Twisted K-theory with coefficients in a C^*-algebra and obstructions against positive scalar curvature metrics

Dissertation

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To my parents

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Chapter 1

Introduction

Twisted K-theory first appeared in the early 1970s in a paper by DONOVAN and KAROUBI [20] as "K-theory with local coefficients". As the name suggests, it is an elaboration of topological K-theory, invented by ATIYAH and HIRZEBRUCH in [4], that supports a notion of POINARÉ duality even in the case of non K-orientable (i.e. non spin$^{(c)}$-) manifolds [72]. Like the local coefficients for ordinary cohomology are described by a class in $H^1(M, \mathbb{Z}/2\mathbb{Z})$ for a paracompact Hausdorff space M, the extra data needed in case of K-theory is a twist represented by a certain (torsion) element in $H^3(M, \mathbb{Z})$. The latter group can also be seen to classify isomorphism classes of bundles $\mathcal{A}_{\mathbb{K}} \to M$ with fibers the compact operators on a separable Hilbert space, which was the starting point of a paper by ROSENBERG [57], who showed that twisted K-theory as defined in [20] is isomorphic to $K_*(C_0(M, \mathcal{A}_{\mathbb{K}}))$ – the K-theory of the C^*-algebra of continuous sections of $\mathcal{A}_{\mathbb{K}}$ tending to zero at infinity. This description needs no restriction to torsion elements in $H^3(M, \mathbb{Z})$, but in case the twist is of finite order BOUWKNEGT, CAREY, MATHAI, MURRAY and STEVENSON developed a geometric description of $K_0(C_0(M, \mathcal{A}_{\mathbb{K}}))$ in terms of modules over bundle gerbes [12, 19], which provide a replacement for vector bundles in the twisted case.

Recently, the rise of string theory gave some fresh impetus to the development of twisted K-theory [64, 13, 41, 77]: In string theory, space-time is modelled in such a way that its classical limit, in which quantum effects are neglected, is not just a Riemannian manifold M, but also carries the extra data of a B-field β on M, which supports topologically non-trivial dynamical structures of the stringy space-time, called D-branes. Now, β coincides with the data needed to define a twisted K-theory group on M and the elements of this group can be identified with D-brane charges. Furthermore, there also is a duality principle, called T-duality, relating two different types of string theories, which amounts on the mathematical side to an interesting involution on twisted K-theory [18, 15].

Another application of twisted K-theory emanates from the representation theory of loop groups, in particular from a beautiful theorem proven by FREED, HOPKINS and TELEMAN [22]: Let G be a simple, simply connected, compact Lie group and $k \in \mathbb{Z} \simeq H^3(G, \mathbb{Z})$ be a level. For each such k there is a canonical central extension $\widehat{\Omega}_k G$ of the loop group ΩG. Let $R^k(\widehat{\Omega} G)$ be the free abelian group generated by isomorphism classes of positive energy representations of $\widehat{\Omega}_k G$. There is a level preserving product, known as fusion, on $R^k(\widehat{\Omega} G)$ turning it into the Verlinde ring. The theorem says that it is isomorphic to the twisted equivariant K-theory $K_G^{d,[k+h^\vee]}(G)$, where $h^\vee$ is the dual Coxeter number, $[k+h^\vee] \in H^3(G,\mathbb{Z})$ represents the twist, d is the dimension of G and G acts on itself by conjugation. Fusion turns into the Pontrjagin product on the K-theoretic side. Another reference is [16], where the formulation of this theorem as a commutative diagram is considered.

As will be seen in the next chapter, twisted K-theory is closely related to higher algebraic structures, like bundle gerbes [12]. Therefore the development of the theory for singular spaces described by stacks or orbifolds comes in very natural. Results in this direction can be found in [1, 73].

Let A be a unital C^*-algebra, M be a compact Hausdorff space, then the continuous functions with values in A, denoted by $C(M,A) \simeq C(M) \otimes A$, will again be a C^*-algebra when equipped with the obvious supremum norm. There is a geometric description of $K_0(C(M,A))$ as the Grothendieck group of isomorphism classes of bundles with fibers finitely generated, projective Hilbert A-modules [61]. The present work takes this geometric picture as a starting point and develops a twisted version of it: Let $\mathcal{A}$ be a locally trivial *bundle* of C^*-algebras with structure group $PU(A) = U(A)/U(1)$ and denote by $C(M,\mathcal{A})$ the C^*-algebra of continuous sections. Replacing the above Hilbert module bundles by modules over a suitable bundle gerbe and thereby merging the above picture with the one developed in [12, 19], we arrive at a geometric interpretation of $K_0(C(M,\mathcal{A}))$ in terms of twisted Hilbert A-module bundles (see definition 3.0.13). In particular, we prove a twisted Serre-Swan-theorem (3.2.8) saying that the category of twisted Hilbert A-module bundles is naturally equivalent to the one of finitely generated, projective $C(M,\mathcal{A})$-modules. A similar proposition holds in the case of locally compact spaces M, if some care is taken about the extendability of $\mathcal{A}$ to compactifications.

A twisted Hilbert A-module bundle E lives over an auxiliary principal bundle $\mathcal{P}$ with structure group $PU(A)$. From a geometric point of view, $\mathcal{P}$ might seem to be a somewhat bulky, even though not unmanageable object. Luckily, in all geometric applications we have in mind, $\mathcal{P}$ reduces to a principal Γ-bundle P, such that Γ is either a compact Lie group or a discrete group. Even without a reduction, but with restriction to torsion twists, the algebra $C(M,\mathcal{A})$ is Morita equivalent to $C(M,\mathcal{K}) \otimes A$ for a bundle of matrix algebras $\mathcal{K} \to M$ (see theorem 3.4.1). Using the isomorphism induced on K-theory, each twisted K-cycle may be represented by a twisted Hilbert A-module bundle over some principal $PU(n)$-bundle P. We exploit this fact by developing a theory of connections on modules over certain lifting bundle gerbes that arise from flat central S^1-extensions of Lie groups:

$$1 \longrightarrow S^1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1 \ .$$

In this case there is a notion of Chern character in the sense of Chern-Weil-theory, which takes values in the ordinary (untwisted) cohomology. It extends like in the setup of Mishchenko-Fomenko-index theory [47] to a Chern character with values in $H^{\text{even}}(M, K_0(A) \otimes \mathbb{R})$.

The usual approach to index theory via pseudodifferential operators does not transfer directly to the twisted case due to a crucial analytic difficulty: Symbols, representing K-cycles in the twisted K-group $K^0_{\pi^*\mathcal{A}}(T^*M)$, just yield *transversally elliptic* pseudodifferential operators on the corresponding twisted

Hilbert A-module bundles (with respect to the covertical subbundle $H^* \subset T^*P$). Nevertheless, the machinery developed by Atiyah in [7] is extendable to this case if the algebra in question is supposed to come equipped with a trace. We follow this idea up to some point, define an analytic index and prove that it only depends on the K-theory class of the principal symbol.

All analytic obstacles can be circumvented in a nice way for the subclass of generalized projective Dirac operators. Let $D : \Gamma(S) \to \Gamma(S)$ be such an operator acting on smooth sections of a twisted Hilbert A-module bundle S. D can be twisted with a bundle gerbe module E in such a way that both – the resulting bundle $S \boxtimes E$ and the operator $D^E : \Gamma(S \boxtimes E) \to \Gamma(S \boxtimes E)$ – descend to analogue structures on the base manifold instead of the auxiliary principal bundle (since E and S are allowed to live over different bundles, we need to use the exterior tensor product $\boxtimes$ here). We extend the index theorem of Mishchenko and Fomenko and also Kasparov's KK-theoretical index theorem to the countertwisted case (see (4.36), (4.37), (4.43) and theorem 4.5.9). However, there is no canonical choice for the countertwisting bundle E and we discuss different possibilities and their index-theoretical consequences. The case when E is flat is of particular interest, since it perturbs the index of the generalized projective Dirac operator as little as possible. Therefore we classify flat countertwisting bundles via their holonomy representations, which now turn out to be *projective* representations of the fundamental group for a *fixed* lifting cocycle instead of honest ones like in the non-twisted case. We also draw some conclusions about the relation between the spectrum of possible dimensions of these representations in case of the projective Dirac operator and the denominators of the $\widehat{A}$-genus.

In the final part of this work, we apply the index theory of twisted Hilbert A-module bundles to the problem of index theoretical obstructions against positive scalar curvature metrics (psc-metrics for short). This subject is based on the pioneering work of Atiyah, Hitchin, Lichnerowicz and Singer [37, 29]. The argument exploits the Bochner-formula to conclude that if a *spin* manifold M allows a metric of positive scalar curvature, then its $\widehat{A}$-genus has to vanish. Rosenberg refined this index to an invariant $\alpha_r(M) \in KO_n(C^*_r(\pi_1(M)))$ [56] by twisting the Dirac operator with the Mishchenko-Fomenko-line bundle $\mathcal{V}_r$, which is a Hilbert C^*-module bundle associated to the universal cover $\widetilde{M}$ with fibers the reduced C^*-algebra of the fundamental group $C^*_r(\pi_1(M))$. This operator has an index in the K-theory of this algebra. Since $\mathcal{V}_r$ is still flat, the argument of Lichnerowicz remains valid in this case and $\alpha_r(M)$ is an obstruction against psc-metrics. In fact, this invariant is quite strong, which led to the following conjecture in [58]:

Conjecture 1.0.1. (Gromov-Lawson-Rosenberg conjecture) Let M be a closed, connected, n-dimensional spin manifold with $n \geq 5$, then M admits a psc-metric if and only if $\alpha_r(M) \in KO_n(C^*_r(\pi_1(M)))$ vanishes.

There is a corresponding stable version of this conjecture proposed in [59]:

Conjecture 1.0.2. (stable Gromov-Lawson-Rosenberg conjecture) Let M be a closed, connected, n-dimensional spin manifold with $n \geq 5$. Let J be a simply connected, spin manifold of dimension 8 with $\widehat{A}(J) = 1$. If $\alpha_r(M) \in KO_n(C^*_r(\pi_1(M)))$ vanishes, then M stably admits a psc-metric, i.e. $M \times J \times \cdots \times J$ admits a psc-metrics for sufficiently many factors of J.

The unstable conjecture is supported by the fact, that for a simply connected manifold, i.e. in the case $\alpha_r(M)$ reduces to the $\widehat{A}$-genus, there is a positive result by Stolz [67] (see also the paper by Gromov and Lawson [25]):

Theorem 1.0.3. *Let M be a connected, simply connected, closed spin manifold of dimension ≥ 5. Then M admits a psc-metric if and only if $\alpha_r(M) = 0 \in KO_n(pt)$.*

Nevertheless, it was shown by Schick in [60] using minimal hypersurface techniques of Schoen and Yau [63] that counterexamples exist:

Theorem 1.0.4. *There exists a 5-dimensional spin manifold with $\pi_1(M) = \mathbb{Z}^4 \times \mathbb{Z}/3\mathbb{Z}$ such that $\alpha_r(M) = 0$, but M does not admit a metric of positive scalar curvature.*

The situation in case of the stable conjecture is quite different. In fact, Stolz showed in [68] that if the Baum-Connes assembly map is injective for some fundamental group π, then the conjecture holds for π as well.

Note that all the results require M to be a spin manifold. In particular, this is necessary for the Rosenberg index $\alpha_r(M)$ to make sense. Twisted K-theory with coefficients in a C^*-algebra provides a way to extend the index obstruction to the case, where M itself need not be spin, but only its universal cover is. This is done by replacing the absent Dirac operator on M by the projective Dirac operator and the Mishchenko-Fomenko-line bundle by a certain twisted Hilbert A-module bundle, where A can either be the reduced or maximal *twisted* group C^*-algebra of the fundamental group centrally extended by a cocycle $c_{\widehat{\pi}} \in H^2_{\mathrm{gr}}(\pi_1(M), S^1)$ related to the spin structure on the universal cover: $C^*_{r,\max}(\pi_1(M), c_{\widehat{\pi}})$. This way, we define an invariant

$$\alpha_{r,\max}(M) \in K_0(C^*_{r,\max}(\pi_1(M), c_{\widehat{\pi}}))\ ,$$

which can be understood as the direct replacement of the (complex version of the) Rosenberg index in case the universal cover is spin. We prove a Lichnerowicz type formula for the twisted case that also reveals, where the argument fails in case the universal cover is not spin (see theorem 4.4.6). This point of view not only sheds a new light on the obstruction considered by Stolz in [66], but also allows us to transfer many of the arguments given in the spin case more or less directly, proving the machinery developed in the prior chapters to be valuable.

In particular, we extend a result by Hanke and Schick about enlargeable manifolds [27]. Gromov and Lawson used geometric methods to prove that these provide examples of manifolds that do not admit a psc-metric [26]. One of the main results in [27] is, that also the (complex version of the) Rosenberg index does not vanish in this case and therefore detects enlargeability. In the spirit of their proof, we construct a flat twisted Hilbert A-module bundle W from a sequence of almost flat twisted bundle gerbe modules with non-vanishing top Chern classes. It is a consequence of enlargeability that such gadgets exist. The projective holonomy representation deduced from this bundle, yields a C^*-algebra homomorphism

$$C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \longrightarrow Q\ ,$$

into a C^*-algebra Q in such a way that the induced map on K-theory sends $\alpha_{\max}(M)$ to the index of the Dirac operator twisted with W. Even though W is flat, it remembers the Chern classes of its ingredients. The K-theory of Q is well-understood and $\mathrm{ind}(D_W) \in K_0(Q)$ reflects the indices of the almost flat ingredients, which were non-vanishing by hypothesis.

We finish this work with some perspectives about the twisted K-theory in the non-torsion case and possible extensions of the theory to bundles of C^*-algebras with different structure groups.

The guided tour through this thesis is as follows:

- Section 2 contains the necessary preliminaries about bundle gerbes and their modules. It explains the notion of stable isomorphism and relates it to Morita equivalence. Furthermore, the Dixmier-Douady-class is defined and its properties are summarized. Aside from this, we explain some aspects of the wide field of higher algebraic structures and higher gauge theory, which bundle gerbes are a part of.

- Section 3 introduces twisted Hilbert A-module bundles and their morphisms. It defines twisted K-theory with coefficients in A on a compact space M as the Grothendieck group of such gadgets and we prove a twisted Serre-Swan-theorem (3.2.8). Exploiting the Banach category structure of twisted Hilbert A-module bundles, the notion of K-theory is extended to locally compact spaces. The twist shifting theorem 3.4.1 will not only play a crucial role in the countertwisting techniques of section 4, but also relates twisted Hilbert A-module bundles over a principal $PU(A)$-bundle to those over $PU(n)$-bundles. We define the analogue of the frame bundle in the twisted case, which will be an important ingredient in the classification of flat bundle gerbe modules, followed by a short note how bundle gerbe modules can be considered as twisted Hilbert $M_n(\mathbb{C})$-bundles. Sections 3 ends with a toolbox explaining the product structure on twisted K-theory and the Künneth formula.

- Section 4 starts with some basics about flat central S^1-extensions of Lie groups, i.e. short exact sequences $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ with $\widehat{\Gamma}$ associated to the universal cover of Γ. The right notion of connection on their lifting bundle gerbes is defined, before focussing on modules over them. The notion of bundle gerbe module connection (bgm-connection for short) is given in terms of connection forms, horizontal subbundles and covariant derivatives. A section about trivial bundle gerbes explains how they are related to untwisted connections on bundles over M and how this fits into the picture of the spinor module. Parallel transport on twisted Hilbert A-module bundles is explained in the next section. It is shown to be equivariant with respect to an action of $\widehat{\Gamma}$. The notion of curvature transformation is given and a Chern character is defined via Chern-Weil-theory in two ways: One using a trace on the algebra if present, the other using the Künneth formula. This section ends by proving the additive and multiplicative properties of this character.

- Before digging into index theory, the next part of section 4 continues with a short digression about Sobolev spaces, especially in the case when the auxiliary principal bundle P is non-compact. Symbols are defined in the next paragraph as sections of the homomorphism bundle $\hom(\pi^*E, \pi^*F)$ for $\pi : T^*M \to M$ and two twisted Hilbert A-module bundles E and F. It is proven that each symbol σ over T^*M lifts to a transversally elliptic one, denoted by $\widehat{\sigma}$, over T^*P by multiplication with some regularization function. Using an intrinsic Fourier transform like in [11], these can be turned into $\widehat{\Gamma}$-equivariant transversally elliptic operators. With the help of parametrices, which are shown to exist, an "analytic" index is constructed in the case the algebra in play carries a trace. It is proven to yield a map from the twisted K-group $K^0_{\pi^*\mathcal{A}}(T^*M)$ into the complex numbers and its relation with the fractional analytic index of [38] is clarified.

- The most important part of section 4 introduces generalized projective Dirac operators and relates them to operators on bundles over M in case the twist is trivial. This is the starting point of the countertwisting technique alluded to in the introduction. Different ways of countertwisting are explained with a special emphasis on spinor and flat countertwisting bundles. The different variations of the Mishchenko-Fomenko-index theorem can be found in (4.36), (4.37) and (4.43). After introducing the notion of covering bundle gerbes, we classify flat bundle gerbe modules in theorem 4.3.14 as those, for which the frame bundle gerbe reduces to a covering one. This implies that they are completely determined by a projective (holonomy) representation of the fundamental group associated to a certain lifting cocycle deduced from the bundle gerbe as amplified in section 4.3.4. For the projective Dirac operator the dimension spectrum corresponding to this particular cocycle carries some information about the denominator of the $\widehat{A}$-genus as stated in corollary 4.3.23.

- The last part of section 4 finally introduces our replacement of the Rosenberg index and we prove a twisted Bochner formula in theorem 4.4.4. Besides the definition of the twisted Mishchenko-Fomenko-line bundle and some remarks about the twisted group C^*-algebras, it also contains a KK-theoretic viewpoint on countertwisting, which shows that it interacts with the Kasparov product in a very nice way, which enables us on one hand to easily prove some naturality statement about the projective Dirac operator and on the other to state the analogue of Kasparov's index theorem in the twisted case (see theorem 4.5.9).

- Section 5 finally contains the application of the whole theory to the case of enlargeable manifolds, which are defined in the beginning. After showing that – with the stated notion of enlargeability – almost flat twisted bundle gerbe modules exist, the core of the argument given in [27] is transferred into our setup in theorem 5.1.1: Here the sequence of almost flat twisted bundles is assembled into an infinite dimensional bundle as explained in the introduction. Having accomplished this, it is a rather small step to arrive at the generalization of the result from [27] in theorem 5.2.5.

- Section 6 contains some concluding remarks and perspectives about possible further generalizations of the theory of twisted Hilbert A-module bundles in the index theoretical direction as well as in the direction of C^*-algebra bundles with more complicated structure group.

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Chapter 2

Preliminaries

2.1 Bundle gerbes and their modules

In this section we will present the basic properties of bundle gerbes, which were first treated by Murray in [49]. Among the many different motivations for them, ranging from a generalization of line bundles to bundles of 2-groups, let us first explore, how they come up in the context of spin structures on non-spin manifolds. More general, we will start with the example of lifting bundle gerbes, which play a fundamental role throughout this paper. Let

$$1 \longrightarrow S^1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

be a central S^1-extension of Lie groups (which may be understood in a very broad context, for example we could think of Banach, or even Fréchet Lie groups here), such that $\widehat{\Gamma} \to \Gamma$ is a principal S^1-bundle. Take a principal Γ-bundle P over a paracompact Hausdorff space X. Any vector bundle $F \to X$ pulls back to a Γ-equivariant vector bundle $E = \pi^*F \to P$. The extra data we have to remember is the group action of Γ on E. This can be formulated in – at first sight far too – elaborate terms by saying that E is a module over the pair groupoid

$$P^{[2]} \rightrightarrows P$$

where source and target maps are given by the projections π_i and $P^{[2]} = P \times_X P = \{(p_1, p_2) \in P^2 \mid \pi(p_1) = \pi(p_2)\}$ denotes the fiber product over X. The module structure of E over $P^{[2]}$ is then expressed as a map

$$P^{[2]} \times \pi_2^* E \longrightarrow \pi_1^* E \quad ; \quad (p_1, p_2, v) \mapsto (p_1, p_2, g_{12} \cdot v)$$

with $v \in E_{p_2}$. The element g_{12} is uniquely defined by the property that $p_1\, g_{12} = p_2$, i.e. it represents the morphism from p_2 to p_1, and acts via the isomorphism $E_{p_2} \to E_{p_1}$. Note the ordering convention, which is due to the fact, that we would like to consider π_2 as the source map and π_1 as the target.

Now suppose that we have a representation ϱ of $\widehat{\Gamma}$. Imagine we would like to have some vector bundle E associated to a lift of P to a principal $\widehat{\Gamma}$-bundle with the slight defect that we do not know whether this lift exists or not. In the case $\Gamma = SO(n)$, $\widehat{\Gamma} = \mathrm{Spin}^c$ this asks for the existence of a complex spinor bundle and therefore is the complex version of the initial problem described above. Even though the lift $\widehat{P}$ may not exist, there still is a central extension of the pair groupoid, given by:

$$\begin{array}{ccc} \widehat{L} & & \\ \downarrow & & \\ P^{[2]} & \rightrightarrows & P \\ & & \downarrow \\ & & X \end{array}$$

where $\widehat{L} = \kappa^*(\widehat{\Gamma})$ is the pullback of the principal S^1-bundle $\widehat{\Gamma} \to \Gamma$ via the map $\kappa : P^{[2]} \to \Gamma$ with $\kappa(p_1, p_2) = g_{12}$, where g_{12} is like above. This way, the fiber $\widehat{L}_{(p_1,p_2)}$ consists of all possible lifts of $g_{12} \in \Gamma$ to an element $\widehat{g}_{12} \in \widehat{\Gamma}$. Observe that we can identify $\widehat{L}$ with a line bundle $L \to P^{[2]}$. The group multiplication of $\widehat{\Gamma}$ is now stretched out diagonally over the pair groupoid and leads to a product structure over $P^{[3]}$:

$$\pi_{12}^* L \otimes \pi_{23}^* L \longrightarrow \pi_{13}^* L \quad ; \quad [\,\widehat{g}_{12},\, \lambda\,] \otimes [\,\widehat{g}_{23},\, \mu\,] \mapsto [\,\widehat{g}_{12}\widehat{g}_{23},\, \lambda\,\mu\,] \ ,$$

where $\pi_{ij} : P^{[3]} \to P^{[2]}$ are the canonical projections and the fiber $L_{(p_1,p_2)} = \widehat{L}_{(p_1,p_2)} \times_{S^1} \mathbb{C}$ is written in the form $[\widehat{g}, \lambda]$ for $\widehat{g} \in \widehat{\Gamma}$ and $\lambda \in \mathbb{C}$. This action covers the multiplicative structure on the pair groupoid and the obvious diagrams for associativity commute. Note that $1 \in \Gamma$ has the canonical lift $1 \in \widehat{\Gamma}$, therefore we will often identify the fiber over the diagonal $L_{(p,p)}$ with the trivial line bundle. It plays the role of an identity for the product this way.

The above point of view on equivariant vector bundles is still applicable to this situation. Indeed, we can ask for a vector bundle $E \to P$ to carry an action[1] of the bundle gerbe L, i.e. a bundle isomorphism:

$$L \otimes \pi_2^* E \longrightarrow \pi_1^* E$$

together with certain diagrams for associativity. E then becomes an almost equivariant vector bundle up to the defect described by the central extension of the pair groupoid $P^{[2]}$.

After this motivation, we can phrase the precise definitions of bundle gerbes and bundle gerbe modules:

Definition 2.1.1. Let $Y \to X$ be a fibration over a paracompact Hausdorff space X. A line bundle $L \to Y^{[2]}$ will be called a *bundle gerbe*, if it there exists a product over $Y^{[3]}$, i.e. an isomorphism of line bundles

$$\mu \ : \ \pi_{12}^* L \otimes \pi_{23}^* L \longrightarrow \pi_{13}^* L \ ,$$

where $\pi_{ij} : Y^{[3]} \to Y^{[2]}$ denotes the canonical projection, such that over $Y^{[4]}$

[1] This will later be called a twisting.

the following diagram commutes:

$$\begin{tikzcd}
(\pi_{12}^*L \otimes \pi_{23}^*L) \otimes \pi_{34}^*L \arrow[rr, equal] \arrow[d, "\mu \otimes \mathrm{id}"] & & \pi_{12}^*L \otimes (\pi_{23}^*L \otimes \pi_{34}^*L) \arrow[d, "\mathrm{id} \otimes \mu"] \\
\pi_{13}^*L \otimes \pi_{34}^*L \arrow[dr, "\mu"] & & \pi_{12}^*L \otimes \pi_{24}^*L \arrow[dl, "\mu"] \\
& \pi_{14}^*L &
\end{tikzcd}$$

In case $Y = P$ is a principal Γ-bundle and L is the bundle gerbe described above, it will be called *lifting bundle gerbe*. Morphisms of bundle gerbes are maps between line bundles compatible with μ.

Remark Let $\Delta \colon Y \to Y^{[2]}$ be the diagonal embedding $\Delta(y) = (y, y)$ and let $L \to Y^{[2]}$ be a bundle gerbe as defined above, then Δ^*L is a trivial line bundle over Y playing the role of the unit for the bundle gerbe multiplication μ. Therefore we will identify it with the trivial line bundle $\underline{\mathbb{C}} \to Y$ using the following canonical isomorphism:

$$\underline{\mathbb{C}} \longrightarrow (\Delta^*L)^* \otimes \Delta^*L \longrightarrow (\Delta^*L)^* \otimes \Delta^*L \otimes \Delta^*L \longrightarrow \Delta^*L \ ,$$

where the second map is induced by the bundle gerbe multiplication, for which Δ^*L is an idempotent.

Likewise we have already met a module over a bundle gerbe above, from which it is easy to derive the precise definition:

Definition 2.1.2. Let $L \to Y^{[2]}$ be a bundle gerbe, and let $E \to Y$ be a finite dimensional vector bundle. E is called a *bundle gerbe module* if it comes equipped with a *twisting*, i.e. a bundle isomorphism

$$\gamma \ : \ L \otimes \pi_2^*E \longrightarrow \pi_1^*E \ ,$$

such that over $Y^{[3]}$ the following associativity diagram commutes:

$$\begin{tikzcd}
(\pi_{12}^*L \otimes \pi_{23}^*L) \otimes \pi_3^*E \arrow[rr, equal] \arrow[d, "\mu \otimes \mathrm{id}"] & & \pi_{12}^*L \otimes (\pi_{23}^*L \otimes \pi_3^*E) \arrow[d, "\mathrm{id} \otimes \gamma"] \\
\pi_{13}^*L \otimes \pi_3^*E \arrow[dr, "\gamma"] & & \pi_{12}^*L \otimes \pi_2^*E \arrow[dl, "\gamma"] \\
& \pi_1^*E &
\end{tikzcd}$$

Remark Here and in everything that follows we will often work with *good covers*, i.e. a covering $\bigcup_i U_i \supset X$, such that $U_{ij} = U_i \cap U_j$ and all higher intersections (U_{ijk}, etc.) are contractible. We will assume that these exist for all *base* spaces X or M without mentioning it. Note that every smooth manifold carries a good cover, as does every simplicial complex.

2.1.1 The Dixmier-Douady class and stable isomorphism

When viewing bundle gerbes as a generalization of line bundles, it comes as no surprise that they also carry a product structure. The notion of triviality used in this context, however, is different from just being trivial as a line bundle over $Y^{[2]}$. It is taylored, as we will see below, to be detected by a certain element in third cohomology, which plays the role of the first Chern class in this theory and is called the DIXMIER-DOUADY-class.

Definition 2.1.3. Given a line bundle $Q \to Y$ over the total space of a fibration $Y \to X$, we can always form the bundle gerbe:

$$L = \pi_1^* Q \otimes \pi_2^* Q^* \longrightarrow Y^{[2]}$$

with the product induced by the paring $Q^* \otimes Q \to \underline{\mathbb{C}}$. This will be called the *trivial bundle gerbe*. Likewise, an arbitrary L will be called *trivial*, if there exists an isomorphism of bundle gerbes $L \to \pi_1^* Q \otimes \pi_2^* Q^*$ for some *trivialization* Q.

Notice that Q itself can be seen as a rank 1-module over L, since $L \to \pi_1^* Q \otimes \pi_2^* Q^*$ corresponds to an isomorphism $L \otimes \pi_2^* Q \to \pi_1^* Q$. In fact, this argument shows that there is a one-to-one correspondence between trivializations and rank 1-modules over L.

Definition 2.1.4. Given bundle gerbes $L_1 \to Y_1,\ L_2 \to Y_2$ over some common space X their *product* is defined to be

$$L_1 \boxtimes L_2 = \pi^*_{Y_1^{[2]}} L_1 \otimes \pi^*_{Y_2^{[2]}} L_2 \longrightarrow (Y_1 \times_X Y_2)^{[2]} = Y_1^{[2]} \times_X Y_2^{[2]}$$

with the multiplication acting factorwise over $(Y_1 \times_X Y_2)^{[2]}$.

Note that for a bundle gerbe L its dual L^* is again a bundle gerbe with the multiplication induced by the inverse of the dual map μ^*. Since we always assume L to be associated to a principal S^1-bundle and therefore being equipped with a hermitian structure, we can identify L^* with the conjugate line bundle $\overline{L}$. Besides the notion of isomorphism explained above, there is a more general notion of equivalence tightly bound to MORITA equivalence.

Definition 2.1.5. Two bundle gerbes $L_1 \to Y_1,\ L_2 \to Y_2$ over some common space X will be called *stably isomorphic* if $L_1^* \boxtimes L_2$ is trivial in the sense above, i.e. there exists a line bundle $Q \to Y_1 \times_X Y_2$ and an isomorphism:

$$(L_1^* \boxtimes L_2) \otimes \pi_2^* Q \longrightarrow \pi_1^* Q$$

compatible with the multiplicative structure on $L_1^* \boxtimes L_2$. Q is a *stable isomorphism* between L_1 and L_2.

Alternatively, we can see Q as an *intertwiner* between the multiplications on L_1 and L_2, since it can be turned into an (associative) isomorphism:

$$\pi^*_{Y_2^{[2]}} L_2 \otimes \pi_2^* Q \longrightarrow \pi_1^* Q \otimes \pi^*_{Y_1^{[2]}} L_1 \ .$$

This has another interpretation in terms of MORITA-equivalence: If we view the frame bundle $\widehat{L}_i$ of L_i as a central S^1-extension of the pair groupoid $Y_i \times_M Y_i$, then the principal S^1-bundle $\widehat{Q} \to Y_1 \times_M Y_2$ fits into the diagram

$$\begin{tikzcd}
\widehat{L}_1 \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr, bend left] & & \widehat{L}_2 \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dl, bend right] \\
& \widehat{Q} \arrow[dl] \arrow[dr] & \\
Y_1 & & Y_2
\end{tikzcd} \tag{2.1}$$

where the curved arrows denote the action of L_i on Q induced by the isomorphism of line bundles. In fact, this turns $\widehat{Q}$ into an $\widehat{L}_1$-$\widehat{L}_2$-bibundle with free and proper actions and since $\widehat{Q}/\widehat{L}_1$ is the space Y_2 and vice versa, $\widehat{Q}$ induces a Morita equivalence of groupoids.

As announced above, there is a characteristic class associated to a bundle gerbe. It is most easily constructed in Čech cohomology in the following way: Choose some good cover of X by open sets U_i (where "good" always refers to the contractibility of non-empty intersections of arbitrary order) and sections $\sigma_i : U_i \to Y$. Over double intersections $U_{ij} = U_i \cap U_j$ the pairs $(\sigma_i, \sigma_j) : U_{ij} \to Y^{[2]}$ induce maps that land in the fiber product. Therefore $L \to Y^{[2]}$ can be pulled back to give $L_{ij} = (\sigma_i, \sigma_j)^* L$ over U_{ij}. Choose sections $\kappa_{ij} : U_{ij} \to L_{ij}$, such that the image at every point has length 1. Note that the product yields isomorphisms $\mu_{ijk} : L_{ij} \otimes L_{jk} \to L_{ik}$ over triple intersections U_{ijk}. Now the maps $\mu_{ijk} \circ (\kappa_{ij}, \kappa_{jk})$ and κ_{ik} differ by a function

$$\omega_{ijk} \; : \; U_{ijk} \longrightarrow S^1 \; ,$$

which, when running through the details of this construction, can be seen to be a Čech 2-cocycle with values in S^1 that, up to a change by a coboundary, does not depend on all the choices made during its construction [49].

Definition 2.1.6. The class $[\omega_{ijk}] \in \check{H}^2(X, S^1)$ will also be denoted by $dd(L)$ and is called the Dixmier-Douady-class of L.

For reasonably well-behaved spaces, like the smooth, closed manifolds we are going to work with, $\check{H}^2(X, S^1)$ is isomorphic to $H^3(X, \mathbb{Z})$. We will summarize the properties of this class in the following theorem, which is proven in any introductory source about bundle gerbes, see e.g. in [49, 12, 48].

Theorem 2.1.7. *Let L, L_1, L_2 be bundle gerbes over X, then*

i) $dd(L) = 0$ if and only if L is trivial.

ii) $dd(L_1 \boxtimes L_2) = dd(L_1) + dd(L_2)$ and $dd(L^*) = -dd(L)$.

iii) Stable isomorphism classes of bundle gerbes are in $1:1$-correspondence with elements in $\check{H}^2(X, S^1)$.

iv) If $L \to P^{[2]}$ is the lifting bundle gerbe of a central S^1-extension we have that $dd(L) = 0$ if and only if P lifts to a principal $\widehat{\Gamma}$-bundle. The frame bundle $\widehat{Q}$ of any trivialization of L provides a lift.

Converse to *iv*), every lift $\widehat{P}$ of P to a $\widehat{\Gamma}$-bundle yields a trivialization. Indeed, the conjugate of the line bundle $Q = \widehat{P} \times_{S^1} \mathbb{C}$ over P is a rank 1-module with twisting:

$$\gamma \; : \; L \otimes \pi_2^* Q^* \longrightarrow \pi_1^* Q^* \quad ; \quad [\,\widehat{g},\, \lambda\,] \otimes [\,\widehat{p},\, \mu\,] \mapsto [\,\widehat{p}\,\widehat{g}^{-1},\, \lambda\, \mu\,] \; .$$

2.1.2 Bundle gerbes in higher gauge theory

This section aims at giving a glimpse into the greater context, in which bundle gerbes appear by explaining what they look like from a different point of view. As we have introduced them above, they occur as S^1-extensions of the pair groupoid. To understand their different flavours and generalizations, we first need some basics about higher algebraic structures.

Definition 2.1.8. A *strict* 2*-group* is a small monoidal category such that every object and every morphism has a strict inverse with respect to the monoidal structure or – phrased equivalently – a group object in groupoids (or equivalently, which is a funny thing to check: a group object in the category of categories).

Definition 2.1.9. A *crossed module* is given by a pair of two groups H, G together with a homomorphism $\alpha : H \to G$ and a right action of G on H denoted by h^g for $h \in H$ and $g \in G$, such that: $\alpha(h^g) = g^{-1}\alpha(h)g$ and $h_2^{\alpha(h_1)} = h_1^{-1}h_2h_1$.

These two notions are equivalent: Given a strict 2-group $\mathfrak{G}$ we take G to be the group of objects $\mathrm{obj}(\mathfrak{G})$ and set H to be the set of morphisms with the monoidal identity as source. This is a group with respect to the monoidal structure, which we will denote by $\otimes$. The action of G on h is given by $h^g = \mathrm{id}_{g^{-1}} \otimes h \otimes \mathrm{id}_g$ and $\alpha(h) = t(h)$. Starting from a crossed module $\alpha : H \to G$, we can construct the groupoid $\mathfrak{G}$ with objects G and arrows $G \times H$ with source $s(g, h) = g$ and target $t(g, h) = g\alpha(h)$. It becomes a group object with the multiplication $(g_1, h_1)\,(g_2, h_2) = (g_1g_2, h_1^{g_2}h_2)$.

Given a central S^1-extension $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$, consider the associated lifting bundle gerbe $L \to \Gamma^2$ over a *point*. This turns out to be nothing else, but the 2-group associated to the homomorphism $\alpha : \widehat{\Gamma} \to \Gamma$, where Γ acts on $\widehat{\Gamma}$ via conjugation. Indeed, the object space in this case is Γ, the arrows are given by L with source and target the two projections π_i. The monoidal structure uses the multiplication on L. Therefore we may see a general lifting bundle gerbe as a (principal) bundle of 2-groups given by the crossed module $\alpha : \widehat{\Gamma} \to \Gamma$.

Observe that the trivial homomorphism $\alpha : S^1 \to \{1\}$ also gives rise to a 2-group, which can be understood as the structure group of a general bundle gerbe $L \to Y^{[2]}$ *over* $Y^{[2]}$ for some fibration $Y \to X$. [3] takes this as a starting point to extend the theory to non-abelian bundle gerbes with more general 2-groups as a fiber.

2.1.3 Twisted K-theory and bundle gerbes

In the same way an orientation on a vector bundle E over a space X induces a THOM class $\tau \in H_c^*(E)$ and a ring isomorphism $H_c^*(E) \simeq H^*(X)$ via cup product with τ, the existence of a spin^c structure on E, i.e. a lift of the frame bundle P_E to a principal Spin^c-bundle will provide similar objects in K-theory.

For ordinary cohomology we can work around the absence of orientability by changing from constant coefficients to local ones. Looking at this procedure from a homotopy theoretic point of view, it corresponds to replacing homotopy classes of maps into the spectrum $E_n = K(\mathbb{Z}, n)$ by homotopy classes of *sections* in a *bundle of spectra* over the space X in question, leading the way into the world of parametrized homotopy theory as explained in the book by MAY and SIGURDSSON [40].

Like twists for ordinary cohomology can be described as $\mathbb{Z}/2$-torsors, that is homotopy classes of maps $X \to B\mathbb{Z}/2$, twists for K-theory are classified by $BU_\otimes$-torsors, i.e. maps $X \to BBU_\otimes$, where $BU_\otimes$ is the classifying space BU of vector bundles with virtual dimension 0 with H-space structure induced by the tensor product. $BBU_\otimes$ splits on the level of spaces to give:

$$BBU_\otimes = K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BSU_\otimes \ .$$

The first factor describes orientation twists of much the same flavour as those for ordinary cohomology, the last one yields the so-called higher twists in K-theory (see for example TELEMAN [69]). The second factor has nice geometric interpretations, which shall be explained on the zeroth part of the spectrum, i.e. for K^0. By the ATIYAH-JÄNICH-theorem, this group can be understood as homotopy classes of maps:

$$K^0(X) = [X, \mathrm{Fred}(H)]$$

into the Fredholm operators on some separable Hilbert space H. Conjugation by an element of the unitary group $U(H)$ maps $\mathrm{Fred}(H)$ into itself. Given a principal bundle $\mathcal{P}$ with structure group $PU(H) = U(H)/U(1)$, we can associate $\mathrm{Fred}(H)$ to get $\mathcal{F} = \mathcal{P} \times_{Ad} \mathrm{Fred}(H)$ and define

$$K^0_{\mathcal{P}}(X) = [X, \mathcal{F}] = [\mathcal{P}, \mathrm{Fred}(H)]^{PU(H)}$$

as the homotopy classes either of sections or of $PU(H)$-equivariant maps [5, 6]. Indeed, $BPU(H)$ is a $K(\mathbb{Z}, 3)$-space, the isomorphism class of $\mathcal{P}$ represents an element in $H^3(X, \mathbb{Z})$ and the above definition yields K-theory twisted with $\mathcal{P}$. Observe that the K-groups for different bundles representing the same class $[\mathcal{P}] \in H^3(X, \mathbb{Z})$ are isomorphic, but the isomorphism is *non-canonical*. Therefore we will only talk about K-theory twisted with $\mathcal{P}$, not with $[\mathcal{P}] \in H^3(X, \mathbb{Z})$.

A little closer to non-commutative geometry ROSENBERG gave a description of this group in terms of the K-theory of C^*-algebras in [57]. For a principal $PU(H)$-bundle $\mathcal{P}$, consider the bundle $\mathcal{A}_{\mathbb{K}} = \mathcal{P} \times_{Ad} \mathbb{K}$ with fibers the compact operators on a separable Hilbert space. The continuous sections $C(M, \mathcal{A}_{\mathbb{K}})$ form a C^*-algebra and we have

Theorem 2.1.10. *For the K^0-group twisted with $\mathcal{P}$ representing the element $[\mathcal{P}] \in H^3(M, \mathbb{Z})$ over a paracompact Hausdorff space X there is a canonical isomorphism:*

$$K^0_{\mathcal{P}}(X) \simeq K_0(C(X, \mathcal{A}_{\mathbb{K}})) \ .$$

In case $\mathcal{P}$ represents a torsion class, it reduces to a bundle with structure group $PU(n)$ for some $n \in \mathbb{N}$. Therefore $\mathcal{A}_{\mathbb{K}}$ reduces to a bundle of matrix algebras $\mathcal{K}$. Associated to $PU(n)$ we have the short exact sequence:

$$1 \longrightarrow S^1 \longrightarrow U(n) \longrightarrow PU(n) \longrightarrow 1$$

and the class represented by $[\mathcal{P}]$ coincides with the DIXMIER-DOUADY-class of the corresponding lifting bundle gerbe. Indeed, as was shown in [12], virtual modules over this bundle gerbe replace virtual vector bundles in their role as geometric representation of K-cycles for twisted K-theory. They fit into the framework in a particularly nice way:

Theorem 2.1.11. *Given a bundle gerbe $L \to Y^{[2]}$ with $dd(L)$ torsion, then the* GROTHENDIECK *group of bundle gerbe modules over L, denoted by $K^0_L(X)$ depends up to isomorphism only on the stable isomorphism class of L. It is isomorphic to the twisted K^0-group for the twist $dd(L)$.*

Chapter 3

Twisted Hilbert A-module bundles

One of the crucial insights of the last section was a geometric representation of the twisted K-theory group $K^0_{\mathcal{P}}(M)$ with torsion twist in terms of virtual bundle gerbe modules over some lifting bundle gerbe L, which had its counterpart in the world of C^*-algebras as the K_0-group of sections in a matrix algebra bundle $\mathcal{K}$. Let A be a unital C^*-algebra and consider continuous A-valued functions $C(M, A)$ equipped with the supremum norm. In view of the previous observations we can see the K-group $K_0(C(M, A))$ as the non-twisted version of some more general setup. Indeed, $K_0(C(M, A))$ can be described by virtual Hilbert A-module bundles over M with projective fibers. So, there is hope to find a similar description for the C^*-algebra $C(M, \mathcal{A})$ of sections in some (locally trivial) bundle with C^*-fibers in terms of twisted bundles, at least in case the structure group of $\mathcal{A}$ is not too wild. This point of view will be developed in the following chapter, which starts with a definition motivated by the twisted case with coefficients in $\mathbb{C}$.

Definition 3.0.12. Given a C^*-algebra A we denote by $PU(A)$ the group of unitary elements in the associated multiplier algebra $M(A)$ modulo the phase factors $U(1) \subset U(M(A))$. Let M be a compact manifold, then a $PU(A)$-*bundle* $\mathcal{A}$ over M is a locally trivial bundle of C^*-algebras with typical fiber A, such that its associated $\mathrm{Aut}(A)$-bundle restricts to $PU(A)$ (that is, the associated Čech 1-cocycle $U_{ij} \longrightarrow \mathrm{Aut}(A)$ factors through $PU(A)$).

Remark At this point, it might seem unreasonable to consider only the quotient by $U(1)$ instead of $Z(A)$, the center of $U(A)$, which would also be closer to deserve the name projective unitary group. The reason is more a simplification than a restriction. In fact, we would like to deal with line bundle gerbes, especially when treating trivializations and connections, which are most easily phrased in terms of tensor products and forms. In case the center of A is a compactly generated abelian group of Lie type, i.e. if it is isomorphic to

$$\mathbb{R}^n \times \mathbb{Z}^m \times (S^1)^k \times F$$

for some finite group F and $k, n, m \in \mathbb{N}_0$, the step towards an extension of the whole theory should be possible with a slightly enhanced version of bundle gerbes. This point will also be addressed in the outlook given at the end of this paper.

Remark Although most of the propositions, in particular lemma 3.2.2 and theorem 3.2.6, also hold for non-unital C^*-algebras A, we will from now on assume that A has a unit.

Definition 3.0.13. A *twisted Hilbert A-module bundle E* associated to a $PU(A)$-bundle $\mathcal{A}$ is a locally trivial (right) Hilbert A-module bundle over the $PU(A)$-principal bundle $\mathcal{P}$ of $\mathcal{A}$ together with an isometric (left) action of the lifting bundle gerbe $L \longrightarrow \mathcal{P}^{[2]}$, i.e.

- a fiberwise A-linear isometric isomorphism
$$\gamma \;:\; L \otimes \pi_2^* E \xrightarrow{\simeq} \pi_1^* E \ ,$$
which is associative with respect to the bundle gerbe product on L.

Alternatively one could describe γ by isometric isomorphisms that *shift the fibers* by $g^{-1} \in PU(A)$, i.e.
$$\gamma_g \;:\; L^g \otimes E \longrightarrow E \ ,$$
where L^g is the pullback of L via $p \mapsto (p, pg)$.

If the typical fiber of E is a projective Hilbert A-module, then E itself will be called *projective*.

Example 3.0.14. Let U be a contractible manifold and $t \in M_n(A)$ be a projection in the matrix algebra of a unital C^*-algebra A. Set
$$F = \{(x, g, v) \in U \times PU(A) \times A^n \mid \hat{g}^* t \hat{g}\, v = v\} \ ,$$
where $\hat{g}$ here and in the following denotes a lift of $g \in PU(A)$ to $U(A)$. Due to the conjugation, the fibers are independent of the choice of lift. Now F is a projective twisted Hilbert A-module bundle over U associated to the $PU(A)$-bundle $U \times A$ with γ given by
$$\gamma \;:\; L \otimes \pi_2^* F \longrightarrow \pi_1^* F \quad , \quad [\hat{g}, \lambda] \otimes v \mapsto \lambda\, \hat{g} v \ .$$
Since every principal $PU(A)$-bundle $\mathcal{P}$ over U is trivial due to its contractibility, this is a quite general situation. Indeed, we will see in the following that every projective twisted Hilbert A-module bundle locally looks like this example, which is as less twisted as possible and therefore shall be called the *slightly twisted Hilbert A-module bundle.* Let $V = t\, A^n$, then F will also be denoted by $\underline{V}$ expressing that the fiber over the identity element is canonically isomorphic to V.

Example 3.0.15. Let $\mathcal{A}$ be a $PU(A)$-bundle over a closed manifold M and let $\mathcal{P}$ be its associated principal bundle. Consider $\underline{A^n} := \mathcal{P} \times A^n$ for some $n \in \mathbb{N}$ together with γ given by
$$\gamma \;:\; L \otimes \pi_2^* \underline{A^n} \longrightarrow \pi_1^* \underline{A^n} \quad , \quad [\hat{g}, \lambda] \otimes v \mapsto \lambda\, \hat{g} v \ . \tag{3.1}$$
This is a projective twisted Hilbert A-module bundle closely connected to the slightly twisted one, but defined globally. We will call this the *trivial twisted Hilbert A-module bundle.*

3.1 The projective unitary group of a C^*-algebra

Since the world of infinite dimensional topological groups has a landscape full of craters and volcanoes, it is a rather lucky fact, that the unitary group $U(A)$ decided to settle down in a quiet and peaceful part of it. When equipped with the norm topology, it is a real Banach Lie group modelled on the real Banach space of skew-adjoint operators, which will be denoted by $i\mathfrak{a}$, see for example Corollary 15.22 in [74] for a proof of this fact. As such, it is also a regular Lie group in the sense of [34], which will help us later, when we need to solve differential equations on $U(A)$, for example to get parallel transport. Due to the quotient theorem proven by Glöckner and Neeb in [23], $PU(A)$ turns out to be a real Banach Lie group as well modelled on the quotient $i\mathfrak{a}/i\mathbb{R}1$, where $i\mathbb{R}1 \subset i\mathfrak{a}$ is the one-dimensional real subspace spanned by $i1 \in A$.

Theorem 3.1.1. *In the short exact sequence*

$$1 \longrightarrow S^1 \longrightarrow U(A) \xrightarrow{q} PU(A) \longrightarrow 1$$

the group $U(A)$ is a (locally trivial) principal S^1-bundle over $PU(A)$.

Proof. Since $U(A)$, $PU(A)$ are Banach Lie groups, their exponential maps are local diffeomorphisms around 0. To find a splitting for q in some open neighborhood of the identity, it is thus enough to find a linear split for $q_* : i\mathfrak{a} \to i\mathfrak{a}/i\mathbb{R}1$. Since $i\mathbb{R}1 \subset i\mathfrak{a}$ is finite-dimensional and therefore complementable, there is a direct sum decomposition $i\mathfrak{a} = \mathfrak{e} \oplus i\mathbb{R}$ together with an isomorphism of Banach spaces: $\mathfrak{e} \simeq i\mathfrak{a}/i\mathbb{R}$. This yields a continuous linear split

$$i\mathfrak{a}/i\mathbb{R} \longrightarrow i\mathfrak{a} \ . \qquad \square$$

3.2 Morphisms of twisted Hilbert A-module bundles

Now consider two twisted Hilbert A-module bundles E and F living over the same $PU(A)$-principal bundle $\mathcal{P} \longrightarrow M$. Let $\mathrm{Hom}(E,F)$ denote the bundle with fiber at $p \in \mathcal{P}$ given by the Hilbert A-module morphisms from E_p to F_p. There is a left action of $PU(A)$ on this bundle shifting its fibers. It maps $\phi_p \in \mathrm{Hom}(E_p, F_p)$ to $g \cdot \phi_p \in \mathrm{Hom}(E_{pg^{-1}}, F_{pg^{-1}})$ given by:

$$(g \cdot \phi_p) \ : E_{pg^{-1}} \xrightarrow{\gamma_E^{-1}} L^g \otimes E_p \xrightarrow{\mathrm{id} \otimes \phi_p} L^g \otimes F_p \xrightarrow{\gamma_F} F_{pg^{-1}} \tag{3.2}$$

Commutativity of the following diagram ensures associativity of this group action:

$$\begin{tikzcd}
 & E_{p(g_1 g_2)^{-1}} \arrow[dl] \arrow[dr] & \\
L^{g_1} \otimes L^{g_2} \otimes E_p \arrow[rr] \arrow[d, "\mathrm{id}_L \otimes \mathrm{id}_L \otimes \phi_p"] & & L^{g_1 g_2} \otimes E_p \arrow[d, "\mathrm{id}_L \otimes \phi_p"] \\
L^{g_1} \otimes L^{g_2} \otimes F_p \arrow[rr] \arrow[dr] & & L^{g_1 g_2} \otimes F_p \arrow[dl] \\
 & F_{p(g_1 g_2)^{-1}} &
\end{tikzcd}$$

The horizontal maps correspond to the bundle gerbe product on the second tensor factor. Since we can identify L^e with $\mathbb{C}$, the trivial element of $PU(A)$ acts trivially.

The projection map $\mathrm{Hom}(E,F) \longrightarrow \mathcal{P}$ factorizes over the action of $PU(A)$ to give $\mathrm{hom}(E,F) \longrightarrow M$, which is again a locally trivial bundle. To see this, choose for an arbitrary point $x \in M$ a contractible neighborhood U. Since $\mathcal{P}$ is trivial over U, the latter maps into $\mathcal{P}$ via the identity section $\sigma \ : \ U \longrightarrow \mathcal{P}|_U$. There are two canonical maps

$$\begin{aligned}
\iota_1 \ &: \ \mathrm{hom}(E,F)|_U \longrightarrow \sigma^* \, \mathrm{Hom}(E,F)|_U \quad , \quad [v] \mapsto g \cdot v \ , \\
\iota_2 \ &: \ \sigma^* \, \mathrm{Hom}(E,F)|_U \longrightarrow \mathrm{hom}(E,F)|_U \quad , \quad w \mapsto [w]
\end{aligned}$$

(with $g \in PU(A)$ chosen in such a way that $g \cdot v$ lies above the identity), which are easily seen to be inverse to each other. But since $\mathrm{Hom}(E,F)$ is locally trivial and U is contractible, we have established a local isomorphism to the trivial bundle.

Global sections of the bundle $\mathrm{hom}(E,F)$ correspond to morphisms between twisted Hilbert A-module bundles, by which we mean a fiberwise A-linear map ϕ intertwining the actions of the bundle gerbe, i.e. the following diagram commutes:

$$\begin{tikzcd} L \otimes \pi_2^* E \arrow[r, "\mathrm{id}_L \otimes \pi_2^*\phi"] \arrow[d, "\gamma_E"] & L \otimes \pi_2^* F \arrow[d, "\gamma_F"] \\ \pi_1^* E \arrow[r, "\pi_1^*\phi"] & \pi_1^* F \end{tikzcd}$$

Note especially that, although the bundle E itself needs the intermediate space $\mathcal{P}$ to be well-defined, its endomorphism bundle $\mathrm{end}(E)$ does not, even better: it is a locally trivial bundle of C^*-algebras over M. Therefore, equipped with the sup-norm, the sections $\Gamma(\mathrm{end}(E))$ form a C^*-algebra.

Example 3.2.1. An important example of the above construction is the endomorphism bundle of the trivial twisted Hilbert A-module bundle $\underline{A^n}$ for some $n \in \mathbb{N}$. Note that right A-linear endomorphisms of A^n are always maps of the form $\phi : v \mapsto T\, v$ for some A-valued $n \times n$-matrix $T \in M_n(A) \simeq A \otimes M_n(\mathbb{C})$. Therefore (3.1) turns (3.2) into the conjugation action on T, i.e.

$$(g \cdot \phi_p)(v) = \hat{g}\, T\, \hat{g}^{-1}\, v\ .$$

Thus, taking the quotient yields an identity that lies at the heart of the theory of twisted Hilbert A-bundles:

$$\mathrm{end}(\underline{A^n}) \;\simeq\; \mathcal{P} \times_{\mathrm{Ad}} M_n(A) \;\simeq\; \mathcal{A} \otimes M_n(\mathbb{C})\ .$$

Especially we regain the PU(A)-bundle $\mathcal{A}$ as endomorphism bundle of the trivial twisted "line" bundle $\underline{A}$.

Lemma 3.2.2. *Let E be a twisted Hilbert A-module bundle over M. Every $x \in M$ has a contractible neighborhood U', such that there exists an isomorphism $\Psi : E|_{U'} \longrightarrow \underline{V}$ of twisted Hilbert A-module bundles to the slightly twisted bundle over U'.*

Proof. Choose a contractible neighborhood U of x, such that $\mathcal{P}$ is trivial over U and let $\bar{E}$ be the pullback:

$$\begin{tikzcd} \bar{E} \arrow[r, "\bar{\varphi}"] \arrow[d] & E \arrow[d] \\ U \times PU(A) \arrow[r, "\varphi"] & \mathcal{P} \end{tikzcd}$$

Pulling back the bundle gerbe L to $\bar{L}$ over $U \times PU(A) \times PU(A)$ as well, there is an action:

$$\bar{\gamma} \;:\; \bar{L} \otimes \bar{\pi}_2^* \bar{E} \longrightarrow \bar{\pi}_1^* \overline{E}\ .$$

Let $V := t\, A^n \simeq E_{(x,1)}$ be an isomorphic image of the fiber $E_{(x,1)}$. Since E is locally trivial over $\mathcal{P}$, there is a neighborhood $U' \times W \subset U \times PU(A)$ containing $(x, 1)$ and a trivialization $\psi \;:\; \bar{E}|_{U' \times W} \longrightarrow U' \times W \times V$. From now on we will identify $\bar{E}$ with its restriction to $U' \times PU(A)$. Now let $\underline{V}$ be the slightly twisted bundle over U' associated to t and $\bar{\delta}$ be the corresponding action of $\overline{L}$ on $\underline{V}$. Let $U'' = U' \times PU(A) \times \{1\}$, then $\bar{E}$ can be identified with $\bar{\pi}_1^* \bar{E}|_{U''}$, likewise $\underline{V}$ is a similar subbundle in $\bar{\pi}_1^* \underline{V}$. Therefore

$$\Psi \;:\; \bar{\pi}_1^* \bar{E}|_{U''} \xrightarrow{\overline{\gamma}^{-1}} \bar{L} \otimes \bar{\pi}_2^* \left(\bar{E}|_{U' \times \{1\}} \right) \xrightarrow{\mathrm{id}_L \otimes \psi} \bar{L} \otimes \bar{\pi}_2^* \left(\underline{V}|_{U' \times \{1\}} \right) \xrightarrow{\bar{\delta}} \bar{\pi}_1^* \underline{V}|_{U''}$$

can be interpreted as an A-linear map between $\bar{E}$ and $\underline{V}$. It is also isometric, since all maps involved preserve the fiberwise scalar product.

What remains to show is that Ψ is a morphism of bundle gerbe modules, i.e. that the diagram:

$$\begin{tikzcd} L \otimes \pi_2^* \bar{E} \arrow[r, "\bar{\gamma}"] \arrow[d, "\mathrm{id}_L \otimes \pi_2^* \Psi"] & \pi_1^* \bar{E} \arrow[d, "\pi_1^* \Psi"] \\ L \otimes \pi_2^* \underline{V} \arrow[r, "\bar{\delta}"] & \pi_1^* \underline{V} \end{tikzcd} \tag{3.3}$$

commutes. Let $\widetilde{\pi}_i, \widetilde{\pi}_{ij}$ for $i, j \in \{1, 2, 3\}$ be the canonical projections of $U' \times PU(A) \times PU(A) \times W$ to one or a product of two factors. Note that the two diagrams

$$\begin{tikzcd} L \otimes \pi_2^* \bar{E} \arrow[r, "\bar{\gamma}"] \arrow[d, "\widetilde{\iota}_1"] & \pi_1^* \bar{E} \arrow[d, "\iota_1"] \\ \widetilde{\pi}_{12}^* L \otimes \widetilde{\pi}_2^* \bar{E} \arrow[r, "\widetilde{\pi}_{12}^* \bar{\gamma}"] & \widetilde{\pi}_1^* \bar{E} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\pi}_{12}^* L \otimes \widetilde{\pi}_2^* \underline{V} \arrow[r, "\widetilde{\pi}_{23}^* \bar{\delta}"] \arrow[d, "\widetilde{\mathrm{pr}}"] & \widetilde{\pi}_1^* \underline{V} \arrow[d, "\mathrm{pr}"] \\ L \otimes \pi_2^* \underline{V} \arrow[r, "\bar{\delta}"] & \pi_1^* \underline{V} \end{tikzcd}$$

commute, where the maps $\widetilde{\iota}_1$ and ι_1 inject the bundles to their pullbacks over $PU(A) \times PU(A) \times \{1\}$, while $\widetilde{\mathrm{pr}}$ and pr project to their restriction to $PU(A) \times PU(A)$. Furthermore all squares in

$$\begin{tikzcd} \widetilde{\pi}_{12}^* L \otimes \widetilde{\pi}_2^* \bar{E} \arrow[r, "\widetilde{\pi}_{12}^* \bar{\gamma}"] \arrow[d, "\mathrm{id}_L \otimes \widetilde{\pi}_{23}^* \bar{\gamma}^{-1}"] & \widetilde{\pi}_1^* \bar{E} \arrow[d, "\widetilde{\pi}_{13}^* \bar{\gamma}^{-1}"] \\ \widetilde{\pi}_{12}^* L \otimes \widetilde{\pi}_{23}^* L \otimes \widetilde{\pi}_3^* \bar{E} \arrow[r] \arrow[d, "\mathrm{id}_L \otimes \mathrm{id}_L \otimes \widetilde{\pi}_3^* \psi"] & \widetilde{\pi}_{13}^* L \otimes \widetilde{\pi}_3^* \bar{E} \arrow[d, "\mathrm{id}_L \otimes \widetilde{\pi}_3^* \psi"] \\ \widetilde{\pi}_{12}^* L \otimes \widetilde{\pi}_{23}^* L \otimes \widetilde{\pi}_3^* \underline{V} \arrow[r] \arrow[d, "\mathrm{id}_L \otimes \widetilde{\pi}_{23}^* \bar{\delta}"] & \widetilde{\pi}_{13}^* L \otimes \widetilde{\pi}_3^* \underline{V} \arrow[d, "\widetilde{\pi}_{13}^* \bar{\delta}"] \\ \widetilde{\pi}_{12}^* L \otimes \widetilde{\pi}_2^* \underline{V} \arrow[r, "\widetilde{\pi}_{12}^* \bar{\delta}"] & \widetilde{\pi}_1^* \underline{V} \end{tikzcd}$$

commute – the upper and lower one because of associativity of the bundle gerbe product, the middle one trivially. Sticking the previous diagrams to top and bottom of this rectangle we read off the commutativity of (3.3). □

Definition 3.2.3. The above construction of twisted bundle morphisms turns the class of twisted Hilbert A-module bundles into a category $\mathsf{TwBun}_{\mathcal{A}}$.

Having at hand an endomorphism bundle of C^*-algebras living above the base manifold instead of the topologically huge space $\mathcal{P}$, enables us to carry over some well known results from the theory of non-twisted Hilbert A-module bundles like the following:

Lemma 3.2.4. *Given two twisted Hilbert A-module bundles E, F, which are isomorphic as twisted A-module bundles, i.e. via a twisted bundle map φ* not necessarily *preserving the scalar product. Then there exists an isomorphism $\bar{\varphi}$ which respects the inner product as well.*

Proof. Assume that $\varphi \in C(M, \hom(E,F))$ is an isomorphism. Since $\varphi^* \varphi \in C(M, \mathrm{end}(E))$ and the latter space is a C^*-algebra, we take the polar decomposition $\varphi = u\,|\varphi|$ (with $|\varphi| = \sqrt{\varphi^* \varphi}$). The element $u \in C(M, \mathrm{iso}(E,F))$ is the isometry we were looking for. □

Lemma 3.2.5. (Clutching construction) *Given a space M, a $PU(A)$-bundle $\mathcal{A}$ over M with frame bundle $\rho : \mathcal{P} \to M$ and an open cover $U_i \subset M$. If E_i over $\rho^{-1}(U_i)$ are twisted Hilbert A-module bundles together with twisted bundle isomorphisms $\varphi_{ij} : E_i \longrightarrow E_j$ over $U_{ij} = U_i \cap U_j$, which satisfy the cocycle identity $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ over U_{ijk}, then there exists a twisted Hilbert A-module bundle E over M and twisted bundle isomorphisms $\psi_i : E_i \longrightarrow E|_{U_i}$ such that the following diagram commutes:*

$$\begin{tikzcd} E_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_i"] & & E_j \arrow[dl, "\psi_j"] \\ & E|_{U_{ij}} & \end{tikzcd}$$

Furthermore, if there exist twisted bundle morphism $\kappa_i : E_i \longrightarrow F$ into another twisted bundle F, such that $\kappa_i = \kappa_j \circ \varphi_{ij}$, then there is a twisted bundle morphism $\kappa : E \longrightarrow F$ restricting to κ_i over U_i.

Proof. The construction can be carried out word by word like in the untwisted case with

$$E = \coprod_i E_i / \sim \quad \text{with} \quad v_j \sim \varphi_{ij}(v_i)$$

and the obvious embeddings ψ_i of E_i into E. Local triviality of E over $\mathcal{P}$ then follows from the local triviality of the E_i. The intertwining condition for twisted bundle morphisms now implies that there is a well-defined twisting

$$\gamma : L \otimes \pi_2^* E \longrightarrow \pi_1^* E \ .$$

Since $\kappa = \coprod_i \kappa_i$ factors over the equivalence relation by assumption, this yields the twisted bundle morphism. □

Theorem 3.2.6. *Let E be a projective twisted Hilbert A-module bundle, then there exists $n \in \mathbb{N}$ such that E is isomorphic (as a twisted Hilbert A-module bundle) to a direct summand of the trivial twisted Hilbert A-module bundle $\underline{A^n}$.*

Proof. Following the trivialization lemma 3.2.2, there exists a finite open cover U_i, $i = 1 \dots m$ of M, such that each U_i is contractible and $E|_{\rho^{-1}(U_i)}$ is twistedly isomorphic to $\underline{V}$ over U_i, where V is the typical fiber of E. We will assume U_i to be a good cover, that is all intersections are either empty or contractible. Embedding $\underline{V}$ into the globally defined trivial bundle $\underline{A^k}$, yields a map

$$\Psi \ : \ E \longrightarrow \underline{A^{km}} \quad ; \quad v \mapsto \sum_{i=1}^{m} h_i(\rho \circ \pi(v)) \cdot \phi_i(v) \ , \tag{3.4}$$

where $(h_i)_{i=1\dots m}$ is a subordinate partition of unity and $\phi_i : E_{\rho^{-1}(U_i)} \longrightarrow \underline{A^k}$. Taking fiberwise orthogonal complements, produces a bundle over $\mathcal{P}$ of the form

$$E^{\perp} = \{(p,v) \in \mathcal{P} \times A^{km} \mid \langle v, \Psi(w) \rangle = 0 \ \forall w \in E_p\} \ .$$

Let δ be the bundle gerbe action of L on $\underline{A^{km}}$, γ be the action on E, then

$$\begin{aligned} & \langle \delta([\hat{g},\lambda]\otimes v), \Psi(w)\rangle \\ = \; & \langle \delta([\hat{g},\lambda]\otimes v), \Psi(\gamma([\hat{g},\mu]\otimes w'))\rangle \\ = \; & \langle \delta([\hat{g},\lambda]\otimes v), \delta([\hat{g},\mu]\otimes \Psi(w'))\rangle \\ = \; & \bar{\lambda}\mu \, \langle v, \Psi(w')\rangle = 0 \end{aligned}$$

for all $v \in E_p^{\perp}, w \in E_{pg^{-1}}$. Thus, it is possible to restrict δ to $E^{\perp}$ to get an action of the bundle gerbe L there as well.

The next step is to show that $E^{\perp}$ is indeed an orthogonal complement of E in $\underline{A^{km}}$, which cannot be taken for granted in the case of Hilbert modules. Since we only need to show this locally, we fix some $p \in \mathcal{P}$ and consider the contractible open set $\bar{U} = \bigcap_{p\in\rho^{-1}(U_i)} U_i$. Now note that Ψ factors as

$$\Psi \; : \; E|_{\rho^{-1}(\bar{U})} \xrightarrow{\overline{\Psi}} \underline{V^{\bar{m}}}|_{\bar{U}\times PU(A)} \longrightarrow \underline{A^{k\bar{m}}} \oplus \underline{A^{k(m-\bar{m})}}\Big|_{\rho^{-1}(\bar{U})} \; ,$$

where $\overline{\Psi}$ is the same map as in (3.4) with ϕ_i replaced by $\overline{\phi}_i : E_{\rho^{-1}(U_i)} \longrightarrow \underline{V}|_{\bar{U}\times PU(A)}$ the index running over all $j \in \{1 \dots m\}$ such that $U_j \cap \bar{U} \neq \emptyset$. In the second step this is embedded into $\underline{A^{k\bar{m}}}$ using the same local trivializations of $\mathcal{P}$ as for the definition of the map to $\underline{V}$. This again factors as

$$\underline{V^{\bar{m}}}\Big|_{\bar{U}\times PU(A)} \longrightarrow \underline{(V\oplus V^{\perp})^{\bar{m}}}\Big|_{\bar{U}\times PU(A)} \longrightarrow \underline{A^{k\bar{m}}} \oplus \underline{A^{k(m-\bar{m})}}\Big|_{\rho^{-1}(\bar{U})} \; .$$

$E^{\perp}\big|_{\rho^{-1}(\bar{U})}$ splits as $\widehat{E}^{\perp}\Big|_{\rho^{-1}(\bar{U})} \oplus \underline{A^{k(m-\bar{m})}}$, since the image of Ψ restricted to $\rho^{-1}(\bar{U})$ lies in $\underline{A^{k\bar{m}}}$. Pulling back $\widehat{E}^{\perp}$ over $\bar{U} \times PU(A)$, it splits off $\underline{(V^{\perp})^{\bar{m}}}$ as a direct summand, since the image of $\overline{\Psi}$ lies completely in $\underline{V^{\bar{m}}}$. The restriction of this pullback to the other summand will be denoted $\overline{E}^{\perp}$.

Therefore proving that $\overline{E}^{\perp}$ is a complement of $\overline{\Psi}(E)$ in $\underline{V^{\bar{m}}}$, finishes the theorem. But this part is very similar to the non-twisted case. For the sake of completeness we will repeat the main points here. It suffices to show that $\overline{\Psi}(E_p) + \overline{E}_p^{\perp} = V^{\bar{m}}$, since directness of the sum easily follows from the positivity of the inner product. The definition of $\overline{\Psi}$ implies

$$\begin{aligned} \overline{\Psi}(E_p) &= \{(\lambda_1 \,\overline{\phi}_1(w), \dots, \lambda_{\bar{m}}\, \overline{\phi}_{\bar{m}}(w)) \in V^{\bar{m}} \mid w \in E_p\} \\ &= \{(v, \kappa_2(v) \dots, \kappa_{\bar{m}}(v)) \in V^{\bar{m}} \mid v \in V\} \end{aligned}$$

with $\lambda_i \in \mathbb{R}$, at least one $\lambda_j \neq 0$. Without loss of generality, we assume that $\lambda_1 \neq 0$, then $\kappa_i = \lambda_1^{-1}\,\lambda_i\,\overline{\phi}_i \circ \overline{\phi}_1^{-1} \in \mathrm{End}(V)$ ($\overline{\phi}_i$ is an isometry and therefore adjointable).

Consider now the Hilbert A-module F_p spanned by the set

$$\{(-\kappa_j^*(v), 0, \dots, 0, v, 0, \dots, 0) \in V^{\bar{m}} \mid v \in V, j \in \{1, \dots \bar{m}\}\}$$

with v at the jth position. A short calculation shows $F_p = \overline{E}_p^{\perp}$. Indeed, the equality $F_p + \overline{\Psi}(E_p) = V^{\bar{m}}$ holds, since the system of equations

$$\begin{aligned}
v_1 - \kappa_2^*(v_2) - \cdots - \kappa_{\bar{m}}^*(v_{\bar{m}}) &= w_1 \\
\kappa_2(v_1) + v_2 &= w_2 \\
&\vdots \\
\kappa_{\bar{m}}(v_1) + v_{\bar{m}} &= w_{\bar{m}}
\end{aligned}$$

or equivalently

$$\begin{aligned}
v_1 + \kappa_2^*(\kappa_2(v_2)) + \cdots + \kappa_{\bar{m}}^*(\kappa_{\bar{m}}(v_{\bar{m}})) &= w_1 + \kappa_2^*(w_2) + \cdots + \kappa_{\bar{m}}^*(v_{\bar{m}}) \\
v_2 &= w_2 - \kappa_2(v_1) \\
&\vdots \\
v_{\bar{m}} &= w_{\bar{m}} - \kappa_{\bar{m}}(v_{\bar{m}})
\end{aligned}$$

has a unique solution for an arbitrary vector $(w_1, \ldots, w_{\bar{m}}) \in V^{\bar{m}}$, because $1 + \kappa_2^* \kappa_2 + \cdots + \kappa_{\bar{m}}^* \kappa_{\bar{m}}$ is an invertible element in the C^*-algebra $\mathrm{End}(V)$.

A trivialization of $\overline{E}^{\perp}$ in this description is easily constructed from

$$\alpha_1 : \overline{E}^{\perp}\Big|_{\{\lambda_1 \neq 0\}} \longrightarrow \underline{V^{k-1}} \quad ; \quad (v_1, \ldots, v_m) \mapsto (v_2, \ldots, v_m)$$

(note that we always assume $\lambda_1 \neq 0$). Using lemma 3.2.4 these maps can be modified to isometric ones. Since the pullback of $E^{\perp}$ to $\bar{U} \times PU(A)$ is isomorphic to $\overline{E}^{\perp} \oplus \underline{(V^{\perp})^{\bar{m}}} \oplus \underline{A^{m-\bar{m}}}$, local triviality holds for $E^{\perp}$ as well.

The outcome is an embedding of E as a direct summand in $E \oplus E^{\perp} = \underline{A^m}$. □

Definition 3.2.7. A virtual twisted Hilbert A-module bundle is a class of pairs (E_+, E_-), denoted $E_+ - E_-$, with respect to the following equivalence relation $(E_+, E_-) \sim (F_+, F_-) \Leftrightarrow \exists G \in \mathsf{TwBun}_{\mathcal{A}}$, such that $E_+ \oplus F_- \oplus G \simeq F_+ \oplus E_- \oplus G$.

Due to theorem 3.2.6, the equivalence relation stays the same if we replace the twisted Hilbert A-module bundle G by $\underline{A^m}$, since we can simply add the stable inverse $G^{\perp}$ on both sides. The operation of direct sum turns the virtual bundles into an abelian group, which will be denoted by $K^0_{\mathcal{A}}(M)$.

Theorem 3.2.8. (Twisted Serre-Swan Theorem) *Let $\mathcal{A}$ be a $PU(A)$-bundle, then there is an isomorphism*

$$K_0(C(M, \mathcal{A})) \simeq K^0_{\mathcal{A}}(M) \ .$$

Indeed, the category of projective twisted Hilbert A-module bundles is naturally equivalent to the one of finitely generated, projective $C(M, \mathcal{A})$-modules.

Proof. Let t be a projection in $M_n(C(M, \mathcal{A})) \simeq C(M, M_n(\mathcal{A}))$ for some $n \in \mathbb{N}$. To get a twisted Hilbert A-module bundle, consider

$$E = \{(p, v) \in \mathcal{P} \times A^n \mid (M_n(p) \circ t)(\rho(p))\, v = v\}$$

where $\rho : \mathcal{P} \longrightarrow M$ denotes the bundle projection and p is considered as a $*$-isomorphism $p : \mathcal{A}_{\rho(p)} \longrightarrow A$. The twisting γ is given by

$$\gamma \ : \ L \otimes \pi_2^*(E) \longrightarrow \pi_1^*(E) \quad , \quad [\lambda, \widehat{g}] \otimes v \mapsto \lambda \widehat{g} v \ .$$

Due to the calculation

$$\begin{aligned} & (M_n(pg^{-1}) \circ t)(\rho(pg^{-1}))\lambda\widehat{g}v = \lambda\widehat{g}(M_n(p) \circ t)(\rho(p))\widehat{g}^{-1}\widehat{g}v \\ = \ & \lambda\widehat{g}(M_n(p) \circ t)(\rho(p))v = \lambda\widehat{g}v \ , \end{aligned}$$

γ is well-defined and maps $L^g \otimes E_p$ to $E_{pg^{-1}}$. This is an isometry with respect to the inner product attained by restricting the product on $\mathcal{P} \times A^n$ to E.

To see that E is locally trivial, fix a point p_0 and the corresponding projection $t_0 = (M_n(p_0) \circ t)(\rho(p_0)) \in M_n(A)$. Set $t_p = (M_n(p) \circ t)(\rho(p)) \in M_n(A)$. By continuity of $p \mapsto t_p$, there is an open neighborhood $\mathcal{P} \supset U \ni p_0$ with $\|t_p - t_0\| < 1$ for all $p \in U$, which implies that t_p and t_0 are unitarily equivalent. Let $u : U \longrightarrow U(M_n(A))$ be such that $ut_p = t_0 u$. Then

$$\Phi : E|_U \longrightarrow U \times t_0 A^n \quad ; \quad (p, v) \mapsto (p, uv)$$

provides a trivialization of E over U.

Note that for two projections $t \in C(M, M_m(\mathcal{A}))$ and $s \in C(M, M_k(\mathcal{A}))$ the block sum $t \oplus s \in C(M, M_{m+k}(\mathcal{A}))$ yields the direct sum of the corresponding twisted Hilbert A-module bundles.

Let E be a projective twisted Hilbert A-module bundle over M. By theorem 3.2.6 there exists an embedding $E \longrightarrow \underline{A^m}$ for some $m \in \mathbb{N}$. The bundle map

$$t \ : \ \underline{A^m} \longrightarrow \underline{A^m}$$

consisting fiberwise of projections with E_p as their image clearly is a morphism of bundle gerbe modules and therefore descends to a projection-valued section in $C(M, M_m(\mathcal{A})) = C(M, \mathrm{end}(\underline{A^m}))$.

Since E is twistedly isomorphic to its image in $\underline{A^m}$ the above operations are inverse to each other. $\square$

3.3 Twisted K-theory of locally compact spaces and $K^0_{\mathcal{A}}(X, Y)$

Since we are going to deal with symbol classes, which live in the K-theory of the cotangent bundle, we have to define twisted K-theory with local coefficients in a unital C^*-algebra A for non-compact spaces as well. According to the non-twisted case, where we have

$$\begin{aligned} & K^0_A(X) \\ \simeq \ & \mathrm{kern}(K^0_A(X^+) \longrightarrow K^0_A(\infty)) \\ \simeq \ & \mathrm{kern}(K_0(C_0(X, A)^+) \longrightarrow \mathbb{Z}) \\ \simeq \ & K_0(C_0(X) \otimes A) \ , \end{aligned}$$

K-theory *with compact supports* should be the right notion. The subtle difficulty in showing that the geometric and the algebraic picture, which we want to be $K(C_0(X,\mathcal{A}))$, agree, is that the bundle of C^*-algebras $\mathcal{A}$ lives over X. Unlike the non-twisted case there need *not* be an extension of $\mathcal{A}$ to the one-point-compactification X^+ (whereas the trivial bundle can always be extended). Nevertheless, we start with the most natural definition:

Definition 3.3.1. Let M be a locally compact, Hausdorff space, then the twisted K-theory $K^0_{\mathcal{A}}(M)$ with local coefficients in the unital C^*-algebra A is given by triples (E_+, E_-, φ_E), where E_+ and E_- are twisted Hilbert-A-module bundles and $\varphi_E \in C_b(M, \hom(E_+, E_-))$ is a twisted bundle morphism that is an isomorphism on the complement of a compact subset $K \subset M$, subject to the equivalence relation generated by the following: (E_+, E_-, φ_E) is equivalent to (F_+, F_-, φ_F) if there exist isomorphisms $\psi_\pm \in C_b(M, \mathrm{iso}(E_\pm, F_\pm))$ and a compact subset $\widehat{K}$ such that on the complement of $\widehat{K}$ the morphisms φ_E, φ_F are isomorphisms and the following diagram commutes:

$$\begin{tikzcd} E_+|_{M\setminus\widehat{K}} \arrow[r, "\varphi_E"] \arrow[d, "\psi_+"] & E_-|_{M\setminus\widehat{K}} \arrow[d, "\psi_-"] \\ F_+|_{M\setminus\widehat{K}} \arrow[r, "\varphi_F"] & F_-|_{M\setminus\widehat{K}} \end{tikzcd}$$

Furthermore, we demand $(E_+, E_-, \varphi_E) \sim (E_+ \oplus G, E_- \oplus G, \varphi_E \oplus \mathrm{id})$ for another twisted Hilbert A-module bundle G. In the compact case the morphism φ_E becomes obsolete and we retrieve the old definition.

Remark A detailed description of the K-theory for Banach categories, which can be applied in our case, since $C_b(M, \hom(E,F))$ are Banach spaces, can be found in the book by KAROUBI [32]. Let us briefly describe how inverses are constructed. We need the following lemma proven in [32]:

Lemma 3.3.2. *Let A, A' be unital Banach algebras and $f : A \longrightarrow A'$ be a surjective homomorphism. Let $GL(A)$, $GL(A')$ denote the groups of invertible elements in A, A' respectively. Let $\sigma : I \longrightarrow GL(A')$ be a continuous path such that $\sigma(0) = f(\alpha)$ for some $\alpha \in GL(A)$. Then there is an element $\beta \in GL(A)$ such that $f(\beta) = \sigma(1)$.*

This implies:

Lemma 3.3.3. *Let (F, F, φ_F) be a triple as described above. If the twisted bundle morphism $\varphi_F \in C_b(M, iso(F))$ is homotopic to the identity outside a compact subset, then:*

$$(F, F, \varphi_F) \sim (F, F, id_F) \ .$$

Proof. By a partition of unity argument it is easy to see that the restriction map $C_b(M, \mathrm{end}(F)) \longrightarrow C_b(U, \mathrm{end}(F))$ for $U = M\setminus W$ and $W \supset K$ a precompact, open neighborhood of the compact set K is a surjective homomorphism of C^*-algebras. By assumption there is a path $h : I \longrightarrow C_b(U, \mathrm{iso}(F))$ with $h(0) = \mathrm{id}_F|_U$ and $h(1) = \varphi_F|_U$. By the previous lemma there exists $\psi \in C_b(M, \mathrm{iso}(F))$ such that $\psi|_U = \varphi_F|_U$. Now the lemma follows from the diagram:

$$\begin{tikzcd} F|_U \arrow[r, "\varphi_F"] \arrow[d, "\psi"] & F|_U \arrow[d, "\mathrm{id}"] \\ F|_U \arrow[r, "\mathrm{id}"] & F|_U \end{tikzcd}$$

□

The inverse of a triple (E_+, E_-, φ_E) is given by $(E_-, E_+, -\varphi_E^{-1})$, where $\varphi_E^{-1} \in C_b(M, \mathrm{end}(E))$ is any section of $C_b(M, \mathrm{end}(E))$ that coincides with the inverse of φ_E on the complement U of a compact subset, on which the latter is invertible. To see this, note that there is a homotopy:

$$h(t) = \begin{pmatrix} 1 & -t\varphi_E^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\varphi_E & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\varphi_E^{-1} \\ 0 & 1 \end{pmatrix}$$

$$h(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad h(1) = \begin{pmatrix} 0 & -\varphi_E^{-1} \\ \varphi_E & 0 \end{pmatrix}$$

connecting $\varphi_E \oplus (-\varphi_E^{-1})$ to the identity in $C_b(U, \mathrm{iso}(E))$.

Lemma 3.3.4. *Every class in $K^0_{\mathcal{A}}(M)$ is represented by a triple $(G, \underline{A^N}, \psi)$, where $\underline{A^N}$ denotes the trivial twisted bundle.*

Proof. Let (E_+, E_-, φ_E) be a triple representing an element in $K^0_{\mathcal{A}}(M)$. Let $K \subset M$ be a compact set such that φ_E is invertible on $M \backslash K$. Choose a precompact open neighborhood $W \supset K$ and a twisted Hilbert-A-module bundle F such that $E_- \oplus F|_W \simeq \underline{A^N}|_W$. Denote the isomorphism by α. This yields:

$$\alpha \circ (\varphi_E \oplus \mathrm{id}) : E_+ \oplus F|_W \longrightarrow \underline{A^N}|_W$$

We now have $E_+ \oplus F$ over W and $\underline{A^N}$ over $M \backslash K$. The previous map yields an isomorphism in $C_b(W \backslash K, \mathrm{hom}(E_+ \oplus F, \underline{A^N}))$. By the clutching construction there exists a twisted Hilbert-A-module bundle G over M and a morphism $\psi \in C_b(M, \mathrm{hom}(G, \underline{A^N}))$ restricting to $\alpha \circ (\varphi_E \oplus \mathrm{id})$ over W. Now consider the restriction

$$\psi \oplus \varphi_E^{-1}|_{W \backslash K} : G \oplus E_-|_{W \backslash K} \longrightarrow \underline{A^N} \oplus E_+|_{W \backslash K} \ ,$$

which is isomorphic to

$$\varphi_E \oplus \mathrm{id} \oplus \varphi_E^{-1}|_{W \backslash K} : E_+ \oplus F \oplus E_-|_{W \backslash K} \longrightarrow E_- \oplus F \oplus E_+|_{W \backslash K} \ .$$

The operator homotopy

$$h(t) = \begin{pmatrix} \sin(t) & \cos(t)\varphi_E \\ \cos(t)\varphi_E^{-1} & -\sin(t) \end{pmatrix} : E_+ \oplus E_- \longrightarrow E_+ \oplus E_-$$

connects

$$h(0) = \begin{pmatrix} 0 & \varphi_E \\ \varphi_E^{-1} & 0 \end{pmatrix} \quad \text{to} \quad h\left(\frac{\pi}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The last map extends over all of W. By lemma 3.3.2 there is an element $\widehat{\psi} \in C_b(W, \mathrm{iso}(E_+ \oplus F \oplus E_-))$ restricting to $\varphi_E \oplus \mathrm{id} \oplus \varphi_E^{-1}$ on $W \backslash V$, where $W \supset V \supset K$ is an open neighborhood of K. This means that we can extend $\psi \oplus \varphi_E^{-1}$ to a map $\widetilde{\psi}$, which is an isomorphism:

$$\widetilde{\psi} : G \oplus E_- \xrightarrow{\simeq} \underline{A^N} \oplus E_+ \ .$$

Thus in $K^0_{\mathcal{A}}(M)$:

$$0 = [G \oplus E_-, \underline{A^N} \oplus E_+, \widetilde{\psi}] = [G, \underline{A^N}, \psi] - [E_+, E_-, \varphi_E] \ .$$

□

Definition 3.3.5. We call a $PU(A)$-bundle $\mathcal{A}$ over some locally compact space M *extendable* if there exists a compactification M^c of M together with a $PU(A)$-bundle $\mathcal{A}^c$ over M^c such that $\mathcal{A}^c|_M = \mathcal{A}$.

Theorem 3.3.6. *Let $\mathcal{A}$ be an extendable $PU(A)$-bundle over M, then*

$$K^0_{\mathcal{A}}(M) \simeq K_0(C_0(M,\mathcal{A})) \ .$$

Proof. There is a canonical embedding of $C_0(M,\mathcal{A})^+$ into $C(M^c,\mathcal{A}^c)$ via extending the sectional part by zero and mapping $1 \in C_0(M,\mathcal{A})^+$ to the constant 1-section of $\mathcal{A}^c$ (note that all transition functions of $\mathcal{A}^c$ are inner). Thus, given $t \in M_n(C_0(M,\mathcal{A})^+)$, we get a twisted Hilbert A-module bundle E^c over M^c like in theorem 3.2.8. Let $s \in M_n(\mathbb{C}) \subset M_n(\mathcal{A}^c)_x \ , \ \forall x \in M^c$ be the value t takes along $\partial M^c = M^c \backslash M$ and set $V = sA^n$. The bundle $\underline{V} = \mathcal{P}^c \times V$ with the twisting $[\lambda, \widehat{g}] \otimes v \mapsto \lambda\, \widehat{g}\, v$ is a twisted Hilbert A-module bundle since $s \in M_n(\mathbb{C})$ commutes with every $\widehat{g}\,1 \in M_n(A)$. This also implies that the map

$$\Psi : E^c \longrightarrow \underline{V} \quad , \quad (q,w) \mapsto (q, s\, w) = (q, M_n(q)(s)w) \ .$$

is a twisted bundle morphism $\Psi \in C(M^c, \hom(E^c, \underline{V}))$. Reversing the direction we get:

$$\Phi : \underline{V} \longrightarrow E^c \quad , \quad (p,v) \mapsto (p, (M_n(p) \circ t)(\rho(p))v)$$

with $\Phi \in C(M^c, \hom(\underline{V}, E^c))$. Now for every $p \in \mathcal{P}^c$ mapping to some $w \in \partial M^c$:

$$\Psi \circ \Phi|_p = 1_{\mathrm{End}(V)} \quad \text{and} \quad \Phi \circ \Psi|_p = 1_{\mathrm{End}(E^c_w)} \ .$$

Since $C(M^c, \mathrm{end}(\underline{V}))$ and $C(M^c, \mathrm{end}(E^c))$ are C^*-algebras, in which invertibility is an open condition, there is an open neighborhood $U \subset M^c$ of ∂M^c such that Φ and Ψ are invertible over $\rho^{-1}(U) \subset \mathcal{P}^c$.

An arbitrary element in $K_0(C_0(M,\mathcal{A}))$ is represented by two projections $t, r \in M_n(C_0(M,\mathcal{A})^+)$ such that $t - r \in M_n(C_0(M,\mathcal{A}))$. Using the notation of the previous construction, this condition translates to $s_t = s_r$. Therefore we end up with two twisted Hilbert A-module bundles E^c and F^c together with morphisms Ψ_E and Ψ_F mapping them into the *same* bundle $\underline{V}$, both invertible outside some compact subset $K \subset M \subset M^c$. Choose a function $\theta : M^c \longrightarrow \mathbb{R}$ which is zero inside of K and 1 outside some larger compact set $\widehat{K}$. Now $[t] - [r] \in K_0(C_0(M,\mathcal{A}))$ is mapped to $[E, F, \rho^*\theta \cdot (\Psi_F^{-1} \circ \Psi_E)] \in K^0_{\mathcal{A}}(M)$.

Starting from a triple $[E_+, E_-, \varphi_E] \in K^0_{\mathcal{A}}(M)$, we can assume w.l.o.g. that it is of the form $[F, \underline{A^N}, \varphi_F]$ by the previous lemma. Therefore there is a compact set $K \subset M$ such that φ_F trivializes F outside K. Choose a precompact open subset $K \subset W \subset M$ and an inverse bundle H over W. Adding the trivial bundle $\underline{A^N}$, yields the isomorphism $F \oplus H \oplus \underline{A^N}|_W \simeq \underline{A^k} \oplus \underline{A^N}|_W$, whereas over $M \backslash K$ we have $F \oplus \underline{A^k}|_{M\backslash K} \simeq \underline{A^N} \oplus \underline{A^k}|_{M\backslash K}$. To get the isomorphism $H \oplus \underline{A^N}|_{W\backslash K} \longrightarrow \underline{A^k}|_{W\backslash K}$ needed to apply the clutching construction just identify $\underline{A^N}$ with F via φ_F^{-1} and apply the trivialization κ of $F \oplus H$ over W. Let G be the resulting twisted Hilbert A-module bundle. Note that $\varphi_F \oplus \mathrm{id} \oplus \varphi_F^{-1} \in C_b(W\backslash K, \mathrm{iso}(F \oplus H \oplus \underline{A^N}))$ extends to an isomorphism τ over all of W, since it is operator homotopic to the identity, which clearly extends. Now the trivialization map $F \oplus G \longrightarrow \underline{A^N} \oplus \underline{A^k}$ is given by $(\kappa \oplus \mathrm{id}_{\underline{A^N}}) \circ \tau$ over W and by $\varphi_F \oplus \mathrm{id}_{\underline{A^k}}$ over the complement of K. By construction the clutching map transforms these two into one another over $W \backslash K$. The projections p_F and $p_{\underline{A^N}}$ in $C(M, \mathrm{end}(\underline{A^N} \oplus \underline{A^k}))$ agree outside K. Thus, we map $(F, \underline{A^N}, \varphi_F)$ to $[p_F] - [p_{\underline{A^N}}]$.

It follows from the standard arguments of non-twisted K-theory that these two maps are inverse to each other. $\square$

Likewise we can treat relative twisted K-theory:

Definition 3.3.7. Let M be compact Hausdorff space, $N \subset M$ be a closed subspace, then the relative twisted K-theory $K^0_{\mathcal{A}}(M, N)$ with local coefficients in the unital C^*-algebra A is given by triples (E_+, E_-, φ_E), where E_+ and E_- are twisted Hilbert-A-module bundles and $\varphi_E \in C(M, \mathrm{hom}(E_+, E_-))$ is a twisted bundle morphism that is an isomorphism over N, subject to the equivalence relation generated by the following: (E_+, E_-, φ_E) is equivalent to (F_+, F_-, φ_F) if there exist isomorphisms $\psi_\pm \in C(M, \mathrm{iso}(E_\pm, F_\pm))$ such that the following diagram commutes:

$$\begin{array}{ccc} E_+|_N & \xrightarrow{\varphi_E} & E_-|_N \\ \downarrow{\scriptstyle \psi_+} & & \downarrow{\scriptstyle \psi_-} \\ F_+|_N & \xrightarrow{\varphi_F} & F_-|_N \end{array}$$

Furthermore, we demand $(E_+, E_-, \varphi_E) \sim (E_+ \oplus G, E_- \oplus G, \varphi_E \oplus \mathrm{id})$ for another twisted Hilbert A-module bundle G.

3.4 Stable isomorphism and Morita equivalence

As was already pointed out in the first chapter, stable isomorphism of bundle gerbes is tightly connected to MORITA equivalence of central S^1-extensions of the pair groupoid, as shown in diagram (2.1). In the first part of this section, we will exploit this fact a little further to explain, why the twisted K-theory defined above can be understood as usual twisted K-theory with coefficients in some C^*-algebra A. In fact, this will lead to the notion of twisted Hilbert A-module bundle over some principal Γ-bundle P, where Γ in our cases will be either be a compact Lie group or some discrete group instead of $PU(A)$. These structures will open the gate to the differential geometry of twisted bundles, which would be much more complicated, if we had to deal with Banach Lie groups.

In the spirit of this goal, the next theorem shows that the (torsion-)twist of a $PU(A)$-bundle can be shifted to a bundle of matrix algebras $\mathcal{K}$ without changing the K-theory involved. Thus $K_0(C(M,\mathcal{A}))$ can be understood as a blend between K-theory with coefficients in A, which would be $K_0(C(M)\otimes A)$, and twisted K-theory $K_0(C(M,\mathcal{K}))$:

Theorem 3.4.1. *Let $\mathcal{A}$ be a $PU(A)$-bundle with $dd(\mathcal{A})$ torsion. Let $\mathcal{K}$ be an $PU(n)$-bundle with $dd(\mathcal{K}) = dd(\mathcal{A})$, then*

$$C(M,\mathcal{K}\otimes A) \quad \simeq_{Mor} \quad C(M,\mathcal{A}) \ .$$

Proof. Let $L_A \to \mathcal{P}_A^{[2]}$ be the lifting bundle gerbe of $\mathcal{A}$, $L_{M_n} \to \mathcal{P}_{M_n}^{[2]}$ that of $\mathcal{K}$. Since

$$dd(\pi_{M_n}^* L_{M_n}^* \otimes \pi_A^* L_A) = -dd(L_{M_n}) + dd(L_A) = 0$$

the bundle gerbe $\pi_{M_n}^* L_{M_n}^* \otimes \pi_A^* L_A \to (\mathcal{P}_{M_n} \times_M \mathcal{P}_A)^{[2]}$ is trivial (here, π_A denotes the canonical projection $(\mathcal{P}_{M_n} \times_M \mathcal{P}_A)^{[2]} \to \mathcal{P}_A^{[2]}$ and π_{M_n} likewise). The latter is a lifting bundle gerbe associated to the exact sequence

$$1 \longrightarrow U(1) \longrightarrow U(n)\bar{\otimes}U(A) \longrightarrow PU(n) \times PU(A) \longrightarrow 1 \ ,$$

where $U(n)\bar{\otimes}U(A) = U(n)\times_{U(1)}U(A)$ is the quotient of the product with respect to the equivalence relation:

$$(\hat{g}_1, \hat{g}_2) \quad \simeq \quad (\hat{g}_1\, e^{i\varphi},\, e^{i\varphi}\hat{g}_2)$$

This implies that the $PU(n)\times PU(A)$ principal-bundle $\mathcal{P} := (\mathcal{P}_{M_n} \times_M \mathcal{P}_A)$ lifts to a $U(n)\bar{\otimes}U(A)$ principal bundle $\widehat{\mathcal{P}}$.

Now consider the action of $U(n)\bar{\otimes}U(A)$ on $\mathbb{C}^n \otimes A$ given by

$$\tau([U,u])(v\otimes a) = Uv \otimes au^*$$

for $U \in U(n), u \in U(A)$ and $v\otimes a \in \mathbb{C}^n\otimes A$ and the bundle of Hilbert $M_n(\mathbb{C})\otimes A$-A-bimodules $V = \widehat{\mathcal{P}} \times_\tau (\mathbb{C}^n \otimes A)$ (the inner product given fiberwise).

Taking continuous functions yields $C(M,V)$, which carries a $C(M,\mathcal{K}\otimes A)$-$C(M,\mathcal{A})$-bimodule structure. Note that the actions induced by

$$(T\otimes t)\cdot(v\otimes a) \quad := \quad (Tv)\otimes(ta) \ , \tag{3.5}$$

$$(v\otimes a)\cdot b \quad := \quad v\otimes(ab) \tag{3.6}$$

are well defined due to the following commutative diagrams

$$\begin{tikzcd}
(M_n(\mathbb{C})\otimes A)\times(\mathbb{C}^n\otimes A) \arrow[r] \arrow[d, "\mathrm{Ad}_U\otimes\mathrm{id}\times\tau_{[U,u]}"] & \mathbb{C}^n\otimes A \arrow[d, "\tau_{[U,u]}"] \\
(M_n(\mathbb{C})\otimes A)\times(\mathbb{C}^n\otimes A) \arrow[r] & \mathbb{C}^n\otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathbb{C}^n\otimes A)\times A \arrow[r] \arrow[d, "\tau_{[U,u]}\times\mathrm{Ad}_u"] & \mathbb{C}^n\otimes A \arrow[d, "\tau_{[U,u]}"] \\
(\mathbb{C}^n\otimes A)\times A \arrow[r] & \mathbb{C}^n\otimes A
\end{tikzcd}$$

The horizontal maps are given by the above actions (3.5) and (3.6) respectively.

V carries two fiberwise scalar products:

$$\begin{aligned}
\langle v\otimes a, w\otimes b\rangle_A &:= \langle v,w\rangle_{\mathbb{C}^n}\, a^*b\ , \\
\langle v\otimes a, w\otimes b\rangle_{M_n(A)} &:= vw^*\, ab^*
\end{aligned}$$

that turn $C(M,V)$ into a $C(M,\mathcal{K}\otimes A)$-$C(M,\mathcal{A})$-Hilbert-bimodule. Indeed:

$$\begin{aligned}
\langle Uv\otimes au^*, Uw\otimes bu^*\rangle_A &= \langle Uv,Uw\rangle\ ua^*bu^* = u\ \langle v,w\rangle\, a^*b\,u^* \\
\langle Uv\otimes au^*, Uw\otimes bu^*\rangle_{M_n(A)} &= U(vw^*)U^*\ au^*ub^* = U\,vw^*\,ab^*\,U^*\ ,
\end{aligned}$$

which implies that the two scalar products on $C(M,V)$ take their values in $C(M,\mathcal{A})$ and $C(M,\mathcal{K}\otimes A)$. Completeness is easily checked by a simple norm estimate.

Note that, since $\mathbb{C}^n\otimes A$ is the bimodule representing the Morita equivalence of $M_n(A)$ and A, we have:

$$\begin{aligned}
\langle v\otimes a, w\otimes b\rangle_{M_n(A)}\ x\otimes c &= (vw^*\,ab^*)(x\otimes c) = (vw^*x\otimes ab^*c) \\
&= v\otimes a\ \langle w\otimes b, x\otimes c\rangle_A\ ,
\end{aligned}$$

which transfers to sections in $C(M,V)$. Similarly, the fullness-conditions:

$$\begin{aligned}
\overline{\mathrm{span}\left\{\langle C(M,V),C(M,V)\rangle_{C(M,\mathcal{K}\otimes A)}\right\}} &= C(M,\mathcal{K}\otimes A) \\
\overline{\mathrm{span}\left\{\langle C(M,V),C(M,V)\rangle_{C(M,\mathcal{A})}\right\}} &= C(M,\mathcal{A})
\end{aligned}$$

are direct consequences of their local analogues. □

Theorem 3.4.2. *Let $\mathcal{A}$ be a $PU(A)$-bundle, $\mathcal{B}$ be a $PU(B)$-bundle and denote by $\mathcal{P}_A$, $\mathcal{P}_B$ the associated principal bundles. If $\mathcal{A}$ and $\mathcal{B}$ have* Morita*-equivalent fibers and the* Dixmier-Douady*-classes of the lifting bundle gerbes $L_A\to\mathcal{P}_A^{[2]}$ and $L_B\to\mathcal{P}_B^{[2]}$ agree, then the C^*-algebras $C(M,\mathcal{A})$ and $C(M,\mathcal{B})$ are* Morita*-equivalent as well.*

Proof. Since $C(M,\mathcal{K})$ is nuclear, we deduce that $C(M,\mathcal{K}\otimes A)\simeq C(M,\mathcal{K})\otimes A$. Therefore

$$C(M,\mathcal{A})\ \simeq_{\mathrm{Mor}}\ C(M,\mathcal{K})\otimes A\ \simeq_{\mathrm{Mor}}\ C(M,\mathcal{K})\otimes B\ \simeq_{\mathrm{Mor}}\ C(M,\mathcal{B})\ .$$

□

Since the algebras $C(M,\mathcal{K}\otimes A)$ and $C(M,\mathcal{A})$ are both unital, the bimodule $C(M,V)$ constructed in theorem 3.4.1, which moderates the Morita equivalence between them, has to be finitely generated and projective as a Hilbert

$C(M, \mathcal{A})$-module. In fact, unitality implies that the identity on $C(M, V)$ is a compact adjointable $C(M, \mathcal{A})$-linear operator, from which the last statement follows. Therefore the isomorphism of the corresponding K-theory groups, which abstractly exists due to the MORITA equivalence, takes the following very concrete form (see also [21] for far more general results)

$$K_0(C(M, \mathcal{K} \otimes A)) \xrightarrow{\sim} K_0(C(M, \mathcal{A})) \quad ; \quad [W] \mapsto [W \otimes_{C(M, \mathcal{K} \otimes A)} C(M, V)] \ .$$

The way back from twisted K^0-groups into their non-twisted analogues, i.e. the transfer of twisted Hilbert A-module bundles over $\mathcal{P}$ to non-twisted bundles over M in case the twist is trivial (and a trivialization is chosen), needs the following lemma describing the descent of equivariant bundles.

Lemma 3.4.3. *Let $E \to P$ be a continuous Hilbert A-module bundle over some principal Γ-bundle P for a (Banach) Lie group Γ carrying a continuous left action of Γ covering the one on P, i.e. fitting into the following diagram:*

$$\begin{tikzcd} E \arrow[r, "(g\cdot)"] \arrow[d, "\rho"] & E \arrow[d, "\rho"] \\ P \arrow[r, "(\cdot g^{-1})"] & P \end{tikzcd}$$

Then E is isomorphic to the pullback of a continuous Hilbert A-module bundle $F \to M$. In case E and P are smooth bundles and the action on E is smooth, then F can also be chosen to be a smooth bundle.

Proof. Choose a good open cover U_i of M with P trivial over each U_i and trivializations $\psi_i : U_i \times \Gamma \to P|_{U_i}$. Set $\alpha_i = \mathrm{pr}_\Gamma \circ \psi_i^{-1} : P|_{U_i} \to \Gamma$ and $W_i = \pi^{-1}(U_i)$, where $\pi : P \to M$. Choose a trivialization

$$\phi_i : E|_{\psi_i(U_i \times \{1\})} \longrightarrow U_i \times V \ ,$$

where V is the typical fiber of E, and extend it to all of $E|_{W_i}$ in the following way:

$$E|_{W_i} \longrightarrow W_i \times V \quad ; \quad w \mapsto (p, \phi_i(\alpha_i(p) \cdot w))$$

with $w \in E_p$. The transition functions derived from this are:

$$\beta_{ij} : W_{ij} \longrightarrow \mathrm{End}(V) \quad ; \quad p \mapsto \phi_i \circ (\alpha_i(p) \cdot \alpha_j(p)^{-1} \cdot \phi_j^{-1}) \ .$$

Since $\alpha_i(pg) = \alpha_i(p)g$, the maps β_{ij} are invariant under the action of Γ on W_{ij} and therefore yield transition data on the quotient, which is just U_{ij}. Applying the clutching construction, yields a bundle F over M together with an isomorphism $E \simeq \pi^* F$. If the data we started with was smooth, then β_{ij} will be smooth maps, therefore F will be smooth as well. □

The intermediate principal $PU(A)$-bundle in the construction of twisted bundles is of course quite an obstacle to geometric applications, where often compactness or at least local compactness is needed. Luckily, in many situations the machinery of stable isomorphism can be used to reduce to twisted Hilbert A-bundles over a much more tractable principal bundle.

Consider a principal $PU(A)$-bundle $\mathcal{P}$, a group Γ (which we assume to be either a Lie group or a finitely generated discrete group – at least in any way nicer than $PU(A)$), a central extension $\widehat{\Gamma}$ of Γ by S^1, such that there is a morphism of extensions:

$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r] & U(A) \arrow[r] & PU(A) \arrow[r] & 1 \\ 1 \arrow[r] & S^1 \arrow[r] \arrow[u, equal] & \widehat{\Gamma} \arrow[r, "q"] \arrow[u, "\widehat{\delta}"] & \Gamma \arrow[r] \arrow[u, "\delta"'] & 1 \end{tikzcd}$$

Assume that $\mathcal{P}$ has a reduction to a principal Γ-bundle P, i.e. $\mathcal{P} = P \times_\delta PU(A)$. Due to the above diagram, P has the same DIXMIER-DOUADY-class as $\mathcal{P}$. Therefore the corresponding lifting bundle gerbes are stably isomorphic and the following definition comes up naturally.

Definition 3.4.4. A *twisted Hilbert A-module bundle E* over the principal Γ-bundle P is a locally trivial (right) Hilbert A-module bundle over P together with an isometric (left) action of the lifting bundle gerbe $L \longrightarrow P^{[2]}$, i.e.

- a fiberwise A-linear isometric isomorphism

$$\gamma \,:\, L \otimes \pi_2^* E \xrightarrow{\simeq} \pi_1^* E \ ,$$

which is associative with respect to the bundle gerbe product on L.

Using the same argument as in [12], we deduce

Theorem 3.4.5. *The monoid of twisted Hilbert A-module bundles over $\mathcal{P}$ with respect to direct sum is isomorphic to the corresponding monoid over P.*

In the case of lifting bundle gerbes there are two canonical maps back and forth. Let L be the lifting bundle gerbe of P, $\mathcal{L}$ be that of $\mathcal{P}$. The tensor product $L^* \otimes \mathcal{L}$ is the lifting bundle gerbe corresponding to the group extension

$$1 \longrightarrow S^1 \longrightarrow \widehat{\Gamma} \,\bar{\otimes}\, U(A) \longrightarrow \Gamma \times PU(A) \longrightarrow 1 \ .$$

where $\widehat{\Gamma} \,\bar{\otimes}\, U(A)$ denotes the product of the two groups modulo the *diagonal* S^1-action. There is a homomorphism

$$\theta \,:\, \Gamma \longrightarrow \widehat{\Gamma} \,\bar{\otimes}\, U(A) \quad , \quad g \mapsto \widehat{g} \,\bar{\otimes}\, \widehat{\delta}(\widehat{g})$$

and $P \times_M \mathcal{P}$ lifts to the S^1-principal bundle $P \times_\theta \widehat{\Gamma} \,\bar{\otimes}\, U(A) \simeq P \times U(A) = \widehat{Q}$ where the action of $\widehat{\Gamma} \,\bar{\otimes}\, U(A)$ on $\widehat{Q}$ is given by

$$\left(p, \widehat{f}\right) \cdot \widehat{g} \,\bar{\otimes}\, \widehat{h} = \left(pg, \delta(\widehat{g}^{-1})\, \widehat{f}\, \widehat{h}\right) \ .$$

The associated line bundle Q is a trivialization for the bundle gerbe $L^* \otimes \mathcal{L}$. Given a twisted Hilbert A-module bundle $\mathcal{E}$ over $\mathcal{P}$, we can form $Q \otimes \pi_{\mathcal{P}}^* \mathcal{E}$. This descends to a twisted bundle over P by the following map (which is written down fiberwise to save some notation):

$$Q_{(p,q_2)} \otimes \mathcal{E}_{q_2} \longrightarrow Q_{(p,q_1)} \otimes \mathcal{L}_{(q_1,q_2)} \otimes \mathcal{E}_{q_2} \longrightarrow Q_{(p,q_1)} \otimes \mathcal{E}_{q_1} \ . \tag{3.7}$$

Given a twisted Hilbert A-module bundle E over P, we form $Q^* \otimes \pi_P^* E$ and get a descent to $\mathcal{P}$ via

$$Q^*_{(p_2,q)} \otimes E_{p_2} \longrightarrow Q^*_{(p_1,q)} \otimes L_{(p_1,p_2)} \otimes E_{p_2} \longrightarrow Q^*_{(p_1,q)} \otimes E_{p_1} \ . \tag{3.8}$$

In the above situation two other constructions come up very naturally: Given a twisted Hilbert A-module bundle $\mathcal{E}$ over $\mathcal{P}$, we can use the canonical map $P \longrightarrow P \times_\delta PU(A) = \mathcal{P}$ to pull $\mathcal{E}$ back to P. We will call this the *restriction* to P even though the map may not be injective.

On the other hand we can form $E \otimes U(A)^*$ over $P \times PU(A)$, since $U(A)$ is a line bundle over its projectivization. There is an action of $\widehat{\Gamma}$ on this bundle covering $(p,a) \mapsto (pg^{-1}, \delta(g)\,a)$ given by: $v \otimes \widehat{a} \mapsto \gamma([\lambda, \widehat{g}] \otimes v) \otimes \widehat{\delta}(\widehat{g})\,\widehat{a}$. The quotient is a twisted Hilbert A-module bundle over $\mathcal{P}$ that will be called the *extension* of E.

Theorem 3.4.6. *The result using the descent in (3.7) is isomorphic to the restriction of $\mathcal{E}$ to P, whereas (3.8) yields a twisted bundle isomorphic to the extension of E to $\mathcal{P}$.*

Proof. Let δ be the action of $\mathcal{L}$ on $\mathcal{E}$ and note that Q is the pullback of $\mathcal{L}$ via $P \times_M \mathcal{P} \longrightarrow \mathcal{P}^{[2]}$; $(p,q) \mapsto ([p,1],q)$. Therefore δ may be considered as a map $Q_{(p,q)} \otimes \mathcal{E}_q \longrightarrow \mathcal{E}_{[p,1]}$, which factors over the descent and is the identity on $\mathcal{E}_{[p,1]} = Q_{(p,[p,1])} \otimes \mathcal{E}_{[p,1]}$. Thus it yields an isomorphism.

For the other half of the theorem the pullback of $Q^* \otimes \pi_P^* E$ to $P \times PU(A)$ via the map $P \times PU(A) \longrightarrow P \times_M \mathcal{P}$; $(p,a) \mapsto (p,[p,a])$ reveals itself to be just the first step of the extension. The descent map turns into the group action described above, so the quotients are isomorphic. $\square$

Remark 3.4.7. Generally, a trivialization of a tensor product $L_1 \boxtimes L_2^*$ of two lifting bundle gerbes can be thought of as a morphism between the corresponding twists in the sense that we change from twisted bundles over one principal bundle to twisted bundles over the other. The trivialization intertwines the two twisted actions. From this point of view the Morita equivalence described in the twist shifting theorem 3.4.1 identifies the monoid of twisted Hilbert-A-module bundles over P with its counterpart of twisted Hilbert $M_n(\mathbb{C}) \otimes A$-module bundles over a $PU(n)$-bundle $\widetilde{P}$ with $dd(P) = dd(\widetilde{P})$ via a trivialization.

The effect of the equivalence on the level of twisted bundles is therefore just shifting a twisted Hilbert A-module bundle E over P via the above procedure to one over $\widetilde{P}$ using a trivialization and tensoring the result with the imprimitivity $\mathbb{C}$-$M_n(\mathbb{C})$-bimodule $\mathbb{C}^*$ to end up with a twisted Hilbert $M_n(\mathbb{C}) \otimes A$-module bundle representing an element in $K^0_{\mathcal{K}\otimes A}(M)$. To be precise, the lift of $P \times_M \widetilde{P}$ to a $\widehat{\Gamma}\bar{\otimes}U(n)$-bundle $\widehat{P}$ corresponds to a line bundle $Q = \widehat{P} \times_{S^1} \mathbb{C}$ over $P \times_M \widetilde{P}$. Let $\pi_P \colon P \times_M \widetilde{P} \to P$ be the projection, then $E \to P$ is mapped to $\pi_P^* E \otimes Q$, which descends to $\widetilde{P}$ (see also part ii) of theorem 4.1.31). This *descent* of E to $\widetilde{P}$ will sometimes be denoted by $\pi_!(\pi_P^* E \otimes Q)$.

3.4.1 The frame bundle gerbe

Let V be a (right) Hilbert A-module and denote by $U(V)$ the unitary operators in the C*-algebra $\mathrm{End}(V)$. For a central S^1-extension $\widehat{\Gamma} \longrightarrow \Gamma$ we can consider representations $\rho : \widehat{\Gamma} \longrightarrow U(V)$ that satisfy $\rho(e^{i\varphi}\,\widehat{g}) = e^{i\varphi}\,\rho(\widehat{g})$. This always yields a twisted Hilbert A-module bundle over a principal Γ-bundle P via

$$\begin{aligned} E &= P \times V \\ \gamma &: L \otimes \pi_2^* E \longrightarrow \pi_1^* E \\ & \quad [\lambda, \widehat{g}] \otimes v \mapsto \lambda\, \rho(\widehat{g})\, v \ . \end{aligned}$$

For a non-twisted bundle B of Hilbert A-modules with typical fiber W one could consider the frame bundle P_B consisting of all unitary A-linear maps $f : W \longrightarrow (P_B)_b$. These form a principal $U(W)$-bundle and B is associated to it via the standard action of $U(W)$ on W.

The corresponding question concerning a twisted Hilbert A-module bundle E would be: Can we vary L in its stable isomorphism class such that E takes the form described above for some representation $\rho : \widehat{\Gamma} \longrightarrow U(V)$. This is indeed the case!

Lemma 3.4.8. *Given a twisted Hilbert A-module bundle E over P, let P_E be its frame bundle. Its projectivization $\widehat{H} = P_E/S^1 \longrightarrow P$ descends to a principal $PU(V)$-bundle $H \longrightarrow M$. The associated lifting bundle gerbe L_H has the same* Dixmier-Douady-*class as L.*

Proof. The action of L on E induces a corresponding map of frame bundles, which, after quotiening out the S^1-action boils down to

$$\overline{\gamma} \ : \ \pi_2^* \widehat{H} = \pi_2^* \left(P_E/S^1\right) \longrightarrow \pi_1^* \left(P_E/S^1\right) = \pi_1^* \widehat{H}$$

providing the descent data for $\widehat{H}$. Therefore $\widehat{H}$ can be identified with $\pi_P^* H = H \times_M P$. Since $\widehat{H}$ has a lift to a $U(V)$-bundle P_E over P, its lifting bundle gerbe $S \longrightarrow \widehat{H}^{[2]}$ is trivial, i.e.:

$$(P_E)_{(q_1,p)} \otimes S_{((q_1,p),(q_2,p))} \simeq (P_E)_{(q_2,p)} \quad \text{for } (q_i,p) \in H \times_M P \ .$$

Here and in the next theorem we identify P_E with the associated line bundle over H. The action of L on P_E takes the form:

$$L_{(p_1,p_2)} \otimes (P_E)_{(q,p_2)} \simeq (P_E)_{(q,p_1)}$$

Let $Q \longrightarrow H^{[2]}$ be the lifting bundle gerbe of H and consider the product $Q^* \boxtimes L \longrightarrow (H \times_M P)^{[2]}$. Note that P_E, when viewed as a line bundle over $H \times_M P$, yields a trivialization of $Q^* \boxtimes L$. Indeed, the pullback of Q to $(H \times_M P)^{[2]}$ coincides with S, therefore

$$\begin{aligned} (Q^* \boxtimes L)_{((q_1,p_1),(q_2,p_2))} \otimes (P_E)_{(q_2,p_2)} &\simeq S^*_{(q_1,p_1),(q_2,p_1)} \otimes L_{(p_1,p_2)} \otimes (P_E)_{(q_2,p_2)} \\ &\simeq S^*_{(q_1,p_1),(q_2,p_1)} \otimes (P_E)_{(q_2,p_1)} \\ &\simeq (P_E)_{(q_1,p_1)} \ . \end{aligned}$$

But this implies $dd(Q^*) + dd(L) = dd(L) - dd(Q) = 0$. □

By the lemma above the pushed down projectivized frame bundle of E yields a $PU(V)$-bundle that is stably isomorphic to L. An explicit isomorphism is given by the frame bundle P_E of E.

Theorem 3.4.9. *The bijection on twisted Hilbert-A-module bundles induced by the frame bundle trivialization P_E maps E to the twisted Hilbert A-module bundle*

$$H \times V \longrightarrow H \ .$$

The action of Q is given by the canonical left action of $U(V)$ on V.

Proof. The descent of $(P_E)^* \otimes \pi_P^* E$ is given by

$$(P_E)^*_{(q,p_2)} \otimes E_{p_2} \longrightarrow (P_E)^*_{(q,p_1)} \otimes L_{(p_1,p_2)} \otimes E_{p_2} \longrightarrow (P_E)^*_{(q,p_1)} \otimes E_{p_1} \ .$$

But $(P_E)^*_{(q,p)} \otimes E_p$ is canonically isomorphic to V via

$$\begin{array}{lcl} (P_E)^*_{(q,p)} \otimes E_p \longrightarrow V & ; & [\widehat{q}, \lambda] \otimes v \mapsto \lambda\, \widehat{q}^{-1}(v) \ , \\ V \longrightarrow (P_E)^*_{(q,p)} \otimes E_p & ; & w \mapsto [\widehat{q}, 1] \otimes \widehat{q}(w) \ , \end{array}$$

where $\widehat{q}$ is an arbitrarily chosen lift of q. Since the action of L on $(P_E)^*$ as well as on E are both given by the same map γ, this isomorphism factors over the descent. □

Definition 3.4.10. The bundle gerbe corresponding to the projectivized and pushed-down frame bundle P_E of a twisted Hilbert-A-module bundle E will be called *frame bundle gerbe*.

3.4.2 Bundle gerbe modules and twisted Hilbert $M_n(\mathbb{C})$-bundles

A $PU(n)$-bundle P defines a (torsion) twist in $H^3(M, \mathbb{Z})$ and we have a matrix bundle (often called Azumaya bundle) $\mathcal{K}$ over M associated to it. Following the above, we arrive at two descriptions of the group $K^0_{\mathcal{K}}(M)$: One by bundle gerbe modules for the lifting bundle gerbe $L \to P^{[2]}$ from [12], the other by twisted Hilbert $M_n(\mathbb{C})$-bundles over P. The former description uses vector bundles over P where in the latter we find bundles of Hilbert $M_n(\mathbb{C})$-modules. Luckily, there is a Morita equivalence between $\mathbb{C}$ and $M_n(\mathbb{C})$ that allows us to switch between the two pictures by the following transformations:

$$\begin{array}{lcl} K^{0,\mathrm{bgm}}_{\mathcal{K}}(M) \longrightarrow K^{0,\mathrm{tw}}_{\mathcal{K}}(M) & ; & [E] \mapsto [E \otimes_{\mathbb{C}} \underline{\mathbb{C}^{n*}}] \\ K^{0,\mathrm{tw}}_{\mathcal{K}}(M) \longrightarrow K^{0,\mathrm{bgm}}_{\mathcal{K}}(M) & ; & [F] \mapsto [F \otimes_{M_n(\mathbb{C})} \underline{\mathbb{C}^n}] \ , \end{array}$$

where the twisting in both cases operates only on the first factor (for the second map this is well-defined because of $M_n(\mathbb{C})$-linearity). The right action of $M_n(\mathbb{C})$ on $\mathbb{C}^{n*}$ is via pullback, let f_i be the basis dual to the canonical one (e_i), then the scalar product of two base covectors is $\langle f_i | f_j \rangle = e_i\, f_j \in M_n(\mathbb{C})$. We can without loss of generality assume that the complex vector bundle E is equipped with a hermitian metric, such that the first map yields indeed a twisted Hilbert $M_n(\mathbb{C})$-bundle. Since $\mathbb{C}^{n*} \otimes_{M_n(\mathbb{C})} \mathbb{C}^n \simeq \mathbb{C}$ as vector spaces and $\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^{n*} \simeq M_n(\mathbb{C})$ as Hilbert $M_n(\mathbb{C})$-modules we even get a stronger result: The categories of twisted Hilbert $M_n(\mathbb{C})$-bundles and bundle gerbe modules are naturally equivalent. Most of the time we will think of $K^0_{\mathcal{K}}(M)$ as the group of virtual bundle gerbe modules, even though this forces us to switch the viewpoint (slightly), when comparing this group with $K^0_{\mathcal{A}}(M)$ like in the product below.

3.4.3 Twisted product

We also have an exterior product on twisted Hilbert A-module bundles, but contrary to ordinary K-theory this combines different groups with each other because the twists that are involved add up during the process. We will formulate this twisted product just for the case that we need involving a twisted Hilbert A- and a twisted Hilbert $M_n(\mathbb{C})$-module bundle. Take a $PU(A)$-bundle $\mathcal{A}$ associated to a principal Γ-bundle P and a $PU(n)$-bundle $\mathcal{K}$ associated to $\widetilde{P}$. Then their fiberwise tensor product $\mathcal{A} \otimes \mathcal{K}$ is a bundle of C^*-algebras with fiber $M_n(A) = M_n(\mathbb{C}) \otimes A$ associated to $P \times_M \widetilde{P}$.

Let E be a twisted Hilbert A-module bundle over P and F be a bundle gerbe module over $\widetilde{P}$. Their exterior tensor product $E \boxtimes (F \otimes \underline{\mathbb{C}^*}) \to P \times_M \widetilde{P}$ defines a twisted Hilbert $A \otimes M_n(\mathbb{C})$-module bundle over $P \times_M \widetilde{P}$ simply by tensoring their twistings $\gamma_E \boxtimes (\gamma_F \otimes \mathrm{id})$. This map extends to formal differences in the obvious way and therefore yields

$$K^0_{\mathcal{A}}(M) \otimes K^0_{\mathcal{K}}(M) \longrightarrow K^0_{\mathcal{A}\otimes\mathcal{K}}(M)$$

for a compact manifold M. For non-compact M take two triples $[E_+, E_-, \varphi_E]$ and $[F_+, F_-, \varphi_F]$ and choose a precompact subset W, such that φ_E^{-1} and φ_F^{-1} exist on the complement of W. Choose a continuous function $\rho : M \to [0, 1]$ vanishing on the closure of W and being 1 on the complement of a larger compact subset. Then the tensor product $[E] \boxtimes [F]$ is defined to be the triple

$$[(E_+ \boxtimes F_+) \oplus (E_- \boxtimes F_-), (E_- \boxtimes F_+) \oplus (E_+ \boxtimes F_-), \varphi_{E\otimes F}] \ ,$$

where $\varphi_{E\otimes F}$ is the twisted bundle morphism given by the matrix:

$$\varphi_{E\otimes F} = \begin{pmatrix} \varphi_E \otimes \mathrm{id}_{F_+} & -\mathrm{id}_{E_-} \otimes \widetilde{\varphi}_F^{-1} \\ \mathrm{id}_{E_+} \otimes \varphi_F & \widetilde{\varphi}_E^{-1} \otimes \mathrm{id}_{F_-} \end{pmatrix}$$

with $\widetilde{\varphi}_E^{-1} = \rho\, \varphi_E^{-1}$ and similarly for $\widetilde{\varphi}_F^{-1}$. As we have seen above, the K-theory class does not depend on the choice of regularization ρ and therefore yields a well-defined product.

3.4.4 Künneth theorem

The above decomposition via Morita equivalence $C(M, \mathcal{A}) \simeq_{\mathrm{Mor}} C(M, \mathcal{K}) \otimes A$ suggests a decomposition on the level of K-theory in the spirit of the Künneth formula known from ordinary cohomology. This, however, does not hold in general, but only for tensor products, in which one of the factors belongs to a nice class of C^*-algebras constructed from certain building blocks and therefore often called the *bootstrap* class N.

Definition 3.4.11. N is the smallest class of separable, nuclear C^*-algebras with the following properties:

(N1) N contains $\mathbb{C}$.

(N2) N is closed under countable inductive limits.

(N3) If $0 \to A \to D \to B \to 0$ is an exact sequence and two of the terms are in N, then so is the third.

(N4) N is closed under KK-equivalence.

We note that N contains $C_0(X)$ for every locally compact space X and also every matrix algebra $M_n(\mathbb{C})$ [10]. Since it is also closed under tensor products, this enables us to use the next lemma on $\mathcal{K}$.

Lemma 3.4.12. *Let $\mathcal{K}$ be a locally trivial bundle of C^*-algebras over a compact space M such that its fibers K belong to the bootstrap class N, then the C^*-algebra of sections $C(M,\mathcal{K})$ also belongs to it. For a locally compact, countably paracompact space M we have $C_0(M,\mathcal{K}) \in N$.*

Proof. We prove this by induction. Choose a good countable trivializing cover (i.e. the sets are contractible and all their higher intersections are contractible) $\bigcup_{i\in I} U_i \supset M$ for $\mathcal{K}$. Let $J \subset I$ be a finite index subset and let $U_J = \bigcap_{j\in J} U_j$. Note that $C_0(U_J,\mathcal{K}) \simeq C_0(U_J,K) \simeq C_0(U_J)\otimes K$ due to the nuclearity of K. Therefore $C_0(U_J,\mathcal{K})$ belongs to the bootstrap class which is closed under tensor products.

Now take an arbitrary family of finite index subsets $J_k \subset I, k \in \mathbb{N}$. Set $A_k = \bigcup_{i=1}^{k} U_{J_i},\ B_k = U_{J_{k+1}}$ and suppose that $C_0(A_k,\mathcal{K})$ as well as $C_0(A_k\cap B_k,\mathcal{K})$ belong to the bootstrap class, which clearly holds for $k = 1$. Consider the following exact sequence of C^*-algebras

$$0 \longrightarrow C_0(A_k\cap B_k,\mathcal{K}) \longrightarrow C_0(B_k,\mathcal{K}) \longrightarrow C_0(B_k\backslash(A_k\cap B_k),\mathcal{K}) \longrightarrow 0\ ,$$

in which the first arrow is continuation by 0 and the last arrow is restriction to the set $B_k\backslash(A_k\cap B_k)$, which is closed in B_k. Keep in mind that C_0 refers to sections vanishing outside some compact subset, which does *not* imply that they also vanish on the boundary of $B_k\backslash(A_k\cap B_k)$. Since the first two algebras are in the bootstrap category by hypotheses, the last one is as well. Now by the exact sequence

$$0 \longrightarrow C_0(A_k,\mathcal{K}) \longrightarrow C_0(A_k\cup B_k,\mathcal{K}) \longrightarrow C_0((A_k\cup B_k)\backslash A_k,\mathcal{K}) \longrightarrow 0$$

$C_0(A_{k+1},\mathcal{K})$ belongs to the bootstrap category as well, since $(A_k\cup B_k)\backslash A_k = B_k\backslash(A_k\cap B_k)$.

To continue we need to show that the algebra over $A_{k+1}\cap B_{k+1}$ is again in the bootstrap category. But this holds, since $A_{k+1}\cap B_{k+1} = \bigcup_{i=1}^{k} U_{J_i\cup J_{k+1}}$ and the union is taken over k sets. For non-compact spaces we get $C_0(M,\mathcal{K})$ as the (countable) inductive limit $\lim_k C_0(\bigcup_{i=1}^{k} U_i,\mathcal{K})$. □

The lemma trims the Künneth theorem [10, 62] to fit our purposes:

Theorem 3.4.13. *Let A be a C^*-algebra and $\Gamma\mathcal{K} = C_0(M,\mathcal{K})$ over M like above. There is a short exact sequence:*

$$0 \to K_*(\Gamma\mathcal{K})\otimes K_*(A) \to K_*(\Gamma\mathcal{K}\otimes A) \to \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\Gamma\mathcal{K}),K_*(A)) \to 0\ ,$$

in which the first map has degree 0 *and the second has degree* 1*. In particular, there is a rational isomorphism:*

$$\Psi\ :\ K^0_{\mathcal{K}}(M)\otimes K_0(A)\otimes\mathbb{Q}\ \oplus\ K_1(\Gamma\mathcal{K})\otimes K_1(A)\otimes\mathbb{Q} \longrightarrow K^0_{\mathcal{A}}(M)\otimes\mathbb{Q}\ . \quad (3.9)$$

When we identify $K_0(\Gamma\mathcal{K} \otimes A) = K^0_{\mathcal{K}\otimes A}(M)$ with $K^0_{\mathcal{A}}(M)$ via MORITA equivalence to exploit the above sequence, it will be more convenient to think of $K^0_{\mathcal{K}}(M)$ as bundle gerbe modules over the principal Γ-bundle P to which $\mathcal{A}$ is associated instead of modules over the $PU(n)$-bundle of $\mathcal{K}$. Keeping this in mind, we get the following explicit description of the above map.

Restricted to the first summand, the isomorphism Ψ is induced by the tensor product of finitely generated projective modules. An element in $K^0_{\mathcal{K}}(M)$ can be represented by a bundle gerbe module E over P, whereas an element in $K_0(A)$ is represented by a finitely generated projective Hilbert A-module V. Now the tensor product on the level of algebras coincides with the tensor product $E \otimes \underline{V}$ over P, with twisting $\gamma \otimes \mathrm{id}_V$ on the level of twisted Hilbert A-module bundles, i.e. $\Psi([E,\gamma] \otimes [V]) = [E \otimes \underline{V}, \gamma \otimes \mathrm{id}_V]$.

The K-groups that appear in the second summand can be written as

$$\begin{aligned} K_1(\Gamma\mathcal{K}) &= K_0(C_0(\mathbb{R}) \otimes C(M,\mathcal{K})) = K_0(C_0(\mathbb{R} \times M, \pi_M^*\mathcal{K}))\ , \\ K_1(A) &= K_0(C_0(\mathbb{R}) \otimes A) = K_0(C_0(\mathbb{R}, A))\ . \end{aligned}$$

The first group describes virtual bundle gerbe modules over $\pi_M^* P \to \mathbb{R} \times M$, elements of the second are represented by virtual Hilbert A-module bundles over $\mathbb{R}$. Their exterior tensor product yields an element in

$$K_0(C_0(\mathbb{R}^2 \times M, \pi_M^*\mathcal{K} \otimes A)) = K_0(C_0(\mathbb{R}^2) \otimes C(M, \mathcal{K} \otimes A)) \simeq K_0(C(M, \mathcal{K} \otimes A))\ ,$$

where the last isomorphism is induced by Bott periodicity. This is the explicit description of Ψ restricted to the second summand. Expressed in terms of C^*-algebra K-theory, the Bott isomorphism is the inverse of the map given by the tensor product with the class $[B] - [\mathrm{pr}_2] \in K_0(C_0(\mathbb{R}))$, where pr_2 is the projection of $\mathbb{C}^2$ to the second summand and

$$B : \mathbb{R}^2 \longrightarrow M_2(\mathbb{C}) \quad ; \quad (x,y) \mapsto \frac{1}{1+|z|^2} \begin{pmatrix} 1 & \overline{z} \\ z & |z|^2 \end{pmatrix} \quad \text{with } z = x + iy$$

is the graph projection of the multiplication by z, i.e. $b(x,y)w = (x+iy)w$. In terms of bundles this corresponds to the triple $[\,\underline{\mathbb{C}}, \underline{\mathbb{C}}, b\,]$. Thus, the Bott map takes the form

$$K^0_{\mathcal{A}}(M) \longrightarrow K^0_{\mathcal{A}}(\mathbb{R}^2 \times M) \quad ; \quad [E] \mapsto [\pi_M^* E, \pi_M^* E, \widetilde{b}]\ , \tag{3.10}$$

where $\widetilde{b} : \pi_M^* E \longrightarrow \pi_M^* E$ is the multiplication by $x + iy$ for $(x,y) \in \mathbb{R}^2$.

The Künneth decomposition is a module map with respect to the twisted product on $K^0_{\mathcal{A}}(M)$ in the sense that the following two diagrams commute:

$$\begin{array}{ccc} K^0_{\mathcal{K}}(M) \otimes K_0(A) \otimes K^0_{\mathcal{K}_2}(M) & \longrightarrow & K^0_{\mathcal{K}\otimes\mathcal{K}_2}(M) \otimes K_0(A) \\ \downarrow & & \downarrow \\ K^0_{\mathcal{A}}(M) \otimes K^0_{\mathcal{K}_2}(M) & \longrightarrow & K^0_{\mathcal{A}\otimes\mathcal{K}_2}(M) \end{array}$$

$$\begin{array}{ccc} K^0_{\pi_M^*\mathcal{K}}(\mathbb{R} \times M) \otimes K^0_A(\mathbb{R}) \otimes K^0_{\mathcal{K}_2}(M) & \longrightarrow & K^0_{\pi_M^*(\mathcal{K}\otimes\mathcal{K}_2)}(\mathbb{R} \times M) \otimes K^0_A(\mathbb{R}) \\ \downarrow & & \downarrow \\ K^0_{\mathcal{A}}(M) \otimes K^0_{\mathcal{K}_2}(M) & \longrightarrow & K^0_{\mathcal{A}\otimes\mathcal{K}_2}(M) \end{array}$$

To see this for the latter diagram, note that the Bott map also commutes with the product. Indeed, it *is* just a pullback composed with the product by an element in $K^0(\mathbb{R}^2 \times M)$. After rationalization we therefore get a module isomorphism.

Chapter 4

Twisted K-homology in the torsion case

4.1 Connections on bundle gerbe modules

4.1.1 Flat extensions of Lie groups

The following section deals with the lifting bundle gerbe of a *flat* central S^1-extension $\widehat{\Gamma}$ of a connected Lie group Γ. To understand this notion, consider the following excerpt of an exact sequence (see e.g. [51]) in group cohomology:

$$\mathrm{Hom}(\pi_1(\Gamma), S^1) \longrightarrow H^2_{\mathrm{gr}}(\Gamma, S^1) \longrightarrow H^2_{\mathrm{alg}}(\mathfrak{g}, i\mathbb{R}) \ ,$$

in which the first map assigns to a homomorphism $\gamma\colon \pi_1(\Gamma) \to S^1$ the extension $\widetilde{\Gamma} \times_\gamma S^1$ where $\widetilde{\Gamma}$ denotes the universal cover. Such extensions can be described by cocycles $\Gamma \times \Gamma \to S^1$, which are continuous in a neighborhood of the identity $(1,1) \in \Gamma \times \Gamma$. There is such a cocycle $\xi\colon \Gamma \times \Gamma \longrightarrow \pi_1(\Gamma)$ describing the central extension $\widetilde{\Gamma} \to \Gamma$. Thus, the above map can be understood as composition:

$$\mathrm{Hom}(\pi_1(\Gamma), S^1) \longrightarrow H^2_{\mathrm{gr}}(\Gamma, S^1) \quad ; \quad \rho \mapsto \rho \circ \xi \ ,$$

which is a group homomorphism.

The second arrow maps an extension $\widehat{\Gamma}$ to the curvature F_ν of a canonical connection ν_σ on the line bundle associated to $\widehat{\Gamma} \to \Gamma$: Indeed, choose a linear section $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$, then $\nu_\sigma \in \Omega^1(\widehat{\Gamma}, i\mathbb{R})$ is given by

$$\nu_\sigma = \widehat{\mu} - \sigma(q^*\mu) \ ,$$

where $\widehat{\mu}$ and μ denote the Maurer-Cartan-forms on $\widehat{\Gamma}$, Γ respectively and $q\colon \widehat{\Gamma} \to \Gamma$ is the projection. The Lie algebra cocycle is now given by:

$$\omega(X,Y) = 2\, d\nu_\sigma(\sigma(X), \sigma(Y)) = \sigma([X,Y]) - [\sigma(X), \sigma(Y)] \quad , \quad X, Y \in \mathfrak{g} \ .$$

Definition 4.1.1. An extension of the above form shall be called *flat* if the associated extension of Lie algebras has a Lie algebra split, or equivalently if the curvature of the associated connection ν_σ vanishes for some split σ, or equivalently if it is associated to the universal cover $\widetilde{\Gamma}$ of Γ.

Example 4.1.2. Our main example will be of fundamental importance for the torsion case of twisted K-homology:

$$1 \longrightarrow S^1 \longrightarrow U(n) \longrightarrow PU(n) \longrightarrow 1 \ .$$

From the short exact sequence:

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow SU(n) \longrightarrow PU(n) \longrightarrow 1$$

it follows that $\pi_1(PU(n)) = \mathbb{Z}/n\mathbb{Z}$. Embedding $\mathbb{Z}/n\mathbb{Z}$ into S^1 via the exponential map yields a homomorphism $\gamma \colon \pi_1(PU(n)) \longrightarrow S^1$ and it is easy to see that $U(n) \simeq SU(n) \times_\gamma S^1$.

Note that the analogous extension in the infinite-dimensional case, i.e. the unitary group of a separable Hilbert space:

$$1 \longrightarrow S^1 \longrightarrow U(H) \longrightarrow PU(H) \longrightarrow 1 \ .$$

can be treated in much the same way as the case above. However, since $\pi_1(PU(H))$ is trivial, it is *not flat* anymore.

Another example sharing properties similar to $U(n) \to PU(n)$ is of course $\mathrm{Spin}^c(n) \to SO(n)$. They both arise from homomorphisms that factor over some finite cyclic abelian group. We generalize this case in our next example.

Example 4.1.3. Suppose that $\widehat{\Gamma}$ is a flat central S^1-extension of Γ, associated to a homomorphism

$$\rho \colon \pi_1(\Gamma) \xrightarrow{\overline{\rho}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\exp} S^1$$

Set $\overline{\Gamma} = \widetilde{\Gamma} \times_{\overline{\rho}} \mathbb{Z}/n\mathbb{Z}$. This is an n-fold cover of Γ with cyclic deck transformation group corresponding to the following short exact sequence

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \overline{\Gamma} \longrightarrow \Gamma \longrightarrow 1 \ ,$$

in which the first map is given by $m \mapsto [1, m]$.

Since ρ factors over $\mathbb{Z}/n\mathbb{Z}$ there is a map

$$\beta \colon \widehat{\Gamma} \longrightarrow S^1 \quad ; \quad [g, z] \mapsto z^n$$

which induces a short exact sequence of the form

$$1 \longrightarrow \overline{\Gamma} \longrightarrow \widehat{\Gamma} \xrightarrow{\beta} S^1 \longrightarrow 1 \tag{4.1}$$

Combining the projection $\alpha \colon \widehat{\Gamma} \longrightarrow \Gamma$ with β yields

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \widehat{\Gamma} \xrightarrow{(\alpha,\beta)} \Gamma \times S^1 \longrightarrow 1$$

and the last map induces an isomorphism on Lie algebra level identifying $\widehat{\mathfrak{g}}$ with $\mathfrak{g} \oplus i\mathbb{R}$. This is the canonical choice for a splitting that we will use.

This will be called the *generalized spin^c case*. It will become clear later how ordinary spin^c-connections are similar to connections on bundle gerbe modules associated to generalized spin^c lifting bundle gerbes.

Note that one could consider the "generalization" of the above factorization over $\mathbb{Z}/n\mathbb{Z}$ to one of the form

$$\rho \colon \pi_1(\Gamma) \xrightarrow{\overline{\rho}} A \longrightarrow S^1$$

for some finite abelian group A, i.e. the extension is represented by a torsion element in $\mathrm{Hom}(\pi_1(\Gamma), S^1)$. But since the image is a finite abelian group in S^1, it automatically has to be cyclic and therefore reduces to the case above. To summarize:

Definition 4.1.4. Let $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ be a central extension of the connected Lie group Γ. This will be called a *generalized spin^c extension* if it is represented by a torsion element $\rho \in \mathrm{Hom}(\pi_1(\Gamma), S^1)$ or equivalently if the image of ρ is finite or equivalently if $\widehat{\Gamma}$ is associated to some *finite* covering $\overline{\Gamma}$ of Γ.

4.1.2 Connections

In the case of a vector bundle $F \to M$ there are many different ways to think about connections, of which the most popular might be: a covariant derivative mapping sections $\Gamma(F)$ to F-valued 1-forms $\Gamma(T^*M \otimes F)$, a connection form ω on the principal bundle $P_{O(n)}$ of frames in F taking values in the Lie algebra $\mathfrak{o}(n)$ and a distribution of horizontal subspaces of the tangent bundle $TP_{O(n)}$ (often we will (mis-)use the term "connection" for all of them). Luckily, the transfer of these notions to connections on bundle gerbe modules is very natural and reveals these structures to be "equivariant up to the bundle gerbe".

Remark From now on we will assume that every principal Γ-bundle P comes equipped with a smooth structure and that the projection map to the smooth base manifold M is a surjective submersion, which implies that the fiber product $P^{[2]}$ is again a smooth manifold.

Let L be the lifting bundle gerbe associated to a *flat* extension:

$$1 \longrightarrow S^1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

and identify the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{\Gamma}$ with the direct sum $\mathfrak{g} \oplus i\mathbb{R}$ by means of some chosen *Lie algebra split.*

Definition 4.1.5. A covariant derivative ∇_L on L is called a *bundle gerbe connection* if the product

$$\mu\colon \pi_{12}^* L \otimes \pi_{23}^* L \longrightarrow \pi_{13}^* L$$

pulls it back to the canonical connection on the left hand side, i.e.

$$\mu^* \pi_{13}^* \nabla_L \;=\; \pi_{12}^* \nabla_L \otimes 1 \;+\; 1 \otimes \pi_{23}^* \nabla_L \;. \tag{4.2}$$

Alternatively, one could describe ∇_L giving the *connection form* $\theta_L \in \Omega^1(\widehat{L}, i\mathbb{R})$, where $\widehat{L}$ denotes the S^1-principal bundle to which L is associated. Aside from the conditions of equivariance and reproduction of generators of fundamental vector fields, θ_L has to satisfy

$$\mu^* \pi_{13}^* \theta_L \;=\; \pi_{12}^* \theta_L \;+\; \pi_{23}^* \theta_L \;, \tag{4.3}$$

which is just the replacement of equation (4.2) (now μ of course denotes the induced product map on $\widehat{L}$).

The passage back and forth between the two uses the well-known correspondence between antiequivariant maps $\widehat{L} \to \mathbb{C}$ and sections of L (see for example [33]).

Now for L as above there is a canonical connection obtained by pulling back ν (see last section and note that we can drop the index σ now) via the defining map $\widehat{\kappa}\colon \widehat{L} \longrightarrow \widehat{\Gamma}$. It is a consequence of the calculations in [24] that (4.3) holds, if one assumes flatness of the extension. Indeed, the horizontal subspaces are of course just induced by the splitting $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus i\mathbb{R}$ of Lie algebras:

$$
\begin{aligned}
&\left(T\widehat{L}\right)_{(p_1,p_2,\widehat{g}_{12})} = \\
&\left\{(V_1,V_2,\widehat{X}_{12}) \in T_{p_1}P \oplus T_{p_2}P \oplus T_{\widehat{g}_{12}}\widehat{\Gamma} \mid \kappa_*(V_1,V_2) = q_*\widehat{X}_{12}, \pi_*V_1 = \pi_*V_2\right\} \\
&\left(H_{\widehat{L}}\right)_{(p_1,p_2,\widehat{g}_{12})} = \left\{(V_1,V_2,\widehat{X}_{12}) \in \left(T\widehat{L}\right)_{(p_1,p_2,\widehat{g}_{12})} \mid \widehat{\mu}(\widehat{X}_{12}) \in \mathfrak{g}\right\} \ , \\
&\left(V_{\widehat{L}}\right)_{(p_1,p_2,\widehat{g}_{12})} = \left\{(0,0,\widehat{X}_{12}) \in \left(T\widehat{L}\right)_{(p_1,p_2,\widehat{g}_{12})} \mid \widehat{\mu}(\widehat{X}_{12}) \in i\mathbb{R}\right\} \ ,
\end{aligned}
$$

where $\widehat{\mu}$ denotes the Maurer-Cartan-form on $\widehat{\Gamma}$ and q is the projection just as in the last section.

Regarding connections on modules over bundle gerbes, we take covariant derivatives as a starting point again. In view of (4.2) the following definition seems to be sensible:

Definition 4.1.6. Let E be a module over the bundle gerbe L with connection ∇_L. A covariant derivative ∇_E is called *bundle gerbe module connection* (or *bgm-connection* for short) if the bundle gerbe action pulls it back to the canonical connection on the tensor product, i.e. for $\gamma\colon L \otimes \pi_2^* E \longrightarrow \pi_1^* E$:

$$\gamma^* \pi_1^* \nabla_E \ = \ \nabla_L \otimes 1 \ + \ 1 \otimes \pi_2^* \nabla_E \ . \tag{4.4}$$

This can again be described by a *connection form* $\eta_E \in \Omega^1(P_E, \mathfrak{u}(n))$ that has to satisfy the analogue condition:

$$\gamma^* \pi_1^* \eta_E \ = \ \theta_L + \ \pi_2^* \eta_E \ . \tag{4.5}$$

The embedding of $i\mathbb{R}$ indirectly needed for this equation to make sense is just given by differentiating the canonical map $U(1) \longrightarrow U(n)$.

The twisting γ yields an action of $\widehat{\Gamma}$ on the sections $u \in C^\infty(P, E)$ via $(\widehat{g} \cdot u)(p) = \gamma([\widehat{g}, 1] \otimes u(pg))$. If L is equipped with its canonical flat connection, then condition (4.4) implies that ∇_E acts covariantly with respect to this action, i.e. for a vector field X on P:

$$\nabla_X(\widehat{g} \cdot u) = \widehat{g} \cdot \nabla_{R_{g*}X} u \ . \tag{4.6}$$

Example 4.1.7. Let L be a lifting bundle gerbe of a flat extension $\widehat{\Gamma} \to \Gamma$ associated to a principal Γ-bundle P over M and equipped with its canonical connection ν. Let $\alpha\colon \widehat{\Gamma} \longrightarrow U(n)$ be a unitary representation of $\widehat{\Gamma}$, such that

$$\alpha(e^{i\varphi}\widehat{g}) = e^{i\varphi}\,\alpha(\widehat{g}) \quad , \forall \widehat{g} \in \widehat{\Gamma} \text{ and } \forall e^{i\varphi} \in U(1) \ .$$

Analogously to the above mentioned trivial twisted bundle there is a canonical module over L, defined by

$$\begin{aligned} E &= P \times \mathbb{C}^n \\ \gamma &: L \otimes \pi_2^* E \longrightarrow \pi_1^* E \\ & \quad [\lambda, \widehat{g}] \otimes v \mapsto \lambda\, \alpha(\widehat{g})\, v \ . \end{aligned}$$

Now suppose that $\eta \in \Omega^1(P, \mathfrak{g})$ is a connection on the principal Γ-bundle P. The form η induces a canonical connection on E via

$$\begin{aligned} \eta_E &= \mathrm{Ad}_{\pi_{U(n)}^{-1}} \alpha_* \, \pi_P^* \eta \ + \ \pi_{U(n)}^* \mu_{U(n)} \quad , \text{ i.e.} \\ \eta_E(V, \widehat{X}) &= \mathrm{Ad}_{a^{-1}} \alpha_* \, \eta(V) \ + \ \mu_{U(n)}(\widehat{X}) \quad \forall V \in T_p P, \widehat{X} \in T_a U(n) \ . \end{aligned}$$

Again $\mu_{U(n)}$ denotes the MAURER-CARTAN-form on the group in question. Let $m \colon P \times \Gamma \to P$ be the right action of Γ, denote the action of $U(n)$ on $\mathbb{C}^n$ by $\widetilde{m}$, then the equivariance with respect to the bundle gerbe product (4.5) takes the following form in this case:

$$\eta_E(V, \widetilde{m}_*(\alpha_* \widehat{Y}, \widehat{X})) - \eta_E(m_*(V, q_* \widehat{Y}), \widehat{X}) = \nu(\widehat{Y}) \ . \tag{4.7}$$

An easy calculation reveals indeed that:

$$\begin{aligned} & \eta_E(V, \widetilde{m}_*(\alpha_* \widehat{Y}, \widehat{X})) - \eta_E(m_*(V, q_* \widehat{Y}), \widehat{X}) \\ = \ & \mathrm{Ad}_{a^{-1}} \, \alpha_* \left(\mu_{\widehat{\Gamma}}(\widehat{Y}) - \mu_\Gamma(q_* \widehat{Y}) \right) \\ = \ & \mathrm{Ad}_{a^{-1}} \alpha_* \nu(\widehat{Y}) \ . \end{aligned}$$

Since α is equivariant with respect to the embedding of $U(1)$ in $\widehat{\Gamma}$ and $U(n)$, we can drop the conjugation and α_* from the last line.

Switching from connection forms to *horizontal distributions*, equation (4.5) turns into a corresponding isomorphism condition. Let P be a principal Γ-bundle, let $L \to P^{[2]}$ be a lifting bundle gerbe of a flat extension $\widehat{\Gamma} \to \Gamma$ equipped with its canonical flat connection and denote by E a module over L. Denote the principal S^1-bundle of L by $\widehat{L}$ and the frame bundle of E by P_E. Note that $\widehat{L}$ is diffeomorphic to $P \times \widehat{\Gamma}$ via $(p_1, p_2, \widehat{g}) \mapsto (p_2, \widehat{g})$. Therefore $\widehat{L} \times_{P^{[2]}} \pi_2^* P_E$ coincides with $P_E \times \widehat{\Gamma}$. Its tangent bundle is given by

$$T(\widehat{L} \times_{P^{[2]}} \pi_2^* P_E) \simeq TP_E \oplus T\widehat{\Gamma} \ .$$

The tensor product on the level of vector bundles yields the following tensor product of principal bundles:

$$\widehat{L} \otimes \pi_2^* P_E = \widehat{L} \times_{P^{[2]}} \pi_2^* P_E \ / \sim \ . \tag{4.8}$$

Using the identification with $P_E \times \widehat{\Gamma}$ the equivalence relation for $(r, \widehat{g}) \in P_E \times \widehat{\Gamma}$ is given by $(r, \widehat{g}\, e^{i\varphi}) \sim (r\, e^{i\varphi}, \widehat{g})$. This is the principal $U(n)$-frame bundle of $L \otimes \pi_2^* E$ as the notation suggests.

Let pr be the projection $\widehat{L} \times_{P^{[2]}} \pi_2^* P_E \to \widehat{L} \otimes \pi_2^* P_E$. The free group action by S^1 induces a short exact sequence of the tangent bundles

$$0 \longrightarrow \underline{i\mathbb{R}} \longrightarrow T(\widehat{L} \times_{P^{[2]}} \pi_2^* P_E) \longrightarrow \mathrm{pr}^* T(\widehat{L} \otimes \pi_2^* P_E) \longrightarrow 0 \ .$$

Let $(r,\widehat{g}) \in \widehat{L} \times_{P^{[2]}} \pi_2^* P_E$, let $l_{\widehat{g}}$ be the left action of $\widehat{\Gamma}$ on itself and let $\beta_r \colon U(n) \to P_E$ be defined by $\beta_r(a) = r\,a$, then the trivial $i\mathbb{R}$-bundle is a subbundle of $T(\widehat{L} \times_{P^{[2]}} \pi_2^* P_E)$ via $(r,\widehat{g},X_0) \mapsto (\beta_{r*}\,X_0, -l_{\widehat{g}*}\,X_0)$ for $X_0 \in i\mathbb{R}$. Thus, $\mathrm{pr}^* T(\widehat{L} \otimes \pi_2^* P_E)$ is isomorphic to $T(\widehat{L} \times_{P^{[2]}} \pi_2^* P_E) \,/\, \underline{i\mathbb{R}} \simeq (TP_E \oplus T\widehat{\Gamma})/\underline{i\mathbb{R}}$. This isomorphism turns the right hand side into an S^1-equivariant vector bundle over $\widehat{L} \times_{P^{[2]}} \pi_2^* P_E$, such that there is an isomorphism

$$T_{[r,\widehat{g}]}\left(\widehat{L} \otimes \pi_2^* P_E\right) = \left\{(W,X) \in T_r P_E \oplus T_{\widehat{g}}\widehat{\Gamma}\right\} / \sim \tag{4.9}$$

with $(m_*(W,X_\varphi),X) \sim (W, m_*(X_\varphi,X))$ for $X_\varphi \in T_\varphi U(1)$ (here m denotes the action on P_E on the left hand side and – by slight abuse of notation – the group multiplication on the right hand side). Choose now a horizontal distribution $H_r \subset T_r P_E$ and define

$$\begin{aligned}\left(H_{\widehat{L}} \mathbin{\widehat{\oplus}} H\right)_{[r,\widehat{g}]} &= \left\{[W,X] \in T_{[r,\widehat{g}]}\left(\widehat{L} \otimes \pi_2^* P_E\right) \mid \widehat{\mu}(X) \in \mathfrak{g}, W \in H_r\right\} \\ &\subset T_{[r,\widehat{g}]}\left(\widehat{L} \otimes \pi_2^* P_E\right) .\end{aligned}$$

In a similar fashion, we can identify $\pi_1^* P_E$ with $P_E \times \Gamma$ via $(r,p_2) \mapsto (r,g)$ where g is such that $\pi_{P_E}(r)\,g = p_2$. As a consequence $T(\pi_1^* P_E)$ is identified with $TP_E \oplus T\Gamma$. Now, γ_* restricted to the second summand is just the map $[W,X] \mapsto q_* X$. Keeping this in mind, we have:

Definition 4.1.8. A horizontal distribution $H_r \subset T_r P_E$ is called a *bundle gerbe module connection* (or *bgm-connection*) for the canonical flat connection on L if

$$\gamma_*\left(\left(H_{\widehat{L}} \mathbin{\widehat{\oplus}} H\right)_{(p_1,[r,\widehat{g}])}\right) \subset H_{\gamma(p_1,[r,\widehat{g}])} \oplus T_g\Gamma \subset T(\pi_1^* P_E).$$

(Since γ_* is an isomorphism and both subspaces have the same dimension we could replace the first $\subset$ by $=$ without changing anything.)

Lemma 4.1.9. *Let $H_r \subset T_r P_E$ be a bgm-connection and $s : T_r P_E \longrightarrow H_r$ the canonical projection with respect to the vertical subspace. Then γ_* commutes with s in the following way:*

$$\begin{tikzcd} T_{(p_1,[r,\widehat{g}])}\left(\widehat{L} \otimes \pi_2^* P_E\right) \arrow[r, "{[s,q_*]}"] \arrow[d, "\gamma_*"] & \left(\left(H_{\widehat{L}} \mathbin{\widehat{\oplus}} H\right)_{(p_1,[r,\widehat{g}])}\right) \arrow[d, "\gamma_*"] \\ T_{\gamma(p_1,[r,\widehat{g}])} P_E \oplus T_g\Gamma \arrow[r, "{(s,\mathrm{id}_{T\Gamma})}"] & H_{\gamma(p_1,[r,\widehat{g}])} \oplus T_g\Gamma \end{tikzcd}$$

Proof. Since by assumption γ_* maps the horizontal subspaces $\left(H_{\widehat{L}} \mathbin{\widehat{\oplus}} H\right)_{(p_1,[r,\widehat{g}])}$ and $H_{\gamma(p_1,[r,\widehat{g}])} \oplus T_g\Gamma$ isomorphically onto each other, we only need to show that the same holds for the vertical ones, then the lemma follows.

Let $\beta_r \colon U(n) \longrightarrow P_E$ be given by $a \mapsto r\,a$ and denote the inclusion $t \mapsto [t,\widehat{g}]$ for $t \in P_E$ by $\iota_{\widehat{g}}$. Since γ is equivariant with respect to the $U(n)$-action on P_E, the following diagram commutes:

$$\begin{tikzcd} U(n) \arrow[r, "\beta_r"] \arrow[rrrr, bend right, "\beta_{\gamma(p_1,[r,\widehat{g}])}"'] & P_E \arrow[r, "\iota_{\widehat{g}}"] & \widehat{L} \otimes \pi_2^* P_E \arrow[r, "\gamma"] & \pi_1^* P_E \arrow[r] & P_E \end{tikzcd}$$

with $p_1 = \pi_{P_E}(r)\, q(\widehat{g})^{-1}$. By differentiating we deduce $\gamma_*([\beta_{r*}(X), 0]) = \beta_{\widetilde{r}*}(X)$ for $X \in \mathfrak{u}(n)$ and $\widetilde{r} = \gamma(p_1, [r, \widehat{g}])$ proving that γ_* restricts to an isomorphism on the vertical subspaces as well. □

The way back from distributions to connection forms is the usual one (see for example [33]). Given $H_r \subset T_r P_E$ define η via $\eta(W) = X$, where $\varrho(W) = \beta_{r*}(X)$ for $W \in T_r P_E$. Here, ϱ denotes the projection onto the vertical subspace with respect to the horizontal one.

Theorem 4.1.10. *(i) If H is a bgm-connection for the canonical flat connection on L, then η is a bgm-connection form.*

(ii) If η is a bgm-connection form for the canonical flat connection on L, then $H_r = \mathrm{kern}(\eta)$ is a bgm-connection.

Proof. Let $Y_1 = \eta(\gamma_*([W, X]))$, $Y_2 = \eta(W)$, set $\widetilde{r} = \gamma(p_1, [r, \widehat{g}])$ and denote the projection from $T_r P_E$ to H_r by s, then

$$\begin{aligned}
\beta_{\widetilde{r}*}(Y_1) &= \varrho(\gamma_*([W, X])) = \gamma_*([W, X]) - s(\gamma_*([W, X])) \\
&= \gamma_*([W, X]) - \gamma_*([s(W), q_* X]) = \gamma_*([\varrho(W), X - q_*(X)]) \\
&= \gamma_*([\beta_{r*}(Y_2), 0]) + \gamma_*([0, X - q_*(X)]) \\
&= \beta_{\widetilde{r}*}(Y_2) + \gamma_*([0, L_{\widehat{g}*}\nu(X)])
\end{aligned}$$

Since $L_{\widehat{g}*}\nu(X) = R_{\widehat{g}*}\nu(X) = m_*(\nu(X), 0)$, the last summand is equal to $\gamma_*[m_*(0, \nu(X)), 0] = \gamma_*[\beta_{r*}(\nu(X)), 0] = \beta_{\widetilde{r}*}(\nu(X))$. Injectivity of $\beta_{\widetilde{r}*}$ now proves that (4.5) holds and therefore also (i).

For (ii) it suffices to show that $\eta(\gamma_*([W, X])) = 0$ for $W \in \mathrm{kern}(\eta)$ and $\widehat{\mu}(X) \in \mathfrak{g}$, but this is a direct consequence of (4.5). □

4.1.3 Flat connections on general S^1-bundle gerbes

As described in section 2.1 the concept of bundle gerbes is by no means limited to the case of *lifting* bundle gerbes. In this section we will consider a smooth fibration $\pi \colon Y \to M$, which is a surjective submersion, and a bundle gerbe $L \to Y^{[2]}$ like in definition 2.1.1. We will analyze the topological obstructions to the existence of a flat connection on L. Note that a *line bundle* ℓ over M carries a flat connection if and only if its first Chern class $c_1(\ell) \in H^2(M, \mathbb{Z})$ is torsion, i.e. $c_1^{\mathbb{R}}(\ell) \in H^2(M, \mathbb{R})$ vanishes. In the case of bundle gerbes, $c_1^{\mathbb{R}}(\ell)$ should be replaced by the real Dixmier-Douady-class $dd^{\mathbb{R}}(L) \in H^3(M, \mathbb{R})$. Even though vanishing of $dd^{\mathbb{R}}(L)$ is necessary, it is not sufficient to ensure the existence of a flat connection as we will see presently.

Connections on bundle gerbes were already studied by Murray in [49], from which we review the main arguments. At the heart of the analysis lies the following *exact* complex of real valued k-forms:

$$\Omega^k(M) \xrightarrow{\pi^*} \Omega^k(Y) \xrightarrow{\delta} \Omega^k(Y^{[2]}) \xrightarrow{\delta} \Omega^k(Y^{[3]}) \xrightarrow{\delta} \cdots$$

Denote by $\pi_i \colon Y^{[p]} \to Y^{[p-1]}$ the projection leaving out the ith factor, then the differentials δ are defined via

$$\delta(\omega) = \sum_{i=1}^{p} (-1)^i \pi_i^* \omega \ .$$

The curvature Ω_L of a bundle gerbe connection on L provides a 2-form on $Y^{[2]}$ and it was proven in [49] (using local trivializations) that $\delta(\Omega_L) = 0$. Therefore there is an element $f \in \Omega^2(Y)$, such that

$$\pi_2^* f - \pi_1^* f = \delta(f) = \Omega_L \ .$$

Definition 4.1.11. A choice of $f \in \Omega^2(Y)$ with $\delta(f) = \Omega_L$ is called a *curving* for the connection on L.

By the BIANCHI identity and since d and δ commute, we have $\delta(df) = 0$, which implies that there is a 3-form $\omega \in \Omega^3(M)$, such that

$$df = \pi^* \omega \ . \tag{4.10}$$

Another fundamental theorem proven in [49] is the following.

Theorem 4.1.12. *Let L be a bundle gerbe with connection. Let f be a curving for this connection and let $\omega \in \Omega^3(M)$ be a 3-form with $df = \pi^*\omega$. Then ω is a closed form representing the* DIXMIER*-Douady-class, i.e.*

$$[\omega] = dd^{\mathbb{R}}(L) \in H^3(M, \mathbb{R}) \ .$$

In particular, this class does not depend on the choice of f.

An immediate corollary of this observation is that the DIXMIER-DOUADY-form can be seen as the obstruction to the existence of a closed curving.

Corollary 4.1.13. *Let L be a bundle gerbe with connection. Then the condition $dd^{\mathbb{R}}(L) = 0$ is equivalent to the existence of a closed curving $f \in \Omega^2(Y)$ for the connection on L.*

Proof. If there is a closed curving f, then equation (4.10) and the injectivity of π^* on forms imply that ω can be chosen to vanish. Therefore $dd^{\mathbb{R}}(L) = [\omega] = 0$.

On the other hand, let $f' \in \Omega^2(Y)$ be a curving for the connection on L and $\omega \in \Omega^3(M)$ be a 3-form satisfying (4.10) and $[\omega] = dd^{\mathbb{R}}(L) = 0$. Denote the curvature of the connection by Ω_L. Then $\omega = d\tau$ for some $\tau \in \Omega^2(M)$. Set $f = f' - \pi^*\tau$. Then

$$\begin{aligned} df &= df' - \pi^* d\tau = \pi^*\omega - \pi^*\omega = 0 \ , \\ \delta(f) &= \delta(f') - \delta(\pi^*\tau) = \Omega_L \ . \end{aligned}$$

Therefore f is a closed curving for the connection. □

If the curvature vanishes, f can be chosen to vanish as well, a choice, which is clearly closed. Therefore $dd^{\mathbb{R}}(L) = 0$ for flat bundle gerbe modules. On the other hand, non-vanishing curvings provide an obstruction to flat connections.

To avoid working with de Rham cohomology groups of infinite dimensional spaces, which is nevertheless possible (see e.g. [9]), we now stick to the case of finite-dimensional smooth fibrations Y. According to [50, Proposition 5.6] if M and the fibers of $Y \to M$ are simply connected a finite-dimensional Y automatically implies that $dd^{\mathbb{R}}(L) = 0$. Note that f provides a class $[f] \in H^2(Y, \mathbb{R})$ in case the curving is closed.

Lemma 4.1.14. *Let L be a bundle gerbe with connection, such that $dd^{\mathbb{R}}(L) = 0$. Denote by $\pi^*\colon H^2(M,\mathbb{R}) \to H^2(Y,\mathbb{R})$ the map induced by the projection on cohomology. Then a choice of closed curving $f \in \Omega^2(Y)$ provides a class*

$$f^{\mathbb{R}}(L) = [f] \in H^2(Y,\mathbb{R})/\mathrm{Im}(\pi^*) \ ,$$

which neither depends on the particular choice of curving f nor on the bundle gerbe connection itself and therefore provides an invariant of the bundle gerbe.

Proof. Two possible choices of curvings f_i, $i \in \{1,2\}$ for the same connection differ by the pullback of a form $\rho \in \Omega^2(M)$ to Y. If both curvings are closed, then ρ is closed as well by the injectivity of π^*. Therefore $[f_1] - [f_2] = \pi^*[\rho]$.

Denote by $\widehat{L}$ the principal S^1-bundle of L and by $\mathrm{pr}\colon \widehat{L} \to Y^{[2]}$ the projection. As proven in [49] two bundle gerbe connections $\theta_1, \theta_2 \in \Omega^1(\widehat{L}, i\,\mathbb{R})$ differ by

$$\theta_2 - \theta_1 = i\,\mathrm{pr}^*\delta(\tau)$$

for some form $\tau \in \Omega^1(Y)$. Denote by Ω_L^j for $j \in \{1,2\}$ the corresponding curvatures. Let f_1 be a closed curving for θ_1 and set

$$f_2 = f_1 + d\tau \ .$$

Clearly f_2 is closed and $\delta(f_2) = \Omega_L^1 + \Omega_L^2 - \Omega_L^1 = \Omega_L^2$. Thus, f_2 is a closed curving for θ_2. But f_1 and f_2 differ by an exact form. Thus $[f_1] = [f_2] \in H^2(Y,\mathbb{R})$. □

Theorem 4.1.15. *Let L be a bundle gerbe with $dd^{\mathbb{R}}(L) = 0$. L allows a flat bundle gerbe connection if and only if*

$$f^{\mathbb{R}}(L) = 0 \in H^2(Y,\mathbb{R})/\mathrm{Im}(\pi^*) \ .$$

Proof. If there is a connection on L with $\Omega_L = 0$, then for any choice of curving $f \in \Omega^2(Y)$ we have

$$\delta(f) = \Omega_L = 0 \ .$$

Therefore $f = \pi^*\rho$ for some $\rho \in \Omega^2(M)$, which implies $f^{\mathbb{R}}(L) = 0$.

Suppose on the other hand that $f^{\mathbb{R}}(L) = 0$ and choose an arbitrary bundle gerbe connection θ on L with curvature Ω_L and curving f. Since by assumption there is a closed form $\rho \in \Omega^2(M)$ with

$$[f - \pi^*\rho] = 0 \in H^2(Y,\mathbb{R})$$

there is a closed form $\eta \in \Omega^1(Y)$ with $f - \pi^*\rho = d\eta$. Denote by $\widehat{L}$ again the principal S^1-bundle of L and by $\mathrm{pr}\colon \widehat{L} \to Y^{[2]}$ the projection. Define θ' by

$$\theta' = \theta - i\,\mathrm{pr}^*\delta(\eta) \in \Omega^1(\widehat{L}, i\,\mathbb{R}) \ .$$

This is the connection form of a bundle gerbe connection on L. Its curvature $\Omega'_L \in \Omega^2(Y^{[2]})$ is

$$\Omega'_L = \Omega_L - \delta(d\eta) = \Omega_L - \delta(f) = 0 \ .$$

Therefore θ' provides a flat bundle gerbe connection on L. □

Example 4.1.16. If L is the trivial bundle gerbe, i.e. $L = \pi_1^* Q \otimes \pi_2^* Q^*$ for a line bundle $Q \to Y$, then the tensor product of a connection pulled back from one on Q is a bundle gerbe connection on L. A possible curving for this choice is of course the curvature $\Omega_Q \in \Omega^2(Y)$ of the connection on Q. Therefore we have $f^{\mathbb{R}}(L) = [c_1(Q)] \in H^2(Y, \mathbb{R})/\mathrm{Im}(\pi^*)$ in this case.

In some cases the invariant $f^{\mathbb{R}}(L)$ only depends on the cohomology of the fiber $F \to Y \to M$ instead of the total space Y.

Theorem 4.1.17. *If $F \to Y \to M$ is a fibration with M and F path-connected, $H^1(F, \mathbb{R}) = 0$ and $\pi_1(M)$ acting trivially on $H^2(F, \mathbb{R})$ and $H^0(F, \mathbb{R})$. Then there is an exact sequence*

$$0 \to H^2(M, \mathbb{R}) \xrightarrow{\pi^*} H^2(Y, \mathbb{R}) \xrightarrow{i^*} H^2(F, \mathbb{R}) \xrightarrow{\tau} H^3(M, \mathbb{R}) \ ,$$

in which i^* and π^* are induced by the inclusion of the fiber and the projection to the base and τ is the transgression map.

Proof. This follows directly from the SERRE spectral sequence. □

In cases matching the conditions of theorem 4.1.17 vanishing of $f^{\mathbb{R}}(L)$ is equivalent to $i^*(f^{\mathbb{R}}(L)) = 0 \in H^2(F, \mathbb{R})$.

Example 4.1.18. In the case of lifting bundle gerbes the fibration $Y \to M$ is given by a principal Γ-bundle $P \to M$. Suppose that the fiber Γ is a compact connected Lie group, then $H^1(\Gamma, \mathbb{R}) = 0$ already implies $H^2(\Gamma, \mathbb{R}) = 0$ (see for example [14, Corollary 12.9]). This immediately yields

Corollary 4.1.19. *If L is a lifting bundle gerbe over a path-connected M for a central S^1-extension*

$$1 \longrightarrow S^1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1 \ ,$$

where Γ is a compact connected Lie group, such that $H^1(\Gamma, \mathbb{R}) = 0$ ($\Leftrightarrow \pi_1(\Gamma)$ finite) and $\pi_1(M)$ acts trivially on $H^0(\Gamma, \mathbb{R})$. Then there is a flat bundle gerbe connection on L.

Denote by $\widehat{\mu}$ and μ the MAURER-CARTAN forms on $\widehat{\Gamma}$ and Γ respectively and choose a split $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ of the corresponding Lie algebras. Let $\kappa\colon P^{[2]} \longrightarrow \Gamma$ be the defining map of the lifting bundle gerbe and denote the projection $\widehat{\Gamma} \to \Gamma$ by q. By [24, Lemma 6.4] there is a bundle gerbe connection on the lifting bundle gerbe L with connection form

$$\nu = \kappa^* \left(\widehat{\mu} - \sigma \circ q^* \mu\right) \ . \tag{4.11}$$

Its curvature is given by

$$\Omega_L = -\frac{1}{2} \kappa^* \omega_\sigma(\mu(\cdot), \mu(\cdot)) \ .$$

where $\omega_\sigma(X, Y) = [\sigma(X), \sigma(Y)]_{\widehat{\Gamma}} - \sigma([X, Y]_\Gamma)$ is the Lie algebra cocycle of the extension. By restricting to local trivializations it is easy to see that in this case $i^*(f^{\mathbb{R}}(L)) = [\omega_\sigma(\mu, \mu)] \in H^2(\Gamma, \mathbb{R})$, which corresponds to the curvature $\omega_\sigma(\mu, \mu)$ of the connection induced by the split on the principal S^1-bundle $\widehat{\Gamma} \to \Gamma$.

Remark 4.1.20. Let $\Delta\colon P \to P^{[2]}$ be the diagonal embedding $p \mapsto (p,p)$. As we have seen in section 2.1 the line bundle $\Delta^* L$ is trivialized via a canonical isomorphism induced by the bundle gerbe multiplication. Denote the principal S^1-bundle of L by $\widehat{L}$. Using the identification $\widehat{L} = P \times \widehat{\Gamma}$, the bundle homomorphism $P \times S^1 \to \Delta^* \widehat{L} \to \widehat{L}$ corresponds to the map $P \times S^1 \to P \times \widehat{\Gamma}$, which is just the inclusion on the second factor. This observation shows that ν from (4.11) pulls back to the canonical *flat* connection on the trivial bundle. Any other bundle gerbe connection on L can be obtained as

$$\theta = \nu + i\mathrm{pr}^*\delta(\eta)$$

for a form $\eta \in \Omega^1(P)$ and since $\pi_j \circ \mathrm{pr} \circ \Delta$ coincides with the bundle projection $\Delta^* \widehat{L} \to P$ for $j \in \{1, 2\}$ we see that the above remark also holds for any other bundle gerbe connection on L: Any bundle gerbe connection pulls back to a *flat* connection over the diagonal.

4.1.4 Trivial bundle gerbes

Since the Dixmier-Douady-class of a lifting bundle gerbe $L \longrightarrow P^{[2]}$ coincides with the obstruction to lifting P to a $\widehat{\Gamma}$-bundle $\widehat{P}$, connections on modules over L should reduce to ordinary connections on the pushed down bundles over M. This section will reveal that this is indeed the case. It will also clarify some properties of connections in the generalized spin^c-case.

Whenever the lifting bundle gerbe L is trivial, there is a line bundle $Q^* = \widehat{P} \times_{S^1} \mathbb{C}$. Its conjugate Q is a bundle gerbe module. To see this, note that on the level of principal bundles switching to conjugates corresponds to twisting the S^1-action by the inversion (i.e. $\widehat{p} \cdot e^{i\varphi} = \widehat{p} e^{-i\varphi}$ where the dot denotes the new action). Therefore the module structure on Q is induced by:

$$\gamma\colon \widehat{L} \otimes \pi_2^* \widehat{P} \longrightarrow \pi_1^* \widehat{P} \quad ; \quad [\widehat{g}\,, \widehat{p}] \mapsto \widehat{p}\,\widehat{g}^{-1}\ .$$

Lemma 4.1.21. *Suppose that P is established with a fixed connection η and L carries the canonical flat connection. Giving a bundle gerbe module connection on $\widehat{P}$ is the same as choosing a connection $\widehat{\eta}$ on $\widehat{P}$ that is compatible with η.*

Proof. Let $\theta \in \Omega^1(\widehat{P}, i\mathbb{R})$ be a bgm-connection on $\widehat{P}$ and denote the Maurer-Cartan form on $\widehat{\Gamma}$ by ν, then equation (4.5) yields

$$\theta(\gamma_*([W, X])) = \theta(W) + \nu(X) \quad \text{for } W \in T_{\widehat{p}}\widehat{P}, X \in T_{\widehat{g}}\widehat{\Gamma}\ .$$

In particular

$$\theta(R_{\widehat{g}*}W) = \theta(W) \quad , \quad \theta(\beta_{\widehat{p}*}X) = -\nu(X) \tag{4.12}$$

with $\beta_{\widehat{p}}(\widehat{g}) = \widehat{p}\,\widehat{g}$ and $X \in \widehat{\mathfrak{g}}$ (note that β multiplies with $\widehat{g}$, whereas γ uses the inverse, hence the minus sign in front of ν!). Let $\pi_{\widehat{P}}\colon \widehat{P} \to P$ and $q\colon \widehat{\Gamma} \to \Gamma$ be the canonical projections and set $\widehat{\eta} = \pi_{\widehat{P}}^* \eta - \theta$. Due to (4.12)

$$\begin{aligned}
\widehat{\eta}(R_{\widehat{g}*}W) &= \eta(R_{q(\widehat{g})*}\pi_{\widehat{P}*}W) - \theta(R_{\widehat{g}*}W) = \mathrm{Ad}_{q(\widehat{g})^{-1}}\,\eta(\pi_{\widehat{P}*}W) - \theta(W) \\
&= \mathrm{Ad}_{\widehat{g}^{-1}}\widehat{\eta}(W) \\
\widehat{\eta}(\beta_{\widehat{p}*}X) &= \eta(\beta_{\pi_{\widehat{P}}(\widehat{p})*}q_*X) - \theta(\beta_{\widehat{p}*}X) = q_*X + \nu(X) = X\ .
\end{aligned}$$

Note that $\pi_{\widehat{P}}^* \eta = q_* \widehat{\eta}$, hence $\widehat{\eta}$ is a connection compatible with η.

A connection $\widehat{\eta}$ compatible with η on $\widehat{P}$ induces a bgm-connection via $\theta = \pi^*_{\widehat{P}}\eta - \widehat{\eta}$. Indeed, since $q_*\widehat{\eta} = \pi^*_{\widehat{P}}\eta$, θ takes values in $i\mathbb{R}$. Furthermore:

$$\begin{aligned} \widehat{\eta}(\gamma_*([W,X])) &= -\widehat{\eta}(\beta_{\widehat{p}*} R_{\widehat{g}^{-1}*} L_{\widehat{g}^{-1}*} X) + \widehat{\eta}(R_{\widehat{g}^{-1}*} W) \\ &= -\operatorname{Ad}_{\widehat{g}}(\widehat{\mu}(X)) + \operatorname{Ad}_{\widehat{g}} \widehat{\eta}(W)\ , \\ \eta(\pi_{\widehat{P}*}\gamma_*([W,X])) &= -\operatorname{Ad}_{q(\widehat{g})}(\mu(q_*X)) + \operatorname{Ad}_{q(\widehat{g})} \eta(\pi_{\widehat{P}*} W) \end{aligned} \tag{4.13}$$

and therefore

$$\begin{aligned} \theta(\gamma_*([W,X])) &= \operatorname{Ad}_{q(\widehat{g})}\big(\widehat{\mu}(X) - \mu(q_*X) - \widehat{\eta}(W) + \eta(\pi_{\widehat{P}*}W)\big) \\ &= \nu(X) + \theta(W)\ . \end{aligned}$$

Invariance under right translation by elements of S^1 is evident. Fundamental vector fields on $\widehat{P}$ with respect to the above action are generated by $\alpha_{\widehat{p}}(e^{i\varphi}) = \widehat{p}\, e^{-i\varphi} = \beta_{\widehat{p}} \circ \operatorname{inv}(e^{i\varphi})$. Thus for $X_\varphi \in i\,\mathbb{R}$

$$\theta(\alpha_{\widehat{p}*} X_\varphi) = -\widehat{\eta}(\beta_{\widehat{p}*} \operatorname{inv}_* X_\varphi) = X_\varphi\ . \qquad \square$$

Theorem 4.1.22. *Let $\widehat{P}$ be the bundle gerbe module for the trivial lifting bundle gerbe L equipped with its canonical flat connection as above, θ a bgm-connection on $\widehat{P}$. Let P_E be the principal $U(n)$-bundle associated to a bundle gerbe module E over L with connection η_E. Denote by $\widehat{P} \otimes P_E$ the tensor product of the principal bundles as defined in (4.8) and let $\pi_!(\widehat{P} \otimes P_E)$ be the pushed down $U(n)$-principal bundle over M. Then $\omega = \pi_!(\eta_E - \theta)$ defines a connection on $\pi_!(\widehat{P} \otimes P_E)$.*

Proof. Using equation (4.5) for elements $X_\varphi \in T_\varphi S^1$, it is easy to check that $\eta_E - \theta$ is a well defined form in $\Omega^1(\widehat{P} \otimes P_E, \mathfrak{u}(n))$. The same argument can be used for the action of elements $X \in T_{\widehat{g}}\widehat{\Gamma}$ to show that $\eta_E - \theta$ can be pushed down to $\pi_!(\widehat{P} \otimes P_E)$. Since η_E is a connection form, $\eta_E - \theta$ inherits this property. $\square$

In the generalized spin^c case we have the short exact sequence

$$1 \longrightarrow \overline{\Gamma} \xrightarrow{\ \delta\ } \widehat{\Gamma} \longrightarrow S^1 \longrightarrow 1\ .$$

In this situation the bundle gerbe module $\widehat{P}$ can be pushed down to a line bundle $\widehat{P}/\overline{\Gamma}$ over M. In the case $\widehat{\Gamma} = \mathrm{Spin}^c$ this is the line bundle corresponding to the inverse canonical class of the spin^c-structure. Since the choice of splitting that we made above implies $\nu(\delta_*X) = 0$ for all $X \in T_{\overline{g}}\overline{\Gamma}$, θ pushes down as well. In the other direction the pullback via $\widehat{P} \longrightarrow \widehat{P}/\overline{\Gamma}$ of a connection form on the inverse canonical line bundle yields a bgm-connection due to a calculation similar to (4.13) and the canonical splitting that we chose above.

To summarize: The trivial bundle gerbe with its connection contains *every* spin^c structure on the manifold. A choice of *trivialization* of the gerbe together with a bgm-connection on it fixes both: the structure as well as a spin^c-connection. It is in this way that bundle gerbe modules in the above case are like generalized spinor bundles without having a spin^c-structure!

4.1.5 Connections on tensor products

Whenever we have two bundle gerbes $L_1 \longrightarrow X_1^{[2]}$ and $L_2 \longrightarrow X_2^{[2]}$ we can form their external tensor product

$$L = L_1 \boxtimes L_2 := \pi^*_{X_1^{[2]}} L_1 \otimes \pi^*_{X_2^{[2]}} L_2 \longrightarrow X_1^{[2]} \times_M X_2^{[2]}$$

where the $\pi_{X_i^{[2]}}$ denote the canonical projections. This is again a bundle gerbe with $dd(L_1 \boxtimes L_2) = dd(L_1) + dd(L_2)$. If each L_i carries a bundle gerbe connection θ_i, then

$$\theta_L = \pi^*_{X_1^{[2]}} \theta_1 + \pi^*_{X_2^{[2]}} \theta_2 \in \Omega^1(\widehat{L_1 \boxtimes L_2}, i\mathbb{R})$$

defines one on the tensor product. Now take two bundle gerbe modules E_i over L_i for $i \in \{1,2\}$, both carrying bgm-connection forms, denoted by $\eta_i \in \Omega^1(P_{E_i}, \mathfrak{u}(n_i))$. The external tensor product

$$E_1 \boxtimes E_2 = \pi^*_{X_1} E_1 \otimes \pi^*_{X_2} E_2$$

is a bundle gerbe module for L with respect to the tensor product action

$$\gamma = \gamma_{E_1} \boxtimes \gamma_{E_2} \colon L \otimes \pi_2^*(E_1 \boxtimes E_2) \longrightarrow \pi_1^*(E_1 \boxtimes E_2) \ ,$$

since $(L_1 \boxtimes L_2) \otimes \pi_k^*(E_1 \boxtimes E_2) = (L_1 \otimes \pi^*_{X_1,k} E_1) \boxtimes (L_2 \otimes \pi^*_{X_2,k} E_2)$. Fix a monomorphism $\tau \colon U(n_1) \otimes U(n_2) \longrightarrow U(n_1 n_2)$. The first group is the quotient of the product by the antidiagonal $U(1)$-action. Using the same notation $P_{E_1 \boxtimes E_2}$ can be written as the $U(n_1 n_2)$-principal bundle $(P_{E_1} \otimes P_{E_2}) \times_\tau U(n_1 n_2)$. The $\mathfrak{u}(n_1 n_2)$-valued 1-form

$$\eta([[W_1, W_2], X]) = \mathrm{Ad}_{a^{-1}}(\tau_* [\eta_1(W_1), \eta_2(W_2)]) + \mu_{U(n_1 n_2)}(X) \tag{4.14}$$

is well-defined on $P_{E_1 \boxtimes E_2}$ and yields a connection. Here we have $W_i \in T_{p_i} E_i$ for $i \in \{1,2\}$, such that $[W_1, W_2] \in T_{[p_1,p_2]}(P_{E_1} \otimes P_{E_2})$ and $X \in T_a U(n_1 n_2)$. The outer brackets denote the quotient by τ_*. Now, $\tau_*([\theta_1(Y_1), \theta_2(Y_2)]) = \theta_1(Y_1) + \theta_2(Y_2)$ for $Y_i \in T_{x_i} \widehat{L}_i$. Therefore η is a bgm-connection.

4.1.6 Parallel transport in twisted Hilbert A-module bundles

Let V be a Hilbert C^*-module over A, then $\mathrm{End}(V)$ turns out to be a C^*-algebra. It contains the group $U(V)$ of unitary elements, which becomes a topological group when equipped with the norm topology. In fact, it is a Banach Lie group in the sense of [34] modelled on the Lie algebra $i\mathfrak{a}$ of anti-selfadjoint elements in $\mathrm{End}(V)$. It therefore is regular in the sense that smooth curves in $i\mathfrak{a}$ can be integrated to smooth curves in $U(V)$ in a smooth way. Surprisingly, the geometry of principal bundles with regular structure groups can be treated in much the same way as the finite-dimensional case. In particular, there is a sensible notion of parallel transport, see [34].

We will apply the above setup to the frame bundle P_E of a smooth twisted Hilbert A-module bundle E over a principal Γ-bundle P to see that parallel transport acts equivariantly with respect to shifting the starting point by the action of $\widehat{\Gamma}$. Definition 4.1.6 extends canonically to our situation:

Definition 4.1.23. Let E be a twisted Hilbert A-module bundle with respect to the bundle gerbe (L, θ_L). A *twisted connection* on E is a connection form $\eta_E \in \Omega^1(P_E, i\mathfrak{a})$ satisfying the following condition:

$$\gamma^* \pi_1^* \eta_E \;=\; \theta_L + \;\pi_2^* \eta_E \; . \tag{4.15}$$

The embedding of $i\mathbb{R}$ implicitly needed for this equation to make sense is just the differential at 1 of the canonical map $U(1) \longrightarrow U(V)$.

At the heart of the theory of parallel transport lies the following observation:

Lemma 4.1.24. *Let E be a twisted Hilbert A-module bundle over P with a connection η. Let $V \in T_pP$ and $r \in P_E$ be a lift of p. There exists a unique vector $W \in T_rP_E$ such that $\pi_{P_E *}(W) = V$ and $\eta(W) = 0$. W is called the* horizontal lift *of V.*

Proof. This is clear from the fact that η provides a splitting of the short exact sequence of Banach spaces:

$$0 \longrightarrow i\mathfrak{a} \longrightarrow T_rP_E \longrightarrow T_pP \longrightarrow 0 \; . \qquad \square$$

Theorem 4.1.25. *Let $E \to P$ be as above. For each smooth curve $c \colon \mathbb{R} \longrightarrow P$ with $c(0) = p \in P$ there is a unique smooth mapping*

$$\mathcal{P}_c \colon \mathbb{R} \times (P_E)_p \longrightarrow P_E$$

with the following properties:

1. $\mathcal{P}_c(t, r) \in (P_E)_{c(t)}$
2. $\mathcal{P}_c(0, \cdot) = \mathrm{id}_{(P_E)_p}$
3. $\eta(\frac{d}{dt}\mathcal{P}_c(t, r)) = 0$

(i.e. each curve c in P has a lift $\widehat{c}(t) = \mathcal{P}_c(t, r)$ to P_E such that it starts at r and the tangent at each point is lifted horizontally).

Proof. see Theorem 6.1 in [34] for the details. In a trivialization $\varphi_i : U_i \times U(V) \longrightarrow P_E$ the curve takes the form $\widehat{c}(t) = (c(t), \tau(t))$, furthermore for $(V, Y) \in T_pU_i \oplus T_gU(V)$:

$$\eta(\varphi_{i*}(V, Y)) = L_{g^{-1}*}(Y) + \mathrm{Ad}_{g^{-1}} \eta_i(V) \; ,$$

where η_i is the pullback of η with respect to the canonical section $U_i \to U_i \times \{1\} \stackrel{\varphi_i}{\to} P_E$. Existence now follows from the solvability of the differential equation

$$\dot{\tau} = -R_{\tau(t)*}(\eta_i(\dot{c}))$$

in $U(V)$ and is therefore reduced to its regularity. $\square$

The twisted action γ induces a true left action of $\widehat{\Gamma}$ on P_E via

$$\widehat{g} \cdot r = \gamma(pg^{-1}, [r, \widehat{g}]) \; ,$$

where $(pg^{-1}, [r, \widehat{g}]) \in \widehat{L} \otimes \pi_2^* P_E$ like in (4.8). It covers the left action of Γ on P in the sense that $\pi_{P_E}(\widehat{g} \cdot r) = pg^{-1} =: g \cdot p$.

Lemma 4.1.26. *Parallel transport is $\widehat{\Gamma}$-equivariant in the following sense:*

$$\mathcal{P}_{g\cdot c}(t, \widehat{g}\cdot r) = \widehat{g}\cdot \mathcal{P}_c(t,r)\ .$$

Proof. Set $\widehat{c}(t) = \mathcal{P}_c(t,r)$. We have to show that $\widehat{g}\cdot\widehat{c}$ satisfies conditions (1) to (3) in theorem 4.1.25 with respect to the curve $g\cdot c$. (1) and (2) follow from the fact that the action of $\widehat{\Gamma}$ on P_E covers that of Γ on P. For (3) we calculate:

$$\frac{d}{dt}\left(\widehat{g}\cdot\widehat{c}\right)(t) = \frac{d}{dt}\left(\gamma(g\cdot c, [\widehat{c},\widehat{g}])\right)(t) = \gamma_*([\dot{\widehat{c}}(t), 0])\ ,$$

where the dot denotes the derivative by t and we used the notation from (4.9) for tangent vectors. But since η is a twisted connection form we have $\eta(\gamma_*([\dot{\widehat{c}}(t),0])) = \theta_L(0) + \eta(\dot{\widehat{c}}) = \eta(\dot{\widehat{c}}) = 0$ by hypothesis. □

4.1.7 Curvature and Chern character of twisted Hilbert A-module bundles

We continue unravelling further classical notions of differential geometry by defining what the curvature of a twisted Hilbert A-module bundle should be. There is no surprise about the next definition:

Definition 4.1.27. Let E be a twisted Hilbert A-module bundle with connection form η_E. Then its *curvature* is defined to be the $i\mathfrak{a}$-valued 2-form

$$\Omega_E = d\eta_E + [\eta_E, \eta_E] \in \Omega^2(P_E, i\mathfrak{a})\ .$$

Here we have $[\eta_E,\eta_E](X,Y) = [\eta_E(X), \eta_E(Y)]$, where the Lie bracket of $i\mathfrak{a}$ is used.

Even the main properties of Ω_E hold in the infinite dimensional case as was proven in [34].

Lemma 4.1.28. *If Ω_E is the curvature of a twisted connection, then it is horizontal, i.e. it kills vertical vector fields. Furthermore it is $U(V)$-equivariant in the following sense: $R_a^*\Omega_E = Ad_{a^{-1}}\Omega_E$ for all $a \in U(V)$.*

The above lemma allows us to see Ω_E as a 2-form over P taking values in the bundle $\mathrm{Ad}(P_E)^{\downarrow} = P_E \times_{\mathrm{Ad}} i\mathfrak{a}$. Since the map induced by the twisting isomorphism on the bundle P_E is equivariant with respect to the action of $U(V)$, $\mathrm{Ad}(P_E)^{\downarrow}$ transforms equivariantly under the right action of $\widehat{\Gamma}$ on P. Ad_a is trivial for elements $a \in U(1) \subset U(V)$, so the latter does not depend on the choice of lift from Γ to $\widehat{\Gamma}$. Thus, $\mathrm{Ad}(P_E)^{\downarrow}$ descends one step further to form $\mathrm{Ad}(P_E)^{\Downarrow}$ over M.

The definition of the curvature 2-form Ω_E together with (4.15) yields

$$\gamma^*\pi_1^*\Omega_E = \Omega_L + \pi_2^*\Omega_E\ , \tag{4.16}$$

where Ω_L denotes the curvature of the bundle gerbe connection on L. But since we only consider flat extensions of Lie groups, Ω_L vanishes, because it is just the pullback of the curvature form on $\widehat{\Gamma}$ over Γ. In explicit terms (4.15) looks like $\Omega_E(\gamma_*[W,X], \gamma_*[V,Y]) = \Omega_E(W,V)$ for $W,V \in T_rP_E$ and $X,Y \in T_aU(V)$. This is just the condition needed to get a 2-form taking values in the bundle $\mathrm{Ad}(P_E)^{\Downarrow}$, i.e. $\Omega_E \in \Omega^2(T^*M, \mathrm{Ad}(P_E)^{\Downarrow})$. The standard representation of $U(V)$ on V enables us to see $\mathrm{Ad}(P_E)^{\downarrow}$ as a subbundle of $\mathrm{End}(E)$, which turns $\mathrm{Ad}(P_E)^{\Downarrow}$ into a subbundle of $\mathrm{end}(E)$ (for the definition see section 3.2), therefore $\Omega_E \in \Omega^2(T^*M, \mathrm{end}(E))$ is a C^*-algebra valued 2-form.

Remark As we have seen in section 4.1.3 the curvature Ω_L of a bundle gerbe connection on L in general satisfies $\delta(\Omega_L) = 0$. Therefore there exists a curving form $f \in \Omega^2(P)$ such that $\Omega_L = \delta(f) = \pi_1^* f - \pi_2^* f$. The choice of f is unique up to 2-forms pulled back from M [49]. If the real DIXMIER-DOUADY-class $dd^{\mathbb{R}}(L) = 0 \in H^3(M, \mathbb{R})$ then f can be chosen to be a closed form. As follows from (4.16) with such a choice of curving f we can push down $\Omega_E - f$ to a 2-form over M taking values in $\Omega^2(T^*M, \mathrm{Ad}(P_E)^{\Downarrow})$.

In the case of bundle gerbe modules the standard trace on matrix algebras extends to a pointwise functional $\mathrm{tr}\colon C(M, \mathrm{end}(E)) \longrightarrow C(M, \mathbb{C})$, which allows us to apply the machinery of CHERN-WEIL-theory.

Lemma 4.1.29. *Let E be a bundle gerbe module over the principal Γ-bundle P with connection η_E having curvature Ω_E. If $T\colon M_n(\mathbb{C}) \to \mathbb{C}$ is an* invariant polynomial *in the sense of [44], then $T(\Omega_E)$ is a closed form in $\Omega^{even}(M, \mathbb{C})$, whose cohomology class does not depend on the choice of connection on E.*

Proof. The proof of closedness given in [44] just uses the expression $d\Omega_E = \Omega_E \wedge \eta_E - \eta_E \wedge \Omega_E$ and properties of the trace and therefore applies to our case as well. The independence of a connection uses the fact that the space of all connections is convex together with a homotopy argument. It is also contained in [44] and applies to the twisted case without any change. $\square$

Definition 4.1.30. Let M be a compact manifold and E a bundle gerbe module over the principal Γ-bundle $P \to M$ with a bgm-connection $\eta_E \in \Omega^1(P_E, \mathfrak{u}(n))$ and curvature $\Omega_E \in \Omega^2(M, \mathrm{end}(E))$, then

$$\mathrm{ch}(E) = \mathrm{tr}\left(\exp\left(\frac{i\Omega_E}{2\pi}\right)\right) \in \Omega^{\mathrm{even}}(M)$$

with $\exp\left(\frac{i\Omega_E}{2\pi}\right) = \sum_{n=0}^{\infty} \left(\frac{i}{2\pi}\right)^n \frac{\Omega_E \wedge \cdots \wedge \Omega_E}{n!}$ is closed by the above lemma. Its class in $H^{\mathrm{even}}(M, \mathbb{R})$ is called the *Chern character* of E. It only depends on the stable isomorphism class of E and therefore extends to a homomorphism:

$$\mathrm{ch}\colon K^0_{\mathcal{K}}(M) \longrightarrow H^{\mathrm{even}}_{\mathrm{dR}}(M, \mathbb{R}) \ .$$

In case M is non-compact, let $[E_+, E_-, \varphi_E] \in K^0_{\mathcal{K}}(M)$ and denote by K a compact set, such that φ_E^{-1} exists on the complement of K. Now, choose a connection ∇_- on E_- and $\widehat{\nabla}$ on E_+, such that it coincides with $\varphi_E^{-1} \circ \nabla_- \circ \varphi_E$ outside a compact subset. This can always be constructed from an arbitrary connection ∇_+ on E_+ in the following way: Let $\rho\colon M \longrightarrow [0,1]$ be a smooth, compactly supported function with $\rho|_K = 1$. Set

$$\widehat{\nabla} = \rho\, \nabla_+ + (1-\rho)\, \varphi_E^{-1} \circ \nabla_- \circ \varphi_E \ .$$

Denote the curvature of $\widehat{\nabla}$ by $\widehat{\Omega}$ and define

$$\mathrm{ch}([E_+, E_-, \varphi_E]) = \mathrm{tr}\left(\exp\left(\frac{i\widehat{\Omega}}{2\pi}\right)\right) - \mathrm{tr}\left(\exp\left(\frac{i\Omega_-}{2\pi}\right)\right) \ ,$$

which is a class in cohomology with compact supports (since $\widehat{\nabla}$ coincides with ∇_- outside a compact subset) and does not depend on the choice of $\widehat{\eta}$ under the above restrictions.

Theorem 4.1.31. *The Chern character has the following properties:*

i) It is additive in the sense that for $[E_1], [E_2] \in K^0_{\mathcal{K}}(M)$ *with the same twisting we have*

$$\mathrm{ch}([E_1 \oplus E_2]) = \mathrm{ch}([E_1]) + \mathrm{ch}([E_2]) \in H^{\mathrm{even}}_{c,\mathrm{dR}}(M, \mathbb{R})$$

ii) If L_1 *and* L_2 *are two lifting bundle gerbes with* $dd(L_1) = dd(L_2)$ *corresponding to the matrix bundles* $\mathcal{K}_1$ *and* $\mathcal{K}_2$*, then a trivialization* F *of* $L_2 \boxtimes L_1^*$ *yields*

$$\begin{tikzcd} K^0_{\mathcal{K}_1}(M) \arrow[r, "\otimes F"] \arrow[d, "\mathrm{ch}"] & K^0_{\mathcal{K}_2}(M) \arrow[d, "\mathrm{ch}"] \\ H^{\mathrm{even}}_{c,\mathrm{dR}}(M, \mathbb{R}) \arrow[r, "\cup \mathrm{ch}(F)"] & H^{\mathrm{even}}_{c,\mathrm{dR}}(M, \mathbb{R}) \end{tikzcd}$$

In particular, it coincides with the ordinary Chern character if there exists a flat *trivialization* F.

iii) It is multiplicative, i.e. if $[E_1] \in K^0_{\mathcal{K}_1}(M)$ *and* $[E_2] \in K^0_{\mathcal{K}_2}(M)$*, then*

$$\mathrm{ch}([E_1 \boxtimes E_2]) = \mathrm{ch}([E_1]) \cup \mathrm{ch}([E_2]) \in H^{\mathrm{even}}_{c,\mathrm{dR}}(M, \mathbb{R})$$

for $[E_1 \boxtimes E_2] \in K^0_{\mathcal{K}_1 \otimes \mathcal{K}_2}$.

iv) It commutes with the Bott map, i.e.

$$\begin{tikzcd} K^0_{\mathcal{K}}(M) \arrow[r, "B"] \arrow[d, "\mathrm{ch}"] & K^0_{\pi_M^* \mathcal{K}}(\mathbb{R}^2 \times M) \arrow[d, "\mathrm{ch}"] \\ H^{\mathrm{even}}_{c,\mathrm{dR}}(M, \mathbb{R}) \arrow[r, "\times(-e_2)"] & H^{\mathrm{even}}_{c,\mathrm{dR}}(\mathbb{R}^2 \times M, \mathbb{R}) \end{tikzcd}$$

where $e_2 \in H^2_c(\mathbb{R}^2) \simeq \mathbb{Z}$ *denotes the generator, i.e. the cohomology class of a differential form that integrates to* 1.

Proof. The first statement is trivial since all constructions involved are well-behaved with respect to direct sums: For the connection $\eta_{E_1} \oplus \eta_{E_2}$ we have $\Omega_{E_1 \oplus E_2} = \Omega_{E_1} \oplus \Omega_{E_2}$. Thus, exp splits into a direct sum of two exponentials, which is turned into a sum of forms by the trace.

To prove iii) we first see that we can regard $\mathrm{end}(E_1) \otimes \mathrm{end}(E_2)$ as a subbundle of $\mathrm{end}(E_1 \boxtimes E_2)$, since both are associated to $P_{E_1} \otimes P_{E_2}$, the latter via an embedding $\tau \colon U(n_1) \otimes U(n_2) \to U(n_1 n_2)$. Two connections η_{E_i} on E_i induce a canonical one on the tensor product by (4.14) denoted by $\eta_{E_1 \boxtimes E_2}$. From (4.14) we deduce that $\Omega_{E_1 \boxtimes E_2} = \Omega_{E_1} \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \Omega_{E_2}$ using the above identification. Since the summands commute, we have:

$$\begin{aligned} \exp\left(\frac{i\Omega_{E_1 \boxtimes E_2}}{2\pi}\right) &= \exp\left(\frac{i(\Omega_{E_1} \otimes \mathrm{id}_{E_2})}{2\pi}\right) \wedge \exp\left(\frac{i(\mathrm{id}_{E_1} \otimes \Omega_{E_2})}{2\pi}\right) \\ &= \left[\exp\left(\frac{i\Omega_{E_1}}{2\pi}\right) \otimes \mathrm{id}_{E_2}\right] \wedge \left[\mathrm{id}_{E_1} \otimes \exp\left(\frac{i\Omega_{E_2}}{2\pi}\right)\right] \\ &= \exp\left(\frac{i\Omega_{E_1}}{2\pi}\right) \wedge \exp\left(\frac{i\Omega_{E_2}}{2\pi}\right) , \end{aligned}$$

where in the last expression the wedge product is defined by

$$\begin{aligned} & (\mu_p \wedge \nu_q)(X_1, \dots X_{p+q}) \\ = \; & \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mu_p(X_{\sigma(1)}, \dots X_{\sigma(p)}) \otimes \nu_q(X_{\sigma(p+1)}, \dots X_{\sigma(p+q)}) \end{aligned} \tag{4.17}$$

for the p-, respectively q-degree part of $\exp\left(\frac{i\Omega_{E_i}}{2\pi}\right)$. Applying the trace on $M_{n_1}(\mathbb{C}) \otimes M_{n_2}(\mathbb{C})$ to the line above turns the tensor product in (4.17) into a product of traces and thus yields the wedge product of the corresponding forms.

In case M is non-compact, note that $[E_+, E_-, \varphi_E] = [F, \underline{\mathbb{C}^n}, \alpha] \in K^0_{\mathcal{K}}(M)$ for a bundle gerbe module F, some $n \in \mathbb{N}$ and twisted morphism α. Using the convexity of the space of connections we can find ∇ on F such that it coincides with the pullback of the flat connection d outside a compact subset. The tensor product of two classes $[F_i, \underline{\mathbb{C}^{n_i}}, \alpha_i] \in K^0_{\mathcal{K}}(M)$ yields

$$[F_1 \otimes F_2 \oplus \underline{\mathbb{C}^{n_1 n_2}}, F_1 \otimes \underline{\mathbb{C}^{n_2}} \oplus \underline{\mathbb{C}^{n_1}} \otimes F_2, \widehat{\alpha}]$$

But now the connection $\nabla_{F_1 \otimes F_2} \oplus d$ agrees with the pullback of $\nabla_{F_1 \otimes \underline{\mathbb{C}^{n_2}}} \oplus \nabla_{\underline{\mathbb{C}^{n_2}} \otimes F_1}$ via $\widehat{\alpha}$ outside a compact subset. Thus, using them to compute the Chern character we arrive at:

$$\begin{aligned} \mathrm{ch}([F_1, \underline{\mathbb{C}^{n_1}}] \otimes [F_2, \underline{\mathbb{C}^{n_2}}]) &= \mathrm{ch}(F_1)\mathrm{ch}(F_2) + n_1\, n_2 - \mathrm{ch}(F_1) n_2 - \mathrm{ch}(F_2) n_1 \\ &= (\mathrm{ch}(F_1) - n_1)\,(\mathrm{ch}(F_2) - n_2) \\ &= \mathrm{ch}([F_1, \underline{\mathbb{C}^{n_1}}]) \cup \mathrm{ch}([F_2, \underline{\mathbb{C}^{n_2}}]) \;. \end{aligned}$$

For the proof of ii) remember that F being a trivialization implies that there is an isomorphism

$$\mathrm{pr}_2^* L_2 \otimes \mathrm{pr}_1^* L_1^* \otimes \pi_2^* F \longrightarrow \pi_1^* F \;, \tag{4.18}$$

where pr_i denotes the projection $P_1^{[2]} \times_M P_2^{[2]} \to P_i^{[2]}$ and π_i the ith projection onto one of the two factors $P_1 \times_M P_2$. Let $\pi_k^{(1)} \colon P_1^{[2]} \times_M P_2 \to P_1 \times_M P_2$ be the projection to the kth factor and $\Delta^{(j)} \colon P_j \to P_j^{[2]}$ be the diagonal inclusion, then

$$\begin{aligned} \pi_k \circ \left(\mathrm{id}_{P_1^{[2]}} \times_M \Delta^{(2)}\right) &= \pi_k^{(1)} \;, \\ \mathrm{pr}_1 \circ \left(\mathrm{id}_{P_1^{[2]}} \times_M \Delta^{(2)}\right) &= \pi_{P_1^{[2]}} \;, \\ \mathrm{pr}_2 \circ \left(\mathrm{id}_{P_1^{[2]}} \times_M \Delta^{(2)}\right) &= \Delta^{(2)} \circ \pi_{P_2} \;. \end{aligned}$$

Moreover $\Delta^{(2)*} L_2$ can be canonically identified with the trivial line bundle. Therefore (4.18) turns F into a bundle gerbe module for $\widetilde{L}_1 = \pi_{P_1^{[2]}}^* L_1^*$ using the pullback via $(\mathrm{id}_{P_1^{[2]}} \times_M \Delta^{(2)})$. The twisting is given by

$$\gamma_F \colon \widetilde{L}_1^* \otimes \pi_2^{(1)*} F \longrightarrow \pi_{P_2}^* \Delta^{(2)*} L_2 \otimes \pi_{P_1^{[2]}}^* L_1^* \otimes \pi_2^{(1)*} F \longrightarrow \pi_1^{(1)*} F$$

and the associativity follows from the fact that the isomorphism (4.18) is an isomorphism of bundle gerbes when seen as a map $\mathrm{pr}_2^* L_2 \otimes \mathrm{pr}_1^* L_1^* \to \pi_1^* F \otimes \otimes \pi_2^* F^*$.

Let $\rho_i\colon P_1 \times_M P_2 \to P_i$ be the canonical maps and consider $\rho_1^* E \otimes F$. This is a Γ_1-equivariant vector bundle over the principal Γ_1-bundle $P_1 \times_M P_2 \to P_2$ via the isomorphism

$$\delta\colon \pi_2^{(1)*}(\rho_1^* E \otimes F) \to \left(\widetilde{L}_1 \otimes \pi_2^{(1)*}\rho_1^* E\right) \otimes \left(\widetilde{L}_1^* \otimes \pi_2^{(1)*} F\right) \to \pi_1^{(1)*}(\rho_1^* E \otimes F)$$

where the first map inserts the trivial line bundle $\widetilde{L}_1 \otimes \widetilde{L}_1^*$ via the canonical isomorphism and the second is induced by the twisting of E and the above argument. The transfer of $[E] \in K^0_{\mathcal{K}_1}(M)$ to $K^0_{\mathcal{K}_2}(M)$ is the descent of the above vector bundle to P_2. Let θ_j be a bundle gerbe connection on L_j and let L_1^* be equipped with the connection $-\theta_1$. These induce a tensor product connection on $\mathrm{pr}_2^* L_2 \otimes \mathrm{pr}_1^* L_1^*$, which is again a bundle gerbe connection. Note that (4.18) turns F into a bundle gerbe module for this tensor product. Choose a bundle gerbe module connection η_F on F for $\mathrm{pr}_2^* L_2 \otimes \mathrm{pr}_1^* L_1^*$. In particular, it is a bgm-connection for the action γ_F of $\widetilde{L}_1$, since the pullback of θ_2 with respect to the trivialization $\underline{\mathbb{C}} \to \Delta^{(2)*} L_2$ is the canonical flat connection (see remark 4.1.20). In the notation introduced in (4.9) condition (4.5) reads

$$\eta_F\left(\gamma_{F*}\left[W, \widehat{X}_1\right]\right) = \eta_F(W) - \theta_1(\widehat{X}_1)$$

for $W \in T_r P_F$ and $\widehat{X}_1 \in T_{\widehat{g}_1}\widehat{\Gamma}_1$. If η_E is a bundle gerbe module connection on E for the action of L_1, then $\rho_1^*\eta_E + \eta_F$ satisfies

$$\pi_2^{(1)*}(\rho_1^*\eta_E + \eta_F) = \hat{\delta}^* \circ \pi_1^{(1)*}(\rho_1^*\eta_E + \eta_F) \circ \hat{\delta}^{-1*}$$

since the θ_1-terms cancel (here $\hat{\delta}$ denotes the morphism of principal bundles induced by the vector bundle isomorphism δ). Thus, $\rho_1^*\eta_E + \eta_F$ descends to a well-defined connection on the transferred bundle.

Denote by $\pi_k^{(2)}\colon P_1 \times_M P_2^{[2]} \to P_1 \times_M P_2$ the projection to the kth factor and let $\widetilde{L}_2 = \pi^*_{P_2^{[2]}} L_2$ be the pullback of L_2 to $P_1 \times P_2^{[2]}$. A similar reasoning as the one given above yields another twisting on F

$$\delta_F\colon \widetilde{L}_2 \otimes \pi_2^{(2)*} F \longrightarrow \pi_1^{(2)*} F$$

turning it into a bundle gerbe module for $\widetilde{L}_2$. Since $\rho_1 \circ \pi_1^{(2)} = \rho_1 \circ \pi_2^{(2)}$ we have that $\pi_1^{(2)*}\rho_1^* E$ is canonically isomorphic to $\pi_2^{(2)*}\rho_1^* E$, therefore $\rho_1^* E \otimes F$ is a bundle gerbe module for $\widetilde{L}_2$ via the isomorphism induced by δ_F

$$\widetilde{L}_2 \otimes \pi_2^{(2)*}(\rho_1^* E \otimes F) \longrightarrow \pi_1^{(2)*}(\rho_1^* E \otimes F)\ .$$

Since η_F was chosen to be a bundle gerbe module connection for the tensor product $\mathrm{pr}_2^* L_2 \otimes \mathrm{pr}_1^* L_1^*$ we have:

$$\eta_F\left(\delta_{F*}\left[W, \widehat{X}_2\right]\right) = \eta_F(W) + \theta_2(\widehat{X}_2)$$

for $W \in T_r P_F$ and $\widehat{X}_2 \in T_{\widehat{g}_2}\widehat{\Gamma}_2$. This shows that $\rho_1^*\eta_E + \eta_F$ is actually a bgm-connection. Since its curvature is $\Omega_E + \Omega_F \in \Omega^2(M, \mathrm{Ad}(P_{\rho_1^* E \otimes F})_{\Downarrow}) \subset \Omega^2(M, \mathrm{end}(\rho_1^* E \otimes F))$, the claim follows from the multiplicativity proven in iii). It is applicable to the non-compact case without any change, since there we tensor both bundles in the triple with F, which implies that if ∇_{E_+} agrees with the pullback of ∇_{E_-} outside some compact set, then the same holds for $\nabla_{\rho_1^* E_+ \otimes F}$ and $\nabla_{\rho_1^* E_- \otimes F}$.

Recall that the Bott map sends $[E]$ to the triple $[\pi_M^* E, \pi_M^* E, b]$, i.e. it pulls E back to a bundle over $\mathbb{R}^2 \times M$ and multiplies with the Bott element $[\underline{\mathbb{C}}, \underline{\mathbb{C}}, b] \in K^0(\mathbb{R}^2 \times M)$, where b is the map of fiberwise multiplication by $(x+iy)$ for $(x,y) \in \mathbb{R}^2$. Since the Chern character is natural with respect to pullbacks and due to its multiplicativity, we only need to check that $\mathrm{ch}([\underline{\mathbb{C}}, \underline{\mathbb{C}}, b]) = -e_2 \in H_c^2(\mathbb{R}^2 \times M)$. We evaluate $\widehat{\nabla}$ from above for the canonical flat connection d on $\underline{\mathbb{C}}$:

$$\widehat{\nabla} = d + (1-\rho) g^{-1} dg \ ,$$

where g is the function $g(x,y,m) = x + iy =: z$. Its curvature is $\Omega_z = d((1-\rho)g^{-1}dg)$, which is a 2-form with vanishing higher powers, therefore

$$\exp\left(\frac{i\Omega_z}{2\pi}\right) = 1 + \frac{i\Omega_z}{2\pi}$$

Using STOKES' theorem on circles with growing radius in $\mathbb{R}^2$ one can check that $\left[\frac{i\Omega_z}{2\pi}\right] = -e_2$. Thus:

$$\mathrm{ch}([\underline{\mathbb{C}}, \underline{\mathbb{C}}, b]) = (1 - e_2) - 1 = -e_2 \ . \qquad \square$$

In a similar fashion we can use CHERN-WEIL-theory to define the *real* Chern classes c_k by

$$\det\left(1 + \frac{it\Omega_E}{2\pi}\right) = \sum_k c_k(E) t^k \in H_{\mathrm{dR}}^{\mathrm{even}}(M, \mathbb{R}) \ . \tag{4.19}$$

As in the untwisted case, the c_k are natural characteristic classes.

Using the Künneth theorem we can easily extend the Chern character to a transformation that takes values in $H_c^{\mathrm{even}}(M, K_0(A) \otimes \mathbb{R})$.

Definition 4.1.32. Let $\mathcal{A}$ be bundle of C^*-algebras associated to a principal Γ-bundle P. Let $\mathcal{K}$ be a bundle of matrix algebras associated to the principal $PU(n)$-bundle $\widetilde{P}$ such that $dd(\mathcal{A}) = dd(\mathcal{K})$. Let Q be a choice of trivialization like in theorem 3.4.1 as explained in remark 3.4.7 and consider

$$\mathrm{ch}_Q : K_{\mathcal{A}}^0(M) \stackrel{\otimes Q}{\longrightarrow} K_{\mathcal{K}}^0(M) \otimes K_0(A) \otimes \mathbb{R} \longrightarrow H^{\mathrm{even}}(M, K_0(A) \otimes \mathbb{R}) \ ,$$

where the first map is the projection to the first Künneth summand and the second is just ch on $K_{\mathcal{K}}^0(M)$. This is called the $K_0(A)$*-valued Chern character.*

Remark 4.1.33. As defined above the $K_0(A)$-valued Chern character *depends* on the particular choice of the bundle of matrix algebras $\mathcal{K}$ and the trivialization Q involved. In case $dd(\mathcal{A}) = 0$ at least $\mathcal{K}$ can be chosen in a canonical way, i.e. $\mathcal{K} = M \times \mathbb{C}$. Note that the following diagram commutes:

$$\begin{tikzcd}
K_{\mathcal{A}}^0(M) \arrow[r, "\otimes Q"] \arrow[d, "\otimes Q"] & K_A^0(M) \arrow[r, "\mathrm{ch}"] & H^{\mathrm{even}}(M, K_0(A) \otimes \mathbb{R}) \arrow[d, "\mathrm{id}"] \\
K_A^0(M) \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{ch}"] & & H^{\mathrm{even}}(M, K_0(A) \otimes \mathbb{R})
\end{tikzcd}$$

If we denote the descend of $E \otimes Q$ by $\pi_!$, then $\mathrm{ch}(\pi_!(E \otimes Q)) = \mathrm{ch}_Q(E)$. If $\mathcal{A}$ is itself a bundle of matrix algebras with $dd(\mathcal{A}) = 0$, then by part ii) of theorem 4.1.31 we have $\mathrm{ch}_Q(E) = \mathrm{ch}(Q) \cup \mathrm{ch}(E) = \exp(c_1(Q)) \cup \mathrm{ch}(E)$.

Theorem 4.1.34. *The Chern character has the following properties:*

i) It is additive.

ii) It is well-behaved with respect to change of twisting in the following sense (compare with theorem 4.1.31 ii). Let $\mathcal{A}_j$ for $j \in \{1, 2\}$ be two C^*-algebra bundles with typical fiber A. Denote the corresponding lifting bundle gerbes by L_j. Let $\mathcal{K}$ be a bundle of matrix algebras with lifting bundle gerbe $\widetilde{L}$, such that $dd(\widetilde{L}) = dd(L_1) = dd(L_2)$. Choose trivializations F of $L_2 \boxtimes L_1^*$ and Q of $\widetilde{L} \boxtimes L_1^*$. F and Q induce a trivialization $Q \boxtimes F^*$ of $\widetilde{L} \boxtimes L_2^*$. Then the following diagram commutes:

$$\begin{tikzcd}
K^0_{\mathcal{A}_1}(M) \arrow[rr, "\otimes F"] \arrow[d, "\mathrm{ch}_Q"] & & K^0_{\mathcal{A}_2}(M) \arrow[d, "\mathrm{ch}_{Q \boxtimes F^*}"] \\
H^{\mathrm{even}}_{c,\mathrm{dR}}(M, K_0(A) \otimes \mathbb{R}) \arrow[rr, "\cup \mathrm{ch}(F)"] & & H^{\mathrm{even}}_{c,\mathrm{dR}}(M, K_0(A) \otimes \mathbb{R})
\end{tikzcd}$$

iii) It is multiplicative with respect to the twisted product, i.e. if $[E] \in K^0_{\mathcal{A}}(M)$ and $[F] \in K^0_{\mathcal{K}}(M)$, then

$$\mathrm{ch}_Q([E \boxtimes F]) = \mathrm{ch}_Q([E]) \cup \mathrm{ch}([F]) \in H^{\mathrm{even}}_{c,\mathrm{dR}}(M, K_0(A) \otimes \mathbb{R})$$

for $[E \boxtimes F] \in K^0_{\mathcal{A} \otimes \mathcal{K}}(M)$, where Q induces the Morita *equivalence between $\mathcal{A}$ and $\mathcal{K}' \otimes A$ for some bundle of matrix algebras $\mathcal{K}'$ with $dd(\mathcal{A}) = dd(\mathcal{K}')$ and therefore also a* Morita *equivalence between $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{K}' \otimes A \otimes \mathcal{K}$.*

iv) It commutes with the Bott map like in theorem 4.1.31 iv).

Proof. All properties follow directly from theorem 4.1.31, since the Künneth decomposition commutes with forming twisted products. □

If the algebra A is unital and comes equipped with a continuous trace τ, then the latter induces a trace on the adjointable endomorphisms $\mathrm{End}(V)$ of a finitely generated Hilbert A-module V, because we have:

$$\mathrm{End}(V) = \mathcal{K}(V) = V \otimes_A \mathcal{K}(V, A) \ ,$$

where $\mathcal{K}$ denotes the compact adjointable operators and we use the tensor product of Hilbert A-modules. The left action of A on $\mathcal{K}(V, A)$ maps T to aT. On elementary tensors we define $\tau_V(v \otimes T) = \tau_V(T(v))$, which is easily seen to have the trace property.

Let E be a twisted Hilbert A-module bundle over P. Since $\mathrm{end}(E)$ is associated to P via the adjoint action of Γ on $\mathrm{End}(V)$, the trace extends to a map on $\mathrm{end}(E)$-valued forms. By a slight abuse of notation we will also denote this map by τ. Now the curvature Ω_E of a twisted connection η_E on E is a 2-form taking values in $\mathrm{end}(E)$. So, having the trace at hand, there is a more direct approach to the Chern character.

First note that a twisted connection η_E on E with covariant derivative ∇_E induces a connection ∇^* on the bundle $\mathrm{Hom}(E, \underline{A})$, such that $d(\varphi(u)) = (\nabla^*\varphi)\, u + \varphi(\nabla_E u)$ for $\varphi \in C^\infty(P, \mathrm{Hom}(E, \underline{A}))$ and $u \in C^\infty(P, E)$. Furthermore, it induces a connection ∇ on the bundle $\mathrm{end}(E)$ in such a way that if a section of $\mathrm{end}(E)$ is interpreted as one of $\mathrm{End}(E)$ over P we have $\nabla_E(\psi(u)) = \nabla(\psi)u + \psi\nabla_E(u)$ for $\psi \in C^\infty(M, \mathrm{end}(E))$. The latter can be extended to forms by demanding the graded Leibniz rule as usual. The isomorphism $E \otimes_A \mathrm{Hom}(E, \underline{A}) \simeq \mathrm{End}(E)$ pulls back ∇ to the connection $\nabla_E \otimes 1 + 1 \otimes \nabla^*$. This is used in the following lemma.

Lemma 4.1.35. *If τ is the trace extended to* $\mathrm{end}(E)$*-valued forms over M, then*

$$d\tau(\omega) = \tau(\nabla\omega)$$

for an arbitrary connection ∇ on $\mathrm{end}(E)$ *induced by a twisted connection ∇_E on E.*

Proof. We can identify $\Omega^k(M, \mathrm{end}(E))$ with horizontal k-forms $\Omega^k_{\mathrm{hor}}(P, \mathrm{End}(E))$ and restrict to elements $\omega \in \Omega^k_{\mathrm{hor}}(P, \mathrm{End}(E))$, which are of the form $\omega = \alpha\, u \otimes \psi$ for $\alpha \in \Omega^k_{\mathrm{hor}}(P), u \in C^\infty(P, E), \psi \in C^\infty(P, \mathrm{Hom}(E, \underline{A}))$ by a partition of unity argument. Now

$$\nabla(\omega) = d\alpha\, u \otimes \psi + (-1)^{\deg(\alpha)}\alpha\,(\nabla_E u \otimes \psi + u \otimes \nabla^*\psi)\ .$$

Taking the trace yields:

$$\begin{aligned}\tau(\nabla(\omega)) &= \tau(d\alpha\, \psi(u) + (-1)^{\deg(\alpha)}\alpha\,(\psi(\nabla_E u) + \nabla^*\psi(u))) \\ &= \tau(d\alpha\, \psi(u) + (-1)^{\deg(\alpha)}\alpha\, d(\psi(u))) = d(\tau(\alpha\, \psi(u)))\ .\end{aligned}$$ □

As a corollary of the above lemma we immediately get:

Lemma 4.1.36. *Let E be a (finitely generated, projective) twisted Hilbert A-module bundle over the principal Γ-bundle P with connection η_E having curvature Ω_E. Let V be the typical fiber of E. If $T\colon \mathrm{End}(V) \to \mathbb{C}$ is an* invariant polynomial *in the sense that it is the trace of a polynomial in* $\mathrm{End}(V)$ *and that $T(USU^*) = T(S)$ for $U \in U(V)$, then $T(\Omega_E)$ is a closed form in $\Omega^{even}(M, \mathbb{C})$, whose cohomology class does not depend on the choice of connection on E.*

Proof. Because closedness is a local property we can restrict to sections supported in a region where the bundle is trivializable. But here we can use the flat connection, together with the last lemma to apply the trick from lemma 4.1.29. The independence of the chosen connection again follows from convexity of the space of connections. □

Definition 4.1.37. Let M be a compact manifold and E be a twisted Hilbert A-module bundle over a principal Γ-bundle $P \to M$ with a twisted connection $\eta_E \in \Omega^1(P_E, i\mathfrak{a})$ and curvature $\Omega_E \in \Omega^2(M, \mathrm{end}(E))$, then

$$\mathrm{ch}_\tau(E) = \tau\left(\exp\left(\frac{i\Omega_E}{2\pi}\right)\right) \in \Omega^{\mathrm{even}}(M)$$

is a closed form. Its class in cohomology is called the *τ-Chern character* of E. Like in definition 4.1.30 there is also a *τ-Chern character* on non-compact manifolds.

Remember that there are two descriptions of the group $K^0_{\mathcal{K}}(M)$, one by virtual bundle gerbe modules, one by twisted Hilbert $M_n(\mathbb{C})$-bundles, which are connected by a MORITA-equivalence that maps the endomorphism spaces $\mathrm{End}_{\mathbb{C}}(V)$ and $\mathrm{End}_{M_n(\mathbb{C})}(V \otimes \mathbb{C}^{n*})$ onto each other. The standard, *non*-normalized trace on $\mathrm{End}_{\mathbb{C}}(V)$ for a vector space V corresponds to the one on $\mathrm{End}_{M_n(\mathbb{C})}(W)$ for a Hilbert $M_n(\mathbb{C})$-module W induced by the *normalized* trace τ_0 on $M_n(\mathbb{C})$. In this way the above construction is consistent with the Chern character we started with, i.e. ch corresponds to ch_{τ_0}.

The trace τ on A extends to a functional $K_0(A) \longrightarrow \mathbb{C}$ generalizing the dimension function for $A = \mathbb{C}$, therefore:

Definition 4.1.38. The *dimension* of a finitely generated, projective Hilbert A-module V is defined to be $\dim_\tau(V) = \tau_V(\mathrm{id}_V)$, where τ_V is the extension of the trace on A to $\mathrm{End}(V)$. This map is well-defined on $K_0(A)$ and coincides with $(\mathrm{tr} \otimes \tau)(p)$ for $p \in M_n(A) = M_n(\mathbb{C}) \otimes A$, where $p\,A^n \simeq V$.

Theorem 4.1.39. *Let E be a twisted Hilbert A-module bundle. Then both Chern characters are related by:*

$$\dim_\tau(\mathrm{ch}_Q(E)) = \mathrm{ch}_\tau(E) \cup \mathrm{ch}(Q) \in H^{\mathrm{even}}(M, \mathbb{R}) \ ,$$

which also implies that ch_τ inherits all properties from ch.

Proof. We first need to prove that ch_τ is multiplicative with respect to the twisted product $K^0_{\mathcal{A}}(M) \times K^0_{\mathcal{K}}(M) \to K^0_{\mathcal{A}\otimes\mathcal{K}}(M)$. First note that for the tensor product of a Hilbert A-module V and a Hilbert $M_n(\mathbb{C})$-module $W \otimes \mathbb{C}^{n*}$ where $W \simeq \mathbb{C}^m$ is a vector space of finite dimension we have

$$\mathrm{End}_{A\otimes M_n(\mathbb{C})}(V \otimes W \otimes \mathbb{C}^{n*}) = \mathrm{End}_A(V) \otimes M_m(\mathbb{C})$$

with the trace induced by $\tau \otimes \tau_0$ being $\tau_V \otimes \mathrm{tr}$.

Using the same argument given in theorem 4.1.31, we see that for a twisted Hilbert A-module bundle E and a bundle gerbe module F corresponding to the twisted Hilbert $M_n(\mathbb{C})$-bundle $F \otimes \underline{\mathbb{C}^{n*}}$ we have

$$\mathrm{ch}_{\tau\otimes\tau_0}(E \boxtimes (F \otimes \underline{\mathbb{C}^{n*}})) = \mathrm{ch}_\tau(E) \cup \mathrm{ch}_{\tau_0}(F \otimes \underline{\mathbb{C}^{n*}}) = \mathrm{ch}_\tau(E) \cup \mathrm{ch}(F) \ .$$

Since the Chern character transforms naturally with respect to pullbacks, this immediately implies that $\mathrm{ch}_{\tau\otimes\tau_0}$ is well-behaved with respect to Bott periodicity in the sense of the diagram in theorem 4.1.31 iii).

Next we consider the case, where $\mathcal{A} = \mathcal{K} \otimes A$ for some bundle of matrix algebras $\mathcal{K}$ and the trivialization $Q = \underline{\mathbb{C}}$ is just given by the trivial line bundle. The trace on $\mathcal{K} \otimes A$ is $\tau_0 \otimes \tau$. Let $[F] \in K^0_{\mathcal{K}}(M)$ be represented by a bundle gerbe module F and represent the class $[V] \in K_0(A)$ by a finitely generated projective Hilbert A-module V. We have $\mathrm{ch}_{\underline{\mathbb{C}}}([F] \otimes [V]) = \mathrm{ch}(F) \otimes [V] \in H^{\mathrm{even}}(M, K_0(A) \otimes \mathbb{R}) = H^{\mathrm{even}}(M, \mathbb{R}) \otimes K_0(A) \otimes \mathbb{R}$, but on the other hand

$$\mathrm{ch}_{\tau_0\otimes\tau}((F \otimes \mathbb{C}^*) \boxtimes \underline{V}) = \mathrm{ch}(F) \cup \mathrm{ch}_\tau(\underline{V}) \ ,$$

where we are exploiting the multiplicativity again. Now $\underline{V}$ comes equipped with a canonical flat connection implying $\mathrm{ch}_\tau(\underline{V}) = \tau(\mathrm{id}_V) = \dim_\tau(V)$. To finish this case we therefore just have to check that ch_τ vanishes on the image of the second

Künneth summand $K^0_{\mathcal{K}}(\mathbb{R}\times M)\otimes K^0_A(\mathbb{R})$ in $K^0_{\mathcal{A}}(M)$. So let $[F]\in K^0_{\mathcal{K}}(\mathbb{R}\times M)$, $[W]\in K^0_A(\mathbb{R})$, then

$$\mathrm{ch}_\tau([F]\otimes[W])\cup(-e_2)=\mathrm{ch}_\tau(F\otimes W)=\mathrm{ch}(F)\cup\mathrm{ch}_\tau(W)\ ,$$

but since $\mathrm{ch}_\tau(W)\in H^{\mathrm{even}}_c(\mathbb{R})=0$ and the Bott map is an isomorphism, we have $\mathrm{ch}_\tau([F]\otimes[W])=0$.

In case of an arbitrary bundle $\mathcal{A}$, the Chern character is defined using a transfer to $\mathcal{K}\otimes A$ via a trivialization Q, where $\mathcal{K}$ has the same Dixmier-Douady-class as $\mathcal{A}$. The argument that lead to the proof of 4.1.31 ii) also shows that the following diagram commutes

$$\begin{tikzcd} K^0_{\mathcal{A}}(M) \arrow[r, "\otimes Q"] \arrow[d, "\mathrm{ch}_\tau"] & K^0_{\mathcal{K}\otimes A}(M) \arrow[d, "\mathrm{ch}_{\tau_0\otimes\tau}"] \\ H^{\mathrm{even}}_{\mathrm{dR}}(M,\mathbb{R}) \arrow[r, "\cup\mathrm{ch}(Q)"] & H^{\mathrm{even}}_{\mathrm{dR}}(M,\mathbb{R}) \end{tikzcd}$$

Thus, if we denote the pushdown of E along Q by $\pi_!(E\otimes Q)$

$$\begin{aligned}\mathrm{ch}_\tau(E)\cup\mathrm{ch}(Q)&=\mathrm{ch}_{\tau_0\otimes\tau}(\pi_!(E\otimes Q))\\&=\dim_\tau(\mathrm{ch}_{\underline{\mathbb{C}}}(\pi_!(E\otimes Q)))=\dim_\tau(\mathrm{ch}_Q(E))\ .\end{aligned}\qquad\square$$

4.2 Index theory on twisted Hilbert A-module bundles

The main disadvantage of twisted Hilbert A-module bundles when focussing on twisted K-homology is the fact that they don't possess well-defined sections. Of course $E \to P$ has sections as a bundle over P, but we would like to consider E as an object above the base manifold M, i.e. sections being equivariant maps $P \to E$. Since the action is twisted by the bundle gerbe L this makes no sense. Rather one would like to have a substitute for E living above M, but still carrying an action of the endomorphism bundle $\mathrm{end}(E)$.

To accomplish this, one idea would be just to tensor E with some trivial twisted Hilbert A-module bundle of opposite twisting and then pushing it down. As we will see soon the advantage of taking trivial bundles as a countertwisting is that sections in the resulting bundle over M take a very nice form.

All throughout this section we will assume Γ to be a *Lie group admitting a bi-invariant metric* and the Γ-bundle P to be a *smooth manifold* in the *finite-dimensional* sense equipped with a smooth action of Γ. If one demands connectedness of Γ, then the existence of a bi-invariant metric already implies that Γ is the product of a compact and an abelian group (see [43]). Since the connected component of the identity always is a normal subgroup in Γ, this restriction implies that Γ is an extension of the form:

$$1 \longrightarrow \Gamma_c \times \Gamma_a \longrightarrow \Gamma \longrightarrow \Gamma_d \longrightarrow 1$$

for a discrete group Γ_d, an abelian Lie group Γ_a and a compact Lie group Γ_c. This includes of course the cases where Γ is itself compact, abelian or discrete.

4.2.1 Sobolev spaces

We follow the lines of Mishchenko and Fomenko [47] in defining Sobolev spaces for A- and V-valued functions. Before considering the situation on manifolds, we will focus on the local case. Therefore let $U \subset \mathbb{R}^n$ be a bounded open subset, let $P = U \times \Gamma$ be the trivial Γ-bundle over U. Let $C_c^\infty(U \times \Gamma, A)$ be the algebra of smooth functions with compact support in $U \times \Gamma$ and values in the C^*-algebra A. Choose an orthonormal basis $X_i \in \mathrm{Lie}(\Gamma)$ (the Lie algebra of Γ) with respect to the invariant metric on $\mathrm{Lie}(\Gamma)$. It acts on $C_c^\infty(U \times \Gamma, A)$ via

$$(X_i \cdot f)(x, g) = \left.\frac{d}{dt}\right|_{t=0} f(x, g \exp tX_i)$$

Now $\Delta_\Gamma = -\sum_i X_i^2$ is the (non-negative) Laplace operator (or Casimir operator) of the group and does not depend on the choice of orthonormal basis. Combining it with $\Delta_{\mathbb{R}^n} = -\sum_i \frac{\partial^2}{\partial x_i^2}$ we set $\Delta = \Delta_\Gamma + \Delta_{\mathbb{R}^n}$ and note that $1 + \Delta$ is a *positive definite* operator on $C_c^\infty(U, A)$. Note that in the discrete case we can simply forget about the Lie algebra and set $\Delta = \Delta_{\mathbb{R}^n}$.

Definition 4.2.1. Let $s \in \mathbb{N}$ and $H_0^s(U \times \Gamma, A)$ be the completion of $C_c^\infty(U \times \Gamma, A)$ with respect to the norm

$$||f||_s^2 = \left|\left|\int_{U \times \Gamma} f(x, g)^* \left((1 + \Delta)^s f(x, g)\right) dx\, dg\right|\right|$$

This will be called the SOBOLEV s-norm and $H_0^s(U \times \Gamma, A)$ is a SOBOLEV space. We set $L_0^2(U \times \Gamma, A) = H_0^0(U \times \Gamma, A)$ for those are the square integrable A-valued functions with respect to the A-valued scalar product of A considered as a Hilbert A-module over itself. Replacing $f(x,g)^*(1+\Delta)^s f(x,g)$ by the A-valued scalar product in a Hilbert A-module V, i.e. by $\langle f(x,g), (1+\Delta)^s f(x,g)\rangle$ we define $H_0^s(U \times \Gamma, V)$ analogously.

Lemma 4.2.2. *If $\kappa : U \times \Gamma \longrightarrow U \times \Gamma$ is a diffeomorphism of the form $\kappa(x,g) = (h(x), \tau(x)\cdot g)$ for some diffeomorphism $h : U \longrightarrow U$ and a smooth map $\tau : U \longrightarrow \Gamma$ such that all partial derivatives of h and τ are bounded (in the latter case with respect to the metric on* $\mathrm{Lie}(\Gamma)$*), then*

$$f \in H^s(U \times \Gamma, V) \quad \Leftrightarrow \quad f \circ \kappa \in H^s(U \times \Gamma, V)\ .$$

Proof. The proof is based on the fact that the SOBOLEV norm is equivalent to the one defined by

$$||f||_s^\partial = \sum_{|\alpha|+|\beta|\leq s} \left|\left| \frac{\partial^\alpha}{\partial x^\alpha} X_\beta\, f \right|\right|_{L^2}\ .$$

Derivates with respect to x_i produce the vector field

$$d\kappa\left(\frac{\partial}{\partial x_i}\right) = \sum_j \left(\frac{\partial h^j}{\partial x_i}\frac{\partial}{\partial x_j} + \mu_\Gamma\left(R_{g*}\frac{\partial \tau}{\partial x_i}\right)\right) = \sum_j \left(\frac{\partial h^j}{\partial x_i}\frac{\partial}{\partial x_j} + \alpha_i^j(x,g) X_j\right)$$

where $\alpha_i^j(x,g)$ is chosen such that $\sum_j \alpha_i^j X_j = \mu_\Gamma\Big(R_{g*}\frac{\partial\tau}{\partial x_i}\Big)$. Now note that

$$\sum_j (\alpha_i^j(x,g))^2 = \left\langle \mu_\Gamma\left(R_{g*}\frac{\partial\tau}{\partial x_i}\right), \mu_\Gamma\left(R_{g*}\frac{\partial\tau}{\partial x_i}\right)\right\rangle_{\mathrm{Lie}(\Gamma)} = \left\langle \frac{\partial\tau}{\partial x_i}(x), \frac{\partial\tau}{\partial x_i}(x)\right\rangle_{\mathrm{Lie}(\Gamma)}$$

is independent of g due to the bi-invariance and bounded by hypothesis. Thus, all derivatives of $\alpha_i^j = \Big\langle X_j, \mu_\Gamma\Big(R_{g*}\frac{\partial\tau}{\partial x_i}\Big)\Big\rangle_{\mathrm{Lie}(\Gamma)}$ with respect to x_k are bounded as well. Furthermore $d\kappa(X_j) = X_j$ by the left invariance of the vector field X_j. The determinant that appears when changing the integration variables to $\kappa(x,g)$ only depends on derivatives of h, since the HAAR measure is invariant. Since all quantities in sight are bounded and the s-norm of $f \circ \kappa$ contains only derivatives up to order s again, the result follows. The discrete case is even simpler, since $||f||_s^\partial = \sum_{|\alpha|\leq s} \big|\big|\frac{\partial^\alpha}{\partial x^\alpha} f\big|\big|_{L^2}$. ☐

Definition 4.2.3. Let E be a twisted Hilbert-A-module bundle over a principal Γ-bundle P. Let $V_\alpha \subset M$ be a trivializing cover for P such that each V_α is a coordinate neighborhood with a chart map $h_\alpha : U_\alpha \longrightarrow V_\alpha$ for some bounded $U_\alpha \subset \mathbb{R}^n$. Denote the trivialization by φ_α and set

$$\kappa_\alpha = \varphi_\alpha \circ (h_\alpha \times \mathrm{id})\ :\ U_\alpha \times \Gamma \longrightarrow P|_{V_\alpha}\ .$$

Choose a subordinate partition of unity ψ_α for V_α and set $\widehat{\psi}_\alpha = \psi_\alpha \circ h_\alpha$. For two smooth sections $\sigma_1, \sigma_2 \in C_c^\infty(P,E)$ we define the A-valued SOBOLEV s-product and the SOBOLEV s-norm via

$$\begin{aligned}(\sigma_1,\sigma_2)_s &= \sum_\alpha \int_{U_\alpha\times\Gamma} \Big\langle (1+\Delta_\alpha)^s\, \widehat{\psi}_\alpha(x)(\kappa_\alpha^*\sigma_1)(x,g), \widehat{\psi}_\alpha(x)(\kappa_\alpha^*\sigma_2)(x,g)\Big\rangle\, dx dg \\ ||\sigma||_s^2 &= ||(\sigma,\sigma)_s||\end{aligned}$$

The completion of $C_c^\infty(P,E)$ with respect to the s-norm will be called the Sobolev s-space and denoted by $H_0^s(P,E)$.

Since the maps treated by lemma 4.2.2 capture exactly what happens when changing the trivialization and the coordinates, we see that the equivalence class of the s-norm does not depend on the choices made to define them. Indeed, they form a chain of topological C^*-Hilbert A-modules in the sense of [65]. In case the group Γ is compact this turns out to be a Rellich chain by the usual argument, but of course we cannot expect this in general. What continues to hold is the Sobolev embedding theorem.

Lemma 4.2.4. *For any integer $p > \frac{n}{2}$, where n denotes the dimension of P, the space $H_0^{k+p}(P,E)$ is continuously included in $C_0^k(P,E)$.*

Proof. Choose an open neighborhood W of the identity in Γ, such that

$$\kappa : U_\Gamma \longrightarrow W \ ; \ (t_1,\dots,t_m) \mapsto \exp(t_1X_1)\cdot\ldots\cdot\exp(t_mX_m)$$

maps $U_\Gamma \subset \mathbb{R}^m$ diffeomorphically onto $W \subset \Gamma$. Note that for $y \in \mathbb{R}^m$

$$d_y\kappa\left(\frac{\partial}{\partial x_i}\right) = L_{g_1^y *}R_{g_2^y *}X_i$$

where $g_1^y = \exp(y_1X_1)\cdot\ldots\cdot\exp(y_iX_i)$ and $g_2^y = \exp(y_{i+1}X_{i+1})\cdot\ldots\cdot\exp(y_mX_m)$. Therefore for $f : U \times \Gamma \longrightarrow V$:

$$\frac{\partial}{\partial x_i}(f \circ (\mathrm{id}\times\kappa)) = \left(\mu_\Gamma\left(R_{g_2^y *}X_i\right) f\right) \circ (\mathrm{id}\times\kappa) \ .$$

Cover Γ by sets of the form $W_i = g_iW$, choose a smooth partition of unity ψ_i and let $\kappa_i = \mathrm{id}\times(L_{g_i}\circ\kappa)$. By a similar reasoning as in 4.2.2 we see that the Sobolev s-norm is equivalent to:

$$||f||_s^{\mathrm{loc}} = \sum_i \sum_{|\alpha|+|\beta|\le s} \left|\left| \frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial y^\beta}\,\psi_i\cdot(f\circ\kappa_i)\right|\right|_{L^2}$$

Now that we have localized the norm the classical argument for example given in [55] applies. $\square$

Corollary 4.2.5. $H_0^\infty(P,E) = \bigcap_k H_0^k(P,E) = C_0^\infty(P,E)$

The following lemmata show that twisting a twisted Hilbert A-module bundle E with a trivial twisted Hilbert B-module bundle $\underline{W}$ yields sensible results from the perspective of Sobolev spaces.

Lemma 4.2.6. *Let V be a Hilbert A-module over a C^*-algebra A, then the inclusion*

$$H_0^s(U\times\Gamma)\otimes V \longrightarrow H_0^s(U\times\Gamma,V) \quad , \quad f\otimes v \mapsto f\cdot v \ .$$

(where the tensor product on the left hand side is the exterior product of a Hilbert $\mathbb{C}$-module and a Hilbert A-module) is a unitary isomorphism of Hilbert A-modules.

Proof. The left hand side of the map above is the completion of $H^s_0(U\times\Gamma)\otimes_{\mathrm{alg}} V$ with respect to the norm induced by the A-valued inner product:

$$\left\langle \sum_i f_i\otimes v_i, \sum_j g_j\otimes w_j\right\rangle = \sum_{i,j} \langle f_i, g_j\rangle_{H^s_0(U\times\Gamma)}\otimes \langle v_i, w_j\rangle_V$$

(since $\mathbb{C}\otimes_{\mathrm{alg}} A\simeq A$, the algebra in play is already complete). This is easily seen to coincide with the product on the right hand side evaluated on elements of the form $\sum_i f_i(x)v_i$ with $f_i\in H^s_0(U\times\Gamma,\mathbb{C})$, $v_i\in V$. Therefore the inclusion is isometric. It is also A-linear. Thus, it only remains to check surjectivity of the map, adjointability is then automatic (see [35]).

This will certainly be the case, if we can show that the submodule

$$C^\infty_c(U\times\Gamma)V = \operatorname{span}\{f\,v \mid f\in C^\infty_c(U\times\Gamma), v\in V\}$$

is dense in $H^s_0(U\times\Gamma,V)$ with respect to the s-norm.

Let $g\in C^\infty_c(U\times\Gamma,V)$ and let $K\subset U\times\Gamma$ be the support of g. Since $g(K)$ is compact and therefore totally bounded, there exist elements $x_1,\dots x_n\in g(K)$, such that for $\varepsilon>0$ the open subsets of K

$$U_j = \{p\in K \mid \ ||g(p)-x_j|| < \varepsilon\}$$

cover K. By definition of the subspace topology there are open sets $V_j\subset U$, s.th. $V_j\cap K = U_j$. By adding $V_0 = U\backslash K$ we get an open cover of U. We choose a smooth subordinate partition of unity h_j for this cover and set $x_0=0$. Hence,

$$\left\| g(x) - \sum_{j=1}^n h_j(x)x_j\right\| = \left\|\sum_{j=0}^n h_j(x)\ (g(x)-x_j)\right\| \leq \sum_{j=0}^n h_j(x)\ ||(g(x)-x_j)|| \leq \varepsilon\ .$$

But $C^\infty_c(U\times\Gamma,V)$ is by definition dense in $H^s_0(U\times\Gamma,V)$, so we are done. □

Corollary 4.2.7. *Let A, B be C^*-algebras, V be a Hilbert A-module, W likewise a Hilbert B-module, then*

$$H^s_0(U\times\Gamma,V)\otimes W \xrightarrow{\sim} H^s_0(U\times\Gamma, V\otimes W)$$

is an isomorphism of Hilbert $A\otimes B$-modules.

Proof. In the diagram

$$\begin{tikzcd} H^s_0(U\times\Gamma)\otimes(V\otimes W) \arrow[r, "\iota_V\otimes\mathrm{id}"] \arrow[rr, hook, bend right, "\iota_{V\otimes W}"'] & H^s_0(U\times\Gamma,V)\otimes W \arrow[r] & H^s_0(U\times\Gamma, V\otimes W) \end{tikzcd}$$

the left and lower Hilbert A-module morphisms are isomorphisms. □

Corollary 4.2.8. *Let E be a twisted Hilbert A-module bundle, $\underline{W}$ be a trivial twisted Hilbert B-module bundle, both over the same principal bundle P. Then*

$$H^s_0(P,E)\otimes W \longrightarrow H^s_0(P, E\otimes\underline{W}) \quad , \quad f\otimes w\mapsto f\cdot w$$

is a unitary isomorphism of Hilbert A-modules.

Proof. Choose a finite trivializing cover $U_i\subset M$, such that $H^s_0(P,E)$ can be identified with a closed subspace of $\bigoplus_i H^s_0(U_i\times\Gamma,V)$, where V is the typical fiber of E. Likewise, $H^s_0(P,E\otimes\underline{W})$ can be embedded isometrically in $\bigoplus_i H^s_0(U_i\times\Gamma,V\otimes W)$. By the previous corollary $\bigoplus_i H^s_0(U_i\times\Gamma,V)\otimes W$ is unitarily isomorphic to $\bigoplus_i H^s_0(U_i\times\Gamma,V\otimes W)$ preserving these subspaces. □

4.2.2 Pseudodifferential operators on twisted Hilbert A-module bundles

Pseudodifferential operators arose quite naturally from the basic observation that differential operators correspond to multiplication by polynomials under Fourier transform: Changing the polynomial expressions to more general functions obeying some growth condition in the Fourier transformed variable leads to a filtered algebra of operators with nice properties. For details about the classic theory, the reader should start with the book by Hörmander [31]. As it turns out, the calculus is rather robust when changing from operators on smooth functions $C^\infty(M)$ to those on sections of vector bundles $C^\infty(M,\xi)$ and finally those on sections of Hilbert A-module bundles as considered by Mishchenko and Fomenko in [47]. As will be seen below, the right notion of ellipticity for pseudodifferential operators on twisted Hilbert A-module bundles leads to examples of *transversally elliptic operators* that were studied first by Atiyah in [7]. Therefore to get a cohomological index formula, one would either have to study the representation theory of (compact) Lie groups on Hilbert A-modules or assume that A possesses a trace. For the sake of simplicity we will assume the latter and restrict ourselves to central extensions of the form

$$1 \longrightarrow \Gamma_f \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1 \ , \tag{4.20}$$

such that Γ_f is a finite group and Γ is compact. In this case we have all the results from [47] (including the Rellich theorem) at hand.

Let P be a smooth Γ-bundle over the compact smooth manifold M, let E,F be twisted Hilbert A-module bundles over P. Denote the projection $T^*M \longrightarrow M$ by π.

Definition 4.2.9. A section $\sigma \in C^\infty(T^*M, \hom(\pi^*E,\pi^*F))$ is called a *symbol* if the estimate

$$\left\| D_x^\alpha \, D_\xi^\beta \, \sigma(x,\xi) \right\| \leq C_{\alpha,\beta} \, (1+|\xi|)^{m-|\beta|} \tag{4.21}$$

holds for all multi-indices α,β and some constants $C_{\alpha,\beta}>0$. The integer $m\in\mathbb{Z}$ is called the *order* of σ. Note that we can identify σ with a section of the form $C^\infty(T^*M\times_M P, \mathrm{Hom}(\pi^*E,\pi^*F))$. We denote the space of all symbols of order m by $S^m(E,F)$.

A symbol $\sigma \in C^\infty(T^*M, \hom(\pi^*E,\pi^*F))$ will be called *transversally elliptic* if it is invertible outside some neighborhood of the zero section in T^*M up to elements in S^{-1}, i.e. there exists $\tau \in S^{-m}(F,E)$, such that $\sigma\circ\tau - 1 \in S^{-1}(F,F)$ and $\tau\circ\sigma - 1 \in S^{-1}(E,E)$.

An element $\sigma \in S^m(E,F)$ is called *homogeneous*, if $\sigma(x,t\xi) = t^m\,\sigma(x,\xi)$ for all $\xi \geq \frac{1}{2}$. It is called *polyhomogeneous* if there exists a formal series $\sum_{k\in\mathbb{Z}} a_k$, with a_k a homogeneous symbol in $S^k_{\mathrm{hom}}(E,F)$ and $a_k = 0$ for $k>m$, such that $\sigma - \sum_{k=m-r}^m a_k \in S^{m-r-1}(E,F)$ for all $r>0$. A sum like $\sum_{k\in\mathbb{Z}} a_k$ is called *asymptotically summable* and we denote the previous relation by

$$\sigma \sim \sum_{k\in\mathbb{Z}} a_k \ .$$

Our notation for the subclass of homogeneous symbols of order m will be $S^m_{\mathrm{hom}}(E,F)$ and $S^m_{\mathrm{ph}}(E,F)$ for polyhomogeneous symbols. For those the limit:

$$\lim_{\lambda\to\infty} \frac{\sigma(x,\lambda\,\xi)}{\lambda^m} = a_m(x,\xi) = \sigma_p(x,\xi) \quad \in S^m_{\mathrm{hom}}(E,F)$$

exists and will be called the *principal symbol*.

Remark For a homogeneous symbol σ the estimate (4.21) is equivalent to the condition that σ should be bounded along the unit sphere, since

$$\sigma(x,\xi) = |\xi|^m \, \sigma(x,\widehat{\xi}) \quad \text{with} \quad \widehat{\xi} = \frac{\xi}{|\xi|} \ ,$$

for $|\xi| \geq \frac{1}{2}$ which implies the symbol estimate. An element $\sigma \in S^m_{\mathrm{ph}}(E,F)$ is transversally elliptic if and only if its principal symbol is invertible outside some neighborhood of the zero section.

The proof of the following classical theorem for the scalar case can be found in [31, Proposition 18.1.3] and transfers to the C^*-algebra case without any difficulties. It will become important in the construction of parametrices.

Theorem 4.2.10. *Every formal sum $\sum_{k\in\mathbb{Z}} a_k$ with $a_k \in S^k_{hom}(E,F)$ and $a_k = 0$ for $k > m$ is asymptotically summable, i.e. there exists a symbol*

$$\sigma \in S_{ph}(E,F) \ ,$$

such that

$$\sigma - \sum_{k=m-r}^{m} a_k \in S^{m-r-1}(E,F)$$

for all $r > 0$.

The transfer to ordinary symbols on the principal bundle P clearly requires a way to embed the cotangent bundle T^*M in T^*P. Note that the covertical subbundle $H^* = \{\xi \in T^*P \mid \xi(\alpha_*(X)) = 0 \ \forall X \in \mathfrak{g}\}$, where $\alpha_{p*} : \mathfrak{g} \longrightarrow T_pP$ denotes the linearized action of Γ, can be identified with π^*T^*M after choosing a connection ω on P. This also yields the following isomorphism:

$$\pi^*(T^*M \oplus \underline{\mathfrak{g}^*}) \longrightarrow T^*P \quad ; \quad (p,\xi,\eta) \mapsto \omega_p^*\eta + \pi^*\xi \ .$$

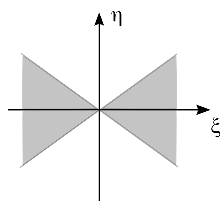

Figure 4.1: C_λ

Γ acts on the left hand side by $g \cdot (p,\xi,\eta) = (pg^{-1}, \xi, \mathrm{Ad}^*_{g^{-1}}\eta)$ and on the right one via the pullback with R_{g*}. The above map is equivariant with respect to these actions. Its inverse yields projections:

$$\pi_{\mathfrak{g}^*} : T^*P \longrightarrow M \times \mathfrak{g}^* \quad \text{and} \quad \pi_{PM} : T^*P \longrightarrow T^*M.$$

Take a homogeneous symbol $\sigma \in S^m_{\mathrm{hom}}(E,F)$. Its pullback via π_{PM} not necessarily satisfies (4.21) due to the new directions $\eta \in \underline{\mathfrak{g}}$ that appear. Nevertheless, it is still homogeneous for $|\xi| \geq \frac{1}{2}$. Therefore we have to control the directions, in which ξ is small in length, but $|\eta|$ is large.

To each smooth function $\lambda : M \longrightarrow (0,1)$ into the open unit interval corresponds an open neighborhood of $T^*M\backslash\{0\} \subset (T^*M \oplus \underline{\mathfrak{g}^*})\backslash\{(0,0)\}$ given by:

$$C_\lambda = \left\{(\xi,\eta) \in T^*M \oplus \underline{\mathfrak{g}^*}\backslash\{(0,0)\} \mid |\xi_m|^2 > \lambda(m)^2 \left(|\xi_m|^2 + |\eta_m|^2\right) \ \forall m \in M\right\}$$

Note that, if $\lambda_1 \geq \lambda_2$, then $C_{\lambda_1} \subseteq C_{\lambda_2}$. In this case, $C_{\lambda_1}/\mathbb{R}_+$ is relatively compact in $C_{\lambda_2}/\mathbb{R}_+$. Since the metric we chose on $\mathfrak{g}$ is invariant under the action of Γ via Ad_g, all C_λ are invariant as well. The grey-colored region in figure 4.1 gives an impression of what C_λ looks like, when restricted to a point $m \in M$.

Lemma 4.2.11. *Let $C_{\lambda_1} \subsetneq C_{\lambda_2}$ be two open neighborhoods like above. There is a smooth function $\chi : T^*M \oplus \underline{\mathfrak{g}^*} \longrightarrow [0,1]$ with the following properties:*

(i) $\chi(\zeta) = \chi\left(\frac{\zeta}{|\zeta|}\right)$ *for $\zeta \in T^*M \oplus \underline{\mathfrak{g}}^*, |\zeta| \geq 1$. Therefore χ is homogeneous of degree* 0.

(ii) $\chi(\zeta) = 1$ *if $\zeta \in C_{\lambda_1}$ and $|\zeta| \geq 1$.*

(iii) $\chi(\zeta) = 0$ *if $\zeta \notin C_{\lambda_2}$.*

(iv) $\chi(\zeta) = 0$ *if $|\zeta| \leq \frac{1}{2}$.*

(v) χ is invariant under the action of Γ on $T^*M \oplus \underline{\mathfrak{g}}$.

Proof. Choose a trivializing cover $W_i \subset M$ of T^*M and a smooth subordinate partition of unity φ_i. If there are functions $\chi_i : W_i \times (\mathbb{R}^n \oplus \mathfrak{g}) \longrightarrow [0,1]$ satisfying all the properties with respect to the images of C_{λ_1} and C_{λ_2} in the trivialization, then we can patch these together to form $\chi = \sum_i \varphi_i\, \chi_i$, which is what we are looking for.

Choose a smooth function on the unit sphere bundle $W_i \times S^{n+m} \subset W_i \times ((\mathbb{R}^n \oplus \mathfrak{g}) \backslash \{(0,0)\})$ satisfying conditions (ii) and (iii) on the restricted images of C_{λ_1} and C_{λ_2} in $W_i \times S^{n+m}$. Since all C_λ and their complements are invariant under the group action, averaging this function over Γ does not disturb (ii) and (iii) and ensures (v). By condition (i) we end up with a $\widehat{\chi}_i$ determined for $|\zeta| \geq 1$. Extend it homogeneously for $1 \geq |\zeta| \geq \frac{1}{2}$ and choose another smooth function $\varphi_i : [0,1] \longrightarrow \mathbb{R}_+$, which is zero on $[0, \frac{1}{2}]$ and 1 on $[1, \infty)$. Then set $\chi_i(\zeta) = \varphi_i(|\zeta|)\, \widehat{\chi}_i(\zeta)$. ☐

Definition 4.2.12. Let $\sigma \in S^m_{\mathrm{hom}}(E,F)$ be a homogeneous symbol. Choose neighborhoods $C_{\lambda_1} \subsetneq C_{\lambda_2}$ with $\lambda_2 < \frac{1}{2}$ like above and a smooth function χ like in the previous lemma. Identify $T^*P \simeq \pi^*(T^*M \oplus \mathfrak{g})$. We call the map:

$$\widehat{\sigma}(p, \xi, \eta) = \chi(\xi, \eta)\, \sigma(\xi)$$

the *regularized pullback* of σ and C_{λ_1} its *regularization domain.*

Definition 4.2.13. A section $\widehat{\sigma} \in C^\infty(T^*P, \mathrm{Hom}(\pi^*E, \pi^*F))$ is called a *symbol* on T^*P if the estimate

$$\left\| D_p^\alpha\, D_\zeta^\beta\, \widehat{\sigma}(p,\zeta) \right\| \leq C_{\alpha,\beta}\ (1 + |\zeta|)^{m-|\beta|} \tag{4.22}$$

holds for all multi-indices α, β and some constants $C_{\alpha,\beta} > 0$. The integer $m \in \mathbb{Z}$ is called the *order* of $\widehat{\sigma}$. We denote the space of all symbols of order m by $S^{m,P}(E,F)$. A symbol $\widehat{\sigma} \in S^{m,P}(E,F)$ will be called *transversally elliptic* for a covertical subbundle $H^* \subset T^*P$ if $\widehat{\sigma}$ is invertible up to elements in $S^{-1,P}(E,F)$ when restricted to H^* without the zero section (see [7]).

The definition of *homogeneous* and *polyhomogeneous* symbols can be deduced directly from definition 4.2.9. The corresponding notations will be $S^{m,P}_{\mathrm{hom}}(E,F)$ and $S^{m,P}_{\mathrm{ph}}(E,F)$.

Lemma 4.2.14. *The regularized pullback $\widehat{\sigma}$ of σ is a symbol on T^*P, which is homogeneous outside the unit sphere bundle.*

Proof. Let $\zeta = (\xi, \eta) \in T^*P$ be a vector of length 1. In directions with $|\xi| > \frac{1}{2}$ we have homogeneity by definition. But in directions ζ with $|\xi| < \frac{1}{2}$ the function χ is zero by our choice of λ_2. □

Remark 4.2.15. If σ is transversally elliptic in the sense of definition 4.2.9, then the pullback $\widehat{\sigma}$ turns out to be transversally elliptic in the sense of definition 4.2.13 with respect to the covertical subbundle $H^* \subset T^*P$.

For the definition of pseudodifferential operators, we first restrict to the (local) case of trivial bundles over open subsets of $\mathbb{R}^s$. The crucial tool will be the Fourier transform, of which we briefly review the definition: Let $f : \mathbb{R}^s \longrightarrow X$ be a Schwartz function taking values in the projective Hilbert A-module X, then the Fourier transform

$$F_{y\mapsto\xi}(f)(\xi) = \widehat{f}(\xi) = \left(\frac{1}{2\pi}\right)^{\frac{s}{2}} \int_{\mathbb{R}^s} f(y)\, e^{-i\langle y,\xi\rangle} dy \tag{4.23}$$

$$F_{\xi\mapsto y}(h)(y) = \left(\frac{1}{2\pi}\right)^{\frac{s}{2}} \int_{\mathbb{R}^s} h(\xi)\, e^{i\langle y,\xi\rangle} d\xi \tag{4.24}$$

yields an automorphism of the Schwartz space $S(\mathbb{R}^s, X)$. This automorphism extends to an isometry of $L^2(\mathbb{R}^s, X)$ (see [65, theorem 2.1.86, theorem 2.1.87]). Let Y be another projective Hilbert A-module. The estimate (4.21) is still applicable to maps of the form $\bar{\sigma} \in C^\infty(W \times \mathbb{R}^s, \mathrm{Hom}(X, Y))$, where $\mathrm{Hom}(X, Y)$ denotes the Banach space of A-linear adjointable operators. Every such $\bar{\sigma}$ satisfying (4.21) for some $m \in \mathbb{Z}$ defines a pseudodifferential operator

$$\bar{\sigma}(D) \colon C^\infty(W, X) \to C^\infty(W, Y) \quad ; \quad u \mapsto \bar{\sigma}(D)u$$
$$(\bar{\sigma}(D)u)(x) = \left(\frac{1}{2\pi}\right)^{\frac{s}{2}} \int_{\mathbb{R}^s} \bar{\sigma}(x, \xi)\, \widehat{u}(\xi)\, e^{i\langle x,\xi\rangle} d\xi \ .$$

Denote by $\Psi^m(W; X, Y)$ the space of all pseudodifferential operators of order m obtained from symbols of the form above. To extend this definition to pseudodifferential operators acting on sections, two more constructions are needed: Note that if $E \to W$ is a trivializable Hilbert A-module bundle with fiber X, then a trivialization $\psi_E \colon E \to W \times X$ induces an isomorphism

$$\psi_{E*} \colon C^\infty(W, E) \to C^\infty(W, X) \quad ; \quad u \mapsto \psi_E \circ u \ .$$

Moreover, if P is a manifold, $U \subset P$ a coordinate neighborhood with chart map $\kappa \colon W \to U$ and $E \to P$ a bundle of Hilbert A-modules for some open subset $W \subset \mathbb{R}^s$, then κ induces a pullback

$$\kappa^* \colon C^\infty(U, E|_U) \to C^\infty(W, E|_U) \quad ; \quad u \mapsto u \circ \kappa \ .$$

Definition 4.2.16. Let $E \to P$ and $F \to P$ be Hilbert A-module bundles over the principal Γ-bundle P with $\dim(P) = s$. Denote the typical fiber of E and F by X and Y respectively. A pseudodifferential operator D of order m is an A-linear map

$$D \colon C^\infty(P, E) \longrightarrow C^\infty(P, F) \ ,$$

which is continuous with respect to the Fréchet topology on the source and target space such that for every local coordinate diffeomorphism $\kappa\colon W \to U$ for open subsets $W \subset \mathbb{R}^s$ and $U \subset P$, over which E and F are trivializable, every trivialization $\psi_E\colon\ E|_U \to W \times X$ and $\psi_F\colon\ F|_U \to W \times Y$ and every $\varphi_1, \varphi_2 \in C_0^\infty(W)$ the map

$$C^\infty(W,X) \ni u \mapsto \varphi_1 \cdot \kappa^* \psi_{F*}^{-1} D\left(\psi_{E*}\kappa^{-1*}(\varphi_2 \cdot u)\right) \in C^\infty(W,Y) \tag{4.25}$$

is an element in $\Psi^m(W;X,Y)$. The space of all such operators will be denoted by $\Psi^m(E,F)$.

Remark 4.2.17. By the coordinate invariance of $\Psi^m(W;X,Y)$ which is proven just like in the scalar case of [31, theorem 18.1.17] (see also [65, theorem 2.1.109]) we only need to check (4.25) for one choice of coordinate neighborhoods and trivializations.

Remark 4.2.18. If $D \in \Psi^m(E,F)$, it is a consequence of (4.25) that D has a Schwartz kernel

$$k \in C^\infty(P \times P)' \otimes_{C^\infty(P\times P)} C^\infty(P \times P, \mathrm{Hom}(\pi_2^* E, \pi_1^* F))\ ,$$

where the dash denotes the dual space equipped with the weak* topology, $\pi_i\colon P \times P \to P$ are projections and the projective tensor product is used. This is the case if D is restricted to $C_0^\infty(U, E|_U)$ for small enough open sets $U \subset P$. Let U_i be a cover of P by such sets and choose a partition of unity ϕ_i subordinate to the cover. Then $D(u) = \sum_i D(\phi_i\, u)$ and each $D_i = D(\phi_i\,\cdot)$ has a kernel k_i. Therefore $k(p_1,p_2) = \sum_i k_i(p_1,p_2)\phi_i(p_2)$ is the kernel of D.

In the scalar case the Schwartz kernel theorem provides a topological isomorphism

$$C^\infty(P \times P)' \simeq \mathcal{L}(C^\infty(P), C^\infty(P)')\ ,$$

where $\mathcal{L}$ denotes the continuous linear operators equipped with the topology of bounded convergence. This implies that every continuous linear operator $S\colon C^\infty(P) \to C^\infty(P)'$ can be obtained from a distributional kernel [70, theorem 51.6 and corollary thereafter]. Let B be a Banach space having a predual. Let X_j for $j \in \{1,2\}$ be open sets in $\mathbb{R}^{n_j}$, then it is proven in [2, theorem 1.8.9] that

$$C^\infty(X_1 \times X_2)' \otimes B \;\simeq\; \mathcal{L}(C^\infty(X_1), C^\infty(X_2)' \otimes B)\ ,$$

where again the tensor products are completed with respect to the projective topology. In view of this result it might be expected that a suitable generalization of the Schwartz kernel theorem is still valid in the case of Hilbert A-module bundles (in fact, if we could set $B = \mathrm{Hom}(X,Y)$ for Hilbert modules X and Y, this is nearly what we are aiming at), but we won't pursue this any further at this point.

In view of remark 4.2.17 the regularized pullback $\widehat{\sigma}$ defines a pseudodifferential operator $\widehat{\sigma}(D)$ in the sense of [65, 47] for the Hilbert A-module bundles E and F over P. Choose a cover U_i of P over which E and T^*P are trivializable, chart maps $\kappa_i : W_i \longrightarrow U_i$ for some open subsets $W_i \subset \mathbb{R}^s$, a subordinate partition of unity ϕ_i and functions ψ_i with $\psi_i|_{\mathrm{supp}(\phi_i)} = 1$. Define $\widehat{\sigma}_i(D)$ for sections $u \in C^\infty(U_i, E)$ via

$$(\widehat{\sigma}_i(D)u)(p) = F_{\xi\mapsto y}(\widehat{\sigma}_i \cdot \widehat{u \circ \kappa_i})(\kappa_i^{-1}(p)) \tag{4.26}$$

with $\widehat{\sigma}_i(p,\xi) = \psi_i(p)\,\widehat{\sigma}(p,\xi)$. We have dropped the trivializations of E and T^*P for the sake of notational clarity. For general $u \in C^\infty(P,E)$ we simply grind the situation down to the local case using ϕ_i and set

$$(\widehat{\sigma}(D)u)(p) = \sum_i \widehat{\sigma}_i(D)(\phi_i\, u)(p)\ .$$

Following [47] the pseudodifferential operator $\widehat{\sigma}(D)\colon C^\infty(P,E) \to C^\infty(P,F)$ extends to

$$\widehat{\sigma}(D)\ :\ H^s(P,E) \longrightarrow H^{s-m}(P,F)\ .$$

Like in [47] we summarize the main properties of the operators $\widehat{\sigma}(D)$ that enable us to use symbol calculus. The proofs are basically the same as in [31]. They still hold in the C^*-case since they don't involve any critical operations on the symbol. In fact, the most elaborate notion needed is that of TAYLOR expansion, which still holds for functions with values in a Banach space.

Theorem 4.2.19. *(i) When the functions ϕ_i and ψ_i and the local coordinates are changed in the definition of $\widehat{\sigma}(D)$, the operator is changed by a lower order summand.*

(ii) If $h : U_i \longrightarrow U_i$ is a diffeomorphism of the coordinate domain $U_i \subset P$ and $D_p h : T_pU_i \longrightarrow T_{h(p)}U_i$ its Jacobian matrix, then the symbol τ of $h^*(\widehat{\sigma}_i(D))$ defined via $(h^*(\widehat{\sigma}_i(D))u)(x) = (\widehat{\sigma}_i(D)(u \circ h))(h^{-1}(x))$ can be expressed by a formal sum as follows:

$$\tau(p,\zeta) \sim \widehat{\sigma}(h(p),(D_p h)^{-1*}\,\zeta) + \sum_{\substack{2 \leq |\alpha| \\ 2|\beta| \leq |\alpha|}} w_{\alpha\beta}(p) D_\zeta^\alpha \widehat{\sigma}(h(p),(D_p h)^{-1*}\,\zeta)\ , \tag{4.27}$$

where the functions $w_{\alpha\beta}$ only depend on h.

(iii) Let $\sigma_1 \in S^m_{hom}(E_1,E_2)$ and $\sigma_2 \in S^m_{hom}(E_2,E_3)$ be homogeneous symbols, then the operators $(\widehat{\sigma}_2\,\widehat{\sigma}_1)(D)$ and $\widehat{\sigma}_2(D)\,\widehat{\sigma}_1(D)$, where the same *regularization is used for σ_i, differ by an operator of lower order.*

(iv) The full symbol τ of the composition $\widehat{\sigma}_2(D)\,\widehat{\sigma}_1(D)$ can be expressed as a formal sum in the following way:

$$\tau(p,\zeta) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} D_\zeta^\alpha \widehat{\sigma}_2(p,\zeta) D_p^\alpha \widehat{\sigma}_1(p,\zeta)\ . \tag{4.28}$$

Remark 4.2.20. The operators in $\Psi^*(E,E)$ gained by this procedure are elements of the filtered algebra of CALDERON-ZYGMUND-SEELEY-A-operators, also known as the polyhomogeneous pseudodifferential A-operators. A definition using jet bundles can be found in [65], where the operators of degree k are denoted by $CZ_k(E,F)$, whereas we will rather stick to the notation $\Psi^k(E,F)$. In each degree there is a principal symbol map $\sigma^k_{\text{prin}} : \Psi^k(E,F) \longrightarrow S^{k,P}_{\text{hom}}(E,F)$, such that the sequence:

$$0 \longrightarrow \Psi^{k-1}(E,F) \longrightarrow \Psi^k(E,F) \xrightarrow{\sigma^k_{\text{prin}}} S^{k,P}_{\text{hom}}(E,F) \longrightarrow 0$$

is exact and for $T \in \Psi^k(E_1,E_2)$ and $S \in \Psi^s(E_2,E_3)$ we have

$$\sigma^{k+s}_{\text{prin}}(ST) = \sigma^s_{\text{prin}}(S)\,\sigma^k_{\text{prin}}(T)\ .$$

Although the symbol $\widehat{\sigma}$ is equivariant with respect to the group action of $\widehat{\Gamma}$, this does not need to be the case for the operators as defined above, since they depend on the choice of coordinate systems. The main problem lies in the notion of FOURIER transform, which requires the linearization of the space coordinate $p \in P$ and therefore seems to be far away from being equivariant. Nevertheless, having chosen connections on E and P, which shall be denoted by η and ω, we are able to define a $\widehat{\Gamma}$-invariant pseudodifferential operator with the same principal symbol as $\widehat{\sigma}$. We will adopt ideas from BOKOBZA-HAGGIAG [11] and WIDOM [76] to accomplish this.

Like above we identify TP with $\pi^*(TM \oplus \underline{\mathfrak{g}})$ via the map (π_*, ω), likewise for the dual bundle T^*P and $\pi^*(T^*M \oplus \underline{\mathfrak{g}^*})$. Note that, with this identification, P inherits an invariant RIEMANNIAN metric from one chosen on M and our invariant scalar product on $\mathfrak{g}$. Due to the bi-invariance of the latter, the action of Γ on P is isometric. Therefore the exponential map

$$\exp_p : T_pP \longrightarrow P$$

is equivariant in the sense that $\exp_{pg}(R_{g*}X) = \exp_p(X)\, g$ for all $X \in T_pP$. Since its differential at p is the identity on T_pP, there exists a neighborhood of 0, such that $\exp_p$ is a diffeomorphism.

Definition 4.2.21. Let $u \in C_c^\infty(P,E)$ be a smooth section of the Hilbert A-module bundle E over P. Choose a twisted connection η on E, a real constant $\rho > 0$, such that $\exp_p$ embeds the disk of radius ρ smoothly into P for all $p \in P$, and a smooth cut-off function $\psi : \mathbb{R}_+ \longrightarrow [0,1]$, which is 1 near 0 and 0 in a neighborhood of $\{t \in \mathbb{R}_+ \mid t \geq \rho\}$. We denote the parallel transport induced by η from y to x for points inside this disk along the (unique, shortest) geodesic by $\mathcal{P}_{y\to x} : E_y \longrightarrow E_x$. For $\zeta \in T_p^*P$ we set

$$\mathcal{F}_{p\mapsto\zeta}(u)(\zeta) = \widehat{u}(\zeta) = \int_{T_pP} e^{-i\zeta(V)}\, U(V)\, dV\ ,$$

where $U(V) = \psi(|V|)\, \mathcal{P}_{\exp_p(V)\to p} u(\exp_p(V))$, and call this the *intrinsic Fourier transform* of u.

Lemma 4.2.22. *The intrinsic Fourier transform is $\widehat{\Gamma}$-equivariant in the sense that $\mathcal{F}_{p\mapsto\zeta}(\widehat{g}\cdot u)(\zeta) = \widehat{g}\,\widehat{u}(R^*_{g^{-1}}\zeta)$.*

Proof. Lemma 4.1.26 implies that $\mathcal{P}_{y\to x}(\widehat{g}\cdot r) = \widehat{g}\cdot \mathcal{P}_{yg\to xg}(r)$. Thus,

$$\begin{aligned} \mathcal{P}_{\exp_p(V)\to p}\,\widehat{g}\,u(\exp_p(V)g) &= \widehat{g}\,\mathcal{P}_{\exp_p(V)g\to pg} u(\exp_p(V)g) \\ &= \widehat{g}\,\mathcal{P}_{\exp_{pg}(R_{g*}V)\to pg} u(\exp_{pg}(R_{g*}V))\ . \end{aligned}$$

Since Γ acts isometrically, this implies $(\widehat{g}\cdot U)(V) = \widehat{g}\, U(R_{g*}V)$. Now:

$$\begin{aligned} \mathcal{F}_{p\mapsto\zeta}(\widehat{g}\cdot u)(\zeta) &= \int_{T_pP} e^{-i\zeta(V)}\, (\widehat{g}\cdot U)(V)\, dV = \int_{T_pP} e^{-i\zeta(V)}\, \widehat{g}\, U(R_{g*}V)\, dV \\ &= \widehat{g} \int_{T_{pg^{-1}}P} e^{-i(R^*_{g^{-1}}\zeta)(V)}\, U(V)\, dV = \widehat{g}\,\widehat{u}(R^*_{g^{-1}}\zeta)\ . \end{aligned}$$

□

We now define the operator $\widehat{\sigma}(\nabla)$ by

$$\widehat{\sigma}(\nabla)u(p) = \left(\frac{1}{2\pi}\right)^s \int_{T_p^*P} \widehat{\sigma}(p,\zeta)\,\widehat{u}(\zeta)\,d\zeta\ .$$

Lemma 4.2.23. *$\widehat{\sigma}(\nabla)$ is equivariant with respect to the $\widehat{\Gamma}$-action on sections $u \in C^\infty(P,E)$.*

Proof.

$$\begin{aligned}
(2\pi)^s\,\widehat{\sigma}(\nabla)(\widehat{g}\cdot u)(p) &= \int_{T_p^*P} \widehat{\sigma}(p,\zeta)\,\mathcal{F}_{p\mapsto\zeta}(\widehat{g}\cdot u)(\zeta)\,d\zeta \\
&= \int_{T_p^*P} \widehat{\sigma}(p,\zeta)\widehat{g}\,\widehat{u}(R_{g^{-1}}^*\zeta)\,d\zeta \\
&= \int_{T_p^*P} \widehat{g}\,\widehat{\sigma}(pg,R_{g^{-1}}^*\zeta)\widehat{u}(R_{g^{-1}}^*\zeta)\,d\zeta \\
&= \int_{T_{pg}^*P} \widehat{g}\,\widehat{\sigma}(pg,\zeta)\widehat{u}(\zeta)\,d\zeta = (2\pi)^s\,\left(\widehat{g}\cdot\widehat{\sigma}(\nabla)(u)\right)(p)
\end{aligned}$$

□

Lemma 4.2.24. *$\widehat{\sigma}(\nabla)$ is a pseudodifferential operator in the sense of definition 4.2.16 (i.e. $\widehat{\sigma}(\nabla)$ is locally of the form (4.26)). If $\sigma \in S^m_{hom}(E,F)$, then the symbol of $\widehat{\sigma}(\nabla)$ is an element in $S^{m,P}_{ph}(E,F)$ and its principal part coincides with $\widehat{\sigma}$.*

Proof. We start with an observation that can also be found in [53]. Fix a point $p_0 \in P$, choose a metric on P and an isometric isomorphism $\varphi : \mathbb{R}^s \xrightarrow{\simeq} T_{p_0}P$. Let

$$\kappa : C \xrightarrow{\varphi} T_{p_0}P \xrightarrow{\exp_{p_0}} P$$

be normal coordinates around p_0 mapping $C \subset \mathbb{R}^s$ diffeomorphically onto $U \subset P$, over which the twisted Hilbert A-module bundle E should be trivial. There is a subset $W \subset TP$, such that $(\pi_P, \exp) : W \longrightarrow P \times P$ is a diffeomorphism. $U \times U$ can be chosen to lie in the range of this map. For $p \in U$ set

$$d(p) = \varphi^{-1} \circ D_p \exp_{p_0}^{-1} : T_pP \longrightarrow T_{p_0}P \longrightarrow \mathbb{R}^s\ ,$$

which is clearly invertible. We will also assume that $\psi|_U = 1$ and drop it from the calculation below. Now for $u \in C_0^\infty(U, E|_U)$:

$$\begin{aligned}
&(2\pi)^s\,\widehat{\sigma}(\nabla)u(p) = \int_{T_p^*P} \widehat{\sigma}(\zeta)\,\widehat{u}(\zeta)\,d\zeta \\
=\ & \int_{\mathbb{R}^{s*}} \widehat{\sigma}(d(p)^*(w))\,|\det(d(p))|\ \widehat{u}(d(p)^*w)\,dw \\
=\ & \int_{\mathbb{R}^{s*}} \int_{T_pP} \widehat{\sigma}(d(p)^*(w))\,|\det(d(p))|\ e^{-i(d(p)^*w)(V)}\mathcal{P}_p u(\exp_p(V))dw\,dV\ ,
\end{aligned}$$

where we have shortened the notation for parallel transport to $\mathcal{P}_p$. Set $\widetilde{\sigma}(p,w) = \widehat{\sigma}(d(p)^*(w))\,|\det(d(p))|$. Let $\widetilde{\kappa}(p,\cdot) = \exp_p^{-1} \circ \kappa : C \longrightarrow T_pP$. This maps C

diffeomorphically onto some subset $U_p \subset T_pP$. Note that $u(\exp_p V)$ has support in U_p, which enables us to use the transform $V = \widetilde{\kappa}(p, v)$. This yields:

$$\int_{\mathbb{R}^{s*}} \int_{\mathbb{R}^s} \widetilde{\sigma}(p,w)\, |\det(D\widetilde{\kappa}(p,v))| e^{-i\langle d(p)^* w, \widetilde{\kappa}(p,v)\rangle} \mathcal{P}_{\kappa(v)\to p} u(\kappa(v)) dw\, dv$$

Note that $\widetilde{\kappa}(p, v) = 0$ is equivalent to $\kappa(v) = p$. We can therefore apply proposition 2.1.3 in [30], which is also known as the KURANISHI trick, to get $G : U \times C \longrightarrow GL(s, \mathbb{R})$ such that

$$\langle d(p)^* w, \widetilde{\kappa}(p,v)\rangle = \big\langle G(p,v)(w), v - \kappa^{-1}(p)\big\rangle$$

With $\tau(p, v, w) = |\det G^{-1}(p, v)||\det(D\widetilde{\kappa}(p, v))|\, \widetilde{\sigma}(p, G^{-1}(p, v)(w))\, \mathcal{P}_{\kappa(v)\to p}$ we get the expression:

$$\int_{\mathbb{R}^{s*}} \int_{\mathbb{R}^s} \tau(p,v,w)\, e^{-i\langle w, v-\kappa^{-1}(p)\rangle} u(\kappa(v)) dw\, dv\ . \tag{4.29}$$

Since τ differs from σ just by multiplication of smooth functions in v and p, a linear transformation in w and the parallel transport, the symbol inequality still holds. It is even homogeneous in w of degree m. We therefore get a symbol on the product $U \times C$, i.e. in $S^m(U \times C; E, F)$, which is just a slightly broader sense of what we have defined above. Using the common reduction argument, which can be found for example in the book by PETERSEN [52], (4.29) can be rewritten as a pseudodifferential operator in the usual sense in such a way that the asymptotic expansion of its symbol $\widetilde{\tau}$ looks like:

$$\widetilde{\tau}(p,w) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} (D_v^\alpha\, D_w^\alpha\, \tau)(p, \kappa^{-1}(p), w)\ .$$

Since τ is homogeneous in w, this is indeed polyhomogeneous. As the principal symbol transforms like a covector, it is independent of the choice of coordinate system. Thus, to calculate its value at p, we can without loss of generality assume $p = p_0 = \kappa(0)$. But then $d(p)^{-1} = \varphi = \widetilde{\kappa}(p, \cdot)$, $G(p, \cdot) = 1_{\mathbb{R}^s}$ and $\tau(p, v, w) = \widetilde{\tau}(p, w) = \widehat{\sigma}(p, \varphi^{-1*}(w))$. □

Since the pseudodifferential operator $\widehat{\sigma}(D)$ is only transversally elliptic, it is not possible to construct a parametrix, i.e. an inverse up to smoothing operators. On the contrary its principal symbol $\sigma : T^*M \longrightarrow \hom(E, F)$ is invertible outside the zero section, which allows us to construct the operator $\widehat{\sigma}^{-1}(D)$, which is an inverse to $\widehat{\sigma}(D)$ on $H^* \subset T^*P$ up to operators of order -1 on H^*. As we see, to exploit this partial invertibility, we have to localize the notion of order, ellipticity and smoothness. At this point, a natural ideal in $\Psi^m(E, F)$ comes up, which turns out to be the replacement for smoothing operators in the transversally elliptic case.

Definition 4.2.25. Let $\widehat{\sigma}$ be a symbol in $S^{m,P}_{ph}$, i.e. a polyhomogeneous symbol over P. Then we say it is *of order k on $C_\lambda \subset T^*P$*, if the estimate

$$\Big\| D_p^\alpha\, D_\zeta^\beta\, \widehat{\sigma}(p,\zeta) \Big\| \le C_{\alpha,\beta}\ (1 + |\zeta|)^{k-|\beta|}$$

holds for all points $(p, \zeta) \in C_\lambda$. If the above equation is valid for all $k \in \mathbb{Z}$, then the symbol will be called *transversally smoothing*. From equation (4.27) we see that, even though the symbol of a pseudodifferential operator does not transform like a covector under the change of coordinates for T^*P, the error terms are all of lower order on C_λ than the untransformed symbol. Therefore it is sensible to speak about pseudodifferential operators that are of order k on C_λ. In particular, there is a well defined, i.e. coordinate invariant, notion of *transversally smoothing operators*. The composition expansion (4.28) tells us that these operators even form an ideal in the algebra of all pseudodifferential operators. Denote by $\Psi^m(C_\lambda; E, F)$ the operators that are transversally of order m on C_λ.

Lemma 4.2.26. *Let $P \in \Psi^m(E,F)$ be a transversally elliptic pseudodifferential operator, such that its principal symbol is a regularized pullback of $\sigma \in S^m_{hom}(E,F)$, then there exists a transversally elliptic pseudodifferential operator $Q \in \Psi^{-m}(F,E)$, such that $PQ-1 \in \Psi^0(F,F)$ and $QP-1 \in \Psi^0(E,E)$ are both transversally smoothing.*

Proof. By ellipticity σ is invertible with inverse given by a symbol $\sigma^{-1} \in S^{-m}_{\text{hom}}(F,E)$. Let Q_0 be a pseudodifferential operator with symbol $\widehat{\sigma^{-1}}$, where we use the same regularization as for the principal symbol of P. Denote by C_λ the common regularization domain of P and Q_0, then Q_0P is transversally of negative order over C_λ. Now set

$$Q_k = \sum_{i=0}^{k} (1 - Q_0 P)^i \, Q_0 = \sum_{i=0}^{k} Q_0 \, (1 - P Q_0)^i \ ,$$

with $Q_k \in \Psi^{-m}(E,F)$ and $Q_k - Q_{k-1} = (1-Q_0P)^k \, Q_0 \in \Psi^{-m-k}(C_\lambda; E, F)$. By theorem 4.2.10 there exists an operator $Q \in \Psi^{-m}(E,F)$ with $Q - Q_k \in \Psi^{-m-k}(C_\lambda; E, F)$. But then:

$$\begin{aligned} PQ - 1 &= P(Q - Q_k) + (PQ_k - 1) \\ &= P(Q - Q_k) - (1 - PQ_0)^{k+1} \in \Psi^{-(k+1)}(C_\lambda; E, F) \end{aligned}$$

for all $k \in \mathbb{N}$, i.e. $PQ - 1 \in \Psi^{-\infty}(C_\lambda; E, F)$. Similarly, $QP - 1 = (Q - Q_k)P - (1 - Q_0P)^{k+1} \in \Psi^{-(k+1)}(C_\lambda; E, F)$. □

We will now define the best replacement we can get for the analytic index of the transversally elliptic pseudodifferential operator $\sigma(\nabla)$. By the last lemma there exists a transversally smoothing parametrix. In the spirit of [7] the analytic index shows up as a distributional character on the group $\widehat{\Gamma}$. Recall that the action of $\widehat{\Gamma}$ on $H^s(P,E)$ is given by:

$$(\widehat{g} \cdot u)(p) = \widehat{g}\,(u(pg)) \ . \tag{4.30}$$

Since we assume to have a bi-invariant scalar product on Γ, (4.30) is isometric in each of the Sobolev s-norms. Furthermore, the map

$$\widehat{\Gamma} \longrightarrow \mathrm{End}(H^s(P,E)) \quad ; \quad \widehat{g} \mapsto (u \mapsto \widehat{g} \cdot u)$$

is continuous, if $\mathrm{End}(H^s(P,E))$ is equipped with the strong topology. Let $\chi \in C^\infty(\widehat{\Gamma})$ be a smooth function on the group and set

$$T^s_\chi : H^s(P,E) \longrightarrow H^s(P,E) \quad ; \quad u \mapsto \int_{\widehat{\Gamma}} \chi(\widehat{g})\,(\widehat{g} \cdot u) \ .$$

In particular, we get $T_\chi = T^{-\infty}_\chi : C^\infty(P,E) \longrightarrow C^\infty(P,E)$.

Definition 4.2.27. If $R : C^\infty(P,E) \longrightarrow C^\infty(P,E)$ is an operator with integral kernel $K : P\times P \longrightarrow \mathrm{End}(E)$ that is continuous along the diagonal $P \longrightarrow P\times P$, then it will be called *trace-class*. In this case we define the *trace* of R to be

$$\mathrm{Tr}(R) = \int_P \tau(K(p,p))\, dp \ ,$$

where τ denotes the extension of the trace on A to $\mathrm{End}(E)$. Indeed, this satisfies the trace property, i.e. $\mathrm{Tr}(RS) = \mathrm{Tr}(SR)$, whenever Tr makes sense.

Lemma 4.2.28. *Let $r = \dim(P)$ and $R \in \Psi^{-(r+2)}(C_\lambda, E, E) \cap \Psi^0(E,E)$ be a pseudodifferential operator, which is transversally of order $-(r+2)$, let $\chi \in C^\infty(\widehat{\Gamma})$ and T_χ be like above, then $T_\chi R$ is a trace-class operator.*

Proof. Fix $p \in P$, choose a trivialization $U' \times \Gamma \longrightarrow P$, such that p lies in the image of $U' \times \{1\}$ and U' is diffeomorphic to some open subset $U \subset \mathbb{R}^n$. Choose some open neighborhood $W' \subset \Gamma$ of $1 \in \Gamma$, such that W' is diffeomorphic to some open subset W of the Lie algebra $\mathfrak{g}$ in such a way that 1 is mapped to 0 and E is trivializable over $U' \times W'$. Note that we assume $\widehat{\Gamma}$ to be an extension of Γ by a finite group Γ_f (see (4.20)). We may assume that the preimage of W' in $\widehat{\Gamma}$ is diffeomorphic to $W' \times \Gamma_f$. A trivialization of E yields an identification of $C_0^\infty(U' \times W', E)$ with $C_0^\infty(U \times W, X)$, where X is the typical fiber of E.

Choose some open, but precompact subset $V \subset W$ with $m(V,V) \subset W$ and $V^{-1} = V$, where m is the map induced by the group multiplication and V^{-1} is gained by taking inverses. By linearity of T_χ in χ we may without loss of generality assume that χ is supported in the image of $V \times \{a\}$ in $\widehat{\Gamma}$ for some $a \in \Gamma_f$. It suffices to show that the induced operator $\widetilde{T}_\chi \widetilde{R} : C_0^\infty(U \times W, X) \to C_0^\infty(U \times W, X)$ has a kernel $K(x,v,y,w)$ that is continuous along the diagonal, where $(x,v),(y,w) \in U \times V$. But the latter is of the form:

$$K(x,v,x,v) = \int_{\mathbb{R}^n}\int_{\mathfrak{g}}\int_V \chi(v')\,\widehat{\tau}(x,m(v,v'),\xi,\eta)e^{i\langle v-m(v,v'),\eta\rangle}dv'\,d\eta\,d\xi \ ,$$

where $\widehat{\tau} : U \times V \times \mathbb{R}^n \times \mathfrak{g} \to \mathrm{End}(E)$ is a smooth map composed of the symbol of R and the action of $\widehat{g}$ that is part of T_χ. By our choice of V, we can find v^{-1}, such that $m(v, m(v^{-1}, v'')) = v''$ for all $v'' \in V$. Now apply the transformation $v' = m(v^{-1}, v'')$. Subsuming the functional determinant and χ into a new function $\widetilde{\chi}$ we end up with:

$$K(x,v,x,v) = \int_{\mathbb{R}^n}\int_{\mathfrak{g}}\left(\int_{\widetilde{V}} \widetilde{\chi}(v,v'')\,\widehat{\tau}(x,v'',\xi,\eta)e^{-i\langle v'',\eta\rangle}dv''\right)\,e^{i\langle v,\eta\rangle}d\eta\,d\xi \ .$$

Let $I(x,v,\xi,\eta)$ be the integral in brackets. Since $R \in \Psi^0(E,E)$, its symbol is bounded in $p \in P$ by an upper bound, which is independent of $\zeta = (\xi,\eta)$ (see (4.22)) thus, $\widehat{\tau}$ is in particular bounded in v'', i.e.

$$\sup_{v''\in\widetilde{V}} \|\widehat{\tau}(x,v'',\xi,\eta)\| \leq c_1$$

for a constant c_1, which is independent of x, ξ and η. Therefore

$$\begin{aligned}\sup_{\eta\in\mathfrak{g}}\|I(x,v,\xi,\eta)\| &\leq \sup_{\eta\in\mathfrak{g}}\int_{\widetilde{V}}\|\widetilde{\chi}(v,v'')\,\widehat{\tau}(x,v'',\xi,\eta)\|dv'' \\ &\leq c_1\int_{\widetilde{V}}\|\widetilde{\chi}(v,v'')\|dv'' \leq c_2 \ . \qquad (4.31)\end{aligned}$$

Products of $I(x,v,\xi,\eta)$ with η_i^α turn into derivatives by the corresponding components of v'' due to the factor $e^{-i\langle v'',\eta\rangle}$, likewise derivatives by the jth η-coordinate turn into multiplications with the jth component of v''. Since $\widetilde{\chi}$ is a Schwartz function in v'' estimates similar to (4.31) show that $I(x,v,\xi,\eta)$ is actually a Schwartz function in η. In particular

$$\|I(x,v,\xi,\eta)\| \leq \frac{c_N}{(1+|\eta|)^N} \ . \tag{4.32}$$

Now denote by $C \subset \mathbb{R}^n \times \mathfrak{g} \simeq \mathbb{R}^r$ the region corresponding to C_λ at x, let $\bar{C}$ be its complement. C is the region, in which the symbol of R is transversally of order $-(r+2)$, thus

$$\int_C \|I(x,v,\xi,\eta)\|\, d\zeta \leq \int_C \frac{c_3}{(1+|\zeta|)^{r+2}}\, d\zeta \leq \int_{\mathbb{R}^r} \frac{c_3}{(1+|\zeta|)^{r+2}}\, d\zeta < \infty \ .$$

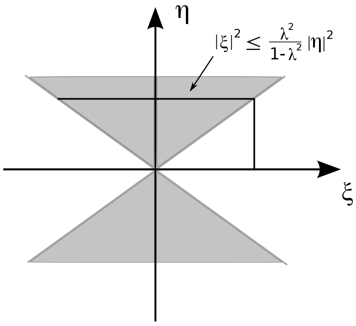

Figure 4.2: Integration over $\bar{C}$

For the integration over the complement $\bar{C}$ let $s \in \mathbb{R}_+$, let λ be the value of $\lambda\colon M \to (0,1)$ at (x,v) and define $B_{\lambda,s}$ by

$$B_{\lambda,s} = \left\{ \xi \in \mathbb{R}^n \mid |\xi| \leq \frac{\lambda}{\sqrt{1-\lambda^2}}\, s \right\} \ .$$

As indicated by figure 4.2, $\bar{C}$ can be parametrized by letting η run over all of $\mathfrak{g}$, while integrating over the balls $B_{\lambda,|\eta|}$. From equation (4.32) we obtain

$$\begin{aligned} \int_{\bar{C}} \|I(x,v,\xi,\eta)\|\, d\zeta &= \int_{\mathfrak{g}} \int_{B_{\lambda,|\eta|}} \|I(x,v,\xi,\eta)\|\, d\xi\, d\eta \\ &\leq \int_{\mathfrak{g}} \int_{B_{\lambda,|\eta|}} \frac{c_{2n+2}}{(1+|\eta|)^{2n+2}}\, d\xi\, d\eta \\ &\leq \widetilde{c} \left(\frac{\lambda}{\sqrt{1-\lambda^2}} \right)^n \int_{\mathfrak{g}} \frac{|\eta|^n}{(1+|\eta|)^{2n+2}}\, d\eta < \infty \ . \end{aligned}$$

Since we have found an integrable function dominating $\|I(x,v,\xi,\eta)\|$, the defining integral of $K(x,v,x,v)$ exists and $K(x,v,x,v)$ is continuous at (x,v). The multiplication with T_χ has smoothed the kernel in the direction of η. $\square$

Definition 4.2.29. Let $\sigma \in S^m_{\mathrm{hom}}(E,F)$ be a homogeneous symbol and set $P = \widehat{\sigma}(\nabla) \in \Psi^m(E,F)$ for some regularization, let $\chi \in C^\infty(\widehat{\Gamma})$. We define the χ-*index* of σ to be

$$\mathrm{ind}_\chi(\sigma) = \mathrm{Tr}\,(T_\chi(PQ - \mathrm{id}_F)) - \mathrm{Tr}\,(T_\chi(QP - \mathrm{id}_E)) \in \mathbb{C}$$

for some parametrix $Q \in \Psi^{-m}(F,E)$.

To see that this is well-defined, we need the following crucial lemma:

Lemma 4.2.30. *The χ-index of σ does not depend on the choice of parametrix and regularization involved. Indeed, if $a \in \Psi^m(E,F)$ is a $\widehat{\Gamma}$-invariant operator, which is* transversally *of lower order than P, then $(P+a)$ has the same χ-index as P.*

Proof. If Q_1, Q_2 are parametrices for P, then the curve $Q_t = t\,Q_1 + (1-t)Q_2$ runs through parametrices for P. Besides that, parametrices are unique up to transversally smoothing operators, since

$$Q_1 - Q_2 = Q_1 - Q_1PQ_2 + Q_1PQ_2 - Q_2 = Q_1(1-PQ_2) + (Q_1P-1)Q_2$$

is transversally smoothing. So:

$$\begin{aligned} & \frac{d}{dt}\left(\mathrm{Tr}\,(T_\chi(PQ_t - \mathrm{id}_F)) - \mathrm{Tr}\,(T_\chi(Q_tP - \mathrm{id}_E))\right) \\ = \;& \mathrm{Tr}\,(T_\chi P(Q_1 - Q_2)) - \mathrm{Tr}\,(T_\chi(Q_1 - Q_2)P) \\ = \;& \mathrm{Tr}\,(T_\chi P(Q_1 - Q_2)) - \mathrm{Tr}\,(T_\chi P(Q_1 - Q_2)) = 0\;, \end{aligned}$$

where we have used that $P = \widehat{\sigma}(\nabla)$ is $\widehat{\Gamma}$-invariant and therefore commutes with T_χ (which is not necessarily the case for Q!).

Let Q be a parametrix for P. When changing P by the lower order summand a, we can replace Q by Q_a, which is defined by the finite sum:

$$Q_a = Q \cdot \sum_{k=0}^{N-1} (-1)^k (aQ)^k = \sum_{k=0}^{N-1} (-1)^k (Qa)^k \cdot Q\;.$$

Note that $aQ \in \Psi^0(F,F)$ and $Qa \in \Psi^0(E,E)$. Both are transversally of negative order. This implies that $T_\chi(aQ)^N$ and $T_\chi(Qa)^N$ are of trace-class if N is chosen large enough, by lemma 4.2.28. But,

$$(P+a)Q_a - 1 = PQ - 1 - (-1)^N(aQ)^N\;,\; Q_a(P+a) - 1 = QP - 1 - (-1)^N(Qa)^N$$

so the trace-property and the equivariance of a imply the statement.

Changing the regularization corresponds to a change of P by an operator, which is equivariant and transversally of lower order, so the invariance follows directly from the previous calculation. ☐

Denote by $D(T^*M)$ the disc bundle of the cotangent space, by $S(T^*M)$ the corresponding sphere bundle. Let $\mathcal{A}$ be a bundle of C^*-algebras over M like above. Every element $[\sigma] \in K^0_{\pi_M^*\mathcal{A}}(D(T^*M), S(T^*M))$ is given by triple $[\pi_M^*E, \pi_M^*F, \sigma]$. σ can be extended to a homogeneous symbol of arbitrary order. It is easily checked that ind_χ is additive with respect to direct sums of symbols and invariant under the equivalence relations from definition 3.3.7. Furthermore, we have the isomorphism $K^0_{\pi_M^*\mathcal{A}}(D(T^*M), S(T^*M)) \simeq K^0_{\pi_M^*\mathcal{A}}(T^*M)$. To summarize we get a homomorphism:

$$\mathrm{ind}_\chi : K_{\pi_M^*\mathcal{A}}(T^*M) \longrightarrow \mathbb{C}$$

or a distribution valued map:

$$\mathrm{ind} : K_{\pi_M^*\mathcal{A}}(T^*M) \longrightarrow C^\infty(\widehat{\Gamma})' \ .$$

We will close this chapter with some justification, why this quantity deserves to be called analytic index. For this, we will reduce to the case of bundle gerbe modules, i.e. when $A = M_n(\mathbb{C})$. Then $\mathrm{ind}(P)$ coincides with the distributional character

$$\mathrm{ind}(P) = \mathrm{char}(\mathrm{kern}(P)) - \mathrm{char}(\mathrm{kern}(P^*))$$

defined in [7]. Let $\Delta_{\widehat{\Gamma}} = 1 - \sum_i X_i^2$ be the Laplace operator on $C^\infty(P, E)$ induced by the action of $\widehat{\Gamma}$ on sections, then set

$$\mathrm{kern}(P)_\lambda = \left\{ u \in C^\infty(P, E) \mid Pu = 0 \text{ and } \Delta_{\widehat{\Gamma}}\, u = \lambda\, u \right\} \ .$$

Likewise, denote by $C^\infty(P, E)_\lambda$ the kernel of $\Delta_{\widehat{\Gamma}} - \lambda$. Atiyah proved that $P_\lambda = P|_{C^\infty(P,E)_\lambda}$ is a Fredholm operator, therefore $\mathrm{char}(\mathrm{kern}(P_\lambda))$ is a well defined function on the group. Furthermore,

$$\sum_\lambda \mathrm{ind}(P_\lambda) = \sum_\lambda \left(\mathrm{char}(\mathrm{kern}(P_\lambda)) - \mathrm{char}(\mathrm{kern}(P^*_\lambda))\right) \tag{4.33}$$

exists in a distributional sense and coincides with $\mathrm{ind}(P)$. Thus, it generalizes the equivariant index of *elliptic* pseudodifferential operators, which is defined in precisely the same way, but yields a *function* on the group [7].

For the case of bundle gerbe modules, the relation of $\mathrm{ind}(P)$ to the topological index has been studied in [39, 38] with the following result:

Theorem 4.2.31. *Let $P = \widehat{\sigma}(\nabla) : C^\infty(P, E) \to C^\infty(P, F)$ be a transversally elliptic pseudodifferential operator of order m with homogeneous principal symbol $\sigma \in S^m_{hom}(E, F)$. If $\phi \in C^\infty(SU(N))$ has support sufficiently close to the identity and is equal to 1 in a neighborhood of it, then*

$$\mathrm{ind}(P)(\phi) = (-1)^{\frac{n(n+1)}{2}} \langle\, \pi_! \mathrm{ch}(\sigma)\mathrm{Td}(M), [M]\,\rangle \in \mathbb{Q} \ .$$

In particular, we have for the projective Dirac operator $D : \Gamma(S) \to \Gamma(S)$ (see definition 4.3.2):

Corollary 4.2.32. *If $\phi \in C^\infty(SU(N))$ has support sufficiently close to the identity and is equal to 1 in a neighborhood of it, then*

$$\mathrm{ind}(D_+)(\phi) = \langle\, \widehat{A}(M), [M]\,\rangle \in \mathbb{Q} \ .$$

We expect similar formulas to hold for twisted Hilbert A-module bundles over more general C^*-algebras A. The crucial point is to find a replacement for the decomposition (4.33) in the flavour of Mishchenko-Fomenko-index theory [47]. Surprisingly, the decomposition of $H^s(P, E)$ into irreducible representations of the group, i.e. with respect to the Casimir-operator $\Delta_{\widehat{\Gamma}}$ is still possible [46, 71]:

Theorem 4.2.33. *Let V be a countably generated Hilbert A-module with a strongly continuous, unitary representation of the group $\widehat{\Gamma}$. Then there is an equivariant isomorphism of Hilbert A-modules*

$$V \simeq \bigoplus_{\pi} \mathrm{Hom}(W_\pi, V) \otimes_{\mathbb{C}} W_\pi \ ,$$

where $\{W_\pi\}$ is a complete collection of unitary, complex, finite dimensional representations of $\widehat{\Gamma}$, which are non-isomorphic to each other. In the decomposition A acts on the first factor, $\widehat{\Gamma}$ on the second one.

The previous theorem allows us to split $H^s(P, E)$ into sub-Hilbert A-modules $H^s(P, E)_\lambda$ like above and a similar argument as the one given in [7] shows that the restrictions P_λ are still elliptic. The equivariant index theory developed by Troitsky [71] now yields an element $\mathrm{ind}(P_\lambda) \in K_0^{\widehat{\Gamma}}(A)$, which the trace on A sends to the representation ring, where we can form the character. It remains to show that the sum of these indices still form a distribution on $\widehat{\Gamma}$. In [7] this problem is solved using some functional analysis, which we do not directly have at hand here.

An alternative approach would be to show directly that ind is well-behaved with respect to Bott periodicity and then apply an argument like in [47] to deduce a result similar to theorem 4.2.31. It can be outlined as follows: Use the Künneth theorem 3.4.13 to split the rational K-theory classes in $K^0_{\pi^*\mathcal{A}}(T^*M) \otimes \mathbb{R}$ into a summand represented by a bundle gerbe module tensored with some fixed Hilbert A-module, for which the analogue of theorem 4.2.31 can easily seen to be true and another factor, on which the index vanishes. We refrain from digging into the details here, since the applications, we have in mind, can be treated in a far more elegant way by countertwisting methods.

4.2.3 Some remarks about the symbol class

Despite the fact that the index of the transversally elliptic operators discussed above is a fairly complicated object, the K-theoretic symbol class still makes sense and can be constructed in a very elegant way, based on an idea by Quillen as we will discuss in this section. Let $\sigma \in S^m_{\mathrm{hom}}(E, F)$ be a homogeneous symbol. As such it yields a twisted bundle morphism:

$$\sigma : \pi^*E \longrightarrow \pi^*F$$

where $\pi : T^*M \longrightarrow M$ denotes the projection, i.e. π^*E is a bundle over $\pi^*P = P \times_M T^*M$ (not T^*P!). Transversal ellipticity now implies that it is invertible away from the zero section. Therefore $[\pi^*E, \pi^*F, \sigma]$ yields a K-cycle in

$$K^0_{\pi^*\mathcal{A}}(T^*M) \simeq K^0_{\pi^*\mathcal{A}}(D(T^*M), S(T^*M)) \simeq K_0(C_0(T^*M, \pi^*\mathcal{A})) \ .$$

The last isomorphism uses the extendability of $\pi^*\mathcal{A}$ to the compactification that is diffeomorphic to the closed disc bundle $D(T^*M)$. It sends $[\pi^*E, \pi^*F, \sigma]$ to a formal difference $[p_E] - [p_F] \in K_0(C_0(T^*M, \pi^*\mathcal{A}))$, such that $p_E - p_F \in C_0(T^*M, \pi^*M_n(\mathcal{A}))$ and the twisted Hilbert A-module bundle generated by p_E via theorem 3.2.8 is isomorphic to $\pi^*E \oplus G$ and likewise for p_F using the same stabilization G.

Following [54] we get another nice description of this class. Form the adjoint σ^* of σ and consider the self-adjoint endomorphism

$$\widetilde{\sigma} = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix} : \pi^*E \oplus \pi^*F \longrightarrow \pi^*E \oplus \pi^*F \ .$$

Lemma 4.2.34. *If $\widetilde{\sigma}$ is a symbol of positive order, then the resolvents $(\widetilde{\sigma} \pm i)^{-1}$ vanish at infinity in the operator norm.*

Proof. By ellipticity the spectrum of $\widetilde{\sigma}$ is bounded below in absolute value on the complement of a neighborhood of the zero section. But by homogeneity we have $\sigma(x, t\xi) = t^k \sigma(x, \xi)$ and therefore for any $C > 0$ there is a compact neighborhood of $M \subset T^*M$ outside which the absolute spectrum of $\widetilde{\sigma}$ is bounded below by C. Therefore $\left|\left|(\widetilde{\sigma}(x,\xi) \pm i)^{-1}\right|\right| \leq \frac{1}{C}$ for arbitrary $C > 0$ if ξ is chosen large enough. □

The Cayley transform yields a unitary operator

$$u = (\widetilde{\sigma} + i)(\widetilde{\sigma} - i)^{-1} = 1 + 2i(\widetilde{\sigma} - i)^{-1} \ .$$

After embedding E and F in the trivial twisted Hilbert A-module bundle $\underline{A}^n$ for $n \in \mathbb{N}$ large enough, we can extend u to a bundle morphism $\pi^*\underline{A}^n \oplus \pi^*\underline{A}^n \longrightarrow \pi^*\underline{A}^n \oplus \pi^*\underline{A}^n$ by setting it to 1 on the complement of $\pi^*(E \oplus F)$. The above lemma now implies that it extends even further to the disc-bundle compactification of T^*M. Let

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \pi^*\underline{A}^n \oplus \pi^*\underline{A}^n \longrightarrow \pi^*\underline{A}^n \oplus \pi^*\underline{A}^n \ .$$

The operators ε and $(u\varepsilon)$ are self-adjoint involutions. Let $p(\varepsilon)$, $p(u\varepsilon)$ be the projection to their $+1$-eigenspaces. Note that $p(\varepsilon) - p(u\varepsilon)$ vanishes at infinity and thus:

$$[p(\varepsilon)] - [p(u\varepsilon)] \in K(C_0(T^*M, \pi^*\mathcal{A})) \ .$$

A short calculation shows that:

$$p(u\varepsilon)(\widetilde{\sigma} + i) = \frac{1}{2}\left(1 + u\varepsilon\right)\right)(\widetilde{\sigma} + i) = (\widetilde{\sigma} + i)\frac{1}{2}(1 - \varepsilon) = (\widetilde{\sigma} + i)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

which implies that $p(u\varepsilon)$ restricted to $\pi^*(E \oplus F)$ projects to the subbundle $G \simeq \pi^*F$ in the decomposition $\pi^*(E \oplus F) = G \oplus G'$ with

$$\begin{aligned} G &= \{(\sigma^*(w), iw) \in \pi^*(E \oplus F) \mid w \in \pi^*F\} \\ G' &= \{(iv, \sigma(v)) \in \pi^*(E \oplus F) \mid v \in \pi^*E\} \ . \end{aligned}$$

Whereas, when restricted to the complement of $\pi^*(E \oplus F)$ in $\pi^*(\underline{A}^n \oplus \underline{A}^n)$, the element $p(u\varepsilon)$ projects to the complement $\pi^*E^\perp$ of π^*E in a single summand $\pi^*\underline{A}^n$. Therefore $p(\varepsilon)$ is the projection generating the bundle $\pi^*\underline{A}^n = \pi^*E \oplus \pi^*E^\perp$, whereas $p(u\varepsilon)$ yields $\pi^*F \oplus \pi^*E^\perp$ and we have:

$$[p(\varepsilon)] - [p(u\varepsilon)] = [p_E] - [p_F] \ .$$

4.3 Generalized projective Dirac operators

When dealing with twisted K-homology, transversally elliptic operators will occur naturally as we have seen above. In this chapter we will treat a class of first-order differential operators that allow to circumvent all the difficulties that arise by non-ellipticity. In particular, they allow the construction of classes in KK-theory and the usual MISHCHENKO-FOMENKO index theorem is applicable to them.

But, let us first focus on the replacement for spinor bundles in the non-spin^c-case. Given a compact, closed, orientable Riemannian n-manifold M with n even, let P_{SO} be its frame bundle. As desribed in [36] there is an injective group homomorphism

$$\theta \colon SO(n) \longrightarrow PU(N) = \mathrm{Aut}(\mathbb{C}l(n)) \ ,$$

such that the CLIFFORD-bundle $\mathbb{C}l(M)$ associated to M is $\mathbb{C}l(M) = P_{\mathrm{SO}} \times_\theta \mathbb{C}l(n)$. This also yields a principal $PU(N)$-bundle $P = P_{\mathrm{SO}} \times_\theta PU(N)$. The DIXMIER-DOUADY-class of its lifting bundle gerbe is $W_3(M) \in H^3(M, \mathbb{Z})$ (i.e. the BOCKSTEIN of the second STIEFEL-WHITNEY-class $w_2(M)$) and therefore 2-torsion.

Denote by g_M the RIEMANNIAN metric on M. We can turn P itself into a Riemannian manifold: For this purpose denote by $\omega \in \Omega^1(P, \mathfrak{pu}(n))$ the connection on P induced by the LEVI-CIVITA-connection on M (indeed, we could have chosen any connection on P). Since the Lie algebra $\mathfrak{pu}(n) \simeq \mathfrak{su}(n)$ is semi-simple, its CARTAN-KILLING-form κ is non-degenerate (and negative-definite), therefore

$$g = \pi^* g_M - \kappa \circ (\omega \otimes \omega) \tag{4.34}$$

is a Riemannian metric on P invariant under right translation. Note that $\pi_* : TP \longrightarrow TM$ restricted to horizontal vector fields is an isometry with respect to g and g_M respectively.

Definition 4.3.1. The lifting bundle gerbe L of P is the frame bundle gerbe (see section 3.4.1) of the following module

$$\begin{aligned} S &= P \times \mathbb{C}^N = \underline{\mathbb{C}^N} \\ \gamma &: L \otimes \pi_2^* \underline{\mathbb{C}^N} \longrightarrow \pi_1^* \underline{\mathbb{C}^N} \\ & \quad [\lambda, \widehat{g}] \otimes v \mapsto \lambda \, \widehat{g} \, v \ , \end{aligned}$$

which we will call the *spinor module.*

Since multiplication by tangent vectors yields a bundle embedding $TM \longrightarrow \mathbb{C}l(M)$, this map pulls back to give $\pi^* TM \longrightarrow P \times \mathbb{C}l(n)$. Since $\mathbb{C}l(n) \simeq M_N(\mathbb{C})$, $\mathbb{C}^N$ gives the irreducible representation of the Clifford algebra (note that n is even).

Definition 4.3.2. Let ∇ be a bgm-connection on S and denote the smooth sections of S by $\Gamma(S)$. The differential operator

$$D \colon \Gamma(S) \xrightarrow{\nabla} \Gamma(S \otimes T^*P) \xrightarrow{g} \Gamma(S \otimes TP) \xrightarrow{\pi_*} \Gamma(S \otimes \pi^* TM) \longrightarrow \Gamma(S) \ .$$

will be called the *projective* DIRAC *operator* (see also [19]).

Choose a local orthonormal frame $\widehat{e}_i \in \Gamma(TP)$ at $p \in P$, then D can be expressed at x via:

$$D\sigma = \sum_i \pi_* \widehat{e}_i \cdot \nabla_{\widehat{e}_i} \sigma \ .$$

We could as well choose an orthonormal basis $e_i \in \Gamma(TM)$ and lift it horizontally through the connection ω chosen above to get $\widehat{e}_i \in \Gamma(TP)$. Note that the latter is still orthonormal due to our choice of g on P. Then we have:

$$D\sigma = \sum_i e_i \cdot \nabla_{\widehat{e}_i} \sigma \ .$$

The symbol of D is

$$\begin{aligned} \widehat{\sigma}_D \quad : \quad & T^*P \xrightarrow{g} TP \xrightarrow{\pi_*} \pi^*TM \longrightarrow P \times \mathbb{C}l(n) = \mathrm{End}(S) \\ & \xi \mapsto ((\pi_* g(\xi)) \cdot\ ) = \sum_i \xi(\widehat{e}_i)\ (e_i \cdot\ ) \ . \end{aligned}$$

The observation that D is *not* elliptic should not be able to cause heart attacks any more by what we have learned from the earlier chapters. We take a closer look at the map $\pi^*TM \longrightarrow P \times \mathbb{C}l(n)$. Denote by κ the embedding $\mathbb{R}^n \longrightarrow \mathbb{C}l(n)$. The former map is explicitely given by

$$\begin{aligned} cl \ : \ \pi^*TM = P \times_M P_{\mathrm{SO}} \times_\rho \mathbb{R}^n \quad & \longrightarrow \quad P \times \mathbb{C}l(n) \\ (p, [f, v]) \quad & \mapsto \quad (p, \widehat{g}^{-1} \kappa(v) \widehat{g}) \ , \end{aligned}$$

where $p = [f, g] \in P = P_{\mathrm{SO}} \times_\theta PU(N)$ and $\widehat{g}$ is an arbitrary lift of g to $U(N)$. So for $X \in T_xM$ and $p \in P$ over $x \in M$ we have $cl(pg, X) = \widehat{g}^{-1}\, cl(p, X)\, \widehat{g}$ for all $g \in PU(N)$. Thus:

$$\widehat{\sigma}_D(R_g^*\xi) = cl\left(pg, \pi_*(g(R_g^*\xi))\right) = cl\left(pg, \pi_* g(\xi)\right) = \widehat{g}^{-1}\, \sigma_D(\xi)\, \widehat{g} \ ,$$

which implies that $\widehat{\sigma}_D$ restricted to $\pi^*T^*M \subset T^*P$ factors to yield a symbol:

$$\sigma_D \colon T^*M \longrightarrow \mathrm{end}(S) = \mathbb{C}l(M) \ .$$

This is of course just left Clifford multiplication with $g_M(\overline{\xi})$ for $\overline{\xi} \in T^*M$. So, the pushed down version of the symbol is invertible away from the zero-section of T^*M, which shows that the projective Dirac operator is transversally elliptic.

Extracting the main feature of Clifford multiplication, we arrive at the following class of operators:

Definition 4.3.3. Let E be a $\mathbb{Z}/2\mathbb{Z}$-graded twisted Hilbert A-module bundle over a Γ-principal bundle P. A twisted bundle morphism

$$c \colon T^*M \longrightarrow \mathrm{end}(E)$$

will be called a *Clifford symbol* if it takes values in the anti-self-adjoint ($c^* = -c$), odd part of $\mathrm{End}(E)$, it squares to the symbol of the Laplace operator

$$c(\xi)^2 = -\left|\left|\xi\right|\right|^2 \mathrm{id}_E \ .$$

and satisfies a product rule with respect to the connection on E:

$$\nabla^E_{\widehat{X}}\left(c(Y)\, u\right) = c\left(\nabla^{T^*M}_{\pi_*\widehat{X}} Y\right) u + c(Y)\left(\nabla^E_{\widehat{X}} u\right)$$

for $\widehat{X} \in T_pP$, $Y \in T^*_{\pi(p)}M$, $u \in \Gamma(E)$. A first-order differential operator $D\colon \Gamma(E) \longrightarrow \Gamma(E)$ will be called a *generalized projective Dirac operator* if its principal symbol restricted to π^*T^*M factors over the action of Γ and its push-downed version is Clifford.

For every Clifford symbol c we can construct a generalized projective Dirac operator D having c as its principal symbol. To achieve this we choose a twisted connection on E and a principal connection on P. The latter yields a projection $T^*P \longrightarrow \pi^*T^*M$. Thus,

$$D\colon \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*P) \longrightarrow \Gamma(E \otimes \pi^*T^*M) \xrightarrow{c} \Gamma(E) \ .$$

is the operator we are looking for.

At a point $m \in M$ choose a local orthonormal frame e_i for T_mM and lift it horizontally to the orthonormal set $\widehat{e}_i \in T_pP$. D can be written at p in the following form:

$$(Du)(p) = \left(\sum_i c(e_i^*)\nabla_{\widehat{e}_i} u \right)(p) \ ,$$

where e_i^* denotes the dual basis. From this, equation (4.6) and the fact that the right action transforms horizontal lifts to horizontal lifts we see that D is $\widehat{\Gamma}$-equivariant.

4.3.1 Trivial bundle gerbes and countertwisting

In case the bundle gerbe L is trivial, there exists a principal $\widehat{\Gamma}$-bundle $\widehat{P}$ and – associated to it – a line bundle $Q^* = \widehat{P} \times_{S^1} \mathbb{C}$, such that its dual carries the structure of a bundle gerbe module over L. The twisted Hilbert A-module bundle $E \otimes Q^*$ is then Γ-equivariant with respect to the action:

$$E_p \otimes Q_p^* \longrightarrow E_{pg} \otimes L^g \otimes Q_p^* \longrightarrow E_{pg} \otimes Q_{pg}^* \ .$$

where the first map is the inverse of the twisting and the second is the twisting of L on Q^* from the right. We denote its push-down by $\pi_!(E \otimes Q^*)$. Note that $\mathrm{end}(E)$ in this case turns out to be isomorphic to the endomorphism bundle of $\pi_!(E \otimes Q^*)$. Therefore a Clifford symbol c defines an ordinary Clifford symbol $c \otimes \mathrm{id}_{Q^*}\colon T^*M \longrightarrow \mathrm{End}(\pi_!(E \otimes Q^*)) = \mathrm{end}(E)$.

To get the Dirac operator corresponding to c we have to choose a connection ω. By lemma 4.1.21 the choice of a bgm-connection θ on Q corresponds to the choice of a connection $\widehat{\omega}$ on $\widehat{P}$ compatible with ω. By theorem 4.1.22 we also know that $\eta_E - \theta$ induces a covariant derivative on $\pi_!(E \otimes Q^*)$, which we denote by $\nabla^{\pi_!(E \otimes Q^*)}$.

The left action of $\widehat{\Gamma}$ on Q given by the twisting induces a right action of $\widehat{\Gamma}$ on the sections of Q^* by setting $(\tau \cdot \widehat{g})(p) = \tau(pg^{-1}) \circ (\widehat{g}\cdot)\colon Q_p \to Q_{pg^{-1}} \to \mathbb{C}$. The actions of $\widehat{\Gamma}$ on sections of E and Q^* compose to give an action of Γ on sections of $E \otimes Q^*$, which on elementary tensors $u \otimes \tau$ looks like

$$(g \cdot u \otimes \tau)(p) = (\widehat{g} \cdot u)(p) \otimes (\tau \cdot \widehat{g}^{-1})(p) \ . \qquad (4.35)$$

This is clearly independent of the choice of lift of g. Being a fixpoint with respect to (4.35) implies:

$$\begin{aligned} u(pg) \otimes \tau(pg) &= \gamma_E([\widehat{g}^{-1}, 1] \otimes u(p)) \otimes \gamma_{Q^*}(\tau(p) \otimes [\widehat{g}, 1]) \\ &= \mathrm{id}_E \otimes \gamma_{Q^*}\left(\gamma_{E,\widehat{g}}^{-1}(u(p)) \otimes \tau(p)\right) , \end{aligned}$$

where γ_{Q^*} is the *right* action of L on Q^*. The last relation is exactly the identification of the fibers by which we get $\pi_!(E \otimes Q^*)$. Thus, *fixpoints* of (4.35) correspond to *sections* of $\pi_!(E \otimes Q^*)$.

Using ω we can as well identify vector fields on M with sections of $TP \longrightarrow P$ that factor over π^*TM, i.e. they fit into the following diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \pi^*TM \arrow[r, "h"] \arrow[d, "\pi_{TM}"] & TP \\ M \arrow[r, "X"] & TM & \end{tikzcd}$$

where h is the horizontal lift. We will denote this vector field by $\widehat{X}$. By the properties of h, $\widehat{X}$ is invariant under the right action of Γ.

When we apply these identifications to the connection $\nabla^{\pi_!(E\otimes Q^*)}$ we see that it corresponds to

$$\nabla^{E\otimes Q^*} = \nabla^E \otimes 1 + 1 \otimes \nabla^{Q^*}$$

on $\Gamma(E \otimes Q^*)$ restricted to the fixpoints of (4.35). Here ∇^{Q^*} denotes the connection belonging to the form $-\theta$ dual to the one on Q. Indeed, by equation (4.6) and the corresponding property for ∇^{Q^*} we see that it commutes with the action (4.35) when restricted to vector fields $\widehat{X}$ as described above and its connection form is $\eta_E - \theta$.

The Dirac operator on E associated to c can be *twisted* with the bundle Q^*. This is defined to be:

$$D\colon \Gamma(E\otimes Q^*) \overset{\nabla^{E\otimes Q^*}}{\longrightarrow} \Gamma(T^*P\otimes E\otimes Q^*) \to \Gamma(\pi^*T^*M\otimes E\otimes Q^*) \overset{c\otimes\mathrm{id}_{Q^*}}{\longrightarrow} \Gamma(E\otimes Q^*) \,.$$

Likewise, we have on the bundle over M:

$$D\colon \Gamma(\pi_!(E \otimes Q^*)) \overset{\nabla^{\pi_!(E\otimes Q^*)}}{\longrightarrow} \Gamma(T^*M \otimes \pi_!(E \otimes Q^*)) \overset{c\otimes\mathrm{id}_{Q^*}}{\longrightarrow} \Gamma(\pi_!(E \otimes Q^*)) \,.$$

Summarizing the above, we have proven:

Theorem 4.3.4. *Let E be a $\mathbb{Z}/2\mathbb{Z}$-graded twisted Hilbert A-module bundle over a trivial bundle gerbe L and c be a Clifford symbol on E. Choose a trivialization Q of L and a bgm-connection θ on Q. Then the Dirac operator D associated to c twisted with the trivializing bundle Q^* can be identified in a canonical way with a Dirac operator D^Q on the push-down $\pi_!(E \otimes Q^*)$ with symbol $c \otimes \mathrm{id}_{Q^*}$ and connection $\nabla^{\pi_!(E\otimes Q^*)}$.*

The problem in defining the analytic index of A-linear operators like D^Q is that its kernel and cokernel, even though they are still modules over the algebra are rarely projective, because A is just closed with respect to continuous functional calculus, but not under the measurable one. It therefore contains less projections than in the case of von Neumann-algebras, in particular $\chi_0(D^*D)$

for the characteristic function χ_0 of $\{0\}$ would not make any sense. Nevertheless it is still possible to define the notion of a Fredholm operator via invertibility modulo compacts, i.e. the existence of a parametrix. This was worked out in [47] by Mishchenko and Fomenko starting with:

Definition 4.3.5. Let A be a C^*-algebra, $H_A = H \otimes_{\mathbb{C}} A$ the canonical Hilbert A-module over A, let $D\colon H_A \to H_A$ be a continuous, A-linear operator. D is called a *Fredholm A-operator* if there are two decomposition $H_A = M_1 \oplus N_1$ and $H_A = M_2 \oplus N_2$, where N_i are finitely generated Hilbert A-modules, such that the matrix representation of $D\colon M_1 \oplus N_1 \to M_2 \oplus N_2$ looks like

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

and $D_1\colon M_1 \to M_2$ is an isomorphism. The element $[N_1] - [N_2] \in K_0(A)$ is called the *(analytic) index* of D.

See [47] for the proof that the index is independent of the choice of decomposition. A Fredholm A-operator has a parametrix, given by

$$Q = \begin{pmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} ,$$

since $1 - DQ = \mathrm{pr}_{N_2}$ and $1 - QD = \mathrm{pr}_{N_1}$ are finite rank projections. The converse is also true, i.e. if we can find an operator Q with $DQ = 1 + K_1$ and $QD = 1 + K_2$ for $K_i \in \mathcal{K}(H_A)$, then D is Fredholm in the sense above.

Mishchenko and Fomenko used this to show that an elliptic pseudodifferential A-operator D of degree m is in fact a Fredholm A-operator when identified with its bounded extension $D\colon H^s(X,E) \to H^{s-m}(X,F)$. Furthermore, they proved the following index theorem:

Theorem 4.3.6. *Let D be as above. Denote its symbol by σ, the dimension of X by n, the* Todd *class by $\mathrm{Td}(X)$ and the fundamental class of X by $[X]$. Moreover denote by $\pi_!\colon H_c^*(T^*X,\mathbb{R}) \to H_c^{*-n}(X,\mathbb{R})$ the* Thom *isomorphism in cohomology. Then:*

$$\mathrm{ind}(D) = (-1)^{\frac{n(n+1)}{2}} \langle\, \pi_! \mathrm{ch}(\sigma)\mathrm{Td}(X), [X]\,\rangle \in K_0(A) \otimes \mathbb{R}\ .$$

Let D be a generalized projective Dirac operator over P associated to a Clifford symbol $c\colon T^*M \to E_+ \oplus E_-$. Suppose the lifting bundle gerbe of P is trivial and choose a trivialization Q. Denote by

$$D_+^Q\colon \Gamma(\pi_!(E_+ \otimes Q^*)) \longrightarrow \Gamma(\pi_!(E_- \otimes Q^*))$$

one part of the decomposition of the odd operator D^Q. Theorem 4.3.6 combined with remark 4.1.33 about Chern characters yields immediately:

$$\mathrm{ind}(D_+^Q) = (-1)^{\frac{n(n+1)}{2}} \langle\, \pi_! \mathrm{ch}_Q(c)\mathrm{Td}(M), [M]\,\rangle \in K_0(A) \otimes \mathbb{R}\ . \tag{4.36}$$

In view of part ii) of theorem 4.1.34 flat trivializations are the most preferable ones, since the Morita equivalences induced by them are not recognized by the Chern character. The topological obstructions to such *flat trivializations* are summarized in the next theorem. The special case of generalized spin^c extensions is again similar to the classical one.

Theorem 4.3.7. *Let $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ be a flat central extension like above. Let P be a principal Γ-bundle, $dd(P) \in H^3(M,\mathbb{Z})$ its* DIXMIER-DOUADY-*class. Suppose $dd(P) = 0$, then:*

i) The lifting bundle gerbe L corresponding to P possesses a flat trivialization if and only if there is a lift $\widehat{P}$ such that $ch(Q) = 1$, where $Q^* = \widehat{P} \times_{S^1} \mathbb{C}$ is the line bundle associated to $\widehat{P} \to P$.

ii) If the extension is generalized spin^c (see section 4.1.1 for the definition and notation), then P lifts to a $\overline{\Gamma}$-bundle if and only if there is a lift $\widehat{P}$ such that $c_1(\bar{Q})$ is n-torsion, where $\bar{Q}$ is the line bundle associated to $\widehat{P}/\overline{\Gamma} \to M$ and $\overline{\Gamma}/(\mathbb{Z}/n\mathbb{Z}) = \Gamma$.

Let $\overline{dd}(P) \in H^2(M,\mathbb{Z}/n\mathbb{Z})$ be the DIXMIER-DOUADY-*class of the lifting bundle gerbe corresponding to $1 \to \mathbb{Z}/n\mathbb{Z} \to \overline{\Gamma} \to \Gamma \to 1$, then $dd(P) = \beta(\overline{dd}(P))$, where β is the* BOCKSTEIN *homomorphism $\beta\colon H^2(M,\mathbb{Z}/n\mathbb{Z}) \to H^3(M,\mathbb{Z})$.*

Proof. $dd(P)$ is exactly the obstruction to the existence of a lift $\widehat{P}$. If the trivialization is flat, then $\widehat{P} \to P$ has a connection with vanishing curvature, therefore $\mathrm{ch}(Q) = 1$. On the other hand

$$\mathrm{ch}(Q) = \exp\left(\frac{i\Omega_Q}{2\pi}\right) = \sum_{k=0}^{\infty} \left(\frac{i}{2\pi}\right)^k \frac{\Omega^k}{k!} = 1$$

implies that all the higher classes in $H^{2k}(M,\mathbb{R})$ for $k \geq 1$ vanish, in particular $[\Omega_Q] = 0 \in H^2(M,\mathbb{R})$. Therefore $\Omega_Q = d\eta$ for a form $\eta \in \Omega^1(M)$. If $\widehat{\pi}\colon \widehat{P} \to M$ denotes the bundle projection, then the connection form on Q may be changed by the summand $-i\widehat{\pi}^*\eta$, which yields a flat bundle gerbe connection.

For the second statement consider the exact sequence induced by $1 \to \mathbb{Z}/n\mathbb{Z} \to \widehat{\Gamma} \to \Gamma \times S^1 \to 1$ in Čech cohomology:

$$H^1(M,\underline{\widehat{\Gamma}}) \longrightarrow H^1(M,\underline{\Gamma}) \oplus H^1(M,\underline{S^1}) \xrightarrow{\delta} H^2(M,\mathbb{Z}/n\mathbb{Z})$$

After identifying $H^1(M,\underline{S^1})$ with $H^2(M,\mathbb{Z})$ via the exponential map, a short calculation shows that the boundary map is actually $\delta([P],[L]) = \overline{dd}(P) - (c_1(L) \mod n)$ for $[P] \in H^1(M,\underline{\Gamma})$ and a line bundle L. $[\widehat{P}]$ is mapped to $([P],[\bar{Q}])$ under the first map, consequently $\overline{dd}(P) = (c_1(\bar{Q}) \mod n)$. For the other direction we use $\widehat{P} = \overline{P} \times_i \widehat{\Gamma}$ for $i\colon \overline{\Gamma} \to \widehat{\Gamma}$, which yields a trivial $\bar{Q}$.

The last statement follows from the diagram:

$$\begin{tikzcd}
H^1(M,\underline{\overline{\Gamma}}) \arrow[r] \arrow[d] & H^1(M,\underline{\Gamma}) \arrow[r, "\bar{\delta}"] \arrow[d, equal] & H^2(M,\mathbb{Z}/n\mathbb{Z}) \arrow[d] \arrow[dr, "\beta"] & \\
H^1(M,\underline{\widehat{\Gamma}}) \arrow[r] & H^1(M,\underline{\Gamma}) \arrow[r, "\delta"] & H^2(M,\underline{S^1}) \simeq & H^3(M,\mathbb{Z})
\end{tikzcd}$$

where $\delta([P]) = dd(P)$ and $\bar{\delta}(P) = \overline{dd}(P)$. □

Definition 4.3.8. Let $D\colon \Gamma(S) \to \Gamma(S)$ be a generalized projective Dirac operator acting on sections of a twisted Hilbert $M_n(\mathbb{C})$-bundle S over a principal $PU(n)$-bundle $\widetilde{P}$. Let E be a twisted Hilbert A-module bundle over a principal

$\widehat{\Gamma}$-bundle P. Set $\bar{P} = \widetilde{P} \times_M P$ and denote by $S \boxtimes E$ the exterior tensor product of both bundles over $\widetilde{P} \times_M P$. The operator

$$D^E : \Gamma(S \boxtimes E) \xrightarrow{\nabla^{S\boxtimes E}} \Gamma(T^*\bar{P} \otimes S \boxtimes E) \to \Gamma(\pi^*T^*M \otimes E \boxtimes S) \xrightarrow{c\otimes \mathrm{id}_E} \Gamma(E \boxtimes S) \ .$$

will be called *D twisted by E*. Analogously, we can switch the roles of the algebras A and $M_n(\mathbb{C})$ and twist a generalized projective Dirac operator over a twisted Hilbert A-module bundle with a bundle gerbe module.

We now turn to the case where the lifting bundle gerbe is non-trivial, i.e. $dd(L) \in H^3(M, \mathbb{Z})$ is a non-vanishing (torsion) class. In this setup we can easily reduce generalized projective Dirac operators to those we have seen in the trivial case just by twisting them with a bundle gerbe module of opposite dd-class. However, this, of course, corresponds to a non-canonical choice to make, on which the index will heavily depend as we will see in the examples. By what we have learned above about the transversal ellipticity of projective operators, this is nevertheless the only sensible way to get a $K_0(A)$-valued invariant. We will call this procedure *countertwisting*. In the following we will discuss some choices for countertwisting bundles.

4.3.2 Spinor countertwisting

Every torsion class in $H^3(M, \mathbb{Z})$ is the Dixmier-Douady-class of a $PU(N)$-bundle $\mathcal{K}$ even though the latter is not uniquely defined by $dd(\widetilde{P})$. So, for an arbitrary generalized projective Dirac operator $D : \Gamma(E) \to \Gamma(E)$ over a Γ-principal bundle P with $dd(P)$ torsion, there exists a matrix bundle $\mathcal{K}$, such that $dd(\widetilde{P}) = -dd(P)$. The latter comes along with a canonical bundle gerbe module S associated to it like the spinor module in definition 4.3.1:

$$\begin{aligned} S &= \widetilde{P} \times \mathbb{C}^N \\ \gamma &: \widetilde{L} \otimes \pi_2^*\underline{\mathbb{C}^N} \longrightarrow \pi_1^*\underline{\mathbb{C}^N} \\ &\quad [\lambda, \widehat{a}] \otimes v \mapsto \lambda\, \widehat{a}\, v \ , \end{aligned}$$

for the lifting bundle gerbe $\widetilde{L}$ of $\widetilde{P}$. Choosing a trivialization Q of $L \boxtimes \widetilde{L}$, we get from (4.36):

$$\mathrm{ind}(D_+^{S,Q}) = (-1)^{\frac{n(n+1)}{2}} \langle\, \pi_! \mathrm{ch}_Q(c)\mathrm{ch}(S)\mathrm{Td}(M), [M]\,\rangle \in K_0(A) \otimes \mathbb{R} \ . \qquad (4.37)$$

Example 4.3.9. In this example we will examine the real counterpart of the above method. Let M be a $2n$-dimensional manifold. Suppose D is the projective Dirac operator from definition 4.3.2, where we consider the spinor module S to live over the frame bundle P_{SO} of TM. We will think of S as a bundle gerbe module over the lifting bundle gerbe L^{spin} for the $\mathrm{Spin}(2n)$-group, i.e. $S = P_{SO} \times (W \otimes_{\mathbb{R}} \mathbb{C})$, where W is the standard representation of $\mathrm{Spin}(2n)$ decomposing into $W_\pm$ with respect to the complex volume element $\omega_{\mathbb{C}} = i^n e_1 \cdot \cdots \cdot e_n$ (see [36, page 34]). Since $\overline{dd}(P_{SO}) = w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ is 2-torsion, we can take $\widetilde{P} = P_{SO}$ and S will be the countertwisting. Note that the bundle gerbe $L \boxtimes L$ over $P^{[2]} \times_M P^{[2]}$ has a canonical trivialization, which is given by L itself. Since the bundle gerbe connection on L is flat as a bgm-connection, we have $\mathrm{ch}(L) = 1$, thus, following remark 4.1.33, $\mathrm{ch}_L = \mathrm{ch}$.

By the usual splitting argument (see proposition 11.2 in [36]), there is a smooth proper fibration $\pi\colon \mathcal{S}_{TM} = N \longrightarrow M$ such that $\pi^*\colon H^*(M) \longrightarrow H^*(\mathcal{S}_{TM})$ is injective and

$$\pi^*(TM \otimes \mathbb{C}) \simeq l_1 \oplus \bar{l}_1 \oplus \cdots \oplus l_n \oplus \bar{l}_n$$

where the l_i are complex line bundles and $\bar{l}_i$ their conjugates. From this we gain a reduction of $\pi^* P_{SO}$ to $P_{l_1} \times_N \cdots \times_N P_{l_n}$, which induces a splitting on the level of bundle gerbes:

$$L^{S^1} \boxtimes_{\mathbb{R}} \cdots \boxtimes_{\mathbb{R}} L^{S^1} \longrightarrow \pi^* L^{\mathrm{spin}} \ , \tag{4.38}$$

where $L^{S^1} \longrightarrow P_{l_i}^{[2]}$ is the lifting bundle gerbe of $S^1 \longrightarrow S^1 \ ; \ z \mapsto z^2$. This decomposition is a result of the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spin}(2) \times_{\mathbb{Z}/2\mathbb{Z}} \cdots \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{Spin}(2) \arrow[r, "\kappa"] \arrow[d] & \mathrm{Spin}(2n) \arrow[d] \\ SO(2) \times \cdots \times SO(2) \arrow[r, "\bar{\kappa}"] & SO(2n) \end{tikzcd}$$

where κ uses $S^1 \to \mathrm{Spin}(2n) \ ; \ \varphi \mapsto \cos(\frac{\varphi}{2}) + e_{2i-1}\, e_{2i} \sin(\frac{\varphi}{2})$ in each factor and $\bar{\kappa}$ is the inclusion via block matrices. Since both lifting bundle gerbes have the same dd-class, (4.38) is a stable isomorphism. Restricting $\pi^* S$ to $P_{l_1} \times_N \cdots \times_N P_{l_n}$ it factors as

$$\pi^* S = (P_{l_1} \times_N \cdots \times_N P_{l_n}) \times (\mathbb{C} \oplus \overline{\mathbb{C}}) \otimes \cdots \otimes (\mathbb{C} \oplus \overline{\mathbb{C}}) = S_1 \boxtimes \cdots \boxtimes S_n$$

where $S_i = P_{l_i} \times (\mathbb{C} \oplus \overline{\mathbb{C}})$ is the spinor module over L^{S^1} with the canonical action on the first summand and conjugate action on the second summand. Choosing a connection η on P_{l_i}, we can construct a bgm-connection on S_i like in example 4.1.7. Keeping in mind that the split identifies the Lie algebras $\mathfrak{g} = i\mathbb{R}$ of $\Gamma = S^1$ and $\widehat{\mathfrak{g}} = i\mathbb{R}$ of $\widehat{\Gamma} = S^1$ via the isomorphism $X \mapsto \frac{1}{2}X$, we get $\Omega_{S_i} = (\frac{1}{2}\Omega_{l_i}) \oplus (-\frac{1}{2}\Omega_{l_i})$, which yields the following relationship between the Chern characters:

$$\mathrm{ch}(\pi^* S) = \prod_{i=1}^{n} \mathrm{ch}(S_i) = \prod_{i=1}^{n} (e^{\frac{1}{2}c_1(l_i)} + e^{-\frac{1}{2}c_1(l_i)}) = \prod_{i=1}^{n} 2\cosh\left(\frac{c_1(l_i)}{2}\right) \ . \tag{4.39}$$

If the natural grading via the complex volume element $\omega_{\mathbb{C}}$ is taken into account (see [36, page 34]), the spinor module splits into S^+ and S^- giving rise to a difference element $[S^+] - [S^-] \in K^0_{\mathcal{K}}(M)$. After applying the splitting principle the grading manifests as $S_i^- = P_{l_i} \times \mathbb{C}$ and $S_i^+ = P_{l_i} \times \overline{\mathbb{C}}$. Therefore:

$$\mathrm{ch}([\pi^* S^+] - [\pi^* S^-]) = \prod_{i=1}^{n} (e^{-\frac{1}{2}c_1(l_i)} - e^{\frac{1}{2}c_1(l_i)}) = \prod_{i=1}^{n} (-2)\sinh\left(\frac{c_1(l_i)}{2}\right) \ . \tag{4.40}$$

Comparing (4.39) and (4.40) with the definitions for the $\widehat{A}$-genus, the Hirzebruch-$\widehat{L}$-class and the Euler-class:

$$\widehat{A}(M) = \prod_{i=1}^{n} \frac{\frac{c_1(l_i)}{2}}{\sinh\left(\frac{c_1(l_i)}{2}\right)} \ , \ \widehat{L}(M) = \prod_{i=1}^{n} \frac{\frac{c_1(l_i)}{2}}{\tanh\left(\frac{c_1(l_i)}{2}\right)} \ , \ e(M) = \prod_{i=1}^{n} c_1(l_i)$$

we see

$$\mathrm{ch}(S) = 2^n \widehat{L}(M)\widehat{A}(M)^{-1} \quad , \quad \mathrm{ch}([S^+] - [S^-]) = (-1)^n e(M)\widehat{A}(M)^{-1} \ .$$

Let c be the symbol of the projective Dirac operator D_+. The image of $\mathrm{ch}(c) \in H^*_c(T^*M)$ under the THOM isomorphism $\pi_!\colon H^*_c(T^*M) \to H^{*-2n}(M)$ can be calculated from

$$e(M)\pi_!\mathrm{ch}(c) = i^* i_! \pi_! \mathrm{ch}(c) = i^*\mathrm{ch}(c) = \mathrm{ch}([S^+] - [S^-]) = (-1)^n e(M)\widehat{A}(M)^{-1}$$

where $i\colon M \to T^*M$ denotes the zero section. Both sides of the equation behave naturally with respect to pullbacks, i.e. come from characteristic classes pulled back from $H^*(BU(2n), \mathbb{R}) \simeq \mathbb{R}[c_1, \dots, c_{2n}]$. In this polynomial ring we can divide by the expression corresponding to $e(M)$ and deduce $\pi_!\mathrm{ch}(c) = (-1)^n \widehat{A}(M)^{-1}$. Combining this with $\mathrm{Td}(M) = \widehat{A}(M)^2$ (see [36, Proposition 11.14]) the cohomological index formula (4.37) takes the form

$$\mathrm{ind}(D_+^{S,L}) = \langle\, 2^n \widehat{L}(M), [M]\,\rangle = \langle\, L(M), [M]\,\rangle \in \mathbb{Z} \ ,$$

which is the HIRZEBRUCH signature theorem. Indeed, theorem 4.3.4 identifies $D_+^{S,L}$ with

$$D_+^{\mathbb{C}l} : \mathbb{C}l(M)_+ \longrightarrow \mathbb{C}l(M)_- \quad ; \quad u \mapsto \sum_i e_i \cdot \nabla_{e_i} u \ , \tag{4.41}$$

where the grading is given by left multiplication with the volume element $\omega_{\mathbb{C}}$. So, $D_+^{S,L}$ is the signature operator. We could have used the graded tensor product in the countertwisting above, i.e. $D^{S,\mathrm{gr}}\colon S \mathbin{\widehat{\boxtimes}} S \longrightarrow S \mathbin{\widehat{\boxtimes}} S$. This changes the grading in (4.41) to the natural odd/even-grading on $\mathbb{C}l(M)$ turning the signature operator into the EULER characteristic operator. Evaluating the right-hand side we have:

$$\mathrm{ind}(D_{\mathrm{odd}}^{S,L}) = \langle\, e(M), [M]\,\rangle = \chi(M) \in \mathbb{Z} \ . \tag{4.42}$$

Example 4.3.10. A special case of spinor countertwisting is possible, if there exists a finite-dimensional unitary representation $\alpha\colon \widehat{\Gamma} \longrightarrow U(N)$ like in example 4.1.7. In this case we can take $\widetilde{L} = L^*$ and the bundle gerbe module:

$$\begin{array}{rcl} S & = & P \times \mathbb{C}^N \\ \gamma & : & L^* \otimes \pi_2^* S \longrightarrow \pi_1^* S \\ & & [\lambda, \widehat{g}] \otimes v \mapsto \lambda\, \overline{\alpha(\widehat{g})}\, v \end{array}$$

that uses the conjugate representation. Since $L \boxtimes L^*$ has a canonical flat trivialization, we get:

$$\mathrm{ind}(D_+^{S,Q}) = (-1)^{\frac{n(n+1)}{2}} \langle\, \pi_!\mathrm{ch}(c)\mathrm{ch}(S)\mathrm{Td}(M), [M]\,\rangle \in K_0(A) \ .$$

As a consequence of the PETER-WEYL-theorem, α always exists in the case of compact groups:

Theorem 4.3.11. *If $\widehat{\Gamma}$ is a compact Lie group which is the central S^1-extension of a Lie-group Γ, then there exists $n \in \mathbb{N}$ and a faithful unitary representation*

$$\widehat{\varrho} \ : \ \widehat{\Gamma} \longrightarrow U(n) \ ,$$

such that $\widehat{\varrho}(tg) = t\widehat{\varrho}(g)\ \forall g \in \widehat{\Gamma}, t \in S^1$.

Proof. As a corollary of the PETER-WEYL-theorem there exists a unitary, faithful representation $\rho\colon \widehat{\Gamma} \longrightarrow U(n)$ for some $n \in \mathbb{N}$. We decompose this into irreducible components $\rho_i\colon \widehat{\Gamma} \longrightarrow U(n_i)$, such that $\rho = \oplus_{i=1}^N \rho_i$. By SCHUR's lemma, $\rho_i(t)$ for $t \in S^1$ has to act like a multiple of the identity, since the extension is central, so ρ_i restricted to S^1 is built from a single one-dimensional, and therefore irreducible, representation of S^1, i.e.

$$\rho_j(e^{i\varphi}) = e^{im_j\varphi}\,1$$

for some $m_j \in \mathbb{Z}$.

Now ρ is a faithful representation. Suppose there were an $m \in \mathbb{Z}, m \neq 1$ dividing all of the m_j, i.e. $m\,l_j = m_j$ for $l_j \in \mathbb{Z}$, then

$$\rho_j(e^{i\frac{2\pi}{m}}) = e^{2\pi i l_j} = 1 \quad \forall j \in \{1,\dots N\} \ .$$

Thus, $e^{i\frac{2\pi}{m}}$ would be an element in the kernel of ρ different from the identity, which is a contradiction. Therefore $\gcd(m_1,\dots,m_N) = 1$. Iterating the EUCLIDEAN algorithm, we get integer numbers $r_1,\dots,r_N$ such that

$$\sum_{i=1}^N r_i m_i = 1 \ .$$

Now set

$$\varrho = \underbrace{\widetilde{\rho}_1 \otimes \cdots \otimes \widetilde{\rho}_1}_{|r_1|\text{times}} \otimes \cdots \otimes \underbrace{\widetilde{\rho}_N \otimes \cdots \otimes \widetilde{\rho}_N}_{|r_N|\text{times}} \ ,$$

where $\widetilde{\rho}_j$ is equal to ρ_j if r_j is positive and equal to the conjugate representation if r_j is negative, and note that

$$\varrho(e^{i\varphi}\,\widehat{g}) = e^{i\,\sum_{j=1}^N r_j m_j \varphi}\,\varrho(\widehat{g}) = e^{i\varphi}\,\varrho(\widehat{g}) \ .$$

This representation can be turned into a faithful one by applying the following trick: By the observation above there is a some tensor power of ϱ (or its conjugate), denoted by $\widehat{\rho}_j$, such that $\widehat{\rho}_j \otimes \rho_j$ preserves S^1-factors. Set

$$\widehat{\varrho} = \bigoplus_{j=1}^N \widehat{\rho}_j \otimes \rho_j \ .$$

Now $\widehat{\varrho}(e^{i\varphi}g) = e^{i\varphi}\widehat{\varrho}(g)$ still holds and implies $S^1 \cap \operatorname{kern}(\widehat{\varrho}) = 1$. But $(\widehat{\rho}_j \otimes \rho_j)(g) = 1$ yields $g \in S^1 \cdot \operatorname{kern}(\rho_j)\ \forall j \in \{1,\dots,N\}$. So

$$\operatorname{kern}(\widehat{\varrho}) \subset S^1 \cdot \bigcap_{j=1}^N \operatorname{kern}(\rho_j) = S^1 \ .$$

Therefore $\widehat{\varrho}$ is faithful. □

This theorem yields a direct proof of a result, which has also been noticed by MURRAY in [49]:

Corollary 4.3.12. *Every lifting bundle gerbe of an S^1-extension of a compact Lie group has a* DIXMIER-DOUADY-*class which is torsion.*

Proof. By the previous theorem, there is a $PU(n)$-bundle extending the Γ-principal bundle P, which has the same DIXMIER-DOUADY-class, therefore the latter has to be torsion. □

4.3.3 Flat countertwisting

When dealing with countertwisting bundles a question that arises naturally is, whether there exists a countertwisting disturbing the cohomological side of equation (4.37) as little as possible. This is definitely the case when the bundle S is flat, since then the Chern character is just a multiple of the identity in $H^0(M)$. In the classical case flat bundles are characterized by the holonomy reduction of their frame bundle, which is just a cover of M. In the twisted case we expect a similar result to hold. With the right notion of what we mean by discrete holonomy bundle, this is indeed true:

Definition 4.3.13. We will call the lifting bundle gerbe $L \to \bar{M}^{[2]}$ of a central S^1-extension of a discrete group Γ_d

$$1 \to S^1 \to \widehat{\Gamma}_d \to \Gamma_d \to 1$$

with a principal Γ_d-bundle $\bar{M}$ a *covering bundle gerbe*.

Theorem 4.3.14. *Let $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ be a flat central extension with Γ connected. Let E be a bundle gerbe module over the principal Γ-bundle P. E carries a flat connection if and only if the frame bundle gerbe reduces to a covering bundle gerbe and the frame trivialization $P_E \to P_E/S^1$ is flat.*

Proof. First suppose that E possesses a flat connection and denote by η_E the connection form on the frame bundle $P_E \to P$. It is a well-known result of classical differential geometry that, due to flatness, the holonomy subbundle $P_E^{\mathrm{hol}}(r, \eta_E) = \bar{P}$ of P_E at any point $r \in P_E$ has a discrete structure group [42, 33]. Therefore $\bar{P} \to P$ is a regular covering with normal classifying subgroup $\bar{\pi} \subset \pi_1(P)$ and deck transformations $D = \pi_1(P)/\bar{\pi}$.

$\widehat{\Gamma} = \widetilde{\Gamma} \times_\rho S^1$ acts on P_E covering the action of Γ on P. This induces an action of $\widetilde{\Gamma} \to \widehat{\Gamma}$ on P_E that transforms $\bar{P}$ into itself. To see this, we need to check that $s \in P_E^{\mathrm{hol}}(r, \eta_E)$ and $\widehat{g}s = [\widetilde{g}, 1]\, s$ for $\widetilde{g} \in \widetilde{\Gamma}$ can be connected by a horizontal path. But any smooth curve $\kappa(t) \in \widetilde{\Gamma}$ connecting the identity with $\widetilde{g}$ gives rise to $\widehat{g}(t) = [\kappa(t), 1] \in \widehat{\Gamma}$, i.e. $\widehat{\mu}\left(\dot{\widehat{g}}\right) \in \mathfrak{g} \subset \mathfrak{g} \oplus i\mathbb{R}$, which implies $\eta_E(\frac{d}{dt}(\widehat{g}(t)s)) = 0$.

Restricting the action to $\pi_1(\Gamma) \subset \widetilde{\Gamma}$, yields a homomorphism $\alpha\colon \pi_1(\Gamma) \to D$. The quotient bundle P_E/S^1 is associated to $\bar{P}$ via $\tau\colon D \longrightarrow PU(N)$. Now $\mathrm{Im}(\alpha) \subset \mathrm{kern}(\tau)$, since the image of $\pi_1(\Gamma) \subset \widetilde{\Gamma}$ in $\widehat{\Gamma}$ lies in $S^1 \subset \widehat{\Gamma}$, which acts by scalars on P_E. Therefore $\bar{\tau}\colon D/\mathrm{Im}(\alpha) \to PU(N)$ exists (note that $\mathrm{Im}(\alpha)$ is abelian). On $\bar{P}/\mathrm{Im}(\alpha)$ the action reduces to one of $\Gamma = \widetilde{\Gamma}/\pi_1(\Gamma)$ and turns $\bar{P} \to P$ into a Γ-equivariant $D/\mathrm{Im}(\alpha)$-principal bundle over P. Denote its push-down by $\bar{M}$ and note that $\bar{M} \to M$ is a connected principal bundle with discrete structure group $D/\mathrm{Im}(\alpha)$, in particular a covering. The push-down of P_E/S^1 is associated to $\bar{M}$ and the lifting bundle gerbe for P_E reduces to a covering bundle gerbe via the following pullback:

$$\begin{array}{ccc} \widehat{D} & \longrightarrow & U(N) \\ \downarrow & & \downarrow \\ D/\mathrm{Im}(\alpha) & \xrightarrow{\bar{\tau}} & PU(N) \end{array}$$

Due to the splitting $\mathfrak{u}(N) = \mathfrak{su}(N) \oplus i\mathbb{R}$, the connection form $\eta_E \in \Omega^1(P_E, \mathfrak{u}(N))$ induces a connection on $P_E \to P_E/S^1$ by projecting it down to $i\mathbb{R}$. The proof that this has the right properties is similar to the one in theorem 4.1.31 ii). By construction its curvature vanishes.

Suppose now that the frame bundle gerbe of $E \to P$ reduces to one of the form $L \to \bar{M}^{[2]}$ for a covering $\bar{M} \to M$, then E shifted back to $\bar{M}$ looks like $\bar{E} = \bar{M} \times \mathbb{C}^N$ for a representation $\tau \colon \widehat{\pi} \to U(N)$ such that $\tau(e^{i\varphi}\widehat{g}) = e^{i\varphi}\tau(\widehat{g})$. Now the connection from example 4.1.7 reduces to the pullback of the Maurer-Cartan form of $U(N)$ to $\bar{M} \times U(N)$ and is therefore flat. Tensoring it with the (flat) connection on the trivialization yields one on $E \to P$. □

Definition 4.3.15. Let D be a generalized projective Dirac operator over P_1. We say that D possesses a *flat countertwisting*, if there exists a lifting bundle gerbe $L_2 \to P_2^{[2]}$ with $dd(P_2) = -dd(P_1)$ and a flat bundle gerbe module $E \to P_2$ for L_2.

Corollary 4.3.16. *Let D be a generalized projective Dirac operator over P for the flat central S^1-extension $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ with Γ connected and lifting bundle gerbe L. D has a flat countertwisting if and only if there exists a normal subgroup $\pi \lhd \pi_1(M)$, a central S^1-extension*

$$1 \longrightarrow S^1 \longrightarrow \widehat{\pi} \longrightarrow \pi_1(M)/\pi \longrightarrow 1$$

such that the lifting bundle gerbe $\bar{L} \to \bar{M}^{[2]}$ (where $\bar{M}$ is the covering classified by π) satisfies $dd(\bar{L}) = -dd(L)$ and a representation $\varrho \colon \widehat{\pi} \to U(N)$ with $\varrho(e^{i\varphi}\widehat{g}) = e^{i\varphi}\varrho(\widehat{g})$. In this case we have a flat bundle gerbe module E and a trivialization Q such that

$$\operatorname{ind}(D_+^{E,Q}) = N \cdot (-1)^{\frac{n(n+1)}{2}} \langle\, \pi_! \mathrm{ch}_Q(c) \mathrm{Td}(M), [M]\,\rangle \in K_0(A) \otimes \mathbb{R}\ . \qquad (4.43)$$

For $\mathcal{A} = \mathcal{K}$ a matrix bundle, we see that the right-hand side has to be an integer.

Proof. Suppose π, $\widehat{\pi}$ and ϱ are given. Set $E = \bar{M} \times \mathbb{C}^N$ and turn this into a bundle gerbe module via the representation ϱ. E is flat by theorem 4.3.14. Since $dd(\bar{M}) = -dd(P)$ by hypothesis, a trivialization Q exists. The cohomological formula (4.43) is now a direct consequence of (4.36). On the other hand, if E is a flat countertwisting, then its frame bundle gerbe reduces to a covering $\bar{M}$ classified by some normal subgroup $\pi \lhd \pi_1(M)$. Over $\bar{M}$ it takes the form $\bar{E} = \bar{M} \times \mathbb{C}^N$ for some representation ϱ satisfying the conditions above. □

Remark Central extensions are classified by second group cohomology. Thus, $\widehat{\pi}$ corresponds to some element $c_{\widehat{\pi}} \in H^2_{\mathrm{gr}}(\pi_1(M)/\pi, S^1)$. Whenever we have a projective representation $\bar{\varrho} \colon \pi_1(M)/\pi \to PU(N)$, its lifting obstruction to be a linear representation is also contained in this group. The existence of ϱ with the above property actually is equivalent to having a projective representation, such that its class coincides with $c_{\widehat{\pi}}$.

Remark Note that the integrality statement is not trivial, since we know from our observations about transversal ellipticity and calculations of the $\widehat{A}$-genus for certain manifolds that $\langle\, \pi_! \mathrm{ch}_Q(c) \mathrm{Td}(M), [M]\,\rangle$ will in general be rational.

Definition 4.3.17. Let $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ be a flat central S^1-extension with Γ connected and let $P \to M$ be a principal Γ-bundle over M. Let $\widehat{\Gamma} = \widetilde{\Gamma} \times_\rho S^1$ for $\rho \colon \pi_1(\Gamma) \to S^1$ and set $\overline{\Gamma} = \widetilde{\Gamma}/\mathrm{kern}(\rho)$. We say that P *allows a $\overline{\Gamma}$-structure* on the universal cover $\widetilde{M}$ if $\pi_M^* P \to \widetilde{M}$ lifts to a $\overline{\Gamma}$-bundle.

Lemma 4.3.18. *P allows a $\overline{\Gamma}$-structure on $\widetilde{M}$ if and only if $kern(\alpha) \subset kern(\rho)$, where α is the second map in the following excerpt of the exact fibration sequence:*

$$\pi_2(M) \longrightarrow \pi_1(\Gamma) \xrightarrow{\alpha} \pi_1(P) \longrightarrow \pi_1(M) \longrightarrow 1 \ .$$

Proof. α factors as $\pi_1(\Gamma) \xrightarrow{\widetilde{\alpha}} \pi_1(\pi_M^* P) \longrightarrow \pi_1(P)$. Since the second map is a covering, it is injective on π_1, i.e. $\mathrm{kern}(\alpha) = \mathrm{kern}(\widetilde{\alpha})$. $\mathrm{kern}(\alpha)$ is the classifying group of some connected regular cover $\dot{\Gamma}$ of Γ. The universal cover $\widetilde{P} = \widetilde{\pi_M^* P}$ of $\pi_M^* P$ (which coincides with the universal cover of P) is a principal $\dot{\Gamma}$-bundle over $\widetilde{M}$. To see this, note that it follows from the fibration sequence that $\widetilde{P}$ restricts to $\dot{\Gamma}$ on each fiber over $\widetilde{M}$. There is a natural action of $\widetilde{\Gamma}$ on $\widetilde{P}$: Let $\beta\colon I \to P$ be a path with $\beta(0) = p_0$ representing a point in $\widetilde{P}$, let $\gamma\colon I \longrightarrow \Gamma$ with $\gamma(0) = 1$ represent a point in $\widetilde{\Gamma}$, then we define

$$(\beta \cdot \gamma)(t) = \beta(t)\, \gamma(t) \ ,$$

which is easily seen to give a well-defined action on homotopy classes. The stabilizer of $\widetilde{p}_0 \in \widetilde{P}$ (represented by the constant path) is given by loops in Γ that become contractible after embedding them into P by $p_0 \cdot \gamma(t)$, which is just the explicit description of $\mathrm{kern}(\alpha)$. Therefore $\widetilde{\Gamma}/\mathrm{kern}(\alpha) = \dot{\Gamma}$ acts freely on $\widetilde{P}$. This action is also transitive on the fibers over $\widetilde{M}$. The hypothesis now implies that the homomorphism $\widetilde{\Gamma} \longrightarrow \widetilde{\Gamma}/\mathrm{kern}(\rho)$ factors over $\dot{\Gamma} = \widetilde{\Gamma}/\mathrm{kern}(\alpha)$, which means that there exists a $\overline{\Gamma}$-bundle associated to $\widetilde{P}$. On the other hand, if we have a principal $\overline{\Gamma}$-bundle $\overline{P}$ over $\widetilde{M}$, then it is covered by $\widetilde{P}$. This map restricts to a covering $\dot{\Gamma} \to \overline{\Gamma}$ on the fibers, which induces an injective map $\mathrm{kern}(\alpha) = \pi_1(\dot{\Gamma}) \to \pi_1(\overline{\Gamma}) = \mathrm{kern}(\rho)$ on π_1. □

There are two actions of $\pi_1(\Gamma)$ on $\widetilde{P}$: the one given above using group multiplication. The other by pre-concatenating a path in P with the loop defined by $p_0\, \gamma$. By the usual Eckmann-Hilton-type argument these two coincide. Similarly, we see that $\pi_1(\Gamma)$ is mapped into the center of $\pi_1(P)$ in the fibration sequence. Thus, $\pi_1(\Gamma)/\mathrm{kern}(\alpha)$ can be seen as a central subgroup of $\pi_1(P)$. If P allows a $\overline{\Gamma}$-structure on $\widetilde{M}$, then ρ factors over $\bar{\rho} : \pi_1(\Gamma)/\mathrm{kern}(\alpha) \to S^1$. Therefore we can form

$$\widehat{\pi} = \pi_1(P) \times S^1/(\pi_1(\Gamma)/\mathrm{kern}(\alpha)) \ . \tag{4.44}$$

$\pi_1(\Gamma)/\mathrm{kern}(\alpha)$ maps to a central *anti-diagonal* subgroup in $\pi_1(P) \times S^1$ via $\gamma \mapsto (\gamma, \bar{\rho}(\gamma)^{-1})$ (note that $\bar{\rho}$ is not necessarily injective). $\widehat{\pi}$ fits into a short exact sequence:

$$1 \longrightarrow S^1 \longrightarrow \widehat{\pi} \longrightarrow \pi_1(M) \longrightarrow 1 \ .$$

Lemma 4.3.19. *Suppose P allows a $\overline{\Gamma}$-structure on $\widetilde{M}$, then the lifting bundle gerbe $\bar{L} \to \widetilde{M}^{[2]}$ of the above central extension over the universal cover $\widetilde{M}$ satisfies $dd(\bar{L}) = -dd(L)$, where L is the lifting bundle gerbe for P. Furthermore, there exists a flat trivialization of $\bar{L} \boxtimes L$.*

Proof. It suffices to show that $\bar{L} \boxtimes L \to (\widetilde{M} \times_M P)^{[2]} = (\pi_M^* P)^{[2]}$ has a trivialization, which is the case if the principal $\pi_1(M) \times \Gamma$-bundle $\pi_M^* P \to M$ lifts to a principal $\widehat{\pi} \otimes \widehat{\Gamma}$-bundle. The group $\overline{\pi} = \pi_1(\Gamma)/\mathrm{kern}(\alpha)$ injects into the center of

$\overline{\Gamma}$, as well as into that of $\pi_1(P)$. Taking the quotient by the antidiagonal action like above, yields the group $\overline{\Gamma} \times_{\overline{\pi}} \pi_1(P) = (\overline{\Gamma} \times \pi_1(P))/\overline{\pi}$. Now the universal cover $\widetilde{P}$ is not only a $\overline{\Gamma}$-bundle over $\widetilde{M}$, but also a $\overline{\Gamma} \times_{\overline{\pi}} \pi_1(P)$ over M. The action of $\overline{\Gamma}$ on $\widetilde{P}$ is described in the last lemma. $\pi_1(P)$ acts on $\widetilde{P}$ by deck transformations. If β represents a point in $\widetilde{P}$, γ an element of $\widetilde{\Gamma}$ and δ one in $\pi_1(P)$, then $(\beta \cdot \gamma) * \delta$ (where the star denotes concatenation) is homotopic to $(\beta * \delta) \cdot \gamma$ by letting γ rest on 1 for the time the path runs through δ. Thus, the two actions commute. Similarly, we have for ϵ representing an element in $\pi_1(\Gamma)/\mathrm{kern}(\alpha)$ that $(\beta \cdot \epsilon) * (p_0\, \epsilon^{-1})$ (ϵ^{-1} denoting the reverse loop) is homotopic to β. Altogether this yields a well-defined action of $\overline{\Gamma} \times_{\overline{\pi}} \pi_1(P)$ on $\widetilde{P}$.

To see transitivity on the fibers, take two paths β_1, β_2 in P and denote the projection $P \to M$ by π_P. The condition that the β_i represent points in the same fiber boils down to $(\pi_P \circ \beta_1)(1) = (\pi_P \circ \beta_2)(1) = m \in M$. Thus, $(\pi_P \circ \beta_i)$ represent two points in $\widetilde{M}$ lying in the same fiber over M, which implies that there exists an element $a \in \pi_1(M)$ mapping the first to the second by a deck transformation. Since $\pi_1(P)$ surjects onto $\pi_1(M)$ there is an element $[\delta] \in \pi_1(P)$ mapping to a. But now $\pi_P \circ (\beta_1 * \delta)$ and $\pi_P \circ \beta_2$ represent the same point in $\widetilde{M}$ implying that there is an element $[\gamma] \in \widetilde{\Gamma}$, such that $(\beta_1 * \delta) \cdot \gamma$ is homotopic to β_2. If p_0 is the constant path on the basepoint, then the condition that $(p_0 * \delta) \cdot \gamma$ is homotopic to p_0 relative endpoints implies that $[\gamma] \in \pi_1(\Gamma)/\mathrm{kern}(\alpha)$ and $[\delta] \cdot [\gamma] = 1 \in \pi_1(P)$ proving freeness.

Since by our hypothesis and the previous lemma we have $\mathrm{kern}(\alpha) \subset \mathrm{kern}(\rho)$ the homomorphism $\widetilde{\Gamma} \to \widehat{\Gamma}$ factors over $\overline{\Gamma} = \widetilde{\Gamma}/\mathrm{kern}(\alpha)$. Likewise, we have $\pi_1(P) \to \widehat{\pi}$ and putting these morphisms together we get $\overline{\Gamma} \times \pi_1(P) \to \widehat{\Gamma} \otimes \widehat{\pi}$, which factors over the antidiagonal embedding of $\pi_1(\Gamma)/\mathrm{kern}(\alpha)$. Thus, we have a principal $\widehat{\Gamma} \otimes \widehat{\pi}$-bundle Q associated to $\widetilde{P}$. Since Q as an S^1-bundle over $\pi_M^* P = P \times_M \widetilde{M}$ reduces to the $\pi_1(\Gamma)/\mathrm{kern}(\alpha)$-bundle $\widetilde{P}$, it is indeed a flat trivialization. □

Theorem 4.3.20. *Suppose P allows a $\overline{\Gamma}$-structure on $\widetilde{M}$ and let D be a generalized projective Dirac operator over P like in corollary 4.3.16. Let $\widehat{\pi}$ be the central S^1-extension from the last lemma and denote its classifying 2-cocycle by $c_{\widehat{\pi}} \in H^2_{gr}(\pi_1(M), S^1)$. Then every representation $\varrho\colon \widehat{\pi} \to U(N)$ with $\varrho(e^{i\varphi}\widehat{a}) = e^{i\varphi}\varrho(\widehat{a})$, i.e. every projective representation of $\pi_1(M)$ with cocycle $c_{\widehat{\pi}}$, yields a flat countertwisting E with flat trivialization Q and in this case*

$$\mathrm{ind}(D_+^{E,Q}) = N \cdot (-1)^{\frac{n(n+1)}{2}} \langle\, \pi_! \mathrm{ch}_Q(c) \mathrm{Td}(M), [M]\,\rangle \in K_0(A) \otimes \mathbb{R}\ . \tag{4.45}$$

For $\mathcal{A} = \mathcal{K}$ a matrix bundle $ch_Q = ch$ (see remark 4.1.33) and the right-hand side has to be an integer.

Proof. By lemma 4.3.19 the trivial group $\pi = \{1\} \lhd \pi_1(M)$, the central S^1-extension $\widehat{\pi}$ and the representation ϱ match the conditions in corollary 4.3.16. □

Example 4.3.21. In case D is the projective Dirac operator over the frame bundle $P = P_{SO}$, corresponding to the central extension

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n) \longrightarrow 1$$

we have $\overline{\Gamma} = \mathrm{Spin}(n)$. Therefore the existence of a $\overline{\Gamma}$-structure on $\widetilde{M}$ corresponds to a spin-structure on the universal cover. In this case $\rho\colon \pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z} \to S^1$ is injective, implying that

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_1(P_{SO}) \longrightarrow \pi_1(M) \longrightarrow 1$$

is exact by lemma 4.3.18 and $\widehat{\pi} = (\pi_1(P_{SO}) \times S^1)/(\mathbb{Z}/2\mathbb{Z})$. Denote by $c_{\mathrm{spin}} \in H^2_{\mathrm{gr}}(\pi_1(M), S^1)$ the cocycle classifying this extension. As follows from example 4.3.9, the class $(-1)^{n(n+1)/2}\pi_!\mathrm{ch}(c)\mathrm{Td}(M)$ just yields $\widehat{A}(M)$ for an even-dimensional manifold M.

Corollary 4.3.22. *If there exists a projective representation $\varrho\colon \pi_1(M) \longrightarrow PU(N)$ of dimension N corresponding to the cocycle $c_{spin} \in H^2_{gr}(\pi_1(M), S^1)$, then*

$$N \cdot \langle\, \widehat{A}(M), [M]\,\rangle \in \mathbb{Z}\ .$$

Denote by $n_{\widehat{\pi}} \in \mathbb{N}$ the minimum over the possible dimensions of projective unitary representations of $(\pi_1(M), c_{\widehat{\pi}})$ with *fixed* classifying cocycle $c_{\widehat{\pi}}$ (note that $n_{\widehat{\pi}}$ can be greater than 1).

Corollary 4.3.23. *Let M be a closed manifold with $\widetilde{M}$ spin, then $n_{\widehat{\pi}}$ is some integer multiple of the denominator of $\widehat{A}(M)$.*

Proof. Let $\widehat{A}(M) = \frac{r}{s}$. By the previous theorem: $n_{\widehat{\pi}}\,\frac{r}{s} = x \in \mathbb{Z}$. Without loss of generality we can assume that r and s are coprime, i.e. $\exists a, b \in \mathbb{Z} : ar + bs = 1$, but then:

$$n_{\widehat{\pi}}\,\frac{ar}{s} = n_{\widehat{\pi}}\,\frac{1-bs}{s} = ax \quad\Leftrightarrow\quad n_{\widehat{\pi}} = (ax + n_{\widehat{\pi}}b)\,s\ .$$

$\square$

Conjectures Now let M be an arbitrary even-dimensional manifold. Then this corollary rises the question, whether some multiple of the $\widehat{A}$-genus of M can be obtained as the index of the projective Dirac operator countertwisted by some flat bundle. This conjecture is certainly false: For example $\mathbb{C}P^2$ has $\widehat{A}(\mathbb{C}P^2) = -\frac{1}{8}$, which implies that $w_2(\mathbb{C}P^2)$ can not be 0. Since $\pi_1(\mathbb{C}P^2) = 1$ every central extension of this group is trivial and we have $\widehat{\pi} = S^1$ giving rise to the trivial lifting bundle gerbe $\bar{L} = \underline{\mathbb{C}} \to \mathbb{C}P^2 \times \mathbb{C}P^2$. This still satisfies $dd(\bar{L}) = -dd(L) = 0$, since $\mathbb{C}P^2$ is spin^c. However, there is *no* flat trivialization, since this would correspond to a *spin-structure.* Therefore the above corollary yields the result

$$\langle\, \exp(-c_1(Q))\widehat{A}(\mathbb{C}P^2), [\mathbb{C}P^2]\,\rangle \in \mathbb{Z}$$

for some trivialization Q which is induced by a line bundle F over $\mathbb{C}P^2$, such that $c_1(F) \mod 2 = w_2(\mathbb{C}P^2)$. Still, the conjecture can be stated for manifolds allowing a spin structure on the universal cover. In other cases the right conjecture would be that there is a class $c \in H^2(M, \mathbb{Z})$ such that some multiple of $\langle\, \exp(c)\widehat{A}(M), [M]\,\rangle$ is represented by a flatly countertwisted projective Dirac operator.

In view of the above conjectures, we could ask for necessary conditions for the existence of a flat countertwisting bundle, i.e. a projective representation of $\pi_1(M)$. Before we present some results in this direction, we analyze the fibration sequence a little further:

Lemma 4.3.24. *Let $\dot{\Gamma}$ be the group from lemma 4.3.18. Then P lifts to a $\dot{\Gamma}$-bundle $\dot{P}$ if and only if the short exact sequence:*

$$1 \longrightarrow \pi_1(\Gamma)/\mathrm{kern}(\alpha) \longrightarrow \pi_1(P) \longrightarrow \pi_1(M) \longrightarrow 1$$

splits. Furthermore, the possible splittings are in $1:1$-correspondence with the different $\dot{\Gamma}$-lifts of P.

Proof. Let $\sigma\colon \pi_1(M) \longrightarrow \pi_1(P)$ be a split, then the cover of P classified by the subgroup $\sigma(\pi_1(M))$ corresponds to a principal $\dot{\Gamma}$-bundle $\dot{P}$ over M, since $\pi_1(\Gamma)/\mathrm{kern}(\alpha) \cap \sigma(\pi_1(M)) = \{1\}$, which implies that the cover induced on each fiber is described by the subgroup $\mathrm{kern}(\alpha) \subset \pi_1(\Gamma)$. Let $\dot{P}$ be a principal $\dot{\Gamma}$-bundle over M, then $\dot{P} \to P$ is a covering map with deck transformation group $\pi_1(\Gamma)/\mathrm{kern}(\alpha)$. The exact sequence:

$$1 \longrightarrow \pi_1(\dot{P}) \longrightarrow \pi_1(P) \xrightarrow{\delta} \pi_1(\Gamma)/\mathrm{kern}(\alpha) \longrightarrow 1$$

yields a map δ, which is easily seen to split the above sequence. In fact, the inclusion $\pi_1(\dot{P}) \longrightarrow \pi_1(P)$ has as image the classifying subgroup of the cover, i.e. it coincides with $\sigma(\pi_1(M))$ if a split is given. Thus, both constructions are inverse to each other. $\square$

Ordinary representations of $\pi_1(M)$ are modules over the group algebra in a natural way. The projective counterpart has the cocycle $c_{\widehat{\pi}}$ built into its multiplication.

Definition 4.3.25. Let $c_{\widehat{\pi}} \in H^2_{\mathrm{gr}}(\pi_1(M), S^1)$ be a cocycle classifying an extension of $\pi_1(M)$. We define the *twisted group algebra* $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$ to be the set of mappings $\lambda\colon \pi_1(M) \to \mathbb{C}$ with finite support, written as

$$\lambda = \sum_{a \in \pi_1(M)} \lambda(a)\, a = \sum_{a \in \pi_1(M)} \lambda_a\, a \ ,$$

with multiplication defined by

$$\sum_{a \in \pi_1(M)} \lambda_a\, a \cdot \sum_{b \in \pi_1(M)} \mu_b\, b = \sum_{a,b \in \pi_1(M)} c_{\widehat{\pi}}(a,b) \lambda_a \mu_b\, ab \ .$$

The cocycle condition ensures associativity of this ring. Cohomologous cocycles give rise to isomorphic rings.

Definition 4.3.26. Let V be a left-module over $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$, let R be a subalgebra of the latter, let W be a left R-module. We define the restriction $\mathrm{Res}_R(V)$ to be the module V considered as a left R-module and the induction $\mathrm{Ind}(W) = \mathbb{C}[\pi_1(M), c_{\widehat{\pi}}] \otimes_R W$ as a tensor product over R.

The following is a well-known result lying at the heart of the theory of modules over rings:

Theorem 4.3.27. *Let V and W be as in definition 4.3.26. There is a canonical isomorphism:*

$$\mathrm{Hom}_R(W, \mathrm{Res}_R(V)) \ \simeq \ \mathrm{Hom}_{\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]}(\mathrm{Ind}(W), V) \ .$$

For every finite cover $\overline{M}$ over M allowing a $\dot{\Gamma}$-structure (and therefore also a $\widehat{\Gamma}$-structure) we expect a representation $\widehat{\pi} \longrightarrow U(r)$ like above, where $r \in \mathbb{N}$ is the number of leaves of $\overline{M} \to M$. This should be related to the permutation group on r elements. In fact, we have

Theorem 4.3.28. *Let $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ be a generalized spin^c-extension, $L \to P^{[2]}$ be a corresponding lifting bundle gerbe. Each finite covering $\pi_M : \overline{M} \to M$ with r leaves, such that $\pi_M^* P$ lifts to a principal $\dot{\Gamma}$-bundle, yields a representation $\widehat{\pi} \longrightarrow U(r)$ restricting to the identity on S^1.*

Proof. Set $V = \mathbb{C}[\pi_1(M)/\pi_1(\overline{M})]$, i.e. the free vector space generated by the finite number of cosets coinciding with the number of leaves r. Lemma 4.3.24 provides us with a splitting of the short exact sequence:

$$1 \longrightarrow \pi_1(\Gamma)/\mathrm{kern}(\alpha) \longrightarrow \pi_1(\pi_M^* P) \longrightarrow \pi_1(\overline{M}) \longrightarrow 1$$

and therefore with a homomorphism $\tau \colon \pi_1(\overline{M}) \longrightarrow \pi_1(\pi_M^* P) \subset \pi_1(P) \longrightarrow \widehat{\pi}$. This means that the cocycle defining the central extension is homologous to one that is trivial on $\pi_1(\overline{M}) \times \pi_1(\overline{M})$. Choosing such a representative yields an inclusion of $\mathbb{C}[\pi_1(\overline{M})]$ as a subalgebra of $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$. But we can do even better: Choose representatives $a_1, \dots a_r \in \pi_1(M)$ for the r cosets in $\pi_1(M)/\pi_1(\overline{M})$ and lifts $\widehat{a}_1, \dots, \widehat{a}_r$ of these elements to $\widehat{\pi}$. Let $x \in \pi_1(M)$, then $x = a_i y$ uniquely for an element $y \in \pi_1(\overline{M})$. Set $\widehat{x} = \widehat{a}_i \tau(y)$, which is a lift of x to $\widehat{\pi}$. Note in particular that for $x a_j = a_k y'$ with $y' \in \pi_1(\overline{M})$ we have $\widehat{x a_j} = \widehat{a}_k \tau(y')$ and thus for $x a_j y = a_k y' y$:

$$\widehat{x a_j y} = \widehat{a}_k \tau(y' y) = \widehat{a}_k \tau(y') \tau(y) = \widehat{x a_j} \tau(y) \ .$$

Keeping this in mind, we define $c_{\widehat{\pi}}$ via

$$c_{\widehat{\pi}}(x, a_j y) = \widehat{x}\, \widehat{a}_j \tau(y)\, (\widehat{x a}_j \tau(y))^{-1} = \widehat{x}\, \widehat{a}_j\, (\widehat{x a}_j)^{-1}$$

for $x \in \pi_1(M), y \in \pi_1(\overline{M})$. The cocycle given above still represents the same extension, but its value is now independent of the choice of $y \in \pi_1(\overline{M})$. Thus, we get a well-defined action of $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$ on V induced by:

$$x\,[b] = c(x, b)[xb]$$

for $x \in \pi_1(M)$, $[b] \in \pi_1(M)/\pi_1(\overline{M})$. V can be identified with $\mathbb{C}^r$ by our choice of representatives, i.e. with the basis $\{[a_j], j = 1, \dots, r\} \in V$. The above representation consists of $r \times r$-permutation matrices with the 1s replaced by S^1-values. With respect to the standard scalar product on $\mathbb{C}^r$ this is unitary. □

We have seen that finite covers with $\dot{\Gamma}$-structures give rise to projective representations. On the other hand, the next theorem shows that the existence of a finite cover with $\overline{\Gamma}$-structure is a necessary condition to have a flat countertwisting. Apart from that, it provides a somewhat recursive relationship between the dimension of the representation and the dimension of its intertwiner space with the permutation representation.

Theorem 4.3.29. *Let $1 \to S^1 \to \widehat{\Gamma} \to \Gamma \to 1$ be a generalized spin^c-extension, $L \to P^{[2]}$ be a corresponding lifting bundle gerbe. Let $\widehat{\varrho} \colon \widehat{\pi} \to U(N)$ be a representation on the vector space $W = \mathbb{C}^N$ with closed image. Then there exists* *a finite covering* $\pi_M \colon \overline{M} \longrightarrow M$*, such that* $\pi_M^* P$ *lifts to a principal* $\overline{\Gamma}$*-bundle. Furthermore,*

$$N = \dim(\mathrm{Hom}_{\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]}(\mathbb{C}[\pi_1(M)/\pi_1(\overline{M})], W)) \ .$$

Proof. Let $\widetilde{\pi} \subset \pi_1(P)$ be the kernel of the representation $\widetilde{\varrho} \colon \pi_1(P) \longrightarrow \widehat{\pi} \xrightarrow{\varrho} U(N)$. Since the extension is generalized spin^c, the hypothesis implies that $\widetilde{\varrho}$ has closed image. Thus, $\pi_1(P)/\widetilde{\pi}$ being a closed subgroup of a compact one is finite, i.e. $\widetilde{\pi}$ classifies a finite regular cover of P. Now we have the following diagram with exact columns:

$$\begin{tikzcd}
\mathrm{kern}(\rho) \arrow[r] \arrow[d, "\alpha"] & \pi_1(\Gamma) \arrow[r, "\rho"] \arrow[d, "\alpha"] & S^1 \arrow[d] \\
\widetilde{\pi} \arrow[r] \arrow[d] & \pi_1(P) \arrow[r, "\widetilde{\varrho}"] \arrow[d] & U(N) \arrow[d] \\
\mathrm{kern}(\overline{\varrho}) \arrow[r] & \pi_1(M) \arrow[r, "\overline{\varrho}"] & PU(N)
\end{tikzcd}$$

Since $\alpha^{-1}(\widetilde{\pi}) = \alpha^{-1}(\widetilde{\varrho}^{-1}(1)) = (\widetilde{\varrho} \circ \alpha)^{-1}(1) = \mathrm{kern}(\rho)$, the cover $\overline{P} \to P$ restricts to $\overline{\Gamma} \to \Gamma$ on each fiber. Like in the proof of lemma 4.3.18 we see that $\overline{P} \to P$ is a principal $\overline{\Gamma}$-bundle, actually $\overline{P} = \widetilde{P}/\widetilde{\pi}$. $\widetilde{P} \longrightarrow \widetilde{M}$ is equivariant with respect to the right action of $\widetilde{\pi}$ on $\widetilde{P}$ and $\mathrm{kern}(\overline{\varrho})$ on $\widetilde{M}$, therefore the base of the above bundle is $\overline{M} = \widetilde{M}/\mathrm{kern}(\overline{\varrho})$, which is a finite cover. As the above diagram shows $\overline{\varrho}|_{\pi_1(\overline{M})} \colon \pi_1(\overline{M}) \to PU(N)$ is trivial and therefore lifts to an honest representation of $\pi_1(\overline{M})$. Thus, like in the previous lemma, the cocycle $c_{\widehat{\pi}} \in H^2(\pi_1(M), S^1)$ is cohomologous to one that is trivial over $\mathbb{C}[\pi_1(\overline{M})] \times \mathbb{C}[\pi_1(\overline{M})]$ ensuring that $R = \mathbb{C}[\pi_1(\overline{M})]$ embeds into $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$. The restriction of the $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$-module W to R is $\mathbb{C}^N$ considered as trivial representation. Furthermore:

$$\mathrm{Ind}(\mathbb{C}) = \mathbb{C}[\pi_1(M), c_{\widehat{\pi}}] \otimes_R \mathbb{C} = \mathbb{C}[\pi_1(M)/\pi_1(\overline{M})] \ ,$$

which yields

$$N = \dim(\mathrm{Hom}_R(\mathbb{C}, \mathrm{Res}_R(W))) = \dim(\mathrm{Hom}_{\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]}(\mathbb{C}[\pi_1(M)/\pi_1(\overline{M})], W)) \ .$$

□

4.3.4 Covering bundle gerbes and holonomy

Let $L \to \bar{M}$ be a covering bundle gerbe corresponding to an S^1-extension of a discrete group Γ_d classified by the cocycle $c_\Gamma \in H^2_{\mathrm{gr}}(\Gamma_d, S^1)$. Since L is the pullback of the trivial line bundle over Γ_d, it is itself trivial *as a line bundle* over $\bar{M}$. Still, it need not be trivializable as a bundle gerbe, since the product is twisted by c_Γ, i.e. for $L = \bar{M} \times_M \bar{M} \times \mathbb{C}$:

$$\begin{array}{rcl} \pi_{12}^* L \otimes \pi_{23}^* L & \longrightarrow & \pi_{13}^* L \\ (m_1, m_2, \lambda) \otimes (m_2, m_3, \mu) & \mapsto & (m_1, m_3, c_\Gamma(g_{12}, g_{23}) \lambda \mu) \ . \end{array}$$

where $g_{ij} \in \Gamma_d$ denotes the group element connecting $(m_i, m_j) \in \bar{M} \times_M \bar{M}$. A bundle gerbe module $E \to \bar{M}$ therefore consists of a vector bundle over $\bar{M}$ together with fiber isomorphisms:

$$\gamma^g : E_{\bar{m}} \xrightarrow{\sim} E_{\bar{m}g^{-1}}$$

such that $\gamma^g \circ \gamma^h = c_\Gamma(g,h)\,\gamma^{gh}$, i.e. γ^g acts like a projective representation of (Γ_d, c_Γ) on E. Likewise, a bgm-connection corresponds to a γ-*invariant* form $\eta_E \in \Omega^1(P_E, \mathfrak{u}(n))$.

Now let $E \to \widetilde{M}$ be a twisted Hilbert A-module bundle over the universal cover, i.e. it carries an action of the lifting bundle gerbe induced by a central S^1-extension of the fundamental group. Choose a twisted connection η_E on E. Let $\tau\colon S^1 \to M$ be a loop in M and denote by $\tau^{\widetilde{m}}$ the unique lift of τ to a curve in $\widetilde{M}$ starting at $\widetilde{m} \in \widetilde{M}$. Let

$$\mathrm{hol}(\widetilde{m}, \tau) \;:\; E_{\widetilde{m}} \longrightarrow E_{\widetilde{m}[\tau]}$$

be the holonomy along $\tau^{\widetilde{m}}$, where $[\tau] \in \pi_1(M)$ denotes the homotopy class of the loop. For *flat* connections the holonomy along a path in the universal cover is already fixed by its endpoints. Based on this, we arrive at the following intertwiner relation:

Lemma 4.3.30. *If η_E is a flat connection, then for $g = [\sigma], h = [\tau] \in \pi_1(M)$:*

$$\gamma^h \circ \mathrm{hol}(\widetilde{m}g, \tau) \circ \mathrm{hol}(\widetilde{m}, \sigma) = \mathrm{hol}(\widetilde{m}, \sigma) \circ \gamma^h \circ \mathrm{hol}(\widetilde{m}, \tau)\ .$$

Proof. The holonomy $\mathrm{hol}(\widetilde{m}, \sigma)$ is defined via parallel transport along $\sigma^{\widetilde{m}}$, so by lemma 4.1.26 the homomorphism $\gamma^{h^{-1}} \circ \mathrm{hol}(\widetilde{m}, \sigma) \circ \gamma^h$ is the parallel transport along $\sigma^{\widetilde{m}}h$ where $\pi_1(M)$ acts *pointwise* via deck transformations. The path $\tau^{\widetilde{m}} * (\sigma^{\widetilde{m}}h)$ runs from $\widetilde{m}$ through τ to $\widetilde{m}h$ and then through $(\sigma^{\widetilde{m}}h)$ to $\widetilde{m}gh$. The curve $\sigma^{\widetilde{m}} * \tau^{\widetilde{m}g}$ also starts at $\widetilde{m}$ and ends at $\widetilde{m}gh$, therefore simply connectedness implies that it is homotopic relative endpoints to the former one. Since η_E is flat the two parallel transports agree, which proves the lemma. □

The main application of this small technicality lies in proving that there is a natural way to get projective holonomy representations from flat twisted bundles:

Theorem 4.3.31. *Let η_E be a flat twisted connection on the twisted Hilbert A-module bundle $E \to \widetilde{M}$ corresponding to the cocycle $c_{\widehat{\pi}}$. Fix $\widetilde{m} \in \widetilde{M}$. Then:*

$$h\colon \pi_1(M) \longrightarrow \mathrm{End}(E_{\widetilde{m}}) \quad ; \quad [\tau] \mapsto \gamma^{[\tau]} \circ \mathrm{hol}(\widetilde{m}, \tau)$$

yields a projective representation of the fundamental group with respect to the cocycle $c_{\widehat{\pi}}$.

Proof. Observe that $\mathrm{hol}(\widetilde{m}, \tau)$ only depends on the homotopy type relative endpoints of τ, so h is indeed well-defined. Choose representatives $\tau_i, i \in \{1, 2\}$ for $g_i \in \pi_1(M)$. Using the previous lemma, we get:

$$\begin{aligned} c_{\widehat{\pi}}(g_1, g_2)\, h(g_1 g_2) &= (c_{\widehat{\pi}}(g_1, g_2)\, \gamma^{g_1 g_2}) \circ \mathrm{hol}(\widetilde{m}, \tau_1 * \tau_2) \\ &= \gamma^{g_1} \circ \gamma^{g_2} \circ \mathrm{hol}(\widetilde{m}g_1, \tau_2) \circ \mathrm{hol}(\widetilde{m}, \tau_1) \\ &= \gamma^{g_1} \circ \mathrm{hol}(\widetilde{m}, \tau_1) \circ \gamma^{g_2} \circ \mathrm{hol}(\widetilde{m}, \tau_2) = h(g_1)\, h(g_2)\ . \end{aligned}$$ □

4.4 Obstructions to positive scalar curvature

Let (M, g) be a Riemannian manifold with Levi-Civita-connection ∇. The Riemannian *curvature transformation* R is defined to be

$$R \in \Omega^2(M, \mathrm{End}(TM)) \quad , \quad R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W \ .$$

R coincides with the curvature $\Omega_{TM} \in \Omega^2(M, \mathfrak{so}(n))$ when the latter is viewed as an endomorphism valued 2-form. It can also be defined by extending the covariant derivative to k-forms

$$\nabla : \Omega^k(M, TM) \longrightarrow \Omega^{k+1}(M, TM)$$

by $\nabla(\omega \otimes u) = d\omega \otimes u + (-1)^{\deg(\omega)} \omega \wedge \nabla u$ and setting $R = \nabla \circ \nabla$. The scalar curvature κ_{TM} is the simplest of all curvature invariants that can be obtained from R:

$$\kappa_{TM} = - \sum_{i,j=1}^{n} \langle R(e_i, e_j) e_i, e_j \rangle \ .$$

In this expression e_i is a local orthonormal frame at a point $m \in M$, a choice on which the value of $\kappa_{TM} : M \to \mathbb{R}$ does not depend.

We are first going to analyze how κ_{TM} is related to the projective Dirac operator. Therefore let η_S be the bgm-connection on S induced by the Levi-Civita-connection η_{SO} on P_{SO} like in example 4.1.7. The canonical Lie algebra split

$$\mathfrak{spin}(n) \oplus i\mathbb{R} \longrightarrow \mathfrak{so}(n)$$

is induced by the Lie algebra isomorphism $\Xi_0 : \mathfrak{spin}(n) \xrightarrow{\simeq} \mathfrak{so}(n)$, where we see the left-hand side as a subvector space of $Cl(n)$ and identify the right-hand side with the skew-symmetric traceless matrices on $\mathbb{R}$. On basis elements $\{e_i e_j\}_{i<j} \subset Cl(n)$ the Lie algebra homomorphism Ξ_0 is given explicitly by [36]:

$$\Xi_0(e_i e_j) = 2 e_i \wedge e_j \quad , \quad \Xi_0^{-1}(v \wedge w) = \frac{1}{4}[v, w] \ .$$

The brackets $[\cdot,\cdot]$ denote the commutator in the algebra $Cl(n)$. Now we choose a local orthonormal frame e_i for TM at the point $m \in M$. The curvature $\Omega_{TM} \in \Omega^2(M, \mathrm{Ad}(P_{SO}))$ of η_{SO} can be expressed by a matrix Ω_{ij}:

$$\Omega_{TM} = \sum_{i<j} \Omega_{ij} \otimes (e_i \wedge e_j) \ .$$

The Lie algebra homomorphism Ξ_0^{-1} maps Ω_{TM} to $\Omega_S \in \Omega^2(M, \mathrm{end}(S)) = \Omega^2(M, \mathbb{C}l(M))$, which takes the form:

$$\Omega_S = \Xi_0^{-1} \Omega_{TM} = \frac{1}{2} \sum_{i<j} \Omega_{ij} \otimes e_i e_j \in \Omega^2(M, \mathbb{C}l(n)) \ . \tag{4.46}$$

There is a canonical connection $\nabla_{\mathbb{C}l(M)}$ on $\mathbb{C}l(M)$, which coincides with η_{SO} on $TM \subset \mathbb{C}l(M)$. When the pullback of this bundle is identified with $P \times \mathbb{C}l(n)$, the corresponding pullback connection maps a $\mathbb{C}l(n)$-valued function f to $df + [\Xi_0^{-1}\eta_{SO}, f]$. Therefore we have for $u \in \Gamma(S)$ and $f \in \Gamma(\underline{\mathbb{C}l(n)})$:

$$\nabla_S(f \cdot u) = ((\pi_M^* \nabla_{\mathbb{C}l(M)}) f) \cdot u + f \cdot \nabla_S(u) \ .$$

In particular for $f \in \Gamma(\pi_M^* TM) \subset \Gamma(\underline{\mathbb{C}l(n)})$:

$$\nabla_S(f \cdot u) = ((\pi_M^* \nabla_{TM}) f) \cdot u + f \cdot \nabla_S(u) \ . \tag{4.47}$$

Identify TP with $\pi^* TM \oplus \underline{\mathfrak{g}}$ via η_{SO}. This bundle carries a metric g_P like in (4.34) and a natural connection, which is induced by the pullback of η_{SO} to $\pi^* TM$ and the canonical flat metric on $\underline{\mathfrak{g}}$. Even though the latter is compatible with g_P, it is *not* the Levi-Civita-connection on P due to its torsion:

Lemma 4.4.1. *Let ∇^P be the covariant derivative corresponding to the connection described above, let $v, w \in \Gamma(TM)$ be vector fields on TM and denote by $\widehat{v}$, $\widehat{w}$ their horizontal lifts with respect to η_{SO}, then*

$$\nabla^P_{\widehat{v}} \widehat{w} - \nabla^P_{\widehat{w}} \widehat{v} = [\widehat{v}, \widehat{w}] + \alpha_*(\Omega_{TM}(v, w)) \ ,$$

where $\alpha_* : P \times \mathfrak{g} \longrightarrow TP$ is the map generating the vertical vector fields.

Proof. With respect to the splitting $TP \simeq \pi^* TM \oplus \underline{\mathfrak{g}}$ the horizontal lift of a vector field corresponds to its pullback to $\pi^* TM$. Thus,

$$\nabla^P_{\widehat{v}} \widehat{w} - \nabla^P_{\widehat{w}} \widehat{v} = \widehat{[v, w]} = [\widehat{v}, \widehat{w}] + \alpha_*(\Omega_{TM}(v, w)) \ .$$

□

In the untwisted case the following operators are of fundamental importance:

Definition 4.4.2. Let v, w be vector fields on M, let $E \longrightarrow M$ be a vector bundle with a connection ∇. We define the *invariant second derivative* by

$$\nabla^2_{v,w} u = \nabla_v \nabla_w u - \nabla_{\nabla^{TM}_v w} u \ ,$$

where ∇^{TM} denotes the Levi-Civita-connection on M. Since it is torsion-free, we have $\nabla^2_{v,w} - \nabla^2_{w,v} = \Omega_E(v, w)$. The *connection Laplacian* is defined by taking the trace, i.e.

$$\nabla^* \nabla u = -\mathrm{tr}(\nabla^2_{\cdot,\cdot} u) = -\sum_i \nabla^2_{e_i, e_i} u$$

for some orthonormal tangent frame field e_i of TM.

In the twisted case they show up in the following disguise:

Definition 4.4.3. Let $E \longrightarrow P$ be a bundle gerbe module with connection ∇ and denote by ∇^P the connection discussed above. Let V, W be vector fields on P. We define the *second derivative* by

$$\nabla^2_{V,W} u = \nabla_V \nabla_W u - \nabla_{\nabla^P_V W} u \ ,$$

The *reduced connection Laplacian* is the operator

$$\nabla^* \nabla_{\mathrm{red}} : \Gamma(E) \longrightarrow \Gamma(E) \quad , \quad \nabla^* \nabla_{\mathrm{red}} = -\sum_i \nabla^2_{\widehat{e}_i, \widehat{e}_i} \ ,$$

where $\widehat{e}_i$ denotes the horizontal lift of an orthonormal frame field e_i of TM to TP. It is a differential operator independent of the choice of orthonormal frame. Its symbol is $\sigma(p, \xi) = ||\pi_* \xi||^2$, where $\pi_* : T^* P \to T^* M$ is induced by the connection. In particular, it is only transversally elliptic, but still non-negative and formally self-adjoint.

As predicted by lemma 4.4.1, an additional torsion term will enter the Bochner-Lichnerowicz-formula for the twisted case. We might call it the bad operator and denote it by:

$$\mathfrak{T} = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \nabla_{\alpha_*(\Omega_{TM}(e_i,e_j))}$$

Theorem 4.4.4. *Let $D : \Gamma(S) \longrightarrow \Gamma(S)$ be the projective Dirac operator over the frame bundle of an even-dimensional manifold M. Then*

$$D^2 = \nabla^* \nabla_{\text{red}} + \frac{1}{4} \kappa_{TM} \circ \pi_M + \mathfrak{T} \ ,$$

with the projection $\pi_M : P \to M$. Let E be a flat bundle gerbe module with a flat trivialization Q, then we have:

$$D^2_{E,Q} = \nabla^* \nabla + \frac{1}{4} \kappa_{TM} \ .$$

Proof. Fix a point $p \in P$ that projects to $m \in M$ and choose a local orthonormal frame field e_i of TM, such that $\nabla^{TM}_{e_i} e_j = 0$. Observe that with this choice we also have $\nabla^P_{\widehat{e}_i} \widehat{e}_j = 0$. The general Bochner identity applied to the twisted case takes the following form:

$$\begin{aligned} D^2 &= \sum_{i,j} e_i \cdot \nabla_{\widehat{e}_i} \left(e_i \cdot \nabla_{\widehat{e}_j} \right) = \sum_{i,j} e_i \cdot e_j \cdot \nabla_{\widehat{e}_i} \nabla_{\widehat{e}_j} = \sum_{i,j} e_i \cdot e_j \cdot \nabla^2_{\widehat{e}_i,\widehat{e}_j} \\ &= -\sum_i \nabla^2_{\widehat{e}_i,\widehat{e}_i} + \sum_{i<j} e_i \cdot e_j \cdot \left(\nabla^2_{\widehat{e}_i,\widehat{e}_j} - \nabla^2_{\widehat{e}_j,\widehat{e}_i} \right) \\ &= \nabla^* \nabla_{\text{red}} + \mathfrak{R} + \mathfrak{T} \ . \end{aligned} \tag{4.48}$$

where $\mathfrak{R} \in \Gamma(M, \text{end}(S))$ is the curvature term given by

$$\mathfrak{R} = \frac{1}{2} \sum_{j,k=1}^{n} e_j \cdot e_k \cdot \Omega_S(e_j, e_k) \ . \tag{4.49}$$

Here, $\Omega_S \in \Omega^2(M, \text{end}(S))$ is the curvature of the connection. The dots denote clifford multiplication. Compare the proof of (4.48) with the classical case in [36]. In the first step we use the fact that ∇ acts as a derivative with respect to the Clifford module structure stated in (4.47). In our case the curvature term $\mathfrak{R}$ takes the form

$$\mathfrak{R} = \frac{1}{4} \kappa_{TM}$$

as can be seen after plugging (4.46) into (4.49) by exploiting the Bianchi identities and the symmetries of the curvature tensor, for details see theorem 8.8 in chapter 2 of [36].

For the second statement observe that $\mathfrak{T}$ just contains covariant derivatives in the vertical direction. Let $\alpha_*(X)$ be the fundamental vector field generated by some $X \in \mathfrak{g}$. The horizontal lift $\widehat{X}$ with respect to $\eta_S + \eta_E - \theta_Q$ is the fundamental vector field we get by deriving the group action of $\widehat{\Gamma}$ on $P_{E \otimes S \otimes Q^*}$, since this covers the one of Γ on P. But being a fixpoint of (4.35) implies that the section, when viewed as an equivariant map on $P_{E \otimes S \otimes Q^*}$, is covariantly constant in the directions given by $\widehat{X}$. Apart from that, due to the flatness of E, the curvature remains untouched, i.e. $\Omega_{S \otimes E} = \Omega_S$, so $\mathfrak{R}$ again evaluates to $\frac{1}{4} \kappa_{TM}$. □

Corollary 4.4.5. *Let M be an even-dimensional Riemannian manifold with a metric g, such that κ_{TM} is a positive function (g is then called a positive scalar curvature metric or psc-metric for short). Suppose that the projective Dirac operator D over S has a flat countertwisting E and a flat trivialization Q, then*

$$\langle \widehat{A}(M), [M] \rangle = 0 \ ,$$

i.e. the $\widehat{A}$-genus is an obstruction to positive scalar curvature.

Proof. By theorem 4.4.4, the kernel of the positive operator $D_{E,Q}^2$ vanishes, so does the kernel of $D_{E,Q}$ by formal self-adjointness. Thus, the index of $D_{E,Q}^+$ vanishes. The latter coincides with $N \langle \widehat{A}(M), [M] \rangle$ for some $N \in \mathbb{N}$. □

Remark Note that the existence of a flat countertwisting and a flat trivialization for D are essential. Take again $\mathbb{C}P^2$, which carries a psc-metric, since S^5 does in such a way that S^1 acts isometrically. But we have $\widehat{A}(\mathbb{C}P^2) = -\frac{1}{8}$.

Remark If there is a way to control the bad operator $\mathfrak{T}$ in the sense that either it is non-negative itself or at least $\nabla^*\nabla_{\text{red}} + \mathfrak{T}$ is non-negative, then the distributional index theorem, in particular corollary 4.2.32, still yields some conclusions about the existence of psc-metrics.

Theorem 4.4.6. *Let M be an even-dimensional manifold admitting a metric of positive scalar curvature and assume that $\nabla^*\nabla_{\text{red}} + \mathfrak{T}$ is a non-negative operator in the sense that*

$$\langle (\nabla^*\nabla_{\text{red}} + \mathfrak{T})\, u, u \rangle_{L^2} \geq 0 \quad \forall u \in \Gamma(S) \ ,$$

then $\widehat{A}(M) = 0$.

Proof. We first show that $D : \Gamma(S) \to \Gamma(S)$ still is a formally self-adjoint operator. Let $p \in P$ be a point over $m \in M$ and choose a local orthonormal frame e_i of T_mM with $\nabla^{TM}_{e_j} e_i = 0$ like in theorem 4.4.4. Then:

$$\begin{aligned} \langle Du, v \rangle &= \sum_i \langle e_i \cdot \nabla_{\widehat{e}_i} u, v \rangle = -\sum_i \langle \nabla_{\widehat{e}_i} u, e_i \cdot v \rangle \\ &= \sum_i \left(-\widehat{e}_i \, \langle u, e_i \cdot v \rangle + \langle u, e_i \cdot \nabla_{\widehat{e}_i} v \rangle \right) \\ &= -\sum_i \left(\widehat{e}_i \, \langle u, e_i \cdot v \rangle \right) + \langle u, Dv \rangle \ . \end{aligned}$$

The first term is the divergence of the vector field X defined by

$$\langle X, W \rangle = -\langle u, \pi_* W \cdot v \rangle$$

and therefore vanishes, when integrating over P. Thus, $\text{kern}(D) = \text{kern}(D^2)$, where $\text{kern}(D)$ means the kernel in smooth sections. But the hypothesis implies that $\text{kern}(D^2) = \text{kern}(\nabla^*\nabla_{\text{red}} + \mathfrak{T} + \frac{1}{4}\kappa_{TM} \circ \pi_M) = \{0\}$. Therefore $\text{kern}(D_+) = \text{kern}(D_-) = \{0\}$ and in the notation of equation (4.33), $\text{ind}(D_{+,\lambda}) = \{0\}$ for all λ. This implies that the distributional index vanishes. Note that it can be computed without forming a continuation to Sobolev spaces, but just by evaluating D on *smooth* sections, which is a point to worry about, since D is only transversally elliptic. By corollary 4.2.32 we can choose a function on $\mathrm{Spin}(n)$ with support small enough, which takes the value 1 close to the identity, such that the index coincides with the $\widehat{A}$-genus. $\square$

In the case of manifolds that allow a spin-structure on the universal cover, we have seen that flat countertwistings correspond to projective representations of the fundamental group with respect to the cocycle $c_{\widehat{\pi}} = c_{\mathrm{spin}} \in H^2(\pi_1(M), S^1)$. The drawback of the above theorem is that the existence of flat countertwistings implies that M already has a *finite* cover allowing a spin-structure. We would rather like to deal with the *universal* cover itself, but this is of course either finite or not a compact manifold anymore. Twisted Hilbert A-module bundles provide a very elegant way not only to provide a new obstruction with values in $K_0(A)$, but also to treat all projective representations of $\pi_1(M)$ at once. This is encapsulated in the following definition:

Definition 4.4.7. Let G be a discrete group, $c_G \in H^2_{\mathrm{gr}}(G, S^1)$ a 2-cocycle. Denote by $\mathbb{C}[G, c_G]$ the twisted group algebra. It becomes a $*$-algebra with the involution:

$$\left(\sum_{g \in G} \lambda_g g \right)^* = \sum_{g \in G} \overline{\lambda_{g^{-1}}}\, c_G \left(g, g^{-1}\right)^{-1} g$$

Let $L^2(G)$ be the Hilbert space of elements, such that $\sum_{g\in G} |\lambda_g|^2 < \infty$ with the obvious scalar product. There is a twisted action of $\mathbb{C}[G, c_G]$ on $L^2(G)$ induced by $h \cdot \lambda_g g = c_G(h,g)\lambda_g hg$ and the above involution coincides with taking adjoints with respect to the scalar product. The closure of $\mathbb{C}[G, c_G]$ with respect to the operator norm on $L^2(G)$ is called the *reduced twisted group C^*-algebra $C^*_r(G, c_G)$.*

There is another norm the twisted group algebra can be endowed with, defined by:

$$||g|| = \sup_{\varrho} ||\varrho(g)|| \ ,$$

where the supremum is taken over all projective non-degenerate $*$-representations on Hilbert spaces corresponding to the lifting cocycle c_G. Since $||\varrho(g)||$ is bounded by the l^1-norm, which follows from the triangle inequality, the supremum exists. The closure with respect to this norm is called the *universal twisted group C^*-algebra $C^*_{\max}(G, c_G)$.* By construction it has the universal property, that any $*$-homomorphism from $\mathbb{C}[G, c_G]$ to some $B(H)$ for a Hilbert space H factors through the inclusion $\mathbb{C}[G, c_G] \to C^*_{\max}(G, c_G)$.

The group algebras $C^*(\pi_1(P))$ and $C^*(\pi_1(M), c_{\widehat{\pi}})$

Every projective representation of a group yields an honest representation of some central S^1-extension of it. This relationship should reflect in certain properties of the corresponding group algebras. We will analyze this for the following setup: Let

$$1 \longrightarrow S^1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

be a generalized spin^c-extension. For simplicity we will assume that the homorphism $\overline{\rho} : \pi_1(\Gamma) \longrightarrow \mathbb{Z}/n\mathbb{Z}$ is an isomorphism, but most of the observations also hold in a far more general setting. Let M be a manifold which allows a Γ-structure on its universal cover $\widetilde{M}$ and let P be a principal Γ-bundle over M. The central S^1-extension

$$1 \longrightarrow S^1 \longrightarrow \widehat{\pi} \longrightarrow \pi_1(M) \longrightarrow 1 \tag{4.50}$$

is defined via $\widehat{\pi} = \pi_1(P) \times S^1/\pi_1(\Gamma)$ and by our hypothesis, $\pi_1(\Gamma) = \mathbb{Z}/n\mathbb{Z}$, this sequence reduces to the finite abelian central extension:

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \pi_1(P) \longrightarrow \pi_1(M) \longrightarrow 1 \tag{4.51}$$

(by lemma 4.3.18 kern(α) is trivial). Therefore the commutative group ring $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ is a central subalgebra of $\mathbb{C}[\pi_1(P)]$. Denote by $x \in \mathbb{Z}/n\mathbb{Z}$ a generator of the cyclic group. Considered as an operator on $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$, it gives rise to n projections onto the eigenspaces corresponding to the eigenvalues $z_j = \exp\left(\frac{2\pi i j}{n}\right)$ for $j = 0 \dots n-1$:

$$p_j = \frac{1}{n}\sum_{k=0}^{n-1}(z_j^{-1}\, x)^k \ .$$

(with $\mathbb{Z}/n\mathbb{Z}$ written multiplicatively). Indeed, $x\, p_j = z_j\, p_j$, $p_j^2 = p_j$ and $p_j^* = p_j$. As elements in $\mathbb{C}[\pi_1(P)]$ the p_j are therefore central projections. Let $c_{\pi_1(P)} \in H^2_{\mathrm{gr}}(\pi_1(M), \mathbb{Z}/n\mathbb{Z})$ be the cocycle representing the extension (4.51). We will identify $\pi_1(P)$ with $\pi_1(M) \times_{c_{\pi_1(P)}} \mathbb{Z}/n\mathbb{Z}$, which coincides with the product as a set, but carries the multiplication $(g,a)\cdot(h,b) = (gh, a+b+c_{\pi_1(P)}(g,h))$ twisted by the cocycle. Given $c_{\pi_1(P)}$ there is a canonical representative for $c_{\widehat{\pi}} = \exp(c_{\pi_1(P)}) \in H^2_{\mathrm{gr}}(\pi_1(M), S^1)$. With these identifications we have an algebra homomorphism:

$$\alpha : \mathbb{C}[\pi_1(P)] \longrightarrow \mathbb{C}[\pi_1(M), c_{\widehat{\pi}}] \quad ; \quad \sum_{(g,a)\in\pi_1(P)} \lambda_{(g,a)}\,(g,a) \mapsto \sum_{(g,a)} \lambda_{(g,a)} \exp(a)\, g$$

Let $p = p_1$ be the first of the projections above, then α restricted to the image of p has an inverse β:

$$\beta : \mathbb{C}[\pi_1(M), c_{\widehat{\pi}}] \longrightarrow p\,\mathbb{C}[\pi_1(P)] \quad ; \quad \sum_{g\in\pi_1(M)} \lambda_g\, g \mapsto p\left(\sum_g \lambda_g (g,0)\right) \ .$$

That β is multiplicative is easily seen from the following calculation:

$$\begin{aligned} c_{\widehat{\pi}}(g,h)\, p\,(gh,0) &= \exp(c_{\pi_1(P)}(g,h))\, p\,(gh,0) = p\,(1, c_{\pi_1(P)}(g,h))\,(gh,0) \\ &= p\,(gh, c_{\pi_1(P)}(g,h)) \ , \end{aligned}$$

where the last equality in the first row uses $\exp(a)\, p = p\,(1,a)$. Thus, the twisted group algebra $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$ is isomorphic to the corner $p\,\mathbb{C}[\pi_1(P)]$ with the central projection p. Furthermore α and β induce intertwiners of the left regular representations:

$$\begin{tikzcd} L^2(\pi_1(P)) \arrow[r, "{(g,a)}"] \arrow[d, "\alpha"] & L^2(\pi_1(P)) \arrow[d, "\alpha"] \\ L^2(\pi_1(M)) \arrow[r, "\exp(a)g"] \arrow[d, "\beta"] & L^2(\pi_1(M)) \arrow[d, "\beta"] \\ p\,L^2(\pi_1(P)) \arrow[r, "{p(g,a)}"] & p\,L^2(\pi_1(P)) \end{tikzcd}$$

Therefore $L^2(\pi_1(M))$ can be seen as a subrepresentation of $L^2(\pi_1(P))$, which restricts to the standard left regular representation of $\mathbb{C}[\pi_1(M), c_{\widehat{\pi}}]$. Summarizing we get an isomorphism of the reduced C^*-algebras:

$$C_r^*(\pi_1(M), c_{\widehat{\pi}}) \;\simeq\; p\,C_r^*(\pi_1(P)) \;.$$

Since every projective, non-degenerate $*$-representation ϱ of $\pi_1(M)$ pulls back to an honest non-degenerate $*$-representation via α, the latter induces a continuous $*$-homomorphism

$$\alpha : C^*_{\max}(\pi_1(P)) \longrightarrow C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \;.$$

A similar argument shows, that β extends to

$$\beta : C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \longrightarrow p\,C^*_{\max}(\pi_1(P)) \;.$$

Altogether we get an isomorphism:

$$C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \;\simeq\; p\,C^*_{\max}(\pi_1(P)) \;.$$

4.4.1 The twisted Mishchenko-Fomenko line bundle

In this section we introduce the definition of the universal version of a flat countertwisting. With the setup described in the last section, we have an action of $\widehat{\pi} = \pi_1(P) \times S^1/\pi_1(\Gamma)$ on the reduced as well as the maximal group C^*-algebra $C^*(\pi_1(M), c_{\widehat{\pi}}) \simeq p\,C^*(\pi_1(P))$, which is basically just left multiplication:

$$[\overline{g}, z]\,h = z\,\overline{g}\,h \quad \text{for } \overline{g} \in \pi_1(P), z \in S^1 \text{ and } h \in p\,C^*(\pi_1(P)) \;.$$

Note the well-definedness of the above action:

$$[\overline{g}\,a, \exp a^{-1}\,z]\,h = \exp a^{-1}\,z\,\overline{g}\,a\,h = \exp a^{-1}\,z\,\overline{g}\,\exp a\,h = z\,\overline{g}\,h$$

for $a \in \pi_1(\Gamma) = \mathbb{Z}/n\mathbb{Z}$, which is due to the fact that $ah = a\,p\,h = \exp(a)h$.

Definition 4.4.8. Let M be a manifold, such that its universal cover $\widetilde{M}$ allows a $\overline{\Gamma}$-structure, then $\mathcal{V}_{r,\max} = \widetilde{M} \times C^*_{r,\max}(\pi_1(M), c_{\widehat{\pi}})$ is a twisted Hilbert A-module bundle for the lifting bundle gerbe $L \to \widetilde{M}^{[2]}$ of the central extension:

$$1 \longrightarrow S^1 \longrightarrow \widehat{\pi} \longrightarrow \pi_1(M) \longrightarrow 1 \;.$$

It is called the *(reduced/maximal) universal flat countertwisting bundle* or in case we have $\Gamma = SO(n)$ and $\overline{\Gamma} = \mathrm{Spin}(n)$ the *twisted* MISHCHENKO-FOMENKO *line bundle.*

Theorem 4.4.9. *Let (M, g) be an even-dimensional Riemannian manifold, such that its universal cover $\widetilde{M}$ carries a spin-structure and g is a positive scalar curvature metric. Then*

$$\mathrm{ind}(D_+^{\mathcal{V}_{r,\max}}) = \langle\, \mathrm{ch}_Q(\mathcal{V}_{r,\max})\widehat{A}(M), [M]\,\rangle = 0 \in K_0(C^*_{r,\max}(\pi_1(M), c_{\widehat{\pi}})) \otimes \mathbb{R}\ ,$$

where we used the flat trivialization Q from lemma 4.3.19. In other words: The projective Dirac operator twisted with the twisted Mishchenko-Fomenko *line bundle is an obstruction to the existence of psc-metrics.*

Proof. The index formula is an application of corollary 4.3.16. Since the Mishchenko-Fomenko line bundle is flat, the obstruction property follows from the Bochner-Lichnerowicz argument given in theorem 4.4.4. □

Remark The above obstruction to positive scalar curvature metrics coincides with the θ-index Stolz used in [66]. The latter is defined using a super-group $\gamma(E)$ associated to a vector bundle: It consists of the extension $\pi_1(P_O/r)$, where P_O is the (unoriented) frame bundle of E and $r \in O(n)$ is the reflection about the hyperplane perpendicular to $e_1 = (1, 0, \ldots, 0)$. There is a $\mathbb{Z}/2\mathbb{Z}$-grading on the group induced by the orientation character $w_1 \in H^1(M, \mathbb{Z}/2\mathbb{Z})$. In the oriented case that we treat here, w_1 is trivial and P_O/r coincides with P_{SO}. In fact, we have neglected orientation twists living in $H^1(M, \mathbb{Z}/2\mathbb{Z})$ throughout the theory, even though it is possible, though tedious to fit them in. The group $G(n, \gamma)$ that appears in [66] coincides with $\mathrm{Spin}(n)\bar{\otimes}\pi_1(P_{SO})$. The fact that E has a distinguished $\gamma(E)$-structure translates in the language of twisted K-theory to the existence of a flat trivialization for the lifting bundle gerbe $L \longrightarrow P_{SO} \times_M \widetilde{M}$ associated to the short exact sequence:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(n)\bar{\otimes}\pi_1(P_{SO}) \longrightarrow SO(n) \times \pi_1(M) \longrightarrow 1\ .$$

The new insight gained from the above description of $\theta(M)$ is that it splits into a twisted K-theory class and one in twisted K-homology like in the untwisted case. As we will see in the next chapter, we can exploit this splitting to transfer proofs given in the untwisted case to our setting.

4.5 KK-theory and projective Dirac operators

This section will show that the spinor countertwisting of a generalized projective Dirac operator $D : \Gamma(F) \longrightarrow \Gamma(F)$ yields a sensible K-homology class, i.e. an element in $KK(C(M, \mathcal{K}), A)$, where $\mathcal{K}$ is a matrix bundle with $dd(\mathcal{K}) = dd(\mathcal{A})$. The Kasparov intersection product

$$\begin{array}{rcl} KK(\mathbb{C}, C(M, \mathcal{A})) \times KK(C(M, \mathcal{K}), \mathbb{C}) & \longrightarrow & KK(\mathbb{C}, A) \ \simeq \ K_0(A) \\ ([E], [D]) & \mapsto & [E] \otimes_{C(M,\mathcal{K})} [D] \end{array}$$

then coincides with the index class $[D_+^{E,Q}] \in K_0(A)$ of D twisted with E. The trivialization Q is hidden in the choice of Morita equivalence between $C(M, \mathcal{K}) \otimes A$ and $C(M, \mathcal{A})$ needed to evaluate the product and, as we have seen, the index class *depends* on that choice.

The twisted K-theory class of a twisted Hilbert A-module bundle $[E] \in K^0_{\mathcal{A}}(M)$ translates into a projection valued section $t \in C(M, M_m(\mathcal{A}))$ with $[t] \in$

$K_0(C(M,\mathcal{A}))$ as was proven in theorem 3.2.8. A minor step further we arrive at the FREDHOLM module $[tC(M,\mathcal{A}^m),0] \in KK(\mathbb{C}, C(M,\mathcal{A})) \simeq K_0(C(M,\mathcal{A}))$ representing $[E]$ in KK-theory (we have dropped the canonical unital action of $\mathbb{C}$). Now choose a matrix bundle $\mathcal{K}$ associated to a principal $PU(n)$-bundle $\widetilde{P}$ with $dd(\mathcal{K}) = dd(\mathcal{A})$. As follows from theorem 3.4.1, a trivialization Q of the tensor product $\widetilde{L}^* \boxtimes L \longrightarrow (\widetilde{P} \times_M P)^{[2]}$ corresponds to an imprimitivity bimodule $C(M,V)$, where $V = \widehat{\mathcal{P}} \times_\tau (\mathbb{C}^n \otimes A)$, on which $C(M, \mathcal{K} \otimes A)$ acts on the left and $C(M,\mathcal{A})$ acts on the right. Apart from this we will need the conjugate bimodule $C(M,\overline{V})$ of $C(M,V)$, where $\overline{V} = \widehat{\mathcal{P}} \times_{\bar\tau} (\mathbb{C}^{n*} \otimes A)$ with

$$\bar\tau([U,\widehat{g}])(\xi \otimes a) = (\xi \circ U^* \otimes \widehat{g}a)$$

for $[U,\widehat{g}] \in U(n)\bar\otimes\widehat{\Gamma}$. The latter is a $C(M,\mathcal{A})$–$C(M,\mathcal{K}\otimes A)$-bimodule representing the inverse MORITA equivalence. To summarize: Our choice of countertwisting and trivialization induces a map

$$\begin{array}{ccc} KK(\mathbb{C}, C(M,\mathcal{A})) & \xrightarrow{\sim} & KK(\mathbb{C}, C(M,\mathcal{K}\otimes A)) \\ [tC(M,\mathcal{A}^m),0] & \mapsto & [tC(M,\overline{V}^m),0]\ . \end{array}$$

There is another way to describe this class: Note that $S^* = \widetilde{P} \times \mathbb{C}^{n*}$ is a bundle gerbe module over $\widetilde{L}^*$, so $S^* \boxtimes E$ can be pushed down to a Hilbert A-module bundle over M, denoted by $\pi_!(S^* \boxtimes E)$. The sections $C(M, \pi_!(S^* \boxtimes E))$ form a Hilbert $C(M,\mathcal{K}\otimes A)$-module: The action of $C(M,\mathcal{K})$ is induced by the following commutative diagram:

$$\begin{tikzcd} \widetilde{L}^* \otimes \pi_2^*(S^* \otimes \underline{M_n(\mathbb{C})}) \arrow[r, "\kappa"] \arrow[d, "\mathrm{id} \otimes \pi_2^* m"] & \pi_1^*(S^* \otimes \underline{M_n(\mathbb{C})}) \arrow[d, "\pi_1^* m"] \\ \widetilde{L}^* \otimes \pi_2^* S^* \arrow[r] & \pi_1^* S^* \end{tikzcd}$$

where $\kappa([U,\lambda]\otimes\xi\otimes T) = \lambda\,\xi\circ U^* \otimes UTU^*$ for $[U,\lambda] \in \widetilde{L}^*, \xi \in S^*, T \in \underline{M_n(\mathbb{C})}$ and $m(\xi \otimes T) = \xi \circ T$ is just right multiplication. The $M_n(\mathbb{C})$-valued scalar product is $\langle \xi, \eta\rangle_{M_n\mathbb{C}} = v_\xi\, \eta$, where v_ξ is the unique vector such that $\langle v_\xi, \cdot\rangle_{\mathbb{C}} = \xi(\cdot)$. Combined with the A-valued scalar product on E, this is easily seen to descend to a Hilbert $C(M,\mathcal{K}\otimes A)$-module structure. So, we are just a short step away from proving:

Theorem 4.5.1. *Let E be a twisted Hilbert A-module bundle and S^* a spinor countertwisting like above with trivialization inducing the bundle $\overline{V} \longrightarrow M$. Let $t \in C(M, M_m(\mathcal{A}))$ be a projection representing $[E] \in K_0(C(M,\mathcal{A}))$. Then $tC(M,\overline{V}^m)$ is isomorphic to $C(M,\pi_!(S^*\boxtimes E))$ as a right Hilbert $C(M,\mathcal{K}\otimes A)$-module.*

Proof. This is certainly true, if E is the trivial twisted Hilbert A-module bundle, since for $E = \underline{A^m}$ the push-down $\pi_!(S^* \boxtimes E)$ along the fixed trivialization coincides with $\overline{V}^m$. For the general case we have $E \simeq t\underline{A^m}$, where t is considered as a twisted bundle morphism. Thus,

$$C(M,\pi_!(S^*\boxtimes E)) \simeq C(M,\pi_!(S^*\boxtimes t\underline{A^m})) = C(M, t\,\pi_!(S^*\boxtimes\underline{A^m})) = C(M, t\,\overline{V}^m)\ .$$

□

We have the following identification of twisted K-theory with coefficients in the bundle $\mathcal{A}$ with the group $KK(\mathbb{C}, C(M, \mathcal{K} \otimes A))$ depending on the choice of countertwisting and trivialization:

$$\begin{array}{rcl} K^0_{\mathcal{A}}(M) & \xrightarrow{\sim} & KK(\mathbb{C}, C(M, \mathcal{K} \otimes A)) \\ {[E]} & \mapsto & [C(M, \pi_!(S^* \boxtimes E)), 0] =: [E^{S,Q}] \; . \end{array} \tag{4.52}$$

Let us now turn to a similar description of the K-homology class in question. Let $D : \Gamma(F) \longrightarrow \Gamma(F)$ be a generalized projective Dirac operator with Clifford symbol $c_F : T^*M \longrightarrow \mathrm{end}(F)$ over the twisted Hilbert A-module bundle F. Let E be a countertwisting and Q be a trivialization, then $D^{E,Q} : \Gamma(\pi_!(F \boxtimes E)) \longrightarrow \Gamma(\pi_!(F \boxtimes E))$ is an *elliptic* first-order A-linear pseudodifferential operator acting on the Hilbert A-module bundle $\pi_!(F \boxtimes E)$ with symbol $c_F \otimes \mathrm{id}_E \otimes \mathrm{id}_Q : T^*M \longrightarrow \mathrm{End}(\pi_!(F \boxtimes E))$. Before we can summarize the analytic properties of $D^{E,Q}$, we need the following notion:

Definition 4.5.2. A densely defined operator $T : V \longrightarrow V'$ on a Hilbert A-module with densely defined adjoint is called *regular* if $1 + T^*T$ is surjective or equivalently if its graph $G(T)$ is orthocomplemented.

The next theorem forms the bridge to KK-theory, which only treats bounded operators. It is proven for example in proposition 21 in [75].

Theorem 4.5.3. *Let D, E, Q be as above, then $D^{E,Q}$ extends to an unbounded, self-adjoint, regular operator on the Hilbert A-module $L^2(\pi_!(F \boxtimes E))$ of square integrable sections:*

$$D^{E,Q} : L^2(\pi_!(F \boxtimes E)) \longrightarrow L^2(\pi_!(F \boxtimes E)) \; .$$

Remark Formal self-adjointness of $D^{E,Q}$ is a direct consequence of the properties of the Clifford symbol. Regularity then follows from ellipticity, since a parametrix allows the construction of a bounded operator having the graph of $D^{E,Q}$ as its closed image, which implies that the latter is orthocomplemented.

The fact making regular operators on Hilbert A-modules invaluable for KK-theory is that they can be turned into bounded ones using the WORONOWICZ- (or bounded-) transform [35]:

$$T \mapsto T(1 + T^*T)^{-\frac{1}{2}}$$

We now focus on a spinor countertwisting S for D. By the same argument as for K-theory, we can equip $L^2(\pi_!(F \boxtimes S))$ with a left action of $C(M, \mathcal{K})$, i.e. a homomorphism:

$$\varphi_{\mathcal{K}} : C(M, \mathcal{K}) \longrightarrow \mathrm{End}(L^2(\pi_!(F \boxtimes S))) \; .$$

which is induced by the canonical left action of $M_n(\mathbb{C})$ on S. Since we assume S to be graded trivially, $\varphi_{\mathcal{K}}$ maps into the *even* part of the endomorphisms. So far, we have set up the scene for the following theorem:

Theorem 4.5.4. *Let D be a generalized projective Dirac operator, S be a spinor countertwisting, $\mathcal{K} = \mathrm{end}(S)$ its twisted endomorphism bundle and Q a trivialization. There is a* FREDHOLM *module representing $D^{S,Q}$ in KK-theory defined* *by:*

$$\left[D^{S,Q}\right] = \left[L^2(\pi_!(F \boxtimes S)), \varphi_{\mathcal{K}}, D^{S,Q}\left(1 + \left(D^{S,Q}\right)^2\right)^{-\frac{1}{2}}\right] \in KK(C(M,\mathcal{K}), A) \ .$$

Proof. A Fredholm module $[H, \varphi, T] \in KK(B, A)$ has to satisfy $[T, \varphi(b)] \in \mathcal{K}(H)$, $(T^2 - 1)\varphi(b) \in \mathcal{K}(H)$ and $(T - T^*)\varphi(b) \in \mathcal{K}(H)$ for all $b \in b$. In our case

$$T = D^{S,Q}\left(1 + \left(D^{S,Q}\right)^2\right)^{-\frac{1}{2}} \ .$$

Since T is self-adjoint and $1 - T^2$ is the extension of $\left(1 + \left(D^{S,Q}\right)^2\right)^{-1}$, which is compact, the latter two conditions hold. Next, note that for $f \in C^\infty(M, \mathcal{K})$:

$$\left[D^{S,Q}, \varphi_{\mathcal{K}}(f)\right] = -\sum_i c_F(e_i^*) \boxtimes \varphi_{\mathcal{K}}\left(\nabla^{\mathcal{K}}_{e_i} f\right) \ ,$$

i.e. the commutator is a bounded operator on $L^2(\pi_!(F \boxtimes S))$. Now we apply a technique used in the classical case in [8] and explained in detail for the C^*-algebra case in [17] (see also [61, 10]). We can express T by the integral:

$$T = \frac{2}{\pi}\int_0^\infty D^{S,Q}\left(\left(D^{S,Q}\right)^2 + 1 + \lambda^2\right)^{-1} d\lambda \ ,$$

where Tu converges in norm if $u \in H^1(\pi_!(F \boxtimes S))$ – the first Sobolev space of sections. Apply $[T, S^{-1}] = -S^{-1}[T, S]S^{-1}$ to get:

$$\begin{aligned} [T, \varphi_{\mathcal{K}}(f)] &= \frac{2}{\pi}\int_0^\infty K\big((1+\lambda^2)[D^{S,Q}, \varphi_{\mathcal{K}}(f)] - D^{S,Q}[D^{S,Q}, \varphi_{\mathcal{K}}(f)]D^{S,Q}\big)K d\lambda \\ K &= \left(\left(D^{S,Q}\right)^2 + 1 + \lambda^2\right)^{-1} \ . \end{aligned}$$

The bounds $\|D^{S,Q}K\| \leq C(d + \lambda^2)^{-\frac{1}{2}}$ and $\|K\| \leq (d + \lambda^2)^{-1}$ (see [17]) for positive constants C and d show that the commutator integral actually converges in norm. Since $D^{S,Q}K$ is compact and $[D^{S,Q}, \varphi_{\mathcal{K}}(f)]$ is bounded, the term under the integral sign is compact, therefore the commutator $[T, \varphi_{\mathcal{K}}(f)]$ is as well. $\square$

Let $E_i \to P_i$ be twisted Hilbert A_i-module bundles over principal Γ_i-bundle P_i for $i \in \{1, 2\}$. Suppose $dd(P_1) = -dd(P_2)$. Let $S \to \widetilde{P}$ be a spinor countertwisting for E_2 over a principal $PU(n)$-bundle $\widetilde{P}$. Then S^* is a spinor countertwisting for E_1. Choose trivializations Q_i for $E_1 \boxtimes S^*$ and $S \boxtimes E_2$ respectively. They correspond to a principal $\widehat{\Gamma}_1 \otimes U(n)^*$-bundle $\widehat{\mathcal{P}}_1$ and a principal $U(n) \otimes \widehat{\Gamma}_2$-bundle $\widehat{\mathcal{P}}_2$ and induce a trivialization Q_{12} of $E_1 \boxtimes E_2$ in the following way: Consider the pullback diagram

$$\begin{tikzcd} \widehat{\mathcal{P}}_3 \arrow[rr] \arrow[d] & & \widehat{\mathcal{P}}_1 \boxtimes \widehat{\mathcal{P}}_2 \arrow[d] \\ P_1 \times_M \widetilde{P} \times_M P_2 \arrow[rr, "\Delta"] & & P_1 \times_M \widetilde{P} \times_M \widetilde{P} \times_M P_2 \end{tikzcd}$$

defining $\widehat{\mathcal{P}}_3$ via $\Delta(p_1, \widetilde{p}, p_2) = (p_1, \widetilde{p}, \widetilde{p}, p_2)$. Note that $\widehat{\mathcal{P}}_3$ is a principal $(\widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2) \times PU(n)$-bundle over M, since the line bundle $U(n)^* \boxtimes U(n) \to PU(n) \times PU(n)$

is trivial over the diagonal. $\widehat{P}_3 = \widehat{\mathcal{P}}_3/PU(n)$ therefore defines a $\widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2$-bundle over M and yields a trivialization Q_{12}. Likewise, given bgm-connections on both of the Q_i's they induce one on $Q_{12} = \Delta^*(Q_1 \boxtimes Q_2)/PU(n)$.

Q_{12} is natural with respect to imprimitivity bimodules in the following sense: Let $V_1 = \pi_!(\underline{A_1} \boxtimes S^*)$. Then $C(M, V_1)$ is a $C(M, \mathcal{A}_1)$–$C(M, A_1 \otimes \mathcal{K})$ bimodule. Similarly $V_2 = \pi_!(S \boxtimes \underline{A_2})$ induces the $C(M, \mathcal{K} \otimes \mathcal{A}_2)$–$C(M, A_2)$ bimodule $C(M, V_2)$. Both of them are incarnations of Morita equivalences. Likewise we set $V_{12} = \pi_!(\underline{A_1} \boxtimes \underline{A_2})$, which is a $C(M, \mathcal{A}_1 \otimes \mathcal{A}_2)$–$C(M, A_1 \otimes A_2)$ bimodule.

Lemma 4.5.5. *With V_i and V_{12} as above there is a Hilbert bimodule isomorphism:*

$$C(M, V_1) \otimes_{C(M,\mathcal{K}) \otimes A_1} C(M, V_2) \otimes A_1 \xrightarrow{\sim} C(M, V_{12})$$

where the tensor product is the inner one taken over the inclusion $C(M, \mathcal{K}) \otimes A_1 \to C(M, \mathcal{K} \otimes \mathcal{A}_2) \otimes A_1$.

Proof. A continuous section u_1 of V_1 can be identified with an equivariant map $f_1 : \widehat{\mathcal{P}}_1 \longrightarrow A_1 \otimes \mathbb{C}^{n*}$. Likewise, there exists $f_2 : \widehat{\mathcal{P}}_2 \longrightarrow \mathbb{C}^n \otimes A_2$ corresponding to $u_2 \in C(M, V_2)$. The canonical $A_1 \otimes A_2$–$A_1 \otimes A_2$ bimodule isomorpism

$$\begin{array}{rcl} \Psi : (A_1 \otimes \mathbb{C}^{n*}) \otimes_{M_n(\mathbb{C}) \otimes A_1} (\mathbb{C}^n \otimes A_1 \otimes A_2) & \longrightarrow & A_1 \otimes A_2 \\ (a_1 \otimes \xi) \otimes_{M_n(\mathbb{C}) \otimes A_1} (v \otimes a_1' \otimes a_2) & \mapsto & \xi(v)\, a_1 a_1' \otimes a_2 \end{array}$$

satisfies $\Psi((\widehat{g}_1 a_1 \otimes \xi \circ \widehat{T}^*) \otimes (\widehat{T} v \otimes a_1' \otimes \widehat{g}_2 a_2)) = \xi(v)\, \widehat{g}_1 a_1 a_1' \otimes \widehat{g}_2 a_2$ for $\widehat{g}_i \in \widehat{\Gamma}_i$, $\widehat{T} \in U(n)$. Therefore the induced map

$$f_3 : \widehat{\mathcal{P}}_3 \longrightarrow A_1 \otimes A_2 \quad ; \quad [\widehat{p}_1, \widehat{p}_2] \mapsto \Psi(f_1(\widehat{p}_1) \otimes f_2(\widehat{p}_2))$$

is $\widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2$-equivariant and $PU(n)$-invariant, thus corresponds to a section $u_3 \in C(M, V_{12})$. Using a common trivializing cover U_i for the bundles V_i and V_{12}, the bimodule homomorphism

$$\begin{array}{rcl} C(M, V_1) \otimes_{C(M,\mathcal{K}) \otimes A_1} C(M, V_2) \otimes A_1 & \longrightarrow & C(M, V_{12}) \\ u_1 \otimes u_2 & \mapsto & u_3 \end{array}$$

restricts to

$$C_0(U_i, A_1 \otimes \mathbb{C}^{n*}) \otimes_{C(U_i, M_n(\mathbb{C})) \otimes A_1} C_0(U_i, \mathbb{C}^n \otimes A_2) \otimes A_1 \longrightarrow C_0(U_i, A_1 \otimes A_2)\ ,$$

which means that a partition of unity argument basically reduces it to an application of Ψ. This proves that it is an isomorphism. □

Corollary 4.5.6. *Let $E_i \to P_i$ be twisted Hilbert A_i-module bundles and S be a countertwisting for E_2 like above. Then there is an isomorphism θ of right Hilbert $A_1 \otimes A_2$-modules:*

$$C(M, \pi_!(E_1 \boxtimes S^*)) \otimes_{C(M,\mathcal{K}) \otimes A_1} L^2(M, \pi_!(S \boxtimes E_2)) \otimes A_1 \xrightarrow{\sim} L^2(M, \pi_!(E_1 \boxtimes E_2))\ ,$$

where the push-down on the left hand side is with respect to Q_i and on the right hand side is given by Q_{12}.

Proof. In case $E_i = \underline{A_i}$ the statement is a direct consequence of the previous lemma and the observation that

$$\begin{aligned} C(M,V_2)\otimes A_1 \otimes_{C(M)\otimes A_2\otimes A_1} L^2(M,A_2\otimes A_1) &= L^2(M,V_2)\otimes A_1 \\ C(M,V_{12}) \otimes_{C(M)\otimes A_2\otimes A_1} L^2(M,A_2\otimes A_1) &= L^2(M,V_{12}) \ . \end{aligned}$$

as $C(M, A_1\otimes A_2)$–$A_1\otimes A_2$ bimodules. In the general case we have projections $t_i \in C(M, M_m(\mathcal{A}_i))$ such that

$$\begin{aligned} & C(M,\pi_!(E_1\boxtimes S^*)) \otimes_{C(M,\mathcal{K})\otimes A_1} L^2(M,\pi_!(S\boxtimes E_2))\otimes A_1 \\ \simeq\ & C(M, t_1V_1^m) \otimes_{C(M,\mathcal{K})\otimes A_1} L^2(M, t_2V_2^m)\otimes A_1 \\ \simeq\ & (t_1\otimes t_2)\, C(M,V_1^m)\otimes_{C(M,\mathcal{K})\otimes A_1} L^2(M,V_2^m)\otimes A_1 \\ \simeq\ & (t_1\otimes t_2)\, L^2(M,V_{12}^m) \ \simeq\ L^2(M,\pi_!(E_1\boxtimes E_2)) \ . \end{aligned}$$

□

This is the first brick on the road to the following theorem announced at the beginning of this section:

Theorem 4.5.7. *Let D be a generalized projective Dirac operator acting on a bundle gerbe module $E_2 \to P_2$. Let S be a spinor countertwisting and Q_2 be a trivialization of $S\boxtimes E_2$. If $E_1 \to P_1$ is a twisted Hilbert A_1-module bundle such that $dd(P_1) = -dd(P_2)$ with countertwisting S^* and trivialization Q_1, then:*

$$[E_1^{S,Q_1}] \otimes_{C(M,\mathcal{K})} [D^{S,Q_2}] \in KK(\mathbb{C}, A_1)$$

is the class representing the MISHCHENKO-FOMENKO*-index of $D^{E_1,Q_{12}}$, where Q_{12} is the trivialization induced by Q_1 and Q_2. In particular, the intersection product depends only on the choices of trivializations involved and not directly on the countertwisting.*

Proof. It suffices to show that the intersection product coincides with the FREDHOLM module:

$$\left[L^2(\pi_!(E_1\boxtimes E_2)), D^{E_1,Q_{12}}\left(1+\left(D^{E_1,Q_{12}}\right)^2\right)^{-\frac{1}{2}}\right] \ . \tag{4.53}$$

The isomorphism $KK(\mathbb{C},A_1) \xrightarrow{\sim} K_0(A_1)$ is defined by taking the MISHCHENKO-FOMENKO-index of the odd part of the operator involved, but since the latter is not changed under composition with invertible bounded operators, the index of $D_+^{E_1,Q_{12}} : H^s(M,\pi_!(E_1\boxtimes E_2^+)) \to H^{s-1}(M,\pi_!(E_1\boxtimes E_2^-))$ coincides with that deduced from (4.53). Set

$$F = D^{E_1,Q_{12}}\left(1+\left(D^{E_1,Q_{12}}\right)^2\right)^{-\frac{1}{2}} \quad , \quad F_2 = D^{S,Q_2}\left(1+\left(D^{S,Q_2}\right)^2\right)^{-\frac{1}{2}} \ .$$

We prove that F is an F_2-connection from which the above assertion will follow, since our representative of $[E_1^{S,Q_1}]$ contains the zero operator. Using the isomorphism θ from corollary 4.5.6, we form an operator

$$T_{u_1} : L^2(M,\pi_!(S\boxtimes E_2))\otimes A_1 \longrightarrow L^2(M,\pi_!(E_1\boxtimes E_2)) \quad ; \quad u_2 \mapsto \theta(u_1\otimes u_2)$$

for every $u_1 \in C(M, \pi_!(E_1 \boxtimes S^*))$. By definition F is an F_2-connection if

$$T_{u_1} \circ F_2 - F \circ T_{u_1} \in \mathrm{Hom}(L^2(M, \pi_!(S \boxtimes E_2)) \otimes A_1, L^2(M, \pi_!(E_1 \boxtimes E_2)))$$

is a compact A_1-linear operator. The crucial observation in proving this is that for $u_1 \in C^\infty(M, \pi_!(E_1 \boxtimes S^*))$ we have the following commutator identity, which holds (at least) on $C^\infty(M, \pi_!(S \boxtimes E_2) \otimes A_1)$

$$T_{u_1} \circ D^{S,Q_2} - D^{E_1,Q_{12}} \circ T_{u_1} = -\sum_i T_{f_i} c(e_i^*) =: -S \tag{4.54}$$

with $f_i = \nabla^{\pi_!(E_1 \boxtimes S^*)}_{e_i} u_1$, where c denotes the symbol of D. The right hand side of (4.54) is a bounded operator. Now the proof runs along the same lines as the one given in theorem 6.22 in [61]. Set

$$K_1(\lambda) = \left(\left(D^{E_1,Q_{12}}\right)^2 + 1 + \lambda^2\right)^{-1} \quad , \quad K_2(\lambda) = \left(\left(D^{S,Q_2}\right)^2 + 1 + \lambda^2\right)^{-1} .$$

Replacing F and F_2 by their integral representation introduced in theorem 4.5.4, we have that up to compact operators:

$$\begin{aligned}
& T_{u_1} \circ F_2 - F \circ T_{u_1} \\
\equiv \ & \int_0^\infty (T_{u_1} K_2(\lambda) D^{S,Q_2} - K_1(\lambda) T_{u_1} D^{S,Q_2}) d\lambda \\
= \ & \int_0^\infty K_1(\lambda) \left(\left(D^{E_1,Q_{12}}\right)^2 T_{u_1} - T_{u_1} \left(D^{S,Q_2}\right)^2\right) K_2(\lambda) D^{S,Q_2} d\lambda \\
= \ & \int_0^\infty K_1(\lambda) \left(D^{E_1,Q_{12}} S + S D^{S,Q_2}\right) K_2(\lambda) D^{S,Q_2} d\lambda
\end{aligned}$$

Since $K_1(\lambda)$, $K_1(\lambda) D^{E_1,Q_{12}}$ and $K_2(\lambda) D^{S,Q_2}$ are compact operators, the integrand is as well. The estimates

$$\begin{aligned}
\|K_1(\lambda)\| \leq (d_1 + \lambda^2)^{-1} \quad &, \quad \|K_1(\lambda) D^{E_1,Q_{12}}\| \leq C_1 (d_1 + \lambda^2)^{-\frac{1}{2}} \\
\|K_2(\lambda) D^{S,Q_2}\| \leq C_2 (d_2 + \lambda^2)^{-\frac{1}{2}} \quad &, \quad \|K_2(\lambda) \left(D^{S,Q_2}\right)^2\| \leq C_2
\end{aligned}$$

proven in [17] yield the convergence of the integral in norm, finally showing that the commutator is a compact operator. □

Remark Given D, E_1 and some trivialization Q_{12} of $E_1 \boxtimes E_2$, there are always two line bundles Q_i, such that Q_{12} is induced by them. To see this, choose arbitrary trivializations $\widetilde{Q}_1$ of $E_1 \boxtimes S^*$ and Q_2 of $S \boxtimes E_2$. The difference between $\Delta^*(\widetilde{Q}_1 \boxtimes Q_2)/PU(n)$ and Q_{12} can be expressed by a line bundle $L_{12} \to M$, i.e.

$$\rho_{12}^* L_{12} \otimes \Delta^*(\widetilde{Q}_1 \boxtimes Q_2)/PU(n) = Q_{12} \quad \text{for } \rho_{12} : P_1 \times_M P_2 \to M \ .$$

Now change $\widetilde{Q}_1$ to $Q_1 = \rho^* L_{12} \otimes \widetilde{Q}_1$, where $\rho : P_1 \times_M \widetilde{P} \to M$ is the projection. Q_1 will do the job.

Furthermore, we could phrase a similar theorem for D acting on a twisted Hilbert A_2-module bundle and some bundle gerbe module E_1 or even for two twisted Hilbert A_i-module bundles, the index taking values in $K_0(A_1 \otimes A_2)$, but we won't need this here.

The decomposition of the index in twisted K-theory directly yields a nice proof of the following naturality result, which will play an important role in the application presented in the next chapter (compare with the untwisted case presented in lemma 3.1 in [27]).

Corollary 4.5.8. *Let D be a generalized projective Dirac operator acting on a bundle gerbe module E_2 and let E_1 be a twisted Hilbert A_1-module bundle. Given a C^*-algebra homomorphism $\varphi : A_1 \longrightarrow B_1$ define the twisted Hilbert B_1-module bundle F_1 via*

$$F_1 = E_1 \otimes_\varphi B_1 \ .$$

Then: $\varphi_*\big(\big[D^{E_1,Q_{12}}\big]\big) = \big[D^{F_1,Q_{12}}\big]$, where $\varphi_* : K_0(A_1) \longrightarrow K_0(B_1)$ denotes the induced map on K-theory and Q_{12} is a trivialization.

Proof. Choose a countertwisting bundle S and trivializations Q_i, such that Q_{12} is the induced by them (see previous remark). Applying φ_* to $[E_1^{S,Q_1}]$ yields

$$\begin{aligned}(\mathrm{id}_{C(M,\mathcal{K})} \otimes \varphi_*)[E_1^{S,Q_1}] &= [C(M, \pi_!(S^* \boxtimes E_1)) \otimes_{\mathrm{id}\otimes\varphi_*} B_1, 0] \\ &= [C(M, \pi_!(S^* \boxtimes (E_1 \otimes_{\varphi_*} B_1))), 0] = [F_1^{S,Q_1}] \ .\end{aligned}$$

Therefore, by naturality of the Kasparov product, we get

$$\begin{aligned}\varphi_*\big(\big[D^{E_1,Q_{12}}\big]\big) &= \varphi_*\Big([E_1^{S,Q_1}] \otimes_{C(M,\mathcal{K})} [D^{S,Q_2}]\Big) \\ &= \Big((\mathrm{id}_{C(M,\mathcal{K})} \otimes \varphi_*)[E_1^{S,Q_1}]\Big) \otimes_{C(M,\mathcal{K})} [D^{S,Q_2}] \\ &= [F_1^{S,Q_1}] \otimes_{C(M,\mathcal{K})} [D^{S,Q_2}] = \big[D^{F_1,Q_{12}}\big] \ .\end{aligned}$$

□

We conclude this chapter with the formulation of Kasparov's index theorem in the case of Dirac operators. Let $\sigma : \Gamma(F) = \Gamma(F_+) \oplus \Gamma(F_-) \rightarrow \Gamma(F)$ be a Clifford symbol. Since it is odd, it decomposes into $\sigma_\pm$. In section 4.2.3 the symbol class $[\sigma] \in K^0_{\rho^*\mathcal{A}}(T^*M)$ was introduced, which is represented by the tripel $[\rho^*F_+, \rho^*F_-, \sigma_+]$. We changed the notation of the projection to $\rho : T^*M \longrightarrow M$ to avoid confusion with $\pi : P \longrightarrow M$. Let S be a countertwisting for F. Since σ_+ is a morphism of twisted bundles, $\sigma_+ \otimes \mathrm{id}_S$ descends to a homomorphism

$$\pi_!(\sigma_+ \otimes \mathrm{id}_S) : \rho^*\pi_!(F_+ \boxtimes S) \longrightarrow \rho^*\pi_!(F_- \boxtimes S) \ .$$

By polar decomposition, $i\,\sigma_+$ can be deformed in its K-theory class to $i\,\widetilde{\sigma}_+$, such that the latter is unitary outside a neighborhood of the zero section. This induces a KK-cycle of the form:

$$[\sigma^{S,Q}] = [C(T^*M, \rho^*\pi_!(F \boxtimes S)), i\,\widetilde{\sigma}] \in KK(C(M,\mathcal{K}), C_0(T^*M) \otimes A) \ .$$

The action of $C(M,\mathcal{K})$ on $C(T^*M, \rho^*\pi_!(F \boxtimes S))$ is given by pullback to T^*M followed by a multiplication like above. Note that skew-adjointness and the above deformation ensure that $\sigma^2 - 1$ and $\sigma - \sigma^*$ are compact operators, which in this case means they vanish, when ζ is send to infinity. Like in corollary 4.5.6 we deduce from lemma 4.5.5 that

$$\big[E^{S,Q_1}\big] \otimes_{C(M,\mathcal{K})} [\sigma^{S,Q_2}] = [\sigma^{E,Q_{12}}] \in KK(\mathbb{C}, C_0(T^*M) \otimes A)$$

is a KK-class representing the symbol of $D^{E,Q_{12}}$. T^*M is equipped with a canonical almost complex structure. Let $\overline{\partial}$ be the DOLBEAULT operator associated to it and denote by $[\overline{\partial}] \in KK(C_0(T^*M), \mathbb{C})$ the element it represents in K-homology. Then KASPAROV's index theorem follows directly from the classical one and takes the form:

Theorem 4.5.9. *Let $\sigma : \rho^*F \to \rho^*F$ be a Clifford symbol, S a countertwisting for F and E a twisted Hilbert A-module bundle. Choose trivializations Q_1 for $E \boxtimes S^*$ and Q_2 for $F \boxtimes S$, then*

$$\left[E^{S,Q_1}\right] \otimes_{C(M,\mathcal{K})} [\sigma^{S,Q_2}] \otimes_{C_0(T^*M)} [\overline{\partial}] = \mathrm{ind}(D^{E,Q_{12}}) \in KK(\mathbb{C}, A) \ ,$$

where the right hand side represents the MISHCHENKO-FOMENKO-*index in KK-theory.*

Chapter 5

An application: enlargeability and positive scalar curvature

As we have seen in chapter 4.4, projective Dirac operators countertwisted with flat bundle gerbe modules yield obstructions against the existence of positive scalar curvature metrics. The index $\mathrm{ind}(D_+^{\mathcal{V}}) \in K_0(C^*(\pi_1(M), c_{\widehat{\pi}}))$ defined in section 4.4.1 can be seen as a universal version of these obstructions. We will denote this invariant, in analogy with the nomenclature introduced by Rosenberg [56] for the untwisted case, by $\alpha(M)$ (i.e. $\alpha_r(M)$ and $\alpha_{\max}(M)$ for the reduced, respectively maximal C^*-algebra). Based on geometric observations Gromov and Lawson proved that a certain type of manifold – the so-called enlargeable ones (see the definition below) – do not allow a metric of positive scalar curvature [26]. But until the work of Schick and Hanke [27] it remained an open question, whether $\alpha(M)$ vanishes in case of enlargeability or not. Using a sequence of vector bundles with curvature vanishing in the limit, called almost flat bundles, they proved that in case M is an enlargeable spin manifold indeed $\alpha(M) \neq 0$. We take their results as a starting point not only to extend it to enlargeable non-spin manifolds allowing a spin structure on the universal cover, but also to show the way, how arguments – worked out for vector bundles – can often be transferred to the framework developed above.

Definition 5.0.10. Let M be a closed oriented manifold. Fix some Riemannian metric g on M. If for all $\epsilon > 0$ there is a finite, connected cover $\overline{M} \to M$, by a spin manifold $\overline{M}$ and an ϵ-contracting map $(\overline{M}, \overline{g}) \to (S^n, g_{S^n})$ of non-zero degree, then M is called *enlargeable.* Here, $\overline{g}$ is induced by g and g_{S^n} is the standard metric on the sphere. If there exist ϵ-*area* contracting maps (i.e. $\|\Lambda^2 D_x f\| \leq \epsilon$ for all $x \in M$), then M is called *area-enlargeable.*

All throughout this section M will be a closed oriented n-manifold with fixed Riemannian metric g that allows a spin structure on its universal cover $\widetilde{M}$. $P = P_{SO}$ will denote the frame bundle, L the lifting bundle gerbe corresponding to $1 \to S^1 \to \mathrm{Spin}^c(n) \to SO(n) \to 1$ and D the projective Dirac operator defined in 4.3.2. Recall that flat countertwistings in this case are induced by projective representations of $(\pi_1(M), c_{\widehat{\pi}})$, where $c_{\widehat{\pi}} \in H^2_{\mathrm{gr}}(\pi_1(M), S^1)$ denotes the cocycle classifying the extension

$$1 \longrightarrow S^1 \longrightarrow \widehat{\pi} \longrightarrow \pi_1(M) \longrightarrow 1 \tag{5.1}$$

with $\widehat{\pi}$ defined in (4.44). As mentioned above, we need a weaker form of flatness:

Definition 5.0.11. Let $\widetilde{L} \to \widetilde{M}^{[2]}$ be the lifting bundle gerbe corresponding to (5.1). A sequence $E_i \to \widetilde{M}$, $i \in \mathbb{N}$ of smooth bundle gerbe modules for $\widetilde{L}$ with bgm-connections ∇^{E_i} will be called *a sequence of almost flat twisted bundles*, if

$$\lim_{i\to\infty} \|\Omega_{E_i}\| = 0 \ ,$$

where the norm on $\Omega^2(M, \mathrm{end}(E_i))$ is induced by the natural pointwise norm on $\mathrm{end}(E_i) \to M$ and the maximum norm on the unit sphere bundle in $\Lambda^2(M)$. Furthermore, we demand the twistings $\gamma_i^g : E_i \to E_i$ considered as sections $C(\widetilde{M}, \mathrm{Hom}(E_i, g^*E_i))$ to be locally Lipschitz continuous maps for a global Lipschitz constant C independent of i, i.e. each point $\widetilde{m}_1 \in \widetilde{M}$ has a neighborhood $U \subset \widetilde{M}$, such that for all $\widetilde{m}_2 \in U$:

$$\|\gamma_i^g(\widetilde{m}_1) - \gamma_i^g(\widetilde{m}_2)\| \leq C\, d(\widetilde{m}_1, \widetilde{m}_2) \ ,$$

where the metric on the right hand side is induced by the Riemannian structure pulled back from M and the norm is the operator norm on $\mathrm{Hom}(E_i, g^*E_i)$.

Remark Note that any sequence $E_i \to P_i$ of bundle gerbe modules for lifting bundle gerbes L_i with bgm-connections ∇^{E_i}, bounded twistings and asymptotically vanishing curvature in the above sense can be turned into a sequence of almost flat twisted bundles as long as $dd(L_i) = dd(\widetilde{L})$ and $L_i^* \boxtimes \widetilde{L}$ has a flat trivialization for every $i \in \mathbb{N}$. These are the conditions to shift all of the E_i to twisted bundles over $\widetilde{M}$ without changing the curvature Ω_{E_i}.

The bridge between sequences of almost flat twisted bundles and enlargeability is built in the following fundamental theorem. The analogoue version for the untwisted case can be found in the midst of the proof of theorem 4.2 in [27].

Theorem 5.0.12. *Let M be an area-enlargeable manifold. There exists a sequence of almost flat twisted bundles $E_i \to \widetilde{M}$ of rank d_i, such that*

$$\begin{aligned} c_k(E_i) &= 0 && \text{if } 0 < k < n \\ \langle c_k(E_i), [M] \rangle &\neq 0 && \text{if } k = n \end{aligned}$$

together with a sequence of flat bundle gerbe modules F_i of corresponding rank, i.e. $\dim(F_i) = d_i$. *(For the definition of Chern classes, see (4.19).)*

Proof. Since the Chern character $\mathrm{ch} : K^0(S^{2n}) \otimes \mathbb{Q} \to H^{\mathrm{even}}(S^{2n}, \mathbb{Q})$ is rationally an isomorphism, there is a vector bundle $E \to S^{2n}$ with non-vanishing top Chern class $c_n(E) \neq 0$. Choose a connection η_E on E and fix $i \in \mathbb{N}$. Since M is enlargeable, there exists a finite, spin covering space $\overline{M} \to M$ with deck transformation group G, such that there is a $\frac{1}{i}$-area contracting map $\varphi : \overline{M} \to S^{2n}$. By passing to a finite cover if necessary, we can without loss of generality assume that $\overline{M}$ is regular. Note that this does not disturb the fact that it is spin, by naturality of the second Stiefel-Whitney-class. Therefore $G = \pi_1(M)/\pi_1(\overline{M})$.

As was shown in theorem 4.3.28, $\mathbb{C}[G]$ yields a projective representation of $(\pi_1(M), c_{\widehat{\pi}})$ (in our case $\dot{\Gamma} = \overline{\Gamma} = \mathrm{Spin}(n)$). Using the canonical map $\widetilde{M} \to \overline{M}$, we form

$$E_i = \widetilde{M} \times_{\overline{M}} \left(\bigoplus_{g \in G} g^* \varphi^* E \right) ,$$

which is a bundle gerbe module with respect to $\widetilde{L}$ by means of the projective action of $\pi_1(M)$ on G. Indeed, thinking of $\widehat{\pi} = \pi_1(M) \times_{c_{\widehat{\pi}}} S^1$ as cartesian product with multiplication altered by the cocycle, the twisting γ_i of E_i is defined via

$$\gamma_i^{\widetilde{h}}(v_g) = c_{\widehat{\pi}}(\widetilde{h}, \widetilde{g})\, v_g \ ,$$

where $v_g \in E_{\varphi(\overline{m}g)}$ on the left is mapped via the identity to $E_{\varphi(\overline{m}h^{-1}hg)}$ on the right and $h \in G$ denotes the image of $\widetilde{h}$ (see section 4.3.4 for the notation used for the twisting). Likewise,

$$F_i = \widetilde{M} \times \left(\mathbb{C}[G] \otimes \mathbb{C}^d \right)$$

is a flat bundle gerbe module with $\dim(F_i) = d_i = \dim(E_i)$. The connection η_{E_i} induced by η_E on E_i turns out to be a bgm-connection such that

$$\|\Omega_{E_i}\| = \|\varphi^* \Omega_E\| \leq \frac{1}{i} \|\Omega_E\| \ ,$$

proving that the E_i actually form a sequence of asymptotically flat twisted bundles. Since $\pi^* : H^k(M, \mathbb{Z}) \to H^k(\overline{M}, \mathbb{Z})$ is injective, it suffices to prove $\pi^*(c_k(E_i)) = c_k(\pi^* E_i) = 0$ for $0 < k < n$ and $\langle \pi^*(c_n(E_i)), [M] \rangle \neq 0$. Since $\overline{M}$ admits a spin structure, there exists a splitting $\pi_1(\overline{M}) \to \widehat{\pi}$, that defines a trivialization of $\widetilde{L} \to \widetilde{M} \times_{\overline{M}} \widetilde{M}$. Pushing down $\pi^* E_i$ along this to a vector bundle over $\overline{M}$, yields

$$\bar{E}_i = \left(\bigoplus_{g \in G} g^* \varphi^* E \right) .$$

But $\bar{E}_i$ clearly matches our assumptions about the Chern classes, so E_i does as well. The Lipschitz continuity of the twisting $\gamma_i^{\widetilde{h}}$ is obvious, since it is actually constant considered as a section of $\mathrm{Hom}(E_i, \widetilde{h}^* E_i)$. □

5.1 Assembling almost flat twisted bundles

The crucial property about almost flat twisted bundles is that they can be stacked up to form a twisted Hilbert Q-module bundle (for a C^*-algebra Q still to define). The latter will be flat, but still carries all the information about the non-vanishing Chern classes of its ingredients. Let $E_i \to \widetilde{M}$ be a sequence of almost flat twisted bundles. Since the dimensions vary throughout the E_i's, we first stabilize for convenience to get fibers that are projective Hilbert $\mathbb{K}$-modules, where $\mathbb{K}$ denotes the compact operators on some separable Hilbert space H. Let d_i be the rank of E_i, choose an embedding $\kappa_i : M_{d_i}(\mathbb{C}) \to \mathbb{K}$ and set $t_i = \kappa_i(1)$. From the frame bundle P_{E_i} we construct

$$E_i^{\mathbb{K}} = P_{E_i} \times_{\kappa_i} \mathbb{K} \ ,$$

which is a twisted Hilbert $\mathbb{K}$-module bundle with fibers isomorphic to $t_i\mathbb{K}$. The twisting $\gamma_i : \widetilde{L} \otimes \pi_2^* E_i \to \pi_1^* E_i$ induces a map on principal bundles. Since $E_i^{\mathbb{K}}$ is associated to P_{E_i} as well and κ_i identifies $\mathrm{End}_{\mathbb{K}}(t_i\mathbb{K})$ with $M_{d_i}(\mathbb{C})$ preserving scalars, γ_i induces a twisting $\gamma_i^{\mathbb{K}}$ on $E_i^{\mathbb{K}}$. Define

$$A = \prod_{i=0}^{\infty} \mathbb{K}$$

to be the C^*-algebra of norm-bounded sequences with values in $\mathbb{K}$. Let A_i be the ith factor in A and set $t = (t_i)_{i \in \mathbb{N}} \in A$. Observe that $H = L^2(\pi_1(M))$ is a projective representation of $(\pi_1(M), c_{\widehat{\pi}})$ with respect to the multiplication:

$$h \cdot \sum_{g \in \pi_1(M)} \lambda_g g = \sum_{g \in \pi_1(M)} \lambda_g \, c_{\widehat{\pi}}(h,g) \, hg \ .$$

It corresponds to a unitary representation $\widehat{\pi} \to U(H)$ that restricts to the identity on $S^1 \subset U(H)$ or equivalently to a projective representation $\pi_1(M) \to PU(H)$ with lifting cocycle $c_{\widehat{\pi}}$. Applying it componentwise, induces a homomorphism

$$\pi_1(M) \longrightarrow PU(M(A))$$

into the multiplier algebra $M(A)$ of A together with the C^*-algebra bundle

$$\mathcal{A} = \widetilde{M} \times_{\mathrm{Ad}} A \ .$$

Since $\widetilde{M}$ is a smooth manifold, $\mathcal{A}$ is a smooth fiber bundle. The technical part now consists of the following theorem, which is a twisted analogue of theorem 2.1 in [27].

Theorem 5.1.1. *There is a smooth twisted Hilbert A-module bundle $V \to \widetilde{M}$ together with a twisted connection*

$$\nabla^V : \Gamma(V) \to \Gamma(T^*\widetilde{M} \otimes V)$$

such that the following holds:

- *$V_i = V \cdot A_i$ is isomorphic to $E_i^{\mathbb{K}}$ as a twisted Hilbert $\mathbb{K}$-module bundle.*
- *The connection preserves the subbundles V_i.*
- *Let Ω_{V_i} be the curvature of the connection induced on V_i by ∇^V, then*

$$\lim_{i \to \infty} \|\Omega_{V_i}\| = 0 \ .$$

Proof. We will first construct a bundle V_L of Hilbert A-modules over $\widetilde{M}$ from the sequence $E_i^{\mathbb{K}}$ in such a way that its transition functions are Lipschitz continuous. The bound that we demand on the twistings γ_i^g will ensure that they can be assembled to form a continuous twisting γ^g on V_L. We can then approximate V_L by a smooth bundle V.

Let $I^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1\} \subset \mathbb{R}^n$ be the n-dimensional cube. Cover M by a finite family of open sets W_j, such that each of them is diffeomorphic to I^n and $\widetilde{M} \to M$ is trivial over W_j via

$$\phi_j : W_j \times \pi_1(M) \xrightarrow{\sim} \widetilde{M}\Big|_{W_j} \ .$$

By induction we can now construct a trivialization of $\phi_j^* E_i^{\mathbb{K}}$ over $W_j \times \{1\}$:

$$\psi_{i,j}^1 : \phi_j^* E_i^{\mathbb{K}}\big|_{W_j \times \{1\}} \xrightarrow{\sim} I^n \times t_i \mathbb{K}$$

such that the constant sections of $\phi_j^* E_i^{\mathbb{K}}$ over $I^k \times \{0\}$ are parallel with respect to ∇_{∂_l} for $1 \leq l \leq k$, which means that for $v \in t_i \mathbb{K}$ we have

$$\begin{aligned} \tau_{k,v} : I^n \longrightarrow \phi_j^* E_i^{\mathbb{K}} \; ; \; \tau_{k,v}(x_1, \ldots, x_n) &= \left(\psi_{i,j}^1\right)^{-1}(x_1, \ldots, x_k, 0, \ldots, 0, v) \; , \\ \nabla_{\partial_l} \tau_{k,v} &= 0 \; , \end{aligned} \tag{5.2}$$

where ∇ now is the connection on $\phi_j^* E_i^{\mathbb{K}}$ pulled back via the diffeomorphism $I^n \longrightarrow W_j$. We can extend $\psi_{i,j}^1$ to a trivialization of $\phi_j^* E_i^{\mathbb{K}}\big|_{W_j \times \pi_1(M)}$ by first shifting back using γ^g, i.e. we get $\psi_{i,j}$ composed of

$$\psi_{i,j}^g : \phi_j^* E_i^{\mathbb{K}}\big|_{W_j \times \{g\}} \xrightarrow{\sim} I^n \times t_i \mathbb{K} \quad ; \quad \psi_{i,j}^g = \psi_{i,j}^1 \circ \gamma^g \; .$$

Forming the constant sections for these trivializations like above, yields:

$$\begin{aligned} \tau_{k,v}^g(x_1, \ldots, x_n) &= \left(\psi_{i,j}^g\right)^{-1}(x_1, \ldots, x_k, 0, \ldots, 0, v) \\ &= c_{\widehat{\pi}}(g^{-1}, g)^{-1} \gamma^{g^{-1}} \circ \tau_{k,v}(x_1, \ldots, x_n) \; . \end{aligned}$$

Since ∇ is a twisted connection on $\phi_j^* E_i^{\mathbb{K}}$, which implies (see section 4.3.4) invariance with respect to γ^g, the bundle isomorphisms $\psi_{i,j}^g$ still satisfy the analogue of (5.2). The maps $\psi_{i,j}^g$ induce sections of the frame bundle $P_{E_i^{\mathbb{K}}}$ pulled back to I^n. Let

$$\eta_{E_i^{\mathbb{K}}} \in \Omega^1(P_{E_i^{\mathbb{K}}}, \mathrm{End}_{\mathbb{K}}(t_i \mathbb{K}))$$

be the connection 1-form associated to $\nabla^{E_i^{\mathbb{K}}}$ taking values in the skew-adjoint operators in $t_i \mathbb{K} t_i = \mathrm{End}_{\mathbb{K}}(t_i \mathbb{K})$. Set

$$\eta_{i,j}^g = \left(\psi_{i,j}^g\right)^{-1*} \eta_{E_i^{\mathbb{K}}} \in \Omega^1(I^n, t_i \mathbb{K} t_i)$$

and observe that, since γ^g acts isometrically, the norm $\|\eta_{i,j}^g\|$, which is induced by the Euclidean metric on I^n and the operator norm on $t_i \mathbb{K} t_i$ does *not* depend on $g \in \pi_1(M)$. Denote by $\Omega_{i,j}^g \in \Omega^2(I^n, t_i \mathbb{K} t_i)$ the curvature of $\eta_{i,j}^g$, which coincides with the pullback of $\Omega_{E_i^{\mathbb{K}}}$ via $\psi_{i,j}^g$. Due to our choice of trivializations, we have

$$\left(\eta_{i,j}^g\right)_{(x_1, \ldots, x_k, 0, \ldots, 0)}(\partial_l) = 0$$

for $1 \leq l \leq k$ by (5.2). This implies the following estimate on the norms of $\eta_{i,j}^g$, which is proven in [27]:

Lemma 5.1.2. *For each i and j, we have $\|\eta_{i,j}^g\| \leq n \cdot \|\Omega_{i,j}^g\|$.*

This means, that our control of the curvature $\Omega_{E_i^{\mathbb{K}}}$ directly carries over to a bound on the local connection 1-forms $\eta_{i,j}^g$, which is independent of $g \in \pi_1(M)$. This crucial estimate will enable us to control the transition functions as well as can be seen in the next lemma also proven in [27].

Lemma 5.1.3. *Let $l \geq 0$. There is a constant $C(l)$ (independent of i, j and g) such that if $\phi : [0,1] \to I^n \times t_i\mathbb{K}$ is a parallel vector field (with respect to the connection $\omega^g_{i,j}$) along a piecewise smooth path $\gamma : [0,1] \to I^n$ of length $l(\gamma) \leq l$, then*

$$\|\phi(1) - \phi(0)\| \leq C(l) \cdot \|\eta^g_{i,j}\| \cdot l(\gamma) \cdot \|\phi(0)\|$$

for all i, j.

The next theorem will show that the transition functions of the bundles $E_i^{\mathbb{K}} \to \widetilde{M}$ are Lipschitz continuous with a (global) Lipschitz constant not depending on the particular index i of the bundle. We will therefore consider the trivialization of $E_i^{\mathbb{K}}$ over the sets $U_\alpha = \phi_\alpha(W_\alpha \times \pi_1(M)) \subset \widetilde{M}$ defined by:

$$\Psi_{\alpha,i} : U_\alpha \times t_i\mathbb{K} \longrightarrow W_\alpha \times \pi_1(M) \times t_i\mathbb{K} \longrightarrow \phi_\alpha^* E_i^{\mathbb{K}} \longrightarrow E_i^{\mathbb{K}}\big|_{U_\alpha}$$
$$\Psi_{\alpha,i} = \Phi_\alpha \circ (\psi_{i,\alpha})^{-1} \circ (\phi_\alpha^{-1} \times \mathrm{id}_{t_i\mathbb{K}})$$
$$\Psi_{\alpha,\beta,i} = \Psi_{\beta,i}^{-1} \circ \Psi_{\alpha,i} : (U_\alpha \cap U_\beta) \times t_i\mathbb{K} \longrightarrow (U_\alpha \cap U_\beta) \times t_i\mathbb{K} \ ,$$

where Φ_α denotes the canonical map from the pullback. Observe that, due to equivariance of ϕ_α and our choice of trivialization over the *compact* sets W_α, the norm of $d\phi_\alpha$ is bounded. Since M is compact, there even exists a bound that is independent of α. Therefore we will drop this part of the trivialization and think of the transition functions $\Psi_{\alpha,\beta,i}$ as maps:

$$\psi_{\alpha,\beta,i} : (W_\alpha \cap W_\beta) \times \pi_1(M) \longrightarrow t_i\mathbb{K}t_i \ .$$

Theorem 5.1.4. *There is a constant $C \in \mathbb{R}$, independent of i, α and β such that with $\psi_{\alpha,\beta,i}$ like above:*

$$\|D_x\psi_{\alpha,\beta,i}\| \leq C$$

for all $(x,g) \in (W_\alpha \cap W_\beta) \times \pi_1(M)$.

Proof. Denote by $\varphi_\alpha : W_\alpha \to I^n$ the diffeomorphism used in the beginning and by $\varphi_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ the chart changing map in M. Choose a point

$$y \in \varphi_\alpha(W_\alpha \cap W_\beta) \cap \overset{\circ}{I^n} \quad \text{and} \quad g \in \pi_1(M) \ .$$

Like in [27] we define

$$f_\nu : (-\varepsilon, \varepsilon) \longrightarrow t_i\mathbb{K}t_i \quad ; \quad f(t) = \psi_{\alpha,\beta,i}(\varphi_\alpha^{-1}(y + te_\nu), g) \ .$$

Note that it only takes values in the unitary group of the algebra $t_i\mathbb{K}t_i$, in particular $f(0)$ is an isometry. Let $X = d\varphi_\alpha^{-1}(e_\nu)$, then

$$D_{(\varphi_\alpha^{-1}(y),g)}\psi_{\alpha,\beta,i}(X) = \left.\frac{df}{dt}\right|_{t=0} \ .$$

Let $v \in t_i\mathbb{K}$. $f(t)v$ can be described by two successive parallel transports along the curve $\tau : [0,t] \to I^n$; $\xi \mapsto y + \xi e_\nu$: First transport v along τ^{-1} using the connection $\eta^g_{i,\alpha}$ to get a vector $w \in t_i\mathbb{K}$. Then transport $f(0)w$ along the curve $\varphi_{\alpha,\beta} \circ \tau$ using the connection $\eta^{g_{\alpha,\beta}g}_{i,\beta}$ where $g_{\alpha,\beta}$ is the group element that appears when changing from $W_\alpha \times \pi_1(M)$ to $W_\beta \times \pi_1(M)$. Observe that in $E_i^{\mathbb{K}}$ this operation corresponds to shifting a vector back and forth along τ, which we just expressed in local coordinates on two differrent charts. The right hand side of

$$\begin{aligned}\|f(t)v - f(0)v\| &\leq \|f(t)v - f(0)w\| + \|f(0)w - f(0)v\| \\ &= \|f(t)v - f(0)w\| + \|w - v\|\end{aligned}$$

has a universal upper bound of the form Ct. Indeed, the last summand is bounded by the length of τ times a constant independent of i, α, β by the previous lemma. Similarly, the first one is bounded by the length of $\varphi_{\alpha,\beta} \circ \tau$ times a universal constant. But since $d\varphi_{\alpha,\beta}$ is bounded by the same reasoning as above, the result follows. □

As a consequence of the last theorem we are able to assemble the $E_i^{\mathbb{K}}$ into a locally trivial Hilbert A-module bundle V_L over the universal cover in such a way that its transition functions are *Lipschitz continuous*. Its fibers are

$$(V_L)_{\widetilde{m}} = \prod_{i \in \mathbb{N}} \left(E_i^{\mathbb{K}}\right)_{\widetilde{m}} \ .$$

The universal bound on the twistings γ_i, that was part of the definition of a sequence of almost flat twisted bundles, ensures that they form a Lipschitz continuous map $\gamma^g : (V_L)_{\widetilde{m}} \longrightarrow (V_L)_{\widetilde{m}g^{-1}}$. Therefore V_L is a *twisted* Hilbert A-module bundle with Lipschitz continuous transition functions and twisting.

As we have seen in theorem 3.2.8 the bundle V_L corresponds to a Lipschitz continuous, projection valued section t_L of $M_k(\mathcal{A})$, which can be approximated arbitrarily close in norm by a *smooth* projection valued section $t_V \in C^\infty(M, M_k(\mathcal{A}))$. When we view t_V as an equivariant smooth projection valued function

$$t_V : \widetilde{M} \longrightarrow M_k(A) \ ,$$

we can recover a *smooth* twisted Hilbert A-module bundle V as was suggested by theorem 3.2.8, that is:

$$V = \left\{(\widetilde{m}, v) \in \widetilde{M} \times A^k \mid t_V(\widetilde{m})v = v\right\}$$

together with a smooth, even constant twisting

$$\gamma_V^g : V \longrightarrow V \quad ; \quad (\widetilde{m}, v) \mapsto (\widetilde{m}, g \cdot v) \ ,$$

where g acts on A^k by the projective representation mentioned prior to this theorem. In particular t_V can be chosen so close to t_L that they are (Lipschitz) isomorphic as finitely generated, projective Hilbert $C(M, \mathcal{A})$-modules and therefore as twisted Hilbert A-module bundles as well. Due to the algebraic structure of A, V also contains blocks defined via $V_i = V \cdot A_i$, which are sent to $V_L \cdot A_i$ by the isomorphism $V \to V_L$ by A-linearity. Since the latter are isomorphic to $E_i^{\mathbb{K}}$, this is true for V_i as well. This map might just be a Lipschitz isomorphism, but those can be smoothed. This finishes the construction of V.

Let $t = (t_i)_{i \in \mathbb{N}} \in \mathrm{End}(A)$, then V has typical fibers tA. Set $C = tAt = \mathrm{End}(tA)$. A twisted connection on V is defined by a γ^g-invariant connection form $\eta^V \in \Omega^1(P_V, C)$ on the principal $U(C)$-bundle over $\widetilde{M}$. η^V can be reconstructed from local data in the following way: Given a family of one-forms $\eta_\alpha \in \Omega^1(W_\alpha, C)$ taking values in the skew-adjoint part of C, we set

$$\eta_\alpha^V = \mathrm{Ad}_{\pi_{U(C)}^{-1}{}^*}(\pi_{W_\alpha}^* \eta_\alpha) + \pi_{U(C)}^* \mu_{U(C)} \ \in \Omega^1(W_\alpha \times U(C), C) \ ,$$

where $\mu_{U(C)}$ denotes the MAURER-CARTAN-form on $U(C)$. η_α^V is a connection form on the trivial $U(C)$-bundle over W_α with values in the skew-adjoint elements of C. Choose a trivialization

$$\psi_V^1 : \phi_\alpha^* P_V|_{W_\alpha \times \{1\}} \xrightarrow{\sim} W_\alpha \times U(C)$$

and extend it via

$$\psi_V^g : \phi_\alpha^* P_V|_{W_\alpha \times \{g\}} \xrightarrow{\sim} W_\alpha \times U(C) \quad ; \quad \psi_V^g = \psi_V^1 \circ \gamma_V^g \ .$$

By constructon $\psi_V^* \eta_\alpha^V$ is a γ^g-invariant connection form on

$$\phi_\alpha^* P_V \simeq P_V|_{\phi_\alpha(W_\alpha \times \pi_1(M))} \ .$$

Let ϱ_α be a smooth partition of unity subordinate to W_α on M. Then

$$\eta^V = \sum_\alpha (\varrho_\alpha \circ \pi_{W_\alpha}) \cdot \psi_V^* \eta_\alpha^V \tag{5.3}$$

is an invariant connection form on P_V. It is smooth if the η_α were so. Thus, what remains to construct are local forms η_α such that η^V has the desired properties. This can be done just like in [27]. Observe that the Lipschitz isomorphism between L and V can, in terms of the trivializations over W_α, be expressed by Lipschitz continuous maps:

$$\kappa_\alpha : I^n \longrightarrow tAt$$

with values in the unitary elements of tAt. The connection forms $\eta_{i,\alpha}^1 \in \Omega^1(I^n, t_i \mathbb{K} t_i)$, which we can see as smooth functions $\eta_{i,\alpha}^1 : I^n \to (t_i \mathbb{K} t_i)^n$ have universally bounded C^1-norms (i.e. the bound is indepent of i and α) by lemma 5.1.2. Therefore they can be assembled into a Lipschitz continuous map

$$\eta_\alpha^L : I^n \longrightarrow (tAt)^n \ .$$

Together with κ_α we get induced forms on V defined by:

$$\eta_\alpha(x) = \kappa_\alpha(x)\, \eta_\alpha^L(x)\, \kappa_\alpha(x)^* \ .$$

Choosing a bump function $\delta_\varepsilon : I^n \to \mathbb{R}$ with total integral 1 and support in the ε-ball around 0, the local forms η_α can be smoothed via convolution:

$$\widetilde{\eta}_\alpha(x) = \int_{I^n} \delta_\varepsilon(x-t) \eta_\alpha(t) dt \ .$$

Denote by $\eta_{i,\alpha}^V$ the form induced by η_α on the ith block of V, i.e. by the projection $A \to A_i$, analogously define $\eta_{i,\alpha}^L$ (which then coincides with $\eta_{i,\alpha}^1$). The L^1-norms of $\eta_{i,\alpha}^V$ and $\eta_{i,\alpha}^L$ satisfy:

$$\|\eta_{i,\alpha}^V\|_1 \leq C\, \|\eta_{i,\alpha}^L\|_1$$

for some (global) constant C. Therefore $\widetilde{\eta}_\alpha$ and its dth derivative are bounded from above by the supremum norms of δ_ε up to its dth derivative and the L^1-norm of $\eta_{i,\alpha}^1$, but the latter tends to 0, if $i \to \infty$ by lemma 5.1.2. Since $\Omega_{V_i} \in \Omega^2(M, \mathrm{end}(V))$ can be calculated in terms of η^{V_i} and its first derivative, which involves only universally bounded quantities in front of terms tending to 0 by (5.3) (note in particular, that the derivative of the twisting map vanishes), its supremum norm also converges to 0 for $i \to \infty$ as stated in the theorem. $\square$

5.2 Index theory

This section will exploit the existence of V established in the last section. From V we will be able to construct a flat twisted Hilbert Q-module bundle W for some quotient algebra Q of A we are now going to construct. Inside A we have the closed two-sided ideal A' of sequences in $\mathbb{K}$ converging to 0, i.e.

$$A' = \overline{\bigoplus_{i\in\mathbb{N}} \mathbb{K}}^{\|\cdot\|} \subset A \ .$$

Let $Q = A/A'$. The subbundle $V \cdot A'$ of $V \to \widetilde{M}$ is mapped into itself by the twisting γ^g, therefore the quotient:

$$W = V/(V \cdot A')$$

is a smooth, twisted Hilbert Q-module bundle over M with fiber $[t]Q$, where $[t]$ denotes the image of $t \in A$ in Q. ∇^V induces a connection on W and from the last theorem we immediately get:

Corollary 5.2.1. $\nabla^V \in \Omega^2(M, \mathrm{end}(V))$ *yields a form in* $\Omega^2(M, \hom(V, V \cdot A'))$*, therefore it induces a flat connection on* W*.*

The K-theory of Q is completely understood and the calculations of $K_0(Q)$ can be found in [27].

Theorem 5.2.2. *Let Q and A be as above, then*

$$K_0(A) \ \simeq \ \prod_{i\in\mathbb{N}} \mathbb{Z} \quad , \quad K_0(Q) \ \simeq \ \prod_{i\in\mathbb{N}} \mathbb{Z} / \bigoplus_{i\in\mathbb{N}} \mathbb{Z} \ .$$

The above isomorphism sends the projection $A \to Q$ to the projection $K_0(A) \to K_0(Q)$, i.e. the canonical homomorphism:

$$\prod_{i\in\mathbb{N}} \mathbb{Z} \longrightarrow \prod_{i\in\mathbb{N}} \mathbb{Z} / \bigoplus_{i\in\mathbb{N}} \mathbb{Z} \ .$$

5.2.1 Projective holonomy representations

As we have seen in theorem 4.3.31 of section 4.3.4 a flat twisted Hilbert Q-module bundle $W \to \widetilde{M}$ gives rise to a *projective* representation of the fundamental group with lifting cocycle $c_{\widehat{\pi}} \in H^2_{\mathrm{gr}}(\pi_1(M), S^1)$ after fixing a point $\widetilde{m} \in \widetilde{M}$. In the case of W, this yields a homomorphism:

$$(\pi_1(M), c_{\widehat{\pi}}) \longrightarrow \mathrm{End}_Q(W_{\widetilde{m}}, W_{\widetilde{m}}) = [t]Q[t] \ .$$

By the universal property of the twisted maximal C^*-algebra, we end up with a C^*-homomorphism:

$$\phi : C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \longrightarrow [t]Q[t] \to Q \ .$$

Corollary 4.5.8 about the naturality of the index with respect to C^*-homomorphisms immediately yields:

Theorem 5.2.3. *Let V be the twisted Hilbert A-module bundle constructed from the sequence of almost flat twisted bundles E_i. Let ϕ be the C^*-algebra homomorphism gained from V (via the flat twisted bundle W) as described above. Let*

$$D_+^{E_i} : \Gamma(\pi_!(S_+ \boxtimes E_i)) \longrightarrow \Gamma(\pi_!(S_- \boxtimes E_i))$$

(where we suppress the flat trivialization in our notation). Denote the index of $D_+^{E_i}$ by $\mathrm{ind}(D_+^{E_i}) \in K_0(\mathbb{K}) \simeq \mathbb{Z}$. Then

$$\phi_*(\alpha_{\max}(M)) = [\mathrm{ind}(D_+^{E_i})]_{i \in \mathbb{N}} ,$$

where the right hand side uses the identification of $K_0(Q)$ with the group from theorem 5.2.2.

Proof. Let $p_i : A \to A_i = \mathbb{K}$ be the projection to the ith factor in A. The block $V_i = V \cdot A_i$ of V is isomorphic to $E_i^{\mathbb{K}}$ by theorem 5.1.1. Therefore corollary 4.5.8 yields:

$$p_{i*}\left(\mathrm{ind}(D_+^V)\right) = \mathrm{ind}(D_+^{E_i^{\mathbb{K}}}) = \mathrm{ind}(D_+^{E_i}) .$$

On the other hand:

$$\phi_*(\alpha_{\max}) = \phi_*\left(\mathrm{ind}\left(D_+^{\mathcal{V}_{\max}}\right)\right) = \mathrm{ind}(D_+^W) = q_*(\mathrm{ind}(D_+^V) ,$$

where $q : A \to Q$ denotes the canonical projection. To see the middle equality, observe that

$$\mathcal{V}_{\max} \otimes_\phi Q = \widetilde{M} \times C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \otimes_\phi Q \simeq \widetilde{M} \times [t]\, Q \tag{5.4}$$

is an isomorphism of twisted Hilbert Q-module bundles, where the action of the bundle gerbe on the right hand side is induced by the holonomy representation

$$(\pi_1(M), c_{\widehat{\pi}}) \longrightarrow [t]Q[t]$$

constructed above. But reducing the frame bundle of W to its holonomy subbundle over $\widetilde{M}$, we see that both of them are trivial and that W is twistedly isomorphic to the right hand side of (5.4). Since $q_* : K_0(A) \to K_0(Q)$ coincides with the projection the result follows. □

Theorem 5.2.4. *Let M be an even-dimensional, (area-)enlargeable manifold, then*

$$\alpha_{\max}(M) \neq 0 .$$

Proof. By theorem 5.0.12 there exists a sequence $(E_i)_{i \in \mathbb{N}}$ of almost flat twisted bundles with non-vanishing nth Chern number and another sequence $(F_i)_{i \in \mathbb{N}}$ of flat twisted bundles, such that $d_i = \mathrm{rank}(E_i) = \mathrm{rank}(F_i)$. Stacking up the bundles E_i into a twisted Hilbert A-module bundle V and the F_i into a similar bundle V', we can form the associated C^*-algebra homomorphisms:

$$\phi, \phi' : C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \longrightarrow Q .$$

Let $\Phi = \phi_* - \phi'_* : K_0(C^*_{\max}(\pi_1(M), c_{\widehat{\pi}})) \to K_0(Q)$, then

$$\Phi(\alpha_{\max}) = [z_i]_{i \in \mathbb{N}} \in K_0(Q) \; \simeq \; \prod_{i \in \mathbb{N}} \mathbb{Z} / \bigoplus_{i \in \mathbb{N}} \mathbb{Z} ,$$

where $z_i = \mathrm{ind}(D^{E_i}) - \mathrm{ind}(D^{F_i})$ by the last theorem. But, since $\mathrm{ch}(E_i) - \mathrm{ch}(F_i)$ is concentrated in degree n, we have:

$$\begin{aligned} \mathrm{ind}(D^{E_i}) - \mathrm{ind}(D^{F_i}) &= \langle \widehat{A}(M) \cup (\mathrm{ch}(E_i) - \mathrm{ch}(F_i)), [M] \rangle \\ &= C \langle c_n(E_i), [M] \rangle \neq 0 \end{aligned}$$

for all $i \in \mathbb{N}$. So, in particular $\Phi(\alpha_{\max}(M))$ is non-zero in $K_0(Q)$, therefore $\alpha_{\max}(M)$ is non-zero as well. □

In case the manifold is of odd dimension, observe that for the second Stiefel-Whitney-classes we have $\pi_M^* w_2(M) = w_2(M \times S^1)$, so that $\widehat{\pi}_{S^1} = \widehat{\pi} \times \mathbb{Z}$, which fits into the short exact sequence

$$1 \longrightarrow S^1 \longrightarrow \widehat{\pi}_{S^1} \longrightarrow \pi_1(M) \times \mathbb{Z} \longrightarrow 1 .$$

As in [27], the decomposition:

$$\begin{aligned} & K_0(C^*_{\max}(\pi_1(M \times S^1), c_{\widehat{\pi}_{S^1}})) \simeq K_0(C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}) \otimes C^*(\mathbb{Z})) \\ \simeq\ & K_0(C^*_{\max}(\pi_1(M), c_{\widehat{\pi}})) \otimes 1 \oplus K_1(C^*_{\max}(\pi_1(M), c_{\widehat{\pi}})) \otimes e \end{aligned}$$

can be used to define $\alpha_{\max}(M) = \mathrm{pr}_e(\alpha_{\max}(M \times S^1)) \in K_1(C^*_{\max}(\pi_1(M), c_{\widehat{\pi}}))$. In this expression $1 \in K_0(C^*(\mathbb{Z})) \simeq K^0(S^1)$ and $e \in K_1(C^*(\mathbb{Z}))$ are the canonical generators and pr_e is the projection to the second summand. In fact, by the product formula from [66] we have $\alpha_{\max}(M) \otimes e = \alpha_{\max}(M \times S^1)$ using the exterior Kasparov product. Since $M \times S^1$ is enlargeable if M is, we immediately get:

Theorem 5.2.5. *Let M be an (area-)enlargeable manifold, then $\alpha_{\max}(M) \neq 0$.*

Chapter 6

Perspectives

Nichts ist getan, wenn noch etwas zu tun übrig ist.
Carl-Friedrich Gauß

The last chapters have shown that the theory of twisted Hilbert A-module bundles merges the non-twisted K-theory with coefficients in a C^*-algebra with twisted K-theory as defined by ROSENBERG. For the rest of this thesis we will discuss further generalizations in the direction of twisted K-theory as well as that of obstructions against positive scalar curvature metrics.

6.1 Central extensions

As mentioned in the beginning, it is easy to generalize the results given above to the case of central extensions of the form

$$1 \longrightarrow B \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow 1 \ .$$

for an arbitrary abelian group B. To give an example, why this might be interesting, suppose $\widehat{\Gamma}$ is a discrete group and $B = Z(\widehat{\Gamma})$ the center of $\widehat{\Gamma}$. Γ acts by conjugation on the reduced C^*-algebra $A = C^*_{\mathrm{red}}(\widehat{\Gamma})$. Thus, given a principal Γ-bundle P over M, we may form the C^*-algebra bundle

$$\mathcal{A} = P \times_{\mathrm{Ad}} C^*_{\mathrm{red}}(\widehat{\Gamma}) \ .$$

Note that B maps into the center of A and this induces a representation of B on every finitely generated Hilbert A-module W. $K_0(C(M, \mathcal{A}))$ is now described by an appropriate generalization of twisted Hilbert A-module bundles: There is a lifting bundle gerbe with a principal B-bundle $\widehat{L} \longrightarrow P^{[2]}$ and a corresponding multiplication over $P^{[3]}$. A twisted Hilbert A-module bundle now carries an action of $\widehat{L}$ in the sense of groupoids, i.e. there is a map

$$\widehat{L} \otimes \pi_2^* E \longrightarrow \pi_1^* E$$

where the tensor product denotes the fiberwise quotient by the antidiagonal action of B. It should restrict to an isomorphism of Hilbert modules on the fiber over $(p_1, p_2) \in P^{[2]}$ and be well-behaved with respect to the multiplication on the bundle gerbe. The DIXMIER-DOUADY-class will be an element of $\check{H}^2(M, \underline{B})$ and if it is trivial, P lifts to a principal $\widehat{\Gamma}$-bundle $\widehat{P}$. In that case we form:

$$V = \widehat{P} \times_\rho C^*_{\mathrm{red}}(\widehat{\Gamma}) ,$$

with the standard representation of $\widehat{\Gamma}$. $C(M, V)$ is our candidate for a MORITA equivalence between $C(M, \mathcal{A})$ and $C(M, A)$.

A similar setup works for more general C^*-algebras A as long as $\widehat{\Gamma}$ injects into the unitaries of A in such a way that B is mapped into the center, as we have seen for $A = C^*(\pi_1(M), c_{\widehat{\pi}})$ and $\widehat{\Gamma} = \widehat{\pi}$.

6.2 Enlargeability and infinite covers

The notion of enlargeability we use is still restricted in that it demands the finite covers $\overline{M} \to M$ to be spin, whereas there may be spaces, for which no finite spin cover exists, but which are enlargeable in a wider sense. It was shown in [28] how to handle enlargeability with infinite covers in the case of spin manifolds. The general idea is not very different from the above construction: Assemble almost flat Hilbert bundles into a large one, which remembers the Chern character of its parts. The ingredients, however, are different. The main proposition is:

Theorem 6.2.1. *Let M be an even-dimensional, (area)-enlargeable manifold and let $i \in \mathbb{N}$. There is a C^*-algebra C_i and a Hilbert C_i-module bundle F_i with connection ∇_i, such that the curvature Ω_i of F_i satisfies*

$$\|\Omega_i\| \leq C \, \frac{1}{i}$$

with C only depending on the dimension of M. Furthermore, there is a split extension of the form:

$$0 \longrightarrow \mathbb{K} \longrightarrow C_i \longrightarrow D_i \longrightarrow 0$$

for a certain C^*-algebra D_i. Let $a_i \in K_0(C_i)$ denote the index of the Dirac operator twisted with F_i, then the $\mathbb{Z} = K_0(\mathbb{K})$-component of a_i is different from 0.

Each F_i is again constructed from a vector bundle $E \to S^n$ pulled back to the cover $\overline{M} \to M$. Set $\pi = \pi_1(M)$, $\overline{\pi} = \pi_1(\overline{M})$. C_i is obtained from two C^*-algebras C_S and C_T contained in $B(H)$ for $H = l^2(\pi/\overline{\pi}) \otimes \mathbb{C}^d$ and some $d \in \mathbb{N}$ (the rank of E_i in the sequence of almost flat bundles). C_S is generated by the isometric actions of the permutations of $\pi/\overline{\pi}$ on $l^2(\pi/\overline{\pi})$. In the twisted case, this should presumably be replaced by the algebra generated by the isometries induced by the *projective* representation of π on $l^2(\pi/\overline{\pi})$. This ensures that π still acts in a projective way on the algebras C_i, which is crucial to get a twisting. C_T can be identified with the compact operators and should be left untouched.

Most of the construction of the F_i should survive the transfer to the twisted case, for which one would built a twisted Hilbert C_i-module bundle over $\widetilde{M}$ together with a bgm-connection ∇_i. Thus, it should be possible to drop the finiteness condition of the covers as well.

6.3 Non-torsion twists

As we have seen in corollary 4.3.12, the lifting bundle gerbes corresponding to central S^1-extensions of *compact* groups always yield finite order twists, even worse – as was mentioned by MURRAY in [49] – any finite dimensional fiber bundle $Y \to M$ will only lead to torsion twists. Therefore infinite order forces us to consider extensions of infinite groups. We fill focus on the case:

$$1 \longrightarrow S^1 \longrightarrow U(H) \longrightarrow PU(H) \longrightarrow 1$$

where H is a separable Hilbert space. It is a well-known fact that for each element $x \in H^3(M, \mathbb{Z})$ there is up to isomorphism exactly one principal $PU(H)$-bundle P over M with $dd(P) = x$. In [12] the authors proposed a way to get a geometric representation of non-torsion twists via modules over bundle gerbes. Suprisingly, much of their program also works in case of K-theory with coefficients in a C^*-algebra and we will sketch some ideas in that direction in this section. Let A be a unital C^*-algebra, $\mathbb{K}$ the compact operators on a separable Hilbert space, and denote by $M(A \otimes \mathbb{K})$ the stable multiplier algebra. It was shown by MINGO in [45] that KUIPER's theorem generalizes and that bounded FREDHOLM operators still form a spectrum for K^0_A.

Theorem 6.3.1. *With A as above, the stable unitary group $U_s(A) = U(M(A \otimes \mathbb{K}))$ of $M(A \otimes \mathbb{K})$ is contractible.*

Let $\mathrm{Fred}(A)$ be the bounded FREDHOLM operators in the sense of MISHCHENKO and FOMENKO [47] acting on the countable Hilbert A-module $l^2(A) = H \otimes_{\mathbb{C}} A$. Now the theorem of ATIYAH and JÄNICH generalizes to:

Theorem 6.3.2. *For unital A and a compact Hausdorff space M the homotopy classes of maps $[M, \mathrm{Fred}(A)]$ form a group under pointwise composition and there is a group isomorphism:*

$$K^0_A(M) = K_0(C(M, A)) = [M, \mathrm{Fred}(A)] \ .$$

From now on, we will take M to be a connected space. The inclusion of a point $pt \to M$ induces a map in K-theory

$$K^0_A(M) \to K^0_A(pt) = K_0(A)$$

evaluating the fiber over pt as a K-cycle in $K_0(A)$.

Definition 6.3.3. Let $\widetilde{K}^0_A(M) = \mathrm{kern}(K^0_A(M) \to K^0_A(pt))$, then $\widetilde{K}^0_A(M)$ will be called the *reduced K-theory* of M with coefficients in A.

By [47] FREDHOLM operators are elements in $M(A \otimes \mathbb{K}) = \mathrm{End}(l^2(A))$ that are invertible modulo the compact operators $\mathcal{K}(l^2(A)) = A \otimes \mathbb{K}$. Denote the CALKIN *algebra* by $Q = M(A \otimes \mathbb{K})/A \otimes \mathbb{K}$ and the invertible elements in Q by $GL(Q)$.

Theorem 6.3.4. *The sequence of topological spaces*

$$A \otimes \mathbb{K} \longrightarrow \mathrm{Fred}(A) \longrightarrow GL(Q)$$

is a fibration.

Proof. Consider the short exact sequence of C^*-algebras:

$$1 \longrightarrow A \otimes \mathbb{K} \longrightarrow M(A \otimes \mathbb{K}) \xrightarrow{q} Q \longrightarrow 1$$

as an exact sequence of Banach spaces. By the BARTLE-GRAVES selection theorem there is a (possibly non-linear!) continuous section $\sigma : Q \longrightarrow M(A \otimes \mathbb{K})$ with $q \circ \sigma = \mathrm{id}_Q$. Now consider the homotopy lifting diagram

$$\begin{tikzcd} M \times \{0\} \arrow[r, "f"] \arrow[d] & \mathrm{Fred}(A) \arrow[d, "q"] \\ M \times I \arrow[r, "h"] & GL(Q) \end{tikzcd}$$

The map $\widetilde{h}(m,t) = \sigma \circ h(m,t) + (f(m) - \sigma \circ h(m,0)) \in M(A \otimes \mathbb{K})$ runs through FREDHOLM operators since the term in brackets is compact and lifts h. □

Corollary 6.3.5. *The canonical map* $\mathrm{Fred}(A) \longrightarrow GL(Q)$ *is a weak equivalence.*

Proof. $A \otimes \mathbb{K}$ is contractible, since it is a Banach space. □

From theorem 6.3.2 we see that $\pi_0(\mathrm{Fred}(A)) = K_0(A)$ and therefore $\widetilde{K}^0_A(M)$ coincides with the homotopy classes of maps into $\mathrm{Fred}(A)_0$ the zero index component of $\mathrm{Fred}(A)$. But by another proposition cited in [45]:

Theorem 6.3.6. *If* $F \in \mathrm{Fred}(A)$ *has index* 0*, then there is* $G \in \mathrm{End}(l^2(A)) = M(A \otimes \mathbb{K})$ *invertible with* $F - G \in \mathcal{K}(l^2(A))$*, where* $\mathcal{K}$ *denotes the compact operators on that Hilbert* C^**-module.*

Therefore the map $\mathrm{Fred}(A) \longrightarrow GL(Q)$ sends $\mathrm{Fred}(A)_0$ to the subgroup $GL(M(A \otimes \mathbb{K}))/GL_{\mathcal{K}} \subset GL(Q)$, where

$$GL_{\mathcal{K}} = \{T \in GL(M(A \otimes \mathbb{K})) \mid T = 1 + K \text{ with } K \in A \otimes \mathbb{K}\}$$

and since the group of invertible elements in a C^*-algebra retracts to the unitaries, we can work as well with $U_s(A)/U_{\mathcal{K}}$ where $U_{\mathcal{K}}$ are the unitary elements in $GL_{\mathcal{K}}$. By theorem 6.3.1 the space $U_s(A)/U_{\mathcal{K}}$ is a model for $BU_{\mathcal{K}}$. Thus,

$$\widetilde{K}^0_A(M) = [M, BU_{\mathcal{K}}] \ .$$

Observe that $U(H) \longrightarrow U_s(A)$, since A is unital. Therefore $U(H)$ acts on $\mathrm{Fred}(A)$ and on $BU_{\mathcal{K}}$ by conjugation. This leads to the following definition:

Definition 6.3.7. Given a $PU(H)$-bundle P, we can form the associated $BU_{\mathcal{K}}$-bundle $\mathcal{B}U_{\mathcal{K}} = P \times_{\mathrm{Ad}} BU_{\mathcal{K}}$. The *reduced twisted K-theory with coefficients in A and twist P* is given by

$$\widetilde{K}^0_{A,P}(M) = [M, \mathcal{B}U_{\mathcal{K}}] = [P, BU_{\mathcal{K}}]^{PU(H)} \ ,$$

i.e. either by homotopy classes of sections of $\mathcal{B}U_{\mathcal{K}}$ or by homotopy classes of $PU(H)$-equivariant maps $P \to BU_{\mathcal{K}}$.

Note that $\widetilde{K}^0_{A,P}(M)$ still is a group with respect to pointwise multiplication like in [12], since our model of $BU_{\mathcal{K}}$ is a group.

Definition 6.3.8. A principal $U_{\mathcal{K}}$-bundle $Q \to P$ is called *covariant* if there is a right action of $PU(H)$ on Q covering the one on P, such that

$$(rg) \cdot a = (r \cdot a)\, \widehat{a}^{-1} g \widehat{a} \qquad \text{for } g \in U_{\mathcal{K}}, a \in PU(H) \text{ and a lift } \widehat{a} \in U(H) \ .$$

Let $E \to P$ be a twisted Hilbert A-module bundle with fiber $l^2(A)$. Denote by P_E the principal $U_s(A)$-frame bundle of E, i.e. each $r \in P_E$ over $p \in P$ represents an isometric isomorphism of Hilbert A-modules $l^2(A) \to E_p$. We call E *a twisted $U_{\mathcal{K}}$-bundle*, if P_E reduces to a principal $U_{\mathcal{K}}$-bundle Q, which is transformed into itself by the action of $U(H)$ induced by the bundle gerbe on P_E.

If E is a twisted $U_{\mathcal{K}}$-bundle, then $PU(H)$ acts on Q as follows:

$$r \cdot a = \widehat{a}^{-1}\ (r \circ \widehat{a}) \ : \ l^2(A) \longrightarrow E_{pa}$$

for $r \in Q$ and $a \in PU(H)$, where $\widehat{a} \in U(H)$ denotes a lift of a and the action on the left of r is the one induced by the bundle gerbe. Observe that Q is a covariant $U_{\mathcal{K}}$-bundle with this action. On the other hand every covariant principal $U_{\mathcal{K}}$-bundle Q yields a twisted $U_{\mathcal{K}}$-bundle E. Indeed, let $E = Q \times_{U_{\mathcal{K}}} l^2(A)$ and set:

$$\gamma_E \ : \ L \otimes \pi_2^* E \longrightarrow \pi_1^* E \quad ; \quad [\widehat{a}, \lambda] \otimes [r, w] \mapsto [r \cdot a^{-1}, \lambda\, \widehat{a}\, v] \ .$$

Now choose an isometry $l^2(A) \times l^2(A) \longrightarrow l^2(A)$ inducing a group monomorphism:

$$\phi \ : \ U_{\mathcal{K}}(A) \times U_{\mathcal{K}}(A) \longrightarrow U_{\mathcal{K}}(M_2(A)) \xrightarrow{\sim} U_{\mathcal{K}}(A) \ .$$

This turns the set of isomorphism classes of covariant principal $U_{\mathcal{K}}$-bundles into a semi-group via

$$Q_1 \oplus Q_2 = (Q_1 \times_P Q_2) \times_\phi U_{\mathcal{K}} \ .$$

Note that this structure reflects the direct sum operation on twisted $U_{\mathcal{K}}$-bundles. In fact, the semi-group of covariant principal $U_{\mathcal{K}}$-bundles is isomorphic to the semi-group of isomorphism classes of twisted $U_{\mathcal{K}}$-bundles with the direct sum operation, denoted by $\mathrm{Mod}_{U_{\mathcal{K}}}(M, P)$, if we identify $l^2(A) \oplus l^2(A)$ with $l^2(A)$ via the above isometry. A stabilization argument shows that the group multiplication on $BU_{\mathcal{K}}$ yields the same H-space structure as the direct sum.

Now the proof of the following result is completely analogues to the proof of proposition 7.2 and 7.3 in [12].

Theorem 6.3.9. *The map*

$$\widetilde{K}^0_{A,P}(M) \longrightarrow \mathrm{Mod}_{U_{\mathcal{K}}}(M, P)$$

given by pulling back the principal $U_{\mathcal{K}}$-bundle $U_s(A) \to BU_{\mathcal{K}}$ via the given map $P \to BU_{\mathcal{K}}$ and adjoining $l^2(A)$ is an isomorphism of semi-groups inducing a group structure on the right hand side.

In this way we can represent the *reduced* twisted K-theory by twisted $U_{\mathcal{K}}$-bundles. In case $A = M_n(\mathbb{C})$, a spectral sequence argument given in [6] shows that for infinite order twists there are no sections of the $\mathrm{Fred}(H)$-bundle over M of non-zero index. This implies that the reduced twisted K-theory actually coincides with its non-reduced counterpart. It is an interesting question, whether one can transfer this argument for some more complicated C^*-algebras. This, however, requires some knowledge about the cohomology $H^2(\mathrm{Fred}(A)_x)$, where $x \in K_0(A)$ labels the path component $\mathrm{Fred}(A)_x$ of $\mathrm{Fred}(A)$.

As we have already noted in the chapter about connections, the extension

$$1 \longrightarrow S^1 \longrightarrow U(H) \longrightarrow PU(H) \longrightarrow 1$$

is not flat. It is still possible to find a connection on corresponding lifting bundle gerbes that are compatible with the gerbe multiplication as was shown by Gomi in [24]. The Chern character in this case takes values in a *twisted* form of cohomology $H^{\mathrm{even}}(M, [P])$, since the rationalization of the Dixmier-Douady-class no longer vanishes [12]. Another question would be to extend this to a Chern character with values in $H^{\mathrm{even}}(M, [P]) \otimes K_0(A)$ like above. This would require an identification of the reduced twisted K-theory with some C^*-algebraic K-group, i.e. a result similar to the theorem proven by Rosenberg in [57].

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