Approximation of center-valued-Betti-numbers and the center-valued Atiyah-conjecture.

Thesis

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Introduction

For a finite CW-complex X with fundamental group π, the L^2-homology of the universal covering $\widetilde{X}$ is given as the kernel of the combinatorial Laplacians Δ_* on $C_*^{(2)}(\widetilde{X}) = C_*^{(cell)}(\widetilde{X}) \otimes_{\mathbb{Z}\pi} \ell^{(2)}(\pi)$. After a choice of a cellular base, this complex, is isomorphic to a complex of finite direct sums of $\ell^2(\pi)$, on which the Laplacian $\Delta_p = (c_p \otimes \mathrm{id})^*(c_p \otimes \mathrm{id}) + (c_{p-1} \otimes \mathrm{id})(c_{p-1} \otimes \mathrm{id})^*$ acts by left multiplication with a matrix over $\mathbb{Z}\pi \subset \mathcal{N}(\pi)$. Here, $\mathcal{N}(\pi) \subset \mathcal{B}(\ell^2(\pi))$ is the group von Neumann algebra of π: it is the von Neumann algebra generated by the left regular representation of π. L^2-Betti-numbers measure the dimension of the L^2-homology and can be defined as

$$\beta_p^2(X) := \dim_{\mathcal{N}(G)}^{\mathbb{C}}(\ker(\Delta_p))$$

.

W. Lück shows in [17] that the L^2-Betti-numbers $\beta_n^{(2)}(\widetilde{X})$ of the universal covering $\widetilde{X}$ of a CW-complex X, with residually finite fundamental group π, can be approximated by their finite dimensional analogons $\beta_n^{(2)}(\widetilde{X}/\pi_i)$.

Using these ideas in a different context, J. Dodziuk and V. Mathai prove in [4] a similar approximation result for amenable groups. In [25], T. Schick combines both ideas and extends the result to a more general class $\mathcal{G}$ of groups containing in particular amenable and residually finite groups.

Later in [5] G. Elek and E. Sabó proved the approximation result also for sofic groups.

These proofs rely on showing that the kernel of a matrix $A \in M_d(\mathbb{Z}G)$ can be approximated via the kernels of the matrices $p_i(A) \in M_d(\mathbb{Z}G)$, where the p_i are coming from some limit or extension process of G. Finally in [3] J. Dodziuk, P. Linnell, T. Schick and S. Yates extend the coefficient ring $\mathbb{Z}G$ to $\overline{\mathbb{Q}}G$, especially to prove the Atiyah conjecture over $\overline{\mathbb{Q}}G$ and G from a subclass of $\mathcal{G}$.

In this thesis, the approximation theorem will be generalized to an approximation theorem for the center-valued Betti-numbers

$$\beta_p^u(X) := \dim^u_{\mathcal{N}(G)}(\ker(\Delta_p)) \,.$$

More precisely, we show that their Fourier coefficients (which are multiples of the so called delocalized Betti-numbers introduced by Lott in [16]) can be approximated.

In the second part of this thesis, we state the center-valued-Atiyah-conjecture. It can be obtained for amenable groups from Linnell's corresponding proof of the (classical) Atiyah-conjecture. We will then use the approximation theorem, to extend the center-valued Atiyah-conjecture to limits of groups which are finite extensions of a torsion free group. The center valued Atiyah-conjecture gives a formula, for the decomposition of the center-valued trace of a projection, relative to minimal central projections, corresponds with the finite subgroups of G. This part is a joint work with Peter Linnell and Thomas Schick.

Chapter 1

Basic Theory of L^2-Invariants

The chapters two and three contain the results of the thesis and are both more or less self contained. The purpose of the first chapter is to give basic definitions, some background information and motivations for the problems treated in the later chapters.

1.1 Basics

In this section we shortly introduce the basic definitions and terminology. For further details and proofs we refer to [7] chapter 5.

Definition 1.1.1. (*-algebra)
A *-algebra A is an algebra possessing an involution $* : A \to A$, i.e. for all $a, b \in A$, $\lambda \in \mathbb{C}$ we have

- $(a^*)^* = a$
- $(ab)^* = b^* a^*$
- $(\lambda a)^* = \overline{\lambda} a^*$

We can define many important topologies on the $*$-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert-space $\mathcal{H}$. For our purpose we need at the moment the weak-operator topology, defined as follows.

Definition 1.1.2. (Weak topology)
The weak-operator topology $\mathcal{T}_{weak}$ on $\mathcal{B}(\mathcal{H})$ is given by the basis of neighborhoods containing the following elements

$$\begin{aligned} &V(a_0 : \omega_{\eta_1,\xi_1}, \dots, \omega_{\eta_n,\xi_n}, \varepsilon) \\ &:= \{a \in \mathcal{B}(\mathcal{H}) \,|\, \omega_{\eta_j,\xi_j}(a - a_0) \leq \varepsilon \text{ für } j = 1 \dots n\} \end{aligned}$$

Here denote $\eta_j, \xi_j \in \mathcal{H}$ and $\omega_{\eta_j,\xi_j}(a) := |\langle a\eta_j, \xi\rangle|$.

So the weak-operator topology is the locally-convex topology defined by the separating family of semi-norms ω_{η_j,ξ_j} .

Using the weak-operator topology, we can define a crucial object. The von Neumann-algebra, named after John von Neumann.

Definition 1.1.3. (von Neumann-algebra)
Let $\mathcal{H}$ be a Hilbert space, if $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a weakly closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is called von Neumann-algebra.

We are specially interested in a special type of von Neumann-algebras, which are generated by a group.

Definition 1.1.4. (Group von-Neumann-algebra)
Let G be a discrete group, and $\mathbb{C}G$ the corresponding group ring acting on the Hilbert space $\ell^2(G) := \{\sum_{g\in G} \lambda_g u_g \mid \lambda_g \in \mathbb{C} \wedge \sum_{g\in G} |\lambda_g|^2 < \infty\}$, where u_g denote the unitaries induced by $g \in G$, then the group von Neumann-algebra is defined as the weak closure of $\mathbb{C}(G) \subset \mathcal{B}(\ell^2(G))$.

A von Neumann-algebra with trivial center is called factor. For a group von Neumann-algebra, this is equivalent to the fact that the group G has no elements with finite conjugacy class.

Definition 1.1.5. (Factor)
A von Neumann-algebra $\mathcal{A}$ with trivial center (i.e. $\mathcal{Z}(\mathcal{A}) = \mathbb{C} \cdot \mathrm{Id}$) is called factor.

Definition 1.1.6. (Commutant)
Given a Hilbert-space $\mathcal{H}$ and $M \subset \mathcal{B}(\mathcal{H})$, the commutant M' of M is defined as $M' := \{a \in \mathcal{B}(\mathcal{H}) \mid \forall\ m \in M\,; am = ma\}$.

An important characterization of von Neumann-algebras is that they are stable under taking the double-commutant.

Theorem 1.1.7. *(Double-commutant theorem)*
If $M \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra of operators, containing the identity, then the weak-operator closure of M (and the strong-operator closure) coincides with the double commutant M'' of M .

Proof.
See [7] , page 326, theorem 5.3.1 . □

Certain $*$-algebras (e.g. C^*-algebras) already carry a special Hilbert-space representation within there structure.

Theorem 1.1.8. *(GNS-construction)*
Assume $\mathcal{A}$ is a $*$-algebra with a positive state ϕ satisfying

$$\forall\ a \in \mathcal{A}\ \exists\ C_a \in \mathbb{R}_+\ \forall\ b \in \mathcal{A}\ :\ \phi((ab)^*ab) \leq C_a \phi(b^*b)\,, \tag{1.1.9}$$

then $\mathcal{A}$ already carries a representation within its structure. It is obtained as follows. The set $\mathcal{L}_\phi := \{a \in \mathcal{A} | \phi(a^*a) = 0\}$ is a left ideal in $\mathcal{A}$ that is closed with respect to the semi-norm $\|a\|_\phi := \phi(a^*a)$. Taking the closure $\mathcal{H}_\mathcal{A}$ of $\mathcal{A}/\mathcal{L}_\phi$ with respect to $\langle a, b\rangle_\phi := \phi(b^*a)$ gives rise to a Hilbert-space $\mathcal{H}_\mathcal{A}$ with $\mathcal{A} \curvearrowright \mathcal{H}_\mathcal{A}$ such that $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_\mathcal{A})$.

(If $\mathcal{A}$ is a C^*-algebra, then the property (1.1.9) *is redundant.)*

Proof.
See [7] , page 277, proposition 4.5.1 and page 278, theorem 4.5.2 . □

1.2 Projections and Types of von Neumann-Algebras

An important tool to study von Neumann-algebras is "comparison" of the projections in a von Neumann-algebra. We introduce an equivalence relation on the projections and a partial ordering on these equivalence classes. According to the structure of the lattice, von Neumann-algebras can be distinguished in "finite" and "infinite" types, more precisely in the finite types

I_n, II_1 and infinite types I_∞, II_∞, III. It is possible to show that every von Neumann-algebra can be decomposed in a direct sum of von Neumann-algebras of these types.

This section is mainly based on [8] chapter 6. We give a short overview on results in this topic. Details and proofs can be found in [8] chapter 6 or alternatively in [28] chapter 5.1.

Proposition 1.2.1.
If $\mathcal{A}$ is a von Neumann-algebra and $\mathcal{P}_\mathcal{A}$ is the set of all projections in $\mathcal{A}$, then $\mathcal{P}_\mathcal{A}$ is a complete lattice, where the patrial ordering $\leq$ is given by the image subspaces of $\mathcal{H}$.

Given a family $\{e_i\}_{i\in I}$, we denote by $\bigwedge_{i\in I} e_i$ its greatest lower bound and by $\bigvee_{i\in I} e_i$ its least upper bound.

Proof.
See [28] page 290, proposition 5.1.1. □

Definition 1.2.2. (Equivalence and partial ordering of projections)
Two projections $e, f \in \mathcal{A}$ are said to be equivalent if there exists an element $u \in \mathcal{A}$ such that $uu^* = e$ and $u^*u = f$. We write $e \sim f$.

If e is equivalent to f_1 and $f_1 \leq f$ we write $e \precsim f$. Obviously $\sim$ gives an equivalence relation on $\mathcal{P}_\mathcal{A}$, further $\precsim$ gives a partial ordering on these equivalence classes.

Theorem 1.2.3. *(Comparison theorem)*
For any pair of projections e, f in a von Neumann-algebra $\mathcal{A}$ there is central projection c such that

$$ce \precsim cf \quad \text{and} \quad (1-c)f \precsim (1-c)e$$

As a direct consequence it follows, that in case of $\mathcal{A}$ is a factor, $\precsim$ gives a total ordering on $\mathcal{P}_\mathcal{A}$.

Proof.
See [8] page 409, theorem 6.27. □

Definition 1.2.4. ((In-)finite projection)
A projection e in $\mathcal{A}$ is said to be infinite (relative to $\mathcal{A}$), if there is a projection $e_1 \in \mathcal{A}$ such that $e \sim e_1 < e$. Otherwise e is said to be finite

(relative to $\mathcal{A}$). A projection e is called purely infinite if there is no finite projection $f \leq e$ in $\mathcal{A}$ other than zero, and e is called properly infinite, if for any central projection $c \in \mathcal{A}$, with $ce \neq 0$, the projection ce is infinite.

We use this to define finite and infinite von Neumann-algebras.

Definition 1.2.5. ((In-)finite von Neumann-algebra)
A von Neumann-algebra $\mathcal{A}$ is called finite, infinite, purely infinite, resp. properly infinite, according to the property of the identity in $\mathcal{A}$.

We will be interested in finite von Neumann-algebras, since they admit a trace and hence a suitable dimension theory.

Definition 1.2.6. (Central-carrier)
A central-carrier c_a of an operator $a \in \mathcal{A}$ is the projection $\mathrm{Id} - p$, where p is the union of all central projections $p_a \in \mathcal{A}$ such that $p_a a = 0$.

Definition 1.2.7. (Abelian projection)
A projection $e \in \mathcal{A}$ is called abelian if $e\mathcal{A}e$ is an abelian von Neumann-algebra.

We are now able to define the different types of von Neumann-algebras.

Definition 1.2.8. (Type classification)
A von Neumann-algebra $\mathcal{A}$ is of type

- I, if $\mathcal{A}$ has an abelian projection with central carrier Id,
- I_n, if Id is the sum of n equivalent abelian projections,
- II, if $\mathcal{A}$ has no non-zero abelian projections, but has a finite projection with central-carrier Id,
- II_1, if $\mathcal{A}$ is of type II and finite,
- II_∞, if $\mathcal{A}$ is of type II and properly infinite,
- III, if $\mathcal{A}$ is purely infinite.

Theorem 1.2.9. *(Type decomposition)*
Every von Neumann-algebra $\mathcal{A}$ is uniquely decomposable into a direct sum of those of type I, type II_1, type II_∞ and type III. In case of $\mathcal{A}$ is a factor it is either one of those types.

Proof.
See [8] page 422, theorem 6.5.2 , and page 424, corollary 6.5.3 . □

1.3 The Trace on Finite von Neumann-Algebras

A characteristic property of a finite von Neumann-algebra is, that it possesses the so called center-valued trace. In this section we will examine the basic properties of this trace and the corresponding dimension function. The proof of the existence of a center valued trace, in a finite von Neumann-algebra, is a very technical task, for details we refer to [8] chapter 8 or [28] chapter 5.2. .

Definition 1.3.1. (Trace)
In this section we denote by $\mathcal{A}$ a von Neumann-algebra with center $\mathcal{Z}$, then the center-valued trace of $\mathcal{A}$ is defined as a linear map

$$\operatorname{tr}^u_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}$$

such that for $a, b \in \mathcal{A}$ $c \in \mathcal{Z}$ we have:

- $\operatorname{tr}^u_{\mathcal{A}}(ab) = \operatorname{tr}^u_{\mathcal{A}}(ba)$;
- $\operatorname{tr}^u_{\mathcal{A}}(c) = c$;
- $\operatorname{tr}^u_{\mathcal{A}}(a) \in \mathcal{Z}^+$ if $a \in A^+$.

If such a mapping $\operatorname{tr}^u_{\mathcal{A}}$ exists, it is unique and the von Neumann-algebra $\mathcal{A}$ is finite. Further the trace possesses some additional properties.

Proposition 1.3.2.
If $\operatorname{tr}^u_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}$ *is the center-valued trace, we have for* $a \in \mathcal{A}$ $c \in \mathcal{Z}$ *, that*

- $\operatorname{tr}^u_{\mathcal{A}}(ca) = c \operatorname{tr}^u_{\mathcal{A}}(a)$ *;*
- $\| \operatorname{tr}^u_{\mathcal{A}}(a)\| \leq \|a\|$ *;*
- $\operatorname{tr}^u_{\mathcal{A}}$ *is ultra-weakly continuous;*

- *every trace $\tau : \mathcal{A} \to \mathbb{C}$ factors via the center i.e. given τ, there exists a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\operatorname{tr}^u_{\mathcal{A}}} & \mathcal{Z} \\ & \searrow^{\tau} & \downarrow{\scriptstyle \phi} \\ & & \mathbb{C} \end{array}$$

Proof.
See [8] page 517, theorem 8.2.8 . □

Theorem 1.3.3.
A von Neumann-algebra is finite if and only if it admits a center-valued trace.

Proof.
The "only if" part is trivial and follows directly from the trace property. The proof of the converse is technical, see [8] page 517 , theorem 8.2.8 . □

Definition 1.3.4. (Standard trace)
Given a discrete group G, the von Neumann-algebra $\mathcal{N}(G)$ possesses also the so-called standard trace, given by

$$\begin{aligned} \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}}(G) : \mathcal{N}(G) &\longrightarrow \mathbb{C} \\ a &\mapsto \langle a \cdot e, e \rangle \end{aligned}$$

(it is derived from $\operatorname{tr}^u_{\mathcal{N}(G)}$ by applying the functional $\langle - \cdot e, e\rangle$) .

Remark 1.3.5.
These traces can be extended to $M_d(\mathcal{A})$ by taking $\operatorname{tr}^u_{\mathcal{A}} := \operatorname{tr}^u_{\mathcal{A}} \otimes \operatorname{tr}_{M_d(\mathbb{C})}$ resp. $\operatorname{tr}^{\mathbb{C}}_{\mathcal{A}} := \operatorname{tr}^{\mathbb{C}}_{\mathcal{A}} \otimes \operatorname{tr}_{M_d(\mathbb{C})}$ (by abuse of notation), with $\operatorname{tr}_{M_d(\mathbb{C})}$ the non-normalized trace on $M_d(\mathbb{C})$.

A good tool to calculate the center valued is the Dixmier approximation theorem.

Theorem 1.3.6. *(Dixmier approximation theorem)*
Denote by $\mathcal{U}$ the group of all unitary elements $u \in \mathcal{A}$. Define for $a \in \mathcal{A}$, $co_{\mathcal{A}}(a)$ the convex hull $\{uau^* \mid u \in \mathcal{U}\}$ of a. Denote with $co_{\mathcal{A}}(a)^=$ its norm closure. Then

$$co_{\mathcal{A}}(a)^= \cap \mathcal{Z} = \operatorname{tr}^u_{\mathcal{A}}(a) \,.$$

Proof.
See [8] page 532, theorem 8.3.5 and 8.3.6. □

We can now easily compute the center-valued trace for group von Neumann-algebras.

Example 1.3.7.
Assume G is a discrete group. Define $\Delta(G) := \{g \in G \mid |<g>| < \infty\}$, where $<g>$ denotes the conjugacy class of the element $g \in G$. The group von Neumann-lgebra $\mathcal{N}(G)$ is given as the left regular representation of $G \curvearrowright \ell^2(G)$. The center of $\mathcal{N}(G)$ is given by the elements constant on the finite conjugacy classes $\big(\mathcal{Z} = \{a := \sum \lambda_g u_g \in \mathcal{N}(G) \mid \forall\, g \in \Delta(G)\,, \forall\, i,j \in <g>\,, \lambda_i = \lambda_j \wedge \forall\, g \in G - \Delta(G)\,, \lambda_g = 0\}\big)$.

The center-valued trace on $\mathcal{N}(G)$ is given by:

$$\begin{aligned} \mathrm{tr}^u_{\mathcal{N}(G)} &: \mathcal{N}(G) \longrightarrow \mathcal{Z}(G) \\ &\sum_{g \in G} \lambda_g u_g \mapsto \sum_{h \in \Delta(G)} \frac{1}{|<h>|} \big(\sum_{g \in <h>} \lambda_g \big) u_h \,. \end{aligned}$$

Proposition 1.3.8. *(Dimension Function)*
Suppose $\mathcal{A}$ is a finite von Neumann-algebra with center $\mathcal{Z}$ and let $\mathcal{P}$ be the set of all projections in $M_d(\mathcal{A})$. Restricting $\mathrm{tr}^u_{\mathcal{A}}$ to $\mathcal{P}$ we obtain a center-valued dimension function $\dim^u_{\mathcal{A}}$ with the following properties:

- $\dim^u_{\mathcal{A}}(p) > 0$ *if* $p \neq 0$,
- $\dim^u_{\mathcal{A}}(p+q) = \dim^u_{\mathcal{A}}(p) + \dim^u_{\mathcal{A}}(q)$ *if* $pq = 0$,
- $\dim^u_{\mathcal{A}}(p) = \dim^u_{\mathcal{A}}(q)$ *if and only if* $p \sim q$.

Proof.
The claim follows directly from the corresponding properties of the trace. □

Remark 1.3.9.
The third property is very important because it ensures that all projective modules with equivalent center-valued dimensions are isomorphic. This ensures later that universal Betti-numbers fully classify the L^2-homology modules.

1.4 The Fuglede-Kadison-determinant

In this section we introduce functional calculus, define spectral-density functions and derive from those the Fuglede-Kadison determinant. Details about basic spectral theory can be found for example in [24] chapter 12, in [7] chapter 5.2 , or in [32] chapter 7 , details about Spectral density functions can be found in [19] chapter 2.1 , Fuglede-Kadison Determinants are treated in detail in [19] chapter 3.2 .

Given a selfadjoint operator A one can define continuous-functional-calculus by taking limits of polynomials in A . A more general concept is given by the measurable-functional-calculus. A proof for of its existence can be found for example in [32] theorem 7.1.6.

Theorem 1.4.1. *(Measurable-functional-calculus)*
Let $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator acting on a Hilbert-space $\mathcal{H}$. Denote by $\sigma(A) \subset \mathbb{R}$ the spectrum of A (i.e. $\lambda \in \sigma(A) \Leftrightarrow (\lambda \operatorname{Id}_H - A)$ is not invertible) . There exists a unique homomorphism

$$\Psi : B(\sigma(A)) \to \mathcal{B}(\mathcal{H})$$

from the Borel-functions on the spectrum of A into the bounded operators on H , satisfying the following properties

- $\Psi(t) = A$, $\Psi(1_{|\sigma(A)}) = \operatorname{Id}$,
- $\Psi(\overline{f}) = \Psi(f)^*$,
- Ψ *is continuous* ,
- $f_n \in B(\sigma(A))$, $\sup_n \|f_n\|_\infty < \infty$ *and* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in \sigma(A)$ *implies* $\langle \Psi(f_n)x, y\rangle \to \langle \Psi(f)x, y\rangle$ *for all* $x, y \in H$.

We abbreviate $f(A)$ for $\Psi(f)$.

Definition 1.4.2. (Spectral measure)
Let Σ be a σ-algebra on $\mathbb{R}$. A spectral measure is a map

$$E : \Sigma \longrightarrow \mathcal{B}(\mathcal{H})$$
$$M \mapsto E_M$$

such that

- all E_M are projections,
- $E_\varnothing = 0\,, E_\mathbb{R} = \mathrm{Id}\,,$
- for pairwise disjoint sets $M_1\,, M_2\,, \dots \in \Sigma$ we have

$$\sum_{i=1}^{\infty} E_{M_i}(x) = E_{\cup M_i}(x) \;\; \forall\; x \in H\,,$$

- $E_M E_N = E_N E_M = E_{M\cap N}\,.$

Theorem 1.4.3. *(Spectral measure)*
Given a selfadjoint operator $A \in \mathcal{B}(\mathcal{H})$ we obtain a spectral measure

$$\begin{aligned} E : Bo(\mathbb{R}) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ M &\mapsto \chi_{M\cap\sigma(A)}\,. \end{aligned}$$

where $Bo(\mathbb{R})$ denotes the Borel-sets on $\mathbb{R}$.

Definition 1.4.4. (Spectral-density function)
Assume $\mathcal{A}$ is a finite von Neumann-algebra, let $A \in M_d(\mathcal{A})$ a positive operator. Define

- the spectral-density function

$$F_A : [0,\infty) \longrightarrow [0,\infty) \;:\; \varepsilon \mapsto \mathrm{tr}^{\mathbb{C}}_{\mathcal{A}}\big(\chi_{[0,\varepsilon]}(A)\big)\,,$$

- and the center valued spectral-density function as

$$F^u_A : [0,\infty) \longrightarrow \mathcal{Z}(\mathcal{A}) \;:\; \varepsilon \mapsto \mathrm{tr}^{u}_{\mathcal{A}}\big(\chi_{[0,\varepsilon]}(A)\big)\,,$$

where $\chi_{[0,\varepsilon]}$ denotes the characteristic function of the interval $[0,\varepsilon]$.

Using this notation we have $F_A(0) = \dim^{\mathbb{C}}_{\mathcal{A}}(\ker(A))$ and $F^u_A(0) = \dim^u_{\mathcal{A}}(\ker(A))$.

Definition 1.4.5. (Fuglede-Kadison determinant)
Given $A \in M_n(\mathcal{A})^+$, the spectral-density function F_A, is a monotone increasing function. It induces a Riemann-Stieltjes-measure. Using this we define the Fuglede-Kadison determinant as follows:

$$\mathrm{lndet}(A) := \int_{0^+}^{\infty} \ln(\lambda)\mathrm{d}F_A(\lambda) \;\in\; \mathbb{R}\cup\{-\infty\}\,.$$

Remark 1.4.6.
If we look at the formula we see that the integral can only diverge at 0 since the measure coming form A vanishes above $\|A\|$. If the operator is invertible the spectrum has a gap around zero, hence in this case the determinant will always be bounded. In general, if the Fuglede-Kadison determinant is bounded, this means that there is not too much spectrum near by zero. This means that the operator somehow behaves well, this observation is a crucial ingredient for approximation of Betti-numbers.

1.5 L^2-Betti-numbers

In this section we introduce L^2-Betti numbers. From the technical viewpoint we deal, in the next section, with kernels of certain operators. This section shows where these operators occur, and hence is meant to give some motivation and topological background.

The topic is quite complex, but well treated in the literature, we only give a brief definition of L^2-Betti-numbers and mention some main properties. For further details and proofs we refer to [19] chapter 1.

Definition 1.5.1. (G-CW-complex)
A G-CW-complex X is a G-space together with a G-invariant filtration $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset \bigcup_{n\geq 0} X_n = X$ such that X carries the colimit topology with respect to this filtration (i.e. a set $C \subset X$ is closed if and only if $C \cap X_n$ is closed in X_n for all $n \geq 0$) and X_{n+1} is obtained from X_n for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e. there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i\in I_n} q_i} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i\in I_n} Q_i} & X_n \end{array}$$

Remark 1.5.2.
Provided a discrete group G, a G-CW-complex X is the same as a CW-complex X with a G-action such that for any open cell $e \subset X$ and $g \in G$ with $ge \cap e \neq \emptyset$, left multiplication with g induces the identity on e.

The canonical examples of G-CW-complexes are given by regular coverings of a CW-complex of finite type (i.e. all skeleta are finite).

Now suppose that G is discrete. The cellular $\mathbb{Z}G$-chain complex $C_*(X)$ of a G-CW-complex has as n-th chain group the singular homology $H_n(X_n; X_{n-1})$ and its n-th differential is the boundary homomorphism associated to the triple $(X_n; X_{n-1}; X_{n-2})$. If one has chosen a G-pushout as in Definition 1.5.1, then there is a preferred $\mathbb{Z}G$-isomorphism

$$\bigoplus_{i \in I_n} \mathbb{Z}[G/H_i] \cong C_n(X) .$$

If we choose a different G-pushout, we obtain another isomorphism, but the two differ only by the composition of an automorphism which permutes the summands appearing in the direct sum and an automorphism of the shape

$$\bigoplus_{i \in I_n} \mathbb{Z}[G/H_i] \xrightarrow{\oplus_{i \in I_n} \varepsilon \cdot r_{g_i}} \bigoplus_{i \in I_n} \mathbb{Z}[G/H_i] ,$$

where $g_i \in G$, $\varepsilon_i \in \pm 1$ and $\varepsilon_i \cdot r_{g_i}$ sends gH_i to $\varepsilon_i \cdot gg_iH_i$. In particular we obtain for a free G-CW-complex X a cellular $\mathbb{Z}G$-basis B_n for $C_n(X)$, which is unique up to permutation and multiplication with trivial units in $\mathbb{Z}G$, i.e. elements of the shape $\pm g \in \mathbb{Z}G$ for $g \in G$.

Definition 1.5.3. (L^2-chain complex)
Let X be a free G-CW-complex of finite type. Denote its cellular L^2-chain complex by

$$C_*^{(2)}(X) := \ell^2(G) \otimes_{\mathbb{Z}G} C_*(X) ,$$

where $C_*(X)$ is the cellular $\mathbb{Z}G$-chain complex.

Remark 1.5.4.
Fixing a cellular basis for $\mathbb{C}_n(X)$ we obtain an explicit isomorphism

$$C_i^{(2)}(X) \cong \bigoplus_{i=1}^{k_i} \ell^2(G)$$

for some $k \in \mathbb{N}_0$. The differentials $\delta_i^{(2)} := \mathrm{id} \otimes \delta_i$ are then given as elements in $M_{k_i \times k_{i-1}}(\mathbb{Z}G) \subset \mathcal{B}(\ell^2(G)^{k_{i-1}}, \ell^2(G)^{k_i})$.

Definition 1.5.5. (L^2-homology and L^2-Betti numbers)
Let X be a free G-CW-complex of finite type. Denote its (reduced) n-th L^2-homology and n-th L^2-Betti number by the corresponding notions of the cellular L^2-chain complexes

$$H_n^{(2)}(X; \mathcal{N}(G)) := H_n^{(2)}(C_*^{(2)}(X)) ,$$
$$\beta_n^{(2)}(X; \mathcal{N}(G)) := \beta_n^{(2)}(C_*^{(2)}(X)) .$$

Remark 1.5.6.
The i-th L^2-homology module $H_i^{(2)}(X, \mathcal{N}(G))$ is given as the kernel of the Laplacian $\Delta_i^{(2)}$:

$$H_i^{(2)}(X, \mathcal{N}(G)) = \ker(\Delta_i^{(2)}) := \ker(\delta_{i+1}^{(2)} \delta_{i+1}^{(2)*} + \delta_i^{(2)*} \delta_i^{(2)})$$

Hence examining L^2-Betti is equivalent to studying the kernels of certain positive operators.

Theorem 1.5.7. *(Some properties of L^2-Betti numbers)*

- *Homotopy invariance: Let $f : X \to Y$ be a G-map of free G-CW-complexes of finite type. If the map induced on homology with complex coefficients $H_n(f; \mathbb{C}) : H_n(X; \mathbb{C}) \to H_n(Y; \mathbb{C})$ is bijective for $n \le d$, then*
$$\beta_n^{(2)}(X) = \beta_n^{(2)}(Y) \quad \text{for } n < d .$$
In particular, if f is a weak homotopy equivalence (i.e. induces a bijection on π_n for all base points and $n \ge 0$), we get for all $p \ge 0$
$$\beta_n^{(2)}(X) = \beta_n^{(2)}(Y) .$$

- *Euler-Poincaré formula: Let X be a free finite G-CW-complex. Let $\chi(G \backslash X)$ be the Euler characteristic of the finite CW-complex $G \backslash X$, i.e.*
$$\chi(G \backslash X) := \sum_{n \ge 0} (-1)^n \beta_n(G \backslash X) ,$$
where $\beta_n(G \backslash X)$ is the number of n-cells of $G \backslash X$. Then
$$\chi(G \backslash X) = \sum_{n \ge 0} (-1)^n \beta_n^{(2)}(X) .$$

- *Poincaré duality: Let M be a cocompact (i.e. M/G is compact) free proper G-manifold of dimension d which is orientable. Then*

$$\beta_n^{(2)}(M) = \beta_{d-n}^{(2)}(M, \partial M)\,.$$

- *Restriction: Let X be a free G-CW-complex of finite type and let $H < G$ be a subgroup of finite index $[G : H]$. Let $\mathrm{res}_G^H(X)$be the H-space obtained from X restricting the G-action to an H-action. This is a free H-CW-complex of finite type. Then we get for $n \geq 0$*

$$[G : H] \cdot \beta_n^{(2)}(X; \mathcal{N}(G)) = \beta_n^{(2)}(res_H^G(X); \mathcal{N}(H))\,.$$

- *Induction: Let H be a subgroup of G and let X be a free H-CW-complex of finite type. Then $G \times_H X$ is a G-CW-complex of finite type and*

$$\beta_n^{(2)}(G \times_H X; \mathcal{N}(G)) = \beta_n^{(2)}(X; \mathcal{N}(H))\,.$$

Proof.
See [19] page 37, theorem 1.35. □

1.6 Approximation of L^2-Betti numbers

In this section we give a brief overview on W. Lücks approach on approximating L^2-Betti numbers in the case G is a residually finite Group. His result was generalized in many steps to sofic groups and algebraic coefficients, but the key ideas were always reused. In the next chapter we will look at approximation of center-valued Betti numbers, we will then adapt the ideas shown here to our new situation. This section is taken from [17] and [25].

Situation 1.6.1.
Assume the following situation:

- Let X be a finite connected CW-complex with fundamental group G. Let $p : \widetilde{X} \to X$ be the universal covering. We let G operate from the left on the universal covering and on its cellular chain complex.

- Let G be a countable residually finite group, with
$$\cdots \subset G_{m+1} \subset G_m \subset \ldots G_1 \subset G$$
a nested sequence of normal subgroups with finite index and $\bigcap_{m=0}^{\infty} G_i = \{1\}$.
- Let X_i be the finite subcover corresponding to quotient G/G_i (i.e with decktransformation group G/G_i).

Theorem 1.6.2.
In the situation just described we have
$$\lim_{i\to\infty} \beta_k^{(2)}(X_i) = \beta_k^{(2)}(X).$$

Remark 1.6.3.
We recall some facts we use in the following:

- The k-th L^2-homology module of X is given (independent from the choice of base) as the kernel of $\Delta_k^{(2)} : \oplus_{n=1}^{i_k} \ell^2(G) \to \oplus_{j=1}^{i_k} \ell^2(G)$, which is a $\mathbb{Z}G$-linear map.
- The projections $p_i : G \to G/G_i$ extend canonically to $p_i : M_k(\mathbb{Z}G) \to M_k(\mathbb{Z}(G/G_i))$. By applying p_i to the Laplacian $\Delta_k^{(2)}$ we obtain the Laplacian $\Delta_{k,i}^{(2)}$ of the finite subcover X_i.
- We have
$$\beta_k^{(2)}(X) = F_{\Delta_k^{(2)}}(0),$$
where $F_{\Delta_k^{(2)}}(\lambda)$ denotes the spectral density function of $\Delta_k^{(2)}$.
- Given $\Delta^{(2)}$ and $\Delta_i^{(2)}$, there is a common upper bound $K < \infty$ of $\|\Delta^{(2)}\|$ and all $\|\Delta_i^{(2)}\|$ (this is given as a multiple of $\|\Delta^{(2)}\|_1$, see [17] Lemma 2.5).
- In our situation we have for all i, that the Fuglede-Kadison determinant $\mathrm{lndet}_{G_i}(\Delta_i^{(2)})$ is positive (see [25] Theorem 6.9).
- In the above situation, given any polynomial p, we have
$$\lim_{i\to\infty} \mathrm{tr}_{\mathcal{N}(G_i)}^{\mathbb{C}}(p(\Delta)_i) = \mathrm{tr}_{\mathcal{N}(G)}^{\mathbb{C}}(\Delta).$$
(This simply follows from the fact that there are only finitely many coefficients in Δ nonzero.)

Notation 1.6.4.
Define for $\Delta^{(2)}$ and $\Delta_i^{(2)}$

- $\overline{F}_{\Delta^{(2)}}(\lambda) := \limsup_i F_{\Delta_i^{(2)}}(\lambda)$ (pointwise) ,
- $\underline{F}_{\Delta^{(2)}}(\lambda) := \liminf_i F_{\Delta_i^{(2)}}(\lambda)$,
- for monotone increasing F we define

$$F^+(\lambda) := \lim_{\varepsilon \to 0^+} F(\lambda + \varepsilon)$$

the right-continuous approximation of F . In particular we defined $\overline{F}^+$ and $\underline{F}^+$.

We now give the core elements of Lück's proof.

Lemma 1.6.5.
Let $\mathcal{A}$ be a finite von Neumann algebra with positive normal and normalized trace $\mathrm{tr}_{\mathcal{A}}^{\mathbb{C}}$. Choose $\Delta \in M_d(\mathcal{A})$ positive and self-adjoint. Given $K \in \mathbb{R}^+$ and functions $p_n : \mathbb{R} \to \mathbb{R}$, if for all $0 \le x \le K$, we have

$$\chi_{[0,\lambda]}(x) \le p_n(x) \le \frac{1}{n}\chi_{[0,K]}(x) + \chi_{[0,\lambda+\frac{1}{n}]}(x) \tag{1.6.6}$$

and if $\|\Delta\| \le K$, then

$$F_\Delta(\lambda) \le \mathrm{tr}_{\mathcal{A}}^{\mathbb{C}}(p_n(\Delta)) \le \frac{d}{n} + F_\Delta(\lambda + \frac{1}{n}) . \tag{1.6.7}$$

Proof.
This is a direct consequence of the positivity of the trace and the definition of the spectral-density-function . □

Proposition 1.6.8.
For all $\lambda \in \mathbb{R}$ we have

$$\overline{F_\Delta}(\lambda) \le F_\Delta(\lambda) \le \underline{F_\Delta}(\lambda)^+$$

and

$$F_\Delta(\lambda) = \underline{F_\Delta}(\lambda)^+ = \overline{F_\Delta}(\lambda)^+ .$$

Proof.
Take $\lambda \in \mathbb{R}$ and $\infty > K \geq \sup\{\|\Delta\|, \|\Delta_i\|\}$, choose a polynomial $p_n \in \mathbb{R}[X]$ such, that (1.6.6) is satisfied for K .

$$F_{\Delta_i}(\lambda) \leq \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G_i)}(p_n(\Delta_i)) \leq \frac{d}{n} + F_{\Delta_i}(\lambda + \frac{1}{n}) \,.$$

Applying lim sup resp. lim inf provides

$$\overline{F_\Delta}(\lambda) \leq \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(p_n(\Delta)) \leq \frac{d}{n} + \underline{F_\Delta}(\lambda + \frac{1}{n}) \,.$$

Further $p_n(\Delta)$ converges strongly to $\chi_{[0,\lambda]}(\Delta)$ and hence it converges also weakly. Taking $n \to \infty$ we obtain $\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(p_n(\Delta)) \to \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(\chi_{[0,\lambda]}(\Delta)) = F_\Delta(\lambda)$, and hence

$$\overline{F_\Delta}(\lambda) \leq F_\Delta(\lambda) \leq \underline{F_\Delta}^+(\lambda).$$

For $\varepsilon > 0$, it follows from the monotony of $\underline{F_\Delta}$ and $\overline{F_\Delta}$, that

$$F_\Delta(\lambda) \leq \underline{F_\Delta}(\lambda + \varepsilon) \leq \overline{F_\Delta}(\lambda + \varepsilon) \leq F_\Delta(\lambda + \varepsilon).$$

Taking limit $\varepsilon \to 0^+$ provides

$$F_\Delta(\lambda) = \underline{F_\Delta}(\lambda)^+ = \overline{F_\Delta}(\lambda)^+.$$

□

The following construction finishes the proof of Lück's approximation Theorem

Proof of Theorem 1.6.2.
Take $K \geq 0$ so, that $K > \|\Delta_i\|$ for all i . Since $\operatorname{lndet}_{G_i}(\Delta_i) \geq 0$ we have

$$0 \leq \operatorname{lndet}_{G_i}(\Delta_i) = \ln(K)(F_{\Delta_i}(K) - F_{\Delta_i}(0)) - \int_{0^+}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, \mathrm{d}\lambda$$

since $F_{\Delta_i}(K) = d$

$$\int_{0^+}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda} \, \mathrm{d}\lambda \leq \ln(K)(d - F_{\Delta_i}(0) \leq \ln(K)d.$$

Take $\varepsilon > 0$

$$\int_{\varepsilon}^{K} \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda}\, \mathrm{d}\lambda = \int_{\varepsilon}^{K} \frac{\underline{\underline{F_{\Delta}}}^{+}(\lambda) - F_{\Delta}(0)}{\lambda}\, \mathrm{d}\lambda = \int_{\varepsilon}^{K} \frac{\underline{\underline{F_{\Delta}}}(\lambda) - F_{\Delta}(0)}{\lambda}\, \mathrm{d}\lambda\,.$$

(Since the integrand is bounded, the integral over the left continuous approximation is equal to the integral over the original function.)

$$\begin{aligned}
&\leq \int_{\varepsilon}^{K} \frac{\underline{\underline{F_{\Delta}}}(\lambda) - \overline{F_{\Delta}}(0)}{\lambda}\, \mathrm{d}\lambda \\
&= \int_{\varepsilon}^{K} \frac{\liminf_i F_{\Delta_i}(\lambda) - \limsup_i F_{\Delta_i}(0)}{\lambda}\, \mathrm{d}\lambda \\
&\leq \int_{\varepsilon}^{K} \frac{\liminf_i (F_{\Delta_i}(\lambda) - F_{\Delta_i}(0))}{\lambda}\, \mathrm{d}\lambda \\
&\leq \liminf_i \int_{\varepsilon}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, \mathrm{d}\lambda \\
&\leq d \ln(K).
\end{aligned}$$

Since this holds for all $\varepsilon > 0$ we have

$$\begin{aligned}
&\int_{0^+}^{K} \frac{F_{\Delta}(\lambda) - F_{\Delta}(0)}{\lambda}\, \mathrm{d}\lambda \\
&\leq \int_{0^+}^{K} \frac{\underline{\underline{F_{\Delta}}}(\lambda) - \overline{F_{\Delta}}(0)}{\lambda}\, \mathrm{d}\lambda \\
&\leq \sup_{\varepsilon > 0} \liminf_i \int_{\varepsilon}^{K} \frac{F_{\Delta_i}(\lambda) - F_{\Delta_i}(0)}{\lambda}\, \mathrm{d}\lambda \leq d \ln(K)\ .
\end{aligned}$$

If $\lim_{\delta \to 0} \underline{F_{\Delta}}(\delta) \neq \overline{F_{\Delta}}(0)$, the second integral would be infinite. Hence from prop. 1.6.8 follows $\limsup_i F_{\Delta_i}(0) = F_{\Delta}(0)$. Since the above inequalities hold, also if we pass to a subnet, we have $\liminf_i F_{\Delta_i}(0) = F_{\Delta}(0)$.

$\square$

Chapter 2

Approximation of center-valued Betti-numbers

In this chapter we state and prove our first main result. It is an extension of Lück's approximation theorem for L^2-Betti-numbers to the finer center-valued-Betti-numbers. The main advantage of center-valued-Betti-numbers, is that they classify the homology up to isomorphisms. The new technique which allows us to extend previous results, is to see delocalized traces as perturbations of the regular trace. The results from this chapter are published in [9] .

2.1 Notation

We first introduce some notations. Let G be a discrete group, we write $\Delta(G)$ for the set of elements $g \in G$ with finite conjugacy class $\langle g \rangle$. The center of a von Neumann-algebra $\mathfrak{A}$ is denoted by $\mathcal{Z}(\mathfrak{A}) := \mathfrak{A} \cap \mathfrak{A}'$. The matrix ring $M_d(\mathcal{N}(G))$ is defined as $M_d(\mathcal{N}(G)) := \mathcal{N}(G) \otimes_{\mathbb{C}} M_d(\mathbb{C})$ and we let these operators act on $\ell^2(G)^d := \ell^2(G) \otimes \mathbb{C}^d$.

Definition 2.1.1.
Let J be an index set. For $A := (a_{i,j})_{i,j \in J}$ with $a_{i,j} \in \mathbb{C}$, define

$$S(A) := \sup_{i \in J} |\mathrm{supp}(z_i)| \,,$$

where z_i is the vector $z_i := (a_{i,j})_{j \in J}$ and $\mathrm{supp}(z_i) := |\{j \in J \mid a_{i,j} \neq 0\}|$.

Now let $|A|_\infty := \sup_{i,j} |a_{i,j}|$ and $A^* := (\overline{a}_{j,i})_{i,j\in J}$. Define

$$\kappa(A) := \begin{cases} \sqrt{S(A)S(A^*)}\cdot |A|_\infty & \text{if } S(A)+S(A^*)+|A|_\infty < \infty \\ \infty & \text{else} \end{cases}$$

Elements of $M_d(\mathbb{C}G)$ are identified with degenerated matrices, indexed by $J\times J$ where $J := \{1,\ldots,d\}\times G$. For more details we refer to [3] .

Definition 2.1.2.
Let G be a discrete group and take $A \in M_d(o(\overline{\mathbb{Q}})G)$ positive (where $o(\overline{\mathbb{Q}})$ denotes the algebraic integers), choose a finite Galois extension $L \subset \mathbb{C}$ of $\mathbb{Q}$, such that $A \in M_d(LG)$. Let $\sigma_1,\ldots,\sigma_r : L \to \mathbb{C}$ be the different embeddings of L in $\mathbb{C}$ with σ_1 the natural inclusion $L\subset\mathbb{C}$. If

$$\operatorname{lndet}(A) \geq -d\sum_{k=2}^{r} \ln\bigl(\kappa(\sigma_k(A))\bigr)\,, \tag{2.1.3}$$

we say A has the bounded determinant property. A discrete group G is said to have the bounded determinant property, if all $A\in M_d(\overline{\mathbb{Q}}G)$ satisfy property (2.1.3).

Lemma 2.1.4.
Given $A\in M_d(\mathbb{C}G)$ and let $A[i]$ be as described in 2.2.1 , then there exists an $i_0 \in I$ such that for all $i\geq i_0$ we have

$$\|A\| \leq \kappa(A) < \infty \qquad \textit{and} \tag{2.1.5}$$

$$\|A[i]\| \leq \kappa(A)\,. \tag{2.1.6}$$

Proof.
This is proven in [3] lemmas 3.31 , 3.22 , 3.28 . □

Definition 2.1.7.
Let $U < G$ be a normal subgroup of G . We call G/U an amenable homogenous space, and G an extension of U with amenable quotient, if we have a G-invariant metric $d : G/U \times G/U \to \mathbb{N}$ such that sets of finite diameter are finite and such that for all $K>0$ and $\varepsilon>0$ there exists some finite subset $\emptyset \neq X \subset G/U$ with

$$|N_K(X)| := |\{x\in G/U\,;\, d(x,X)\leq K \text{ and } d(x, G/U - X)\leq K\}| \leq \varepsilon|K|\,.$$

Remark 2.1.8.
In [25] this definition is made without the assumption that U is normal, but the approximation result in [25] is only proved for this case. Since we adapt the proof form [25] we also need the assumption that U is normal.

Lemma 2.1.9.
A nested sequence of finite subsets $X_1 \subset X_2 \subset \cdots \subset G/U$ is called Følner exhaustion of G/U if $\bigcup X_i = G/U$ and for all $K > 0$ and $\varepsilon > 0$ there exists an $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have

$$N_K(X_i) \leq \varepsilon |X_i| \,.$$

Every amenable homogenous space admits such an exhaustion.

Proof.
Compare for example Lemma 4.2 in [25] . □

2.2 Main Result

Situation 2.2.1.
Let G be a discrete group that can be constructed out of groups satisfying the bounded determinant property, in one of the following ways:

- $U < G$ with $\Delta(G) \subset \Delta(U)$ and G/U admits a G-invariant metric making it an amenable homogenous space.
- If G is the direct or inverse limit of a directed system of groups G_i .

In [3] the bounded determinant property is proven for a large class $\mathcal{G}$ of groups which is based on the above constructions. Most common examples with this property are amenable groups and residually finite groups.

A bigger class of groups satisfying the determinant bound property are *sofic* groups. A brief description about *sofic* groups and a proof for the determinant bound property is done in the next section. For more details, about *sofic* groups we refer to [5] where the slightly different *semi-integral-determinant* property is proven for sofic groups.

We now introduce a uniform notation for the three constructions. Let $A \in M_d(\overline{\mathbb{Q}}G)$, where $\overline{\mathbb{Q}}$ denotes the field of algebraic numbers. The approximating matrices denoted by $A[i]$ will have different meanings depending on how G is constructed. We have three cases.

1. The group G is the inverse limit of a directed system of groups G_i. Define $A[i] \in M_d(\overline{\mathbb{Q}}G)$ to be the image $p_i(A)$ of A under the natural map $p_i : G \to G_i$. In this case $\mathrm{tr}_i^{\mathbb{C}}$, tr_i^{u} and $\mathrm{tr}_i^{\langle g \rangle}$ will denote $\mathrm{tr}^{\mathbb{C}}_{\mathcal{N}(G_i)}$, $\mathrm{tr}^{u}_{\mathcal{N}(G_i)}$ and $\mathrm{tr}^{\langle g \rangle}_{\mathcal{N}(G_i)}$.

2. The group G is the direct limit of a directed system of groups G_i. Denote by $p_i : G_i \to G$ the corresponding maps.

   In order to define the approximating matrices $A[i]$ we need to make some choices. Write $A = (a_{k,l})$ with $a_{k,l} = \sum_{g \in G} \lambda^g_{k,l} g$. Then, only finitely many of the $\lambda^g_{k,l}$ are non-zero. Let V be the corresponding finite collection of $g \in G$. Since G is a direct limit of G_i we can find $j_0 \in I$ such that $V \subset p_{j_0}(G_{j_0})$. Choose an inverse image for each g in G_{j_0}. This gives a matrix $A[j_0] \in M_d(\overline{\mathbb{Q}}G_{j_0})$ which is mapped to $A[i] := p_{j_0 i}(A[j_0]) \in M_d(\overline{\mathbb{Q}}G_i)$ for $i > j_0$. In this case, $\mathrm{tr}_i^{\mathbb{C}}$, tr_i^{u} and $\mathrm{tr}_i^{\langle g \rangle}$ will denote $\mathrm{tr}^{\mathbb{C}}_{\mathcal{N}(G_i)}$, $\mathrm{tr}^{u}_{\mathcal{N}(G_i)}$ and $\mathrm{tr}^{\langle g \rangle}_{\mathcal{N}(G_i)}$. Keep in mind that the values of the traces can depend on the choices made to define $A[i]$.

3. The group G is an amenable extension of U with Følner exhaustion $X_1 \subset X_2 \subset \cdots \subset G/U$. Let $P_i = p_i \otimes \mathrm{id}_d$ with $p_i : \ell^2(G) \to \ell^2(G)$ the projection on the closed subspace generated by the inverse image of X_i in G. The image of P_i is isomorphic to $\ell^2(U)^{|X_i| d}$ as $\mathcal{N}(U)$-module. We define $A[i] := P_i A P_i$ considered as an operator on the image of P_i.

   With this definition, $A[i]$ is no longer an element of $M_d(\mathcal{N}(G))$ but can be seen as an element in $M_{d|X_i|}(\mathcal{N}(U))$. In this case, $\mathrm{tr}_i^{\mathbb{C}}$, tr_i^{u} and $\mathrm{tr}_i^{\langle g \rangle}$ denote the following

$$\mathrm{tr}_i^{(\cdot)}(A[i]) := \frac{1}{|X_i|} \, \mathrm{tr}^{(\cdot)}_{M_{d|X_i|}(\mathcal{N}(U))}(A[i]) \, .$$

Throughout the rest of the paper, G_i will denote the obvious groups in the limit cases (1) and (2). In the amenable case we take $G_i = U$ constantly. We use $\mathrm{tr}_i^{\mathbb{C}}$, tr_i^{u} to define $F_{A[i]}$ and $F^u_{A[i]}$.

Betti-numbers are given as the dimension of the kernel of the Laplacian Δ_p. Since the value of the spectral density functions at zero is exactly the dimension of the kernel, we can state our approximation theorem as follows.

Theorem 2.2.2.
Let $A \in M_d(\overline{\mathbb{Q}}G)$ and $g \in \Delta(G)$. Then, for any $\varepsilon > 0$ and any choice of matrices $A[i]$, there exists an $i_0 \in I$ such that for all $i \geq i_0$,

$$|\langle F^u_A(0) \cdot \delta_e, \delta_g \rangle - \langle F^u_{A[i]}(0) \cdot \delta_{[e_i]}, \delta_{[g_i]} \rangle| < \varepsilon \,.$$

where we denote by $\delta_{[g_i]}$ the unit vector corresponding to

- *the group element $p_i(g) \in G_i$, in the inverse limit case (1) of (2.2.1),*
- *a chosen preimage of $g \in G_i$, according to the choices made to define $A[i]$ in the direct limit case (2) of (2.2.1),*
- *$g \in \Delta(G)$ in the amenable case (3) of (2.2.1). Without the assumption that $\Delta(G) \subset \Delta(U)$ approximation is still possible but then only for $g \in \Delta(U)$.*

Remark 2.2.3.
The original approximation theorem (Theorem 3.12 in [3]) is contained in the above result if we set $g = e$.

Examples 2.2.4.
As a direct consequence, one can use the center-valued approximation theorem to show the vanishing of β^u for a closed manifold X with fundamental group π_1 in certain cases. One has

1. $\beta^u_0(\widetilde{X}) = \beta^2_0(\widetilde{X})e$, for residually finite π_1 and
2. $\beta^u_p(\widetilde{X}) = \beta^2_p(\widetilde{X})e$, for all $p \in \mathbb{N}$, if π_1 is free abelian.

This follows directly using [16] (example 8 and proposition 2).

2.3 Bounded Determinant for Sofic Groups

In this section we describe the method of G. Elek and E. Szabó in [5] to show that sofic groups have the *semi-integral-determinant property* and show how we can use this to prove that they also have the determinant bound property. We use a general method that can be used to show that the semi-integral determinant property implies that determinant bound property if we have approximations with matrices over finite groups.

Definition 2.3.1. (Semi-integral-determinant property)
A group G has the semi-integral-determinant property if for any matrix $A \in M_d(\mathbb{Z}G)^+$ we have

$$\operatorname{lndet}(A) \geq 0\,.$$

Definition 2.3.2. (Sofic group)
Let G be a finitely generated group and $S \subset G$ be a finite set of generators. Then the group G is called sofic, if there is a sequence of finite directed graphs $\{V_n, E_n\}_{n\geq 1}$ edge-labeled by S and subsets $V_0 \subset V_n$ with the following property:

For any $\delta > 0$ and $r \in \mathbb{N}$, there is an integer $n_{r,\delta}$ such that if $m \geq n_{r,\delta} > 0$ and $B_{(G,S)}(r)$ denotes the r-ball in the Cayley-graph, then

- For each $v \in V_m^0$, there is a map $\psi : B_{(G,S)}(r) \to V_m$, which is an isomorphism (of labeled graphs) between $B_{(G,S)}(r)$ and the r-ball in V_m around v,
- $|V_m^0| \geq (1-\delta)|V_m|$.

Remark 2.3.3.
This definition for sofic groups is equivalent to the more common description using maps $\psi_n : G \to S_n$ and looking at the fixed-point-sets.

Further sofic groups are characterized by the following. A group is sofic if and only if every finitely generated subgroup is sofic.

Theorem 2.3.4.
Sofic groups have the determinant bound property (Def. 2.1.2).

Let G be sofic and $A = (a_{i,j})_{1\leq i,j\leq d} \in M_d(o(\overline{\mathbb{Q}})G)$ be a positive operator. Consider the operator kernel of A, that is the function $K_A : G \times G \to M_d(o(\overline{\mathbb{Q}}))$ such that for $f : G \to \ell^2(G)^d$ we have

$$Af(x) = \sum_{y\in G} K_A(x,y)f(y)\,.$$

This just means $K_A(x,y) = A_g$ if $x = gy$ and $A = \sum_{g\in G} A_g g\,, A_g \in (a^g_{i,j})_{1\leq i,j\leq d} \in M_d o(\overline{\mathbb{Q}})$. There is a constant ω_A, the width of A such that $K_A(x,y) = 0$ if $d(x,y) > \omega_A$ in the word metric of G with respect to the generating system S.

The approximating kernel is constructed as follows. For $m > n_{(\omega_A,\frac{1}{2})}$, define $K^m_A : V_m \times V_m \to M_d(o(\overline{\mathbb{Q}}))$, let $K^m_A(x,y) = 0$ if $y \notin V^0_m$ and $K^m_A(x,y) = K^m_A(g,e)$ if $y \in V^0_m$, $x = \psi_y(g)$.

Lemma 2.3.5.
Let G be a sofic group, $A \in M_d(o(\overline{\mathbb{Q}})G)$ a positive operator. Denote by A_m the bounded linear transformations on $\ell^2(V_m)^d$ defined by the kernel functions K^m_A and denote with $\det^*(K^m_A)$ the product of the non-zero eigenvalues of K^m_A.

$$\lim_{m\to\infty} \frac{\ln(\det^*(A))}{|V_m|} = \operatorname{lndet}(A)$$

Proof.
This is proven in [5] Lemma (6.1). □

G. Elek and E. Szabó prove the semi-integral-determinant property (Theorem 6 in [5]) by using that the product of the positive eigenvalues of the A_m are integers and hence by applying the lemma the claim follows. Given $A \in M_d(o(\overline{\mathbb{Q}})G)$, choose a finite Galois extension $\mathbb{Q} \subset L \subset \mathbb{C}$ such that $A \in M_d(LG)$. Let $\sigma_{i=1,\dots,n} : L \hookrightarrow \mathbb{C}$ be the different embeddings of L in $\mathbb{C}$ and denote with σ_1 the natural inclusion. We set $\widetilde{A} := \bigoplus_{i=1}^d \sigma_i(A)$. For $\widetilde{A}$ Lemma 2.3.5 obviously still holds. The product of the non-zero eigenvalues of $\widetilde{A}_m$ is the lowest non-zero coefficient c of the characteristic polynomial. Since $o(\overline{\mathbb{Q}})$ is a ring, $c \in o(\overline{\mathbb{Q}})$ and c is stable under all σ_i, hence c is in $\mathbb{Q}$ and also is an algebraic integer. This implies $c \in \mathbb{Z}$.

Lemma 2.3.6.
If A and B are positive injective operators in $M_d(\mathbb{C}G)$ and $A \leq B$ we have

$$\operatorname{lndet}(A) \leq \operatorname{lndet}(B)$$

Proof.
This is proven in [19] , Lemma 3.15 . □

Lemma 2.3.7.
Let A be a positive operator in $M_d(\mathbb{C}G)$ and let $A^\perp : \ker(A)^\perp \to \overline{\operatorname{Im}(A)}$ be the weak isomorphism obtained by restricting A to $\ker(A)^\perp$. Then

$$\operatorname{lndet}(\sqrt{(A^\perp)^* A^\perp}) = \operatorname{lndet}(A)\,.$$

Proof.
This is also proven in [19] , Lemma 3.15 . □

We have $\sqrt{(A^\perp)^* A^\perp} \le \|A\| \operatorname{id} \le \kappa(A) \operatorname{id}$. By applying Lemma 2.3.6 , Lemma 2.3.7 and Lemma 2.3.5 we get

$$0 \le \operatorname{lndet}(\widetilde{A}) = d \sum_{i=1}^{n} \operatorname{lndet} \sigma_i(A)$$
$$\implies -\, d \sum_{i=2}^{n} \ln \kappa(\sigma_i(A)) \le - \sum_{i=2}^{n} \operatorname{lndet} \sigma_i(A) \le \operatorname{lndet}(A)\,.$$

This proves Theorem 2.3.4 .

2.4 Some Key Lemmas

The Fourier coefficients of $F^u_A(0)$ are given by

$$\langle F^u_A(0) \cdot \delta_e, \delta_g \rangle = \begin{cases} \frac{1}{|\langle g \rangle|} \operatorname{tr}^{\langle g \rangle}_{\mathcal{N}(G)}(\operatorname{pr}_{|\ker(A)}) & \text{if } g \in \Delta(G) \\ 0 & \text{otherwise.} \end{cases}$$

This can be easily seen using Dixmier's approximation theorem (see e.g. [8]). In the rest of the paper g is always taken in $\Delta(G)$.

The proof of the $\mathbb{C}$-valued approximation theorem in [3] is based on the following three major facts.

1. $\|A\|$ and $\|A[i]\|$ have an upper bound,
2. $\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}$ is positive,

3. the *Fuglede-Kadison determinant* $\operatorname{lndet}(A)$ has a lower bound.

For the center-valued approximation theorem that we prove in this paper, fact (1) is obviously still valid. The facts (2) and (3) of course do not apply to our situation, since they involve the $\mathbb{C}$-valued trace $\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}$, but the main ideas of Lück's method work in general for any positive functional if in addition the Fuglede-Kadison determinant derived from it has a lower bound for A and all approximating $A[i]$. In Definition 2.4.2, we define traces which are derived from delocalized traces and are positive. Using these traces we also define deviated Fuglede-Kadison determinants and prove the existence of a lower bound. Using our method, it would also be possible to directly approximate the Fourier coefficients of the projections on the homology. These coefficients depend on the choice of the basis, hence we do not see any application for this general approximation and restrict to functionals derived from delocalized traces.

A key ingredient of our method is the following simple lemma.

Lemma 2.4.1.
If $a \in \mathcal{N}(G)$ is a positive element, then for all $g \in G$ we have

$$\langle a \cdot \delta_e, \delta_e \rangle \geq |\langle a \cdot \delta_g, \delta_e \rangle|$$

Proof.
$a = b^*b$ then, using Cauchy-Schwarz inequality we get

$$\langle a \cdot \delta_e, \delta_e \rangle = \|b \cdot \delta_e\| \cdot \|b \cdot \delta_e\| = \|b \cdot \delta_e\| \cdot \|b \cdot \delta_g\| \geq |\langle b \cdot \delta_e, b \cdot \delta_g \rangle| = |\langle a \cdot \delta_g, \delta_e \rangle|$$

□

Definition 2.4.2. (Perturbated traces)
Take $A \in M_d(\mathcal{N}(G))$ and $g \in \Delta(G) - \{e\}$, define

$$\operatorname{Tr}^{\langle g \rangle, \mathrm{Re}}_{\mathcal{N}(G)}(A) := \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(A) + \frac{1}{2|\langle g \rangle|}\Big(\operatorname{tr}^{\langle g \rangle}_{\mathcal{N}(G)}(A) + \operatorname{tr}^{\langle g^{-1} \rangle}_{\mathcal{N}(G)}(A)\Big)\,, \tag{2.4.3}$$

$$\operatorname{Tr}^{\langle g \rangle, \mathrm{Im}}_{\mathcal{N}(G)}(A) := \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(A) + \frac{1}{2i|\langle g \rangle|}\Big(\operatorname{tr}^{\langle g \rangle}_{\mathcal{N}(G)}(A) - \operatorname{tr}^{\langle g^{-1} \rangle}_{\mathcal{N}(G)}(A)\Big)\,. \tag{2.4.4}$$

It follows from Lemma 2.4.1 that both traces are positive. The next lemma shows that for a selfadjoint $A \in M_d(\mathcal{N}(G))$ we have

$$\begin{aligned}\langle F^u_A(0) \cdot \delta_e, \delta_g \rangle &= \frac{1}{|\langle g \rangle|} \operatorname{tr}^{\langle g \rangle}_{\mathcal{N}(G)}(A) \\ &= \operatorname{Tr}^{\langle g \rangle, \mathrm{Re}}_{\mathcal{N}(G)}(A) + i \operatorname{Tr}^{\langle g \rangle, \mathrm{Im}}_{\mathcal{N}(G)}(A) - \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(A) - i \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(A)\,.\end{aligned}$$

This is one of the main tricks in our paper. We prove the approximation theorem for $\mathrm{Tr}^{\langle g\rangle,\mathrm{Re}}_{\mathcal{N}(G)}$ and $\mathrm{Tr}^{\langle g\rangle,\mathrm{Im}}_{\mathcal{N}(G)}$. Then we finally prove Theorem 2.2.2 by applying this approximation and the classical approximation theorem (Theorem 3.12 in [3]) to the above equation.

Lemma 2.4.5.
For all $g \in \Delta(G)$ and selfadjoint $A \in M_d(\mathcal{N}(G))$, the traces $\mathrm{Tr}^{\langle g\rangle,\mathrm{Re}}_{\mathcal{N}(G)}(A)$ and $\mathrm{Tr}^{\langle g\rangle,\mathrm{Im}}_{\mathcal{N}(G)}(A)$ are given by the following real numbers

$$\mathrm{Tr}^{\langle g\rangle,\mathrm{Re}}_{\mathcal{N}(G)}(A) = \mathrm{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(A) + \mathrm{Re}\left(\frac{1}{|\langle g\rangle|}\,\mathrm{tr}^{\langle g\rangle}_{\mathcal{N}(G)}(A)\right),$$
$$\mathrm{Tr}^{\langle g\rangle,\mathrm{Im}}_{\mathcal{N}(G)}(A) = \mathrm{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(A) + \mathrm{Im}\left(\frac{1}{|\langle g\rangle|}\,\mathrm{tr}^{\langle g\rangle}_{\mathcal{N}(G)}(A)\right).$$

Proof.
Since the trace on $M_d(\mathcal{N}(G))$ is just a summation of traces on $\mathcal{N}(G)$ it is sufficient to treat the case $d=1$. Write $A = \sum_{h\in G}\lambda_h h \in \mathcal{N}(G)$. We have $\langle g\rangle^{-1} = \langle g^{-1}\rangle$ and selfadjointness of A yields $\lambda_h = \overline{\lambda_{h^{-1}}}$. Hence

$$\mathrm{tr}^{\langle g\rangle}_{\mathcal{N}(G)}(A) = \overline{\mathrm{tr}^{\langle g^{-1}\rangle}_{\mathcal{N}(G)}(A)}.$$

□

2.5 Lower Bound for Determinants

Definition 2.5.1.
Take a positive operator $A \in M_d(\mathcal{N}(G))$ and denote by $\{E^A_\lambda := \chi_{[0,\lambda]}(A) \mid \lambda \in \mathbb{R}^+_0\}$ the spectral family of A. Then define the following spectral density functions:

$$F_A(\lambda) := \mathrm{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(E^A_\lambda),$$
$$F^{\langle g\rangle,\mathrm{Re}}_A(\lambda) := \mathrm{Tr}^{\langle g\rangle,\mathrm{Re}}_{\mathcal{N}(G)}(E^A_\lambda),$$
$$F^{\langle g\rangle,\mathrm{Im}}_A(\lambda) := \mathrm{Tr}^{\langle g\rangle,\mathrm{Im}}_{\mathcal{N}(G)}(E^A_\lambda).$$

For positive $A \in M_d(\mathcal{N}(G))$, the spectral density functions F_A, $F^{\langle g\rangle,\mathrm{Re}}_A$ and $F^{\langle g\rangle,\mathrm{Im}}_A$ are monotonically increasing and induce Riemann-Stieltjes measures $\mathrm{d}F_A(\lambda)$, $\mathrm{d}F^{\langle g\rangle,\mathrm{Re}}_A(\lambda)$ and $\mathrm{d}F^{\langle g\rangle,\mathrm{Im}}_A(\lambda)$, allowing us to define the following (deviations of the) Fuglede-Kadison determinant.

Definition 2.5.2.
Take $A \in M_d(\mathcal{N}(G))$ positive and define

$$\begin{aligned}\operatorname{lndet}(A) &:= \int_{0^+}^{\infty} \ln(\lambda) \mathrm{d}F_A(\lambda)\,,\\ \operatorname{lndet}^{\langle g\rangle,\mathrm{Re}}(A) &:= \int_{0^+}^{\infty} \ln(\lambda) \mathrm{d}F_A^{\langle g\rangle,\mathrm{Re}}(\lambda)\,,\\ \operatorname{lndet}^{\langle g\rangle,\mathrm{Im}}(A) &:= \int_{0^+}^{\infty} \ln(\lambda) \mathrm{d}F_A^{\langle g\rangle,\mathrm{Im}}(\lambda)\,.\end{aligned}$$

In order to prove Theorem 2.2.2 we need lower bounds for the deviated Fuglede-Kadison determinants $\operatorname{lndet}^{\langle g\rangle,\mathrm{Re}}(A)$ and $\operatorname{lndet}^{\langle g\rangle,\mathrm{Re}}(A)$. We obtain it using the fact that the perturbation caused by the delocalized trace is controlled by the standard trace.

Lemma 2.5.3.
Let G be a group that satisfies the determinant bound property and is constructed as described in 2.2.1. Take $A \in M_d\big(o(\overline{\mathbb{Q}})G\big)$ positive (where $o(\overline{\mathbb{Q}})$ denotes the algebraic integers), choose a finite Galois extension $L \subset \mathbb{C}$ of $\mathbb{Q}$, such that $A \in M_d(LG)$. Let $\sigma_1, \dots, \sigma_r : L \to \mathbb{C}$ be the different embeddings of L in $\mathbb{C}$ with σ_1 the natural inclusion $\sigma_1 : L \subset \mathbb{C}$. Then

$$\operatorname{lndet}^{\langle g\rangle,\mathrm{Re}}(A) \geq -2d \Big| \sum_{k=2}^{r} \ln\big(\kappa(\sigma_k(A))\big)\Big| - 2d \ln\Big(\max\big(1, \kappa(A)\big)\Big)\,,$$
$$\operatorname{lndet}^{\langle g\rangle,\mathrm{Im}}(A) \geq -2d \Big| \sum_{k=2}^{r} \ln\big(\kappa(\sigma_k(A))\big)\Big| - 2d \ln\Big(\max\big(1, \kappa(A)\big)\Big)\,.$$

Proof.
We prove the lemma only for $\operatorname{lndet}^{\langle g\rangle,\mathrm{Re}}$, the case $\operatorname{lndet}^{\langle g\rangle,\mathrm{Im}}$ being identical. Using Lemmas 2.4.1 and 2.4.5 we get for any $B \in \mathcal{N}(G)$

$$\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(B) \geq \frac{1}{|\langle g\rangle|} \big|\operatorname{Re}\big(\operatorname{tr}^{\langle g\rangle}_{\mathcal{N}(G)}(B)\big)\big|\,. \tag{2.5.4}$$

Define the function $f_A^{\langle g\rangle,\mathrm{Re}}(\lambda) := \frac{1}{|\langle g\rangle|} \operatorname{Re}\big(\operatorname{tr}^{\langle g\rangle}_{\mathcal{N}(G)}(E^A_\lambda)\big)$. For $a \leq b \in \mathbb{R}_0^+$, inequality (2.5.4) yields

$$\underbrace{F_A(b) - F_A(a)}_{=\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(\chi_{(a,b]}(A))} \geq \Big|\underbrace{f_A^{\langle g\rangle,\mathrm{Re}}(b) - f_A^{\langle g\rangle,\mathrm{Re}}(a)}_{=\operatorname{Re}\left(\frac{1}{|\langle g\rangle|}\operatorname{tr}^{\langle g\rangle}_{\mathcal{N}(G)}(\chi_{(a,b]}(A))\right)}\Big|\,.$$

The Riemann-Stieltjes measure induced by $F_A(\lambda)$ dominates in absolute values the (possibly signed) measure induced by $f_A(\lambda)$. Hence, we have

$$\begin{aligned}
\operatorname{lndet}^{\langle g\rangle,\mathrm{Re}}(A) &= \int_{0^+}^{\infty} \ln(\lambda)\mathrm{d}F_A^{\langle g\rangle,\mathrm{Re}}(\lambda) \\
&= \int_{0^+}^{1} \ln(\lambda)\mathrm{d}F_A^{\langle g\rangle,\mathrm{Re}}(\lambda) + \int_{1}^{\infty} \ln(\lambda)\mathrm{d}F_A^{\langle g\rangle,\mathrm{Re}}(\lambda) \\
&= \int_{0^+}^{1} \ln(\lambda)\mathrm{d}F_A(\lambda) + \int_{0^+}^{1} \ln(\lambda)\mathrm{d}f_A^{\langle g\rangle,\mathrm{Re}}(\lambda) \\
&\quad + \int_{1}^{\infty} \ln(\lambda)\mathrm{d}F_A(\lambda) + \int_{1}^{\infty} \ln(\lambda)\mathrm{d}f_A^{\langle g\rangle,\mathrm{Re}}(\lambda) \\
&\geq -\Big|\int_{0^+}^{1} \ln(\lambda)\mathrm{d}F_A(\lambda)\Big| - \Big|\int_{0^+}^{1} \ln(\lambda)\mathrm{d}f_A^{\langle g\rangle,\mathrm{Re}}(\lambda)\Big| \\
&\quad - \Big|\int_{1}^{\infty} \ln(\lambda)\mathrm{d}F_A(\lambda)\Big| - \Big|\int_{1}^{\infty} \ln(\lambda)\mathrm{d}f_A^{\langle g\rangle,\mathrm{Re}}(\lambda)\Big| \\
&\geq -2\Big|\int_{0^+}^{1} \ln(\lambda)\mathrm{d}F_A(\lambda)\Big| - 2\Big|\int_{1}^{\infty} \ln(\lambda)\mathrm{d}F_A(\lambda)\Big| \\
&\geq -2d\Big|\sum_{k=2}^{r} \ln\big(\kappa(\sigma_k(A))\big)\Big| - 2d\ln\Big(\max\big(1,\kappa(A)\big)\Big). \qquad \square
\end{aligned}$$

2.6 Convergence of the Trace

In this section we basically use the ideas from [3] and [25] to prove the following equalities, for all G constructed as described in Situation 2.2.1 , $g \in \Delta(G)$, $A \in M_d(\mathbb{C}G)$ and every polynomial $p \in \mathbb{C}[x]$:

$$\lim_{i\to\infty} \operatorname{Tr}_i^{\langle g\rangle,\mathrm{Re}}(p(A[i])) = \operatorname{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Re}}(p(A))\,, \tag{2.6.1}$$

$$\lim_{i\to\infty} \operatorname{Tr}_i^{\langle g\rangle,\mathrm{Im}}(p(A[i])) = \operatorname{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Im}}(p(A))\,. \tag{2.6.2}$$

The traces Tr_i depend on the construction of G . We deal with the limit cases (1) and (2) of 2.2.1 first.

Lemma 2.6.3.
Take $A \in M_d(\mathbb{C}G)$, $p \in \mathbb{C}[x]$ and $g \in \Delta(G)$. If G is the direct or inverse limit of groups $(G_i)_{i\in I}$ then there is an $i_0 \in I$ such that for all $i \geq i_0$:

$$\mathrm{Tr}_i^{\langle g\rangle,\mathrm{Re}}(p(A[i])) = \mathrm{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Re}}(p(A)),$$
$$\mathrm{Tr}_i^{\langle g\rangle,\mathrm{Im}}(p(A[i])) = \mathrm{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Im}}(p(A)).$$

Proof.
The proof follows directly from the fact that the *support*

$$\mathrm{supp}(p(A)) := \Big\{\lambda_g^{k,l} \neq 0 \mid 1 \leq k,l \leq d\,,\ (p(A))_{k,l} = \sum_{g\in G} \lambda_g^{k,l} g\Big\}$$

of $p(A) \in M_d(\mathbb{C}G)$ is finite. Since G is an inverse or direct limit, choosing i_0 big enough, we have, for all $i \geq i_0$:

$$\mathrm{supp}(p(A[i])) = \mathrm{supp}(p(A)).$$

As a consequence, the traces coincide. □

To prove (2.6.1) and (2.6.2) in the amenable case (3) of 2.2.1, we adapt ideas from [25] (Lemma 4.6) to our situation.

Lemma 2.6.4.
Let G be an amenable extension of U with Følner exhaustion $X_1 \subset X_2 \subset \cdots \subset G/U$. Then, for all $g \in \Delta(U)$, $A \in M_d(\mathbb{C}G)$ and every polynomial $p \in \mathbb{C}[x]$ we have

$$\lim_{i\to\infty} \mathrm{Tr}_i^{\langle g\rangle,\mathrm{Re}}(p(A[i])) = \mathrm{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Re}}(p(A)),$$
$$\lim_{i\to\infty} \mathrm{Tr}_i^{\langle g\rangle,\mathrm{Im}}(p(A[i])) = \mathrm{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Im}}(p(A)).$$

Proof.
Again we only treat the case $\mathrm{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Re}}$ and assume $d=1$, since the general case follows by summing up the traces. Let $A \in \mathcal{N}(G)$ and denote $A[i] := P_i A P_i^*$, as described in 2.2.1. By linearity of the trace, it also suffices to treat the case where p is a monomial. Pull back the metric on G/U in order to get a semi-metric on G. Denote the inverse image of X_i by X_i'. For $g \in X_i'$ and $h \in U$ we have $P_i(h\cdot\delta_g) = h\cdot\delta_g$. Selfadjointness of P_i implies

for $g \in X_i'$ and $h \in U$, that $\langle (P_iAP_i)^n\delta_g, h\cdot\delta_g\rangle = \langle AP_iAP_i\dots P_iA\delta_g, h\cdot\delta_g\rangle$ and we have the following telescope sum:

$$AP_iA\cdots P_iA = A^n - A(1-P_i)A^{n-1}\cdots - AP_i\cdots A(1-P_i)A\,. \quad (2.6.5)$$

We now compute for $s \in \Delta(U)$,

$$\begin{aligned}
&\left|\operatorname{Tr}^{\langle s\rangle,\mathrm{Re}}_{\mathcal{N}(G)}(A^n) - \operatorname{Tr}^{\langle s\rangle,\mathrm{Re}}_i(A[i]^n)\right| \\
&\quad = \left| \begin{aligned} &\langle A_n\delta_e,\delta_e\rangle + \frac{1}{2|\langle s\rangle|}\sum_{h\in\langle s\rangle\cup\langle s^{-1}\rangle}\langle A^n\delta_e, h\cdot\delta_e\rangle \\ &- \frac{1}{|X_i|}\sum_{[g]\in X_i}\Big(\langle A^n\delta_g,\delta_g\rangle - \frac{1}{2|\langle s\rangle|}\sum_{h\in\langle s\rangle\cup\langle s^{-1}\rangle}\langle A[i]^n\delta_g, h\cdot\delta_g\rangle\Big)\end{aligned}\right| \\
&\quad \le \frac{1}{|X_i|}\sum_{[g]\in X_i}\left|\begin{aligned} &\langle A_n\delta_g,\delta_g\rangle + \frac{1}{2|\langle s\rangle|}\sum_{h\in\langle s\rangle\cup\langle s^{-1}\rangle}\langle A^n\delta_g, h\cdot\delta_g\rangle \\ &- \langle A^n\delta_g,\delta_g\rangle - \frac{1}{2|\langle s\rangle|}\sum_{h\in\langle s\rangle\cup\langle s^{-1}\rangle}\langle A[i]^n\delta_g, h\cdot\delta_g\rangle\end{aligned}\right| \\
&\quad = \frac{1}{|X_i|}\sum_{[g]\in X_i}\left|\begin{aligned} &\Big(\langle A^n\delta_g,\delta_g\rangle - \langle A^n\delta_g,\delta_g\rangle\Big) \\ &+ \frac{1}{2|\langle s\rangle|}\sum_{h\in\langle s\rangle\cup\langle s^{-1}\rangle}\Big(\langle A^n\delta_g, h\cdot\delta_g\rangle - \langle A[i]^n\delta_g, h\cdot\delta_g\rangle\Big)\end{aligned}\right|.
\end{aligned}$$

Using (2.6.5) and applying Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&\left|\operatorname{Tr}^{\langle g\rangle,\mathrm{Re}}_{\mathcal{N}(G)}(A^n) - \operatorname{Tr}^{\langle g\rangle,\mathrm{Re}}_i(A[i]^n)\right| \\
&\quad \le \frac{1}{|X_i|}\sum_{j=1}^{n-1}\sum_{[g]\in X_i}\left|\begin{aligned} &\langle (1-P_i)A^j\delta_g, (A^*P_i)^{n-j}\delta_g\rangle \\ &+ \frac{1}{2|\langle s\rangle|}\sum_{h\in\langle s\rangle\cup\langle s^{-1}\rangle}\langle (1-P_i)A^j\delta_g, (A^*P_i)^{n-j}h\cdot\delta_g\rangle\end{aligned}\right| \\
&\quad \le \frac{1}{|X_i|}\sum_{j=1}^{n-1}\sum_{[g]\in X_i/U}\left(\begin{aligned} &\|(1-P_i)A^j\delta_g\|\cdot\|A^*\|^{n-j} \\ &+ \frac{1}{2|\langle s\rangle|}\sum_{h\in\langle s\rangle\cup\langle s^{-1}\rangle}\|(1-P_i)A^j\delta_g\|\cdot\|A^*\|^{n-j}\end{aligned}\right) \\
&\quad \le \frac{2}{|X_i|}\sum_{j=1}^{n-1}\sum_{[g]\in X_i}\|(1-P_i)A^j\delta_g)\|\cdot\|A^*\|^{n-j}\,.
\end{aligned}$$

Define for $i \in \mathbb{N}$

$$T_i := \Big\{g \in G \mid \lambda^{k,l}_{[i],g} \neq 0 \text{ where } (A[i])_{k,l} := \sum_{g\in G}\lambda^{k,l}_{[i],g}g \text{ and } 1 \le k,l \le d\Big\}.$$

Then the set $T := \bigcup_{i=1}^{\infty} T_i$ is a finite subset of G. Hence if we take $R \in \mathbb{N}$ big enough and let $B_R(g)$ be the ball with radius R around g, we have

$$(1 - P_{B_R(g)})A^j \delta_g = 0\,.$$

The integer R is independent of g, since the semi-metric is G-invariant. Now if $B_R(g) \subset X_i'$, which means $[g] \in X_i - N_R(X_i)$ (see Definition 2.1.7), we have $\mathrm{Im}(P_{B_R}) \subset \mathrm{Im}(P_i)$ and hence

$$(1 - P_i)A^j \delta_g = 0\,.$$

Now we have

$$\begin{aligned}
\left| \mathrm{Tr}_{\mathcal{N}(G)}^{\langle s \rangle,\mathrm{Re}}(A^n) - \mathrm{Tr}_i^{\langle s \rangle,\mathrm{Re}}(A[i]^n) \right| &\leq \frac{2}{|X_i|} \sum_{j=1}^{n-1} \sum_{[g]\in X_i} \|(1-P_i)A^j\delta_g\| \cdot \|A^*\|^{n-j} \\
&= \frac{2}{|X_i|} \sum_{j=1}^{n-1} \sum_{[g]\in N_R(X_i)} \|(1-P_i)A^j\delta_g\| \cdot \|A^*\|^{n-j} \\
&\leq 2 \cdot \frac{|N_R(X_i)|}{|X_i|} \sum_{j=1}^{n-1} \|(1-P_i)A^j\| \cdot \|A^*\|^{n-j} \\
&\leq \frac{|N_R(X_i)|}{|X_i|} \underbrace{2n \max_{j=1,\dots,n} \{\|A\|^j \cdot \|A^*\|^{n-j}\}}_{c_n}\,.
\end{aligned}$$

The quantity c_n is independent of i and Lemma 2.1.9 shows that

$$\lim_{i\to\infty} \frac{|N_R(X_i)|}{|X_i|} = 0\,;$$

hence the claim follows. □

2.7 Finalization of the Proof

Now we are finally ready to prove our theorem. The main idea in this section is to use the lower bound of the Fuglede-Kadison determinant and is due to W. Lück in [17].

Define for the spectral density functions $F_A^{\langle g\rangle,\mathrm{Re}}$ and $F_A^{\langle g\rangle,\mathrm{Im}}$:

$$\overline{F}_A^{(\cdot)}(\lambda) := \limsup_{i\to\infty}(F_{A[i]}^{(\cdot)})(\lambda)\,,$$
$$\underline{F}_A^{(\cdot)}(\lambda) := \liminf_{i\to\infty}(F_{A[i]}^{(\cdot)})(\lambda)\,,$$

and denote their right-continuous approximations by

$$\overline{F}_A^{(\cdot),+}(\lambda) := \lim_{\varepsilon\to 0^+}(\overline{F}_A^{(\cdot)})(\lambda+\varepsilon)\,,$$
$$\underline{F}_A^{(\cdot),+}(\lambda) := \lim_{\varepsilon\to 0^+}(\underline{F}_A^{(\cdot)})(\lambda+\varepsilon)\,.$$

Theorem 2.7.1.
Let $g \in \Delta(G)$ and $A \in M_d(\overline{\mathbb{Q}}G)$. Then

$$F_A^{\langle g\rangle,\mathrm{Re}}(0) = \lim_{i\to\infty} F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0)\,,$$
$$F_A^{\langle g\rangle,\mathrm{Im}}(0) = \lim_{i\to\infty} F_{A[i]}^{\langle g\rangle,\mathrm{Im}}(0)\,.$$

Proof.
We only prove $F_A^{\langle g\rangle,\mathrm{Re}}(0) = \lim_{i\to\infty} F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0)$. The other case can be done identically.

Fix $\lambda \geq 0$ and take a sequence P_n of polynomials converging pointwise to $\chi_{[0,\lambda]}$, such that for $0 \leq x \leq \kappa(A)$

$$\chi_{[0,\lambda]}(x) \leq P_n(x) \leq \chi_{[0,\lambda+\frac{1}{n}]}(x) + \frac{1}{n}\chi_{[0,\kappa(A)]}(x)\,.$$

Applying functional calculus preserves the inequality and since for all $i \in I$ we have $\|A[i]\| \leq \kappa(A)$ we get

$$E_\lambda^{A[i]} \leq P_n(A[i]) \leq E_{\lambda+\frac{1}{n}}^{A[i]} + \frac{1}{n}\,\mathrm{id}\,.$$

Then we apply the positive and hence order preserving trace $\mathrm{Tr}_{\mathcal{N}(G_i)}^{\langle g\rangle,\mathrm{Re}}$ and use the fact that $\mathrm{Tr}_{\mathcal{N}(G_i)}^{\langle g\rangle,\mathrm{Re}}(\mathrm{id}) \leq 2\,\mathrm{tr}_{\mathcal{N}(G)}^{\mathbb{C}}(\mathrm{id}) = 2d$. We get for all $i \in I$

$$F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(\lambda) \leq \mathrm{Tr}_{\mathcal{N}(G_i)}^{\langle g\rangle,\mathrm{Re}}(P_n(A[i])) \leq F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(\lambda+\frac{1}{n}) + \frac{2d}{n}\,.$$

Taking lim sup on the left side and lim inf on the right side leads to:

$$\overline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) \le \mathrm{Tr}_{\mathcal{N}(G)}^{\langle g\rangle,\mathrm{Re}}(P_n(A)) \le \underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda + \frac{1}{n}) + \frac{2d}{n}\,.$$

The sequence $P_n(A)$ converges strongly in a norm bounded set, hence it converges already in the ultra-strong topology. Taking $n \to \infty$ and using normality of $\mathrm{Tr}_{\mathcal{N}(G_i)}^{\langle g\rangle,\mathrm{Re}}$ yields:

$$\overline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) \le F_A^{\langle g\rangle,\mathrm{Re}}(\lambda) \le \underline{F}_A^{\langle g\rangle,\mathrm{Re},+}(\lambda)\,. \tag{2.7.2}$$

Setting $\lambda = 0$ gives us the first half of the proof:

$$\limsup_{i\in I} F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0) \le F_A^{\langle g\rangle,\mathrm{Re}}(0)\,.$$

We now prove that $F_A^{\langle g\rangle,\mathrm{Re}}(0) \le \liminf_{i\in I} F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0)$, which finishes the proof. We first pass from I to a subnet $J \subset I$, such that

$$\limsup_{i\in J} F_A^{\langle g\rangle,\mathrm{Re}}(0) = \liminf_{i\in I} F_A^{\langle g\rangle,\mathrm{Re}}(0)$$

. Equation (2.7.2) still holds and we keep our notation $\overline{F}^{(\cdot)}$, $\underline{F}^{(\cdot)}$ but using J instead of I. Moreover we need the Fatou lemma and the fact that the (deviated) Fuglede-Kadison determinant is bounded. For this we restrict to the case $A \in M_d(\overline{\mathbb{Q}}G)$ to the case $M_d(o(\overline{\mathbb{Q}})G)$, since Lemma 2.5.3 only holds for $A \in M_d(o(\overline{\mathbb{Q}})G)$. But every algebraic number z can be written as a quotient y/k with $y \in o(\overline{\mathbb{Q}})$ and $k \in \mathbb{N}$. We then work with $sA \in M_d(o(\overline{\mathbb{Q}})G)$ instead of $A \in M_d(\overline{\mathbb{Q}}G)$, where s is an appropriate integer. Of course this does not change the kernel and we do not lose any generality.

Recall that $\kappa(A) \ge \|A\|, \|A[i]\|$. Using partial integration, we get

$$\begin{aligned}\mathrm{lndet}^{\langle g\rangle,\mathrm{Re}}(A) =& \ln(\kappa(A))(F_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)) \\ & - \int_{0^+}^{\kappa(A)} \frac{F_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda} \mathrm{d}\lambda\end{aligned}$$

Lemma 2.5.3 yields a $C \in \mathbb{R}$, independent of $i \in I$, such that $\mathrm{lndet}^{\langle g\rangle,\mathrm{Re}}(A[i]) \ge C$ and since $F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(\lambda) \le \mathrm{Tr}_{\mathcal{N}(G_i)}^{\langle g\rangle,\mathrm{Re}}(\mathrm{id}) \le 2d$, it follows that

$$\begin{aligned}\int_{0^+}^{\kappa(A)} \frac{F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda} \mathrm{d}\lambda &\le \ln(\kappa(A))(F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0)) \\ &\le 2d \cdot \ln(\kappa(A)) - C\,.\end{aligned} \tag{2.7.3}$$

Moreover for $\varepsilon \geq 0$ we get

$$\left|\int_{\varepsilon}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re},+}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda - \int_{\varepsilon}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda\right|$$
$$= \lim_{n\to\infty}\left|\int_{\varepsilon}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda+\frac{1}{n}) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda - \int_{\varepsilon}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda\right|$$
$$= \lim_{n\to\infty}\left|\int_{\varepsilon+\frac{1}{n}}^{\kappa(A)+\frac{1}{n}} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda - \int_{\varepsilon}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda\right|$$
$$= \lim_{n\to\infty}\left|\int_{\kappa(A)}^{\kappa(A)+\frac{1}{n}} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda - \int_{\varepsilon}^{\varepsilon+\frac{1}{n}} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda\right|$$
$$\leq \lim_{n\to\infty}\left(\frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\kappa(A)) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{n\varepsilon} - \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\kappa(A)) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{n\kappa(A)}\right) = 0\,.$$

Since this holds for every $\varepsilon > 0$ we can now use equation (2.7.3) to finish the proof

$$\begin{aligned}
\int_{0^+}^{\kappa(A)} \frac{F_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda &\leq \int_{0^+}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re},+}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda \\
&= \int_{0^+}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda \\
&\underset{(*)}{\leq} \int_{0^+}^{\kappa(A)} \frac{\underline{F}_A^{\langle g\rangle,\mathrm{Re}}(\lambda) - \overline{F}_A^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda \\
&\leq \int_{0^+}^{\kappa(A)} \frac{\liminf_{i\in J}\left(F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0)\right)}{\lambda}\mathrm{d}\lambda \\
&\leq \liminf_{i\in J}\int_{0^+}^{\kappa(A)} \frac{F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(\lambda) - F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0)}{\lambda}\mathrm{d}\lambda \\
&\leq 2d\ln(\kappa(A)) - C\,.
\end{aligned}$$

From this follows that $\underline{F}_A^{\langle g\rangle,\mathrm{Re},+}(0) = \overline{F}_A^{\langle g\rangle,\mathrm{Re}}(0)$, otherwise the third integral $(*)$ would not be finite. So we have, using equation (2.7.2)

$$\liminf_{i\in I} F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0) = \limsup_{i\in J} F_{A[i]}^{\langle g\rangle,\mathrm{Re}}(0) = F_A^{\langle g\rangle,\mathrm{Re}}(0)\,,$$

hence the second part is proven. □

Chapter 3

The center-valued Atiyah Conjecture

3.1 Representation Theory of Finite Groups

In this section we will recapitulate some results on representation theory of finite groups, some of this will be used in the following sections. Representation theory of finite groups is a well developed theory, for proofs or further detail see for example [26] . In the following, let K be an algebraic closed subfield of $\mathbb{C}$ and G a finite group.

Definition 3.1.1. (Representation of a finite group)
Let $W \subset V$ be K vector spaces. A linear representation of G is a homomorphism

$$\rho : G \to \mathrm{GL}(V)\,.$$

Assume $\rho(G)W = W$, then we can obtain a representation

$$\rho^W : G \to \mathrm{GL}(W)$$

from ρ . This is said to be a sub-representation of V .

Definition 3.1.2. (Irreducible representations)
Assume $\rho : G \to \mathrm{GL}(V)$ is a representation of G , and V has no nontrivial subspaces invariant under G , then ρ is called an irreducible representation.

Theorem 3.1.3.
Every representation is the direct sum of irreducible representations.

Proof.
This is easily proved, using induction on the dimension of V. See for example [26], theorem 2. □

Definition 3.1.4. (Character)
Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G and tr the standard trace on $GL(V)$, then

$$\chi_\rho := \mathrm{tr} \circ \rho : G \to K$$

is called a character of G.

Proposition 3.1.5.
If χ is a character of a representation of degree n (i.e. $\dim(V) = n$), then

1. $\chi(1) = n$,
2. $\chi(g^{-1}) = \overline{\chi(g)}$ *for* $g \in G$,
3. $\chi(tgt^{-1}) = \chi(g)$ *for* $g, t \in G$.

Proof.
Follows directly from the definition. □

Remark 3.1.6.
Let G be a finite group, ρ a representation of degree n and χ_ρ a character. The possible values the characters can take are algebraic integers.

Proof.
Since g has finite order k, we have $\rho(g)^k = \mathrm{id}$. Hence the eigenvalues are roots of unity of a degree dividing k. □

Proposition 3.1.7.
Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be two representations of G, and χ_1 and χ_2 their characters, then we have:

- *the character of* $V_1 \oplus V_2$ *is* $\chi_1 + \chi_2$,
- *the character of* $V_1 \otimes V_2$ *is* $\chi_1 \cdot \chi_2$.

Proposition 3.1.8. *(Schur's lemma)*
Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be two irreducible representations of G, and let f be a linear map from V_1 to V_2, such that $\rho_2(s) \circ f = f \circ \rho_1(s)$ for all s.

1. *If ρ_1 and ρ_2 are not isomorphic (i.e. it exists no isomorphism with the above property) then $f = 0$ and*

2. *if $\rho_1 \cong \rho_2$, then f is a scalar multiple of* id_V .

Proof.
See for example [26] prop. 4 . □

Definition 3.1.9. (Scalar product)
Let ψ and ϕ be complex valued functions on G . With

$$(\psi \mid \phi) := \frac{1}{|G|} \sum_{g \in G} \psi(g)\overline{\phi(g)}$$

we define a scalar product on $\mathrm{Map}(G; C)$.

Theorem 3.1.10. *(Orthogonality of characters)*

1. *If χ is the character of an irreducible representation we have $(\chi \mid \chi) = 1$.*

2. *If χ and χ' are characters of two non-isomorphic representations, then $(\chi \mid \chi') = 0$.*

Proof.
This follows from the matrix representation of the ρ_i . See for example [26] theorem 3 . □

Theorem 3.1.11.
Let H be the set of class functions on G (i.e. $f(gh) = f(hg)$ for all $g, h \in G$). There are h non-isomorphic irreducible representations of G where h is the number of conjugacy classes in G . The characters $\chi_1, \ldots \chi_h$ of irreducible representations form an orthogonal basis of H .

Proof.
See for example [26] theorem 6 and theorem 7 . □

Proposition 3.1.12.
For $K \subset \mathbb{C}$ an algebraically closed subfield, $K[G]$ is a product of matrix algebras over K .

Proof.
This is a direct consequence of the fact that $K[G]$ is semi-simple, and the structure theorem for semi-simple algebras. See for example [11] . □

Proposition 3.1.13.
Let $\rho_i : G \to \mathrm{GL}(W_i)$ for $1 \leq i \leq h$ be the distinct irreducible representation of degree n_i . The ρ_i extend by linearity to a homomorphism

$$\tilde{\rho}_i : K[G] \to \mathrm{End}(W_i)\,.$$

The family (ρ_i) defines an isomorphism

$$\tilde{\rho} : K[G] \to \prod_{i=1}^{h} \mathrm{End}(W_i) \cong \prod_{i=1}^{h} M_{n_i}(K)\,.$$

Proof.
First $\tilde{\rho}$ is surjective, since otherwise there would be a nonzero linear form on $\prod M_{n_i}(K)$ vanishing on the image of $\tilde{\rho}$. But this contradicts the orthogonality properties of theorem 3.1.10 . Now $K[G]$ and $\prod M_{n_i}(K)$ both have dimension $|G| = \sum n_i^2$, hence the claim follows. □

Proposition 3.1.14. *(Decomposition of the center)*
If we restrict $\tilde{\rho}_i$ to the center $\mathcal{Z}(K[G])$ of $K[G]$, we obtain an algebra homomorphism from $\mathcal{Z}(K[G])$ to the algebra of scalar multiplies of the identity on W_i . It defines a homomorphism

$$\begin{aligned} \omega_i : \mathcal{Z}(K[G]) &\longrightarrow K \\ \sum_{s \in G} \lambda_s s &\mapsto \frac{1}{n_i} \sum_{s \in G} \lambda_s \chi_i(s)\,. \end{aligned}$$

The family $(\omega_i)_{1 \leq i \leq h}$ defines an isomorphism

$$\omega : \mathcal{Z}(K[G]) \longrightarrow \bigoplus_{i=1}^{h} K\,.$$

Proof.
Assume $c := \sum_{s \in G} \lambda_s s \in \mathcal{Z}(K[G])$, we define

$$\begin{aligned} f : G &\longrightarrow \mathbb{K} \\ s &\mapsto \lambda_s\,. \end{aligned}$$

The function f is a class function on G. We have

$$\tilde{\rho}_i(c) = \sum_{s\in G} f(s)\rho_i(s) \in \mathrm{End}(V)\,.$$

We compute $\rho_i(t^{-1})\tilde{\rho}_i(c)\rho_i(t)$:

$$\begin{aligned}\rho_i(t^{-1})\tilde{\rho}_i(c)\rho_i(t) &= \sum_{s\in G} f(s)\rho_i(t^{-1})\rho_i(s)\rho_i(t) \\ &= \sum_{s\in G} f(s)\rho_i(t^{-1}st) \\ &= \sum_{s\in G} f(tst^{-1})\rho_i(s) \\ &= \sum_{s\in G} f(s)\rho_i(s) \\ &= \tilde{\rho}_i(c)\,.\end{aligned}$$

Now from Schur's Lemma (lemma 3.1.8) follows that $\tilde{\rho}_i(c)$ is a multiple of id_{W_i}. □

Corollary 3.1.15.
Define

$$p_i := \frac{n_i}{|G|}\sum_{s\in G}\chi_i(s^{-1})s\,.$$

The p_i with $1 \le i \le h$, and h the number of conjugacy classes in G, form a basis of $\mathcal{Z}(K[G])$, we further have $p_i^2 = p_i$ and $p_ip_j = 0$ for $i \ne j$.

Proof.
The p_i are the preimages of the standard unit vectors of $\bigoplus_{i=1}^h K$ under ω. Hence they form a basis. The orthogonality follows from the fact that ω is ring-isomorphism. □

Definition 3.1.16. (Induced representation)
Let H be a subgroup of G. Let $\theta : H \to \mathrm{GL}(W)$ be a representation of H and $\rho : G \to \mathrm{GL}(V)$ be a representation of G, with $W \subset V$. We say ρ is induced by the representation θ, if $\theta = \rho_{|H}$ and

$$V = \bigoplus_{[\sigma]\in G/H} \rho(\sigma)W\,.$$

(This is well defined, since $\rho(s)W = \rho(t)W$ for $[s] = [t] \in G/H$).

Theorem 3.1.17. *(Existence of an induced representation)*
Assume $\theta : H \to \mathrm{GL}(W)$ is a representation of $H \leq G$. Then there exists a unique representation (up to isomorphism) $\rho : G \to \mathrm{GL}(V)$ of G induced by θ.

Proof.
See for example [26] theorem 12. □

Proposition 3.1.18.
In order that V is induced by W, it is necessary and sufficient, that

$$V \cong K[G] \otimes_{K[H]} W \,.$$

Proof.
This is a consequence of the fact, that a set of representatives of G/H forms a basis of $K[G]$ as $K[H]$ module. □

Theorem 3.1.19. *(Character of an induced representation)*
Let $\theta : H \to \mathrm{GL}(W)$ be a representation of $H \leq G$, and let $\rho : G \to \mathrm{GL}(V)$ be the induced representation. Then for $u \in G$ we have

$$\chi(u) = \sum_{[\sigma] \in G/H \,,\, \sigma^{-1} u \sigma \in H} \chi_\theta(\sigma^{-1} u \sigma) = \frac{1}{|H|} \sum_{s \in G \,,\, s^{-1} u s \in H} \chi_\theta(s^{-1} u s) \,.$$

Proof.
This is proven by direct calculation. □

Theorem 3.1.20. *(Frobenius reciprocity)*
Let ψ be a class function on H and ϕ a class function on G. Denote by $\mathrm{Res}_H^G(\phi)$ the restriction of ϕ to H. Then

$$(\psi \,,\, \mathrm{Res}_H^G(\phi))_H = (\mathrm{Ind}_H^G(\psi) \,,\, \phi)_G \,.$$

Proof.
See for example [26] theorem 13. □

Remark 3.1.21.
If V is induced by W and if E is a $K[G]$-module, we have

1. $\mathrm{Hom}^H(W, E) \cong \mathrm{Hom}^G(V, E)$,

2. $\dim(\mathrm{Hom}^H(W, \mathrm{Res}^G_H(E))) = \dim(\mathrm{Hom}^G(\mathrm{Ind}^G_H(W), E))$,

3. $\mathrm{Ind}^G_H(\psi \,\mathrm{Res}^G_H(\phi)) = \mathrm{Ind}^G_H(\psi)\phi$.

4. Further induction is transitive: Assume $H \leq G \leq S$, then

$$\mathrm{Ind}^S_G(\mathrm{Ind}^G_H(W)) \cong \mathrm{Ind}^S_H(W) .$$

Proposition 3.1.22.
Let H, K be subgroups of G . Denote by $H_s := sHs^{-1} \cap K$ and set

$$\rho^s(x) := \rho(s^{-1}xs) , \quad \textit{for } x \in H_s .$$

We obtain a representation of H_s , denoted W_s . We have

$$\mathrm{Res}^G_K(\mathrm{Ind}^G_H(W)) \cong \bigoplus_{[s]\in K\backslash G/H} \mathrm{Ind}^K_{H_s}(W_s) .$$

Proposition 3.1.23. *(Mackey's irreducibility criterion)*
We apply the preceding result to the case $K = H$. For $s \in G$ we still denote by H_s the subgroup $sHs^{-1} \cap H$ of H . Denote by $\rho : H \to \mathrm{GL}(W)$ a representation of H . In order that the induced representations $V = \mathrm{Ind}^G_H(W)$ be irreducible, it is necessary and sufficient that the following two conditions be satisfied:

- *W is irreducible.*
- *For each $s \in G - H$ the two representations ρ^s and $Res^H_{H_s}(\rho)$ are disjoint (i.e. they don't have a common subrepresentation).*

Proof.
See [26] Prop. 23. □

Corollary 3.1.24.
Suppose H is normal in G . In order that $\mathrm{Ind}^G_H(\rho)$ is irreducible, it is necessary and sufficient that ρ is irreducible and not isomorphic to any of its conjugates ρ^s for $s \notin H$.

3.2 Linnell's Proof of the Atiyah Conjecture for Elementary Amenable Groups

In this section we give a sketch of Linnell's proof of the Atiyah conjecture over $\mathbb{C}$ for elementary amenable groups. The Atiyah conjecture is also known for free groups (also over $\mathbb{C}$) and certain constructions with the previous classes, but then only over $\overline{\mathbb{Q}}$ instead of $\mathbb{C}$. For more detail, see [3].

We will use one part of this proof in the next section to formulate and prove the center-valued Atiyah conjecture. A crucial ingredient is Moody's induction theorem (see. [22]). This section mainly is based on [12] and [14].

Definition 3.2.1. (Atiyah conjecture)
Let G be a group, such that the orders of the finite subgroups have a bounded least common multiple $\mathrm{lcm(G)}$. We say that G satisfies the Atiyah-conjecture over $\Lambda \subset \mathbb{C}$, if for any operator $a : \ell^2(G)^n \to \ell^2(G)^m$ with $a \in M(\Lambda G|n \times m)$ we have

$$\mathrm{lcm(G)} \cdot \dim_{\mathcal{N}(G)}(\ker(a)) \in \mathbb{Z} \,.$$

Lemma 3.2.2.
Let R be a ring, let $m, n \in \mathbb{N}_+$, and let P, Q be finitely generated projective right R-modules such that $P \cong Q$. If P and Q correspond to the idempotents $e \in M_n(R)$ and $f \in M_m(R)$ respectively, then there exists $u \in \mathrm{GL}_{m+n}(R)$ such that $u \,\mathrm{diag}(e, 0_m)u^{-1} = \mathrm{diag}(f, 0_n)$.

Theorem 3.2.3. *(Atiyah conjecture for elementary amenable groups)*
Let G be an elementary amenable group, such that the orders of the finite subgroups have a bounded least common multiple $\mathrm{lcm(G)}$. *Then G satisfies the Atiyah conjecture.*

Proof.
We only give a sketch of the proof here. For more detail we refer to [12] theorem 6, lemma 11 and lemma 17. Write $N = \Delta^+(G)$ the torsion subgroup of the finite conjugate subgroup of G. Write $\mathbb{C}N = R_1 \oplus \cdots \oplus R_m$ where the R_i are matrix rings over $\mathbb{C}$. Then G/N permutes the R_i by conjugation, and, by renumbering if necessary, we may assume that $\{R_1, ..., R_t\}$ is a set

of orbit representatives for this action. Let G_i/N be the stabilizer of R_i, and write $n_i = [G : G_i]$. By Clifford's theorem we have

$$\mathbb{C}G = \mathbb{C}N * G/N = \bigoplus_{i=1}^{t} M_{n_i}(R_i * G_i/N)\,.$$

Let Q_i be the simple Artinian quotient ring of $M_{n_i}(R_i * G_i/N) = M_{n_i}(R_i) * G_i/N$ $(1 < i < t)$, which exists by lemma 4.1(i) of [10]. Then $D(\mathbb{C}G) = \oplus_{i=1}^{t} Q_i$. Using lemma 4.1 (ii) of [10], we see that the natural induction map

$$\bigoplus_{F \in \mathcal{F}(G_i)} G_0(M_{n_i}(R_i) * FN/N) \to G_0(Q_i)$$

is onto $(1 \leq i \leq t)$, where $\mathcal{F}(G)$ is the set of finite subgroups in G. We can now infer that the natural induction map

$$\bigoplus_{F \in \mathcal{F}(G)} G_0(\mathbb{C}F) \to G_0(D(\mathbb{C}G))$$

is also onto. Furthermore, all $D(\mathbb{C}G)$-modules are projective. This means that if P is the projective $D(\mathbb{C}G)$-module corresponding to e, then there exist $r, s \in \mathbb{N}_+$, finite subgroups $F_1, \ldots, F_s$ of G and finitely generated $\mathbb{C}F_i$-modules P_i with $(1 < i < s)$ such that

$$P \oplus D(\mathbb{C}G)^r \cong \bigoplus_{i=1}^{s} P_i \otimes_{\mathbb{C}F_i} D(\mathbb{C}G) \text{ (use [21] prop 12.1.4)}\,.$$

Since a finitely generated $\mathbb{C}F_i$-module is isomorphic to a direct sum of right ideals of $\mathbb{C}F_i$, we may assume that $P_i \cong f_i \mathbb{C}F_i$ for some projection $f_i(1 < i < s)$. Then $\operatorname{diag}(e, 1_r)(D(\mathbb{C}G)^{n+r}) \cong \operatorname{diag}(f_1, \ldots, f_s)(D(\mathbb{C}G)^s)$, as $D(\mathbb{C}G)$-modules. Hence by lemma 3.2.2

$$\operatorname{diag}(e, 1_r, 0_s) = u \operatorname{diag}(f_1, \ldots, f_s, 0_{n+r}) u^{-1}\,,$$

for some $u \in \mathrm{GL}_{n+r+s}(D(\mathbb{C}G))$ hence

$$\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(e) + r = \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(f_1) + \cdots + \operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(f_s)\,,$$

with $\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(f_i) = \frac{1}{|F_i|}$ for all i. □

3.3 $D(K[G])$ is Semisimple Artinian

In this section we give an overview of the proof, that the division closure $D(K[G])$ of $K[G]$ is semisimple Artinian for algebraic closed $K \subset \mathbb{C}$ and a discrete group G with $\Delta^+(G)$ trivial. In the next section we will prove this, without the assumption Δ^+ is trivial, for elementary amenable groups. The proof that $D(K[G])$ is semisimple Artinian for trivial $\Delta^+(G)$ can be found in full detail in [15].

Remark 3.3.1.
Let $\mathcal{U}(G)$ denote the algebra of unbounded operators on $\ell^2(G)$ affiliated to $\mathcal{N}(G)$ Then the involution on $\mathcal{N}(G)$ extends to an involution on $\mathcal{U}(G)$, and $\mathcal{U}(G)$ is a finite $*$-regular algebra. Also if M is a right $\mathcal{N}(G)$-module, then

$$\dim^*_{\mathcal{N}(G)} M = \dim^*_{\mathcal{N}(G)} M \otimes_{\mathcal{N}(G)} \mathcal{U}(G)$$

in particular $\dim^*_{\mathcal{N}(G)} e\mathcal{U}(G) = \operatorname{tr}^*_{\mathcal{N}(G)}(e)$.

More detail on the algebra $\mathcal{U}(G)$ can be found in [19] chapter 8.

Definition 3.3.2.
The extended division closure $\mathcal{E}(K[G])$ of $K[G]$ in $\mathcal{U}(G)$ is defined as the smallest subring of $\mathcal{U}(G)$ satisfying

- $x \in \mathcal{E}(K[G])$ and $x^{-1} \in \mathcal{U}(G) \Longrightarrow x^{-1} \in \mathcal{E}(K[G])$,
- $x \in \mathcal{E}(K[G])$ and $x\mathcal{U}(G) = e\mathcal{U}(G)$ with e a central idempotent in $\mathcal{U}(G)$ implies $e \in \mathcal{E}(K[G])$.

Lemma 3.3.3.
Let G be a group and K an algebraically closed subfield of $\mathbb{C}$, then

$$\langle \dim_{\mathcal{N}(G)} x\mathcal{U}(G)^n \mid a \in M_n(K[G])\rangle = \langle \dim_{\mathcal{N}(G)} x\mathcal{U}(G)^n \mid a \in M_n(\mathcal{E}(K[G]))\rangle$$

Proof.
See [15] lemma 2.4. □

Remark 3.3.4.
Lemma 3.3.3 can be extended to the center-valued dimension. The proof in [15] works without modification.

Proposition 3.3.5.
Let G be a group with $\Delta(G)$ finite and let K be an algebraically closed subfield of $\mathbb{C}$. Then $\mathcal{E}(K[G]) = D(K[G])$.

Proof.
With $\Delta(G)$ finite, the center of $\mathcal{N}(G)$ is finite dimensional and hence it is already contained in $K[G]$. □

Theorem 3.3.6.
Let G be a group and let K be a subfield of $\mathbb{C}$ which is closed under complex conjugation. Suppose there is an $L \in \mathbb{N}$ such that $L \dim_{\mathcal{N}(G)} a\mathcal{U}(G)^n \in \mathbb{Z}$ for all $a \in M_n(KG)$ and for all $n \in \mathbb{N}$. Then $\mathcal{E}(K[G])$ is a semisimple Artinian ring.

Proof.
First observe that the above lemma tells us that $L \dim_{\mathcal{N}(G)} a\mathcal{U}(G) \in \mathbb{Z}$ for all $a \in \mathcal{E}(K[G])$. This tells us that $\mathcal{E}(K[G])$ has at most L primitive central idempotents. Indeed, if $e_1, \dots, e_{L+1}$ are (nonzero distinct) primitive central idempotents, then $e_i e_j = 0$ for $i \neq j$ and we see that the sum $\oplus_{i=1}^{L+1} e_i\mathcal{U}(G)$ is direct. But

$$\dim_{\mathcal{N}(G)} \bigoplus_{i=1}^{L+1} e_i\mathcal{U}(G) = \sum_{i=1}^{L+1} \dim_{\mathcal{N}(G)}(e_i\mathcal{U}(G)) \geq (L+1)/L \geq 1\,,$$

which is a contradiction. Thus $\mathcal{E}(K[G])$ has n primitive central idempotents $e_1, \dots, e_n$ with $n \leq L$. For each $1 \leq i \leq n$ chose $0 \neq a_i \in e_i\mathcal{E}(K[G])$ such that $\dim_{\mathcal{N}(G)} a_i\mathcal{U}(G)$ is minimal. Fix $m \in \{1, 2, \dots, n\}$. Since $L \dim_{\mathcal{N}(G)} a\mathcal{U}(G) \in \mathbb{Z}$ for all $a \in \mathcal{E}(K[G])$, we may choose $g_1, ..., g_r \in G$ with $\dim_{\mathcal{N}(G)}(\sum_{i=1}^r g_i a_m a_m^* g_i^{-1})\mathcal{U}(G)$ maximal. Note that if $g_{r+1} \in G$, then (using [15] lemma 2.5), we get

$$(\sum_{i=1}^{r+1} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G) \supset (\sum_{i=1}^{r} g_i a_m)\mathcal{U}(G) \supset (\sum_{i=1}^{r} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G)\,,$$

and hence

$$\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r+1} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G) \geq \dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G)\,.$$

By the maximality of $\dim_{\mathcal{N}(G)}\big(\sum_{i=1}^{r}(g_i a_m a_m^* g_i^{-1})\big)\mathcal{U}(G)$ we have

$$\dim_{\mathcal{N}(G)}(\sum_{i=1}^{r+1} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G) = \dim_{\mathcal{N}(G)}(\sum_{i=1}^{r} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G)\,.$$

It follows that

$$(\sum_{i=1}^{r+1} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G) = (\sum_{i=1}^{r} g_i a_m a_m^* g_i^{-1})\mathcal{U}(G)\,.$$

Using again [15] lemma 2.5 , we obtain that $g a_m \mathcal{U}(G) \subset \sum_{i=1}^{r}(g_i a_m a_m^* g_i^{-1})\mathcal{U}(G)$, for all $g \in G$. Let $f \in \mathcal{U}(G)$ be the unique projection such that

$$f\mathcal{U}(G) = \big(\sum_{i=1}^{r} g_i a_m a_m^* g_i^{-1}\big)\mathcal{U}(G)\,.$$

Then $gf\mathcal{U}(G) = \big(\sum_{i=1}^{r} g g_i a_m a_m^* g_i^{-1}\big)\mathcal{U}(G) \subset \sum g g_i a_m \mathcal{U}(G) \subset f\mathcal{U}(G)$ for all $g \in G$, thus $gf\mathcal{U}(G) = f\mathcal{U}(G)$ and we deduce that $gfg^{-1}\mathcal{U}(G) = f\mathcal{U}(G)$. Also gfg^{-1} is a projection thus we $gfg^{-1} = f$. We conclude that f is a central projection in $\mathcal{E}(K[G])$. Since $f \neq 0$, $f\mathcal{U}(G) \subset e_m\mathcal{U}(G)$ and e_m is primitive, we conclude that $f = e_m$ and consequently $\sum_{i=1}^{r} g_i a_m \mathcal{U}(G) = e_m\mathcal{U}(G)$.

By omitting some of the terms in this sum, if necessary, we may assume that

$$\sum_{1\le i\le r\,,\, i\neq s} g_i a_m \mathcal{U}(G) \neq e_m \mathcal{U}(G) \tag{3.3.7}$$

for all $1 \le s \le r$. We make the following observation:

$$\text{if} \quad 0 \neq x \in g_s a_m \mathcal{E}(K[G])\,, \quad \text{then} \quad x\mathcal{U}(G) = g_s a_m \mathcal{U}(G)\,, \tag{3.3.8}$$

where $1 \le s \le r$. This is obtained because $0 \neq x\mathcal{U}(G) \subset g_s a_m \mathcal{U}(G)$ and consequently $x\mathcal{U}(G) = g_s a_m \mathcal{U}(G)$.

We claim that $e_m\mathcal{E}(K[G]) = \sum_{i=1}^{r} g_i a_m \mathcal{E}(K[G])$. Set $\sigma := \sum_{i=1}^{r} g_i a_m a_m^* g_i^{-1}$. Since $\sigma\mathcal{U}(G) = e_m\mathcal{U}(G)$, we see that

$$(\sigma + (1-e_m))\mathcal{U}(G) \supset \sigma\mathcal{U}(G) + (1-e_m)\mathcal{U}(G) = e_m\mathcal{U}(G) + (1-e_m)\mathcal{U}(G) = \mathcal{U}(G)\,.$$

Therefore $\sigma+(1-e_m)$ is invertible in $\mathcal{U}(G)$ and hence $\sigma+(1-e)$ is invertible in $\mathcal{E}(K[G])$. Thus

$$e_m\sigma\mathcal{E}(K[G]) = e_m(\sigma+1-e_m)\mathcal{E}(K[G]) = e_m\mathcal{E}(K[G])\,.$$

Moreover, $\sigma\mathcal{E}(K[G]) \subset e_m\mathcal{E}(K[G])$ and therefore $e_m\sigma\mathcal{E}(K[G]) = \sigma\mathcal{E}(K[G])$, hence

$$e_m\mathcal{E}(K[G]) = \sigma\mathcal{E}(K[G]) = \sum_{i=1}^{r} g_i a_m\mathcal{E}(K[G])\,.$$

If this sum is not direct, then for some s with $1 \le s \le r$ we have

$$g_s a_m\mathcal{E}(K[G]) \cap \sum_{i\neq s} g_i a_m\mathcal{E}(K[G]) \neq 0\,,$$

and without loss of generality we may assume that $s = 1$. So let

$$0 \neq x \in g_1 a_m\mathcal{E}(K[G]) \cap \sum_{i=2}^{r} g_i a_m\mathcal{E}(K[G])\,.$$

Then $0 \neq x\mathcal{U}(G) \subset g_1 a_m\mathcal{U}(G)$ and (3.3.7) shows that $x\mathcal{U}(G) = g_1 a_m\mathcal{U}(G)$. It follows that $g_1 a_m\mathcal{U}(G) \subset \sum_{i=2}^{r} g_i a_m\mathcal{U}(G)$ consequently

$$\sum_{i=2}^{r} g_i a_m\mathcal{U}(G) = e_m\mathcal{U}(G)$$

which contradicts (3.3.6) and our claim is established. Now we show that $g_1 a_m\mathcal{E}(K[G])$ is an irreducible $\mathcal{E}(K[G])$-module. Suppose $0 \neq x \in g_1 a_m\mathcal{E}(K[G])$. Then $x\mathcal{U}(G) = g_1 a_m\mathcal{U}(G)$ by (3.3.7) and using lemma 3.3.3, we see as before that $xx^* + \sum_{i=2}^{r} g_i a_i a_i^* + 1 - e_m$ is a unit in $\mathcal{U}(G)$ and hence is also a unit in $\mathcal{E}(K[G])$. This proves that $x\mathcal{E}(K[G]) = g_1 a_m\mathcal{E}(K[G])$ and we deduce that $\mathcal{E}(K[G])$ is a finite direct sum of irreducible $\mathcal{E}(K[G])$-modules. It follows that $\mathcal{E}(K[G])$ is a semisimple Artinian ring. □

3.4 Center-valued Atiyah conjecture

The following result is the fruit of a joint work with Peter Linnell and Thomas Schick. Let G be a group with lcm(G) $< \infty$, satisfying the Atiyah-conjecture over K, with $K \subset \mathbb{C}$ algebraically closed. Denote by Δ^+ the normal subgroup of all elements having finite conjugacy classes and finite order. Assume without loss of generality that any finite subgroup $E \le G$ is containing Δ^+ (otherwise take $E \cdot \Delta^+$).

Proposition 3.4.1. *(Δ^+ is a finite group)*
The finite conjugacy class group Δ^+ of a group with lcm(G) $< \infty$ *(i.e. it has an upper bound on the orders of its finite subgroups) is a finite normal subgroup.*

Proof.
Let H be a finitely generated subgroup of $\Delta(G)$. We obtain from [23] lemma 2.1 and 2.2 that the commutator group $H' := [G : H]$ is finite. Denote by N the finite normal subgroup obtained by the product of all finite normal subgroups in G (there are only finitely many since lcm(G) $\leq \infty$). Notice that $N \subset \Delta^+$. Now take $g \in \Delta^+ - N$ and denote by H the subgroup generated by the finite conjugacy class $\langle g \rangle$. Now H is a normal subgroup in G and H is finitely generated. We have H' is finite, and H/H' is a finitely generated abelian group, generated by elements of finite order. Hence H/H' is finite and so is H. This is a contradiction to $g \in \Delta^+ - N$ and hence $\Delta^+ = N$.(The key argument for this proof is taken from [23] lemma 19.3). $\square$

Lemma 3.4.2.
A basis of orthogonal irreducible projections $\{P^1 \dots \mathcal{P}^{C_G}\} \in \mathcal{Z}(\mathcal{N}(G))$, of $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(K\Delta^+) \subset \mathcal{Z}(K\Delta^+)$ is given by

$$P^i := \sum_{\left\{\begin{array}{l} k \ s.t. \ gp_ig^{-1} = p_k \\ for \ a \ g \in G \end{array}\right\}} p_k$$

with

$$p_i := \frac{n_i}{|\Delta^+|} \sum_{s \in G} \chi_i(s^{-1})s \, .$$

with n_i the dimensions of the irreducible representations of Δ^+ , χ_i the corresponding characters.

Proof.

We have to check that they form a basis. The dimension of $\mathcal{Z}(\mathcal{N}(G)) \cap \mathcal{Z}(K\Delta^+)$ is equal to the number C_G of finite conjugacy classes in G . Two projections p_i and p_k are conjugate in G , iff they can be identified by conjugation. This means that p_i and p_k are identified, iff the coefficients coincide after certain permutations within each conjugacy class in G. Hence we get for every finite conjugacy class in G a set of projections p_i^j. After possible renumbering, we can assume that $P^1, \dots, P^{C_G}$ are distinct and hence, because of dimension reasons, they form a basis. $\square$

Lemma 3.4.3.
Let $\mathrm{pr} : \mathcal{N}(G) \to \mathcal{Z}(K\Delta^+)$ *be the projection onto the subspace* $\mathcal{Z}(K\Delta^+)$. *Denote by* $\mathrm{tr}^u_{\Delta^+} := \mathrm{pr} \circ \mathrm{tr}^u_{\mathcal{N}(G)}$, *and let* E *be a finite subgroup of* G *containing* Δ^+. *For an irreducible projection* $Q \in K[E]$ *we obtain*

$$\mathrm{tr}^u_{\Delta^+}(Q) = \sum_{j=1}^{C_G} \frac{\langle Q, P^j \rangle}{\langle P^j, P^j \rangle} P^j \tag{3.4.4}$$

$$= \frac{\dim_{\mathbb{C}}(Im(Q))|\Delta^+|}{\dim_{\mathbb{C}}(Im(P^i))|E|} P^i \tag{3.4.5}$$

$$= \frac{\dim_{\mathcal{N}(G)}(Im(Q))}{\dim_{\mathcal{N}(G)}(Im(P^i))} P^i \tag{3.4.6}$$

where P^i *is the central carrier of* Q.

Proof.
We have $QP^j + Q(1 - P^j) = Q$ and $QP^jQ(1 - P^j) = 0$. Since Q is irreducible, we get either $QP^j = Q$ (hence P^j is the central carrier of Q), or $QP^j = 0$. In the case $QP^j = Q$ we have

$$\begin{aligned} \langle Q, P^j \rangle &= \langle QP^j, 1 \rangle \\ &= \langle Q, 1 \rangle \\ &= \frac{\dim_{\mathbb{C}}(Im(Q))}{|E|} \\ \langle P^j, P^j \rangle &= \frac{\dim_{\mathbb{C}}(Im(P^j))}{|\Delta^+|}. \end{aligned}$$

$\square$

Theorem 3.4.7.
Let G *be a discrete group, with* $\mathrm{lcm(G)} < \infty$ *and let* $K \subset \mathbb{C}$ *be a algebraically closed subfield. The following statements are equivalent.*

1. $D(K[G])$ *is a semisimple Artinian ring. The primitive central idempotents are central idempotents* $P^1, \dots, P^{C_G}$ *in* $K\Delta^+$. *Each* $P^iD(K[G])P^i$

is an $L_i \times L_i$ matrix ring over a division ring, and L_i is defined as follows:

$$L := \frac{\dim_{\mathbb{C}}(P^j)\frac{\mathrm{lcm(G)}}{|\Delta^+|}}{\left(\begin{array}{l}\gcd\Big(\dim_{\mathbb{C}}(Q_1)\,,\dots\,,\dim_{\mathbb{C}}(Q_n)\,,\,\dfrac{\mathrm{lcm(G)}}{|\Delta^+|}\Big)\\ \cdot\gcd\Big(\dfrac{\gcd\big(\dim_{\mathbb{C}}(Q_1)\,,\dots\,,\dim_{\mathbb{C}}(Q_n)\big)}{\gcd\big(\dim_{\mathbb{C}}(Q_1)\,,\dots\,,\dim_{\mathbb{C}}(Q_n)\,,\,\frac{\mathrm{lcm(G)}}{|\Delta^+|}\big)}\,,\,\dim_{\mathbb{C}}(P^j)\Big)\,,\end{array}\right)} \in \mathbb{Z}\,,$$

where the Q_i are irreducible sub-projections of P^j in $K[E_i]$ where E_i is running through all isomorphism classes of finite subgroups in G. (This will be obtained as follows. We first combine the occurring projections Q_i to a projection having their gcd *as dimension (in the denominator). Then we reduce the fraction and use the same procedure to obtain a projection with the same denominator but one in the counter.)*

2. $\mathrm{Colim}_{E\leq G\,:\,|E|<\infty} K_0(KE) \to K_0(D(K[G]))$ *is surjective and DG is semisimple Artinian.*

3. $\mathrm{Colim}_{E\leq G\,;\,|E|<\infty} G_0(KE) \to G_0(DG)$ *is surjective.*

4. *For each finitely presented KG-module M, the center-valued dimension is quantized. It is linear combination of the dimensions, induced up from projections over KE, where E runs through the finite subgroups of G, and dimensions are taking values according to* (3.4.4) *.*

We prove the equivalence of these statements later in this paragraph.

Conjecture 3.4.8. (Center-valued Atiyah-conjecture)
We say G satisfies the center-valued Atiyah conjecture, if one (and hence all) statements are true over G.

Theorem 3.4.9.
The center-valued Atiyah conjecture is true for elementary amenable groups.

Proof.
Linnell proves statement (3) in [12] theorem 6. Since the statements are equivalent (as we will see) the claim follows. □

Corollary 3.4.10.
Assume G is a subgroup of a inverse limit of a inverse system of groups $(G_i)_{i\in I}$. Such that all groups satisfy $\mathrm{lcm(G)} < \infty$. *Assume that the G_i are elementary amenable and that the finite subgroups in G_i are images of the finite subgroups in G. Then G also satisfies the center-valued Atiyah-conjecture over $K = \overline{\mathbb{Q}}$.*

Proof of corollary 3.4.10.
For large enough i we can assume that $\Delta^+(G) = \Delta^+(G_i)$, this follows since we only have finitely many finite normal subgroups in G and in all G_i (see prop. 3.4.1). Denote by Q_i the projection on the kernel $\ker(A[i])$ of $A[i] := p_i(A)$, with $p_i : M_n(\overline{\mathbb{Q}}[G]) \to M_n(\overline{\mathbb{Q}}[G_i])$ obtained from the corresponding maps $G \to G_i$ as described in 2.2.1.

Since Δ^+ is finite and amenable groups satisfy the determinant bound property. Using the approximation theorem we obtain

$$\lim_{i\to\infty} \mathrm{tr}^u_{\Delta^+}(Q_i) = \mathrm{tr}^u_{\Delta^+}(Q)\,.$$

On the other hand we obtain from the center-valued-Atiyah conjecture for elementary amenable groups that the coefficients of $\mathrm{tr}^u_{\mathcal{N}(G_i)}(Q_i)$ are trivial outside of Δ^+. Hence applying the approximation theorem provides this also for $\mathrm{tr}^u_{\mathcal{N}(G_i)}(Q)$ and we get

$$\lim_{i\to\infty} \mathrm{tr}^u_{\mathcal{N}(G)}(Q_i) = \mathrm{tr}^u_{\mathcal{N}(G)}(Q)\,.$$

From the quantization of the center (lemma 3.4.3) it follows that G satisfies (4) in the Atiyah-conjecture. □

Proof of the equivalence in 3.4.7.
(1) $\Longrightarrow$ (2): We look at the following map

$$\begin{aligned} \sigma : K_0(D(K[G])) &\longrightarrow \mathcal{Z}(NG) \\ [p] &\mapsto \mathrm{tr}^u_{\mathcal{N}(G)}(p) \end{aligned}$$

This map is welldefined and injective. From this we get the following commuting diagram.

$$\begin{array}{ccc}
\operatorname{Colim}_{E\le G:|E|<\infty} K_0(KE) & \xrightarrow{\phi} & K_0(D(K[G])) \\
\downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} \\
\bigoplus_{E\le G:|E|<\infty} \mathcal{Z}(KE) & \xrightarrow{\psi} & \mathcal{Z}(\mathcal{N}(G))
\end{array}$$

From (1) it follows that $\psi\circ\sigma_1 : \operatorname{Colim}_{E\le G:|E|<\infty} K_0(KE) \to \mathcal{Z}(\mathcal{N}(G))$ maps surjective onto the projections in $\mathcal{Z}(\mathcal{N}(G))\cap D(K[G])$. Which is the image of σ_2. So the diagram actually commutes and we obtain from the injectivity of σ_2 the requested surjectivity of ϕ.

(2) $\Longrightarrow$ (3): For a semisimple Artinian ring every finitely generated module is projective, therefore $G_0 = K_0$. (G_0 are the equivalence classes of finitely generated modules).

(3) $\Longrightarrow$ (4) is evident.

(4) $\Longrightarrow$ (1): We look at the following sum of the sub-projections Q_i of P^j, with integral coefficients a_i, where the Q_i are irreducible projections supported on $K[E_i]$ and E_i runs through the isomorphism classes of finite groups in G.

$$\begin{aligned}
\frac{\dim_{\mathbb{C}}(P^j)\,\mathrm{lcm(G)}}{|\Delta^+|}\,\mathrm{tr}^u_{\mathcal{N}(G)}(a_i\sum_{i=1}^n Q_iP^j) = \\
&= \frac{\dim_{\mathbb{C}}(P^j)\,\mathrm{lcm(G)}}{|\Delta^+|}\sum_{i=1}^n \frac{a_i\dim_{\mathbb{C}}(Q_i)|\Delta^+|}{|E_i|\dim_{\mathbb{C}}(P^j)}P^j \\
&= \sum_{i=1}^n a_i\dim_{\mathbb{C}}(Q_i)\frac{\mathrm{lcm(G)}}{|E_i|}P^j\,.
\end{aligned}$$

Since $\gcd(\frac{\mathrm{lcm(G)}}{E_1},\ldots,\frac{\mathrm{lcm(G)}}{E_n}) = 1$, (this is already true for the Sylow-subgroups) we obtain from elementary number theory, that we can find coefficients $a_i\in\mathbb{Z}$ such that

$$\sum_{i=1}^n a_i\dim_{\mathbb{C}}(Q_i)\frac{\mathrm{lcm(G)}}{|E_i|} = \gcd\big(\dim_{\mathbb{C}}(Q_1),\ldots,\dim_{\mathbb{C}}(Q_n)\big)\,. \qquad (3.4.11)$$

We use this to construct a projection $\tilde{e}$ as follows. We first add all projections with positive coefficients by taking direct sums and obtain a projection $\tilde{e}^+$, then we add all projections with negative coefficients the same way (ignoring the sign), obtaining $\tilde{e}'$. From (3.4.11) follows that $\dim^u(\tilde{e}') \leq \dim^u(\tilde{e}^+)$ .Now it follows from [8] theorem 8.4.3 that $\tilde{e}' \preceq \tilde{e}^+$. Hence we can find $\tilde{e}' \sim \tilde{e}^- < \tilde{e}^+$.

We know that $\tilde{e}', \tilde{e}^+ \in D(K[G])$ and we want to deduce that this also holds for $\tilde{e}^-$. By [27], exercise 13.15A, there exists a similarity (that is self-adjoint unitary) $u \in \mathcal{U}(G)$ such that $\tilde{e}^- = u\tilde{e}'u$ (regard that $\tilde{e}' \perp \tilde{e}^-$ which is necessary for [27], 13.15A). There is a countable subgroup F of G, with $\Delta(G) = \Delta(F)$, such that $u \in N(F)$. We take the smallest subgroup containing F with the desired Δ. This group is still countable. By the Kaplansky density theorem [33] p. 8, there exists a sequence $u_k \in KF$ such that $u_k \to u$ as $k \to \infty$ in the strong operator topology in $\mathcal{N}(F)$. We have

$$\dim^u_{\mathcal{N}(F)}(u_k\tilde{e}'u_k) \to \dim^u_{\mathcal{N}(F)}(u\tilde{e}'u)$$

strongly. From the quantization of the dimension we assume in (4), it follows that already for a finite n we have

$$\dim^u_{\mathcal{N}(F)}(u_n\tilde{e}'u_n) = \dim^u_{\mathcal{N}(F)}(u\tilde{e}'u)$$

We have constructed F so that $\mathcal{Z}(\mathcal{N}(G)) = \mathcal{Z}(\mathcal{N}(F))$ and so we get that

$$\dim^u_{\mathcal{N}(G)}(u_n\tilde{e}'u_n) = \dim^u_{\mathcal{N}(G)}(u\tilde{e}'u)$$

with $u_n\tilde{e}'u_n \in DKG$. We now define the projection $\tilde{e}$ as

$$\tilde{e} := \tilde{e}^+ - \tilde{e}^- \in D(K[G]).$$

From this we get

$$\begin{aligned}\operatorname{tr}^u_{\mathcal{N}(G)}(\tilde{e}) &= \frac{|\Delta^+| \gcd\big(\dim_{\mathbb{C}}(Q_1), \ldots, \dim_{\mathbb{C}}(Q_n)\big)}{\operatorname{lcm}(\mathrm{G}) \dim_{\mathbb{C}}(P^j)} P^j \\ &= \frac{\gcd\big(\dim_{\mathbb{C}}(Q_1), \ldots, \dim_{\mathbb{C}}(Q_n)\big)}{\frac{\operatorname{lcm}(\mathrm{G})}{|\Delta^+|} \dim_{\mathbb{C}}(P^j)} P^j\end{aligned}$$

We want to reduce this fraction regarding the general formula

$$\frac{a}{b \cdot c} = \frac{\frac{a}{\gcd(a,b)\gcd(\frac{a}{\gcd(a,b)},c)}}{\frac{b \cdot c}{\gcd(a,b)\gcd(\frac{a}{\gcd(a,b)},c)}}.$$

We obtain the reduced fraction

$$\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(\tilde{e}) = \frac{R}{L}$$

$$L := \frac{\dim_{\mathbb{C}}(P^j)\frac{\mathrm{lcm(G)}}{|\Delta^+|}}{\left(\begin{array}{l}\gcd\Big(\dim_{\mathbb{C}}(Q_1)\,,\ldots\,,\dim_{\mathbb{C}}(Q_n)\,,\,\dfrac{\mathrm{lcm(G)}}{|\Delta^+|}\Big)\\ \cdot\gcd\Big(\dfrac{\gcd\big(\dim_{\mathbb{C}}(Q_1)\,,\ldots\,,\dim_{\mathbb{C}}(Q_n)\big)}{\gcd\big(\dim_{\mathbb{C}}(Q_1)\,,\ldots\,,\dim_{\mathbb{C}}(Q_n)\,,\,\frac{\mathrm{lcm(G)}}{|\Delta^+|}\big)}\,,\,\dim_C(P^j)\Big)\,,\end{array}\right)}$$

on the other hand we have

$$\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(1-\tilde{e}) = \frac{L-R}{L}$$

with $\gcd(L, L-R) = 1$. Using again the above argument, we obtain the required projection P with

$$\operatorname{tr}^{\mathbb{C}}_{\mathcal{N}(G)}(e'') = \frac{1}{L}$$
$$= 1/\underbrace{\frac{\dim_{\mathbb{C}}(P^j)\frac{\mathrm{lcm(G)}}{|\Delta^+|}}{\left(\begin{array}{l}\gcd\Big(\dim_{\mathbb{C}}(Q_1)\,,\ldots\,,\dim_{\mathbb{C}}(Q_n)\,,\,\dfrac{\mathrm{lcm(G)}}{|\Delta^+|}\Big)\\ \cdot\gcd\Big(\dfrac{\gcd\big(\dim_{\mathbb{C}}(Q_1)\,,\ldots\,,\dim_{\mathbb{C}}(Q_n)\big)}{\gcd\big(\dim_{\mathbb{C}}(Q_1)\,,\ldots\,,\dim_{\mathbb{C}}(Q_n)\,,\,\frac{\mathrm{lcm(G)}}{|\Delta^+|}\big)}\,,\,\dim_{\mathbb{C}}(P^j)\Big)\,,\end{array}\right)}}_{\in\mathbb{Z}}$$

We used direct sums of projections hence he projection we just constructed is not an element in $P^j\mathcal{N}(G)$ but in $M_n(P^j\mathcal{N}(G))$ for suitable n. Take now the projection $(a := (a_{i,j})_{i,j=1\ldots n} \in M_n(P^j\mathcal{N}(G))$ with $a_{1,1} = \mathrm{id}$ and $a_{i,j} = 0$ elsewhere. We have $\dim^u(a) \geq \dim^u(e'')$. Using again the above argument, we can find a sub-projection with dimension $\frac{1}{L}$. This is our desired projection $e \in P^j\mathcal{N}(G)$ with dimension $\frac{1}{L}$.

We have

$$\operatorname{tr}^u_{\mathcal{N}(G)}((1-e)P^iU(G)) = \frac{L-1}{L}P^i\,.$$

Therefore

$$(1-e)P^iU(G) \cong e(P^iU(G))^{L-1}$$

and we deduce that there exist orthogonal projections $e = e_1, e_2, ..., e_L \in \mathcal{U}(G)$ (with $e_i e_j = 0$ for $i \neq j$), such that $\sum_{j=1}^{L} e_i = P^i$ and $e_j P^i U(G) \cong eP^j U(G)$ for all j. By [27], exercise 13.15A, there exist similarities (that is self-adjoint unitaries) $u_i \in \mathcal{U}(G)$ with $u_1 = 1$ such that $e_j = u_j e u_j$. There is a countable subgroup F of G, with $\Delta(G) = \Delta(F)$, such that $u_i \in N(F)$ for all i. We take the smallest subgroup containing F with the desired Δ. This group is still countable. By the Kaplansky density theorem [33] p. 8, for each j $(1 \leq j \leq L)$ there exists a sequence $u_{j,k} \in KF$ such that $u_{j,k} \to u_j$ as $j \to \infty$ in the strong operator topology in $\mathcal{N}(F)$ with $u_{1,k} = 1$ for all k. Set $v_k = \sum_{j=1}^{L} u_{j,k} e u_{j,k}$. Then $v_k \to \sum_{j=1}^{L} e_j = P^i$ strongly, We have constructed F so that $\mathcal{Z}(\mathcal{N}(G)) = \mathcal{Z}(\mathcal{N}(F))$ so we get that

$$\dim^u_{\mathcal{N}(F)}(v_k P^i \mathcal{U}(F)) \to \dim^u_{\mathcal{N}(F)}(\sum_{j=1}^{L} e_j \mathcal{U}(F)) = P^i$$

strongly. We have constructed F so that $\mathcal{Z}(\mathcal{N}(G)) = \mathcal{Z}(\mathcal{N}(F))$ so we get that $\operatorname{Ind}_F^G \dim^u_{\mathcal{N}(F)}(x\mathcal{U}(F)) = \dim^u_{\mathcal{N}(G)}(x\mathcal{U}(G))$ for all $x \in \mathcal{U}(F)$, consequently

$$\dim^u_{\mathcal{N}(G)}(v_k P^i \mathcal{U}(G)) \to \dim^u_{\mathcal{N}(G)}(\sum_{j=1}^{L} e_j \mathcal{U}(G)) = P^i$$

strongly. Since G satisfies (4), we have already for some $n \in \mathbb{N}_+$ that for all $k \geq n$

$$\dim^u_{\mathcal{N}(G)}(v_k P^i \mathcal{U}(G)) = \dim^u_{\mathcal{N}(G)}(\sum_{j=1}^{L} e_j \mathcal{U}(G)) .$$

From this follows for $k \geq n$ that we have $v_k = v_n$, $u_{k,j} = u_j$ and that v_k and $u_{k,j}$ are units in $P^i \mathcal{U}(G)$ and hence in $P^i D(K[G])$. We see that $\oplus_{j=1}^{L} u_{j,n} e P^i \mathcal{U}(G) = P^i \mathcal{U}(G)$ and deduce that

$$P^i D(K[G]) = \bigoplus_{j=1}^{L} u_{j,n} e P^i D(K[G])$$

is a direct sum. Now let c be an central idempotent in $P^j \mathcal{E}(K[G])$. We want to show that $c = 0$ or 1. From the above observations, we obtain $cu_{i,n} e\mathcal{U}(G) \cong ce\mathcal{U}(G)$ for all i. It follows from 3.3.3 and 3.3.4 that

$$\operatorname{tr}^u_{\mathcal{N}(G)}(cP^j \mathcal{U}(G)) = L \operatorname{tr}^u_{\mathcal{N}(G)}(ceP^j \mathcal{U}(G)) \in \mathbb{Z} P^j .$$

But $\mathrm{tr}^u_{\mathcal{N}(G)}(cP^j\mathcal{U}(G)) \leq P^j$ and hence is either 0 or 1 . Using theorem 3.3.6 it follows that $P^jD(K[G])$ is a semisimple Artinian ring, that contains no nontrivial idempotents.

Further from (3.4.4) it follows that no projections exist in $P^iD(K[G])$ with smaller trace than $\frac{1}{L_i}$. Hence we can not partition smaller, otherwise we can construct a projection in $\mathcal{U}(G)$, with trace bigger than one, which is a contradiction. Hence it is a $L_i \times L_i$ matrix ring over a skew field. $\square$

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