Noncommutative manifolds and Seiberg–Witten equations

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Abstract

In this thesis we study differential geometry of noncommutative manifolds. We introduce a general framework of noncommutative manifolds based on Poincaré duality and study the notions of differential forms and Sobolev spaces for noncommutative manifolds. We introduce conditions under which the noncommutative manifolds have reasonable differential calculus and Sobolev theory. Furthermore, we study the properties of the Laplace operator on differential forms, proving that in certain cases it has compact resolvent similarly to the commutative situation. This allows us to address the question of comparison of the "de Rham cohomology" and periodic cyclic cohomology.

In the second part of the thesis, we introduce an analogue of the Seiberg–Witten equations for noncommutative manifolds and prove that the known properties of the Seiberg–Witten gauge theory continue to hold in the noncommutative situation. For instance, as far as the noncommutative manifold has Sobolev theory which has nice mutiplicative properties, the moduli space of smooth solutions coincides with the moduli spaces of the Sobolev solutions for large values of the Sobolev parameter. We also derive the holomorphic description of the moduli space for toric deformations of Kähler manifolds, which allows us to compute the moduli spaces for a family of such toric deformations where the underlying manifold has constant scalar curvature.

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Contents

Introduction

The idea of noncommutative manifolds relies on the observation that certain operator algebras, being noncommutative and very far from algebras of functions (having, for instance, representations of type II), allow structures very similar to that arizing on smooth functions on a manifold. This leads to the attempt to describe differential-geometric constructions in functional-analytic and operator-algebraic terms. Recently, the reconstruction theorem was proved by A. Connes, which allows to say that a certain set of "axioms" characterizes noncommutative manifolds in the right way.

The basic idea can be formulated as follows: a compact oriented manifold is a topological space with a smooth structure and therefore has the fundamental class in cohomology. If we switch from manifolds to noncommutative algebras, we are thus led to the idea of saying that a noncommutative manifold is a dense subalgebra of a C^*-algebra having a "fundamental class". However, we also have to switch from cohomology to K-theory and thus forced to consider a fundamental class in K-homology, which in the case of an ordinary manifold is given by the Dirac operator. The Dirac operator on a manifold is a certain unbounded selfadjoint operator acting on a vector bundle over the manifold. It is well-known that vector bundles over the manifold M correspond to finitely generated projective modules over the algebra $\mathcal{A} = C^\infty(M)$, and the latter have obvious noncommutative counterparts. Developing these ideas further, one can obtain a characterization of manifolds in functional-analytic and operator-algebraic terms which allow to drop the assumption on the commutativity of the algebra in question. Moreover, one obtains a series of examples of such "noncommutative manifolds" given by deformations of ordinary manifolds. It is very striking that the operator algebras which arise in this way fail to be of type I, thus they have a quite complicated representation theory and algebraic structure, being far away from commutative algebras.

In this context, several natural questions emerge. First of all, there is a general question: which properties of the usual, commutative manifolds continue to hold for noncommutative manifolds? What about calculus of differential forms, connections, curvature, Sobolev theory? Finally, what about invariants of gauge-theoretic nature?

The author's work on these questions lead to this thesis, whose results can be summarized as follows. We start with the observation that the Poincaré duality assumption is a crucial ingredient in the axiomatics for noncommutative manifolds and that allowing the algebra $\mathcal{A}$ to have an abstract Poincaré dual $\mathcal{B}$ relaxes the situation of the spin manifolds to a general setting, still allowing to develop the analytic machinery but leading to a bit more general point of view, allowing to include toric deformations of not necessarily spin manifolds to the list of noncommutative manifolds. Then we proceed to the analytic properties of noncommutative manifolds, proving that the algebras $\mathcal{A}$ and $\mathcal{B}$ are automatically Fréchet algebras and introducing a precursor of the Sobolev topology. Then we abstractly introduce the spaces of differential forms and the Laplace operator on them. Unfortunately, we don't know how to ensure that these spaces automatically form finitely generated projective modules, and therefore we have to assume this for certain constructions.

Then we proceed to the proofs of certain theorems concerning Laplace operators. It turns out that in the case where the differential forms behave algebraically like they do on an ordinary manifold, the Laplace operator automatically has compact resolvent, thus yielding a finite-dimensional space of harmonic forms ("de Rham cohomology"). We address the question of comparison between this "de Rham cohomology" and periodic cyclic cohomology by means of a natural map introduced by A. Connes and prove that this map is surjective in certain cases. We also give counterexamples to the injectivity of this map.

After that, we give an affirmative answer to the question whether an analogue of the "twisting procedure" for spin^c structures on manifolds can be introduced in the noncommutative setting. It turns out that we indeed can "twist" a given structure on a noncommutative manifold by a "line bundle" (i.e. a bimodule whose left and right dimensions equal 1), obtaining another structure of a noncommutative manifold.

The last part of the thesis consists in studying a certain gauge theory on noncommutative 4-manifolds, which is a straightforward generalization of the celebrated Seiberg–Witten gauge theory on ordinary manifolds. It turns out that many interesting properties of the Seiberg-Witten theory still hold in the noncommutative case (provided that one has a certain multiplicativity property of Sobolev spaces, analogous to that of an ordinary manifold – it is, for instance, the case for toric deformations of ordinary manifolds). For instance, elliptic regularity still yields the independence of the moduli space of the degree of Sobolev completion, and the virtual dimension of the moduli space is still given by the same formula as in the commutative case. Finally, the holomorphic description of the moduli space for Kähler manifolds is still valid for their toric deformations, and if the scalar curvature of the deformed manifold is constant, we can actually compute the moduli space – it then consists of one point exactly as in the commutative situation.

CHAPTER 1

Toolbox

In this chapter we collect some technical results needed in the sequel. Some of these results are well-known and we basically just cite them here, some other are modifications of well-known results or even new, but technical in nature.

1. Pre-Hilbert modules over pre-C^*-algebras

Definition 1.1. Let $\mathcal{A}$ be a unital pre-C^*-algebra, i.e. a dense unital Fréchet $*$-subalgebra of a unital C^*-algebra A, stable under holomorphic functional calculus. We denote by $\|\cdot\|$ the norm on $\mathcal{A}$ inherited from A. A Fréchet space $\mathcal{X}$ is called a (right) Fréchet pre-Hilbert module over $\mathcal{A}$ if the following holds:

- i) $\mathcal{X}$ is a right $\mathcal{A}$-module,
- ii) the multiplication map $\mathcal{X} \times \mathcal{A} \to \mathcal{X}$, $(x, a) \to x \cdot a$ is continuous,
- iii) $\mathcal{X}$ is equipped with a continuous $\mathcal{A}$-valued scalar product $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ with following properties:
  - (a) it is linear: $\langle y, x + z \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$,
  - (b) it is $\mathcal{A}$-linear and involutive:
$$\langle y, x \cdot a \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}} a,$$
$$\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^* ,$$
  - (c) it is positive and nondegenerate:
$$\langle x, x \rangle_{\mathcal{A}} \geq 0,$$
$$\langle x, x \rangle = 0 \ \Leftrightarrow x = 0.$$

For $x \in \mathcal{X}$, we let $\|x\| := \sqrt{\|\langle x, x \rangle\|}$. This is a continuous norm on $\mathcal{X}$.

We will not develop any general theory of modules over pre-C^*-algebras, but we will need some simple facts about them which are very much analogous to those about Hilbert modules over C^*-algebras. The ideas of the proofs are taken from the proofs of corresponding statements for Hilbert C^*-modules, cf. [**MT05**]. The following statement is a key to this.

Definition 1.2. If $\mathcal{X}$, $\mathcal{Y}$ are two Fréchet pre-Hilbert modules and $T \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$, then T is called adjointable if there exists $T^* \colon \mathcal{Y} \to \mathcal{X}$ such that for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$
$$\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^* y \rangle_{\mathcal{A}}$$

Proposition 1.3.
- i) *Let $\mathcal{X}$ be a Fréchet pre-Hilbert module over a pre-C^*-algebra $\mathcal{A}$ dense in A. Let X be its completion with respect to the norm $\|x\|$. Then X is a Hilbert module over A.*
- ii) *Let $T \colon \mathcal{X} \to \mathcal{Y}$ be an element in $\mathrm{Hom}^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. Consider its image $\overline{T} \in \mathcal{B}(X, Y)$ under the completion functor. Then $T \in \mathrm{Hom}^*_A(X, Y)$.*

Proof. The first statement is well-known, see, for instance, [**GBVF01**, Sect. 3.8]. For the second, let $\xi \in X$, $a \in A$. Take $\xi_n \in \mathcal{X}$, $a_n \in \mathcal{A}$ with $\xi_n \to \xi$, $a_n \to a$. Then
$$T(\xi a) = \lim_{n \to \infty} T(\xi_n a_n) = \lim_{n \to \infty} T(\xi_n) a_n = T(\xi) a.$$
□

Definition 1.4. Let $\mathcal{X}$ be a right Fréchet pre-Hilbert module over $\mathcal{A}$. Denote by $\langle \mathcal{X}, \mathcal{X} \rangle$ the closure inside $\mathcal{A}$ of the space

$$\mathrm{span}\{\langle y, x \rangle \mid x, y \in \mathcal{A}\}.$$

It is a Fréchet $*$-subalgebra of $\mathcal{A}$. It is a closed two-sided ideal in $\mathcal{A}$. If $\langle \mathcal{X}, \mathcal{X} \rangle = \mathcal{A}$, then the module $\mathcal{X}$ is called full.

Example 1.5.
i) $\mathcal{A}$ is a full Fréchet pre-Hilbert $\mathcal{A}$-module with the scalar product $\langle b, a \rangle_{\mathcal{A}} = b^* a$,
ii) a direct sum of Fréchet pre-Hilbert $\mathcal{A}$-modules is a Fréchet pre-Hilbert $\mathcal{A}$-module in a natural way,
iii) if $e \in \mathbb{M}_k(\mathcal{A})$ is a projection, then the space $e\mathcal{A}^k$ is a Fréchet pre-Hilbert $\mathcal{A}$-module when equipped with the scalar product inherited from $\mathcal{A}^k$. A module isomorphic to such is called a finitely generated projective Fréchet pre-Hilbert module.

The following proposition is well-known, cf. [**Con94**, VI.1].

Proposition 1.6. *Let $\mathcal{A} \subset A$ be a pre-C^*-algebra (i.e. a dense Fréchet subalgebra stable under holomorphic functional calculus) and let $\mathcal{X}$ be a finitely generated projective right $\mathcal{A}$-module. Then there is a structure of a self-dual Fréchet pre-Hilbert module on $\mathcal{X}$ and such structure is unique up to an isomorphism.*

Proposition 1.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Fréchet pre-Hilbert modules over $\mathcal{A}$ and let $T \in \mathrm{Hom}^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. Then T is a continuous linear $\mathcal{A}$-module morphism.*

Proof. The $\mathcal{A}$-linearity follows easily from the adjointability and the linearity of the scalar product:

$$\langle y, T(x \cdot a) \rangle = \langle T^* y, x \cdot a \rangle = \langle y, Tx \rangle a = \langle y, Tx \cdot a \rangle .$$

To prove continuity, we use the closed graph theorem: if $x_\alpha \to x$ and $Tx_\alpha \to y$, then for every $z \in \mathcal{Y}$,

$$\langle y - Tx, z \rangle = \langle y, z \rangle - \langle Tx, z \rangle = \lim \langle Tx_\alpha, z \rangle - \langle Tx, z \rangle = \lim \langle x_\alpha, T^* z \rangle - \langle x, T^* z \rangle = 0.$$

□

Lemma 1.8. *Let $\mathcal{X}$ be a full Fréchet pre-Hilbert module over $\mathcal{A}$. Consider the set*

$$S = \{c \in \mathcal{A} \mid \|c\| \leq 1,\, c = \sum_{i=1}^{k} \langle x_i, x_i \rangle \; k \in \mathbb{N},\, x_i \in \mathcal{X}\}.$$

Then for every $a \in \mathcal{A}$, $a \geq 0$ and for every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ and an element $c \in S$ such that $\|(1 - c)a\| < \varepsilon$.

Proof. As $\mathcal{X}$ is a full module, there are finitely many elements $y_i \in \mathcal{X}$ with

$$\left\| a - \sum_{i=1}^{k} \langle y_i, y_i \rangle \right\| < \varepsilon/2.$$

Let

$$x_i = \left(\varepsilon^2 + \sum_{i=1}^{k} \langle y_i, y_i \rangle \right)^{-1/2} y_i, i = 1, \ldots, k,$$

$$c = \sum_{i=1}^{k} \langle x_i, x_i \rangle , \quad b = \sum_{i=1}^{k} \langle y_i, y_i \rangle .$$

Then

$$\|c\| = \left\| (\varepsilon^2 + b)^{-1/2} b (\varepsilon^2 + b)^{-1/2} \right\| \leq 1,$$

thus $c \in S$. Taking $f(t) = \varepsilon^4 t(\varepsilon^2 + t)^{-2}$ and applying f to b, we obtain $\|f(b)\| = \|(1-c)b(1-c)\| \leqslant \varepsilon^2/4$, thus $\|(1-c)b\| \leqslant \varepsilon/2$, which proves the lemma. □

Proposition 1.9. *Let $\mathcal{A}$ be a unital pre-C^*-algebra and let $\mathcal{X}$ be a full Fréchet Hilbert module over it which is finitely generated and projective. Then there are elements $\xi_1, \dots, \xi_k \in \mathcal{X}$ such that*

$$\sum_{i=1}^{k} \langle \xi_i, \xi_i \rangle_{\mathcal{A}} = 1.$$

Proof. The previous lemma gives us an element $c \in S$ such that $\|1 - c\| < 1/2$ and $c = \sum_{i=1}^{k} \langle y_k, y_k \rangle$ for some $y_k \in \mathcal{X}$. Thus the elements c is invertible in the C^*-algebra A, and as c is selfadjoint, its spectrum lies on the positive real line and is separated from zero. As $\mathcal{A}$ is stable under holomorphic functional calculus, $c^{-1/2} \in \mathcal{A}$. Thus, taking $x_i = y_i c^{-1/2}$, we obtain the conclusion. □

Corollary 1.10. *Under the assumptions of the proposition, there is a $k \in \mathbb{N}$ such that the direct sum of k copies of $\mathcal{X}$ contains a copy of $\mathcal{A}$:*

$$(\mathcal{X})^{\oplus k} = \mathcal{A} \oplus \mathcal{Y}$$

Proof. Using the proposition, we find elements $x_1, \dots, x_k$ and consider the closed submodule $\mathcal{Z}$ of $(\mathcal{X})^{\oplus k}$ generated by $\iota_i(x_i)$, $i = 1, \dots, k$, where $\iota_i \colon \mathcal{X} \to (\mathcal{X})^k$ is the canonical inclusion into the ith component. Then the map $a \mapsto (x_1 a, \dots, x_k a)$ yields an isomorphism of right $\mathcal{A}$-modules because

$$\sum_{i=1}^{k} \langle x_i b, x_i a \rangle_{\mathcal{A}} = \sum_{i=1}^{k} b^* \langle x_i, x_i \rangle\, a = b^* a.$$

□

Let us observe the following obvious proposition.

Proposition 1.11. *Let $\mathcal{X}$ be a right Fréchet pre-Hilbert module over $\mathcal{A}$ which is finitely generated and projective and let $P \colon \mathcal{X} \to \mathcal{X}$ be an $\mathcal{A}$-endomorphism which is a projection: $P^2 = P = P^*$. Then $\mathcal{Y} := P\mathcal{X}$ is a finitely generated projective Fréchet pre-Hilbert module over $\mathcal{A}$.*

Proof. If $\mathcal{X} = e\mathcal{A}^n$, then P is given by the left multiplication with a selfadjoint matrix $p \in e\mathbb{M}_n(\mathcal{A})e$, and $\mathcal{Y}$ is therefore isomorphic to $p\mathcal{A}^n$. □

Definition 1.12. An $\mathcal{A}$-basis for $\mathcal{X}$ is a finite set $\{v_j\}_{j=1}^{m} \subset \mathcal{X}$ such that

$$\forall x \in \mathcal{X} \quad x = \sum_{j=1}^{m} v_j \langle v_j, x \rangle .$$

Proposition 1.13. *A right Fréchet pre-Hilbert module over $\mathcal{A}$ is finitely generated and projective if and only if it has a finite $\mathcal{A}$-basis.*

Proof. If $\{v_j\}$ is an $\mathcal{A}$-basis, then $p := \{\langle v_i, v_j \rangle\} \in \mathcal{A}^m$ is a projection, and $\mathcal{X} \cong p\mathcal{A}^m$, the isomorphism given by

$$\alpha \colon \mathcal{X} \to p\mathcal{A}^m,$$

$$x \mapsto (\langle v_j, x \rangle)_{j=1,\dots,m}.$$

On the other hand, if $\mathcal{X} \cong p\mathcal{A}^m$, then the columns of p, regarded as elements in $\mathcal{A}^m$, consitute an $\mathcal{A}$-basis. □

The following construction is well-known in the case of C^*-algebras, cf. [**MT05**].

Definition 1.14. Let $\mathcal{A} \subset \mathcal{B}$ be a pair of Fréchet pre-C^*-algebras, and let $E\colon \mathcal{B} \to \mathcal{A}$ be a linear map. E is called a conditional expectation, if E is a projection of norm 1 onto $\mathcal{A}$.

Proposition 1.15. *If $E\colon \mathcal{B} \to \mathcal{A}$ is a conditional expectation, then it is a bimodule map: $E(aba') = aE(b)a'$ for $a, a' \in \mathcal{A}$ and $b \in \mathcal{B}$.*

Proof. Let $\overline{E}\colon B \to A$ be the corresponding map between the C^*-completions of $\mathcal{B}$ and $\mathcal{A}$. Then $\overline{E}$ is a conditional expectation in the sense of C^*-algebras, and therefore it is a bimodule map [**MT05**]. The claim follows by restriction to $\mathcal{B}$. □

A conditional expectation $E\colon \mathcal{B} \to \mathcal{A}$ gives rise to an $\mathcal{A}$-valued scalar product on $\mathcal{B}$ defined by

$$\langle b, b' \rangle_{\mathcal{A}} := E(b^* b').$$

We will be mainly interested in conditional expectations of algebraically finite index, which allow to endow $\mathcal{B}$ with a structure of a finitely generated projective $\mathcal{A}$-module.

Definition 1.16. A conditional expectation $E\colon \mathcal{B} \to \mathcal{A}$ is said to be of algebraically finite index if there is a finite set $\{u_i\}_{i=1}^n$ in $\mathcal{B}$ such that for every $b \in \mathcal{B}$

$$b = \sum_{i=1}^{n} u_i E(u_i^* x).$$

Such a finite set is called a quasi-basis for E.

Proposition 1.17. *Let $E\colon \mathcal{B} \to \mathcal{A}$ be a conditional expectation of algebraically finite index. Then $\mathcal{B}$ is a finitely generated projective Fréchet pre-Hilbert module over $\mathcal{A}$.*

Proof. The statement follows from the fact that $\{u_i\}$ form a basis for $\mathcal{B}$ as for a pre-Hilbert module over $\mathcal{A}$, which in turn follows directly from the definition of a quasi-basis. □

2. Pre-Hilbert bimodules

Here we develop the notions of Hermitian bimodules of finite type ovre pre-C^*-algebras. The proofs are direct adaptaions of the corresponding statements from [**KW00**], where an analogous theory was developed for bimodules over C^*-algebras.

In this section, $\mathcal{A}$, $\mathcal{B}$ etc. will denote Fréchet pre-C^*-algebras.

Definition 1.18. A Fréchet space $\mathcal{X}$ is called a Fréchet pre-Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type iff $\mathcal{X}$ is equipped with the structures of a left pre-Hilbert $\mathcal{A}$-module and a right pre-Hilbert $\mathcal{B}$-module with corresponding scalar products ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ such that:

i) the actions of $\mathcal{A}$ and $\mathcal{B}$ commute;
ii) with these structures $\mathcal{X}$ is a finitely generated projective pre-Hilbert $\mathcal{A}$-module and a finitely generated projective pre-Hilbert $\mathcal{B}$-module;
iii) $\mathcal{A}$ acts on $\mathcal{X}$ through an $*$-representation into $\mathcal{B}$-adjointable operators, and $\mathcal{B}$ acts on $\mathcal{X}$ through an $*$-representation into $\mathcal{A}$-adjointable operators.

Proposition 1.19. *Let $\mathcal{X}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type. Then the elements $\sum_{i=1}^{n} {}_{\mathcal{A}}\langle u_i, u_i \rangle$ and $\sum_{j=1}^{m} \langle v_j, v_j \rangle_{\mathcal{B}}$ are contained in $\mathcal{Z}(\mathcal{A})$ resp. $\mathcal{Z}(\mathcal{B})$ and are independent of the choices of a right $\mathcal{B}$-basis $\{u_i\}$ resp. left $\mathcal{A}$-basis $\{v_j\}$, respectively.*

Proof. Take another $\mathcal{B}$-basis $\{u_1',\dots,u_p'\}$. Then

$$\sum_{i=1}^n {}_{\mathcal{A}}\langle u_i,u_i\rangle = \sum_{i=1}^n {}_{\mathcal{A}}\left\langle \sum_{k=1}^p u_k' \langle u_k',u_i\rangle_{\mathcal{B}}, u_i\right\rangle = \\ = \sum_{i,k} {}_{\mathcal{A}}\langle u_k', u_i\langle u_i,u_k'\rangle_{\mathcal{B}}\rangle = \sum_{k=1}^p {}_{\mathcal{A}}\langle u_k',u_k'\rangle.$$

Similarly, with $a\in\mathcal{A}$,

$$a\sum_{i=1}^n {}_{\mathcal{A}}\langle u_i,u_i\rangle = \sum_{i=1}^n {}_{\mathcal{A}}\langle au_i,u_i\rangle = \sum_{i=1}^n {}_{\mathcal{A}}\left\langle \sum_{k=1}^p u_k' \langle u_k',au_i\rangle_{\mathcal{B}}, u_i\right\rangle = \\ = \sum_{i,k} {}_{\mathcal{A}}\langle u_k', u_i\langle au_i,u_k'\rangle_{\mathcal{B}}\rangle = \sum_{i,k} {}_{\mathcal{A}}\langle u_k', u_i\langle u_i,a^*u_k'\rangle_{\mathcal{B}}\rangle = \\ = \sum_{k=1}^p {}_{\mathcal{A}}\langle u_k',a^*u_k'\rangle = \sum_{k=1}^p {}_{\mathcal{A}}\langle u_k',u_k'\rangle a.$$

□

Lemma 1.20. *If $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type, and $\mathcal{Y}$ is a $\mathcal{B}$-$\mathcal{C}$ bimodule of finite type, then $\mathcal{X}\otimes_{\mathcal{B}}\mathcal{C}$ is an $\mathcal{A}$-$\mathcal{C}$ bimodule of finite type.*

Proof. Let $\{u_1,\dots,u_n\}$ be a right $\mathcal{B}$-basis in $\mathcal{X}$ and $\{v_1,\dots,v_m\}$ a right $\mathcal{C}$-basis in $\mathcal{Y}$. Then

$$x = \sum_{i=1}^n u_i\langle u_i,x\rangle_{\mathcal{B}}, \quad y = \sum_{j=1}^m v_j\langle v_j,y\rangle_{\mathcal{C}}.$$

Consider the set $\{u_i\otimes v_j\}\in\mathcal{X}\otimes_{\mathcal{B}}\mathcal{Y}$ and calculate:

$$\sum_{i=1}^n\sum_{j=1}^m (u_i\otimes v_j)\langle u_i\otimes v_j, x\otimes y\rangle = \sum_{i=1}^n\sum_{j=1}^m (u_i\otimes v_j)\langle v_j\langle u_i,x\rangle_{\mathcal{B}}, y\rangle_{\mathcal{C}} \\ = \sum_{i=1}^n u_i\otimes\left(\sum_{j=1}^m v_j\langle v_j,\langle u_i,x\rangle_{\mathcal{B}}^* y\rangle_{\mathcal{C}}\right) = \\ = \sum_{i=1}^n u_i\otimes\langle u_i,x\rangle_{\mathcal{B}}^* y = \sum_{i=1}^n u_i\langle u_i,x\rangle_{\mathcal{B}}\otimes y = x\otimes y.$$

□

Lemma 1.21. *Let $\mathcal{X}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type which is full as a left $\mathcal{A}$-module. Then there exists a conditional expectation $E\colon \mathrm{End}^*_{\mathcal{B}}(\mathcal{X})\to\mathcal{A}$ which is of algebraically finite index.*

Proof. We take a right $\mathcal{B}$-basis $\{u_1,\dots,u_n\}$. For $T\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{X})$, we set

$$F(T) := \sum_{i=1}^n {}_{\mathcal{A}}\langle Tu_i,u_i\rangle.$$

This is independent of the choice of a right $\mathcal{B}$-basis in $\mathcal{X}$, because if $\{u'_k\}$ is another $\mathcal{B}$-basis, then

$$(1.1)\quad \sum_{i=1}^{n} {}_{\mathcal{A}}\langle Tu_i, u_i\rangle = \sum_{i=1}^{n} {}_{\mathcal{A}}\left\langle \sum_{k=1}^{p} u'_k \langle u'_k, Tu_i\rangle_{\mathcal{B}}, u_i \right\rangle =$$
$$= \sum_{i,k} {}_{\mathcal{A}}\langle u'_k, u_i \langle Tu_i, u'_k\rangle_{\mathcal{B}}\rangle = \sum_{i,k} {}_{\mathcal{A}}\langle u'_k, u_i \langle u_i, T^* u'_k\rangle_{\mathcal{B}}\rangle =$$
$$= \sum_{k=1}^{p} {}_{\mathcal{A}}\langle u'_k, T^* u'_k\rangle = \sum_{k=1}^{p} {}_{\mathcal{A}}\langle Tu'_k, u'_k\rangle.$$

Moreover, if we take $T = \Theta_{x,y}$, then we obtain

$$F(T) = \sum_{i=1}^{n} {}_{\mathcal{A}}\langle x \langle y, u_i\rangle_{\mathcal{B}}, u_i\rangle = \sum_{i=1}^{n} {}_{\mathcal{A}}\langle x, u_i \langle u_i, y\rangle_{\mathcal{B}}\rangle = {}_{\mathcal{A}}\langle x, y\rangle,$$

and therefore

$$F(aTa') = aF(T)a', \quad T \in \mathrm{End}_{\mathcal{B}}(\mathcal{X}),\ a, a' \in \mathcal{A}.$$

The element $z := F(\mathrm{id})$ is contained in $\mathcal{Z}(\mathcal{A})$. Let us show that it is invertible. By Proposition 1.9, there exists a finite set of elements $\{x_j\}_{j=1}^{m} \subset \mathcal{X}$ such that

$$\sum_{j=1}^{m} {}_{\mathcal{A}}\langle x_i, x_i\rangle = \mathrm{id}\,.$$

The operator

$$S := \sum_{j=1}^{m} \Theta_{x_i, x_i}$$

is selfadjoint and positive, and therefore there is a constant $C > 0$ such that

$$C\,\mathrm{id} \geqslant \sum_{j=1}^{m} \Theta_{x_i, x_i}.$$

Now,

$$\mathrm{id} = \sum_{i=1}^{n} \Theta_{u_i, u_i},$$

and therefore

$$F(C\,\mathrm{id}) \geqslant \sum_{j=1}^{m} F(\Theta_{x_i, x_i}) = \sum_{j=1}^{m} {}_{\mathcal{A}}\langle x_i, x_i\rangle.$$

This shows that

$$\sum_{i=1}^{n} {}_{\mathcal{A}}\langle u_i, u_i\rangle \geqslant \frac{1}{C} \sum_{j=1}^{m} {}_{\mathcal{A}}\langle x_i, x_i\rangle = \frac{1}{C}\,\mathrm{id},$$

and this establishes the claim.

Now, we put

$$E(T) := z^{-1/2} F(T) z^{-1/2} = z^{-1} F(T) = F(T) z^{-1}.$$

Now, let $\{u_i\}$ and $\{v_j\}$ be the right $\mathcal{B}$-basis and the left $\mathcal{A}$-basis of $\mathcal{X}$, respectively. We show that Θ_{u_i,v_j} forms a quasi-basis for F:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{u_i,v_j}F(\Theta^*_{u_i,v_j}\Theta_{p,q}) = \sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{u_i,v_j}\,{}_{\mathcal{A}}\langle v_j\,\langle u_i,p\rangle_{\mathcal{B}}\,,q\rangle =$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{u_i,v_j}\,{}_{\mathcal{A}}\langle v_j, q\,\langle p,u_i\rangle_{\mathcal{B}}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{u_i,\,{}_{\mathcal{A}}\langle q\langle p,u_i\rangle_{\mathcal{B}},v_j\rangle v_j} = \sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{u_i,q\langle p,u_i\rangle_{\mathcal{B}}} =$$
$$= \Theta_{p,q}.$$

We put $U_{ij} := \Theta_{u_i,v_j}z^{1/2}$. Then, using $z\in\mathcal{Z}(\mathcal{A})$, we obtain

$$E(\Theta_{p,q}) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{m}U_{ij}E(U^*_{ij}\Theta_{p,q}) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{m}\Theta_{u_i,v_j}F(\Theta^*_{u_i,v_j}\Theta_{p,q}) \;=\; \Theta_{p,q},$$

and therefore U_{ij} form a quasi-basis for E. □

Given a pre-Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type, we just saw that $\mathrm{End}_{\mathcal{B}}$ becomes a Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule of finite type. It is interesting to observe that this is a particular case of the tensor product construction, if we use conjugate bimodules.

Definition 1.22. Let $\mathcal{X}$ be a pre-Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodule. Let $\overline{\mathcal{X}}$ be the conjugate space of $\mathcal{X}$, equipped with the $\mathcal{B}$-$\mathcal{A}$-bimodule structure as follows:

$$b\cdot\overline{x} := \overline{x\cdot b^*},$$

$$\overline{x}\cdot a := \overline{a^*\cdot x}$$

and with the scalar products as follows:

$$\langle\overline{x},\overline{y}\rangle_{\mathcal{A}} := {}_{\mathcal{A}}\langle y,x\rangle^*,$$

$${}_{\mathcal{B}}\langle\overline{x},\overline{y}\rangle := \langle y,x\rangle^*_{\mathcal{B}}\,.$$

Proposition 1.23. *Let $\mathcal{X}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type. Then*

$$\mathrm{End}_{\mathcal{B}}(\mathcal{X}) \cong \mathcal{X}\otimes_{\mathcal{B}}\overline{\mathcal{X}}$$

as a pre-Hilbert $\mathcal{A}$-$\mathcal{A}$-bimodule via the map

$$\Theta_{x,y}\mapsto x\otimes\overline{y}.$$

Proof. The mapping in question is $\mathcal{A}$-linear:

$$a'\Theta_{x,y}a(z) = a'x\,\langle y,az\rangle_{\mathcal{B}} = \Theta_{a'x,a^*y}$$

and the constructions of Lemma 1.20 and Lemma 1.21 identify the bases for $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ in different pictures. □

Proposition 1.24. *Let $\mathcal{X}$ be a full $\mathcal{A}$-$\mathcal{A}$-bimodule of finite type. Then $\mathcal{X}\otimes_{\mathcal{A}}\overline{\mathcal{X}}$ contains the trivial $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A}$ as a direct summand.*

Proof. This follows from a direct adaptation of [**KW00**, Prop. 2.5] to our situation. □

3. Quasi-completions and Hilbert modules

Here we recall the notion of a quasi-completion of a topological vector space from [**Köt79**].

Definition 1.25 ([**Köt79**, 18.4])**.** A locally convex space E is called quasi-complete iff every bounded closed set of E is complete.

Example 1.26 ([**Köt79**, 23.3])**.** The dual E^* of a barreled locally convex space E is quasi-complete with respect to the weak-$*$ topology. In particular, every von Neumann algebra is quasi-complete.

Definition 1.27. A subset F of a locally convex space E is called quasi-closed if it contains all the closure points in E of its bounded subsets. The quasi-closure of an arbitrary subset G of E is defined to be the intersection of all quasi-closed subsets of E containing G. The quasi-completion of E is defined to be the quasi-closure of E in its completion $\overline{E}$.

Lemma 1.28. *Let E be a Banach space and E^* its dual Banach space. Let $F \subseteq E^*$ be a subspace such that $F \cap (E^*)_1$ is weakly-$*$ dense in $(E^*)_1$. Then the quasi-completion of F coincides with E^*.*

Proof. As E^* is quasi-complete with respect to the weak-$*$ topology, the quasi-completion $\widehat{F} \subseteq E^*$. But every element in E^* is a weak-$*$ limit of a bounded net from F, thus $E^* \subset \widehat{F}$. This proves the claim. □

Proposition 1.29. *Let M be a von Neumann algebra and $\mathscr{E}$ be a Hilbert module over M. Then the following conditions are equivalent:*

i) *$\mathscr{E}$ is self-dual;*
ii) *$\mathscr{E}$ is quasi-complete with respect to the topology defined by the family of neighbourhoods of zero*

$$U_W := \{\xi \in X \mid {}_A\langle \xi, \xi\rangle \in W\},$$

where W is an ultraweak neighbourhood of zero in M;
iii) *the unit ball of $\mathscr{E}$ is complete with respect to the topology defined by the seminorms*

$$\xi \mapsto \varphi(\langle \xi, \xi\rangle)^{1/2},$$

where φ runs over normal states of M.

Proof. ii) implies iii) by the definition of quasi-completion and the ultraweak topology (as given by normal functionals). To see that iii) implies ii), it suffices to observe that bounded sets in the s-topology coincide with norm-bounded sets.

The equivalence of i) and iii) is established in [**MT05**, Thm. 3.5.1]. □

4. Perturbations of unbounded operators

In this section we recollect some facts about perturbations of unbounded operators which will be used in the sequel. The main reference for these results is [**RS80**, Ch. VIII, X], [**RS78**, Ch. XII, XIII].

Definition 1.30 ([**RS80**, VII.6])**.** Let $\mathcal{H}$ be a Hilbert space. A quadratic form q on $\mathcal{H}$ is a map $q\colon Q(q) \times Q(q) \to \mathbb{C}$, where $Q(q)$ is a dense subspace in $\mathcal{H}$, which is called the domain of q, such that $q(\psi, \cdot)$ is antilinear and $q(\cdot, \varphi)$ is linear. If for all $\psi, \varphi \in \mathcal{H}$ $q(\psi, \varphi) = \overline{q(\varphi, \psi)}$, then q is called symmetric. If $q(\varphi, \varphi) \geqslant 0$ for all $\varphi \in Q(q)$, then q is called positive. If there exists an M such that for all $\psi \in \mathcal{H}$ $q(\psi, \psi) \geqslant -M\,\langle \psi, \psi\rangle$, then q is called semi-bounded.

Example 1.31. If $A\colon \mathrm{Dom}(A) \to \mathcal{H}$ is an unbounded selfadjoint operator on $\mathcal{H}$, then by the spectral theorem A is unitarily equivalent to the multiplication operator with x on the direct sum $\bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$ for some $N \in \mathbb{N} \cup \{\infty\}$ and some measures μ_n on $\mathbb{R}$. We let

$$Q(q) = \left\{ \{\psi_n(x)\}_{n=1}^N \,\middle|\, \sum_{n=1}^{N} \int_{\mathbb{R}} |x||\psi_n(x)|^2 \, d\mu_n(x) < \infty \right\}$$

and for each $\psi, \varphi \in Q(q)$ we let

$$q(\psi, \varphi) := \sum_{n=1}^{N} \int_{\mathbb{R}} x\psi_n(x)\overline{\varphi_n(x)} \, d\mu_n(x).$$

We refer to q as to the quadratic form induced by A and let $Q(A) := Q(q)$.

Definition 1.32. Let q be a semi-bounded quadratic form on $\mathcal{H}$ with $q(\psi, \psi) \geqslant -M \langle \psi, \psi \rangle$ for $\psi \in Q(q)$. The form q is called closed iff $Q(q)$ is complete with respect to the norm

$$\|\psi\|_{+1} := \sqrt{q(\psi, \psi) + (M + 1) \|\psi\|^2}.$$

Theorem 1.33 ([**RS80**, Thm. VIII.15])**.** *Every semi-bounded closed quadratic form is induced by some uniquely determined selfadjoint operator A.*

Theorem 1.34 (Minimax principle, [**RS78**, Thm. XIII.1, Thm. XIII.2])**.** *Let A be an unbounded bounded from below selfadjoint operator on a separable Hilbert space $\mathcal{H}$. We let*

$$\mu_n(A) := \sup_{\varphi_1, \ldots, \varphi_{n-1}} U_A(\varphi_1, \ldots, \varphi_{n-1}),$$

where

$$U_A(\varphi_1, \ldots, \varphi_{n-1}) = \inf_{\substack{\psi \in \mathrm{Dom}(A),\, \|\psi\|=1 \\ \psi \in [\varphi_1, \ldots, \varphi_n]^\perp}} \langle A\psi, \psi \rangle = \inf_{\substack{\psi \in Q(A),\, \|\psi\|=1 \\ \psi \in [\varphi_1, \ldots, \varphi_n]^\perp}} \langle A\psi, \psi \rangle .$$

Then for each fixed n

i) *either there are n eigenvalues of A, counted with multiplicity, which are below the lower boundary of the essential spectrum of A, and $\mu_n(A)$ is the n-th eigenvalue (counting multiplicity)*
ii) *or μ_n is the lower boundary of the essential spectrum of A, i.e. $\mu_n(A) = \inf\{\lambda \,|\, \lambda \in \sigma_{\mathrm{ess}}(A)\}$; in this case $\mu_n(A) = \mu_{n+1}(A) = \ldots$ and there are at most $n-1$ eigenvalues below μ_n, counted with multiplicity.*

Lemma 1.35 ([**RS78**, Thm. XIII.64, i), ii)])**.** *Let A be an unbounded operator on a Hilbert space and let $R_\mu(A) := (A - \mu)^{-1}$ for $\mu \in \rho(A)$. The following statements are equivalent:*

i) *$(A - \mu)^{-1}$ is compact for some $\mu \in \rho(A)$;*
ii) *$(A - \mu)^{-1}$ is compact for all $\mu \in \rho(A)$.*

Proof. This is a trivial consequence of the resolvent formula

$$R_\mu(A) - R_\lambda(A) = -(\mu - \lambda) R_\lambda(A) R_\mu(A)$$

and the fact that compact operators form an ideal in $\mathcal{B}(\mathcal{H})$). □

Theorem 1.36 ([**RS78**, Thm. XIII.64])**.** *Let A be an unbounded selfadjoint operator on a separable Hilbert space $\mathcal{H}$ which is bounded from below: $\langle A\psi, \psi \rangle \geqslant M \langle \psi, \psi \rangle$ for some $M \in \mathbb{R}$. The following statements are equivalent:*

i) *$(A - \mu)^{-1}$ is compact for some $\mu \in \rho(A)$;*
ii) *$(A - \mu)^{-1}$ is compact for all $\mu \in \rho(A)$.*

iii) *for all* $b \in \mathbb{R}$ *the set* $\{\psi \in \mathrm{Dom}(A) \mid \|\psi\| \leqslant 1, \|A\psi\| \leqslant b\}$ *is compact;*
iv) *for all* $b \in \mathbb{R}$ *the set* $\{\psi \in Q(A) \mid \|\psi\| \leqslant 1, \langle A\psi, \psi\rangle \leqslant b\}$ *is compact;*
v) *there is an orthonormal basis* $\{\varphi_n\}_{n=1}^\infty$ *in* $\mathcal{H}$ *such that* $A\varphi_n = \mu_n\varphi_n$ *with* $\mu_1 \leqslant \mu_2 \leqslant \dots$ *and* $\mu_n \to \infty$*;*
vi) $\mu_n(A) \to \infty$*, where* $\mu_n(A)$ *is the number obtained from the minimax principle.*

Definition 1.37. Let A be an unbounded positive selfadjoint operator on $\mathcal{H}$ and let B be a selfadjoint operator on $\mathcal{H}$ such that

i) $Q(B) \supset Q(A)$;
ii) $|\langle B\varphi, \varphi\rangle| \leqslant a\langle A\varphi, \varphi\rangle + b\langle \varphi, \varphi\rangle, \ \varphi \in Q(A)$

for some $a < 1$ and $b \in \mathbb{R}$. Then B is called a perturbation of A which is bounded in the sense of quadratic forms.

Theorem 1.38 ([**RS78**, Thm. XIII.68])**.** *Let A be an unbounded selfadjoint operator on a separable Hilbert space $\mathcal{H}$ which is bounded from below. Let b be a symmetric perturbation of A which is bounded in the sense of quadratic forms. Let $C = A + b$ be defined as the sum of the corresponding quadratic forms. Then C has compact resolvent iff A does.*

Proof. By assumption,

$$|b(\psi, \psi)| \leqslant \alpha\langle A\psi, \psi\rangle + \beta\langle \psi, \psi\rangle$$

for some $\alpha < 1$. Thus for every $\psi \in Q(C) = Q(A)$

$$\langle C\psi, \psi\rangle \geqslant (1-\alpha)\langle A\psi, \psi\rangle - \beta\langle \psi, \psi\rangle$$

and

$$\langle C\psi, \psi\rangle \leqslant (\alpha+1)\langle A\psi, \psi\rangle + \beta\langle \psi, \psi\rangle .$$

From the minimax principle for forms [**RS78**, Thm. XIII.2] it follows that

$$\mu_n(C) \geqslant (1-\alpha)\mu_n(A) - \beta$$

and

$$\mu_n(C) \leqslant (1+\alpha)\mu_n(A) + \beta$$

Thus $\mu_n(A) \to \infty$ iff $\mu_n(C) \to \infty$, and the claim follows from Theorem 1.36. □

Lemma 1.39. *Let $A\colon D(A) \to \mathcal{H}$ be an unbounded positive operator with compact resolvent and let $T \in \mathcal{B}(H)$ be a finite rank selfadjoint operator. Then $A + T$ has compact resolvent.*

Proof. We use the above characterization of positive operators with compact resolvent (cf. Theorem 1.36): an unbounded positive operator A has compact resolvent iff the set

$$F_b := \{\psi \in D(A) \mid \|\psi\| \leqslant 1, \|A\psi\| \leqslant b\}$$

is compact for all $b \in \mathbb{R}$. Obviuosly,

$$F_b^T := \{\psi \in D(A) \mid \|\psi\| \leqslant 1, \|(A+T)\psi\| \leqslant b\} \subset F_{b+\|T\|},$$

and as F_b^T is closed, this proves the claim. □

Lemma 1.40. *Let $A\colon \mathrm{Dom}(A) \to \mathcal{H}_2$ be a closed unbounded operator between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T\mathcal{H}_1 \cap \mathrm{Dom}(A^*)$ is dense in $\mathcal{H}_2$. Suppose that A^*A has compact resolvent. Then $(A+T)^*(A+T)$ has compact resolvent as well.*

PROOF. A direct computation shows that

$$(A+T)^*(A+T) = A^*A + A^*T + T^*A + T^*T =: A^*A + R.$$

In terms of the corresponding quadratic forms the difference is equal to

$$\langle R\xi, \xi\rangle = 2\,\mathrm{Re}\,\langle T\xi, A\xi\rangle + \|T\xi\|^2 \leqslant \|A\xi\|^2 \left(2\frac{\|T\xi\|}{\|A\xi\|} + \frac{\|T\xi\|^2}{\|A\xi\|^2} \right).$$

If $\|A\xi\| \geqslant 4\,\|T\xi\|$ for all ξ, then the multiplier on the right-hand side is less than 1, and we are done by Theorem 1.38. In general, we have to achieve it artificially.

Consider the finite-dimensional subspace $\mathcal{F} \subset \mathcal{H}_1$ spanned by the eigenvectors of A^*A with eigenvalues $\lambda \leqslant 16\,\|T\|^2$. Define a new operator B as follows: $B := 16\,\|T\|^2$ on $\mathcal{F}$ and $B = A^*A$ on $\mathcal{F}^\perp$. Obviously, B still has compact resolvent.

Thus,

$$\langle B\xi, \xi\rangle \geqslant 16\,\|T\|^2\,\|\xi\|^2 \geq 16\,\|T\xi\|^2$$

and

$$\langle B\xi, \xi\rangle \geqslant \|A\xi\|^2\,.$$

Thus

$$\langle R\xi, \xi\rangle = 2\,\mathrm{Re}\,\langle T\xi, A\xi\rangle + \|T\xi\|^2 \leqslant \langle B\xi, \xi\rangle \left(2\frac{\|T\xi\|\,\|A\xi\|}{\langle B\xi, \xi\rangle} + \frac{\|T\xi\|^2}{\langle B\xi, \xi\rangle} \right) \leqslant \frac{5}{16}\,\langle B\xi, \xi\rangle\,.$$

Therefore the operator $B+R$ has compact resolvent by Theorem 1.38, and so does its finite-rank perturbation A. □

Lemma 1.41. *Let $\mathcal{H}$ be a Hilbert space and $A\colon D(A) \to \mathcal{H}$ be an unbounded positive operator with compact resolvent. Let $P \in \mathcal{B}(\mathcal{H})$ be a projection such that $\overline{D(A) \cap P\mathcal{H}} = P\mathcal{H}$. Then the operator $PAP\colon P\mathcal{H} \to P\mathcal{H}$ is a selfadjoint operator with compact resolvent.*

PROOF. Once again we use the above characterization of positive operators with compact resolvent (cf. Theorem 1.36): an unbounded positive operator A has compact resolvent iff the set

$$F_b := \{\psi \in D(A) \mid \|\psi\| \leqslant 1,\ \langle A\psi, \psi\rangle \leqslant b\}$$

is compact for all $b \in \mathbb{R}$.

Let

$$F_b^P := \{\psi \in D(A) \cap P\mathcal{H} \mid \|\psi\| \leqslant 1,\ \langle A\psi, \psi\rangle \leqslant b\}$$

Obviously, $F_b^P = F_b \cap P\mathcal{H}$, thus F_b^P is compact if F_b is. This proves the lemma. □

5. The Dixmier trace

The Dixmier trace plays a major role in noncommutative geometry, mainly because it provides the right generalization of the integration over a Riemannian manifold to the noncommutative case. In this section we recall the basic properties of the Dixmier trace, following [**Con94**, IV.2.β]. We refer to [**Con94**, IV.2.β] and [**GBVF01**] where other relevant properties of the Dixmier trace are described.

Let T be a compact operator on a separable Hilbert space $\mathcal{H}$ and let $|T| = (T^*T)^{1/2}$ be its absolute value. Let

$$\mu_0(T) \geqslant \mu_1(T) \geqslant \dots$$

be the eigenvalues of $|T|$, sorted in descending order and repeated according to their multiplicity. As T is compact, $\mu_n(T) \to 0$, $n \to \infty$ (these are also called s-numbers of T).

We set
$$\sigma_N(T) = \sum_{k=0}^{N-1} \mu_k(T).$$

There is a wide variety of ideals in $\mathcal{K}(\mathcal{H})$ consisting of operators T with certain asymptotics of $\sigma_N(T)$ and we refer to [**Sim05**] for the detailed treatment of them. We will focus only on one of them, namely,
$$\mathcal{L}^{(1,\infty)} := \{T \in \mathcal{K} \,|\, \sigma_N(T) = O(\log N)\}.$$

The natural norm on $\mathcal{L}^{(1,\infty)}$ is given by
$$\|T\|_{1,\infty} = \sup_{N \geqslant 2} \frac{\sigma_N(T)}{\log N}.$$
With this norm $\mathcal{L}^{(1,\infty)}$ becomes a symmetrically normed ideal in $\mathcal{K}$, i.e.
$$\|ATB\| \leqslant \|A\|_\infty \|T\|_{1,\infty} \|B\|_\infty$$
for any $A, B \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{L}^{(1,\infty)}$. This is an easy consequence of the fact that the s-numbers of AT resp. TB are bounded by the s-numbers of T multiplied with the norm of A resp. B.

We would like to define a functional on $\mathcal{L}^{(1,\infty)}$ for $T \geqslant 0$ as the limit of the expressions
$$\frac{\sigma_N(T)}{\log N}$$
for $N \to \infty$. The problem is that this sequence need not be convergent, and therefore one has to choose a certain kind of limiting procedure (applying a functional in $(\ell^\infty/c_0)^*$). If one wants linearity of the obtained mapping (as defined on operators), one is only allowed to deal with certain kinds of limits. Namely, one chooses a functional $\omega \in (\ell^\infty)^*$ with the following properties:

i) $\omega(\alpha) \geqslant 0$ if $\alpha \geq 0$,
ii) $\omega(\alpha) = \lim_n \alpha_n$ if α_n is convergent,
iii) $\omega(\alpha_1, \alpha_1, \dots, \alpha_n, \alpha_n, \dots) = \omega(\alpha)$.

We call such an ω a scale-invariant limit and we introduce the notation $\mathrm{Lim}_\omega(\alpha_n) := \omega(\alpha)$.

Remark 1.42. The very existence of the functionals $\omega \in (\ell)^\infty$ with the third property of scale invariance is not really obvious, so we sketch the construction here. From a bounded sequence (α_n) one can get a bounded function $f_\alpha(\lambda)$ on $\mathbb{R}_+^*$ by setting $f_\alpha(\lambda) = \alpha_n$ for $\lambda \in (n-1, n]$. Now we consider the Cesàro means of f_α with respect to the Haar measure on $\mathbb{R}_+^*$:
$$M(f_\alpha)(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda f_\alpha(u) \frac{du}{u}.$$
Thus, any linear L on $C_b(\mathbb{R}_+^*)$ vanishing on $C_0(\mathbb{R}_+^*)$ and coinciding with the limit if it exists gives a desired form ω by setting
$$\omega(\alpha) := L(M(f_\alpha)).$$

After this preparatory work, we can introduce the following definition:

Definition 1.43. For a fixed scale-invariant limit ω and $0 \leqslant T \in \mathcal{L}^{(1,\infty)}$, we let
$$\mathrm{Tr}_\omega(T) := \mathrm{Lim}_\omega \frac{\sigma_N(T)}{\log N}$$
and call it the Dixmier trace of T.

To understand the need of the scale invariance condition, we give the proof of lineaity.

Proposition 1.44 ([**Con94**, IV.2.β])**.** *The Dixmier trace is linear: if* $T_1, T_2 \geqslant 0$ *and* $T_1, T_2 \in \mathcal{L}^{(1,\infty)}$*, then*

$$\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2).$$

Proof. It follows easily from the minimax principle that

$$\sigma_N(T) = \sup\{\mathrm{Tr}(TP) \mid P = P^* = P^2,\, \mathrm{Tr}(P) = N\},$$

and therefore

$$\sigma_N(T_1 + T_2) \leqslant \sigma_N(T_1) + \sigma_N(T_2)$$

and

$$\sigma_N(T_1) + \sigma_N(T_2) \leqslant \sigma_{2N}(T_1 + T_2).$$

Now, let $\alpha_N := \frac{\sigma_N(T_1)}{\log N}$, $\beta_N := \frac{\sigma_N(T_2)}{\log N}$, $\gamma_N := \frac{\sigma_N(T_1+T_2)}{\log N}$; then the above inequalities translate into

$$\alpha_N + \beta_N \leqslant \frac{\log 2N}{\log N}\gamma_{2N}$$

and

$$\gamma_N \leqslant \alpha_N + \beta_N.$$

Now, as $\frac{\log 2N}{\log N} \to 1$ as $N \to \infty$, the linearity follows after applying Lim_ω using its scale invariance. □

Corollary 1.45. *The Dixmier trace uniquely extends to the whole ideal* $\mathcal{L}^{(1,\infty)}$.

Proposition 1.46 ([**Con94**, Prop. IV.2.3])**.** *Let* $T \in \mathcal{L}^{(1,\infty)}$

i) *If* $T \geqslant 0$*, then* $\mathrm{Tr}_\omega(T) \geqslant 0$.
ii) *If* $S \in \mathcal{B}(\mathcal{H})$*, then* $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$.
iii) $\mathrm{Tr}_\omega(T)$ *is independent of the choice of the inner product on* $\mathcal{H}$.
iv) Tr_ω *vanishes on the ideal* $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$*, which is the closure of the ideal of finite-rank operators in the norm* $\|\cdot\|_{1,\infty}$.

Proof. The property i) is obvious by construction. Now, if $S \in \mathcal{B}(\mathcal{H})$ is invertible, then $\mathrm{Tr}_\omega(S^{-1}TS) = \mathrm{Tr}_\omega(T)$, which implies ii) and iii). Property iv) follows from the fact that Tr_ω vanishes on finite-rank operators. □

At this point, it is natural to ask to what extent the Dixmier trace depends on the choice of the functional ω. As it turns out, in certain cases we can obtain the Dixmier trace by computing a residue.

Let $T \geqslant 0$ be an operator from $\mathcal{L}^{(1,\infty)}$. Then its complex powers T^s are of trace class for $\mathrm{Re}\, s > 1$, and thus the equality

$$\zeta_T(s) := \mathrm{Tr}(T^s) = \sum_{n=0}^{\infty} \mu_n(T)^s$$

defines a holomorphic function on the half-plane $\mathrm{Re}(s) > 1$. Now, we recall the Tauberian theorem of Hardy and Littlewood:

Theorem 1.47 (Hardy-Littlewood)**.** *If* $a_n = O(1/n)$*, and as* $x \to 1$ *we have*

$$\sum_{n=1}^{\infty} a_n x^n \to s,$$

then

$$\sum_{n=1}^{\infty} a_n = s.$$

In our situation, this theorem leads to the following result:

Theorem 1.48. *For $T \geqslant 0$, $T \in \mathcal{L}^{(1,\infty)}$, the following two conditions are equivalent:*

i)
$$(s-1)\zeta(s) \to L,\ s \to 1+0,$$

ii)
$$\frac{1}{\log N}\sum_{n=0}^{N-1}\mu_n(T) \to L, \quad N \to \infty.$$

Thus, in this case the value of $\mathrm{Tr}_\omega(T)$ is independent of the choice of ω, and, if $\zeta(s)$ has a simple pole at $s=1$, then this value is equal to the residue of $\zeta(s)$ at $s=1$.

It is exactly this point which allows us to make the connection between the Dixmier trace and certain numerical values naturally obtained in differential geometry as "residues of pseudodifferential operators" [**Wod87**].

Proposition 1.49. *Let M be a d-dimensional compact Riemannian manifold, $E \to M$ a Hermitian vector bundle over M and let T be a classical pseudodifferential operator of order $-d$ acting on sections of E. Then:*

i) *the corresponding operator T acting on the space $L^2(M,E)$ of square-integrable sections of E belongs to $\mathcal{L}^{(1,\infty)}$;*
ii) *the Dixmier trace $\mathrm{Tr}_\omega(T)$ is independent of ω and given by the following expression:*
$$\mathrm{Tr}_\omega(T) = \frac{1}{d(2\pi)^d}\int_{S^*M}\mathrm{Tr}_E(\sigma_{-d}(T))\,ds,$$
where $S^*M = \{\xi \in T^*M \mid \|\xi\|\}$ is the cosphere bundle of M, ds is the volume form associated with the metric on S^*M induced by the Riemannian metric on M and $\mathrm{Tr}_E(\sigma_{-d}(T))$ is the trace of the principal symbol of T viewed as an endomorphism of the pullback of E to S^*M.

Thus, there are many naturally arising measurable operators:

Definition 1.50. Let $T \in \mathcal{L}^{(1,\infty)}$. We say that T is measurable if $\mathrm{Tr}_\omega(T)$ is independent of ω.

Let us observe the following easy

Proposition 1.51 ([**Con94**, Prop. IV.2.6])**.**

i) *Let $T \geqslant 0$, $T \in \mathcal{L}^{(1,\infty)}$. Then T is measurable iff the Cesàro means $M(f_\alpha)(\lambda)$ of the sequence $\alpha_N = \frac{\sigma_N(T)}{\log N}$ are convergent for $\lambda \to \infty$.*
ii) *The subset $\mathcal{M} \subset \mathcal{L}^{(1,\infty)}$ of measurable operators is a subspace which contains $\mathcal{L}_0^{(1,\infty)}$ and is closed in the $(1,\infty)$-norm.*

For our later purposes, we will need the following observation.

Proposition 1.52. *Let D be an unbounded selfadjoint operator with compact resolvent such that $|D|^{-d} \in \mathcal{L}^{(1,\infty)}$ for a $d \in \mathbb{N}$ ($|D|^{-1}$ is hereby defined to be 0 on $\ker D$). Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra such that $[|D|,a]$ is bounded for every $a \in \mathcal{A}$. Then the functional*
$$a \mapsto \mathrm{Tr}_\omega(a|D|^{-d})$$
extends to a positive trace on $\mathcal{A}$.

PROOF. Let $a, b \in \mathcal{A}$. We consider

$$\mathrm{Tr}_\omega((ab - ba)|D|^{-d}) = \mathrm{Tr}_\omega(ab|D|^{-d} - a|D|^{-d}b) = \mathrm{Tr}_\omega(a[b, |D|^{-d}])$$

Now we want to prove that $[b, |D|^{-d}]$ is a trace-class operator, and for this we use induction:

$$[|D|^{-1}, b] = -|D|^{-1}[|D|, b]|D|^{-1},$$

and

$$[|D|^{-k-1}, b] = |D|^{-1}[|D|^{-k}, b] + [|D|^{-1}, b]|D|^{-k},$$

which clearly implies the statement, because an operator of the form $|D|^{-i}S|D|^{-j}$ with $i + j > d$ is of trace class. The positivity of the trace follows from the computation

$$\mathrm{Tr}_\omega(a^*a|D|^{-d}) = \mathrm{Tr}_\omega(a^*|D|^{-d}a),$$

and the observation that the operator $a^*|D|^{-d}a$ is positive. □

The crucial observation about the Dixmier trace is that it depends only on the "large-scale behaviour" of the operator; indeed, any finite-rank perturbation doesn't change it. In the above case of certain functionals on a $*$-algebra emerging from an unbounded operator $|D|$ and the Dixmier trace, we have better control on the functional.

Lemma 1.53. *Let D be an unbounded selfadjoint operator with compact resolvent such that $|D|^{-d} \in \mathcal{L}^{(1,\infty)}$ for a $d \in \mathbb{N}$ ($|D|^{-1}$ is defined to be 0 on $\ker D$). Then*

$$\mathrm{Tr}_\omega(T|D|^{-d}) = 0$$

for $T \in \mathcal{K}$.

PROOF. It clearly suffices to prove the statement in the case $T \geqslant 0$. In this case, we take a sequence T_n of positive finite-rank operators converging to T in norm. In view of the equality

$$\mathrm{Tr}_\omega(T_i|D|^{-d}) = 0$$

and in view of the inequality

$$\left\|T|D|^{-d}\right\|_{1,\infty} \leqslant \|T\| \left\||D|^{-d}\right\|_{1,\infty}$$

the functional $T \mapsto \mathrm{Tr}_\omega(T|D|^{-d})$ is norm-continuous, thus the statement follows. □

The following lemma shows that the trace we just constructed is independent of certain "regular" perturbations of D. To get the idea of why this is true, we advise the reader to think about the abovementioned example of an operator with scalar principal symbol on a manifold: if we perturb, say, a Dirac operator by lower-order terms, the principal symbol does not change, thus the Dixmier trace, being equal to the Wodzicki residue, remains invariant.

Lemma 1.54. *Let D be an unbounded selfadjoint operator with compact resolvent such that $|D|^{-d} \in \mathcal{L}^{(1,\infty)}$ for a $d \in \mathbb{N}$ ($|D|^{-1}$ is hereby defined to be 0 on $\ker D$). Let $T \in \mathcal{B}(\mathcal{H})$ be a regular operator (belonging to the domain of δ^m for every $m \in \mathbb{N}$, where $\delta(\cdot) = [|D|, \cdot]$) and $a \in \mathcal{B}(\mathcal{H})$ be such that $a|D|^{-d}$ is measurable. Then $a|D + T|^{-d}$ is measurable as well, and*

$$\mathrm{Tr}_\omega(a|D|^{-d}) = \mathrm{Tr}_\omega(a|D + T|^{-d}).$$

PROOF. We begin with the equality

$$(D+T)^*(D+T) = D^2 + T^*D + DT + T^*T$$

and compute

$$|D+T|^2 = |D|(1 + |D|^{-1}T^*F + FT|D|^{-1} + |D|^{-1}T^*T|D|^{-1})|D|,$$

where $F = \operatorname{sign} D$. Now, $(1+|D|^{-1}T^*F+FT|D|^{-1}+|D|^{-1}T^*T|D|^{-1})$ is a positive operator of the form $(1+Q)$, where Q is a compact operator; in view of this,

$$|D+T|^{-2} = |D|^{-1}(1+Q)^{-1}|D|^{-1}$$

(here we once again use the convention that $(1+Q)^{-1}$ is equal to 0 on $\ker(1+Q)$). Now,

$$[|D|, Q] = |D|^{-1}[|D|, T^*]F + F[|D|, T]|D|^{-1} + |D|^{-1}[|D|, T^*T]|D|^{-1}$$

where R is a compact operator; in view of this,

$$|D+T|^{-2} = (1+S)|D|^{-2},$$

where S is a compact operator. Thus, the statement is true for $d = 2$. For $d = 2k$, we obtain

$$|D+T|^{2k} = |D|(1+Q)|D| \dots |D|(1+Q)|D|,$$

and, as the multiple commutators of $|D|$ and $Q]$ are bounded,

$$|D+T|^{-2k} = (1+S_k)|D|^{-2k},$$

where S_k is a compact operator. The statement for $d = 2k$ follows from Lemma 1.53.

To conclude that the lemma is true for odd d as well, we need to prove that $|D+T| - |D| = |D|S'$, where S' is compact. For this, we use the formula

$$|A| = \pi^{-1} \int_0^\infty \lambda^{-1/2} \frac{A^2}{A^2+\lambda}\, d\lambda,$$

valid for a selfadjoint operator A.

Thus,

$$|D+T| - |D| = \pi^{-1} \int_0^\infty \lambda^{-1/2} \left((|D+T|^2+\lambda)^{-1}|D+T|^2 - |D|^2(|D|^2+\lambda)^{-1}\right) d\lambda =$$

$$= \pi^{-1} \int_0^\infty \lambda^{-1/2}(|D+T|^2+\lambda)^{-1}(|D+T|^2(|D|^2+\lambda) - (|D+T|^2+\lambda)|D|^2)(|D|^2+\lambda)^{-1}\, d\lambda =$$

$$\pi^{-1} \int_0^\infty \lambda^{1/2}(|D+T|^2+\lambda)^{-1}(|D+T|^2 - |D|^2)(|D|^2+\lambda)^{-1}\, d\lambda =$$

$$= \pi^{-1} \int_0^\infty \lambda^{1/2}(|D+T|^2+\lambda)^{-1}|D|Q|D|(|D|^2+\lambda)^{-1}\, d\lambda.$$

Now,

$$[Q, (|D|^2+\lambda)^{-1}] = -(|D|^2+\lambda)^{-1}[Q, |D|^2](|D|^2+\lambda)^{-1} =$$
$$-(|D|^2+\lambda)^{-1}|D|[Q, |D|](|D|^2+\lambda)^{-1} - (|D|^2+\lambda)^{-1}[Q, |D|]|D|(|D|^2+\lambda)^{-1}$$

is a compact operator, and the integral converges uniformly, because

$$\left\|(|D+T|^2+\lambda)^{-1}\right\| \leqslant \lambda^{-1},$$
$$\left\|(|D|^2+\lambda)^{-1}\right\| \leqslant \lambda^{-1},$$

thus the terms under the integral have order $\lambda^{-3/2}$ at infinity. In view of this,

$$|D+T| - |D| = S'|D|,$$

where S' is a compact operator. This proves the lemma. □

6. Hochschild and cyclic homology and cohomology

In this section we review the definitions and properties of the Hochschild and cyclic homology and cohomology. The references for this section are [**CST04**], [**Lod98**].

Let $\mathcal{A}$ be a unital algebra. We denote by $\Omega^*(\mathcal{A})$ the universal differential graded algebra over $\mathcal{A}$, which is generated by $x \in \mathcal{A}$ with relations from $\mathcal{A}$ and additional symbols dx, $x \in \mathcal{A}$, where dx is linear in x and satisfies

$$d(xy) = xd(y) + d(x)y.$$

We also impose the relation $d1 = 0$, because we will be dealing with unital algebras only. Sometimes this relation isn't imposed, in particular, when there's a need to treat nonnital algebras. This doesn't change the corresponding homologies, as explained in [**CST04**, Rem. 2.15].

Thus, $\Omega^*(\mathcal{A})$ is a direct sum of subspaces

$$\Omega^n(\mathcal{A}) = \operatorname{span}\{x_0 dx_1 \dots dx_n \,|\, x_j \in \mathcal{A}, j = 1, \dots, n\}.$$

This decomposition makes $\Omega^*(\mathcal{A})$ a graded algebra.

We define several operators on $\Omega^*(\mathcal{A})$. The operator d is defined by

$$d(x_0 dx_1 \dots dx_n) := dx_0 \dots dx_n,$$

the operator b is defined by

$$b(\omega dx) := (-1)^{\deg \omega}[\omega, x], \quad \omega \in \Omega^*(\mathcal{A}), x \in \mathcal{A},$$

$$b(dx) = 0,$$

$$b(x) = 0.$$

Clearly, $d^2 = 0$, and one can prove that $b^2 = 0$.

We also define the number operator

$$N(\omega) := \deg(\omega)\omega.$$

We define the Karoubi operator

$$\kappa := 1 - db - bd.$$

One can verify that κ satisfies

$$(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$$

on $\Omega^n(\mathcal{A})$. Thus, there exists an operator P on $\Omega^*(\mathcal{A})$ which projects onto the generalized eigenspace of κ for the eigenvalue 1.

By construction, the operator P commutes with b, d, and N. Thus, setting $B := NPd$, we find

$$Bb + bB = 0, \quad B^2 = 0.$$

Explicitly, B is given on $\Omega^n(\mathcal{A})$ by

$$B(\omega) = \sum_{j=0}^{n} \kappa^j d\omega.$$

Thus, we obtain a bicomplex

$$
\begin{array}{ccccccc}
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
\Omega^3(\mathcal{A}) & \xleftarrow{B} & \Omega^2(\mathcal{A}) & \xleftarrow{B} & \Omega^1(\mathcal{A}) & \xleftarrow{B} & \Omega^0(\mathcal{A}) \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
\Omega^2(\mathcal{A}) & \xleftarrow{B} & \Omega^1(\mathcal{A}) & \xleftarrow{B} & \Omega^0(\mathcal{A}) & & \\
\downarrow b & & \downarrow b & & & & \\
\Omega^1(\mathcal{A}) & \xleftarrow{B} & \Omega^0(\mathcal{A}) & & & & \\
\downarrow b & & & & & & \\
\Omega^0(\mathcal{A}) & & & & & &
\end{array} \tag{1.2}
$$

Definition 1.55. The homology of the total complex of the bicomplex (1.2) is called the cyclic homology $\mathsf{HC}_*(\mathcal{A})$.

Definition 1.56. The homology of the first column of the bicomplex (1.2) is called the Hochschild homology $\mathsf{HH}_*(\mathcal{A})$.

Dualizing the above complex we get the bicomplex

$$
\begin{array}{ccccccc}
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
\Omega^3(\mathcal{A})^* & \xleftarrow{B} & \Omega^2(\mathcal{A})^* & \xleftarrow{B} & \Omega^1(\mathcal{A})^* & \xleftarrow{B} & \Omega^0(\mathcal{A})^* \\
\uparrow b & & \uparrow b & & \uparrow b & & \\
\Omega^2(\mathcal{A})^* & \xleftarrow{B} & \Omega^1(\mathcal{A})^* & \xleftarrow{B} & \Omega^0(\mathcal{A})^* & & \\
\uparrow b & & \uparrow b & & & & \\
\Omega^1(\mathcal{A})^* & \xleftarrow{B} & \Omega^0(\mathcal{A})^* & & & & \\
\uparrow b & & & & & & \\
\Omega^0(\mathcal{A})^* & & & & & &
\end{array} \tag{1.3}
$$

We omit the stars at the operators b, B and others because it will be always clear from the context where they act.

Definition 1.57. The homology of the total complex of the bicomplex (1.3) is called the cyclic cohomology $\mathsf{HC}^*(\mathcal{A})$.

Definition 1.58. The homology of the first column of the bicomplex (1.3) is called the Hochschild cohomology $\mathsf{HH}^*(\mathcal{A})$.

We denote by $\widehat{\Omega}(\mathcal{A})$ the infinite product

$$\widehat{\Omega}(\mathcal{A}) := \prod \Omega^n(\mathcal{A})$$

We denote by

$$\operatorname{Hom}(\widehat{\Omega}(\mathcal{A}), \widehat{\Omega}(\mathcal{B})) = \varprojlim_m \varinjlim_n \operatorname{Hom}\left(\bigoplus_{i \leqslant n} \Omega^i(\mathcal{A}), \bigoplus_{j \leqslant m} \Omega^j(\mathcal{B}) \right).$$

It is a $\mathbb{Z}/2$-graded complex with the boundary map

$$\partial\varphi = \varphi \circ \partial - (-1)^{\deg \varphi} \partial \circ \varphi,$$

where $\partial = B - b$.

Definition 1.59. The bivariant periodic cyclic homology $\mathsf{HP}_*(\mathcal{A},\mathcal{B})$ is defined as the homology of the Hom-complex

$$\mathsf{HP}_*(\mathcal{A},\mathcal{B}) = \mathsf{H}_*(\mathrm{Hom}(\widehat{\Omega}(\mathcal{A}),\widehat{\Omega}(\mathcal{B}))), \quad * = 0,1.$$

The periodic cyclic homology of $\mathcal{A}$ is defined to be

$$\mathsf{HP}_*(\mathcal{A}) := \mathsf{HP}_*(\mathbb{C},\mathcal{A}).$$

The periodic cyclic cohomology of $\mathcal{A}$ is defined to be

$$\mathsf{HP}^*(\mathcal{A}) := \mathsf{HP}^*(\mathcal{A},\mathbb{C}).$$

We let S be the shift operator on the cyclic bicomplex and I be the operator including the Hochschild complex as the first column into the cyclic bicomplex; we denote the corresponding induced operators on (co)homologies by the same letters.

Theorem 1.60 ([**Lod98**, Thm. 2.2.1], [**Con94**, Thm. III.1.26])**.** *There are natural long exact sequences*

$$\ldots \to \mathsf{HH}_n(\mathcal{A}) \xrightarrow{I} \mathsf{HC}_n(\mathcal{A}) \xrightarrow{S} \mathsf{HC}_{n-2}(\mathcal{A}) \xrightarrow{B} \mathsf{HH}_{n-1}(\mathcal{A}) \to \ldots$$

and

$$\ldots \to \mathsf{HH}^{n-1}(\mathcal{A}) \xrightarrow{B} \mathsf{HC}^{n-2}(\mathcal{A}) \xrightarrow{S} \mathsf{HC}^{n}(\mathcal{A}) \xrightarrow{I} \mathsf{HH}^{n}(\mathcal{A}) \to \ldots$$

Proposition 1.61 ([**CST04**, Prop. 2.17])**.** *We let*

$$S\colon \mathsf{HC}_n \to \mathsf{HC}_{n-2}$$

and

$$S\colon \mathsf{HC}^n \to \mathsf{HC}^{n+2}$$

induced by S on the level of cyclic homology resp. cohomology. Then we have

$$\mathsf{HP}^*(\mathcal{A}) = \varinjlim_S \mathsf{HC}^{2n+*}(\mathcal{A})$$

and an exact sequence

$$0 \to \varprojlim_S{}^1 \mathsf{HC}_{2n+*+1}(\mathcal{A}) \to \mathsf{HP}_*(\mathcal{A}) \to \varprojlim_S \mathsf{HC}_{2n+*}(\mathcal{A}) \to 0.$$

Corollary 1.62. *If $\mathsf{HH}_*(\mathcal{A}) \cong 0$ for $* > d$, then*

$$\mathsf{HP}^{d \bmod 2}(\mathcal{A}) = \mathsf{HC}^{d}(\mathcal{A}),$$
$$\mathsf{HP}_{d \bmod 2}(\mathcal{A}) = \mathsf{HC}_{d}(\mathcal{A}),$$
$$\mathsf{HP}^{d \bmod 2+1}(\mathcal{A}) = \mathsf{HC}^{d+1}(\mathcal{A}),$$
$$\mathsf{HP}_{d \bmod 2+1}(\mathcal{A}) = \mathsf{HC}_{d+1}(\mathcal{A}),$$

While it is generally hard to understand the intersection product in bivariant periodic cyclic cohomology [**Pus98**], in a particular case which will be interesting for us it can be defined explicitly, and we give the construction below. Let $[\Phi] \in \mathsf{HP}(\mathcal{A}\otimes\mathcal{B},\mathbb{C})$ be the class of $\Phi \in \mathrm{Hom}(\widehat{\Omega}(\mathcal{A}\otimes\mathcal{B}),\Omega(\mathbb{C}))$. We would like to define a mapping

$$\cap[\Phi]\colon \mathsf{HP}(\mathbb{C},\mathcal{A}) \to \mathsf{HP}(\mathcal{B},\mathbb{C}).$$

This is defined as follows. Consider

$$\alpha \in \mathrm{Hom}(\widehat{\Omega}(\mathbb{C}),\widehat{\Omega}(\mathcal{A}))$$

and form $\alpha \otimes \mathrm{id} \in \mathrm{Hom}(\widehat{\Omega}(\mathcal{B}),\widehat{\Omega}(\mathcal{A})\otimes\widehat{\Omega}(\mathcal{B}))$.

There is a quasiisomorphism $G\colon \widehat{\Omega}(\mathcal{A})\otimes\widehat{\Omega}(\mathcal{B}) \to \widehat{\Omega}(\mathcal{A}\otimes\mathcal{B})$ which we will describe below. The class $[\alpha]\cap[\Phi]$ is given by the class of $\Phi\circ G\circ(\alpha\otimes\mathrm{id}) \in \mathrm{Hom}(\widehat{\Omega}(\mathcal{B}),\widehat{\Omega}(\mathbb{C}))$.

The quasiisomomorphism G is defined using the shuffle product sh and the cyclic shuffle product sh', whose definitions we briefly recall here, refering to [**Lod98**, Sect. 4.3] for details.

By definition, a (p, q)-shuffle is a permutation $\sigma \in \Sigma_{p+q}$, the symmetric group on $(p + q)$ letters, such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p + 1) < \cdots < \sigma(p + q).$$

For any algebra $\mathcal{A}$, we let Σ_n act on $\Omega^n(\mathcal{A})$ by

$$\sigma(a_0 da_1 \dots da_n) = a_0 da_{\sigma^{-1}(1)} \dots da_{\sigma^{-1}(n)}.$$

Let $\mathcal{B}$ be another algebra. The shuffle product

$$sh \colon \Omega^p(\mathcal{A}) \otimes \Omega^q(\mathcal{B}) \to \Omega^{p+q}(\mathcal{A} \otimes \mathcal{B})$$

is defined by

$$sh(a_0 da_1 \dots da_p, b_0 db_1 \dots b_p) = \sum_\sigma \operatorname{sgn} \sigma \cdot \sigma(a_0 b_0 da_1 \dots da_p db_1 \dots db_q),$$

where the sum runs over all (p, q)-shuffles.

Now, by definition, (p, q)-cyclic shuffles are elements of Σ_{p+q} obtained as follows. Perform a cyclic permutation of the set $\{1, \dots p\}$ and a cyclic permutation of the set $\{p+1, \dots, p+q\}$. Then shuffle the two results to obtain $\{\sigma(1), \dots, \sigma(p+q)\}$. This is a cyclic permutation iff 1 appears before $p + 1$ in this sequence.

Thus, we can define the cyclic shuffle product

$$sh' \colon \Omega^p(\mathcal{A}) \otimes \Omega^q(\mathcal{B}) \to \Omega^{p+q+2}(\mathcal{A} \otimes \mathcal{B})$$

by

$$sh'(a_0 da_1 \dots da_p, b_0 db_1 \dots b_p) = \sum_\sigma \operatorname{sgn} \sigma \cdot \sigma(da_0 da_1 \dots da_p db_0 db_1 \dots db_q),$$

where the sum runs over all cyclic (p, q)-shuffles.

Now, the abovementioned map G is matricially defined as follows:

$$G = \begin{pmatrix} sh & sh' & 0 & \dots \\ 0 & sh & sh' & 0 \\ \dots & 0 & sh & sh' \\ \dots & \dots & 0 & sh \end{pmatrix}$$

The Hochschild and cyclic homology and cohomology can be naturally generalized to the case where our algebra is $\mathbb{Z}/2$-graded (as explained in [**Kas86**]) or is a m-algebra, i.e. a locally convex algebra equipped with a submultiplicative family of seminorms (as explained in [**CST04**, Ch. 3]); all the above constructions can be naturally generalized to this case and the above statements continue to hold. We refer the skeptical reader to [**Kas86**] and [**CST04**] for the thorough treatment of those generalizations, just mentioning that the necessary modifications are completely natural: in the $\mathbb{Z}/2$-graded one naturally induces the grading from $\mathcal{A}$ to $\Omega^*(\mathcal{A})$ and uses graded commutators instead of usual commutators, and in the case where $\mathcal{A}$ is an m-algebra one uses completed tensor products to form $\Omega^n\mathcal{A}$. We will use these theories on $\mathbb{Z}/2$-graded m-algebras (cf. Definition 2.3, Proposition 2.20) omitting the technical particularities and bearing in mind that all the constructions work "as they should".

CHAPTER 2

Differential geometry of noncommutative manifolds

In this chapter we discuss the notion of a noncommutative manifold, based on a spectral triple with additional structures and prove various results concerning their "differential geometry". The "axioms" for spectral triples were first proposed in [**Con96**]. We first discuss the content of the axioms in the commutative case, and then pass to the noncommutative generalization.

1. Riemannian manifolds and spectral triples

In this section we consider "commutative" spectral triples, i.e. these associated with compact Riemannian manifolds.

Let (M, g) be a compact spin^c manifold of dimension d. Denote by $\mathcal{A}$ the algebra $C^\infty(M)$ of complex-valued smooth functions on M. The existence of a spin^c structure can be translated to the language of Morita equivalence as follows. Let $\mathcal{C} := \Gamma^\infty(\mathrm{Cliff}_\mathbb{C}(T^*M))$ be the algebra of sections of the bundle of complex Clifford algebras associated with the cotangent bundle of M. The existence of a spin^c structure is equivalent to the fact that there is a Morita equivalence of pre-C*-algebras $\mathcal{C} \sim \mathcal{A}$ provided by a finitely generated projective Hermitian $\mathcal{A}$-module $\mathcal{H}^\infty$ [**GBVF01**, Def. 9.7]. By the Serre–Swan Theorem [**GBVF01**, Thm. 2.10], this bimodule corresponds to a vector bundle S over M which is called the spinor bundle. It naturally carries a representation of $\mathcal{C}$. If moreover M admits a spin structure, it yields a real structure on the module $\mathcal{H}^\infty$.

The main interest in considering spin^c structures is that there is a naturally defined unbounded selfadjoint operator D such that the triple $(\mathcal{A}, \mathcal{H}, D)$ gives the fundamental class in K-homology $KK(A, \mathbb{C})$. Thus, while in general there are many spectral triples associated with a manifold M, the existence of a spin^c structure allows us to focus on fundamental classes only. The operator D is the Dirac operator, and we refer to [**GBVF01**, Def. 9.10] for its precise construction.

Thus, from a compact spin^c manifold of dimension d we can obtain the following data:

i) a unital $*$-algebra $\mathcal{A}$ represented by bounded operators on a Hilbert space $\mathcal{H}$,
ii) an unbounded selfadjoint operator D with compact resolvent acting on $\mathcal{H}$

subject to following conditions:

i) the eigenvalues $\{\lambda_k\}$ of $|D|$, sorted in ascending order, satisfy $\lambda_k = O(k^{1/d})$,
ii) $[D, a]$ is bounded for every $a \in \mathcal{A}$,
iii) $[[D, a], b] = 0$ for every $a, b \in \mathcal{A}$,
iv) for $a \in A$, $[D, \mathcal{A}]$, the map $t \mapsto e^{it|D|} a e^{-it|D|}$ is C^∞ in norm topology;
v) the $\mathcal{A}$-module $\mathcal{H}^\infty := \bigcap_{n\in\mathbb{N}} \mathrm{Dom}\, |D|^n$ is Hermitian, projective and finitely generated;
vi) the Hermitian structure on $\mathcal{H}^\infty$ is compatible with the scalar product:

$$\langle \xi, \eta \rangle = \mathrm{Tr}_\omega(\langle \xi, \eta \rangle_\mathcal{A} \, |D|^{-d}) \quad \forall \xi, \eta \in \mathcal{H}^\infty$$

vii) the class $\Delta^*[(\mathcal{A}, \mathcal{H}, D)] \in \mathsf{KK}(A \otimes A^{\mathrm{op}}, \mathbb{C})$ yields the Poincaré duality isomorphism $\mathsf{KK}(\mathbb{C}, A) \to \mathsf{KK}(A, \mathbb{C})$ via the Kasparov intersection product.

We observe that these properties are all formulated in a way which is completely independent of the commutativity of the algebra $\mathcal{A}$ and doesn't appeal to any geometric constructions. These conditions are the starting point of the proof of the reconstruction theorem, which allows to get the manifold M back from the above data. More precisely, one has:

Theorem 2.1 ([**Con08**]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be as above with $\mathcal{A}$ commutative and let there be an anticommutative Hochschild cycle $c \in Z_d(\mathcal{A}, \mathcal{A})$ such that $\pi_d(c) = \gamma$ ($\gamma = 1$ in the odd case). Suppose that the following regularity condition is satisfied: for every $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$, the map $t \mapsto e^{it|D|} a e^{-it|D|}$ is C^∞ in norm topology.*

Then there exists a compact oriented Riemannian manifold M with $\mathcal{A} = C^\infty(M)$.

2. Axioms for noncommutative spectral triples

The axioms for noncommutative spectral triples were proposed by A. Connes in [**Con96**]. We will discuss them and then propose a slight generalization, based on the Poincaré duality.

Definition 2.2. A noncommutative spin manifold (noncommutative spin geometry) of dimension d is given by the following data:

i) a unital $*$-algebra $\mathcal{A}$ represented by bounded operators on a Hilbert space $\mathcal{H}$,
ii) an unbounded selfadjoint operator D with compact resolvent acting on $\mathcal{H}$,
iii) if d is even, an bounded operator $\gamma = \gamma^*$, $\gamma^2 = 1$ acting on $\mathcal{H}$,
iv) an antilinear operator $J\colon \mathcal{H} \to \mathcal{H}$, $J^2 = \varepsilon \in \{\pm 1\}$ depending on $d \bmod 8$.

subject to following conditions:

i) the eigenvalues $\{\lambda_k\}$ of $|D|$, sorted in ascending order, satisfy $\lambda_k = O(k^{1/d})$,
ii) $[D, a]$ is bounded for every $a \in \mathcal{A}$,
iii) $\gamma a = a\gamma$ for $a \in \mathcal{A}$, $\gamma D = -D\gamma$, $DJ = \varepsilon' JD$, $\gamma J = \varepsilon'' J\gamma$, where $\varepsilon', \varepsilon'' \in \{\pm 1\}$ depend on $d \bmod 8$,
iv) $[[D, a], JbJ] = 0$ for every $a, b \in \mathcal{A}$,
v) for $a \in \mathcal{A}, [D, \mathcal{A}]$, the map $t \mapsto e^{it|D|} a e^{-it|D|}$ is C^∞ in norm topology;
vi) the $\mathcal{A}$-module $\mathcal{H}^\infty := \bigcap_{n \in \mathbb{N}} \mathrm{Dom}\, |D|^n$ is Hermitian, projective and finitely generated;
vii) the Hermitian structure on $\mathcal{H}^\infty$ is compatible with the scalar product:

$$\langle \xi, \eta \rangle = \mathrm{Tr}_\omega(\langle \xi, \eta \rangle_{\mathcal{A}} |D|^{-d}) \quad \forall \xi, \eta \in \mathcal{H}^\infty$$

viii) the class $[D] \in [(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, \mathcal{H}, D, J)] \in \mathsf{KR}(A \otimes A^{\mathrm{op}}, \mathbb{C})$ yields the Poincaré duality isomorphism $\mathsf{KK}(\mathbb{C}, A) \to \mathsf{KK}(A, \mathbb{C})$ via the Kasparov intersection product.

The presence of the antilinear involution J and the choice of signs $\varepsilon, \varepsilon', \varepsilon''$ correspond to a spin manifold and therefore restrict the class of possible examples; in particular, we will see that the toric deformations of manifolds need not be restricted to the spin case. Besides that, one would certainly like to have a "canonical" noncommutative geometry associated to every manifold, regardless of the existence of a spin structure.

We propose a slight generalization of the axioms above, rich enough to give a straightforward analogy between the commutative and noncommutative cases and allow a slight extension of existing examples.

Definition 2.3. A noncommutative manifold (noncommutative geometry) of dimension d is given by the following data:

i) two unital $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ with commuting injective representations of $\mathcal{A}$ and $\mathcal{B}^{\mathrm{op}}$ by bounded operators on a Hilbert space $\mathcal{H}$,
ii) an unbounded selfadjoint operator D with compact resolvent acting on $\mathcal{H}$,
iii) if d is even, a bounded operator $\gamma = \gamma^*$, $\gamma^2 = 1$ acting on $\mathcal{H}$,

subject to following conditions:

i) the eigenvalues $\{\lambda_k\}$ of $|D|$, sorted in ascending order, satisfy $\lambda_k = O(k^{1/d})$,
ii) $[D, a]$ is bounded for every $a \in \mathcal{A}$, $[D, b]$ is bounded for every $b \in \mathcal{B}$
iii) $[[D, a], b] = 0$ for every $a \in \mathcal{A}$, $b \in \mathcal{B}$,
iv) $\gamma a = a\gamma$ for $a \in \mathcal{A}$, $\gamma D = -D\gamma$;
v) for $S \in \mathcal{A} \cup \mathcal{B} \cup [D, \mathcal{A}] \cup [D, \mathcal{B}]$, the map $t \mapsto e^{it|D|} S e^{-it|D|}$ is of class C^∞ in the norm topology;
vi) the module $\mathcal{H}^\infty = \mathcal{X} := \bigcap_{n \in \mathbb{N}} \operatorname{Dom} |D|^n$ is Hermitian, projective and finitely generated as a left $\mathcal{A}$-module and a right $\mathcal{B}$-module;
vii) the Hermitian structures on $\mathcal{X}$ are compatible with the scalar product:

$$\langle \xi, \eta \rangle = \mathrm{Tr}_\omega({}_{\mathcal{A}}\langle \xi, \eta \rangle |D|^{-d}) = \mathrm{Tr}_\omega(\langle \xi, \eta \rangle^*{}_{\mathcal{B}} |D|^{-d}) \quad \forall \xi, \eta \in \mathcal{X} \tag{2.1}$$

viii) the class $[D] \in [(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{H}, D, J)] \in \mathsf{KK}(A \otimes B^{\mathrm{op}}, \mathbb{C})$ yields Poincaré duality between A and B in KK-theory. Hereby $\mathcal{B}$ and B are endowed with the grading induced by γ: $\mathcal{B}^{(0)} = \{\gamma b = b\gamma\}$, $\mathcal{B}^{(1)} = \{\gamma b = -b\gamma\}$.

This definition is certainly wider than the preceding one, because there we can just take $\mathcal{B} = J\mathcal{A}J^{-1}$ and define the $\mathcal{B}$-valued scalar product via

$$\langle \xi, \eta \rangle_{\mathcal{B}} := J_{\mathcal{A}}\langle J\xi, J\eta \rangle J^{-1}.$$

On the other hand, now we can canonically encorporate any smooth manifold M and its toric deformations in this framework. It is also known that the class of C^*-algebras which satisfy Poincaré duality in KK-theory with themselves is much smaller than the class of C^*-algebras which have a Poincaré dual in KK-theory (even in the commutative case).

3. “Smooth”, “continuous” and “measurable” modules and algebras

There are several “levels” we can look at our spectral triple at: the fundamental of them is the “smooth” level included into definitions. On this level we get the following structure:

- two unital $*$-algebras $\mathcal{A}$ and $\mathcal{B}$,
- an $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type $\mathcal{X} \subset \mathcal{H}$ over $\mathcal{A}$,
- a trace τ on $\mathcal{A}$ and $\mathcal{B}$ which is compatible with the scalar product in $\mathcal{H}$: $\langle \cdot, \cdot \rangle = \tau\left({}_{\mathcal{A}}\langle \cdot, \cdot \rangle\right) = \tau\left(\langle \cdot, \cdot \rangle_{\mathcal{B}}\right)$

Hereby we denote by τ the linear functional on $\mathcal{B}(\mathcal{H})$ given by

$$\tau(T) = \mathrm{Tr}_\omega(T|D|^{-d}).$$

It restricts to a trace on $\mathcal{A}$ and $\mathcal{B}$ by the regularity assumption for elements of $\mathcal{A}$ resp. $\mathcal{B}$.

We can also look at the “continuous” and “measurable” level; the first corresponds to the C^*-completions, the second corresponds to the von Neumann completions of the objects involved. To distinguish different “levels”, we introduce the following notational convention: the “smooth” algebras and modules will be denoted by `mathcal`-letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$ etc., the C^*-algebras in question as well as C^*-modules will be denoted by the letters A, B, X etc.; the von Neumann algebras and modules over them over them will be denoted by the `mathscr`-letters $\mathscr{A}$, $\mathscr{B}$, $\mathscr{X}$ etc. The Hilbert space will always be denoted by $\mathcal{H}$.

Proposition 2.4. *Let ι be the inclusion $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, and let $e = e^* = e^2 \in \mathbb{M}_m(\mathcal{A})$ be the projection such that $\mathcal{X} \cong \mathcal{A}^n e$ as left Hermitian $\mathcal{A}$-modules. Then this isomorphism can be continued to an isomorphism of $*$-representations ι and μ of $\mathcal{A}$ on $\mathcal{H}$ and $L^2(\mathcal{A},\tau)^m e$ respectively.*

Similarly, if $f = f^* = f^2 \in \mathbb{M}_n(\mathcal{B})$ is the projection such that $\mathcal{X} \cong f\mathcal{B}^n$ as right Hermitian $\mathcal{B}$-modules, then this isomorphism can be continued to an isomorphism of $*$-representations ι and μ of $\mathcal{B}^{\mathrm{op}}$ on $\mathcal{H}$ and $fL^2(\mathcal{B},\tau)^n$ respectively.

Hereby $L^2(\mathcal{A},\tau)$ and $L^2(\mathcal{B},\tau)$ denote the GNS spaces of $\mathcal{A}$ resp. $\mathcal{B}$ with respect to τ.

PROOF. The absolute continuity condition (2.1) tells precisely that the scalar products on these modules become equal under the abovementioned isomorphisms. □

Thus our Hilbert space comes equipped with a dense $\mathcal{A}$–$\mathcal{B}$-bimodule. The "continuous" and "measurable" variants of these modules can be obtained by replacing $\mathcal{A}$ or $\mathcal{B}$ by their C^*- or von Neumann completions — and they can still be regarded as subspaces in $\mathcal{H}$. The main issue here is, however, that *a priori* there is no guarantee that $A^m e \cong fB^n$ after identifying both with subspaces in $\mathcal{H}$.

To circumvent this issue, we make the following observation which is essentially a repetition of [**KW00**, Prop. 1.18], although there it was formulated for the C^*-case only.

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two normed $*$-algebras and let $\mathcal{X}$ be a left Hermitian $\mathcal{A}$-module and a right Hermitian $\mathcal{B}$-module such that the actions of $\mathcal{A}$ and $\mathcal{B}$ commute and yield $*$-representations of $\mathcal{A}$ in $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ and of $\mathcal{B}$ in $\mathrm{End}_{\mathcal{A}}(\mathcal{X})$, respectively.*

If $\mathcal{X}$ has a right $\mathcal{B}$-basis $\{u_1,\dots,u_n\}$, then

$$\|{}_{\mathcal{A}}\langle\xi,\xi\rangle\| \leqslant \left\|\sum_{i=1}^n {}_{\mathcal{A}}\langle u_i,u_i\rangle\right\| \|\langle\xi,\xi\rangle_{\mathcal{B}}\|.$$

If $\mathcal{X}$ has a left $\mathcal{A}$-basis $\{v_1,\dots,v_m\}$, then

$$\|\langle\xi,\xi\rangle_{\mathcal{B}}\| \leqslant \left\|\sum_{j=1}^m {}_{\mathcal{A}}\langle v_j,v_j\rangle\right\| \|{}_{\mathcal{A}}\langle\xi,\xi\rangle\|.$$

In particular, if $\mathcal{X}$ has a left $\mathcal{A}$-basis and a right $\mathcal{B}$-basis, then the norms coming from the $\mathcal{A}$-valued and the $\mathcal{B}$-valued scalar product are equivalent.

PROOF. If $n = 1$ and u is the basis element, then $\xi = u\,\langle u,\xi\rangle_{\mathcal{B}}$, and

$$\langle\xi,\xi\rangle_{\mathcal{B}} = \langle u,\xi\rangle_{\mathcal{B}}^*\,\langle u,\xi\rangle_{\mathcal{B}}.$$

Moreover,

$$ {}_{\mathcal{A}}\langle\xi,\xi\rangle = {}_{\mathcal{A}}\langle u\,\langle u,\xi\rangle_{\mathcal{B}}\,, u\,\langle u,\xi\rangle_{\mathcal{B}}\rangle \leqslant {}_{\mathcal{A}}\langle u,u\rangle\,\|\langle u,\xi\rangle_{\mathcal{B}}\|^2, \tag{2.2}$$

which proves the claim in this case.

In the general case we make the following observation (cf. [**KW00**, Lemma 1.17]). Put $\mathcal{Y} := \mathcal{X} \otimes \mathbb{C}^n$ and regard it as an $\mathcal{A}$–$\mathcal{B} \otimes \mathbb{M}_n(\mathbb{C})$-bimodule, equipped with the scalar products

$$ {}_{\mathcal{A}}\langle\xi,\eta\rangle := \sum_{i=1}^n {}_{\mathcal{A}}\langle\xi_i,\eta_i\rangle,$$

$$\langle\xi,\eta\rangle_{\mathcal{B}\otimes\mathbb{M}_n(\mathbb{C})} := (\langle\xi_i,\eta_j\rangle_{\mathcal{B}})_{i,j=1}^n.$$

Then $\widetilde{u} := (u_1, \dots, u_n)$ is a right $\mathcal{B} \otimes \mathbb{M}_n(\mathbb{C})$-basis for $\mathcal{Y}$.

Thus, putting $\widetilde{\xi} := (\xi, 0, \dots, 0)$ and applying the first part of the proof, we obtain the conclusion. □

This proposition shows that the following definition is correct.

Definition 2.6. We denote by A resp. B the C^*-completions of $\mathcal{A}$ resp. $\mathcal{B}$ in $\mathcal{B}(\mathcal{H})$. We denote by X the completion of $\mathcal{X}$ with respect to either of the norms

$$_{\mathcal{A}}\|\cdot\|\colon \xi \mapsto \|_{\mathcal{A}}\langle \xi, \xi\rangle\|^{1/2}$$

or

$$\|\cdot\|_{\mathcal{B}} : \xi \mapsto \|\langle \xi, \xi\rangle_{\mathcal{B}}\|^{1/2}$$

An easy observation is

Proposition 2.7. *X is a Hilbert A–B-bimodule, which is finitely generated and projective as a left A-module and as a right B-module. Moreover, we can identify X with a subspace in $\mathcal{H}$.*

Proof. Let $\xi = \lim \xi_n$, $a = \lim a_n$ with $\xi_n \in \mathcal{X}$, $a_n \in \mathcal{A}$. In view of the inequality

$$\|a_n\xi_n - a_m\xi_m\| \leqslant \|a_n - a_m\|\, _{\mathcal{A}}\|\xi_n\| + \|a_m\|\, _{\mathcal{A}}\|\xi_n - \xi_m\|$$

the sequence $\{a_n\xi_n\}$ is a Cauchy sequence, and thus converges. We define $a\xi := \lim a_n\xi_n$. A standard argument shows that the limit is independent of the choice of the sequences $\{a_n\}$ and $\{\xi_n\}$, and the continuity of the norms yields $\|a\xi\| \leqslant \|a\|\,\|\xi\|$.

Analogously, we set $_A\langle \xi, \xi\rangle := \lim {}_{\mathcal{A}}\langle \xi_n, \xi_n\rangle$. The convergence of the sequence is guaranteed by the inequality

$$\|_{\mathcal{A}}\langle \xi_n, \xi_n\rangle - {}_{\mathcal{A}}\langle \xi_m, \xi_m\rangle\| \leqslant ({}_{\mathcal{A}}\|\xi_n\| + {}_{\mathcal{A}}\|\xi_m\|)\, {}_{\mathcal{A}}\|\xi_n - \xi_m\|,$$

and once again a standard argument shows that the limit is independent of the choice of the sequence.

Now, a left $\mathcal{A}$-basis for $\mathcal{X}$ is also a left A-basis for X, which proves that X is finitely generated and projective. The corresponding statements involving B are proved in a similar manner. □

Now we turn to the "measurable" variants of the statements above.

Definition 2.8. We let $\mathscr{A}$ be the von Neumann algebra generated by $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ and let $\mathscr{B}$ be the opposite of the von Neumann algebra generated by $\mathcal{B}$ in $\mathcal{B}(\mathcal{H})$.

An easy consequence of Proposition 2.4 is that the trace τ has a normal continuation to both $\mathscr{A}$ and $\mathscr{B}$ and that the Hilbert space $\mathcal{H}$ becomes a finitely generated projective normal left module over $\mathcal{A}$ and a finitely generated projective normal right module over $\mathcal{B}$. As our von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ are equipped with a finite normal trace τ, from the classical work of Lück [**Lüc02**, Sect. 6.2] it is known that such normal modules alway contain dense finitely generated projective modules over $\mathscr{A}$ resp. $\mathscr{B}$. Once again, the point is that these are in fact one single module.

Proposition 2.9. *Consider the following vector space topologies on X:*

i) *the topology (τ_A) defined by the following system of neighbourhoods of zero:*

$$U_W := \{\xi \in X \mid {}_A\langle \xi, \xi\rangle \in W\},$$

where W is a ultraweak neighbourhood of zero in $A \subset \mathscr{A}$;

ii) *the topology (τ_B) defined by the following system of neighbourhoods of zero:*

$$V_W := \{\xi \in X \mid \langle \xi, \xi\rangle_B \in W\},$$

where W is a ultraweak neighbourhood of zero in $B \subset \mathscr{B}$.

These topologies coincide on bounded subsets of X, and the quasi-completions of X with respect to these topologies are isomorphic topological vector spaces.

Remark 2.10. It is easy to see that these topologies are defined by the family of seminorms

$$\xi \mapsto \varphi(\langle \xi, \xi \rangle)^{1/2},$$

where φ runs over normal states on $\mathcal{A}$ resp. $\mathcal{B}$ and therefore coincide with the so-called s-topologies used in the characterization of self-dual modules over von Neumann algebras.

PROOF. Let ξ_i be a bounded net in X converging to zero in (τ_B). Then

$$\tau\left({}_A\langle \xi_i, \xi_i \rangle\right) = \tau\left(\langle \xi_i, \xi_i \rangle_B\right) \to 0,$$

because τ is normal. We have to show that $a_i := {}_A\langle \xi_i, \xi_i \rangle \to 0$ in the ultraweak topology. It suffices to show that every subnet of $\{a_i\}$ has a subnet converging to 0. We take an arbitrary subnet of a_i; as it is a bounded net, it has a ultraweakly convergent subnet by the Banach–Alaoglu theorem, and the limit of the latter is a positive operator $a \in \mathscr{A}$ with $\tau(a) = 0$, because τ is normal. Thus $a = 0$, which proves the first claim.

To prove the second claim, denote the quasi-completions of X with respect to (τ_A) and (τ_B) by $\mathscr{X}_A$ resp. $\mathscr{X}_B$. The second claim will now follow from the fact that both $\mathscr{X}_A$ and $\mathscr{X}_B$ have the property that any element of them is a limit of a bounded net from X (this is not automatic for quasi-completions in general). This is in turn established by the following observation: let $X \cong A^m e$ as a left A-module and $X \cong fB^n$ as a right B-module. Then $\mathscr{X}_A \cong \mathscr{A}^m e$, $\mathscr{X}_B \cong f\mathscr{B}^n$ in view of Lemma 1.28 and the following facts:

- any self-dual Hilbert module over a von Neumann algebra is a dual Banach space [**MT05**, Prop. 3.3.3],
- any finitely generated projective Hilbert module is self-dual [**MT05**].

□

Definition 2.11. We denote by $\mathscr{X}$ the quasi-completion of X with respect to one of the topologies (τ_A) or (τ_B).

Proposition 2.12. *$\mathscr{X}$ is naturally an $\mathscr{A}$–$\mathscr{B}$ Hilbert bimodule, which is finitely generated and projective over both $\mathscr{A}$ and $\mathscr{B}$. Moreover, $\mathscr{X}$ can be identified with a subspace in $\mathcal{H}$.*

PROOF. We take an A-basis $\{v_1, \ldots, v_m\}$ for X. If $\xi_i = \sum_{j=1}^m a_j^{(i)} v_j$ is a bounded Cauchy net for the topology (τ_A), convergent to $\xi \in \mathscr{X}$, then for each j $a_j^{(i)}$ is a bounded net which is Cauchy for the ultraweak topology, thus convergent. Therefore for each $a \in \mathscr{A}$ the nets $aa_j^{(i)}$ are bounded and ultraweakly Cauchy. Thus, the net $\sum_{j=1}^m a_j^{(i)} v_j$ is a bounded Cauchy net in $\mathscr{X}$ and therefore convergent. We define $a\xi$ to be the limit of this net. As before, a standard argument shows that this definition is independent of the choice of a bounded ξ_i. Thus, $\mathscr{X}$ is a finitelu generated projective left $\mathscr{A}$-module (isomorphic to $\mathscr{A}^m e$ as established the previous proposition). Analogously, $\mathscr{X}$ is a finitely generated right $\mathscr{B}$-module.

To define a Hilbert module structure, we observe that for a net ξ_i as above the net ${}_A\langle \xi_i, \xi_i \rangle$ is a bounded Cauchy net in the ultraweak topology, thus convergent to some $a \in \mathscr{A}$, which we define to be ${}_{\mathscr{A}}\langle \xi, \xi \rangle$, $\xi = \lim \xi_i$. As before, this choice is independent of ξ_i. The first claim of the proposition follows.

The second claim of the proposition follows from the fact that the topology on X inherited from $\mathcal{H}$ is weaker than the topologies (τ_A) and (τ_B). □

Proposition 2.13. *$\mathcal{X}$ is a full Hermitian $\mathcal{A}$-module and a full Hermitian $\mathcal{B}$-module.*

PROOF. It follows from the proof of Lemma 1.21 that $\mathcal{X}$ is full if and only if the element

$$z = \sum_{i=1}^{n} {}_{\mathcal{A}}\langle u_i, u_i\rangle$$

is invertible, where $\{u_i\}_{i=1}^n$ is a right $\mathcal{B}$-basis for $\mathcal{X}$. It follows that $\mathcal{X}$ is full iff $\mathscr{X}$ is full.

It follows from Proposition 1.19 that $z \in \mathcal{Z}(\mathcal{A})$. Consider the support projection $p \in \mathscr{A}$ of z. As in the proof of Lemma 1.21, for any finite set $\{x_j\}_{j=1}^m \subset \mathcal{X}$ there exists a $C > 0$ such that

$$z \geqslant C^{-1} \sum_{j=1}^{m} {}_{\mathcal{A}}\langle x_i, x_i\rangle.$$

It follows that the central supports of all $\mathscr{A}$-valued scalar products are majorized by p, which gives $p = 1$, because otherwise the representation of $\mathscr{A}$ on $\mathcal{H}$ cannot be injective.

If $n = 1$, then [**MT05**, Prop. 3.4.1] gives us an element $\zeta \in \mathscr{X}$ such that ${}_{\mathscr{A}}\langle \zeta, \zeta\rangle = p = 1$, which proves fullness of $\mathscr{X}$. In general we have to adapt the argument from [**MT05**, Prop. 3.4.1] to our situation.

We let for $k \in \mathbb{N}$

$$z_k = (z + 1/k)^{-1/2}, \quad u_i^{(k)} = z_k u_i.$$

Then we get

$$\sum_{i=1}^{n} {}_{\mathscr{A}}\left\langle u_i^{(k)}, u_i^{(k)}\right\rangle \leqslant 1.$$

Let y_i be a weak* accumulation point of $u_i^{(k)}$. It follows that $u_i = z^{1/2} y_i$, hence

$$z = z^{1/2} r z^{1/2},$$

where

$$r = \sum_{i=1}^{n} {}_{\mathscr{A}}\langle y_i, y_i\rangle \leqslant 1.$$

Thus,

$$z^{1/2}(1 - r)z^{1/2} = 0,$$

and it follows that $r = 1$. □

4. Frechét topology on $\mathcal{A}$ and relevant modules

Now we will adapt some propositions from [**Con08**] from the commutative case to our general situation. Most of them will be similar for $\mathcal{A}$ and $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$, although there will be some slight differences.

For these constructions to work, we have to make some additional assumptions. Let us strat with the definition of a regular operator.

Definition 2.14. For $T \in \mathcal{B}(\mathcal{H})$, we let $\delta(T) := [|D|, T]$. T is called regular iff $\delta^m(T)$ is bounded for all $m \in \mathbb{N}$.

It is easy to see that regularity of T is equivalent to the fact that the map $t \mapsto e^{-it|D|} T e^{it|D|}$ is C^∞ in the norm topology.

In this section, we assume that one of the following two assumptions hold for the $\mathcal{B}$-endomorphisms of $\mathcal{X}$:

i) the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ is equal to $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ or
ii) every $\mathcal{B}$-endomorphism is regular.

If we add to these one of two assumptions

i') the algebra generated by $\mathcal{B}$ and $[D,\mathcal{B}]$ is equal to $\mathrm{End}_{\mathcal{A}}(\mathcal{X})$ or
ii') every $\mathcal{A}$-endomorphism is regular,

then all the statements below remain true after exchanging $\mathcal{A}$ and $\mathcal{B}$ (or their relevant completions).

By the regularity assumptions in the axioms for noncommutative geometry, i) implies ii) and i') implies ii'). These assumptions are not unnatural; in fact, the assumptions ii) and ii') were used by A. Connes in the proof of the Reconstruction Theorem and seem to be crucial.

We equip the algebras $\mathrm{End}_{\mathcal{A}}(\mathcal{X})$ and $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ with the trace τ given by

$$\tau(T) = \mathrm{Tr}_{\omega}(T|D|^{-d}).$$

Lemma 2.15. *The following conditions are equivalent for $T \in \mathscr{B}'$:*

i) $T \in \mathrm{End}_{\mathcal{B}}(\mathcal{X})$;
ii) *T belongs to the domain of δ^m for any natural m;*

PROOF. The implication i) ⇒ ii) is a direct consequence of the above assumptions and the regularity condition. The implication ii) ⇒ i) can be proven along the lines [**Con08**, Lemma 2.1] using $\mathcal{X} = \bigcap_n \mathrm{Dom}\,|D|^n$ and the equality

$$|D|^m T\xi = \sum_{k=0}^{m} \binom{m}{k} \delta^k(T)|D|^{m-k}\xi, \quad \xi \in \mathrm{Dom}\,|D|^m.$$

□

Lemma 2.16. *The following conditions are equivalent for $T \in \mathscr{A}$:*

i) $T \in \mathcal{A}$;
ii) *$[D,T]$ is bounded and both T and $[D,T]$ belong to the domain of δ^m for any natural m;*
iii) *T belongs to the domain of δ^m for any natural m;*
iv) $T\mathcal{X} \subset \mathcal{X}$.

PROOF. iv) ⇒ i): Let us assume iv). Then T is a $\mathcal{B}$-endomorphism of $\mathcal{X} \cong f\mathcal{B}^m$. Applying the conditional expectation on the von Neumann completion $E\colon \mathrm{End}_{\mathscr{B}}(\mathscr{X}) \to \mathscr{A}$ is by construction equal to $E\colon \mathrm{End}_{\mathcal{B}}(\mathcal{X}) \to \mathcal{A}$, thus $E(T) = T \in \mathcal{A}$, which shows i).

i) ⇒ ii) follows from regularity, ii) ⇒ iii) is immediate and iii) ⇒ iv) follows analogously to the proof of Lemma 2.15. □

We shall now show that $\mathcal{A}$ is a Fréchet algebra when endowed with the submultiplicative norms

$$\|x\|_k := \|\rho_k(x)\|, \quad \rho_k(x) = \begin{pmatrix} x & \delta(x) & \dots & \delta^k(x)/k! \\ 0 & x & \dots & \dots \\ \dots & \dots & x & \delta(x) \\ 0 & \dots & 0 & x \end{pmatrix}.$$

Proposition 2.17.

i) *$\delta\colon \mathrm{Dom}\,\delta \to \mathcal{B}(\mathcal{H})$ is a closed unbounded derivation;*
ii) *$\mathcal{A}$ is a Fréchet algebra when endowed with the norms $\|\cdot\|_k$;*
iii) *$\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ is a Fréchet algebra when endowed with the norms $\|\cdot\|_k$;*
iv) *the semi-norms $\|a\|'_k := \|[D,a]\|_k$ are continuous on $\mathcal{A}$.*

PROOF. The proof of [**Con08**, Prop. 2.2] applies *mutatis mutandis*, and we sketch it here.

i) Let $T_n \in \operatorname{Dom}\delta$ be such that $T_n \to T$ in norm and $\delta(T_n)$ is norm convergent. We have to show that $T \in \operatorname{Dom}\delta$ and that $\delta(T_n)$ converges to $\delta(T)$ in norm. Take $\xi \in \operatorname{Dom}|D|$ and observe that

$$\delta(T_n)\xi = |D|T_n\xi - T_n|D|\xi,$$

so $|D|T_n\xi$ converges to $|D|T\xi$, because $|D|$ is closed. Thus, $\delta(T_n) \to \delta(T)$ strongly on $\operatorname{Dom}|D|$, $T \in \operatorname{Dom}\delta$, and $\delta(T_n) \to \delta(T)$ in norm, since the limit of this sequence coincides with $\delta(T)$ on a dense subspace.

ii) A Cauchy sequence $\{a_n\}$ with respect to the norms $\|\cdot\|_k$ is necessarily Cauchy in A and thus converges to a $T \in A \subset \mathcal{A}''$. Since δ is closed, $T \in \operatorname{Dom}\delta$ and $\delta(a_n) \to \delta(T)$ in norm. By induction $T \in \bigcap_n \operatorname{Dom}\delta^n$ and thus $T \in \mathcal{A}$ by the previous lemma. Furthermore, it follows that $\delta^k(a_n) \to \delta^k(T)$ in norm, and thus $\mathcal{A}$ is a Fréchet space.

iii) is proven analogously to ii).

iv) The derivation $T \mapsto [D, T]$ is closed similarly to δ, and thus the above proof of completeness applies. The result then follows from the open mapping theorem. □

Corollary 2.18. $\operatorname{End}_{\mathcal{B}}(\mathcal{X}) \subset \operatorname{End}_B(X)$ *and* $\mathcal{A} \subset A$ *are pre-*C^**-algebras.*

Proof. If $T \in \operatorname{End}_{\mathcal{B}}(\mathcal{X})$ is invertible in $\operatorname{End}_B(X)$, then the resolvent formula

$$\delta(T^{-1}) = -T^{-1}\delta(T)T^{-1}$$

and similar formulae obtained by applying δ^m to this show that properties ii) resp. iii) from Lemma 2.15 resp. Lemma 2.16 are satisfied. □

Moreover, we get “Sobolev estimates” using that $\mathcal{X}$ is a finitely generated $\mathcal{B}$-module.

Definition 2.19. Define a family of norms on $\operatorname{End}_{\mathcal{B}}(\mathcal{X})$ as follows:

$$\|T\|''_s = \sum_i \left\|(1 + D^2)^{s/2} T u_i\right\|_2,$$

where $\{u_i\}$ is a right $\mathcal{B}$-basis for $\mathcal{X}$.

Proposition 2.20.

i) *When endowed with the norms* $\|\cdot\|''_s$, $\operatorname{End}_{\mathcal{B}}(\mathcal{X})$ *is a Fréchet separable nuclear space and* $\mathcal{A}$ *is its closed subspace;*
ii) *The algebraic isomorphism* $\mathcal{X} \cong \mathcal{A}^n e$ *is topological.*
iii) *The maps* $(T, \xi) \to a\xi$ *and the* $\mathcal{A}$*-valued inner product on* $\mathcal{X}$ *are jointly continuous as maps* $\operatorname{End}_{\mathcal{B}}(\mathcal{X}) \times \mathcal{X} \to \mathcal{X}$ *and* $\mathcal{X} \times \mathcal{X} \to \mathcal{A}$.

Proof. i) Let $\{T_n\}$ be a sequence in $\operatorname{End}_{\mathcal{B}}(\mathcal{X})$ such that $(1 + D^2)^{s/2} T_n u_i$ converge for all s and i. We thus obtain vectors

$$z_i := \lim T_n u_i \in \mathcal{X}.$$

Define an operator $T\colon X \to X$ by

$$T(u_i b_i) = \lim T_n u_i b_i = z_i b_i.$$

This is well-defined since b_i are continuous with respect to the Fréchet space topology on $\mathcal{X}$. Thus T is an element of $\operatorname{End}_{\mathcal{B}}(\mathcal{X})$. Moreover, if $T_n \in \mathcal{A}$, $T \in \mathscr{A}$ by construction, and Lemma 2.16 shows that $T \in \mathcal{A}$. Besides that, by construction $T_n \to T$ in the topology defined by the norms $\|\cdot\|''_s$, and therefore $\operatorname{End}_{\mathcal{B}}(\mathcal{X})$ and $\mathcal{A}$ are complete in this topology. Thus $\operatorname{End}_{\mathcal{B}}(\mathcal{X})$ is a Fréchet space in this topology, and being a closed subspace of the finite direct sum of finitely many separable nuclear spaces $\mathcal{X}$, it is itself separable and nuclear.

ii) The product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is jointly continuous because the norms $\|\cdot\|_k$ are submultiplicative. Thus $e\mathcal{A}^n$ is a closed subspace and hence complete. The mapping $(a, \xi) \to a \cdot \xi$ is continuous because a is regular, and thus the open mapping theorem gives the result.

iii) follows from ii) applied to $\mathcal{B}$ and the joint continuity of the product map. □

Corollary 2.21. *The topologies on $\mathcal{A}$ given by three families of seminorms $\{\|\cdot\|_k\}$, $\{\|\cdot\|_k, \|\cdot\|'_k\}$, $\{\|\cdot\|''_s\}$ coincide.*

Corollary 2.22. *The C^*-algebra A is separable.*

Proposition 2.23. *Let $a = a^*$ be a selfadjoint element of $\mathbb{M}_n(\mathcal{A})$ and $f\colon \mathbb{R} \to \mathbb{C}$ be a smooth function defined on a neighbourhood of the spectrum of a. Then $f(a) \in \mathbb{M}_n(\mathcal{A})$.*

Proof. We first consider the case $n = 1$. Let us denote $\beta_u(T) = e^{iusa} T e^{-iusa}$. Then

$$\frac{1}{n!}\delta^n(e^{isa})e^{-isa} = \sum_{\substack{k_j>0,\\ \sum k_j = n}} i^\ell s^\ell \int_{S_\ell} \beta_{u_1}\left(\frac{\delta^{k_1}(a)}{k_1!}\right) \cdot \dots \cdot \beta_{u_\ell}\left(\frac{\delta^{k_\ell}(a)}{k_\ell!}\right) du,$$

where $S_\ell = \{u \in \mathbb{R}^\ell \mid 0 \leqslant u_1 \leqslant \dots \leqslant u_\ell \leqslant 1\}$ is the standard ℓ-simplex. Thus,

$$\left\|\delta^k(e^{isa})\right\| = O(|s|^k), \quad |s| \to \infty.$$

Therefore, after extending f to a compactly supported function, we obtain

$$f(a) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(s) e^{isa}\, ds,$$

and this integral converges in the $\|\cdot\|_k$-topology on $\mathcal{A}$. For the general case, we consider the derivation $\delta \otimes 1$ on $\mathbb{M}_n(\mathcal{A})$ and repeat the argument. □

5. The bimodule of endomorphisms

In this section we consider algebras of $\mathscr{A}$- and $\mathscr{B}$-endomorphisms of $\mathscr{X}$, mainly concentrating on the "measurable level".

The von Neumann algebras $\mathrm{End}_{\mathscr{B}}(\mathscr{X}) \cong \mathscr{B}'$ and $\mathrm{End}_{\mathscr{A}}(\mathscr{X}) \cong \mathscr{A}$ come equipped with natural finite traces which we denote τ' [**Tak02**, Ch. V]. They can be defined as follows: representing $T \in \mathrm{End}_{\mathscr{B}}(\mathscr{X})$ as a matrix $[T_{ij}]$, we set

$$\tau'(T) := \sum_i \tau(T_{ii}),$$

analogously for $S \in \mathrm{End}_{\mathscr{A}}(\mathscr{X})$. A direct computation shows that this defines a normal trace on $\mathrm{End}_{\mathscr{B}}(\mathscr{X})$, and one can show [**AK11**] that it is well defined on $\mathrm{End}_{\mathcal{B}}(\mathscr{X})$ (i.e. independent of the choice of a matrix representation).

Proposition 2.24. *There is a faithful conditional expectation $E\colon \mathrm{End}_{\mathscr{B}}(\mathscr{X}) \to \mathscr{A}$, which is of algebraically finite index. Moreover, this conditional expectation maps $\mathrm{End}_B(X)$ to A, and $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ to $\mathcal{A}$.*

Proof. We use the conditional expectation obtained from Lemma 1.21: for $T \in \mathrm{End}_{\mathscr{B}}(\mathscr{X})$, we let

$$E(T) := F(I)^{-1/2} F(T) F(I)^{-1/2}, \tag{2.3}$$

where

$$F(T) := \sum_{i=1}^n {}_{\mathscr{A}}\langle Tu_i, u_i\rangle \tag{2.4}$$

and $\{u_i\}$ is a right $\mathscr{B}$-basis of $\mathscr{X}$. E is a faithful conditional expectation, because $\tau(F(T)) = \tau'_B(T)$, and both traces are faithful.

Now, if we choose $\{u_i\}$ to be the right $\mathcal{B}$-basis of $\mathcal{X}$ and $T \in \mathrm{End}_{\mathcal{B}}(\mathcal{X})$, then $E(T) \in \mathcal{A}$; analogously, if $T \in \mathrm{End}_B(X)$, then $E(T) \in A$. This finishes the proof. □

Proposition 2.25. *The trace τ on the von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$, the trace τ' on $\mathscr{B}'$ and $\mathscr{A}'$, the conditional expectation E and the operator-valued weight F and different operator-valued scalar products on $\mathscr{X}$ are related as follows:*

i) $\tau' = \tau \circ F$,
ii) ${}_{\mathscr{A}}\langle \xi, \eta\rangle = F\left({}_{\mathscr{B}'}\langle \xi, \eta\rangle\right)$, $\xi, \eta \in \mathscr{X}$,
iii) $\langle \xi, \eta\rangle_{\mathscr{B}} = F\left(\langle \xi, \eta\rangle_{\mathscr{A}'}\right)$, $\xi, \eta \in \mathscr{X}$.

Proof. The claim follows because of the computations

$$\tau \circ F(T) = \tau\left(\sum_{i=1}^{n} {}_{\mathscr{A}}\langle Tu_i, u_i\rangle\right) = \tau\left(\sum_{i=1}^{n} \langle Tu_i, u_i\rangle_{\mathscr{B}}\right) = \tau'(T)$$

and

$$\tau\left({}_{\mathscr{A}}\langle \xi, \eta\rangle\right) = \tau\left(\langle \xi, \eta\rangle_{\mathscr{B}}\right) = \tau'\left({}_{\mathscr{B}'}\langle \xi, \eta\rangle\right)$$

and the fact that the inner products of the form ${}_{\mathscr{A}}\langle \xi, \xi\rangle$ resp. ${}_{\mathscr{B}'}\langle \xi, \xi\rangle$ span $\mathscr{A}$ resp. $\mathscr{B}$, and the traces τ and τ' are faithful. □

Proposition 2.26.

i) *The algebras $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ and $\mathrm{End}_{\mathcal{A}}(\mathcal{X})$ are Hermitian $\mathcal{A}$–$\mathcal{A}$ resp. $\mathcal{B}$–$\mathcal{B}$-bimodules, which are finitely generated and projective for both left and right module structures.*
ii) *The algebras $\mathrm{End}_B(X)$ and $\mathrm{End}_A(X)$ are Hilbert A–A resp. B–B-bimodules of finite type.*
iii) *The algebras $\mathrm{End}_{\mathscr{B}}(\mathscr{X})$ and $\mathrm{End}_{\mathscr{A}}(\mathscr{X})$ are Hilbert $\mathscr{A}$–$\mathscr{A}$ resp. $\mathscr{B}$–$\mathscr{B}$-bimodules of finite type.*

Proof. By Lemma 1.21, the conditional expectation E is of algebraically finite index. Thus, the formulae

$${}_A\langle T_1, T_2\rangle := F(T_1T_2^*) = z^{1/2}E(T_1T_2^*)z^{1/2}$$

and

$$\langle T_1, T_2\rangle_A := F(T_1^*T_2) = z^{1/2}E(T_1^*T_2)z^{1/2}$$

define the structures of a finitely generated left resp. right Hilbert A-module on $\mathrm{End}_B(H)$ [**MT05**, Sect. 4.5]. The "smooth" and "measurable" variants are analogous. □

Proposition 2.27. *There exists an $L \geqslant 1$ such that the mapping*

$$L \cdot F - \mathrm{id}_{\mathrm{End}_{\mathscr{B}}(\mathscr{X})}$$

is completely positive on $\mathrm{End}_{\mathscr{B}}(\mathscr{X})$.

Proof. By [**FK98**], there is a $K \geqslant 1$ such that the mapping $K \cdot E - \mathrm{id}_{\mathrm{End}_B(H)}$ is completely positive, or equivalently,

$$K \cdot E \otimes \mathrm{id}_{\mathbb{M}_n(\mathbb{C})} - \mathrm{id}_{\mathrm{End}_B(X)} \otimes \mathrm{id}_{\mathbb{M}_n(\mathbb{C})} \geqslant 0.$$

Thus,

$$K \cdot F \otimes \mathrm{id}_{\mathbb{M}_n(\mathbb{C})} = K \cdot zE \otimes \mathrm{id}_{\mathbb{M}_n(\mathbb{C})} \geqslant \left\|z^{-1}\right\|^{-1} \cdot \mathrm{id}_{\mathrm{End}_B(X)} \otimes \mathrm{id}_{\mathbb{M}_n(\mathbb{C})},$$

which proves the claim. □

Proposition 2.28. *Suppose that the trace τ has a normal faithful continuation to the von Neumann algebras $\mathscr{A}'$ and $\mathscr{B}'$, and $\mathscr{Z}(\mathscr{A}) = \mathscr{Z}(\mathscr{B})$. Then the faithful conditional expectation $E : \mathrm{End}_{\mathscr{B}}(H) \to \mathscr{A}$ constructed above preserves τ and is of algebraically finite index. Moreover, this conditional expectation maps $\mathrm{End}_B(X)$ to A, and $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ to $\mathcal{A}$.*

Proof. We first suppose that $\mathscr{Z}(\mathscr{A}) = \mathscr{Z}(\mathscr{B}) = \mathbb{C}$. Then $\mathscr{A}$ and $\mathscr{B}'$ are finite von Neumann factors, τ coincides with the unique trace on both of them, and $F(I)$ is a scalar. By the theorem of Kadison about diagonalization of matrices over von Neumann algebras [**MT05**, Thm. 6.1.1], $\mathscr{X} \cong p\mathscr{B}^n$, where $p = \mathrm{diag}(p_{11}, \dots, p_{nn})$ is a projection in $M_n(\mathscr{B})$. Now, every endomorphism T of $\mathscr{X}$ can be written as a matrix $(t_{ij}) \in M_n(\mathscr{B})$, and therefore

$$\tau \circ E(T) = F(I)^{-1} \sum_{i=1}^{n} \langle Tu_i, u_i \rangle = F(I)^{-1} \sum_{i=1}^{n} \tau(t_{ii}).$$

But $\mathscr{B}'$ is a factor and therefore the trace $\tau \circ E$ has to coincide with the trace τ, because it has the right normalization.

For the general case we set $\mathscr{Z} = \mathscr{Z}(\mathscr{A}) = \mathscr{Z}(\mathscr{B})$ and decompose everything into direct integrals:

$$\mathcal{H} \cong \int_X^{\oplus} \mathcal{H}_x \, d\mu(x),$$
$$\mathscr{A} \cong \int_X^{\oplus} \mathscr{A}_x \, d\mu(x),$$
$$\mathscr{B} \cong \int_X^{\oplus} \mathscr{B}_x \, d\mu(x)$$

such that $\mathscr{Z}$ coincides with the diagonal operators and $\tau|_{\mathscr{Z}}$ corresponds to μ [**Tak02**, Thm. 8.21, Theorem 8.23]. Almost every $\mathscr{A}_x$ and almost every $\mathscr{B}_x$ is a factor, and τ decomposes as

$$\tau = \int_X^{\oplus} \tau_x \, d\mu(x)$$

where τ_x is the unique normalized trace on $\mathscr{A}'_x$ or $\mathscr{B}'_x$.

Now, we take a left A-basis $\{v_j\}$ and a right B-basis $\{u_i\}$ in $\mathcal{X}$ and consider its direct integral decompositions via the above isomorphism:

$$v_j = \int_X^{\oplus} v_j(x) \, d\mu(x)$$
$$u_i = \int_X^{\oplus} u_i(x) \, d\mu(x)$$

The operator-valued weight F has the property that for every $z \in \mathscr{Z}$

$$F(zT) = zF(T).$$

Therefore the conditional expectation E decomposes as a direct integral of conditional expectations

$$E = \int_X^{\oplus} E_x \, d\mu(x),$$

where

$$E_x \colon \mathscr{B}'_x \to \mathscr{A}_x.$$

In view of the first part of the proposition, each E_x preserves the normalized trace τ_x on $\mathscr{B}'_x$; thus, the conditional expectation E preserves τ. □

Corollary 2.29. *If the trace τ has a normal continuation to $\mathscr{A}'$ and $\mathscr{B}'$, and $\mathscr{Z}(\mathscr{A}) = \mathscr{Z}(\mathscr{B})$, then*

$$\tau(T) = \tau'(z^{-1}T), \quad T \in \mathscr{B}',$$

$$\tau(T) = \tau'(w^{-1}T), \quad T \in \mathscr{A}'.$$

In particular, the norms induced on $\mathscr{B}'$ resp. $\mathscr{A}'$ by the scalar products $\langle\cdot,\cdot\rangle_\tau$ and $\langle\cdot,\cdot\rangle_{\tau'}$ are equivalent.

Proof. Indeed,

$$\tau'(z^{-1}T) = \tau(z^{-1}F(T)) = \tau(E(T)) = \tau(T),$$

analogously for w. □

In the sequel, we will consider only the case where the assumptions of Proposition 2.28 are true, but our conclusions will also be true in general, if we consider only the scalar product $\langle\cdot,\cdot\rangle_{\tau'}$ on the endomorphisms. On the other hand, the scalar product $\langle\cdot,\cdot\rangle_\tau$ has been considered so far as the main and most natural scalar product on $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$; thus we wanted to address the question of their comparison.

6. Differential forms

In the previous section we have established the fundamental property following from the axioms for noncommutative manifolds; namely, the Fréchet $*$-algebra generated by $\mathcal{A}$ and $[D,\mathcal{A}]$ in $\mathcal{B}(\mathcal{H})$, coincides with the endomorphisms of the Hilbert bimodule $\mathcal{X} = \mathcal{H}_\infty$. In the commutative case it reflects precisely the statement that the endomorphisms of the spinors are given by the Clifford algebra.

Now, in the commutative case, differential forms can be built out of the Clifford algebra. As they are important for differential geometry, it would be desirable to have an analogous construction in the noncommutative case. This can be done with some algebraic considerations [**Con94**, V.1]. The outcome is as follows:

Proposition 2.30 ([**Con94**, Prop. V.1.4, Formula V.1.3])**.**

i) *The following equality defines a $*$-representation π of the reduced universal algebra $\Omega^*(\mathcal{A})$ on $\mathcal{H}$:*

$$\pi(a_0 da_1 \dots da_n) = a_0[D,a_1]\dots[D,a_n], \quad a_j \in A. \tag{2.5}$$

ii) *Let $J_0 = \ker\pi \subset \Omega^*(\mathcal{A})$ be the graded two-sided ideal of $\Omega^*(\mathcal{A})$ given by $J_0^{(k)} = \{\omega \in \Omega^k(\mathcal{A}) \,|\, \pi(\omega) = 0\}$; then $J := J_0 + dJ_0$ is a graded differential two-sided ideal of $\Omega^*(\mathcal{A})$. If we denote $\Omega_D^* := \Omega^*(\mathcal{A})/J$, then*

$$\Omega_D^k \cong \pi(\Omega^k)/\pi(d(J_0 \cap \Omega^{k-1}), \tag{2.6}$$

and it is an isomorphism of differential graded algebras;

iii) *If $(\mathcal{A},\mathcal{H},D) = (C^\infty(M), L^2(M,S), \partial_M)$, where M is a spin^c manifold of dimension d, $L^2(M,S)$ the L^2-sections of the spinor bundle, ∂_M the Dirac operator, then there is a canonical isomorphism of Ω_D^* with the (graded) de Rham algebra of differential forms acting by Clifford multiplication on spinors, and under this isomorphism*

$$\mathrm{Tr}_\omega(\eta_1\eta_2^*|D|^{-d}) = c(d)^{-1}\int_M \eta_1 \wedge *\eta_2, \tag{2.7}$$

where $c(d) = 2^{d-\lfloor d/2\rfloor}\pi^{d/2}\Gamma\left(\frac{d}{2}+1\right)$.

PROOF. The proof is taken from [**Con94**]. i) is a matter of direct computation. To prove ii), take $J = J_0 + dJ_0$ and observe that $dJ \subset J$, because d^2=0. Thus, we only need to show that J is an ideal. For that, take a homogeneous element $\omega \in J^{(k)}$ and decompose it as

$$\omega = \omega_1 + d\omega_2, \quad \omega_1 \in J_0 \cap \Omega^k,\ \omega_2 \in J_0 \cap \Omega^{k-1}$$

and let $\omega' \in \Omega^{k'}$. We compute

$$\omega\omega' = \omega_1\omega' + (d\omega_2)\omega' = \omega_1\omega' + d(\omega_2\omega') - (-1)^{k-1}\omega_2 d\omega' = \\ = \big(\omega_1\omega' + (-1)^k\omega_2 d\omega'\big) + d(\omega_2\omega'),$$

which establishes ii).

The proof of iii) is omitted. □

This proposition leads to the following construction. Each $\pi(\Omega^k(\mathcal{A}))$ is a closed sub-bimodule of the $\mathcal{A}$-bimodule $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$. Thus, if we want to represent our differential forms in the endomorphisms of $\mathcal{X}$, the following definition is natural:

Definition 2.31. We denote by $\Omega^k_{\mathcal{D}}$ the $\mathcal{A}$-orthogonal complement of $\pi(d(J_0 \cap \Omega^{k-1})$ in $\pi(\Omega^k)$:

$$\Omega^k_{\mathcal{D}} := \pi(\Omega^k) \cap \pi(d(J_0 \cap \Omega^{k-1}))^{\perp}$$

By construction, $\Omega^k_{\mathcal{D}}$ is a closed $\mathcal{A}$-sub-bimodule in $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$. We do not have any direct argument showing that it is finitely generated and projective (as a left or right $\mathcal{A}$-module) in general, although this holds in all examples known to the author. Moreover, as we will see below, its "von Neumann version" is finitely generated and projective in full generality.

Definition 2.32. We denote by $\Omega^k_{\mathscr{D}}$ the quasi-completion of $\Omega^k_{\mathcal{D}}$ in $\mathrm{End}_{\mathscr{B}}(\mathscr{X})$.

Proposition 2.33. *$\Omega^k_{\mathscr{D}}$ is an $\mathscr{A}$-$\mathscr{A}$ bimodule which is finitely generated and projective as a left $\mathscr{A}$-module as well as a right $\mathscr{A}$-module.*

PROOF. Denote by $\mathscr{E}_k$ the quasi-completions of $\pi(\Omega^k(\mathcal{A}))$ in $\mathrm{End}_{\mathscr{B}}(\mathscr{X})$. As $\mathrm{End}_{\mathscr{B}}(\mathscr{X})$ is a finitely generated and projective $\mathscr{A}$-module, $\mathscr{E}_k$ also are finitely generated and projective. Now, the orthogonal complement of an arbitrary submodule of finitely generated projective $\mathscr{A}$-module is finitely generated projective by [**MT05**, Lemma 3.6.1]. □

Definition 2.34. Let $\mathcal{H}_\Omega = L^2(\mathrm{End}_{\mathcal{B}}(\mathcal{X}))$ be the completion of $\mathrm{End}_{\mathcal{B}}(\mathcal{X})$ with respect to the following inner product:

$$\langle T_1, T_2\rangle_k := \tau(T_1^* T_2) \tag{2.8}$$

and let $\mathcal{H}_k$ be the closure of $\pi(\Omega^k(\mathcal{A}))$ in $\mathcal{H}_\Omega$.

Let P_k be the orthogonal projection of $\mathcal{H}_k$ onto the orthogonal complement of the subspace $\pi(d(J_0 \cap \Omega^{k-1}))$. We denote Λ_k the Hilbert space $P_k\mathcal{H}_k$. $\Lambda^0 \cong \mathcal{H}_0$ will be also denoted $L^2(\mathcal{A})$.

Let us make an easy but useful observation.

Proposition 2.35.

i) *the left and right actions of $\mathcal{A}$ on Λ^k define commuting $*$-representations.*
ii) *P_k are $\mathscr{A}$-bimodule maps: $P_k(a\xi b) = aP_k(\xi)b$, $a, b \in \mathscr{A}$.*
iii) *Suppose that $\pi(\Omega^k(\mathcal{A}))$ is an $\mathcal{A}$-$\mathcal{A}$-bimodule of finite type. The property $P_k(\Omega^k(\mathcal{A})) \subset \Omega^k(\mathcal{A})$ is equivalent to $\Omega^k_{\mathcal{D}}$ being an $\mathcal{A}$-$\mathcal{A}$-bimodule of finite type.*

Proof. i) and ii) were shown in [**Con94**, Prop. V.1.5]: the left and the right $\mathcal{A}$-actions on $\Omega^k_{\mathcal{D}}$ define $*$-representations and they descend to Λ^k because $\pi(d(J_0 \cap \Omega^{k-1}))$ is a sub-bimodule of $\pi(\Omega^k)$, which shows i) and ii). To show iii), first observe that by ii) P_k is a projection which commutes with the $\mathcal{A}$-action. Thus, if $P_k \in \mathrm{End}_{\mathcal{A}}(\Omega^k_{\mathcal{D}})$, then its image splits off as an orthogonal direct summand. On the other hand, if $\Omega^k_{\mathcal{D}}$ is a bimodule of finite type, then it splits off as a direct summand in $\Omega^k(\mathcal{A})$, and the projection onto it has to coincide with P_k; thus, $P_k(\Omega^k(\mathcal{A})) \subset \Omega^k(\mathcal{A})$. □

We will encounter the assumption of part iii) of the preceding proposition several times, and it is therefore natural to introduce a name for this property.

Definition 2.36. A subspace $\mathcal{W} \subset \pi(\Omega^*(\mathcal{A}))$ is called regular, if the projection $P_{\mathcal{W}}$ on its L^2-closure maps regular elements to regular elements, i.e. $P_{\mathcal{W}}(\pi(\Omega^*(\mathcal{A}))) = \mathcal{W}$.

7. Connections and curvature

In this section we define the notion of a (Clifford) connection on a Fréchet pre-Hilbert module over $\mathcal{A}$. There is a notion of an abstract connection on a module over an algebra $\mathcal{A}$ using the notion of abstract differential forms over $\mathcal{A}$, but in differential geometry one usually encounters another type of connection, namely, a connection viewed as an operator sending sections of a vector bundle to differential forms with values in this vector bundle. Thus, having defined differential forms in our context, we are naturally led to the notion of a "differential-geometric" connection. The reference for this section is [**Con94**].

Definition 2.37. Let $\mathcal{E}$ be Fréchet right pre-Hilbert module over $\mathcal{A}$. A connection on $\mathcal{E}$ is a linear operator

$$\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}(\mathcal{A})$$

which satisfies the Leibniz rule

$$\nabla(\xi a) = \xi \otimes [D, a] + (\nabla\xi)a, \quad \xi \in \mathcal{E},\ a \in \mathcal{A}.$$

Connections on $\mathcal{E}$ naturally form an affine space over $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}(\mathcal{A}))$.
We define for $\omega \in \Omega^1_{\mathcal{D}}(\mathcal{A})$, $\xi, \eta \in \mathcal{E}$

$$\langle \xi \otimes \omega, \eta \rangle_{\Omega^1_{\mathcal{D}}} := \omega^* \langle \xi, \eta \rangle_{\mathcal{A}},$$

$$\langle \xi, \eta \otimes \omega \rangle_{\Omega^1_{\mathcal{D}}} := \langle \xi, \eta \rangle_{\mathcal{A}} \omega.$$

This defines a pairing between $\mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}(\mathcal{A})$ with values in $\Omega^1_{\mathcal{D}}(\mathcal{A})$.

Definition 2.38. A connection on $\mathcal{E}$ is said to be metric-compatible iff

$$\langle \nabla\xi, \eta \rangle_{\Omega^1_{\mathcal{D}}} - \langle \xi, \nabla\eta \rangle_{\Omega^1_{\mathcal{D}}} = -d \langle \xi, \eta \rangle.$$

Metric-compatible connections on $\mathcal{E}$ naturally form an affine space over $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}(\mathcal{A}))_{sa}$, where

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}(\mathcal{A}))_{sa} = \left\{ T \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}(\mathcal{A})) \,|\, \forall \xi, \eta \in \mathcal{E}\ \langle T\xi, \eta \rangle_{\Omega^1_{\mathcal{D}}} = \langle \xi, T\eta \rangle_{\Omega^1_{\mathcal{D}}} \right\}.$$

Definition 2.39. Let $\mathcal{E}$ be Fréchet left pre-Hilbert module over $\mathcal{A}$. A connection on $\mathcal{E}$ is a linear operator

$$\nabla\colon \mathcal{E} \to \Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$

which satisfies the Leibniz rule

$$\nabla(a\xi) = [D, a] \otimes \xi + a(\nabla\xi), \quad \xi \in \mathcal{E},\ a \in \mathcal{A}.$$

We define for $\omega \in \Omega^1_{\mathcal{D}}(\mathcal{A})$, $\xi, \eta \in \mathcal{E}$

$$_{\Omega^1_{\mathcal{D}}}\langle \omega \otimes \xi, \eta \rangle := \omega\, _{\mathcal{A}}\langle \xi, \eta \rangle,$$

$$_{\Omega^1_{\mathcal{D}}}\langle \xi, \omega \otimes \eta \rangle := {}_{\mathcal{A}}\langle \xi, \eta \rangle \omega^*.$$

This defines a pairing between $\mathcal{E}$ and $\Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ with values in $\Omega^1_{\mathcal{D}}(\mathcal{A})$.

Definition 2.40. A connection on $\mathcal{E}$ is said to be metric-compatible iff

$$_{\Omega^1_{\mathcal{D}}}\langle \nabla\xi, \eta \rangle - {}_{\Omega^1_{\mathcal{D}}}\langle \xi, \nabla\eta \rangle = -d\, \langle \xi, \eta \rangle\, .$$

Given a connection ∇ on a right module $\mathcal{E}$, there is a unique continuation of ∇ to an operator

$$\widetilde{\nabla}\colon \mathcal{E} \otimes_{\mathcal{A}} \Omega^*_{\mathcal{D}} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^*_{\mathcal{D}}$$

given on $\mathcal{E} \otimes_{\mathcal{A}} \Omega^k_{\mathcal{D}}$ by

$$\widetilde{\nabla}(\xi \otimes \omega) := (\nabla\xi)\omega + (-1)^k \xi \otimes d\omega.$$

Definition 2.41. The curvature of the (right) connection ∇ is the operator $\widetilde{\nabla}^2\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D$. It is a right $\mathcal{A}$-homomorphism: $\widetilde{\nabla}^2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D)$.

Definition 2.42. Let $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{A}$-bimodule with fixed isomorphism of $\mathcal{A}$-$\mathcal{A}$-bimodules $\sigma\colon \Omega^*_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^*_{\mathcal{D}}$. A biconnection on $\mathcal{E}$ (with respect to σ) is a right connection ∇ such that

i) $\sigma \circ \nabla$ is a left $\mathcal{A}$-connection,
ii) $\widetilde{(\sigma \circ \nabla)} = \sigma \circ \widetilde{\nabla} \circ \sigma^{-1}$.

The curvature of a biconnection is a bimodule morphism:

$$\nabla^2 \in {}_{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D).$$

The unitary group of the endomorphisms of $\mathcal{E}$ acts on connections as follows. Let $u \in \mathcal{U}(\mathrm{End}_{\mathcal{A}}(\mathcal{E}))$. Then u naturally acts on $\Omega^1_D \otimes_{\mathcal{A}} \mathcal{E}$ as $1 \otimes u$, and this induces an action on connections

$$\nabla \mapsto u\nabla u^*.$$

The curvature $F_\nabla = \widetilde{\nabla}^2$ transforms under this action as

$$F_{u\nabla u^*} = uF_\nabla u^*.$$

Remark 2.43. Let $\mathcal{E}$ is an $\mathcal{A}$-$\mathcal{A}$-bimodule such that for all $k \in \mathbb{N}$

$$\Omega^k_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_{\mathcal{D}}$$

and

$$\Omega^k_{\mathcal{D}} \otimes_{\mathcal{A}} \overline{\mathcal{E}} \cong \overline{\mathcal{E}} \otimes_{\mathcal{A}} \Omega^k_{\mathcal{D}}$$

as $\mathcal{A}$-$\mathcal{A}$-bimodules, where $\overline{\mathcal{E}}$ is the conjugate bimodule, then we have following identifications:

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_{\mathcal{D}}) \cong \overline{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_{\mathcal{D}} \cong \Omega^k_{\mathcal{D}} \otimes_{\mathcal{A}} \overline{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{E}.$$

Thus, if $\overline{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{A}$, then $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_{\mathcal{D}}) \cong \Omega^k_{\mathcal{D}}$, the identifications being given by the rule

$$\omega \mapsto (\xi \mapsto \omega \otimes \xi).$$

8. Complex structures

A complex structure on a manifold M yields a bigrading of the algebra of differential forms on that manifold and a splitting of the differential d into two differentials: $d = \partial + \overline{\partial}$. In this section we define this sort of additional structure for noncommutative manifolds and discuss related notions.

Definition 2.44. A complex structure on the noncommutative manifold over the pair $(\mathcal{A}, \mathcal{B})$ is the decomposition of the graded involutive differential algebra $\Omega^*_D(\mathcal{A})$ as an associate to a bigraded differential algebra:

$$\Omega^n_D(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{(p,q)}_D(\mathcal{A}),$$

$$d = \partial + \overline{\partial},$$

where

$$\partial\colon \Omega^{(p,q)}_D(\mathcal{A}) \to \Omega^{(p+1,q)}_D(\mathcal{A})$$

and

$$\overline{\partial}\colon \Omega^{(p,q)}_D(\mathcal{A}) \to \Omega^{(p,q+1)}_D(\mathcal{A})$$

with

$$\partial(a)^* = \overline{\partial}(a^*)$$

and $*(\Omega^{(p,q)}_D(\mathcal{A})) = \Omega^{(q,p)}_D(\mathcal{A})$

Formally, this definition resembles only the case of an *almost complex structure*, but to our knowledge at present no satisfactory description of the integrability conditions in the noncommutative setting has been worked out; on the other hand, in the examples we have at hand the additional structure defined above will come from an honest complex structure.

The existence of a complex structure allows us to decompose connections and define the Cauchy–Riemann operators accociated with them.

Definition 2.45. Let $\nabla\colon \mathcal{E} \to \Omega^1_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{E}$ be a connection. The Cauchy–Riemann operator associated with ∇ is the operator

$$\overline{\partial}_\nabla := \pi^{(0,1)} \circ \nabla,$$

where $\pi^{(0,1)}$ is the projection onto the $(0,1)$-forms.

Definition 2.46. The connection $\nabla\colon \mathcal{E} \to \Omega^1_{\mathcal{D}} \otimes \mathcal{E}$ is said to be holomorphic (or induce a holomorphic structure on $\mathcal{E}$) if its the Cauchy–Riemann operator satisfies

$$\overline{\partial}^2_\nabla = 0.$$

The following proposition is straightforward.

Proposition 2.47. *A connection ∇ is holomorphic if and only if the $(0,2)$-part of its curvature, $\pi^{(0,2)} \circ F_\nabla$, is equal to zero.*

9. Laplacian on functions and differential forms

After we have defined the differential forms, it is natural to pose the question whether the usual properties of the exterior derivative and its adjoint hold in our situation. In this section we assume that all $\pi(\Omega^k_{\mathcal{D}})$ and $\Omega^k_{\mathcal{D}}$ are regular, and hence $\mathcal{A}$-$\mathcal{A}$-bimodules of finite type.

Consider the following sequence of spaces:

$$\Lambda_0 \xrightarrow{d_0} \Lambda_1 \xrightarrow{d_1} \cdots \to \Lambda_k \xrightarrow{d_k} \Lambda_{k+1} \xrightarrow{d_{k+1}} \dots, \tag{2.9}$$

where the differentials correspond to the restrictions of d to $\Omega^k_{\mathcal{D}}$. This is a chain complex of Hilbert spaces with densely defined closable differentials.

Let $d_k^* : D(d_k^*) \to \mathcal{H}_k$ be the adjoints of d_k; they are closed densely defined operators. We consider the Laplacian

$$\Delta_k := d_k^* \circ d_k + d_{k-1} \circ d_{k-1}^*. \tag{2.10}$$

It is a selfadjoint unbounded operator on $\mathcal{H}_k$.

Now we establish generalizations of some classical results about the Laplacian. These will be obtained in some steps: first, one has to analyze the connection between the Laplacian and the square of D, which will give the right properties of Δ.

Now we want to compare the square of the Dirac operator with the Laplacian on $L^2(\mathcal{A})$. For this, we define the following operator:

$$\nabla_e : \mathcal{H}^\infty \cong \mathcal{A}^n e \to \Omega_D^1(\mathcal{A})^n e \cong \Omega_D^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}^\infty,$$

$$(a_1, \dots, a_n) \mapsto ([D, a_1], \dots, [D, a_n])\, e \tag{2.11}$$

It is obviously a closable operator, and therefore it has an adjoint ∇_e^*. Observe that ∇_e is a connection on $\mathcal{X}$ compatible with the metric. It is called the Grassmanian connection associated to the isomorphism $\mathcal{X} \cong \mathcal{A}^n e$, whence the notation.

In order to compare D and $\Delta_e := \nabla_e^* \nabla_e$ we need to relate the connection and the "Dirac operator" D. This is done in the following proposition.

Proposition 2.48. *There is an odd*[1] *$\mathcal{A}$-endomorphism T of $\mathcal{H}^\infty$ such that $D = D' + T$, where*

$$D' = m \circ \nabla_e.$$

Hereby m denotes the Clifford multiplication

$$m \colon \Omega_{\mathcal{D}}^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{X} \to \mathcal{X},$$
$$a[D, b] \otimes \xi \mapsto a[D, b]\xi.$$

Proof. Take an $\mathcal{A}$-basis $\{u_i\}$ of $\mathcal{H}^\infty$ and set $T := D - m \circ \nabla_e$. For $\xi = a_i u_i$

$$T\xi = D\xi - m \circ \nabla_e \xi = [D, a_i]u_i + a_i D u_i - [D, a_i]u_i - a_i m \circ \nabla_e u_i = a_i T u_i,$$

from which the statement follows. □

Remark 2.49. If a connection is compatible with the metric (cf. Definition 2.40), the corresponding "Dirac operator" $D_\nabla := m \circ \nabla$ is selfadjoint.

$$\langle D_\nabla \xi, \eta \rangle = \langle m \circ \nabla \xi, \eta \rangle = \left\langle m(\sum \omega_i \otimes v_i), \eta \right\rangle =$$
$$= \tau(\omega_i\ {}_{\mathrm{End}_B(H)}\langle v_i, \eta \rangle) = \tau(\langle \nabla \xi, \eta \rangle) = \tau(\langle \xi, \nabla \eta \rangle) = \langle \xi, D_\nabla \eta \rangle .$$

This shows that the difference term T is selfadjoint as well.

Lemma 2.50. *The operator D' is selfadjoint and has compact resolvent.*

Proof. This is clear from the equality

$$(D' + \lambda)^{-1} = (D + T + \lambda)^{-1} = (1 + (D + \lambda)^{-1} T)^{-1} (D + \lambda)^{-1}.$$

with λ big enough to ensure that $1 + (D + \lambda)^{-1} T$ is invertible, and the fact that it is enough to establish that the resolvent is compact for one single value of $\lambda \in \mathbb{C}$ by Theorem 1.36. □

Proposition 2.51. *The operator $\Delta_e := \nabla_e^* \nabla_e$ is a selfadjoint perturbation of D'^2 which is bounded in the sense of quadratic forms. In particular, it is selfadjoint and has compact resolvent.*

[1] in the even-dimensional situation

PROOF. If $\{u_i\}$ is an $\mathcal{A}$-basis of $\mathcal{H}^\infty$ and $\xi = \sum_i a_i u_i$, then

$$D'\xi = \sum_i [D, a_i] u_i$$

and

$$\nabla\xi = \sum_i [D, a_i] \otimes u_i,$$

because for the Grassmann connection

$$\sum_i a_i \nabla_e u_i = 0,$$

if $(a_1, \ldots, a_n) = (a_1, \ldots, a_n)e$. Then

$$\langle D'^2\xi, \xi\rangle = \sum_{i,j} \tau \circ F({}_{\mathrm{End}_B(H)}\langle [D, a_i]u_i, [D, a_j]u_j\rangle) =$$

$$= \sum_{i,j} \tau \circ F([D, a_i]_{\mathrm{End}_B(H)}\langle u_i, u_j\rangle [D, a_j]^*)$$

$$\langle \nabla_e\xi, \nabla_e\xi\rangle = \sum_{i,j} \tau \circ F([D, a_i]_A\langle u_i, u_j\rangle [D, a_j]^*)$$

Now, we observe that ${}_A\langle u_i, u_j\rangle = E\left({}_{\mathrm{End}_B(H)}\langle u_i, u_j\rangle\right)$. Thus, by Proposition 2.27

$$\|D'\xi\|^2 - \|\nabla_e\xi\|^2 \leqslant \left(1 - \frac{1}{L}\right)\|D'\xi\|^2 < \alpha^2 \|D'\xi\|^2$$

for some $\alpha < 1$. Therefore the operator $D'^2 - \nabla_e^*\nabla_e$ is a selfadjoint perturbation of D'^2 which is bounded in the sense of quadratic forms, and by Theorem 1.38, $\nabla_e^*\nabla_e$ is a selfadjoint operator with compact resolvent. □

Proposition 2.52. *Let $\mathcal{E}$ be a finitely generated projective Hermitian left $\mathcal{A}$-module, let $\nabla\colon \mathcal{E} \to \Omega^1_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{E}$ be a metric compatible connection on $\mathcal{E}$. Suppose that there is a connection ∇_e on $\mathcal{E}$ such that $\nabla_e^*\nabla_e$ has compact resolvent as a densely defined operator on $L^2(\mathcal{E})$. Then $\nabla^*\nabla$ is also a densely defined operator with compact resolvent.*

PROOF. The connection ∇ differs from the Grassmann connection by an element $T \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^1_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{E})$, which is bounded, and therefore we are done by Lemma 1.40. □

Theorem 2.53. *The Laplace operator on functions $\Delta := d^*d$ is an unbounded selfadjoint operator with compact resolvent.*

PROOF. As $\mathcal{X}$ is a full pre-Hilbert module over $\mathcal{A}$, there exists a $k \in \mathbb{N}$ such that $\mathcal{A}$ is a direct summand of $\bigoplus_{i=1}^k \mathcal{X}$. Consider this inclusion ι and the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \iota\ } & \bigoplus_{i=1}^k \mathcal{X} \\ \big\downarrow d & & \big\downarrow \nabla \\ \Omega^1_D(\mathcal{A}) & \xrightarrow{\ \iota\ } & \bigoplus_{i=1}^k \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{X} \end{array}$$

As the compression of an operator with compact resolvent is an operator with compact resolvent by Lemma 1.41, the claim follows. □

Corollary 2.54. *Let $\mathcal{E}$ be a finitely generated projective Hermitian $\mathcal{A}$-module. Consider any metric compatible connection $\nabla\colon \mathcal{E} \to \Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$. Then $\nabla^*\nabla$ is a selfadjoint operator with compact resolvent.*

Proof. In view of Lemma 1.40, it is sufficient to consider the Grassmann connection. In this case the statement follows using Lemma 1.41. ☐

To prove the statement about the Laplacian on forms, we need to make some preparatory observations and additional properties.

Lemma 2.55. *The following formulas are valid for* $\omega \in \operatorname{Dom} d_k^*$*:*

$$d_{k-1}^*(a\omega) = P_{k-1}([D,a]\omega) + a d_{k-1}^*\omega,$$

$$d_{k-1}^*(\omega \cdot a) = P_{k-1}(\omega[D,a]) + (-1)^{k-1} d_{k-1}^*\omega \cdot a.$$

Proof. For an arbitrary $\eta \in \Omega_{\mathcal{D}}^k(\mathcal{A})$ the following holds:

$$\begin{aligned}\left\langle d_{k-1}^*(a\omega), \eta \right\rangle &= \langle a\omega, d_{k-1}\eta\rangle = \tau(a\omega \cdot (d_{k-1}\eta)^*) = \tau(\omega \cdot (a^* d_{k-1}\eta)^*) = \\ &= \langle \omega, a^* d_{k-1}\eta\rangle = \langle \omega, d_{k-1}(a^*\eta)\rangle - \langle \omega, P_k([D,a^*]\eta)\rangle = \\ &= \tau(a d_{k-1}^*\omega \cdot \eta^*) + \langle P_{k-1}([D,a]\omega), \eta\rangle = \left\langle a d_{k-1}^*\omega, \eta\right\rangle + \langle P_{k-1}([D,a]\omega), \eta\rangle .\end{aligned}$$

The proof of the second statement is analogous. ☐

Next we introduce the following definition.

Definition 2.56. We say that the differential calculus on a noncommutative manifold is Clifford-like if the following conditions are satisfied for all $k \in \mathbb{N}$: $\Omega_{\mathcal{D}}^k$ are $\mathcal{A}$-$\mathcal{A}$-bimodules of finite type,

$$\pi(\Omega^k(\mathcal{A})) \subset \pi(\Omega^{k+2}(\mathcal{A})),$$

and

$$\Omega_{\mathcal{D}}^{k+2} = \pi(\Omega^k(\mathcal{A}))^{\perp} \cap \pi(\Omega^{k+2}(\mathcal{A})).$$

Theorem 2.57. *Suppose that the k-forms* $\Omega_{\mathcal{D}}^k$ *constitute a finitely generated projective left* $\mathcal{A}$*-module, and the differential forms are Clifford-like. Then the operator*

$$\Delta_k := d_k^* \circ d_k + d_{k-1} \circ d_{k-1}^* \colon \Lambda^k \to \Lambda^k$$

is selfadjoint and has compact resolvent.

Proof. Take a basis $\{\omega_1, \ldots, \omega_\ell\}$ for $\Omega_{\mathcal{D}}^k$ and consider the Grassmann connection

$$\nabla \colon \Omega_{\mathcal{D}}^k \to \Omega_{\mathcal{D}}^1 \otimes_{\mathcal{A}} \Omega_{\mathcal{D}}^k$$

associated with this basis. For an arbitrary form

$$\omega = \sum a_i \omega_i$$

we obtain

$$\nabla\omega = \sum [D, a_i] \otimes_{\mathcal{A}} \omega_i$$

whereas

$$d\omega = \sum [D, a_i]\omega_i + \sum a_i d\omega_i$$

and

$$d^*\omega = P_{k-1}\left(\sum [D, a_i]\omega_i\right) + \sum a_i d^*\omega_i$$

in view of Lemma 2.55.

Now we use the same technique as before: we perturb the operators d and d^* such that it doesn't affect the compactness of the resolvent of $d \oplus d^*$ and compare this new operator with $\nabla^*\nabla$. To be more specific, we consider the densely defined operators

$$d' \colon \Lambda^k \to \Lambda^{k+1},$$

$$\sum a_i\omega_i \mapsto P_{k+1}\left(\sum [D, a_i]\omega_i\right)$$

and

$$\delta' : \Lambda^k \to \Lambda^{k-1},$$

$$\sum a_i\omega_i \mapsto P_{k-1}\left([D, a_i]\omega_i\right).$$

These operators are well-defined because of the following: if $\omega = \sum a_i\omega_i = 0$, then $\nabla\omega = \sum [D, a_i] \otimes_{\mathcal{A}} \omega_i = 0$.

It is easy to see that $R := d - d' \in \mathrm{Hom}_{\mathcal{A}}(\Omega^k_{\mathcal{D}}, \Omega^{k+1}_{\mathcal{D}})$ and $S := d^* - \delta' \in \mathrm{Hom}_{\mathcal{A}}(\Omega^k_{\mathcal{D}}, \Omega^{k-1}_{\mathcal{D}})$. Thus, by Theorem 1.38, $(d \oplus d^*)^*(d \oplus d^*)$ has compact resolvent iff $(d' \oplus \delta')^*(d' \oplus \delta')$ does.

Now we make use of the formulae

$$\|\nabla\omega\|^2 = \left\|\sum [D, a_i] \otimes_{\mathcal{A}} \omega_i\right\|^2 = \sum_{i,j} \tau\left([D, a_i]\,{}_A\langle\omega_i, \omega_j\rangle [D, a_j]^*\right)$$

and

$$\|(d' \oplus \delta')\omega\|^2 = \left\|\sum [D, a_i]\omega_i\right\|^2 = \sum_{i,j} \tau\left([D, a_i]\,{}_{\mathrm{End}_B(X)}\langle\omega_i, \omega_j\rangle [D, a_j]^*\right).$$

The second formula follows from the assumption that the differential forms are Clifford-like. Indeed, in this case $\omega \in \Omega^k_{\mathcal{D}}$ implies that $\omega \perp \pi(\Omega^{k-2j}(\mathcal{A}))$ for $j \in \mathbb{N}^*$ and for each $a \in \mathcal{A}$ it follows that $[D, a]\omega \perp \pi(\Omega^{k-2j-1}(\mathcal{A}))$ for $j \in \mathbb{N}^*$.

The statement of the theorem follows with the same technique as before: we have ${}_A\langle\omega_i, \omega_j\rangle = E\left({}_{\mathrm{End}_B(H)}\langle\omega_i, \omega_j\rangle\right)$ and thus

$$\|\nabla\omega\|^2 - \|(d' \oplus \delta')\omega\|^2 \leqslant \left(1 - \frac{1}{L}\right)\|\nabla\omega\|^2 < \alpha^2 \|\nabla\omega\|^2$$

for some $\alpha < 1$. Therefore the operator $\nabla^*\nabla - (d' \oplus \delta')^*(d' \oplus \delta')$ is a relatively infinitely small selfadjoint perturbation of $(d' \oplus \delta')^*(d' \oplus \delta')$, and by Theorem 1.38, is a selfadjoint operator with compact resolvent. □

Corollary 2.58. *The "de Rham cohomology" of* $\mathcal{A}$

$$\mathsf{H}^k_d = \ker d_k / \operatorname{im} d_{k-1}$$

is isomorphic to the finite-dimensional space of harmonic forms:

$$\mathsf{H}^k_d = \ker \Delta_k.$$

10. Sobolev topologies and elliptic regularity

10.1. Sobolev topologies. In the previous section we have observed that certain naturally arising operators acting on the space $\mathcal{X}$ and finitely generated projective modules over $\mathcal{X}$ differ by a "relatively bounded perturbation". Bearing in mind the analogy to the commutative case, where the Laplacians are used to define Sobolev norms on $\mathcal{A}$ and vector bundles, one may ask what happens with the Sobolev topologies in our general case.

First of all we define the Sobolev topology of order $s \in \mathbb{N}$ on $\mathcal{X}$ as the topology given by the scalar product

$$\langle \xi, \eta \rangle_s := \left\langle (1 + D^2)^s \xi, \eta \right\rangle.$$

We denote by $\mathcal{H}^s$ or $H^s(\mathcal{X})$ the completion of $\mathcal{X}$ with respect to this topology and call it the s-th Sobolev space of spinors. It is obviously equal to $\mathrm{Dom}\,|D|^s$.

We introduce the *order* of operators acting on $\mathcal{X}$:

Definition 2.59. A linear operator $T\colon \mathcal{X} \to \mathcal{X}$ is said to be of order (at most) $\alpha \in \mathbb{Z}$ iff for each s it extends to a continuous operator

$$T\colon \mathcal{H}^s \to \mathcal{H}^{s-\alpha}.$$

We denote this by $\operatorname{ord}(T) \leqslant \alpha$.

Obviously, an operator of finite order is continuous as an operator $\mathcal{X} \to \mathcal{X}$, and $\operatorname{ord}(T)\operatorname{ord}(S) \leqslant \operatorname{ord}(T) + \operatorname{ord}(S)$, the inequality having the obvious meaning.

Proposition 2.60. *If a bounded operator $T\colon \mathcal{X} \to \mathcal{X}$ is regular, i.e. $T \in \bigcap_n \operatorname{Dom}(\delta^n)$, then T is of order 0.*

Proof. Obvious using the equality

$$|D|^m T\xi = \sum_{k=0}^{m} \binom{m}{k} \delta^k(T)|D|^{m-k}\xi, \quad \xi \in \operatorname{Dom}|D|^m.$$

□

Lemma 2.61. *If T is a selfadjoint operator of order 1 such that D^2+T is positive, then the topology on $\mathcal{X}$ defined by the scalar product*

$$\langle \xi, \eta\rangle'_s := \big\langle (1 + D^2 + T)^s \xi, \eta\big\rangle$$

coincides with the Sobolev topology of order s.

Proof. Let $s = 2k$, $k \in \mathbb{N}$. Then

$$\langle \xi, \xi\rangle'_s = \big\|(1 + D^2 + T)^k \xi\big\|^2 \leqslant C \left\|\xi\right\|_{2k}^2 = C \left\langle (1 + D^2)^s \xi, \xi\right\rangle$$

for some $C > 0$.

Let now $s = 2k + 1$, $k \in \mathbb{N}$. Denote $Q = \sqrt{1 + D^2 + T}$. Then

$$\|Q\xi\|^2 = \big\langle (1 + D^2 + T)\xi, \xi\big\rangle \leqslant \|\xi\|_1^2 + \|T\xi\| \,\|\xi\| \leqslant (\|T\|_{1\to 0} + 1) \,\|\xi\|_1^2 ,$$

where $\|T\|_{1\to 0}$ is the norm of the operator $T\colon H^1 \to H^0$. Thus, $\|Q\|_{1\to 0} < \infty$, and

$$\langle \xi, \xi\rangle'_s = \big\|Q(1 + D^2 + T)^k \xi\big\|^2 \leqslant \|Q\|_{1\to 0}^2 \big\|(1 + D^2 + T)^k \xi\big\|_1^2 \leqslant C \,\|Q\|_{1\to 0}^2 \,\|\xi\|_s^2 .$$

On the other hand,

$$\|\xi\|_1^2 = \big\langle (1 + D^2)\xi, \xi\big\rangle \leqslant \|Q\xi\|^2 + |\left\langle T\xi, \xi\right\rangle| \leqslant \|\xi\|_1'^2 + \|\xi\|_1 \,\|\xi\|_1' ,$$

and it follows that $\|\xi\|_1^2 \leqslant \gamma \,\|\xi\|_1'^2$. For suppose the contrary and choose a sequence ξ_n with $\|\xi_n\|_1^2 \to \infty$ and $\|\xi_n\|_1'^2 = 1$. Then

$$\|\xi_n\|_1 \leqslant 1 + \frac{1}{\|\xi_n\|_1} \to 1,$$

yielding a contradiction.

Let us now argue by induction. If $s = 2k$,

$$\langle \xi, \xi\rangle_{2k} = \big\|(1 + D^2)^k \xi\big\|^2 = \big\|(1 + D^2 + T - T)^k \xi\big\|^2 \leqslant \|\xi\|_{2k}'^2 + C' \,\|\xi\|_{2k-1}^2 .$$

and if $s = 2k + 1$,

$$\begin{aligned} &\langle \xi, \xi\rangle_{2k+1} = \big\langle (1 + D^2 + T - T)(1 + D^2)^k \xi, (1 + D^2)^k \xi\big\rangle \\ &\leqslant \big\|(1 + D^2 + T - T)^k \xi\big\|_1^2 + \|T\|_{1\to 0}^2 \,\|\xi\|_{2k}^2 \leqslant \|\xi\|_{2k+1}'^2 + \gamma^2 \,\|\xi\|_{2k}^2 \leqslant C'' \,\|\xi\|_{2k+1}'^2 .\end{aligned}$$

for some $C', C'' > 0$. This finishes the proof. □

We recall from Proposition 2.20 that our algebra $\mathcal{A}$ as well as the space $\mathrm{End}_{\mathcal{B}}$ is complete when endowed with the topology given by the family of seminorms

$$\|a\|_s = \left(\sum_i \left\| (1+D^2)^{s/2} a u_i \right\|_2 \right),$$

where $\{u_i\}$ is a right $\mathcal{B}$-basis for $\mathcal{X}$. Equivalently, it is complete when endowed with the topology given by the family of scalar products

$$\langle T, S \rangle_s = \sum_i \left\langle (1+D^2)^s T^* S u_i, u_i \right\rangle .$$

If R is a regular selfadjoint operator, then the topology given by $\langle \cdot, \cdot \rangle_s$ coincides with the topology given by the scalar product

$$\langle T, S \rangle'_s = \sum_i \left\langle (1+(D+R)^2)^s T^* S u_i, u_i \right\rangle ,$$

in view of Lemma 2.61.

Bearing in mind the result that the "connection Laplacian" $\nabla^* \nabla$ on $\mathcal{X}$ is not very far from D^2, we would like to investigate the family of scalar products on $\mathcal{A}$ given by

$$\langle b, a \rangle_{\nabla, s} = \sum_i \langle (1+\nabla^*\nabla)^s b^* a u_i \rangle ,$$

where ∇ is a metric compatible connection on $\mathcal{X}$.

To do this, we introduce the following definition.

Definition 2.62. We say that the connections on $\mathcal{X}$ are compatible with the order calculus if for any metric compatible connection

$$\nabla \colon \mathcal{X} \to \Omega^1_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{X}$$

the operator $\nabla^*\nabla \colon \mathcal{X} \to \mathcal{X}$ is an order one perturbation of D^2:

$$\nabla^*\nabla = D^2 + T,$$

where T is of order 1.

Remark 2.63. It is reasonable to expect this condition to hold in general for noncommutative manifolds, and it indeed does hold in known examples; however, we are not aware of any proof of such a statement in general.

Proposition 2.64. *Suppose that connections on $\mathcal{X}$ are compatible with the order calculus. Consider the family of norms on $\mathcal{A}$ defined by the scalar products*

$$\langle a, a \rangle_s = \langle (1+\Delta)^s a, a \rangle .$$

Then $\mathcal{A}$ is complete with respect to the locally convex topology defined by all these norms, and this topology on $\mathcal{A}$ coincides with the topology given by the s-th Sobolev scalar product.

Proof. Consider the right $\mathcal{B}$-basis $\{u_1, \dots, u_n\} \subset \mathcal{X}$ involved in the definition of the Sobolev topology and the invertible element $z = \sum_{i=1}^n {}_{\mathcal{A}}\langle u_i, u_i \rangle$. Consider the closed $\mathcal{A}$-submodule $\mathcal{Y} \subset \oplus_{k=1}^n \mathcal{X}$ defined as the closure of the image of the map

$$\varphi \colon \mathcal{A} \to \oplus_{k=1}^n \mathcal{X},$$
$$1 \mapsto (z^{-1/2} u_1, \dots, z^{-1/2} u_n).$$

For $a, b \in \mathcal{A}$

$$\sum_{k=1}^n {}_{\mathcal{A}}\left\langle z^{-1/2} a u_k, z^{-1/2} b u_k \right\rangle = ab^*,$$

thus, the module $\mathcal{Y}$ is isomorphic to $\mathcal{A}$ via φ.

Thus, the Laplace operator Δ on $\mathcal{A}$ corresponds under this isomorphism to the connection Laplacian $\nabla^*\nabla$ on $\mathcal{Y}$ for some metric compatible connection ∇.

By assumption and in view of Lemma 2.61 the s-th Sobolev topology on $\mathcal{Y}$ is given by the scalar product

$$\langle \xi, \xi \rangle_s = \langle (1 + \nabla^*\nabla)^s \xi, \xi \rangle, \quad \xi \in \mathcal{Y},$$

which after using the isomorphism φ corresponds to the scalar product

$$\langle a, a \rangle_s = \langle (1 + \Delta)^s a, a \rangle$$

on $\mathcal{A}$. This proves the first claim.

Now, again using Lemma 2.61, we may define the s-th Sobolev topology on $\mathcal{Y}$ using the scalar product

$$\langle \xi, \xi \rangle_s = \sum_{k=1}^{n} \left\langle (1 + D^2)^s z^{-1/2} u_k, z^{-1/2} u_k \right\rangle = \sum_{k=1}^{n} \left\| z^{-1/2} u_k \right\|_s^2 .$$

As z is regular, the topology defined by this scalar product coincides with the s-th Sobolev topology. □

Corollary 2.65. *If connections on $\mathcal{X}$ are compatible with the order calculus, then all elements in* $\ker \Delta \subset L^2(\mathcal{A})$ *belong to* $\mathcal{A}$.

Corollary 2.66. *Let $\mathcal{E}$ be a finitely generated projective Hermitian $\mathcal{A}$-module. Consider any metric compatible connection $\nabla\colon \mathcal{E} \to \Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$. Consider the topology on $\mathcal{E}$ given by the scalar product*

$$\langle \xi, \xi \rangle_s = \langle (1 + \nabla^*\nabla)^s \xi, \xi \rangle .$$

Then this topology coincides with the topology inherited from $\mathcal{A}$ using the isomorphism $\mathcal{E} \cong \mathcal{A}^n p$ for $p = p^* = p^2 \in \mathbb{M}_n(\mathcal{A})$.

Proof. This follows from the fact $\mathcal{E}$ can be considered as a closed submodule of a direct sum of several copies of $\mathcal{X}$. □

Corollary 2.67. *In the assumptions of the Theorem 2.57 and if the connections on $\mathcal{X}$ are compatible with the order calculus, then the following holds true. Consider the family of scalar products on $\Omega^k_{\mathcal{D}}(\mathcal{A})$ defined by*

$$\langle \omega, \eta \rangle_s := \langle (1 + \Delta_k)^s \omega, \eta \rangle .$$

Then the topology induced by this scalar product coincides with the Sobolev topology induced on $\Omega^k_{\mathcal{D}}$ as on a finitely generated projective module. In particular, $\Omega^k_{\mathcal{D}}(\mathcal{A})$ is complete with respect to the locally convex topology defined by these scalar products.

Proof. Using the proof of Theorem 2.57, one observes that the Laplace operator Δ_k is an order one perturbation of $\nabla^*\nabla$ (with respect to the Sobolev topology inherited from the structure of a finitely generated projective module). Now, Lemma 2.61 gives the statement. □

Corollary 2.68. *In the assumptions of the previous Corollary, all elements in* $\ker \Delta_k \subset \Lambda^k$ *are regular, i.e. belong to $\Omega^k_{\mathcal{D}}$.*

It is natural to call the topologies given by the above scalar products *Sobolev topologies* on the corresponding spaces (finitely generated modules, differential forms etc.). This explains the notion of *reasonable Sobolev theory*: all "reasonable" definitions of Sobolev topologies coincide. Similarly to the definitions above, we may then introduce Sobolev spaces, denoting by $H^s(\mathcal{A})$ the Hilbert space completion of $\mathcal{A}$ with respect to the s-th Sobolev topology. Notice that these norms induce the corresponding Sobolev topology on any finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and endomorphisms of any finitely generated projective module over $\mathcal{A}$; we denote the corresponding completions by $H^s(\mathcal{E})$ and $H^s(\mathrm{End}_{\mathcal{A}}(\mathcal{E}))$. It then makes sense to speak about orders of operators on these Sobolev spaces as we did above for $H^s(\mathcal{X})$, and Lemma 2.61 remains applicable in this context (with D^2 replaced by Δ or $\nabla^*\nabla$).

10.2. Elliptic regularity. In the development of the Seiberg-Witten gauge theory we will use elliptic regularity in its abstract form, given by the following simple observation.

Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces and $A\colon \mathcal{H} \to \mathcal{K}$ an unbounded closable operator, let $\mathcal{H}_1 = \mathrm{Dom}\,\overline{A}$. Consider the equation

$$A\psi = \eta$$

and observe that

$$\langle (1 + A^*A)\psi, \psi\rangle = \|\psi\| + \|\eta\|\,.$$

Consider some scalar product $\langle\cdot,\cdot\rangle'$ on $\mathrm{Dom}\,\overline{A}$ equivalent to that naturally given by the above scalar product $\langle (1 + A^*A)\psi, \eta\rangle$ making it to a Hilbert space $\mathcal{H}_1$. It follows that we can *estimate* the norm of ψ in $\mathcal{H}_1$ in terms of the norms of ψ and $A\psi$ on $\mathcal{H}$:

$$\|\psi\|' \leqslant C(\|A\psi\| + \|\psi\|).$$

Using Lemma 2.61, we will use this principle in following cases:

i) $\mathcal{H} = H^s(\mathcal{X})$, the s-th Sobolev space of spinors, $A = D + T$, where T is a perturbation of order (at most) 1; $\mathcal{H}_1 = H^{s+1}(\mathcal{X})$,
ii) $\mathcal{H} = H^s(\Omega^k_{\mathcal{D}})$, the Sobolev space of k-forms, $\mathcal{K} = H^s(\Omega^{k+1}_{\mathcal{D}}) \oplus H^s(\Omega^{k-1}_{\mathcal{D}})$, $A = d + d^* + T$, where T is a perturbation of order (at most) 1,

concluding that the solution of the equation $A\psi = \eta$ lies in the $(s + 1)$-th Sobolev space provided that η lies in H^s.

11. Poincaré duality and cyclic homology

In this section we discuss some maps which naturally arise in our context and connect the cohomology of the differential algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ with the periodic cyclic cohomology of $\mathcal{B}$ (hereby $\mathcal{B}$ is considered as a $\mathbb{Z}/2$-graded algebra and we use the periodic cyclic cohomology for $\mathbb{Z}/2$-graded algebras as in [**Kas86**]). The constructions of the relevant maps is due to A. Connes [**Con94**, VI.4.γ]; however, nothing more was known about these maps. We investigate this question more thoroughly. Unfortunately, several technical conditions come into play here and the result we obtain, although reflecting the situation in known examples, is not completely satisfactory.

In this section we will assume that our noncommutative manifold satisfies the following closedness condition:

Definition 2.69. The noncommutative manifold is said to satisfy the closendess condition if for any $x_0, \dots, x_d \in \mathcal{A} \otimes \mathcal{B}$

$$\mathrm{Tr}_\omega(\gamma[D, x_0] \dots [D, x_d]|D|^{-d}) = 0.$$

Proposition 2.70. *The closedness condition is equivalent to the following:*

$$B\Phi = 0 \quad \textit{as a cochain,}$$

where Φ is the Hochschild cycle on $\mathcal{A}\widehat{\otimes}\mathcal{B}$ given by

$$\Phi(x^0, \dots, x^d) = \tau(\gamma x^0[D, x^1] \dots [D, x^d]).$$

Proof. This is a matter of direct computation. ∎

We recapitulate some lemmas from [**Con95**, VI.4] which give an interesting construction allowing to compare the space of "harmonic forms" with the periodic cyclic homology.

Lemma 2.71 ([**Con94**, Lemma VI.4.3])**.**

i) *For every* $k \leqslant d$ *and* $\alpha \in \Omega^k(\mathcal{A})$*, a Hochschild cocycle* $C_\alpha \in Z^{d-k}(\mathcal{B}, \mathcal{B}^*)$ *is defined by*

$$C_\alpha(b^0, \dots, b^{d-k}) = \tau(\gamma\pi(\alpha)b^0[D, b_1]\dots[D, b^{d-k}]).$$

ii) *If the closedness condition is satisfied, then* C_α *depends only on the class of* α *in* $\Omega^k_{\mathcal{D}}(\mathcal{A})$*, and we have*

$$B_0 C_\alpha = (-1)^k C_{d\alpha},$$

where $B_0\varphi(\omega) = \varphi(d\omega)$*.*

Proof. i) follows by direct computation:

$$bC_\alpha(b^0, \dots, b^{d-k+1}) = \tau(\gamma\pi(\alpha)b^0[D, b_1]\dots[D, b^{d-k}]b^{d-k+1}) - \\ - \tau(\gamma\pi(\alpha)b^{d-k+1}b^0[D, b_1]\dots[D, b^{d-k}]) = 0.$$

For ii), we consider the differential graded algebra $\Omega^*_{\mathcal{D}}(\mathcal{A}\otimes\mathcal{B})$. We observe that for $\alpha \in \Omega^d_{\mathcal{D}}(\mathcal{A}\otimes\mathcal{B})$, the value of $\tau(\gamma\pi(\alpha_0))$ doesn't depend on the choice of the representative α_0 of α and we denote this value by $\int \alpha$. $\int$ is a closed graded trace on $\Omega^*_{\mathcal{D}}(\mathcal{A}\otimes\mathcal{B})$. Using the natural homomorphism $\Omega^*_{\mathcal{D}}(\mathcal{A}) \to \Omega^*_{\mathcal{D}}(\mathcal{A}\otimes\mathcal{B})$, we see that C_α depends only on the class of α. By construction, the Hochschild cocycle C_α vanishes if one of b^j is equal to 1. Thus,

$$(B_0C_\alpha)(b^0, \dots, b^{d-k-1}) = \int \alpha db^0 \dots db^{d-k-1} = \\ = (-1)^k \int (d\alpha) b^0 db^1 \dots db^{d-k-1} = (-1)^k C_{d\alpha}(b^0, \dots, b^{d-k-1}).$$

□

Proposition 2.72 ([**Con94**, Prop. VI.4.4])**.**

i) *For* $0 \leqslant k \leqslant d$*, the mapping* $\alpha \to C_\alpha$ *is well-defined from* $\Omega^k_{\mathcal{D}}(\mathcal{A})$ *to the Hochschild cocycles* $Z^{d-k}(\mathcal{B}, \mathcal{B}^*)$*.*
ii) *The image under* C *of* $\ker d \subset \Omega^k_{\mathcal{D}}(\mathcal{A})$ *is contained in* $Z^{d-k}_\lambda(\mathcal{B})$*, i.e., if* $d\alpha = 0$*, then* C_α *is a cyclic cocycle.*
iii) *The image under* C *of* $\operatorname{im} d \subset \Omega^k_{\mathcal{D}}(\mathcal{A})$ *is contained in* $\operatorname{im} B$*, where* $B\colon \mathsf{HH}^{d-k+1} \to \mathsf{HC}^{d-k}(\mathcal{B})$ *is the cyclic cohomology operation.*
iv) C *defines a map* $\mathsf{H}^*(\Omega^*_{\mathcal{D}}) \to \mathsf{HP}^*(\mathcal{B})$*.*

Proof. The first assertion follows from the lemma. The second assertion follows from the lemma and the fact that a Hochschild cocycle C is cyclic iff $B_0C = 0$. The third assertion can be proved as follows: first, if $\alpha = d\beta$, then $d\alpha = 0$, and thus C_α is a cyclic cocycle. Then $C_\alpha = (-1)^{k-1}B_0C_\beta$ and, since C_α is cyclic, $AC_\alpha = (d-k+1)C_\alpha$, where A is the cyclic antisymmetrization; thus, $C_\alpha = \frac{(-1)^{k-1}}{d-k+1}BC_\beta$ belongs to the range of B. iv) follows from ii) and iii). □

In view of the results in the previous section, the spaces $\mathsf{H}^k_d(\Omega^*_{\mathcal{D}})$ are finite-dimensional, because they coincide with the kernels of the corresponding Laplace operator Δ_k on forms. It is a well-known result in differential geometry that in the case of a manifold these spaces are independent of the choice of the metric and coincide with the cohomology of the manifold (with complex coefficients).

Now, we analyze the mapping we have around. We observe that the mapping C defined above is well-defined on the level of $\Omega^k(\mathcal{A})$ and descends to $\Omega^k_{\mathcal{D}}(\mathcal{A})$. Therefore one source of degeneration is the kernel of C on the level of $\Omega^k_{\mathcal{D}}(\mathcal{A})$. We observe the following easy

Lemma 2.73. *Let $\omega \in \Omega^k(\mathcal{A})$ be in the image of the operator b: $\omega = b\eta$. Then $C_\omega = 0$.*

Proof. Obvious by order one condition and vanishing of the trace on commutators. ☐

Thus, we see that in the noncommutative case there is a major possibility for C to have a kernel which wasn't there in the commutative case, as there $\pi(b\eta) = 0$ for any $\eta \in \Omega^*(\mathcal{A})$. We will see later that this indeed happens in examples. Nevertheless, often it is the only condition which characterizes the kernel of the mapping C. To understand the situation better, let us introduce the following subspace:

Definition 2.74. We let $\mathcal{C}_k$ be the subspace of $\Omega^k_{\mathcal{D}}$ generated by commutators with $a \in \mathcal{A}$:

$$\mathcal{C}^k := \overline{\text{span}}\{[\omega, a] \,|\, \omega \in \Omega^k_{\mathcal{D}},\ a \in \mathcal{A}\}.$$

Obviously, $\mathcal{C}^k = \overline{\pi(b(\Omega^{k+1}(\mathcal{A})))}$.

Definition 2.75. We say that the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is graded-commutative modulo $\operatorname{im} b$, if the graded commutator of every two differential forms belongs to $\mathcal{C}$:

$$[\omega, \eta]_{\text{gr}} := \omega\eta - (-1)^{\deg\omega\cdot\deg\eta}\eta\omega \in \mathcal{C}^{\deg\eta+\deg\omega}.$$

Graded-commutativity modulo $\operatorname{im} b$ implies the following interesting property.

Proposition 2.76. *Suppose that the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is graded-commutative modulo $\operatorname{im} b$ and the subspace $\mathcal{C}$ is regular. Then the decomposition*

$$\Omega^*_{\mathcal{D}} = \mathcal{C}^{(k)} \oplus \mathcal{C}^{(k)\perp}$$

is d-invariant.

Proof. This follows from the formulas

$$d[\omega, a] = d[\omega, a]_{\text{gr}} = [d\omega, a] + [\omega, da]_{\text{gr}}.$$

and

$$d^*[\omega, a] = d^*[\omega, a]_{\text{gr}} = [d^*\omega, a] + P_{k-1}([\omega, [D, a]]_{\text{gr}}),$$

the latter being a consequence of Lemma 2.55. ☐

Theorem 2.77. *Suppose the following:*

i) *the class of the cocycle $[\Phi] \in \mathsf{HP}^*(\mathcal{A}\widehat{\otimes}\mathcal{B})$, $* = d \mod 2$, gives the fundamental class in periodic cyclic cohomology,*
ii) *the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A})$ is graded-commutative modulo $\operatorname{im} b$,*
iii) *$\mathcal{C}$ is a regular subspace of $\pi(\Omega^*_{\mathcal{D}})$,*
iv) *$\Omega^k_{\mathcal{D}}(\mathcal{A} \otimes \mathcal{B})$ is Clifford-like and for $k < d$, $\alpha \in \Omega^k(\mathcal{A} \otimes \mathcal{B})$, one has $\text{Tr}_\omega(\gamma\pi(\alpha)|D|^{-d}) = 0$,*
v) *the Hochschild dimension of $\mathcal{A}$ is at most d.*

Then the mapping $C\colon \mathsf{H}^*_d(\mathcal{A}) \to \mathsf{HP}^*(\mathcal{B})$ is surjective.

Proof. By assumption, every element in $\mathsf{HP}^*(\mathcal{B})$ is obtained from an element in $\mathsf{HC}_*(\mathcal{A})$ via the product with $[\Phi]$. From the condition on the Hochschild dimension we know that we can represent every element in $\mathsf{HP}_*(\mathcal{A})$ through its representative in $\mathsf{HC}_d(\mathcal{A})$ or $\mathsf{HC}_{d+1}(\mathcal{A})$, depending on the parity of d and $*$, i.e. as a sum of forms. If $* = 0$ and d is even, this sum is written as

$$\omega = \omega_0 \oplus \omega_2 \oplus \cdots \oplus \omega_d.$$

As ω represents a class in $\mathsf{HC}_d(\mathcal{A})$, $(b+B)\omega = 0$, or, equivalently, $B\omega_j = b\omega_{j+2}$ for $j = 0, 2, \ldots$. Writing $B = \sum_{i=0}^{j} d\kappa^i$ on $\Omega^j(\mathcal{A})$, we conclude that the classes of the forms

$$\omega'_j = \frac{1}{j}\sum_{i=0}^{j} \kappa^i \omega_j$$

in $\Omega^j_{\mathcal{D}}(\mathcal{A})$ satisfy $d\omega'_j \in C^{(j+1)}$.

Consider the decomposition $\Omega^j_{\mathcal{D}}(\mathcal{A}) = \mathcal{C}^{(j)} \oplus \mathcal{C}^{(j)\perp}$. It is d-invariant by the previous proposition, and thus we may conclude that the projections η_j of ω'_j onto $(C)^{(j)\perp}$ satisfy $d\eta_j = 0$.

By construction and Lemma 2.73, C_η coincides with the cocycle with components

$$\theta_{d-j}(b_0, \ldots, b_{d-j}) = \mathrm{Tr}_\omega(\pi(\omega_j) b_0 [D, b_1] \ldots [D, b_{d-j}]).$$

We have to compare this with $\Theta := [\omega] \cap [\Phi]$. The components of the cocycle Θ are given by the shuffle product and cyclic shuffle product:

$$\Theta_{d-j}(b_0, \ldots, b_{d-j}) = \Phi(sh(\omega_j, b_0 db_1 \ldots db_{d-j}) + sh'(\omega_{j-2}, b_0 db_1 \ldots db_{d-j})).$$

Now, by the closedness condition, Φ vanishes on any cyclic shuffle product, so the term with sh' vanishes. The remaining term is equal to

$$\Phi(sh(\omega_j, b_0 db_1 \ldots db_{d-j})).$$

By definition of the shuffle product, the factors of ω_j and $b_0 db_1 \ldots db_{d-j}$ appear in the same order as they did in ω_j and $b_0 db_1 \ldots db_{d-j}$. Thus all we have to do is to permute da's and db's in all the terms in a way to get the first term, which is equal to

$$\mathrm{Tr}_\omega(\pi(\omega_j) b_0 [D, b_1] \ldots [D, b_{d-j}]) = \mathrm{Tr}_\omega(\pi(\omega'_j) b_0 [D, b_1] \ldots [D, b_{d-j}]) = \\ = \mathrm{Tr}_\omega(\pi(\eta_j) b_0 [D, b_1] \ldots [D, b_{d-j}]).$$

Observe that as a commutes with $[D, b]$ and b commutes with $[D, a]$, we only get interesting terms when we permute $[D, a]$ with $[D, b]$. Consider now the differential forms over $\mathcal{A} \otimes \mathcal{B}$ and observe that

$$[D, a \otimes b] = a[D, b] + b[D, a]$$

by the order one condition. Thus, the class of $[D, a][D, b] + [D, b][D, a]$ in $\Omega^2_D(\mathcal{A} \otimes B)$ vanishes, and we get that

$$[D, a][D, b] + [D, b][D, a] = f \in \mathcal{A} \otimes \mathcal{B}.$$

Therefore, all additional terms we get from commutators vanish after taking Tr_ω by the condition iv). This finishes the proof. □

Unfortunately, in the preceding theorem there are many technical conditions we couldn't get rid of yet. However, we expect the condition on the Hochschild dimension and vanishing of the trace to follow in general from the axioms.

We also observe the following simple fact connecting the property of Φ to be the fundamental class and the orientability condition.

Proposition 2.78. *Suppose that the class $[\Phi] \in \mathsf{HP}^*(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}})$ of the cocycle Φ is the fundamental class in periodic cyclic cohomology and the orientability condition is satisfied. Then the Hochschild cycle c from the orientability condition is the dual of the trace τ on $\mathcal{B}$.*

PROOF. By assumption, the Hochschild cycle c is antisymmetric, i.e. $\lambda c = c$; thus, it gives a nontrivial element in $\mathsf{HC}_d(\mathcal{A}) \cong \mathsf{HP}_d(\mathcal{A})$. As $\pi(c) = \gamma$,

$$\mathrm{Tr}_\omega(\gamma\pi(c)b|D|^{-d}) = \mathrm{Tr}_\omega(b|D|^{-d}), \quad b \in \mathcal{B},$$

which proves the assertion. □

It is natural to ask, whether the converse is true: does the condition "Φ gives the fundamental class in HP" imply the orientability condition? This seems plausible, because one can look at the dual of the trace τ on $\mathcal{B}$, obtaining some class in $\mathsf{HP}_*(\mathcal{A})$, whose representative in $\mathsf{HC}_d(\mathcal{A})$ is a natural candidate for the cycle c. However, at present we don't have any argument which would show this.

12. Twisting by bimodules

On a Riemannian spin^c manifold M there are in general many spin^c structures. The difference between two different spin^c structures consists in twisting by a $U(1)$-bundle. In the present section we describe an analogous construction for noncommutative geometries.

Definition 2.79. Let $\mathcal{E}$ be a Hermitian $\mathcal{A}$-$\mathcal{A}$-bimodule of finite type. Consider its von Neumann completion $\mathscr{E}$. It is an $\mathscr{A}$-$\mathscr{A}$-bimodule of finite type over a von Neumann algebra, and therefore has dimensions [**Lüc02**, Ch. 6]: d_ℓ – as a left $\mathscr{A}$-module and d_r – as a right $\mathscr{A}$-module. We say that $\mathcal{E}$ is of dimension d if $d_r = d_\ell = d$.

Definition 2.80. Let $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{A}$-bimodule of finite type of dimension 1 which as full as a left $\mathcal{A}$-module and as a right $\mathcal{A}$-module and such that for any $\eta_1, \eta_2 \in \mathcal{E}$

$$\tau({}_{\mathcal{A}}\langle\eta_1, \eta_2\rangle) = \tau(\langle\eta_2, \eta_1\rangle_{\mathcal{A}}).$$

Then we call $\mathcal{E}$ a twisting bimodule.

Theorem 2.81. *Let $(\mathcal{A}, \mathcal{A}, \mathcal{H}, D)$ be a noncommutative geometry over $(\mathcal{A}, \mathcal{A})$ such that for all $a \in \mathcal{A}$*

$$\mathrm{Tr}_\omega(\lambda(a)|D|^{-d}) = \mathrm{Tr}_\omega(\rho(a)|D|^{-d}),$$

where λ, ρ are left and right actions of $\mathcal{A}$ on $\mathcal{X}$, respectively, and let $\mathcal{E}$ be a twisting bimodule. Choose a metric compatible connection $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}(\mathcal{A})$. Define a new noncommutative geometry as follows:

- *$\mathcal{H}'$ is the L^2-completion of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{X}$,*
- *the left action of $\mathcal{A}$ is induced by its left action on $\mathcal{E}$,*
- *the right action of $\mathcal{A}$ is induced by its right action on $\mathcal{X}$,*
- *the operator $D' = D_\nabla$ is given by $D_\nabla(\eta \otimes \xi) := (\nabla\eta)\xi + \eta \otimes D\xi$,*
- *if d is even, the operator γ' is defined to be $1 \otimes \gamma$.*

Then $(\mathcal{A}, \mathcal{A}, \mathcal{H}', D')$ is a noncommutative geometry over $(\mathcal{A}, \mathcal{A})$ of the same dimension as $(\mathcal{A}, \mathcal{A}, \mathcal{H}, D)$.

PROOF. We have to check the conditions of Definition 2.3. In view of Theorem 1.38 and estimates therein, D' has compact resolvent and satisfies condition i). Conditions ii)–iv) can be checked by direct computations.

The compatibility of the scalar product with the inner products (condition vii)) is less trivial. To establish it, we have to analyze the situation more thoroughly. First of all, we observe that the scalar product on $\mathcal{H}'$ is compatible with both $\mathcal{A}$-valued inner products via the original trace τ:

$$\langle\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \tau(\lambda({}_{\mathcal{A}}\langle\xi_1{}_{\mathcal{A}}\langle\eta_1, \eta_2\rangle, \xi_2\rangle)) = \\ = \tau(\rho(\langle\xi_2, \xi_1{}_{\mathcal{A}}\langle\eta_1, \eta_2\rangle\rangle_{\mathcal{A}})) = \tau(\rho({}_{\mathcal{A}}\langle\eta_1, \eta_2\rangle\,\langle\xi_2, \xi_1\rangle_{\mathcal{A}})),$$

and

$$\begin{aligned}\tau(\rho(\langle\xi_1\otimes\eta_1,\xi_2\otimes\eta_2\rangle_{\mathcal{A}})) &= \tau(\rho(\langle\eta_1,\langle\xi_1,\xi_2\rangle_{\mathcal{A}}\,\eta_2\rangle_{\mathcal{A}}))\\ &= \tau(\lambda({}_{\mathcal{A}}\langle\langle\xi_1,\xi_2\rangle_{\mathcal{A}}\,\eta_2,\eta_1\rangle)) = \tau(\lambda({}_{\mathcal{A}}\langle\eta_2,\eta_1\rangle\,\langle\xi_1,\xi_2\rangle_{\mathcal{A}})).\end{aligned}$$

Thus, the only thing we have to prove is that the original trace τ coincides with the trace

$$a\mapsto \operatorname{Tr}_\omega(\rho(a)|D'|^{-d}),$$

where we are denoting the right action of a on $\mathcal{H}'$ (coming from the right action on X) by ρ.

To prove this, we want to show that as a right $\mathcal{A}$-representation, $\mathcal{H}'$ is isomorphic to $\mathcal{H}$ via the isomorphism

$$\varphi\colon \xi\mapsto\sum_i v_i\otimes\xi,$$

where $\{v_i\}$ is a right $\mathcal{A}$-basis of $\mathcal{E}$. For this, consider the isomorphism $\mathcal{E}\cong p\mathcal{A}^n$ as right Hermitian $\mathcal{A}$-modules, where $p=p^*=p^2$ is a projection in $\mathbb{M}_n(\mathcal{A})$. Then it follows that $\mathcal{H}'\cong p\mathcal{H}^n$ as right $\mathcal{A}$-representations, and the above mapping maps $\xi\in\mathcal{H}$ to $p(\xi,\dots,\xi)^t\in p\mathcal{H}^n$.

Now we pass to the level of the von Neumann algebras. By the theorem of Kadison, there is a unitary $u\in\mathbb{M}_n(\mathscr{A})$ such that $u^*pu=q$ is a diagonal matrix. Therefore, $\mathcal{H}'\cong q\mathcal{H}^n$ as left $\mathcal{A}$-representations. Now, the diagonal entries q_{ii} of q are projections elements in $\mathscr{A}$, and

$$\sum_i\tau(q_{ii})=1$$

because the dimension of $\mathcal{E}$ is 1. But as the action of $\mathcal{A}$ on $\mathcal{E}$ is faithful, q_{ii} must be orthogonal projections with

$$\sum_i q_{ii}=1$$

which proves the isomorphism $\mathcal{H}'\cong\mathcal{H}$ of right $\mathcal{A}$-representations.

Using the isomorphism φ, we may compare the operators D and $\varphi^{-1}D'\varphi$. To do this, we make the following computation:

$$\begin{aligned}\varphi^{-1}D'\varphi(\xi) = \varphi^{-1}\left(\sum_i D'(v_i\otimes\xi)\right) &= \sum_i\varphi^{-1}(\nabla v_i\xi+v_i\otimes D\xi)\\ &= D\xi+\sum_{i,j}p_{ij}[D,p_{ji}]\xi = D\xi+T\xi,\end{aligned}$$

where T is a bounded regular operator. Thus, Lemma 1.54 tells us that the traces given by

$$a\mapsto\operatorname{Tr}_\omega(\rho(a)|D|^{-d})$$

and

$$a\mapsto\operatorname{Tr}_\omega(\rho(a)|D'|^{-d})$$

coincide. Thus, the trace-compatibility property (condition vii)) is satisfied.

We also have to argue why the mapping

$$t\mapsto e^{it|D'|}Se^{-it|D'|}$$

is of class C^∞, S being the operator given by the left or right action of $\mathcal{A}$ on $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{X}$. This follows from the following equivalent characterization of it, cf. [**Con08**], [**CM95**]: it is enough to prove that $S\in\operatorname{Dom}L^mR^n$, where

$$L(S)=|D+T|^{-1}[(D+T)^2,S],\quad R(S)=[(D+T)^2,S]|D+T|^{-1}.$$

Now, the fact that T is regular yields $S \in \operatorname{Dom} L^m R^n$ and $\bigcap_n \operatorname{Dom}|D+T|^n = \bigcap_n \operatorname{Dom}|D|^n$. Thus, conditions v) and vi) are also satisfied.

Let us now show that the twisted noncommutative geometry also defines a fundamental class in $\mathsf{KK}(A \otimes A^{\mathrm{op}}, \mathbb{C})$. This follows because its KK-class can be written as the Kasparov product of the original fundamental class $\Theta \in \mathsf{KK}(A \otimes A^{\mathrm{op}}, \mathbb{C})$ with the class $[E] \otimes \mathbf{1} \in \mathsf{KK}(A \otimes A^{\mathrm{op}}, A \otimes A^{\mathrm{op}})$, where $[E] \in \mathsf{KK}(A, A)$ is the class of the A-A-bimodule E. Thus, condition viii) is satisfied. This finishes the proof. □

Observe that the new operator D' depends on the choice of a metric compatible connection on $\mathcal{E}$, which we emphasize by writing $D' = D_\nabla$. This freedom will be crucial for the further investigation of Seiberg–Witten equations in our situation.

Of course, this construction resembles the classical construction for spin^c structures on manifolds obtained by twisting a spin or spin^c structure with a line bundle. The difference between spin and spin^c structure is rephrased in the language of noncommutative geometry via the conjugation operator J which we already encountered in the discussion of the axioms for noncommutative geometries, which implies, for instance, $\mathcal{X} \cong \overline{\mathcal{X}}$ for the $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{X}$ and its conjugate bimodule $\overline{\mathcal{X}}$.

Another possible description of spin structures identifies them with spin^c structures having trivial determinant bundle. Therefore we now turn our attention towards the possible ways to describe the "determinant bundle" of a given spin^c structure.

Let us first analyze the "spin" case, where $\overline{\mathcal{X}} \cong \mathcal{X}$ as an $\mathcal{A}$-$\mathcal{A}$-bimodule, the isomorphism being given by the mapping

$$\overline{\mathcal{X}} \to \mathcal{X},$$
$$\bar{\xi} \mapsto J\xi.$$

We consider the algebra $\mathcal{B} := \operatorname{End}_{\mathcal{A}}(\mathcal{X})$. Using the isomorphism $\overline{\mathcal{X}} \cong \mathcal{X}$, we obtain that

$$\mathcal{B} \cong \operatorname{End}_{\mathcal{A}}(\overline{\mathcal{X}}) \cong {}_{\mathcal{A}}\operatorname{End}(\mathcal{X})^{\mathrm{op}},$$

where the last isomorphism (reading it from the right to the left) is given by the formula

$$T \mapsto (\bar{\xi} \mapsto \overline{T^*\xi}).$$

Thus, we may consider the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{T} := \mathcal{X} \otimes_{\mathcal{B}} \mathcal{X} \cong \overline{\mathcal{X}} \otimes_{\operatorname{End}_{\mathcal{A}}(\mathcal{X})} \mathcal{X},$$

where $b \in \operatorname{End}_{\mathcal{A}}(\mathcal{X})$ acts from the right on $\overline{\mathcal{X}}$ as

$$\bar{\xi} \cdot b := \overline{b^*\xi}.$$

Now, $\mathcal{T}$ is isomorphic to the trivial $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A}$ via the map

$$(\bar{\xi}, \eta) \mapsto \langle \xi, \eta \rangle_{\mathcal{A}}\,.$$

We also observe in this case that the algebra $\mathcal{B}$, considered as an $\mathcal{A}$-$\mathcal{A}$-bimodule, is naturally isomorphic to the bimodule

$$\overline{\mathcal{X}} \otimes_{\mathcal{A}} \mathcal{X} \cong \mathcal{X} \otimes_{\mathcal{A}} \overline{\mathcal{X}} \cong \mathcal{X} \otimes_{\mathcal{A}} \mathcal{X}.$$

Let us now turn to the general case. Given a biconnection ∇ on $\mathcal{X}$, we would like to define a biconnection on $\mathcal{T}$ by setting

$$\nabla(\xi \otimes \xi') = \nabla\xi \otimes \xi' + \xi \otimes \nabla\xi'.$$

We can do it under the following assumption.

Definition 2.82. Let $\mathcal{X}$ be an $\mathcal{A}$-$\mathcal{A}$-bimodule and ∇ be a biconnection on $\mathcal{X}$. We say that ∇ is compatible with the endomorphisms, if the following conditions are satisfied:

i) the algebra $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{X})$ is isomorphic to the algebra $\mathcal{B}' = {}_{\mathcal{A}}\mathrm{End}(\mathcal{X})^{\mathrm{op}}$ via a fixed isomorphism $\theta\colon \mathcal{B} \to \mathcal{B}'$;
ii) if one views $\mathcal{B}$ and $\mathcal{B}'$ as $\mathcal{A}$-$\mathcal{A}$-bimodules $\mathcal{B} \cong \mathcal{X} \otimes_{\mathcal{A}} \overline{\mathcal{X}}$, $\mathcal{B}' \cong \overline{\mathcal{X}} \otimes_{\mathcal{A}} \mathcal{X}$, then the induced biconnections coincide using the isomorphism θ.

Given an endomorphism-compatible biconnection ∇ on $\mathcal{X}$, we naturally obtain a biconnection on $\mathcal{B}$, viewed as an $\mathcal{A}$-$\mathcal{A}$-bimodule, such that

$$\nabla(T\xi) = \nabla T \cdot \xi + T\nabla\xi, \quad T \in \mathcal{B},\ \xi \in \mathcal{X}$$

and

$$\nabla(\eta \cdot T) = \nabla\eta \cdot T + \eta \cdot \nabla T, \quad T \in \mathcal{B},\ \xi \in \mathcal{X}.$$

Now, the compatibility of the biconnection with endomorphisms implies that we can still define the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{T} := \mathcal{X} \otimes_{\mathcal{B}} \mathcal{X},$$

as well as the biconnection ∇ on it by setting

$$\nabla(\xi \otimes \xi') = \nabla\xi \otimes \xi' + \xi \otimes \nabla\xi'.$$

We observe that if our noncommutative geometry is obtained by twisting a "spin" noncommutative geometry by a bimodule $\mathcal{E}$ such that $\mathcal{X} \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{A}} \mathcal{X}$, then we obtain

$$\mathcal{T} \cong \mathcal{E} \otimes_{\mathcal{A}} \overline{\mathcal{X}} \otimes_{\mathcal{B}} \mathcal{X} \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E},$$

which reflects exactly the fact that the determinant line bundle of a spin structure twisted with a line bundle is the square of this line bundle. Thus, in the commutative case our construction gives the well-known description of the determinant line bundle. To see that our construction gives precisely the induced connection on the determinant line bundle, one has to construct the isomorphism θ using of the canonical antiautomorphism α of the Clifford algebra induced by the order and sign reversal, cf. [**GBVF01**, Sect. 9.2, Sect. 9.3].

CHAPTER 3

Examples of noncommutative geometries

In this chapter we discuss the known examples of noncommutative manifolds with respect to the results and conditions appeared in the last chapter. Using the more general approach to the conditions for noncommutative manifolds based on Poincaré duality, we can slightly expand the amount of known examples, encompassing toric deformations of arbitrary (not just spin) manifolds.

1. Oriented Riemannian manifolds

Let M be a compact oriented Riemannian manifold. Denote by $\mathcal{A}$ the algebra $C^\infty(M)$ of complex-valued smooth functions on M. Let $\mathcal{B}_0 := \Gamma^\infty(\mathrm{Cliff}_{\mathbb{C}}(T^*M))$ be the algebra of sections of the bundle of complex Clifford algebras associated with the cotangent bundle of M, considered as a Fréchet $*$-algebra. It is well-known that the isomorphism class of the complex Clifford algebra C_d of a d-dimensional vector space is determined as follows:

$$C_d \cong \begin{cases} \mathbb{M}_{2^{d/2}}(\mathbb{C}), & d \text{ even}, \\ \mathbb{M}_{2^{\lfloor d/2 \rfloor}}(\mathbb{C}) \oplus M_{2^{\lfloor d/2 \rfloor}}(\mathbb{C}), & d \text{ odd}, \end{cases}$$

and the decomposition in the odd case corresponds to the decomposition of the Clifford algebra in the even and odd parts. We let $\mathcal{B} := \mathcal{B}_0$ if d is even, and we let $\mathcal{B}$ be the algebra of sections of the even part of the complex Clifford algebra bundle if d is odd.

We let $\mathcal{X}$ be the underlying Fréchet space of $\mathcal{B}$ endowed with a left action of $\mathcal{A}$ coming from the inclusion $\mathcal{A} \subset \mathcal{B}$ and a natural right action of $\mathcal{B}$. Then by the Serre-Swan Theorem, $\mathcal{X}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule of finite type. We endow $\mathcal{X}$ with the Hermitian structures given by

$${}_{\mathcal{A}}\langle \xi, \eta \rangle(x) = \mathrm{Tr}(\xi(x)\eta^*(x)), \quad x \in M,$$

where Tr is the natural trace on matrices, and

$$\langle \xi, \eta \rangle_{\mathcal{B}} := \xi^* \eta.$$

If d is even, we let γ be the grading operator on $\mathcal{X}$, having the even part of the Clifford algebra bundle as the $+1$-eigenspace and the odd part of the Clifford algebra bundle as the (-1)-eigenspace.

The algebras $\mathcal{A}$ and $\mathcal{B}$ are endowed with the naturally arising trace given on $\mathcal{A}$ by

$$\tau(a) := \int_M a \text{ vol},$$

and on b by

$$\tau(b) := \int_M \mathrm{Tr}(b) \text{ vol},$$

where vol is the volume form associated with the given Riemannian metric, and Tr is the natural trace on matrices, and we let $\mathcal{H}$ be the completion of $\mathcal{X}$ with respect to the scalar product

$$\langle \xi, \eta \rangle := \tau({}_{\mathcal{A}}\langle \xi, \eta \rangle) = \tau(\langle \xi, \eta \rangle_{\mathcal{B}}).$$

The Levi-Civita connection on M lifts naturally to a connection

$$\nabla\colon \mathcal{X} \to \Omega^1(M) \otimes_{\mathcal{A}} \mathcal{X},$$

where $\Omega^1(M)$ is the $\mathcal{A}$-$\mathcal{A}$-bimodule of differential 1-forms on M. Notice that there is a natural identification of $\Omega^1(M)$ with a subspace of $\mathcal{B}$. Using this identification, we obtain a Clifford multiplication map

$$m\colon \Omega^1(M) \otimes_{\mathcal{A}} \mathcal{X} \to \mathcal{X}.$$

Hereby $\mathcal{B}$ acts on $\mathcal{X}$ from the left by multiplication.

We define the Clifford-Dirac operator as

$$D := m \circ \nabla\colon \mathcal{X} \to \mathcal{X}.$$

It is an unbounded selfadjoint operator on $\mathcal{H}$, which has compact resolvent and whose eigenvalues have the asymptotics $|\lambda_k| = O(k^{1/d})$.

Moreover, the spectral triple $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D)$ determines the fundamental class in $\mathsf{KK}(A \otimes B)$, where A and B are the C^*-completions of $\mathcal{A}$ and $\mathcal{B}$ [**Con94**, VI.4.β].

Therefore, this noncommutative geometry canonically associated with an oriented Riemannian manifold is subject to all conditions of Definition 2.3.

Let us discuss other conditions of the previous chapter for this example. For instance, the bimodules $\Omega^k_{\mathcal{D}}$ are identified with the bimodules of differential forms [**Con94**, VI.1]. Therefore the condition for these modules to be finitely generated and projective is satisfied. The theorem about the Laplace operator and connection Laplacians having compact resolvent is in this situation well-known from classical differential geometry.

If the manifold M is spin or spin^c, one can as well consider the "noncommutative" geometries over $\mathcal{A}$ given by the spin or spin^c Dirac operators.

In this case $\mathcal{X}$ will be the $\mathcal{A}$-$\mathcal{A}$-bimodule of the sections of the corresponding spinor bundle equipped with the canonical Hermitian structure. In the spin case it will be additionally equipped with the involution J coming from the real structure of the spinor representation. In the even case the grading operator γ will be given by the action of the volume form on spinors.

In the spin case the Levi-Civita connection lifts canonically to the spinor bundle, allowing to define the Dirac operator as the composition of the connection operator and the Clifford multiplication. In the spin^c case the lift is not quite canonical; namely, one has to choose a connection on the determinant line bundle of the spin^c structure, defined in exactly the same manner as we did before: if $\mathcal{X}$ is a spinor bundle, we define

$$\mathcal{T} := \mathcal{X} \otimes_{\mathrm{End}_{\mathcal{A}}(\mathcal{X})} \mathcal{X},$$

where the right action of $\mathrm{End}_{\mathcal{A}}(\mathcal{X})$ on $\mathcal{X}$ is defined using the canonical antiautomorphism α of the Clifford algebra induced by the order and sign reversal, cf. [**GBVF01**, Sect. 9.2, Sect. 9.3].

It is easy to see that the $\mathcal{A}$-module $\mathcal{T}$ is of finite type, and thus corresponds to a vector bundle, and one can easily check that it is of dimension 1. Thus, it corresponds to a line bundle, which is called the determinant line bundle of the given spin^c structure. Using an arbitrary connection on the determinant line bundle and the Levi-Civita connection, one can define a Clifford connection on the spinor bundle and define the Dirac operator [**GBVF01**, Sect. 9].

It is interesting to observe that in this case the condition of Poincaré duality in cyclic cohomology is satisfied: the cocycle Φ is given by

$$\Phi(a_0 \otimes b_0, \dots, a_d \otimes b_d) = \int_M a_0 b_0 d(a_1 b_1) \wedge d(a_d b_d)$$

and thus does coincide with the fundamental class in $\mathsf{HP}(\mathcal{A} \otimes \mathcal{A}, \mathbb{C})$ [**BMRS08**].

2. Stabilizations

The "commutative case" of the previous section admits a straightforward generalization by "tensoring everything with matrices". This is achieved as follows: let for simplicity M be an odd-dimensional spin manifold; take the noncommutative geometry $(\mathcal{A}, \mathcal{A}, \mathcal{H}, D)$ as in the previous section and consider the noncommutative geometry

$$(\mathcal{A} \otimes \mathbb{M}_n(\mathbb{C}), \mathcal{A} \otimes \mathbb{M}_n(\mathbb{C}), \mathcal{H} \otimes \mathbb{M}_n(\mathbb{C}), D \otimes 1),$$

where $M_n(\mathbb{C})$ acts on itself by left resp. right multiplication and is equipped with the Hilbert-Schmidt scalar product. Then it is easy to see that we haven't changed anything in the asymptotic and commuting behaviour of D, and therefore we have still obtained a noncommutative manifold in our sense. The interesting thing about this simple example is the following: it shows that the natural map defined in Proposition 2.72 may fail to be injective. Indeed, the periodic cyclic cohomology is insensitive to tensoring with matrices, whereas the Laplace operator Δ_k in this example is easily seen to be equal to

$$\Delta_k \otimes 1 \colon \Omega^k(M) \otimes \mathbb{M}_n(\mathbb{C}) \to \Omega^k(M) \otimes \mathbb{M}_n(\mathbb{C}),$$

and therefore $\dim \ker(\Delta_k \otimes 1) = n^2 \dim \ker \Delta_k$.

This shows that in general we may only hope that this map is surjective, which explains our interest in proving Theorem 2.77.

3. Toric deformations

We recapitulate the basic facts about toric deformations of manifolds, following [**CDV02**, Sect. 12], and propose a slight generalization of these deformations to encompass not only the spin case, but also the spin^c and the general case, where the Poincaré dual is given by the sections of the deformed Clifford bundle.

3.1. The general setup. The locally convex $*$-algebra $C^\infty(\mathbb{T}^n_\theta)$ of smooth functions on the noncommutative torus $\mathbb{T}^n_\theta$ was defined in [**Con80**]. It is a completion of the algebra $\mathrm{Pol}(\mathbb{T}^n_\theta)$ of "polynomials" on the noncommutative torus, which is by definition the unital $*$-algebra generated by n unitaries $U^1, \dots, U^n$ with relations

$$U^\mu U^\nu = \lambda^{\mu\nu} U^\nu U^\mu, \tag{3.1}$$

where $\lambda^{\mu\nu} = e^{i\theta^{\mu\nu}}$ for some antisymmetric matrix $\theta \in \mathbb{M}_n(\mathbb{R})$.

We denote by $s \to \tau_s$ the natural action of $\mathbb{T}^n$ on this algebra, defined by

$$\tau_s(U^\mu) = e^{2\pi i s_\mu} U^\mu, \quad \mu = 1, \dots, n.$$

To obtain "smooth functions" on the noncommutative torus, we endow $\mathrm{Pol}(\mathbb{T}^n_\theta)$ with the locally convex topology generated by the seminorms

$$|u|_r = \sup_{r_1 + \dots + r_n \le r} \| X_1^{r_1} \dots X_n^{r_n}(u) \|$$

where $\| \cdot \|$ is the C^*-norm (which is the sup of the C^*-seminorms) and where the X_μ are the infinitesimal generators of the action $s \mapsto \tau_s$ of $\mathbb{T}^n$ on $\mathbb{T}^n_\theta$. They are the unique derivations of $\mathrm{Pol}(\mathbb{T}^n_\theta)$ satisfying

$$X_\mu(U^\nu) = 2\pi i \delta^\nu_\mu U^\nu \tag{3.2}$$

for $\mu, \nu = 1, \dots, n$. These derivations are $*$-derivations and they commute:

$$X_\mu(u^*) = (X_\mu(u))^*$$

and

$$X_\mu X_\nu - X_\nu X_\mu = 0.$$

This locally convex $*$-algebra is a nuclear Fréchet space and it follows from the general theory of topological tensor products that the projective and injective topologies coincide [**Gro55**] on any tensor product,

$$E \otimes_\pi C^\infty(\mathbb{T}^n_\theta) = E \otimes_\varepsilon C^\infty(\mathbb{T}^n_\theta)$$

so that on $E \otimes C^\infty(\mathbb{T}^n_\theta)$ there is essentially one reasonable locally convex topology and we denote by $E \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$ the corresponding completion.

Let M be a smooth d-dimensional compact manifold endowed with a smooth action $s \mapsto \sigma_s$ of the Lie group $\mathbb{T}^n$. We also denote by $s \mapsto \sigma_s$ the corresponding action of $\mathbb{T}^n$ on complex smooth functions on M with its standard topology and on the graded-involutive differential graded algebra $\Omega(M)$ of smooth differential forms.

We will now give a direct description of $\mathcal{A} = C^\infty(M_\theta)$ as a fixed point algebra.

The completed tensor product $C^\infty(M)\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$ is unambiguously defined by nuclearity and is a unital locally convex $*$-algebra which is a complete nuclear space. We define by duality the noncommutative smooth manifold $M \times \mathbb{T}^n_\theta$ by setting $C^\infty(M \times \mathbb{T}^n_\theta) = C^\infty(M)\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$; elements of $C^\infty(M \times \mathbb{T}^n_\theta)$ will be refered to as *the smooth functions on* $M \times \mathbb{T}^n_\theta$.

Let $C^\infty(M \times \mathbb{T}^n_\theta)^{\sigma\times\tau^{-1}}$ be the subalgebra of the $f \in C^\infty(M \times \mathbb{T}^n_\theta)$ which are invariant by the diagonal action $\sigma \times \tau^{-1}$ of $\mathbb{T}^n$, that is such that $\sigma_s \otimes \tau_{-s}(f) = f$ for any $s \in T^n$. One defines $\mathcal{A} = C^\infty(M_\theta) = C^\infty(M \times \mathbb{T}^n_\theta)^{\sigma\times\tau^{-1}}$; the elements of $\mathcal{A}$ will be refered to as *the smooth functions on* M_θ.

Now we give a direct construction of smooth differential forms on M_θ. Let $\Omega(M_\theta)$ be the graded-involutive subalgebra $(\Omega(M)\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta))^{\sigma\times\tau^{-1}}$ of $\Omega(M)\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$ consisting of elements which are invariant by the diagonal action $\sigma \times \tau^{-1}$ of $\mathbb{T}^n$. This subalgebra is stable by $d \otimes 1$ and therefore $\Omega(M_\theta)$ is a locally convex graded-involutive differential algebra. The action $s \mapsto \sigma_s$ of $\mathbb{T}^n$ on $\Omega(M)$ induces the action $s \mapsto \sigma_s \otimes 1$ on $\Omega(M)\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$ which gives an action, again denoted $s \mapsto \sigma_s$, of $\mathbb{T}^n$ on the graded-involutive differential algebra $\Omega(M_\theta)$.

Proposition 3.1 ([**CDV02**, Prop. 12.3])**.** *The graded-involutive differential subalgebra $\Omega(M_\theta)^\sigma$ of σ-invariant elements of $\Omega(M_\theta)$ is in the graded center of $\Omega(M_\theta)$ and identifies canonically with the graded-involutive differential subalgebra $\Omega(M)^\sigma$ of σ-invariant elements of $\Omega(M)$.*

In other words the subalgebra of σ-invariant elements of $\Omega(M_\theta)$ is not deformed (i.e. independent of θ). One has $\Omega(M_\theta)^\sigma = \Omega(M)^\sigma \otimes 1$.

The notation M_θ, $C^\infty(M_\theta)$ introduced here is coherent with the standard one $\mathbb{T}^n_\theta$, $C^\infty(T^n_\theta)$ used for the noncommutative torus. Indeed, we have $C^\infty(\mathbb{T}^n_\theta) = (C^\infty(\mathbb{T}^n)\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta))^{\sigma\times\tau^{-1}}$ where σ is the canonical action of $\mathbb{T}^n$ on itself.

The construction of $\Omega(M_\theta)$ admits the following generalization which will be crucial for our later purposes. Let E be a smooth complex vector bundle of finite rank over M and let $\Gamma^\infty(M,E)$ be the $C^\infty(M)$-module of its smooth sections, endowed with its usual topology of complete nuclear space. The vector bundle E will be called σ-equivariant if there is an action $s \mapsto \sigma_s$ of $\mathbb{T}^n$ on E which covers the action $s \mapsto \sigma_s$ of T^n on M. In terms of smooth sections this means that one has

$$V_s(f\psi) = \sigma_s(f)V_s(\psi) \tag{3.3}$$

for $f \in C^\infty(M)$ and $\psi \in \Gamma^\infty(M,E)$. Let $\mathcal{E} = C^\infty(M_\theta, E)$ be the closed subspace of $\Gamma^\infty(M,E)\widehat{\otimes} C^\infty(T^n_\theta)$ consisting of elements which are invariant by the diagonal action $V \times \tau^{-1}$ of $\mathbb{T}^n$.

The locally convex space $\mathcal{E} = C^\infty(M_\theta, E)$ is canonically a bimodule over $\mathcal{A}$.

Proposition 3.2 ([**CDV02**, Prop. 12.5]). *The bimodule $\mathcal{E}$ is a finitely generated projective module over $\mathcal{A}$ as a left and as a right module.*

Moreover, any invariant Hermitian metric on E (which can be easily obtained by averaging) gives two Hermitian structures on $\mathcal{E}$: a left one and a right one. They are defined by the rules

$$
{}_{\mathcal{A}}\langle \xi \otimes a, \eta \otimes b \rangle := \langle \xi, \eta \rangle \otimes ab^*,
$$
$$
\langle \xi \otimes a, \eta \otimes b \rangle_{\mathcal{A}} := \overline{\langle \xi, \eta \rangle} \otimes a^* b,
$$

Let T be a continuous $\mathbb{C}$-linear operator on $C^\infty(M, S)$ such that

$$
TV_s = V_s T \tag{3.4}
$$

for any $s \in \mathbb{T}^n$. Then $C^\infty(M_\theta, S)$ ($\subset C^\infty(M, S) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$) is stable by $T \otimes 1$ which defines the operator T_θ on $C^\infty(M_\theta, S)$. If T is a module homomorphism over $C^\infty(M)$, then it is obvious that T_θ is a bimodule homomorphism over $C^\infty(M_\theta)$.

An important (and slightly more general) example of this construction is the following one: let $\omega \mapsto *\omega$ be the (conjugate-linear) Hodge operator on $\Omega(M)$ corresponding to a σ-invariant Riemannian metric on M. One has $* \circ \sigma_s = \sigma_s \circ *$. Thus we obtain a conjugate-linear map $*_\theta = * \otimes (\cdot)^*$ of $\Omega(M_\theta)$ which is an antiendomorphism: $*(f\omega f') = f'^* * \omega f^*$. We denote $*_\theta$ simply by $*$ in the following and observe that $*\Omega^p(M_\theta) \subset \Omega^{m-p}(M_\theta)$.

Theorem 3.3 ([**CDV02**, Thm. 12.8]). *Let M_θ be a θ-deformation of M then one has $\dim(M_\theta) = \dim(M)$, that is the Hochschild dimension d_θ of $C^\infty(M_\theta)$ coincides with the dimension d of M.*

In the case where M is a spin manifold, the Dirac operator can be used to obtain a noncommutative geometry over $(\mathcal{A}, \mathcal{A})$ together with the antilinear involution J. Namely, one proves the following theorem:

Theorem 3.4 ([**CDV02**, Thm. 12.9]). *Let M_θ be a θ-deformation of a compact spin manifold M. Then there is a noncommutative spin geometry over $\mathcal{A} = C^\infty(M_\theta)$ satisfying all axioms of Definition 2.2.*

We will now generalize this theorem to the case where M is no longer a spin manifold, obtaining a noncommutative geometry in the sense of Definition 2.3. The proof used in [**CDV02**] applies *mutatis mutandis*, and therefore we only sketch it.

3.2. The general case. Now we will generalize the constructions from [**CDV02**], giving a noncommutative geometry for any toric deformation, regardless of whether the underlying manifold is spin.

It is well-known and easy to check that we can average any Riemannian metric on M under the action of σ and obtain one for which the action $s \mapsto \sigma_s$ of $\mathbb{T}^n$ on M is isometric. Taking this Riemannian metric, we can first observe that the Clifford bundle associated to the cotangent bundle is equivariant and thus the above construction applies to the Clifford bundle in the even case and the even part of the Clifford bundle in the odd case, giving a finitely generated projective left Hermitian $\mathcal{A}$-module $\mathcal{X}$. Moreover, if we take the bundle B of complex Clifford algebras, then the above construction will provide us with a Fréchet $*$-algebra $\mathcal{B}$ defined as

$$
\mathcal{B} := (\Gamma(M, B) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta))^{\sigma \times \tau^{-1}}.
$$

This algebra canonically acts on $\mathcal{X}$ from the right. The Clifford-Dirac operator D constructed in Section 1 gives by the above construction an operator $D \otimes 1$ on $\mathcal{X}$. Completing $\mathcal{X}$ with respect to the scalar product coming from the Hermitian structure and the canonical trace

$$
\tau = \int \cdot \operatorname{vol} \otimes \tau_{\mathbb{T}^n_\theta} : \mathcal{A} \to \mathbb{C},
$$

one obtains a Hilbert space $\mathcal{H}$ with commuting injective representations of $\mathcal{A}$ and $\mathcal{B}$. The compatibility of the scalar product with both Hermitian structures follows from the compatibility in the non-deformed case. The order one condition and smoothness conditions v), vi) can be proved in the same manner as in [**CDV02**]. The $\mathcal{A}$ and $\mathcal{B}$-valued scalar products on $\mathcal{X}$ are given by

$$_{\mathcal{A}}\langle \psi \otimes f, \psi' \otimes f' \rangle = {}_{C^\infty(M)}\langle \psi, \psi' \rangle \otimes f f'^*$$

and

$$\langle \psi \otimes f, \psi' \otimes f' \rangle_{\mathcal{B}} = \psi^* \psi' \otimes f^* f'.$$

As we are dealing with deformations of Poincaré dual algebras, the results of [**Rie93**] apply, yielding the Poincaré duality condition. Thus, this toric deformation satisfies the axioms for noncommutative geometry over the pair $(\mathcal{A}, \mathcal{B})$ from Definition 2.3.

In the even case, the grading operator γ from M is σ-invariant and therefore yields an element $\gamma \otimes 1$ which commutes with the $\mathcal{A}$ action and endows $\mathcal{B}$ with the grading which coincides with the usual grading inherited from the Clifford bundle.

3.3. The spin^c case. Let M now be a spin^c manifold such that the determinant line bundle of the spin^c structure is σ-equivariant. We will now generalize the deformation of [**CDV02**] to this case.

Let S be the spinor bundle corresponding th the chosen spin^c structure. The bundle S is not exactly σ-equivariant, so we cannot apply the results of the last section directly. It is, however, equivariant in a generalized sense which was exploited in [**CDV02**] and which we also apply here. Namely, the isometric action σ of $\mathbb{T}^n$ on M does not lift directly to S but lifts only modulo $\{\pm 1\} \subset \mathbb{T}^n$. More precisely, one has a twofold covering $p : \widetilde{\mathbb{T}^n} \twoheadrightarrow \mathbb{T}^n$ of the group $\mathbb{T}^n$, and a group homomorphism $\widetilde{s} \mapsto V_{\widetilde{s}}$ of $\widetilde{\mathbb{T}}^n$ into the group $\operatorname{Aut}(S)$ which covers the action $s \mapsto \sigma_s$ of $\mathbb{T}^n$ on M. In terms of smooth sections, it means that

$$V_{\widetilde{s}}(f\psi) = \sigma_s(f) V_{\widetilde{s}}(\psi) \tag{3.5}$$

where $f \in C^\infty(M)$ and $\psi \in C^\infty(M, S)$ with $s = p(\widetilde{s})$. The bundle S is Hermitian and one has

$$\langle V_{\widetilde{s}}(\psi), V_{\widetilde{s}}(\psi') \rangle = \sigma_s \langle \psi, \psi' \rangle \tag{3.6}$$

for $\psi, \psi' \in C^\infty(M, S)$, $\widetilde{s} \in \widetilde{\mathbb{T}}^n$ and $s = p(\widetilde{s})$. Furthermore, the Dirac operator D commutes with the $V_{\widetilde{s}}$.

To the projection $p : \widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ corresponds an injective homomorphism of $C^\infty(\mathbb{T}^n)$ into $C^\infty(\widetilde{\mathbb{T}}^n)$ which identifies $C^\infty(\mathbb{T}^n)$ with the subalgebra $C^\infty(\widetilde{\mathbb{T}}^n)^{\ker(p)}$ of $C^\infty(\widetilde{\mathbb{T}}^n)$ of elements which are invariant by the action of the subgroup $\ker(p) \simeq \mathbb{Z}_2$ of $\widetilde{\mathbb{T}}^n$. Let $\widetilde{\mathbb{T}}^n_\theta$ be the noncommutative n-torus $\mathbb{T}^n_{\frac{1}{2}\theta}$ and let $\widetilde{s} \mapsto \widetilde{\tau}_{\widetilde{s}}$ be the canonical action of the n-torus $\widetilde{\mathbb{T}}^n$ on it. The very reason for this notation is that $C^\infty(\mathbb{T}^n_\theta)$ identifies with the subalgebra $C^\infty(\widetilde{\mathbb{T}}^n_\theta)^{\ker(p)}$ of $C^\infty(\widetilde{\mathbb{T}}^n_\theta)$ of elements which are invariant by the $\widetilde{\tau}_{\widetilde{s}}$ for $\widetilde{s} \in \ker(p) \cong \mathbb{Z}_2$. Under this identification, one has $\widetilde{\tau}_{\widetilde{s}}(f) = \tau_s(f)$ for $f \in C^\infty(\mathbb{T}^n_\theta)$ and $s = p(\widetilde{s}) \in \mathbb{T}^n$.

Define $\mathcal{X} = C^\infty(M_\theta, S)$ to be the closed subspace of $C^\infty(M, S) \widehat{\otimes} C^\infty(\widetilde{\mathbb{T}}_\theta)$ consisting of elements which are invariant by the diagonal action $V \times \widetilde{\tau}^{-1}$ of $\widetilde{\mathbb{T}}^n$; this is canonically a topological bimodule over $C^\infty(M_\theta)$. Since the Dirac operator commutes with the $V_{\widetilde{s}}$, $C^\infty(M_\theta, S)$ is stable under $D \otimes \mathrm{id}$ and we denote by D_θ the corresponding operator on $C^\infty(M_\theta, S)$. Again, D_θ is a first-order operator of the bimodule $C^\infty(M_\theta, S)$ over $C^\infty(M_\theta)$ into itself. The space $C^\infty(M, S) \widehat{\otimes} C^\infty(\widetilde{\mathbb{T}}^n_\theta)$ is canonically a bimodule over $C^\infty(M) \widehat{\otimes} C^\infty(\widetilde{\mathbb{T}}^n_\theta)$ (and therefore also on $C^\infty(M) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)$).

One defines a Hermitian structure on $C^\infty(M,S)\widehat{\otimes} C^\infty(\widetilde{\mathbb{T}}^n_\theta)$ for its left module structure over $C^\infty(M)\widehat{\otimes} C^\infty(\widetilde{\mathbb{T}}^n_\theta)$ by setting

$$_{\mathcal{A}}\langle \psi\otimes t, \psi'\otimes t'\rangle = \langle \psi,\psi'\rangle \otimes tt'^*$$

for $\psi,\psi'\in C^\infty(M,S)$ and $t,t'\in C^\infty(\widetilde{\mathbb{T}}^n_\theta)$. This gives by restriction the Hermitian structure of $C^\infty(M_\theta,S)$ considered as a left $C^\infty(M_\theta)$-module. Analogously, one defines a right Hermitian structure by setting

$$\langle \psi\otimes t, \psi'\otimes t'\rangle_{\mathcal{A}} = \overline{\langle \psi,\psi'\rangle} \otimes t^*t'$$

and obtains a right Hermitian structure on $C^\infty(M_\theta,S)$ by restriction.

All axioms of Definition 2.3 can be checked here similarly to the above general case, yielding a noncommutative geometry over the pair $(\mathcal{A},\mathcal{A})$ (possibly with $\mathcal{X}\not\cong\overline{\mathcal{X}}$).

Notice that when $\dim(M)$ is even, one has a $\mathbb{Z}_2$-grading γ of $C^\infty(M,S)$ as Hermitian module which induces a $\mathbb{Z}_2$-grading, again denoted by γ, of $C^\infty(M_\theta,S)$ as Hermitian right $C^\infty(M_\theta)$-module.

Moreover, the algebras

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{X})$$

and

$$\mathcal{B}^o = {}_{\mathcal{A}}\mathrm{End}(\mathcal{X})^{\mathrm{op}}$$

are canonically antiisomorphic. Indeed, one has

$$\mathcal{B} \cong (\Gamma(\mathrm{Cliff}_{\mathbb{C}}(T^*M))\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta))^{\sigma\times\tau^{-1}},$$

$$\mathcal{B}^o \cong (\Gamma(\mathrm{Cliff}_{\mathbb{C}}(T^*M))^{\mathrm{op}}\widehat{\otimes} C^\infty(\mathbb{T}^n_\theta))^{\sigma\times\tau^{-1}},$$

and we have the canonical isomorphism

$$\theta\colon \mathcal{B}\to\mathcal{B}^o$$

given by

$$\theta = \alpha\otimes 1,$$

where α is the canonical antiautomorphism of the complex Clifford algebras given by the order and sign reversal, [**GBVF01**, Sect. 9.2]. Moreover, any invariant spin^c connection on S gives an endomorphism-compatible biconnection on $\mathcal{X}$, and the construction of the bimodule $\mathcal{T}$ and the induced biconnection on it yields exactly the right result: $\mathcal{T}$ is the bimodule canonically associated with the σ-invariant determinant line bundle, and the induced biconnection is given by the invariant connection we started with, cf. [**GBVF01**, Sect. 9.3, p. 386].

3.4. Fourier decomposition. Let us introduce an important tool in studying the toric deformations of manifolds, the Fourier decomposition. Let L be a σ-equivariant vector bundle over M and let $\mathcal{L}$ be the corresponding finitely generated projective module, as explained above (the construction works equally well for modules such as spinor bundles, although they are merely almost σ-equivariant). Then any element of $\mathcal{L}$ can be written as a Fourier series

$$\xi = \sum_{m\in\mathbb{Z}^n} \xi_m\otimes U^m,$$

where $U^{(m_1,\dots,m_n)} = U_1^{m_1}\dots U_n^{m_n}$, and the elements $\xi_m\in\Gamma(L)_m$, where $\Gamma(L)_m$ is the subspace of $\Gamma(L)$ consisting of the sections of L which transform corresponding to the character $m=(m_1,\dots m_n)$ of $\mathbb{Z}^n$ in the representation $\sigma\colon\mathbb{T}^n\to\mathrm{Aut}(\Gamma(L))$ arising from the action of $\mathbb{T}^n$.

In particular, there is a map

$$E_0\colon \mathcal{L}\to\Gamma(L)^\sigma,$$

$$\xi \mapsto \xi_0.$$

If $L = M \times \mathbb{C}$ is a trivial bundle, then $\mathcal{L} = \mathcal{A}$, and the map E_0 enjoys an additional property of being a *trace-preserving conditional expectation*:

$$E_0(afb) = aE_0(f)b, \quad a, b \in C^\infty(M)^\sigma,$$

and

$$\tau \circ E_0(f) = \int_M f_0 = \tau(f).$$

We also observe that if we equip the bimodule $\mathcal{L}$ coming from the Hermitian vector bundle L with two $\mathcal{A}$-valued scalar products,

$${}_{\mathcal{A}}\langle \chi \otimes f, \zeta \otimes g\rangle = \langle \chi, \zeta\rangle \otimes fg^*$$

and

$$\langle \chi \otimes f, \zeta \otimes g\rangle_{\mathcal{A}} = \overline{\langle \chi, \zeta\rangle} \otimes f^* g,$$

then $E_0({}_{\mathcal{A}}\langle \xi, \eta\rangle) = E_0(\langle \xi, \eta\rangle_{\mathcal{A}})$. The proof is straightforward using the fact that $E_0 = 1 \otimes \tau$ on $C^\infty(M_\theta)$ and the tracial property of τ.

3.5. Differential forms and de Rham cohomology. Let us discuss further the conditions which we encountered in Chapter 2 in the case of this toric deformation. First of all, the following result is mentioned in [**CDV02**] in the spin case. It can be obtained in the same way in general:

Proposition 3.5. *In the situation of a toric deformation, the bimodules* $\Omega^k_{\mathcal{D}}(\mathcal{A})$ *coincide with the bimodules of differential forms* $\Omega^k(M_\theta)$ *constructed above.*

In particular, they are $\mathcal{A}$-$\mathcal{A}$-bimodules of finite type and the differential calculus on this noncommutative manifold is Clifford-like (Definition 2.56). Using the above observations, it is also easy to see that the Laplace operator Δ_k on k-forms coincides with the operator $\Delta^M_k \otimes 1$, here Δ^M_k being the classical Laplacian on differential forms on M. Therefore

$$\dim\ker \Delta_k = \dim\ker \Delta^M_k,$$

and thus $\mathsf{H}^k_d(\mathcal{A}) \cong \mathsf{H}^k_{\mathrm{dR}}(M)$. As the periodic cyclic cohomology is insensitive to deformations [**CDV02**], we conclude that the de Rham cohomology and the periodic cyclic cohomology coincide in the case of toric deformations.

It is also interesting to analyze the mapping $\alpha \mapsto C_\alpha$ which we encountered in Ch. 2 and which compares the de Rham cohomology with the periodic cyclic cohomology and different conditions entering Theorem 2.77.

First of all, the subspace

$$\mathcal{C}^k = \overline{\mathrm{span}}\{[\omega, a] \mid \omega \in \Omega^*,\ a \in \mathcal{A}\},$$

is easily seen to be equal to the regular subspace of differential forms whose Fourier decomposition has no invariant term:

$$\mathcal{C}^k = \{\omega \mid E(\omega) = 0\}.$$

Indeed, as the identity commutes with every element in the noncommutative torus, the inclusion $\subset$ is obvious. On the other hand, using the Fourier decomposition, it is enough to prove that

$$\omega = \omega_m \otimes U^m, \quad m \neq 0,$$

is in $\mathcal{C}^k$, which easily follows from the fact that for $m \neq 0$ one can write U^m as a commutator.

Using the characterization of the differential forms $\Omega^*(M_\theta)$ given above, we easily see that the algebra $\Omega^*_{\mathcal{D}}$ is indeed graded-commutative modulo $\mathrm{im}\, b$ in our case:

$$[\omega \otimes f, \eta \otimes g]_{\mathrm{gr}} = \omega \wedge \eta \otimes fg - (-1)^{\deg\omega\deg\eta}\eta \wedge \omega \otimes gf.$$

Now, the algebra $\Omega^*_{\mathcal{D}}(\mathcal{A} \otimes \mathcal{B})$ is easily seen to be isomorphic to the algebra

$$(\Omega^*(M) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)^{op})^{\sigma \otimes \tau^{-1} \otimes \tau^{-1}},$$

and therefore is Clifford-like. For $k < d$, $\alpha \in \Omega^k(\mathcal{A} \otimes \mathcal{B})$, the element

$$\pi(\alpha) \in (\Gamma(\mathrm{Cliff}^{(k)}_{\mathbb{C}}(M)) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)^{op})^{\sigma \otimes \tau^{-1} \otimes \tau^{-1}},$$

where $\mathrm{Cliff}^{(k)}_{\mathbb{C}}(M)$ is the subbundle of elements of the Clifford algebra bundle with length at most k (given by the canonical length filtering on the Clifford algebra), and therefore orthogonal to γ.

3.6. Sobolev theory. It is quite natural to expect that toric deformations of ordinary manifolds inherit reasonable Sobolev theory from them, and this is indeed the case: if we consider some operator $P \colon \mathcal{X} \to \mathcal{X}$ given by

$$P = \sum_i D_i \otimes f_i,$$

where D_i is a differential operator of order k, then P has order k on the Sobolev spaces on $\mathcal{X}$. This is easily seen by repeating the standard arguments from classical Sobolev theory on manifolds on the first tensor factor and using boundedness of multiplication on $C^\infty(\mathbb{T}^n_\theta)$ on the second tensor factor. Thus, in particular, connection operators on $\mathcal{X}$ are compatible with the order calculus, and thus toric deformations have reasonable Sobolev theory. Moreover, as the modules of differential forms are finitely generated and projective and the differential calculus is Clifford-like, the Sobolev norms on differential forms defined using the Laplacians coincide with the Sobolev norms induced from the structure of a finitely generated projective module.

CHAPTER 4

Seiberg–Witten equations for noncommutative manifolds

1. Classical Seiberg–Witten equations

In this section we briefly review the basics of the Seiberg–Witten theory. The constructions are well-known and therefore we omit proofs and provide only references. We refer to [**Mor96**], [**Moo96**], [**Mar99**], [**Nic00**] for systematic exposure.

Let M be a smooth closed orientable four-manifold with a Riemannian metric g. Choose a spin^c structure with a determinant line bundle L and let a be a connection on L. Consider the Dirac operator,

$$D_a \colon \Gamma(S^+) \to \Gamma(S^-),$$

where $S^\pm$ are the bundles of positive and negative spinors, respectively. Let $q(\psi)$ denote a self-dual 2-form associated to a spinor ψ in the following way. The endomorphisms of the positive part of the spinor bundle decompose as

$$\mathrm{End}_{C^\infty(M)}(\Gamma(S^+)) \cong C^\infty(M) \oplus \Omega^{2,+}(M).$$

Consider the endomorphism $\Theta_{\psi,\psi}$ of the spinor bundle. $q(\psi)$ is defined to be the projection of $\Theta_{\psi,\psi}$ to $\Omega^{2,+}(M)$.

The following nonlinear first order partial differential equations for a pair (a, ψ)

$$D_a\psi = 0,$$
$$F_a^+ = q(\psi),$$

are called the Seiberg–Witten equations. The gauge group $\mathcal{G} = C^\infty(M, U(1))$ naturally acts on the configuration space, and the Seiberg–Witten equations are invariant under this action.

One considers the Sobolev versions of the Seiberg–Witten equations completing the gauge group $\mathcal{G}$, the spinors and the differential forms on M with respect to the Sobolev topology, thus obtaining the Sobolev gauge group $\mathcal{G}_{s+1}$ and Sobolev moduli spaces $\mathcal{M}_s$. Using the multiplicative properties of the Sobolev spaces, one obtains

Proposition 4.1 ([**Mor96**, Prop. 2.2.11])**.** *For s big enough, any H^s-solution to the Seiberg–Witten equations is $\mathcal{G}_{s+1}$-gauge equivalent to a smooth solution.*

Thus, for s big enough, the Sobolev moduli space coincides with the smooth moduli space. The advantage of switching to the Sobolev moduli space is that we can use index theory to obtain information about it.

Proposition 4.2 ([**Mor96**, Lemma 4.6.1])**.** *The linearization of the Seiberg–Witten equations at a solution (a, ψ) yields an elliptic complex*

$$0 \to H^{s+1}(M, i\mathbb{R}) \xrightarrow{D_1} H^s(T^*M \otimes i\mathbb{R}) \oplus H^s(S^+) \xrightarrow{D_2} H^{s-1}(\Lambda^2_+ T^*M \otimes i\mathbb{R}) \oplus H^{s-1}(S^-),$$

where $D_1(f) = (2df, -f \cdot \psi)$, $D_2 = D(SW)_{(a,\psi)} = \begin{pmatrix} d^+ & -Dq_\psi \\ \frac{1}{2}(\cdot)\psi & D_a \end{pmatrix}$. *Its Euler characteristic is equal to the index of the elliptic operator*

$$\chi(C) = \mathrm{Ind}(D_a \oplus (d^+ + d^*)),$$

where $D_a\colon \Gamma(S^+) \to \Gamma(S^-)$, *and* $d^+ + d^*\colon \Omega^1(M) \to \Omega^{2,+}(M) \oplus \Omega^0(M)$.

In the case where the manifold M is Kähler, then any spin^c structure on M has determinant line bundle $K_M^{-1} \otimes L_0^2$ for some line bundle L_0 over M, and the space of positive spinors is equal to

$$\Gamma(S^+) = \Omega^0(M, L_0) \oplus \Omega^{(0,2)}(M, L_0),$$

hence a spinor can be written as $\psi = (\alpha, \beta)$, and the connection $a = A_0 \otimes A$, where A_0 is the canonical holomorphic connection on K_M^{-1}. In this case there is a holomorphic description of the moduli space which looks as follows.

Theorem 4.3 ([**Moo96**, Thm. 3.9])**.** *Assume that M is Kähler and let (a, ψ) be a solution to the Seiberg–Witten equations. Then the following holds:*

i) *the connection A defines a holomorphic structure on L_0, i.e. $(F_A)^{(0,2)} = 0$,*
ii) $\alpha\overline{\beta} = 0$,
iii) *α is a holomorphic section of L_0 and β is an antiholomorphic section of $\Omega^{(0,2)}(L_0)$ with respect to the abovementioned holomorphic structure.*

Moreover, in this case one can actually compute the whole moduli space.

Theorem 4.4 ([**Mor96**, Thm. 7.3.1])**.** *Let M be a Kähler surface such that $\deg K_X > 0$. Then for the choice of the spin^c structure whose determinant line bundle is equal to K_X^{-1}, there is a unique solution of the Seiberg–Witten equations. Moreover, if the scalar curvature s of M is a constant and $s < 0$, this solution is given by $(\alpha, \beta) = (\sqrt{-s}, 0)$ in the above picture.*

The main goal of this chapter will be to provide the generalizations of the abovementioned statements to the noncommutative context.

2. Noncommutative Seiberg–Witten equations

We consider a noncommutative geometry of dimension 4 over the pair $(\mathcal{A}, \mathcal{A})$ with the smooth bimodule $\mathcal{H}^\infty = \mathcal{X}_0$ such that the bimodules of differential forms are finitely generated and projective. Moreover, we assume that

$$\operatorname{End}_{\mathcal{A}}(\mathcal{X}_0) = \bigoplus_{k=1}^{4} \Omega^k_{\mathcal{D}}$$

and that

$$\Omega^4_{\mathcal{D}} \cong \mathcal{A}\gamma.$$

Thus, the endomorphisms of $\mathcal{X}_0$ decompose in the even and odd parts: the even endomorphisms commute with the grading γ, and the odd endomorphisms anticommute with γ. The even endomorphisms decompose as

$$\operatorname{End}_{\mathcal{A}}(\mathcal{X}_0)^{\text{even}} \cong \operatorname{End}_{\mathcal{A}}(\mathcal{X}_0^+) \oplus \operatorname{End}_{\mathcal{A}}(\mathcal{X}_0^-)$$

Let us consider the mapping

$$\varepsilon : \operatorname{End}_{\mathcal{A}}(\mathcal{X}_0)^{\text{even}} \to \operatorname{End}_{\mathcal{A}}(\mathcal{X}_0)^{\text{even}},$$
$$T \mapsto \gamma T = T\gamma.$$

In view of the assumptions above this mapping interchanges $\Omega^4_{\mathcal{D}}$ and $\mathcal{A}$ and therefore leaves $\Omega^2_{\mathcal{D}}$ invariant. Thus, $\Omega^2_{\mathcal{D}}$ decomposes as

$$\Omega^2_{\mathcal{D}} \cong \Omega^{2,+}_{\mathcal{D}} \oplus \Omega^{2,-}_{\mathcal{D}},$$

where $\Omega^{2,+}_{\mathcal{D}} = \{T \in \Omega^2_{\mathcal{D}} \mid T\gamma = T\}$.

We take a twisting bimodule $\mathcal{E}$ and consider the twisted noncommutative geometry as in Theorem 2.81, using the metric-compatible connection as a parameter. We denote the twisted bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{X}_0$ by $\mathcal{X}$ and assume our bimodule $\mathcal{E}$ to carry a metric-compatible biconnection ∇_0 which we will use as a reference connection. The role of the twisting bimodule is twofold: first of all, it provides the freedom in the choice of the "spin^c structure", but even if we take $\mathcal{E} = \mathcal{A}$ to be the trivial $\mathcal{A}$-$\mathcal{A}$-bimodule, it will still provide the necessary degree of freedom in the cohice od a connection.

2.1. "Spin" version. We first describe the version of the Seiberg–Witten equations starting with a spin noncommutative geometry because in this case there is no additional term in the second equation and we therefore need no additional assumptions on the bimodule $\mathcal{X}_0$.

We introduce the *configuration space* as

$$C = \{(\nabla, \psi) \,|\, \nabla \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}} \text{ a metric compatible connection}, \psi \in \mathcal{X}^+\},$$

where $\mathcal{X}^+ = \{\psi \in \mathcal{X} \,|\, \psi = \gamma\psi\}$. This is an infinite-dimensional affine Fréchet space modeled on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}})_{sa} \oplus \mathcal{X}^+$. Using the metric compatible biconnection ∇_0 on $\mathcal{E}$ and the isomorphism $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}})_{sa} \cong \Omega^1_{\mathcal{D},sa}$, we reparametrize the configuration space as

$$C = \{(A, \psi) \,|\, A \in \Omega^1_{\mathcal{D},sa}, \psi \in \mathcal{X}^+\},$$

where $\Omega^1_{\mathcal{D},sa} = \{A \in \Omega^1_{\mathcal{D}} \,|\, A = A^*\}$ is the subspace of self-adjoint forms.

We introduce the mapping $q \colon \mathcal{X}^+ \to \Omega^{2,+}_D$,

$$q(\psi) := {\pi^{2,+}}_{\mathrm{End}_{\mathcal{A}}(\mathcal{X})}\langle \psi, \psi \rangle$$

where $\pi^{2,+}$ is the projection from $\mathrm{End}_{\mathcal{A}}(\mathcal{X}^+)$ to $\Omega^{2,+}_D$ obtained from the decomposition

$$\mathrm{End}_{\mathcal{A}}(\mathcal{X}^+) \cong \mathcal{A} \oplus \Omega^{2,+}_D.$$

This is automatically an $\mathcal{A}$-$\mathcal{A}$-bimodule map, and it coincides with $(1 - E)$, where E is the canonical $\mathcal{A}$-valued conditional expectation.

Notice that q satisfies

$$q(a\psi) = aq(\psi)a^*, a \in \mathcal{A}$$

because $\mathcal{A} \subset \mathrm{End}_{\mathcal{A}}(\mathcal{X})$ and $\pi^{2,+}$ is a bimodule map.

Definition 4.5. The Seiberg–Witten equations are given by

$$D_\nabla \psi = 0, \tag{4.1}$$

$$2F^+_\nabla = q(\psi), \tag{4.2}$$

where D_∇ is the Dirac operator twisted with the connection ∇, cf. Theorem 2.81, and $F^+_\nabla := \pi^+ F_\nabla$, where π^+ is the projection onto $\Omega^{2,+}_D$.

Proposition 4.6. *Let the group $\mathcal{G} = \mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \,|\, u^*u = uu^* = 1\}$ act in the configuration space C by*

$$u \colon \psi \mapsto u\psi, \quad \nabla \mapsto u\nabla u^*.$$

The Seiberg–Witten equations are invariant unter the action of $\mathcal{G}$.

Proof. This is a straightforward computation: the first equation transforms as

$$D_{u\nabla u^*} u(\eta \otimes \chi) = u\nabla u^* u\eta \cdot \chi + u\eta \otimes D\chi = uD_\nabla \psi = 0,$$

and the second equation transforms as

$$2F^+_{u\nabla u^*} - q(u\psi) = 2uF^+_\nabla u^* - uq(\psi)u^* = 0,$$

because the curvature transforms covariantly.This finishes the proof. □

Remark 4.7. We introduced the factor 2 in front of the curvature in the second equation because in the classical Seiberg–Witten equations one makes use of the curvature of the connection on the determinant line bundle and not on the twisting bundle.

2.2. General version. If we have a general 4-dimensional noncommutative geometry over $(\mathcal{A}, \mathcal{A})$, then the bimodule $\mathcal{X}$ is not necessarily self-dual. However, as we have seen in Section 12, it makes sense to consider the "determinant line bundle" in this case, which potentially makes it possible to consider the Seiberg–Witten equations in this case as well. In this subsection, we make an additional assumption that the bimodule $\mathcal{X}$ is equipped with a biconnection ∇ compatible with the endomorpisms and such that $D = m \circ \nabla$.

In the case of an ordinary spin^c manifold there is a natural correspondence between connections on the spinor bundle which are compatible with the Levi-Civita connection and connections on the determinant bundle. Here we don't have such a direct correspondence and therefore have to use an ad hoc construction: we demand that we have a starting biconnection on $\mathcal{X}$ and induce with it a biconnection on the "determinant bundle" whose curvature then enters the second equation.

Let $\mathcal{T} := \overline{\mathcal{X}} \otimes_{\mathrm{End}_{\mathcal{A}}(\mathcal{X})} \mathcal{X}$ be the determinant $\mathcal{A}$-$\mathcal{A}$-bimodule. Recall that an endomorphism-compatible biconnection on $\mathcal{X}$ induces a biconnection on $\mathcal{T}$ by

$$\nabla(\xi \otimes \eta) = \nabla\xi \otimes \eta + \xi \otimes \nabla\eta.$$

We introduce the *configuration space* as

$$C = \{(\nabla, \psi) \,|\, \nabla \colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}} \text{ a metric compatible connection}, \psi \in \mathcal{X}^+\}.$$

This is an infinite-dimensional Fréchet manifold modeled on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{D}}) \oplus \mathcal{X}^+$. Using the metric compatible biconnection ∇_0 on $\mathcal{E}$ as a reference connection and repeating the construction of the preceding subsection, we reparametrize the configuration space as

$$C = \{(A, \psi) \,|\, A \in \Omega^1_{\mathcal{D},sa}, \psi \in \mathcal{X}^+\}.$$

Definition 4.8. Let ∇_0 be a biconnection on $\mathcal{X}$ such that $D = m \circ \nabla_0$ and let $\nabla_{\mathcal{T}}$ be the corresponding connection on $\mathcal{T}$. Let $F_{\nabla_{\mathcal{T}}} \in \mathrm{Hom}(\mathcal{T}, \mathcal{T} \otimes_{\mathcal{A}} \Omega^2_{\mathcal{D}}) \cong \Omega^2_{\mathcal{D}}$ be the curvature of $\nabla_{\mathcal{T}}$. The Seiberg–Witten equations are given by

$$D_\nabla \psi = 0, \tag{4.3}$$

$$2F^+_\nabla + F^+_{\nabla_{\mathcal{T}}} = q(\psi). \tag{4.4}$$

As before, we prove

Proposition 4.9. *Let the group $\mathcal{G} = \mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \,|\, u^*u = uu^* = 1\}$ act in the configuration space C by*

$$u \colon \psi \mapsto u\psi, \quad \nabla \mapsto u\nabla u^*.$$

The Seiberg–Witten equations are invariant unter the action of $\mathcal{G}$.

Proof. This is a straightforward computation: the first equation transforms as

$$D_{u\nabla u^*} u(\eta \otimes \chi) = u\nabla u^* u\eta \cdot \chi + u\eta \otimes D\chi = uD_\nabla \psi = 0,$$

and the second equation transforms as

$$2F^+_{u\nabla u^*} + F^+_{\nabla_{\mathcal{T}}} - q(u\psi) = 2uF^+_A u^* + uF^+_{\nabla_{\mathcal{T}}} u^* - uq(\psi)u^* = 0,$$

because the curvature transforms covariantly, and the curvature of a biconnection commutes with both actions of $\mathcal{A}$. This finishes the proof. $\square$

Notice that in the commutative case this construction yields the classical Seiberg–Witten equations with the gauge group $\mathcal{G} = C^\infty(M, U(1))$, using the line bundles as twisting bimodules.

We define the Seiberg–Witten functional

$$SW\colon C \to \mathcal{X}^- \oplus \Omega_D^{2,+}(\mathcal{A})$$

by

$$(\nabla, \psi) \mapsto (D_\nabla \psi, 2F_\nabla^+ + F_{\nabla_\mathcal{T}}^+ - q(\psi)).$$

3. Moduli space and the deformation complex

The C^∞-moduli space of the Seiberg–Witten equations is defined to be $\mathcal{M} := SW^{-1}(0)/\mathcal{G}$. However, for technical reasons it is often more convenient to work with Hilbert manifolds, and therefore typically one carries out the analysis of the Seiberg–Witten equations working with Sobolev spaces. As the Sobolev theory on noncommutative manifolds is not yet available in full generality, the proof of the regularity theorem is now restricted to the case of toric deformations, although we expect a similar argument to work in full generality.

Analogous to the classical case, we introduce several other variants of the moduli space and the deformation complex. The proofs we give here are based on the ideas of proof in the Seiberg–Witten and Donaldson gauge theories as well as their generalizations in $PU(2)$-monopole theory. The proofs used can be directly applied to our situation, because the properties of all ingredients involved in the proofs are still at hand in our situation.

Recall that the Sobolev norms are defined on $\mathcal{A}$ as follows:

$$\|a\|_s = \langle (1+\Delta)^s a, a \rangle ,$$

where Δ is the Laplace operator on $L^2(\mathcal{A})$.

As Hilbert manifolds are generally easier to deal with as Fréchet manifolds, we replace the configuration space and the moduli space by their Sobolev counterparts. Then an elliptic regularity argument is used to prove that the moduli space is from some point on independent of the choice of a particular Sobolev completion (the parameter s above), and thus coincides with the smooth moduli space. This is intimately related with the following question about multiplicative properties of the Sobolev norms: for which values of s does the multiplication map $m\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ extend by continuity to a continuous map $m\colon H^s \times H^l \to H^l$, $0 \leqslant l \leqslant s$? It is well-known that in the commutative situation it does so for $s \geqslant d/2$, and for noncommutative tori it is known to be the case for $s > d$, and it is known that for $s \geqslant d$ the C^*-algebra $C(\mathbb{T}^n_\theta) \subset H^s(C^\infty(\mathbb{T}^n_\theta))$ [**Spe91**]. From this it is easy to deduce the following result:

Proposition 4.10. *Let $\mathcal{A} = C^\infty(M_\theta)$ be a toric deformation of the manifold M of dimension d, and let $s > d$. Then the multiplication map $m\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ extends by continuity to a map $m\colon H^s \times H^l \to H^l$, for $0 \leqslant l \leqslant s$ giving the space H^s the structure of a Banach $*$-algebra and the space H^l the structure of a Banach module over H^s.*

The following two observations are easy but crucial for our analysis of the situation.

Proposition 4.11. *If the multiplication map $m\colon \mathcal{A} \times \mathcal{A} \to A$ extends to a map $m\colon H^s \times H^s \to H^s$, then the following holds true: for any finitely generated projective right module $\mathcal{E}$ over $\mathcal{A}$ the s-th Sobolev completion of the endomorphism algebra $\mathrm{End}_\mathcal{A}(\mathcal{E})$ is a Banach $*$-algebra. Moreover, the multiplication map $\mathrm{End}_\mathcal{A}(\mathcal{E}) \times \mathcal{E} \to \mathcal{E}$ extends to a map $H^s(\mathrm{End}_\mathcal{A}(\mathcal{E})) \times H^l(\mathcal{E}) \to H^l(\mathcal{E})$, $0 \leqslant l \leqslant s$.*

PROOF. Obvious after identification of endomorphisms with a subalgebra of matrices over $\mathcal{A}$. □

Proposition 4.12. *In the assumptions of the previous proposition, the quadratic map* $\mathcal{E} \to \mathrm{End}_{\mathcal{A}}(\mathcal{E})$, $\psi \mapsto \Theta_{\psi,\psi}$, *extends to a map* $H^s(\mathcal{E}) \times H^s(\mathcal{E}) \to H^s(\mathrm{End}_{\mathcal{A}}(\mathcal{E}))$.

PROOF. Using the isomorphism $\mathcal{E} \cong e\mathcal{A}^n$, one identifies the quadratic map with

$$(a_1, \dots, a_n) \mapsto (a_i a_j^*)_{i,j=1,\dots,n},$$

and the claim follows. □

For the rest of the section, we will assume the following property, which is satisfied in examples due to the above observations.

Definition 4.13. We call the noncommutative manifold Sobolev-multiplicative, if it has reasonable Sobolev theory and the multiplication map $m\colon \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ extends to the Sobolev space H^s for s big enough.

Now it makes sense to consider the Sobolev configuration space

$$C_s = \{(A,\psi) \mid A \in H^s(\Omega^1_D)_{sa}, \psi \in H^s(\mathcal{X}^+)\}$$

and the "Sobolev version" of the Seiberg–Witten equations (4.3).

We introduce the Sobolev version of the gauge group $\mathcal{G}_s$ as the completion of the group $\mathcal{G}$ endowed with the Sobolev topology. It is a Hilbert Lie group modeled on the Sobolev completion

$$H^s(\mathcal{A})_{sa} = \{f \in H^s(\mathcal{A}) \mid f = f^*\}.$$

Repeating the arguments in [**Mor96**], one obtains

Proposition 4.14. *The configuration space* C_s *is an affine Hilbert space modeled on* $H^s(\Omega^1_{\mathcal{D}}) \oplus H^s(\mathcal{X}^+)$, *hence its tangent space canonically identifies with this Hilbert space. The Hilbert Lie group* $\mathcal{G}_{s+1}$ *acts smoothly on the configuration space* C_s, *and the Seiberg–Witten functional extends to a smooth function on the configuration space* C_s.

Now we turn to a technical result which will be useful when comparing the moduli spaces corresponding to different values of s. Its proof is a straightforward adaptation of the well-known proof of the existence of a local Coulomb gauge for $SU(2)$ gauge theory, and we follow the strategy outlined in [**DK90**, Ch. 3]. To prove it, we have to introduce the Coulomb gauge condition.

Using the reference biconnection ∇_0 on $\mathcal{E}$, the connections on $\mathcal{E}$ will be parameterized by one-forms $A \in \Omega^1_{\mathcal{D}}$. This allows us to consider the metric on connections given by

$$\rho(\nabla, \nabla') = \langle \nabla - \nabla', \nabla - \nabla' \rangle^2,$$

It is straightforward to check that this metric is gauge-invariant with respect to the diagonal action of $\mathcal{U}(\mathcal{A})$:

$$\rho(u\nabla u^*, u\nabla' u^*) = \rho(\nabla, \nabla').$$

We will also write $\rho(A, B)$ for $\rho(\nabla_0 + A, \nabla_0 + B)$. Obviously,

$$\rho(A, B) = \|A - B\|_2^2.$$

We observe that, as ∇_0 is a biconnection,

$$u\nabla_0 u^* = u[D, u^*] + \nabla_0$$

Given a connection with a connection form A, let us define the operator

$$d_A = d + [A, \cdot]\colon \mathcal{A} \to \Omega^1(\mathcal{A}),$$

and if $u \in \mathcal{U}(\mathcal{A})$ is a gauge transformation, we use the abbreviation $u(A)$ to denote $u(\nabla_0 + A)u^* - \nabla_0$.

Definition 4.15. We say that a connection B is in Coulomb gauge with respect to A if

$$d_A^*(B - A) = 0.$$

We observe that the Coulomb gauge condition is given by the Euler-Lagrange equation for the functional

$$B \mapsto \|B - A\|^2$$

on the space of connections. Indeed, if we consider the one-parameter family of gauge transformations of the form

$$u_t = \exp(t\chi),$$

then

$$u_t(\nabla_0 + B)u_t^* = \nabla_0 + u_t[D, u_t^*] + u_t B u_t^*,$$

and, as

$$\frac{d}{dt}\Big|_{t=0} (u_t[D, u_t^*] + u_t B u_t^*) = -[D, \chi] - [B, \chi] = -d_B\chi,$$

and thus we obtain

$$\frac{d}{dt}\Big|_{t=0} \|u_t(B) - A\|^2 = -\langle \chi, d_B^*(B - A)\rangle .$$

This description and the gauge invariance of the metric yield the following observation: the Coulomb gauge condition is symmetric and gauge invariant, i.e.

$$d_A^*(B - A) = 0 \;\Leftrightarrow d_B^*(A - B) = 0.$$

Proposition 4.16. *Let $\nabla = \nabla_0 + A$ be a metric compatible connection on $\mathcal{E}$. There is a constant $c(A)$ such that if $\nabla' = \nabla_0 + B$ is another connection on $\mathcal{E}$ and if $a = B - A$ satisfies*

$$\|a\|_{H^s} < c(A),$$

then there is a gauge transformation u such that $u\nabla' u^*$ is in the Coulomb gauge with respect to ∇:

$$d_A^*(u\nabla' u^* - \nabla) = 0.$$

Proof. We have to solve the equation

$$d_A^*((d_A u)u^* - uau^*) = 0.$$

We use the ansatz $u = \exp(\chi)$ and set

$$G(\chi, a) = d_A^*(d_A e^{\chi})e^{-\chi} - e^{\chi} a e^{-\chi}).$$

The map G extends to a smooth map $H^{s+1} \times H^s \to H^{s-1}$, and the image of G is contained in the H^{s-1}-closure of $\operatorname{im} d_A^*$. The derivative of G at the point $\chi = 0$, $a = 0$ is given by

$$DG(\xi, b) = d_A^* d_A \xi - d_A^* b - d_A^*[\chi, a],$$

and thus the implicit function theorem gives a small solution χ to the equation $G(\chi, a) = 0$ for all a small enough, because the map $\xi \mapsto d_A^* d_A \xi$ maps onto $\operatorname{im} d_A^*$ by the Fredholm alternative for the operator $d_A^* d_A$. □

The following lemma is an adaptation of [**Mor96**, Lemma 4.5.3] and has the same purpose: it will guarantee that the moduli space is Hausdorff.

Proposition 4.17. *Let (A_n, ψ_n), $(B_n, \mu_n) \in C_s$ be two sequences which converge to (A, ψ) resp. (B, μ) in C_s. Let $u_n \in \mathcal{G}_{s+1}$ be the gauge transformations such that for all $n \in \mathbb{N}$*

$$u_n(A_n, \psi_n) = (B_n, \mu_n).$$

Then the sequence $u_n \in \mathcal{G}_{s+1}$ contains a convergent subsequence with a limit $u \in \mathcal{G}_{s+1}$ such that

$$u(A, \psi) = (B, \mu).$$

Proof. By assumption, we have for all $n \in \mathbb{N}$

$$[D, u_n] = u_n A_n - B_n u_n.$$

The norms $\|B_n\|_{H^s}$ and $\|A_n\|_{H^s}$ are uniformly bounded, and so is $\|u_n\|_{H^0}$; thus, by continuity of multiplication, $\|[D, u_n]\|_{H^0}$ is uniformly bounded, which means exactly that $\|u_n\|_{H^1}$ is uniformly bounded. Now, a bootstrapping argument using the above equation implies that $\|u_n\|_{H^{s+1}}$ is uniformly bounded, and, as the embedding $H^{s+1} \subset H^s$ is compact, after passing to a subsequence we may assume that $u_n \to u$ in H^s. Clearly, $[D, u] = uA - Bu \in H^s$, and $[D, u_n] \to [D, u]$ in H^s, which implies $u \in H^{s+1}$ and $u_n \to u$ in H^{s+1}. Passing to the limit and using the continuity of the action on the configuration space, we obtain $u(A, \psi) = (B, \mu)$, as desired. □

Corollary 4.18. *The quotient $C_s/\mathcal{G}_{s+1}$ is Hausdorff.*

Proof. As all the spaces in question are first-countable, the non-Hausdorffness would imply the existence of sequences $(A_n, \psi_n) \in C_s$ and $u_n \in \mathcal{G}_{s+1}$ such that $(A_n, \psi_n) \to (A, \psi)$, $(A_n, \psi_n)u_n \to (B, \mu)$ such that (A, ψ) and (B, μ) are not gauge equivalent. But this is impossible by the above lemma. □

We now arrive at the key result, which is proved similar to the classical case using the elliptic bootstrapping technique.

Proposition 4.19. *For s big enough, any H^s-solution of the Seiberg–Witten equations is $\mathcal{G}_{s+1}$-equivalent to a smooth solution.*

Proof. Let us fix a smooth connection ∇_0 on $\mathcal{E}$ and write a H^s-solution (∇, ψ) as (A, ψ), where $A \in H^s(\Omega^1_{\mathcal{D}})$. We know by Proposition 4.16 that every H^s-connection B which is close to A can be tranformed into the Coulomb gauge relative to A, i.e. we can find $u \in H^{s+1}$ such that

$$d_A^*(u^{-1}(B) - A) = 0.$$

By the symmetry of the Coulomb gauge condition,

$$d^*_{u^{-1}B}(A - u^{-1}(B)) = 0.$$

By the invariance of the Coulomb gauge condition, writing $A' = u(A) = B + a$, we have

$$d_B^* a = 0. \tag{4.5}$$

Now we choose such a smooth connection B, and obtain for the difference form a the equation

$$d_B^+ a + (a \wedge a)^+ = -F^+_{\nabla_0 + B} - \frac{1}{2} F^+_{\nabla_{\mathcal{T}}} + \frac{1}{2} u q(\psi) u^*, \tag{4.6}$$

this being the gauge-transformed second Seiberg–Witten equation. The first equation gives

$$(D + u(A)) u\psi = 0. \tag{4.7}$$

Now, as s is in the multiplicative range for Sobolev spaces, the right-hand side as well as the term $(a \wedge a)^+$ in the equation (4.6) are in H^s and thus by the elliptic regularity of the operator $d_B^* \oplus d_B^+$ we obtain that $a \in H^{s+1}$. The elliptic regularity of the operator D in (4.7) yields $u\psi \in H^{s+1}$. But then $uq(\psi)u^* \in H^{s+1}$, and elliptic regularity of the operator in (4.6) gives $a \in H^{s+2}$. The elliptic regularity of $(D+a)$ in (4.7) then implies $u\psi \in H^{s+2}$. By a bootstrapping argument we conclude that $u\psi$ and a are indeed smooth. □

Finally, we consider the deformation complex associated with the Sobolev version of Seiberg–Witten equations. For any $\mathcal{G}_{s+1}$-invariant functional $F\colon C_s \to \mathcal{H}$ on the configuration space C_s and for any $a \in F^{-1}(0)$ one can consider the complex

$$T\mathcal{G}_{s+1}(e) \xrightarrow{D\sigma} TC_s(a) \xrightarrow{DF} \mathcal{H},$$

where the first map is induced by the infinitesimal action of $\mathcal{G}_{s+1}$, and the second map is induced by the differential of F at the solution a. The idea behind this is to encode the "tangent space to the moduli space" in this deformation complex. Indeed, it is natural to expect that the tangent space to the moduli space is equal to the first cohomology of the complex above. The zeroth cohomology measures the lack of freeness in the gauge action, and the second cohomology measures the lack of transversality at a. Although we are yet unable to solve the transversality issue in the noncommutative case, it is still interesting to see that the ellipticity holds true, thus giving a hope to achieve smoothness of the moduli space.

Theorem 4.20. *The linearization of the Seiberg–Witten equations at a solution* (A,ψ) *yields an elliptic complex*

$$0 \to H^{s+1}(\mathcal{A}_{sa}) \xrightarrow{D_1} H^s(\Omega^1_{\mathcal{D},sa}) \oplus H^s(S^+) \xrightarrow{D_2} H^{s-1}(\Omega^{2,+}(\mathcal{A})) \oplus H^{s-1}(S^-),$$

where $D_1(f) = (-d_A f, f\cdot\psi)$, $D_2 = D(SW)_{(a,\psi)} = \begin{pmatrix} d^+ + A^2 & -Dq_\psi \\ (\cdot)\psi & D+A \end{pmatrix}$. *Its Euler characteristic is equal to the index of the operator*

$$\chi(C) = \operatorname{Ind}(D_A \oplus (d^+ + d^*)),$$

where $D_A\colon \mathcal{X}^+ \to \mathcal{X}^-$, *and* $d^+ + d^*\colon \Omega^1_{\mathcal{D}} \to \Omega^{2,+}_{\mathcal{D}} \oplus \mathcal{A}$.

Proof. To compute the first differential, we have to compute the infinitesimal action of the gauge group on (A,ψ). Taking $u_t = e^{tf}$, we obtain

$$\frac{d}{dt}\Big|_{t=0} u_t\psi = f\psi,$$

$$\frac{d}{dt}\Big|_{t=0} (u_t A u_t^* + u_t[D,u_t^*]) = -d_A f.$$

To compute the second map, we have to differentiate the functional SW at (A,ψ). For this, consider a tangent vector (α,η) and compute

$$\frac{d}{dt_i}\Big|_{t_i=0} (D_{\nabla+A+t_1\alpha})(\psi+t_2\eta) - (D_{\nabla+A}\psi), \quad i=1,2.$$

A short computation gives us exactly the second row in the matrix above.

Now we have to compute the first row in the matrix. For this, we have to understand the map $\nabla \to F^+_\nabla$ in terms of the "connection form" A. This is given by identifying

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^1_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{E}) \cong \Omega^1_{\mathcal{D}}$$

and writing $\nabla = \nabla_0 + A$, where ∇_0 is a biconnection. We get

$$(\nabla_0 + A)(\nabla_0 + A)\xi = F_{\nabla_0}\xi + (dA + A^2)\xi,$$

where $dA + A^2$ is understood in terms of differential forms (A^2 being a 2-form). Thus, we get the desired desult.

Now, by the gauge invariance of the functional SW, the above sequence is a complex, and we are interested in ints Euler characteristic. It is well-known that the Euler characteristic of a complex

$$C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_2$$

is given by the index $\operatorname{Ind}(d_0^* \oplus d_1 \colon C_1 \to C_0 \oplus C_2)$. Thus, all we have to do is to analyze the operators which enter the complex. But it is easy to see that up to operators of order 0 D_1 coincides with the differential $d \colon \mathcal{A} \to \Omega^1_{\mathcal{D}}$, and D_2 coincides with the operator $d^+ \oplus (D + A)$. Thus, the Euler characteristic of this complex is given by

$$\chi(C) = \operatorname{Ind}(D_A \oplus (d^+ \oplus d^*)),$$

as desired. □

4. Toric deformations of Kähler manifolds

In this chapter we consider a 4-dimensional compact Kähler manifold M with a holomorphic action of $\mathbb{T}^2$. We analyze the Seiberg–Witten equations in this case, deriving a holomorphic description of the moduli space, analogous to that of [**Mor96**].

As the $\mathbb{T}^2$-action is supposed to be holomorphic, the canonical bundle K_M is equivariant with respect to this action, and the Kähler form $\omega \in \Omega^{1,1}$ is invariant. There is a canonical spin^c structure on M for which the determinant line bundle coincides with K_M^{-1}, and the decomposition in positive and negative spinors is as follows:

$$\Gamma(S^+) = \Omega^0(M) \oplus \Omega^{0,2}(M),$$
$$\Gamma(S^-) = \Omega^{(0,1)}(M),$$

and the Clifford multiplication with a form $a \in \Omega^1(M, C)$ is equal to the difference of the exterior product and the insertion of $\sqrt{2}\pi^{0,1}(a) \in \Omega^{0,1}(M)$. Moreover, the Dirac operator is given by

$$D = \sqrt{2}\left(\overline{\partial} + \overline{\partial}^*\right) : \Omega^0(M) \oplus \Omega^{0,2}(M) \to \Omega^{(0,1)}(M).$$

If $L_0 \to M$ is a $\mathbb{T}^2$-equivariant line bundle over M we can twist the canonical spin^c-structure over M by L_0 and obtain another spin^c-structure, whose spaces of positive and negative spinors are given by

$$\Gamma(S^+_{L_0}) = \Omega^0(M, L_0) \oplus \Omega^{0,2}(M, L_0),$$
$$\Gamma(S^-_{L_0}) = \Omega^{(0,1)}(M, L_0).$$

and the Clifford multiplication with a form $a \in \Omega^1(M, C)$ is as before equal to the difference of the exterior product and the insertion of $\sqrt{2}\pi^{0,1}(a) \in \Omega^{0,1}(M)$.

Let us now consider the toric deformation M_θ of M, twisted by the bimodule $\mathcal{L} = (\Gamma(M, L_0)\widehat{\otimes}C^\infty(\mathbb{T}^2))^{\sigma\times\tau^{-1}}$. First of all, we observe that by holomorphicity of the action and the above decomposition of the spinor bundle, the spinor bundle is actually σ-equivariant, and therefore the construction of the previous chapter yields $\mathcal{X} = (\Gamma(M, S_{L_0})\widehat{\otimes}C^\infty(\mathbb{T}^2))^{\sigma\times\tau^{-1}})$. Moreover, as the canonical holomorphic connection ∇ on the determinant bundle was $\mathbb{T}^2$-invariant by assumption, we get a canonical biconnection on the $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{X}$, which induces a connection on the determinant bimodule $\mathcal{T}$; the latter is here isomorphic to

$$\mathcal{T} \cong (\Gamma(M, K_M)\widehat{\otimes}C^\infty(\mathbb{T}^2_\theta))^{\sigma\times\tau^{-1}}.$$

Let us now consider the curvature equations. On a Kähler manifold there is the following decomposition of the self-dual 2-forms:

$$\Omega^{2,+}(M) = \Omega^0(M)\cdot\omega \oplus \left(\Omega^{2,0}(M) \oplus \Omega^{0,2}(M)\right)$$

We now recollect the facts about the action of these forms on spinors. These can be proven by direct computation and are as follows [**Mor96**, Sect. 7.1]:

i) the action of $(2,0)$-forms on $\Gamma(S^+)$ is trivial;
ii) the Clifford multiplication with a $(0,2)$-form μ acts on $\Omega^0(M)$ as exterior product with 2μ;
iii) the Clifford multiplication with a $(0,2)$-form μ acts on $\Omega^{(0,2)}(M)$ as insertion of μ, given by $(\mu,\lambda)\mapsto 2*(\mu\wedge\overline{\lambda})$, $*$ being the Hermitian continuation of the standard Hodge operator: $*f\,\mathrm{vol}=\overline{f}$;
iv) the action of the Kähler form ω is given by the multiplication with $2i$ on $\Omega^{(0,2)}(M)$ and with $-2i$ on $\Omega^0(M)$.

Thus, in the deformed case we naturally have the following description of the relevant spaces and actions:

$$\mathcal{X}^+\cong\mathcal{L}\oplus\mathcal{L}\otimes\Omega^{(0,2)}(\mathcal{A}),$$

$$\Omega^{2,+}=(\mathcal{A}\cdot(\omega\otimes 1))\oplus\left(\Omega^{(2,0)}(M_\theta)\oplus\Omega^{(0,2)}(M_\theta)\right).$$

The Clifford multiplication by a 2-form $\eta=if\omega+\mu$, $f\in\mathcal{A}$, $\mu\in\Omega^{(0,2)}$ is given by

$$\begin{pmatrix} 2f & *2\,(\mu\wedge\bar{\cdot})\\ 2(\mu\wedge\cdot) & -2f\end{pmatrix}.$$

On the other hand, the matrix representation for $\Theta_{\psi,\psi}$ is given by

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\overline{\alpha} & \overline{\beta}\end{pmatrix}=\begin{pmatrix}\alpha\overline{\alpha} & \alpha\overline{\beta}\\ \beta\overline{\alpha} & \beta\overline{\beta}\end{pmatrix}.$$

Therefore, $q(\psi)$ is given by the matrix

$$\begin{pmatrix}\frac{1}{2}\left(\alpha\overline{\alpha}-\beta\overline{\beta}\right) & \alpha\overline{\beta}\\ \beta\overline{\alpha} & \frac{1}{2}\left(\beta\overline{\beta}-\alpha\overline{\alpha}\right)\end{pmatrix}$$

Theorem 4.21. *Let (∇,ψ) be a solution of the Seiberg–Witten equations on the deformed compact Kähler manifold M_θ. Let $\psi=(\alpha,\beta)$ with $\alpha\in\Omega^0(M_\theta)\otimes\mathcal{L}_0$, $\beta\in\Omega^{(0,2)}(M_\theta)\widehat{\otimes}\mathcal{L}_0$. Then*

i) *∇ is a holomorphic connection on $\mathcal{L}_0$,*
ii) *α is a holomorphic section of $\mathcal{L}_0$ with respect to this holomorphic structure,*
iii) *β is an antiholomorphic section of $\mathcal{K}_M\otimes\mathcal{L}_0$,*
iv) *$\alpha\overline{\beta}=0$.*

Proof. The equation for the spinor is written as

$$\overline{\partial}_\nabla(\alpha)+\overline{\partial}^*_\nabla(\beta)=0.$$

If we apply $\overline{\partial}_\nabla$ to this equation, we get

$$\overline{\partial}_\nabla\overline{\partial}_\nabla(\alpha)+\overline{\partial}_\nabla\overline{\partial}^*_\nabla(\beta)=0. \tag{4.8}$$

Now,

$$\overline{\partial}_\nabla\overline{\partial}_\nabla(\alpha)=F^{(0,2)}_\nabla\cdot\alpha.$$

Thus, from our equations we get

$$F^{(0,2)}_\nabla=\frac{1}{4}\beta\overline{\alpha},$$

and, inserting this into the above equation, we obtain

$$\frac{1}{4}\beta\overline{\alpha}\alpha+\overline{\partial}_\nabla\overline{\partial}^*_\nabla(\beta)=0.$$

Taking the L^2-scalar product with β (by multiplying with $\overline{\beta}$ from the right and taking the trace), we get

$$\frac{1}{2}\left\|\alpha\overline{\beta}\right\|_2^2 + \left\|\overline{\partial}^*_\nabla(\beta)\right\|_2^2 = 0.$$

Thus, we conclude that $\alpha\overline{\beta} = 0$, and $\overline{\partial}^*_\nabla(\beta)$. But now we observe that then $F_\nabla^{(0,2)} = 0$, thus ∇ is a holomorphic connection in this case, and the condition $\overline{\partial}^*_\nabla(\beta) = 0$ means exactly that β is antiholomorphic. □

In the classical Seiberg–Witten theory one could at this point conclude that one of the forms α and β vanishes identically, because one of them has to vanish on an open subset of M. It is not clear yet whether a similar condition holds in the noncommutative situation.

We now give an example of the computation of the moduli space. To carry out the computation, we first need to prove an analogue of the Weitzenböck formula.

Lemma 4.22. *Consider a toric deformation M_θ of a compact spin^c manifold M with an equivariant determinant bundle L. Let $\nabla_0\colon \mathcal{X} \to \Omega^1_{\mathcal{D}}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{X}$ be the biconnection on $\mathcal{X}$ given by the invariant connection on L, and let $\nabla = \nabla_0 + A$ be a connection given by twisting ∇_0 with a metric compatible connection $\nabla' = d + A$ on the trivial right $\mathcal{A}$-module $\mathcal{A}$, and let D_A be the corresponding twisted Dirac operator. Then the following formula holds true:*

$$D_A^2 = \nabla^*\nabla + \frac{s}{4} + \frac{F_{\det,\nabla_0}}{2} + F_A,$$

where $F_{\det,\nabla_0}$ denotes the curvature of the induced connection on the determinant line bundle and s denotes the scalar curvature of M.

Proof. The twisted Dirac operator is given by

$$D_A = D + A,$$

where $A \in \Omega^1_{\mathcal{D},sa}(\mathcal{A})$ is the operator of Clifford multiplication with a 1-form.

Identifying $\Omega^1_{\mathcal{D}}(\mathcal{A})$ with a subspace in $\Omega^1(M, C^\infty(\mathbb{T}^2_\theta))$, we may consider the "local form" of such a differential form. Namely, A is locally represented as

$$A = \sum_{i=1}^4 \alpha_i e_i \otimes f_i,$$

where $\alpha_i \in C^\infty(M, i\mathbb{R})$, $f_i \in C^\infty(\mathbb{T}^2_\theta)$, $f_i^* = f_i$, and e_i is a local orthonormal frame for M, which we may choose normal at a given point $x \in M$, i.e. with $\nabla_i e_k(x) = 0$.

Analogously, the operator $\nabla^*\nabla$ is represented as

$$\nabla^*\nabla = (\nabla_0^* + A_\otimes^*)(\nabla_0 + A_\otimes),$$

where

$$A_\otimes\colon \mathcal{X} \to \Omega^1_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{X}, \qquad \psi \mapsto A \otimes \psi.$$

Now, $A_\otimes^*\colon \Omega^1_{\mathcal{D}} \otimes_{\mathcal{A}} \mathcal{X} \to \mathcal{X}$ is given locally by

$$\omega \otimes \psi \otimes f \mapsto -\sum_{i=1}^4 \alpha_i \langle e_i, \omega\rangle \psi \otimes f_i f,$$

and the operator ∇_0^* is given by $-\operatorname{Tr}\circ(\nabla_0^{LC} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_0)$, where ∇_0^{LC} is induced by the Levi-Civita connection; abusing notation slightly, we still denote ∇_0^{LC} by ∇_0. Thus, we get

$$\nabla_0^*\left(\sum_i \alpha_i e_i \otimes \psi \otimes f\right) = -\sum_i (\nabla_{0,i}(\alpha_i e_i)\psi + \alpha_i \nabla_{0,i}\psi) \otimes f.$$

Thus,

$$(\nabla_0 + A_\otimes)(\psi \otimes f) = \nabla_0\psi \otimes f + \sum_i \alpha_i e_i \otimes \psi \otimes f_i f,$$

and

$$(\nabla_0^* + A_\otimes^*)(\nabla_0 + A_\otimes)(\psi \otimes f) = \nabla_0^*\nabla_0\psi \otimes f - \sum_i \alpha_i \nabla_{0,i}\psi \otimes f_i f - \sum_i (\nabla_i(\alpha_i e_i)\psi + \alpha_i \nabla_{0,i}\psi) \otimes f_i f - \sum_i \alpha_i^2 \psi \otimes f_i^2 f = \nabla_0^*\nabla_0\psi \otimes f - 2\sum_i \alpha_i \nabla_{0,i}\psi \otimes f_i f - \sum_i \partial_i(\alpha_i)\psi \otimes f_i f - \sum_i \alpha_i^2 \psi \otimes f_i^2 f. \tag{4.9}$$

On the other hand,

$$(D+A)^2(\psi \otimes f) = (D+A)(D\psi \otimes f + \sum_i \alpha_i e_i \cdot \psi \otimes f_i f) = D^2\psi \otimes f + \sum_i \alpha_i e_i \cdot D\psi \otimes f_i f + D\sum_i (\alpha_i e_i \cdot \psi \otimes f_i f) + \sum_{i,j} \alpha_i \alpha_j e_i e_j \cdot \psi \otimes f_i f_j f.$$

Let us now consider the Dirac operator on M and a connection on L locally given by the same expression $\sum_i \alpha_i e_i$. We know by classical Weitzenböck formula from differetial geometry [**Mor96**, Sect. 5.1] that

$$D^2\psi - \nabla_0^*\nabla_0 = \frac{s}{4}\psi + \frac{F_{\det,\nabla_0}}{2}\psi$$

and compute

$$\sum_i (\alpha_i e_i \cdot D\psi + D(\alpha_i e_i \cdot \psi)) = \sum_{j,i} (\alpha_i e_i e_j \cdot \nabla_{0,j}\psi + \alpha_i e_j e_i \cdot \nabla_{0,j}\psi + (\partial_j \alpha_i) e_j e_i \cdot \psi) = \sum_{i,j} (\partial_j \alpha_i) e_j e_i \cdot \psi - 2\sum_i \alpha_i \nabla_{0,i}\psi.$$

Thus,

$$((D+A)^2 - \nabla^*\nabla)(\psi \otimes f) = \frac{F_{\nabla_0}}{2} + \frac{s}{4} + \sum_{i \neq j} (\partial_j \alpha_i) e_j e_i \cdot \psi \otimes f_i f + \sum_{i<j} \alpha_i \alpha_j e_i \wedge e_j \cdot \psi \otimes [f_i, f_j] f$$

which is exactly the formula in the claim. □

Now we want to prove an estimate on the spinor part of a solution to Seiberg–Witten equations which follows from the Weitzenböck formula. To do this, we will need an analogue of a classical formula from differential geometry.

Proposition 4.23. *In the assumptions of Lemma 4.22 the following equality holds true:*

$$\nabla^*\nabla\psi = -\sum_i \nabla_i \nabla_i \psi,$$

where

$$\nabla_i \psi = \langle e_i, \nabla\psi \rangle,$$

and e_i is a local orthonormal frame on M.

Remark 4.24. Notice that the mapping ∇_i as such doesn't preserve $\mathcal{X}$ – it maps to the space $C^\infty(M)\widehat{\otimes}C^\infty(\mathbb{T}^2_\theta)$ and doesn't preserve invariance in general (because the local frame need not be $\mathbb{T}^2$-invariant).

PROOF. Writing as before

$$A = \sum_i \alpha_i e_i \otimes f_i,$$

we observe

$$\nabla_i(\psi \otimes f) = \nabla_{0,i}\psi \otimes f - \alpha_i \psi \otimes f_i f.$$

Thus,

$$\nabla_i\nabla_i(\psi\otimes f) = \nabla_{0,i}\nabla_{0,i}\psi\otimes f - \nabla_{0,i}(\alpha_i\psi)\otimes f_i f - \alpha_i\nabla_{0,i}\psi\otimes f_i f - \alpha_i^2\psi\otimes f_i^2 f.$$

The equality now follows from (4.9). □

Lemma 4.25. *Let (A,ψ) be a solution to the Seiberg–Witten equations on a toric deformation M_θ. Then the spinor part is uniformly bounded in the following sense:*

$$E_0(\langle\psi,\psi\rangle_{\mathcal{A}}) \leqslant -s.$$

PROOF. Let (∇,ψ) be a solution to the Seiberg–Witten equations. The Weitzenböck formula (Lemma 4.22) implies

$$\nabla^*\nabla\psi + \frac{s}{4}\psi + \frac{q(\psi)}{2}\psi = 0.$$

Thus, we obtain

$$\langle\nabla^*\nabla\psi,\psi\rangle_{\mathcal{A}} + \frac{s}{4}\langle\psi,\psi\rangle_{\mathcal{A}} + \frac{\langle q(\psi)\psi,\psi\rangle_{\mathcal{A}}}{2} = 0. \tag{4.10}$$

Moreover, because the connection ∇ is compatible with the metric, we obtain the equality

$$-\sum_i \partial_i^2\langle\psi,\psi\rangle_{\mathcal{A}} = -\sum_i\langle\nabla_i\nabla_i\psi,\psi\rangle_{\mathcal{A}} - 2\sum_i\langle\nabla_i\psi,\nabla_i\psi\rangle_{\mathcal{A}} - \sum_i\langle\psi,\nabla_i\nabla_i\psi\rangle_{\mathcal{A}},$$

or, equivalently,

$$\Delta\langle\psi,\psi\rangle_{\mathcal{A}} + 2\sum_i\langle\nabla_i\psi,\nabla_i\psi\rangle_{\mathcal{A}} = \langle\nabla^*\nabla\psi,\psi\rangle_{\mathcal{A}} + \langle\psi,\nabla^*\nabla\psi\rangle_{\mathcal{A}}.$$

Applying the conditional expectation $E_0\colon \mathcal{A}\to C^\infty(M)$, we obtain

$$\Delta E_0(\langle\psi,\psi\rangle_{\mathcal{A}}) + 2\sum_i E_0(\langle\nabla_i\psi,\nabla_i\psi\rangle_{\mathcal{A}}) = E_0(\langle\nabla^*\nabla\psi,\psi\rangle_{\mathcal{A}}) + E_0(\langle\psi,\nabla^*\nabla\psi\rangle_{\mathcal{A}}).$$

Now, $E_0(\langle\psi,\psi\rangle_{\mathcal{A}})$ is a function on a compact manifold M. To obtain the desired estimate, let x_0 be a local maximum of $E_0(\langle\psi,\psi\rangle_{\mathcal{A}})$. Then $\Delta(E_0(\langle\psi,\psi\rangle_{\mathcal{A}}))(x_0)\geqslant 0$, and we obtain

$$2\operatorname{Re}E_0(\langle\nabla^*\nabla\psi,\psi\rangle_{\mathcal{A}})\geqslant 0.$$

Thus, applying E_0 to the above equation (4.10), we obtain

$$\frac{s(x_0)}{4}E_0(\langle\psi,\psi\rangle_{\mathcal{A}})(x_0) + \frac{E_0(\langle q(\psi)\psi,\psi\rangle_{\mathcal{A}})(x_0)}{2}\leqslant 0. \tag{4.11}$$

Let us analyze the second term. We recall that if $F\colon \operatorname{End}_{\mathcal{A}}(\mathcal{X}^+)\to\mathcal{A}$ is the canonical operator-valued weight, then $F(\Theta_{\eta,\zeta}) = {}_{\mathcal{A}}\langle\eta,\zeta\rangle$; in particular,

$$F(\Theta_{\eta,\zeta}\Theta_{\xi,\xi}) = F(\Theta_{\eta\langle\zeta,\xi\rangle_{\mathcal{A}},\xi}) = {}_{\mathcal{A}}\langle\eta\langle\zeta,\xi\rangle_{\mathcal{A}},\xi\rangle,$$

and therefore

$$\begin{aligned}E_0(F(\Theta_{\eta,\zeta}\Theta_{\xi,\xi})) = E_0({}_{\mathcal{A}}\langle\eta\langle\zeta,\xi\rangle_{\mathcal{A}},\xi\rangle) = E_0(\langle\xi,\eta\rangle_{\mathcal{A}}\langle\zeta,\xi\rangle_{\mathcal{A}}) &= E_0(\langle\xi,\eta\langle\zeta,\xi\rangle_{\mathcal{A}}\rangle_{\mathcal{A}})\\ &= E_0(\langle\xi,\Theta_{\eta,\zeta}\xi\rangle_{\mathcal{A}}).\end{aligned}$$

In view of this,

$$E_0(\langle q(\psi)\psi,\psi\rangle_{\mathcal{A}}) = E_0(\langle \psi, q(\psi)\psi\rangle_{\mathcal{A}}) = E_0({}_{\mathcal{A}}\langle q(\psi)\psi,\psi\rangle)$$
$$= E_0(F(q(\psi)\Theta)) = E_0\left(F\left(\left(\Theta - \frac{F(\Theta)}{2}\right)\Theta\right)\right),$$

where $\Theta = \Theta_{\psi,\psi}$, and F is the canonical operator-valued weight $F\colon \operatorname{End}_{\mathcal{A}}(\mathfrak{X}^+) \to \mathcal{A}$ (recall that $q(\psi) = \Theta - \frac{F}{2}(\Theta)$, the factor 2 being the dimension of the positive spinors). Let us now observe that Θ is selfadjoint and compute

$$(F(\Theta)-\Theta)^2 = F(\Theta)^2 - F(\Theta)\Theta - \Theta F(\Theta) + \Theta^2 \geqslant 0,$$

from which it follows

$$F(\Theta^2) \geqslant F(\Theta)^2.$$

Thus,

$$F\left(\left(\Theta - \frac{F(\Theta)}{2}\right)\Theta\right) = F(\Theta^2) - \frac{F(\Theta)^2}{2} \geqslant \frac{F(\Theta)^2}{2} = \frac{{}_{\mathcal{A}}\langle \xi,\xi\rangle^2}{2}$$

Now, using the Schwarz inequality for conditional expectations $E_0(a^*a) \geqslant E_0(a)^*E_0(a)$, we obtain

$$\frac{E_0({}_{\mathcal{A}}\langle \psi,\psi\rangle^2)}{2} \geqslant \frac{E_0({}_{\mathcal{A}}\langle \psi,\psi\rangle)^2}{2} = \frac{E_0(\langle \psi,\psi\rangle_{\mathcal{A}})^2}{2}.$$

Thus,

$$s(x_0)E_0(\langle \psi,\psi\rangle_{\mathcal{A}})(x_0) + E_0(\langle \psi,\psi\rangle_{\mathcal{A}}))^2(x_0) \leqslant 0.$$

Thus, either $E_0(\langle \psi,\psi\rangle_{\mathcal{A}})(x_0) = 0$, which implies $\psi = 0$, or $E_0(\langle \psi,\psi\rangle_{\mathcal{A}})(x_0) \leqslant -s(x_0)$. This proves the lemma. □

Now let us come to the result which allows to compute the moduli space in some interesting examples. The outline of the proof follows the approach in [**LeB95**].

Theorem 4.26. *Let M be a compact complex surface with a holomorphic $\mathbb{T}^2$-action and a Kähler metric of constant scalar curvature s. Consider the Seiberg–Witten equations for the noncommutative geometry associated with the canonical spin^c structure on M. If $s \geqslant 0$, then there are no irreducible solutions of the Seiberg–Witten equations; if $s < 0$, then the solution to the Seiberg–Witten equations is unique and is given by*

$$\alpha = -\sqrt{s}, \quad \beta = 0, \quad A = 0.$$

Thus, the moduli space consists of one point.

Proof. As the $\mathbb{T}^2$-action is holomorphic, the Chern connection on the anticanonical bundle, which is equal to the determinant bundle, is invariant. The self-dual part of its curvature is known to be equal to $is\omega/4$, and therefore the pair above is a solution to the Seiberg–Witten equations (another way to see it consists in observing that if A has only an invariant component, the solution of the Seiberg–Witten equations on M_θ coincides with the solution of the Seiberg–Witten equations on M).

Now, the Weitzenböck formula (Lemma 4.22) implies ($\|\cdot\|$ denoting the L^2-norm)

$$\left\|2F_A^+ + \frac{is\omega}{4}\right\|^2 = \tau\left(\left\langle 2F_A^+ + \frac{is\omega}{4}, 2F_A^+ + \frac{is\omega}{4}\right\rangle_{\mathcal{A}}\right) = \frac{\tau\left(\langle q(\psi),q(\psi)\rangle_{\mathcal{A}}\right)}{2}.$$

Now, $q(\psi) = \Theta_{\psi,\psi} - \frac{{}_{\mathcal{A}}\langle\psi,\psi\rangle}{2}$, and thus

$$\langle q(\psi),q(\psi)\rangle_{\mathcal{A}} = \left(\langle q(\psi),\Theta_{\psi,\psi}\rangle_{\mathcal{A}}\right) = \frac{{}_{\mathcal{A}}\langle q(\psi)\psi,\psi\rangle}{2}.$$

and it follows from (4.10) that

$$\tau(\langle q(\psi), q(\psi)\rangle_{\mathcal{A}}) \leqslant -\frac{s}{2}\tau(\langle \psi, \psi\rangle_{\mathcal{A}}) \leqslant \frac{s^2}{2}\operatorname{vol}(M).$$

Thus,

$$\left\|2F_A^+ + \rho\right\|^2 \leqslant \int_M \left(\frac{s}{4}\|\omega\|\right)^2 \operatorname{vol} = \int_M \left\|\rho^+\right\|^2 \operatorname{vol},$$

where ρ is the Ricci form of M. Its cohomology class coincides with that of $2\pi c_1(L)$, where $L = K_X^{-1}$ is the anticanonical bundle. As the curvature s is constant, ρ is harmonic, and therefore

$$\begin{aligned}
&\int_M \left\|\rho^+\right\|^2 \operatorname{vol} = 2\pi^2 c_1(L)^2 + \frac{1}{2}\int_M \|\rho\|^2 \operatorname{vol} \\
&\leqslant 2\pi^2 c_1(L)^2 + \frac{1}{2}\int_M \|2E_0(F_A)+\rho\|^2 \operatorname{vol} \leqslant 2\pi^2 c_1(L)^2 + \frac{1}{2}\int_M \|2F_A+\rho\|^2 \operatorname{vol} \\
&\qquad = \int_M \left\|2F_A^+ + \rho\right\|^2 \operatorname{vol},
\end{aligned}$$

where $E_0(F_A)$ is the invariant part of the curvature. Hereby we used that $E_0(F_A)$ is a closed form in the same cohomology class as ρ and the fact that the pairing between the K-theory class of $\mathcal{K}_X$ and the fundamental cyclic cohomology class is expressed in the terms of the curvature [**Con94**, Prop. 3.8].

Now, in view of the inequality

$$\|2E_0(F_A)+\rho\|^2 \leqslant \|2F_A+\rho\|^2,$$

the equality being equivalent to $F_A = E_0(F_A)$, we obtain

$$2F_A + \rho = \rho,$$

and thus, A is a connection on a trivial module with curvature equal to 0.

Moreover, the above inequalities imply $E_0(\langle\psi,\psi\rangle_{\mathcal{A}}) = -s$, and thus (4.10) implies $\nabla\psi = 0$. Thus, $\nabla^2\psi = 0$. But the curvature on the second component is necessarily nontrivial (coming from the curvature of $L = K_X^{-1}$), and therefore we must have $\psi = (\alpha, 0)$. But then $\alpha = \sqrt{-s}u$, where u is a unitary, and, taking u as the gauge transformation, we arrive at the described solution. □

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