

**On the resolvent
of the Laplacian on functions
for degenerating surfaces
of finite geometry**

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Abstract

We consider families (Y_n) of degenerating hyperbolic surfaces. The surfaces are geometrically finite of fixed topological type. Let \mathcal{Z}_n be the Selberg Zeta function of Y_n , and let \mathcal{Z}_n^d be the contribution of the pinched geodesics to \mathcal{Z}_n . Extending a result of Wolpert's, we prove that $\mathcal{Z}_n(s)/\mathcal{Z}_n^d(s)$ converges to the Zeta function of the limit surface for all s with $\operatorname{Re}(s) > 1/2$. The technique is an examination of resolvent of the Laplacian, which is composed from that for elementary surfaces via meromorphic Fredholm theory. The resolvent $(\Delta_n - t)^{-1}$ is shown to converge for all $t \notin [1/4, \infty)$. We also use this property to define approximate Eisenstein functions and scattering matrices.

Contents

0	Introduction	2
1	Surface geometry	5
1.1	A degenerating family of pairs of pants	6
1.2	Assembling a surface from pairs of pants	8
1.3	Choice of a trivialisation	11
1.4	The Selberg Zeta function in its domain of convergence	12
2	The resolvent for geometrically finite surfaces	15
2.1	Auxiliary surfaces	15
2.1.1	Compact surfaces	15
2.1.2	Elementary surfaces	17
2.1.3	Proof of proposition 2.3	18
2.2	Geometrically finite surfaces	28
3	Applications	32
3.1	Eisenstein functions and the scattering matrix	32
3.1.1	Definitions and fundamental properties	32
3.1.2	Approximating the scattering theory of finite area surfaces	42
3.2	The Selberg Zeta function	44
A	Families of Fuchsian groups	49
B	The Selberg trace formula	52

0 Introduction

A family of *degenerating Riemannian manifolds* consists of a manifold M and a family $(g_\ell)_{\ell>0}$ of Riemannian metrics on M that meet the following assumptions:

- There are finitely many disjoint open subsets $Z_i \subset M$ that are diffeomorphic to cylinders $F_i \times J_i$. The fibre F_i is a compact manifold and $J_i \subset \mathbb{R}$ is a neighbourhood of 0.
- The restriction of each metric g_ℓ to $Z_i = F_i \times J_i$ is a product metric

$$(x, a) \mapsto \nu_{i,\ell}(a) \cdot g_{F_i}(x) + \mu_{i,\ell}(a) \cdot da^2$$

such that $\nu_{i,\ell}(0) \rightarrow 0$ and $\mu_{i,\ell}(0) \rightarrow \infty$ as $\ell \rightarrow 0$.

- On the complement of $\bigcup_i F_i \times \{0\}$ in M , the metrics g_ℓ converge to a Riemannian metric g_0 .

Spectral geometric properties of certain types of degenerating manifolds have been examined by several authors, let us mention Colbois-Courtois [4], Chavel-Dodziuk [2] and Judge [14].

We consider the case of hyperbolic surfaces, which is the fundamental example of such a degeneration. Here M is an oriented surface of negative Euler characteristic, and the metrics g_ℓ are hyperbolic, chosen in such a way that there are finitely many closed curves c_i , geodesic with respect to all metrics, with the length ℓ_i of each curve converging to 0 as ℓ decreases. On the complement of the distinguished curves, the sequence of metrics is required to converge to a hyperbolic metric.

In the description above, the geodesics c_i correspond to the central fibres $F_i \times \{0\}$. The collar lemma of hyperbolic geometry ensures that each c_i has got a collar neighbourhood $Z_i = \mathbb{R}/\mathbb{Z} \times (-\epsilon, \epsilon)$, with the Riemannian metric on Z_i being given by

$$(x, a) \mapsto (\ell_i^2 + a^2)dx^2 + (\ell_i^2 + a^2)^{-1}da^2.$$

Let M_ℓ denote the surface M equipped with the metric g_ℓ if $\ell > 0$, and let $M_0 = M \setminus \bigcup_i c_i$ carry the limit metric $\lim_{\ell \rightarrow 0} g_\ell$. Note that M_0 is a complete hyperbolic surface by definition, which contains a pair of cusps for each i .

From the point of view of spectral theory, this example was initially studied by Schoen-Wolpert-Yau [22], Colbois-Courtois [3] and by Hejhal [10], Ji [12] and Wolpert [25]. To exemplify how the spectrum may behave during this process, assume that M is compact for the moment. Then M_0 is of finite area but not compact. The spectrum of compact manifolds is purely discrete, whereas that of M_0 is the union of finitely many eigenvalues in $[0, 1/4)$, and the essential spectrum is $[1/4, \infty)$. It was observed that the small eigenvalues of M_0 are limits of eigenvalues of M_ℓ as $\ell \rightarrow 0$, and the eigenvalues of M_ℓ accumulate at each point in $[1/4, \infty)$. One also tries to obtain information on embedded eigenvalues in the essential spectrum of M_0 by means of such an approximation [26].

Hejhal and Wolpert proved results on the behaviour of the Selberg Zeta function. Our motivation for this work was to extend one of these results.

Let us recall the definition of the Selberg Zeta function. It is a meromorphic function \mathcal{Z} on the complex plane, associated with a hyperbolic surface. In the domain $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$ it is given by an absolutely convergent product

$$\mathcal{Z}: s \mapsto \prod_c \mathcal{Z}_c(s), \quad \text{where} \quad \mathcal{Z}_c(s) := \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell(c)})^2.$$

The product ranges over the set of all unoriented, simple, closed geodesics c of the surface, and $\ell(c)$ denotes the length of c . Now if $\ell(c)$ decreases as the metric changes, we have (the precise asymptotics are given in lemma 3.22)

$$\mathcal{Z}_c(s) = \mathcal{O}(\ell(c)^{1-2\operatorname{Re}(s)} e^{-\pi^2/3\ell(c)}), \quad \ell(c) \rightarrow 0.$$

Therefore, one cannot expect to Selberg Zeta function of M_ℓ to converge to that of the limit M_0 . In section 3.2 we prove

Theorem. *Consider a family M_ℓ of degenerating surfaces, and let $\{c_i\}$ be the set of distinguished geodesics that are pinched. Then $\mathcal{Z}(s)/\prod_i \mathcal{Z}_{c_i}(s)$ converges to the Zeta function of the limit surface M_0 if $\operatorname{Re}(s) > 1/2$.*

This is Wolpert's *Conjecture 2* [25]. Hejhal proved that it holds in the domain of convergence $\operatorname{Re}(s) > 1$, and this was extended by Wolpert to a neighbourhood of $s = 1$. He also concluded from the functional equation of \mathcal{Z} that the same cannot be true in any domain that intersects $\{s \in \mathbb{C} \mid \operatorname{Re}(s) < 1/2\}$, at least if the surfaces M_ℓ are compact.

Each zero s of the Zeta function with $\operatorname{Re}(s) > 1/2$ corresponds to an eigenvalue $s(1-s) \in [0, 1/4)$ of the Laplacian. We also prove Wolpert's *Conjecture 1*, which states that one may divide out these zeroes from the quotient above to obtain uniform boundedness from below. It applies to geometrically finite surfaces of both finite and infinite area.

Let us elaborate a little on the proof of these statements. Analysis of the Selberg Zeta function is based on the trace formula, which relates the logarithmic derivative of \mathcal{Z} with the resolvent kernel of the Laplacian. Our primary object of investigation is the resolvent operator. The following theorem, proved throughout section 2, extends a result of Jorgenson and Lundelius [13] that arose as a consequence of their discussion of the heat kernel.

Theorem. *If $t \in \mathbb{C} \setminus [1/4, \infty)$, let $R_\ell(t)$ denote the pull-back of the resolvent $(\Delta - t)^{-1}$ from M_ℓ to M_0 . Then $R_\ell(t)$ converges to $R_0(t)$ in the topology of continuous linear maps $L^2(M_0) \rightarrow L^2_{\text{loc}}(M_0)$.*

Restriction to the vector space $L^2_{\text{loc}}(M_0)$ in the image of the resolvent means that we prove convergence of the resolvent kernel on the complement of small neighbourhoods of the pinched geodesics. This restriction may be dropped if $(\Delta - t)^{-1}$ is replaced with

$$(\Delta - t)^{-1} - \sum_i \psi_i (\Delta_{\bar{Z}_i} - t)^{-1} \phi_i, \quad (1)$$

where $(\Delta_{\bar{Z}_i} - t)^{-1}$ denotes the resolvent on an infinite cylinder \bar{Z}_i that admits an isometric embedding $Z_i \rightarrow \bar{Z}_i$. The functions ψ_i, ϕ_i are suitable cut-off functions such that the operator is defined. We prove that if two operators like (1) with different values of t are considered, their difference converges in the trace class topology. This immediately implies our results on the Zeta function.

The notion of convergence in the previous theorem is to be understood in the following sense: The map $t \mapsto R_\ell(t)$ is meromorphic on $\mathbb{C} \setminus [1/4, \infty)$ with possibly finitely many poles of finite rank in $[0, 1/4)$. If t_0 is not a pole of R_0 , then there exist neighbourhoods V of t_0 and \mathcal{U} of 0 such that R_ℓ has no poles in V for all $\ell \in \mathcal{U}$, and convergence holds uniformly in V . So if \mathcal{C} is a closed curve in the complement of the discrete spectrum of R_0 , then the Riesz-projector

$$-\frac{1}{2\pi i} \oint_{\mathcal{C}} R_\ell(t) dt$$

converges to that of R_0 as $\ell \rightarrow 0$. In particular, we obtain an alternative proof of the convergence of small eigenvalues for degenerating surfaces [3].

To prove the previous theorem, we apply a technique that is typically used to establish a continuation of the resolvent kernel for M_0 , across the essential spectrum $[1/4, \infty)$, to a branched cover of \mathbb{C} . More precisely, we use meromorphic Fredholm theory to compose the resolvent of a geometrically finite surface from those for a compact surface and for elementary quotients of the hyperbolic plane. If $\operatorname{Re}(s) > 1/2$, and $s(1-s)$ is not an eigenvalue of Δ , then the following equation holds, where Δ_i denotes the Laplacian of an auxiliary surface:

$$(\Delta - s(1-s))^{-1} = \left(\sum_i \psi_i (\Delta_i - s(1-s))^{-1} \phi_i \right) (1 + K(s))^{-1}. \quad (2)$$

Here $(1 + K(s))$ denotes a meromorphically invertible family of Fredholm operators. This formula gives the information needed to deduce convergence of the resolvent from that of its constituent parts.

Now the right hand side of equation (2) is known to have a continuation in s to the complex plane. In view of this analytic continuation, it would be interesting to obtain similar results on the left of the critical axis $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = 1/2\}$. But obviously the continuation is symmetric in s and $1-s$ for compact surfaces, while there is no such trivial relation in the other cases. To overcome this symmetry, we introduce *approximate Eisenstein functions* for M_ℓ as follows.

Recall that the definition of Eisenstein functions is based on certain eigenfunctions of the Laplacian on a cusp $Z_k^- \subset M_0$, where

$$Z_k^- := \mathbb{R}/\mathbb{Z} \times (-\epsilon, 0) \subset Z_k$$

carries the metric $(x, a) \mapsto a^2 dx^2 + a^{-2} da^2$. In these coordinates, the functions are given by

$$h(0, s): Z_k^- \longrightarrow \mathbb{C}, \quad (x, a) \longmapsto |a|^{-s}.$$

The number 0 in $h(0, s)$ refers to the ‘diameter’ of a cusp. In section 3.1 we define functions $h(\ell(k), s): Z_k^- \rightarrow \mathbb{C}$ that depend on the length $\ell(k) > 0$ of the closed geodesic in Z_k . The corresponding notion of approximate Eisenstein functions on M_ℓ then consists of a meromorphic family of functions associated with each half-cylinder $Z_k^\pm \subset Z_k$. Up to a jump discontinuity on the respective closed geodesic, they are eigenfunctions of the Laplacian on M_ℓ . In this introduction they shall be denoted by $E_i(s)$, where i runs through the set \tilde{S} of half-cylinders.

In this context, the classical scattering matrix is replaced with a pair of matrices $(C_{ij}(s))_{i,j \in \tilde{S}}$ and $(D_{ij}(s))_{i,j \in \tilde{S}}$ with the following properties: On each half-cylinder Z_j^- we may define fibrewise Fourier coefficients F_j^n , and the approximate Eisenstein functions $E_i(s)$ satisfy

$$F_j^0 E_i(s) = D_{ij}(s) \cdot h(\ell(j), s) + C_{ij}(s) \cdot h(\ell(j), 1-s).$$

Then, in consequence of convergence of the resolvent, we obtain

Theorem. 1. As $\ell \rightarrow 0$, the approximate Eisenstein function $E_i(s)$ on M_ℓ converge to Eisenstein functions for M_0 if $\operatorname{Re}(s) > 1/2$.

2. If $\operatorname{Re}(s) > 1/2$, the matrix $(C_{ij}(s))_{ij}$ converges to the scattering matrix of M_0 .

Let us mention that the first part of this theorem in particular implies that the Eisenstein functions $E_i(s)$ on M_ℓ , defined if the latter has cusps, converge to Eisenstein functions of the limit if $\operatorname{Re}(s) > 1/2$. Results of this kind were used by Obitsu [19] to study the geometry of Teichmüller spaces.

But the approximate Eisenstein functions do not immediately accomplish the problem of extending the convergence results to $\{s \in \mathbb{C} \mid \operatorname{Re}(s) < 1/2\}$. Rather, they satisfy a Mass-Selberg relation that gives rise to functional equations for surfaces of finite area. These can be used to prove the following assertion. Here $E(s)$ denotes the column vector that has the approximate Eisenstein functions as entries.

Corollary. *Assume that the surfaces M_ℓ are of finite area. For ℓ near 0, the meromorphic family of matrices $D: s \mapsto (D_{ij}(s))$ is meromorphically invertible. Then $D(s)^{-1} \cdot E(s)$ and $D(s)^{-1} \cdot C(s)$ converge on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \neq 1/2\}$ to the Eisenstein functions and the scattering matrix of M_0 , respectively.*

The corollary suggests a replacement for the quotient $\mathcal{Z}(s)/\prod_i \mathcal{Z}_{c_i}(s)$ in the first theorem, namely

$$\det D(s) \cdot \mathcal{Z}(s) / \prod_i \mathcal{Z}_{c_i}(s).$$

The additional factor does not alter the limit if $\operatorname{Re}(s) > 1/2$, and we prove in Theorem 3.23 that this expression also converges to the Zeta function of M_0 if $\operatorname{Re}(s) < 1/2$ for a degenerating family of compact surfaces. It is in this sense, that the approximate scattering data admit an extension of the first theorem to the left of the critical axis. Unfortunately we do not know how this term behaves on the critical axis itself.

The text is arranged as follows: In section 1, we give an explicit description of the metric degeneration in terms of Fenchel-Nielsen coordinates. It comes with a convenient choice of coordinates for elementary cylinders that are embedded in the surfaces, and these coordinates are the basis for our comparison of integral kernels later on. Section 2 is divided into two parts. After a few remarks on compact surfaces, the first part uses the resolvent kernel of the Laplacian on the hyperbolic plane to compare the kernels for elementary surfaces of different diameters. The main difficulty is to obtain trace class estimates that carry over to general surfaces of finite geometry. The second part applies meromorphic Fredholm theory to examine these surfaces. Approximate Eisenstein series are described in section 3.1, and in section 3.2 we apply our results to the Selberg Zeta function.

From the year 2000 on I was a member of the DFG research group “Zetafunktionen und lokalsymmetrische Räume”, located at Clausthal-Zellerfeld and Göttingen, and of the Graduate Program “Gruppen und Geometrie”. I am very grateful to the members of these groups for their support. But first and foremost I want to express my thanks to Prof. Dr. Ulrich Bunke. He raised my interest in this subject, and he was of great influence on me during the past years. I also want to thank Dr. Martin Olbrich and Dr. Margit Rösler for their help.

1 Surface geometry

A collection of $3p - 3$ incontractible, simple, closed curves that are disjoint and homotopically distinct may be used to dissect a closed, oriented surface of genus $p \geq 2$ into pairs of pants. Fenchel-Nielsen coordinates as gluing data for hyperbolic pairs of pants determine a hyperbolic surface of the same genus. Such a decomposition of hyperbolic surfaces will be applied to examine their degeneration in the sections to come.

To introduce some notation, we begin with giving an explicit construction for a family of extended pairs of pants. These are extended in the sense that they are not compact, but the removal of infinite cylinders will result in the common pairs or pants, which have a boundary consisting of three closed geodesics. The family provides the building blocks for general surfaces of finite geometry, and it also serves as the fundamental example for the degeneration process.

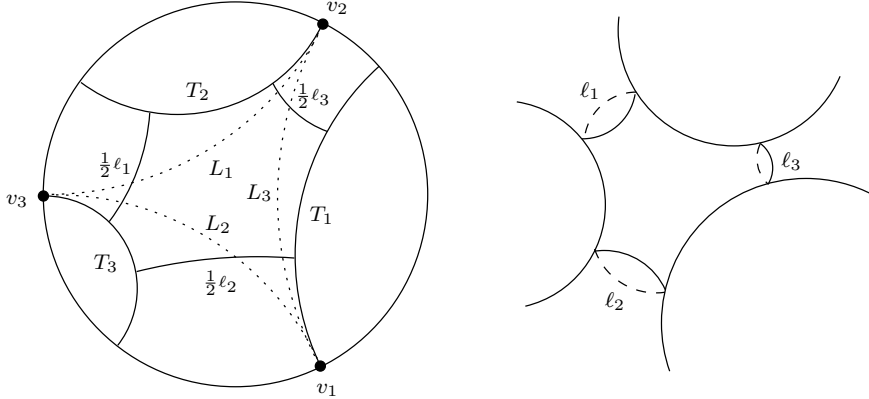


Figure 1: Construction of a pair of pants with prescribed circumference of funnels.

1.1 A degenerating family of pairs of pants

Topologically, the surfaces are thrice punctured spheres, so their fundamental group is the free group on two generators $\mathbb{Z} * \mathbb{Z}$. To provide such a one carrying a hyperbolic structure, we specify the generators of a Fuchsian group that is isomorphic to $\mathbb{Z} * \mathbb{Z}$. Up to conjugation within the orientation preserving isometries of the hyperbolic plane, the Fuchsian group is determined by a triple of three nonnegative numbers, of which every positive stands for the length of a closed geodesic. In case at least one of the lengths vanishes, we speak of a degeneration.

For the time being, we use the unit disc model D of the hyperbolic plane. Hyperbolic distance between $z_1, z_2 \in D$ is denoted by $d(z_1, z_2)$. To determine a conjugacy class for the groups to be constructed, fix three distinct points that occur in the order v_1, v_2, v_3 on ∂D , the indices will be taken as elements of $\mathbb{Z}/3\mathbb{Z}$. Let L_i denote the geodesic line that joins v_{i+1} with v_{i+2} .

1.1 Lemma. *Any triple (ℓ_1, ℓ_2, ℓ_3) of non-negative reals determines disjoint geodesics $T_1, T_2, T_3 \subset D$ with the following properties (cf. figure 1).*

1. Each T_i meets ∂D at v_i and the geodesic L_{i-1} separates T_i from L_i .
2. The number $\ell_i/2$ is equal to the hyperbolic distance $d(T_{i+1}, T_{i+2})$.

In fact, T_i is the unique geodesic with the first property that satisfies

$$\cosh(d(T_i, L_i)) = (\cosh(\ell_i/2) + 1)^{-1} [m(\ell_1, \ell_2, \ell_3) + \cosh(\ell_{i+2}/2) - \cosh(\ell_{i+1}/2)] \quad (3)$$

where

$$m(\ell_1, \ell_2, \ell_3) = (\cosh^2(\ell_1/2) + \cosh^2(\ell_2/2) + \cosh^2(\ell_3/2) + 2 \cosh(\ell_1/2) \cosh(\ell_2/2) \cosh(\ell_3/2) - 1)^{1/2}.$$

Proof. It is convenient to consider this as a statement on the inversive product of spheres in the extended plane $\hat{\mathbb{R}}^2$. The set of spheres is identified with a subset of $\mathbb{R}P^3$ by mapping $\{x \in \mathbb{R}^2 \mid a_0 \|x\|^2 - 2\langle x, (a_1, a_2) \rangle + a_3 = 0\}$ to the equivalence class of (a_0, a_1, a_2, a_3) . With the bilinear form $q(a, b) = 2(a_1 b_1 + a_2 b_2) - a_0 b_3 - a_3 b_0$, the inversive product of two spheres S and T , represented by $a, b \in \mathbb{R}P^3$, is defined to be (cf. Beardon [1, p. 28ff])

$$(S, T) = \frac{|q(a, b)|}{|q(a, a)|^{1/2} |q(b, b)|^{1/2}}.$$

Now, given spheres L_1, L_2, L_3 and T_1, T_2, T_3 as in the proposition with property **1**, one can compute (T_i, L_i) from (T_i, T_{i+1}) and (T_{i+1}, L_{i+1}) :

$$(T_i, L_i) = \frac{2((T_i, T_{i+1}) + (T_{i+1}, L_{i+1}))}{1 + (T_{i+1}, L_{i+1})} - 1.$$

The formula can be verified conveniently in the upper half plane with $\{v_1, v_2, v_3\} = \{0, 1, \infty\}$ since the inversive product is invariant under Möbius transformations.

This gives a system of three equations, solved by the (non-negative) inversive products (T_i, L_i) if and only if

$$(T_i, L_i) = ((T_{i+1}, T_{i+2}) + 1)^{-1} \cdot \left[(T_i, T_{i+1}) - (T_i, T_{i+2}) + \sqrt{(T_1, T_2)^2 + (T_2, T_3)^2 + (T_3, T_1)^2 + 2(T_1, T_2)(T_2, T_3)(T_3, T_1) - 1} \right].$$

Finally, if the intersections of two spheres S, T with D are disjoint geodesics, we have $(S, T) = \cosh d(S \cap D, T \cap D)$. So the previous equation calculates (T_i, L_i) from ℓ_1, ℓ_2, ℓ_3 , and this in turn determines T_i . \square

In view of this lemma, the set $\mathcal{B}_0 := [0, \infty)^3$ now serves as parameter space for Fuchsian groups. Each point $(\ell_1, \ell_2, \ell_3) \in \mathcal{B}_0$ gives rise to geodesics T_i . Let σ_i denote reflection in T_i and $\gamma_i = \sigma_{i+2}\sigma_{i+1}$. Then γ_i, γ_{i+1} act as side-pairing transformations on the convex polygon bounded by $T_i, T_{i+1}, \sigma_{i+2}T_i$ and $\sigma_{i+2}T_{i+1}$. Observe that γ_i is a parabolic isometry of D if T_{i+2} and T_{i+1} meet in v_{i+2} , i.e. if $\ell_i = 0$. Otherwise it is a hyperbolic transformation of translation length ℓ_i , its axis being the common orthogonal of T_{i+2} and T_{i+1} .

The group generated by $\gamma_1, \gamma_2, \gamma_3$ has a presentation $\langle \gamma_1, \gamma_2, \gamma_3 \mid 1 = \gamma_3\gamma_2\gamma_1 \rangle$, and Poincaré's theorem implies that it is a discrete group of isometries. We fix the isomorphism of $\mathbb{Z} * \mathbb{Z}$ onto this group that maps the natural generators to γ_1, γ_2 . Equation (3) shows that we constructed a family

$$\phi: \mathcal{B}_0 \longrightarrow \text{hom}(\mathbb{Z} * \mathbb{Z}, \text{isom}(D))$$

that is smooth even on the boundary of \mathcal{B}_0 .

Each of the generators $\gamma_1, \gamma_2, \gamma_3$ corresponds to a cylinder embedded in the quotient $Y_0(\ell) := \phi(\ell) \setminus D$. General surfaces will be defined by a gluing procedure along the cylinders, and we need to choose suitable coordinates. For this purpose, the following models of the hyperbolic plane are used. The Riemannian metrics are special cases of those employed by Judge [14] in his study of the spectrum of degenerating manifolds of more general geometry.

1.2 Definition. For each positive real number t , let X_t denote the manifold \mathbb{R}^2 endowed with the Riemannian metric

$$(x, a) \longmapsto (t^2 + a^2)dx^2 + (t^2 + a^2)^{-1}da^2.$$

For $t = 0$ we consider the disconnected manifold

$$X_t = X_0 = \{(x, a) \in \mathbb{R}^2 \mid a \neq 0\}, \quad (x, a) \longmapsto a^2 dx^2 + a^{-2} da^2.$$

We shall also use the notation $X_t^\pm = \{(x, a) \in X_t \mid \pm a > 0\}$, and more generally, for arbitrary subsets $A \subset \mathbb{R}$, the set $\{(x, a) \in X_t \mid a \in A\}$ is denoted by X_t^A .

The plane X_t , if $t \neq 0$, is mapped isometrically onto the upper half-plane model for the hyperbolic plane by

$$(x, a) \longmapsto e^{tx} \left(\frac{a}{\sqrt{t^2 + a^2}}, \frac{t}{\sqrt{t^2 + a^2}} \right),$$

and so is X_0^\pm by $(x, a) \mapsto (x, \pm a^{-1})$. The map

$$\gamma: X_t \longrightarrow X_t, \quad (x, a) \longmapsto (x + 1, a)$$

corresponds to a hyperbolic isometry of translation t in the first case and to a parabolic one in the second. So the quotient $Z_t := \langle \gamma \rangle \backslash X_t$ is either an elementary hyperbolic cylinder or the disjoint union of two cusps.

It is convenient to denote subsets of Z_t by Z_t^\pm and Z_t^A as in definition 1.2. The following lemma is essentially the collar lemma from hyperbolic geometry. The definition of $A(t)$ therein is only a preliminary, cf. (4).

1.3 Lemma. *Let $\ell = (\ell_1, \ell_2, \ell_3) \in \mathcal{B}_0$, and for all $t \geq 0$ we put*

$$A(t) = \begin{cases} \left(-\frac{t}{2 \sinh(t/2)}, \infty\right) & \text{if } t > 0, \\ (-1, 0) & \text{if } t = 0. \end{cases}$$

Then the construction of lemma 1.1 gives rise to canonical embeddings of cylinders

$$\Psi_i: Z_{\ell_i}^{A(\ell_i)} \longrightarrow Y_0(\ell)$$

such that the images $C_i(\ell) := \Psi_i(Z_{\ell_i}^{A(\ell_i)})$ are mutually disjoint.

Proof. Any pair (S_1, S_2) of disjoint geodesics with positive distance $t/2$ determines an orientation-preserving isometry of X_t onto D . It maps $\{(x, a) \mid x = 0\}$ to S_1 and $\{(x, a) \mid x = 1/2\}$ to S_2 . In case the geodesics are disjoint with distance $t/2 = 0$, it may be necessary to interchange S_1 and S_2 , but then there is a unique isometry $X_0^- \rightarrow D$ with the same property. We apply this to the geodesics that are associated with (ℓ_1, ℓ_2, ℓ_3) by Lemma 1.1: Each pair (T_{i+1}, T_{i+2}) of geodesics gives rise to $\tilde{\Psi}_i: X_{\ell_i} \rightarrow D$ (or $\tilde{\Psi}_i: X_0^- \rightarrow D$, respectively). The set $\tilde{\Psi}_i(X_{\ell_i}^{A(\ell_i)}) \subset D$ is precisely invariant under the action of $\langle \gamma_i \rangle \subset \mathbb{Z} * \mathbb{Z}$ (this is one way of proving the collar lemma). Then the $\tilde{\Psi}_i$ induce maps Ψ_i between the quotients as proposed. \square

So we defined a connected hyperbolic surface $Y_0(\ell)$ for each $\ell \in \mathcal{B}_0$ with distinguished subsets C_1, C_2, C_3 . Each C_i is diffeomorphic to a cylinder $Z_{\ell_i}^{A(\ell_i)}$, and a point $p \in C_i$ is given, via the map Ψ_i , by coordinates $p = (x, a)$ where $x \in \mathbb{R}$ is well-defined modulo \mathbb{Z} . The image of $Z_{\ell_i}^+ \cap Z_{\ell_i}^{A(\ell_i)}$ under Ψ_i is

$$C_i^+ = \{(x, a) \in C_i \mid a > 0\},$$

and the interior $P(\ell)$ of the set $Y_0(\ell) \setminus \bigcup_{i=1}^3 C_i^+$ is an open pair of pants in the customary sense of this word. Note that C_i^+ is empty by definition if $\ell_i = 0$.

1.2 Assembling a surface from pairs of pants

We describe the gluing procedure that composes a geometrically finite surface from pairs of pants. Our presentation follows that of Kra's article on horocyclic coordinates [15], with focus on Riemannian geometry rather than complex structure. It incorporates a choice of local coordinates for embedded cylinders, which are modelled after lemma 1.3.

Starting point is an *admissible graph* G of type (p, n) . Basically, it determines the topological structure of the surface to be constructed, which will be of genus p with n deleted discs in the non-degenerate case. The graph consists of a set G_0 of vertices, a set \tilde{G}_1 of oriented edges, an involution ι of \tilde{G}_1 without fixed elements, and a map $s: \tilde{G}_1 \rightarrow G_0$ that associates the source to every edge. The set of unoriented

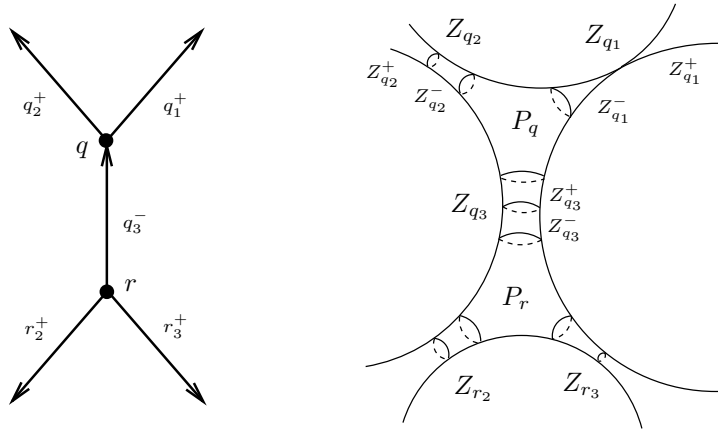


Figure 2: A surface for the graph of type $(0,4)$, $\ell(q_1) = 0$.

edges \tilde{G}_1/ι is denoted by G_1 . The numbers $p, n \in \mathbb{N}_0$ satisfy $2p - 2 + n > 0$ and $3p - 3 + n \geq 0$ by definition. The number of vertices is $2p - 2 + n$, the number of unoriented edges is $3p - 3 + n$, and the preimage of any $q \in G_0$ under s consists of three edges at most.

It is convenient to introduce additional edges that do not belong to the domain of definition of the involution ι . They correspond to infinite funnels, i.e. cylinders with only one end attached to a pair of pants.

1.4 Definition. An *augmented admissible graph* G^* consists of a set G_0 of vertices, a set \tilde{G}_1^* of oriented edges, a source map $s : \tilde{G}_1^* \rightarrow G_0$, and a distinguished subset $\tilde{G}_1 \subset \tilde{G}_1^*$ with an involution ι of \tilde{G}_1 such that

- the tuple $(\tilde{G}_1, G_0, s|_{\tilde{G}_1}, \iota)$ is an admissible graph,
- for each vertex $q \in G_0$, the preimage $s^{-1}(q) \subset \tilde{G}_1^*$ consists of exactly three elements.

The set G_1^* of *unoriented edges* is the quotient of \tilde{G}_1^* under the equivalence relation generated by $e \sim \iota(e)$, and the elements of $\tilde{G}_1^* \setminus \tilde{G}_1$ are called *phantom edges*.

If G^* is an augmented admissible graph, we may choose a map

$$G_0 \longrightarrow \tilde{G}_1^* \times \tilde{G}_1^* \times \tilde{G}_1^*, \quad q \longmapsto (q_1, q_2, q_3),$$

that assigns all three adjacent edges to each vertex. Now a surface $Y_G(\lambda)$ with a hyperbolic metric is determined by a labelling λ of the edges of G^* with length and twist parameters as follows. Let

$$\lambda: G_1^* \longrightarrow [0, \infty) \times \mathbb{R}, \quad d \longmapsto (\ell(d), \tau(d))$$

be an arbitrary map. The lift of λ to \tilde{G}_1^* is also denoted by λ . We consider a pair of pants

$$P_q := \{q\} \times P(\ell(q_1), \ell(q_2), \ell(q_3))$$

for each vertex $q \in G_0$ as defined at the end of section 1.1, and an infinite elementary half-cylinder or a cusp

$$Z_e := \{e\} \times \left(Z_{\ell(e)}^{A(\ell(e))} \cup Z_{\ell(e)}^+ \right)$$

for each *phantom* edge $e \in \tilde{G}_1^* \setminus \tilde{G}_1$. These half-cylinders will be attached to ends of the P_q , while additional collars will serve as connectors between the remaining ends of pairs of pants. To define these connectors, we alter the definition of the interval $A(t)$ from previous subsection into a more symmetric one,

$$A(t) := \begin{cases} \left(-\frac{t}{2 \sinh(t/2)}, \frac{t}{2 \sinh(t/2)}\right) & \text{if } t \neq 0, \\ (-1, 1) & \text{if } t = 0. \end{cases} \quad (4)$$

This allows for the definition of a collar Z_e , for each *proper* edge $e \in \tilde{G}_1$, by

$$Z_e := \{e\} \times Z_{\ell(e)}^{A(\ell(e))},$$

which is indeed a collar around a closed geodesic if $\ell(e) \neq 0$.

Now we define $Y_G(\lambda)$ by an equivalence relation, where the coordinates we use on Z_e and on $C_i^- \subset P_q$ are the canonical ones induced by definition 1.2 and lemma 1.3.

1.5 Definition. Let \mathcal{B} be the set of maps $G_1^* \rightarrow [0, \infty) \times \mathbb{R}$. For each $\lambda = (\ell, \tau) \in \mathcal{B}$, let $Y_G(\lambda)$ be the surface defined by

$$Y_G(\lambda) = \left(\bigcup_{q \in G_0} P_q \cup \bigcup_{e \in \tilde{G}_1^*} Z_e \right) / \sim, \quad (5)$$

with the equivalence relation being generated as follows:

$$\begin{aligned} (x, a) \sim (x - \tau(q_i)/2, a) & \quad \text{where } (x, a) \in C_i^- \subset P_q, \\ & \quad (x - \tau(q_i)/2, a) \in Z_{q_i}^- \subset Z_{q_i}; \\ (x, a) \sim (-x, -a) & \quad \text{where } (x, a) \in Z_e, (-x, -a) \in Z_{\iota(e)}. \end{aligned}$$

The first relation glues a cylinder to each end C_i^- of a pair of pants P_q , and the second identifies two such cylinders $Z_e, Z_{\iota(e)}$ according to the structure of the graph. Recall that the involution ι is defined on the *proper* edges in \tilde{G}_1 only, so the condition for the second generator is never satisfied if e is a phantom edge.

From now on, the notation P_q and Z_e will refer to the respective images in the quotient $Y_G(\lambda)$. Then $Z_e = Z_{\iota(e)}$ for each proper edge e , and we may speak of a subset $Z_d \subset Y_G(\lambda)$ for each unoriented edge $d \in G_1^*$. Note the following: The choice of an orientation $e \in \tilde{G}_1^*$ of d endows the cylinder Z_d with a canonical quotient structure $\langle \gamma \rangle \backslash X_{\ell(e)}^A$. Replacing e with $\iota(e)$ corresponds to the coordinate change $(x, a) \mapsto (-x, -a)$. In particular, we may use oriented edges to specify half-cylinders $Z_e^\pm \subset Z_d$, and $Z_e^\pm = Z_{\iota(e)}^\mp$ holds. This will be of importance in the definition of approximate Eisenstein series in section 3.1.

The P_q and Z_d provide an open cover of $Y_G(\lambda)$ as illustrated in figure 2. If the admissible graph is of type $(0, 3)$, then the extended pairs of pants $Y_0(\ell)$, as defined in the previous paragraph, occur as the non-elementary component of $Y_G(\lambda)$, and $Y_G(\lambda)$ is not connected if a length $\ell(d)$ vanishes. In this instance, the edge d gives rise to an isolated cusp $Z_d^+ \subset Y_G$. The reader is advised to have a look at Kra's text [15] for a discussion of several graph types.

The quotient $Y_G(\lambda)$ inherits a hyperbolic metric. Obviously, there exists an isometry between the emerging surfaces if a twist $\tau(d)$ is replaced with $\tau(d) + 1$ (or with $\tau(d) + \delta$ for arbitrary δ if $d \in G_1^* \setminus G_1$).

1.3 Choice of a trivialisation

In section 2 we will need maps between surfaces $Y_G(\lambda)$ for different values of λ , and these will be constructed here. In definition 1.2, the identity mapping on \mathbb{R}^2 descends to diffeomorphisms between the surfaces $Z_t^+ \cup Z_t^-$ as t varies in $[0, \infty)$. We want our maps to coincide with these canonical maps on the cylinders embedded in $Y_G(\lambda)$. What we do is to define such maps for each pair of pants P_q (and possibly its adjacent infinite half-cylinders) separately. They do not necessarily extend to a diffeomorphism near a closed geodesic where two pairs of pants meet. But this is sufficient for our needs as it yields operators on L^2 -spaces that are strongly continuous with respect to the geometry.

So we start with choosing suitable diffeomorphisms for the extended pairs of pants $Y_0(\ell)$ of subsection 1.1. Recall that we have $\mathcal{B}_0 = [0, \infty)^3$, and ϕ maps \mathcal{B}_0 to the monomorphisms of $\mathbb{Z} * \mathbb{Z}$ into the isometries of the hyperbolic plane D . There is a fibre space in analogy with the Bers fibre space in Teichmüller theory:

1.6 Definition. Let $\tilde{\mathcal{E}}_0 := \mathcal{B}_0 \times D$ and $\tilde{\rho}: \tilde{\mathcal{E}}_0 \rightarrow \mathcal{B}_0$ be the canonical map. Then we define $\mathcal{E}_0 := (\mathbb{Z} * \mathbb{Z}) \backslash \tilde{\mathcal{E}}_0$ and $\rho: \mathcal{E}_0 \rightarrow \mathcal{B}_0$ to be the map induced by $\tilde{\rho}$, where $\mathbb{Z} * \mathbb{Z}$ acts on $\tilde{\mathcal{E}}_0$ by $\gamma(\ell, z) = (\ell, \phi(\ell)(\gamma)z)$.

The $\mathbb{Z} * \mathbb{Z}$ -action on $\tilde{\mathcal{E}}_0$ is smooth, and we refer to lemma A.1 for a proof that it is freely discontinuous. So \mathcal{E}_0 is a smooth manifold. The group acts on each fibre of $\tilde{\rho}$, so ρ is well-defined, and $\rho^{-1}(\ell) \cong Y_0(\ell)$.

The aim is to equip ρ locally with a trivialisation that exhibits a specific behaviour on the infinite ends of the fibres. This behaviour is modelled on the canonical map induced by the identity on $Z_t^+ \cup Z_t^-$ by means of lemma 1.3:

Let us fix $j \in \{1, 2, 3\}$ for a moment. Differentiation of each map Ψ_i with respect to ℓ_j defines a smooth vector field s_i^j on the open subset

$$\bigcup_{\ell} \{\ell\} \times C_i(\ell) \subset \mathcal{E}_0$$

such that $s_i^j - \partial_{\ell_j}$ is tangent to the fibre of ρ . Then discontinuity of the $\mathbb{Z} * \mathbb{Z}$ -action on $\tilde{\mathcal{E}}_0$ implies the existence of a smooth vector field s^j on \mathcal{E}_0 such that $s^j - \partial_{\ell_j}$ is again vertical with respect to ρ , the restriction of s^j to each subset $\{\ell\} \times C_i(\ell)$ being equal to s_i^j .

If we agree on the order, in which the integral flows of s^1 , s^2 and s^3 are to be applied, they can be used to “trivialise” ρ . With reference to appendix A for the details, we only state the outcome in the proposition below. Here $P(\ell) \subset \rho^{-1}(\ell)$ again denotes the embedded pair of pants.

1.7 Proposition. *For each $\ell \in \mathcal{B}_0$ there exists a neighbourhood \mathcal{U} of ℓ and a smooth map*

$$\Psi: \mathcal{U} \times P(\ell) \longrightarrow \mathcal{E}_0 \tag{6}$$

with the following properties:

- For all $\ell' \in \mathcal{U}$, the set $P(\ell') \subset \rho^{-1}(\ell')$ is the diffeomorphic image of $\{\ell'\} \times P(\ell)$.
- Via the identification of $C_i(\ell')$ with $Z_{\ell'_i}^{A(\ell'_i)}$ in lemma 1.3, the map

$$\Psi(\ell', \cdot): C_i(\ell) \longrightarrow P(\ell')$$

corresponds to the canonical identification of $Z_{\ell_i}^I$ with $Z_{\ell'_i}^I$, where I is the intersection of $A(\ell_i)$ with $A(\ell'_i)$. In particular, this restriction to $Z_{\ell_i}^I$ is area-preserving, since the volume form on $X_{\ell'}$ is the Euclidean one for all ℓ' .

- If the degeneracies of $\rho^{-1}(\ell')$ coincide with those of $\rho^{-1}(\ell)$, that is to say, if $\ell'_i = 0$ implies $\ell_i = 0$ and vice versa, then the integral flows extend this map to a diffeomorphism $\Psi(\ell'): \rho^{-1}(\ell) \rightarrow \rho^{-1}(\ell')$.

Now let G be an arbitrary admissible graph. To compare $Y_G(\lambda')$ with $Y_G(\lambda)$, we apply the trivialising maps from proposition 1.7 to each P_q with its adjacent cylinders $Z_{q_i}^-$ separately. Thus, if $U(\lambda) \subset Y_G(\lambda)$ is the complement of all closed geodesics that are associated with elements of G_1^* , there is a distinguished map $U(\lambda) \rightarrow Y_G(\lambda')$. This map is a diffeomorphism onto the corresponding set $U(\lambda')$. The pull-back metric has a unique extension to $Y_G(\lambda)$ if there are no additional degeneracies in $Y_G(\lambda')$, and these metrics match up to form a continuous family with respect to λ' .

1.4 The Selberg Zeta function in its domain of convergence

The result proved here is a prerequisite for section 3.2. It only requires the construction from the first two sections, so we include this assertion here. It makes use of a standard procedure to estimate the number of prime closed geodesics in a hyperbolic surface that also proves convergence of the infinite product in the definition of the Zeta function. The argument is essentially due to Hejhal [10], but we skip the notion of regular b-groups and consider arbitrary surfaces of finite geometry.

Let G be an admissible graph of type (p, n) and G^* its augmentation. The graph will remain fixed. Let \mathcal{Z}_λ denote the Selberg Zeta function of the surface $Y(\lambda) = Y_G(\lambda)$, which is the product of the Zeta functions of all connected components. The contribution of an elementary cusp to \mathcal{Z}_λ is trivial by definition.

For each $d \in G_1^*$ and $\lambda = (\ell, \tau) \in \mathcal{B}$ with $\ell(d) \neq 0$, an entire function $\mathcal{Z}_{d,\lambda}$ is defined by the infinite product

$$\mathcal{Z}_{d,\lambda}: s \mapsto \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell(d)}\right)^2. \quad (7)$$

This product is absolutely convergent. We extend this definition to $\ell(d) = 0$ by putting $\mathcal{Z}_{d,\lambda} = 1$ in that case. The purpose of this section is to prove the following assertion.

1.8 Proposition. *With respect to the topology of locally uniform convergence of analytic functions on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$, the map $\lambda \mapsto \mathcal{Z}_\lambda / \prod_{d \in G_1^*} \mathcal{Z}_{d,\lambda}$ is continuous on \mathcal{B} .*

The basic idea is the following elementary estimate: If Y is a compact, connected hyperbolic surface, let $N(r)$ be the number of closed geodesics on Y of length at most r . Then there exists $C > 0$ such that $N(r) \leq Ce^r$ holds for all $r > 0$. The constant C is given explicitly in terms of area and diameter of a fundamental domain for a uniformising group.

In our situation we must not assume that a family of Fuchsian groups has fundamental domains of uniformly bounded diameter as cusps emerge. But each closed geodesic that is not entirely contained in a cylinder must meet a certain fixed subset of the surface, and it is sufficient that this subset has uniformly bounded diameter. To make this precise, we choose uniformising groups and fundamental domains for all $Y_G(\lambda)$.

As $Y_G(\lambda)$ may have several non-elementary components, it is convenient to have one group $\Gamma_q(\lambda)$ at hand, for each vertex $q \in G_0$, that is a uniformizing group for the component that contain a pair of pants P_q . So let $q \in G_0$ be a vertex with the

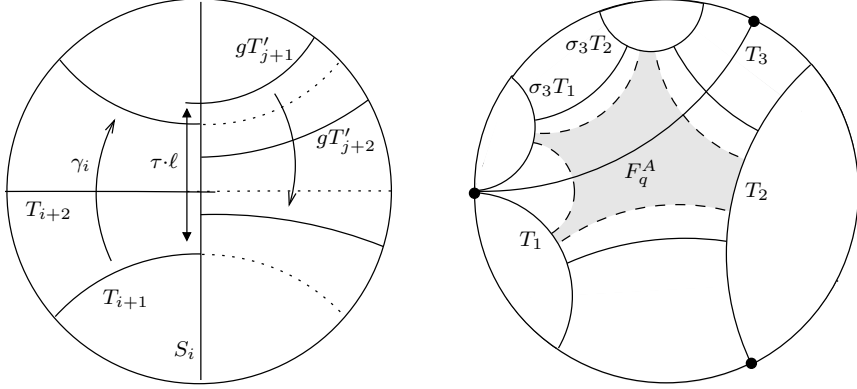


Figure 3: Amalgamated free product/HNN-extension and the set F_q^A .

adjacent edges $q_1, q_2, q_3 \in \tilde{G}_1^*$. For each $\lambda = (\ell, \tau) \in \mathcal{B}$ we defined in 1.1 geodesics $T_1, T_2, T_3 \subset D$ such that their distances satisfy

$$d(T_{i+1}, T_{i+2}) = \ell(q_i)/2,$$

and a Fuchsian group $\Delta_q = \phi(\ell(q_1), \ell(q_2), \ell(q_3))$ with generators $\gamma_1, \gamma_2, \gamma_3$. If $\ell(q_i) \neq 0$ and $\iota(q_i) = q'_j$ for some $q' \in G_0$, we use the combination theorems to form a new Fuchsian group [18, ch. 7]. This group is either an amalgamated free product of Δ_q and $\Delta_{q'}$, where γ_i is identified with $\gamma_j'^{-1}$ if $q \neq q'$, or an HNN-extension of Δ_q if $q = q'$. An iteration of this procedure will yield the group $\Gamma_q(\lambda)$. For the definition consider

$$X_G := \coprod_{q \in G_0} \{q\} \times D.$$

We define an equivalence relation on X_G such that the quotient is canonically isometric to $Y_G(\lambda)$. The equivalence relation has the following generators:

1. $(q, z) \sim (q', z')$ if $q = q'$ and $z' = \delta z$ for some $\delta \in \Delta_q$.
2. Let $e \in \tilde{G}_1$ with $\ell(e) \neq 0$. Let $q = s(e)$, so $e = q_i$ for some $i \in \mathbb{Z}/3\mathbb{Z}$ in the notation of section 1.2. The common orthogonal of T_{i+1}, T_{i+2} is denoted by S_i . Then S_i is the axis of $\gamma_i \in \Delta_q$. We have $\iota(e) = q'_j$ for some $q' \in G_0$, and if S'_j is the corresponding axis of $\gamma'_j \in \Delta_{q'}$, there exists a unique isometry g of the hyperbolic plane such that (see figure 3)

- $g(S'_j) = S_i$,
- $g^{-1}\gamma'_j g = \gamma_i^{-1}$,
- $\gamma_i^{\tau(d)}$ maps T_{i+1} to $g(T'_{j+1})$.

Then we put $(q, gz) \sim (q', z)$.

This gives $Y_G(\lambda) \cong X_G/\sim$ and

$$\Gamma_q(\lambda) = \{\gamma \in \text{isom}(D) \mid (q, z) \sim (q, \gamma z) \text{ for all } z \in D\}.$$

Now the crucial property is that we may choose a fundamental polygon F_q in $\{q\} \times D$ for the action of Δ_q on its Nielsen domain. Then the union of these polygons is a fundamental domain for the equivalence relation on X_G up to a set of measure zero. In particular, let F_q be the set bounded by $T_1, T_2, \sigma_3 T_1, \sigma_3 T_2$ and the common

orthogonal of T_1, T_2 and $\sigma_3 T_1, \sigma_3 T_2$ as on page 7. If $\mathcal{U} \subset \mathcal{B}$ is relatively compact, there exists $A > 0$ such that the cylinders $Z_d^A \subset Y_G(\lambda)$ of area $2A$ exist for all $d \in G_1^*$ and all $\lambda \in \mathcal{U}$. Then the subset $F_q^A \subset F_q$ that is mapped onto the complement of $\bigcup_d Z_d^A$ in $Y_G(\lambda)$ has uniformly bounded diameter for $\lambda \in \mathcal{U}$. (Note the cusp on the left of F_q^A in figure 3. It belongs to F_q , so F_q is of infinite diameter.) This observation allows to give a uniform estimate on the number of closed geodesics in $Y_G(\lambda)$.

1.9 Lemma. *Let $N(\lambda; r)$ denote the number of unoriented, primitive, closed geodesics in $Y_G(\lambda)$ of length at most r . Then there exist a neighbourhood $\mathcal{U} \subset \mathcal{B}$ for each $\lambda_0 \in \mathcal{B}$ and $C > 0$ such that $N(\lambda; r) \leq C \cdot e^r$ holds for all $\lambda \in \mathcal{U}$ and all $r > 0$.*

Proof. Fix \mathcal{U} and $A > 0$ as above. Let $\lambda \in \mathcal{U}$. If $c \subset Y_G(\lambda)$ is a closed geodesic that is not the central geodesic in one of the cylinders Z_d , then its intersection with $Y_G(\lambda) \setminus \bigcup_d Z_d^A$ is nonempty. So there exists $q \in G_0$ and a geodesic in $\{q\} \times D$ that is mapped onto c and has nonempty intersection with the set F_q^A . Moreover, this geodesic is the axis of some hyperbolic element in $\Gamma_q(\lambda)$ with translation length equal to the length of c . So $N(\lambda; r) - (3p - 3 + 2n)$ is bounded from above by the number of hyperbolic isometries γ such that $\gamma \in \Gamma_q(\lambda)$ for some q and $d(F_q^A, \gamma F_q^A) \leq r$ holds. $3p - 3 + 2n$ is the number of edges. As explained above, there is an upper bound D for the diameter of F_q^A , so

$$N(\lambda; r) - (3p - 3 + 2n) \leq \#\{\gamma \mid \gamma F_q^A \subset B_{r+D}(F_q^A) \text{ for some } q\}.$$

If $v > 0$ is a uniform lower bound for the area of F_q^A for all q and all $\lambda \in \mathcal{U}$, this implies

$$N(\lambda; r) - (3p - 3 + 2n) \leq (2p - 2 + n) \cdot \max\{k \in \mathbb{N} \mid kv \leq \text{vol } B_{r+2D}\},$$

where B_r denotes a hyperbolic ball of radius r , and $2p - 2 + n$ is the number of vertices. This shows

$$\begin{aligned} N(\lambda; r) &\leq 3p - 3 + 2n + (2p - 2 + n) \cdot \max\{k \mid kv \leq 4\pi \sinh^2(D + r/2)\} \\ &\leq 3p - 3 + 2n + 4\pi(2p - 2 + n) \frac{\sinh^2(D + r/2)}{v} \\ &\leq 3p - 3 + 2n + \frac{\pi(2p - 2 + n)}{v} e^{2D} \cdot e^r. \end{aligned}$$

□

Now we prove proposition 1.8. Let $\lambda_0 = \{\ell_0, \tau_0\} \in \mathcal{B}$. Let \mathcal{U} be a relatively compact neighbourhood of λ_0 such that $\ell(d) = 0$ implies $\ell_0(d) = 0$ for all $(\ell, \tau) \in \mathcal{U}$. There are two kinds of geodesics that contribute to the quotient (7) for $\lambda \in \mathcal{U}$:

1. Those that cross a cylinder $Z_d \subset Y_G(\lambda)$ with $\ell_0(d) = 0$. As λ converges to λ_0 , their length goes to infinity, so the contribution of each of these to the zeta function converges to 1.
2. Those that do not cross such a cylinder. They are given by conjugacy classes of isometries in $\Gamma_q(\lambda)$ that converge to elements of $\Gamma_q(\lambda_0)$, and each hyperbolic isometry in $\Gamma_q(\lambda_0)$ arises as such a limit.

So if the Zeta function is replaced with a finite product over all geodesics with length less than a fixed constant, the resulting quotient in (7) is continuous at λ_0 . Lemma 1.9 implies a uniform estimate in \mathcal{U} for the remainder.

2 The resolvent for geometrically finite surfaces

In this section, the resolvent of the Laplacian on functions defined on $Y_G(\lambda)$ is examined for varying λ . In particular, we address its behaviour as $\lambda \in \mathcal{B}$ approaches the boundary. The main assertion, theorem 2.14, states that the resolvent is continuous in λ if the spectral parameter does not belong to $[1/4, \infty)$.

The proof uses meromorphic Fredholm theory to compose the resolvent from those of auxiliary surfaces, where the latter are rather easy to describe. So the first subsection is concerned with the auxiliary surfaces, and its results are combined in the second subsection to examine general surfaces of finite geometry.

This technique of applying meromorphic Fredholm theory has often been applied to prove the existence of a continuation of the resolvent across the essential spectrum of the Laplacian. More precisely, the map $s \mapsto (\Delta - s(1 - s))^{-1}$, defined for all $s \in \mathbb{C}$ such that $s(1 - s)$ belongs to the resolvent set and $\operatorname{Re}(s) > 1/2$, has a meromorphic continuation to \mathbb{C} as a family in, say, $B(L^2_c, H^2_{\text{loc}})$. We will make use of the continuation in section 3.1. Yet the continuity results of this section only address the *physical domain* $\operatorname{Re}(s) > 1/2$, and similar results fail to hold for the continued resolvent even in the case of elementary surfaces. We refer to Guillopé's text [7] for further details on the continuation, our description of the resolvent closely follows his presentation.

2.1 Auxiliary surfaces

We provide all statements on the resolvent operators for model surfaces that are needed to deduce similar results for a geometrically finite surface. The models are either compact or quotients of the hyperbolic plane by an elementary group of isometries. In the compact case, Hilbert-Schmidt and trace class properties of the resolvent are immediate, and so is its continuity with respect to an arbitrary family of Riemannian metrics. We will give the arguments below. In the case of elementary quotients of the hyperbolic plane, continuous dependency on the metric refers to the identification of a pair of cusps with the complement of a closed geodesic in a hyperbolic cylinder of arbitrary diameter, as it is induced by the identity map of \mathbb{R}^2 via definition 1.2. Note that this identification preserves the Riemannian volume, so it gives an isometry of L^2 -spaces. Continuity of the resolvent as a bounded map into (local) Sobolev spaces is then obtained from inspection of its integral kernel (proposition 2.6). The major part of this section is aimed at a trace class property for a truncated resolvent and, in particular, uniform boundedness of its trace norm for cylinders of small diameter (proposition 2.12).

2.1.1 Compact surfaces

Let (Y, g) be a compact Riemannian manifold of dimension 2. The spectrum of the Laplacian Δ is purely discrete, and there exists a complete set of orthonormal eigenfunctions for Δ in $L^2(Y, \operatorname{vol}_g)$. Weyl's asymptotic law states that the spectral counting function $N(\lambda)$, i.e. the number of eigenvalues below λ according to their multiplicities, is asymptotic to $\lambda \operatorname{vol}(Y)/4\pi$ as λ increases. This implies that the resolvent $(\Delta + 1)^{-1}$ is of Hilbert-Schmidt class, and $(\Delta + 1)^{-2}$ is of trace class.

We use the notion of trace class mappings between different Hilbert spaces to reformulate this observation. A bounded linear map $T: E \rightarrow F$ of separable Hilbert spaces is of trace class if the supremum over all sums

$$\sum_{i=1}^{\infty} |\langle Te_i, f_i \rangle|$$

is finite, where $(e_i)_{i \in \mathbb{N}}$ runs through the complete orthonormal systems of E , and (f_i) through those of F . The supremum is denoted by $\|T\|_1$ or $\|T\|_{B_1(E,F)}$. The vector space $B_1(E, F)$ of trace class operators is complete with respect to this norm. If $T \in B_1(E, E)$, the trace of T is defined by the series

$$\operatorname{tr} T := \sum_i \langle T e_i, e_i \rangle,$$

which is independent of the chosen orthonormal system.

There is a related concept of Hilbert-Schmidt operators, we refer to Hörmander [11, pp. 185-193] for the particulars.

In the present situation, the Hilbert spaces we are interested in are Sobolev spaces of functions on Y . As a vector subspace of $L^2(Y, \operatorname{vol}_g)$, for any $k \geq 0$, the Sobolev space $H^k(Y, g)$ is the image of $(\Delta + 1)^{-k/2}$. A Hilbert space structure is defined by the requirement that

$$(\Delta + 1)^{-k/2}: L^2(Y, \operatorname{vol}_g) \longrightarrow H^k(Y, g)$$

be isometric. Then $(\Delta + 1)^{-2}$, if considered as a bounded operator on $L^2(Y, \operatorname{vol}_g)$, is equal to the composition of bounded maps

$$L^2(Y, \operatorname{vol}_g) \xrightarrow{(\Delta+1)^{-2}} H^4(Y, g) \xrightarrow{\iota} L^2(Y, \operatorname{vol}_g).$$

Since the first of these is isometric, trace class property for the square of the resolvent on $L^2(Y, \operatorname{vol}_g)$ means that the inclusion ι is of trace class and

$$\|\iota\|_{B_1(H^4, L^2)} = \|(\Delta + 1)^{-2}\|_{B_1(L^2, L^2)}.$$

The following lemma is an immediate consequence.

2.1 Lemma. *Let $A(t): L^2(Y, \operatorname{vol}_g) \rightarrow H^4(Y, g)$ be a continuous family of bounded linear mappings. Then $\iota \circ A(t) \in B_1(L^2(Y, \operatorname{vol}_g), L^2(Y, \operatorname{vol}_g))$ also depends continuously on t .*

Proof. We have

$$\|\iota \circ (A(t) - A(t_0))\|_{B_1(L^2, L^2)} \leq \|\iota\|_{B_1(H^4, L^2)} \cdot \|A(t) - A(t_0)\|.$$

□

2.2 Corollary. *Let Δ_h denote the Laplacian on functions that is associated with a Riemannian metric h on Y . Then*

$$h \longmapsto \operatorname{tr} \left((\Delta_h - \lambda)^{-1} - (\Delta_h - \lambda_0)^{-1} \right)$$

is continuous on the open set of metrics where it is defined.

Proof. Let g be a Riemannian metric on Y . A linear isometry $\delta_h: L^2(Y, \operatorname{vol}_g) \rightarrow L^2(Y, \operatorname{vol}_h)$ is defined by multiplication of functions with the square root of an appropriate Radon-Nikodym derivative. The above trace is equal to the trace of the following operator on $L^2(Y, \operatorname{vol}_g)$:

$$\begin{aligned} \delta_h^{-1} \circ \left((\Delta_h - \lambda)^{-1} - (\Delta_h - \lambda_0)^{-1} \right) \circ \delta_h \\ = (\lambda - \lambda_0) \cdot \delta_h^{-1} \circ (\Delta_h - \lambda)^{-1} (\Delta_h - \lambda_0)^{-1} \circ \delta_h. \end{aligned}$$

The coefficients of the differential operator $(\Delta_h - \lambda)(\Delta_h - \lambda_0)$ depend continuously on h , so it defines a continuous family of bounded maps $H^4(Y, g) \rightarrow L^2(Y, \operatorname{vol}_g)$. Therefore, the inverse maps also depend continuously on h , and the previous lemma is applicable. □

2.1.2 Elementary surfaces

An elementary hyperbolic surface is a quotient $Z = \Gamma \backslash D$ of the hyperbolic plane, where the group Γ of isometries has a cyclic subgroup of finite index. We only consider elementary groups that are cyclic, generated by a hyperbolic or a parabolic isometry of X .

Our aim is to compare the resolvent of the Laplacians for different surfaces. Define

$$X := \{(x, a) \in \mathbb{R}^2 \mid a \neq 0\},$$

and

$$Z := \langle \gamma \rangle \backslash X, \quad \text{where } \gamma: (x, a) \mapsto (x + 1, a).$$

Recall that, for each $\ell \geq 0$, the split plane X may be identified with a subset of the model X_ℓ for the hyperbolic plane (definition 1.2). If we fix the Euclidean volume on X , this gives rise to an isometry of L^2 -spaces. So there is a corresponding family of self-adjoint Laplacians Δ_ℓ acting on $L^2(X)$ and on $L^2(Z)$. In each case, the spectrum of Δ_ℓ is $[1/4, \infty)$. We may thus consider the resolvent operators as an analytic family of bounded operators

$$s \mapsto (\Delta_\ell - s(1 - s))^{-1}, \quad \operatorname{Re}(s) > 1/2,$$

and the half-plane $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$ is a maximal domain where it is defined.

We shall prove the following statement.

2.3 Proposition. 1. *Let ψ be a smooth function of compact support on Z . Then*

$$s \mapsto \psi(\Delta_\ell - s(1 - s))^{-1}$$

depends continuously on $\ell \in [0, \infty)$ in the sense of locally uniform convergence of analytic families of bounded linear maps $L^2(Z) \rightarrow H^2(Z)$. If $\operatorname{Re}(s_0) > 1/2$, the same holds true for

$$\psi\left((\Delta_\ell - s(1 - s))^{-1} - (\Delta_\ell - s_0(1 - s_0))^{-1}\right): L^2(Z) \longrightarrow H^4(Z).$$

2. *Let ψ be bounded and supported in a cylindrical subset $Z^S \subset Z$ of finite area, and let χ be of compact support in Z . Then*

$$\psi\left((\Delta_\ell - s(1 - s))^{-1} - (\Delta_\ell - s_0(1 - s_0))^{-1}\right)\chi$$

is of trace class, and it depends continuously on $\ell \in [0, \infty)$ as an analytic family of trace class operators.

2.4 Remark. The integral kernel of the resolvent $(\Delta_\ell - s(1 - s))^{-1}$ can be continued meromorphically in s to the complex plane. The poles, which are called *resonances*, are

$$\begin{aligned} &\{1/2\} \quad \text{for a cusp } Z_0^+, \\ &\{-n + 2\pi im/\ell \mid n \in \mathbb{N}_0, m \in \mathbb{Z}\} \quad \text{for a cylinder } Z_\ell, \ell \neq 0. \end{aligned}$$

This indicates that one cannot expect a result like proposition 2.3 to hold for the continuation on the left of the critical axis $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = 1/2\}$.

The Sobolev spaces in part 1 of proposition 2.3 don't need to be specified because the support of ψ is compact. In the second part, the support of ψ is not necessarily compact. We note that the methods of section 2.1.1 apply to deduce the second part from the first if the degenerate case $\ell = 0$ is excluded. The proof for small values of ℓ is given in the forthcoming subsection.

2.1.3 Proof of proposition 2.3

A continuity statement. Let K_s^ℓ denote the Schwartz kernel of $(\Delta_\ell - s(1-s))^{-1}$, acting on $L^2(Z)$. It is a smooth function on the complement of the diagonal in $Z \times Z$.

The following is a well-known property of K_s^ℓ .

2.5 Lemma. *Let*

$$k_s(t) := \frac{4^{s-1}}{\pi} \int_0^1 (x(1-x))^{s-1} (4x+t)^{-s}, \quad t > 0,$$

and

$$\sigma_\ell: X \times X \longrightarrow \mathbb{R}_+ \cup \{\infty\}, \quad (z_1, z_2) \longmapsto 2(\cosh(d(z_1, z_2)) - 1),$$

where d denotes hyperbolic distance if X is identified with X_ℓ . Then, for all s with $\operatorname{Re}(s) > 1/2$, the kernel K_s^ℓ is given by the function on $X \times X$

$$(z_1, z_2) \longmapsto \sum_{n \in \mathbb{Z}} k_s \circ \sigma_\ell(z_1, \gamma^n z_2).$$

By this formulation we mean that the series is absolutely convergent on $\{(z_1, z_2) \in X \times X \mid z_1 \neq \gamma^n z_2 \text{ for all } n \in \mathbb{Z}\}$, it is γ -invariant in each argument, and the induced function on $Z \times Z \setminus \{(z, z) \mid z \in Z\}$ coincides with the kernel K_s^ℓ .

Proof. The analogous statement holds for an arbitrary hyperbolic surface if $\operatorname{Re}(s) > 1/2$ holds and if the critical exponent of a uniformising group is less than $\operatorname{Re}(s)$. We refer to Elstrodt [5] for a proof. In the present situation, the critical exponent is $1/2$ if $\ell = 0$, and 0 otherwise. \square

Now we use lemma 2.5 to deduce the first statement of proposition 2.3.

2.6 Proposition. *Let ψ be a smooth function on Z of compact support. Then $(\ell, s) \mapsto \psi(\Delta_\ell - s(1-s))^{-1}$ is a continuous mapping of $[0, \infty) \times \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$ into the normed vector space of bounded linear maps $L^2(Z) \rightarrow H^2(Z)$.*

Proof. First of all, we observe that $k_s \circ \sigma_\ell$ converges to $k_s \circ \sigma_{\ell_0}$ as $\ell \rightarrow \ell_0$, uniformly on every compact subset of $X \times X$. This even holds on the diagonal in the sense that the difference $k_s \circ \sigma_\ell - k_s \circ \sigma_{\ell_0}$ has a continuous extension, which converges locally uniformly to 0. Lemma 2.7 below shows that the same holds true for the kernel K_s^ℓ on the quotient.

Now let B denote a compact neighbourhood of $\operatorname{supp} \psi$. There exists a positive number C such that

$$\|\psi f\|_{H^2} \leq C \left(\|f\|_{L^2(B)} + \|(\Delta_{\ell_0} - s(1-s))f\|_{L^2(B)} \right) \quad (8)$$

for any smooth function f , locally uniform in s . Since the metric on B converges as $\ell \rightarrow \ell_0$, there exist C_ℓ with $C_\ell \rightarrow 0$ such that

$$\|(\Delta_{\ell_0} - \Delta_\ell)f\|_{L^2(B)} \leq C_\ell \left(\|(\Delta_\ell - s(1-s))f\|_{L^2(B)} + \|f\|_{L^2(B)} \right).$$

Therefore, if f is a smooth function of compact support,

$$\begin{aligned}
& \left\| \psi (\Delta_\ell - s(1-s))^{-1} f - \psi (\Delta_{\ell_0} - s(1-s))^{-1} f \right\|_{\mathbb{H}^2} \\
& \leq C \left(\left\| \left((\Delta_\ell - s(1-s))^{-1} - (\Delta_{\ell_0} - s(1-s))^{-1} \right) f \right\|_{\mathbb{L}^2(B)} \right. \\
& \quad \left. + \left\| (\Delta_{\ell_0} - \Delta_\ell) (\Delta_\ell - s(1-s))^{-1} f \right\|_{\mathbb{L}^2(B)} \right) \\
& \leq C \left\| \left((\Delta_\ell - s(1-s))^{-1} - (\Delta_{\ell_0} - s(1-s))^{-1} \right) f \right\|_{\mathbb{L}^2(B)} \\
& \quad + CC_\ell \left(\|f\|_{\mathbb{L}^2(B)} + \left\| (\Delta_\ell - s(1-s))^{-1} f \right\|_{\mathbb{L}^2(B)} \right).
\end{aligned}$$

The last summand can be estimated in terms of the Euclidean distance between $s(1-s)$ and the spectrum of Δ_ℓ .

To estimate the first term, we exhaust Z by subsets $\text{supp } \chi_n$ for compactly supported, smooth functions χ_n . The above-mentioned locally uniform convergence of the kernel K_s^ℓ proves for each n

$$\left\| \left((\Delta_\ell - s(1-s))^{-1} - (\Delta_{\ell_0} - s(1-s))^{-1} \right) \chi_n \right\|_{\mathbb{L}^2(B)} \longrightarrow 0, \quad \ell \rightarrow \ell_0.$$

So we need to find an upper bound for

$$\left\| \left((\Delta_\ell - s(1-s))^{-1} - (\Delta_{\ell_0} - s(1-s))^{-1} \right) (1 - \chi_n) \right\|$$

that decreases with growing n . We may require that the distance of B from the support of $1 - \chi_n$ be greater than n when measured with respect to each of the given metrics for ℓ near ℓ_0 . Now the spectral theorem admits to define the resolvent by

$$(\Delta_\ell - s(1-s))^{-1} = \int_0^\infty \frac{\sin(t\sqrt{\Delta_\ell - 1/4})}{\sqrt{\Delta_\ell - 1/4}} e^{-(s-1/2)t} dt.$$

The operator $(\Delta_\ell - 1/4)^{-1/2} \sin(t\sqrt{\Delta_\ell - 1/4})$ solves the wave equation

$$\left[\frac{\partial^2}{\partial t^2} + (\Delta_\ell - 1/4) \right] \frac{\sin(t\sqrt{\Delta_\ell - 1/4})}{\sqrt{\Delta_\ell - 1/4}} = 0,$$

so it has unit propagation speed. Again by the spectral theorem,

$$\begin{aligned}
& \left\| (\Delta_\ell - s(1-s))^{-1} (1 - \chi_n) f \right\|_{\mathbb{L}^2(B)} \\
& = \left\| \int_n^\infty \frac{\sin(t\sqrt{\Delta_\ell - 1/4})}{\sqrt{\Delta_\ell - 1/4}} e^{-(s-1/2)t} dt (1 - \chi_n) f \right\|_{\mathbb{L}^2(B)} \\
& \leq \frac{n-1}{\text{Re}(s) - 1/2} e^{-(\text{Re}(s)-1/2)n} \|(1 - \chi_n) f\|_{\mathbb{L}^2(X_0)},
\end{aligned}$$

and this vanishes as $n \rightarrow \infty$. □

The proof of proposition 2.6 is completed with the following observation.

2.7 Lemma. *The kernel K_s^ℓ , considered as a smooth function on the complement of the diagonal in $Z \times Z$, depends continuously on $(s, \ell) \in \{s \in \mathbb{C} \mid \text{Re}(s) > 1/2\} \times [0, \infty)$.*

Proof. It is easy to see that a finite sum $(z_1, z_2) \mapsto \sum_{|n| \leq N} k_s(\sigma_\ell(z_1, \gamma^n z_2))$ depends continuously on s and ℓ . So we only need to give a bound for the derivatives of the remainder in the series. We will show below

$$\frac{d^n}{dt^n} k_s(t) = (4+t)^{-s-n} f_n(s, t) \quad (9)$$

with a function f_n that is bounded on any compact subset of $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\} \times (0, \infty]$. Therefore it is sufficient to replace $k_s(t)$ with $(4+t)^{-s}$ and to give a bound in the C^0 -topology. Now

$$\begin{aligned} \sigma_\ell(x_1, a_1, x_2, a_2) = 2 \left[\frac{\cosh(\ell(x_1 - x_2)) - 1}{\ell^2} \sqrt{\ell^2 + a_1^2} \sqrt{\ell^2 + a_2^2} \right. \\ \left. + \frac{1}{\ell^2} \left(\sqrt{\ell^2 + a_1^2} \sqrt{\ell^2 + a_2^2} - a_1 a_2 \right) - 1 \right] \quad (\ell > 0), \end{aligned}$$

and if $(x_1, a_1), (x_2, a_2)$ belong to the same component of X

$$\sigma_0(x_1, a_1, x_2, a_2) = \frac{(a_1 - a_2)^2}{a_1 a_2} + a_1 a_2 (x_1 - x_2)^2.$$

In particular, the inequality

$$\sigma_\ell(x_1, a_1, x_2, a_2) \geq |a_1 a_2| (x_1 - x_2)^2 \quad (10)$$

holds for all $\ell \geq 0$, so

$$\left| \sum_{|n| > N} (4 + \sigma_\ell(x, a_1, x + n, a_2))^{-s} \right| \leq \sum_{|n| > N} (4 + |a_1 a_2| n^2)^{-\operatorname{Re}(s)}.$$

The last expression is of order $N^{1-2\operatorname{Re}(s)}$ as $N \rightarrow \infty$, uniformly if $|a_1 a_2|$ is bounded from below by a positive number.

We still need to verify (9). This follows from

$$k_s(t) = \frac{4^{s-1}}{\pi} \sum_{j=0}^{\infty} \frac{4^j \Gamma(s+j)^2}{j! \Gamma(2s+j)} (4+t)^{-s-j},$$

which is proved by substituting

$$(4x+t)^{-s} = (4+t)^{-s} \left(1 + \frac{4x-4}{4+t} \right)^{-s}$$

in the definition of k_s and expanding the second factor into a power series in $\frac{4x-4}{4+t}$. \square

The proof of proposition 2.6 can easily be modified to prove the first statement in proposition 2.3 concerning the difference

$$\psi\left((\Delta_\ell - s(1-s))^{-1} - (\Delta_\ell - s_0(1-s_0))^{-1}\right).$$

We shall not do this here. Rather, before we proceed with the second part of proposition 2.3, we want to include an observation that will not be needed until the Selberg trace formula is applied in section 3.2. The expression arises as a remainder therein that has to be examined separately.

2.8 Lemma. For all $A > 0$ the following map is continuous:

$$(\ell, s) \mapsto \int_A^\infty \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} k_s(\sigma_\ell(0, a, n, a)) da.$$

Proof. Consider the two estimates

$$\begin{aligned} \int_B^\infty \sum_{n \neq 0} \sigma_\ell(0, a, n, a)^{-\operatorname{Re}(s)} da &\leq \int_B^\infty a^{-2\operatorname{Re}(s)} da \sum_{n \neq 0} |n|^{-2\operatorname{Re}(s)} \\ &= \frac{\zeta(2\operatorname{Re}(s))}{\operatorname{Re}(s) - 1/2} B^{1-2\operatorname{Re}(s)} \end{aligned}$$

and

$$\begin{aligned} \int_A^B \sum_{|n| > N} \sigma_\ell(0, a, n, a)^{-\operatorname{Re}(s)} da &\leq \int_A^B a^{-2\operatorname{Re}(s)} da \sum_{|n| > N} |n|^{-2\operatorname{Re}(s)} \\ &\leq \frac{A^{1-2\operatorname{Re}(s)} - B^{1-2\operatorname{Re}(s)}}{(\operatorname{Re}(s) - 1/2)^2} N^{1-2\operatorname{Re}(s)}. \end{aligned}$$

They show that it is sufficient to replace the series with a finite sum and the domain of integration with a finite interval. Then the statement follows from the continuity of $\ell \mapsto \sigma_\ell$. \square

Trace class estimates. We prove the second part of proposition 2.3. Essentially the proof is split into two parts: Trace class property is immediate if $\ell > 0$, for in this case ψ corresponds to a compactly supported function on X_ℓ . The first point is to show that the operator under consideration is of trace class for $\ell = 0$. Then proposition 2.12 states that the trace norm is uniformly bounded in ℓ if the support of ψ is small, which reduces the proof to the case where ψ is compactly supported, i.e. part 1 of proposition 2.3 applies.

As the elementary cylinder Z is a quotient of $X = \{(x, a) \in \mathbb{R}^2 \mid a \neq 0\}$, the support condition on χ implies that there is a gap between the support and the deleted closed curve in Z . The support of ψ in contrast may include a neighbourhood of this curve, so ψ corresponds to a function of unbounded support on the disjoint union Z_0 of two cusps. Thus our first issue is to prove that in this situation

$$\psi \left((\Delta_0 - s(1-s))^{-1} - (\Delta_0 - s_0(1-s_0))^{-1} \right) \chi$$

is of trace class. Let us explain how the propositions below are applied to prove this fact. We may assume that ψ and χ are supported in the component Z^+ . In this paragraph we will use Z^A synonymously with $Z^{[-A, A]}$ for numbers $A > 0$. The first resolvent formula implies that the operator is the strong limit of

$$(s(1-s) - s_0(1-s_0)) \psi \left((\Delta_0 - s(1-s))^{-1} \kappa_A (\Delta_0 - s_0(1-s_0))^{-1} \right) \chi$$

as $A \rightarrow 0$, where κ_A denotes the characteristic function of $Z^+ \setminus Z^A$. Lemma 2.10 below states that these operators are indeed of trace class if $A > 0$. Then proposition 2.11 implies that, if A converges to 0, the sequence of operators is a Cauchy sequence in the trace class topology. The limit of this sequence coincides with the strong limit, so the latter is of trace class.

The crucial observation in this argument will be a growth estimate for the kernel that implies proposition 2.11 and convergence in the trace class topology. Essentially

the same estimate will be used again, in proposition 2.12, to examine the trace norm of

$$\psi\left((\Delta_\ell - s(1-s))^{-1}\right) - (\Delta_\ell - s_0(1-s_0))^{-1}\chi$$

locally uniformly in ℓ . There it proves that the norm

$$\left\|\psi_\epsilon\left((\Delta_\ell - s(1-s))^{-1} - (\Delta_\ell - s_0(1-s_0))^{-1}\right)\chi\right\|_1$$

becomes uniformly small as $\epsilon \rightarrow 0$, where ψ_ϵ is the characteristic function of Z^ϵ . Then we may apply the first part of proposition 2.3 to conclude the proof of the second.

We proceed with the technical assertions that contribute to the proof. They rely on Hilbert-Schmidt properties for Δ_ℓ , which are proved using the description of the resolvent kernel in lemma 2.5. As it turns out, the logarithmic singularity of the resolvent kernel is irrelevant. The decisive property is its asymptotic

$$k_s(t) = O(t^{-\operatorname{Re}(s)}), \quad t \rightarrow \infty,$$

and therefore the following lemma will be useful. It is purely technical but provides a convenient formula for our estimates.

2.9 Lemma. *Let $\ell \geq 0$ and $r > 1/2$. Consider the function h_ℓ on $X \times X$ defined by*

$$h_\ell(z_1, z_2) = \sum_{m \in \mathbb{Z}} (1 + \sigma_\ell(z_1, \gamma^m z_2))^{-r}.$$

Then

$$\begin{aligned} & \int_0^1 \int_0^1 |h_\ell(x_1, a_1, x_2, a_2)|^2 dx_2 dx_1 \\ & \leq \sqrt{\pi} \frac{\Gamma(2r - 1/2)}{\Gamma(2r)} g_\ell(a_1, a_2, 2r) + \pi \frac{\Gamma(r - 1/2)^2}{\Gamma(r)^2} g_\ell(a_1, a_2, r)^2 \end{aligned} \quad (11)$$

holds, where g_ℓ is defined for all positive ℓ by

$$\begin{aligned} g_\ell(a_1, a_2, r) &= \left[1 + \frac{2}{\ell^2} \left(\sqrt{(\ell^2 + a_1 a_2)^2 + \ell^2 (a_1 - a_2)^2} - (\ell^2 + a_1 a_2) \right) \right]^{1/2-r} \\ & \quad \cdot \left((\ell^2 + a_1 a_2)^2 + \ell^2 (a_1 - a_2)^2 \right)^{-1/4}, \end{aligned}$$

and g_0 by

$$g_0(a_1, a_2, r) = \begin{cases} [a_1 a_2 + (a_1 - a_2)^2]^{1/2-r} (a_1 a_2)^{-1+r} & \text{if } a_1 a_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If the product $|h_\ell(x_1, a_1, x_2, a_2)|^2$ of the two series in (11) is expanded, the integral becomes

$$\begin{aligned} & \sum_{m,n} \int_0^1 \int_0^1 (1 + \sigma_\ell(x_1, a_1, x_2 + m + n, a_2))^{-r} (1 + \sigma_\ell(x_1, a_1, x_2 + m, a_2))^{-r} dx_2 dx_1 \\ & = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} (1 + \sigma_\ell(0, a_1, x + n, a_2))^{-r} (1 + \sigma_\ell(0, a_1, x, a_2))^{-r} dx. \end{aligned} \quad (12)$$

The $n = 0$ contribution is immediately estimated by substituting the definition of σ_ℓ . This gives for $\ell > 0$

$$\begin{aligned}\sigma_\ell(0, a_1, x, a_2) &= 2 \frac{\cosh(\ell x) - 1}{\ell^2} \sqrt{(\ell^2 + a_1 a_2)^2 + \ell^2 (a_1 - a_2)^2} \\ &\quad + \frac{2}{\ell^2} \left(\sqrt{(\ell^2 + a_1 a_2)^2 + \ell^2 (a_1 - a_2)^2} - (\ell^2 + a_1 a_2) \right) \\ &\geq x^2 \sqrt{(\ell^2 + a_1 a_2)^2 + \ell^2 (a_1 - a_2)^2} \\ &\quad + \frac{2}{\ell^2} \left(\sqrt{(\ell^2 + a_1 a_2)^2 + \ell^2 (a_1 - a_2)^2} - (\ell^2 + a_1 a_2) \right),\end{aligned}$$

so the formula

$$\int_{-\infty}^{\infty} (\alpha + x^2)^{-r} dx = \sqrt{\pi} \frac{\Gamma(r - 1/2)}{\Gamma(r)} \alpha^{1/2-r} \quad (13)$$

yields a contribution of

$$\int_{-\infty}^{\infty} (1 + \sigma_\ell(0, a_1, x, a_2))^{-2r} dx \leq \sqrt{\pi} \frac{\Gamma(2r - 1/2)}{\Gamma(2r)} g_\ell(a_1, a_2, 2r),$$

where the function g_ℓ is as proposed. If $\ell = 0$, by definition of σ_0 we obtain from equation (13)

$$\int_{-\infty}^{\infty} (1 + \sigma_0(0, a_1, x, a_2))^{-2r} dx = \sqrt{\pi} \frac{\Gamma(2r - 1/2)}{\Gamma(2r)} g_0(a_1, a_2, 2r).$$

The previous two formulae give the first summand in (11). With respect to the remaining part of the series in equation (12), we observe that the integral decreases as $|n|$ increases, and therefore

$$\begin{aligned}\sum_{n \neq 0} \int_{-\infty}^{\infty} (1 + \sigma_\ell(0, a_1, x + n, a_2))^{-r} (1 + \sigma_\ell(0, a_1, x, a_2))^{-r} dx \\ \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + \sigma_\ell(0, a_1, x + y, a_2))^{-r} (1 + \sigma_\ell(0, a_1, x, a_2))^{-r} dx dy \\ = \left(\int_{-\infty}^{\infty} (1 + \sigma_\ell(0, a_1, x, a_2))^{-r} dx \right)^2.\end{aligned}$$

Another application of (13) completes the proof. \square

We continue with the first trace class property.

2.10 Lemma. *Let $s, s_0 \in \{w \in \mathbb{C} \mid \operatorname{Re}(w) > 1/2\}$. If κ is a bounded function supported off the deleted curve, then*

$$\psi(\Delta_0 - s(1 - s))^{-1} \kappa(\Delta_0 - s_0(1 - s_0))^{-1} \chi$$

is of trace class.

Proof. We show that $\psi(\Delta_0 - s(1 - s))^{-1} \kappa$ is a Hilbert-Schmidt operator, the same argument proves that this also holds true for $\kappa(\Delta_0 - s_0(1 - s_0))^{-1} \chi$. We may further assume that

$$\operatorname{supp} \psi \subset Z^S \cap Z^+, \quad \operatorname{supp} \kappa \subset Z^+ \setminus Z^A$$

for some $A > 0$, so the aim is to see

$$\int_{Z^S \cap Z^+} \int_{Z^+ \setminus Z^A} |K_s^0(z_1, z_2)|^2 \operatorname{vol}(z_1) \operatorname{vol}(z_2) < \infty.$$

According to lemma 2.5, the kernel is given by

$$(z_1, z_2) \mapsto \sum_{m \in \mathbb{Z}} k_s \circ \sigma_0(\tilde{z}_1, \gamma^m \tilde{z}_2),$$

where the function k_s satisfies

$$k_s(t) = O(t^{-\operatorname{Re}(s)}), \quad t \rightarrow \infty, \quad \text{and} \quad k_s(t) \sim -\frac{1}{4\pi} \log t, \quad t \rightarrow 0.$$

Integration of the logarithmic singularity over a two-dimensional domain causes no difficulties, therefore $k_s \circ \sigma_0$ may be replaced with $(1 + \sigma_0)^{-\operatorname{Re}(s)}$. By lemma 2.9 the proof is reduced to

$$\int_0^S \int_A^\infty g_0(a_1, a_2, 2r) da_2 da_1 < \infty \quad \text{and} \quad \int_0^S \int_A^\infty g_0(a_1, a_2, r)^2 da_2 da_1 < \infty.$$

The explicit formula for g_0 shows that the integrands here are of order

$$O(a_1^{-2+2r}), \quad a_1 \rightarrow 0, \quad O(a_2^{-2r}), \quad a_2 \rightarrow \infty;$$

and therefore the integrals do exist. \square

As explained at the beginning, the next statement establishes the analogue of lemma 2.10 if the cut-off function κ is replaced with the constant function 1.

2.11 Proposition. *For each $r \in (0, \operatorname{Re}(s_0) - 1/2)$ there exists $C > 0$ such that*

$$\left\| \psi(\Delta_0 - s(1-s))^{-1}(\kappa_B - \kappa_A)(\Delta_0 - s_0(1-s_0))^{-1} \chi \right\|_1 \leq C \cdot \int_B^A a^{-1+r} da$$

holds for all $A > 0$ sufficiently small and $A > B > 0$. In particular, if $A - B$ converges to 0, then the norm converges to 0 locally uniformly in A .

Proof. We begin with a consideration that is local in A and B . The operators are of trace class by lemma 2.10, and the trace norm can be estimated by a product of Hilbert-Schmidt norms

$$\left\| \psi(\Delta_0 - s(1-s))^{-1}(\kappa_B - \kappa_A) \right\|_2 \cdot \left\| (\kappa_B - \kappa_A)(\Delta_0 - s_0(1-s_0))^{-1} \chi \right\|_2. \quad (14)$$

The first factor is the square root of

$$\int_{Z^A \setminus Z^B} \int_{\operatorname{supp} \psi} |\psi(z_1) K_s^0(z_1, z_2)|^2 \operatorname{vol}(z_1) \operatorname{vol}(z_2).$$

Observe that, for each $B > 0$, the function

$$f: (0, \infty) \rightarrow \mathbb{R}, \quad A \mapsto \int_{Z^A \setminus Z^B} \int_{\operatorname{supp} \psi} |\psi(z_1) K_s^0(z_1, z_2)|^2 \operatorname{vol}(z_1) \operatorname{vol}(z_2)$$

is continuously differentiable. This implies

$$\left\| \psi(\Delta_0 - s(1-s))^{-1}(\kappa_B - \kappa_A) \right\|_2^2 \leq (A - B)(f'(A) + \epsilon)$$

for arbitrarily small positive ϵ if A belongs to a fixed compact subset of $(0, \infty)$, and if B is sufficiently close to A . The same argument is applied to the second Hilbert-Schmidt norm in (14), so

$$\begin{aligned} \left\| \psi(\Delta_0 - s(1-s))^{-1}(\kappa_B - \kappa_A)(\Delta_0 - s_0(1-s_0))^{-1} \chi \right\|_1 \\ \leq (A - B) \sqrt{f'(A) + \epsilon} \sqrt{g'(A) + \epsilon}, \end{aligned}$$

where

$$g(A) = \int_{Z^A \setminus Z^B} \int_{\text{supp } \chi} |K_{s_0}^0(z_1, z_2) \chi(z_2)|^2 \text{vol}(z_1) \text{vol}(z_2).$$

This formula leads to the desired ‘global’ estimate as follows: For fixed numbers A and B , we choose a subdivision $B = C_0 < C_1 < \dots < C_k = A$, and if this subdivision is sufficiently small, we get

$$\begin{aligned} & \left\| \psi(\Delta_0 - s(1-s))^{-1} (\kappa_B - \kappa_A) (\Delta_0 - s_0(1-s_0))^{-1} \chi \right\|_1 \\ & \leq \sum_{j=0}^{k-1} \left\| \psi(\Delta_0 - s(1-s))^{-1} (\kappa_{C_j} - \kappa_{C_{j+1}}) (\Delta_0 - s_0(1-s_0))^{-1} \chi \right\|_1 \\ & \leq \sum_{j=0}^{k-1} (C_{j+1} - C_j) \sqrt{f'(C_{j+1}) + \epsilon} \sqrt{g'(C_{j+1}) + \epsilon}. \end{aligned}$$

The last sum can be considered as a Riemann sum, and by refining the subdivision, we see

$$\left\| \psi(\Delta_0 - s(1-s))^{-1} (\kappa_B - \kappa_A) (\Delta_0 - s_0(1-s_0))^{-1} \chi \right\|_1 \leq \int_A^B \sqrt{f'(a)} \sqrt{g'(a)} da. \quad (15)$$

We finally need to estimate the integrand in (15). The fundamental theorem of calculus gives

$$f'(a) = \int_{\text{supp } \psi} |\psi(z) K_s^0(z, (x, a))|^2 \text{vol}(z),$$

where (x, a) denotes an arbitrary point on the boundary of Z^a . The kernel K_s^0 is given by an infinite summation of k_s . As $a \rightarrow 0$, the logarithmic singularity of k_s can only contribute a term of order $O(a^{-1})$ to this integral, because the Euclidean diameter of hyperbolic balls in X_0 grows linearly as the centre approaches the line $a = 0$. The remaining contribution of k_s is bounded by $(1 + \sigma_0)^{-\text{Re}(s)}$, and lemma 2.9 yields

$$\begin{aligned} f'(a) & \leq O(a^{-1}) + C_1 a^{-1+2\text{Re}(s)} \int_0^S (a_1 a + (a_1 - a)^2)^{1/2-2\text{Re}(s)} a_1^{-1+2\text{Re}(s)} da_1 \\ & \quad + C_2 a^{-2+2\text{Re}(s)} \int_0^S \left((a_1 a + (a_1 - a)^2)^{1/2-\text{Re}(s)} a_1^{-1+\text{Re}(s)} \right)^2 da_1 \end{aligned} \quad (16)$$

The second summand is of order $a^{-1/2}$. In view of the third, we assume $a < \min(1, S)$ and split the integral into two parts to see

$$\begin{aligned} & \int_0^S \left((a_1 a + (a_1 - a)^2)^{1/2-\text{Re}(s)} a_1^{-2+2\text{Re}(s)} \right)^2 da_1 \\ & \leq \int_0^{a^2} (a_1 - a)^{2-4\text{Re}(s)} a_1^{-2+2\text{Re}(s)} da_1 + a^{1-2\text{Re}(s)} \int_{a^2}^S a_1^{-1} da_1 \\ & \leq a^{2-4\text{Re}(s)} |1-a|^2 \int_0^{a^2} a_1^{-2+2\text{Re}(s)} da_1 + a^{1-2\text{Re}(s)} \log(Sa^{-2}) \\ & = O(a^{1-2\text{Re}(s)-2\epsilon}) \end{aligned}$$

for each $\epsilon > 0$, so $f'(a) = O(a^{-1-2\epsilon})$.

The estimate for g' is analogous, but the compact support of χ implies that the corresponding integrands in (16) are actually bounded in this case, so $g'(a) = O(a^{-2+2\text{Re}(s_0)})$. This proves $\sqrt{f'(a)g'(a)} = O(a^{-3/2+\text{Re}(s_0)-\epsilon})$. \square

To complete the second part of proposition 2.3, we need to examine the dependency of the operators on ℓ . In case ψ vanishes in a neighbourhood of the deleted curve, continuity in ℓ as a family of bounded maps $L^2(Z) \rightarrow H^4(\text{supp } \psi)$ was proved in the first part of proposition 2.3. This in turn implies continuity in the trace class topology under this assumption on ψ , so it is sufficient to prove the following.

2.12 Proposition. *For each $\epsilon > 0$ let ψ_ϵ denote the characteristic function of the cylindrical set Z^ϵ of area 2ϵ . Given $\delta > 0$ and an arbitrary compact subset R of $\{s \in \mathbb{C} \mid \text{Re}(s) > 1/2\}$, there exists ϵ such that*

$$\left\| \psi_\epsilon \left((\Delta_\ell - s(1-s))^{-1} - (\Delta_\ell - s_0(1-s_0))^{-1} \right) \chi \right\|_1 \leq \delta$$

holds for sufficiently small ℓ and for all $s, s_0 \in R$.

Proof. The proof is similar to the trace class estimates for $\ell = 0$, i.e. the operator is approximated by variation of the cut-off parameter A in

$$\psi_\epsilon (\Delta_\ell - s(1-s))^{-1} \kappa_A (\Delta_\ell - s_0(1-s_0))^{-1} \chi.$$

The next two equations follow from the explicit formulae in lemma 2.9, we will go into further detail at the end of this proof. Firstly the resolvent has the following property for fixed $A > 0$:

$$\left\| \psi_\epsilon (\Delta_\ell - s(1-s))^{-1} \kappa_A \right\|_2 = O(\epsilon^{\text{Re}(s)-1/2}), \quad (17)$$

and the implicit constant may be chosen uniformly for small ℓ . Secondly, if we assume that χ is supported in Z^+ , we observe that

$$\left\| (1 - \kappa_{-A}) (\Delta_\ell - s_0(1-s_0))^{-1} \chi \right\|_2 = O(\ell^{\text{Re}(s_0)-1/2}), \quad \ell \rightarrow 0 \quad (18)$$

holds for fixed $A > 0$.

Taking these properties for granted, we need to examine the behaviour of

$$\psi_\epsilon (\Delta_\ell - s(1-s))^{-1} \kappa (\Delta_\ell - s_0(1-s_0))^{-1} \chi$$

if κ is supported in a small neighbourhood of the deleted curve. Here we proceed exactly as in the proof of proposition 2.11 to see

$$\left\| \psi_\epsilon (\Delta_\ell - s(1-s))^{-1} (\kappa_B - \kappa_A) (\Delta_\ell - s_0(1-s_0))^{-1} \chi \right\|_1 \leq C \int_B^A a^{-1+r} da$$

for small A with $B < A < 0$, and C is locally independent of ℓ and ϵ . Note that, although the function g_0 that appeared in the proof has to be replaced with g_ℓ a priori, monotonicity of $g_\ell(a_1, a_2, r)$ in ℓ if $a_1 a_2 > 0$ and $\lim_{\ell \rightarrow 0} g_\ell = g_0$ imply that the original estimates remain valid.

Now given δ and R as proposed, the last estimate together with a symmetry consideration imply that we can choose $A > 0$ such that

$$\left\| \psi_\epsilon (\Delta_\ell - s(1-s))^{-1} (\kappa_{-A} - \kappa_A) (\Delta_\ell - s_0(1-s_0))^{-1} \chi \right\|_1 < \delta/3$$

holds. By (18) and (17)

$$\left\| \psi_\epsilon (\Delta_\ell - s(1-s))^{-1} (1 - \kappa_{-A}) (\Delta_\ell - s_0(1-s_0))^{-1} \chi \right\|_1 < \delta/3$$

holds if ℓ is sufficiently small, and then a suitable choice of ϵ gives

$$\left\| \psi_\epsilon (\Delta_\ell - s(1-s))^{-1} \kappa_A (\Delta_\ell - s_0(1-s_0))^{-1} \chi \right\|_1 < \delta/3.$$

We proved the proposition except for equations (17) and (18). The first of these reduces to the Hilbert-Schmidt estimate in lemma 2.10 because of

$$g_\ell(a_1, a_2, r) \leq g_\ell(a_1, -a_2, r) \quad \text{if } a_1 a_2 \geq 0$$

and by monotonicity in ℓ . Equation (18) follows from

$$\begin{aligned} g_\ell(a_1, a_2, r) &\leq \left[\frac{2}{\ell^2} (|\ell^2 + a_1 a_2| - (\ell^2 + a_1 a_2)) \right]^{1/2-r} |\ell^2 + a_1 a_2|^{-1/2} \\ &\leq C \ell^{2r-1} |a_1 a_2|^{-r}, \end{aligned}$$

where we assume that $|a_1 a_2|$ is bounded from below by a positive number. \square

We close the subsection with an additional observation that will allow surfaces of infinite area to be covered by theorem 2.14. Here the support of ψ may be unbounded, but ψ and χ are supported in different components of Z .

2.13 Proposition. *Let ψ be bounded and supported in $Z^- \setminus Z^\epsilon$ for some $\epsilon > 0$, and let χ be of compact support in Z^+ . Then*

$$\lim_{\ell \rightarrow 0} \left\| \psi \left((\Delta_\ell - s(1-s))^{-1} - (\Delta_\ell - s_0(1-s_0))^{-1} \right) \chi \right\|_1 = 0$$

holds with locally uniform convergence in s and s_0 .

Proof. For each $A < 0$

$$\lim_{\ell \rightarrow 0} \left\| \psi (\Delta_\ell - s(1-s))^{-1} \kappa_A (\Delta_\ell - s_0(1-s_0))^{-1} \chi \right\|_1 = 0$$

holds by the proof of proposition 2.12. We let $A \rightarrow -\infty$ and estimate the remainder. More precisely, we show (cf. (15))

$$\int_{-\infty}^A \sqrt{f'(a)} \sqrt{g'(a)} da = O(\ell^{\operatorname{Re}(s_0)-1/2}).$$

The restrictions $a < A < 0$ and $\epsilon > 0$ imply

$$\int_{-\infty}^{-\epsilon} g_\ell(a_1, a, 2 \operatorname{Re}(s)) da_1 = O(1), \quad a \rightarrow -\infty,$$

and

$$\int_{-\infty}^{-\epsilon} g_\ell(a_1, a, \operatorname{Re}(s))^2 da_1 = O(|a|^{-1}), \quad a \rightarrow -\infty,$$

so $\sqrt{f'(a)}$ is bounded. The asymptotic of g' is governed by

$$\int_{\epsilon}^{\infty} g_\ell(a, a_1, \operatorname{Re}(s_0))^2 da_1 \leq C \ell^{4 \operatorname{Re}(s_0)-2} |a|^{-2 \operatorname{Re}(s_0)} \int_{\epsilon}^{\infty} |a_1|^{-2 \operatorname{Re}(s_0)} da_1,$$

so that $\int_{-\infty}^A \sqrt{f'(a)} \sqrt{g'(a)} da$ exists at least provided that $\operatorname{Re}(s_0) > 1$ holds. \square

2.2 Geometrically finite surfaces

We take up the nomenclature introduced in section 1. Associated with an admissible graph G and its augmentation G^* , there is a parameter space for hyperbolic structures

$$\mathcal{B} = \{\lambda = (\ell, \tau) \mid \ell: G_1^* \rightarrow [0, \infty), \tau: G_1^* \rightarrow \mathbb{R}\},$$

and a total space of hyperbolic surfaces

$$\mathcal{E} = \bigcup_{\lambda \in \mathcal{B}} \{\lambda\} \times Y_G(\lambda).$$

We fix $\lambda_0 = (\ell_0, \tau_0) \in \mathcal{B}$, and the surfaces $Y_G(\lambda)$ for λ near λ_0 will be compared to $Y = Y_G(\lambda_0)$ via the trivialisation of \mathcal{E} from section 1. Recall the open cover defined on page 10

$$Y = \bigcup_{q \in G_0} P_q \cup \bigcup_{d \in G_1^*} Z_d.$$

The cylindrical parts Z_d of Y can be embedded isometrically into elementary cylinders $\langle \gamma \rangle \backslash X_{\ell_0(d)}$. To remove the ambiguity in this statement, we note that Y is defined in such a way that each orientation $e \in \tilde{G}_1^*$ of $d \in G_1^*$ gives rise to such an embedding. If d is a proper edge, replacing e with $\iota(e)$ corresponds to composition of that embedding with $(x, a) \mapsto (-x, -a)$, so there is a canonical inclusion of Z_d for each $d \in G_1^*$ into the cylinder

$$\bar{Z}_d := \begin{cases} (\{e\} \times \langle \gamma \rangle \backslash X_{\ell_0(d)} \cup \{\iota(e)\} \times \langle \gamma \rangle \backslash X_{\ell_0(d)}) / \sim & \text{if } d \in G_1, \\ \{d\} \times \langle \gamma \rangle \backslash X_{\ell_0(d)} & \text{if } d \in G_1^* \setminus G_1, \end{cases}$$

where the equivalence relation is generated by $(e, (x, a)) \sim (\iota(e), (-x, -a))$. If d is a proper edge, the image of this inclusion is a cylindrical subset $\bar{Z}_d^{A(\ell_0(d))}$ for the interval

$$A(t) = \begin{cases} \left(-\frac{t}{2 \sinh(t/2)}, \frac{t}{2 \sinh(t/2)}\right) & \text{if } t \neq 0, \\ (-1, 1) & \text{if } t = 0. \end{cases}$$

In case of a phantom edge the image is the set $\bar{Z}_d^{A(\ell(d))} \cup \bar{Z}_d^+$.

We want to use the embeddings to pull back operators acting on $L^2(\bar{Z}_d)$. For this purpose, we choose a collection of smooth functions

$$\phi_d: Y \longrightarrow [0, 1] \quad \text{for } d \in G_1^*, \quad \text{supp } \phi_d \subset Z_d,$$

that satisfy for some $\epsilon > 0$

$$\begin{aligned} \phi_d(z) &= 1 & \text{if } d \in G_1 \text{ and } z \in Z_d^{A(\ell_0(d)) \cdot \epsilon}, \\ \phi_d(z) &= 1 & \text{if } d \in G_1^* \setminus G_1 \text{ and } z \in Z_d^+ \cup Z_d^{A(\ell_0(d)) \cdot \epsilon}. \end{aligned}$$

Let the smooth functions ψ_d , $d \in G_1^*$, fulfil the same support conditions and in addition $\psi_d \cdot \phi_d = \phi_d$. If $\text{Re}(s) > 1/2$, the resolvent operators

$$R_d(s) := (\Delta_{\bar{Z}_d} - s(1-s))^{-1}$$

exist. Then $\psi_d R_d(s) \phi_d$ is considered to be acting on $L^2(Y)$, where functions that are supported in Z_d are identified with functions on \bar{Z}_d via the embeddings just described.

This construction for $Y = Y_G(\lambda_0)$ is carried over to $Y_G(\lambda)$ for all λ near λ_0 . There exists a neighbourhood \mathcal{U} of λ_0 such that, with respect to a trivialisation of $\mathcal{E} \rightarrow \mathcal{B}$

over \mathcal{U} as constructed in section 1, the cut-off functions can be chosen simultaneously for all $\lambda \in \mathcal{U}$. The trivialising map of $\mathcal{E}|_{\mathcal{U}}$ is denoted by

$$\Psi_\lambda: Y = Y_G(\lambda_0) \longrightarrow Y_G(\lambda).$$

It maps the complement of all closed geodesics in the cylinders Z_d diffeomorphically onto its image, and there is an induced linear homeomorphism $L^2(Y, \text{vol}) \rightarrow L^2(Y_G(\lambda), \text{vol})$. The pull-back of an operator to $L^2(Y, \text{vol})$ will be labelled with a subscript λ .

2.14 Theorem. 1. *Let ψ be a smooth function on $Y_G(\lambda_0)$, compactly supported in the complement of the closed geodesics in the Z_d . If $\text{Re}(s) > 1/2$, the bounded linear maps*

$$\psi \cdot (\Delta_\lambda - s(1-s))^{-1}: L^2(Y_G(\lambda_0)) \longrightarrow H^2(Y_G(\lambda_0))$$

depend continuously on λ in the strong topology.

2. *Let $\text{Re}(s) > 1/2$, $\text{Re}(s_0) > 1$, and let $s(1-s)$ belong to the resolvent set. Then the operator on $L^2(Y_G(\lambda))$*

$$T(s) := (\Delta - s(1-s))^{-1} - (\Delta - s_0(1-s_0))^{-1} - \sum_{d \in G_1^*} \psi_d (R_d(s) - R_d(s_0)) \phi_d \quad (19)$$

is of trace class, and the trace depends continuously on λ .

2.15 Remark. One concludes from part 1 of this theorem that the same result holds if the resolvent is considered as a map $H_c^k(Y') \rightarrow H_{\text{loc}}^{k+2}(Y')$, where Y' is the complement of the distinguished closed geodesics. The proof is analogous to that of proposition 2.6.

Appearance of the strong operator topology in theorem 2.14 is a consequence of how the trivialisation Ψ_λ was chosen. If $Z_d(\lambda)$ and $\bar{Z}_d(\lambda)$ are the cylinders defined for $Y_G(\lambda)$ in analogy with Z_d , $\bar{Z}_d \subset Y_G(\lambda_0)$, there are canonical diffeomorphisms

$$\bar{\Psi}_{d,\lambda}: \bar{Z}_d \longrightarrow \bar{Z}_d(\lambda) \quad (20)$$

induced by the identity of \mathbb{R}^2 . It does not coincide with the map induced by the trivialisation Ψ_λ . Instead, the composition

$$\bar{Z}_d^\epsilon \cong Z_d^\epsilon \xrightarrow{\Psi_\lambda} Z_d^\epsilon(\lambda) \cong \bar{Z}_d^\epsilon(\lambda)$$

is given by

$$(x, a) \longmapsto \begin{cases} (x + (\tau_0(d) - \tau(d))/2, a) & \text{if } a < 0, \\ (x - (\tau_0(d) - \tau(d))/2, a) & \text{if } a > 0. \end{cases}$$

If we denote by Ψ_λ the obvious extension of this map to $\bar{Z}_d \rightarrow \bar{Z}_d(\lambda)$, the automorphism of $L^2(\bar{Z}_d)$ induced by $\bar{\Psi}_{d,\lambda}^{-1} \circ \Psi_\lambda$ depends continuously on λ in the strong operator topology, but not necessarily in the norm topology. It is the identity mapping if we restrict λ to a subset of \mathcal{B} where all twist-parameters are constant. The proof of 2.14 will show that under this restriction strong continuity may be replaced with norm continuity.

Proof of the theorem, part 1. The approach is analogous to that described by Guillopé [7], where the resolvent is continued analytically across the essential spectrum of Δ by using meromorphic Fredholm theory. See the book of Reed and Simon [21] for a statement of the Fredholm theorem.

Let P denote the interior of

$$Y \setminus \left(\bigcup_{d \in G_1^*} Z_d^{A(\ell(d))-\epsilon/2} \cup \bigcup_{d \notin G_1} Z_d^+ \right).$$

The family ϕ_d of functions on Y is supplemented with an additional function ϕ_0 to form a partition of unity. The support of ϕ_0 is contained in P , and there exists a smooth function ψ_0 of support in P with $\psi_0 \cdot \phi_0 = \phi_0$.

Just as the cylindrical subsets Z_d are embedded into \bar{Z}_d , we embed P isometrically into an auxiliary surface \bar{P} . This surface is required to be compact. Composition of this map with Ψ_λ^{-1} gives an inclusion of the corresponding subset of $Y_G(\lambda)$ into \bar{P} , and we may assume that there is a continuous family g_λ of Riemannian metrics given on \bar{P} such that this inclusion is isometric in each case.

Suppressing its dependence on λ in our notation, we consider the resolvent operators

$$R_0(s) := (\Delta_{\bar{P}} + t)^{-1} \quad \text{and} \quad R_d(s) := (\Delta_{\bar{Z}_d} - s(1-s))^{-1}.$$

The parameter t will be chosen suitably below. In the formulae to come, the summation index j will run through G_1^* and $\{0\}$. On $L^2(Y_G(\lambda))$ there is the equation of operators

$$(\Delta - s(1-s)) \circ \sum_j \psi_j R_j(s) \phi_j = (1 + Q(s)), \quad (21)$$

with a compact operator $Q(s)$ given by

$$Q(s) = (-t - s(1-s))\psi_0(\Delta_{\bar{P}} + t)^{-1}\phi_0 + \sum_j [\Delta, \psi_j] R_j(s) \phi_j.$$

If $1 + Q(s)$ is invertible and the resolvent operators exist, by (21) we have

$$\left(\sum_j \psi_j R_j(s) \phi_j \right) \circ (1 + Q(s))^{-1} = (\Delta - s(1-s))^{-1}.$$

So continuity of the resolvent in λ is reduced to that of the operators on the left hand side. And invertibility of $1 + Q(s)$ indeed holds: If $t = -s(1-s)$, the norm of $Q(s)$ becomes arbitrarily small as the real part of s increases. Thus there exists t such that $1 + Q(s)$ is invertible for some s . We fix such a t , and analytic Fredholm theory implies that the map $s \mapsto (1 + Q(s))^{-1}$ is meromorphic on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$.

We want to see that the pull-back $Q(s)_\lambda$ to $L^2(Y)$ depends continuously on λ in the norm topology of $B(L^2(Y))$, and so do the $\psi_j R_j(s)_\lambda \phi_j$ in the strong topology of $B(L^2(Y), \mathbb{H}_{\text{loc}}^2(Y))$.

In both cases, this is obvious for the contribution of the compact part \bar{P} , as these operators are associated with the Laplacian for a continuous family of metrics on a compact surface. So let us consider a contribution $[\Delta_\lambda, \psi_d] R_d(s)_\lambda \phi_d$ to $Q(s)_\lambda$ for some $d \in G_1^*$. In terms of the canonical map $\bar{\Psi}_{d,\lambda}$ between elementary cylinders, cf. (20), this factorises into

$$\begin{aligned} & \Psi_\lambda^* \circ [\Delta_{\bar{Z}_d(\lambda)}, \psi_d] \circ (\Delta_{\bar{Z}_d(\lambda)} - s(1-s))^{-1} \circ \Psi_\lambda^{-1*} \circ \phi_d \\ &= [\Delta_\lambda, \psi_d] \circ \Psi_\lambda^* \circ (\Delta_{\bar{Z}_d(\lambda)} - s(1-s))^{-1} \circ \Psi_\lambda^{-1*} \circ \phi_d \\ &= [\Delta_\lambda, \psi_d] \circ \Psi_\lambda^* \bar{\Psi}_{d,\lambda}^{-1*} \circ \bar{\Psi}_{d,\lambda}^* (\Delta_{\bar{Z}_d(\lambda)} - s(1-s))^{-1} \bar{\Psi}_{d,\lambda}^{-1*} \circ \bar{\Psi}_{d,\lambda}^* \Psi_\lambda^{-1*} \circ \phi_d. \end{aligned}$$

In proposition 2.6 we proved that $\chi \bar{\Psi}_{d,\lambda}^* (\Delta_{\bar{Z}_d(\lambda)} - s(1-s))^{-1} \bar{\Psi}_{d,\lambda}^{-1*}$ is a continuous family of compact linear maps $L^2(\bar{Z}_d) \rightarrow H^1(\bar{Z}_d)$ if χ is compactly supported off the closed geodesic. The support of the differential operator on the left also satisfies this condition. This and the strong continuity of $\bar{\Psi}_{d,\lambda}^* \circ \Psi_\lambda^{-1*}$ proves norm continuity of the composition and that of $Q(s)_\lambda$.

The remaining operators $R_j(s)_\lambda$ are treated in the same manner, but we must stick to strong continuity since they are not composed with a compact operator.

Proof of the theorem, part 2. Using the first resolvent formula, we may proceed as in the first part to see

$$\begin{aligned} & (\Delta - s_0(1-s_0)) \circ (\Delta - s(1-s)) \circ \left(\sum_{j \neq 0} \psi_j (R_j(s) - R_j(s_0)) \phi_j + \psi_0 R_0(s) \phi_0 \right) \\ & = (s(1-s) - s_0(1-s_0)) \cdot (1 + Q(s)), \end{aligned} \quad (22)$$

where $R_0(s)$ is defined by

$$R_0(s) := \frac{s(1-s) - s_0(1-s_0)}{-t - s_0(1-s_0)} [(\Delta_{\bar{P}} + t)^{-1} - (\Delta_{\bar{P}} - s_0(1-s_0))^{-1}],$$

and, just for completeness, we have

$$\begin{aligned} Q(s) &= (-t - s(1-s)) \psi_0 (\Delta_{\bar{P}} + t)^{-1} \phi_0 + \frac{-t - s(1-s)}{s(1-s) - s_0(1-s_0)} [\Delta, \psi_0] R_0(s) \phi_0 \\ &+ [\Delta, \psi_0] ((\Delta_{\bar{P}} + t)^{-1} + (\Delta_{\bar{P}} - s_0(1-s_0))^{-1}) \phi_0 \\ &+ \sum_{d \in G_1^*} [\Delta, \psi_d] (R_d(s) + R_d(s_0)) \phi_d \\ &+ \frac{1}{-t - s_0(1-s_0)} [\Delta, [\Delta, \psi_0]] R_0(s) \phi_0 \\ &+ \frac{1}{s(1-s) - s_0(1-s_0)} \sum_{d \in G_1^*} [\Delta, [\Delta, \psi_d]] (R_d(s) - R_d(s_0)) \phi_d. \end{aligned}$$

Again $Q(s)$ is compact and $1 + Q(s)$ is meromorphically invertible by Fredholm theory, so that

$$\begin{aligned} & \left(\sum_{d \in G_1^*} \psi_d (R_d(s) - R_d(s_0)) \phi_d + \psi_0 R_0(s) \phi_0 \right) \circ (1 + Q(s))^{-1} \\ & = (s_0(1-s_0) - s(1-s)) \cdot (\Delta - s(1-s))^{-1} \circ (\Delta - s_0(1-s_0))^{-1} \\ & = (\Delta - s(1-s))^{-1} - (\Delta - s_0(1-s_0))^{-1}. \end{aligned}$$

According to this equation, the operator $T(s)$ that we want to examine is given by

$$\begin{aligned} & (\Delta - s(1-s))^{-1} - (\Delta - s_0(1-s_0))^{-1} - \sum_{d \in G_1^*} \psi_d (R_d(s) - R_d(s_0)) \phi_d \\ & = \left(\psi_0 R_0(s) \phi_0 - \sum_{d \in G_1^*} \psi_d (R_d(s) - R_d(s_0)) \phi_d Q(s) \right) \circ (1 + Q(s))^{-1}. \end{aligned} \quad (23)$$

We want to know that the first factor in (23) is of trace class, that its pull-back to $Y = Y_G(\lambda_0)$ depends continuously on λ in the trace class topology, and that the pull-back of the second factor depends continuously on λ in the norm topology. Then

one can make use of Radon-Nikodym derivatives as in corollary 2.2 to complete the proof.

Continuity of $(Q(s))_\lambda$ in λ follows exactly as in part 1 of the theorem. To prove trace class property, we observe that the image of $\phi_d Q(s)$ is supported in the intersection of $\text{supp } \phi_d$ with $\text{supp } \psi_0$. This is a compact subset of the complement of the pinching geodesics, and it was shown in section 2.1 that this implies trace class property for the composition with $\psi_d(R_d(s) - R_d(s_0))$. Since \bar{P} is compact, this property holds for $R_0(s)$ as well, which proves that $T(s)$ is of trace class.

To examine continuity of the first factor in (23), the operators are pulled back to $L^2(Y)$ using Ψ_λ , and then we have in analogy with the elementary contribution in part 1

$$\begin{aligned} & \psi_d(R_d(s) - R_d(s_0))_\lambda \phi_d Q(s)_\lambda \\ &= \psi_d \Psi_\lambda^* [(\Delta_{\bar{Z}_d(\lambda)} - s(1-s))^{-1} - (\Delta_{\bar{Z}_d(\lambda)} - s_0(1-s_0))^{-1}] \Psi_\lambda^{-1*} \phi_d Q(s)_\lambda \\ &= (\Psi_\lambda^* \bar{\Psi}_{d,\lambda}^{-1*}) \circ \psi_d \bar{\Psi}_{d,\lambda}^* [(\Delta_{\bar{Z}_d(\lambda)} - s(1-s))^{-1} - (\Delta_{\bar{Z}_d(\lambda)} - s_0(1-s_0))^{-1}] \bar{\Psi}_{d,\lambda}^{-1*} \\ & \quad \circ (\bar{\Psi}_{d,\lambda}^* \Psi_\lambda^{-1*}) \phi_d \circ Q(s)_\lambda. \end{aligned}$$

Now

$$\psi_d \bar{\Psi}_{d,\lambda}^* [(\Delta_{\bar{Z}_d(\lambda)} - s(1-s))^{-1} - (\Delta - s_0(1-s_0))^{-1}] \bar{\Psi}_{d,\lambda}^{-1*} \chi$$

is a continuous family of trace class operators on $L^2(Y)$ by propositions 2.6 and 2.13. Strong continuity of $\bar{\Psi}_{d,\lambda}^* \Psi_\lambda^{-1*}$ implies that the same holds true for $T(s)_\lambda$.

3 Applications

3.1 Eisenstein functions and the scattering matrix

We saw that the resolvent of the Laplacians for a degenerating family converges to that of the limit surface. For surfaces of finite area, the resolvent is intimately related with the theory of Eisenstein series. It is therefore natural to ask for functions that approximate the Eisenstein series during this process. In the first subsection, we define functions that meet this criterion to some extent, as well as an appropriate notion of approximate scattering matrices. The Maass-Selberg relation and the functional equation generalise to this setting, at least if the surface is of finite area. In the second subsection, we therefore restrict to the finite area case and examine how the introduced structure behaves during degeneration.

3.1.1 Definitions and fundamental properties

This part is not concerned with families of degenerating surfaces, but with a fixed surface $Y = Y_G(\lambda)$. We make use of the meromorphic continuation of the resolvent $(\Delta - s(1-s))^{-1}$ in s from $\text{Re}(s) > 1/2$ to the complex plane [7]. We also use the conventions from section 1 that associate with each oriented edge $j \in \tilde{G}_1^*$ coordinates for an embedded cylinder $Z_j \subset Y$.

The original definition of Eisenstein series for hyperbolic surfaces of finite area is based on certain eigenfunctions of the Laplacian on elementary cusps. We give a definition, in terms of the hypergeometric function, that serves the same purpose for half-cylinders of arbitrary circumference.

3.1 Definition. For each $\ell \geq 0$ and $s \in \mathbb{C} \setminus (-1/2 - \mathbb{N}_0)$ put

$$h(\ell, s): \mathbb{R} \longrightarrow \mathbb{C}, \quad a \longmapsto \begin{cases} |a|^{-s} F(s/2, 1/2 + s/2; 1/2 + s; -\ell^2/a^2) & \text{if } a < 0, \\ 0 & \text{otherwise.} \end{cases}$$

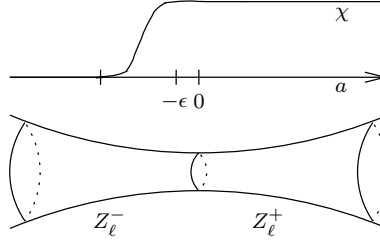


Figure 4: The cut-off function χ .

We identify $h(\ell, s)$ in the obvious manner with a rotationally-symmetric function on the elementary cylinder $Z_\ell = \langle \gamma \rangle \backslash X_\ell$. The Laplacian on Z_ℓ is

$$\Delta_{Z_\ell} = -(\ell^2 + a^2)^{-1} \partial_x^2 - (\ell^2 + a^2) \partial_a^2 - 2a \partial_a.$$

One checks, for example by expanding the hypergeometric function into a power series, that the restriction of $h(\ell, s)$ to the open half-cylinder Z_ℓ^- belongs to the kernel of $\Delta_{Z_\ell} - s(1-s)$. The choice of these functions is motivated by the following properties:

- On Z_ℓ^- they converge smoothly to $h(0, s)$ as $\ell \rightarrow 0$, and this is precisely the function used to define Eisenstein series associated with a cusp.
- For generic s , the functions $h(\ell, s)$ and $h(\ell, 1-s)$ are linearly independent.
- The asymptotic of $h(\ell, s)$ as $a \rightarrow -\infty$ is that of $h(0, s)$ for all ℓ .

As an aside, note that $h(\ell, s)$ depends quite trivially on ℓ in the upper half-plane model: By means of the isometries introduced in section 1, we obtain the function

$$z \mapsto \ell^{-s} |\tan(\arg z)|^s F(s/2, 1/2 + s/2; 1/2 + s; -(\tan(\arg z))^2).$$

Being an eigenfunction of a linear ordinary differential equation, there is of course a continuation of $h(\ell, s)|_{Z_\ell^-}$ to Z_ℓ as an eigenfunction if $\ell > 0$, and this continuation will be used below. But we want to stress out that $h(\ell, s)$ has a jump discontinuity at $a = 0$ by definition.

The asymptotic of $h(\ell, s)$ as $a \rightarrow -\infty$ admit to define approximate Eisenstein functions by summation over elements of a uniformising group, just as in the classical definition of Eisenstein series. We analysed the resolvent of the Laplacian in section 2, so it is preferable to use the resolvent instead. Fix a cut-off function $\chi: Z_\ell \rightarrow [0, 1]$ that is smooth, rotationally symmetric, and supported in a small neighbourhood of Z_ℓ^+ , such that the support of $1 - \chi$ is contained in $Z_\ell^{(-\infty, -\epsilon)}$ for some $\epsilon > 0$. Then $[\Delta_{Z_\ell}, \chi]$ is a differential operator, compactly supported in Z_ℓ^- , and $[\Delta_{Z_\ell}, \chi] h(\ell, s)$ is a smooth function of compact support in Z_ℓ^- . Recall from section 1 how an oriented edge $j \in \tilde{G}_1^*$ of a graph gives rise to an identification of $Z_{\ell(j)}^{A(\ell(j))}$ with a collar in Y . We require the support of $\chi h(\ell(j), s)$ to belong to this subset, so that $\chi h(\ell(j), s)$ and $[\Delta_{Z_{\ell(j)}}, \chi] h(\ell(j), s)$ both can be considered as functions on Y .

3.2 Definition. If $\text{Re}(s) > 1$, the *approximate Eisenstein function* of $j \in \tilde{G}_1^*$ is the function on Y given by

$$E_j(s) := (\Delta - s(1-s))^{-1} ([\chi, \Delta_{Z_{\ell(j)}}] h(\ell(j), s)) + \chi h(\ell(j), s).$$

From the definition we immediately see that $E_j(s)$ specialises to the Eisenstein series if $\ell(j) = 0$. It is an eigenfunction of the Laplacian on the subset of Y where it is smooth, that is, on the complement of a single closed geodesic.

The aforementioned continuation of the resolvent provides a meromorphic continuation of $E_j(s)$ in s to the complex plane. A property that these functions inherit from the classical Eisenstein series is that they satisfy some kind of Maass-Selberg relation, which involves the notion of scattering matrices. We need a replacement for the latter in the present situation.

For each function $f \in L^1_{\text{loc}}(Y)$ and each $j \in \tilde{G}_1^*$, the canonical coordinates of the half-cylinder Z_j^- give rise to fibrewise Fourier coefficients of f by

$$F_j^n f(a) := \int_0^1 f(x, a) \exp(2\pi i n x) dx, \quad -\frac{\ell(j)}{2 \sinh(\ell(j)/2)} < a < 0.$$

In particular, if $s - 1/2 \notin \mathbb{Z}$ and if the restriction of f to Z_j^- is annihilated by $\Delta - s(1 - s)$, then $F_j^0 f$ is a linear combination of $h(\ell(j), s)$ and $h(\ell(j), 1 - s)$. This observation is used to define a pair of matrices that are associated with a distinguished subset S of the edges:

3.3 Definition. Let $S \subset G_1^*$ be a set that contains all edges j with $\ell(j) = 0$ and all phantom edges. Let $\tilde{S} \subset G_1^*$ be the set of oriented representatives of edges in S . Then there are two analytic families $C = (C_{ij})_{i,j \in \tilde{S}}$ and $D = (D_{ij})_{i,j \in \tilde{S}}$ of matrices defined by the equation

$$F_j^0 E_i(s) = D_{ij}(s) h(\ell, s) + C_{ij}(s) h(\ell, 1 - s) \quad (24)$$

if $\text{Re}(s) > 1$ and $s - 1/2 \notin \mathbb{N}$. The matrix $C(s)$ is called an *approximate scattering matrix*.

Again we see that the approximate scattering matrix specialises to the scattering matrix if λ satisfies $\ell(d) = 0$ for all $d \in S$. It has a meromorphic continuation in s to all of \mathbb{C} . The choice of S should be imagined as the choice of a certain stratum in the boundary of the space \mathcal{B} of hyperbolic surfaces constructed from a given graph. Our objective in the next subsection will be to approximate the Eisenstein series and the scattering matrix for a surface $Y_G(\lambda')$, where $\lambda' = (\ell', \tau')$ satisfies $\ell'(j) = 0$ if and only if $j \in S$.

Some fundamental properties of the approximate scattering matrices are deduced in lemma 3.4 and 3.5 below, before the Maass-Selberg relation will be given in theorem 3.7. Then the functional equations for the approximate Eisenstein functions and scattering matrices will be discussed as consequences of the Maass-Selberg-relation. Recall that $h(\ell, s)$ has simple poles in s if $\ell \neq 0$. To clarify the behaviour of C at these points, we give an explicit formula in terms of the constant modes $F_j^0 E_i(s)$.

3.4 Lemma. *Let s be a regular point of the approximate Eisenstein function E_i . Then*

$$\begin{pmatrix} D_{ij}(s) \\ C_{ij}(s) \end{pmatrix} = \frac{\ell(j)^2 + a^2}{1 - 2s} \begin{pmatrix} \partial_a h(\ell(j), 1 - s) & -h(\ell(j), 1 - s) \\ -\partial_a h(\ell(j), s) & h(\ell(j), s) \end{pmatrix} \begin{pmatrix} F_j^0 E_i(s) \\ \partial_a F_j^0 E_i(s) \end{pmatrix}.$$

This formula indicates that there might be poles of the D_{ij} in $\{s \in \mathbb{C} \mid \text{Re}(s) > 1\}$, while the approximate scattering matrix in this domain is holomorphic.

Proof. As $h(\ell(j), s)$ and $h(\ell(j), 1 - s)$ are linearly independent solutions of a second-order ordinary differential equation, the coefficients $D_{ij}(s)$ and $C_{ij}(s)$ in (24) are

determined by $F_j^0 E_i(s)$ and $\partial_a F_j^0 E_i(s)$ for an arbitrary point $a < 0$. The formula then follows from solving the system

$$\begin{pmatrix} F_j^0 E_i(s) \\ \partial_a F_j^0 E_i(s) \end{pmatrix} = \begin{pmatrix} h(\ell(j), s) & h(\ell(j), 1-s) \\ \partial_a h(\ell(j), s) & \partial_a h(\ell(j), 1-s) \end{pmatrix} \begin{pmatrix} D_{ij}(s) \\ C_{ij}(s) \end{pmatrix}$$

by inversion of the Wronskian matrix. The determinant ω of the Wronskian solves

$$\partial_a((\ell^2 + a^2)\omega(a)) = 0.$$

It can be evaluated at $a = 0$ by equation (25) below, which gives $\omega(0) = (1-2s)\ell^{-2}$. So we obtain

$$\omega(a) = \frac{\ell^2 \omega(0)}{\ell^2 + a^2} = \frac{1-2s}{\ell^2 + a^2}.$$

□

If C is the scattering matrix of a finite-area surface, one knows that D is the identity matrix. This does not hold in the more general situation here, but the next lemma shows that D can in fact be calculated from C . The reason for this dependency is that the approximate Eisenstein functions are smooth eigenfunctions up to a single discontinuity that is known explicitly. So one simply has to compare the functions $D_{ij}(s)h(\ell(j), s) + C_{ij}(s)h(\ell(j), 1-s)$, which are associated with Z_j^- , with those for $Z_{\iota(j)}^- = Z_j^+$, to see if they match up correctly.

3.5 Lemma. *For all $s \in \mathbb{C}$ let λ^s and $\Sigma(s)$ denote the matrices with the following entries:*

$$(\lambda^s)_{ij} = \begin{cases} \ell(j)^s & \text{if } i = j \text{ and } \ell(j) \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$(\Sigma(s))_{ij} = \begin{cases} \cos(\pi s)^{-1} & \text{if } i = j, \\ 1 & \text{if } i \in \tilde{G}_1 \text{ and } j = \iota(i), \\ 0 & \text{otherwise.} \end{cases}$$

Then C and D satisfy

$$D(s) = 1 + (2s-1) \frac{4^{-s} \Gamma(s)^2}{\Gamma(1/2+s)^2} C(s) \cdot \Sigma(s) \cdot \lambda^{2s-1}.$$

Proof. We already mentioned that this formula is related with the continuation of $h(\ell, s)$ as a smooth eigenfunction from Z_ℓ^- to Z_ℓ . If $\ell > 0$ and $a < 0$, we have

$$\begin{aligned} h(\ell, s)(a) &= \ell^{-s} \frac{\Gamma(1/2)\Gamma(1/2+s)}{\Gamma(1/2+s/2)^2} F(s/2, 1/2-s/2; 1/2; -a^2/\ell^2) \\ &\quad - \ell^{-(1+s)} \frac{\Gamma(-1/2)\Gamma(1/2+s)}{\Gamma(s/2)^2} a F(1/2+s/2, 1-s/2; 3/2; -a^2/\ell^2), \end{aligned} \quad (25)$$

and each summand on the right hand side is an eigenfunction on the real line. The proof of this equation consists of a functional equation for the hypergeometric function, cf. [6, p. 17, eq. (17)]. Now if a is positive and the same functional equation is applied to either summand in (25) again, we see that it is equal to

$$\left[\cos(\pi s)^{-1} h(\ell, s) + \ell^{1-2s} \frac{4^s \Gamma(1/2+s)^2}{(2s-1)\Gamma(s)^2} h(\ell, 1-s) \right] (-a). \quad (26)$$

For completeness, let us carry out an intermediate step in the derivation of this formula: The functional equation gives

$$\begin{aligned} & \pi\Gamma(1/2+s)\Gamma(1/2-s) \left(\frac{1}{\Gamma(1/2+s/2)^2\Gamma(1/2-s/2)^2} + \frac{1}{\Gamma(s/2)^2\Gamma(1-s/2)^2} \right) \\ & \quad \cdot a^{-s} F(s/2, 1/2+s/2; 1/2+s, -\ell^2/a^2) \\ & + 2\pi\ell^{1-2s} \frac{\Gamma(1/2+s)\Gamma(-1/2+s)}{\Gamma(1/2+s/2)^2\Gamma(s/2)^2} a^{-(1-s)} F(1/2-s/2, 1-s/2; 3/2-s; -\ell^2/a^2). \end{aligned}$$

Then (26) follows from $\cos(\pi z)\Gamma(1/2+z)\Gamma(1/2-z) = \pi$ and from Legendre's duplication formula for the Gamma function.

With this at hand, we determine $D_{ij}(s)$ if j is a phantom edge and $\ell(j) \neq 0$. For each $i \in \tilde{S}$,

$$E_i(s) - \delta_{ij} \chi h(\ell(i), s)$$

is an eigenfunction of Δ in a neighbourhood of the closed geodesic in Z_j , and its constant term on Z_j^- is

$$(D_{ij}(s) - \delta_{ij} h(\ell, s) + C_{ij}(s) h(\ell, 1-s)).$$

According to (26), the constant term on Z_j^+ is

$$\begin{aligned} & (D_{ij}(s) - \delta_{ij}) [\alpha(s) h(\ell, s) + \ell^{1-2s} \beta(s) h(\ell, 1-s)] (-a) \\ & + C_{ij}(s) [\alpha(1-s) h(\ell, 1-s) + \ell^{1-2(1-s)} \beta(s) h(\ell, s)] (-a) \\ & = [(D_{ij}(s) - \delta_{ij})\alpha(s) + C_{ij}(s)\ell^{2s-1}\beta(1-s)] \cdot h(\ell, s)(-a) \\ & \quad + [(D_{ij}(s) - \delta_{ij})\ell^{1-2s}\beta(s) + C_{ij}(s)\alpha(1-s)] \cdot h(\ell, 1-s)(-a), \end{aligned} \quad (27)$$

where $\alpha(s), \beta(s)$ are abbreviations for the coefficients in (26). If $\text{Re}(s) > 1/2$, this function must be square-integrable on the infinite half-cylinder Z_j^+ , so the coefficient of $h(\ell, 1-s)$ must be zero. This means

$$D_{ij}(s) = \delta_{ij} + (2s-1) \frac{4^{-s}\Gamma(s)^2}{\Gamma(1/2+s)^2} C_{ij}(s) \cos(\pi s)^{-1} \ell^{2s-1}.$$

Now suppose that j is a proper edge and $\ell(j) \neq 0$. The procedure to compute $D_{ij}(s)$ in this case is similar: The constant term of $E_i(s) - \delta_{ij} \chi h(\ell(i), s)$ on Z_j^- is

$$(D_{ij}(s) - \delta_{ij}) h(\ell(j), s) + C_{ij}(s) h(\ell(j), 1-s),$$

and the constant term on $Z_{i(j)}^- \cong Z_j^+$ is again given by (27). On the other hand, the latter is equal to

$$(D_{i(j)}(s) - \delta_{i(j)}) h(\ell(j), s)(-a) + C_{i(j)}(s) h(\ell(j), 1-s)(-a)$$

by definition. Equating the coefficients we see

$$\begin{aligned} & \begin{pmatrix} D_{ij}(s) - \delta_{ij} \\ D_{i(j)}(s) - \delta_{i(j)} \end{pmatrix} \\ & = (2s-1) \frac{4^{-s}\Gamma(s)^2}{\Gamma(1/2+s)^2} \ell(j)^{2s-1} \begin{pmatrix} \cos(\pi s)^{-1} & 1 \\ 1 & \cos(\pi s)^{-1} \end{pmatrix} \cdot \begin{pmatrix} C_{ij}(s) \\ C_{i(j)}(s) \end{pmatrix}. \end{aligned}$$

We have determined the value of $D_{ij}(s)$ for all j with $\ell(j) \neq 0$. If $\ell(j) = 0$, then $D_{ij}(s) - \delta_{ij} = 0$ immediately follows from the square-integrability of $E_i(s) - \chi h(\ell(i), s)$ on a cusp if $\text{Re}(s) > 1/2$. \square

To state the Maass-Selberg relation, we need one more definition.

3.6 Definition. If A is a sufficiently small positive number, the *truncated Eisenstein functions* are

$$E_i^A(s): z \mapsto \begin{cases} E_i(s)(z) - (F_j^0 E_i(s))(a) & \text{if } z = (x, a) \in Z_j^-, j \in \tilde{S} \text{ and } |a| < A, \\ E_i(s)(z) - (F_j^0 E_i(s))(a) & \text{if } z = (x, a) \in Z_j^+ \text{ and } j \notin \tilde{G}_1, \\ E_i(s)(z) & \text{otherwise.} \end{cases}$$

3.7 Theorem. *If both s and s' are regular points of the approximate Eisenstein functions, and if the truncated functions are square-integrable, then the following relation holds:*

$$\begin{aligned} & (s(1-s) - s'(1-s')) \cdot \langle E_i^A(s), E_j^A(s') \rangle \\ &= \sum_{e \in \tilde{S}} (\ell(e)^2 + A^2) \left[D_{ie}(s) D_{je}(s') \cdot \omega_{\ell(e)}(s, s'; -A) \right. \\ & \quad + D_{ie}(s) C_{je}(s') \cdot \omega_{\ell(e)}(s, 1-s'; -A) \\ & \quad + C_{ie}(s) D_{je}(s') \cdot \omega_{\ell(e)}(1-s, s'; -A) \\ & \quad \left. + C_{ie}(s) C_{je}(s') \cdot \omega_{\ell(e)}(1-s, 1-s'; -A) \right], \end{aligned}$$

where

$$\omega_{\ell}(s_1, s_2; a) = \det \begin{pmatrix} h(\ell, s_1)(a) & h(\ell, s_2)(a) \\ \partial_a h(\ell, s_1)(a) & \partial_a h(\ell, s_2)(a) \end{pmatrix}.$$

Proof. We proceed exactly as Kubota [16]. Let $f = E_i(s)$, $g = E_j(s')$, and let $V \subset Y$ be the subset where the approximate Eisenstein series are not truncated, i.e.

$$V = Y \setminus \left(\bigcup_{e \in \tilde{S}} Z_e^- \cup \bigcup_{e \in \tilde{G}_1^* \setminus \tilde{G}_1} Z_e^{[0, \infty)} \right).$$

The outer unit normal of V at Z_e^- is $\nu = \sqrt{\ell(e)^2 + A^2} \partial_a$, and the one-dimensional volume form of $\partial V \cap Z_e^-$ is $\text{vol}_{\partial V} = \sqrt{\ell(e)^2 + A^2} dx$. Green's theorem states that

$$-\int_V (f \Delta g - g \Delta f) \text{vol} = \int_{\partial V} (f \cdot \nu(g) - g \cdot \nu(f)) \text{vol}_{\partial V}. \quad (28)$$

Insertion of the Fourier decomposition of f and g on the right leads to

$$\begin{aligned} & \sum_{e \in \tilde{S}} \int_0^1 \left[\left(\sum_{n \in \mathbb{Z}} F_e^n f(-A) \exp(2\pi i n x) \right) \left(\sum_{m \in \mathbb{Z}} \nu(F_e^m g)(-A) \exp(2\pi i m x) \right) \right. \\ & \quad \left. - \left(\sum_{n \in \mathbb{Z}} F_e^n g(-A) \exp(2\pi i n x) \right) \left(\sum_{m \in \mathbb{Z}} \nu(F_e^m f)(-A) \exp(2\pi i m x) \right) \right] \text{vol}_{\partial}(x) \\ &= - \sum_{e \in \tilde{S}} \left[(\ell(e)^2 + A^2) \cdot \sum_{n \in \mathbb{Z}} \left((F_e^n f)'(-A) F_e^{-n} g(-A) - F_e^n f(-A) (F_e^{-n} g)'(-A) \right) \right]. \end{aligned} \quad (29)$$

For each $e \in \tilde{G}_1^* \setminus \tilde{G}_1$ and $n \neq 0$, we assume that the Fourier coefficients $F_e^n f$, $F_e^n g$ are square-integrable eigenfunctions on the infinite half-cylinder $Z_e^{(-A, \infty)}$. The following shows that their respective contribution to the previous expression is given

by an L^2 scalar product over this cylinder:

$$\begin{aligned}
& - (\ell(e)^2 + A^2) \left((F_e^n f)'(-A) F_e^{-n} g(-A) - F_e^n f(-A) (F_e^{-n} g)'(-A) \right) \\
&= \int_{-A}^{\infty} \partial_a \left[(\ell(e)^2 + a^2) \cdot \left((F_e^n f)'(a) F_e^{-n} g(a) - F_e^n f(a) (F_e^{-n} g)'(a) \right) \right] da \\
&= \int_{-A}^{\infty} \left[(\ell(e)^2 + a^2) (F_e^n f)''(a) + 2a (F_e^n f)'(a) \right] F_e^{-n} g(a) da \\
&\quad - \int_{-A}^{\infty} F_e^n f(a) \left[(\ell(e)^2 + a^2) (F_e^{-n} g)''(a) + 2a (F_e^{-n} g)'(a) \right] da \\
&= (-s(1-s) + s'(1-s')) \int_{-A}^{\infty} F_e^n f(a) \cdot F_e^{-n} g(a) da.
\end{aligned} \tag{30}$$

This scalar product is part of the final formula. The same calculation gives for each proper edge $e \in \tilde{G}_1$ and $n \neq 0$

$$\begin{aligned}
& - (\ell(e)^2 + A^2) \left((F_e^n f)'(-A) F_e^{-n} g(-A) - F_e^n f(-A) (F_e^{-n} g)'(-A) \right) \\
&= (-s(1-s) + s'(1-s')) \int_{-A}^0 F_e^n f(a) \cdot F_e^{-n} g(a) da \\
&\quad + \ell(e)^2 \left((F_e^n f)'(0) F_e^{-n} g(0) - F_e^n f(0) (F_e^{-n} g)'(0) \right),
\end{aligned} \tag{31}$$

and we see that the additional summand here cancels with that of $\iota(e)$ in the sum over all $e \in \tilde{S}$ in (29). Combining (28), (29) and (30) with the assumption that $f = E_i^A(s)$ and $g = E_j^A(s')$ are eigenfunctions on the interior of V , we get

$$\begin{aligned}
& \sum_{e \in \tilde{S}} (\ell(e)^2 + A^2) \left[F_e^0 f(-A) (F_e^0 g)'(-A) - (F_e^0 f)'(-A) F_e^0 g(-A) \right] \\
&= (s(1-s) - s'(1-s')) \left[\int_S f g \text{ vol} + \sum_{e \in \tilde{G}_1} \sum_{n \neq 0} \int_{-A}^0 F_e^n f(a) F_e^{-n} g(a) da \right. \\
&\quad \left. + \sum_{e \notin \tilde{G}_1} \sum_{n \neq 0} \int_{-A}^{\infty} F_e^n f(a) F_e^{-n} g(a) da \right] \\
&= (s(1-s) - s'(1-s')) \cdot \langle E_i^A(s), E_j^A(\bar{s}') \rangle.
\end{aligned}$$

The left hand side is

$$\sum_{e \in \tilde{S}} (\ell(e)^2 + A^2) \det \begin{pmatrix} F_e^0 f(-A) & F_e^0 g(-A) \\ (F_e^0 f)'(-A) & (F_e^0 g)'(-A) \end{pmatrix}.$$

What remains is to substitute the definition of $C(s)$ and $D(s)$, and to expand the result using the linearity of the determinant in each column. \square

Of particular interest is the Mass-Selberg relation in the cases $s = s'$ and $s = 1 - s'$. The first yields a relation between C and D that is equivalent to the symmetry of C , the second case implies a functional equation for surfaces of finite area, which generalises the functional equation of the classical scattering matrix.

3.8 Lemma. *The function ω_ℓ of theorem 3.7 satisfies*

$$(\ell^2 + a^2) \cdot \omega_\ell(s, 1 - s; a) = 1 - 2s.$$

Proof. Here ω_ℓ specialises to the Wronskian determinant of $h(\ell, s)$ and $h(\ell, 1 - s)$. We already noted in the proof of lemma 3.4 that

$$a \mapsto (\ell^2 + a^2) \cdot \omega_\ell(s, 1 - s; a)$$

is constant on $(-\infty, 0)$. \square

3.9 Corollary. *The approximate scattering matrix is symmetric, and it satisfies*

$$C(s) \cdot D(s)^t = D(s) \cdot C(s).$$

Proof. We begin with proving $C(s) \cdot D(s)^t = D(s) \cdot C(s)^t$, and this will imply the symmetry of $C(s)$. It is sufficient to consider $\operatorname{Re}(s) > 1$. There are no poles of C or of the approximate Eisenstein functions in this domain, and the truncated Eisenstein series are square-integrable by definition. The Mass-Selberg relation for $s = s'$ gives via the preceding lemma

$$\begin{aligned} 0 &= (1 - 2s) \sum_{e \in \tilde{S}} (D_{ie}(s) C_{je}(s) - C_{ie}(s) D_{je}(s)) \\ &= (1 - 2s) ((D(s) C(s)^t)_{ij} - (C(s) D(s)^t)_{ij}), \end{aligned}$$

so we have $C(s) \cdot D(s)^t = D(s) \cdot C(s)^t$. Lemma 3.5 implies that

$$C(s)(1 + C(s)X)^t = (1 + C(s)X)C(s)^t$$

holds for some symmetric matrix X , and in consequence

$$C(s) + C(s)XC(s)^t = C(s)^t + C(s)XC(s)^t.$$

\square

Apparently, the matrices $D(s)$ need not be symmetric in general. Rather, lemma 3.5 gives

$$\begin{aligned} D(s) - D(s)^t &= 1 + (2s - 1) \frac{4^{-s}\Gamma(s)^2}{\Gamma(1/2 + s)^2} C(s)\Sigma(s)\lambda^{2s-1} \\ &\quad - \left(1 + (2s - 1) \frac{4^{-s}\Gamma(s)^2}{\Gamma(1/2 + s)^2} \lambda^{2s-1}\Sigma(s)C(s) \right) \\ &= (2s - 1) \frac{4^{-s}\Gamma(s)^2}{\Gamma(1/2 + s)^2} [C(s), \Sigma(s)\lambda^{2s-1}]. \end{aligned} \quad (32)$$

In certain special cases, for example if all edges in the distinguished set S are phantom edges and the length $\ell(i)$ is independent of $i \in S$, the commutator in equation (32) vanishes.

As a second consequence of the Maass-Selberg relation, we obtain a functional equation for the approximate scattering matrices if the surface is of finite area. In this situation, there is also a functional equation for the approximate Eisenstein functions, corollary 3.11.

3.10 Corollary. *The following functional equation holds if Y is of finite area:*

$$D(s) \cdot D(1 - s)^t = C(s) \cdot C(1 - s).$$

Proof. It suffices to prove the equation for all regular points $s \neq 1/2$, $\operatorname{Re}(s) = 1/2$ of the approximate Eisenstein functions. Let $(s_n)_n$ and $(t_n)_n$ be sequences in $\{w \in \mathbb{C} \mid \operatorname{Re}(w) > 1/2\}$ that converge to s and $1-s$, respectively. We may assume that the Eisenstein functions $E_i(s_n)$, $E_i(t_n)$ exist. The truncated ones $E_i^A(s_n)$ and $E_i^A(t_n)$ are square-integrable, and the matrices $C(s_n)$, $D(s_n)$, $C(t_n)$ and $D(t_n)$ converge. It is well-known in the classical theory of Eisenstein series that this implies the existence of a uniform bound for the L^2 -norms of $E_i^A(s_n)$ and $E_i^A(t_n)$ (cf. Kubota [16, theorem 4.1.2]). Therefore

$$\lim_{n \rightarrow \infty} (s_n(1-s_n) - t_n(1-t_n)) \cdot \langle E_i^A(s_n), E_j^A(\bar{t}_n) \rangle = 0.$$

The right hand side of the Maass-Selberg relation converges to

$$\begin{aligned} (1-2s) \sum_{e \in \tilde{S}} (D_{ie}(s) D_{je}(1-s) - C_{ie}(s) C_{je}(1-s)) \\ = (1-2s) (D(s) \cdot D(1-s)^t - C(s) \cdot C(1-s)^t)_{ij}. \end{aligned}$$

□

3.11 Corollary. *Assume that Y is of finite area and that $D(s)$ is meromorphically invertible. Let $E(s)$ be the column vector that has one entry $E_j(s)$ for each $j \in \tilde{S}$. Then the functional equation*

$$E(s) = C(s) D(1-s)^{-1} \cdot E(1-s)$$

holds.

Proof. We define a square matrix $Q(s) = (q_{ij}(s))$ from the constant terms of the Eisenstein series, namely by

$$q_{ij}(s) := F_j^0 E_i(s).$$

If $h(s)$ denotes the diagonal matrix with entries $h(\ell(j), s)$, then

$$Q(s) = D(s) h(s) + C(s), h(1-s).$$

The functional equation 3.10 and the relation in 3.9 give

$$\begin{aligned} 0 &= D(s) h(s) + C(s) h(1-s) \\ &\quad - C(s) D(1-s)^{-1} \cdot (D(1-s) h(1-s) + C(1-s) h(s)) \\ &= Q(s) - C(s) D(1-s)^{-1} \cdot Q(1-s). \end{aligned}$$

The latter is the constant term matrix of the column $E(s) - C(s) D(1-s)^{-1} E(1-s)$. This is zero for the following reasons:

- Each entry is a linear combination of Eisenstein functions such that the constant term is 0 on each cylinder. Thus it is a meromorphic family of smooth eigenfunctions of the Laplacian. But if Y is compact, the spectrum is discrete.
- If Y is non-compact, this follows from vanishing of the constant term on the cusps as in Kubota [16, thm 4.4.2].

□

At the end of this subsection, we want to say a few words on the finite-area assumption in the functional equations.

Uniform boundedness of L^2 -norms, which allows to take the limit $\operatorname{Re}(s) \rightarrow 1/2$ of the Maass-Selberg relation, is essential in the proof. In the general case of geometrically finite surface with possibly infinite area, one knows from the definition of the approximate Eisenstein functions in terms of the resolvent

$$\|E_i^A(s)\| = O((\operatorname{Re}(s) - 1/2)^{-1}), \operatorname{Re}(s) \rightarrow 1/2.$$

The Maass-Selberg relation improves this bound:

3.12 Corollary. *Let $s \in \mathbb{C} \setminus \{1/2\}$ with $\operatorname{Re}(s) = 1/2$ be a regular point of the approximate Eisenstein functions, and let A be a positive number such that the truncated functions below are defined. If (s_n) is a sequence in $\{w \in \mathbb{C} \mid \operatorname{Re}(w) > 1/2\}$ that converges to s , then the norms $\|E_i^A(s_n)\|_{L^2(Y)}$ satisfy*

$$\|E_i^A(s_n)\|_{L^2(Y)} = O((\operatorname{Re}(s_n) - 1/2)^{-1/2}), n \rightarrow \infty.$$

Proof. The Maass-Selberg relation for $s = s_n$ and $s' = \bar{s}_n$ reads

$$\begin{aligned} (s_n - \bar{s}_n)(1 - 2\operatorname{Re}(s_n)) \|E_i^A(s)\|^2 \\ = \sum_{e \in \tilde{S}} (\ell(e)^2 + A^2) \left[|D_{ie}(s_n)|^2 \omega_{\ell(e)}(s_n, \bar{s}_n; -A) \right. \\ \quad + D_{ie}(s_n) C_{ie}(\bar{s}_n) \omega_{\ell(e)}(s_n, 1 - \bar{s}_n; -A) \\ \quad + C_{ie}(s_n) D_{ie}(\bar{s}_n) \omega_{\ell(e)}(1 - s_n, \bar{s}_n; -A) \\ \quad \left. + |C_{ie}(s_n)|^2 \omega_{\ell(e)}(1 - s_n, 1 - \bar{s}_n; -A) \right]. \end{aligned}$$

The assumptions now imply that the right hand side of this formula converges to

$$(1 - 2s) \sum_{e \in \tilde{S}} (|D_{ie}(s)|^2 - |C_{ie}(s)|^2).$$

□

One might expect the $E_i^A(s_n)$ in corollary 3.12 to satisfy a stronger estimate like $o((\operatorname{Re}(s_n) - 1/2)^{-1/2})$, so that the previous proof would lead to the same functional equation as in corollary 3.10. But it turns out that the limit

$$\lim_{n \rightarrow \infty} (2\operatorname{Re}(s_n) - 1) \|E_i^A(s_n)\|^2 = \sum_{e \in \tilde{S}} (|D_{ie}(s)|^2 - |C_{ie}(s)|^2).$$

does not vanish.

Up to now, only the *constant* term in the Fourier decomposition of an eigenfunction was given explicitly, namely as a linear combination of $h(\ell, s)$ and $h(\ell, 1 - s)$. The ordinary differential equation that is solved by arbitrary Fourier modes $F_e^n f$ of an eigenfunction f is

$$(\ell^2 + a^2)u''(a) + 2au'(a) + \left(s(1 - s) - \frac{4\pi^2 n^2}{\ell^2 + a^2} \right) u(a) = 0.$$

It is solved (on the complement of $a = 0$) by

$$h^n(\ell, s): a \mapsto |a|^{-s} (1 + \ell^2/a^2)^{-i\frac{\pi n}{\ell}} F(s/2 - i\frac{\pi n}{\ell}, 1/2 + s/2 - i\frac{\pi n}{\ell}; 1/2 + s; -\ell^2/a^2).$$

If the surface Y is of infinite area, and if e is a phantom edge with $\ell = \ell(e) \neq 0$, then $F_e^n(E_j(s))$ is a linear combination of $h^n(\ell, s)$ and $h^n(\ell, 1 - s)$ on the infinite half-cylinder Z_e^+ . It follows from square-integrability that the coefficient of $h^n(\ell, 1 - s)$ vanishes if $\operatorname{Re}(s) > 1/2$ and $n \neq 0$, so

$$F_e^n(E_j(s))(a) = c(s) h^n(\ell, s)(a), \quad a \in (0, \infty).$$

This shows

$$\int_A^\infty |(F_e^n E_j(s))(a)|^2 da \sim |c(s)|^2 \frac{A^{1-2\operatorname{Re}(s)}}{2\operatorname{Re}(s) - 1}, \quad \operatorname{Re}(s) \rightarrow 1/2.$$

Now suppose that $E_j^A(s) = o((\operatorname{Re}(s) - 1/2)^{-1/2})$ holds. The asymptotic shows $\lim_n |c(s_n)|^2 = 0$ for each sequence (s_n) that converges from the right to a point s with $\operatorname{Re}(s) = 1/2$. Thus the coefficient $c(s)$ must vanish on the critical line. But this means that c is the zero function, and therefore all higher Fourier modes $F_e^n E_j(s)$ vanish identically. This is not the case in general, because the approximate Eisenstein functions do approximate the Eisenstein functions at least if $\operatorname{Re}(s) > 1/2$.

3.1.2 Approximating the scattering theory of finite area surfaces

We proved in section 2 that the resolvent of the Laplacian depends continuously on $\lambda \in \mathcal{B}$, and this implies that the approximate Eisenstein functions $E_j(s)$ and scattering matrices $C(s)$ depend continuously on λ if $\operatorname{Re}(s) > 1/2$. In particular, if $\lambda_0 = (\ell_0, \tau_0) \in \partial B$ satisfies $\ell_0(j) = 0$ for all $j \in S$, and if (λ_n) is a sequence in \mathcal{B} that converges to λ_0 , then the approximate Eisenstein functions and scattering matrices converge to Eisenstein series and scattering matrix of the limit surface in the right half-plane. The following theorem summarises what was proved so far.

3.13 Theorem. *Let $\lambda_0 \in \mathcal{B}$ and $\operatorname{Re}(s_0) > 1/2$ such that $s_0(1 - s_0)$ belongs to the resolvent set of the Laplacian on $Y_G(\lambda_0)$.*

1. *There exists a neighbourhood $\mathcal{U} \subset \mathcal{B}$ of λ_0 and a neighbourhood V of s_0 such that each $s \in V$ is not in the spectrum of $Y_G(\lambda)$ for each $\lambda \in \mathcal{U}$. In particular, the approximate Eisenstein series $E_i(\lambda, s)$ and the matrices $C(\lambda, s)$ are always defined, and so is $D(\lambda, s)$ if $s - 1/2 \notin \mathbb{N}$.*
2. *Let $\Phi^* E_i(\lambda, s)$ denote the pull-back of $E_i(\lambda, s)$ to $Y_G(\lambda_0)$ via the trivialisation maps from section 1. Then $\Phi^* E_i(\lambda, s)$ depends continuously on $\lambda \in \mathcal{U}$ and $s \in V$ in the sense of locally uniform convergence on $Y_G(\lambda_0)$, the approximate scattering matrices $C(\lambda, s)$ and the $D(\lambda_k, s)$ depend continuously on λ .*

Proof. Part 1 is only a reminder of theorem 2.14. Additional poles of $D(\lambda, s)$ at $s - 1/2 \in \mathbb{N}$ might appear due to the poles of $s \mapsto h(\ell, 1 - s)$ (cf. lemma 3.4).

Theorem 2.14 and remark 2.15 also show that $\Phi E_i(\lambda, s)$ depends smoothly on s , at least up to the twist caused by a variation of τ . Now to determine the matrices $C(\lambda, s)$ and $D(\lambda, s)$, we only need the constant term of approximate Eisenstein functions and its first derivative in the complement of all closed geodesics that are associated with edges of the graph. \square

Behaviour of the approximate scattering data on the left of $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = 1/2\}$ is completely different. We examine this under the assumption that all surfaces are of finite area, i.e. we fix an admissible graph G of type $(p, 0)$ for the rest of this section. There are no phantom edges, and $Y_G(\lambda)$ is compact for each $\lambda \in \operatorname{int} \mathcal{B}$. If

$\lambda \in \partial\mathcal{B}$ then $Y_G(\lambda)$ is of finite area. We also fix a set $S \subset G_1$ as in definition 3.3, and therefore the parameter space for surfaces must be restricted to

$$\mathcal{B}_S := \{(\ell, \tau) \in \mathcal{B} \mid \ell(d) \neq 0 \text{ if } d \notin S\}.$$

In this situation, we can use the matrices $D(\lambda, s)$ in conjunction with the approximate Eisenstein functions and scattering matrices to achieve convergence on the complement of the critical axis, see the corollary below. Its proof is based on the following asymptotic.

3.14 Proposition. *Let $\lambda_0 = (\ell_0, \tau_0) \in \partial\mathcal{B}$ satisfy $\ell(d) = 0$ for all $d \in S$. Let (λ_k) be a sequence in $\text{int } \mathcal{B}$ that converges to λ_0 . If $\text{Re}(s) < 1/2$, then*

$$C(\lambda_k, s) \sim \lambda_k^{1-2s} (1-2s) \frac{4^{-(1-s)} \Gamma(1-s)^2}{\Gamma(3/2-s)^2} \Sigma(1-s), \quad k \rightarrow \infty,$$

where the notation is that of lemma 3.5.

Proof. If $\lambda = (\ell, \tau) \in \text{int } \mathcal{B}$, the resolvent $s \mapsto (\Delta_\lambda - s(1-s))^{-1}$ on $Y_G(\lambda)$ is meromorphic on the complex plane for the spectrum is purely discrete. Thus the approximate Eisenstein functions are given by

$$E_i(\lambda, s) = (\Delta_\lambda - s(1-s))^{-1}([\chi, \Delta_\lambda]h(\ell(i), s)) + \chi h(\ell(i), s)$$

even if $\text{Re}(s) < 1/2$. Theorem 2.14 implies that the pull-back of $E_i(\lambda_k, s)$ on $Y_G(\lambda_0)$ converges to

$$\tilde{E}_i(s) := (\Delta_{\lambda_0} - s(1-s))^{-1}([\Delta_{\lambda_0}, \chi]h(\ell_0(i), s)) + \chi h(\ell_0(i), s) \quad \text{if } \text{Re}(s) < 1/2.$$

Note that $\tilde{E}_i(s)$ is defined in terms of the resolvent of the Laplacian, we do not need a meromorphic continuation across the critical axis. So $\tilde{E}_i(s) - \chi h(\ell_0(i), s)$ is square-integrable, and the constant term of this function on Z_j^- is

$$F_j^0 \tilde{E}_i(s) = \alpha_{ij}(s) h(\ell_0(j), s) + \beta_{ij}(s) h(\ell_0(j), 1-s)$$

with $\beta_{ij}(s) = 0$ if $\text{Re}(s) < 1/2$. On the other hand we know that $\beta_{ij}(s)$ is the limit of $C_{ij}(\lambda_k, s)$, and this gives

$$\lim_{k \rightarrow \infty} C(\lambda_k, s) = 0, \quad \text{Re}(s) < 1/2. \quad (33)$$

The same can be concluded for $D(\lambda_k, s)$ by the functional equation 3.10: We know that $D(\lambda_0, s)$ is the identity matrix, and that it is the limit of $D(\lambda_k, s)$ if $\text{Re}(s) > 1/2$. This implies that $s \mapsto D(\lambda_k, s)$ is meromorphically invertible if k is sufficiently large, and the functional equation gives

$$D(\lambda_k, s) = C(\lambda_k, s) \cdot C(\lambda_k, 1-s) \cdot (D(\lambda_k, 1-s))^t^{-1}.$$

If $\text{Re}(s) < 1/2$ the rightmost factor converges to the identity and $C(\lambda_k, 1-s)$ converges to the scattering matrix, so $\lim_k D(\lambda_k, s) = 0$ follows from (33).

Now the claim is proved by lemma 3.5:

$$\begin{aligned} C(\lambda_k, s) &= (D(\lambda_k, s) - 1) \left((2s-1) \frac{4^{-s} \Gamma(s)^2}{\Gamma(1/2+s)^2} \Sigma(s) \lambda^{2s-1} \right)^{-1} \\ &= (D(\lambda_k, s) - 1) \left(-\lambda^{1-2s} (1-2s) \frac{4^{-(1-s)} \Gamma(1-s)^2}{\Gamma(3/2-s)^2} \Sigma(1-s) \right). \end{aligned}$$

□

Proposition 3.14 suggests that the approximate scattering matrix should be replaced with the matrix defined now.

3.15 Definition. Let \mathcal{B}'_S be the set of all $\lambda \in \mathcal{B}_S$ such that $s \mapsto D(\lambda, s)$ is meromorphically invertible. Then we define $C'(\lambda, s)$ for each $\lambda \in \mathcal{B}'_S$ to be the meromorphic family in s given by

$$C'(\lambda, s) := D(\lambda, s)^{-1} \cdot C(\lambda, s).$$

Recall that $D(\lambda, s) = 1$ if $\lambda(j) = 0$ holds for each $j \in S$, so $C'(\lambda, s)$ again coincides with the scattering matrix. It turns out that there are additional properties that $C'(\lambda, s)$ shares with scattering matrices.

3.16 Lemma. *The $C'(\lambda, s)$ are symmetric, they are unitary if $\operatorname{Re}(s) = 1/2$, and they satisfy the functional equation*

$$C'(\lambda, 1-s) \cdot C'(\lambda, s) = 1.$$

Proof. We leave out λ in the notation. Symmetry is a consequence of both statements in corollary 3.9:

$$(C'(s))^t = C(s) \cdot (D(s)^t)^{-1} = D(s)^{-1} \cdot C(s) = C'(s).$$

By definition of C' this implies

$$C'(1-s) \cdot C'(s) = C'(1-s)^t \cdot C'(s) = C(1-s) \cdot (D(s) D(1-s)^t)^{-1} \cdot C(s),$$

and this is the identity matrix by 3.10. So if $\operatorname{Re}(s) = 1/2$

$$C'(s)^* = C'(\bar{s})^t = C'(\bar{s}) = C'(1-s) = C'(s)^{-1}.$$

□

We conclude this section with a corollary of theorem 3.13, of the functional equation for C' , and of corollary 3.11.

3.17 Corollary. *Let $E(\lambda, s)$ be the column vector of approximate Eisenstein functions as in corollary 3.11, and let $E'(\lambda, s) := D(\lambda, s)^{-1} \cdot E(\lambda, s)$. Then*

$$s \mapsto \Phi^* E'(\lambda, s) \quad \text{and} \quad s \mapsto C'(\lambda, s)$$

depend continuously on $\lambda \in \mathcal{B}'_S$ as meromorphic families on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \neq 1/2\}$.

3.2 The Selberg Zeta function

We extend proposition 1.8 to the domain $\operatorname{Re}(s) > 1/2$, thus proving Wolpert's second conjecture. As in section 1.4, \mathcal{Z}_λ is the Selberg Zeta function of $Y_G(\lambda)$, and the contribution of a particular edge $d \in G_1^*$ to the Zeta function is denoted by $\mathcal{Z}_{d,\lambda}$.

3.18 Theorem. *With respect to the topology of locally uniform convergence of functions on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$, the map $\lambda \mapsto \mathcal{Z}_\lambda / \prod_{d \in G_1^*} \mathcal{Z}_{d,\lambda}$ is continuous on \mathcal{B} .*

Before we come to the proof, let us state an immediate corollary of this theorem and of the functional equation for \mathcal{Z}_λ for surfaces of finite area. It can be used to show that there is no straight-forward extension of theorem 3.18 to the left of the critical axis [25]. At the end of this section, we will use this corollary to prove theorem 3.23, which covers 3.18 but adds convergence on the left half-plane if the non-degenerate surfaces are compact.

3.19 Corollary. Let $\lambda_0 = (\ell_0, \tau_0) \in \partial B$ be a point where $Y(\lambda_0)$ is the union of elementary cusps and a surface of finite area. The area of the latter surface is denoted by $|F|$. Let (λ_m) be a sequence in \mathcal{B} that converges to λ_0 . Then

$$\lim_{m \rightarrow \infty} \frac{\mathcal{Z}_{\lambda_m}(s)}{\prod_{d \in S} \mathcal{Z}_{d, \lambda_m}(s)} = \pm \mathcal{Z}_{\lambda_0}(1-s) \det(C(\lambda_0, 1-s)) \cdot \left(\frac{\Gamma(1/2+s)}{\Gamma(3/2-s)} \right)^k \\ \cdot \exp\left(|F| \int_0^{s-1/2} t \tanh(\pi t) dt + (2s-1)k \log 2 \right)$$

holds in $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$. Here $C(\lambda_0, s)$ is the scattering matrix of $Y(\lambda_0)$, and k is the number of cusps. In particular, if the $Y(\lambda_n)$ are compact for all $n \neq 0$, we have for $\operatorname{Re}(s) > 1/2$

$$\lim_{m \rightarrow \infty} \frac{\mathcal{Z}_{\lambda_m}(1-s)}{\prod_{d \in S} \mathcal{Z}_{d, \lambda_m}(s)} = \mathcal{Z}_{\lambda_0}(1-s) \det(C(\lambda_0, 1-s)) \cdot 2^{(2s-1)k} \left(\frac{\Gamma(1/2+s)}{\Gamma(3/2-s)} \right)^k.$$

The sign in the first formula above depends on the determinant of the scattering matrix for $Y(\lambda_0)$ at $s = 1/2$. According to Lax and Phillips [17, Prop. 8.14], this determinant equals $(-1)^k$. It vanishes in the second formula as the number of cusps is even and there are no elementary components.

Now we prove Theorem 3.18. The logarithmic derivative of \mathcal{Z}_λ is related with the resolvent of the Laplacian by the trace formula, as stated in appendix B. More precisely, if $\operatorname{Re}(s) > 1/2$ the functions

$$h_s(\xi) = (1/4 + \xi^2 - s(1-s))^{-1}, \quad g_s(u) = (2s-1)^{-1} e^{-(s-1/2)|u|}$$

correspond via the Selberg transform to

$$k_s: t \mapsto \frac{4^{s-1}}{\pi} \int_0^\infty \frac{(\sqrt{w+t} + \sqrt{w+t+4})^{1-2s}}{\sqrt{w}\sqrt{w+t}\sqrt{w+t+4}} dw \\ = \frac{4^{s-1}}{\pi} \int_0^1 (x(1-x))^{s-1} (4x+t)^{-s} dx.$$

The trace formula is applicable to the difference $h_s - h_{s_0}$ if $\operatorname{Re}(s) > 1$ and $\operatorname{Re}(s_0) > 1$, and this yields for $Y = Y(\lambda)$

$$\lim_{A \rightarrow 0} \int_{Y^+} \chi_A K^0 \operatorname{vol} = \frac{1}{2s-1} \sum_{c \in \mathcal{C}} \sum_{n=1}^\infty \frac{\ell(c) e^{-(s-1/2)n\ell(c)}}{e^{n\ell(c)/2} - e^{-n\ell(c)/2}} \\ - \frac{1}{2s_0-1} \sum_{c \in \mathcal{C}} \sum_{n=1}^\infty \frac{\ell(c) e^{-(s_0-1/2)n\ell(c)}}{e^{n\ell(c)/2} - e^{-n\ell(c)/2}} \\ = \frac{1}{2s-1} \frac{\mathcal{Z}'_\lambda(s)}{\mathcal{Z}_\lambda(s)} - \frac{1}{2s_0-1} \frac{\mathcal{Z}'_\lambda(s_0)}{\mathcal{Z}_\lambda(s_0)}.$$

The surface Y^+ is a disjoint union of $Y(\lambda)$ and one pair \bar{Z}_d of elementary cusps for each $d \in G_1^*$ with $\ell(d) = 0$. We add to Y^+ an elementary cylinder \bar{Z}_d for each $d \in G_1^*$ with $\ell(d) \neq 0$. This gives

$$Y^* := Y^+ \cup \bigcup_{\substack{d \in G_1^* \\ \ell(d) \neq 0}} \bar{Z}_d = Y \cup \bigcup_{d \in G_1^*} \bar{Z}_d,$$

where each \bar{Z}_d is an isometric copy of the elementary cylinder $\langle (x, a) \mapsto (x+1, a) \rangle \setminus X_{\ell(d)}$. In analogy with the trace formula above, the logarithmic derivative of

the quotient $\mathcal{Z}_\lambda / \prod_d \mathcal{Z}_{d,\lambda}$ will be realised as an integral over Y^* , so that the newly added \bar{Z}_d yield a logarithmic derivative of $\mathcal{Z}_{d,\lambda}(s_0) / \mathcal{Z}_{d,\lambda}(s)$ each. On the other hand, the surface Y^* already appeared in section 2.2, where a family of trace class operators on $L^2(Y)$ was defined by

$$T(s) = (\Delta - s(1-s))^{-1} - (\Delta - s_0(1-s_0))^{-1} - \sum_{d \in G_1^*} \psi_d(R_d(s) - R_d(s_0))\phi_d.$$

The sum on the right hand side consists of operators on the \bar{Z}_d that are pulled back to Y . The following proposition computes the trace by integration of the operator kernel. The function K^0 on Y^* that appears in the statement is defined, as in appendix B, by the truncated symmetrisation of $k_s - k_{s_0}$ over a uniformising group for each component of Y , and by that of $-(k_s - k_{s_0})$ for each component of $Y^* \setminus Y$, and $\phi_0 = 1 - \sum_{d \in G_1^*} \phi_d$.

3.20 Proposition. *If $\operatorname{Re}(s) > 1$ and $\operatorname{Re}(s_0) > 1$, the trace of*

$$(\Delta - s(1-s))^{-1} - (\Delta - s_0(1-s_0))^{-1} - \sum_{d \in G_1^*} \psi_d(R_d(s) - R_d(s_0))\phi_d$$

is equal to the following expression:

$$\begin{aligned} & \frac{1}{2s-1} \left(\frac{\mathcal{Z}'_\lambda(s)}{\mathcal{Z}_\lambda(s)} - \sum_{d \in G_1^*} \frac{\mathcal{Z}'_{d,\lambda}(s)}{\mathcal{Z}_{d,\lambda}(s)} \right) - \frac{1}{2s_0-1} \left(\frac{\mathcal{Z}'_\lambda(s_0)}{\mathcal{Z}_\lambda(s_0)} - \sum_{d \in G_1^*} \frac{\mathcal{Z}'_{d,\lambda}(s_0)}{\mathcal{Z}_{d,\lambda}(s_0)} \right) \\ & - \sum_{d \in G_1^*} \int_{\bar{Z}_d} (1 - \bar{\phi}_d) K^0 \operatorname{vol} + \int_Y \phi_0 \operatorname{vol} \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} \xi \cdot (h_s(\xi) - h_{s_0}(\xi)) \tanh(\pi\xi) d\xi. \end{aligned} \tag{34}$$

Proof. The operator is known to be of trace class, and the trace can be computed by integrating its integral kernel over the diagonal in $Y \times Y$. If $\operatorname{Re}(s) > 1$ and $\operatorname{Re}(s_0) > 1$ the kernel for each operator in the definition of T is given as the symmetrisation of $\pm(k_s - k_{s_0})$ for all components of Y^* . The overall contribution of the identity is

$$\begin{aligned} & \int_Y \left(1 - \sum_{d \in G_1^*} \psi_d \phi_d \right) \operatorname{vol} \cdot (k_s(0) - k_{s_0}(0)) \\ & = \int_Y \phi_0 \operatorname{vol} \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} \xi \cdot (h_s(\xi) - h_{s_0}(\xi)) \tanh(\pi\xi) d\xi. \end{aligned}$$

If $\ell(d) \neq 0$, the remaining contribution of $\psi_d(R_d(s) - R_d(s_0))\phi_d$ is

$$\int_{\bar{Z}_d} \bar{\phi}_d K^0 \operatorname{vol} = \frac{1}{2s-1} \frac{\mathcal{Z}'_{d,\lambda}(s)}{\mathcal{Z}_{d,\lambda}(s)} - \frac{1}{2s_0-1} \frac{\mathcal{Z}'_{d,\lambda}(s_0)}{\mathcal{Z}_{d,\lambda}(s_0)} - \int_{\bar{Z}_d} (1 - \bar{\phi}_d) K^0 \operatorname{vol},$$

and the trace formula, as applied at the beginning of this paragraph, completes the proof. \square

We proved in theorem 2.14 and in proposition 2.13 that the trace depends continuously on $\lambda \in \mathcal{B}$, and so do the correction terms in equation (34) by lemma 2.8. Hence we see that the logarithmic derivative of $\mathcal{Z}_\lambda / \prod_{d \in G_1^*} \mathcal{Z}_{d,\lambda}$ is continuous in λ . In view of equation (23), proposition 3.20 provides an analytic continuation of \mathcal{Z}_λ from its domain of convergence to the half plane $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$. The theorem follows from proposition 1.8 by integration of the logarithmic derivative.

In a similar way we obtain the following observation, Wolpert's *Conjecture 1*.

3.21 Theorem. If $\lambda \in \mathcal{B}$ let $\mathcal{N}(\lambda, s) := \prod_{t < 1/4} (t - s(1 - s))$, where t runs through the eigenvalues of the Laplacian below $1/4$. Let $K \subset \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$ and $\mathcal{U} \subset \mathcal{B}$ be relatively compact subsets. Then there exist positive numbers α, β such that

$$\alpha \leq \left| \mathcal{N}(\lambda, s)^{-1} \cdot \mathcal{Z}_\lambda(s) / \prod_{d \in G_1^*} \mathcal{Z}_{d, \lambda}(s) \right| \leq \beta$$

holds for all $s \in K$ and $\lambda \in \mathcal{U}$.

Proof. We only need to subtract the singular part of the operator in proposition 3.20, to get a holomorphic family of operators on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1/2\}$ that is continuous in λ . If $t \in [0, 1/4)$ is an eigenvalue of the Laplacian, the corresponding singular part of $s \mapsto (\Delta - s(1 - s))^{-1}$ is

$$\frac{1}{t - s(1 - s)} \cdot \operatorname{pr}_t = \frac{1}{2s - 1} \cdot \frac{2s - 1}{t - s(1 - s)} \cdot \operatorname{pr}_t,$$

where pr_t denotes projection onto the eigenspace. Now $\frac{2s-1}{t-s(1-s)}$ is the contribution of a particular eigenvalue to the logarithmic derivative of $\mathcal{N}(\lambda, s)$. \square

In order to examine the Zeta function on the left, we need the precise asymptotics of the contribution of a single geodesic as its length decreases.

3.22 Lemma. Define $\mathcal{Z}_l(s) := \prod_{k=0}^{\infty} (1 - e^{-(s+k)l})^2$ for all $l > 0$. Then

$$\lim_{l \rightarrow 0} \Gamma(s)^2 \mathcal{Z}_l(s) e^{\pi^2/3l} l^{2s-1} = 2\pi.$$

Proof. It is sufficient to prove the lemma if $\operatorname{Re}(s) > 0$, for it can be continued iteratively to the left using the functional equation of Γ . Like Wolpert and Hejhal we begin with

$$\begin{aligned} \log \mathcal{Z}_l(s) &= 2 \sum_{k=0}^{\infty} \log(1 - e^{-(s+k)l}) = -2 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-(s+k)nl}}{n} \\ &= -2 \sum_{n=1}^{\infty} \frac{e^{-snl}}{n} (1 - e^{-nl})^{-1}. \end{aligned}$$

This expression is split into three part according to

$$\begin{aligned} (1 - e^{-nl})^{-1} &= \frac{1}{2} + \frac{1}{nl} + \left[(1 - e^{-nl})^{-1} - \frac{1}{nl} - \frac{1}{2} \right] \\ &= \frac{1}{2} + \frac{1}{nl} + \left[(e^{nl} - 1)^{-1} - \frac{1}{nl} + \frac{1}{2} \right]. \end{aligned}$$

The contribution of the first term to $\log \mathcal{Z}_l(s)$ is

$$- \sum_{n=1}^{\infty} \frac{e^{-snl}}{n} = \log(1 - e^{-sl}). \tag{35}$$

The second summand yields Euler's dilogarithm:

$$-2l^{-1} \sum_{n=1}^{\infty} \frac{e^{-snl}}{n^2} = -2l^{-1} \operatorname{Li}_2(e^{-sl}),$$

and the dilogarithm satisfies $\text{Li}_2(z) = \pi^2/6 - \log(z) \cdot \log(1-z) - \text{Li}_2(1-z)$, so

$$\begin{aligned} -2l^{-1} \sum_{n=1}^{\infty} \frac{e^{-snl}}{n^2} &= -\frac{\pi^2}{3l} - 2s \log(1 - e^{-sl}) + 2l^{-1} \text{Li}_2(1 - e^{-sl}) \\ &\sim -\frac{\pi^2}{3l} - 2s \log(1 - e^{-sl}) + 2s, \quad l \rightarrow 0. \end{aligned} \quad (36)$$

Finally, as $l \rightarrow 0$, we have [6, p. 21, eq. (4)]

$$\begin{aligned} -2l \sum_{n=1}^{\infty} \frac{e^{-snl}}{nl} \left[(e^{nl} - 1)^{-1} - \frac{1}{nl} + \frac{1}{2} \right] \\ \longrightarrow -2 \int_0^{\infty} t^{-1} e^{-st} \left[(e^t - 1)^{-1} - \frac{1}{t} + \frac{1}{2} \right] dt \\ = -2 \log \Gamma(s) + (2s - 1) \log s - 2s + \log(2\pi). \end{aligned} \quad (37)$$

The sum of (35),(36),(37) gives the result. \square

Now we want to consider the Zeta function on the left of the critical axis. The theorem below relies on the asymptotic of the approximate scattering matrix given in proposition 3.14. We must therefore restrict to a degenerating family of compact surfaces.

3.23 Theorem. *Let G be a graph of type $(p, 0)$ and $\lambda_0 = (\ell_0, \tau_0) \in \partial\mathcal{B}$. Let S be the set of edges d with $\ell_0(d) = 0$. If (λ_m) is a sequence in $\text{int } \mathcal{B}$ that converges to λ_0 , then*

$$\lim_{m \rightarrow \infty} \frac{\det D(\lambda_m, s) \cdot \mathcal{Z}_{\lambda_m}(s)}{\prod_{d \in S} \mathcal{Z}_{d, \lambda_m}(s)} = \mathcal{Z}_{\lambda_0}(s)$$

holds on $\{s \in \mathbb{C} \mid \text{Re}(s) \neq 1/2\}$.

Proof. The statement is an extension of theorem 3.18 to the left of the critical axis: the matrices $D(\lambda_m, s)$ converge to the identity matrix if $\text{Re}(s) > 1/2$, so multiplication of the quotient $\mathcal{Z}(\lambda_m, s) / \prod_d \mathcal{Z}_{d, \lambda_m}$ with $\det D(\lambda_m, s)$ does not alter the limit in this domain.

Then the proof consists in the application of a number of formulas stated earlier. Let k denote the number of cusps in $Y_G(\lambda_0)$. By corollary 3.19 we have for all s with $\text{Re}(s) > 1/2$

$$\lim_{m \rightarrow \infty} \frac{\mathcal{Z}_{\lambda_m}(1-s)}{\prod_{d \in S} \mathcal{Z}_{d, \lambda_m}(s)} = \mathcal{Z}_{\lambda_0}(1-s) \det(C(\lambda_0, 1-s)) \cdot 2^{(2s-1)k} \left(\frac{\Gamma(1/2+s)}{\Gamma(3/2-s)} \right)^k.$$

In corollary 3.17 we proved $\lim_m D(\lambda_m, 1-s)^{-1} C(\lambda_m, 1-s) = C(\lambda_0, 1-s)$, so

$$\lim_{m \rightarrow \infty} \frac{\det D(\lambda_m, 1-s)}{\det C(\lambda_m, 1-s)} \frac{\mathcal{Z}_{\lambda_m}(1-s)}{\prod_d \mathcal{Z}_{d, \lambda_m}(s)} = \mathcal{Z}_{\lambda_0}(1-s) \frac{\Gamma(1/2+s)^k}{\Gamma(3/2-s)^k} 2^{(2s-1)k}. \quad (38)$$

Proposition 3.14 gives

$$\det C(\lambda_m, 1-s) \sim \left(\prod_{d \in \tilde{S}} \ell(d)^{2s-1} \right) (2s-1)^k \frac{4^{-ks} \Gamma(s)^{2k}}{\Gamma(1/2+s)^{2k}} \det \Sigma(s),$$

and

$$\det \Sigma(s) = \left(\frac{1}{\cos(\pi s)^2} - 1 \right)^{k/2} = \frac{\Gamma(1/2+s)^k \Gamma(1/2-s)^k}{\Gamma(s)^k \Gamma(1-s)^k}$$

implies

$$\det C(\lambda_m, 1-s) \sim \left(\prod_{d \in \tilde{S}} \ell(d)^{2s-1} \right) (2s-1)^k \frac{4^{-ks} \Gamma(s)^k}{\Gamma(1/2+s)^k} \frac{\Gamma(1/2-s)^k}{\Gamma(1-s)^k}.$$

We substitute this into (38) to see

$$\lim_{m \rightarrow \infty} \frac{\det D(\lambda_m, 1-s) \mathcal{Z}_{\lambda_m}(1-s) \Gamma(1-s)^k}{\prod_{d \in S} (\mathcal{Z}_{d, \lambda_m}(s) \ell(d)^{4s-2}) \Gamma(s)^k} = \mathcal{Z}_{\lambda_0}(1-s).$$

Thus the proof is reduced to

$$\frac{\mathcal{Z}_{d, \lambda_m}(s) \ell(d)^{4s-2} \Gamma(s)^2}{\mathcal{Z}_{d, \lambda_m}(1-s) \Gamma(1-s)^2} \longrightarrow 1,$$

which follows from lemma 3.22. \square

A Families of Fuchsian groups

Throughout this appendix, the hyperbolic plane is identified with its unit-disc model D . By $d(z_1, z_2)$ we mean the hyperbolic distance between two points. Let Γ denote a finitely generated group without torsion that admits a continuous family $\phi: \mathcal{B} \rightarrow \text{hom}(\Gamma, \text{isom}(D))$ of discrete inclusions into the orientation-preserving isometries of D . The image of $\phi(b)$ is denoted by Γ_b , and the image of any $\gamma \in \Gamma$ under $\phi(b)$ by γ_b . By continuity we mean that $b \mapsto \gamma_b$ is a continuous map for each $\gamma \in \Gamma$. This is equivalent to the continuity of $b \mapsto \Gamma_b$, where the set of closed subgroups of $\text{isom}(D)$ carries the Chabauty topology. We assume \mathcal{B} to be a path-connected, locally compact and metrisable space.

We assume that the Γ_b are non-elementary. The only element of finite order in Γ_b is the identity. An action of Γ on $\mathcal{B} \times D$ is defined by $\gamma(b, z) := (b, \gamma_b z)$. Our first observation is that this action is freely discontinuous, i.e. every element of $\mathcal{B} \times D$ has a neighbourhood U such that $\gamma U \cap U \neq \emptyset$ implies $\gamma = 1$.

A.1 Lemma. *The induced action of Γ on $\mathcal{B} \times D$ is freely discontinuous.*

Proof. We use the fact that the inequality

$$\sinh(d(z, \gamma_1 z)/2) \sinh(d(z, \gamma_2 z)/2) \geq 1$$

is satisfied for any $z \in D$ if the group generated by $\gamma_1, \gamma_2 \in \text{isom}(D)$ is torsion-free and non-elementary (Beardon [1, §8.3, p. 198]).

Let $(b_0, z_0) \in \mathcal{B} \times D$, the aim is to prove that Γ acts freely discontinuously in (b_0, z_0) . Choose $\tau \in \Gamma \setminus \{1\}$. Since \mathcal{B} is locally compact, there exist neighbourhoods $V \subset \mathcal{B}$ of b_0 and $W \subset D$ of z_0 such that $d(\phi(b)(\tau)z, z) > \epsilon > 0$ for all $(b, z) \in V \times W$. The cited inequality implies $d(\phi(b)(\gamma)z, z) > 2 \text{Arsinh}(\epsilon^{-1})$ for $(b, z) \in V \times W$ if γ is not contained in the centraliser $Z(\tau)$. Now the images of $Z(\tau)$ under the homomorphisms considered are elementary and discrete, so we may choose a neighbourhood U' of (b_0, z_0) with the desired property for $Z(\tau)$. Then

$$U' \cap (V \times \{z \in D \mid d(z, z_0) < \text{Arsinh}(\epsilon^{-1})\}) \cap W$$

is a suitable neighbourhood of (b_0, z_0) . \square

The previous lemma allows for a definition of Γ -invariant structures on $\mathcal{B} \times D$, e.g. Riemannian metrics or vector fields if the action of Γ is differentiable. We will stick to the latter in order to define locally a trivialisation of the quotient $\Gamma \backslash \mathcal{E} \rightarrow \mathcal{B}$, where \mathcal{E} denotes the fibre-wise Nielsen domain for the group action. We must restrict to the Nielsen domain as we explicitly want to allow the type of an isometry γ_b to change from parabolic to hyperbolic and vice versa.

Of particular importance in this respect are those primitive elements of a Fuchsian group that are related to the infinite parts of the corresponding quotient of the hyperbolic plane. They are specified in the following definition.

A.2 Definition. An element $\gamma \in \Gamma \setminus \{1\}$ is called a *boundary pairing* if it is not a proper multiple of any other element of Γ , and if there exists $b \in \mathcal{B}$ such that γ_b either is parabolic or it leaves invariant a component of the set of discontinuity in ∂D .

The definition refers to a property of γ_b for some particular $b \in \mathcal{B}$, but the next lemma shows that it is in fact independent of b .

A.3 Lemma. *If γ is a boundary pairing, then the condition in definition A.2 is satisfied by γ_b for all $b \in \mathcal{B}$.*

Proof. Suppose that the condition is satisfied by γ_{b_0} but not by γ_{b_1} . There exists a curve $t \mapsto b_t$ from b_0 to b_1 , and two curves c_1, c_2 in ∂D such that $c_1(t)$ and $c_2(t)$ are the fixed points of γ_{b_t} for all t . The fixed points of non-trivial elements of Γ_{b_1} are dense in the limit set. By assumption, γ_{b_1} is hyperbolic and each component of $\partial D \setminus \{c_1(1), c_2(1)\}$ contains a fixed point of an element η_{b_1} or η'_{b_1} . Now $\partial D \setminus \{c_1(0), c_2(0)\}$ either consists of one component, or one of its components is a subset of the set of discontinuity. This implies that the intersection of $\{c_1(t), c_2(t)\}$ with the set of fixed points of $\eta_{b(t)}$ or $\eta'_{b(t)}$ is not empty for some t . This is a contradiction to discreteness of $\Gamma_{b(t)}$. \square

Now we want to construct a trivialisation of $\Gamma \backslash \mathcal{E} \rightarrow \mathcal{B}$ from an invariant vector field s on $\mathcal{B} \times D$. For convenience we restrict to a smooth family of Fuchsian groups parametrised by a compact interval \mathcal{B} . Let $\Gamma' \subset \Gamma$ be the set of boundary pairings. Let us recall the definition of \mathcal{E} . If $\gamma \in \Gamma'$ and γ_b is hyperbolic, its axis separates D into two components. Precisely one of them, denoted by $D_\gamma^\perp(b)$, is not bounded at ∂D by a component of the set of discontinuity. If γ_b is parabolic, we define $D_\gamma^\perp(b)$ to be empty. Then the fibre-wise Nielsen domain is the following Γ -invariant subset of $\mathcal{B} \times D$:

$$\mathcal{E} := \bigcup_{b \in \mathcal{B}} \bigcap_{\gamma \in \Gamma'} \{b\} \times D_\gamma^\perp(b).$$

The projection $\Gamma \backslash \mathcal{E} \rightarrow \mathcal{B}$ is not a proper map if any of the Fuchsian groups contain parabolic elements. If we want to apply the flow of a vector field to trivialise this map, we must impose additional conditions on the vector field. These ensure that for each compact neighbourhood $\mathcal{U} \subset \text{int } \mathcal{B}$ of a point b_0 there exists a relatively compact set $K \subset D$ such that the following holds:

- If $\Phi_b: \mathcal{U} \times K \rightarrow \mathcal{B} \times D$, $|b - b_0| < \epsilon$, denotes the flow of the vector field, then \mathcal{E} is invariant under Φ_t .
- The map $\Phi_b|_{\{b_0\} \times K}$ extends to a diffeomorphism $\mathcal{E}_{b_0} \rightarrow \mathcal{E}_b$ for all $|b - b_0| < \epsilon$.

Essentially this means that we must require the vector field to be well-behaved near the fibre-wise boundary of $\Gamma \backslash \mathcal{E} \subset \Gamma \backslash (\mathcal{B} \times D)$. We give a precise definition of a neighbourhood of this boundary in terms of covering subsets \tilde{C}_γ of collars (figure 5):

There is an open subset \tilde{C}_γ of \mathcal{E} associated with each $\gamma \in \Gamma'$. If $\ell(\gamma_b)$ denotes the hyperbolic translation of γ_b , resp. $\ell(\gamma_b) = 0$ if γ_b is parabolic, there exists an orientation preserving isometry $X_{\ell(\gamma_b)}^- \rightarrow D_\gamma^\perp(b)$ such that the isometry $(x, a) \mapsto (x \pm 1, a)$ of $X_{\ell(\gamma_b)}^- \rightarrow D_\gamma^\perp(b)$ corresponds to γ_b (cf. p. 7). The isometry is uniquely determined up to composition on the left with maps $(x, a) \mapsto (x + \delta, a)$. So we can define the fibre of \tilde{C}_γ over b to be the the image of $X_{\ell(\gamma_b)}^{A(\ell(\gamma_b))}$, where

$$A(\ell) = \begin{cases} \left(-\frac{\ell}{2 \sinh(\ell/2)}, 0\right) & \text{if } \ell \neq 0 \\ (-1, 0) & \text{if } \ell = 0. \end{cases}$$

If ℓ_0 satisfies $\ell_0 < \ell(\gamma_b)$ for all $b \in \mathcal{B}$, we define $\tilde{C}_\gamma^{\ell_0} \subset \tilde{C}_\gamma$ as the fibre-wise image of $X_{\ell(\gamma_b)}^{A(\ell_0)}$. The Fuchsian groups are required to be geometrically finite, so there are only finitely many conjugacy classes of boundary pairings. This implies that such a number ℓ_0 can be chosen independently of $\gamma \in \Gamma'$.

A.4 Lemma. *Let s be a smooth, Γ -invariant lift of the canonical vector field ∂_b on the interval \mathcal{B} . Provided that the integral flow of s defines a diffeomorphism of the fibres of $\bigcup_\gamma \tilde{C}_\gamma^{\ell_0} \rightarrow \mathcal{B}$ for some $\ell_0 > 0$, it extends these diffeomorphisms to a local trivialisation of $\mathcal{E} \rightarrow \mathcal{B}$.*

Proof. We verify that the two properties above are satisfied. Invariance of \mathcal{E} is immediate from the condition on s .

The set K will be defined in terms of Dirichlet domains for the Fuchsian groups. Choose $z_0 \in D$. If $\gamma \in \Gamma$ and $b \in \mathcal{B}$, the open half-plane

$$D_\gamma(b) := \{z \in D \mid d(z, z_0) < d(z, \gamma_b z_0)\}$$

is bounded by the geodesic

$$L_\gamma(b) := \{z \in D \mid d(z, z_0) = d(z, \gamma_b z_0)\}.$$

The Dirichlet domain $D(b)$ of Γ_b with centre z_0 is $\bigcap_\gamma D_\gamma(b)$. Due to finite geometry, for fixed $b \in \mathcal{B}$ this intersection can be replaced with an intersection of finitely many half-planes, say of those associated with $\gamma_1, \dots, \gamma_n \in \Gamma$.

We need to have a closer look at those points where the Euclidean closure of $D(b)$ meets ∂D . This only happens in a component of the set of discontinuity or in a parabolic fixed point for the following reason: The limit set of a geometrically finite group only consists of conical limit points and parabolic fixed points. But a convex fundamental polygon of such a group cannot meet ∂D in a point of approximation [1, thm 10.2.3 and thm 10.2.5]. This observation implies the existence of finitely many boundary pairings η_1, \dots, η_j and of a compact neighbourhood K of z_0 such that

$$D(b) \setminus K \subset \bigcup_{k=1}^j \left(D_{\eta_k}^\perp(b) \cup \tilde{C}_{\eta_k}^{\ell_0} \right) \quad (39)$$

(this is illustrated in figure 5). Moreover, the Dirichlet domain $D(b')$ is always contained in the intersection $\bigcap_{k=1}^n D_{\gamma_k}(b')$, so the compact set K may be chosen such that (39) holds if b is replaced with any element b' of a suitable neighbourhood of b . If $\mathcal{U} \subset \mathcal{B}$ is a compact neighbourhood with this property, we only need to apply the integral flow to $\mathcal{U} \times K$ and to each $\tilde{C}_\gamma^{\ell_0}$ separately. \square

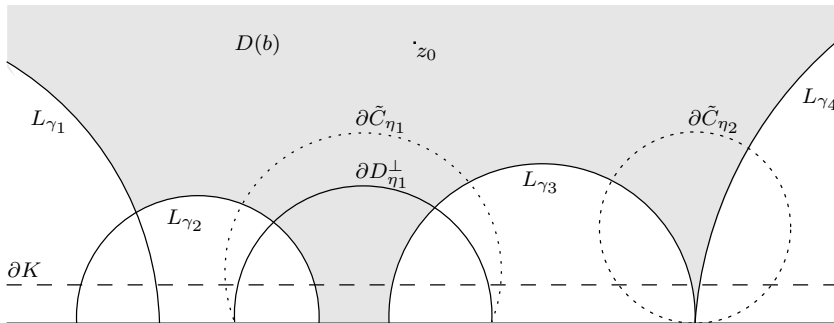


Figure 5: Trimming of a Dirichlet domain

B The Selberg trace formula

The trace formula as formulated by Selberg [23] expresses the trace of certain operators $\tilde{h}(\Delta)$ in terms of the closed geodesics on a geometrically finite surface. Difficulties may arise since the spectrum of Δ is not discrete for non-compact surfaces. In this appendix, we will only state a rudimentary version of this trace formula in the sense that it computes the distributional trace of an operator by integration of its integral kernel over the diagonal, but the result is not known to be related to the trace of a trace class operator a priori. Such a relation is established in a special case only, in section 3.2, where the operator is some kind of relative resolvent of the Laplacian (cf. p. 29 for the definition).

For suitable functions \tilde{h} , the Selberg transform provides an explicit formula for the integral kernel of $\tilde{h}(\Delta)$. The kernel on the hyperbolic plane is given by

$$(z_1, z_2) \longmapsto k(4 \sinh^2(d(z_1, z_2)/2)),$$

where k is related to the function $h: \xi \mapsto \tilde{h}(1/4 + \xi^2)$ by

$$k(t) = -\frac{1}{\pi} \int_t^\infty \frac{dQ(w)}{\sqrt{w-t}}, \quad \int_w^\infty \frac{k(t)}{\sqrt{t-w}} dt = Q(w),$$

$$Q(e^u + e^{-u} - 2) = g(u),$$

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^\infty h(\xi) e^{-i\xi u} d\xi, \quad \int_{-\infty}^\infty g(u) e^{i\xi u} du = h(\xi).$$

For the time being, we only consider those operators with an integral kernel that can be derived from k by summation over a uniformising group. To specify a class of functions h that meet this condition, we prove the following lemma. It is certainly well-known, but to our knowledge the complete proof is only implicit in the literature, so we give it here for the sake of completeness (c.f. Hejhal [8]).

B.1 Lemma. *Let h be an analytic function in $\{\xi \in \mathbb{C} \mid |\operatorname{Im}(\xi)| < 1/2 + \delta\}$ satisfying $h(\xi) = h(-\xi)$ and $|h(\xi)| \leq M(1 + |\operatorname{Re}(\xi)|^2)^{-(1+\alpha)}$. Then k is continuous on $[0, \infty)$ and there exist numbers C_ρ for all $\rho < \delta$ such that*

$$|g(u)| \leq C_\rho e^{-(1/2+\rho)|u|} \quad \text{and} \quad |k(t)| \leq C_\rho (1+t)^{-(1+\rho)}. \quad (40)$$

Conversely, if we start with a continuous function k on $[0, \infty)$ with $|k(t)| \leq C_\rho (1+t)^{-(1+\rho)}$ for $\rho < \delta$ and we apply the inverse transformations, then g satisfies estimates as above and h is analytic in the strip.

Proof. The relation between g and h is an immediate consequence of the Cauchy integral formula. The final assertion is also easy to see: Given k , the function Q is continuous and the estimate for g follows from

$$\int_w^\infty \frac{(1+t)^{-(1+\rho)}}{\sqrt{t-w}} dt \leq (1+w)^{-(1/2+\rho-\epsilon)} \int_0^\infty (1+t+w)^{-(1/2+\epsilon)} t^{-1/2} dt.$$

So we proceed with the main part. A function h that meets the assumptions gives rise to an element g of the Sobolev space $H^{3/2+\alpha}(\mathbb{R})$, because the following function is square-integrable:

$$\xi \longmapsto (1+\xi^2)^{1/2(3/2+\alpha)} h(\xi).$$

Now $3/2 + \alpha > 1 + (\dim \mathbb{R})/2$ holds, so the Sobolev embedding theorem implies $g \in C^{1,\gamma}(\mathbb{R})$ for $\gamma < \alpha$. Hölder continuity of the derivative g' implies continuity of k at 0:

$$k(t) = -\frac{1}{\pi} \int_0^\infty \frac{g'(\operatorname{Arcosh}(1 + \frac{w+t}{2}))}{\sqrt{w(w+t)(w+t+4)}} dw,$$

and using $g'(0) = 0$, the integrand can be estimated near 0 by

$$\frac{(\operatorname{Arcosh}(1 + \frac{w+t}{2}))^\gamma}{\sqrt{w(w+t)(w+t+4)}} \leq \frac{\sqrt{w+t}^\gamma}{\sqrt{w(w+t)(w+t+4)}} \leq w^{-1+\gamma/2}.$$

To derive the bounds for k , we use the fact that

$$f: \{\xi \in \mathbb{C} \mid |\operatorname{Im} \xi| < 1/2 + \delta\} \longrightarrow \mathbb{C}, \quad \xi \longmapsto -i\xi h(\xi)$$

is an analytic function which satisfies

$$\sup_{|\eta| < 1/2 + \delta} \|f(\cdot + i\eta)\|_{L^2(\mathbb{R})} < \infty.$$

The Fourier transform of f is g' . Therefore (cf. Reed-Simon [20, p. 18]) the function $u \mapsto e^{b|u|} g'(u)$ is square-integrable for any $b < 1/2 + \delta$, and we see

$$\begin{aligned} k(t) &= -\frac{1}{\pi} \int_t^\infty \frac{Q'(w) dw}{\sqrt{w-t}} \\ &= -\frac{1}{\pi} \left(\int_t^{t+t^{-2\delta}} \frac{Q'(w) dw}{\sqrt{w-t}} + \int_{\operatorname{Arcosh}(1+(t+t^{-2\delta})/2)}^\infty \frac{g'(u) du}{\sqrt{2(\cosh(u)-1)-t}} \right). \end{aligned}$$

We use boundedness of g' for the first integral, and the second is estimated with the Cauchy-Schwarz inequality for some $b = b_1 + b_2 < 1/2 + \delta$:

$$\begin{aligned} |k(t)| &\leq \frac{\|g'\|_\infty}{\pi} \int_t^{t+t^{-2\delta}} ((w-t)w(w+4))^{-1/2} dw \\ &\quad + \frac{1}{\pi} \left(\int_{\operatorname{Arcosh}(1+(t+t^{-2\delta})/2)}^\infty e^{-(1-2b_1)u} |g'(u)|^2 du \right)^{1/2} \\ &\quad \cdot \left(\int_{\operatorname{Arcosh}(1+(t+t^{-2\delta})/2)}^\infty \frac{e^{(1-2b_1)u} du}{2(\cosh(u)-1)-t} \right)^{1/2} \\ &\leq \frac{\|g'\|_\infty}{\pi} t^{-1} \int_0^{t^{-2\delta}} w^{-1/2} dw \\ &\quad + \frac{1}{\pi} e^{-(1/2+b_2) \operatorname{Arcosh}(1+(t+t^{-2\delta})/2)} \left(\int_{\operatorname{Arcosh}(1+(t+t^{-2\delta})/2)}^\infty e^{2b_1 u} |g'(u)|^2 du \right)^{1/2} \\ &\quad \cdot \left(\int_{t+t^{-2\delta}}^\infty ((w-t)\sqrt{w}\sqrt{w+4})^{-1} e^{(1-2b_1) \operatorname{Arcosh}(1+w/2)} dw \right)^{1/2} \\ &\leq \frac{2\|g'\|_\infty}{\pi} t^{-(1+\delta)} + C(1+t)^{-(1/2+b_2)}. \end{aligned}$$

The last line follows from $e^{\operatorname{Arcosh} v} = v + \sqrt{v^2 - 1}$. This proves the bound for k as b_2 may be chosen arbitrarily close to $1/2 + \delta$. \square

Now assume that $k \in C([0, \infty))$ satisfies $|k(t)| \leq C_\rho(1+t)^{-(1+\rho)}$ for all $\rho < \delta$. Let Y' be a connected, geometrically finite surface. We choose a uniformising group $\Gamma' \subset \operatorname{isom}(D)$ and identify Y' with $\Gamma' \backslash D$. The critical exponent of a Fuchsian group is less than or equal to one, and therefore the series

$$K_{\Gamma'}(z_1, z_2) := \sum_{\gamma \in \Gamma'} k(4 \sinh^2(d(z_1, \gamma z_2)/2))$$

is absolutely convergent for $z_1, z_2 \in D$. It defines a $\Gamma' \times \Gamma'$ -invariant function, so let $K_{Y'}: Y' \times Y' \rightarrow \mathbb{C}$ denote the function induced on the quotient. We would like to integrate $K_{Y'}$ over the diagonal in $Y' \times Y'$, but the integral might diverge for two reasons, either due to the growth of the kernel (if the surface has cusps) or because of the surface having infinite area (in the presence of funnels). We will simply subtract the divergent parts.

Consider Y' to be a component of $Y = Y_G(\lambda)$ as defined in section 1. An edge $d \in G_1^*$ with $\ell(d) = 0$ represents a pair of cusps $Z_d \subset Y$. We adjoin an extra pair of cusps \bar{Z}_d , which is by definition an isometric copy of the elementary quotient $\langle (x, a) \mapsto (x+1, a) \rangle \backslash X_0$. The resulting surface is a disjoint union

$$Y^+ := Y \cup \bigcup_{\substack{d \in G_1^* \\ \ell(d)=0}} \bar{Z}_d,$$

and we will integrate over Y^+ the function K^0 defined component-wise by

$$K^0(z) := \begin{cases} K_{Y'}(z, z) - k(0) & \text{if } z \in Y' \subset Y, \\ -(K_{Y'}(z, z) - k(0)) & \text{if } z \in \bar{Z}_d \subset Y^+ \setminus Y. \end{cases}$$

The constant $k(0)$ is subtracted to avoid difficulties with integration over infinite-area surfaces. Finally, we need a family of cut-off functions $\chi_A: Y^+ \rightarrow \{0, 1\}$ that vanish in the cuspidal ends of area A :

$$\chi_A(z) = 0 \iff z = (x, a) \in \bigcup_{\ell(d)=0} Z_d \cup \bar{Z}_d \text{ and } |a| < A.$$

B.2 Theorem (Selberg). *Let \mathcal{C} denote the set of oriented, simple, closed geodesics in Y , and $\ell(c)$ the length of a curve $c \in \mathcal{C}$. Then the following formula holds:*

$$\lim_{A \rightarrow 0} \int_{Y^+} \chi_A K^0 \operatorname{vol} = \sum_{c \in \mathcal{C}} \sum_{n=1}^{\infty} \frac{\ell(c) g(n\ell(c))}{e^{n\ell(c)/2} - e^{-n\ell(c)/2}}. \quad (41)$$

The proof is given in Selberg's Göttingen lectures, in Hejhal's monograph [9] or by Venkov [24], one only needs to pay attention to the following facts.

Assume the function k to be positive first. For every component Y' of Y we fix a complete set B' of representatives for the conjugacy classes of hyperbolic elements in Γ' . The famous calculation of Selberg's [23, 9], in connection with Beppo Levi's theorem, shows that

$$\begin{aligned} \sum_{Y' \subset Y} \int_F \sum_{\substack{\gamma \in \Gamma' \\ \gamma \neq 1}} \tilde{\chi}_A(z) k(\sigma(z, \gamma z)) \operatorname{vol}(z) &- \sum_{Y' \subset Y} \int_F \sum_{\substack{\gamma \in \Gamma' \\ \text{parabolic}}} \tilde{\chi}_A(z) k(\sigma(z, \gamma z)) \operatorname{vol}(z) \\ &\longrightarrow \sum_{Y' \subset Y} \sum_{\gamma \in B'} \frac{\ell(\gamma_0) g(\ell(\gamma))}{e^{\ell(\gamma)/2} - e^{-\ell(\gamma)/2}}, \quad A \rightarrow 0, \end{aligned} \quad (42)$$

where γ_0 denotes the unique primitive element of Γ' such that $\gamma = \gamma_0^n$ holds for some positive n , and $\ell(\gamma)$ is the hyperbolic translation of γ . The domain F of integration is a suitable fundamental domain for Γ' and $\sigma(z_1, z_2) = 4 \sinh^2(d(z_1, z_2)/2)$. The infinite series on the right hand side converges since

$$\#\{\gamma \in B' \mid e^{\ell(\gamma)} \leq x\} = O(x), \quad x \rightarrow \infty$$

and g satisfies $g(u) \leq Ce^{-(1/2+\rho)|u|}$ by lemma B.1.

If k is not positive, we see that Lebesgue's theorem on dominated convergence implies that the positivity condition on k may be dropped in equation (42). On the other hand (see Hejhal [9, pp. 198, 311])

$$\lim_{A \rightarrow 0} \left(\sum_{\ell(d)=0} \int_{\bar{Z}_d} \chi_A K^0 \text{vol} + \sum_{Y' \subset Y} \int_F \sum_{\substack{\gamma \in \Gamma' \\ \text{parabolic}}} \tilde{\chi}_A(z) k(\sigma(z, \gamma z)) \text{vol}(z) \right) = 0. \quad (43)$$

The primitive elements $\gamma_0 \in B'$ correspond bijectively to the simple closed geodesics of Y' , so the sum of (42) and (43) yields the claim.

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