

for $B \subset E$ and rates $r_{\mathbf{birth}_e} = \alpha$ and $r_{\mathbf{death}_e} = \beta$ for all $e \in E$. It is not difficult to see that the resulting Feller process has the same transition rates as the Feller process constructed with the maps $\mathbf{up}_{x,F}$ and $\mathbf{down}_{x,F}$ with respective rates $r_{\mathbf{up}_{x,F}} = \alpha$ and $r_{\mathbf{down}_{x,F}} = \beta$ for all $x \in V$ and $F \subset \mathcal{N}_x$. The advantage of this simplification of the graphical representation is that it is clear that in case of the dynamical percolation every edge updates independently of all other edges, i.e. the events $\{e \in \mathbf{B}_t\}$ and $\{e' \in \mathbf{B}_t\}$ are independent if $e \neq e'$. On the other hand from the dynamics it is also clear that

$$\mathbb{P}(e \in \mathbf{B}_t^B) = \mathbb{1}_{\{e \in B\}} \exp(-(\alpha+\beta)t) + \frac{\alpha}{\alpha+\beta}(1 - \exp(-(\alpha+\beta)t)) \to \frac{\alpha}{\alpha+\beta},$$

as $t \to \infty$. The first summand is the probability that no update event occurred at e, and thus for e to be open it must already hold that $e \in B$. The second summand is the probability that the edge is in the state open conditioned on the event that the edge was already updated at least once. This shows that the invariant law of $\mathbf{B}$ is $\pi_{\alpha,\beta}$, under which the state of every edge is independent and it is open with probability $\frac{\alpha}{\alpha+\beta}$. Not surprisingly this means that the invariant law $\pi = \pi_{\alpha,\beta}$ depends on the parameters α and β.

Now we turn our attention to the main objective of this chapter, which is to provide an oriented site percolation model which is coupled to the CPDP in such a way that the percolation model survives if and only if the infection process of the CPDP survives. The strategy of this coupling is not new. We define so called "good" blocks, which satisfy certain desirable properties guaranteeing survival throughout a large space-time box and also let the process end in a advantageous state such that it can survive throughout the next good blocks with high probability. Using these good blocks we construct an oriented site percolation on a "macroscopic" grid, where the sites correspond to the space-time boxes.

As already mentioned, this particular block construction was initially developed by [BG90] for the classical contact process, which they then used to show that the contact process dies out at criticality. It can also be used to show complete convergence and an asymptotic shape theorem. We mainly follow [Lig13, Part I.2], since he describes a version of this construction in a neat and detailed manner. We are not the first ones to adapt these techniques to a variation of the contact process. This was already done by several people, for example the already mentioned works [Rem08] and [SW08] did this for a contact process with varying recovery rates and in [Des14] this was done for a contact process with ageing.

This chapter is arranged as follows: In Section 6.1 we will introduce two finite space-time conditions and show that if survival of the CPDP is possible, i.e. a positive survival probability, this implies already that these conditions are satisfied. We use these results to construct the oriented site percolation previously mentioned in Section 6.2. The so constructed coupling yields the equivalence of the finite space-time condition and the possibility of survival. At last we use this comparison tool in Section 6.3, where we prove the equivalent conditions for complete convergence, i.e. that (1.8) and (1.9) are satisfied. Therefore, we can use Theorem 1.4.15 to conclude that for the CPDP complete convergence holds. Furthermore, we will also show that the CPDP dies out at criticality. This enables us to show continuity of the survival probability.

6.1 A finite space-time condition which is equivalent to survival of the CPDP

In this section we formulate the aforementioned finite space-time conditions, which we will show to be equivalent to survival of the CPDP. For this, we introduce a truncated version of the CPDP on a finite space-time box. For an arbitrary but fixed $L \in \mathbb{N}$ set

$$V_L := \mathbb{Z}^d \cap [-L, L]^d \text{ and } E_L := \{e : e \cap V_L \in E\}$$

and denote this truncated version by $({}_L\mathbf{C}, {}_L\mathbf{B})$. This process can again be defined via a graphical representation with the difference that we only consider the finite graph $G_L = (V_L, E_L)$ instead of G. Therefore, only flip events influencing edges in E_L are considered and for the infection process we only consider recovery symbols on sites $x \in [-L, L]^d \cap \mathbb{Z}^d$ and infection events which emanate from a site $x \in (-L, L)^d \cap \mathbb{Z}^d$.

Remark 6.1.1. Note that we abuse notation slightly in the way that if we say $({}_L\mathbf{C}, {}_L\mathbf{B})$ has initial configuration $C \subset V$ and $B \subset E$, we instead consider $C \cap V_L$ and $B \cap E_L$ as the initial configuration. Furthermore, we often consider all sites to be initially infected in a box $[-n, n]^d \cap \mathbb{Z}^d$ to keep the formulas somewhat "cleaner" we omit the intersection with $\mathbb{Z}^d$ and write for example $\mathbf{C}_t^{[-n,n]^d,B}$ instead of $\mathbf{C}_t^{[-n,n]^d \cap \mathbb{Z}^d,B}$.

Now we are ready to formulate the above mentioned conditions on the finite space-time box $[-L, L]^d \times [0, T + 1]$, where $T > 0$. For that we need to consider the events

$$\mathcal{A}_1 = \mathcal{A}_1(n, L, T) := \big\{ {}_{L+n}\mathbf{C}_{T+1}^{[-n,n]^d,\emptyset} \supset x + [-n, n]^d \text{ for some } x \in [0, L)^d \big\}, \tag{6.1}$$

$$\mathcal{A}_2 = \mathcal{A}_2(n, L, T) := \big\{ {}_{L+2n}\mathbf{C}_{t+1}^{[-n,n]^d,\emptyset} \supset x + [-n, n]^d \text{ for some } 0 \leq t < T \text{ and } x \in \{L + n\} \times [0, L)^{d-1} \big\}. \tag{6.2}$$

In words the event $\mathcal{A}_1$ states if we start the truncated CPDP the initial configuration $([-n,n]^d, \emptyset)$ we find a spatial shifted version of the box $[-n,n]^d$ at the top of a bigger space-time box $[-(L+n), L+n]^d \times [0, T+1]$. On the other hand the event $\mathcal{A}_2$ states that we instead find a spatial shifted version of the box $[-n,n]^d$ at the "right" boundary, in direction of the first coordinate, of the bigger space-time box. In broad terms one could say that these events guarantee that throughout this big space-time box the infections survives at least as "strong" as it started. We illustrate the cross section of these events in the direction of the first coordinate axis of the two events in Figure 6.1.

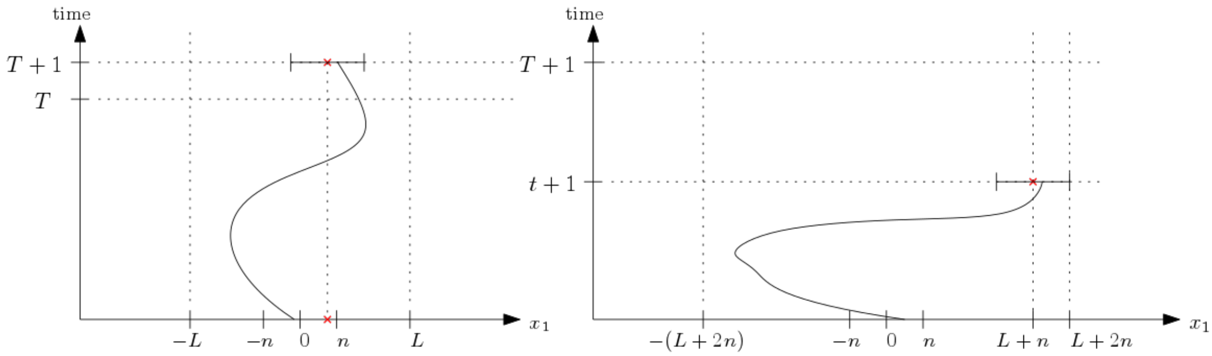


Figure 6.1: Illustration of the events in (6.1) and (6.2)

The finite space-time condition, which we will impose are that we can choose the parameters n, L and T in such a way that these events happen with "high" probability.

Condition 6.1.2. *For all $\varepsilon > 0$ there exist $n, L \geq 1$ and $T > 0$ such that*

$$\mathbb{P}(\mathcal{A}_1) > 1 - \varepsilon \quad \textit{and} \quad \mathbb{P}(\mathcal{A}_2) > 1 - \varepsilon.$$

The goal of this section is to show that if survival is possible, i.e. $\theta(\lambda, r, \alpha, \beta, \{\mathbf{0}\}, \emptyset) > 0$, then Condition 6.1.2 is satisfied. This takes some effort to prove. We start by showing an approximation result for the survival probability.

Proposition 6.1.3. *For every $B \subset E$, $N \geq 1$ and $C \subset \mathbb{Z}^d$ finite*

$$\lim_{t\to\infty} \lim_{L\to\infty} \mathbb{P}^{(C,B)}(|{}_L\mathbf{C}_t| \geq N) = \mathbb{P}^{(C,B)}(\mathbf{C}_t \neq \emptyset \; \forall t \geq 0).$$

Proof. Recall that $\widetilde{\mathbf{C}}$ denotes the classical contact process without recovery. For a given $t \geq 0$, by Lemma 5.1.8 it follows that for any $\varepsilon > 0$ there exists a finite $D \subset V$ such that

$$\mathbb{P}(\widetilde{\mathbf{C}}_t^C \subset D) > 1 - \varepsilon$$

Recall that D_E denotes the set of all edges which are attached to a site in D. Now let $L_0 \in \mathbb{N}$ be large enough such that $D \subset V_L$ and $D_E \subset E_L$ for all $L \geq L_0$. Since we consider a dynamical percolation as background it follows ${}_L\mathbf{B}_s^B = \mathbf{B}_s^B$ on D_E for all $s \geq 0$, since edges do not interact with each other. This implies that

$$\mathbb{P}({}_L\mathbf{C}_t^{C,B} = \mathbf{C}_t^{C,B}) > 1 - \varepsilon$$

for all $L > L_0$. Therefore, we get that for every $\varepsilon > 0$ there exist an $L_0 \in \mathbb{N}$ such that

$$|\mathbb{P}^{(C,B)}(|\mathbf{C}_t| \geq N) - \mathbb{P}^{(C,B)}(|{}_L\mathbf{C}_t| \geq N)| < \varepsilon$$

for all $L > L_0$. This implies $\lim_{L\to\infty} \mathbb{P}^{(C,B)}(|{}_L\mathbf{C}_t| \geq N) = \mathbb{P}^{(C,B)}(|\mathbf{C}_t| \geq N)$. Hence, it remains to show that

$$\lim_{t\to\infty} \mathbb{P}^{(C,B)}(|\mathbf{C}_t| \geq N) = \mathbb{P}^{(C,B)}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0).$$

The idea is to split this up and show for all $N \geq 1$,

$$\lim_{t\to\infty} \mathbb{P}^{(C,B)}(|\mathbf{C}_t| \geq N, \mathbf{C}_s = \emptyset \text{ for some } s > 0) = 0, \tag{6.3}$$

$$\lim_{t\to\infty} \mathbb{P}^{(C,B)}(|\mathbf{C}_t| \geq N, \mathbf{C}_s \neq \emptyset \ \forall s > 0) = \mathbb{P}^{(C,B)}(\mathbf{C}_t \neq \emptyset \ \forall t > 0). \tag{6.4}$$

Now (6.3) follows immediately by Fatou's lemma, since we get that

$$\begin{aligned} &\limsup_{t\to\infty} \mathbb{P}^{(C,B)}(|\mathbf{C}_t| \geq N, \mathbf{C}_s = \emptyset \text{ for some } s > 0) \\ \leq &\mathbb{E}^{(C,B)}\Big[\limsup_{t\to\infty} \mathbb{1}_{\{|\mathbf{C}_t| \geq N, \mathbf{C}_s = \emptyset \text{ for some } s>0\}}\Big] = 0. \end{aligned}$$

Note that obviously the integrand convergences to 0 pointwise, as $\emptyset$ is an absorbing state.

Next we see that by the martingale convergence theorem

$$\mathbb{P}^{(\mathbf{C}_s,\mathbf{B}_s)}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0) = \mathbb{P}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0 | \mathcal{F}_s) \to \mathbb{1}_{\{\mathbf{C}_t \neq \emptyset \ \forall t \geq 0\}} \tag{6.5}$$

as $s \to \infty$ almost surely, where the first equation follows by the Markov property and for the limit we used that the event of survival is a tail event, i.e. measurable by the terminal σ-algebra.

Let us assume that at time s there are N infected sites. Then the probability that all these n sites recover before from any of these N sites an outwards pointing infection arrow occurs is $\left(\frac{r}{r+2d\lambda}\right)^N$. This implies due to the Markov property that

$$\mathbb{P}^{(\mathbf{C}_s,\mathbf{B}_s)}(\mathbf{C}_t \neq \emptyset \ \forall t \geq 0) \leq 1 - \left(\frac{r}{r+2d\lambda}\right)^{|\mathbf{C}_s|}. \tag{6.6}$$

Now note that to show (6.4) it suffices to show that

$$\lim_{t\to\infty} |\mathbf{C}_t| = \infty \text{ almost surely on } \{\mathbf{C}_t \neq \emptyset \ \forall t \geq 0\}.$$

We show this by contradiction by assuming that

$$\mathbb{P}^{(C,B)}\big(\lim_{t\to\infty} |\mathbf{C}_t| \neq \infty, \ \mathbf{C}_t \neq \emptyset \ \forall t \geq 0\big) > 0. \tag{6.7}$$

Now for every $\omega \in \{\lim_{t\to\infty} |\mathbf{C}_t| \neq \infty\}$ we find a $M(\omega) > 0$ and a sequence $(\tau_n(\omega))_{n\in\mathbb{N}}$ such that $\tau_n(\omega) \leq \tau_{n+1}(\omega)$, $\tau_n(\omega) \to \infty$ as $n \to \infty$ and $|\mathbf{C}_{\tau_n}(\omega)| < M(\omega)$. For every $\omega \in \{\lim_{t\to\infty} |\mathbf{C}_t| = \infty\}$ set $\tau_n(\omega) = n$. But this yields for $\omega \in \{\lim_{t\to\infty} |\mathbf{C}_t| \neq \infty\}$ that

$$\mathbb{P}^{(\mathbf{C}_{\tau_n},\mathbf{B}_{\tau_n})}(\mathbf{C}_t \neq \emptyset, \ \forall t \geq 0)(\omega) \leq 1 - \left(\frac{r}{r+2d\lambda}\right)^{M(\omega)} < 1,$$

for every $n \geq 1$. Letting $n \to \infty$ yields together with (6.5) a contradiction to (6.7). Thus the proof is complete. □

Let us recall that a measure μ is said to have positive correlations if

$$\int fg \mathsf{d}\mu \geq \int f \mathsf{d}\mu \int g \mathsf{d}\mu \tag{6.8}$$

for all increasing functions f and g. As already mentioned $({}_L\mathbf{C}, {}_L\mathbf{B})$ is constructed via a graphical representation, and thus it is also a Feller process. Let

$$q^* : \big(\mathcal{P}(V_L) \times \mathcal{P}(E_L)\big)^2 \to \mathbb{R}_+$$

denote the transition rates of $({}_L\mathbf{C}, {}_L\mathbf{B})$. Now we can use [Lig12, Theorem II.2.14]. Preceding this theorem it is nicely described that for interacting particle systems on finite state spaces this theorem ensures that

$$\begin{aligned} & q^*((C,B),(C',B')) > 0 \Rightarrow (C,B) \supset (C',B') \text{ or } (C,B) \subset (C',B') \\ \Leftrightarrow \quad & \mu T^L(t) \text{ has positive correlation whenever } \mu \text{ does,} \end{aligned} \tag{6.9}$$

where T^L denotes the Feller semigroup of $({}_L\mathbf{C}, {}_L\mathbf{B})$. This is easy to check for $({}_L\mathbf{C}, {}_L\mathbf{B})$, since every event in the graphical representation only affects one site or edge at a time.

Proposition 6.1.4. *For every $n, N \geq 1$ and $L \geq n$,*

$$\mathbb{P}^{([-n,n]^d,\emptyset)}(|{}_L\mathbf{C}_t \cap [0,L)^d| \leq N) \leq \big(\mathbb{P}^{([-n,n]^d,\emptyset)}(|{}_L\mathbf{C}_t| \leq 2^d N)\big)^{2^{-d}}.$$

Proof. Let us define $X_1 := |{}_L\mathbf{C}_t^{[-n,n]^d,\emptyset} \cap [0,L)^d|$ and $X_2, \dots, X_{2^d}$ analogously for the other orthants in $\mathbb{R}^d$. Obviously $X_1, \dots, X_{2^d}$ are identically distributed random variables. Furthermore, as functions of ${}_L\mathbf{C}_t^{[-n,n]^d,\emptyset}$ the X_m are increasing functions for every $m \in \{1, \dots, 2^d\}$. Thus, since (6.9) proves that the measure $\mathbb{P}^{([-n,n]^d,\emptyset)}\big(({}_L\mathbf{C}_t, {}_L\mathbf{B}_t) \in \cdot\big)$ has positive correlations, (6.8) yields that the events $(\{X_m \leq N\})_{m \in \{1,\dots,2^d\}}$ are positively correlated. This implies

$$\begin{aligned}\mathbb{P}\big(|{}_L\mathbf{C}_t^{[-n,n]^d,\emptyset}| \leq 2^d N\big) &\geq \mathbb{P}\big(X_1 + \dots + X_{2^d} \leq 2^d N\big) \\ &\geq \mathbb{P}\Big(\bigcap_{m=1}^{2^d} \{X_m \leq N\}\Big) \geq \big(\mathbb{P}(X_1 \leq N)\big)^{2^d}.\end{aligned}$$

□

For $L \in \mathbb{N}$ and $T \geq 0$ we set

$$S(L,T) := \{(x,t) \in \mathbb{Z}^d \times [0,T] : ||x||_\infty = L\}.$$

This is the union of all lateral faces of the space-time box $[-L,L]^d \times [0,T]$. Now we fix a $C \subset (-L,L)^d \cap \mathbb{Z}^d$. We want to consider all points in $S(L,T)$ which can be reached from C through an $\emptyset$-infection path, i.e. an infection path which starts with the background in state $\emptyset$.

Let us define $N_\emptyset^C(L,T)$ to be the maximal number of points in any $D \subset S(L,T) \cap {}_L\mathbf{C}^{C,\emptyset}$, where D has the property that every two points with the same spatial coordinate $(x,t_1) \in D$ and $(x,t_2) \in D$ satisfy $|t_2 - t_1| \geq 1$. Obviously subsets which satisfy this property exist. Since $S(L,T)$ is bounded every subset which satisfies this property can only contain finitely many points, and therefore the maximal number also exists. Of course there might be more than one subset whose cardinality is equal to the maximal number of points.

The next result provides us with a connection of the extinction probability and having "few" infected points at the top and lateral faces of a large space-time box.

Lemma 6.1.5. *Let $L_j \uparrow \infty$ and $T_j \uparrow \infty$. Then for all $M, N \geq 1$ and finite C, we have*

$$\limsup_{j\to\infty} \mathbb{P}(N_\emptyset^C(L_j, T_j) \leq M)\mathbb{P}^{(C,\emptyset)}(|\,_{L_j}\mathbf{C}_{T_j}| \leq N) \leq \mathbb{P}^{(C,\emptyset)}(\mathbf{C}_t = \emptyset \textit{ for some } t \geq 0)$$

Proof. Let $\mathcal{F}_{L,T}$ be the σ-algebra generated by the Poisson point process Ξ of the graphical representation restricted to $V_L \times [0,T]$ and $E_L \times [0,T]$. Let us assume that $L \geq 1$ is large enough such that $C \subset (-L, L)^d \cap \mathbb{Z}^d$ and we already know that

$$\begin{aligned} \mathbb{P}^{(C,\emptyset)}(\mathbf{C}_t = \emptyset \text{ for some } t > 0 | \mathcal{F}_{L,T}) \geq \Big(\frac{r}{r+2d\lambda}\Big)^N \exp(-4d\lambda M) > 0 \\ \text{almost surely on } \{N_\emptyset^C(L,T) \leq M,\ |\,_L\mathbf{C}_T^{C,\emptyset}| \leq N\}. \end{aligned} \tag{6.10}$$

Note that we show (6.10) in the second part of the proof.

Then as in the proof of Proposition 6.1.3 we will make use of the martingale convergence theorem together with using positive correlations of the appropriate events as in Proposition 6.1.4. Let us fix arbitrary $M, N \geq 0$ and set

$$\begin{aligned} G &:= \{\mathbf{C}_t^{C,\emptyset} = \emptyset \text{ for some } t > 0\} \\ H_j &:= \{N_\emptyset^C(L_j, T_j) \leq M,\ |\,_{L_j}\mathbf{C}_{T_j}^{C,\emptyset}| \leq N\} \end{aligned}$$

for all $j \geq 0$. Then again by the martingale convergence theorem we get that

$$\mathbb{P}(G|\mathcal{F}_{L_j,T_j}) \to \mathbb{1}_G \text{ almost surely as } j \to \infty.$$

Now equation (6.10) implies that on H_j the conditional probability $\mathbb{P}(G|\mathcal{F}_{L_j,T_j})$ is bounded from below by a positive constant which is independent of j. Thus, $\mathbb{1}_G = \lim_{j\to\infty} \mathbb{P}(G|\mathcal{F}_{L_j,T_j}) > 0$ on $\{H_j \text{ i.o.}\}$, which implies that $\{H_j \text{ i.o.}\} \subset G$ and therefore

$$\limsup_{j\to\infty} \mathbb{P}(H_j) \leq \mathbb{P}(G).$$

Now we only need to use positive correlations again in order to see that

$$\mathbb{P}(N_\emptyset^C(L,T) \leq M,\ |\,_L\mathbf{C}_T^{C,\emptyset}| \leq N) \geq \mathbb{P}(N_\emptyset^C(L,T) \leq M)\mathbb{P}(|\,_L\mathbf{C}_T^{C,\emptyset}| \leq N).$$

Then putting the two pieces together proves the claim.

Now it remains to show (6.10). We consider the infected sites at the "top" of the space-time box, i.e. the set $_L\mathbf{C}_T^{C,\emptyset}$. Then by the same argument used to obtain (6.6) we can conclude that the probability that all infections originating from ${}_L\mathbf{C}_T^{C,\emptyset}$ conditional on $|{}_L\mathbf{C}_T^{C,\emptyset}| = N$ go extinct, is bounded from below by $\left(\frac{r}{r+2d\lambda}\right)^N$. Here we used that for $x, y \in \mathbb{Z}^d$ the random sets $\{t \geq 0 : (\mathbf{coop}_{x,y}, t) \in \Xi^{\inf}\}$ and $\{t \geq 0 : (\mathbf{coop}_{y,x}, t) \in \Xi^{\inf}\}$ are independent.

Now let us consider the time lines $\{x\} \times [0, T]$ above $(x, 0)$, where $||x||_\infty = L$ and let

$$(x, s_1), \ldots, (x, s_n) \in \{x\} \times [0, T]$$

be points of a maximal set of points on this time line contained in $S(L, T) \cap {}_L\mathbf{C}^{C,\emptyset}$ which satisfy that each pair is separated by at least the distance 1, where $n = n(x)$. Assume that $n \geq 1$ and let

$$I := \bigcup_{i=1}^{n} (\{x\} \times (s_i - 1, s_i + 1)) \cap (\{x\} \times [0, T]).$$

Now all infected points in $\{x\} \times [0, T]$ are contained in I, i.e. if $x \in {}_L\mathbf{C}_s^{C,\emptyset}$ for $s \leq T$ then $(x, s) \in I$. Otherwise there would exist a point $(x, u) \in (\{x\} \times [0, T]) \cap {}_L\mathbf{C}^{C,\emptyset}$ such that $|u - s_i| > 1$ for every $i \in \{1, \ldots n\}$, which would violate the assumption of maximality. The Lebesgue measure of the time coordinate of I is at most $2n$. Let us denote by A_x the event that no infection arrow of x emanates from I towards any of its $2d$ neighbors. The probability $\mathbb{P}(A_x)$ is bounded below by $e^{-4dn\lambda}$. On the other hand, we already concluded that the complement of I with respect to the time line $\{x\} \times [0, T]$ can contain no infected space-time point, so that any infection arrow emanating from it cannot contribute to survival of the infection. Note that the initial set of infections is contained in the large space-time box.

The events of the Poisson point processes used in the graphical representation which happen before and after T are independent, since they take place on disjoint parts. This means that the contributions of the points in ${}_L\mathbf{C}_T^{C,\emptyset}$ and the contributions of the several time lines $\bigcap_{x:||x||_\infty = L} A_x$ are independent. Also note that the events A_x are independent and $\sum_{x:||x||_\infty = L} n(x) = N_\emptyset^C(L, T)$. Thus, we get that

$$\mathbb{P}^{(C,\emptyset)}(\mathbf{C}_t = \emptyset \text{ for some } t > 0 | \mathcal{F}_{L,T}) \geq \left(\frac{r}{r + 2d\lambda}\right)^{|{}_L\mathbf{C}_T^{C,\emptyset}|} \exp(-4d\lambda |N_\emptyset^C(L, T)|),$$

which implies (6.10). We want to remark here on that $\bigcap_{x:||x||_\infty = L} A_x$ the infection cannot leave the large space time box before T, i.e. ${}_L\mathbf{C}_t^{C,\emptyset} = \mathbf{C}_t^{C,\emptyset}$ for $t \leq T$. Therefore, the lower bound on extinction of ${}_L\mathbf{C}^{C,\emptyset}$ is also a lower bound for extinction of $\mathbf{C}^{C,\emptyset}$. □

Set for $L \in \mathbb{N}$ and $T > 0$

$$S_+(L,T) := \{(x,s) \in \mathbb{Z}^d \times [0,T] : x_1 = +L, x_i \geq 0 \text{ for } 2 \leq i \leq d\}.$$

This is the intersection of one particular lateral face of the box $[-L,L]^d \times [0,T]$ with the first orthant. Let $C \subset (-L,L)^d \cap \mathbb{Z}^d$. Similar as before let $N^C_{+,\emptyset}(L,T)$ be the maximal number of points in any subset $D \subset S_+(L,T) \cap {}_L\mathbf{C}^{C,\emptyset}$ such that the points fulfill the following property: If $(x,t_1) \in D$ and $(x,t_2) \in D$ are any two points with the same spatial coordinate, then $|t_2 - t_1| \geq 1$.

Proposition 6.1.6. *For every $n, M \geq 1$ and $L \geq n$,*

$$\big(\mathbb{P}(|N^{[-n,n]^d}_{+,\emptyset}(L,T)| \leq M)\big)^{d2^d} \leq \mathbb{P}(|N^{[-n,n]^d}_{\emptyset}(L,T)| \leq Md2^d)$$

Proof. Note that $S(L,T)$ consists of $2d$-many lateral faces and there exist 2^d orthants. So if we take every non-empty intersection of a lateral faces and an orthants we decompose $S(L,T)$ in $d2^d$ disjoint hypersurfaces. Next let $X_1, \dots, X_{d2^d}$ be the maximal number of infected points contained in the those hypersurfaces, for example $X_1 = N^C_{+,\emptyset}(L,T)$. Analogously to Proposition 6.1.4 we obtain that

$$\big(\mathbb{P}(|N^{[-n,n]^d}_{+,\emptyset}(L,T)| \leq M)\big)^{d2^d} \leq \mathbb{P}\Big(\bigcap_{k=1}^{d2^d}\{X_m \leq M\}\Big).$$

Now we know that on the event on the right-hand side each of the $d2^d$ many disjoint parts of the lateral sides cannot contain more than M elements. Thus, if we add all parts together we know that $S(L,T)$ cannot contain more than $Md2^d$ many infected space-time points on this event and this implies the claim. □

Finally we are able to show the first direction of the desired equivalence.

Theorem 6.1.7. *Suppose $\theta(\lambda, r, \alpha, \beta, \{\mathbf{0}\}, \emptyset) > 0$, then Condition 6.1.2 is satisfied.*

Proof. The proof consists of three parts. First we derive some bounds on the probabilities of crucial events and then we use these results to derive the first and second bound of Condition 6.1.2 successively.

Let $0 < \delta < 1$. We will later on specify how to choose δ exactly. By Lemma 5.1.11 we know that there exists an $n = n(\delta)$ such that

$$\mathbb{P}(\mathbf{C}_t^{[-n,n]^d,\emptyset} \neq \emptyset \ \forall t \geq 0) \geq 1 - \delta^2. \tag{6.11}$$

Given n we can choose $N' = N'(n,\delta)$ such that

$$\Big(1 - \mathbb{P}\big({}_{n+1}\mathbf{C}_1^{\{\mathbf{0}\},\emptyset} \supset [-n,n]^d\big)\Big)^{N'} < \delta, \tag{6.12}$$

since $\mathbb{P}\big({}_{n+1}\mathbf{C}_1^{\{\mathbf{0}\},\emptyset} \supset [-n,n]^d\big) > 0$. In the next step we choose $N = N(N')$ large enough such that for $A \subset \mathbb{Z}^d$ with $|A| > N$, there exists $D = D(A)$ such that $D \subset A$ with $|D| > N'$ and $||x-y||_\infty \geq 2n+1$ for all $x, y \in D$ with $x \neq y$. In words, N needs to be large enough such that any subset, of size at least N contains at least N' elements that are all spaced a distance $2n+1$ apart. Later on we will consider the probability

$$\begin{aligned} a := \mathbb{P}(&\text{there exist } \emptyset\text{-infection paths contained in } [0,2n] \times [-n,n]^{d-1} \times [0,1] \\ &\text{from } (\mathbf{0},0) \text{ to every point in } [0,2n] \times [-n,n]^{d-1} \times \{1\}) \end{aligned} \tag{6.13}$$

where it is clear that $a > 0$. We choose $M' = M'(n,\delta)$ such that

$$(1-a)^{M'} < \delta. \tag{6.14}$$

Then choose $M = M(M')$ such that if $F \subset \mathbb{Z}^d \times \mathbb{R}_+$ is a finite set with $|F| \geq M$ and the distance of points with the same spatial coordinates is at least one, there exists an $H = H(F)$ with $H \subset F$ and $|H| \geq M'$ such that for two points $(x,t) \in H$ and $(y,s) \in H$ it holds that either

$$x = y,\ |t-s| \geq 1 \quad \text{or} \quad ||x-y||_\infty \geq 2n+1. \tag{6.15}$$

Now it obviously holds that $1-\delta < 1-\delta^2$ and we know from Proposition 6.1.3 and (6.11) that

$$\lim_{t\to\infty}\lim_{L\to\infty} \mathbb{P}^{([-n,n]^d,\emptyset)}(|{}_L\mathbf{C}_t^{[-n,n]^d,\emptyset}| \geq 2^d N) = \mathbb{P}(\mathbf{C}_t^{[-n,n]^d,\emptyset} \neq \emptyset\ \forall t \geq 0) \geq 1-\delta^2. \tag{6.16}$$

Next we will construct two strictly increasing sequences $(T_k)_{k\geq 0}$ and $(L_k)_{k\geq 0}$ such that $T_k, L_k \uparrow \infty$ and

$$\mathbb{P}(|{}_{L_k}\mathbf{C}_{T_k}^{[-n,n]^d,\emptyset}| > 2^d N) = 1-\delta \tag{6.17}$$

for all $k \geq 0$. But to construct these sequences we need two more properties. Since $({}_L\mathbf{C}_t, {}_L\mathbf{B}_t)$ is a Feller process, we know that

$$t \mapsto \mathbb{P}(|\,{}_L\mathbf{C}_t^{[-n,n]^d,\emptyset}| > 2^d N) \text{ is continuous} \tag{6.18}$$

and since the contact process on a finite graph dies out almost surely, and therefore also a CPDP on a finite graph, we can conclude that

$$\lim_{t\to\infty} \mathbb{P}(|\,{}_L\mathbf{C}_t^{[-n,n]^d,\emptyset}| > 2^d N) = 0. \tag{6.19}$$

First by (6.16) there exist a $T_0' > 0$ such that

$$\lim_{L\to\infty} \mathbb{P}(|\,{}_L\mathbf{C}_t^{[-n,n]^d,\emptyset}| > 2^d N) > 1 - \delta \tag{6.20}$$

for all $t \geq T_0'$. Obviously there exists an $L_0 \in \mathbb{N}$ such that

$$\mathbb{P}(|\,{}_{L_0}\mathbf{C}_{T_0'}^{[-n,n]^d,\emptyset}| > 2^d N) > 1 - \delta.$$

Now we keep L_0 fixed, then by (6.18) and (6.19) it follows that there exists a $T_0 > T_0'$ such that (6.17) holds for $k = 0$. Now we define the sequences recursively. Now choose $T_1' > T_0 + 1$. Since in particular $T_1' > T_0'$ by (6.20) it follows that

$$\lim_{L\to\infty} \mathbb{P}(|\,{}_L\mathbf{C}_{T_1'}^{[-n,n]^d,\emptyset}| > 2^d N) > 1 - \delta.$$

Again there exists an L_1' such that

$$\mathbb{P}(|\,{}_{L_1'}\mathbf{C}_{T_1'}^{[-n,n]^d,\emptyset}| > 2^d N) > 1 - \delta$$

Now set $L_1 = \max(L_1', L_0 + 1)$, note that by monotonicity the strict inequality still holds with L_1 instead of L_1'. Analogously as before by (6.18) and (6.19) we find an $T_1 > T_1'$ such that (6.17) holds for $k = 1$. We can repeat this procedure recursively such that (6.17) holds for all $k \geq 0$.

Using this particular choice of L_k and T_k's together with Lemma 6.1.5 yields that for some $k \geq 0$

$$\begin{aligned}\delta\mathbb{P}\big(N_\emptyset^{[-n,n]^d}(L_k, T_k) \leq Md2^d\big) =& \mathbb{P}\big(N_\emptyset^{[-n,n]^d}(L_k, T_k) \leq Md2^d\big)\mathbb{P}\big(|\,{}_{L_k}\mathbf{C}_{T_k}^{[-n,n]^d,\emptyset}| \leq 2^d N\big)\\ \leq& \mathbb{P}\big(\mathbf{C}_t^{[-n,n]^d,\emptyset} = \emptyset \text{ for some } t \geq 0\big) + \delta^2 < 2\delta^2,\end{aligned}$$

where we used (6.11) in the last inequality. Thus,

$$\mathbb{P}\big(N_{\emptyset}^{[-n,n]^d}(L_k, T_k) \le Md2^d\big) < 2\delta \tag{6.21}$$

for some $k \ge 0$. Now letting $T = T_k$ and $L = L_k$ for this k we get by using Proposition 6.1.4 and 6.1.6 that

$$\begin{aligned}\mathbb{P}\big(|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d| \le N\big) &\le \big(\mathbb{P}\big(|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset}| \le 2^d N\big)\big)^{2^{-d}},\\ \mathbb{P}\big(|N_{+,\emptyset}^{[-n,n]^d}(L,T)| \le M\big) &\le \big(\mathbb{P}\big(|N_{\emptyset}^{[-n,n]^d}(L,T)| \le Md2^d\big)\big)^{d^{-1}2^{-d}},\end{aligned}$$

which implies due to (6.17) and (6.21) that

$$\mathbb{P}(|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d| > N) \ge 1 - \big(\mathbb{P}(|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset}| \le 2^d N)\big)^{2^{-d}} = 1 - \delta^{2^{-d}}, \tag{6.22}$$

$$\mathbb{P}(|N_{+,\emptyset}^{[-n,n]^d}(L,T)| > M) \ge 1 - \big(\mathbb{P}(|N_{\emptyset}^{[-n,n]^d}(L,T)| \le Md2^d)\big)^{\frac{1}{d2^d}} > 1 - (2\delta)^{\frac{1}{d2^d}}. \tag{6.23}$$

Now we attend to the first inequality in Condition 6.1.2. Let us define for every $D \subset V$ and $T > 0$,

$$\begin{aligned} W_D^T = \{&\exists x \in D \text{ such that there are } \emptyset\text{-infection paths from } (x,T) \text{ to every}\\ &(y, T+1) \text{ with } y \in (x + [-n,n]^d) \text{ that stay in } (x+[-n,n]^d) \times (T, T+1]\}.\end{aligned}$$

Now let $A \subset [0,L)^d$ with $|A| > N$. Recall that $D(A)$ is a subset of A containing at least N' elements, which are all spaced a distance $2n+1$ apart. We see that for any such A

$$\begin{aligned}&\{|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d| > N, {}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d = A\} \cap W_{D(A)}^T\\ \subset&\{{}_{L+n}\mathbf{C}_{T+1}^{[-n,n]^d,\emptyset} \supseteq x + [-n,n]^d \text{ for some } x \in [0,L)^d\}.\end{aligned} \tag{6.24}$$

The inclusion holds since the first event on the left-hand side guarantees that at time T more than N sites contained in $[0,L)^d$ are infected and the second event guarantees that one of the infected sites $x \in D(A)$ infects $x + [-n,n]^d$. Also by the restrictions imposed in the event it is clear that the paths stay in the space box $[-(n+L), n+L]^d$.

Let $x_1, x_2 \in D(A)$ with $x_1 \ne x_2$. Note that by definition of $W_{D(A)}^T$ and $D(A)$ it follows that for $i = 1,2$ the events that (x_i, T) infect the whole set $x_i + [-n,n]^d$ at time $T+1$

are independent since the paths must be contained in $(x_i + [-n,n]^d) \times (T, T+1]$. Now for every $A \subset [0,L)^d$ with $|A| > N$ by choice of $D(A)$ and N' with (6.12) we know that

$$\mathbb{P}\big(W^T_{D(A)}\big|\mathcal{F}_T\big) = \mathbb{P}\big(W^T_{D(A)}\big) > 1 - \Big(1 - \mathbb{P}\big({}_{n+1}\mathbf{C}_1^{\{\mathbf{0}\},\emptyset} \supset [-n,n]^d\big)\Big)^{N'} > 1-\delta. \quad (6.25)$$

Thus, $\mathbb{P}\big(W^T_{D(A)}\big|\mathcal{F}_T\big) > 1-\delta$. Note that we used that $W^T_{D(A)}$ only depends on the graphical representation on the time interval $(T, T+1]$, since disjoint parts of the graphical representation are independent, $W^T_{D(A)}$ is independent of $\mathcal{F}_T$. Obviously we have that

$$\begin{aligned}&\bigcup_{A\subseteq[0,L)^d} \{|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d| > N, {}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d = A\} \\ =&\{|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d| > N\}.\end{aligned} \quad (6.26)$$

Now we choose an arbitrary but fixed subset $A' \subset [0,L)^d$ with $|A'| > N$. By using (6.24), (6.25), (6.26) and the just mentioned independence of disjoint parts of the graphical representation we obtain

$$\begin{aligned}&\mathbb{P}\big({}_{L+n}\mathbf{C}_{T+1}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } x \in [0,L)^d\big) \\ \geq&\mathbb{P}\Big(\bigcup_{A\subseteq[0,L)^d} \{|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d| > N, {}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d = A\} \cap W^T_{D(A)}\Big) \\ =&\sum_{A\subseteq[0,L)^d} \mathbb{E}\Big(\mathbb{1}_{\{|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset}\cap[0,L)^d|>N, {}_L\mathbf{C}_T^{[-n,n]^d,\emptyset}\cap[0,L)^d=A\}} \underbrace{\mathbb{P}(W^T_{D(A)}|\mathcal{F}_T)}_{>1-\delta}\Big) \\ \geq&\mathbb{P}\big(|\,{}_L\mathbf{C}_T^{[-n,n]^d,\emptyset} \cap [0,L)^d| > N\big)(1-\delta).\end{aligned}$$

By using (6.22) we get

$$\mathbb{P}\big({}_{L+n}\mathbf{C}_{T+1}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } x \in [0,L)^d\big) > (1-\delta)(1-\delta^{2^{-d}}).$$

By an adequate choice of δ we obtain the first inequality of Condition 6.1.2. Of course this must be done in accordance with the second inequality, which we attend to next. First of all let us recall that $N_{+,\emptyset}^{[-n,n]^d}(L,T)$ denotes the maximal number of infected time points in the intersection of the first orthant and the lateral face with the first space coordinate being L (see right before Proposition 6.1.6). Let $\{(x_k,t_k)\}_k$ be one possible choice of maximal points counted by $N_{+,\emptyset}^{[-n,n]^d}(L,T)$. Next let $\tilde{Y}_k$ be a variable which is 1 if (x_k,t_k) infects all points in $(x_k + [0,2n]) \times [-n,n]^{d-1} \times \{t_k+1\}$ via $\emptyset$-infection paths which are contained in $(x_k+[0,2n]) \times [-n,n]^{d-1} \times (t_k, t_k+1]$ and otherwise 0.

If $N_{+,\emptyset}^{[-n,n]^d}(L,T) > M$ then we can choose M' space time points distance $2n+1$ apart in space or having the same spatial coordinate and being 1 apart in time by (6.15). We denote the just defined variables by Y_k with $1 \le k \le M'$ for these M' points. Let $\mathcal{F}_{L,T}$ be defined as in the proof of Lemma 6.1.5. It is clear that conditioned on $\mathcal{F}_{L,T}$ and restricted to $\{N_{+,\emptyset}^{[-n,n]^d}(L,T) > M\}$, the M' space-time points are determined and therefore $Y_1, \dots, Y_{M'}$ are independent. Also

$$\mathbb{P}(Y_k = 1 | \mathcal{F}_{L,T}) \mathbb{1}_{\{N_{+,\emptyset}^{[-n,n]^d}(L,T)>M\}} = a \mathbb{1}_{\{N_{+,\emptyset}^{[-n,n]^d}(L,T)>M\}}$$

for every $1 \le k \le M'$ where a was defined in (6.13). A direct conclusion is that

$$\mathbb{P}(Y_k = 1 \text{ for some } k = 1, \dots, M' | \mathcal{F}_{L,T}) = 1 - (1-a)^{M'} \\ \text{on } \{N_{+,\emptyset}^{[-n,n]^d}(L,T) > M\}.$$

Now since

$$\{Y_k = 1 \text{ for some } k = 1, \dots, M'\} \cap \{N_{+,\emptyset}^{[-n,n]^d}(L,T) > M\} \\ \subset \{{}_{L+2n}\mathbf{C}_{t+1}^{[-n,n]^d,\emptyset} \supseteq x + [-n,n]^d \text{ for some } 0 \le t < T \text{ and } x \in \{L+n\} \times [0,L)^{d-1}\}$$

we get by using that disjoint parts of the graphical representation are independent, (6.14) and (6.23) that

$$\mathbb{P}({}_{L+2n}\mathbf{C}_{t+1}^{[-n,n]^d,\emptyset} \supseteq x + [-n,n]^d \text{ for some } 0 \le t < T \text{ and } x \in \{L+n\} \times [0,L)^{d-1}) \\ > (1-\delta)(1-(2\delta)^{d^{-1}2^{-d}}).$$

By choosing δ accordingly the proof is finished and yields the claim. □

6.2 Comparison of CPDP to an oriented site percolation on a macroscopic grid

In the last section we formulated Condition 6.1.2. The events used in this condition only depended on the graphical representation in a large space-time box. We also showed that the possibility of survival of the CPDP implies this condition. The goal of this section is to prove that equivalence holds, i.e. we show that Condition 6.1.2 implies survival of the CPDP.

The strategy is to use Condition 6.1.2 to define so-called good blocks and with that to construct an oriented percolation model which is coupled to the CPDP in the sense that if percolation occurs it implies that the CPDP survives. We "stack" the good blocks in such a way that an $\emptyset$–infection path exists which connects $(\mathbf{0}, 0)$ to ∞. For this argument to work at the end of each block (in time direction) one uses the Markov property to restart the CPDP in an adequate initial state. We will see that every time we restart, we need to set the background to $\emptyset$ as its initial configuration.

But first we need to combine (6.1) and (6.2) into one, since it is more convenient to have a single condition which a good block has to fulfill. We consider the event

$$\begin{aligned}\mathcal{A}_3 = \mathcal{A}_3(n, L, T) := \big\{ {}_{2L+2n}\mathbf{C}_t^{[-n,n]^d,\emptyset} \supset x + [-n, n]^d \text{ for some } T \le t < 2T \\ \text{and } x \in [L+n, 2L+n] \times [0, 2L)^{d-1} \big\}.\end{aligned} \tag{6.27}$$

Similar as before we illustrate in Figure 6.2 the cross section in direction of the first coordinate of the event in (6.27). This event states that we start with a space box of

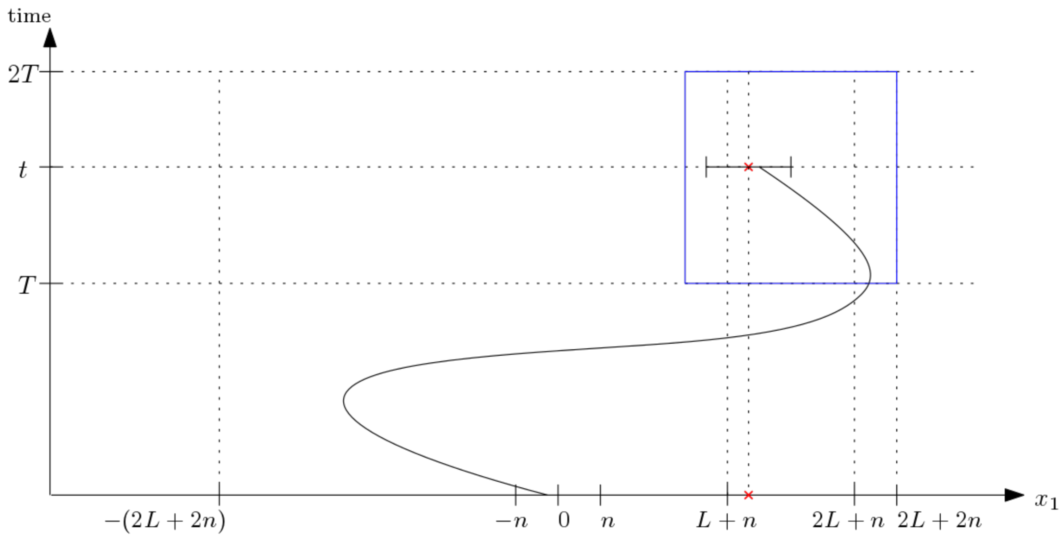


Figure 6.2: Visualization of the events in (6.27). The blue space-time box shows the area where the infected space box of length $2n$ will be contained.

infected sites, here $[-n, n]^d$ in the worst possible background configuration, then we find again such a infected space box at some later time shifted at least by $L + n$ and at most by $2L + n$ to the right along the first spatial coordinate.

Proposition 6.2.1. *Suppose Condition 6.1.2 holds. Then for every $\varepsilon > 0$, there are choices of n, L, T such that $\mathbb{P}(\mathcal{A}_3) > 1 - \varepsilon$.*

Proof. For $\varepsilon > 0$ we choose n, T, L such that Condition 6.1.2 is satisfied. Now let τ be the first hitting time such that

$$_{L+2n}\mathbf{C}_{\tau}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } x \in \{L+n\} \times [0,L)^{d-1}. \tag{6.28}$$

If $\tau < \infty$ we choose $y = y(\tau) \in \{L+n\} \times [0,L)^{d-1}$ to be one site such that $_{L+2n}\mathbf{C}_{\tau}^{[-n,n]^d,\emptyset} \supset y + [-n,n]^d$. If y is not unique, we choose it minimal with respect to an arbitrary order on $\mathbb{Z}^d$, which we picked beforehand. Out of notational convenience we set $I(L,n) := [L+n, 2L+n] \times [0,2L)^{d-1}$. We see that

$$\begin{aligned} &\big\{ {}_{2L+2n}\mathbf{C}_{t}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } T+1 \le t < 2T+2 \text{ and } x \in I(L,n) \big\} \\ \supset &\{\tau \le T+1\big\} \cap \big\{ {}_{2L+2n}\mathbf{C}_{T+\tau+1}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } x \in I(L,n) \big\}, \end{aligned} \tag{6.29}$$

where we used for this inclusion the fact that if the process satisfies the event $\mathcal{A}_2$ and then afterwards, a time and spatially shifted version of $\mathcal{A}_1$ it also satisfies the event $\mathcal{A}_3$ (see Figure 6.3 for a illustration). Furthermore,

$$\begin{aligned} &\mathbb{P}\big(\{\tau \le T+1\big\} \cap \big\{ {}_{2L+2n}\mathbf{C}_{T+\tau+1}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } x \in I(L,n) \big\}\big) \\ =&\mathbb{E}\big[\mathbb{1}_{\{\tau \le T+1\}} \mathbb{P}\big({}_{2L+2n}\mathbf{C}_{T+\tau+1}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } x \in I(L,n) | \mathcal{F}_\tau \big)\big] \\ \ge&\mathbb{P}(\tau \le T+1)\mathbb{P}\big(\underbrace{\big\{ {}_{L+n}\mathbf{C}_{T+1}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } x \in [0,L)^d \big\}}_{=\mathcal{A}_1} \big), \end{aligned} \tag{6.30}$$

where we used in the last inequality (6.28) and the strong Markov property to restart the process at time τ with $(y + [-n,n]^d, \emptyset)$ as initial state, which yields a lower bound by monotonicity. This is possible since we are on the event $\{\tau \le T+1\}$. Note we also used the spatial invariance to shift the process back to the origin. Furthermore, we shrank in the last inequality the truncation of the process from $[-2(L+n), 2(L+n)]^d$ to $[-(L+n), L+n]^d$. This is no problem since by monotonicity the probability only gets smaller. By Condition 6.1.2 we know that $\mathbb{P}(\tau \le T+1) > 1-\varepsilon$ and $\mathbb{P}(\mathcal{A}_1) > 1-\varepsilon$. This fact together with (6.29) and (6.30) yields that

$$\mathbb{P}\left({}_{2L+2n}\mathbf{C}_{t}^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \begin{array}{l} \text{for some } T+1 \le t < 2T+2 \\ \text{and } x \in [L+n, 2L+n] \times [0,2L)^{d-1} \end{array} \right) > (1-\varepsilon)^2.$$

Now set $T' := T+1$ and replacing $(1-\varepsilon)^2$ by $1-\varepsilon$ yields the claim. □

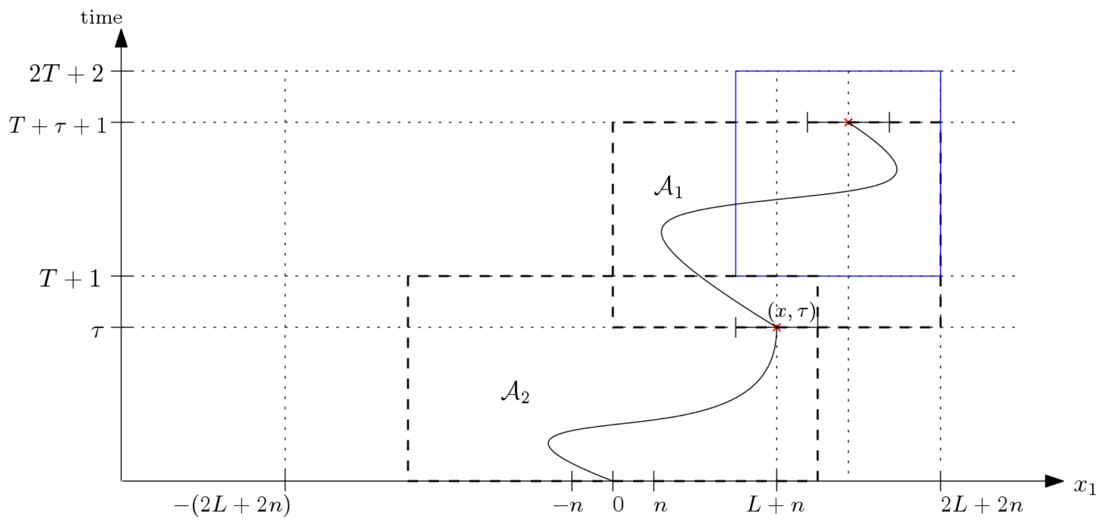


Figure 6.3: Here it is illustrated how $\mathcal{A}_3$ is constructed by first using $\mathcal{A}_2$ and then $\mathcal{A}_1$, where we restart the process with only a copy of $[-n,n]^d$ being infected and the background in the state $\emptyset$.

Since we finally obtained our key ingredient Proposition 6.2.1, we are ready to start with the construction. Let us set

$$\mathcal{D}_{j,k} := [-(1-2j)a,(1+2j)a] \times [-a,a]^{d-1} \times [5kb,(5k+1)b],$$

where $j,k \in \mathbb{Z}$ and $a,b > 0$.

Proposition 6.2.2. *Suppose Condition 6.1.2 holds, then for every $\varepsilon > 0$ there are choices of n,a,b with $n < a$ such that if $(x,s) \in \mathcal{D}_{j,k}$,*

$$\begin{aligned}\mathbb{P}\big(&\exists (y,t) \in \mathcal{D}_{j+1,k+1} \text{ s.t. there are } \emptyset\text{-infection paths that stay in} \\ &([-5a,5a]+2ja) \times [-5a,5a]^{d-1} \times [0,6b] \text{ and goes from} \\ &(x,s)+([-n,n]^d \times \{0\}) \text{ to every point in } (y,t)+([-n,n]^d \times \{0\})\big) > 1-\varepsilon.\end{aligned}$$

Proof. Without loss of generality we will asume that $j,k=0$, since we can obtain the result for arbitrary j,k by shifting the construction which follows below by a suitable space-time shift. One important fact we need to mention is that even though (6.27) is formulated in such a way that x is in the the box $[L+n,2L+n] \times [0,2L)^{d-1}$ by symmetry we can replace this box by every box obtained via reflection about a coordinate plane in $\mathbb{Z}^d$ (see Figure 6.4). The idea is that we apply the event $\mathcal{A}_3$ repeatedly to move the centre (x,s) of the initially fully infected hypercube, where $x \in [-(2L+n),2L+n]^d$, in five to ten steps to a new centre (y,t) with $y \in [2L+n,3(2L+n)] \times [-(2L+n),(2L+n)]^{d-1}$.

We visualized the procedure in Figure 6.5. For $\varepsilon > 0$ let n, L, T be chosen such that (6.27) is satisfied. Let $a = 2L + n$ and $b = 2T$. The construction proceeds as follows:

1. Concerning the coordinates $2 \leq i \leq d$ we will use that we can reflect about the coordinate planes, so if the initial center at any step is (z, r) and $z_i \geq 0$, then we will move the box in the negative direction and if $z_i < 0$ in the positive direction (see Figure 6.5(b)). The centre is moved at most by $2L$, so by choice of a we will never leave $[-a, a]^{d-1}$ with this procedure.
2. Concerning the first coordinate at the beginning we will move it always in the positive direction until the center of the infected box is contained in $[a, 3a]$ (see Figure 6.5(a)). Since we move the centre of the box by at least $L+n$ and at most $2L+n$, by choice of a this is achieved after at most four steps. Then assume that (k, r) is the centre of the fully infected box. If $z_1 > 2a$ we move it towards the negative direction and if $z_1 < 2a$ towards the positive. Again by choice of a the centre will not leave $[a, 3a]$.

This procedure is carried out until the time coordinate r of the centre of the infected box (z, r) is contained in $[5b, 6b]$. By choice of b we see that this happens after five to ten steps. Note that the construction only uses the graphical representation corresponding to the sites and edges in $[5a, 5a]^d$ and the truncated edge set E_{5a}. Furthermore, the subsequent steps take place on disjoint time intervals. Analogously as in the proof of Proposition 6.2.1 after each step we restart the process with $(z, r) + [-n, n]^d$ as initially infected individuals and the background in the state $\emptyset$, i.e. all edges closed. Since disjoint parts of the graphical representation are independent, this yields that we succeed with at least probability $(1-\varepsilon)^{10}$. Change ε accordingly and we are finished. $\square$

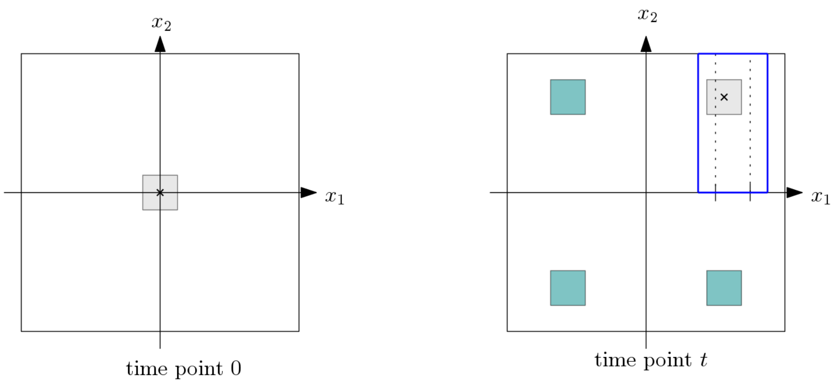
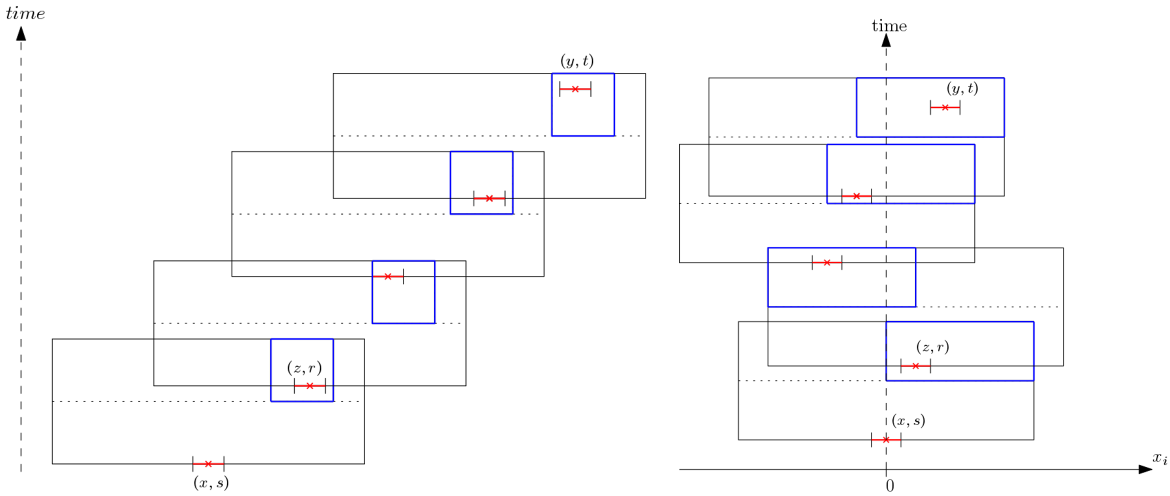


Figure 6.4: Here we visualized for $d = 2$ the space cross-cut at time 0 and t. The green boxes are the reflections about the coordinate planes. The blue box is the area where the infected box of side length $2n$ is contained.



(a) **First coordinate:** Start in (x, s) use (6.27) to find infected box around (z, r) restart at this point and use successively (6.27).

(b) **All other coordinates:** Assume $x_i = 0$, while using successively (6.27) reflect along the coordinate plane if i-th coordinate changes its sign. Note that after achieving $x_1 \in [a, 3a]$ we apply this strategy to the first coordinate as well until $t \in [5b, 6b]$

Figure 6.5: Visualization of the construction in Proposition 6.2.2

Remark 6.2.3. The proof of Proposition 6.2.2 also yields that a "reflection" of the statement holds true, i.e. we reflect the whole construction in the direction of the first coordinate at $(2ja, 0, \ldots, 0) \in \mathbb{Z}^d$ such that at the end $(y, t) \in \mathcal{D}_{j-1,k+1}$.

The idea is to switch to the "macroscopic" grid $\{(j, k) \in \mathbb{Z} \times \mathbb{N}_0 : j + k \text{ even}\}$, where we identify the points (j, k) with the space-time boxes

$$\mathcal{S}_{j,k} := [a(12j - 1), a(12j + 1)] \times [-a, a]^{d-1} \times [30kb, (30k + 1)b] = \mathcal{D}_{6j,6k}.$$

Heuristically speaking, we will declare (j, k) to be open if we find an appropriate translation of $[-n, n]^d$ in this box, which is completely infected. For $a, b > 0$ as in Proposition 6.2.2 let $w(j, k) := ((12ja, 0, \ldots, 0), 30kb) \in \mathbb{Z}^d \times \mathbb{N}_0$ and set

$$\mathcal{M}^{\pm}(j, k) := \left(\bigcup_{l=0}^{6} ([-5a, 5a] \pm 2la) \times [-5a, 5a]^{d-1} \times [5l, (5l + 1)b] \right) + w(j, k).$$

See the solid boxes in Figure 6.6 for a illustration.

Next we formulate the events which are fundamental for our construction. For a point $(x, s) \in \mathcal{S}_{k,m}$ we define

$$\begin{aligned}\mathcal{B}^{\pm} = \mathcal{B}^{\pm}(j, k, (x, s)) := \Big\{ &\exists (y, t) \in \mathcal{S}_{j\pm1,k+1} \text{ and there are} \\ &\emptyset - \text{infection paths that stay in } \mathcal{M}^{\pm}(j, k) \\ &\text{and go from } (x, s) + ([-n, n]^d \times \{0\}) \\ &\text{to every point in } (y, t) + ([-n, n]^d \times \{0\}) \Big\}.\end{aligned} \tag{6.31}$$

For these events, similarly to Proposition 6.2.2, we show the following lemma.

Lemma 6.2.4. *Suppose Condition 6.1.2 holds, then for every $\varepsilon > 0$ there are choices of n, a, b with $n < a$ such that if $(x, s) \in \mathcal{S}_{j,k}$, $\mathbb{P}(\mathcal{B}^{\pm}) > 1 - \varepsilon$, where $(j, k) \in \mathbb{Z} \times \mathbb{N}_0$.*

Proof. This is a direct consequence of Proposition 6.2.2. So if $(x, s) \in \mathcal{S}_{j,k} = \mathcal{D}_{6j,6k}$, then we let for $\varepsilon > 0$, n, a, b be the choice such that we get with a probability larger than $1 - \varepsilon'$ that there exists an $(z, r) \in \mathcal{D}_{6j+1,6k+1}$ such that there exist $\emptyset$-infection paths from $(x, s) + ([-n, n]^d \times \{0\})$ to all points in $(z, r) + ([-n, n]^d \times \{0\})$. There does not necessarily exist a unique point (z, r), if there is more than one point, we just take the earliest and if that does not yield a unique point we minimize the space coordinate according to an arbitrary order on $\mathbb{Z}^d$, which we fixed beforehand. Next we use this procedure again on (z, r). We repeat this procedure in total six times successively. For visualization take a look at the solid lined boxes in Figure 6.6. Then, similarly to the proof of Proposition 6.2.2, by choosing $\varepsilon' > 0$ correctly we get the statement that for every $\varepsilon > 0$ there are choices n, a, b such that $\mathbb{P}(\mathcal{B}^{+}) > 1 - \varepsilon$.

The same statement holds for $\mathcal{B}^{-}$, where we want to point out that one can use the same procedure just with the reflected events, see Remark 6.2.3. □

Note that the boxes $\mathcal{B}^{\pm}$ only depend on a finite sector of the graphical representation and only overlap with the adjacent boxes (see Figure 6.6). At first this last step seems a bit redundant, since we could very well work with the events defined in Proposition 6.2.2, but with this additional step we made the dependency between the respective events clearer. Now we are ready to prove the main theorem of this section.

Theorem 6.2.5. *Suppose Condition 6.1.2 holds. Then for every $q < 1$ there are choices of n, a, b such that if the initial configurations $W_0 \subset 2\mathbb{Z}$ and $\mathbf{C}_0 = C$ satisfy*

$$j \in W_0 \Rightarrow C \supset x + [-n, n]^d \text{ for some } x \in [a(12j - 1), a(12j + 1)] \times [-a, a]^{d-1} \tag{6.32}$$

then $\{(\mathbf{C}_t, \mathbf{B}_t) : t \geq 0\}$ *can be coupled with an oriented site percolation* $\{W_k : k \geq 0\}$ *with parameter* q *such that*

$$j \in W_k \Rightarrow \mathbf{C}_t \supset x + [-n, n]^d \text{ for some } (x, t) \in \mathcal{S}_{j,k} \tag{6.33}$$

In particular this implies that the CPDP survives.

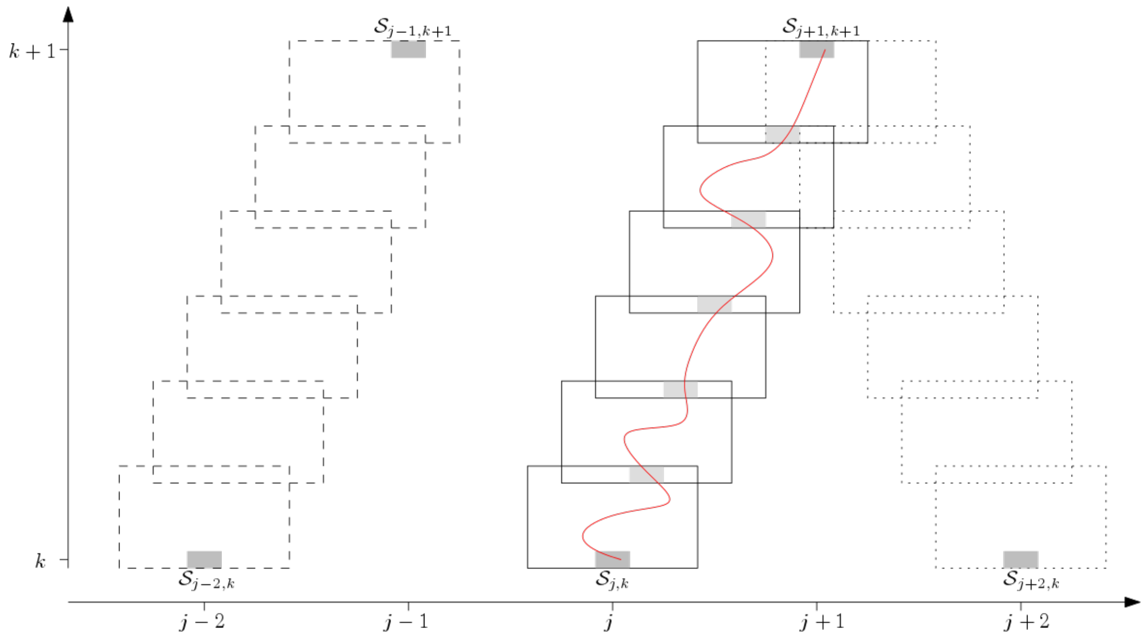


Figure 6.6: Here we see a visualization of the space-time boxes $\mathcal{B}^{\pm}$ (defined in (6.31)), where the solid line visualizes the box $\mathcal{B}^{+}(j, k, \cdot)$. We also see that $\mathcal{B}^{+}(j, k, \cdot)$ only overlaps with $\mathcal{B}^{-}(j, k, \cdot)$ and $\mathcal{B}^{-}(j+2, k, \cdot)$, where the dotted lines visualizes $\mathcal{B}^{-}(j+2, k, \cdot)$ and the dashed $\mathcal{B}^{+}(j-2, k, \cdot)$.

Proof. The construction of the oriented site percolation is similar to [Lig13, Theorem 2.23]. The idea is that we construct our percolation model recursively with the help of Lemma 6.2.4. Thus, let for an arbitrary $\varepsilon > 0$ the numbers n, a, b be the choices done in the Lemma 6.2.4. Note that since the events we use are not independent we need to use a comparison of independent and locally dependent Bernoulli random variables to obtain an independent oriented site percolation in a second step, as we desire.

We will now construct random variables $(X_j(k), Y_j(k))$ with $k \geq 0$ and $j \in \mathbb{Z}$. These variables $X_j(k)$ will either be 1 if there exists a $(x, s) \in S_{j,k}$ such that $(x, s) + ([-n, n]^d \times \{0\})$ is infected and otherwise 0. Additionally if such a point exists we set $Y_j(k) = (x, s)$ and if not $Y_j(k) = \dagger$, where $\dagger$ is a designated state such that the state space of these random variables is $\{0, 1\} \times \big(\mathbb{Z}^d \times [0, \infty)\big) \cup \{\dagger\}$.

Without loss of generality we will assume that $W_0 = \{0\}$. By assumption (6.32) there exists an x_0 such that $(x_0, 0) \in \mathcal{S}_{0,0}$ and $x_0 + [-n, n]^d$ is initially infected. We set $(X_0(0), Y_0(0)) = (1, (x_0, 0))$ and $(X_j(0), Y_j(k)) = (0, \dagger)$ for all $j \neq 0$. Now with respect to k we recursively construct these random variables. Suppose that $(X_j(k), Y_j(k))_{j\in\mathbb{Z}}$ are defined for all $k \leq m$, then we proceed with the step $m \to m+1$.

1. If $X_{j-1}(m) = 0$ and $X_{j+1}(m) = 0$ then we set $(X_j(m+1), Y_j(m+1)) = (0, \dagger)$
2. We set $X_j(m+1) = 1$ if either $X_{j-1}(m) = 1$ and the event $\mathcal{B}^+(j-1, m, Y_{j-1}(m))$ occurs and/or $X_{j+1}(m) = 1$ and $\mathcal{B}^-(j+1, m, Y_{j+1}(m))$ occurs.
   Again the events $\mathcal{B}^+(j-1, m, Y_{j-1}(m))$ and $\mathcal{B}^-(j+1, m, Y_{j+1}(m))$ only guarantee existence of a point $(y, t) \in S_{j,m+1}$ such that $(z, r) + ([-n, n]^d \times \{0\})$ is completely infected, but there might exist more than one. We set $Y_j(m+1)$ as the smallest space-time point (y, t), smallest in the sense that we take the earliest with respect to time and if that does not yield a unique point we minimize according to an arbitrary but beforehand specified order on $\mathbb{Z}^d$.

By this construction for fixed $k \geq 0$ the set $\{j : X_j(k) = 1\}$ obviously satisfies (6.33). Next let $\mathcal{G}_m$ be σ-algebra generated from all $(X_j(k), Y_j(k))_{j\in\mathbb{Z}}$ with $k \leq m$. By the choice of n, a, b made at the beginning of the proof we see that

$$\mathbb{P}(X_j(m+1) = 1 | \mathcal{G}_m) > 1 - \varepsilon \quad \text{on} \quad \{X_j(m) = 1 \text{ or } X_{j-1}(m) = 1\}.$$

Since $\mathcal{B}^\pm$ only overlap with their adjacent boxes, by construction the $(X_j(m+1))_{j\in\mathbb{Z}}$ are conditional on $\mathcal{G}_m$, 3-dependent family of Bernoulli variables (see Definition B.2.1). By Theorem B.2.2 we find a families of independent Bernoulli variables such that we can define a oriented site percolation W_k with parameter $q := (1 - \varepsilon^{-3})^2$ which satisfies (6.32) and (6.33). Since ε was arbitrarily we are finished. $\square$

6.3 Consequences of the percolation comparison

In this section we can finally reap the benefits of all work we have done so far in Chapter 6. First we prove that at criticality, survival is not possible and as direct consequence we gain continuity of the survival probability. Then, we use Theorem 6.2.5 to show that for the CPDP the two conditions (1.8) and (1.9) are satisfied such that by Theorem 1.4.15 it follows that complete convergence for the CPDP holds true. Recall that we defined in (1.12) the survival region as $\mathcal{S} := \{(\lambda, r, \alpha, \beta) \in (0, \infty)^2 : \theta\big(\lambda, r, \alpha, \beta, \{\mathbf{0}\}, \emptyset\big) > 0\}$.

6.3.1 Extinction at criticality and continuity

In Section 3.4 we showed some basic properties of the CPERE. In case of the CPDP we have two additional parameters α and β for which we can easily deduce similar monotonicity and continuity properties as for the infection and recovery rate λ and r.

Lemma 6.3.1 (Monotonicity with respect to the background)**.** *Let* $(\mathbf{C}, \mathbf{B})$ *be a CPDP with parameters* $\lambda, r, \alpha, \beta > 0$. *Let* $\widehat{\alpha} \geq \alpha$, *then there exists a CPDP* $(\widehat{\mathbf{C}}, \widehat{\mathbf{B}})$ *with parameter* $\lambda, r, \widehat{\alpha}, \beta$ *and the same initial configuration such that* $\mathbf{C}_t \subseteq \widehat{\mathbf{C}}_t$ *and* $\mathbf{B}_t \subseteq \widehat{\mathbf{B}}_t$ *for all* $t \geq 0$. *In words* $\mathbf{C}$ *is monotone increasing in* α. *On the other hand* $\mathbf{C}$ *is monotone decreasing in* β.

Proof. This follows with an analogous coupling as in the proof of Lemma 3.4.2. Since if we consider $\widehat{\alpha} \geq \alpha$, then let $\widehat{\Xi}^{\text{birth}}$ be a Poisson point process on $\mathbb{R}_+ \times \{\mathbf{birth}_e : e \in E\}$ with intensity measure $(\widehat{\alpha} - \alpha)\mathsf{d}t$, i.e. all maps $\mathbf{birth}_e$ occur with rate $(\widehat{\alpha} - \alpha)$, and again let $\widehat{\Xi}^{\text{birth}}$ be independent of Ξ, where Ξ is the Poisson point process used in the graphical representation of the original CPDP. Then set $\widehat{\Xi} := \Xi + \widehat{\Xi}^{\text{birth}}$ and proceed as in Lemma 3.4.2. The monotonicity in β follows analogously. □

Remark 6.3.2. Obviously $\pi_{\alpha,\beta} \preceq \pi_{\widehat{\alpha},\beta}$ if $\alpha \leq \widehat{\alpha}$. Thus, if we consider the stationary case, i.e. that $\mathbf{C}_0 = C \subset V$ and $\mathbf{B}_0 \sim \pi_{\alpha,\beta}$, then there exists an CPDP $(\widehat{\mathbf{C}}, \widehat{\mathbf{B}})$ with parameter $\lambda, r, \widehat{\alpha}, \beta$ and $\mathbf{C}_0 = C \subset V$ and $\widehat{\mathbf{B}}_0 \sim \pi_{\widehat{\alpha},\beta}$ such that $\mathbf{C}_t \subseteq \widehat{\mathbf{C}}_t$ and $\mathbf{B}_t \subseteq \widehat{\mathbf{B}}_t$ for all $t \geq 0$. This follows by first coupling the initial state of the background with Theorem 2.1.12 such that $\mathbf{B}_0 \subseteq \widehat{\mathbf{B}}_0$ and then using Lemma 3.4.1 and Lemma 6.3.1.

Lemma 6.3.3 (Continuity for finite times and finite initial infections)**.** *Let* $(\mathbf{C}, \mathbf{B})$ *be a CPDP with initial configuration* $\mathbf{C}_0 = C \subset V$ *with* $|C| < \infty$. *Also let* $\mathcal{A} \subset D_{\mathcal{P}(V)}([0, t])$ *for* $t \geq 0$.

1. *The maps* $\alpha \mapsto \mathbb{P}^{(C,B)}_{\lambda,r,\alpha,\beta}\big((\mathbf{C}_s)_{s \leq t} \in \mathcal{A}\big)$ *and* $\beta \mapsto \mathbb{P}^{(C,B)}_{\lambda,r,\alpha,\beta}\big((\mathbf{C}_s)_{s \leq t} \in \mathcal{A}\big)$ *are continuous, where* $\mathbf{B}_0 = B$.
2. *If* $\mathbf{B}_0 \sim \pi_{\alpha,\beta}$, *then* $\alpha \mapsto \mathbb{P}^{(C,\pi_{\alpha,\beta})}_{\lambda,r,\alpha,\beta}\big((\mathbf{C}_s)_{s \leq t} \in \mathcal{A}\big)$ *and* $\beta \mapsto \mathbb{P}^{(C,\pi_{\alpha,\beta})}_{\lambda,r,\alpha,\beta}\big((\mathbf{C}_s)_{s \leq t} \in \mathcal{A}\big)$ *are continuous.*

Proof. 1. The proof for α and β is similar to the proof of Lemma 3.4.4. Again we will only prove the statement for the function $\alpha \mapsto \mathbb{P}_{\lambda,r,\alpha,\beta}((\mathbf{C}_s)_{s \leq t} \in \mathcal{A})$, since the statement follows similarly when varying β as a variable with just a few obvious changes.

Let $(\widehat{\mathbf{C}}, \widehat{\mathbf{B}})$ be the CPDP that has the same rates and initial configuration as $(\mathbf{C}, \mathbf{B})$ with the exception that the rate α is substituted with $\tilde{\alpha} > \alpha$. This process was constructed in the way such that $\mathbf{C}_t \subseteq \widehat{\mathbf{C}}_t$ and $\mathbf{B}_t \subseteq \widehat{\mathbf{B}}_t$ for all $t \geq 0$ and also $\mathbf{C}_0 = \widehat{\mathbf{C}}_0$ and $\mathbf{B}_0 = \widehat{\mathbf{B}}_0$. Here again it suffices to show that as $\widehat{\alpha} \to \alpha$ it follows that$\mathbb{P}(\mathbf{C}_s \neq \widehat{\mathbf{C}}_s \text{ for some } s \leq t) \to 0$.

Set $Y_t(x) := \#\{y \in \mathbf{C}_s : \{x,y\} \in \widehat{\mathbf{B}}_s, \{x,y\} \notin \mathbf{B}_s\}$, which is the number of infected neighbors of x at time s that could infect x according to $\widehat{\mathbf{B}}_s$, but not with regards to $\mathbf{B}_s$. For the process $\mathbf{C}$ and $\widehat{\mathbf{C}}$ to differ, an additional infection path must have been started by an infection event $(s, \mathbf{inf}_{x,y}) \in \Xi^{\text{inf}}$ such that $\{x,y\} \in \widehat{\mathbf{B}}_s$ and $\{x,y\} \notin \mathbf{B}_s$. Thus it again holds that

$$\mathbb{P}(\mathbf{C}_s \neq \widehat{\mathbf{C}}_s \text{ for some } s \leq t) = 1 - \mathbb{E}\Big[\exp\Big(-\int_0^t \sum_{x\in V} Y_s(x)ds\Big)\Big].$$

Now let $\overline{\mathbf{C}}$ be again the classical contact process with infection rate λ and recovery rate r constructed via Ξ^{inf} and Ξ^{rec} (see Remark 2.3.2), thus $\mathbf{C}_t \subseteq \overline{\mathbf{C}}_t$ for all $t \geq 0$. Obviously the classical contact process $\overline{\mathbf{C}}$ is independent of $\mathbf{B}$ and $\widehat{\mathbf{B}}$, and thus

$$\begin{aligned}\mathbb{E}\Big[\int_0^t \sum_{x\in V} Y_s(x)ds\Big] \leq& \mathbb{E}\Big[\int_0^t \sum_{x\in V}\sum_{\{x,y\}\in E} (\mathbb{1}_{\{\{x,y\}\in\widehat{\mathbf{B}}_s\}} - \mathbb{1}_{\{\{x,y\}\in\mathbf{B}_s\}})\mathbb{1}_{\{x\in\overline{\mathbf{C}}_s\}} ds\Big] \\ =& \sum_{x\in V}\sum_{\{x,y\}\in E}\int_0^t \big(\mathbb{P}(\{x,y\}\in\widehat{\mathbf{B}}_s) - \mathbb{P}(\{x,y\}\in\mathbf{B}_s)\big)\mathbb{P}(x\in\overline{\mathbf{C}}_s)ds \\ \leq& K\mathbb{E}\Big[\int_0^t |\overline{\mathbf{C}}_s| ds\Big] < \infty.\end{aligned}$$

Since every edge e flips from open to closed and vice versa independently we see via the coupling that it follows $\mathbb{P}(e \in \widehat{\mathbf{B}}_s) - \mathbb{P}(e \in \mathbf{B}_s) \to 0$ as $|\widehat{\alpha} - \alpha| \to 0$ for every $e \in E$ and every $s \geq 0$. So by the same inequality as in the first part and by dominated convergence we see that

$$\mathbb{P}(\mathbf{C}_s \neq \widehat{\mathbf{C}}_s \text{ for some } s \leq t) \leq \mathbb{E}\Big[\int_0^t \sum_{x\in V} Y_s(x)ds\Big] \to 0$$

as $|\widehat{\alpha} - \alpha| \to 0$.

The proof for continouity of $\beta \mapsto \mathbb{P}_{\lambda,r,\alpha,\beta}((\mathbf{C}_s)_{s\leq t} \in \mathcal{A})$ follows analogously.

2. The difference to 1. is that the invariant law depends on α and β, and thus in this case the initial distribution of the background also changes if we vary α or β.

So let us assume $\mathbf{B}_0 \sim \pi_{\alpha,\beta}$. Recall that every edges e is open with probability $\frac{\alpha}{\alpha+\beta}$ and closed otherwise. Now it holds for

$$\alpha \leq \widehat{\alpha} \Leftrightarrow \frac{\alpha}{\alpha+\beta} \leq \frac{\widehat{\alpha}}{\widehat{\alpha}+\beta}$$

Let $(Z(e))_{e\in E}$ be family of independent Bernoulli random variables such that

$$\mathbb{P}(Z(e)=1) = \frac{\widehat{\alpha}}{\widehat{\alpha}+\beta} - \frac{\alpha}{\alpha+\beta}$$

Now we set $\widehat{\mathbf{B}}_0 := \mathbf{B}_0 \cup \{e \in E : Z(e) = 1\}$. Obviously $\widehat{\mathbf{B}}_0 \sim \pi_{\widehat{\alpha},\beta}$ and $\mathbf{B}_0 \subset \widehat{\mathbf{B}}_0$ almost surely. From this point we can proceed as we did before by using the coupling out from the proof of Lemma 3.4.2 to construct a CPDP $(\widehat{\mathbf{C}}, \widehat{\mathbf{B}})$ with the desired rates and $\mathbf{C}_t \subseteq \widehat{\mathbf{C}}_t$ and $\mathbf{B}_t \subseteq \widehat{\mathbf{B}}_t$ for all $t \geq 0$. Where we have the slight difference we have $\mathbf{C}_0 = \widehat{\mathbf{C}}_0$ and $\mathbf{B}_0 \subset \widehat{\mathbf{B}}_0$ almost surely instead of equality. Thus the initial state of the two background process are not the same as before. But by the coupling we know that we have again that $\mathbb{P}(e \in \widehat{\mathbf{B}}_s) - \mathbb{P}(e \in \mathbf{B}_s) \to 0$ as $|\widehat{\alpha} - \alpha| \to 0$ for every $e \in E$ and every $s \geq 0$. Thus, from here on we can apply the exact same proof strategy as above. □

We are finally ready to show that survival is impossible at criticality.

Proof of Theorem 1.4.17. As we already mentioned at the end of the proof of Proposition 6.2.2 the "block"-events only depends on a bounded section of the graphical representation, but by Lemma 3.4.4 and Lemma 6.3.3 we get that $\mathbb{P}_{\lambda,r,\alpha,\beta}(\mathcal{B}^{\pm})$ is continuous seen as a function of any of the four parameters. Let us take as usual the infection parameter λ as an example. By Proposition 6.2.2 we know that for every $\varepsilon > 0$ we find a, b, n such that $\mathbb{P}_{\lambda}(\mathcal{B}^{\pm}) > 1 - \varepsilon$, then because of continuity there must exist a $\lambda' < \lambda$ such that $\mathbb{P}_{\lambda'}(\mathcal{B}^{\pm}) > 1 - \varepsilon$ as well and then by Theorem 6.2.5 it follows that the CPDP also survives with λ'. This proves the claim. □

Recall that we call a function $f : \mathbb{R}^d \subset U \to \mathbb{R}$ separately continuous if it is continuous in each coordinate separately. In comparison to that one calls f jointly continuous if it is continuous with respect to the Euclidean topology on $\mathbb{R}^d$.

Proposition 6.3.4. *Let $C \subset V$ with C finite and non-empty and $B \subset E$.*

1. *The survival probability $\theta(\lambda, r, \alpha, \beta, C, B)$ is separately right continuous seen as a function in $(\lambda, r, \alpha, \beta)$ on $(0, \infty)^4$.*

2. *The survival probability $\theta(\lambda, r, \alpha, \beta, C, B)$ is separately left continuous seen as a function in $(\lambda, r, \alpha, \beta)$ on $\mathring{\mathcal{S}}$.*

Proof. We already showed right and left continuity in λ and r on the respective parameter sets in Proposition 5.3.2 and 5.3.4. Right and left continuity in α and β can be shown by the same approach. □

We have seen that Corollary 1.4.17 states that the infection process $\mathbf{C}$ cannot survive at criticality. As a consequence of this fact we can conclude that the survival probability is jointly continuous with respect to its parameters $(\lambda, r, \alpha, \beta)$.

Proof of Corollary 1.4.18. Theorem 6.3.4 shows that the survival probability is separately left continuous seen as a function in the four parameters $(\lambda, r, \alpha, \beta)$ on $\mathcal{S}$ and is separately right continuous on $(0,\infty)^4$. Now let us again exemplarily prove continuity of $\lambda \mapsto \theta(\lambda, r, \alpha, \beta, C, B)$. The proof is analogous for the remaining three parameter. By Proposition 6.3.4 it is clear that the function is every continuous expect at criticality. Now obviously in case of λ the left limit at criticality exists, since we come from the subcritical parameter region where the survival probability is constant 0. But by Theorem 1.4.17 we know that the CPDP almost surely goes extinct at criticality, which means that the survival probability is 0. But with that we have shown that the left limit and the right limit at the critical value are the same since $\lambda \mapsto \theta(\lambda, r, \alpha, \beta, C, B)$ is right continuous on $(0,\infty)$ by Proposition 6.3.4 and thus, the function is continuous.

Now we know that the survival probability is separately continuous seen as a function of the four parameters. But we also know that the function is monotone in each coordinate, so we can use [KD69, Proposition 2], which states that if a function is continuous and monotone in each coordinate, then it is jointly continuous. □

6.3.2 Complete Convergence of the CPDP

We start by showing that the second condition (1.9) holds true, which is proven by the next proposition.

Proposition 6.3.5. *Suppose $(\lambda, r, \alpha, \beta) \in \mathcal{S}$, then for every $x \in \mathbb{Z}^d$,*

$$\lim_{n\to\infty} \liminf_{t\to\infty} \mathbb{P}^{(\mathbb{B}_n(x),\emptyset)}_{\lambda,r,\alpha,\beta}(\mathbf{C}_t \cap \mathbb{B}_n(x) \neq \emptyset) = 1.$$

Proof. By translation invariance it suffices to prove the claim for $x = \mathbf{0}$. For $d \geq 2$ the claim follows analogously as in the second part of the proof of [Lig13, Theorem 2.27]. Hence, we only need to consider $d = 1$.

Again by Theorem 6.2.5 for every $0 < q < 1$ there exists n, a, b such that an oriented site percolation $(W_k)_{k \geq 0}$ with parameter q exist, which satisfies (6.32) and (6.33). Now let us consider the set $D_m = (-15am - 2, 15am + 1)$. By construction of the oriented site percolation in the proof of Theorem 6.2.5 (see Figure 6.6 for a visualization) it follows that for $m > 0$,

$$\liminf_{t \to \infty} \mathbb{P}^{(D_m, \emptyset)}(\mathbf{C}_t \cap D_m \neq \emptyset) \geq \liminf_{k \to \infty} \mathbb{P}^{\{-m, \dots, m\}}(W_k \cap \{-m, \dots, m\} \neq \emptyset), \quad (6.34)$$

since the infection is always contained in the blocks $\mathcal{B}^{\pm}$. By Theorem B.1.2 we get that the right-hand side in (6.34) convergences to 1 as $m \to \infty$. □

Now it is left to prove that (1.8) holds true. We will split the prove of this condition in two parts. First we show that with Theorem 6.2.5 that a positive survival probability already implies that the probability that a single site is infinitely often infected is positive as well.

Proposition 6.3.6. *Suppose* $(\lambda, r, \alpha, \beta) \in \mathcal{S}$, *then* $\mathbb{P}^{(C,B)}_{\lambda,r,\alpha,\beta}(x \in \mathbf{C}_t \text{ i.o.}) > 0$ *for all* $x \in V$ *and all non-empty* $C \subset V$ *and* $B \subset E$.

Proof. We will now show that if $\theta(\{\mathbf{0}\}, \emptyset) > 0$ then

$$\mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}) > 0, \quad (6.35)$$

where $C \subset V$ and $B \subset E$. By monotonicity we see that for any $y \in C$,

$$\mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}) \geq \mathbb{P}^{(\{y\},\emptyset)}(x \in \mathbf{C}_t \text{ i.o.}).$$

Recall that the stopping time $\tau_x = \tau_x(\{y\}, \emptyset)$ was the first time that at least the site x is infected with initial configuration $(\{y\}, \emptyset)$ (see (5.19)). Since we consider a CPDP, we know that $\mathbb{P}^{(\{y\},\emptyset)}(x \in \mathbf{C}_t) > 0$ for all $x, y \in V$, and thus $\mathbb{P}^{(\{y\},\emptyset)}(\tau_x < \infty) > 0$. By the strong Markov property we see that

$$\mathbb{P}^{(\{y\},\emptyset)}(x \in \mathbf{C}_t \text{ i.o.}) \geq \mathbb{P}^{(\{y\},\emptyset)}(\tau_x < \infty)\mathbb{P}^{(\{x\},\emptyset)}(x \in \mathbf{C}_t \text{ i.o.})$$

Thus, by translation invariance to show (6.35) it suffices to show

$$\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(\mathbf{0} \in \mathbf{C}_t \text{ i.o.}) > 0. \tag{6.36}$$

Analogously as we just did let $\tau_n = \tau_{[-n,n]^d}(\{\mathbf{0}\},\emptyset)$ be the first time that at least all sites in $[-n,n]^d$ are infected with initial configuration $(\{\mathbf{0}\},\emptyset)$. We can conclude in the same manner that $\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(\tau_n < \infty) > 0$, and thus by the strong Markov property

$$\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(\mathbf{0} \in \mathbf{C}_t \text{ i.o.}) \geq \mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(\tau_n < \infty)\mathbb{P}^{([-n,n]^d,\emptyset)}(\mathbf{0} \in \mathbf{C}_t \text{ i.o.}).$$

By Theorem 6.2.5 we know that for every $0 < q < 1$ there exist n, a, b and an oriented percolation $(W_k)_{k\geq 0}$ with parameter q such that (6.32) and (6.33) are satisfied. By choosing q close enough to 1 Theorem B.1.1 (i) shows that $\inf_{k\geq 0} \mathbb{P}^{\{0\}}(0 \in W_{2k}) > 0$. Now by Fatou's lemma

$$\mathbb{P}^{\{0\}}(0 \in W_{2k} \text{ i.o.}) \geq \limsup_{k\to\infty} \mathbb{P}^{\{0\}}(0 \in W_{2k}) > 0.$$

Thus, by (6.32) and (6.33) with positive probability for infinitely many k

$$\mathbf{C}_t^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } (x,t) \in \mathcal{S}_{0,k}.$$

It is clear that $\mathbb{P}(x \in \mathbf{C}_t^{[-n,n]^d,\emptyset}) > 0$ for every $x \in \mathbb{Z}^d$ and that this probability is continuous in t, since $(\mathbf{C},\mathbf{B})$ is a Feller process, and therefore for any compact set $\mathcal{K} \subset \mathbb{Z}^d \times [0,\infty)$ we see that

$$\inf_{(x,t)\in\mathcal{K}} \mathbb{P}\big(\mathbf{0} \in \mathbf{C}_t^{x+[-n,n]^d,\emptyset}\big) = \inf_{(x,t)\in\mathcal{K}} \mathbb{P}\big(x \in \mathbf{C}_t^{[-n,n]^d,\emptyset}\big) > 0,$$

where we again used translation invariance and symmetry. This implies that every time a hypercube of side length $2n$, which is bounded away from $\mathbf{0}$, is completely infected, there is a positive probability that $\mathbf{0}$ gets infected from this hypercube after a time step of length 1. Then (6.36) can be shown analogously to Lemma 5.2.4 and 5.3.3, which means that we utilized a generalized version of the Borel-Cantelli Lemma to show that the event $\{\mathbf{0} \in \mathbf{C}_t^{[-n,n]^d,\emptyset} \text{ i.o.}\}$ happens almost surely on the event

$$\{\mathbf{C}_t^{[-n,n]^d,\emptyset} \supset x + [-n,n]^d \text{ for some } (x,t) \in \mathcal{S}_{0,k} \text{ for infinitely many } k\}.$$

With this argument we have shown (6.35), i.e. we have shown that $\theta(\{\mathbf{0}\},\emptyset) > 0$ implies $\mathbb{P}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}) > 0$. $\square$

Next we will show that if we have a positive probability that a single site is infinitely often infected we can already conclude that (1.8) holds true. Note that the following result actually holds true for general CPERE on the d-dimensional integer lattice.

Proposition 6.3.7. *Let $(\mathbf{C},\mathbf{B})$ be a CPERE with infection rate $\lambda > 0$ and recovery rate $r > 0$ on the d-dimensional integer lattice, i.e. $G = (\mathbb{Z}^d, E)$. Suppose that $\mathbb{P}^{(C,B)}_{\lambda,r}(x \in \mathbf{C}_t \text{ i.o.}) > 0$ for all $x \in V$, all non-empty $C \subset V$ and all $B \subset E$, then*

$$\mathbb{P}^{(C,B)}_{\lambda,r}(x \in \mathbf{C}_t \text{ i.o.}) = \theta(\lambda, r, C, B).$$

Proof. First, we observe that $\{x \in \mathbf{C}_t \text{ i.o.}\} \subset \{\mathbf{C}_t \neq \emptyset \; \forall t \geq 0\}$. Thus, to show the claim we need to show the converse inclusion. In principle this can be shown analogous to the first part of the proof of [Lig13, Theorem 2.27]. We will now adapt this prove to our setting, where we need to take the background into consideration. First we set $A := \{\mathbf{0} \in \mathbf{C}_t \text{ i.o.}\}$ and show the inequality

$$\mathbb{P}(A|\mathcal{F}_s) = \mathbb{P}^{(\mathbf{C}_s,\mathbf{B}_s)}(A) \geq \mathbb{P}^{(\{x\},\emptyset)}(\mathbf{0} \in \mathbf{C}_t \text{ for some } t \geq 0)\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(A)\mathbb{1}_{\{x \in \mathbf{C}_s\}} \tag{6.37}$$

for every $x \in \mathbb{Z}^d$. For that let us consider $\tau := \inf\{t \geq 0 : \mathbf{0} \in \mathbf{C}_t\}$, i.e. the first time $\mathbf{0}$ got infected. By the Markov property we know that $\mathbb{P}(A|\mathcal{F}_s) = \mathbb{P}^{(\mathbf{C}_s,\mathbf{B}_s)}(A)$, and thus we see that

$$\mathbb{P}^{(\mathbf{C}_s,\mathbf{B}_s)}(A) \geq \mathbb{P}^{(\{x\},\emptyset)}(A) \geq \mathbb{E}[\mathbb{P}(A|\mathcal{F}_s)\mathbb{1}_{\{\tau<\infty\}}] \quad \text{on } \{x \in \mathbf{C}_s\}, \tag{6.38}$$

where we used monotonicity in the first inequality and the tower property in second. Now using the strong Markov property, by (6.38) it follows that

$$\mathbb{P}^{(\mathbf{C}_s,\mathbf{B}_s)}(A) \geq \mathbb{E}[\mathbb{P}^{(\mathbf{C}_\tau,\mathbf{B}_\tau)}(A)\mathbb{1}_{\{\tau<\infty\}}] \geq \mathbb{E}[\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(A)\mathbb{1}_{\{\tau<\infty\}}] \quad \text{on } \{x \in \mathbf{C}_s\}, \tag{6.39}$$

where we used, in the second inequality, again that the CPERE is monotone and that by definition $\mathbf{0} \in \mathbf{C}_\tau$. Now we see that (6.37) follows by (6.39).

Since we assumed that $\mathbb{P}^{C,B}(\{\mathbf{0} \in \mathbf{C}_t \text{ i.o.}\}) > 0$ for any non empty $C \subset V$ and $B \subset E$, by translation invariance of the background $\mathbf{B}$ we know that $\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(x \in \mathbf{C}_t) > 0$ for any $x \in \mathbb{Z}^d$. Thus, by using symmetry of $\mathbb{Z}^d$ and translation invariance we see that

$$\mathbb{P}^{(\{x\},\emptyset)}(\mathbf{0} \in \mathbf{C}_t \text{ for some } t \geq 0) = \mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(x \in \mathbf{C}_t \text{ for some } t \geq 0) \geq \mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(A), \tag{6.40}$$

where we used Lemma 5.2.4 in the last inequality. Now (6.37) together with (6.40) this implies that

$$\mathbb{P}(A|\mathcal{F}_s) \geq \left(\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(A)\right)^2 \mathbb{1}_{\{\mathbf{C}_s \neq \emptyset\}}.$$

But by assumption we know that $\mathbb{P}^{(\{\mathbf{0}\},\emptyset)}(A) > 0$. Furthermore, by the martingale convergence theorem it follows that $\mathbb{P}(A|\mathcal{F}_s) \to \mathbb{1}_A$, since A is an element of the tail σ-algebra. But this implies that $\{\mathbf{C}_t \neq \emptyset \ \forall t \geq 0\} \subset \{\mathbf{0} \in \mathbf{C}_t \text{ i.o.}\}$ almost surely. $\square$

Finally we are able prove that complete convergence holds for the CPDP on the whole parameter set $(0,\infty)^4$.

Proof of Theorem 1.4.19. Suppose $(\lambda, r, \alpha, \beta) \in \mathcal{S}$, then by Proposition 6.3.5, Proposition 6.3.6 and Proposition 6.3.7 we know that (1.8) and (1.9) are satisfied, and thus by Theorem 1.4.15 it follows that

$$(\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \Rightarrow (1-\theta(C,B))(\delta_\emptyset \otimes \pi) + \theta(C,B)\overline{\nu} \quad \text{as } t \to \infty.$$

On the other hand if $(\lambda, r, \alpha, \beta) \in \mathcal{S}^c$, then by Proposition 5.1.6 it follows that $\overline{\nu} = \delta_\emptyset \otimes \pi$. Thus, by Proposition 5.1.3 follows that $(\mathbf{C}_t^{V,E}, \mathbf{B}_t^E) \Rightarrow \delta_\emptyset \otimes \pi$ as $t \to \infty$. By monotonicity shown in Lemma 3.4.1 we then know that $(\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \Rightarrow \delta_\emptyset \otimes \pi$ as $t \to \infty$ for all $C \subset V$ and $B \subset E$, which proves the claim. $\square$

We conclude this chapter by showing that for a general CPERE on the d-dimensional integer lattice, complete convergence holds on a suitable subset of its survival region. To be precise this subset will be the survival region of a suitable chosen CPDP, which lies “below” the CPERE. Here we will again use the subscript DP since we need to distinguish between a CPERE and a CPDP.

Proof of Theorem 1.4.20. Let $(\mathbf{C}, \mathbf{B})$ be CPERE with infection rate λ, recovery rate r and the background process has spin rate $q(\cdot,\cdot)$, recall from (1.4) the rates $\alpha_{\min} := \min_{F \subset \mathcal{N}_e^L} q(e, F)$ and $\beta_{\max} := \max_{F \subset \mathcal{N}_e^L} q(e, F \cup \{e\})$. By Proposition 3.4.5 there exists a CPDP $(\underline{\mathbf{C}}, \underline{\mathbf{B}})$ with rates $\alpha_{\min}$, $\beta_{\max}$ and the same initial configuration as $(\mathbf{C}, \mathbf{B})$, i.e. $\mathbf{C}_0 = \underline{\mathbf{C}}_0$ and $\mathbf{B}_0 = \underline{\mathbf{B}}_0$, such that $\underline{\mathbf{C}}_t \subset \mathbf{C}_t$ and $\underline{\mathbf{B}}_t \subset \mathbf{B}_t$ for all $t \geq 0$. This implies that

$$\mathbb{P}(x \in \underline{\mathbf{C}}_t \text{ i.o.}) \leq \mathbb{P}(x \in \mathbf{C}_t \text{ i.o.}) \tag{6.41}$$

By assumption $\theta_{\mathrm{DP}}(\lambda, r, \alpha_{\min}, \beta_{\max}, \{\mathbf{0}\}, \emptyset) > 0$, and thus by Proposition 6.3.6 and (6.41) it follows that $\mathbb{P}_{\lambda,r}^{(C,B)}(x \in \mathbf{C}_t \text{ i.o.}) > 0$ for any finite and non-empty set $C \subset \mathbb{Z}^d$ and any $B \subset E$. Furthermore, by Proposition 6.3.7 it follows that the first condition (1.8) holds. Now it follows analogously by the fact that $\underline{\mathbf{C}}_t \subset \mathbf{C}_t$ and $\underline{\mathbf{B}}_t \subset \mathbf{B}_t$ for all $t \geq 0$ and Proposition 6.3.5 that (1.9) is satisfied. Since we assumed that (i)-(iii) of Assumption 1.4.1 are satisfied Theorem 1.4.15 implies that if $\theta_{\text{DP}}(\lambda, r, \alpha_{\min}, \beta_{\max}, \{\mathbf{0}\}, \emptyset) > 0$, then

$$(\mathbf{C}^{C,B}, \mathbf{B}^B) \Rightarrow (1 - \theta(\lambda, r, C, B))(\delta_\emptyset \otimes \pi) + \theta(\lambda, r, C, B)\overline{\nu}$$

for all $C \subset V$ and all $B \subset E$. □

Chapter 7

Contact process on a dynamical long range percolation

7.1 Construction of the CPLDP via a graphical representation and further applications

The CPLDP cannot be constructed in exactly the same way as we constructed the CPERE on graphs with bounded degrees, where we relied on the graphical representation as introduced in Section 2.3. The reason for this is that we want to allow transmission of an infection between each pair of vertices $x, y \in V$ if the edge connecting them is open at the time of transmission. Thus, it is fairly obvious that the rate bound (2.1) is not satisfied since we would draw maps $\mathbf{coop}_{x,y}$ for every $x, y \in V$ with $x \neq y$ with a positive but fixed rate λ.

It is still possible to construct the CPLDP via a graphical representation, if we consider a setting where most connections $\{x, y\} \in \mathcal{E}$ are closed. We basically need to ensure that $|\{y \in V : \{x, y\} \in \mathbf{B}_{t-}\}| < \infty$ for all $t \geq 0$ and all $x \in V$, i.e. all vertices x have almost surely a finite degree at all times. We will see that Assumption 1.4.21 guarantees this. Recall that this assumption states that

$$\sum_{y \in V} v_{\{x,y\}} p_{\{x,y\}} < \infty \quad \text{and} \quad \sum_{y \in V} v_{\{x,y\}}^{-1} < \infty$$

for all $x \in V$. Now we start to construct the CPLDP. The long range dynamical percolation itself can be defined via the graphical representation described in Section 2.3 by considering the maps $\mathbf{birth}_e(B) := B \cup \{e\}$ and $\mathbf{death}_e(B) := B \backslash \{e\}$ with respective rates $r_{\mathbf{birth}_e} = \hat{v}_e \hat{p}_e$ and $r_{\mathbf{death}_e} = \hat{v}_e(1 - \hat{p}_e)$, where $e \in \mathcal{E}$ and $B \subset \mathcal{E}$. Thus the set of maps is $\mathcal{M}_{\mathrm{DP}} := \{\mathbf{birth}_e : e \in \mathcal{E}\} \cup \{\mathbf{death}_e : e \in \mathcal{E}\}$ and we denote the Poisson point process on $\mathcal{M}_{\mathrm{DP}} \times \mathbb{R}$ with the corresponding rates $(r_m)_{m \in \mathcal{M}_{DP}}$ by Ξ_{DP}. Obviously Assumption 1.4.21 implies that the rate bound (2.1) is satisfied since for any $e \in \mathcal{E}$ only two maps m exist such that $e \in \mathcal{D}(m)$, and thus we obtain a Feller process $\mathbf{B}$ on the state space $\mathcal{P}(\mathcal{E})$ with jump rates (1.13), i.e. the process has transitions

$$\mathbf{B}_{t-} = B \to B \cup \{e\} \quad \text{at rate } \hat{v}_e \hat{p}_e \text{ and}$$
$$\mathbf{B}_{t-} = B \to B \backslash \{e\} \quad \text{at rate } \hat{v}_e (1 - \hat{p}_e).$$

Next let $\{x, y\} \in \mathcal{E}$ and define the map

$$\mathbf{inf}^*_{\{x,y\}}(A) := \begin{cases} A \cup \{x, y\} & \text{if } x \in A \text{ or } y \in A \\ A & \text{otherwise ,} \end{cases}$$

where $A \subset V$ and recall the recovery map $\mathbf{rec}_x$ from Example 2.3.2 and let the rates be $r_{\mathbf{inf}^*_{\{x,y\}}} = \lambda > 0$ and $r_{\mathbf{rec}_x} = r > 0$. Now set

$$\mathcal{M}^* := \underbrace{\{\mathbf{inf}^*_{\{x,y\}} : \{x, y\} \in \mathcal{E}\}}_{=\mathcal{M}^*_{\inf}} \cup \underbrace{\{\mathbf{rec}_x : x \in V\}}_{=\mathcal{M}_{\mathrm{rec}}}$$

We denote again by Ξ^{inf^*} the Poisson point process on $\mathcal{M}^*_{\inf} \times \mathbb{R}$ corresponding to the infection events, where the intensity measure is determined through the rates $(r_m)_{m \in \mathcal{M}^*_{\inf}}$ and Ξ^{rec} on $\mathcal{M}_{\mathrm{rec}} \times \mathbb{R}$ for the recovery events and the rates are $(r_m)_{m \in \mathcal{M}_{\mathrm{rec}}}$.

Remark 7.1.1. The difference between the maps $\mathbf{inf}^*_{\{x,y\}}$ and $\mathbf{inf}_{x,y}$ from Example 2.3.2 is that the action of $\mathbf{inf}^*_{\{x,y\}}$ causes x to infect y and vice versa. Thus, if either of x or y is infected afterwards both sites are infected. On the other hand $\mathbf{inf}_{x,y}$ only causes x to infect y. It is not difficult to see that we could also use $\mathbf{inf}^*_{\{x,y\}}$ instead of $\mathbf{inf}_{x,y}$ in Example 2.3.2 and we would still obtain the classical contact process, see Figure 7.1 for a visualization. We change the maps here only for technical reasons. For some results in Chapter 6 it was important that we were able to identify in which direction the infection arrow points. In this section it is more convenient to use the infection maps $\mathbf{inf}^*_{\{x,y\}}$. Since this enables us to use the comparison results developed by [Bro07] in the next section.

Definition 7.1.2 (Infection path)**.** Given space-time points (y, s) and (x, u) with $u > s$ we say that there is an infection path from (y, s) to (x, u) if there is a sequence of times $s = t_0 < t_1 < \cdots < t_n \leq t_{n+1} = u$ and space points $y = x_0, x_1, \ldots, x_n = x$ such that, $(\mathbf{inf}^*_{\{x_{k-1},x_k\}}, t_k) \in \operatorname{supp}(\Xi^{\mathrm{inf}^*})$ and $\{x_k, x_{k+1}\} \in \mathbf{B}_{t_k}$ for all $k \in \{0, \dots, n\}$ and $\operatorname{supp}(\Xi^{\mathrm{rec}}) \cap \big(\{\mathbf{rec}_{x_k}\} \times [t_k, t_{k+1})\big) = \emptyset$ for all $k \in \{0, \dots, n\}$. We write $(y, s) \to (x, u)$ if there exists an infection path.

Now we define the infection process by

$$\mathbf{C}_t^C := \{x \in V : \exists y \in V \text{ such that } (y, 0) \to (x, t)\}, \tag{7.1}$$

where $t \geq 0$ and we set $\mathbf{C}_0^C := C \subset V$. By definition it is not clear yet if this process is well-defined in the sense that if we start with a finite initial set it stays finite for the whole time.

Lemma 7.1.3. *Suppose Assumption 1.4.21 is satisfied. Let $C \subset V$ be finite, then $|\mathbf{C}_t^C| < \infty$ almost surely for all $t \geq 0$.*

Proof. Let us consider $\mathbf{B}'_t := \bigcup_{s \leq t} \mathbf{B}_t$, which is the set of all $e \in \mathcal{E}$, which were open at least once between time 0 and t. The process $\mathbf{B}' = (\mathbf{B}'_t)_{t \geq 0}$ is again a Feller process with transition $\mathbf{B}'_{t-} = B \to B \cup \{e\}$ at rate $\hat{v}_e \hat{p}_e$. This can be seen by just ignoring every $\mathbf{death}_e$ map in the previous construction. Now one can easily calculate that

$$\begin{aligned}\mathbb{E}[|\{y \in V : \{x, y\} \in \mathbf{B}'_t\}|] &= \sum_{y \in V} \mathbb{P}(\{x, y\} \in \mathbf{B}_0) + \mathbb{P}(\{x, y\} \notin \mathbf{B}_0) \underbrace{\left(1 - e^{-\hat{v}_{\{x,y\}} \hat{p}_{\{x,y\}} t}\right)}_{\leq \hat{v}_{\{x,y\}} \hat{p}_{\{x,y\}} t} \\ &\leq \sum_{y \in V} \hat{p}_{\{x,y\}} + t \sum_{y \in V} \hat{v}_{\{x,y\}} \hat{p}_{\{x,y\}} < \infty,\end{aligned}$$

where we used that the events $(\{e \in \mathbf{B}_0\})_{e \in \mathcal{E}}$ are independent and Assumption 1.4.21 provides that the expression is finite. Now we can conclude that for every fixed t the graph $(V, \mathbf{B}'_t)$ is almost surely locally finite. Thus, analogously to Example 2.3.2 we can define a classical contact process $\mathbf{X}^t = (\mathbf{X}^t_s)_{s \leq t}$ on the graph $(V, \mathbf{B}'_t)$ such that we have transitions

$$\begin{aligned}\mathbf{X}^t_{s-} &= A \to A \cup \{x\} \quad \text{at rate } \lambda \cdot |\{y \in A : \{x, y\} \in \mathbf{B}'_t\}|, \text{ and} \\ \mathbf{X}^t_{s-} &= A \to A \backslash \{x\} \quad \text{at rate } r,\end{aligned}$$

where $\mathbf{X}^t_0 = \mathbf{C}_0 = C$. This definition is meant in a quenched sense, i.e. we first fix the realization of $\mathbf{B}'_t$ and then define the classical contact process on the graph $(V, \mathbf{B}'_t)$. By definition $\mathbf{B}_t \subset \mathbf{B}'_t$ for all $t \geq 0$. Thus, we see that $\mathbf{C}_s \subset \mathbf{X}^t_s$ for all $s \leq t$. But since we know that $(V, \mathbf{B}'_t)$ is almost surely finite, we also know that $|\mathbf{C}_s| \leq |\mathbf{X}^t_s| < \infty$

almost surely for all $s \leq t$. This is again a direct consequence of the construction in Example 2.3.2. $\square$

We chose the probability of an edge being open after an update to be of the form $\hat{p}_e = qp_e$ and the update speed to be $\hat{v}_e = \gamma v_e$ for all $e \in \mathcal{E}$, where $\gamma > 0$ and $q \in (0,1)$. Thus, the critical infection rate $\lambda_c(r,\gamma,q)$ can be seen as function of γ, q and the recovery rate. Now we show, via the graphical representation, that the function $\gamma \mapsto \gamma^{-1}\lambda_c(r,q,\gamma)$ is non-increasing. This means that the critical infection rate $\lambda_c(r,q,\gamma)$ can at most increase with linear growth with respect to γ.

Proof of Proposition 1.4.22. Let $(\mathbf{C},\mathbf{B})$ be a CPLDP with parameter $\lambda,\gamma,r > 0$ and $q \in (0,1)$. Suppose that $\lambda > \lambda_c(r,\gamma,q)$. If we rescale the time by sending $t \to \frac{\gamma'}{\gamma}t$, we get a process $(\widehat{\mathbf{C}},\widehat{\mathbf{B}})$ with transitions

$$
\begin{aligned}
\widehat{\mathbf{C}}_t &= C \to C \cup \{x\} \quad \text{ at rate } \tfrac{\lambda\gamma'}{\gamma} \cdot |\{y \in \widehat{\mathbf{C}}_{t-} : \{x,y\} \in \widehat{\mathbf{B}}_{t-}\}|, \\
\widehat{\mathbf{C}}_t &= C \to C\backslash\{x\} \quad \text{ at rate } \tfrac{\gamma'}{\gamma}r, \\
\widehat{\mathbf{B}}_t &= B \to B \cup \{e\} \quad \text{ at rate } \gamma' v_e \hat{p}_e \text{ and} \\
\widehat{\mathbf{B}}_t &= B \to B\backslash\{e\} \quad \text{ at rate } \gamma' v_e(1-\hat{p}_e).
\end{aligned}
$$

Of course the time change has no influence on the survival probability, and thus the critical value stays the same. If we assume that $\gamma' > \gamma$, we see that the recovery rate is bigger than r. Therefore, we can couple $(\widehat{\mathbf{C}},\widehat{\mathbf{B}})$ via the graphical representation with a CPLDP $(\overline{\mathbf{C}},\widehat{\mathbf{B}})$ with parameter $\frac{\lambda\gamma'}{\gamma}$, r, γ' and q such that $\overline{\mathbf{C}}_t \geq \widehat{\mathbf{C}}_t$ for all $t \geq 0$. Since $\lambda > \lambda_c(r,\gamma,q)$ we know that $\widehat{\mathbf{C}}$ has a chance to survive and through the coupling we see that if $\widehat{\mathbf{C}}$ survives, so does $\overline{\mathbf{C}}$. This implies that $\frac{\gamma'\lambda}{\gamma} > \lambda_c(r,\gamma',q)$ for all $\lambda > \lambda_c(r,\gamma,q)$, and thus $\frac{1}{\gamma}\lambda_c(r,\gamma,q) \geq \frac{1}{\gamma'}\lambda_c(r,\gamma',q)$ for $\gamma' > \gamma$. $\square$

Next we formulate a comparison between a long range contact process and the CPLDP. We will see that the long range contact process acts as a lower bound with respect to survival, i.e if the long range contact process survives so does CPLDP.

But first, let us rigorously define a long range contact process. Let $r > 0$ and $(a_e)_{e\in\mathcal{E}}$ be a sequence of positive real numbers such that $\sum_{y\in V} a_{\{x,y\}} < \infty$ for all $x \in V$. We assume translation invariance, i.e. that $a_{\{x,y\}} = a_{\{x',y'\}}$ if $d(x,y) = d(x',y')$, and use the convention that $a_{\{x,x\}} = 0$.

We consider the set $\mathcal{M}^* := \{\mathbf{inf}^*_e : e \in \mathcal{E}\} \cup \{\mathbf{rec}_x : x \in V\}$ as the set of all possible maps. Furthermore, we set $\mathcal{M}^*_{\text{inf}} := \{\mathbf{inf}^*_e : e \in \mathcal{E}\}$ and $\mathcal{M}_{\text{rec}} := \{\mathbf{rec}_x : x \in V\}$. We choose the rates to be $r_{\mathbf{inf}^*_e} = a_e > 0$ and $r_{\mathbf{rec}_x} = r > 0$ for all $e \in \mathcal{E}$ and all $x \in V$. Note that the bound (2.1) on the rates $(r_m)_{m\in\mathcal{M}^*}$ is satified since

$$\sup_{x\in V} \sum_{m\in\mathcal{M},\mathcal{D}(m)\ni x} r_m(|\mathcal{R}_x(m)|+1) \leq 3\sup_{x\in V}\left(\sum_{y\in V} a_{\{x,y\}}\right) + r < \infty, \tag{7.2}$$

where we used in the first inequality that $\mathcal{R}_x(\mathbf{inf}^*_{\{x,y\}}) = \{x,y\}$ for all $y \neq x$ and that $a_{\{x,y\}} = a_{\{x',y'\}}$ if $d(x,y) = d(x',y')$ to conclude that the supremum of the sums is finite. Thus, by the construction discussed in Section 2.3 we obtain a Feller process $\mathbf{X}$ on the state space $\mathcal{P}(V)$ and the jump rates are given by

$$\begin{aligned} \mathbf{X}_{t-} = C \to C \cup \{x\} \quad &\text{at rate } \sum_{y\in C} a_{\{x,y\}} \text{ and} \\ \mathbf{X}_{t-} = C \to C\backslash\{x\} \quad &\text{at rate } r. \end{aligned}$$

Next we show Proposition 1.4.23, which states that we can couple the CPLDP $(\mathbf{C}^C, \mathbf{B})$ with a long range contact process $\overline{\mathbf{X}}^C$ with transition rates

$$\overline{a}_e(\lambda,\gamma,q) = \frac{1}{2}\Big(\lambda + \gamma v_e - \sqrt{(\lambda+\gamma v_e)^2 - 4v_e p_e \lambda\gamma q}\Big).$$

and the same recovery rate r such that $\overline{\mathbf{X}}^C_t \subset \mathbf{C}^C_t$ for all $t \geq 0$.

Proof of Proposition 1.4.23. By Definition 7.1.2 we know that we only use potential infection events $(\mathbf{inf}^*_{\{x,y\}}, t) \in \mathrm{supp}(\Xi^{\mathrm{inf}^*})$ such that $\{x,y\} \in \mathbf{B}_t$ in an infection path, i.e. only infection arrows placed on an open edge are valid. We set for all $e \in \mathcal{E}$

$$Y_t(e) := |\{s \leq t : (\mathbf{inf}^*_e, s) \in \mathrm{supp}(\Xi^{\mathrm{inf}^*}) \text{ and } e \in \mathbf{B}_s\}| \quad \text{and} \quad X_t(e) := \mathbb{1}_{\{e\in\mathbf{B}_t\}}.$$

Now we can identify the transitions and transition rates of the just defined process (Y, X) quite easily. The state of $Y_t(e)$ depends on $X_t(e)$, and thus has transitions

$$Y_{t-}(e) = n \to n+1 \quad \text{at rate } \lambda X_{t-} = 1,$$

The process $X(e)$ is autonomous such that it has transition

$$\begin{aligned} X_{t-}(e) = 0 \to 1 \quad &\text{at rate } \hat{v}_e\hat{p}_e \text{ and} \\ X_{t-}(e) = 1 \to 0 \quad &\text{at rate } \hat{v}_e(1-\hat{p}_e). \end{aligned}$$

Now [Bro07, Theorem 1.4] together with Theorem 2.1.12 yields that there exists an Poisson process $\underline{Y}_t(e)$ on $[0,\infty)$ with rate $\overline{a}_e$ such that $Y_t(e) \geq \underline{Y}_t(e)$ almost surely for all $t \geq 0$. Since e was chosen arbitrarily this holds for every $e \in \mathcal{E}$, where

$$\overline{a}_e = \frac{1}{2}\Big(\lambda + \hat{v}_e - \sqrt{(\lambda+\hat{v}_e)^2 - 4\lambda\hat{v}_e\hat{p}_e}\Big).$$

This means we find a Poisson point process Ξ^{inf^*} on $\mathcal{M}^*_{\text{inf}} \times \mathbb{R}$ with intensity measure $r_m \mathsf{d}t$, where $r_{\mathbf{inf}^*_e} = \overline{a}_e > 0$ for $e \in \mathcal{E}$ such that $(\mathbf{inf}^*_e, t) \in \text{supp}(\underline{\Xi}^{\text{inf}^*})$ already implies that $(\mathbf{inf}^*_e, t) \in \text{supp}(\Xi^{\text{inf}^*})$ such that $e \in \mathbf{B}_t$.

Thus, via the graphical representation we can construct a Feller process $\underline{\mathbf{X}}$ on $\mathcal{P}(V)$ with respect to the Poisson point process $\underline{\Xi}^{\text{inf}^*} + \Xi^{\text{inf}}$ such that it has the required transition rates and $\underline{\mathbf{X}}^C_t \subset \mathbf{C}^C_t$ for all $t \geq 0$. Now it remains to show that $\underline{\mathbf{X}}$ is well-defined. To show this it suffices to verify (7.2). We see that

$$\overline{a}_e = \frac{\lambda+\hat{v}_e}{2}\Big(1 - \sqrt{1 - \tfrac{4\hat{v}_e\hat{p}_e\lambda}{(\lambda+\hat{v}_e)^2}}\Big) \leq \frac{2\lambda\hat{v}_e\hat{p}_e}{\lambda+\hat{v}_e},$$

where we used that $1-x \leq \sqrt{1-x}$ for $0 \leq x \leq 1$. Since $\frac{\hat{v}_e}{\lambda+\hat{v}_e} \leq 1$ we see that $\overline{a}_e \leq 2\lambda\hat{p}_e$. But by Assumption 1.4.21 the sequence $(\hat{p}_{\{x,y\}})_{y\in V}$ is summable for every $x \in V$, and thus (7.2) is satisfied. □

Next we show that the rates $(\overline{a}_e(\lambda,\gamma,q))_{e\in\mathcal{E}}$ chosen in Theorem 1.4.23 converge as $\gamma \to \infty$ and we provide the exact limit.

Lemma 7.1.4. *Let the sequence $(\overline{a}_e(\lambda,\gamma,q))_{e\in\mathcal{E}}$ be chosen as in Theorem 1.4.23. Then, it follows that* $\lim_{\gamma\to\infty}\overline{a}_e(\lambda,\gamma,q) = \lambda q p_e$ *for all* $e \in \mathcal{E}$

Proof. Let us consider the function $x \mapsto \sqrt{1-x}$ for $0 \leq x \leq 1$. The Taylor expansion at $x = 0$ yields that

$$\sqrt{1-x} = 1 - \frac{x}{2} - O(x^2).$$

Since $(\lambda+\hat{v}_e)^2 \geq 4\hat{v}_e\lambda$ is equivalent to $(\lambda-\hat{v}_e)^2 \geq 0$ we know that $\frac{4\hat{v}_e\hat{p}_e\lambda}{(\lambda+\hat{v}_e)^2} \in [0,1]$, where we used that $\hat{p}_e \in [0,1]$. Thus, if we consider γ as variable we see that

$$1 - \sqrt{1 - \tfrac{4\gamma v_e\hat{p}_e\lambda}{(\lambda+\gamma v_e)^2}} = \frac{1}{2}\tfrac{4\gamma v_e\hat{p}_e\lambda}{(\lambda+\gamma v_e)^2} + O(\gamma^{-2}),$$

where $O(\gamma^{-2})$ is meant with respect to $\gamma \to \infty$. This implies that

$$\overline{a}_e(\gamma) = \frac{\lambda+\gamma v_e}{2}\Big(1 - \sqrt{1 - \tfrac{4\gamma v_e\hat{p}_e\lambda}{(\lambda+\gamma v_e)^2}}\Big) = \tfrac{\gamma v_e\hat{p}_e\lambda}{\lambda+\gamma v_e} + O(\gamma^{-1}).$$

Now we see that the remainder vanishes and $\frac{\gamma v_e}{\lambda+\gamma v_e} \to 1$ as $\gamma \to \infty$. Thus, $\overline{a}_e(\gamma) \to \lambda \hat{p}_e$ as $\gamma \to \infty$. □

7.2 Comparison of a long range percolation model with the dynamical long range percolation

In this section we will compare the dynamical long range percolation $\mathbf{B}$ blockwise to a long range percolation model. The idea is that we partition the time axis $[0, \infty)$ at each edge $e \in \mathcal{E}$ into equidistant blocks $[nT, (n+1)T)$, where $T > 0$ and $n \in \mathbb{N}_0$. Now we set

$$w_n(e) := \begin{cases} 1 & \text{if } e \notin \mathbf{B}_t \text{ for all } t \in [nT, (n+1)T) \\ 0 & \text{otherwise,} \end{cases} \tag{7.3}$$

which indicates whether an edge e was closed for the whole time period $[nT, (n+1)T)$. We will simplify notation and write $w_n(x, y)$ instead of $w_n(\{x, y\})$ for $\{x, y\} \in \mathcal{E}$. The idea is that we accept all infection events $(t, \mathbf{inf}_e^*)$ with $t \in [nT, (n+1)T)$ such that $w_n(e) = 0$. This leads to an infection process, which survives more easily than $\mathbf{C}$, see also the visualization of the graphical representation for the CPDP in Figure 7.1. These techniques are not new, they were already used by [LR20] for graphs with bounded degree. Here we adjust the arguments to graphs with unbounded degrees.

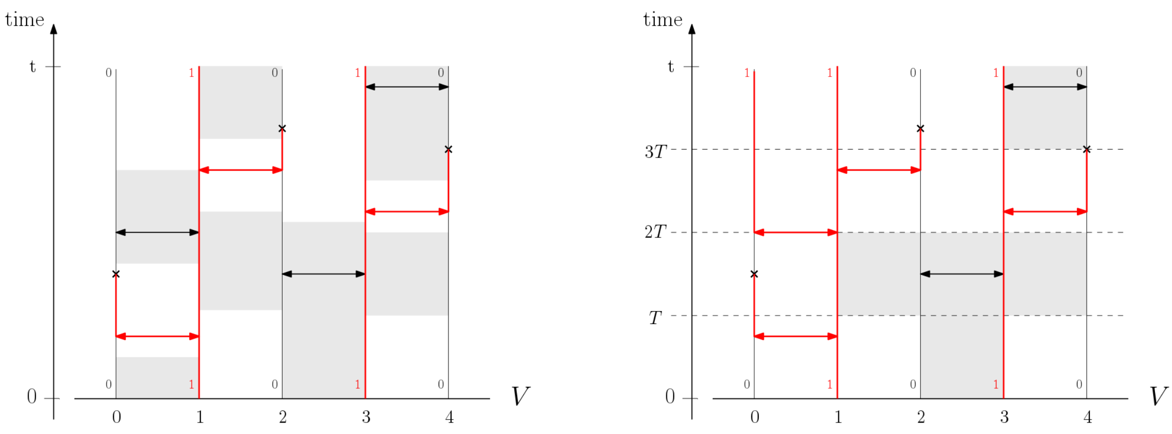


Figure 7.1: Red lines indicate as usual infection paths. On the left hand side we illustrated the graphical representation with respect to the background $\mathbf{B}$. On the right hand side we modified the background in such a way that the edges are only closed if they were closed throughout a whole block of length $[nT, (n+1)T)$.

Obviously $(w_n(e))_{(n,e)\in\mathbb{N}_0\times\mathcal{E}}$ is not a family of independent variables. But at least we know that $w_n(e)$ and $w_m(e')$ are independent as long as $e \neq e'$ for all $n, m \in \mathbb{N}_0$. So dependence only occurs along the time line for a fixed edge. A lower bound on the conditional probability that $w_n(e) = 1$ given all previous states $w_{n-1}(e), \dots, w_0(e)$ already exists and was proven in [LR20].

Proposition 7.2.1. *Let $T > 0$ be fixed, then it holds for all $n \in \mathbb{N}$ that for every $e \in E$*

$$\begin{aligned}
&\mathbb{P}(w_n(e) = 1 | w_{n-1}(e), \dots, w_0(e)) \\
&\geq (1-\hat{p}_e)e^{-\hat{p}_e\hat{v}_eT}\frac{e^{-\hat{v}_eT} + (1-\hat{p}_e)(1-e^{-\hat{v}_eT}) - e^{-\hat{p}_e\hat{v}_eT}}{1-e^{-\hat{p}_e\hat{v}_eT}} \\
&= (1-\hat{p}_e)e^{-\hat{p}_e\hat{v}_eT}\Big(1 - \hat{p}_e\frac{1-e^{-\hat{v}_eT}}{1-e^{-\hat{p}_e\hat{v}_eT}}\Big) := \delta_e(\gamma, q, T) = \delta_e.
\end{aligned}$$

Proof. See [LR20, Proposition 3.8]. □

Lemma 7.2.2. *Let $(X_n)_{n\in\mathbb{N}_0}$ be a family of Bernoulli random variables such that*

$$\mathbb{P}(X_n = 1 | X_{n-1}, \dots, X_0) \geq q,$$

where $q \in (0,1)$. Then there exist an independent and identically distributed family of Bernoulli random variables $(X'_n)_{n\in\mathbb{N}_0}$, such that $\mathbb{P}(X'_n = 1) = q$ and $X_n \geq X'_n$ almost surely for every $n \in \mathbb{N}_0$.

Proof. First of all we set

$$p_n(x_{n-1}, \dots, x_0) := \mathbb{P}(X_n = 1 | X_{n-1} = x_{n-1}, \dots, X_0 = x_0)$$

for $n \geq 1$ and $p_0 = \mathbb{P}(X_0 = 1)$, where $x_{n-1}, \dots x_0 \in \{0,1\}$. Let $(\chi_n)_{n\geq 0}$ be a family of independent and identical uniform distributed random variables on $[0,1]$ which are also independent of the family $(X_n)_{n\geq 0}$.

Next we iteratively define the desired family of random variables $(X'_n)_{n\geq 0}$. For that we need to define a family of auxiliary random variables $(Y_n)_{n\geq 0}$. First let $Y_0 := \mathbb{1}_{\{\chi_0 \leq q_0\}}$, where $q_0 \in [0,1]$. Note that the exact value of q_0 is yet to be determined. This will happen in the next step. Now set $X'_0 := X_0Y_0$. We see that $X'_0 \leq X_0$ and that

$$\mathbb{P}(X'_0 = 1) = \mathbb{P}(X_0 = 1)\mathbb{P}(Y_0 = 1) = p_0q_0.$$

Thus, by choosing $q_0 := \frac{q}{p_0}$ we see that $\mathbb{P}(X_0' = 1) = q$. Next suppose that we already defined $X_{n-1}', \dots, X_0'$. We set

$$p_n'(x_{n-1}, \dots, x_0) := \mathbb{P}(X_n = 1 | X_{n-1}' = x_{n-1}, \dots, X_0' = x_0),$$

where $x_{n-1}, \dots x_0 \in \{0,1\}$ and $p_0' := p_0$. Also let $q_n(\cdot)$ be a function which maps $\{0,1\}^n$ to $[0,1]$ which is yet to be determined. We set $Y_n := \mathbb{1}_{\{\chi_n \leq q_n(X_{n-1}', \dots, X_0')\}}$ and $X_n' := X_n Y_n$. It is again immediately clear that $X_n' \leq X_n$. Now we see that

$$\mathbb{P}(X_n' = 1 | X_{n-1}', \dots, X_0') = \mathbb{P}(Y_n = 1, X_n = 1 | X_{n-1}', \dots, X_0')$$

By choice χ_n is independent of $(X_k')_{k \leq n-1}$ and $(X_k)_{k \leq n}$. The random variable Y_n is a function of χ_n and all $(X_k')_{k \leq n-1}$ and X_n is a function of all $(X_k)_{k \leq n-1}$. This yields that Y_n and X_n are conditional on $(X_k')_{k \leq n-1}$ independent, i.e.

$$\begin{aligned} \mathbb{P}(X_n' = 1 | X_{n-1}', \dots, X_0') =& \mathbb{P}(X_n = 1 | X_{n-1}', \dots, X_0') \mathbb{P}(Y_n = 1 | X_{n-1}', \dots, X_0') \\ =& q_n(X_{n-1}', \dots, X_0') p_n'(X_{n-1}', \dots, X_0'). \end{aligned}$$

Thus, if we choose $q_n(X_{n-1}', \dots, X_0') := q \cdot \left(p_n'(X_{n-1}', \dots, X_0')\right)^{-1}$, we get that

$$\mathbb{P}(X_n' = 1 | X_{n-1}', \dots, X_0') = q.$$

Since the right hand side is independent of the values of $X_0', \dots, X_{n-1}'$ it follows that X_n' is independent of $(X_k')_{k \leq n-1}$. Furthermore, if we take the expectation of both sides, we get that $\mathbb{P}(X_n' = 1) = q$. What is left to show is that $p_n'(X_{n-1}', \dots, X_0') \geq q$, since otherwise $q_n'(X_{n-1}', \dots, X_0') > 1$. By the choice, the family $(\chi_n)_{n \geq 0}$ is independent of the family $(X_n)_{n \geq 0}$, and thus we see that

$$\mathbb{P}(X_n = 1 | X_{n-1}, \dots, X_0, \chi_{n-1}, \dots, \chi_0) = \mathbb{P}(X_n = 1 | X_{n-1}, \dots, X_0) \geq q.$$

But we know that the $(X_k')_{k \leq n-1}$ are functions of the $(X_k)_{k \leq n-1}$ and $(\chi_k)_{k \leq n-1}$. This implies that $\mathcal{G}_{n-1}' := \sigma(X_k' : k \leq n-1) \subset \sigma(X_k, \chi_k : k \leq n-1)$, and therefore by taking the conditional exception with respect to $\mathcal{G}_{n-1}'$ on both sides it follows that

$$p_n'(X_{n-1}', \dots, X_0') = \mathbb{P}(X_n = 1 | X_{n-1}', \dots, X_0') \geq q.$$

This concludes the proof. □

The bounds derived in Proposition 7.2.1 together with Lemma 7.2.2 enable us to compare $(w_n(e))_{n\geq 0}$ with a family of independent and identically distributed Bernoulli random variables.

Corollary 7.2.3. *Let $T > 0$ and $(w_n(e))_{(n,e)\in\mathbb{N}_0\times\mathcal{E}}$ be defined as in (7.3). Then there exists a family of independent Bernoulli variables $(w'_n(e))_{(n,e)\in\mathbb{N}_0\times\mathcal{E}}$ such that $\mathbb{P}(w'_n(e) = 1) = \delta_e$ and $w_n(e) \geq w'_n(e)$ almost surely for all $(n, e) \in \mathbb{N}_0 \times \mathcal{E}$.*

Proof. This is a direct consequence of Proposition 7.2.1, Lemma 7.2.2 and the fact that $w_n(e)$ and $w_m(e')$ are independent as long as $e \neq e'$ for all $n, m \in \mathbb{N}$. □

Since we want to formulate a comparison with a long range percolation model, we will now briefly introduce this model and summarize some fact about it. First let us clarify the notation. Recall that the graph $G = (V, E)$ is transitive, connected and has bounded degree. Furthermore, we again denote the set of edges of all lengths by

$$\mathcal{E} = \{e = \{x, y\} \subset V : x \neq y\}.$$

The long range percolation model is defined on $\big(\bigotimes_{e\in\mathcal{E}}\{0,1\}, \mathcal{F}, \mu\big)$ where $\bigotimes_{e\in\mathcal{E}}\{0,1\}$ is the sample space, $\mathcal{F}$ is the σ-algebra generated by the finite-dimensional cylinders and $\mu := \prod_{e\in\mathcal{E}} \mu_e$ with $\mu_e(\{1\}) = b_e \in [0, 1]$. Now $w \in \bigotimes_{e\in\mathcal{E}}\{0,1\}$ is a realization of the long range percolation model. We declare an edge $e = \{x, y\} \in \mathcal{E}$ to be open if $w(e) = 1$. Then with probability $b_{\{x,y\}} > 0$ the edge between x and y is open. Furthermore we assume for every fixed $x \in V$ that $\sum_{y\in V} b_{\{x,y\}} < \infty$ to guarantee that $\big(V, w^{-1}(\{1\})\big)$ is a locally finite graph, where $w^{-1}(\{1\}) = \{e \in \mathcal{E} : w(e) = 1\}$. Note that we use the convention $b_{\{x,x\}} = 0$ for all $x \in V$. Furthermore, we again assume translation invariance, i.e. that $b_{\{x,y\}} = b_{\{x',y'\}}$ if $d(x, y) = d(x', y')$, where $d(\cdot,\cdot)$ is the graph distance induced by G. We denote by $\mathcal{C}(x)$ the connected component containing $x \in V$. The following result provides a sufficient condition for absence of percolation.

Proposition 7.2.4. *Let $\sum_{y\in V} b_{\{x,y\}} < 1$ for one and hence every $x \in V$. Then almost surely there exists no infinite connected component. In this case $|\mathcal{C}(x)|$ is also integrable for all $x \in V$.*

Proof. This can be proven via a coupling with a branching process. Since V is countable we can index all vertices such that $V = \{x_0, x_1, \dots\}$. Recall that $w = (w(e))_{e\in\mathcal{E}}$ is a family of independent random variables such that $\mathbb{P}(w(e) = 1) = b_e$ for all $e \in \mathcal{E}$. Let $|\mathcal{C}(x_0)|$ be the connected component of x_0 with respect to w. Now let $w^{n,m}$ be an independent copy of w for all $n, m \in \mathbb{N}_0$, i.e. $\mathbb{P}(w^{n,m}(e) = 1) = b_e$ for all $e \in \mathcal{E}$ and all $n, m \in \mathbb{N}_0$. Furthermore, this means that w is independent of all $w^{n,m}$ and the copies $w^{n,m}$ and $w^{n',m'}$ are independent if either $n \neq n'$ or $m \neq m'$. We consider the index set

$$\mathcal{T} := \{(\alpha_0, \alpha_1, \dots, \alpha_n) \in V^{n+1} : n \in \mathbb{N}_0, \alpha_0 = x_0 \text{ and } \alpha_{i-1} \neq \alpha_i \text{ for all } 0 < i \leq n\}.$$

For $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ we define the generation of α as $|\alpha| = n$ (so that $|(x)| = 0$). Furthermore, we equip $\mathcal{T}$ with the lexicographical order with respect to the enumeration of V.

Now we construct a family of random variables $(X_\alpha)_{\alpha \in \mathcal{T}}$ with $X_\alpha \in \{0, 1\}$, where $X_{(x_0)} = 1$. We will define these random variables iteratively according to the lexicographical order. For given $n, m \in \mathbb{N}_0$ we define the set $I_n^m \subset \mathcal{E}$, which contains all edges, which we at least “observed” once until the offsprings of x_m in the n-th generation are drawn. Suppose we already constructed all X_α with $|\alpha| < n$ and all X_α with $|\alpha| = n$ such that $\alpha_{n-1} \in \{x_0, \dots, x_{m-1}\}$. Then, let I_n^m contain all edges $\{y, z\} \in \mathcal{E}$ such that there exists an $\alpha \in \mathcal{T}$ with $|\alpha| = k < n$ and $\alpha_k = y \in V$ or with $|\alpha| = n$ and $\alpha_n = y \in \{x_0, \dots, x_{m-1}\}$, which satisfies $X_\alpha = 1$. Now, we define X_α for all $\alpha \in \mathcal{T}$ with $|\alpha| = n$ and $\alpha_{n-1} = x_m$ by

$$X_\alpha := \begin{cases} 1 & \text{if } \{x_m, \alpha_n\} \notin I_n^m,\ w(\{x_m, \alpha_n\}) = 1 \text{ and } X_{(\alpha_0, \dots, \alpha_{n-2}, x_m)} = 1 \\ 1 & \text{if } \{x_m, \alpha_n\} \in I_n^m,\ w^{n,m}(\{x_m, \alpha_n\}) = 1 \text{ and } X_{(\alpha_0, \dots, \alpha_{n-2}, x_m)} = 1 \\ 0 & \text{otherwise} \end{cases}$$

In words if we have that $X_{(\alpha_0, \dots, \alpha_{n-2}, x_m)} = 1$ and have not observed $\{x_m, \alpha_n\}$ yet, then we set $X_\alpha = 1$ if $w(\{x_m, \alpha_n\}) = 1$. But if $\{x_m, \alpha_n\}$ was previously already observed, then we already know if the edge is open or closed with respect to w. To preserve independence between different generations and between the several offspring of the same generation we use the independent copy $w^{n,m}$ instead of w. Now we set $Z_n := \sum_{\alpha \in \mathcal{T} : |\alpha| = n} X_\alpha$. Because of translation invariance the offspring distribution is the same in every step. Thus, we see that $Z = (Z_n)_{n \in \mathbb{N}_0}$ is a branching process with $Z_0 = 1$ and offspring mean $\mu := \sum_{y \in V} b_{\{x,y\}}$, which is constant over x because of translation invariance.

If $y \in \mathcal{C}(x_0)$ then there exists a collection of edges $\{\{y_i, y_{i+1}\} : i \leq n\}$ such that $y_0 = x_0$ and $y_n = y$. But then all edges $\{y_i, y_{i+1}\}$ must have been observed at least once in the course of the construction of $(X_\alpha)_{\alpha \in \mathcal{T}}$, and thus all y_i will be counted by Z eventually, which implies that $|\mathcal{C}(x_0)| \leq \sum_{n \in \mathbb{N}} Z_n := T$, where T is the total progeny of the branching process. Note that this might not happen in the same order as the original path $\{\{y_i, y_{i+1}\} : i \leq n\}$, since we might take a shortcut via a resampled edge, which was originally closed. But, because of the resampling mechanism, it is not possible for an originally open edge to be closed without being "used" at least once. Thus, the total progeny T of Z can only be larger than $|\mathcal{C}(x_0)|$.

It is well known that for $\mu < 1$ the branching process dies out almost surely which provides the first claim. It also holds that $\mathbb{E}[T] \leq \frac{1}{\mu-1}$ for $\mu < 1$ as for example shown in [Hof16, Theorem 3.5], which provides integrability of $|\mathcal{C}(x_0)|$. Because of translation invariance this result does not depend on the choice of x_0 since $|\mathcal{C}(x_0)| = |\mathcal{C}(y)|$ for all $y \in V$. □

Next we consider the special case $V = \mathbb{Z}$ and $E = \{\{x, y\} \subset \mathbb{Z} : |x - y| = 1\}$. Since we assumed translation invariance we can simplify notation and set $b_{\{n,n+k\}} = b_{\{0,k\}} =: b_k$ for all $k \in \mathbb{N}$ and all $n \in \mathbb{Z}$. In fact only if $\sum_{k\in\mathbb{N}} kb_k = \infty$, is it possible for a infinite component to exists. The reason for this is that if $\sum_{k\in\mathbb{N}} kb_k < \infty$ holds, then the long range percolation is similar to a finite range percolation in the sense that there appear so-called "cut-points", see Figure 7.2, which lead to a partition of the integer lattice $\mathbb{Z}$, which consists of finite connected components. We will briefly show this result for the long range percolation before we continue with our study of the CPLDP.

Definition 7.2.5. Let $V = \mathbb{Z}$. A *cut-point* $m \in \mathbb{Z}$ is a point such that no (unoriented) edge $\{x, y\}$ with $x \leq m < y$ is present in the model, i.e. $\omega(\{x, y\}) = 0$.

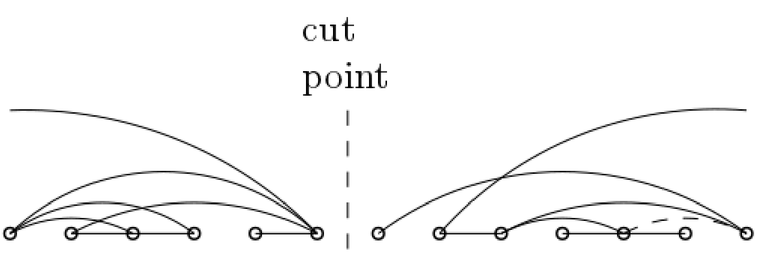


Figure 7.2: Visualization of a cut point

In the proof of the following result ergodic theory is used. A brief summary of some of the important notions can be found right before Remark 5.1.12.

Proposition 7.2.6. *Let* $(b_k)_{k\in\mathbb{N}} \subset [0, 1)$ *with* $\sum_{k\in\mathbb{N}} kb_k < \infty$*, then the following holds:*

1. *For* $m \in \mathbb{Z}$ *the probability* $\mathbb{P}(m$ *is a cut-point*$) = \mathbb{P}(0$ *is a cut-point*$) > 0$*, and as a consequence there exist almost surely infinitely many cut-points.*

2. *The subgraphs induced in the intervals between consecutive cut-points are independent and identically distributed. In particular, this implies that the distances between consecutive cut-points form a sequence of i.i.d. random variables as well.*
3. *There exists no infinite component.*

Proof. By translation invariance we know that

$$\mathbb{P}(m \text{ is a cut-point}) = \mathbb{P}(0 \text{ is a cut-point}) = \prod_{x \leq 0 < y} (1 - b_{\{x,y\}}).$$

The infinite product on the right hand side is strictly positive, since

$$\sum_{x \leq 0 < y} b_{\{x,y\}} = \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} b_{\{-l,-l+k\}} = \sum_{l \in \mathbb{N}} k b_k < \infty,$$

where we used that $b_{\{-l,-l+k\}} = b_k$ for every $l \in \mathbb{Z}$. Thus this yields the first claim. Next let us define $X_m := \mathbb{1}_{\{m \text{ is a cut-point}\}}$. Let S be a shift operator such that

$$(\omega(\{x,y\}))_{\{x,y\} \in \mathcal{E}} \mapsto (\omega(\{x+1,y+1\}))_{\{x,y\} \in \mathcal{E}}.$$

In words we shift all edge by one vertex to the right. Since $(\omega(e))_{e \in \mathcal{E}}$ is a family of independent random variables it is clear that (ω, S) is ergodic. It is not difficult to see that there must exists a measurble function $f : \Omega \to \{0,1\}$ such that $X_k = f(S^{-k}\omega)$ for all $k \in \mathbb{Z}$. Then by Birkhoff's mean ergodic theorem follows that

$$\frac{1}{2n} \sum_{k=-n}^{n} X_k \to \mathbb{E}[X_0] = \mathbb{P}(0 \text{ is a cut-point}) > 0$$

almost surely. This implies that infinitely many X_k are equal to 1 almost surely. The second statement is immediate, since there are no edges between different intervals between consecutive cut-points. This also means that with probability 1 there cannot exist an infinitely large component. □

7.3 Existence of an immunization phase

Throughout this section we assume that Assumption 1.4.21 is satisfied, which states that $\sum_{y \in V} v_{\{x,y\}} p_{\{x,y\}} < \infty$ and $\sum_{y \in V} v_{\{x,y\}}^{-1} < \infty$ for all $x \in V$. In this section we will show Theorem 1.4.24, which means that we prove that for given $r > 0$ and $\gamma > 0$, there exists a $q^* \in (0,1)$ such that $\mathbf{C}$ dies out almost surely for all $q < q^*$ regardless of the choice of $\lambda > 0$, i.e. $\lambda_c(r, \gamma, q) = \infty$ for all $q < q^*$.

The idea is that, if q is small enough, then an arbitrary vertex will eventually be isolated for a long time, and therefore a potential infection cannot spread to another vertex before the isolated vertex is affected by a recovery event. So it is basically a dead end for an infection path. To make this formally precise let us define $X = (X_{e,n})_{(e,n) \in \mathcal{E} \times \mathbb{N}_0}$ and $U = (U_{x,n})_{(x,n) \in V \times \mathbb{N}_0}$ by

$$X_{e,n} := \begin{cases} 1 & \text{if } e \in \mathbf{B}_t \text{ for some } t \in [nT, (n+1)T) \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{x,n} := \begin{cases} 1 & \text{if } \operatorname{supp}(\Xi^{\text{rec}}) \cap \{\mathbf{rec}_x\} \times [nT, (n+1)T) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that $X_{e,n} = 1 - w_n(e)$ from 7.3 for all $e \in \mathcal{E}$, where $w_n(e)$ is defined in 7.3. If $U_{x,n} = 0$ and $\sum_{y \in V} X_{\{x,y\},n} = 0$, then an infection on site x cannot possibly survive in the time interval $[nT, (n+1)T)$. This follows since $\sum_{y \in V} X_{\{x,y\},n} = 0$ implies that for the whole time interval all edges attached to x are closed. Therefore, since $U_{x,n} = 0$ we know that the site x will recover and cannot be reinfected. Furthermore, between time nT and $(n+1)T$ no infection can spread from x. Now we define a random graph G_1 with vertex set $V \times \mathbb{N}_0$ and add edges according to the following rules.

1. If $U_{x,n} = 1$, add an oriented edge from (x, n) to $(x, n+1)$.
2. If $X_{e,n} = 1$ for $e = \{x, y\}$, add edges as if $U_{x,n} = 1$, $U_{y,n} = 1$ and an unoriented edge between (x, n) and (y, n).

The rules are visualized in Figure 7.3. Note that all “horizontal” edges are unoriented such that they can be used in both directions, but all “vertical” edges are oriented and only point upwards.

Definition 7.3.1. (Valid path) Let G_1 be the random graph constructed above and $C \subset V$ be the set of all initially infected individuals. We say that there exists a valid path from $C \times \{0\}$ to a point (x, n), if there exists sequence $x_0, x_1, \ldots, x_m = x$ with $x_0 \in C$ and $0 = n_0 \le n_1 \le \cdots \le n_m = n$ such that there exist an edge in G_1 from (x_k, n_k) to (x_{k+1}, n_{k+1}) for all $k \in \{0, \ldots, m-1\}$.

In Figure 7.4 we visualized how a fragment of the graph could look like. Here, the red path shows a possible valid path. For every $n \in \mathbb{N}$ we denote by $Y_n = Y_n(U, X)$, the set of all points $x \in V$ such that there exists a valid path from $Y_0 \times \{0\}$ to (x, n).

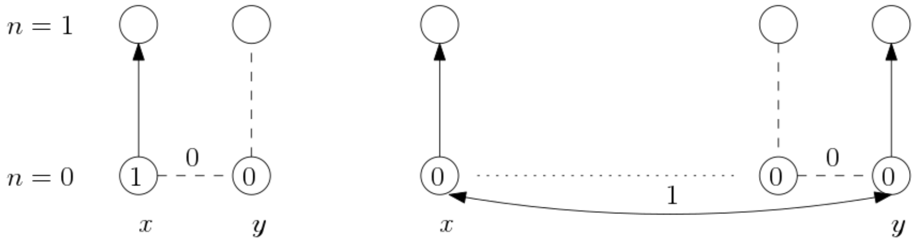


Figure 7.3: Illustration of the first and second rule. Solid lines are present edges and dashed lines indicate absent edges. The numbers in the circles indicate the state of the U variables and the number of above the horizontal edges the state of the X variables.

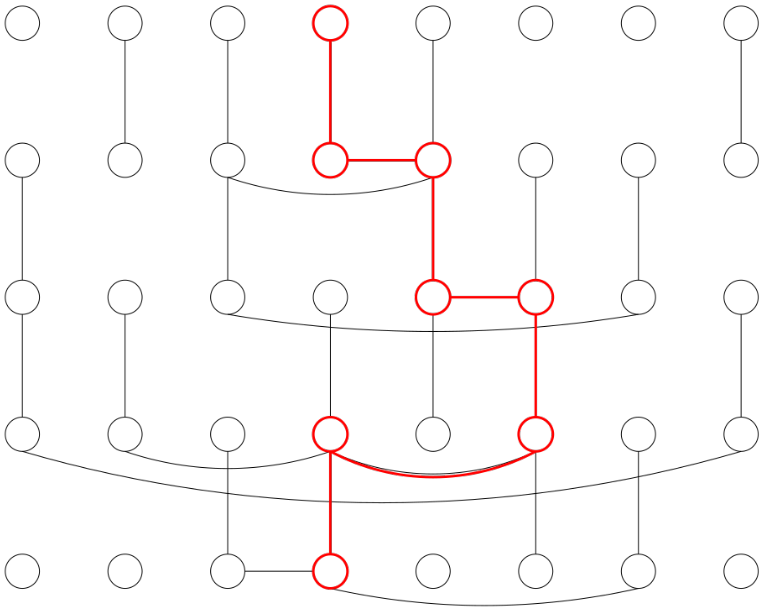


Figure 7.4: Visualization of a fragment of G_1. The red path indicates a possible valid path

Lemma 7.3.2. *Let $T > 0$,$n \in \mathbb{N}_0$ and $C \subset V$. Then $x \in \mathbf{C}^C_{nT}$ and $C = Y_0$ implies that $x \in Y_n$, and thus in particular if $Y_n = \emptyset$, then $\mathbf{C}^C_{nT} = \emptyset$.*

Proof. If $x \in \mathbf{C}^C_{nT}$ then there must at least exist one infection path from $C \times \{0\}$ to (x, nT) as defined in Definition 7.1.2. This means that there exists a sequence of times $0 = t_0 < t_1 < \cdots < t_{n'} < t_{n'+1} = nT$ with $n' \geq n$ and space points $x_0, x_1, \ldots, x_{n'} = x$ with $x_0 \in C$ such that, $(\mathbf{inf}^*_{\{x_{k-1},x_k\}}, t_k) \in \operatorname{supp}(\Xi^{\mathrm{inf}^*})$ and $\{x_{k-1}, x_k\} \in \mathbf{B}_{t_k}$ for all $k \in \{1, \ldots, n'+1\}$ and $\operatorname{supp}(\Xi^{\mathrm{rec}}) \cap \big(\{\mathbf{rec}_{x_k}\} \times [t_k, t_{k+1})\big) = \emptyset$ for all $k \in \{0, \ldots, n'\}$. For any such path there must exist a subsequence of sites $(x_m)_{m \leq n} \subset (x_k)_{k \leq n'}$ (including x_0 and x_n) such that $x_m \in \mathbf{C}_{mT}$ for $m \in \{0, \ldots, n\}$. Now if we can show that $x_{m-1} \in \mathbf{C}_{(m-1)T}$ and $x_m \in \mathbf{C}_{mT}$ imply that $x_m \in Y_m$ the claim follows since $x_0 \in Y_0 = C$ by assumption.

So if $x_{m-1} \neq x_m$ it means that the infection must have spread from x_{m-1} to x_m in the time interval $[(m-1)T, mT)$. But we already assumed the existence of an infection path. Thus, we find a sequence of sites $x_{m-1} = y_0^m, \dots y_l^m = x_m$ and a sequences of times $(m-1)T < t_1^m < \dots < t_l^m < t_{l+1}^m = mT$ with $(y_j^m)_{j\leq l} \subset (x_k)_{k\leq n'}$ and $(t_j^m)_{j\leq l} \subset (t_k)_{k\leq n'+1}$ such that $\big(\mathbf{inf}^*_{\{y_{k-1}^m, y_k^m\}}, t_k^m\big) \in \Xi^{\text{inf}^*}$ and $\{y_{k-1}^m, y_k^m\} \in \mathbf{B}_{t_k^m}$ for all $k \in \{1,\dots,l\}$. In particular this implies that $X_{\{y_{k-1}^m, y_k^m\}, m-1} = 1$ for all $k \in \{1,\dots,l\}$, thus by the second rule $x_m \in Y_m$.

If $x_{m-1} = x_m$ then either there was no recovery event in the whole time interval $\big[(m-1)T, mT\big)$, then by the first rule $x_m \in Y_m$ or the infection must have spread to another site and the site x_m got reinfected. Then there must have been a site x' and a time $t \in [(m-1)T, mT)$ such that $\{x_m, x'\} \in \mathbf{B}_t$ and therefore $x_m \in Y_m$ by the second rule. □

Obviously $(U_{x,n})_{(x,n)\in V\times\mathbb{N}_0}$ is an independent and identically distributed family of random variables with $\mathbb{P}(U_{x,n} = 1) = e^{-rT}$. Furthermore by definition it is independent of the family $(X_{e,n})_{(e,n)\in\mathcal{E}\times\mathbb{N}_0}$. We already mentioned that $X_{e,n} = 1 - w_n(e)$, and thus we get by Corollary 7.2.3 that there exists a family of independent and identically distributed random variables $(X'_{e,n})_{(e,n)\in\mathcal{E}\times\mathbb{N}_0}$ such that $\mathbb{P}(X'_{e,n} = 1) = 1 - \delta_e$ and $X_{e,n} \leq X'_{e,n}$ almost surely for all $(e,n) \in \mathcal{E}\times\mathbb{N}_0$, which are also independent of $(U_{x,n})_{(x,n)\in V\times\mathbb{N}_0}$. Analogously to Y_n we can now define $Y'_n = Y'_n(X', U)$ in the same way with the difference that we use $X'_{e,n}$ instead of $X_{e,n}$. We see immediately that $Y_n \subset Y'_n$ for all $n \in \mathbb{N}_0$. So whenever $(Y'_n)_{n\in\mathbb{N}_0}$ goes extinct, i.e. there exists a $k \in \mathbb{N}_0$ such that $Y'_k = \emptyset$, so does $(Y_n)_{n\in\mathbb{N}_0}$. We will see that Y'_n is much easier to analyse compared to Y_n.

Lemma 7.3.3. *Let $x \in V$. If $\mathbb{E}[|Y'_1||Y'_0 = \{x\}] < 1$, then Y' goes extinct almost surely for any finite $A \subset V$ as initial state.*

Proof. The process $Y' = (Y'_n)_{n\in\mathbb{N}_0}$ is basically a type of oriented percolation model. Thus, it is not difficult to see that Y' is a Markov process and the state $\emptyset$ is an absorbing state. The idea is to consider $N_{\text{ex}} := \inf\{n \geq 0 : Y'_n := \emptyset\}$, which is the extinction time of Y', and set $F_A(n) := \mathbb{P}(N_{\text{ex}} \leq n | Y'_0 = A)$. Note that since the U and X' are all independent the event

$$\{\text{there exists no valid path from } (y,0) \text{ to } \mathbb{N}_0 \times \{n\}\}$$

is a decreasing event with respect to a product measure for every $y \in V$. So we know that by the FKG inequality or rather the reverse of it, see [Gri99, Theorem 2.4], that

$$F_A(n) \geq \prod_{y \in A} \mathbb{P}(\{\text{there exists no valid path from } (y,0) \text{ to } \mathbb{N}_0 \times \{n\}\}) = F_{\{x\}}(n)^{|A|},$$

where we used translation invariance. The aim is to show that if $\mathbb{E}[|Y_1'||Y_0' = \{x\}] < 1$, then $F_{\{x\}}(n) \to 1$ as $n \to \infty$. We will not prove this result in detail since the proof is identical to [LR20, Lemma 3.7]. □

Now Theorem 1.4.24 follows as a corollary.

Proof of Theorem 1.4.24. Let us fix $x \in V$. We can calculate that

$$\mathbb{E}[|Y_1'||Y_0' = \{x\}] = \mathbb{E}[\mathbb{1}_{\{\exists y \in V : X'_{\{x,y\},0} = 1\}}|Y_1'|] + \mathbb{P}\Big(\bigcap_{y \in V}\{X'_{\{x,y\},0} = 0\}\Big)\mathbb{E}[U_{x,0}]. \quad (7.4)$$

Let us choose $0 < \varepsilon < 1$ arbitrarily but fixed. For the last term, we find a $T_1 > 1$ large enough such that

$$\mathbb{E}[U_{x,0}] = e^{-rT} < \frac{\varepsilon}{3} \quad (7.5)$$

for all $T > T_1$. For the first term we see that Y_1' is actually the connected component containing x formed by a long range percolation model with probabilities $(1-\delta_e)_{e \in \mathcal{E}}$, where δ_e is defined in Proposition 7.2.1. First we note that $\delta_e = \delta_e(q,T)$ can be considered as a function of q and T, where we omitted γ since this parameter remains constant throughout this proof. We see that

$$1 - \delta_e = 1 - e^{-\hat{p}_e \hat{v}_e T} + \hat{p}_e e^{-\hat{p}_e \hat{v}_e T} + (1 - \hat{p}_e)\hat{p}_e \frac{1 - e^{-\hat{v}_e T}}{e^{\hat{p}_e \hat{v}_e T} - 1} \leq \hat{p}_e \hat{v}_e T + \hat{p}_e + \frac{1}{\hat{v}_e T}, \quad (7.6)$$

for all $e \in \mathcal{E}$, where we used that $1 - x \leq e^{-x}$ and $1 + x \leq e^x$ for $x \geq 0$. Recall that $\hat{p}_k = qp_k$. For the remainder of this proof we choose $q = q(T) := T^{-2}$ and see that

$$1 - \delta_e(q(T), T) \leq \frac{1}{T} p_e \hat{v}_e + \frac{1}{T^2} p_e + \frac{1}{\hat{v}_e T} =: b_e(T) \quad (7.7)$$

for all $e \in \mathcal{E}$. We attach T as an index to $Y_1'(T)$, since by the choice of q the probabilities $(1 - \delta_e)_{e \in \mathcal{E}}$ determining the connected components only depend on the choice of T. Next we will show that there exists $T_2 > 0$ and an $M(\varepsilon, T_2) = M > 0$ such that

$$\mathbb{E}[\mathbb{1}_{\{|Y_1'(T)| > M\}}|Y_1'(T)|] < \frac{\varepsilon}{3} \quad (7.8)$$

for all $T > T_2$. For this, let $Z(T)$ be the connected component containing x formed by a long range percolation model with probabilities $(b_e(T))_{e\in\mathcal{E}}$ such that $Y_1'(T) \subset Z(T)$ for every $T > 0$. This is possible since (7.7) holds for all $e \in \mathcal{E}$. Furthermore, $b_e(T)$ is decreasing in T and $b_e(T) \to 0$ as $T \to \infty$ for all $e \in \mathcal{E}$. We also see that $b_e(T) \leq b_e(1)$ for all $T \geq 1$ and every $e \in \mathcal{E}$. By Assumption 1.4.21 it follows that $(b_{\{x,y\}}(1))_{y\in V}$ is summable for all $x \in V$. Therefore, by Lebesgues theorem of dominated convergence we see that there exists a $T_2 \geq T_1$ large enough such that

$$\sum_{y\in V} b_{\{x,y\}}(T) < 1$$

for all $T \geq T_2$. For this choice of T_2 the integrability of $|Z(T_2)|$ follows by Proposition 7.2.4, i.e. $\mathbb{E}[|Z(T_2)|] < \infty$. Thus, for every $\varepsilon > 0$ there exist an $M(\varepsilon, T_2) = M > 0$ such that

$$\mathbb{E}[\mathbb{1}_{\{|Z(T_2)|>M\}}|Z(T_2)|] < \frac{\varepsilon}{3}.$$

Now we see that $b_e(T)$ is monotone decreasing in T for all $e \in \mathcal{E}$, and thus

$$\mathbb{E}[\mathbb{1}_{\{|Z(T)|>M\}}|Z(T)|] \leq \mathbb{E}[\mathbb{1}_{\{|Z(T_2)|>M\}}|Z(T_2)|] < \frac{\varepsilon}{3}$$

for all $T > T_2$. Furthermore since by definition $Y_1'(T) \subset Z(T)$ for all T we see that

$$\mathbb{E}[\mathbb{1}_{\{|Y_1'(T)|>M\}}|Y_1'(T)|] < \frac{\varepsilon}{3}.$$

for all $T > T_2$. Now we see that

$$\mathbb{E}[\mathbb{1}_{\{\exists y\in V: X'_{\{x,y\},0}=1\}}|Y_1'|] \leq \mathbb{E}[\mathbb{1}_{\{|Y_1'|>M\}}|Y_1'|] + \mathbb{E}[\mathbb{1}_{\{|Y_1'|\leq M\}}\mathbb{1}_{\{\exists y\in V: X'_{\{x,y\},0}=1\}}|Y_1'|],$$

and therefore we can use (7.4) and conclude with the bounds (7.5) and (7.8) that

$$\mathbb{E}[|Y_1'||Y_0' = \{x\}] < \frac{\varepsilon}{3} + M\underbrace{\mathbb{P}(\{\exists y \in V : X'_{\{x,y\},0} = 1\} \cap \{|Y_1'| \leq M\})}_{\leq \mathbb{P}(\exists y\in V: X'_{\{x,y\},0}=1)} + \frac{\varepsilon}{3} \tag{7.9}$$

for all $T > T_2$. By using subadditivity of the measure $\mathbb{P}$ we get that

$$\mathbb{P}(\exists y \in V : X'_{\{x,y\},0} = 1) = \mathbb{P}\Big(\bigcup_{y\in V}\{X'_{\{x,y\},0} = 1\}\Big) \leq \sum_{y\in V}\big(1 - \delta_{\{x,y\}}(q(T), T)\big),$$

since $\mathbb{P}(X'_{\{x,y\},0} = 1) = 1 - \delta_{\{x,y\}}(q(T), T)$ for every $\{x, y\} \in \mathcal{E}$. Now we can use again that $1 - \delta_{\{x,y\}}(q(T), T) \leq b_e(1)$ for all $T \geq 1$ and $(b_{\{x,y\}}(1))_{y\in V}$ is summable for every

$x \in V$. Together with the fact that $1 - \delta_{\{x,y\}}(q(T), T) \to 0$ as $T \to \infty$ this implies that there exists a $T_3 \geq T_2$ such that

$$\sum_{y \in V} (1 - \delta_{\{x,y\}}(q(T), T)) < \frac{\varepsilon}{3M} \tag{7.10}$$

for all $T > T_3$. Now (7.9) and (7.10) imply that there exists a T and q such that

$$\mathbb{E}[|Y_1'||Y_0' = \{x\}] < \varepsilon < 1.$$

Therefore, by Lemma 7.3.3 we see that Y' goes extinct almost surely and since $Y_n \subset Y_n'$, we know that Y_n goes extinct almost surely. Finally, we can use Lemma 7.3.2 to conclude that this already implies that $\mathbf{C}^{\{x\}}$ goes extinct almost surely as well.
But if $\mathbf{C}^{\{x\}}$ goes extinct almost surely, i.e $\theta(\{x\}) = 0$, then by translation invariance it follows that $\theta(\{y\}) = 0$ for all $y \in V$, and thus if we assume $\theta(C) > 0$ for some finite $C \subset V$ via the graphical representation it would follow that there must exist a $z \in C$ such that $\theta(\{z\}) > 0$ which leads to a contradiction. Thus, $\mathbf{C}^C$ goes extinct almost surely for all finite $C \subset V$. □

7.4 Extinction for slow background speed for $V = \mathbb{Z}$

On general graphs $G = (V, E)$ Proposition 1.4.22 and Theorem 1.4.24 provide partial results on the behaviour of the critical infection rate for slow speed of the background process, which we stated in Corollary 1.4.25. Let us recall the statement of this corollary. For a given $r > 0$ there exists a $q^* = q^*(r) > 0$ such that $\lim_{\gamma \to 0} \lambda_c(r, \gamma, q) = \infty$ for all $q < q^*$. We prove this result now.

Proof of Corollary 1.4.25. Let $r > 0$ be fixed. Now Theorem 1.4.24 provides that for a given $\gamma_0 > 0$ there exists a $q_0 = q_0(r, \gamma_0) > 0$ such $\lambda_c(r, q, \gamma_0) = \infty$ for all $q < q_0$. But by Proposition 1.4.22 it also follows that $\lambda_c(r, q, \gamma) = \infty$ for all $q < q_0$ and all $\gamma \leq \gamma_0$. Another consequence of Proposition 1.4.22 is that if $\gamma_1 < \gamma_0$, then the $q_1 = q_1(r, \gamma_1)$ provide by Theorem 1.4.24 for γ_1 must be bigger or equal to q_0, i.e. $q_1 \geq q_0$. Like this we can recursively construct an increasing sequence $(q_n)_{n \in \mathbb{N}_0}$ such that we can define $q^* := \sup_{n \in \mathbb{N}} q_n$. Now for every $q < q^*$ there must exist an $n \in \mathbb{N}_0$ such that $q \leq q_n$, and thus $\lambda_c(r, q, \gamma) = \infty$ for all $\gamma \leq \gamma_n$. Hence, it follows in particular that $\lim_{\gamma \to 0} \lambda_c(r, q, \gamma) = \infty$ for all $q < q^*$. □

Now we restrict ourselves to the one dimensional integer lattice. Since in this case we can fully characterize the behaviour of the critical infection rate as $\gamma \to 0$. Thus, throughout this section we consider $G = (V, E)$ to be the one dimensional lattice, i.e $V = \mathbb{Z}$ and $E = \{\{x, y\} \subset \mathbb{Z} : |x - y| = 1\}$, and assume that Assumption 1.4.26 is satisfied, i.e.

$$\sum_{y \in \mathbb{N}} y v_{\{0,y\}}^{-1} < \infty \quad \text{and} \quad \sum_{y \in \mathbb{N}} y v_{\{0,y\}} p_{\{0,y\}} < \infty$$

Obviously this assumption already implies Assumption 1.4.21 and by (7.6) we see that

$$\sum_{y \in \mathbb{N}} y(1 - \delta_{\{0,y\}}) \leq \sum_{y \in \mathbb{N}} y\Big(\hat{p}_{\{0,y\}} \hat{v}_{\{0,y\}} T + \hat{p}_{\{0,y\}} + \frac{1}{\hat{v}_{\{0,y\}} T}\Big) < \infty \tag{7.11}$$

for all $x \in \mathbb{Z}$, by taking the stronger Assumption 1.4.26 into consideration. The goal of this section is to show Theorem 1.4.27. We will now modify and adapt the strategy used in [LR20].

This means that as in the previous section we construct a type of oriented long range percolation model, which will be coupled to the CPLDP in such a way that if this model goes extinct so does the CPLDP. Recall that $w_n(e)$, which is defined in (7.3), is the indicator function, that is one if the edge e is closed for the whole time interval $[nT, (n + 1)T)$. By Corollary 7.2.3 we know that there exists a family of independent Bernoulli random variables $(w'_n(e))_{\{(n,e) \in \mathbb{N}_0 \times \mathcal{E}\}}$ such that $w'_n(e) \leq w_n(e)$ almost surely and $\mathbb{P}(w'_n(e) = 1) = \delta_e$ for all $n \in \mathbb{N}_0$ and all $e \in \mathcal{E}$. We define the oriented long range percolation model right away with respect to the family $(w'_n(e))_{\{(n,e) \in \mathbb{N}_0 \times \mathcal{E}\}}$.

One key point of the arguments used in [LR20] was that in an independent percolation model on $\mathbb{Z}$ with $p < 1$ an infinitely large cluster does not occur, and thus the percolation almost surely partitions $\mathbb{Z}$ into finite connected components. As we saw in Proposition 7.2.6 the long range percolation exhibits a similar behaviour, as finite range percolation models on $\mathbb{Z}$, in case that $\sum_{y \in \mathbb{N}} y(1 - \delta_{\{0,y\}}) < \infty$, i.e. the percolation graph is almost surely a union of finite connected components.

Recall from Definition 7.2.5 that a cut-point m is a point such that no edge $\{x, y\}$ with $x \leq m < y$ is present in the model. In comparison to the nearest neighbour case, one major problem is that the presence of cut points at two different vertices is not independent. The events $(\{k \text{ is a cut-point}\})_{k \in \mathbb{Z}}$ are in fact a positively correlated. To be precise $\{k$ is a cut-point$\}$ is a decreasing event, and thus by the FKG inequality, see [Gri99, Theorem 2.4], for any $m \in G$

$$\mathbb{P}(\{0 \text{ is a cut-point}\} \cap \{m \text{ is a cut-point}\}) \geq \mathbb{P}(0 \text{ is a cut-point})\mathbb{P}(m \text{ is a cut-point}).$$

So we need to adjust the construction in such a way that we can deal with this unfavourable correlation.

Definition 7.4.1. Let $n, K_0 \in \mathbb{N}$ and $T > 0$. We call $m \in G$ an (n, K_0)-cut if $w'_n(x, y) = 1$ for all $x \leq m < y$ with, $|x - y| \leq 2K_0$.

The event $\bigcap_{x \leq m < y : |x-y| \leq 2K_0} \{w'_n(x, y) = 1\}$ corresponds to m being an (n, K_0)-cut. Let $r_0 \in \mathbb{N}$ and define

$$\begin{aligned}
M_k &:= [k(2K_0 + r_0), (k+1)(2K_0 + r_0) - 1] \cap \mathbb{Z} \\
M_k^{\text{left}} &:= [k(2K_0 + r_0), k(2K_0 + r_0) + K_0 - 1] \cap \mathbb{Z} \\
M_k^{\text{mid}} &:= [k(2K_0 + r_0) + K_0, k(2K_0 + r_0) + K_0 + r_0 - 1] \cap \mathbb{Z} \\
M_k^{\text{right}} &:= [k(2K_0 + r_0) + K_0 + r_0, (k+1)(2K_0 + r_0) - 1] \cap \mathbb{Z}
\end{aligned}$$

The collection $(M_k)_{k \in \mathbb{Z}}$ forms a disjoint partition of $\mathbb{Z}$. Furthermore, for every $k \in \mathbb{Z}$ the sets $M_k^{\text{mid}}, M_k^{\text{left}}$ and M_k^{right} are disjoint and $M_k = M_k^{\text{mid}} \cup M_k^{\text{left}} \cup M_k^{\text{right}}$. We also want to remark that $|M_k| = 2K_0 + r_0$, $|M_k^{\text{mid}}| = r_0$ and $|M_k^{\text{left}}| = |M_k^{\text{right}}| = K_0$. See Figure 7.5 for a visualization.

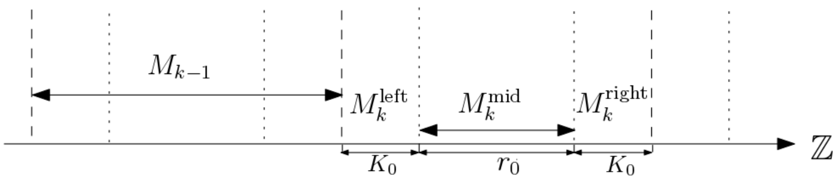


Figure 7.5: Visualization of the sets M_{k-1}, $M_k^{\text{mid}}, M_k^{\text{left}}$ and M_k^{right}.

Next we define the random variables

$$X_{k,n} := \begin{cases} 1 & \text{if no } (n, K_0)\text{-cut lies in } M_k^{\text{mid}} \\ 0 & \text{otherwise.} \end{cases} \tag{7.12}$$

If $X_{k,n} = 0$, then there exists a barrier in M_k^{mid}, which the infection cannot overcome via edges of length shorter than $2K_0$.

We will now partition the space-time strip $\mathbb{Z} \times [nT, (n+1)T)$ for every n, where $T > 0$, according to the presence of (n, K_0)-cuts. Let $c_{k,n}$ be the right most (n, K_0)-cut in $M_k^{\text{mid}} \times [nT, (n+1)T)$ and if none is present, then set it equal to the right boundary of M_k^{mid}. Now set $D_{k,n} := [c_{k-1,n} + 1, c_{k,n}] \cap \mathbb{Z}$. We see that $S_{k,n} := D_{k,n} \times [nT, (n+1)T)$ is a disjoint space-time partition of $\mathbb{Z} \times [0, \infty)$. See Figure 7.6 for an illustration.

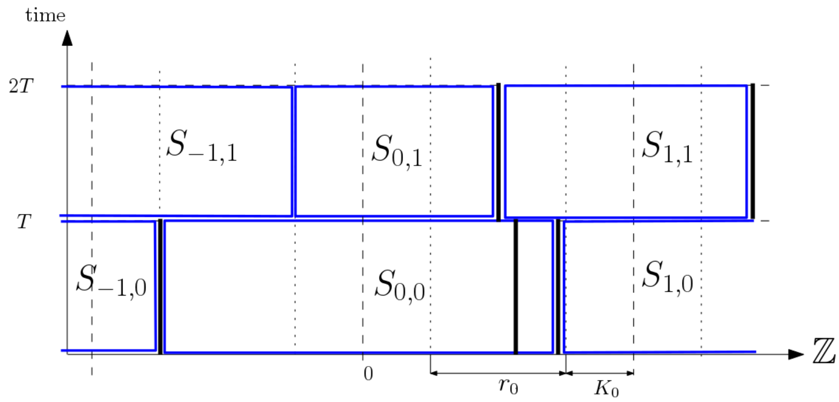


Figure 7.6: A visualization of a possible partition. The thick black lines represent an (n, K_0)-cuts and the blue boxes the resulting partition.

The boxes can only be of bounded size and we see from the construction that

$$\begin{aligned} D_{k,n} &\supset D_k^{\min} := M_{k-1}^{\text{right}} \cup M_k^{\text{left}}, \\ D_{k,n} &\subset D_k^{\max} := M_{k-1}^{\text{mid}} \cup M_{k-1}^{\text{right}} \cup M_k^{\text{left}} \cup M_k^{\text{mid}} = M_{k-1}^{\text{mid}} \cup D_k^{\min} \cup M_k^{\text{mid}} \end{aligned} \tag{7.13}$$

Here $D_k^{\min}$ is the minimal set, in the sense that $D_{k,n}$ must at least contain all vertices contained in $D_k^{\min}$ and $D_k^{\max}$ is maximal, i.e $D_{k,n}$ can at most contain all vertices in $D_k^{\max}$. This provides us with an upper and lower bound on the number of vertices contained $D_{k,n}$, which are $2K_0 \leq |D_{k,n}| \leq 2K_0 + 2r_0$. Thus, we can define $S_{k,n}^{\min} := D_k^{\min} \times [nT, (n+1)T)$ and $S_{k,n}^{\max} := D_k^{\max} \times [nT, (n+1)T)$ as the minimal and maximal possible space-time box with $S_{k,n}^{\min} \subset S_{k,n} \subset S_{k,n}^{\max}$.

Recall that $X_{k,n}$ provides us with the information whether it is possible for the infection to traverse M_k^{mid} via *short* edges. So if $X_{k,n} = 0$ and $X_{k+1,n} = 0$, then the boundaries of $S_{k,n}$ are (n, K_0)-cuts and the infection can only leave this box via *long* edges. Hence, we define

$$W_{\{k,l\},n} := \begin{cases} 1 & \text{if there exists an edge } e = \{x, y\} \text{ with } |x - y| > 2K_0 \\ & \text{which connects } S_{k,n} \text{ to } S_{l,n} \text{ at some } t \in [nT, (n+1)T) \\ 0 & \text{otherwise,} \end{cases}$$

where $k \neq l$. See Figure 7.7 for a visualization.

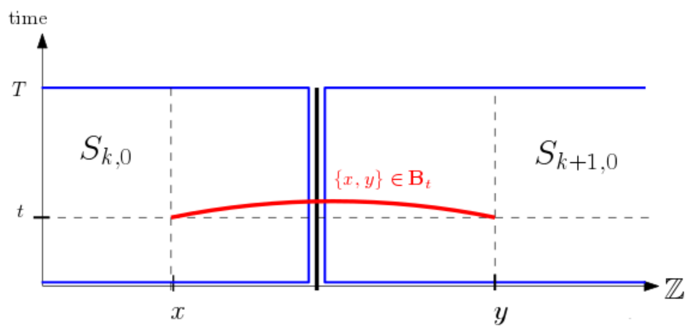


Figure 7.7: The thick black lines represent again $(0, K_0)$-cuts and the blue boxes a part of the resulting partition. Here we visualized the case when $W_{\{k,k+1\},0} = 1$.

These variables provides us with the information whether it is possible for an infection to travel via *long* edges at time step n from box k to l. Note that by definition $W_{\{k,l\},n} = W_{\{l,k\},n}$, and thus we will assume $k < l$. The idea is that for large K_0 a transmission of the infection via a *long* edge will be unlikely since they will most likely not be open. Therefore, we intend to control the survival via short edges in *isolated* boxes. Here isolated means that both boundaries of the boxes are (n, K_0)-cuts. Hence, we need a variable which provides us with the information whether the infection can persist in a box $S_{k,n}$ for a time period of length T.

We will now define random variables to control the survival in a box $S_{k,n}$ by

$$U_{k,n} := \begin{cases} 1 & \text{if there exists an infection path starting at } nT \text{ that} \\ & \text{is ending at } (n+1)T \text{ and is contained in } S_{k,n}, \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 7.8 for a illustration. If $U_{k,n} = 0$ then an infection contained in an isolated box $S_{k,n}$, i.e. $X_{k-1,n} = 0$ and $X_{k,n} = 0$, cannot survive via transmission along short edges only. We denote by $\mathcal{B}_n^{K_0}$ the σ-algebra containing informations of all $w'_n(e)$ of all short edges in time step n, i.e.

$$\mathcal{B}_n^{K_0} := \sigma\Big(\big\{\{w'_n(x,y) = 1\} : d(x,y) \leq 2K_0\big\}\Big). \tag{7.14}$$

Remark 7.4.2. Let us summarize some properties of the variables we just defined.

(*i*) The X variables from (7.12) depend only on edges of maximal length $2K_0$. Since the minimal distance $d(M_k^{\text{mid}}, M_l^{\text{mid}}) > 2K_0$ for $k \neq l$ we see that $X_{k,n}$ and $X_{k',n'}$ are independent if $k \neq k'$ for all $n, n' \in \mathbb{N}_0$.

(*ii*) $U_{k,n}$ only depends on edges $\{x, y\}$ with $x, y \in D_{k,n}$ and $|x - y| \leq 2K_0$. On the other hand $W_{\{k,l\},n}$ only depends on edges $\{x', y'\}$ such that $x' \in D_{k,n}$ and $y' \in D_{l,n}$ with $|x - y| > 2K_0$. Recall that $D_{k,n} \cap D_{l,n} = \emptyset$ for $k \neq l$. Therefore, $U_{k',n'}$ and $W_{\{k,l\},n}$ are independent for all $k', k, l \in \mathbb{Z}$ and $n, n' \in \mathbb{N}$, where $k < l$.

(*iii*) By definition $U_{k,n}$ and $U_{k',n'}$ are independent if $n \neq n'$ and only conditionally independent given $\mathcal{B}_n^{K_0}$ if $n = n'$ and $k \neq k'$.

(*iv*) Analogously the variables $W_{\{l,k\},n}$ and $W_{\{l',k'\},n'}$ are independent if $n \neq n'$ but are only conditionally independent given $\mathcal{B}_n^{K_0}$ if $n = n'$ and $\{l, k\} \neq \{l', k'\}$.

Note that in (*iii*) and (*iv*) conditioning on $\mathcal{B}_n^{K_0}$ serves the purpose of knowing how the partition $(S_{k,n})_{k\in\mathbb{Z}}$ in step n look like.

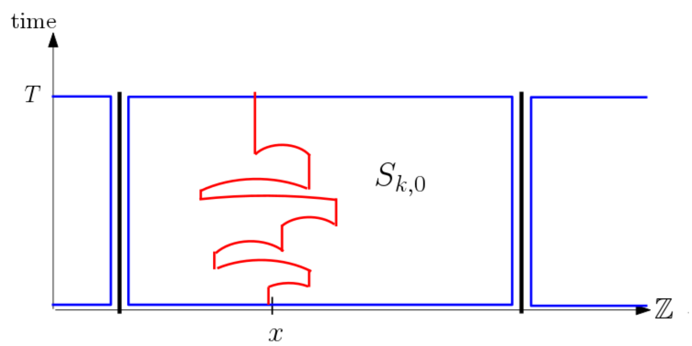


Figure 7.8: The thick black lines represent again $(0, K_0)$-cuts and the blue boxes a part of the resulting partition. Here we visualized the case when $U_{k,0} = 1$.

We will again define a random graph G_2 with vertex set $\mathbb{Z} \times \mathbb{N}_0$ where the edges are placed according to the following rules which are visualized in Figure 7.9:

1. If $U_{k,n} = 1$ add oriented edges from (k, n) to $(k - 1, n + 1)$, $(k, n + 1)$ and $(k + 1, n + 1)$
2. If $X_{k,n} = 1$ add edges as if $U_{k,n} = 1$, $U_{k+1,n} = 1$ and additionally an unoriented edge between (k, n) and $(k + 1, n)$.
3. If $W_{\{k,l\},n} = 1$ add an edge as if $U_{k,n} = 1$, $U_{l,n} = 1$ and additionally an unoriented edge from (k, n) to (l, n).

If $U_{k,n} = 1$ then the infection survives through the space-time box $S_{k,n}$ and it could possibly spread in at least one of the boxes $S_{m,n+1}$ for $m \in \{k-1, k, k+1\}$. If $X_{k,n} = 1$

then it could possibly spread to its neighbors in the time period $[nT, (n+1)T)$. If $W_{\{k,l\},n} = 1$ for any $l \neq k$ the infection could spread to the space-time box $S_{l,n}$. So in this case even if $U_{k,n} = 0$ we add the same edges or rather assume that the infection survives, because it could leave $S_{k,n}$ to some $S_{l,n}$ for $l > k$ and return to $S_{k,n}$ before $(n+1)T$.

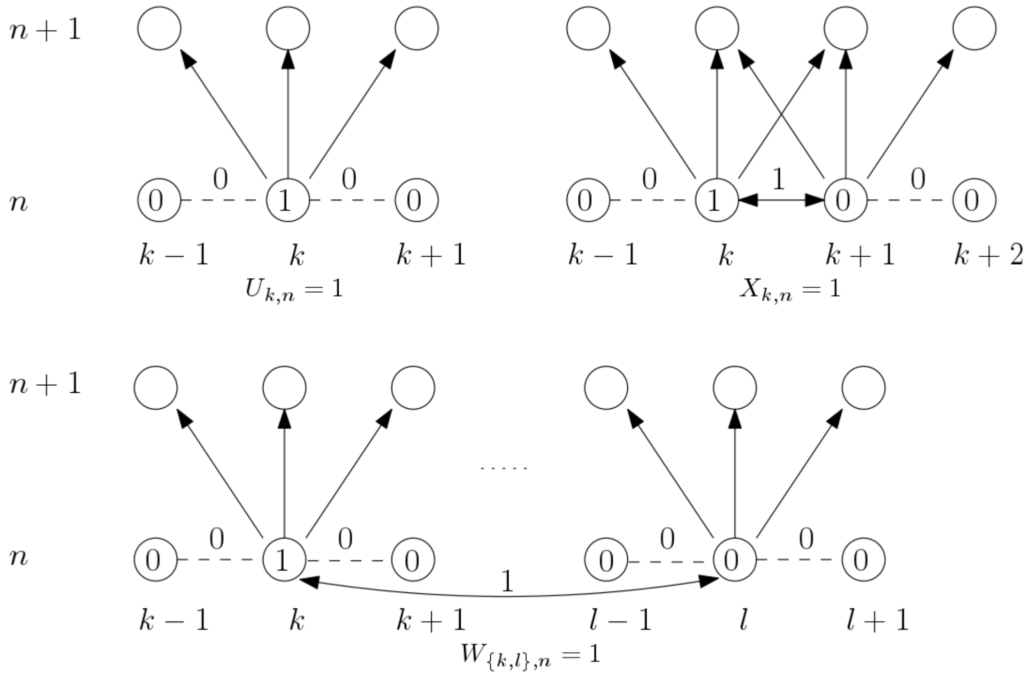


Figure 7.9: Visualization of the three rules. Solid lines are present edges and dashed lines absent edges.

Definition 7.4.3. (valid path in G_2) Let G_2 be the above constructed random graph. Let $Z_0 \subset \mathbb{Z}$ denotes the indices of the boxes which contain the initially infected sites C. We say that there exists a valid path from $Z_0 \times \{0\}$ to a point (k, n) if there exists a sequence $k_0, k_1, \ldots k_m = k$ with $k \in Z_0$ and $0 = n_0 \leq n_1 \leq \cdots \leq n_m = n$ such that there exist an edge in G_2 between (x_k, n_k) and (x_{k+1}, n_{k+1}) for all $k \in \{0, \ldots, m-1\}$.

Similar as in the previous section we define a process $Z = (Z_n)_{n \geq 0}$, where for all $n \geq 0$ the random set $Z_n = Z_n(U, X, W)$ contains all points $x \in \mathbb{Z}$ for which there exists a valid path from $Z_0 \times \{0\}$ to (x, n) in G_2 for $n \geq 1$.

Lemma 7.4.4. *Let $T > 0$, $n \in \mathbb{N}_0$ and $C \subset V$. We choose Z_0 such that $k \in Z_0$ if and only if $C \cap D_{k,0} \neq \emptyset$. If $x \in \mathbf{C}^C_{nT}$ then there exists a $k \in \mathbb{Z}$ such that $x \in S_{k,n}$ and $k \in Z_n$ and thus if $Z_n = \emptyset$, then $\mathbf{C}^C_{nT} = \emptyset$.*

Proof. This proof is similar to Lemma 7.3.2. Again if we assume that $x \in \mathbf{C}^C_{nT}$ then for some $x \in C$ there must exist an infection path from $(x, 0)$ to (y, nT), and thus we find a subsequence of sites $x = x_0, x_1, \ldots, x_n = y$ such that $x_m \in \mathbf{C}^C_{mT}$ for $m \in \{0, \ldots, n\}$. Note that these sites are part of the infection path. Also since the $(S_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$ form a disjoint partition of $\mathbb{Z} \times [0, \infty)$ for every x_m there exists an $k_m = k_m(x_m)$ such that $(x_m, mT) \in S_{k_m,m}$. Here it again suffices to show that $x_m \in \mathbf{C}^C_{mT}$ and $x_{m+1} \in \mathbf{C}^C_{(m+1)T}$ imply that $k_{m+1} \in Z_{m+1}$. Then, the claim follows immediately, since $x \in Z_0$ by definition of Z_0. Now from the way we chose the sequence $(x_m)_{m\leq n}$ there is an infection path from (x_m, mT) to $(x_{m+1}, (m + 1)T)$ for all $m \leq n - 1$. To be precise these paths are just sections of the original infection path.

1. Let us start with the case that $k_m \neq k_{m+1}$. Let $e_1, \ldots e_r$ be the edges present in the infection path from (x_m, mT) to $(x_{m+1}, (m + 1)T)$, where $r \in \mathbb{N}$. We only need to consider the edges which connect vertices in different space-time boxes. Let $e_{m'} = \{x', y'\}$ and $t' \in \{mT, (m + 1)T\}$ with $(x', t') \in S_{k',mT}$ and $(y', t') \in S_{l',mT}$ on the infection path. Then again there exists k', l' such that $x' \in S_{k',m}$ and $y' \in S_{l',m}$

   If $|x' - y'| > 2K_0$ then $W_{\{k',l'\},m} = 1$, since if $(e_{m'}, t')$ is part of the infection path there must have been an infection event, and thus the edge $e_{m'}$ must have been open. Thus, by the third rule $l' \in Z_{m+1}$ if $k' \in Z_m$.

   On the other hand if $|x' - y'| \leq 2K_0$ then $|l' - k'| = 1$. This is because for any space boxes $|D_{k,n}| \geq 2K_0$, so the space time boxes which are connected via $e_{m'}$ must be adjacent. Hence, the boundary between $S_{k',m}$ and $S_{l',m}$ is no (m, K_0)-cut, since this would prevent an infection to spread via the short edge $e_{m'}$. This implies that either $X_{k',m} = 1$ or $X_{l',m} = 1$. Thus, by the second rule $l' \in Z_{m+1}$ if $k' \in Z_m$.

   Since $e_{m'}$ was chosen arbitrarily from $e_1, \ldots, e_m$, by a combination of the second and third rule follows that $l \in Z_{m+1}$.

2. Now we consider the case that $k_m = k_{m+1}$. Now either the infection path is contained in $S_{k,mT}$, this would imply that $U_{k,m} = 1$, or it left the box and returns at a later time. This would mean that either there exist an $l \in \mathbb{Z}$ such that $W_{\{k,l\},m} = 1$, $X_{k,m} = 1$ or $X_{k-1,m} = 1$, since the infection left the box, and therefore an edge connecting two different boxes must have been open. Thus, $k_{m+1} \in Z_{m+1}$ □

We again find ourselves in the situation that Z_n is somewhat easier to handle than the original infection process, but still hides a lot of dependency structure. For the remaining section we will choose $T := \frac{1}{\gamma}$ and let $q \in (0,1)$ be fixed. Recall the definition of δ_e from Lemma 7.2.1. Note that this yields

$$\delta_e(\gamma, q, \gamma^{-1}) = (1 - qp_e)e^{-qp_e v_e}\Big(1 - qp_e \frac{1 - e^{-v_e}}{1 - e^{-qp_e v_e}}\Big), \tag{7.15}$$

which is now independent of γ.

Let us give a short description of what we do now. Next we show that we can choose r_0, K_0 and γ or equivalently T such that the probabilities are small that any of the X, W or U variables are one. With this we will then show that we can choose r_0, K_0 and γ^* in such a way that Z_n goes almost surely extinct for all $\gamma < \gamma^*$. For this we again need the results we derived in Section 7.2.

Bound for the X variables: Let us recall that

$$\{X_{k,n} = 1\} = \{\text{no } (n, K_0)\text{-cut lies in } M_k^{\text{mid}}\} = \bigcap_{m \in M_k^{\text{mid}}} \bigcup_{\substack{x \le m < y: \\ |x-y| \le 2K_0}} \{w_n'(x,y) = 0\}. \tag{7.16}$$

The probability $\mathbb{P}(X_{k,n} = 1)$ does not depend on γ, as already mentioned in (7.15). This is important since later, in order to find a bound on $\mathbb{P}(U_{k,n} = 1)$, we need to vary γ. Thus, changing γ will not affect the probability $\mathbb{P}(X_{k,n} = 1)$. If we remove the restriction $|x - y| \le 2K_0$ we obtain with (7.16) that

$$\{X_{k,n} = 1\} \subset \bigcap_{m \in M_k^{\text{mid}}} \bigcup_{x \le m < y} \{w_n'(x,y) = 0\}.$$

Now consider n to be fixed. Since $(w_n'(e))_{e \in \mathcal{E}}$ is a family of independent Bernoulli random variables, we can interpret these variables as a long range percolation model with probabilities $b_k := (1 - \delta_{\{0,k\}})$ for all $k \in \mathbb{Z}$, where we used that $\delta_{\{x,y\}} = \delta_{\{x',y'\}}$ if $d(x,y) = d(x',y')$. Therefore, we see that in the terms of the long range percolation model it holds that

$$\bigcap_{m \in M_k^{\text{mid}}} \bigcup_{x \le m < y} \{w_n'(x,y) = 0\} = \{\text{no cut point lies in } M_k^{\text{mid}}\}.$$

We set

$$\mathbb{P}(X_{k,n} = 1) \le \mathbb{P}\Big(\bigcap_{m \in M_k^{\text{mid}}} \bigcup_{i \le m < j} \{w_n'(x,y) = 0\}\Big) := \varepsilon_1(r_0), \tag{7.17}$$

where $|M_k^{\text{mid}}| = r_0$. Note that the right hand side only depends on the size of M_k^{mid} and not its exact location. Since $X_{k,n}$ and $X_{k',n'}$ have the same distribution and are independent if either $n \neq n'$ or $k \neq k'$ we see that the right hand side does not depend on n or k. Now by (7.11) we know that $\sum_{k\in\mathbb{Z}}^{\infty} kb_k = \sum_{k\in\mathbb{Z}}^{\infty} k(1-\delta_{\{0,k\}}) < \infty$. Thus, by Theorem 7.2.6 there exist almost surely infinitely many cut points. But this means that

$$\varepsilon_1(r_0) \to 0 \quad \text{as} \quad r_0 \to \infty. \tag{7.18}$$

Note that this bound is independent of the choice of K_0. This is important since in the next step we derive a bound for the probability $\mathbb{P}(W_{\{k,l\},n} = 1)$ by choosing K_0 accordingly. But the choice of K_0 will depend on the choice of r_0.

Bound for the W variables: Next we consider the family describing transmission along long edges, i.e. $\{W_{\{k,l\},n} : k,l \in \mathbb{Z}, k < l, n \in \mathbb{N}_0\}$. By definition it holds $W_{\{l,k\},n} = W_{\{k,l\},n}$, which is why we only need to consider $k < l$. We see that

$$\{W_{\{k,l\},n} = 1\} = \bigcup_{\substack{x\in D_{k,n}, y\in D_{l,n}:\\ |x-y|>2K_0}} \{w_n'(x,y) = 0\}$$

If we now use the sets $D_k^{\max}$ and $D_l^{\max}$ defined in (7.13) we see that

$$\{W_{\{k,l\},n} = 1\} \subset \bigcup_{\substack{x\in D_k^{\max}, y\in D_l^{\max}:\\ |x-y|>2K_0}} \{w_n'(x,y) = 0\}.$$

Note that the right hand side is independent of $\mathcal{B}_n^{K_0}$, where $\mathcal{B}_n^{K_0}$ is defined in 7.14. Thus, for a r_0 given we can conclude that

$$\mathbb{P}(W_{\{k,l\},n} = 1 | \mathcal{B}_n^{K_0}) \leq \sum_{\substack{x\in D_k^{\max}, y\in D_l^{\max}:\\ |x-y|>2K_0}} (1-\delta_{\{x,y\}}) := a_{k,l}(K_0, r_0), \tag{7.19}$$

where again the right hand side is independent of γ. By subadditivity and (7.19) we get that

$$\mathbb{P}(\exists l \neq k : W_{\{k,l\},n} = 1 | \mathcal{B}_n^{K_0}) \leq \sum_{l\neq k} a_{k,l}(K_0, r_0).$$

Next we take a closer look at $D_k^{\max}$ defined in (7.13). We see that $D_k^{\max} \cap D_{k+1}^{\max} = M_k^{\text{mid}}$ and if $l > k+1$ then $D_k^{\max} \cap D_l^{\max} = \emptyset$. Since $|M_k^{\text{mid}}| = r_0$, the neighbouring maximal boxes have an overlap of r_0 many vertices which we count double in the sum $\sum_{l\neq k} a_{k,l}$. If for a given k we just count every edge of length $> 2K_0$ "leaving" $D_k^{\max}$, again the sum only gets larger. Also note that $|D_k^{\max}| = 2(r_0 + K_0)$. By symmetry and the thoughts above, we see that

$$\sum_{l:l\neq k} a_{k,l}(K_0, r_0) = 2\sum_{l:l>k} a_{k,l}(K_0, r_0) \leq 4|D_k^{\max}| \sum_{y>2K_0} (1 - \delta_{\{0,y\}})$$
$$= 8(K_0 + r_0) \sum_{y>2K_0} (1 - \delta_{\{0,y\}}),$$

where we used translation invariance. Summarizing the whole procedure yields that for any $k \in \mathbb{Z}$,

$$\mathbb{P}(\exists l \neq k : W_{\{k,l\},n} = 1|\mathcal{B}_n^{K_0}) \leq 8(K_0 + r_0) \sum_{y>2K_0} (1 - \delta_{\{0,y\}}) := \varepsilon_2(K_0, r_0). \qquad (7.20)$$

But since we know that $\sum_{y\in\mathbb{N}} y(1 - \delta_{\{0,y\}}) < \infty$ from (7.11), it is not difficult to see that also $2N\sum_{y>N}(1 - \delta_{\{0,y\}}) \to 0$ as $N \to \infty$ must hold. Hence, for every r_0, $\varepsilon_2(K_0, r_0) \to 0$ as $K_0 \to \infty$. But in particular if we choose $K_0 = r_0$, then we see that also

$$\varepsilon(r_0, r_0) \to 0 \quad \text{as} \quad r_0 \to \infty \qquad (7.21)$$

Bound for the U variables: Recall that on every finite graph the classical contact process dies out. We denote by $\tau_{\text{ext}}^{r_0,K_0}$ the extinction time of a classical contact process with infection rate and recovery rate as the CPLDP $(\mathbf{C}, \mathbf{B})$ on a complete graph with $2(K_0 + r_0)$ vertices, where every vertex is initially infected. Since $|D_{k,n}| \leq 2(K_0 + r_0)$ it holds that

$$\mathbb{P}(U_{k,n} = 1|\mathcal{B}_n^{K_0}) \leq \mathbb{P}(\tau_{\text{ext}}^{r_0,K_0} > \gamma^{-1}) := \varepsilon_3(K_0, r_0, \gamma). \qquad (7.22)$$

For every $\varepsilon > 0$ we can choose $\gamma^* > 0$ small enough such that $\mathbb{P}(\tau_{\text{ext}}^{r_0,K_0} > \gamma^{-1}) < \varepsilon$ for all $\gamma < \gamma^*$, and thus in particular $\varepsilon_3(K_0, r_0, \gamma) \to 0$ as $\gamma \to 0$.

We have now derived upper bounds on the probability that the X, W and U variables are one. We see that $(X_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$ are independent random variables and $X_{k,n}$ is measurable with respect to $\mathcal{B}_n^{K_0}$ for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}_0$. But the families $(U_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$ and $\{W_{\{k,l\},n} : k, l \in \mathbb{Z}, k < l, n \in \mathbb{N}_0\}$ are only independent in time direction. In spatial direction they are only independent conditionally on $\mathcal{B}_n^{K_0}$, see Remark 7.4.2. Therefore, the aim now is to construct independent upper bounds of the W and U variables, which are also independent of the X variables.

Proposition 7.4.5. *Let $a_{l,k}(K_0, r_0)$ and $\varepsilon_3(K_0, r_0, \gamma)$ be chosen as in* (7.19) *and* (7.22) *and $r_0, K_0, \gamma > 0$ large enough such that $a_{l,k}(K_0, r_0), \varepsilon_3(K_0, r_0, \gamma) < 1$ for all $l \neq k$ and $n \in \mathbb{N}_0$. Then there exist independent families*

$$(U'_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0} \quad \text{and} \quad \{W'_{\{k,l\},n} : k,l \in \mathbb{Z}, k < l, n \in \mathbb{N}_0\}$$

of independent Bernoulli random variables with $\mathbb{P}(W'_{\{l,k\},n} = 1) = a_{l,k}(K_0, r_0)$ and $\mathbb{P}(U'_{k,n} = 1) = \varepsilon_3(K_0, r_0, \gamma)$ for all $k \neq l$ and all $n \in \mathbb{N}_0$ such that they are independent of the family $(X_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$ and such that $W_{\{l,k\},n} \leq W'_{\{k,l\},n}$ and $U_{k,n} \leq U'_{k,n}$ almost surely for all $k \neq l$ and all $n \in \mathbb{N}_0$.

Proof. Recall from (7.14) that $\mathcal{B}_n^{K_0} = \sigma\big(\big\{\{w'_n(x,y) = 1\} : d(x,y) \leq 2K_0\big\}\big)$. We will now explicitly construct the U' variables. For that we define the random variable $p^U_{k,n} := \mathbb{P}(U_{k,n} = 0|\mathcal{B}_n^{K_0})$ for $k \in \mathbb{Z}$ and $n \in \mathbb{Z}_0$. Note that by (7.22) and the assumptions of this proposition $p^U_{k,n} \geq 1 - \varepsilon_3(K_0, r_0, \gamma) > 0$. Now let $(\chi^U_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$ be family of independent uniform random variables on $[0,1]$ and are also independent of the X, U and W variables. Let $s_{k,n}$ be random variables with values in $[0,1]$ which are yet to be determined. Next let $(U'_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$ be random variables in $\{0,1\}$ such that $U'_{k,n} = 0$ if and only if $U_{k,n} = 0$ and $\chi^U_{k,n} \leq s_{k,n}$. By definition it is clear that $U'_{k,n} \geq U_{k,n}$.

Next we set $s_{k,n} := \frac{1-\varepsilon_3(K_0,r_0,\gamma)}{p^U_{k,n}}$ and see that

$$\mathbb{P}(U'_{k,n} = 0) = \mathbb{E}[\mathbb{P}(U_{k,n} = 0, \chi^U_{k,n} \leq s_{k,n}|\mathcal{B}_n^{K_0})] = \mathbb{E}[p^U_{k,n} s_{k,n}] = 1 - \varepsilon_3(K_0, r_0, \gamma),$$

where we used in the second equation conditional independence given $\mathcal{B}_n^{K_0}$, which follows by the same line of arguments as in the proof of Lemma 7.2.2, since we assumed that $(\chi^U_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$ is independent of $(U_{k,n})_{(k,n)\in\mathbb{Z}\times\mathbb{N}_0}$. Analogously follows that

$$\mathbb{P}(U'_{k,n} = 0|\mathcal{B}_n^{K_0}) = \mathbb{P}(U_{k,n} = 0, \chi^U_{k,n} \leq s_{k,n}|\mathcal{B}_n^{K_0}) = 1 - \varepsilon_3(K_0, r_0, \gamma),$$

The right hand side is not random anymore, and thus it follows that the variable $U'_{k,n}$ is independent of $\mathcal{B}_n^{K_0}$ for all $k \in \mathbb{Z}$ and $n \in N_0$.

We already know that $U'_{k,n}$ and $U'_{k',n'}$ are independent if $n \neq n'$. Thus, it suffices to show that $U'_{k,n}$ and $U'_{k',n}$ are independent if $n = n'$ and $k \neq k'$. Let us fix some n and let $k_1 \neq \dots \neq k_l$ be an arbitrary but finite sequence of integers and $u_1, \dots u_l \in \{0,1\}$, where $l \in \mathbb{N}$. Since we fixed n we omit the subscript n in the following. We need to show that

$$\mathbb{P}(U'_{k_1} = u_1, \dots U'_{k_l} = u_l) = \mathbb{P}(U'_{k_1} = u_1) \dots \mathbb{P}(U'_{k_l} = u_l).$$

For this, it suffices to consider $u_1 = \dots = u_l = 0$, since if two events A and B are independent, then so are A and B^c. Now we see that

$$\begin{aligned}&\mathbb{P}(U'_{k_1} = 0, \dots U'_{k_l} = 0)\\ =&\mathbb{E}[(U_{k_1} = 0, \dots U_{k_l} = 0, \chi^U_{k_1} \le s_{k_1}, \dots, \chi^U_{k_l} \le s_{k_l} | \mathcal{B}^{K_0})]\\ =&\mathbb{E}[(U_{k_1} = 0, \dots U_{k_l} = 0 | \mathcal{B}^{K_0})\mathbb{P}(\chi^U_{k_1} \le s_{k_1} | \mathcal{B}^{K_0}) \dots \mathbb{P}(\chi^U_{k_l} \le s_{k_l} | \mathcal{B}^{K_0})],\end{aligned}$$

where we again used conditional independence which follows analogously as before. Thus, we have that

$$\mathbb{P}(U'_{k_1} = 0, \dots U'_{k_l} = 0) = \big(1 - \varepsilon_3(K_0, r_0, \gamma)\big)^l \mathbb{E}\Big[(U_{k_1} = 0, \dots U_{k_l} = 0 | \mathcal{B}^{K_0}) \prod_{i=1}^{l} \frac{1}{p^U_{k_i}}\Big].$$

But since the U variables are conditional independent given $\mathcal{B}^{K_0}$ and $\mathbb{P}(U_{k_i} = 0) = p^U_{k_i}$ it follows that

$$\mathbb{P}(U'_{k_1} = 0, \dots U'_{k_l} = 0) = \big(1 - \varepsilon_3(K_0, r_0, \gamma)\big)^l = \mathbb{P}(U'_{k_1} = 0) \dots \mathbb{P}(U'_{k_l} = 0).$$

The W' variables can be constructed analogously. The only thing we need to mention is that we must choose the family $\{\chi^W_{\{k,l\},n} : k, l \in \mathbb{Z}, k \neq l, n \in \mathbb{N}_0\}$ to be independent of the X, U and W variables and additionally to be independent of the family $(\chi^U_{k,n})_{(k,n) \in \mathbb{Z} \times \mathbb{N}_0}$. □

Analogously as in the previous section we define a process $(Z'_n)_{n \in \mathbb{Z}}$ with respect to the random variables X, U' and W' we obtained in Proposition 7.4.5. It follows that $Z_n \subset Z'_n$ for all $n \in \mathbb{N}_0$. Thus if $(Z'_n)_{n \in \mathbb{Z}}$ goes extinct almost surely, then the same follows for $(Z_n)_{n \in \mathbb{Z}}$.

Lemma 7.4.6. *If $\mathbb{E}[|Z'_1||Z'_0 = \{0\}] < 1$, then Z' dies out almost surely for any finite $A \subset V$ as initial state.*

Proof. Analogously to Lemma 7.3.3. □

Now we are ready to show Theorem 1.4.27. Thus, let $r > 0$, $q \in (0, 1)$ and $C \subset V$ non-empty and finite, for a given $\lambda > 0$ we show that there exists $\gamma^* > 0$ such that $\mathbf{C}^C$ dies out almost surely for all $\gamma \le \gamma^*$, i.e. $\theta(\lambda, r, \gamma, q, C) = 0$ for all $\gamma \le \gamma^*$. This implies in particular that $\lambda_c(r, \gamma, q) \to \infty$ as $\gamma \to 0$.

Proof of Theorem 1.4.27. Again it suffices to consider $Z'_0 = \{0\}$, since the general case follows analogously as shown in the proof of Theorem 1.4.24. Thus, we again fix $Z'_0 = \{0\}$. The proof strategy is similar to the proof of Theorem 1.4.24. We see that $|Z'_1| < 3|Z|$, where Z is a connected component containing 0 of a long range percolation model with probabilities given through

$$b_{\{k,l\}} = \mathbb{P}(W'_{\{k,l\},n} = 1) \text{ and } b_{\{k,k+1\}} = \mathbb{P}(\{W'_{\{k,k+1\},n} = 1\} \cup \{X_{k,n} = 1\})$$

for all $k, l \in \mathbb{Z}$ with $|k - l| = 2$. Note that the constant 3 comes from the fact that if any of the X or W' variable are 1, three blocks will get infected, see Figure 7.9. We see that we can again split up the expectation such that

$$\begin{aligned} \mathbb{E}[|Z|] =& \mathbb{E}\big[(\mathbb{1}_{\{X_{1,0}=1\}\cup\{X_{-1,0}=1\}\cup\{\exists j\in\mathbb{Z}:W'_{\{0,j\},0}=1\}})|Z'_1|\big] \\ &+ \underbrace{\mathbb{E}\big[1_{\{X_{1,0}=0\}\cap\{X_{-1,0}=0\}\cap\bigcap_{j\in\mathbb{Z}}\{W'_{\{0,j\},0}=0\}}U'_{0,0}\big]}_{\leq \mathbb{E}[U'_{0,0}]}. \end{aligned} \tag{7.23}$$

We also know that by (7.17) and (7.19)

$$\mathbb{P}(X_{k,n} = 1) = \varepsilon_1(r_0) \text{ and } \mathbb{P}(W'_{\{k,l\},n} = 1) = a_{k,l}(r_0, K_0).$$

Note that by (7.15) we know that $b_{\{k,l\}}$ is independent of the choice of γ. Thus, the probabilities $b_{\{k,l\}}(K_0, r_0)$ can be seen as functions of the parameters K_0 and r_0 and we see that

$$\sum_{l\neq k} b_{\{k,l\}}(K_0, r_0) \leq 2\varepsilon_1(r_0) + \sum_{l\neq k} a_{k,l}(r_0, K_0).$$

From here onwards for the remainder of the proof we choose $K_0 = r_0$ such that $b_{\{k,l\}}(r_0)$ is only a function of r_0. Now by (7.18) and (7.21) it follows that

$$\sum_{l\neq k} b_{\{k,l\}}(r_0) \leq 2\varepsilon_1(r_0) + \sum_{l\neq k} a_{k,l}(r_0, r_0) \to 0$$

as $r_0 \to \infty$. Thus there exists $R_1 > 0$ such that $\sum_{l\neq k} b_{\{k,l\}}(r_0) < 1$ for all $r_0 \geq R_1$. Thus, by Proposition 7.2.4 we know that $|Z|$ is integrable. We add r_0 as an index, i.e. $Z(r_0)$. We can show analogously as in the proof of Theorem 1.4.24 that for every $\varepsilon > 0$ there exists an $M = M(\varepsilon, R_1)$ such that

$$\mathbb{E}[|Z(r_0)|\mathbb{1}_{\{|Z(r_0)|>M\}}] < \frac{\varepsilon}{3}$$

for all $r_0 \geq R_1$. Thus, we can conclude that

$$\begin{aligned}
&\mathbb{E}[(\mathbb{1}_{\{X_{1,0}=1\}\cup\{X_{-1,0}=1\}\cup\{\exists j\in\mathbb{Z}:W'_{\{0,j\},0}=1\}})|Z(r_0)|] \\
\leq&\mathbb{E}[(\mathbb{1}_{\{|Z(r_0)|>M\}})|Z(r_0)|] + M\big(\mathbb{P}(X_{1,0}=1) + \mathbb{P}(X_{-1,0}=1) + \sum_{j\in\mathbb{Z}}\mathbb{P}(W'_{\{0,j\},0}=1)\big) \\
\leq&\frac{\varepsilon}{3} + M\Big(2\varepsilon_1(r_0) + \sum_{l\neq k} a_{k,l}(r_0,r_0)\Big).
\end{aligned}$$

Next we again use (7.18) and (7.21) and see there must exists a $R_2 > R_1$ such that $M\big(2\varepsilon_1(r_0) + \sum_{l\neq k} a_{k,l}(r_0,r_0)\big) < \frac{\varepsilon}{3}$ for all $r_0 > R_2$. By (7.22) we can choose $\gamma^* > 0$ small enough such that $\mathbb{E}[U_{0,0}] < \frac{\varepsilon}{3}$ for all $\gamma < \gamma^*$, then it follows with (7.23) that $\mathbb{E}[|Z|] < 3\varepsilon$. Thus, if we choose $\varepsilon < \frac{1}{3}$ we see that

$$\mathbb{E}[|Z'_1||Z'_0 = \{x\}] \leq 3\mathbb{E}[|Z|] < 1.$$

By Lemma 7.4.6 it follows that $(Z'_n)_{n\in\mathbb{N}}$ goes extinct almost surely, which implies that $(Z_n)_{n\in\mathbb{N}}$ goes extinct almost certain, since $Z_n \subset Z'_n$ for all n almost surely. Then by Lemma 7.4.4 it follows that $\mathbf{C}^{\{x\}}$ goes extinct almost certain, where $x \in D_{0,0}$. Therefore, it follows that $\mathbf{C}^C$ goes extinct almost certain for all finite $C \subset \mathbb{Z}$ and all $\gamma < \gamma^*$. In formulas this means that $\theta(\lambda, r, \gamma, q) = 0$ for all $\gamma < \gamma^*$.

The infection rate λ was chosen to be fixed, but arbitrary in beginning, and therefore this also implies that $\lim_{\gamma\to 0}\lambda_c(r,\gamma,q) = \infty$. Since assuming otherwise would imply that there must exists a $\lambda_0 > 0$ and $\gamma_0 > 0$ such that $\lambda_c(r,\gamma,q) \leq \lambda_0$ for all $\gamma \in (0,\gamma_0)$. But we just showed that there exists a $\gamma_0^* = \gamma_0^*(\lambda_0)$ such that $\theta(\lambda_0, r, \gamma, q) = 0$ for all $\gamma < \gamma_0^*$, and thus the assumption leads to a contradiction. □

Chapter 8

Conclusion and open problems

In this chapter we briefly recapitulate some of the major results and point out some possible further problems, which might be interesting to tackle. The main focus of this thesis was on a contact process in an evolving random environment, which we abbreviated by CPERE, on a graph $G = (V, E)$ with bounded degree and exponential growth ρ, where the evolving random environment is described by an ergodic and reversible spin system with finite range interactions. Recall that λ is the infection rate, r is the recovery rate and κ is chosen as in Assumption 1.4.1 (*ii*). As usual in this kind of model we focused mainly on the parameter regime where survival of the infection process $\mathbf{C}$ is possible, which we named the survival region and denoted by

$$\mathcal{S}(C, B) = \{(\lambda, r) \in (0, \infty)^2 : \theta(\lambda, r, C, B) > 0\},$$

where (C, B) are the initial configuration of the CPERE. Note that we only consider C non-empty and finite, since otherwise the question whether survival is possible or not is trivial.

We managed to show that if we find a $\lambda > 0$ with $\theta(\lambda, r, C, B) > 0$ for some configuration (C, B), which satisfies the inequality $c_1(\lambda, \rho) > \kappa^{-1}\rho$, then the survival of the infection process $\mathbf{C}$ is independent of the choice of the initial configuration. Recall that $c_1(\lambda, \rho)^{-1}$ is an upper bound on the asymptotic expansion speed of the set of all possible infections and $\kappa\rho^{-1}$ is a lower bound on the asymptotic expansion speed of the permanently coupled region. Furthermore, we were able to show that the survival probability is continuous on the interior of the subset

$$\mathcal{S}_{c_1} = \{(\lambda, r) : \exists \lambda' \leq \lambda \text{ s.t. } (\lambda', r) \in \mathcal{S}(\{x\}, \emptyset) \text{ and } c_1(\lambda', \rho) > \kappa^{-1}\rho\}.$$

We also managed to conclude that the phase transition of survival with the background started stationary, i.e. $\theta^\pi(\lambda, r, \{x\}) = 0$ to $\theta^\pi(\lambda, r, \{x\}) > 0$, agrees with the phase transition of non-triviality of the upper invariant law, i.e. $\overline{\nu} = \delta_\emptyset \otimes \pi$ to $\overline{\nu} \neq \delta_\emptyset \otimes \pi$. Thus, if additionally $c_1(\lambda^\pi(r), \rho) > \kappa^{-1}\rho$ holds the initial configuration of the background is of no importance to the question of non-triviality of $\overline{\nu}$. This in itself is an interesting observation, but we were also able to derive equivalent conditions for complete convergence, i.e.

$$(\mathbf{C}_t^{C,B}, \mathbf{B}_t^B) \Rightarrow \theta(C,B)\overline{\nu} + [1 - \theta(C,B)](\delta_\emptyset \otimes \pi)$$

as $t \to \infty$ on the parameter subset

$$\mathcal{S}_{c_1}^* := \{(\lambda, r) \in \mathcal{S}(\{x\}, \emptyset) : c_1(\lambda', \rho) > \kappa^{-1}\rho\}.$$

Note that if we know that complete convergence holds, then we have also fully characterized all possible invariant laws of the CPERE. We illustrated the survival region $\mathcal{S}(C, B)$ and the two subsets $\mathcal{S}_{c_1}^*$ and $\mathcal{S}_{c_1}$ in Figure 8.1. Note that these three parameter regions are subsets of each other, i.e. $\mathcal{S}_{c_1}^* \subset \mathcal{S}_{c_1} \subset \mathcal{S}(C, B)$. On subexponential graphs, i.e. $\rho = 0$, the inequality $c_1(\lambda, \rho) > \kappa^{-1}\rho$ is trivially satisfied for all $\lambda > 0$ since $c_1(\lambda, \rho) > 0$ for all $\lambda > 0$. Thus, $\mathcal{S}_{c_1}^* = \mathcal{S}_{c_1} = \mathcal{S}(C, B)$ for all (C, B) with C non-empty and finite.

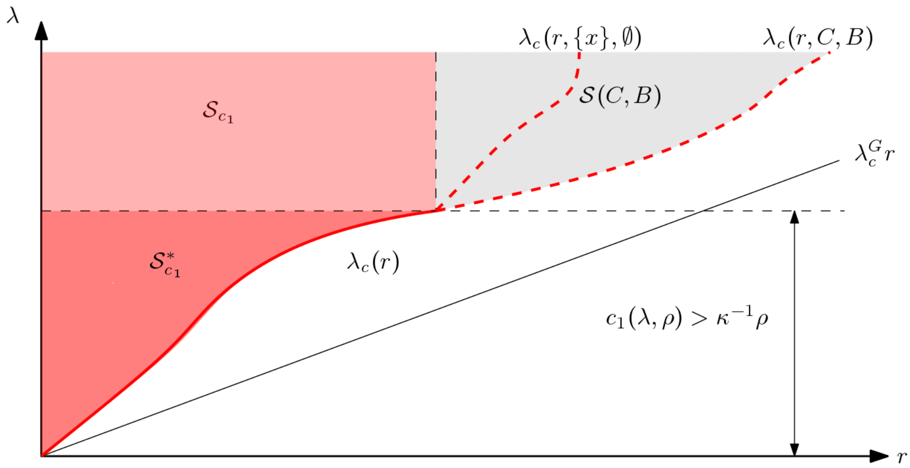


Figure 8.1: The solid and dashed red curve indicates the critical infection rate $\lambda_c(r, \cdot)$ of the CPERE. The solid black curve indicates the critical infection rate of a CP, where λ_c^G is the critical infection rate for $r = 1$.

Influence of the initial configuration on the critical infection rate: This brings us to the first open problem. We already mentioned that the initial configuration (C, B)

has no influence on the critical infection rate if $c_1(\lambda_c(r, C, B), \rho) > \kappa^{-1}\rho$ is satisfied, or to be precise if the asymptotic expansion speed of the permanently coupled region is greater than that of the infection. But is this still the case if $c_1(\lambda, \rho) \leq \kappa^{-1}\rho$? Of course this is only possible if $\rho > 0$. It seems appropriate to mention here that $c_1(\lambda, \rho)^{-1}$ and $\kappa\rho^{-1}$ are only bounds on the asymptotic expansion speeds, but even if we are able to determine the exact constants it might be possible to choose the parameters of the background small enough with respect to the infection rate λ and recovery rate r such that it might happen that the coupled region expands slower than the infection, see for example the dynamical percolation in Example 1.1.2 (i). For this model $\alpha + \beta$ seems to determine the expansion speed, and thus we can just choose $\alpha + \beta$ small enough in comparison to λ and r.

Open problem 1. Let $x \in V$ be arbitrary but fixed and suppose $\rho > 0$. Is the critical infection rate always independent of the initial conditions? In other words is $\lambda_c(r, \{x\}, \emptyset) = \lambda_c(r, C, B)$ for all $r \geq 0$, $C \subset V$ finite and $B \subset E$? Or do $r > 0$, $C \subset V$ finite and $B \subset E$ exists such that $\lambda_c(r, \{x\}, \emptyset) > \lambda_c(r, C, B)$?

Complete convergence of CPERE on general graphs: If $c_1(\lambda, \rho) > \kappa^{-1}\rho$ is satisfied Theorem 1.4.15 states that if the two conditions (1.8) and (1.9) are satisfied, then we get that complete convergence holds for the CPERE. Hence, again the same question arises. What if $c_1(\lambda, \rho) \leq \kappa^{-1}\rho$? In the proof of Theorem 1.4.15 we rely at some crucial steps on the assumption that the asymptotic expansion speed of the permanently coupled region is greater than that of the infection. Thus, we cannot just forgo this assumption.

For the CP Salzano and Schonmann studied the property of complete convergence in [SS97] and [SS99]. Among other things they showed in [SS97, Theorem 1(i)] that on transitive, connected graphs with bounded degree the complete convergence is monotone in the sense that if it holds for some infection rate λ it already holds for all $\lambda' > \lambda$ and if it holds for an infection rate λ on some transitive and connected subgraph $G_0 \subset G$ it holds on G for the same rate λ as well.

For the CP an intermediate phase is possible, where complete convergence does not hold but the survival probability is positive. But because of the above mentioned monotonicity on transitive graphs, this is normally only a bounded parameter region, see [Lig13, Chapter I.4] where among other things this is studied for the CP on regular trees. Hence, if in the case of the CPERE complete convergence fails for large λ, then it might lead to a fourth phase where again infinitely many extremal invariant laws exist.

Open problem 2. Is it possible to extend Theorem 1.4.15 in such a way that it holds for every $(\lambda, r) \in \mathcal{S}_{c_1}$?

Another interesting question could be the following.

Open problem 3. Is complete convergence a monotonous property (as described above) for the CPERE?

CPERE on $\mathbb{Z}^d$: As an application we showed for our main example the contact processes on a dynamical percolation, which we abbreviate with CPDP, on a d-dimensional integer lattice, i.e. $V = \mathbb{Z}^d$ and $E = \{\{x, y\} \subset \mathbb{Z}^d : ||x - y||_1 = 1\}$, that complete convergence holds for all $(\lambda, r, \alpha, \beta) \in (0, \infty)^4$. Furthermore for general CPERE on (V, E), where the background process $\mathbf{B}$ satisfies Assumption 1.4.1, complete convergence holds on the survival region of a CPDP with suitable chosen parameters. Therefore, we might ask the following question.

Open problem 4. Does complete convergence hold for every $(\lambda, r) \in (0, \infty)^2$ for a CPERE on the d-dimensional integer lattice, if the background satisfies Assumption 1.4.1?

Furthermore, it would be interesting to know the behaviour at criticality of a CPERE.

Open problem 5. Does the CPERE on the d-dimensional integer lattice go extinct almost surely at criticality, if the background satisfies Assumption 1.4.1?

Asymptotic shape theorem on $\mathbb{Z}^d$: Closely related to complete convergence is the asymptotic shape theorem. Recall that we denoted by $\tau := \inf\{t \geq 0 : \mathbf{C}_t^{\{\mathbf{0}\},\emptyset} \neq \emptyset\}$ the extinction time of the infection process $\mathbf{C}$ with initial configuration $(\{\mathbf{0}\}, \emptyset)$. Let $\mathbf{H}_t := \bigcup_{s \leq t} \mathbf{C}_s^{\{\mathbf{0}\},\emptyset}$ be the set of all sites which were infected at least once until time t and $\mathbf{K}_t := \{x \in V : x \in \mathbf{C}_s^{\{\mathbf{0}\},\emptyset} \triangle \mathbf{C}_s^{V,E}\ \forall s \geq t\}$ be the permanently coupled region of the infection process $\mathbf{C}$. Furthermore, we set $\mathbf{H}'_t := \mathbf{H}_t + \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ and $\mathbf{K}'_t := \mathbf{K}_t + \left[-\frac{1}{2}, \frac{1}{2}\right]^d$.

Conjecture 6. Let $(\mathbf{C}, \mathbf{B})$ be a CPERE with infection rate $\lambda > 0$ and recovery rate $r > 0$, where $\mathbf{B}$ satisfies Assumption 1.4.1. Suppose that $\theta(\lambda, r, \{\mathbf{0}\}, \emptyset) > 0$ and there exist constants $C_1, C_2, M > 0$ such that

$$\mathbb{P}(t \leq \tau < \infty) \leq C_1 \exp(-C_2 t) \tag{8.1}$$

$$\mathbb{P}(x \notin \mathbf{H}_{M||x||_1 + t}, \tau = \infty) \leq C_1 \exp(-C_2 t) \tag{8.2}$$

$$\mathbb{P}(x \notin \mathbf{K}_{M||x||_1 + t}, \tau = \infty) \leq C_1 \exp(-C_2 t) \tag{8.3}$$

Then there exists a bounded and convex subset $U \subset \mathbb{R}^d$ such that for every $\varepsilon > 0$

$$\mathbb{P}\big(\exists s \geq 0 : t(1-\varepsilon)U \subset (\mathbf{K}'_t \cap \mathbf{H}'_t) \subset \mathbf{H}'_t \subset t(1+\varepsilon)U \ \forall t \geq s \big| \tau = \infty\big) = 1.$$

Let us briefly explain the three conditions mentioned in this conjecture. Condition (8.1) implies that if the infection process $\mathbf{C}$ goes extinct, then this will happen most likely early on. Condition (8.2) basically states that if $\mathbf{C}$ survives, the infection expands asymptotically at least according to some linear speed with high probability. Condition (8.3) has a similar interpretation, i.e that also the permanently coupled region expands at least with some linear speed with high probability. Note that Lemma 4.1.2 already implies that both processes $\mathbf{H}_t$ and $\mathbf{K}_t$ can expand at most according to some linear speed.

As we already mentioned in Section 1.2 Garet and Marchand proved in [GM12] an asymptotic shape theorem for the contact process on $\mathbb{Z}^d$ in a static random environments. Deshayes adapted their techniques in [Des14] to a dynamical setting and showed an asymptotic shape theorem for a contact process with ageing. Furthermore, in [Des15] it was explained that this can also be extended to a broader class of time dynamical contact process, which includes among others the contact process with varying recovery rates studied by [Bro07] and [SW08]. Since the latter model shares a lot of similarities with the CPERE constructed here we believe that Conjecture 6 should hold true.

Both works [GM12] and [Des14] have proven similar conditions to (8.1), (8.2) and (8.3), for the contact process in a static random environment and respectively for the contact process with ageing, by an adaption of the techniques developed in [BG90]. Since we already formulated, for the CPDP, an adaption of these techniques in Chapter 6 we believe that the following conjecture to be true.

Conjecture 7. Let $(\mathbf{C}, \mathbf{B})$ be a CPDP with rates $\lambda, r, \alpha, \beta > 0$ on the d-dimensional integer lattice. Suppose $\theta_{\text{DP}}(\lambda, r, \alpha, \beta) > 0$, then there exists $C_1, C_2, M > 0$ such that (8.1), (8.2) and (8.3) are fulfilled.

CPERE with more general background: In this thesis we focused on a certain type of background, which is described by an ergodic and reversible spin system with finite range interactions. But there are certainly interesting choices for the background which do not satisfy all of Assumption 1.4.1. For example in Remark 1.4.12 we pointed out that a more general version of the noisy voter model see Example 1.1.2 (*ii*), might not satisfy the reversibility assumption, and thus we know nothing about complete convergence or continuity of the survival probability in this case, even though this model seems to be one of the most natural choices for introducing interaction between edges. Note that in Remark 5.1.12 we mentioned an alternative approach for some technical aspects, which do not use reversibility.

Also, if we consider the ferromagnetic Ising model on $\mathbb{Z}^d$ for $d \geq 2$ as the background, see Example 1.1.2 (*iii*), we can choose the inverse temperature β large enough such that this system is no longer ergodic, i.e. there exist more than one invariant law. Another interesting choices for a non-ergodic background would be another contact process or a similar interacting particle system.

Since we strive to formulate a model which is as realistic as possible. One natural extension would be to allow a feedback from the infection process to the background. This seems reasonable, since if an individual is infected and shows symptoms, one would assume that it would distance itself from other people on its own to avoid the spread of the infection. Of course this would lead to vastly different model since the dependency structure is far more complex than in our case.

Further studies on the contact process on a long range dynamical percolation: In the last part of this thesis we studied a contact process on a long range dynamical percolation. This model is basically an extension of the process considered by Linker and Remenik in [LR20]. We have not really studied the long range case in too much depth, and therefore further studies would be necessary to obtain more understanding of this model. We focused on extending some of the results proven in [LR20], for example the existence of an immunization region.

Recall $\hat{p}_e = qp_e$ was the probability of an edge e being open after an update and $\hat{v}_e = \gamma v_e$ was the update speed of this edge, where $q \in (0,1)$, $\gamma > 0$, $(p_e)_{e\in\mathcal{E}} \subset [0,1]$ and $(v_e)_{e\in\mathcal{E}} \subset (0,\infty)$.

Theorem 1.4.23, yields an upper bound on the critical infection rate $\lambda_c(r,\gamma,q)$, since it provides a comparison with a long range contact process. This is of course useful to determine if this system has a positive survival probability. But this theorem is also the first step towards characterizing the asymptotic behaviour for fast speed, i.e. $\gamma \to \infty$. Hence, the next step would be to find a lower bound. The approach which Linker and Remenik used for the CPDP on a graph with bound degree, see [LR20, Theorem 2.3], cannot be extended easily to the long range setting, since it relies heavily on the fact that a graph with bounded degrees is considered.

We studied the asymptotic behaviour as $\gamma \to 0$ under fairly strong assumption, i.e. Assumption 1.4.26. One could also ask what the asymptotic behaviour is if we assume that $\sum_{y\in\mathbb{N}} y v_{\{0,y\}} p_{\{0,y\}} = \infty$. We would expect that the asymptotic behaviour should depend on the choice of the parameter q, since if q is chosen close enough to 1 the long range dynamical percolation model might not partition $\mathbb{Z}$ in finite connected components anymore. Thus, it is reasonable to assume that there exists an $q^* \in (0,1)$ such that $\sup\{\lambda_c(r,\gamma,q) : \gamma > 0, q \in (q^*,1)\} < \infty$, where $r > 0$. A similar result was shown for the CPDP on the d-dimensional integer lattice in [Hil+21].

At the end we want to briefly discuss Assumption 1.4.21 (ii), i.e. $\sum_{y\in V} v_{\{x,y\}}^{-1} < \infty$ for all $x \in V$. This assumption does not seem natural. In fact, the reason for this assumption is of technical nature, since it allowed us to extend the existing results to our setting. It does seem more natural to assume that $v_e = v$ for all $e \in \mathcal{E}$, which means that every edge is updated at the same speed. We would expect that the asymptotic behaviour is similar or even the same in this case. However, without this assumption the situation becomes more complicated, since for example one consequence of this assumption is that all edges attached to a site x can be updated in finite time. But the number of edges attached to x are infinitely many. By setting the speed constant we would lose this property, which we heavily relied on.

Appendix A

$\varepsilon > M$ condition for the background process

Here we calculate the constants ε and M for the processes defined in Example 1.1.2 (i)-(iii), which we already stated Remark 1.4.4. Recall that

$$M := \sum_{a \in \mathcal{N}_e^L} \sup_{B \subset E} |q(e, B) - q(e, B \triangle \{z\})| \quad \text{and} \quad \varepsilon := \inf_{B \subset E} |q(e, B) + q(e, B \triangle \{e\})|.$$

After determining the constants ε and M we will also state for which parameter regime the inequality $\varepsilon - M > \rho$ from Corollary 1.4.3 is satisfied.

Dynamical percolation: We introduced the dynamical percolation in Example 1.1.2 (i) and the spin rate of this model is $q(e, B) = \alpha \mathbb{1}_{\{e \notin B\}} + \beta \mathbb{1}_{\{e \in B\}}$, where $\alpha, \beta > 0$. We see that

$$q(e, B) + q(e, B \triangle \{e\}) = \alpha + \beta \quad \text{and} \quad q(e, B) = q(e, B \triangle \{a\})$$

for all $e \in E$ and all $a \neq e$. Thus, we can conclude that the two constants are $M = 0$ and $\varepsilon = \alpha + \beta$. This shows that $\varepsilon - M > \rho$ if and only if $\alpha + \beta > \rho$.

Noisy voter model: As one can infer from Example 1.1.2 (ii) the spin rate of the noisy voter model is

$$q(e, B) = \beta \big(|B \cap \mathcal{N}_e^L| \mathbb{1}_{\{e \notin B\}} + |B^c \cap \mathcal{N}_e^L| \mathbb{1}_{\{e \in B\}} \big) + \frac{\alpha}{2},$$

where $\alpha, \beta > 0$. We see that

$$q(e, B) + q(e, B \triangle \{e\}) = \alpha + \beta |\mathcal{N}_e| \quad \text{and} \quad |q(e, B) - q(e, B \triangle \{a\})| = \beta$$

for all $e \in E$ and all $a \in \mathcal{N}_e^L$, and thus $M = \beta|\mathcal{N}_e^L|$ and $\varepsilon = \alpha + \beta|\mathcal{N}_e^L|$. Furthermore, $\varepsilon - M > \rho$ if and only if $\alpha > \rho$.

Ferromagnetic stochastic Ising Model: The calculations for this model are a bit more lengthy. Recall from Example 1.1.2 *(iii)* that the spin rate of this model is

$$q(e,B) = 1 - \tanh\left(\beta \sum_{a\in\mathcal{N}_e^L} (-1)^{|B^c\cap\{e,a\}|}\right) = 2\left(1 + \exp\left(2\beta \sum_{a\in\mathcal{N}_e^L} (-1)^{|B^c\cap\{e,a\}|}\right)\right)^{-1}.$$

Let us first introduce the shorthand notation $\chi(e,B) := \sum_{a\in\mathcal{N}_e^L}(-1)^{|B^c\cap\{e,a\}|}$. We start with calculating the constant ε. We see that $\chi(e, B \triangle \{e\}) = -\chi(e,B)$, which yields that

$$q(e,B) + q(e, B \triangle \{e\}) = \frac{2}{1+\exp\big(2\beta\chi(e,B)\big)} + \frac{2\exp\big(2\beta\chi(e,B)\big)}{1+\exp\big(2\beta\chi(e,B)\big)} = 2$$

for all $e \in E$ and all $B \subset E$. Hence, the infimum over all B yields

$$\varepsilon = \inf_{B\subset E} |q(e,B) + q(e, B\triangle\{e\})| = 2.$$

Next we calculate M. For $z \in \mathcal{N}_e^L$ define

$$H(z,B) := \big(q(e,B) - q(e, B\triangle\{z\})\big) \quad \text{and} \quad \chi_z(e,B) := \sum_{a\in\mathcal{N}_e^L\setminus\{z\}} (-1)^{|B^c\cap\{e,a\}|}.$$

Since $\chi(e,B) + \chi(e, B\triangle\{z\}) = 2\chi_z(e,B)$ we see that

$$\begin{aligned}\frac{1}{2}H(z,B) =&\Big(\exp\big(2\beta\chi(e,B)\big) - \exp\big(2\beta\chi(e,B\triangle\{z\})\big)\Big)\\ &\times\Big(1+\exp\big(2\beta\chi(e,B)\big) + \exp\big(2\beta\chi(e,B\triangle\{z\})\big) + \exp\big(4\beta\chi_z(e,B)\big)\Big)^{-1}.\end{aligned}$$

Now we see that the factor $\exp\big(-2\beta\chi_z(e,B)\big)$ in the numerator and denominator and we use again that $(-1)^{|(B\triangle\{z\})^c\cap\{e,z\}|} = -(-1)^{|B^c\cap\{e,z\}|}$ for all $z \in \mathcal{N}_e^L$. This yields

$$\begin{aligned}\frac{1}{2}H(z,B) =&\left(\exp\left(2\beta(-1)^{|B^c\cap\{e,z\}|}\right) - \exp\left(-2\beta(-1)^{|B^c\cap\{e,z\}|}\right)\right)\\ &\times\left(\exp\left(-2\beta\chi_z(e,B)\right) + \exp\left(2\beta(-1)^{|B^c\cap\{e,z\}|}\right)\right.\\ &\left.+\exp\left(-2\beta(-1)^{|B^c\cap\{e,z\}|}\right) + \exp\left(2\beta\chi_z(e,B)\right)\right)^{-1}.\end{aligned}$$

We take the absolute value and use that $\chi(e, B \triangle \{e\}) = -\chi(e, B)$, which provides

$$\frac{1}{2}|H(z,B)| = \frac{\exp(2\beta) - \exp(-2\beta)}{\exp(2\beta) + \exp(-2\beta) + \exp\big(-2\beta\chi_z(e,B)\big) + \exp\big(2\beta\chi_z(e,B)\big)}.$$

Maximizing this term with respect to $B \subset E$ is equivalent to minimizing

$$\exp\big(-2\beta\chi_z(e,B)\big) + \exp\big(2\beta\chi_z(e,B)\big).$$

The function $x \mapsto e^x + e^{-x}$ is continuous, strictly decreasing on $(-\infty, 0]$ and strictly increasing $[0, \infty)$. It is easy to see that it takes its minimum at $x = 0$, which has the function value 2. We see that

$$\chi_z(e,B) \in \{-|\mathcal{N}_e^L| + 1, -|\mathcal{N}_e^L| + 3, \dots, |\mathcal{N}_e^L| - 3, |\mathcal{N}_e^L| - 1\}.$$

Note that the set on the right hand side contains 0 only if $|\mathcal{N}_e^L|$ is odd. Thus, we see that

$$\sup_{B \subset E} |H(z,B)| = \begin{cases} \frac{2(e^{2\beta} - e^{-2\beta})}{e^{2\beta} + e^{-2\beta} + 2} & \text{if } |\mathcal{N}_e^L| \text{ odd} \\ \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}} & \text{if } |\mathcal{N}_e^L| \text{ even.} \end{cases}$$

Therefore, using that $\sup_{B \subset E} |H(z,B)|$ is the same for all $z \in \mathcal{N}_e^L$ inserting this into the definition of M yields

$$M = \sum_{z \in \mathcal{N}_e^L} \sup_{B \subset E} |q(e,B) - q(e, B \triangle \{z\})| = \begin{cases} |\mathcal{N}_e^L| \frac{2(e^{2\beta} - e^{-2\beta})}{e^{2\beta} + e^{-2\beta} + 2} & \text{if } |\mathcal{N}_e^L| \text{ odd} \\ |\mathcal{N}_e^L| \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}} & \text{if } |\mathcal{N}_e^L| \text{ even.} \end{cases}$$

After we calculated the constants ε, M we will now determine for which β the inequality $\varepsilon - M > \rho$ holds. Obviously we need that $\rho < 2 = \varepsilon$, since $M \geq 0$. If we consider $|\mathcal{N}_e^L|$ even, then by inserting ε and M we see that

$$\varepsilon - M = 2 - |\mathcal{N}_e^L| \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}} > \rho \quad \Leftrightarrow \quad \frac{1}{4} \log\left(\frac{|\mathcal{N}_e^L| - \rho + 2}{|\mathcal{N}_e^L| + \rho - 2}\right) > \beta.$$

Thus, we see that all $0 \leq \beta < \frac{1}{4} \log\left(\frac{|\mathcal{N}_e^L| - \rho + 2}{|\mathcal{N}_e^L| + \rho - 2}\right)$ satisfy the inequality. This is actually also true if $|\mathcal{N}_e^L|$ is odd. This follows by the fact that M obviously is smaller if $|\mathcal{N}_e^L|$ is odd. Nevertheless, we are able to obtain a slightly better bound if we consider $|\mathcal{N}_e^L|$

to be odd. Thus, if we again insert ε and M into the inequality $\varepsilon - M > \rho$. After rearranging the terms we get that

$$(2-\rho-2|\mathcal{N}_e^L|)e^{4\beta} + 2(2-\rho)e^{2\beta} + (2-\rho+2|\mathcal{N}_e^L|) > 0.$$

Next we substitute $t = e^{2\beta}$ and calculate the root of

$$(2-\rho-2|\mathcal{N}_e^L|)t^2 + 2(2-\rho)t + (2-\rho+2|\mathcal{N}_e^L|) = 0,$$

which are by square addition

$$t_\pm = \frac{-2(2-\rho) \pm \sqrt{4(2-\rho)^2 - 4(2-\rho-2|\mathcal{N}_e^L|)(2-\rho+2|\mathcal{N}_e^L|)}}{2(2-\rho-2|\mathcal{N}_e^L|)}.$$

Since $(2-\rho-2|\mathcal{N}_e^L|)(2-\rho+2|\mathcal{N}_e^L|) = (2-\rho)^2 - 4|\mathcal{N}_e^L|^2$ we see that

$$t_\pm = \frac{\pm 2|\mathcal{N}_e^L| - \rho + 2}{(2|\mathcal{N}_e^L| + \rho - 2)}.$$

Obviously $e^{2\beta} > 0$ for all $\beta \in \mathbb{R}$, and thus the only root which is possible is t_+. Furthermore, $\beta \mapsto e^{2\beta}$ is monotone increasing, which yields that if $|\mathcal{N}_e^L|$ is odd, then

$$\varepsilon - M > \rho \quad \Leftrightarrow \quad \frac{1}{2}\log\left(\frac{2|\mathcal{N}_e^L| - \rho + 2}{2|\mathcal{N}_e^L| + \rho - 2}\right) > \beta.$$

Appendix B

Oriented percolation and K-dependence

B.1 Oriented percolation

The term oriented percolation is not really uniquely connected to one model. In principle every percolation model defined on a directed graph can be called an oriented percolation model. Here we will only consider a special case. We consider the oriented percolation on $\mathbb{Z}$ or rather $\mathbb{Z} \times \mathbb{N}_0$. For this type of model there is more then one possible representation. Here we will formulate it as a discrete stochastic growth model, as in [Dur84] or [Lig13]. They considered a Markov chain $(X_n)_{n\geq 0}$ with values in $\mathcal{P}(\mathbb{N}_0)$ and the evolution of the process is described through the conditional probability

$$\mathbb{P}(x \in X_{n+1} | X_0, \dots X_n) = \begin{cases} p & \text{if } X_n \cap \{x, x+1\} \\ 0 & \text{otherwise .} \end{cases}$$

We recommend [Dur84] for detailed survey on this model. Note that we will consider a slightly different version. Let

$$\begin{aligned} f : \mathbb{Z} \times \mathbb{N}_0 &\to \mathbb{N}_0 \times \mathbb{N}_0 \\ (x, n) &\mapsto f(x, n) = (2x - n, n). \end{aligned}$$

and set $W_n := f(X_n)$. Since f is bijective, this transformation is a mere reformulation of the state space and does not really change the behaviour of the process. We only use this version since we want to compare the oriented percolation to the CPDP in

Chapter 6, and thus this choice seems more intuitive. See Figure B.1 for a visualization of the two versions. We see that the dynamics of $(W_n)_{n\in\mathbb{N}_0}$ are

$$\mathbb{P}(x \in W_{n+1}|W_0, \dots W_n) = \begin{cases} p & \text{if } X_n \cap \{x-1, x+1\} \\ 0 & \text{otherwise,} \end{cases}$$

and thus $W_{2n} \subset 2\mathbb{Z}$ and $W_{2n-1} \subset 2\mathbb{Z} - 1$ for every $n \in \mathbb{N}_0$. Similar as for the contact process we will indicate the initial state by a superscript, i.e. W^A, where $A \subset 2\mathbb{Z}$. We also see that $W_n^{\{0\}} \subseteq [-n, n]$. Furthermore, we denote by $\tau = \{n \geq 0 : W_n = \emptyset\}$ the "extinction" time of $(W_n)_{n\geq 0}$. In the terminology of percolation models $\{\tau = \infty\}$ is the event that percolation occurs. Now we state some facts, which we need to utilize in Section 6.

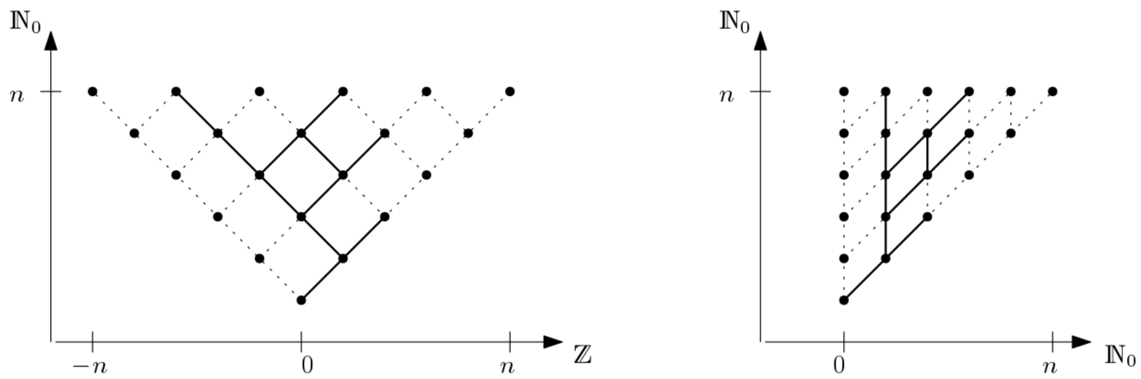


Figure B.1: Here we visualized a possible realization of an oriented percolation. On the left the verison $(W_n)_n$ and on the right $(X_n)_n$.

Theorem B.1.1. *For p close enough to 1 there exist $C > 0$ and $\varepsilon > 0$ such that*

(i) $\inf_{k\geq 0} \mathbb{P}^{\{0\}}(0 \in W_{2k}) > 0$,

(ii) $\mathbb{P}^{\{0\}}(k < \tau < \infty) \leq Ce^{-\varepsilon k}$,

(iii) $\mathbb{P}^A(\tau < \infty) \leq Ce^{-\varepsilon|A|}$, *where* $A \subset 2\mathbb{Z}$.

Proof. This follows from [Lig13, Theorem B24], which proves the equivalent statements for $(X_n)_{n\in\mathbb{N}_0}$ □

Theorem B.1.2. *For p close enough to 1 it holds that*

$$\lim_{m\to\infty} \liminf_{n\to\infty} \mathbb{P}^{\{-m,\dots,m\}}(W_n \cap \{-m, \dots, m\} \neq \emptyset) = 1.$$

Proof. Let us again consider the equivalent version $(X_n)_{n\in\mathbb{N}_0}$. In [DS87, Section 5] it was shown that for a broad class of stochastic growth models complete convergence holds. The oriented percolation $(X_n)_{n\in\mathbb{N}_0}$ is part of this class as mentioned in their Example 2. Thus, by [DS87, Theorem 2] for p close to 1 there exists a law ν on $\mathcal{P}(\mathbb{N}_0)$ such that $X_n^A \Rightarrow \mathbb{P}^A(\tau < \infty)\delta_\emptyset + \mathbb{P}^A(\tau < \infty)\nu$ as $n \to \infty$. Similar as for the contact process they derived a duality relation such that for $A, B \subset \mathbb{N}_0$

$$\mathbb{P}(X_n^A \cap B \neq \emptyset) = \mathbb{P}(X_n^B \cap A \neq \emptyset),$$

and furthermore used this relation to show that

$$\mathbb{P}(X_n^A \neq \emptyset \ \forall n \geq 0) = \nu(B \subset \mathbb{N}_0 : B \cap A \neq \emptyset).$$

Thus, we can conclude that

$$\lim_{|A|\to\infty} \liminf_{n\to\infty} \mathbb{P}(X_n^A \cap A \neq \emptyset) = \lim_{|A|\to\infty} \mathbb{P}(X_n^A \neq \emptyset \ \forall n \geq 0)^2 = 1,$$

where we used Theorem B.1.1 (*iii*) to conclude the last equality. Now by transforming X_n with f and choosing A appropriately we obtain the claim. □

B.2 K-dependence

In this section we introduce the notion of K-dependence. To be more precise we consider a family of Bernoulli variables with a certain dependence structure and state a comparison result with a family of independent Bernoulli variables.

Definition B.2.1 (K-dependence)**.** Let $(X_i)_{i\in\Lambda}$ be a family of Bernoulli random variables, where Λ is a countable index set. We call the family $(X_i)_{i\in\Lambda}$ *K-dependent*, if for every $i \in \Lambda$ there exists a subset $\Lambda_i \subset \Lambda$ with $i \in \Lambda_i$, $|\Lambda_i| \leq K$ and

$$X_i \text{ is independent of } (X_j)_{j\in\Lambda\setminus\Lambda_i}.$$

Note that by this definition 1-dependence is equivalent to $(X_i)_{i\in\Lambda}$ being an independent family of Bernoulli random variables. The next theorem provides that K-dependent families can be coupled with an independent family such that the independent family acts as a lower bound.

Theorem B.2.2. *Let* Λ *be a countable set,* $p \in (0,1)$ *and* $K < \infty$*. Assume that* $(X_i)_{i\in\Lambda}$ *is a* K*-dependent family of Bernoulli random variables with* $\mathbb{P}(X_i = 1) \geq p$ *for all* $i \in \Lambda$ *and that*

$$\tilde{p} := \left(1 - (1-p)^{\frac{1}{K}}\right)^2 \geq \frac{1}{4}.$$

Then there exists a family $(\tilde{X}_i)_{i\in\Lambda}$ *of independent Bernoulli random variables such that*

$$\mathbb{P}(\tilde{X}_i = 1) = \tilde{p}$$

and $X_i \geq \tilde{X}_i$ *for all* $i \in \Lambda$*.*

Proof. See [Swa17, Theorem 7.4] □

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