Some Aspects on Coarse Homotopy Theory

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Abstract

The most important examples of coarse spaces arise from proper metrics on spaces or from metrisable compactifications. In both cases, the bounded subsets are precisely the relatively compact ones. On the other hand, one of the characteristic properties a coarse map required to have is formulated by the bounded subsets. Motivated from the above two facts and in order to be able to develop the notion of locally properness which has been introduced by Viêt-Trung Luu for discrete spaces, we introduce a new notion of compatibility between the coarse structure and the topology when a coarse space also carries a topology. The spirit of this is to let the local part of the theory govern by the topology of spaces. Having established this we are able to introduce some basic notions such as pull-back and push-forward coarse structures, and products and coproducts of coarse spaces which also are carrying a topology with the required compatibility between the topology and the coarse structure.

We use the notion of basepoint projection introduced by Paul Mitchener and Thomas Schick to develop a notion of pointed coarse spaces which leads us to a new notion of collapsing from a coarse point of view. This enables us to introduce some essential notions such as coarse quotient spaces, coarse spaces obtained by coarse collapsing and coarse spaces obtained by coarse attaching via a coarse map.

We introduce a new notion which in a sense is the analogue for coarse geometry of locally compactness for topology and investigate some of its properties.

Then, we develop basic notions in the coarse homotopy theory including some constructions such as coarse smash product, coarse suspensions and coarse mapping cone needed to develop coarse homotopy theory and we prove some of their properties. The coarse homotopy groups are introduced next and then we develop an exact sequence of coarse homotopy groups.

We also give a more complete exposition on the coarse CW-complexes broad enough to provide an appropriate foundation in order to carry over more of the tools from algebraic topology into coarse geometry. Having established that we prove a coarse version of the theorem of J.H.C. Whitehead which allows certain aspects of coarse homotopy classification.

Next, we pursue a big step forward and calculate the coarse homotopy groups of the standard coarse spheres. More precisely, we prove $\pi_k^{crs}(S^n_{\mathbb{R}_+}) \cong \pi_k(S^n)$ for all $k \leq n$. This has some intense applications, namely, this enables us to carry over some important theorems from algebraic topology concerning the coarse homotopy groups of coarse CW-complexes when they are $\mathbb{R}_+$-spaces. Then, as a one result of these theorems, we introduce the coarse Eilenberg-Maclane spaces.

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Contents

Chapter 1

Foundations

Coarse geometry is the study of the "very large scale" properties of spaces. To see what is meant by the "very large scale", let restrict ourselves, for a moment, to metric spaces. For topologists, the significance of the metric lies in the collection of open sets it generates, but this passage from the metric to its associated topology loses a good deal of information; in fact only the 'very small scale structure' of the metric is reflected in its topology. For example the metric

$$d'(x, y) := \min\{d(x, y), 1\}$$

defines the same topology as the metric d itself.

Coarse geometry arises when we consider the dual procedure in which "very large scale" properties of spaces are to be investigated. In coarse geometry, the role of open subsets is played by some subsets of $X \times X$, called entourages. For example, in the metric space (X, d), the entourages are defined to be the subsets of the following sets

$$D_r := \{(x, y)|\ d(x, y) \leq r\}$$

where $r \geq 0$. Now, we can see what we mean by 'very large scale' property of a metric space (X, d); consider two functions $f, g : S \to X$, where S is an arbitrary set. If there is a positive real number $r \geq 0$ with

$$\{(f(s), g(s))|\ s \in S\} \subseteq D_r,$$

then the functions f and g are considered as the same object in the coarse sense, that is, any two functions into a bounded metric space represent the same object as far as coarse geometry is concerned or with the other words, every space of a finite size is coarsely equivalent to a single point.

As one can define the notion of an abstract topological space by axiomatizing the properties of open sets in metric spaces, one can define an abstract coarse space by axiomatizing the properties of entourages in metric spaces.

In section 1.2, we will give an axiomatic description of the structure needed to do coarse geometry.

In this chapter, we redevelop some basic notions in coarse geometry needed to develop some aspects of coarse homotopy theory. In [Luu], Viêt-Trung Luu has introduced the notion of locally properness for the discrete spaces. We first develop this notion for the coarse spaces which also carry a topology with some kind of compatibility between their coarse structure and their topology. Therefore, we often refer to the difference between these two setups. But, I should remark that our approach is more geometrical rather than his approach which is categorical.

1.1 Properness axiom

First, we fix some notations. They mainly come from [Roe03] and [Luu]. Let X be a set. We will use the following notation for subsets of $X \times X$.

(i) If $E \subseteq X \times X$, then E^{-1} denotes the set $\{(y,x) \in X \times X |\ (x,y) \in E\}$, called the *inverse* of E;

(ii) If $E_1, E_2 \subseteq X \times X$, then $E_1 \circ E_2$ denotes the set $\{(x,z) \in X \times X |\ (x,y) \in E_1 \text{ and } (y,z) \in E_2 \text{ for some } y \in X\}$, called the *composition* of E_1 and E_2.

If $E \subseteq X \times X$ and $K \subseteq X$, we define

$$E \cdot K := \{x \in X |\ \exists y \in K,\ (x,y) \in E\},$$
$$K \cdot E := \{x \in X |\ \exists y \in K,\ (y,x) \in E\}.$$

In case K is a singleton $\{x\}$, we use the notations E_x and E^x for *E-balls* $E \cdot \{x\}$ and $\{x\} \cdot E$, respectively.

The following two lemmas are obvious:

Lemma 1.1.1. *Let X be a set. If $E_1, E_2 \subseteq X \times X$, and $K \subseteq X$, then*

$$(E_1 \cup E_2) \cdot K = E_1 \cdot K \cup E_2 \cdot K \quad \textit{and} \quad K \cdot (E_1 \cup E_2) = K \cdot E_1 \cup K \cdot E_2;$$
$$E_1 \circ 1_{E_2 \cdot K} = E_1 \circ E_2 \circ 1_K \quad \textit{and} \quad 1_{K \cdot E_1} \circ E_2 = 1_K \circ E_1 \circ E_2;$$
$$(E_1 \circ E_2) \cdot K = E_1 \cdot (E_2 \cdot K) \quad \textit{and} \quad K \cdot (E_1 \circ E_2) = (K \cdot E_1) \cdot E_2; \quad \textit{and}$$
$$E_1^{-1} \cdot K = K \cdot E_1 \quad \textit{and} \quad K \cdot E_1^{-1} = E_1 \cdot K.$$

Lemma 1.1.2. *Let X be a set. If $E_1, E_2, E_1', E_2' \in \wp(X \times X)$ with $E_1 \subseteq E_2$ and $E_1' \subseteq E_2'$, and $K_1, K_2 \subseteq X$ with $K_1 \subseteq K_2$, then*

$$E_1 \cup E_1' \subseteq E_2 \cup E_2', \qquad E_1 \circ E_1' \subseteq E_2 \circ E_2',$$
$$(E_1)^{-1} \subseteq (E_2)^{-1}, \qquad 1_{K_1} \subseteq 1_{K_2},$$
$$E_1 \cdot K_1 \subseteq E_2 \cdot K_2, \quad \textit{and} \quad K_1 \cdot E_1 \subseteq K_2 \cdot E_2.$$

Definition 1.1.3. Let X and Y be topological spaces[1].

- A subset $E \subseteq X \times X$ satisfies the *Roe properness axiom* if $E \cdot K$ and $K \cdot E$ are relatively compact[2] whenever K is a relatively compact subset of X;
- A map $f : X \to Y$ (not necessarily continuous) is called *topologically proper* if $f^{-1}(K')$ is relatively compact whenever K' is a relatively compact subset of Y.

The following follows directly from Lemmas 1.1.1 and 1.1.2.

Proposition 1.1.4. *If $E, E' \in \wp(X \times X)$ satisfy the Roe properness axiom, then $E \cup E', E \circ E', E^{-1}$, and all subsets of E satisfy the Roe properness axiom. Also, all singleton $\{1_x\}$, $x \in X$, satisfy the Roe properness axiom.*

Note that Proposition 1.2.2 of [Luu] does not hold in this setup, but fortunately, by adding more assumptions on maps which will be fulfilled automatically in the cases we will work on in the future, we get exactly what we need. The following is the corresponding statement:

Proposition 1.1.5. *Let X, Y and Z be topological spaces. Consider the composition of set maps $X \xrightarrow{f} Y \xrightarrow{g} Z$:*

(i) If f and g are topologically proper, then $g \circ f$ is topologically proper.

(ii) If $g \circ f$ is topologically proper and g preserves relatively compact subsets, then f is topologically proper.

(iii) If $g \circ f$ is topologically proper and if f preserves relatively compact subsets and it is surjective, then g is topologically proper.

Proof. The proof is straightforward. □

1.2 Coarse structure and compatibility with topology

In this section, we first give an axiomatic description of the structure needed to do coarse geometry. Compare [Roe96], [Mit01],[Roe03] and [Luu].

Definition 1.2.1. Let X be a set. A collection $\mathcal{E}_X$ of subsets of $X \times X$ is called a *coarse structure* on X, and the elements of $\mathcal{E}_X$ will be called *entourages*, if the following axioms are fulfilled:

[1] In general, we do not assume topological spaces to be Hausdorff unless otherwise stated.

[2] We use the term *relatively compact* in the following sense: a subset K of a topological space X is relatively compact if it is contained in some compact subspace of X.

(i) $\mathcal{E}_X$ is closed under the formation of subsets, inverses, compositions, and finite unions; and

(ii) for all $x \in X$, the singleton $\{1_x\}$ is an entourage.

A *coarse space* is a set X equipped with a coarse structure $\mathcal{E}_X$ on X. We denote such a coarse space by $(X, \mathcal{E}_X)$ or simply by X.

The coarse space X is called *connected* if every singleton $\{e\}$, $e \in X \times X$, is an entourage of $\mathcal{E}_X$. A pair of points $x, x' \in X$ are *connected* (with respect to $\mathcal{E}_X$) if $\{(x, x')\} \in \mathcal{E}_X$. There is also a notion of a subset of a coarse space being of finite size, namely

Definition 1.2.2. Let $(X, \mathcal{E}_X)$ be a coarse space. We call a subset $B \subseteq X$ *bounded* if $B \subseteq E_x$ for some $E \in \mathcal{E}_X$ and for some $x \in X$.

Proposition 1.2.3 (properties of bounded sets). *Let $(X, \mathcal{E}_X)$ be a coarse space.*

(i) Subsets of bounded sets are bounded.

(ii) If $B \subseteq X$ is bounded, then $B \times B \subseteq \mathcal{E}_X$.

(iii) If $B \subseteq X$ is bounded and $E \in \mathcal{E}_X$, then $E \cdot B$ and $B \cdot E$ are bounded.

(iv) Let $B_1, B_2 \subseteq X$ be bounded sets, The following are equivalent.

- *$B_1 \cup B_2$ is bounded;*
- *$B_1 \times B_2 \in \mathcal{E}_X$;*
- *There exists an entourage $E \in \mathcal{E}_X$ such that $E \cap (B_1 \times B_2) \neq \emptyset$.*

(v) If $(X, \mathcal{E}_X)$ is a connected coarse space, then any finite union of bounded sets is bounded.

For a proof consult Proposition 1.7 of [Gra].

Definition 1.2.4. Let X be a set and $\mathcal{E}'$ a collection of subsets of $X \times X$. Since any intersection of coarse structure on X is itself a coarse structure, we can make the following definition. By $\langle \mathcal{E}' \rangle$, we denote the smallest coarse structure containing $\mathcal{E}'$, i.e., the intersection of all coarse structures containing $\mathcal{E}'$. We call $\langle \mathcal{E}' \rangle$ the *coarse structure generated by $\mathcal{E}'$*.
In the same way, we define the connected coarse structure generated by $\mathcal{E}'$ and we denote it by $\langle \mathcal{E}' \rangle_{cn}$.

To give a motivation for the next definition, we review two notions from the classical coarse geometry, although, we do not stay faithful to them. The first is the notion of a coarse map: assuming X and Y to be coarse spaces, a set map $f : X \to Y$ was said to be coarse if

(i) it is *coarsely uniform* in the sense that for every entourage $E \in \mathcal{E}_X$, the image

$$f^{\times 2}(E) = \{(f(x), f(y))|\ (x, y) \in E\}$$

is an entourage, and

(ii) it is *coarsely proper* in the sense that the inverse image of a bounded set is also bounded.

The second notion is the notion of compatibility of coarse structure and topology when a coarse space also carries a topology: suppose we are given a Hausdorff topological space X. A coarse structure $\mathcal{E}_X$ on X was said to be compatible with the topology (or as in [Roe03], the coarse structure on X was said to be proper) if (1) there is a neighborhood of the diagonal Δ_X which is an entourage and (2) every bounded subsets of X is relatively compact. An immediate consequence of this compatibility was that the bounded subsets in proper coarse spaces are exactly the relatively compact ones (see [Roe03], [Mit01] and [Gra]). Motivated from this fact and the fact that one of the property that indicates coarse maps has been formulated by bounded subsets, we are going to let this side of the theory govern by topology of the space. To do it, we first require the following compatibility between the coarse structure and the topology:

Definition 1.2.5. Let $(X, \mathcal{E}_X)$ be a coarse space. Then X is called a *coarse topological space*, if it is equipped with a topology (not necessarily Hausdorff) such that every $E \in \mathcal{E}_X$ satisfies the Roe properness axiom. A coarse topological space X is called *proper* if its bounded subsets are precisely the relatively compact ones. We say the coarse structure and the topology of a space X are *compatible* if X is a proper coarse topological space. A (proper) coarse topological space X is called *unital* if the diagonal $\triangle_X := \{(x, x)|\ x \in X\}$ is an entourage.

Note that the definition of a "proper coarse topological space" is slightly redundant: if the bounded subsets are precisely the relatively compact ones, then every entourage automatically satisfies the Roe properness axiom. As in [Luu], we use the notation $E \in \mathcal{E}_{|X|_1}$ as a convenient abbreviation for $E \in \wp(X \times X)$ satisfying the Roe properness axiom.

Example 1.2.6. Let (X, d) be a proper metric space. Set $D_r := \{(x, y) \in X \times X \mid d(x, y) < r\}$ and define

$$\mathcal{E}_d := \{E \subseteq X \times X \mid E \subseteq D_r \text{ for some } r > 0\}.$$

It is easy to verify that $(X, \mathcal{E}_d)$ is a connected unital proper coarse topological space and it will be called the *bounded coarse structure* coming from the metric d.

Example 1.2.7. Let X be a Hausdorff space and $\overline{X}$ a compactification of X, i.e., X is a dense and open subset of the compact set $\overline{X}$. The collection

$$\mathcal{E}_{\overline{X}} := \{E \subseteq X \times X \mid \overline{E} \subseteq X \times X \cup \Delta_{\overline{X}}\}$$

of all subsets $E \subseteq X \times X$, whose closure meets the boundary $(\overline{X} \times \overline{X}) \backslash (X \times X)$ only in the diagonal, is a connected coarse structure on X, called the *continuously controlled coarse structure*. It is easy to verify that if $\overline{X}$ is metrisable, then the coarse structure $\mathcal{E}_{\overline{X}}$ is compatible with the topology. Compare [Roe03], [Mit01] and [Gra].

Definition 1.2.8. Let X be a coarse topological space and let $X' \subseteq X$ be a subset. Then

$$\mathcal{E}_{X'} := \mathcal{E}_X|_{X'} := \mathcal{E}_X \cap \wp((X')^{\times 2})$$

is a coarse structure on X', called the *subspace coarse structure*. If X is a proper coarse topological space and X' is a closed subset of X, then X' is itself a proper coarse topological space with subspace coarse structure.

1.3 Coarse maps

In [Luu], Viêt-Trung Luu has introduced the notion of locally properness for discrete spaces. In this section, we develop his approach for coarse topological spaces. The goal is to introduce a notion which is weaker than topologically properness when spaces are nonunital. Therefore, from now on, we assume that spaces always carry a topology.

Definition 1.3.1. Let X and Y be topological spaces. A set map $f : X \to Y$ is *locally proper for* $F \in \mathcal{E}_{|X|_1}$ if $E = f^{\times 2}(F) \in \mathcal{E}_{|Y|_1}$ and $f^{-1}(K) \cdot F$ and $F \cdot f^{-1}(K)$ are relatively compact for all relatively compact $K \subseteq Y$. If $(X, \mathcal{E}_X)$ is a coarse topological space, then f is *locally proper* if it is locally proper for all $F \in \mathcal{E}_X$.

Definition 1.3.2. Let $(Y, \mathcal{E}_Y)$ be a coarse topological space. A set map $f : X \to Y$ *preserves* $F \in \mathcal{E}_{|X|_1}$ (with respect to $\mathcal{E}_Y$) if $E = f^{\times 2}(F) \in \mathcal{E}_Y$. If $(X, \mathcal{E}_X)$ is also a coarse topological space, then f *preserves entourages* if f preserves every $F \in \mathcal{E}_X$.

Definition 1.3.3. Let Y be a coarse topological space. A set map $f : X \to Y$ is *coarse for* $F \in \mathcal{E}_{|X|_1}$ if f is locally proper for F and if f preserves F. If $(X, \mathcal{E}_X)$ is also a coarse topological space, then f is *coarse map* if f is coarse for every $F \in \mathcal{E}_X$.

Lemma 1.3.4. *Let X and Y be topological spaces and let Z be a unital coarse topological space.*

- *If a set map $f : X \to Y$ is topologically proper and respects the Roe properness axiom, then f is locally proper for any $F \in \mathcal{E}_{|X|_1}$ (so f is locally proper for any coarse structure on X).*

- *If a set map $g : Z \to Y$ is locally proper, then g is topologically proper.*

Proof. The proof is straightforward. □

Note that for unital proper coarse topological spaces, our notion of a "coarse map" is just the classical notion. Therefore, in this case, we continue using the terminologies "coarsely uniform" and "coarsely proper". But, the question which arises here is whether our coarse maps contain the whole characteristic properties that we expect a coarse map to have. As we will see in the future, being careful enough, we will get what we need from a coarse map even in the cases that coarse topological spaces are not unital or both source and target spaces are not proper.

The only problem that causes some difficulties in this setup in compare with the discrete case is that in some proofs we can not conclude that our maps respect the Roe properness axiom. We will add this as an assumption whenever it is required. But, note that adding this assumption is not a demanding, because, in the future, we will consider only coarse maps which preserve entourages by definition, that is, they respect the Roe properness axiom which means the assumption will be automatically fulfilled. The following is the corresponding proposition to Proposition 1.6.6 of [Luu]:

Proposition 1.3.5. *Let X and Y be topological spaces and $f : X \to Y$ be a set map. If $f : X \to Y$ is locally proper for $F, F' \in \mathcal{E}_{|X|_1}$, then f is locally proper for $F \cup F', F \circ F', F^{-1}$, and all subsets of F. Also, f is locally proper for all singletons $\{e\}, e \in X \times X$.*

Proof. Note that the argument in the proof of Proposition 1.6.6 of [Luu] does not hold here, but the proof is still straightforward applying Lemmas 1.1.1 and 1.1.2. □

Without losing more time, we state the final statement that we need later, namely

Corollary 1.3.6. *Let X be a topological space and let Y be a coarse topological space and assume $f : X \to Y$ to be a set map. If $f : X \to Y$ is coarse for $F, F' \in \mathcal{E}_{|X|_1}$, then f is coarse for $F \cup F', F \circ F', F^{-1}$, and all subsets of F. Also, f is coarse for all singleton $\{1_x\}, x \in X$; if X is connected, then f is coarse for all singletons $\{e\}, e \in X \times X$.*

And finally, although Proposition 1.6.12 and 1.6.13 of [Luu] does not hold in our setup, but we can prove the following:

Proposition 1.3.7. *Let X and Y be topological spaces. If set maps $f, g : X \to Y$ are locally proper for $F \in \mathcal{E}_{|X|_1}$, then*

(i) If f and g preserve relatively compact subsets, then $E := (f \times g)(F) \subseteq Y \times Y$ satisfies the Roe properness axiom; and

(ii) for each relatively compact subset K' of $Y \times Y$, the subset $(f \times g)^{-1}(K') \cap F$ is a relatively compact subset of $X \times X$.

Proof. We omit proofs of the symmetric cases. The part (i) follows from the following relations

$$\begin{aligned}(f \times g)(F) \cdot K \subseteq f(F \cdot g^{-1}(K)),\\ K \cdot (f \times g)(F) \subseteq g(f^{-1}(K) \cdot F),\end{aligned}$$

where $K \subseteq Y$ is a relatively compact subset.

For (ii), one can easily see that

$$(f \times g)^{-1}(K') \cap F \subseteq [F \cdot g^{-1}(\pi_2(K'))] \times [f^{-1}(\pi_1(K')) \cdot F].$$

□

Proposition 1.3.8. *Let X, Y and Z be topological spaces. Consider the composition of set maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, supposing that $F \in \mathcal{E}_{|X|_1}$ and $G := f^{\times 2}(F)$:*

(i) If f is locally proper for F and g is locally proper for G, then $g \circ f$ is locally proper for F.

(ii) If $g \circ f$ is locally proper for F, g preserves relatively compact subsets and $G \in \mathcal{E}_{|Y|_1}$, then f is locally proper for F.

(iii) If $g \circ f$ is locally proper for F, f preserves relatively compact subsets and $G \in \mathcal{E}_{|Y|_1}$, then g is locally proper for G.

Proof. We omit proofs of the symmetric cases. Note that the proof of Proposition 1.6.13 of [Luu] does not hold here. The statements (i)-(iii) easily follow from the above enhanced conditions and the following relations:

$$\begin{aligned}(g \circ f)^{-1}(K') \cdot F \subseteq [F \cdot f^{-1}(g^{-1}(K') \cdot G)] \cdot F,\\ f^{-1}(K) \cdot F \subseteq (g \circ f)^{-1}(g(K)) \cdot F, \quad \text{and}\\ g^{-1}(K') \cdot G \subseteq f((g \circ f)^{-1}(K') \cdot F),\end{aligned}$$

where $K \subseteq Y$ and $K' \subseteq Z$ are relatively compact subsets. □

Definition 1.3.9. Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse topological spaces and let $f, g : X \to Y$ be coarse maps.

- The maps f and g are called *close*, denoted by $f \sim_{cl} g$, if $(f \times g)(F) \in \mathcal{E}_Y$ for all $F \in \mathcal{E}_X$.
- The map f is called a *coarse equivalence* if there is a coarse map $h : Y \to X$ such that the composites $h \circ f$ and $f \circ h$ are close to the identities 1_X and 1_Y, respectively.
- We say that X and Y are *coarsely equivalent*, denoted by $\cong_{crs}$, if there exists a coarse equivalence from X to Y.

Note that for unital coarse topological spaces, our notion of closeness is just the classical one, i.e., for X unital, f and g are close if and only if $(f \times g)(\triangle_X) \in \mathcal{E}_Y$. Compare [Roe96], [Roe03] and [Luu].

Example 1.3.10. The coarse spaces $\mathbb{R}$ and $\mathbb{Z}$ (both equipped with their usual bounded coarse structure) are coarsely equivalent.

Proof. Let $f : \mathbb{R} \to \mathbb{Z}$ be the greatest integer function which assigns to any given number x the biggest integer not exceeding x. Obviously, the map f is coarse and it is easy to see that the inclusion $i : Z \to \mathbb{R}$ provides a coarse inverse for f. □

The following is obvious:

Proposition 1.3.11. *Closeness of coarse maps $X \to Y$ is an equivalence relation.*

1.4 Pull-back coarse topological structure

Let $(Y, \mathcal{E}_Y)$ be a coarse topological space and let $f : X \to Y$ be a set map. The goal is to equip X with a topology and a coarse structure which make f into a coarse map. Consider

As topology: the topology on X which has the elements of the following set as its open subsets:

$$\left\{ f^{-1}(U) |\ U \text{ is open in } X \right\};$$

As coarse structure: the following collection of subsets of $X \times X$ which is actually a coarse structure on X by Corollary 1.3.6:

$$\left\{ F \in \mathcal{E}_{|X|_1} |\ f \text{ is coarse for } F \right\}.$$

The set X equipped with the above topology and coarse structure is obviously a coarse topological space, therefore, we define

Definition 1.4.1. Let $(Y, \mathcal{E}_Y)$ be a coarse topological space and let $f : X \to Y$ be a set map. The coarse topological structure defined as above on X is called the *pull-back coarse topological structure* of $\mathcal{E}_Y$ on X along the map f and will be deoted by $f^*\mathcal{E}_Y$.

Moreover, we can prove

Proposition 1.4.2. *Let $(Y, \mathcal{E}_Y)$ be a proper coarse topological space and let $f : X \to Y$ be a set map. Then the pull-back coarse topological structure of $\mathcal{E}_Y$ on X along f is also proper.*

Proof. First, we show that the bounded subsets of $(X, f^*\mathcal{E}_Y)$ are relatively compact. Suppose that $B \subseteq X$ is bounded. It means that $B \times B$ is an entourage of $f^*\mathcal{E}_Y$, i.e., the map f is coarse for $B \times B$. Since f preserves entourages, so $f(B)$ is bounded which means $f(B)$ is relatively compact because Y is a proper coarse topological space. On the other hand, f is locally proper for $B \times B$ which means $f^{-1}(K) \cdot (B \times B)$ is relatively compact for every relatively compact subset $K \subseteq Y$. Hence $f^{-1}(f(B)) \cdot (B \times B)$ is relatively compact. But $B \subseteq f^{-1}(f(B)) \cdot (B \times B)$, so B is relatively compact. Now we show that the relatively compact subsets of X are bounded. Suppose that B is a relatively compact subset of X. Since for every subset $K \subseteq X$ (therefore also for relatively compact ones), we have $(B \times B) \cdot K \subseteq B$ and $K \cdot (B \times B) \subseteq B$, therefore $B \times B \in \mathcal{E}_{|X|_1}$. On the other hand, since f is continuous, $f(B)$ is a relatively compact subset of Y, but Y is a proper coarse topological space which means $f(B)$ is bounded, i.e., $f(B) \times f(B) = f^{\times 2}(B \times B)$ is an entourage. So, we have shown that f preserves $B \times B$ which also means $f^{\times 2}(B \times B) \in \mathcal{E}_{|Y|_1}$. Similarly, for every subset $K' \subseteq Y$ (so also for relatively compact subsets) we have $f^{-1}(K') \cdot (B \times B) \subseteq B$ and $(B \times B) \cdot f^{-1}(K') \subseteq B$ which means $f^{-1}(K') \cdot (B \times B)$ and $(B \times B) \cdot f^{-1}(K')$ are relatively compact. So we have shown that $B \times B \in \mathcal{E}_{|X|_1}$ and f is coarse for $B \times B$ which means B is bounded. □

If Y is connected, then $f^*\mathcal{E}_Y$ is connected. If Y is unital and f is topologically proper, then $f^*\mathcal{E}_Y$ is unital. The following is obvious.

Proposition 1.4.3. *Let $(Y, \mathcal{E}_Y)$ be a coarse topological space and let $f : X \to Y$ be a set map. If $\mathcal{E}_X$ is a coarse structure on X which makes X into a coarse topological space with respect to the topology on X defined above and which also makes f into a coarse map, then $\mathcal{E}_X \subseteq f^*\mathcal{E}_Y$.*

1.5 Push-forward coarse topological structure

Let $(X, \mathcal{E}_X)$ be a coarse topological space and let $f : X \to Y$ be a surjective set map. This time the goal is to equip Y with a topology and a coarse structure which make f into a coarse map. Consider

As topology: the quotient topology on Y induced by f;

Now, if f is topologically proper and respects the Roe properness axiom, then we go further and define a coarse structure on X as follows:

As coarse structure: the coarse structure generated by the set

$$\left\{ f^{\times 2}(F) \mid F \in \mathcal{E}_X \right\}.$$

The set Y equipped with the above topology and coarse structure is obviously a coarse topological space, therefore, we define

Definition 1.5.1. Let $(X, \mathcal{E}_X)$ be a coarse topological space and let $f : X \to Y$ be a surjective map which is topologically proper and respects the Roe properness axiom after topologizing Y with the quotient topology. The coarse topological structure defined as above on Y is called the *push-forward coarse topological structure* of $\mathcal{E}_X$ on Y along f and will be denoted by $f_*\mathcal{E}_X$.

Also in this case, we can prove

Proposition 1.5.2. *Let $(X, \mathcal{E}_X)$ be a proper coarse topological space and let $f : X \to Y$ be a surjective set map which satisfies the above conditions. Then the push-forward coarse topological structure of $\mathcal{E}_X$ on Y along f is also proper.*

Proof. First, we show that the bounded subsets of $(Y, f_*\mathcal{E}_X)$ are relatively compact. Suppose $B \subseteq Y$ to be bounded. It means that $B \times B$ is an entourage of $f_*\mathcal{E}_X$. Therefore, from the construction of the coarse structure generated by $\left\{ f^{\times 2}(F) \mid F \in \mathcal{E}_X \right\}$ and the fact that $f^{\times 2}(F) \in \mathcal{E}_{|Y|_1}$ for every $F \in \mathcal{E}_X$, follows that $B \times B \in \mathcal{E}_{|Y|_1}$. Now, fix an element b_0 of B. From $B \times B \in \mathcal{E}_{|Y|_1}$ follows that $\{b_0\} \cdot (B \times B)$ is relatively compact. But $B \subseteq \{b_0\} \cdot (B \times B)$ which means that B is relatively compact.
Now, we show that the relatively compact subsets of Y are bounded. Suppose that B is a relatively compact subset of Y. Since f is topologically proper, $f^{-1}(B)$ is a relatively compact subset of X, but X is a proper coarse topological space which means $f^{-1}(B)$ is bounded, i.e., $f^{-1}(B) \times f^{-1}(B)$ is an entourage. Now since f is surjective, one can easily show that $B \times B \subseteq f^{\times 2}(f^{-1}(B) \times f^{-1}(B))$ which means $B \times B$ is an entourage, i.e., B is bounded. □

Under the above assumptions, if X is connected, then $f_*\mathcal{E}_X$ is also connected, and if X is unital, then $f_*\mathcal{E}_X$ is also unital. The following is obvious.

Proposition 1.5.3. *If $(X, \mathcal{E}_X)$ is a coarse topological space and $f : X \to Y$ is a set map with the above properties, then $f_*\mathcal{E}_X$ is the minimum coarse structure on Y which makes f into a coarse map.*

1.6 Products and coproducts

As we mentioned before, we are taking a geometrical point of view, therefore we are not going to investigate too much the categorical aspect of the theory. Indeed, we only concentrate on constructions needed later to develop a notion of coarse homotopy. Nevertheless, we can define

Definition 1.6.1. The *precoarse topological category* has as objects all coarse topological spaces and as arrows coarse maps. The *connected precoarse topological category* is full subcategory of it consisting of the connected coarse topological spaces. Similarly, one can define the *unital precoarse topological category.*

Note that the above category is not actually the proper one, because the locally properness of a morphism as has been formulated for topological coarse spaces cannot provide the characteristic properties that we expect from our morphism in coarse sense (it has been formulated by using the relatively compact subsets, while in coarse topological spaces they are not in general the bounded ones). But, we can also form the subcategory of all proper coarse topological spaces and coarse maps, denoted by **PCrsT** and its full subcategory consisting of connected proper coarse topological spaces, denoted by **CPCrsT**. Then, we can show that the former has finite products while the latter has coproducts.

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be proper coarse topological spaces and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projections. Taking the product topology and the coarse structure

$$\mathcal{E}_{X\times Y} := (\pi_X)^*\mathcal{E}_Y \cap (\pi_Y)^*\mathcal{E}_X$$

on $X \times Y$, we can easily see that $(X \times Y, \mathcal{E}_{X\times Y})$ is a proper coarse topological space. Moreover,

Proposition 1.6.2. *Under the above assumptions,* $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ *is in the categorical sense the product of* X *and* Y *in* **PCrsT**.

Proof. It is obvious that under the above construction π_X and π_Y are coarse maps. Therefore, the only thing that remains to show is the universality. For it, suppose $X \xleftarrow{f} Z \xrightarrow{g} Y$ is another cone in **PCrsT**. Define the map $t : Z \to X \times Y$ by the equation

$$t(z) := (f(z), g(z)).$$

We must show that t is a coarse map (the uniqueness is clear). Suppose that $G \in \mathcal{E}_Z$ and $F := t^{\times 2}(G)$. First, we show $F \in \mathcal{E}_{|X\times Y|_1}$. For it, assume that K is a relatively compact subset of $X \times Y$. But

$$K \cdot F \subseteq (\pi_X(K) \cdot f^{\times 2}(G)) \times (\pi_Y(K) \cdot g^{\times 2}(G))$$

which means that $K \cdot F$ is relatively compact. We omit the proof of the symmetric case. So from Proposition 1.3.8 (ii) directly follows that t is locally proper for $G \in \mathcal{E}_Z$. It only remains to show $F \in \mathcal{E}_{X\times Y}$. For it, we must show that π_X and π_Y are coarse for F. It is clear that they preserve F. To show that π_X and π_Y are locally proper for F, we first show that t preserves bounded subsets: suppose $C \subseteq Z$ is bounded, hence $f(C)$ and $g(C)$ are bounded, but $t(C) \subseteq f(C) \times g(C)$, i.e., $t(C)$ is bounded. Now for every relatively compact subset $K' \subseteq X$, the subset $B := f^{-1}(K') \cdot G$ is relatively compact in Z. But Z is a proper coarse topological space which means B is bounded. So from above follows that $t(B)$ is bounded, i.e., relatively compact. On the other hand, one can easily see that $(\pi_X)^{-1}(K') \cdot F \subseteq t(B)$ which means $(\pi_X)^{-1}(K') \cdot F$ is relatively compact so we have shown that π_X is locally proper for F. Similarly, one can show that π_Y is also locally proper for F and then we are done. □

Remark 1.6.3. Note that being unital in product coarse structure is a fatal problem, because, for instance, $\pi_X^{-1}(K) \cdot \triangle_{X\times Y}$ fails to be relatively compact for each relatively compact subset $K \subseteq X$. Similarly, we can see that each $F \in \mathcal{E}_{|X\times Y|_1}$ for which one of the sets $F \cdot (X \times Y)$ or $(X \times Y) \cdot F$ spreads along one of the axes fails to be an entourage in coarse product structure.

Now, let $\{(X_j, \mathcal{E}_{X_j}) : j \in J\}$ be a family of connected proper coarse topological spaces and let $i_j : X_j \to \amalg X_j$, $j \in J$, be the inclusion maps. Taking the weak topology on $\amalg X_j$, the inclusion maps are obviously topologically proper and respect the Roe properness axiom, hence we can consider the coarse structure $\mathcal{E}_\amalg := \langle (i_j)_* \mathcal{E}_{X_j} \rangle_{cn}$ on $\amalg X_j$. Now, we claim

Proposition 1.6.4. *Under the above assumptions, $((\amalg X_j, \mathcal{E}_\amalg), i_j)$ is in the categorical sense the coproduct of X_j's in* **CPCrsT**.

Proof. It is obvious that under the above construction the inclusions i_j's are coarse maps. We must first show that the weak topology and the coarse structure $\mathcal{E}_\amalg$ on $\amalg X_j$ are compatible, but before it let us introduce some notation: by D^δ, we mean a finite union of subsets of the form $\{(x_j, x_k)\}$ where $x_j \in X_j$ and $x_k \in X_k$. Similarly, for a connected proper topological space Z, by D_Z^δ we mean a finite union of subsets of the form $\{(z, z')\}$ where $z, z' \in Z$. Now, assume that $B \subseteq \amalg X_j$ is bounded, i.e., $B \times B \in \mathcal{E}_\amalg$. So there are entourages $E_{j_{k,l}} \in \mathcal{E}_{X_{j_{k,l}}}$ and $D_l^\delta \in \mathcal{E}_\amalg$, where $j_{k,l} \in J$ for $k = 1, \cdots, n$ and $l = 1, \cdots, m$ such that

$$B \times B \subseteq \mathop{\circ}_{l=1}^{m} (i_{j_{1,l}}^{\times 2}(E_{j_{1,l}}) \cup i_{j_{2,l}}^{\times 2}(E_{j_{2,l}}) \cup \cdots \cup i_{j_{n,l}}^{\times 2}(E_{j_{n,l}})) \cup D_l^\delta.$$

Therefore there are indices $b_1, b_2, \cdots, b_s \in J$ such that

$$B \subseteq (\bigcup_{q=1}^{s} B_{X_{b_q}}) \cup (\bigcup_{w=1}^{r} \{x_w\}),$$

where $B_{X_{b_q}} \subseteq X_{b_q}$ are relatively compact and $x_w \in X_w$. That is, B is relatively compact in $\amalg X_j$. Conversely, assume that $B \subseteq \amalg X_j$ is relatively compact. So there are indecies $j_1, j_2, \cdots, j_s \in J$ such that

$$B \subseteq X_{j_1} \cup X_{j_2} \cup \cdots \cup X_{j_s}$$

by the definition of the weak topology. On the other hand, $B \cap X_{j_t}$ is relatively compact for every $t = 1, \cdots, s$. Hence $B \cap X_{j_t}$ is a bounded subset of X_{j_t} for every $t = 1, \cdots, s$. Therefore $i_{j_t}(B \cap X_{j_t})$ are bounded subsets of $\amalg X_j$ and because the coarse structure $\mathcal{E}_{\amalg}$ is connected, therefore $B = \bigcup_{t=1}^{s} i_{j_t}(B \cap X_{j_t})$ is a bounded subset of $\amalg X_j$. So we have shown that $(\amalg X_j, \mathcal{E}_{\amalg})$ is a proper coarse topological space. Now, to show that it is coproduct of X_j in **CPCrsT**, it remains to show the universality. For it, suppose that $(Z, X_j \xrightarrow{t_j} Z)$ is another cone in **CPCrsT**. Define the map $t : \amalg X_j \to Z$ by the equation

$$t(x) := t_j(x), \text{ if } x \in X_j.$$

We must show that t is a coarse map (the uniqueness is clear). Assume $F \in \mathcal{E}_{\amalg}$, therefore

$$F \subseteq \mathop{\circ}_{l=1}^{m} (i_{j_{1,l}}^{\times 2}(E_{j_{1,l}}) \cup i_{j_{2,l}}^{\times 2}(E_{j_{2,l}}) \cup \cdots \cup i_{j_{n,l}}^{\times 2}(E_{j_{n,l}})) \cup D_l^{\delta},$$

where $E_{j_{k,l}} \in \mathcal{E}_{X_{j_{k,l}}}$ for $k = 1, \cdots, n$ and $l = 1, \cdots, m$. So we have

$$t^{\times 2}(F) \subseteq \mathop{\circ}_{l=1}^{m} (t^{\times 2}(i_{j_{1,l}}^{\times 2}(E_{j_{1,l}})) \cup t^{\times 2}(i_{j_{2,l}}^{\times 2}(E_{j_{2,l}})) \cup \cdots \cup t^{\times 2}(i_{j_{n,l}}^{\times 2}(E_{j_{n,l}}))) \cup t^{\times 2}(D_l^{\delta}).$$

Therefore

$$t^{\times 2}(F) \subseteq \mathop{\circ}_{l=1}^{m} (t_{j_{1,l}}^{\times 2}(E_{j_{1,l}}) \cup (t_{j_{2,l}}^{\times 2}(E_{j_{2,l}})) \cup \cdots \cup (t_{j_{n,l}}^{\times 2}(E_{j_{n,l}}))) \cup (D_l)_Z^{\delta},$$

which means that t preserves entourages. Now we shall show that t is locally proper for every $F \in \mathcal{E}_{\amalg}$. For it, suppose that K is a relatively compact subset of Z. Without lose of generality, we can assume that

$$F \subseteq (i_{j_1}^{\times 2}(E_{j_1}) \cup i_{j_2}^{\times 2}(E_{j_2}) \cup \cdots \cup i_{j_n}^{\times 2}(E_{j_n})) \cup D^{\delta},$$

where $E_{j_k} \in \mathcal{E}_{X_{j_k}}$ for $k = 1 \cdots, n$. If $D^{\delta} = (x_{l_1}, x_{l'_1}) \cup (x_{l_2}, x_{l'_2}) \cup \cdots \cup (x_{l_m}, x_{l'_m})$, then from the fact that $t^{-1}(K) \cap i_j(X_j) = i_j(t_j^{-1}(K))$, one can easily show that

$$t^{-1}(K) \cdot F \subseteq (i_{j_1}(t_{j_1}^{-1}(K) \cdot E_{j_1}) \cup i_{j_2}(t_{j_2}^{-1}(K) \cdot E_{j_2}) \cup \cdots \cup i_{j_n}(t_{j_n}^{-1}(K) \cdot E_{j_n})) \cup (\bigcup_{q=1}^{m} i_{l_q}(t_{l_q}^{-1}(K) \cdot (x_{l_q}, x_{l'_q})),$$

which means $t^{-1}(K) \cdot F$ is relatively compact, i.e., the map t is locally proper for F. So we are done. □

Chapter 2

Coarse topological R-spaces

In this chapter, we first give a more complete exposition on the coarse CW-complexes in order to prepare an appropriate foundation for our later work specially for the chapter 5 in which we will prove the coarse Whitehead theorem. Then, we introduce the coarse topological R-spaces and prove some of their basic properties. Next, we use the notion of basepoint projection introduced in [MS], to define a new notion of collapsing from a coarse point of view. And at the end of this chapter, we introduce a new notion which in a sense is the analogue for coarse geometry of locally compact spaces for topology.

2.1 Coarse CW-complexes

In order to define coarse CW-complexes, we need to describe their building blocks. The main idea is the one introduced in [Mit01], which provided a generalization of the metric space $\mathbb{R}_+ = [0, \infty)$ in the coarse category, but our definition is slightly different, namely, we add one more condition which is again a generalization of a property of the metric space $\mathbb{R}_+$ will be needed later.

Definition 2.1.1. Let $\mathcal{E}_R$ be a unital connected coarse structure on $\mathbb{R}_+$ which is compatible with the standard topology on $\mathbb{R}_+$ (recall that our understanding of the compatibility between the topology and the coarse structure of a space is different from what have been introduced in [Roe03] and [Mit01]). We call $R = (\mathbb{R}_+, \mathcal{E}_R)$ a *generalised ray* if the coarse structure $\mathcal{E}_R$ satisfies the following conditions:

- If M and N are entourages, the same is true for

$$M + N := \{(u + x, v + y) \mid (u, v) \in M, (x, y) \in N\}.$$

- If M is an entourage, so is

$$M^{\boxtimes} := \{(u, v) \mid (x, y) \in M \text{ and } (x \leq u \leq v \leq y \text{ or } y \leq v \leq u \leq x)\}.$$

- If M is an entourage and $q \geq 1$ a real number, then

$$M^{\times q} := \{(xa, ya) | (x, y) \in M \text{ and } a \in [0, q]\}$$

is also an entourage.

Proposition 2.1.2. *By $\mathcal{E}_{\mathbb{R}_+}$, we denote the usual bounded coarse structure on $\mathbb{R}_+$. The coarse space $(\mathbb{R}_+, \mathcal{E}_{\mathbb{R}_+})$ is a generalised ray and $\mathcal{E}_{\mathbb{R}_+} \subseteq \mathcal{E}_R$ whenever $(\mathbb{R}, \mathcal{E}_R)$ is a generalised ray.*

Proof. The coarse space $(\mathbb{R}_+, \mathcal{E}_{\mathbb{R}_+})$ is obviously a generalised ray. For the second statement, because our generalised rays have more entourages in compare with the classical ones, therefore the standard argument presented in Proposition 2.5 of [Mit03] still holds in our setup. □

Proposition 2.1.3. *By $\mathcal{E}_.$, we denote the continuously controlled coarse structure on $\mathbb{R}_+$ induced by the one-point compactification. The coarse space $(\mathbb{R}_+, \mathcal{E}_.)$ is a generalised ray and $\mathcal{E}_R \subseteq \mathcal{E}_.$ whenever $(\mathbb{R}_+, \mathcal{E}_R)$ is a generalised ray.*

Proof. The fact that the coarse space $(\mathbb{R}_+, \mathcal{E}_.)$ is a generalised ray easily follows from Theorem 2.27 of [Roe03]. On the other hand, the inclusion $\mathcal{E}_R \subseteq \mathcal{E}_.$ is true for every coarse structure which is compatible with the topology of $\mathbb{R}_+$. □

Recall that if X is a compact subset of the unit sphere in a normed space then we define the *open cone* on X, denoted by $\mathcal{O}X$, to be the metric space

$$\mathcal{O}X := \{\lambda x | \ \lambda \in [0, \infty), x \in X\}.$$

We know that the coarse geometry of the cone $\mathcal{O}X$ is closely related to the topology of the space X, namely,

Proposition 2.1.4. *Let X and Y be compact metrisable spaces which are bi-Lipschitz homeomorphic*[1] *with respect to the metrics coming from embedding X and Y in the unit sphere of a real Hilbert space H. Then $\mathcal{O}X$ and $\mathcal{O}Y$ are coarsely equivalent.*

For a proof consult Proposition 2.2 of [Roe96].

With the motivation coming from the above proposition and the facts that the cone of the sphere S^{n-1} is the Euclidean space $\mathbb{R}^n$ and the cone of the Euclidean n-cell, $D^n - S^{n-1}$, is the half-space $\mathbb{R}^n \times \mathbb{R}_+$, we define

[1]A map $f : X \to Y$ between metric spaces is said to be *Lipschitz* if there is a constant C such that $d(f(x_1), f(x_2)) \leq Cd(x_1, x_2)$ for all $x_1, x_2 \in X$; a *bi-Lipschitz homeomorphism* is a Lipschitz map with a Lipschitz inverse.

Definition 2.1.5. Let R be a generalised ray and let $n \geq 0$. The *coarse R-sphere of dimension n* is the coarse product $S^n_R = (R \amalg R)^{n+1}$. The *coarse R-cell of dimension n* is the coarse product $D^n_R = S^{n-1}_R \times R$. The coarse R-sphere

$$\{(x, 0) |\ x \in S^n_R\}$$

is called the *boundary* of the coarse cell D^{n+1}_R. A *coarse n-cell*, $n \geq 0$, denoted by e^{crs}_R, is a connected proper coarse topological space which is coarsely equivalent to the coarse R-cell of dimension n, D^n_R, if $n \geq 1$, and which is coarsely equivalent to the generalised ray R, if $n = 0$. For later use, it is more convenient to define a coarse n-cell, for $n \geq 1$, as a coarsely equivalent copy of $D^n_R \setminus S^{n-1}_R$, although there is no difference between them from the coarse point of view. In the future, we will drop the "R" suffix in e^{crs}_R whenever it is clear from the context what is intended. We define the *coarse interval*, denoted by I^{crs}_R, as the coarse product $R \times R$. We should remark here that we always consider the connected coarse coproduct structure on $R \amalg R$.

As a special case, we define the *standard coarse sphere of dimension n*, denoted by $S^n_{\mathbb{R}_+}$, to be the unital proper coarse topological space $\mathbb{R}^{n+1}$ equipped with the bounded coarse structure coming from the metric. Similarly, we define the *standard coarse cell of dimension n* and we will denote it by $D^n_{\mathbb{R}_+}$.

Let $0 \leq s \leq 1$ be a real number. Consider the map $i_s : R \to I^{crs}_R$ defined by the equation $i_s(t) := te^{\frac{\pi}{2}is}$, we have

Lemma 2.1.6. *Let R be a generalised ray. The coarse space $\mathrm{Im}(i_s)$ equipped with the subspace coarse structure is coarsely equivalent to R.*

Proof. Consider the map $i_s : R \to \mathrm{Im}(i_s)$ defined in above. We will show that it is actually a coarse equivalence. For it, suppose that $E \in \mathcal{E}_R$. By the third axiom in the definition of a generalised ray, the subset $E^{\times^1}$ is an entourages in $\mathcal{E}_R$. On the other hand, one can easily show that $E' := (\pi_1^{\times 2})^{-1}(E^{\times^1}) \cap (\pi_2^{\times 2})^{-1}(E^{\times^1})$ is an entourage in $\mathcal{E}_{I^{crs}_R}$. Now we claim $i_s^{\times 2}(E) \subseteq (\mathrm{Im}(i_s))^{\times 2} \cap E'$ which means that i_s preserves entourages. To see this, let $(t, t') \in E$. Therefore, by the third axiom in the definition of a generalised ray, we have

$$(t\cos(\frac{\pi}{2}s), t'\cos(\frac{\pi}{2}s)), (t\sin(\frac{\pi}{2}s), t'\sin(\frac{\pi}{2}s)) \in E^{\times^1},$$

that is, $(te^{\frac{\pi}{2}is}, t'e^{\frac{\pi}{2}is}) \in E'$. On the other hand, $\mathrm{Im}(i_s)$ is a closed subset of I^{crs}_R which means it is also a proper coarse topological space. Hence, since the map $i_s : R \to \mathrm{Im}(i_s)$ is also topologically proper, therefore it is a coarse map. Now, consider the map $j : \mathrm{Im}(i_s) \to R$ defined by the equation $j(te^{\frac{\pi}{2}is}) := t$. To show that it is a coarse map it is enough to show that it preserves entourages. Suppose $E \in \mathcal{E}_{I_R^{crs}}|_{\mathrm{Im}(i_s)}$. Therefore there exists an entourage $F \in \mathcal{E}_{I_R^{crs}}$ such that $E = (\mathrm{Im}(i_s))^{\times 2} \cap F$. Clearly, $\pi_1^{\times 2}(F)$, $\pi_2^{\times 2}(F) \in \mathcal{E}_R$. Hence, $(\pi_1^{\times 2}(F))^{\times^1}$, $(\pi_2^{\times 2}(F))^{\times^1} \in \mathcal{E}_R$, by the third axiom in the definition of a generalised ray. Now, the first axiom implies that $(\pi_1^{\times 2}(F))^{\times^1} + (\pi_2^{\times 2}(F))^{\times^1} \in \mathcal{E}_R$. On the other hand,

$$j^{\times 2}(E) \subseteq (\pi_1^{\times 2}(F))^{\times^1} + (\pi_2^{\times 2}(F))^{\times^1},$$

which means j preserves entourages. Obviously, $i \circ j = 1_{\mathrm{Im}(i_s)}$ and $j \circ i = 1_R$, i.e., i is a coarse equivalence. □

As we can see above, we can think of a coarse sphere as a "sphere at infinity" which agrees with the philosophy of the coarse geometry. We can also think of a generalised ray as a "point at infinity" and this leads us to the following notion of basepoint in the coarse category. The definitions come from [MS], but ours are slightly different.

Definition 2.1.7. Let R be a generalised ray.

- A coarse topological space X is called a *coarse topological R-space* if it is equipped with a map $p_X : X \to R$ which is topologically proper and preserves the entourages. We will call the map $p_X : X \to R$ the *basepoint projection* (we can define a *proper coarse topological R-space* similarly).
- Assume that X has been equipped with a basepoint projection $p_X : X \to R$. A coarse map $i_X : R \to X$ is called a *basepoint inclusion* if the composite $p_X \circ i_X$ is close to the identity 1_R.
- Let X and Y be coarse topological R-spaces with basepoint projections $p_X : X \to R$, $p_Y : Y \to R$ and basepoint inclusions $i_X : R \to X$, $i_Y : R \to Y$. Let $f : X \to Y$ be a coarse map. The map f is said to be *basepoint-preserving* if the composite $f \circ i_X$ and i_Y are close. The map f is said to be *compatible with the basepoint projections* (resp. *strongly compatible with the basepoint projections*) if the composite $p_Y \circ f$ and p_X are close (resp. $p_X(x) = p_Y(f(x))$ for every $x \in X$).

Remark 2.1.8. Note that saying $p_X : X \to R$ is topologically proper and preserves entourages is stronger than saying $p_X : X \to R$ is a coarse map, because we did not assume that X is unital.

Now, we introduce an important class of proper coarse topological R-spaces which are built in stages: attach a (possibly infinite) family of coarse 1-cells to a disjoint union of generalised rays; attach a family of coarse 2-cells to the result; then attach coarse 3-cells, coarse 4-cells, and so on. Since we allow attaching infinitely many coarse cells, let us begin by discussing an appropriate coarse structure.

Definition 2.1.9. Let X be a topological space covered by its subsets A_j, where j lies in some (possibly infinite) index set J, that is, $X = \bigcup_{j \in J} A_j$. Moreover, assume that

(i) each $(A_j, \mathcal{E}_j)$ is a connected coarse topological space such that the following relations hold between the topology of A_j and the topology of X;

  (a) every relatively compact subset K of A_j is relatively compact in X; and

  (b) for every relatively compact subset L of X, the subset $L \cap A_j$ is relatively compact in A_j for every $j \in J$.

(ii) for each $j, k \in J$, the coarse structure of A_j and of A_k agree on $A_j \cap A_k$, that is, for every entourage $E \in \mathcal{E}_{A_j}$ there exists an entourage $F \in \mathcal{E}_{A_k}$ such that $E \cap (A_k \times A_k) = F \cap (A_j \times A_j)$ and vice versa.

Let $\{i_j : A_j \to X | \, j \in J\}$ be the inclusion maps. Then the *weak coarse structure on X determined* by $\{A_j | \; j \in J\}$ is defined to be the following coarse structure

$$\mathcal{E}_w := \left\langle (i_j)_* \mathcal{E}_{A_j} \right\rangle_{cn}.$$

One can show that each A_j, as a coarse subspace of X, retains its original coarse structure. We may remind the fact that we were allowed to make the above definition, because the inclusion maps are topologically proper and respect the Roe properness axiom. Note that in the future, when we say X has the weak coarse structure determined by $\{A_j | \; j \in J\}$, we mean that all above assumptions hold. For instance, in the case that the topology of each coarse topological space A_j coincides with the subspace topology induced by the topology of X and each A_j is closed in X, then the conditions (a) and (b) are automatically fulfilled. Moreover, note that under the above assumptions, the coarse space $(X, \mathcal{E}_w)$ is a coarse topological space. From now on, we stop mentioning the inclusion maps i_j's and we will consider A_j as $i_j(A_j)$ and so on.

Lemma 2.1.10. *Let a topological space X have the weak coarse structure determined by a family of coarse topological subsets $\{A_j | \; j \in J\}$. For any connected coarse topological space Y, a function $f : X \to Y$ is coarse if and only if $f|_{A_j}$ is coarse for every $j \in J$.*

Proof. We omit proofs of the symmetric cases. Assume that f is a coarse map. Let $F \in \mathcal{E}_{A_j}$, hence $F \in \mathcal{E}_w$ by the definition of the weak coarse structure. Therefore $f^{\times 2}(F) \in \mathcal{E}_Y$ which means $(f|_{A_j})^{\times 2} \in \mathcal{E}_Y$, since $F \subseteq A_j \times A_j$. Now assume that K is a relatively compact subset of Y, so $f^{-1}(K) \cdot F$ is a relatively compact subset of X. But, $(f|_{A_j})^{-1}(K) \cdot F \subseteq f^{-1}(K) \cdot F$ which implies that $(f|_{A_j})^{-1}(K) \cdot F$ is relatively compact in A_j by (b), i.e.,

$f|_{A_j}$ is locally proper for F. So we have shown that $f|_{A_j}$ is a coarse map for every $j \in J$. Conversely, suppose that $f|_{A_j}$ is coarse for every $j \in J$. Assume $E \in \mathcal{E}_w$, therefore by the definition of the weak coarse structure, without lose of generality, we can write:

$$E \subseteq \bigcup_{l=1}^{m}(E_{j_{1,l}} \circ E_{j_{2,l}} \circ \cdots \circ E_{j_{n,l}}) \cup D_l^{\delta},$$

where $E_{j_{k,l}} \in \mathcal{E}_{A_{j_{k,l}}}$ for $k = 1, \cdots, n$ and $l = 1, \cdots, m$ and the subsets D_l^{δ} are the ones introduced in the proof of Proposition 1.6.4. Therefore,

$$f^{\times 2}(E) \subseteq \bigcup_{l=1}^{m}(f^{\times 2}(E_{j_{1,l}}) \circ f^{\times 2}(E_{j_{2,l}}) \circ \cdots \circ f^{\times 2}(E_{j_{n,l}})) \cup (D_l)_Y^{\delta}.$$

But, $f^{\times 2}(E_{j_{k,l}}) = (f|_{A_{j_{k,l}}})^{\times 2}(E_{j_{k,l}})$, for all $k = 1, \cdots, n$ and all $l = 1, \cdots, m$ which implies that f preserves entourages. Now assume that K is a relatively compact subset of Y, so

$$f^{-1}(K) \cdot E \subseteq \bigcup_{l=1}^{m}(((f^{-1}(K) \cdot E_{j_{1,l}}) \cdot E_{j_{2,l}}) \cdot \cdots \cdot E_{j_{n,l}}) \cup (f^{-1}(K) \cdot F_l'),$$

where F_l' are some finite subsets of X. But, $f^{-1}(K) \cdot E_{j_{1,l}} = (f|_{A_{j_{1,l}}})^{-1}(K) \cdot E_{j_{1,l}}$, which implies that $f^{-1}(K) \cdot E_{j_{1,l}}$ is relatively compact in $A_{j_{1,l}}$ and therefore in X by (a). Now, from (b) and the fact that entourages satisfy the Roe properness axiom follow that $[(f^{-1}(K) \cdot E_{j_{1,l}}) \cap A_{j_{2,l}}] \cdot E_{j_{2,l}}$ is relatively compact in $A_{j_{2,l}}$ and therefore in X. But,

$$[(f^{-1}(K) \cdot E_{j_{1,l}}) \cap A_{j_{2,l}}] \cdot E_{j_{2,l}} = (f^{-1}(K) \cdot E_{j_{1,l}}) \cdot E_{j_{2,l}}.$$

That is, by repeating this for finitely many times, one can conclude that $f^{-1}(K) \cdot E$ is relatively compact in X, as desired. □

Definition 2.1.11. Assume that a coarse topological space X is a disjoint union of coarse cells: $X = \bigcup \{e_R^{crs} |\ e_R^{crs} \in E\}$. For each $k \geq 0$, the *k-skeleton* $X^{(k)}$ of X is defined by

$$X^{(k)} = \bigcup \{e_R^{crs} \in E |\ \dim(e_R^{crs}) \leq k\}.$$

Of course, $X^{(0)} \subseteq X^{(1)} \subseteq \cdots$ and $X = \bigcup_{k \geq 0} X^{(k)}$.

Definition 2.1.12. A *coarse CW-complex* is an ordered triple (X, E, Φ), where X is a connected coarse topological R-space, E is a family of coarse cells, and $\Phi = \left\{\Phi_{e_R^{crs}} |\ e_R^{crs} \in E\right\}$ is a family of coarse maps, called *coarse characteristic maps*, such that

(1) $X = \bigcup \{e_R^{crs} |\ e_R^{crs} \in E\}$ (disjoint union);

(2) for each coarse k-cell $e_R^{crs} \in E$, the map $\Phi_{e_R^{crs}} : (D_R^k, S_R^{k-1}) \to (e^{crs} \cup X^{(k-1)}, X^{(k-1)})$ is a relative coarse equivalence, i.e., $\Phi_{e_R^{crs}}$ is compatible with the basepoint projections and $\Phi_{e_R^{crs}}\Big|_{(D_R^k \setminus S_R^{k-1})} : D_R^k \setminus S_R^{k-1} \to e^{crs}$ is a coarse equivalence;

(3) if we define $\bar{e}_R^{crs}$ to be $\Phi_{e_R^{crs}}(D_R^k)$, then X has the weak coarse structure determined by $\{\bar{e}_R^{crs} |\ e_R^{crs} \in E\}$;

(4) if $e_R^{crs} \in E$, then $\bar{e}_R^{crs}$ is contained in a finite union of coarse cells in E.

In the future, we say "a coarse CW-complex X is an $\mathbb{R}_+$-space" to indicate that it is a connected coarse topological $\mathbb{R}_+$-space and the domain of all coarse characteristic maps of X are the pairs consisting of the standard coarse cells and the standard coarse spheres. We finish this section by defining a coarse subcomplex of a coarse CW-complex.

Definition 2.1.13. Let (X, E, Φ) be a coarse CW-complex. If $E' \subseteq E$, define

$$\left|E'\right| = \cup \left\{e_R^{crs} |\ e_R^{crs} \in E'\right\} \subseteq X,$$

and define $\Phi' = \left\{\Phi_{e_R^{crs}} : e_R^{crs} \in E'\right\}$. Call $(|E'|, E', \Phi')$ a *coarse subcomplex* if $\mathrm{Im}(\Phi_{e_R^{crs}}) \subseteq |E'|$ for every $e_R^{crs} \in E'$.

2.2 Coarse equivalence relations

In topology, when we have an equivalence relation on a topological space X, then we can define the quotient space $X/\sim$ consisting of the equivalence classes equipped with the quotient topology. But, in coarse topology collapsing all points of an equivalence class to a point may cause that the natural map fails to be topologically proper which then means that we cannot pushforward the coarse structure of $\mathcal{E}_X$ on $X/\sim$ along the natural map. In this section, we define a notion of coarse collapsing which leads us to a notion of coarse quotient spaces. As the first step, we introduce the notion of a pointed coarse topological R-spaces:

Definition 2.2.1. Let X be a coarse topological R-space equipped with a basepoint projection $p_X : X \to R$ and a basepoint inclusion $i_X : R \to X$. We say the basepoint inclusion $i_X : R \to X$ *represents a basepoint of* X if the following holds:

$$\forall r \in R\ \exists\, x \in \mathrm{Im}(i_X) \text{ such that } p_X(x) = r.$$

In the case that such a basepoint inclusion exists, we denote $\mathrm{Im}(i_X)$ by $*_{crs}$ and we call X *pointed* with the basepoint $*_{crs}$, or we shortly write $(X, *_{crs})$ to mean that the coarse topological R-space X is pointed with the basepoint $*_{crs}$.

In the upcoming lines, we will explain the reason why we assumed such a splitting property.

Example 2.2.2. Let R be a generalised ray. In the future, for each $n \geq 0$, we always consider the second copy of the ray R in the first component of S_R^n as its basepoint.

Our next goal is to define a new notion of collapsing from a coarse point of view. In topology, when we collapse a subset A of a topological space X, we get a point in the space X. Therefore, in coarse topology, we expect to get a coarse point which is a point at infinity or more precisely a copy of the generalised ray R for a coarse topological R-space X, when we are collapsing. In the following, we will make this idea more precise.

Definition 2.2.3. Let X be a coarse topological R-space. An equivalence relation $\sim$ on X is called *coarse* if each equivalence class $[x]$, $x \in X$,

- is bounded; or
- is a pointed coarse topological R-space with subspace coarse structure and $p_X|_{[x]}$ as its basepoint projection.

Now, we define a new equivalence relation $\sim_{crs}$ on X, called the *coarse equivalence relation generated* by $\sim$, as follows: $x \sim_{crs} y$ if and only if

(i) $x \sim y$, and

(ii) $p_X(x) = p_X(y)$.

We denote the equivalence class containing x under the new equivalence relation by $[x]_{crs}$. Now, suppose that the map $\upsilon : X \to X/\sim_{crs}$ is the natural map, that is, it carries each point of X to the equivalence class $[x]_{crs}$ and assume $X/\sim_{crs}$ has been equipped with the quotient topology. Obviously, the basepoint projection $p_X : X \to R$ factors through the natural map, that is, there exists a map $p_{X/\sim_{crs}} : X/\sim_{crs} \to R$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\upsilon} & X/\sim_{crs} \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle p_{X/\sim_{crs}}} & \\ & R. & \end{array}$$

This implies that the natural map is topologically proper and respects the Roe properness axiom (with respect to the quotient topology). Therefore we can equip the topological space $X/\sim_{crs}$ with the coarse push-forward structure $\upsilon_*\mathcal{E}_X$. The obtained space is called a *coarse quotient space* of X and will be denoted again by $X/\sim_{crs}$. In fact, the coarse quotient space $X/\sim_{crs}$ is a pointed coarse topological R-space. Moreover, if X is proper, so do $X/\sim_{crs}$.

Now, we can see why we assume such a splitting property for the basepoint. We expect to get a copy of R for each unbounded equivalence class after collapsing it. Assuming each unbounded equivalence class to be pointed, guarantees that in each unbounded equivalence class we are able to collapse all points with the same distance from the origin to a point of the basepoint of that class having the same distance from the origin. Another point should be mentioned is that although our final goal is to get for each unbounded equivalence class a point at infinity but we do not have any choice unless dealing with all points lying in an unbounded equivalence class. Now, to show that the final object we get after collapsing each unbounded equivalence class is indeed a coarse point, we must show the following:

Proposition 2.2.4. *Under the above assumptions, after collapsing an unbounded equivalence class* $[z]$*, we have*

$$[z] / \sim_{crs} \cong_{crs} R.$$

Proof. It is easy to show that the coarse map $v \circ i_{[z]} : R \to [z] / \sim_{crs}$ is indeed a coarse equivalence. □

Now, we consider a special case, namely, let A be a subset of a coarse topological R-space X which is either a pointed coarse topological R-space with the subspace coarse structure, or a bounded subset of X. Then X/A denotes the coarse quotient R-space obtained via the equivalence relation $\sim$ whose equivalence classes are A and the single point sets $\{x\}$, $x \in X \setminus A$. We call X/A the coarse quotient R-space *obtained from* X *by coarse collapsing* A *to the ray* R.

Example 2.2.5. Let $\mathbb{R}$ be the real line with its bounded coarse structure and with the basepoint projection $p_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}_+$ defined by $p_{\mathbb{R}}(x) = |x|$. Suppose that A is the union of the subsets $B = \{(\frac{4k+1}{2}, \frac{4k+3}{2})|\ k \in \mathbb{Z}_+\}$ and $C = \{x \in \mathbb{R}|\ x \leq 0\}$ of $\mathbb{R}$. Then one can easily check that $\mathbb{R}/A \cong_{crs} \mathbb{R}_+$.

The following is an important construction of examples of coarse quotient spaces of proper coarse topological R-spaces.

Definition 2.2.6. Let X, Y and A be coarse topological R-spaces. Suppose $i : A \to X$ and $f : A \to Y$ are coarse maps which are strongly compatible with the basepoint projections. Moreover, assume that the inverse image of bounded subsets under the coarse maps i and f are bounded (for example, this is the case, if X, Y and A are proper coarse topological R-spaces with A unital). Let $Z := X \amalg Y$ be the coarse coproduct of X and Y. Assume that $\sim$ is the equivalence relation on Z generated by the binary relation $\{(i(a), f(a)) \in Z \times Z|\ a \in A\}$. Then the coarse topological quotient R-space $Z/\sim_{crs}$ as defined above, is called the *coarse space obtained from* Y *by weakly coarse attaching* X *via* f and will be denoted by $X \cup_A Y$. Note that we did not assume any splitting property because each equivalence class of $\sim$ is bounded.

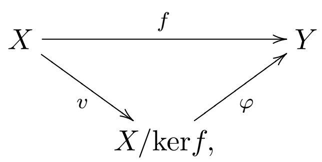
Now, there is a special example that we wish to mention, because it will play an important role in the next chapter.

Example 2.2.7. Let X and Y be pointed proper coarse topological R-spaces. Suppose that $f : X \to Y$ is a coarse surjection map which is also injective unless for a subset of the set $\mathbf{ker} f := \{x \in X | \ f(x) \in *_{crs}\}$. Moreover, assume that the subset $\ker f$ has the following properties:

(i) it consists of the basepoint of the space X;

(ii) for every $x_1, x_2, \in \ker f$, we have $f(x_1) = f(x_2)$ if and only if $p_X(x_1) = p_X(x_2)$.

Therefore, by above, we can consider the coarse quotient R-space $X/\ker f$. Note that, given $f : X \to Y$ with the above properties, there always exists an injection $\varphi : X/\ker f \to Y$ making the following diagram commute:



namely, $\varphi([x]_{crs}) = f(x)$.

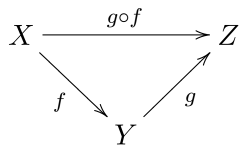
The natural questions which arise here are whether the map $\varphi : X/\ker f \to Y$ above is coarse and under which conditions the map φ is a coarse equivalence. In the upcoming lines, we will give an answer to these questions.

Definition 2.2.8. Let X and Y be coarse topological spaces (not necessarily unital). A coarse surjection map $f : X \to Y$ is called a *coarse identification* if for every entourage $F \in \mathcal{E}_Y$ there are entourages $E_{ij} \in \mathcal{E}_X$, $1 \leq i \leq n$, $1 \leq j \leq m$ and a subset $F' \subseteq Y \times Y$ which is a finite union of subsets of the form $\{1_y\}$, $y \in Y$, such that

$$F \subseteq (\bigcup_{j=1}^{m} f^{\times 2}(E_{1j}) \circ f^{\times 2}(E_{2j}) \circ \cdots \circ f^{\times 2}(E_{nj})) \cup F'.$$

Example 2.2.9. Let $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}^2$ be the pointed proper coarse topological spaces equipped with the bounded coarse structures. The coarse map $\exp : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^2$ defined by $\exp(te^{\frac{\pi}{2}is}) := te^{2\pi is}$, $0 \leq s \leq 1$, is called the *coarse exponential map.* One can easily see that it is actually a coarse identification. Moreover, one can similarly define the exponential map $\exp : I_R^{crs} \to S_R^1$ for a generalised ray R.

Theorem 2.2.10. *Let X and Y be proper coarse topological spaces and let $f : X \to Y$ be a coarse identification. Then, for all coarse topological spaces Z and all functions $g : Y \to Z$, one has g coarse if and only if $g \circ f$ is coarse.*



Proof. If g is a coarse map, then $g \circ f$ is clearly a coarse map. Conversely, let $g \circ f$ be a coarse map and let $F \in \mathcal{E}_Y$. We first show that $g^{\times 2}(F) \in \mathcal{E}_Z$. Without lose of generality, we can assume

$$F \subseteq (f^{\times 2}(E_1) \circ \cdots \circ f^{\times 2}(E_n)) \cup F',$$

where $E_1, \cdots, E_n \in \mathcal{E}_X$ and $F' \subseteq Y \times Y$ is a finite union of subsets of the form $\{1_y\}$, $y \in Y$. Therefore,

$$g^{\times 2}(F) \subseteq ((g^{\times 2} \circ f^{\times 2}(E_1)) \circ \cdots \circ (g^{\times 2} \circ f^{\times 2}(E_n))) \cup g^{\times 2}(F')$$

That is, since $g \circ f$ preserves entourages, therefore $g^{\times 2}(F)$ is entourage. It remains to show that g is locally proper for $F \in \mathcal{E}_Y$. By the first part, we have already shown that $g^{\times 2}(F) \in \mathcal{E}_{|Z|_1}$. Now, suppose that K is a relatively compact subset of Z. Again, without lose of generality, we have

$$F \subseteq (f^{\times 2}(E_1) \circ \cdots \circ f^{\times 2}(E_n)) \cup F',$$

where the subsets $E_1, \cdots, E_n$ and F' are as in above. Therefore,

$$g^{-1}(K) \cdot F \subseteq ((((g^{-1}(K) \cdot f^{\times 2}(E_1)) \cdot f^{\times 2}(E_2)) \cdot \cdots) \cdot f^{\times 2}(E_n)) \cup g^{-1}(K) \cdot F'$$

But, one can easily show that

$$g^{-1}(K) \cdot f^{\times 2}(E_1) \subseteq f((g \circ f)^{-1}(K) \cdot E_1),$$

which by the locally properness of $g \circ f$ and the facts that X and Y are proper means $g^{-1}(K) \cdot f^{\times 2}(E_1)$ is a relatively compact subset of Y. Now the result follows from the fact that for every $i = 2, 3, \cdots, n$, $f^{\times 2}(E_i) \in \mathcal{E}_Y$ and the fact that each entourage satisfies the Roe properness axiom. □

The following is clear:

Lemma 2.2.11. *Let X, Y and Z be coarse topological spaces and let $f : X \to Y$ be a coarse identification. Suppose that $g : Y \to Z$ is a coarse surjection. Then g is a coarse identification if and only if $g \circ f$ is a coarse identification.*

Now go back to Example 2.2.7, we claim

Theorem 2.2.12. *Under the assumptions of Example 2.2.7, the map φ is a coarse equivalence if the coarse map $f : X \to Y$ is a coarse identification.*

Proof. It is clear that the natural map $v : X \to X/\ker f$ is a coarse identification. Consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle v}\searrow \quad \nearrow{\scriptstyle \varphi} & \\ & X/\ker f. & \end{array}$$

That $\varphi \circ v = f$ is a coarse map implies that φ is a coarse map, by Theorem 2.2.10. Note that φ is surjective because f is. Now, consider the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad v \quad} & X/\ker f \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle \varphi^{-1}} & \\ & Y. & \end{array}$$

Again, from Theorem 2.2.10 and the fact that f is a coarse identification follow that φ^{-1} is a coarse map. So we have shown that in the case of Example 2.2.7, the proper coarse topological R-spaces Y and $X/\ker f$ are coarsely equivalent if f is a coarse identification. □

Now, we introduce a class of coarse CW-complexes which are the analogue for coarse topology of the regular CW-complexes for topology. They come to play in different occasions, for example, when we do not want our coarse topological R-spaces to have any hole: examples of such case are, as we have seen before, when we are dealing with the coarse collapsing or as we will see immediately after the next definition when we want to inroduce the notion of coarse product. Another place in which full coarse CW-complexes will show up are when we are dealing with coarse homotopy groups of coarse CW-complexes in Chapter 5.

Definition 2.2.13. a coarse CW-complex (X, E, Φ) is called *full* if

(1) $X^{(0)}$ is a disjoint union of the generalised rays R with the exception that any two generalised rays which are the boundary of the same coarse 1-cell intersect at 0;

(2) each coarse k-cell e_R^{crs} arises from $X^{(k-1)}$ by weakly coarse attaching D_R^k vie a coarse map $f_{e_R^{crs}} : S_R^{k-1} \to X^{(k-1)}$ which is an embedding (that is a homeomorphism onto its image) and which is also strongly compatible with the basepoint projections (the maps $i : S_R^{k-1} \to D_R^k$ are just the inclusions);

(3) the basepoint projection $p_X : X \to R$ coincides with the standard basepoint projection of the generalised rays in $X^{(0)}$ and of the coarse R-cells D^k_R (the basepoint projections of the coarse R-cells D^k_R have been taken to be the standard ones).

Endowing the coarse topological spaces with basepoint projections make it possible to define a special coarse product which plays a crucial role in developing the coarse homotopy theory. More precisely,

Definition 2.2.14. Let X and Y be coarse topological R-spaces, where at least one of them is a full coarse CW-complex. By the *coarse product* of X and Y, denoted by $X \times_R Y$, we mean the set

$$X \times_R Y = \{(x, y)|\ p_X(x) = p_Y(y)\},$$

equipped with the following coarse structure

$$\mathcal{E}_{X\times_R Y} := (\pi_X|_{X\times_R Y})^* \mathcal{E}_X \cap (\pi_Y|_{X\times_R Y})^* \mathcal{E}_Y.$$

Before explaining the reason why we assume at least one of the spaces X or Y to be a full coarse CW-complex, note that

Lemma 2.2.15. *Let R be a generalised ray. If X is a coarse topological R-space, then $X \times_R R \cong_{crs} X$.*

Proof. Consider the maps $i : X \to X \times_R R$ and $\pi_X : X \times_R R \to X$ defined by $i(x) := (x, p_X(x))$ and $\pi_X(x, r) = x$, respectively. They are obviously coarse and $i \circ \pi_X = 1_{X\times_R R}$ and $\pi_X \circ i = 1_X$. □

So, as expected, the coarse product of a coarse topological R-space X with the generalised ray R, that is, with a coarse point, is just the space X itself. Now, one can see why we assumed at least one of the spaces X or Y to be a full coarse CW-complex: roughly speaking, we do not want to kill any coarse points. To see it clearer, let X be the set of positive even integers and let Y be the set of positive odd integers both of them equipped with the bounded coarse structure and the trivial basepoint projection. Obviously, $X \cong_{crs} Y \cong_{crs} \mathbb{R}_+$. Therefore, we expect $X \times_{\mathbb{R}_+} Y \cong_{crs} \mathbb{R}_+$, while $X \times_{\mathbb{R}_+} Y = \emptyset$. In the following, we will give a standard way to construct an entourage of $X \times_R Y$ from entourages of X and Y.

Lemma 2.2.16. *Let X and Y be proper coarse topological R-spaces where at least one of them is a full coarse CW-complex and let $F \in \mathcal{E}_X$ and $E \in \mathcal{E}_Y$. Define $F \times_R E$ as the following subset of $X \times_R Y$*

$$F \times_R E := ((\pi_X|_{X\times_R Y})^{\times 2})^{-1}(F) \cap ((\pi_Y|_{X\times_R Y})^{\times 2})^{-1}(E).$$

Then $F \times_R E$ is an entourage (possibly empty) of $X \times_R Y$.

Proof. We must show

(i) $F \times_R E \in \mathcal{E}_{|X\times_R Y|_1}$;

(ii) the map $\pi_X|_{X\times_R Y} : X \times_R Y \to X$ (resp. $\pi_Y|_{X\times_R Y} : X \times_R Y \to Y$) is coarse for $F \times_R E$.

We omit proofs of the symmetric cases. For (i), let K be a relatively compact subset of $X \times_R Y$. Obviously,

$$(F \times_R E) \cdot K \subseteq (F \cdot (\pi_X|_{X\times_R Y})(K)) \times (E \cdot (\pi_Y|_{X\times_R Y})(K)),$$

which means $(F \times_R E) \cdot K$ is relatively compact. For (ii), first, it is clear that $(\pi_X|_{X\times_R Y})^{\times 2}(F \times_R E) \subseteq F$ which means $\pi_X|_{X\times_R Y}$ preserves $F \times_R E$. It remains to show that $\pi_X|_{X\times_R Y}$ is locally proper for $F \times_R E$. For it, let K' be a relatively compact subset of X, we must show that $(\pi_X|_{X\times_R Y})^{-1}(K') \cdot (F \times_R E)$ is relatively compact. The subset K' is relatively compact in X, so it is bounded, hence $T := p_X(K')$ is relatively compact in R, since R is a proper coarse topological space. On the other hand, since p_Y is topologically proper, therefore $p_Y^{-1}(T) \cdot E$ is relatively compact. But, obviously

$$(\pi_X|_{X\times_R Y})^{-1}(K') \cdot (F \times_R E) \subseteq (K' \cdot F) \times (p_Y^{-1}(T) \cdot E),$$

so we are done. Similarly, one can show that $\pi_Y|_{X\times_R Y} : X \times_R Y \to Y$ is coarse for $F \times_R E$. □

Given an entourage $F \in \mathcal{E}_R$, a question which naturally arises is whether the subset $F_R := (\pi_1^{\times 2})^{-1}(F) \cap (\pi_2^{\times 2})^{-1}(F)$ of $I_R^{crs} \times I_R^{crs}$ is an entourage in $\mathcal{E}_{I_R^{crs}}$ or not, where $\pi_i : R \times R \to R$, $i = 1, 2$, are the projections on the first and the second coordinate, respectively. The answer is in general no, because the projections π_i's fail to be locally proper for F_R. The following lemma will be essential later on.

Lemma 2.2.17. *Let X and Y be coarse topological R-spaces. If $f : X \to Y$ is a coarse map which is compatible with the basepoint projections, then the map $f \times_R 1_{I_R^{crs}} : X \times_R I_R^{crs} \to Y \times_R I_R^{crs}$ defined by*

$$(x, p_X(x)e^{\frac{\pi}{2}is}) \longmapsto (f(x), p_Y(f(x))e^{\frac{\pi}{2}is})$$

is a coarse map.

Proof. Assume that $E \in \mathcal{E}_{X\times_R I_R^{crs}}$. We must show that the map $f \times_R 1_{I_R^{crs}}$ is coarse for E. Set $E_1 := \pi_X^{\times 2}(E)$, $E_2 := \pi_{I_R^{crs}}^{\times 2}(E)$, $E_3 := f^{\times 2}(E_1)$, $D := ((p_Y \circ f) \times p_X)(E_1 \circ E_1^{-1})$ and $D' := (p_X \times (p_Y \circ f))(E_1^{-1} \circ E_1)$. They are obviously entourages. Now, we define the subset $T \subseteq I_R^{crs} \times I_R^{crs}$ as follows:

$$T := \{(re^{\frac{\pi}{2}is}, r'e^{\frac{\pi}{2}is'})|\ \exists\ ((x, p_X(x)e^{\frac{\pi}{2}it}), (x', p_X(x')e^{\frac{\pi}{2}it'})) \in E \text{ such that } (s, s') = (t, t') \text{ and } (r, r') = (p_Y(f(x)), p_Y(f(x')))\}$$

Moreover, define $F := D^{\times^1} \circ (\pi_1^{\times 2}(E_2))^{\times^1} \circ (D')^{\times^1}$ and $G := D^{\times^1} \circ (\pi_2^{\times 2}(E_2))^{\times^1}$ $\circ (D')^{\times^1}$, where $\pi_i : R \times R \to R$, $i = 1, 2$, are the projections on the first and the second coordinate, respectively. We define

$$J := (\pi_1^{\times 2}|_T)^{-1}(F) \cap (\pi_2^{\times 2}|_T)^{-1}(G)$$

and we first claim that the subset J is an entourage in $\mathcal{E}_{I_R^{crs}}$. We must show that

(i) $J \in \mathcal{E}_{|R\times R|_1}$;

(ii) $\pi_1 : R \times R \to R$ (resp. $\pi_2 : R \times R \to R$) is coarse for J.

We omit proofs of the symmetric cases. For (i), suppose that K is a relatively compact subset of $R \times R$. Obviously,

$$J \cdot K \subseteq (F \cdot (K \cdot R)) \times (G \cdot (R \cdot K)),$$

which implies that $J \cdot K$ is relatively compact, as desired. The fact that π_i, $i = 1, 2$, preserve J immediately follows from the way that we have defined J. Now, we must show that $\pi_i : R \times R \to R$, $i = 1, 2$, are locally proper for J. Let K' be a relatively compact subset of R. Suppose that $re^{\frac{\pi}{2}is} \in \pi_1^{-1}(K') \cdot J$, where $0 \leq s \leq 1$ and $r \geq 0$. Therefore, there exists $r'e^{\frac{\pi}{2}is'} \in \pi_1^{-1}(K')$ for some $0 \leq s' \leq 1$ and for some $r' \geq 0$ such that

$$(re^{\frac{\pi}{2}is}, r'e^{\frac{\pi}{2}is'}) \in J.$$

But $J \subseteq T$, which means there are $((x, p_X(x)e^{\frac{\pi}{2}it}), (x', p_X(x')e^{\frac{\pi}{2}it'})) \in E$ such that $(s, s') = (t, t')$ and $(r, r') = (p_Y(f(x)), p_Y(f(x')))$. Hence,

$$(x, x') \in E_1, \text{ and}$$
$$(p(x)e^{\frac{\pi}{2}it}, p(x')e^{\frac{\pi}{2}it'}) \in E_2.$$

Now if $\pi_1^{-1}(K) \cdot J$ is not relatively compact, then there exist an unbounded sequence $\{r_k\}$ of positive real numbers such that $r_k e^{\frac{\pi}{2}is_k} \in \pi_1^{-1}(K) \cdot J$ for some $0 \leq s_k \leq 1$, $k \in \mathbb{N}$. Therefore, there are sequences $\{r'_k\}$, $\{s'_k\}$, $\{x_k\}$ and $\{x'_k\}$ having above properties, that is, for each $k \in \mathbb{N}$,

$$(r_k, r'_k) = (p_Y(f(x_k)), p_Y(f(x'_k))), \ (x_k, x'_k) \in E_1,$$
$$(p_X(x_k)e^{\frac{\pi}{2}is_k}, p_X(x'_k)e^{\frac{\pi}{2}is'_k}) \in E_2 \text{ and } r'_k e^{\frac{\pi}{2}is'_k} \in \pi_1^{-1}(K').$$

From the above assumptions directly follow that

$$\forall\ k \in \mathbb{N},\ (p_X(x'_k)\cos(\frac{\pi}{2}s'_k), p_Y(f(x'_k))\cos(\frac{\pi}{2}s'_k)) \in (D')^{\times^1}.$$

But since $p_X(f(x'_k))\cos(\frac{\pi}{2}s'_k) \in K'$, for every $k \in \mathbb{N}$, therefore

$$\forall\ k \in \mathbb{N},\ p_X(x'_k)\cos(\frac{\pi}{2}s'_k) \in (D')^{\times^1} \cdot K',$$

which means $p_X(x'_k)e^{\frac{\pi}{2}is'_k} \in \pi^{-1}((D')^{\times^1} \cdot K')$, for every $k \in \mathbb{N}$. Then since $(p_X(x_k)e^{\frac{\pi}{2}is_k}, p_X(x'_k)e^{\frac{\pi}{2}is'_k}) \in E_2$, for every $k \in \mathbb{N}$, therefore

$$\forall\ k \in \mathbb{N},\ p_X(x_k)e^{\frac{\pi}{2}is_k} \in E_2 \cdot \pi_1^{-1}((D')^{\times^1} \cdot K').$$

But since $E_2 \in \mathcal{E}_{I_R^{crs}}$ and because $(D')^{\times^1} \cdot K'$ is a relatively compact subset of R, therefore $E_2 \cdot \pi_1^{-1}((D')^{\times^1} \cdot K')$ is a relatively compact subset of $R \times R$. That is, $Q := \{p_X(x_k)|\ k \in \mathbb{N}\}$ is a bounded subset of R. On the other hand, $(p_Y(f(x_k)), p_X(x_k)) \in D$, for every $k \in \mathbb{N}$. Therefore,

$$\{p_Y(f(x_k))|\ k \in \mathbb{N}\} \subseteq D \cdot Q,$$

which implies that the subset $\{p_Y(f(x_k))|\ k \in \mathbb{N}\}$ is bounded in R which is a contradiction since $p_Y(f(x_k)) = r_k$, for every $k \in \mathbb{N}$. So we have shown that J is an entourage in $\mathcal{E}_{I_R^{crs}}$. Now the way is free to show the main statement, namely, the map $f \times_R 1_{I_R^{crs}}$ is coarse for E: taking

$$E_3 \times_R J := ((\pi_Y|_{Y \times_R I_R^{crs}})^{\times 2})^{-1}(E_3) \cap ((\pi_{I_R^{crs}}|_{Y \times_R I_R^{crs}})^{\times 2})^{-1}(J),$$

one can show, as in the proof of Lemma 2.2.16, that $E_3 \times_R G \in \mathcal{E}_{Y \times_R I_R^{crs}}$. Now, we have

$$(f \times_R 1_{I_R^{crs}})^{\times 2}(E) \subseteq E_3 \times_R G.$$

The last thing that remains to show is locally properness of $f \times_R 1_{I_R^{crs}}$ for E. Assume that K'' is a relatively compact subset of $Y \times_R I_R^{crs}$. One can easily show that

$$(f \times_R 1_{I_R^{crs}})^{-1}(K'') \cdot E \subseteq (f^{-1}(\pi_Y(K'')) \cdot E_1) \times (\pi_{I_R^{crs}}(K'') \cdot E_2),$$

which implies that $(f \times_R 1_{I_R^{crs}})^{-1}(K'') \cdot E$ is relatively compact, as desired. So we have shown that $f \times_R 1_{I_R^{crs}}$ preserves entourages and is locally proper, that is, it is a coarse map. □

Corollary 2.2.18. *Let X and Y be coarse topological R-spaces and let Z be a full coarse CW-complex. If $f : X \to Y$ is a coarse map which is strongly compatible with the basepoint projections. Then the map $f \times_R 1_Z : X \times_R Z \to Y \times_R Z$ defined by*

$$(f \times_R 1_Z)(x, z) := (f(x), z),$$

is a coarse map.

2.3 Coarse Hamband spaces

It is a natural question whether the map $f \times_R 1_Z : X \times_R Z \to Y \times_R Z$ is a coarse identification when $f : X \to Y$ is a coarse identification (recall that we defined the coarse product $\times_R$ of two coarse topological R-spaces when at least one of them is a full coarse CW-complex and the function $f\times_R 1_Z$ can be defined if the map f is strongly compatible with the basepoint projections). As we know, in topology, to get from an identification $f : X \to Y$, an identification $f \times 1_Z : X \times Z \to Y \times Z$, we need the space Z to be a locally compact space. So, we are investigating some conditions on the coarse topological R-space Z which imply $f \times_R 1_Z : X \times_R Z \to Y \times_R Z$ to be a coarse identification. Considering coarse geometry as a dual theory to topology, the following is a dual notion of locally compactness in coarse world:

Definition 2.3.1. Let X be a coarse topological R-space with a basepoint projection $p_X : X \to R$. We call X *Hamband*[2] if for every entourage $E \in \mathcal{E}_X$ and for every entourages $M_1, M_2 \cdots, M_n \in \mathcal{E}_R$ with $p_X^{\times 2}(E) \subseteq M_1 \circ M_2 \circ \cdots \circ M_n$, there exist entourages $G_1, G_2, \cdots, G_n \in \mathcal{E}_X$ such that if $(x, y) \in E$, then assuming $k_0 := x$, inductively, for every $r_{i+1} \in (M_{i+1})^{p(k_i)}$, $i = 0, 1, \cdots, n-2$, one can find $k_{i+1} \in X$ with $p(k_{i+1}) = r_{i+1}$ such that the following hold:

(i) $(k_i, k_{i+1}) \in G_{i+1}$ for every $i = 0, 1, \cdots, n-2$; and

(ii) $(k_{n-1}, y) \in G_n$.

Example 2.3.2. The Euclidean space $\mathbb{R}^n$ equipped with the bounded coarse structure and its standard basepoint projection is clearly a coarse Hamband space.

Lemma 2.3.3. *Let X, Y and Z be proper coarse topological R-spaces. Suppose that the coarse products $X \times_R Z$ and $Y \times_R Z$ can be defined (that is, at least one of the component at each coarse product is a full coarse CW-complex). If $f : X \to Y$ is a coarse identification which is strongly compatible with the basepoint projections, and Z is Hamband, then the map $f \times_R 1_Z : X \times_R Z \to Y \times_R Z$ defined by*

$$(f \times_R 1_Z)(x, z) := (f(x), z)$$

is a coarse identification.

Proof. By Corollary 2.2.18, the map $f \times_R 1_Z$ is coarse. Let $F \in \mathcal{E}_{Y\times_R Z}$. Therefore, by definition of the coarse structure $\mathcal{E}_{Y\times_R Z}$, the subsets $E := (\pi_Y|_{Y\times_R Z})^{\times 2}(F)$ and $G := (\pi_Z|_{Y\times_R Z})^{\times 2}(F)$ are entourages of $\mathcal{E}_Y$ and $\mathcal{E}_Z$, respectively. Since f is a coarse identification, without lose of generality, we write

$$E \subseteq (f^{\times 2}(E_1) \circ f^{\times 2}(E_2) \circ \cdots \circ f^{\times 2}(E_n)) \cup F'$$

[2]This is a Persian word means "connected". Because the word "connected" has already been used in the classical setup and since the notion we are going to define bring some kind of connectness in mind, therefore the word "Hamband" has been suggested.

where $E_i \in \mathcal{E}_X$, $i = 1, 2, \cdots, n$, and F' is a subset of $Y \times Y$ which is the finite union of subsets of the form $\{1_y\}$, $y \in Y$. For each i, $i = 1, 2, \cdots, n$, define $M_i := p_Y^{\times 2}(f^{\times 2}(E_i) \cup F')$. First, we show that

$$p_Z^{\times 2}(G) \subseteq M_1 \circ M_2 \circ \cdots \circ M_n.$$

For it, suppose that $(z_1, z_2) \in G$, hence there exists $(y_1, y_2) \in Y \times Y$ such that $((y_1, z_1), (y_2, z_2)) \in F$. Therefore, $p_Y(y_j) = p_Z(z_j)$, $j = 1, 2$, and $(y_1, y_2) \in E$. If $(y_1, y_2) \in F'$, then $y_1 = y_2$ and therefore,

$$(p_Z(z_1), p_Z(z_2)) = (p_Y(y_1), p_Y(y_1)) \in M_i,$$

for all $i = 1, 2, \cdots, n$, that is,

$$(p_Z(z_1), p_Z(z_2)) \in M_1 \circ M_2 \circ \cdots \circ M_n,$$

as desired. Otherwise, there exist $s_k \in Y$, $k = 1, \cdots, n-1$, such that

$$(y_1, s_1) \in f^{\times 2}(E_1),\ (s_1, s_2) \in f^{\times 2}(E_2),\ \cdots,\ (s_{n-1}, y_2) \in f^{\times 2}(E_n).$$

Therefore,

$$(p_Z(z_1), p_Z(z_2)) = (p_Y(y_1), p_Y(y_2)) \in M_1 \circ M_2 \circ \cdots \circ M_n.$$

Now since Z is Hamband and $p_Z^{\times 2}(G) \subseteq M_1 \circ M_2 \circ \cdots \circ M_n$, therefore there exist entourages $G_1, G_2, \cdots, G_n \in \mathcal{E}_Z$ having the properties mentioned in Definition 2.3.1. On the other hand, by Lemma 2.2.16, for each i, $i = 1, 2, \cdots, n$, we can construct the entourage $E_i \times_R G_i$ of $X \times_R Z$. Now, defining $F'' := F' \times_R G$, the final claim is

$$F \subseteq ((f \times_R 1_Z)^{\times 2}(E_1 \times_R G_1) \circ (f \times_R 1_Z)^{\times 2}(E_2 \times_R G_2) \circ \cdots \\ \cdots \circ (f \times_R 1_Z)^{\times 2}(E_n \times_R G_n)) \cup F''.$$

Suppose that $((y_1, z_1), (y_2, z_2)) \in F$, hence $(y_1, y_2) \in E$ and $(z_1, z_2) \in G$. So, $(y_1, y_2) \in F'$, or

$$(y_1, y_2) \in f^{\times 2}(E_1) \circ f^{\times 2}(E_2) \circ \cdots \circ f^{\times 2}(E_n).$$

In the former case, obviously, we have $((y_1, z_1), (y_2, z_2)) \in F''$. In the latter case, there exist $s_k \in Y$, $k = 1, \cdots, n-1$, such that

$$(y_1, s_1) \in f^{\times 2}(E_1),\ (s_1, s_2) \in f^{\times 2}(E_2),\ \cdots,\ (s_{n-1}, y_2) \in f^{\times 2}(E_n).$$

Therefore, there exist $x_j \in X$, $j = 1, 2$, and $s'_l, s''_l \in X$, $l = 1, 2, \cdots, n-1$, such that

$$(x_1, s'_1) \in E_1,\ (s''_1, s'_2) \in E_2,\ (s''_2, s'_3) \in E_3,\ \cdots,\ (s''_{n-1}, x_2) \in E_n$$

with $f(x_j) = y_j$, $j = 1, 2$, and $f(s'_l) = f(s''_l) = s_l$ for all $l = 1, 2, \cdots, n-1$. On the other hand,

$$(p_Y(y_1), p_Y(s_1)) \in M_1, (p_Y(s_1), p_Y(s_2)) \in M_2, \cdots, (p_Y(s_{n-1}), p_Y(y_2)) \in M_n.$$

But, $p_Y(y_j) = p_Z(z_j)$, $j = 1, 2$, that is,

$$p_Y(s_1) \in (M_1)^{p_Z(z_1)}, p_Y(s_2) \in (M_2)^{p_Y(s_1)}, \ \cdots, \ p_Y(s_{n-1}) \in (M_{n-1})^{p_Y(s_{n-2})}.$$

Hence, by the definition of being Hamband, we can inductively find $k_i \in Z$, $i = 1, 2, \cdots, n-1$, such that

(i) $p_Z(k_i) = p_Y(s_i)$ for all $i = 1, 2, \cdots, n-1$; and

(ii) $(z_1, k_1) \in G_1, \ (k_1, k_2) \in G_2, \ \cdots, \ (k_{n-1}, z_2) \in G_n$.

On the other hand, $p_Z(z_j) = p_Y(y_j) = p_Y(f(x_j)) = p_X(x_j)$, $j = 1, 2$. Similarly, for each $i = 1, 2, \cdots, n-1$, we have

$$p_Z(k_i) = p_Y(s_i) = p_Y(f(s'_i)) = p_Y(f(s''_i)) = p_X(s'_i) = p_X(s''_i).$$

That is, setting $(s''_0, k_0) := (x_1, z_1)$ and $(s'_n, k_n) := (x_2, z_2)$, we have

$$((s''_i, k_i), (s'_{i+1}, k_{i+1})) \in E_{i+1} \times_R G_{i+1},$$

for all $i = 0, 1, \cdots, n-1$. Therefore,

$$((f(s''_i), k_i), (f(s'_{i+1}), k_{i+1})) \in (f \times_R 1_Z)^{\times 2}(E_{i+1} \times_R G_{i+1}),$$

for all $i = 0, 1, \cdots, n-1$. Hence,

$$((y_1, z_1), (y_2, z_2)) \in (f \times_R 1_Z)^{\times 2}(E_1 \times_R G_1) \circ (f \times_R 1_Z)^{\times 2}(E_2 \times_R G_2) \circ \cdots$$
$$\cdots \circ (f \times_R 1_Z)^{\times 2}(E_n \times_R G_n),$$

as desired. □

Chapter 3

Coarse homotopy theory

In this chapter we first develop basic notions in the coarse homotopy theory. We will next introduce some constructions such as coarse smash product, coarse suspensions and coarse mapping cone needed to develop coarse homotopy theory and we will prove some of their properties. The coarse homotopy groups will be introduced next and at the end of this chapter, we develop an exact sequence of coarse homotopy groups. Throughout this chapter, we shall assume unless otherwise stated that all coarse topological R-spaces are pointed.

3.1 Coarse homotopy

In order to define a coarse version of homotopy theory, one is for sure: we can not simply use the closed interval I. To see it, let $f, g : X \to Y$ be coarse maps between proper metric spaces X and Y equipped with their bounded coarse structures. If there exists a coarse map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for every $x \in X$, then because $I \times I$ is entourage, therefore, for every two points $t, s \in I$, the maps F_t and F_s are close, that is, $f \sim_{cl} g$ which means the definition is not actually the proper one. Therefore, as the first step we replace I by the coarse interval I_R^{crs}. Recall that given a real number $0 \leq s \leq 1$, we defined the coarse map $i_s : R \to I_R^{crs}$ by the equation $i_s(t) := te^{\frac{\pi}{2}is}$. Inspired from the fact that the subset $\mathrm{Im}(i_s)$ is a copy of R, that is, a coarse point in I_R^{crs}, from now on, we will denote the subset $\mathrm{Im}(i_s)$ by $(s)^{crs}$ representing a copy of R in I_R^{crs}. The following is our understanding of deformation of coarse maps from a coarse point of view:

Definition 3.1.1. Let X and Y be coarse topological R-spaces and let $f, g : X \to Y$ be coarse maps. Then f and g are called *coarsely homotopic*, denoted by $f \simeq_{crs} g$, if there exist a coarse map $H : X \times_R I_R^{crs} \to Y$ such that $H \circ (1_X \times_R i_0) \circ i \sim_{cl} f$ and $H \circ (1_X \times_R i_1) \circ i \sim_{cl} g$, where the map $i : X \to X \times_R R$ is the map defined in Lemma 2.2.15. Such a map H is called a *coarse homotopy*. One often writes $H : f \simeq_{crs} g$ if one wishes to display a coarse homotopy. From now on, for convenience, we will denote the composite $(1_X \times_R i_s) \circ i$ by i_s^*. If A is a subset of X with $1_A \in \mathcal{E}_X$, then a coarse homotopy $H : X \times_R I_R^{crs} \to Y$ is said to be *relative to* A (or rel A) if there exist a coarse map $h : X \to Y$ and an entourage $E \in \mathcal{E}_Y$ such that

$$(H \circ i_s^*(a), h(a)) \in E,$$

for all $a \in A$ and all $0 \leq s \leq 1$.

If $f_s : X \to Y$ is defined by $f_s(x) = H \circ i_s^*(x)$, then a coarse homotopy H gives a one-parameter family of coarse maps deforming f into g from a coarse point of view. One thinks of f_s as describing the coarse deformation at time s.

Theorem 3.1.2. *Coarse homotopy is an equivalence relation on the set of all coarse maps $X \to Y$.*

Proof. Reflexivity and symmetry are trivial. We will show the transitivity. Assume that $H : f \simeq_{crs} g$ and $G : g \simeq_{crs} h$. Define $J : X \times_R I_R^{crs} \to Y$ by

$$J(x, p_X(x)e^{\frac{\pi}{2}is}) := \begin{cases} H(x, p_X(x)e^{\pi is}), & \text{if } 0 \leq s \leq \frac{1}{2} \\ G(x, p_X(x)e^{\frac{\pi}{2}i(2s-1)}), & \text{if } \frac{1}{2} < s \leq 1. \end{cases}$$

We must only show that J is a coarse map. We omit proofs of the symmetric cases. Let $D \in \mathcal{E}_{X \times_R I_R^{crs}}$. First, consider the subset D' of $(X \times_R I_R^{crs})^{\times 2}$ containing the pairs

$$\begin{gathered} ((x, p_X(x)e^{\frac{\pi}{4}i}),(x', p_X(x')e^{\frac{\pi}{4}i})), \\ ((x, p_X(x)e^{\frac{\pi}{2}is}),(x, p_X(x)e^{\frac{\pi}{4}i})), \text{ and} \\ ((x', p_X(x')e^{\frac{\pi}{4}i}),(x', p_X(x')e^{\frac{\pi}{2}is'})), \end{gathered}$$

where $((x, p_X(x)e^{\frac{\pi}{2}is}), (x', p_X(x')e^{\frac{\pi}{2}is'})) \in D$ with

$$(0 \leq s \leq \frac{1}{2}, \frac{1}{2} < s' \leq 1) \text{ or } (\frac{1}{2} < s \leq 1, 0 \leq s' \leq \frac{1}{2}).$$

The claim is that $D' \in \mathcal{E}_{X \times_R I_R^{crs}}$. To prove the claim the only part that is not clear is to show that $\pi_{I_R^{crs}}^{\times 2}(D') \in \mathcal{E}_{I_R^{crs}}$. For it, let $F := \pi_{I_R^{crs}}^{\times 2}(D)$. Taking $M := \pi_1^{\times 2}(F) \cup \pi_2^{\times 2}(F)$, one can see that

$$\pi_1^{\times 2}(\pi_{I_R^{crs}}^{\times 2}(D')),\ \pi_2^{\times 2}(\pi_{I_R^{crs}}^{\times 2}(D')) \subseteq M^{\boxtimes} \cup (M^{\boxtimes})^{-1},$$

that is, $\pi_1^{\times 2}(\pi_{I_R^{crs}}^{\times 2}(D')), \pi_2^{\times 2}(\pi_{I_R^{crs}}^{\times 2}(D')) \in \mathcal{E}_R$. The rest, that is, showing that $\pi_{I_R^{crs}}^{\times 2}(D')$ satisfies the Roe properness axiom and that $\pi_1, \pi_2 : I_R^{crs} \to R$ are locally proper for $\pi^{\times 2}_{I_R^{crs}}(D')$ are straightforward. Set

$$\begin{aligned} D'' &:= D \cup D' \cup (D')^{-1}, \\ E &:= \pi_X^{\times 2}(D'' \cap (X \times_R (\frac{1}{2})^{crs})^{\times 2}) \text{ and} \\ E' &:= E \cup [(E \cup E^{-1}) \circ (E \cup E^{-1})] \cup E^{-1}. \end{aligned}$$

From Lemma 2.1.6 and Lemma 2.2.15 follow that E, and therefore E', is an entourage in $\mathcal{E}_X$, hence the subsets $E_1 := ((H \circ i_1^*) \times g)(E')$ and $E_2 := (g \times (G \circ i_0^*))(E')$ are entourages in $\mathcal{E}_Y$. Taking $E_3 := D'' \cap (X \times_R (\bigcup_{0 \le s \le \frac{1}{2}} (s)^{crs}))^{\times 2}$, $E_4 := D'' \cap (X \times_R (\bigcup_{\frac{1}{2} \le s \le 1} (s)^{crs}))^{\times 2}$,

$$\begin{aligned} T &:= (H^{\times 2} \circ (1_X \times_R \alpha)^{\times 2})(E_3) \circ E_1 \circ E_2 \circ (G^{\times 2} \circ (1_X \times_R \beta)^{\times 2})(E_4), \text{ and} \\ S &:= E_1 \circ E_2 \circ (G^{\times 2} \circ (1_X \times_R \beta)^{\times 2})(E_4), \end{aligned}$$

one can easily see that

$$\begin{aligned} J^{\times 2}(D) \subseteq &(H^{\times 2} \circ (1_X \times_R \alpha)^{\times 2})(E_3) \\ &\cup (G^{\times 2} \circ (1_X \times_R \beta)^{\times 2})(E_4) \\ &\cup S \cup S^{-1} \\ &\cup T \cup T^{-1}, \end{aligned}$$

where the coarse maps $\alpha, \beta : I_R^{crs} \to I_R^{crs}$ are defined as follows:

$$\alpha(re^{\frac{\pi}{2} is}) := \begin{cases} re^{\pi is}, & \text{if } 0 \le s \le \frac{1}{2} \\ re^{\frac{\pi}{2} i}, & \text{if } \frac{1}{2} \le s \le 1, \text{ and} \end{cases}$$

$$\beta(re^{\frac{\pi}{2} is}) := \begin{cases} re^{0i}, & \text{if } 0 \le s \le \frac{1}{2} \\ re^{\frac{\pi}{2} i(2s-1)}, & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

That is, the map J preserves entourages. Note that the maps $1_X \times_R \alpha$ and $1_X \times_R \beta$ are coarse by Lemma 2.2.17, since the coarse maps α and β are compatible with the basepoint projections. Now assume that K is a relatively compact subset of Y. We have

$$J^{-1}(K) \cdot D \subseteq [H \circ (1_X \times_R \alpha)]^{-1}(K) \cdot D \cup [G \circ (1_X \times_R \beta)]^{-1}(K) \cdot D,$$

which implies that J is locally proper for D. We can do the same for the symmetric case, that is, the map J is coarse. □

Example 3.1.3. Let X and Y be coarse topological R-spaces and let $f, g : X \to Y$ be two coarse maps which are close. Then, $f \simeq_{crs} g$, by definition.

Definition 3.1.4. If $f : X \to Y$ is a coarse map, its *coarse homotopy class* is the equivalence class

$$[f]^{crs} := \{\text{coarse map } g : X \to Y \mid g \simeq_{crs} f\}.$$

The family of all such coarse homotopy classes is denoted by $[X;Y]^{crs}$.

Theorem 3.1.5. *Let $f_i : X \to Y$ and $g_i : Y \to Z$, $i = 0, 1$, be coarse maps. If $f_0 \simeq_{crs} f_1$ and $g_0 \simeq_{crs} g_1$, then $g_0 \circ f_0 \simeq_{crs} g_0 \circ f_1$. Moreover, if f_0 is compatible with the basepoint projections, then $g_0 \circ f_0 \simeq_{crs} g_1 \circ f_0$.*

Proof. Let $F : f_0 \simeq_{crs} f_1$ and $G : g_0 \simeq_{crs} g_1$ be coarse homotopies. Obviously,

$$K : g_0 \circ f_0 \simeq_{crs} g_0 \circ f_1,$$

where $K : X \times_R I_R^{crs} \to Z$ is the composite $g_0 \circ F$. For the second part, define $H : X \times_R I^{crs} \to Z$ by $H(x, p_X(x)e^{\frac{\pi}{2}is}) = G(f_0(x), p_Y(f_0(x))e^{\frac{\pi}{2}is})$. The map H is a coarse map, by Lemma 2.2.17. The rest is obvious. □

Definition 3.1.6. Let X and Y be coarse topological R-spaces and let $f : X \to Y$ be a coarse map.

- The map f is called a *coarse homotopy equivalence* if there is a coarse map $g : Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are coarsely homotopic to the identities 1_X and 1_Y, respectively. In the future, we will use the notation $X \simeq_{crs} Y$ to indicate that there exists a coarse homotopy equivalence between X and Y.
- The map f is *coarsely nullhomotopic* if it is coarsely homotopic to a map $c : X \to Y$ whose image is coarsely equivalent to the ray R.
- The coarse topological R-space X is *coarsely contractible* if 1_X is coarsely nullhomotopic.

Example 3.1.7. For each $n > 0$, the standard coarse cell, $D^n_{\mathbb{R}_+}$, is coarsely contractible.

Proof. Let V^n be the subset

$$\{x \in D^n_{\mathbb{R}_+} \mid \|x\| = 1\}$$

of $D^n_{\mathbb{R}_+}$ and let $\varphi : D^n \to V^n$ be a bi-Lipschitz homeomorphism which is identity on S^{n-1}. Let $F : D^n \times I \to D^n$ be a Lipschitz map provided a homotopy between 1_{D^n} and a constant map $c : D^n \to D^n$. Now, consider the map $\widetilde{F} : D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to D^n_{\mathbb{R}_+}$ defined by

$$\widetilde{F}(x, \|x\|e^{\frac{\pi}{2}is}) := \begin{cases} \|x\| \cdot \varphi(F(\psi(\tilde{x}), s)), & \text{if } \|x\| \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

where $\tilde{x}$ is the point where the ray emanating from the origin and passing through the point x meets V^n and where $\psi : V^n \to D^n$ is the inverse Lipschitz of φ. The Lipschitz condition on the maps guarantees that the map $\widetilde{F}$ is a coarse map (see [HR95] for example). On the other hand, it is obvious that $\widetilde{F}$ is the desired coarse homotopy. □

3.2 Coarse suspension

Suppose that X and Y are pointed coarse topological R-spaces such that at least one of them is a full coarse CW-complex. Since the subset

$$(X \times_R *_{crs}) \cup (*_{crs} \times_R Y)$$

of $X \times_R Y$ is with the subspace coarse structure a pointed coarse topological R-space, therefore, as we have seen in the preceding chapter, we can collapse it to the ray R. That is, we can define

Definition 3.2.1. Let X and Y be pointed coarse topological R-spaces such that at least one of them is a full coarse CW-complex. The *coarse smash product* of X and Y, denoted by $X \wedge Y$, is defined to be the pointed coarse quotient R-space

$$X \times_R Y/(X \times_R *_{crs}) \cup (*_{crs} \times_R Y),$$

obtained from $X \times_R Y$ by collapsing $(X \times_R *_{crs}) \cup (*_{crs} \times_R Y)$ to the ray R with the identified subset as its basepoint.

Theorem 3.2.2. *Let X, Y and Z be pointed proper coarse topological R-spaces. If X and Z are Hamband, then $(X \wedge Y) \wedge Z$ is coarsely equivalent to $X \wedge (Y \wedge Z)$.*

Proof. Write v for the various coarse identification maps of the form $X \times_R Y \to X \wedge Y$, and consider the diagram

$$\begin{array}{ccc}
X \times_R Y \times_R Z & \xrightarrow{1} & X \times_R Y \times_R Z \\
{\scriptstyle v \times_R 1}\downarrow & & \downarrow{\scriptstyle 1 \times_R v} \\
(X \wedge Y) \times_R Z & & X \times_R (Y \wedge Z) \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v} \\
(X \wedge Y) \wedge Z & & X \wedge (Y \wedge Z).
\end{array}$$

Now $v \times_R 1$ is a coarse identification by Lemma 2.3.3, since Z is Hamband. Similarly, $1 \times_R v$ is a coarse identification. Therefore the maps $v \circ (1 \times_R v) \circ 1$ and $v \circ (v \times_R 1) \circ 1$ are coarse identifications. Now, the result follows from Theorem 2.2.12 by considering appropriate maps. □

Now, denoting the subset $(0)^{crs} \cup (1)^{crs}$ of I_R^{crs} by $\dot{I}_R^{crs}$, we define

Definition 3.2.3. If $(X, *_{crs})$ is a pointed coarse topological R-space, then the *reduced coarse suspension* of X, denoted by $\Sigma^{crs}X$, is the coarse quotient R-space

$$\Sigma^{crs}X = (X \times_R I_R^{crs})/((X \times_R \dot{I}_R^{crs}) \cup (*_{crs} \times_R I_R^{crs})),$$

where the identified subset is regarded as its basepoint.

We define the *unreduced coarse suspension* (or *coarse double cone*) of a pointed coarse topological R-space X, denoted by $S^{crs}X$, as the coarse quotient space of $X \times_R I_R^{crs}$ obtained by collapsing $X \times_R (0)^{crs}$ to a ray and $X \times_R (1)^{crs}$ to another ray R. The space X then can be imbedded in $S^{crs}X$ as $X \times_R (\frac{1}{2})^{crs}$.

Theorem 3.2.4. *Let R be a generalised ray and assume that S_R^1 has been given a basepoint as in Example 2.2.2. Let X be a pointed proper coarse topological R-space. If X is Hamband, then*

$$\Sigma^{crs}X \cong_{crs} X \wedge S_R^1.$$

Proof. Since X is Hamband, the map $1_X \times_R \exp : X \times_R I_R^{crs} \to X \times_R S_R^1$ is a coarse identification, by Lemma 2.3.3. If $v : X \times_R S_R^1 \to X \wedge S_R^1$ is the natural map, then $h := v \circ (id_X \times_R \exp)$ is also a coarse identification, by Lemma 2.2.11. But it is easy to check that $(X \times_R I_R^{crs})/\ker h = \Sigma^{crs}X$, and so the result follows from Theorem 2.2.12. □

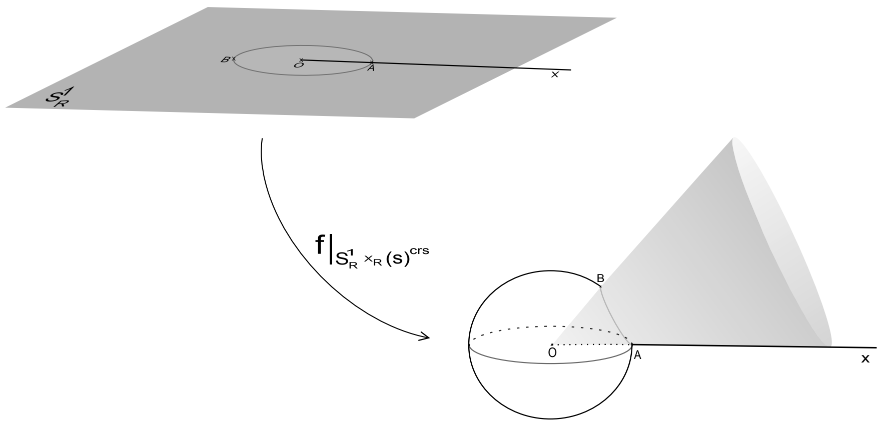
Theorem 3.2.5. $\Sigma^{crs}S_R^n \cong_{crs} S_R^{n+1}$ *for all* $n \geq 0$.

Proof. By the definition of a generalised ray, one can consider the disjoint union $R \amalg R$ as the topological space $\mathbb{R}$ equipped with some coarse structure compatible with the topology coming from the metric such that its restriction on both of the subspaces $[0, \infty)$ and $(-\infty, 0]$ is just the coarse structure on R. Therefore, assuming $e_1, \cdots, e_{n+1}$ to be the standard base of $\mathbb{R}^{n+1}$ and te_1, $t \in [0, \infty)$, the basepoint of S_R^n, we allow ourselves to define the map $f : S_R^n \times_R I_R^{crs} \to S_R^{n+1}$ by the equation

$$(x, \|x\|e^{\frac{\pi}{2}is}) \longmapsto$$

$$\|x\| \cdot \left\{ \frac{1}{2}(e_1 + \frac{x}{\|x\|}) + (\cos 2\pi s)\frac{1}{2}(e_1 - \frac{x}{\|x\|}) + (\sin 2\pi s)\frac{1}{2}\left\| e_1 - \frac{x}{\|x\|} \right\| e_{n+2} \right\}.$$

The map f is indeed a modification of the map is used in the classical situation. To get a clearer picture, one can look at the following figure showing the way that f is defined on $S_R^1 \times_R (s)^{crs}$ (which is, of course, a copy of S_R^1) for some $0 \leq s \leq 1$.



As one can see from the above the map f is a coarse identification which satisfies the conditions of Example 2.2.7. On the other hand, it is easy to check that $S^n_R \times_R I^{crs}_R/\ker f = \Sigma^{crs} S^n_R$. Hence, $\Sigma^{crs} S^n_R \cong_{crs} S^{n+1}_R$, by Theorem 2.2.12. $\square$

Corollary 3.2.6. *$S^m_{\mathbb{R}_+} \wedge S^n_{\mathbb{R}_+} \cong_{crs} S^{m+n}_{\mathbb{R}_+}$ for all $m, n \geq 1$.*

Proof. We prove by induction on n. For $n = 1$, the statement directly follows from Theorem 3.2.4 and Theorem 3.2.5. Now assume that the above statement holds for every $k \leq n - 1$, we shall prove it for $k = n$.

$$\begin{aligned} S^m_{\mathbb{R}_+} \wedge S^n_{\mathbb{R}_+} &\cong_{crs} S^m_{\mathbb{R}_+} \wedge (\Sigma^{crs} S^{n-1}_{\mathbb{R}_+}) \\ &\cong_{crs} S^m_{\mathbb{R}_+} \wedge (S^{n-1}_{\mathbb{R}_+} \wedge S^1_{\mathbb{R}_+}) \\ &\cong_{crs} (S^m_{\mathbb{R}_+} \wedge S^{n-1}_{\mathbb{R}_+}) \wedge S^1_{\mathbb{R}_+} \\ &\cong_{crs} S^{m+n-1}_{\mathbb{R}_+} \wedge S^1_{\mathbb{R}_+} \\ &\cong_{crs} \Sigma^{crs} S^{m+n-1}_{\mathbb{R}_+} \\ &\cong_{crs} S^{m+n}_{\mathbb{R}_+}, \end{aligned}$$

by repeated applications of Theorem 3.2.2, 3.2.4, 3.2.5, and the induction assumption. Note that we could apply Theorem 3.2.2 here since for each $n \geq 0$, $S^n_{\mathbb{R}_+} = \mathbb{R}^{n+1}$ is a full coarse CW-complex. $\square$

As in the classical case, to be able to go further, we need some notion of a pointed coarse map, that is, coarse maps which have some basepoint preserving property. The following is our understanding of a pointed coarse map.

Definition 3.2.7. Let X and Y be pointed coarse topological R-spaces and let $f : X \to Y$ be a coarse map. As we have seen in the preceding chapter the basepoints of X and Y are represented by some fixed coarse maps $i_X : R \to X$ and $i_Y : R \to Y$, respectively, with some splitting properties. We call the map f *pointed* if $f \circ i_X \sim_{cl} i_Y$. We sometimes write $f : (X, *_{crs}) \to (Y, *_{crs})$ to indicate that f is pointed.

Now, we are ready to introduce some important constructions.

Definition 3.2.8. Let X and Y be pointed coarse topological R-spaces and let $f : X \to Y$ be a pointed coarse map. The *coarse reduced cone* on X, denoted by $c^{crs}X$, is defined to be the smash product $X \wedge I_R^{crs}$ (recall that the basepoint of I_R^{crs} is always taken to be $(1)^{crs}$). We denote the class of $(x, p_X(x)e^{\frac{\pi}{2}is})$ in $c^{crs}X$ by $x \wedge p_X(x)e^{\frac{\pi}{2}is}$. Note that X can be identified with the subspace $X \wedge (0)^{crs}$. Moreover, if the inverse image of bounded subsets under f are bounded (for example, this is the case if X and Y are pointed proper coarse topological R-spaces with X unital), and f is strongly compatible with the basepoint projections, then we define the *coarse mapping cone* C_f^{crs} to be the pointed coarse topological R-space obtained from Y by weakly coarse attaching $c^{crs}X$ via f as defined in Definition 2.2.6. The basepoint of C_f^{crs} can be taken to be the coarse point $\{[y]|\ y \in \mathrm{Im}(i_Y)\}$.

Write f' for the "inclusion map" of Y in C_f^{crs}; more precisely, f' is the inclusion of Y in the coarse coproduct of Y and $c^{crs}X$, composed with the natural map onto C_f^{crs}.

Theorem 3.2.9. *Let X and Y be pointed proper coarse topological $\mathbb{R}_+$-spaces with X unital and let $f : X \to Y$ be a pointed coarse map which is strongly compatible with the basepoint projections. Then*

$$C_{f'}^{crs} \simeq_{crs} \Sigma^{crs}X.$$

Proof. Define $\Phi : \Sigma^{crs}X \to C_{f'}^{crs}$ by

$$[x \times_R p_X(x)e^{\frac{\pi}{2}is}] \longmapsto \begin{cases} f(x) \wedge p_Y(f(x))e^{\frac{\pi}{2}i(1-2s)} \text{ in } c^{crs}Y, & \text{if } 0 \le s \le \frac{1}{2} \\ x \wedge p_X(x)e^{\frac{\pi}{2}i(2s-1)} \text{ in } c^{crs}X, & \text{if } \frac{1}{2} < s \le 1, \end{cases}$$

followed by the natural map. This is well-defined, because $p_Y(f(x)) = p_X(x)$ for every $x \in X$ and since $f(x) \wedge (1)^{crs}$ and $x \wedge (1)^{crs}$ both represent the same point at the subset $[f(*_{crs})]$ of $C_{f'}^{crs}$. Moreover, Φ is a coarse map (for the top row follows from Lemma 2.2.17). Now, define $\psi : c^{crs}X \amalg c^{crs}Y \to \Sigma^{crs}X$ by

$$\psi(x \wedge p_X(x)e^{\frac{\pi}{2}is}) = [x \times_R p_X(x)e^{\frac{\pi}{2}is}] \in \Sigma^{crs}X, \tag{3.2.1}$$

for points of $c^{crs}X$ and for points of $c^{crs}Y$ by

$$\psi(y \wedge p_Y(y)e^{\frac{\pi}{2}it}) = [x \times_R p_X(x)e^{\frac{\pi}{2}it}], \tag{3.2.2}$$

where x is a point of $*_{crs}$ with $p_X(x) = p_Y(y)$. Now, let $\Psi : C^{crs}_{f'} \to \Sigma^{crs}X$ be the map induced by ψ. Note that the map ψ does induce Ψ, since $\psi(x \wedge (0)^{crs}) = x \wedge (0)^{crs} = \psi(f(x) \wedge (0)^{crs})$.

It remains to prove that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are coarsely homotopic to the respective identity maps. Now, $\Psi \circ \Phi : \Sigma^{crs}X \to \Sigma^{crs}X$ is given by

$$(\Psi \circ \Phi)([x \times_R p_X(x)e^{\frac{\pi}{2}is}]) := \begin{cases} [x' \times_R p_Y(f(x))e^{\frac{\pi}{2}i0}], & \text{if } 0 \le s \le \frac{1}{2} \\ [x \times_R p_X(x)e^{\frac{\pi}{2}i(2s-1)}], & \text{if } \frac{1}{2} < s \le 1, \end{cases}$$

where x' is a point of $*_{crs}$ with $p_X(x) = p_X(x')$. And, this is obviously coarsely homotopic to $1_{\Sigma^{crs}X}$.

On the other hand, $\Phi \circ \Psi$ is given by the map induced by

$$x \wedge p_X(x)e^{\frac{\pi}{2}is} \longmapsto \begin{cases} f(x) \wedge p_Y(f(x))e^{\frac{\pi}{2}i(1-2s)}, & \text{if } 0 \le s \le \frac{1}{2} \\ x \wedge p_X(x)e^{\frac{\pi}{2}i(2s-1)}, & \text{if } \frac{1}{2} < s \le 1, \end{cases}$$

for points of $c^{crs}X$ and for points of $c^{crs}Y$ by

$$y \wedge p_Y(y)e^{\frac{\pi}{2}it} \longmapsto [z],$$

where z is a point of the subset $f(*_{crs})$ with $p_Y(y) = p_Y(z)$. To construct a coarse homotopy $F : C^{crs}_{f'} \times_R I^{crs}_{\mathbb{R}_+} \to C^{crs}_{f'}$, between $\Phi \circ \Psi$ and $1_{C^{crs}_{f'}}$, define $F_X : c^{crs}X \times_R I^{crs}_{\mathbb{R}_+} \to C^{crs}_{f'}$ by

$$F_X(x \wedge p_X(x)e^{\frac{\pi}{2}is}, p_X(x)e^{\frac{\pi}{2}it}) = f(x) \wedge p_Y(f(x))e^{\frac{\pi}{2}i(1-2s-t(1-s))},$$

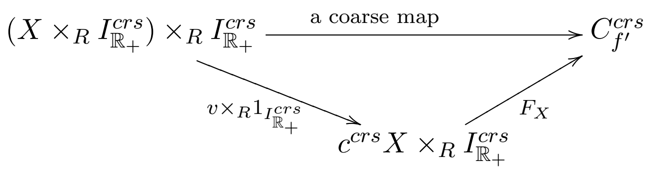
if $0 \le s \le (1-t)/(2-t)$ and by

$$F_X(x \wedge p_X(x)e^{\frac{\pi}{2}is}, p_X(x)e^{\frac{\pi}{2}it}) = x \wedge p_X(x)e^{\frac{\pi}{2}i(2s-1+t(1-s))},$$

if $(1-t)/(2-t) < s \le 1$. Next, we define $F_Y : c^{crs}Y \times_R I^{crs}_{\mathbb{R}_+} \to C^{crs}_{f'}$ by

$$F_Y(y \wedge p_Y(y)e^{\frac{\pi}{2}is}, p_Y(y)e^{\frac{\pi}{2}it}) = y \wedge p_Y(y)e^{\frac{\pi}{2}i(1-t(1-s))}, \ 0 \le s \le 1.$$

Now, to see that F_X is a coarse map, one can first notice that it is induced by a coarse map of $(X \times_R I^{crs}_{\mathbb{R}_+}) \times_R I^{crs}_{\mathbb{R}_+} \to C^{crs}_{f'}$, i.e.,



But, $v \times_R 1_{I^{crs}_{\mathbb{R}_+}}$ is a coarse identification, since $I^{crs}_{\mathbb{R}_+}$ is Hamband. So, Theorem 2.2.10 implies that F_X is a coarse map. Similarly, F_Y is a coarse map; and since

$$\begin{aligned} F_X(x \wedge (0)^{crs}, p_X(x)e^{\frac{\pi}{2}it}) &= f(x) \wedge p_Y(f(x))e^{\frac{\pi}{2}i(1-s)} \\ &= F_Y(f(x) \wedge (0)^{crs}, p_Y(f(x))e^{\frac{\pi}{2}it}), \end{aligned}$$

F_X and F_Y together induce a coarse homotopy $F : C_{f'}^{crs} \times_R I_{\mathbb{R}_+}^{crs} \to C_{f'}^{crs}$, which is a coarse map by an argument similar to that used for F_X and F_Y. Moreover, for $s = 1$,

$$F_X(x \wedge (1)^{crs}, p_X(x)e^{\frac{\pi}{2}it}) = [z],$$

where z is a point of the subset $f(*_{crs})$ with $p_X(x) = p_Y(z)$ and

$$F_Y(y \wedge (1)^{crs}, p_Y(y)e^{\frac{\pi}{2}it}) = [z'],$$

where z' is a point of the subset $f(*_{crs})$ with $p_Y(y) = p_Y(z')$. So, F is a pointed coarse homotopy; and clearly F is a coarse homotopy between $\Phi \circ \Psi$ and $1_{C_{f'}^{crs}}$. Hence Φ and Ψ are coarse homotopy equivalences. □

3.3 Coarse homotopy groups

In this section, we introduce coarse homotopy groups. As in the classical algebraic topology, the central idea is to associate an algebraic situation to a coarse situation, and to study the simpler resulting algebraic setup.

Let X and Y be pointed coarse topological R-spaces. Two coarse homotopy of X into Y can be "concatenated" if the first ends where the second begins, of course, in a coarse sense. More precisely,

Definition 3.3.1. Let X and Y be pointed coarse topological R-spaces. If $F : X \times_R I_R^{crs} \to Y$ and $G : X \times_R I_R^{crs} \to Y$ are two coarse homotopies such that $F \circ i_1^* \sim_{cl} G \circ i_0^*$, then define a coarse homotopy $F * G : X \times_R I_R^{crs} \to Y$, which is called the *coarse concatenation of* F and G, by

$$(F * G)(x, p_X(x)e^{\frac{\pi}{2}is}) := \begin{cases} F(x, p_X(x)e^{\pi is}), & 0 \le s \le \frac{1}{2} \\ G(x, p_X(x)e^{\frac{\pi}{2}i(2s-1)}), & \frac{1}{2} < s \le 1. \end{cases}$$

Showing that $F * G$ is actually a coarse map is based on the same argument we have seen in the proof of Theorem 3.1.2, therefore, we avoid repeating it here.

One does not have to combine these coarse homotopies at $s = \frac{1}{2}$. We can do it at any $(s)^{crs}$, namely,

Lemma 3.3.2. *Let ϕ_1 and ϕ_2 be Lipschitz maps $(I, \partial I) \to (I, \partial I)$ which are equal on ∂I. Let $F : X \times_R I_R^{crs} \to Y$ be a coarse homotopy and let $G_k(x, p_X(x)e^{\frac{\pi}{2}is}) = F(x, p_X(x)e^{\frac{\pi}{2}i\phi_k(s)})$ for $k = 1, 2$. Then $G_1 \simeq_{crs} G_2$ rel $X \times_R (s)^{crs}$ for $s = 0, 1$.*

Proof. Note that the Lipschitz condition has been made to ensure that the maps defined by $re^{\frac{\pi}{2}is} \mapsto re^{\frac{\pi}{2}i\phi_k(s)}$ are coarse (see Example 3.1.7). It is easy to check that the coarse map $H : X \times_R I_R^{crs} \times_R I_R^{crs} \to Y$ defined by

$$H(x, p_X(x)e^{\frac{\pi}{2}it}, p_X(x)e^{\frac{\pi}{2}is}) := F(x, p_X(x)e^{\frac{\pi}{2}i(s\phi_2(t)+(1-s)\phi_1(t))})$$

is the required coarse homotopy. □

We shall use C to denote a constant coarse homotopy, whichever one makes sense in the current context. For example $F * C$ is coarse concatenation with the constant coarse homotopy C for which $C(x, p_X(x)e^{\frac{\pi}{2}is}) = F(x, p_X(x)e^{\frac{\pi}{2}i1})$, but use of $C*F$ will imply the one for which $C(x, p_X(x)e^{\frac{\pi}{2}is}) = F(x, p_X(x)e^{\frac{\pi}{2}i0})$. As in the classical case, we have

Proposition 3.3.3. *Let X and Y be pointed coarse topological R-spaces and let $F_k, G_k : X \times_R I_R^{crs} \to Y$ coarse homotopies for $k = 1, 2$.*

(i) We have $F_1 * C \simeq_{crs} F_1$ rel $X \times_R (s)^{crs}$ for $s = 0, 1$, and, similarly, $C * F_1 \simeq_{crs} F_1$ rel $X \times_R (s)^{crs}$ for $s = 0, 1$.

(ii) Defining $F_1^{-1} : X \times_R I_R^{crs} \to Y$ by

$$F_1^{-1}(x, p_X(x)e^{\frac{\pi}{2}is}) := F_1(x, p_X(x)e^{\frac{\pi}{2}i(1-s)}),$$

we have $F_1 * F_1^{-1} \simeq_{crs} C$ rel $X \times_R (s)^{crs}$ for $s = 0, 1$.

(iii) If the coarse concatenations $F_1 * F_2$ and $F_2 * G_1$ are defined, we have $(F_1 * F_2) * G_1 \simeq_{crs} F_1 * (F_2 * G_1)$ rel $X \times_R (s)^{crs}$ for $s = 0, 1$.

(iv) If $F_1 \simeq_{crs} F_2$ and $G_1 \simeq_{crs} G_2$ rel $X \times_R (s)^{crs}$ for $s = 0, 1$, then

$$F_1 * G_1 \simeq_{crs} F_2 * G_2 \text{ rel } X \times_R (s)^{crs}$$

for $s = 0, 1$.

Proof. They are easy applications of Lemma 3.3.2. □

Let us denote the set of coarse homotopy classes of pointed coarse maps of a pointed coarse topological R-space X to a pointed coarse topological R-space Y, with coarse homotopies preserving the base points (by that, we mean coarse homotopies rel $*_{crs}$), by $[X; Y]_*^{crs}$ (we use this notation for stress here. In the future we will drop the asterisk suffix, depending on the context to make clear what is intended). For the moment let us set $A := X \times_R \dot{I}_R^{crs} \cup *_{crs} \times_R I_R^{crs}$.

If $f, g : \Sigma^{crs} X \to Y$ are pointed coarse maps, then they induce coarse homotopies $f', g' : X \times_R I_R^{crs} \to Y$ by means of composition with the natural map $X \times_R I_R^{crs} \to \Sigma^{crs} X$. Then $f' * g' : X \times_R I_R^{crs} \to Y$ is defined. But this map do not still induce a map $\Sigma^{crs} X \to Y$, because of the way that it has been defined on $s=1$ and on $*_{crs} \times_R (s)^{crs}$ for every $\frac{1}{2} < s \leq 1$. But since the coarse maps f and g are pointed, this can be easily fixed by defining $f' * g'$ at $s=1$ as follows:

$$(f' * g')(x, p_X(x)e^{\frac{\pi}{2}is}) := \begin{cases} f'(x, p_X(x)e^{\pi is}), & 0 \leq s \leq \frac{1}{2} \\ g'(x, p_X(x)e^{\frac{\pi}{2}i(2s-1)}), & \frac{1}{2} < s < 1 \text{ and } x \notin *_{crs} \\ f'(x, p_X(x)e^{\frac{\pi}{2}is}), & s = 1 \text{ or } x \in *_{crs}. \end{cases}$$

The resulting pointed coarse map $\Sigma^{crs}X \to Y$ will be denoted by $f * g$ with little danger of confusion.

For any coarse map $f : (\Sigma^{crs}X, A) \to (Y, *)$, we denote its homotopy class in $[\Sigma^{crs}X; Y]_*^{crs}$ by $[f]^{crs}$. For two such maps f and g we define

$$[f]^{crs} \cdot [g]^{crs} = [f * g]^{crs} .$$

Of course, we must check that $[f_1]^{crs} = [f_2]^{crs}$ and $[g_1]^{crs} = [g_2]^{crs}$ imply that $[f_1 * g_1]^{crs} = [f_2 * g_2]^{crs}$, but this follow from Proposition 3.3.3 (iv).

Given a pointed coarse topological R-space X, we can define the coarse quotient R-space X/X. And, therefore, in a canonical way, we get a map $i_* : R \to X/X$. We define

Definition 3.3.4. Let X and Y be pointed coarse topological R-spaces. A map $c : X/X \to Y$ is called *coarse pre-constant* if $c \circ i_* \sim_{cl} i_Y$. Then, we call the map $X \to Y$ induced by c by means of composition with the natural map $X \to X/X$, the *coarse constant map* to the basepoint of Y and we will denote it by c_*.

Let $c_* : \Sigma^{crs}X \to Y$ be the constant coarse map to the basepoint of Y. Then, from the laws of coarse homotopies developed in Proposition 3.3.3, we easily see that:

(associativity)	$[f]^{crs} \cdot ([g]^{crs} \cdot [h]^{crs}) = ([f]^{crs} \cdot [g]^{crs}) \cdot [h]^{crs}$
(unity element)	$[c_*]^{crs} \cdot [f]^{crs} = [f]^{crs} = [f]^{crs} \cdot [c_*]^{crs}$
(inverse)	$[f]^{crs} \cdot \left[f^{-1}\right]^{crs} = [c_*]^{crs}$.

(Recall that f^{-1} stands here for the "inverse" coarse homotopy with time running the opposite way to that in f, and not to an inverse function.)

Thus, under this operation, the set $[\Sigma^{crs}X; Y]_*^{crs}$ of pointed coarse homotopy classes of pointed coarse maps $\Sigma^{crs}X \to Y$, becomes a group.

The most important special case of the foregoing is that of coarse suspensions of coarse spheres. Let S_R^0 denote the coarse 0-sphere, $R \amalg R$, having the second copy of the ray R as its basepoint. Because of Theorem 3.2.5, we can for the purposes of this section, define the pointed coarse n-sphere, S_R^n, to be the n-fold reduced coarse suspension of S_R^0. Thus, as a special case of the foregoing discussion, we can define

Definition 3.3.5. Let $(X, *_{crs})$ be a pointed coarse topological R-space. For every $n \geq 0$, we define

$$\pi_n^{crs}(X, *_{crs}) := [S_R^n; X]_*^{crs}.$$

We shall usually abbreviate $\pi_n^{crs}(X, *_{crs})$ to $\pi_n^{crs}(X)$. By the above discussion, when $n \geq 1$, $\pi_n^{crs}(X)$ is a group and is called the *nth coarse homotopy group* of X.

As in the classical case, the general problem of computing coarse homotopy groups is very difficult. The most important cases are the groups $\pi_k^{crs}(S_{\mathbb{R}_+}^n)$ which will be our task in the next chapter.

Given a pointed coarse map $h : X \to Y$ between two pointed coarse topological R-spaces X and Y, one can easily see that h induces homomorphisms $h_* : \pi_n^{crs}(X, *_{crs}) \to \pi_n^{crs}(Y, *_{crs})$ defined by $[f]^{crs} \mapsto [h \circ f]^{crs}$. Moreover, if $X \cong_{crs} Y$, then, obviously, $\pi_n^{crs}(X, *_{crs}) \cong \pi_n^{crs}(Y, *_{crs})$. Sadly, if there exists a coarse homotopy equivalence between two pointed coarse topological R-spaces X and Y, we cannot conclude that $\pi_n^{crs}(X, *_{crs}) \cong \pi_n^{crs}(Y, *_{crs})$. To see this, let $f : X \to Y$ be a coarse equivalence between X and Y with the inverse $g : Y \to X$. To show that $f_* : \pi_n^{crs}(X, *_{crs}) \to \pi_n^{crs}(Y, *_{crs})$ is an isomorphism, at some step we need to conclude from $g \circ f \simeq_{crs} 1_X$ that $(g \circ f) \circ \gamma \simeq_{crs} \gamma$, where γ is a representative of an element of $\pi_n^{crs}(X, *_{crs})$. But, from Theorem 3.1.5 we know that in general this is not the case unless the representative γ is compatible with the basepoint projections. Therefore, if for pointed coarse topological R-spaces X and Y, each element of $\pi_n^{crs}(X, *_{crs})$ and $\pi_n^{crs}(X, *_{crs})$ can be represented by a pointed coarse map which is compatible with the basepoint projections, then the existence of a coarse homotopy equivalence between X and Y guarantees that $\pi_n^{crs}(X, *_{crs}) \cong \pi_n^{crs}(Y, *_{crs})$.

Definition 3.3.6. A *pointed coarse pair* (X, A) consists of a pointed coarse topological R-space X together with a closed subset A of X equipped with the subspace coarse structure such that $i_X(R) \subseteq A$, where $i_X : R \to X$ is the basepoint inclusion of X. Given pointed coarse pairs (X, A) and (Y, B), a *map of coarse pairs* $f : (X, A) \to (Y, B)$ is a coarse map $f : X \to Y$ such that $f(A) \subseteq B$. Moreover, we call a map of coarse pairs $f : (X, A) \to (Y, B)$ *pointed* if $f : X \to Y$ is pointed.

All we have done goes over immediately to the case of pointed coarse pairs (X, A), namely

Definition 3.3.7. If (X, A) and (Y, B) are pointed coarse pairs, then

$$[(X, A, *_{crs}); (Y, B, *_{crs})]_*^{crs}$$

is the set of all pointed coarse homotopy classes of pointed maps of coarse pairs $\beta : (X, A, *_{crs}) \to (Y, B, *_{crs})$. We often suppress basepoints and parentheses and write $[X, A; Y, B]_*^{crs}$. In particular, we define the *relative coarse homotopy group* of a pointed coarse pair (X, A) to be

$$\pi_n^{crs}(X, A, *_{crs}) := \left[D_R^n, S_R^{n-1}; X, A\right]_*^{crs} = \left[\Sigma^{crs}(D_R^{n-1}, S_R^{n-2}); X, A)\right]_*^{crs},$$

(we usually abbreviate $\pi_n^{crs}(X, A, *_{crs})$ to $\pi_n^{crs}(X, A)$). This is a group for $n \geq 2$.

3.4 The coarse homotopy sequence of a coarse pair

In this section we develop an exact sequence of coarse homotopy groups analogous to the exact coarse homology sequence of a coarse pair. It is, of course, an indispensable tool in the study of coarse homotopy groups. Everything in this section is in the pointed setup.

Definition 3.4.1. Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence of pointed coarse topological R-spaces (or pointed coarse pairs) such that the pointed coarse maps are compatible with basepoint projections. It is called *coexact* if, for each pointed coarse topological R-space (or coarse pair) Y, the sequence of sets (pointed coarse homotopy classes)

$$[C; Y]_*^{crs} \xrightarrow{g^\sharp} [B; Y]_*^{crs} \xrightarrow{f^\sharp} [A; Y]_*^{crs}$$

is exact, i.e., $\mathrm{Im}(g^\sharp) = (f^\sharp)^{-1}(c_*)$, where c_* is the coarse constant map to the basepoint of Y.

Theorem 3.4.2. *Let A and X be pointed proper coarse topological R-spaces with A unital. For any pointed coarse map $f : A \to X$ strongly compatible with the basepoint projections, the sequence*

$$A \xrightarrow{f} X \xrightarrow{f'} C_f^{crs}$$

is coexact, where $f' : X \longrightarrow C_f^{crs}$ is the inclusion map of X in C_f^{crs}.

Proof. Clearly, $f' \circ f \simeq_{crs} c_*$, the coarse constant map to the basepoint of C_f^{crs}, so $\mathrm{Im}(g^\sharp) \subseteq (f^\sharp)^{-1}(c_*)$. Suppose given a pointed coarse map $\phi : X \to Y$ with $\phi \circ f \simeq_{crs} c_*$ via the pointed coarse homotopy $F : A \times_R I_R^{crs} \to Y$, i.e., $F \circ i_0^* \sim_{cl} \phi \circ f$ and $F \circ i_1^* \sim_{cl} c_*$. Then F on $A \times_R (I_R^{crs} \setminus (0)^{crs})$ and ϕ on X fit together to give a map $C_f^{crs} \to Y$ extending ϕ. □

Note that since the inclusion maps are strongly compatible with the basepoint projections and because the inverse image of bounded subsets under the inclusion maps are bounded, therefore the construction of C_f^{crs} can be iterated, that is we obtain a long sequence of proper coarse topological R-spaces and coarse maps

$$A \xrightarrow{f} X \xrightarrow{f'} C_f^{crs} \xrightarrow{f''} C_{f'}^{crs} \xrightarrow{f^{(3)}} C_{f''}^{crs} \longrightarrow \cdots .$$

Therefore, by above theorem, we have

Corollary 3.4.3. *Let A and X be pointed proper coarse topological R-spaces with A unital. For any pointed coarse map $f : A \to X$ strongly compatible with the basepoint projections, the sequence*

$$A \xrightarrow{f} X \xrightarrow{f'} C_f^{crs} \xrightarrow{f''} C_{f'}^{crs} \xrightarrow{f^{(3)}} C_{f''}^{crs} \longrightarrow \cdots$$

is coexact.

In addition, if all spaces are pointed proper coarse topological $\mathbb{R}_+$-spaces, then from Theorem 3.2.9 follows that $C_{f''}^{crs} \simeq_{crs} \Sigma^{crs} X$, $C_{f^{(3)}}^{crs} \simeq_{crs} \Sigma^{crs} C_f^{crs}$, and so on; in fact each proper coarse topological $\mathbb{R}_+$-space in the sequence

$$A \xrightarrow{f} X \xrightarrow{f'} C_f^{crs} \xrightarrow{f''} C_{f'}^{crs} \xrightarrow{f^{(3)}} C_{f''}^{crs} \longrightarrow \cdots$$

can be identified, up to coarse homotopy equivalence, with an iterated coarse suspension of A, X or C_f^{crs}.

Proposition 3.4.4. *Let A and X be pointed proper coarse topological $\mathbb{R}_+$-spaces with A unital. For any pointed coarse map $f : A \to X$ strongly compatible with the basepoint projections, the diagram*

$$\begin{array}{ccc} C_{f'}^{crs} & \xrightarrow{\quad f^{(3)} \quad} & C_{f''}^{crs} \\ \Phi \uparrow & & \downarrow \Psi \\ \Sigma^{crs} A & \xrightarrow[\quad [f \times_R \upsilon] \quad]{} & \Sigma^{crs} X \end{array}$$

is coarse homotopy commutative, where Φ and Ψ are coarse homotopy equivalences defined as in Theorem 3.2.9, and where $\upsilon : I_{\mathbb{R}_+}^{crs} \to I_{\mathbb{R}_+}^{crs}$ is defined by $\upsilon(re^{\frac{\pi}{2}is}) = re^{\frac{\pi}{2}i(1-s)}$ for $0 \le s \le 1$.

Proof. Looking at the way that Ψ has been defined in Theorem 3.2.9 shows that $\Psi \circ f^{(3)}$ maps points of C_f^{crs} to the basepoint of $\Sigma^{crs} X$ as in Equation (3.2.2), and points of $c^{csr} X$ to $\Sigma^{crs} X$ by the rule $x \wedge p_X(x) e^{\frac{\pi}{2}is} \mapsto [x \times_R p_X(x) e^{\frac{\pi}{2}is}]$ as in Equation (3.2.1). Thus

$$(\Psi \circ f^{(3)} \circ \Phi)([a \times_R p_A(a) e^{\frac{\pi}{2}is}]) = \begin{cases} [f(a) \times_R p_X(f(a)) e^{\frac{\pi}{2}i(1-2s)}], & \text{if } 0 \le s \le \frac{1}{2} \\ [x \times_R p_X(x) e^{\frac{\pi}{2}i1}], & \text{if } \frac{1}{2} < s \le 1, \end{cases}$$

where x is a point of $*_{crs}$ with $p_X(x) = p_A(a)$. Therefore, $\Psi \circ f^{(3)} \circ \Phi = [f \times_R \bar{\upsilon}]$, where $\bar{\upsilon} : I^{crs}_{\mathbb{R}_+} \to I^{crs}_{\mathbb{R}_+}$ is defined by

$$\bar{\upsilon}(re^{\frac{\pi}{2}is}) = \begin{cases} re^{\frac{\pi}{2}i(1-2s)}, & \text{if } 0 \leq s \leq \frac{1}{2} \\ re^{\frac{\pi}{2}i0}, & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

But $\upsilon \simeq_{crs} \bar{\upsilon}$ by an obvious coarse homotopy, so that $[f \times_R \upsilon] \simeq_{crs} [f \times_R \bar{\upsilon}]$. □

Theorem 3.4.5. *Let A and X be pointed proper coarse topological $\mathbb{R}_+$-spaces with A unital. For any pointed coarse map $f : A \to X$ strongly compatible with the basepoint projections, the sequence*

$$A \xrightarrow{f} X \xrightarrow{f_1} C^{crs}_f \xrightarrow{f_2} \Sigma^{crs} A \xrightarrow{[f \times_R 1]} \Sigma^{crs} X \xrightarrow{[f_1 \times_R 1]} \Sigma^{crs} C^{crs}_f \longrightarrow$$

is coexact, where $f_1 := f'$ and $f_2 := \Psi \circ f''$. Similarly, for coarse maps of coarse pairs of pointed proper coarse topological $\mathbb{R}_+$-spaces.

Proof. Let Y be a pointed coarse topological $\mathbb{R}_+$-space. Consider the diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & [\Sigma^{crs} X; Y]^{crs}_* & \xrightarrow{([f \times_R \upsilon])^\sharp} & [\Sigma^{crs} A; Y]^{crs}_* & \xrightarrow{(f_2)^\sharp} & [C^{crs}_f; Y]^{crs}_* \\ & \downarrow{\scriptstyle \Psi^\sharp} & & \downarrow{\scriptstyle \Psi^\sharp} & \nearrow{\scriptstyle (f'')^\sharp} & \\ \cdots \longrightarrow & [C^{crs}_{f''}; Y]^{crs}_* & \xrightarrow[(f^{(3)})^\sharp]{} & [C^{crs}_{f'}; Y]^{crs}_* & & \end{array}$$

$$\xrightarrow{(f_1)^\sharp} [X; Y]^{crs}_* \xrightarrow{(f)^\sharp} [A; Y]^{crs}_*.$$

By Proposition 3.4.4 this diagram is commutative; and each $\Psi^\sharp$ is a one-one correspondence. Therefore, the upper row is an exact sequence. But

$$[f \times_R \upsilon] = [f \times_R 1] \circ [1 \times_R \upsilon] : \Sigma^{crs} A \to \Sigma^{crs} X,$$

and

$$([1 \times_R \upsilon])^\sharp : [\Sigma^{crs} A; Y]^{crs}_* \to [\Sigma^{crs} A; Y]^{crs}_*$$

is the function that sends each element into its inverse (note that $[\Sigma^{crs} A; Y]^{crs}_*$ is a group). Since the image of $([f \times_R 1])^\sharp : [\Sigma^{crs} X; Y]^{crs}_* \to [\Sigma^{crs} A; Y]^{crs}_*$ is a subgroup, this means that $\mathrm{Im}([f \times_R 1])^\sharp = \mathrm{Im}([f \times_R \upsilon])^\sharp$; also, $(([f \times_R 1])^\sharp)^{-1}(c_*) = (([f \times_R \upsilon])^\sharp)^{-1}(c_*)$. Thus $([f \times_R \upsilon])^\sharp$ can be replaced by $([f \times_R 1])^\sharp$ without sacrificing exactness. Note that the upper row is an exact sequence of groups as far as $[\Sigma^{crs} A; Y]^{crs}_*$, and an exact sequence of abelian groups as far as $[\Sigma^{crs}(\Sigma^{crs} A); Y]^{crs}_*$. □

Thus a coarse map of coarse pairs $f : (A, A') \to (X, X')$ with $f' = f|_{A'}$, gives the coarse pair of coarse mapping cones $(C_f^{crs}, C_{f'}^{crs})$, and for a pointed coarse pair (Y, B), there is the exact sequence of sets

$$\begin{aligned}\cdots &\to [(\Sigma^{crs})^2 X, (\Sigma^{crs})^2 X'; Y, B]_*^{crs} \to [(\Sigma^{crs})^2 A, (\Sigma^{crs})^2 A'; Y, B]_*^{crs} \\ &\to [\Sigma^{crs} C_f^{crs}, \Sigma^{crs} C_{f'}^{crs}; Y, B]_*^{crs} \to [\Sigma^{crs} X, \Sigma^{crs}) X'; Y, B]_*^{crs} \\ &\to [\Sigma^{crs} A, \Sigma^{crs} A'; Y, B]_*^{crs} \to [C_f^{crs}, C_{f'}^{crs}; Y, B]_*^{crs} \to [X, X'; Y, B]_*^{crs} \\ &\to [A, A'; Y, B]_*^{crs},\end{aligned}$$

where the terms involving coarse suspensions consists of groups and homomorphisms. The rest contains only pointed sets and maps.

Consider the special case of the inclusion $f : (S^0_{\mathbb{R}_+}, S^0_{\mathbb{R}_+}) \to (D^1_{\mathbb{R}_+}, S^0_{\mathbb{R}_+})$. Clearly, the coarse mapping cone C_f^{crs} is coarse homotopy equivalent to the coarse pair $(S^1_{\mathbb{R}_+}, *_{crs})$. Thus we have the coexact sequence

$$(S^0_{\mathbb{R}_+}, S^0_{\mathbb{R}_+}) \to (D^1_{\mathbb{R}_+}, S^0_{\mathbb{R}_+}) \to (S^1_{\mathbb{R}_+}, *_{crs}) \to (S^1_{\mathbb{R}_+}, S^1_{\mathbb{R}_+}) \to (D^2_{\mathbb{R}_+}, S^1_{\mathbb{R}_+}),$$

where the second map is the result of coarse collapsing $S^0_{\mathbb{R}_+}$ to the basepoint. By coarse suspending this $n-1$ times, we get the coexact sequence

$$(S^{n-1}_{\mathbb{R}_+}, S^{n-1}_{\mathbb{R}_+}) \to (D^n_{\mathbb{R}_+}, S^{n-1}_{\mathbb{R}_+}) \to (S^n_{\mathbb{R}_+}, *_{crs}) \to (S^n_{\mathbb{R}_+}, S^n_{\mathbb{R}_+}) \to (D^{n+1}_{\mathbb{R}_+}, S^n_{\mathbb{R}_+}).$$

All these fit together to give a long coexact sequence. Now

$$\begin{aligned}&[S^n_{\mathbb{R}_+}, S^n_{\mathbb{R}_+}; Y, B]_*^{crs} = [S^n_{\mathbb{R}_+}; B]_*^{crs} = \pi_n^{crs}(B), \\ &[S^n_{\mathbb{R}_+}, *_{crs}; Y, B]_*^{crs} = [S^n_{\mathbb{R}_+}; Y]_*^{crs} = \pi_n^{crs}(Y), \\ &[D^n_{\mathbb{R}_+}, S^{n-1}_{\mathbb{R}_+}; Y, B]_*^{crs} = \pi_n^{crs}(Y, B).\end{aligned}$$

Therefore, we obtain the "exact coarse homotopy sequence" of the coarse pair (Y, B):

$$\begin{aligned}&\cdots \to \pi_{n+1}^{crs}(Y, B) \to \pi_n^{crs}(B) \xrightarrow{i_\sharp} \pi_n^{crs}(Y) \xrightarrow{j_\sharp} \pi_n^{crs}(Y, B) \xrightarrow{\partial_\sharp} \pi_{n-1}^{crs}(B) \to \cdots \\ &\cdots \to \pi_1^{crs}(Y, B) \to \pi_0^{crs}(B) \to \pi_0^{crs}(Y),\end{aligned}$$

where all are groups and homomorphisms until the last three, which are only pointed sets and maps. Tracing through the definitions shows easily that $i_\sharp$ is induced by the inclusion $B \hookrightarrow Y$, $j_\sharp$ is induced by the inclusion $(Y, *_{crs}) \to (Y, B)$, and $\partial_\sharp$ is induced by the restriction to $S^{n-1}_{\mathbb{R}_+} \subseteq D^n_{\mathbb{R}_+}$.

Chapter 4

Calculating of the coarse homotopy groups of the coarse spheres $S^n_{\mathbb{R}_+}$

Chapter 3 was concerned with general results on coarse homotopy theory including some principal notions such as the coarse suspensions, coarse mapping cone, coarse homotopy groups, exact sequences of coarse pairs, and some of their properties. As in the classical case, the general problem of calculating coarse homotopy groups is very difficult, but is reasonably manageable provided that we confine our attention to fairly "well-behaved" coarse topological spaces such as coarse CW-complexes. But, as in the classical case, it is not possible to get very far without knowing the groups $\pi_r^{crs}(S^n_{\mathbb{R}_+})$, at least for $r \leq n$, therefore, as the first step, we shall prove that

$$\pi_r^{crs}(S^n_{\mathbb{R}_+}) \cong \begin{cases} \mathbb{Z}, & r = n \\ 0, & r < n. \end{cases}$$

Then in the next chapter we will pursue these ideas further, so as to obtain more precise results when the spaces involved are coarse CW-complexes.

Our first goal is to introduce for each integer $r \geq 1$, a discrete subset of the coarse sphere $S^r_{\mathbb{R}_+}$, denoted by $(S^r_{\mathbb{R}_+})_{dis}$, such that it sits coarsely dense[1] in $S^r_{\mathbb{R}_+}$, but considering each point x of $(S^r_{\mathbb{R}_+})_{dis}$, there is no other point of $(S^r_{\mathbb{R}_+})_{dis}$ lies within $\pi/24$ of x. Obviously, $(S^r_{\mathbb{R}_+})_{dis}$ and $S^r_{\mathbb{R}_+}$ are coarsely equivalent.

We start with $r = 1$. For each $m \in \mathbb{Z}_+$ (by $\mathbb{Z}_+$, we mean the set of positive integers), let S_m be the circle of radius m centered at the origin and

[1]A subset Z of a metric space Y is called *coarsely dense* if there is some $R > 0$ such that every point of Y lies within R of a point of Z. Then, we say Z sits coarsely dense in Y with constant R.

let $s_{m,0} := (m,0)$. Assume that the points $s_{m,k}$, $k = 1, \cdots, 12m-1$, have been chosen on the circle S_m such that

(i) the inclination of the straight line passing through the origin and the point $s_{m,1}$ is $\pi/6m$;

(ii) the points $s_{m,k}$, $k = 0, 1, \cdots, 12m-1$, form a partition, denoted by $(S_m)_{dis}$, of the circumference of the circle S_m into $12m$ archs with the same length $\pi/6$.

We define $(S^1_{\mathbb{R}_+})_{dis}$ to be the set $\bigcup_{m\in\mathbb{Z}_+} (S_m)_{dis}$. Obviously, the inclusion $i : (S^1_{\mathbb{R}_+})_{dis} \to S^1_{\mathbb{R}_+}$ is a coarse equivalence, i.e., $(S^1_{\mathbb{R}_+})_{dis}$ is coarsely equivalent to $S^1_{\mathbb{R}_+}$. Next, we introduce a special discrete subset of the coarse product $S^1_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$ which we will denote it by $(S^1_{\mathbb{R}_+})_{dis} \times_{\mathbb{R}_+} (I^{crs}_{\mathbb{R}_+})_{dis}$. To do it, we first consider the discrete subset $(I^{crs}_{\mathbb{R}_+})_{dis}$ of $I^{crs}_{\mathbb{R}_+}$ which is a set containing those points of $\bigcup_{m\in\mathbb{Z}_+} (S_m)_{dis}$ having non-negative components, i.e., $(I^{crs}_{\mathbb{R}_+})_{dis} := \bigcup_{m\in\mathbb{Z}_+} I_m$, where $I_m := \{i_{m,j} := s_{m,j}|\ j = 0, 1, \cdots, 3m\}$. Now, by the definition of the coarse product $\times_{\mathbb{R}_+}$:

$$(S^1_{\mathbb{R}_+})_{dis} \times_{\mathbb{R}_+} (I^{crs}_{\mathbb{R}_+})_{dis} := \bigcup_{\substack{m\in\mathbb{Z}_+ \\ j=0,1,\cdots,3m}} S^j_m,$$

where $S^j_m := \big\{s_{m,k} \times_{\mathbb{R}_+} i_{m,j} := (s_{m,k}, i_{m,j})|\ k = 0, 1, \cdots, 12m-1\big\}$. We call each point $i_{m,j}$ a time-unit, that is, for each fixed $m \in \mathbb{Z}_+$, we have $3m+1$ time-units $i_{m,j}$, $j = 0, 1, \cdots, 3m$, each of them represents a copy of $(S_m)_{dis}$ in $(S^1_{\mathbb{R}_+})_{dis} \times_{\mathbb{R}_+} (I^{crs}_{\mathbb{R}_+})_{dis}$. The reason that we wrote the elements of $(S^1_{\mathbb{R}_+})_{dis} \times_{\mathbb{R}_+} (I^{crs}_{\mathbb{R}_+})_{dis}$ as $s_{m,k} \times_{\mathbb{R}_+} i_{m,j}$ is to remind ourselves the structure of entourages in $\times_{\mathbb{R}_+}$. More precisely, by writing $\big\|s_{m,k} \times_{\mathbb{R}_+} i_{m,j} - s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'}\big\|_{\mathbb{R}_+} \leq S$, we mean that $\big\|s_{m,k} - s_{m',k'}\big\| \leq S$ and $\big\|i_{m,j} - i_{m',j'}\big\| \leq S$.

With a little work, one can generalise the above idea to introduce $(S^r_{\mathbb{R}_+})_{dis}$ with the desired properties for each integer $r > 1$. But, to keep this part short, we content ourselves with giving some hints for the case $r = 2$: we first discretize the xy-plane as we did above. Then, we keep doing this for all planes passing through the antipodal points and the z-axis. Then comes some hand works, namely removing some points carefully out such that

(i) for each $m \in \mathbb{Z}_+$, the set consisting of the remaining points lying on the sphere of radius m centered at the origin, S_m, sit coarsely dense in S_m with constant $\pi/6$; and

(ii) for each point x of $(S^2_{\mathbb{R}_+})_{dis}$, there is no other point of $(S^2_{\mathbb{R}_+})_{dis}$ lies within $\pi/24$ of x.

For later use, we fix some notations: for each $m \in \mathbb{Z}_+$, let S_m be the sphere of radius m in $\mathbb{R}^{r+1}$ centered at the origin. We again denote the points of $(S^r_{\mathbb{R}_+})_{dis}$ by $s_{m,k}$ while the index k here is a tuple $(k_1, k_2, \cdots, k_r)$, where $k_i = 1, \cdots, 12m-1$, for each $i = 1, 2, \cdots, r$. Similarly, we continue using the notations S^j_m and $s_{m,k} \times_R i_{m,j}$, while in order to remind ourselves that k is not a positive integer but a tuple we will write $k \in C(S_m)$.

The following lemma comes from [Roe03](Lemma 1.10) and plays an important role in this chapter.

Lemma 4.0.1. *Let X be a length space*[2]*, Y any metric space. Then the following properties of a set map $f : X \to Y$ are equivalent:*

(a) f is large-scale Lipschitz[3]*;*

(b) f preserves entourages;

(c) There exists $S, \mathcal{L} > 0$ such that $d(x, x') < S \Rightarrow d(f(x), f(x')) < \mathcal{L}$.

Therefore, if we define a map F on $(S^r_{\mathbb{R}_+})_{dis} \times_{\mathbb{R}_+} (I^{crs}_{\mathbb{R}_+})_{dis}$ and then extend it affinely on the whole of $S^r_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$, to show that it preserves entourages it is enough to find $S, \mathcal{L} > 0$ such that

$$\left\|s_{m,k} \times_{\mathbb{R}_+} i_{m,j} - s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'}\right\|_R \leq S \;\; \Rightarrow$$
$$\left\|F(s_{m,k} \times_{\mathbb{R}_+} i_{m,j}) - F(s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'})\right\| < \mathcal{L}.$$

For simplicity, we take unless otherwise stated $S_c := 2$ as the positive real number S above up to which we consider a neighboring of each point of $(S^r_{\mathbb{R}_+})_{dis}$. The following are obvious:

$$B(s_{m,k}; S_c) \cap (S_p)_{dis} \neq \emptyset, \text{ if } p = m-1, m, m+1$$
$$B(s_{m,k}; S_c) \cap (S_p)_{dis} = \emptyset, \text{ otherwise,}$$

for each $k \in C(S_m)$. In the future, by $\hat{s}_{m,k}$, we mean an arbitrary element of the set $B(s_{m,k}; S_c) \cap (S^1_R)_{dis}$.

The next simple Example gives us some elementary ideas about what we are going to deal with in this chapter.

[2] A (connected) metric space X is called *length space* if the distance between any two points, of X is equal to the infimum of the lengths of the curves joining them.

[3] One says that $f : X \to Y$ is *large-scale Lipschitz* if there are positive constants c and A such that

$$d(f(x), f(x')) \leq c d(x, x') + A.$$

Example 4.0.2. If $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a coarse map, then η is coarsely homotopic to the identity map.

Proof. Without lose of generality, we can assume that $\eta(0) = 0$ because otherwise we can find a coarse map $\eta' : \mathbb{R}_+ \to \mathbb{R}_+$ which is close to η and $\eta'(0) = 0$, and then we can start from η'. We shall define a coarse map $F : \mathbb{R}_+ \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to \mathbb{R}_+$ such that $F \circ i_0^* \sim_{cl} \eta$ and $F \circ i_1^* \sim_{cl} 1_{\mathbb{R}_+}$. For each integer $n \geq 0$, set $s_n := n$. We use the subset $\{s_n | n \geq 0\}$ as $(\mathbb{R}_+)_{dis}$. Obviously, it is enough to define F on $(\mathbb{R}_+)_{dis} \times_{\mathbb{R}_+} (I^{crs}_{\mathbb{R}_+})_{dis}$ such that

- for every integer $n \geq 0$, $F((s_n \times_{\mathbb{R}_+} i_{n,0})) = \eta(n)$, and $F((s_n \times_{\mathbb{R}_+} i_{n,3n})) = n$; and
- the map $F|_{(\mathbb{R}_+)_{dis} \times_{\mathbb{R}_+} (I^{crs}_{\mathbb{R}_+})_{dis}}$ preserves entourages; and
- the inverse image $(F|_{(\mathbb{R}_+)_{dis} \times_{\mathbb{R}_+} I^{crs}_{dis}})^{-1}(K)$ of every finite subset $K \subseteq \mathbb{R}_+$ is again finite.

Because then we can extend it to a coarse map with the desired properties. For simplicity, set $\eta_n := \eta(n)$, for each integer $n \geq 0$. Since η is a coarse map, so there is a positive real number L such that $\eta_{m+1} \in [\eta_m - L, \eta_m + L]$, for every integer $m \geq 0$. For simplicity, for each integer $n \geq 0$ and each j, $j = 0, 1, \cdots, 3n$, we denote the value of F at $s_n \times_{\mathbb{R}_+} i_{n,j}$ by $F_{n,j}$. Clearly, we define $F_{n,0} := \eta_n$ and $F_{n,3n} := n$, for every integer $n \geq 0$. We define $F_{k,j}$ by induction on k starting from $k = 1$. Choose a positive real number $M > 0$ which is bigger than $L + 1/2$. Define

$$F_{1,1} = F_{1,2} := \begin{cases} F_{1,0} - M, & \text{if } F_{1,0} - F_{1,3} > M \\ F_{1,0} + M, & \text{if } F_{1,3} - F_{1,0} > M \\ F_{1,3}, & \text{otherwise.} \end{cases}$$

Since $|F_{1,0} - F_{1,3}| \leq L + 1$, clearly $|F_{1,0} - F_{1,1}| \leq M$ and $|F_{1,3} - F_{1,1}| \leq M$. Now assume that $F_{k,j}$ has been defined for $k = n$ and every $j = 0, \cdots, 3n$ such that

$$F_{n,j+1} = \begin{cases} F_{n,j} - M, & \text{if } F_{n,j} - F_{n,3n} > M \\ F_{n,j} + M, & \text{if } F_{n,3n} - F_{n,j} > M \\ F_{n,3n}, & \text{otherwise.} \end{cases} \tag{4.0.1}$$

Now we shall define $F_{k,j}$ for $k = n + 1$ and for each $j = 1, \cdots, 3(n+1) - 1$ such that (4.0.1) holds. The critical point is that

$$\big| |F_{n+1,0} - F_{n+1,3(n+1)}| - |F_{n,0} - F_{n,3n}| \big| \leq L + 1,$$

while we have three more time-units in $n + 1$-step in compare with the previous step. Therefore, we can overcome the difference and define $F_{n+1,j}$ for every $j = 1, \cdots, 3(n+1)$ as in (4.0.1) and this complete the induction.

Clearly, for each integer $k \geq 0$ and for each j, $j = 0, \cdots, 3k$, the distance of $F_{k,j}$ from each of the points $F_{k\pm1,j-1}$, $F_{k\pm1,j}$ and $F_{k\pm1,j+1}$ is smaller than $L+M$. In the next step, we can easily extend F affinely on $\mathbb{R}_+ \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$. The inverse image of bounded subsets under F are clearly bounded. On the other hand, choosing $S > 0$ small enough, one can easily see that the only values of F needed to compare are the ones we have already compared above, that is F preserves entourages. Moreover, $F \circ i_0^* \sim_{cl} \eta$ and $F \circ i_1^* \sim_{cl} 1_{\mathbb{R}_+}$. □

Now, suppose given a pointed coarse map $\eta : S^1_{\mathbb{R}_+} \to S^1_{\mathbb{R}_+}$. Consider the restriction of η on $(S^1_{\mathbb{R}_+})_{dis}$ and suppose that $S_c > 0$ is the constant we introduced above. Since η is coarsely uniform, therefore there exists a positive real number $L > 0$ such that $\|\eta(s_{m,k}) - \eta(\hat{s}_{m,k})\| \leq L$ for every $m \in \mathbb{Z}_+$ and $k = 0, \cdots, 12m-1$. On the other hand, since the coarse map η is pointed, therefore there exists a positive real number $M > 0$ such that $\|\eta(s_{m,0}) - s_{m,0}\| \leq M$ for every $m \in \mathbb{Z}_+$. Coarsely properness of η implies that for each $T > 0$, there is a positive integer $m_0 \in \mathbb{Z}_+$ such that $\eta((S_k)_{dis}) \cap S_l = \emptyset$ for every $k \geq m_0$ and $l \leq T$, where S_l is the circle of radius l centered at the origin. Now, we first choose $T_0 > 0$ big enough such that $\pi \times (L+M)/4T_0$ is small enough; and then assume that m_0 is a positive integer having above property with respect to T_0 with this additional property that $1000\pi/m_0$ is small enough. Let us remark that some of the above required conditions have been made for the later use in the proof of the main theorem and not necessarily for what is coming in below.

Now, for each positive integer $t \geq m_0$, we assign to $\eta|_{(S_t)_{dis}}$ a continuous pointed map $\tilde{\eta}_t : S^1 \to S^1$ as follows: Let $\tilde{s}_{t,k}$ be the projection of the point $s_{t,k}$ on S^1 where $k = 0, \cdots, 12t-1$. We define $\tilde{\eta}_t(\tilde{s}_{t,k})$ to be the projection of the point $\eta(s_{t,k})$ on S^1. Since for each k, $k = 0, \cdots, 12t-1$, the point $\eta(s_{t,k})$ lies outside of the circle of radius T_0 centered at the origin and because $\|\eta(s_{t,k}) - \eta(s_{t,k+1})\| \leq L$ and L is small enough relative to T_0, therefore for each pair of points $(\tilde{\eta}_t(\tilde{s}_{t,k}), \tilde{\eta}_t(\tilde{s}_{t,k+1}))$ on S^1 there is a unique path with the smallest length joining them. As the next step, we extend the map $\tilde{\eta}_t$ affinely on each arc joining points $\tilde{s}_{t,k}$ and $\tilde{s}_{t,k+1}$, for each $k = 0, \cdots, 12t-2$. And, finally, to get a pointed continuous map $\tilde{\eta}_t : S^1 \to S^1$, we define $\tilde{\eta}_t$ affinely on the arc joining points $(1,0)$ and $s_{t,0}$ and on the arc joining points $s_{t,12t-1}$ and $(1,0)$. Note that each $\tilde{\eta}_t : S^1 \to S^1$, $t \geq m_0$, can be defined so as to be also Lipschitz. Now the claim is

Lemma 4.0.3. *If $\eta : S^1_{\mathbb{R}_+} \to S^1_{\mathbb{R}_+}$ is a pointed coarse map, then there is a positive integer m_0 such that $deg(\tilde{\eta}_t)$ is a fixed integer for every $t \geq m_0$.*

Proof. Let T_0 and m_0 be the integers appeared above and let $\deg(\tilde{\eta}_{m_0}) = q \in \mathbb{Z}$. One can easily show by induction on t starting from m_0 that

$$deg(\tilde{\eta}_t) = q,$$

for every $t > m_0$. □

Remark 4.0.4. Note that one can similarly assign to any pointed coarse map $\eta : S^r_{\mathbb{R}_+} \to S^n_{\mathbb{R}_+}$ a family of pointed Lipschitz maps $\{\tilde{\eta}_t : S^r \to S^n |\ t \in \mathbb{Z}$ and $t \geq m_0$ for some positive integer $m_0\}$. Moreover, the similar statement as in above holds if $r = n$.

Now we are ready to prove the main theorem of this chapter, namely

Theorem 4.0.5. *Let $S^n_{\mathbb{R}_+}$ be the coarse sphere of dimension n equipped with the bounded coarse structure. Then*

$$\pi^{crs}_r(S^n_{\mathbb{R}_+}) \cong \pi_r(S^n),$$

for all $r \leq n$.

Proof. We shall construct an isomorphism $\Phi : \pi_r(S^n, *) \to \pi^{crs}_r(S^n_{\mathbb{R}_+}, *_{crs})$. For this purpose, we define Φ as follows and we will show that it is actually an isomorphism. Considering any element of $\pi_r(S^n, *)$, we choose a representative $f : (S^r, *) \to (S^n, *)$ of this equivalence class which is Lipschitz. We assign to f a map $\tilde{f} : S^r_{\mathbb{R}_+} \to S^n_{\mathbb{R}_+}$ defined as follows:

$$\tilde{f}(x) := \begin{cases} rf(\tilde{x}), & \text{if } x \neq 0 \text{ and } x = r\tilde{x} \\ 0, & \text{if } x = 0, \end{cases}$$

where $\tilde{x}$ is the point where the ray emanating from the origin and passing through the point x meets S^r. That the map f is Lipschitz guarantees that the map $\tilde{f}$ is a coarse map. Moreover, $\tilde{f}$ is pointed because f is.

To see that Φ is well-defined, suppose that $f, g : (S^r, *) \to (S^n, *)$ are two pointed Lipschitz maps which are homotop. From the classical algebraic topology, we know that there exists a pointed homotopy $F : S^r \times I \to S^n$ between f and g which is Lipschitz. Then, the map $\widetilde{F} : S^r_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to S^n_{\mathbb{R}_+}$ defined as follows clearly provides a coarse homotopy rel $*_{crs}$ between the coarse maps $\tilde{f}$ and $\tilde{g}$.

$$\widetilde{F}(r\tilde{x}, r\tilde{t}) := \begin{cases} rF(\tilde{x}, \tilde{t}), & \text{if } r \neq 0 \\ 0, & \text{if } r = 0, \end{cases}$$

where by $\tilde{x}$ and $\tilde{t}$ we mean exactly what we have introduced above.

The map Φ is surjective. Let $\eta : (S^r_R, *_{crs}) \to (S^n_R, *_{crs})$ be a pointed coarse map. As we have seen before, we can assign to $\eta : (S^r_R, *_{crs}) \to (S^n_R, *_{crs})$ a family of pointed Lipschitz maps $\{\tilde{\eta}_t : S^r \to S^n |\ t \in \mathbb{Z}$ and $t \geq m_0$ for some positive integer $m_0\}$. Moreover, let positive integers T_0 and m_0 be the ones which appeared in the construction of $\tilde{\eta}_t$'s. For simplicity, we denote $\tilde{\eta}_{m_0}$ by f. We claim

$$\Phi([f]) = [\eta]^{crs}.$$

We shall therefore define a coarse map $F : S^r_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to S^n_{\mathbb{R}_+}$ (rel $*_{crs}$) with $F \circ i_0^* \sim_{cl} \eta$ and $F \circ i_1^* \sim_{cl} \tilde{f}$. For it, we will first define F on S^j_m, $j = 0, 1, \cdots, 3m$, by induction on m starting from m_0 and then we will extend it affinely on the whole of $S^r_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$. To be precise, $P(m)$ is the following assertion:

There exist a real number $\mathcal{L} > 0$ (depending only on the real number $S_c > 0$) and an entourage $E \in \mathcal{E}_{S^n_{\mathbb{R}_+}}$ such that for each $m \in \mathbb{Z}_+$, $m \geq m_0$, we can define F on S^j_m, $j = 0, 1, \cdots, 3m$, with the following properties:

(i) $\forall k \in C(S_m)$: $F(s_{m,k} \times_{\mathbb{R}_+} i_{m,0}) = \eta(s_{m,k})$ and $F(s_{m,k} \times_{\mathbb{R}_+} i_{m,3m}) = \tilde{f}(s_{m,k})$;

(ii) if $\left\| s_{m,k} \times_{\mathbb{R}_+} i_{m,j} - s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'} \right\|_R \leq S_c$, then

$$\left\| F(s_{m,k} \times_{\mathbb{R}_+} i_{m,j}) - F(s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'}) \right\| < \mathcal{L},$$

where $m' = m - 1, m$;

(iii) there exists a function $\omega : \mathbb{N} \to \mathbb{R}_+$, defined inductively, which is coarsely proper and for each positive integer m, $m \geq m_0$, the following holds:

$$\forall k \in C(S_m),\ \forall j,\ j = 0, 1, \cdots, 3m :\ \left\| F(s_{m,k} \times_{\mathbb{R}_+} i_{m,j}) \right\| \geq \omega(m);$$

(iv) for each positive integer m, $m \geq m_0$, the following holds:

$$\forall j,\ j = 0, 1, \cdots, 3m,\ (F(s_{m,0} \times_{\mathbb{R}_+} i_{m,j}), s_{m,0}) \in E,$$

where the set $\{s_{m,0} |\ m \in \mathbb{Z}_+\}$ represents the basepoint of $S^r_{\mathbb{R}_+}$ as well as the basepoint of $S^n_{\mathbb{R}_+}$.

The fourth condition implies that the coarse homotopy F is rel $*_{crs}$, the third one guarantees coarsely properness of F, while, by the discussion came after Lemma 4.0.1, the second condition guarantees that F is coarsely uniform.

We first introduce some notations. For each point $s_{m,k} \in (S^r_{\mathbb{R}_+})_{dis}$, $m \geq m_0$, $k \in C(S_m)$, we introduce a finite sequence of points $(a^p_{m,k})^m_{p=m_0}$ of $(S^r_{\mathbb{R}_+})_{dis}$ defined as follows: Let S^r_p be the sphere of radius p centered at the origin in $\mathbb{R}^{r+1}$. Assume that for each p, $m_0 \leq p \leq m$, $b^p_{m,k}$ is the point where the ray emanating from the origin passing through the point $s_{m,k}$ meets the sphere S^r_p. Now, for each p, $m_0 \leq p \leq m$, we define $a^p_{m,k}$ to be a point of $(S_p)_{dis}$ with the minimum distance from $b^p_{m,k}$. Note that it is possible that more than one point of $(S_p)_{dis}$ have the same minimum distance from $b^p_{m,k}$, then in this case we just introduce one of them as the point $a^p_{m,k}$. Clearly, $a^m_{m,k} = s_{m,k}$. Now, for each $m \in \mathbb{Z}_+$, $m \geq m_0$, set

$$c_m := \min_{k \in C(S_m)} \left\{ \left\| \eta(s_{m,k}) \right\|, \left\| \tilde{f}(s_{m,k}) \right\| \right\},$$

and let $S^n_{c_m}$ be the sphere of radius c_m centered at the origin in $\mathbb{R}^{n+1}$. Moreover, assume that $t^l_{m,k}$ is the point where the ray radiating from the origin and passing through the point $\eta(s_{m,k})$ meets the sphere $S^n_{c_l}$ and $(t')^l_{m,k}$ is the point where the ray radiating from the origin and passing through the point $\tilde{f}(s_{m,k})$ meets the sphere $S^n_{c_l}$. We will denote the line segment joining two points $\eta(s_{m,k})$ and $t^l_{m,k}$ by $I^l_{m,k}$ and the line segment joining two points $(t')^l_{m,k}$ and $\tilde{f}(s_{m,k})$ by $(I')^l_{m,k}$. Note that for fixed m and c_l, all line segments $(I')^l_{m,k}$, $k \in C(S_m)$, have the same length. Now, let C be a curve in $\mathbb{R}^{n+1}$ and assume that $S^n_{p_1}$ and $S^n_{p_2}$ are again spheres of radius p_1 and p_2 respectively with $p_1 \leq p_2$. By $C^{p_1 \circledcirc p_2}$, we mean that part of C which is on or between the two spheres $S^n_{p_1}$ and $S^n_{p_2}$, that is

$$C^{p_1 \circledcirc p_2} := \{x \in C |\ p_1 \leq \|x\| \leq p_2\},$$

which, of course, can be an empty set. Furthermore, by $C^{\odot p_i}$, we mean the projection of $C^{p_1 \circledcirc p_2}$ on $S^n_{p_i}$ (by projection, we mean for each point of C which lies on or between the two spheres $S^n_{p_1}$ and $S^n_{p_2}$ finding the point where the ray radiating from the origin and passing through that point of C meets the sphere $S^n_{p_i}$).

As the first step, we define F on $S^j_{m_0}$, $j = 0, 1, \cdots, 3m_0$. Clearly, for each $k \in C(S_{m_0})$, the points $t^{m_0}_{m_0,k}$ and $(t')^{m_0}_{m_0,k}$ coincide. Now, for each $k \in C(S_{m_0})$, let $C_{m_0,k}$ be the curve which joins two points $\eta(s_{m_0,k})$ and $\tilde{f}(s_{m_0,k})$ and consists of two line segments $I^{m_0}_{m_0,k}$ and $(I')^{m_0}_{m_0,k}$. Set

$$L_{m_0} := \max_{k \in C(S_{m_0})} \left\{ \left\| \eta(s_{m_0,k}) - t^{m_0}_{m_0,k} \right\| + \left\| \tilde{f}(s_{m_0,k}) - (t')^{m_0}_{m_0,k} \right\| \right\},$$

and let L and M be the constant which we have introduced before and let L' be the corresponding constant for the coarse map $\tilde{f}$. Next, we take $N > 0$ big enough in compare with $2\pi \times (L + M + L')$ and of course bigger than $\frac{L_{m_0}}{3m_0}$. Now, for each $k \in C(S_{m_0})$, we partition the curve $C_{m_0,k}$ as follows: considering each curve $C_{m_0,k}$ as a straight line segment, we start partitioning $C_{m_0,k}$ from the point $\eta(s_{m_0,k})$ and we choose the points of the partition such that all the partition segments have the same length N except possibly the last segment which has the length smaller than N. That is, $\xi^0_{m_0,k} = \eta(s_{m_0,k})$ and $\xi^{j_k}_{m_0,k} = \tilde{f}(s_{m_0,k})$ for some j_k. Choosing N bigger than $\frac{L_{m_0}}{3m_0}$ guarantees that for each $k \in C(S_{m_0})$ the index j_k is not bigger than $3m_0$ which in the other words means that for each $k \in C(S_{m_0})$, $3m_0 + 1$ time-units that we have available at the step m_0 is enough to get from $\eta(s_{m_0,k})$ to $\tilde{f}(s_{m_0,k})$. For the k's in $C(S_{m_0})$ which $\xi^{j_k}_{m_0,k} = \tilde{f}(s_{m_0,k})$ for some $j_k < 3m_0$, we define $\xi^j_{m_0,k} := \tilde{f}(s_{m_0,k})$ for every j, $j = j_k + 1, \cdots, 3m_0$. Now we define F on $S^j_{m_0}$ as follows:

$$F(s_{m_0,k} \times_{\mathbb{R}_+} i_{m_0,j}) := \xi^j_{m_0,k},$$

for every $k \in C(S_{m_0})$ and every j, $j = 0, 1, \cdots, 3m_0$. Obviously, the conditions (i)-(iv) are fulfilled for $\mathcal{L} := 8N$, $E := \{(x, y) \in S^n_{\mathbb{R}_+} \times S^n_{\mathbb{R}_+} |\ \|x - y\| \leq M\}$ and $\omega(m_0) := c_{m_0}$.

Before going through the details of the proof, let us give a outline of it. Note that having defined F on $S^j_{m_0}$ already suggests a way to define F on $S^j_{m_0+1}$ which is simply by using the points $a^{m_0}_{m_0+1,k}$ to define F on $S^1_{m_0+1}$ and then by using the way that F has been defined on $S^j_{m_0}$ to reach $\tilde{f}|_{(S_{m_0})_{dis}}$ and finally use again the points $a^{m_0}_{m_0+1,k}$ to define F on $S^{3(m_0+1)+1}_{m_0+1}$ in the desired way to reach $\tilde{f}|_{(S_{m_0+1})_{dis}}$ and the important fact here is that going one step further we have three more time-units available which allowed us to do as above. But, obviously, we cannot use this strategy to define F on S^j_m for every m because then it fails to be coarsely proper. And this is where the sequence $(c_m)_{m \geq m_0}$ comes to play. The rough idea is to introduce a coarsely proper function $\omega : \mathbb{N} \to \mathbb{R}_+$ which is the desired function stated in (iii) by using the sequence $(c_m)_{m \geq m_0}$ which is as a function coarsely proper. So, roughly speaking, the main problem that we are facing is introducing the function $\omega : \mathbb{N} \to \mathbb{R}_+$ step by step and then at each step m, $m > m_0$, introducing special curves which for every $k \in C(S_m)$ join the points $\eta(s_m, k)$ and $\tilde{f}(s_{m,k})$, and which live outside and on the sphere $S^n_{\omega(m)}$ and have some nice properties. In the following, we will make the above ideas precise.

First, we introduce a sequence $(\mathcal{M}_i)_{i=0}^{\infty}$ of positive integers, inductively, as follows: setting $\mathcal{M}_{-1} := 0$, we define $\mathcal{M}_i$, $i = 0, 1, 2, \cdots$, to be the smallest positive integer such that

$$\eta((S_m)_{dis}) \cap S_l = \emptyset \qquad \text{and} \qquad \tilde{f}((S_m)_{dis}) \cap S_l = \emptyset,$$

for every integer $m \geq m_0 + \mathcal{M}_i$ and every $l \leq c_{m_0+\mathcal{M}_{i-1}}$, where S_l is the sphere of radius l centered at the origin in $\mathbb{R}^{n+1}$. The coarsely properness of η and $\tilde{f}$ guarantee the existence of $\mathcal{M}_i$'s. Now, choose $\epsilon_0 > 0$ small enough (for our purpose, it is enough to choose for instance $\epsilon_0 := \pi/1000$). Next, we choose the positive integer $\mathcal{N}_0$ so big such that the projection of any given arc of the sphere $S^r_{m_0+\mathcal{N}_0}$ of the length $10^r \times \pi$ on the sphere $S^r_{m_0}$ has the length smaller than ϵ_0. As the next step, we define ω at every m, $m_0 < m \leq m_0 + \max\{\mathcal{N}_0, \mathcal{M}_0\}$, inductively, as follows:

$$\omega(m_0 + p) := \begin{cases} \omega(m_0 + p - 1), & \omega(m_0 + p - 1) < c_{m_0+p} \\ c_{m_0+p}, & \omega(m_0 + p - 1) \geq c_{m_0+p}, \end{cases}$$

where $1 \leq p \leq \max\{\mathcal{N}_0, \mathcal{M}_0\}$. Now, we define F on S^j_m for every m, $m = m_0 + 1, \cdots, m_0 + \max\{\mathcal{N}_0, \mathcal{M}_0\}$. As we said, we first introduce for each $k \in C(S_m)$, a curve joining two points $\eta(s_{m,k})$ and $\tilde{f}(s_{m,k})$. We will denote this curve by $C_{m,k}$. For it, we consider the finite sequence $(a^p_{m,k})^m_{p=m_0}$ of $(S^r_{\mathbb{R}_+})_{dis}$

introduced before. Each curve $C_{m,k}$ consists of three main parts, called η-part, middle part and $\tilde{f}$-part. The η-part starts from the point $\eta(s_{m,k})$ and consisting of the following line segments: the line segment joining two points $\eta(s_{m,k})$ and $\eta(a^{m-1}_{m,k})$, then the line segment joining points $\eta(a^{m-1}_{m,k})$ and $\eta(a^{m-2}_{m,k})$ and so on until the line segment joining two points $\eta(a^{m_0+1}_{m,k})$ and $\eta(a^{m_0}_{m,k})$. Let $s_{m_0,k'}$ be the point with $a^{m_0}_{m,k} = s_{m_0,k'}$. The middle part consists of two line segments $I^{m_0}_{m_0,k'}$ and $(I')^{m_0}_{m_0,k'}$ and finally we define the $\tilde{f}$-part of the curve $C_{m,k}$ to be the curve consisting of the following line segments: the line segment joining two points $\tilde{f}(a^{m_0}_{m,k})$ and $\tilde{f}(a^{m_0+1}_{m,k})$, then the line segment joining two points $\tilde{f}(a^{m_0+1}_{m,k})$ and $\tilde{f}(a^{m_0+2}_{m,k})$ and so on until we reach the point $\tilde{f}(s_{m,k})$. As the curves $C_{m,k}$, $m = m_0+1, \cdots, m_0+\max\{\mathcal{N}_0, \mathcal{M}_0\}$ have been defined, we already have a partition for each of them. For each $k \in C(S_m)$, let $\xi^j_{m,k}$ be the points of the partition of $C_{m,k}$. Obviously, for each $k \in C(S_m)$, the indices j in $\xi^j_{m,k}$ run over $0 \le j \le 3m+1$. Now we simply define F as follows:

$$F(s_{m,k} \times_{\mathbb{R}_+} i_{m,j}) := \xi^j_{m,k}.$$

Obviously, the conditions (i)-(iv) are fulfilled for the constant $\mathcal{L}$ and the entourage E introduced before.

From this step, we change our strategy, because, for instance, we cannot define the value of ω in advance at the whole of some periods as we did for $m_0 < m \le m_0 + \max\{\mathcal{N}_0, \mathcal{M}_0\}$. The reason is that it deponds not only on c_m's but on other factors, therefore there is no way unless define it step by step. Let us first have a look at some highlights of our strategy: the way that the sequence $(\mathcal{M}_i)^{\infty}_{i=0}$ has been defined guarantees that the movement of the points $\eta(s_{m,k})$ and $\tilde{f}(s_{m,k})$ for $m \ge m_0+\mathcal{M}_0$ do not cause any change in those parts of our curves which lie inside of the sphere $S^n_{c_{m_0}}$. This fact encourages us to establish the following:

(i) as long as we are defining F at some step m with $m_0 + \mathcal{M}_i < m \le m_0+\mathcal{M}_{i+1}$, the positive real number $c_{m_0+\mathcal{M}_{i-1}}$ is the upper bound for the values of ω at this period (that is, it is the upper bound for the radius of the spheres that we are going to use as the lower bounds of our curves in this period).

As one can see above, we used the points $s_{m_0,k}$, $k \in C(S_{m_0})$, to determine the middle part of our curves when we were defining F for m, $m_0 < m \le m_0+\max\{\mathcal{N}_0, \mathcal{M}_0\}$. Now, one of the questions which naturally arises is that as we continue defining F for m, $m > m_0+\max\{\mathcal{N}_0, \mathcal{M}_0\}$, regardless of the fact that we have projected some parts of our curves on a bigger sphere or not, which m' we are going to use in order to determine the middle parts of our curves. Clearly, if we use the points of the previous step to determine the middle parts of our curves, then possibly at some steps it may happen that even for two nearby points some parts of their middle parts have been determined by some points in the previous steps which are too far from each other only because of the fact that we do not have enough control over the points of our discrete model of $S^r_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$. To avoid above situation and of course because of another important reason that will be explained later we require that

(ii) after each time of projectiong some parts of our curves on a bigger sphere (we will explain later under which conditions we are allowed to do it), we will stay at that sphere for a special period which is determined as follows: assume that at some step m' we have projected our curves on a bigger sphere. We choose the positive integer $\mathcal{N}_{m'}$ so big such that the projection of any given arc of the sphere $S^r_{m'+\mathcal{N}_{m'}}$ of the length $10^r \times \pi$ on the sphere $S^r_{m'}$ has the length smaller than ϵ_0. Now, we define $\omega(m) := \omega(m')$ for every m with $m' < m \leq m' + \mathcal{N}_{m'}$. We call this period the "*stabilization period*". The way that F is defined in a stabilization period is as above but with this big difference that we use the points $\eta(a^{m'}_{m,k})$, $k \in C(S_m)$, as the last point of the η-parts, then the middle parts are defined by using the curves $C_{m',k'}$ where k''s have been determined by the equations like $a^{m'}_{m,k} = s_{m',k'}$, and finally the $\tilde{f}$-parts are again defined as above but started from the points $\tilde{f}(a^{m'}_{m,k})$.

Obviously, (i) force us to the following:

(iii) after each stabilization period, to define ω at the next step (that is, to define the next lower bound), we should always consider an additional rule, namely, if after a stabilization period we are at some step m'', $m_0 + \mathcal{M}_q < m'' \leq m_0 + \mathcal{M}_{q+1}$, then the value of ω that we are about to define cannot be bigger than $c_{m_0+\mathcal{M}_{q-1}}$.

Now, let us go through the way that F is defined on $S^j_{m_0+\max\{\mathcal{N}_0,\mathcal{M}_0\}+1}$ as an example for the steps we are possibly allowed to project some parts of our curves on a bigger sphere. For simplicity, set $\nu_0 := m_0 + \max\{\mathcal{N}_0 + \mathcal{M}_0\}$. Obviously, $\omega(\nu_0) = \min_{m_0 \leq m \leq \nu_0} \{c_m\}$. Let m_1 be the biggest integer with $m_0 \leq m_1 \leq \nu_0$ such that $c_{m_1} = \omega(\nu_0)$. First, we distinguish two cases:

(a) $m_1 = \nu_0$ which simply means that $\omega(\nu_0) = c_{m_0}$;

(b) $m_1 < \nu_0$ but $c_{m_1+1} > c_{m_0}$.

in both of the above cases, as we said above, the next step is to consider the position of $\nu_0 + 1$, i.e., if $\nu_0 + 1 \leq m_0 + \mathcal{M}_1$, then it means that we are already on the sphere that is our upper bound in this period which means we should define $\omega(\nu_0 + 1) := \omega(\nu_0)$ and we define F in this case as we did for $m_0 < m \leq m_0 + \max\{\mathcal{N}_0, \mathcal{M}_0\}$. But if $\nu_0 + 1 > m_0 + \mathcal{M}_1$, then we will do the same as we will do for case

(c) $m_1 < \nu_0$ and $c_{m_1+1} \leq c_{m_0}$,

namely, we assume that for each $k \in C(S_{\nu_0+1})$, the curve $E_{\nu_0+1,k}$ is the curve defined as above having the points $\eta(a^{m_0}_{\nu_0+1,k})$ and $\tilde{f}(a^{m_0}_{\nu_0+1,k})$ as the last and the first point of its η- and $\tilde{f}$-parts, respectively, and having as its middle part the curve $C_{m_0,k'}$, where k' has been determined by the equation $a^{m_0}_{\nu_0+1,k} = s_{m_0,k'}$. As the next step, we project that part of $E_{\nu_0+1,k}$ which lies on or between the two spheres $S^n_{\omega(\nu_0)}$ and $S^n_{c_{m_1+1}}$ on the sphere $S^n_{c_{m_1+1}}$. As before, we will denote that part of $E_{\nu_0+1,k}$ which is on or between the two spheres $S^n_{\omega(\nu_0)}$ and $S^n_{c_{m_1+1}}$ by $E^{\omega(\nu_0)\circledcirc c_{m_1+1}}_{\nu_0+1,k}$, while we denote its projection on the sphere $S^n_{c_{m_1+1}}$ by $E^{\odot c_{m_1+1}}_{\nu_0+1,k}$. Now, if for $k \in C(S_{\nu_0+1,k})$, $E_{\nu_0+1,k}$ lives outside or on the sphere $S^n_{c_{m_1+1}}$, then we define $D_{\nu_0+1,k}$ to be just E_{ν_0+1} itself. Otherwise, we define the curve $D_{\nu_0+1,k}$ to be the curve constructed from $E_{\nu_0+1,k}$ by replacing $E^{\omega(\nu_0)\circledcirc c_{m_1+1}}_{\nu_0+1,k}$ by $E^{\odot c_{m_1+1}}_{\nu_0+1,k}$. For later use, we will call $E^{\odot c_{m_1+1}}_{\nu_0+1,k}$ the spherical part of the curve $D_{\nu_0+1,k}$ and denote it by $D^{sph}_{\nu_0+1,k}$.

The next step is to partition the curves $D_{\nu_0+1,k}$, $k \in C(S_{\nu_0+1})$. Recall that for every $k \in C(S_{\nu_0+1})$, we already have a partition for the curve $E_{\nu_0+1,k}$ which we denote them by $\chi^j_{\nu_0+1,k}$. Therefore, for each $k \in C(S_{\nu_0+1})$ that the curve $E_{\nu_0+1,k}$ lives outside or on the sphere $S^n_{c_{m_1+1}}$, we already have a partition of $D_{\nu_0+1,k}$ denoted this time by $\zeta^j_{\nu_0+1,k}$, i.e., in this case $\zeta^j_{\nu_0+1,k} := \chi^j_{\nu_0+1,k}$. For $k \in C(S_{\nu_0+1})$ for which the curve $E_{\nu_0+1,k}$ meets the inside of the sphere $S^n_{c_{m_1+1}}$, we just need to define a partition for the spherical part of $D_{\nu_0+1,k}$. For that part, we just take the projection of the points of the partition which lies on $E^{\omega(\nu_0)\circledcirc c_{m_1+1}}_{\nu_0+1,k}$ on the sphere $S^n_{c_{m_1+1}}$. Let again $\zeta^j_{\nu_0+1,k}$ be the points of the partition of $D_{\nu_0+1,k}$. Now, since the values $\omega(\nu_0)$ and c_{m_1+1} are close enough and because it is the first time that we are projecting those parts of our curves which lie on or between the spheres $S_{\omega(\nu_0)}$ and $S^n_{c_{m_1+1}}$ on the sphere $S^n_{c_{m_1+1}}$, therefore the points that are supposed to compare to each other stay close enough to each other, therefore we allow ourselves to define for every $k \in C(S_{\nu_0+1})$, the curve $C_{\nu_0+1,k}$ to be the curve $D_{\nu_0+1,k}$ itself with $\xi^j_{\nu_0+1,k} := \zeta^j_{\nu_0+1,k}$ as the points of its partition. That is, we define $\omega(\nu_0+1) := c_{m_1+1}$. Now, we define F as follows:

$$F(s_{\nu_0+1,k} \times_{\mathbb{R}_+} i_{\nu_0+1,j}) := \xi^j_{\nu_0+1,k}.$$

Clearly, since we have chosen N bigger than $2\pi \times (L + L')$, therefore the conditions (i)-(iv) are again fulfilled for the constant $\mathcal{L}$ and the entourage E.

As we required in (ii), since we have projected some parts of our curves on a bigger sphere at the step ν_0+1, therefore we take a stabilization period which in this case, it will take until $\nu_1 := \nu_0 + 1 + \mathcal{N}_{\nu_0+1}$. Now, to define F at the step $\nu_1 + 1$, we first have some routine works in front of us, namely

(1) we take the integer m_2 which is the first integer coming after $m_1 + 1$ with $c_{m_2} > c_{m_1+1}$ (recall that $c_{m_1+1} = \omega(\nu_1)$);

(2) then we consider the position of $\nu_1 + 1$, i.e., for which integer $\mathcal{M}_q$ the inequality $m_0 + \mathcal{M}_q < \nu_1 + 1 \leq m_0 + \mathcal{M}_{q+1}$ holds. Therefore, the positive real number $c_{m_0+\mathcal{M}_{q-1}}$ is the upper bound for the radius of the sphere that we are allowed to use;

(3) the next step, as we have seen above, is the comparing the values c_{m_2} and $c_{m_0+\mathcal{M}_{q-1}}$.

Now, if as the result of the above process it turns out that we are "possibly" allowed to project on the sphere $S^n_{c_{m_2}}$, then the question which arises is whether we can use the same strategy to define F on $S^j_{\nu_1+1}$ as at the step $\nu_0 + 1$. Knowing that η is coarsely uniform just tells us that the image of the points that are of distance smaller than $S_c > 0$ remains of distance at most L which means the map η may move very wildly which means because we are projecting some parts of our curves on a bigger sphere the following might happen:

"After partitioning the curves $D_{\nu_1+1,k}$, $k \in C(S_{\nu_1+1})$ as above, for some $k \in C(S_{\nu_1+1})$, there are points $\zeta^j_{\nu_1+1,k}$ and $\zeta^{j+1}_{\nu_1+1,k}$ of the partition of the curve $D_{\nu_1+1,k}$ such that

$$\|\zeta^j_{\nu_1+1,k} - \zeta^{j+1}_{\nu_1+1,k}\| > 2N.\text{"} \tag{4.0.2}$$

For a moment, let us assume that (4.0.2) happens just for one $k_0 \in C(S_{\nu_1+1})$ and just for one j, say j_0, in the partition of the curve D_{ν_1+1,k_0}. We call our strategy to overcome this sort of difficulty "*saving time-units*". Roughly speaking, what we understand under saving a time-unit on a point ζ (or at j) is to define F such that it has the same value ζ at both two points $s_{m,k} \times_{\mathbb{R}_+} i_{m,j}$ and $s_{m,k} \times_{\mathbb{R}_+} i_{m,j+1}$. Let us remind the fact that not only the points of partitions are important but the way that we define F by using them because finally we are going to compare the value of F at points $s_{m,k} \times_{\mathbb{R}_+} i_{m,j}$ and $s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'}$ with $\left\|i_j - i_{j'}\right\| \leq S_c$. To be more precise, let us go back to our situation at the step $\nu_1 + 1$. What we wanted was to define $\omega(\nu_1 + 1) := c_{m_2}$ but because (4.0.2) happened, we again define $\omega(\nu_1 + 1) := \omega(\nu_1)$. As we mentioned in (ii), for each $k \in C(S_{\nu_1+1})$, we use the points $s_{\nu_0+1,k'}$, $k' \in C(S_{\nu_0+1})$, to determine the middle part of the curves $C_{\nu_1+1,k}$, that is the middle parts are defined by using the curves $C_{\nu_0+1,k'}$, where k''s have been determined by the equations like $a^{\nu_0+1}_{m,k} = s_{\nu_0+1,k'}$. Now, we define F on $S^j_{\nu_1+1}$, $j = 0, 1, \cdots, 3(\nu_1 + 1) + 1$, as follows: for each

$k \in C(S_{\nu_1+1})$, let $\xi^j_{\nu_1+1,k}$ be the points of the partition of $C_{\nu_1+1,k}$ (recall that they are determined directly by the points $\chi^j_{\nu_1+1,k}$ of the partition of the curve $E_{\nu_1+1,k}$). We define

$$F(s_{\nu_1+1,k} \times_{\mathbb{R}_+} i_{\nu_1+1,j}) := \begin{cases} \xi^j_{\nu_1+1,k}, & 0 \le j \le j_0 \\ \xi^{j-1}_{\nu_1+1,k}, & j > j_0, \end{cases}$$

where, as we said above, j_0 is the only point with $\left\| \zeta^{j_0}_{\nu_1+1,k} - \zeta^{j_0+1}_{\nu_1+1,k} \right\| > 2N$. Therefore,

$$F(s_{\nu_1+1,k} \times_{\mathbb{R}_+} i_{\nu_1+1,j_0}) = F(s_{\nu_1+1,k} \times_{\mathbb{R}_+} i_{\nu_1+1,j_0+1}),$$

and this is what we mean by saving a time-unit. Now, because we have three more time-units at the step ν_1+1, we can spend two of them to overcome the changes and save one as we explained on the point $\zeta^{j_0}_{\nu_1+1,k}$. The important technical point in saving time-units, as we can see above, is that

(iv) if we have to save a time-unit at j for some $k_0 \in C(S_m)$, then we will save a time-unit at j for all $k \in C(S_m)$.

Above, we assumed that (4.0.2) happens just for one j_0 and just for one $k_0 \in C(S_{\nu_1+1})$ but the fact is that since the curve $E_{\nu_1+1,k}$ might has wrapped so many times, so it might happen that $\|\zeta^j_{\nu_1+1,k} - \zeta^{j+1}_{\nu_1+1,k}\| > 2N$ for many points of the partition of the curve $D_{\nu_1+1,k}$ all of a sudden and this may also happen for many $k \in C(S_{\nu_1+1})$. In this case, our strategy is to stay at the sphere $S^n_{\omega(\nu_1)}$ as long as we have saved a time-unit at each j with $\|\zeta^j_{\nu_1+1,k} - \zeta^{j+1}_{\nu_1+1,k}\| > 2N$ for all $k \in C(S_{\nu_1+1})$. Obviously, at some step $m'-1$, we have saved a time-unit on every needed point of the partiton of the curve $C_{m'-1,k}$ for all $k \in C(S_{m'-1})$. Note that

(v) in the whole of a period that we are staying at some sphere in order to save enough time-units we will use the points $s_{m'',k''}$ to determine the middle parts of our curves, where m'' is the last previous step at which we have projected some parts of our curves on a bigger sphere. That is, for example, we will use the curves $C_{\nu_0+1,k}$ to define the middle parts of our curves in the period $\nu_1 + 1 \le m \le m' - 1$.

Now, we explain the way that F is defined at the step m'. First of all, finally, we define $\omega(m') := c_{m_2}$. For each $k \in C(S_{m'})$, let $D_{m',k}$ be the curve defined as above with $\zeta^j_{m',k}$ as the points of its partition. For each $k \in C(S_{m'})$, the curve $C_{m',k}$ is defined to be the curve $D_{m',k}$ itself but we define the points of its partition as follows: let j' be the first index with $\|\zeta^{j'}_{m',k} - \zeta^{j'+1}_{m',k}\| > 2N$, then we define

$$\xi^j_{m',k} := \begin{cases} \zeta^j_{m',k}, & 0 \le j \le j' \\ \vartheta^{j'}_{m',k}, & j = j'+1 \\ \zeta^{j'+1}_{m',k}, & j = j'+2, \end{cases}$$

where $\vartheta^{j'}_{m',k}$ is the middle point of the line segment joining two points $\zeta^{j}_{m',k}$ and $\zeta^{j'+1}_{m',k}$ whenever this middle point lies on or outside of the sphere $S^n_{c_{m_2}}$ or it is the projection of this middle point on the sphere $S^n_{c_{m_2}}$ if it lies inside this sphere. We keep doing this for all the points of the partition on which we have saved a time-unit. So, until now, for each $k \in C(S_{m'})$, we have defined a curve $C_{m',k}$ joining two points $\eta(s_{m',k})$ and $\tilde{f}(s_{m',k})$ partitioned by the points $\xi^j_{m',k}$ such that j's run over $0 \leq j \leq 3m'+1$ and $\|\xi^j_{m',k} - \xi^{j+1}_{m',k}\| \leq 2N$ for every $j = 0, 1, \cdots, 3m'-1$. But, we are not still done, because for each $k \in C(S_{m'})$, the nearby points on the curve $C_{m',k}$ are not the only points are supposed to compare to each other, that is,

"It might happen that for two different k and k', there are points $\xi^j_{m',k}$ and $\xi^{j'}_{m'',k'}$ of the partitions of the curves $C_{m',k}$ and $C_{m'',k'}$, respectively, with this property that they are supposed to compare to each other and $\|\xi^j_{m',k} - \xi^{j'}_{m'',k'}\| > 2N$, where $m'' = m'$ or $m'' = m' - 1$."

This sort of difficulty may arise at the beginning when we were projecting on the bigger sphere or even at the time that we were choosing the new points $\vartheta^{j'}_{m',k}$. But, we have already prepared ourselves to overcome this difficulty, namely, by considering the stabilization periods. Using the stabilization periods guarantee that around each point $\eta(s_{m',k})$ in each direction there are relatively big number of points which have the same middle part as the curve $C_{m',k}$ or a middle part which has been determined by a nearby point of the point that has determined the middle part of the curve $C_{m',k}$ and this means that we can replace some of these middle parts smoothly by appropriate curves (with the same number of points of partition) whenever it is necessary to fill the undesired gap between the middle parts of two curves joining two nearby points for which the above has happened. That is we can finally define F on $S^j_{m'}$ with the desired properties.

The only fact that maybe we should add here is the way that we determine the radius of the next sphere on which we are going to project our curves:

(vi) if the radius of the sphere on which we have projected for the last time is c_{m_p}, then the radius of the next sphere on which we are going to project is m_q which is the first integer coming after m_p with $c_{m_q} > c_{m_p}$.

Obviously, we will not get a stuck at any constant c_m, that is the function $\omega : \mathbb{N} \to \mathbb{R}_+$ is coarsely proper and with this we are done because one can define F with the desired properties (i)-(iv) by repeating this process.

The map Φ is injective. Since in the case that $r < n$ any two pointed continuous map $f, g : (S^r, *) \to (S^n, *)$ are homotopic, therefore, it is enough to consider the case $r = n$. Hence, let $f, g : (S^n, *) \to (S^n, *)$ be two pointed Lipschitz maps such that $\Phi([f]) = \Phi([g])$, i.e., $\tilde{f} \simeq_{crs} \tilde{g}$. Therefore, there is a coarse homotopy $F : S^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to S^n_{\mathbb{R}_+}$ with $F \circ i_0^* \sim_{cl} \tilde{f}$ and $F \circ i_1^* \sim_{cl} \tilde{g}$. Let $L > 0$ be a positive real number such that

$$\left\|F(s_{m,k} \times_{\mathbb{R}_+} i_{m,j}) - F(s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'})\right\| \leq L$$

if $\left\|s_{m,k} \times_{\mathbb{R}_+} i_{m,j} - s_{m',k'} \times_{\mathbb{R}_+} i_{m',j'}\right\|_R \leq S_c$. The map F to be pointed means that there exists a positive real number $M > 0$ such that for every $m \in \mathbb{Z}_+$ the following holds:

$$\forall j,\ j = 0, 1, \cdots, 3m : \ \left\|F(s_{m,0} \times_{\mathbb{R}_+} i_{m,j}) - s_{m,0}\right\| \leq M.$$

On the other hand, F to be coarsely proper implies that for each $T > 0$, there is a positive integer $m \in \mathbb{Z}_+$ such that $F(S_l^j) \cap S_k = \emptyset$ for every integer $l \geq m$ and every $k \leq T$, where S_k is the sphere of radius k centered at the origin in $\mathbb{R}^{n+1}$. Now, we first choose $T_0 > 0$ big enough such that $\pi \times (L+M)/4T_0$ is small enough; and then assume that m_0 is a positive integer having above property with respect to T_0. By definition, $F(S^0_{m_0}) = \tilde{f}|_{S_{m_0}}$ and $F(S^{3m_0}_{m_0}) = \tilde{g}|_{S_{m_0}}$. As we have seen in the first part, for each j, $j = 0, 1, \cdots, 3m_0$, we can assign to $F|_{S^j_{m_0}}$ a pointed continuous map $\widetilde{F}_j : S^n \to S^n$. Clearly, the degree of $\widetilde{F}_0 : S^n \to S^n$ is equal to the degree of f. At the same time, since

$$\left\|F(s_{m_0,k} \times_{\mathbb{R}_+} i_{m_0,j}) - F(s_{m_0,k} \times_{\mathbb{R}_+} i_{m_0,j+1})\right\| \leq L,$$

for every $k \in C(S_{m_0})$ and for every $j = 0, 1, \cdots, 3m_0$ and because F is pointed, therefore the degree of the induced pointed continuous map $\widetilde{F}_j$ is equal to the degree of induced pointed continuous map $\widetilde{F}_{j+1}$. Therefore $\deg(\widetilde{F}_0) = \deg(\widetilde{F}_{3m_0})$ which means $\deg(f) = \deg(g)$ which implies that f and g are homotopic, i.e., Φ is injective. $\square$

Chapter 5

Coarse homotopy groups and coarse CW-complexes

In chapter 3, we have defined the coarse homotopy groups of a pointed coarse topological R-space X, and established some of their properties. Then, in chapter 4, we pursued a big step forward and calculated the coarse homotopy groups of the standard coarse spheres. In this chapter, we will see some applications of the main theorem of the chapter 4: we first prove some important theorems concerning the coarse homotopy groups of coarse CW-complexes. Next, we state and prove a coarse version of the theorem of J. H. C. Whitehead. And at the end of this chapter, we carry over some classical results to the coarse setup which lead us to coarse Eilenberg-Maclane spaces (for the classical results see for example [Mau]). The key point in all proofs is Theorem 4.0.5 directly or the details of its proof. Throughout this chapter, we shall assume unless otherwise stated that all coarse topological spaces including coarse CW-complexes are pointed $\mathbb{R}_+$-spaces.

5.1 Coarse homotopy groups of coarse CW-complexes

In this section, we first prove a general theorem to the effect that $\pi_r^{crs}(X, Y)$ $= 0$ in certain circumstances, at least if X and Y are coarse CW-complexes. Let V^n be the subset

$$\{x \in D^n_{\mathbb{R}_+} |\ \|x\| = 1\}$$

of $D^n_{\mathbb{R}_+}$ and let $\varphi : D^n \to V^n$ be a bi-Lipschitz homeomorphism which is identity on S^{n-1}. Then,

Lemma 5.1.1. *Let X be a full coarse CW-complex of dimension n and let A_n be the indexing set for the coarse n-cells of X. For each $\alpha \in A_n$, let V_α be the subspace $\Phi^n_\alpha(\mathcal{O}(\varphi(\{x \in D^n |\ \|x\| \le \frac{1}{2}\})))$. Let $V = \cup_\alpha V_\alpha$. Then*

$$\pi_k^{crs}(X \setminus V) \cong \pi_k^{crs}(X^{n-1}),$$

for all k.

Proof. To prove this, we first construct a coarse homotopy $F : X \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to X$, starting with the identity map such that $F((X \setminus V) \times_{\mathbb{R}_+} (1)^{crs}) \subseteq X^{n-1}$ and $F(x, p_X(x)e^{\frac{\pi}{2}i1}) = x$ for all $x \in X^{n-1}$. We define F on X^{n-1} to be the identity homotopy, that is,

$$F(x, p_X(x)e^{\frac{\pi}{2}is}) := x, \text{ if } x \in X^{n-1},$$

and on coarse n-cells as follows:

$$F(\Phi^n_\alpha(y), p_X(\Phi^n_\alpha(y))e^{\frac{\pi}{2}is}) := \Phi^n_\alpha(\|y\| \cdot \varphi((1+s)\psi(\tilde{y}))),$$

if $y \in D^n_{\mathbb{R}_+}$, $\|\psi(\tilde{y})\| \leq \frac{1}{(1+s)}$, $\alpha \in A_n$, and

$$F(\Phi^n_\alpha(y), p_X(\Phi^n_\alpha(y))e^{\frac{\pi}{2}is}) := \Phi^n_\alpha(\|y\| \cdot \varphi(\psi(\tilde{y})/\|\psi(\tilde{y})\|)),$$

if $y \in D^n_{\mathbb{R}_+}$, $\|\psi(\tilde{y})\| \geq \frac{1}{(1+s)}$, $\alpha \in A_n$, where $\tilde{y}$ is the point where the ray radiating from the origin and passing through the point y meets V^n and where $\psi : V^n \to D^n$ is the inverse Lipschitz of φ. One can easily see that F has the required properties. Let $r := H \circ i_1^* : X \setminus V \to X \setminus V$. Clearly, $r \circ i = 1_{X^{n-1}}$ and $i \circ r \simeq_{crs} 1_{X \setminus V}$, where $i : X^{n-1} \to X \setminus V$ is the inclusion. The critical point here is that the proof of Theorem 4.0.5 guarantees that each element of $\pi^{crs}_k(X \setminus V)$ (resp. each element of $\pi^{crs}_k(X^{n-1})$) can be represented by a pointed coarse map $f : S^k_{\mathbb{R}_+} \to X \setminus V$ (resp. by a pointed coarse map $g : S^k_{\mathbb{R}_+} \to X^{n-1}$) which is compatible with the basepoint projections. So, we are done. □

Given an n-dimensional full coarse CW-complex X we can consider a regular ordinary CW-complex K by declaring for each coarse k-cell e^{crs}_α of X, $k \leq n$, the subset $e_\alpha := \{x \in e^{crs}_\alpha |\ p_X(x) = 1\}$ as a k-cell attached via the map $f_{e^{crs}_\alpha}|_{S^{k-1}}$ we have available (see Definition 2.2.13). Assuming K to be compact guarantees that X is the cone of K. From now on, when we use the expression "full coarse CW-complex", we shall assume unless otherwise stated that this property holds.

Theorem 5.1.2. *Let X be an n-dimensional full coarse CW-complex ($n \geq 2$), and let Y be a coarse subcomplex that contains X^{n-1}, Then*

$$\pi^{crs}_k(X, Y) = 0,$$

for all $1 \leq k < n$.

Proof. Let the indexing sets of coarse n-cells for X and Y be A_n and B_n, respectively. As we mentioned, we do not need to care about the characteristic maps because in the case of full coarse CW-complexes, they are just the inclusions. Now, let $\eta : (D^k_{\mathbb{R}+}, S^k_{\mathbb{R}_+}) \to (X, Y)$ be a pointed coarse map with $k < n$. The critical point is that the proof of Theorem 4.0.5 guarantees the existence of a pointed Lipschitz map $f : (I^k, \partial I^k) \to (K, L)$ such that $\tilde{f} \simeq_{crs} \eta$ via a coarse homotopy rel $*_{crs}$, where K is a regular CW-complex of dimension n introduced above, L is a subcomplex of K whose cone is Y itself and where $\tilde{f}$ is the pointed coarse map constructed from f as in the proof of Theorem 4.0.5. Obviously, the indexing sets of n-cells for K and L are A_n and B_n themselves, respectively. Moreover, let Ψ^n_α be the characteristic maps for K. For each $\alpha \in A_n - B_n$, let U_α be the open subspace $\Psi^n_\alpha(\{x \in D^n | \ \|x\| < \frac{2}{3}\})$ of K, and let V be the closed subspace $\bigcup_{A_n - B_n} \Psi^n_\alpha(\{x \in D^n | \ \|x\| \leq \frac{1}{3}\})$. Thus $K \setminus V$ is open. Also, write W_α for $(K \setminus V) \cap U_\alpha$. We shall show that the map $\tilde{f} : (D^k_{\mathbb{R}+}, S^k_{\mathbb{R}_+}) \to (X, Y)$ can be 'pushed off' $\mathcal{O}V$ and hence pushed into Y.

Now I^k can be regarded as the product of k copies of I. Since I is a CW-complex with one 1-cell and two 0-cells, Theorem 7.3.16 of [Mau] yields a CW decomposition of I^k, in which there is just one k-cell. Indeed, if I is "subdivided" by introducing a new 0-cell at $\frac{1}{2}$, this has the effect of subdividing I^k into 2^k hypercubes each of side $\frac{1}{2}$, and the corresponding CW decomposition has 2^k k-cell. This process can be iterated: at the next stage we obtain a CW-decomposition with 2^{2^k} k-cells consisting of hypercubes of side $\frac{1}{4}$, and so on. Now, considering the map $f : (I^k, \partial I^k) \to (K, L)$, $k < n$, the sets $f^{-1}(K \setminus V)$, $f^{-1}(U_\alpha)$ form an open covering of I^k, so that we can iterate the subdivision process until I^k is subdivided into a CW-complex M in which each k-cell is mapped by f into $K \setminus V$ or into one of the sets U_α. We denote the characteristic maps for M by ψ^m_β. Notice also that ∂I^k is a subcomplex of M.

The next step is to construct a pointed coarse map $\theta : \mathcal{O}M \to X$ such that

(a) for each m-cell $\psi^m_\beta(D^m)$ of M, $m < n$, $f(\psi^m_\beta(D^m)) \subseteq K \setminus V \Rightarrow \theta|_{\mathcal{O}(\psi^m_\beta(D^m))} = \tilde{f}|_{\mathcal{O}(\psi^m_\beta(D^m))}$; otherwise

$$f(\psi^m_\beta(D^m)) \subseteq U_\alpha \Rightarrow \theta(\mathcal{O}(\psi^m_\beta(D^m))) \subseteq \mathcal{O}W_\alpha;$$

(b) $\tilde{f} \simeq_{crs} \theta$ rel $\mathcal{O}(\partial I^k)$ and the cone of points of M that are mapped by f into U_α remains in $\mathcal{O}U_\alpha$ throughout the coarse homotopy.

This is done by induction on the skeletons of M. Suppose θ has been defined on $\mathcal{O}M^{m-1}$, $m < n$, so as to satisfy (a) and (b) (it is easy to define θ on $\mathcal{O}M^0$, since the cone of each 0-cell that is mapped by f into U_α can be joined by the cone of a straight line to the cone of a point of W_α). Now consider an m-cell $\psi^m_\beta(D^m)$ of M such that $f(\psi^m_\beta(D^m)) \subseteq U_\alpha$; then $f(\psi^m_\beta(S^{m-1})) \subseteq U_\alpha$ and $\theta(\mathcal{O}(\psi^m_\beta(S^{m-1}))) \subseteq \mathcal{O}W_\alpha$. Since each characteristic map of M can be chosen from the beginning bi-Lipschitz homeomorphism, therefore $\theta|_{\mathcal{O}(\psi^m_\beta(S^{m-1}))}$ represents an element of $\pi^{crs}_{m-1}(\mathcal{O}W_\alpha)$ by

Proposition 2.1.4. But $\pi^{crs}_{m-1}(\mathcal{O}W_\alpha) \cong \pi^{crs}_{m-1}(S^{n-1}_{\mathbb{R}_+})$ by Lemma 5.1.1. On the other hand, $\pi^{crs}_{m-1}(S^{n-1}_{\mathbb{R}_+}) = 0$, by Theorem 4.0.5, since $m < n$. Thus $\theta|_{\mathcal{O}(\psi^m_\beta(S^{m-1}))}$ is coarsely nullhomotopic and hence can be extended to a coarse map $\theta : \mathcal{O}(\psi^m_\beta(D^m)) \to \mathcal{O}W_\alpha$. Moreover, the original coarse homotopy between $\tilde{f}$ and θ on $\mathcal{O}(\psi^m_\beta(S^{m-1}))$ can be extended to a coarse homotopy of $\mathcal{O}(\psi^m_\beta(D^m))$ in $\mathcal{O}U_\alpha$ that starts with $\tilde{f}$ and whose final map is θ on $\mathcal{O}(\psi^m_\beta(S^{m-1}))$; and this final map is coarsely homotopic to θ, rel $\mathcal{O}(\psi^m_\beta(S^{m-1}))$. It follows that we can extend θ to $\mathcal{O}M^m$ so as still satisfy (a) and (b), by using this construction on the cone of m-cells mapped into some U_α, and by defining $\theta = \tilde{f}$ (with the coarse constant homotopy) on the cone of m-cells mapped into $K \setminus V$, the resulting θ (and coarse homotopy) being coarse by Lemma 2.1.10. By induction, therefore θ can be extended to $\mathcal{O}M^k = \mathcal{O}M$, since $k < n$.

Because $\tilde{f}$ is pointed and $\tilde{f} \simeq_{crs} \theta$ rel $\mathcal{O}(\partial I^k)$, therefore $[\tilde{f}]^{crs} = [\theta]^{crs}$. On the other hand, if $k < n$, $[\theta]^{crs}$ is the image under the inclusion map of an element of $\pi^{crs}_k(\mathcal{O}(K\setminus V), \mathcal{O}L)$; but $\pi^{crs}_k(\mathcal{O}(K\setminus V), \mathcal{O}L) = 0$ by Lemma 5.1.1, and hence $[\tilde{f}]^{crs} = [\theta]^{crs} = 0$. It follows that $\pi^{crs}_k(X, Y) = 0$ for $k < n$, as desired. □

Theorem 5.1.3. *Let (X, Y) be a full coarse CW-pair; that is, X is a full coarse CW-complex, Y is a coarse subcomplex and (X, Y) is a pointed coarse pair, and let $i : X^n \cup Y \to X$ be the inclusion map with $n \geq 1$. Then*

(i) $i_* : \pi^{crs}_k(X^n \cup Y) \to \pi^{crs}_k(X)$ is onto for $1 \leq k \leq n$ and isomorphism for $1 \leq k < n$;

(ii) $\pi^{crs}_k(X, X^n \cup Y) = 0$ for $1 \leq k \leq n$.

Proof. Consider the exact coarse homotopy sequence of the pointed coarse pair (X^{m+1}, X^m):

$$\cdots \to \pi^{crs}_{k+1}(X^{m+1}, X^m) \to \pi^{crs}_k(X^m) \xrightarrow{i_\sharp} \pi^{crs}_k(X^{m+1}) \longrightarrow \pi^{crs}_k(X^{m+1}, X^m) \to \cdots,$$

where $i : X^m \to X^{m+1}$ denotes the inclusion map. Now by Theorem 5.1.2, $\pi^{crs}_k(X^{m+1}, X^m) = 0$ for $1 \leq k \leq m$, so that $i_\sharp : \pi^{crs}_k(X^m) \to \pi^{crs}_k(X^{m+1})$ is onto for $1 \leq k \leq m$ and isomorphism for $1 \leq k < m$. Moreover, clearly, $i_\sharp : \pi^{crs}_0(X^m) \to \pi^{crs}_0(X^{m+1})$ is always onto, and is one-one correspondence if $m > 0$.

Hence $i_\sharp : \pi^{crs}_k(X^n) \to \pi^{crs}_k(X^m)$ is isomorphic for $1 \leq k < n$ and onto for $k = n$, for all $m > n$. But the elements of $\pi^{crs}_k(X)$ are represented by pointed coarse maps $\eta : S^k_{\mathbb{R}_+} \to X$. And, once again, as in the proof of Theorem 5.1.2, there is a pointed Lipschitz map $f : S^k \to K$ such that $[\tilde{f}]^{crs} = [\eta]^{crs}$, where K is exactly the regular CW-complex we introduced in the proof of Theorem 5.1.2. Now, since S^k is compact the images must be contained in finite skeleton of K which implies that the image under $\tilde{f}$ must be contained in finite skeletons of X. A similar argument applies to coarse homotopies of $S^k_{\mathbb{R}_+}$ in X, so that $i_\sharp : \pi_k^{crs}(X^n) \to \pi_k^{crs}(X)$ is isomorphism for $1 \leq k < n$ and onto for $k = n$. To deduce the first part of (a), observe that $\pi_k^{crs}(X^n \cup Y^{n+1}) \to \pi_k^{crs}(X^n \cup Y)$ is an isomorphism for $1 \leq k \leq n$, and $\pi_k^{crs}(X^n \cup Y^{n+1}) \to \pi_k^{crs}(X^{n+1})$ is isomorphism if $k < n$ and onto if $k = n$, by another application of Theorem 5.1.2. Now, the coarse homotopy sequence of the pair $(X, X^n \cup Y)$ gives (b). □

5.2 The coarse Whitehead's theorem

As in the classical case, since coarse CW-complexes are built using attaching coarse maps whose domains are coarse spheres, it is perhaps not too surprising that coarse homotopy groups of coarse CW-complexes carry a lot of information. The main goal of this section is to prove the coarse version of Whitehead's theorem, that states that if $f : K \to L$ is a pointed coarse map of coarse CW-complexes that induces isomorphisms $f_* : \pi_k^{crs}(K) \to \pi_k^{crs}(L)$ for all $k \geq 0$, then f is a coarse homotopy equivalence. We start this section with the following definition.

Definition 5.2.1. Let X and Y be coarse topological $\mathbb{R}_+$-spaces and let $f : X \to Y$ be a coarse map.

- f is called *strongly pointed* if for each coarse subspace x_{crs} of X for which the restriction $p_X|_{x_{crs}} : x_{crs} \to \mathbb{R}_+$ turns into a coarse equivalence, the restriction $p_Y|_{f(x_{crs})} : f(x_{crs}) \to \mathbb{R}_+$ is also a coarse equivalence and $f \circ i_{x_{crs}} \sim_{cl} i_{f(x_{crs})}$, where $i_{x_{crs}}$ and $i_{f(x_{crs})}$ are coarse inverses of $p_X|_{x_{crs}}$ and $p_Y|_{f(x_{crs})}$, respectively.
- f is called a *weak coarse homotopy equivalence* if it is strongly pointed, $f_* : \pi_0^{crs}(X, x_{crs}) \to \pi_0^{crs}(Y, f(x_{crs}))$ is an one-one correspondence, and $f_* : \pi_k^{crs}(X, x_{crs}) \to \pi_k^{crs}(Y, f(x_{crs}))$ is an isomorphism for all $k \geq 1$ and for all coarse subspace x_{crs} of X for which the restriction $p_X|_{x_{crs}} : x_{crs} \to \mathbb{R}_+$ turns into a coarse equivalence.
- X is called *coarsely path connected* if, for every two coarse subspaces $*_1$ and $*_2$ of X for which there are coarse maps $f_k : \mathbb{R}_+ \to X$, $k = 1, 2$, such that $f_k(\mathbb{R}_+) = *_k$, $k = 1, 2$, there exists a coarse map $\alpha : I^{crs}_{\mathbb{R}_+} \to X$ such that $\alpha \circ i_0 \sim_{cl} f_1$ and $\alpha \circ i_1 \sim_{cl} f_2$, where $i_s : R \to I^{crs}_R$ is defined as before by $i_s(t) = te^{\frac{\pi}{2}is}$. We call such an α a *coarse path joining* $*_1$ and $*_2$.

Theorem 5.2.2. *Let X be a coarse CW-complex and let Y be a coarse subcomplex of X such that the inclusion map $i : Y \to X$ is a weak coarse homotopy equivalence. Let Z be a coarsely path connected coarse CW-complex,* *with a coarse 0-cell as basepoint. Then for any choice of basepoint in Y, $i_* : [Z;Y]_*^{crs} \to [Z;X]_*^{crs}$ is an one-one correspondence.*

Proof. We show first that i_* is onto. Suppose, then, that we have a pointed coarse map $f : Z \to X$; we shall show by induction on the skeletons of Z that f can be coarsely deformed into Y. The map f is regarded as a map of $Z \times_{\mathbb{R}_+} (0)^{crs}$ to X (recall that $Z \times_{\mathbb{R}_+} (0)^{crs} \cong_{crs} Z$, by Lemma 2.2.15), and will be extended to a coarse map $f : Z \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs} \to X$ such that $f(Z \times_{\mathbb{R}_+} (1)^{crs}) \subseteq Y$, and if K is any coarse subcomplex of Z that is mapped by f into Y, then f is coarsely constant on $K \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}$, that is, $f(k, p_Z(k)e^{\frac{\pi}{2}is}) = f(k)$ for all $k \in K$ and $0 \leq s \leq 1$. Moreover, we can assume from the beginning that f maps the basepoint of Z, say z^{crs}, into Y: assume that a coarse 0-cell y^{crs} in Y has been chosen as the basepoint of Y. Obviously, it can be also regarded as the basepoint of X. Now, since $f : Z \to X$ is pointed, therefore a pointed coarse map $f' : Z \to X$ can be defined such that it is close to f (and therefore coarsely homotopic to f) and also maps the basepoint of Z into Y. Thus, in particular the above coarse extension will be pointed.

Given such a coarse subcomplex K (the above argument also shows that such a coarse subcomplex K exists), write $M^n = Z^n \cup K$, and extend f as the coarsely constant homotopy to $(Z \times_{\mathbb{R}_+} (0)^{crs}) \cup (K \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs})$, i.e., $f(k, p_Z(k)e^{\frac{\pi}{2}is}) := f(k)$ for every $k \in K$ and $0 \leq s \leq 1$. If e^{crs} is any coarse 0-cell of $Z \setminus K$, since Z is coarsely path connected there is a coarse path $\alpha : I_{\mathbb{R}_+}^{crs} \to X$ such that $\alpha \circ i_0 \sim_{cl} f \circ \Phi_{e^{crs}}^0$ and $\alpha \circ i_1 \sim_{cl} f \circ \Phi_{z^{crs}}^0 \subseteq Y$ (sometimes we state the second relation simply by writing $\alpha \circ i_1 \overset{cl}{\subseteq} Y$); thus we can extend f to $M^0 \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}$ by setting $f(z, p_Z(z)e^{\frac{\pi}{2}is}) = \alpha(e^{\frac{\pi}{2}is}), 0 \leq s \leq 1$. This serves to start the induction; so we may now assume that f has been extended to a coarse map $f : (Z \times_{\mathbb{R}_+} (0)^{crs}) \cup (M^{n-1} \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}) \to X$ such that $f(M^{n-1} \times_{\mathbb{R}_+} (1)^{crs}) \subseteq Y$. Now, for each coarse n-cell e_n^{crs} of $Z \setminus K$, Lemma 2.2.17 allows us to consider the following composite

$$\begin{array}{c} (D_{\mathbb{R}_+}^n \times_{\mathbb{R}_+} (0)^{crs}) \cup (S_{\mathbb{R}_+}^{n-1} \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}) \\ \Big\downarrow {\scriptstyle \Phi_{e_n^{crs}} \times_{\mathbb{R}_+} 1_{I_{\mathbb{R}_+}^{crs}}} \\ (Z \times_{\mathbb{R}_+} (0)^{crs}) \cup (M^{n-1} \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}) \\ \Big\downarrow {\scriptstyle f} \\ X, \end{array}$$

which sends $S_{\mathbb{R}_+}^{n-1} \times_{\mathbb{R}_+} (1)^{crs}$ to Y. Now we define a coarse equivalence h of $D_{\mathbb{R}_+}^n \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}$ to itself, for simplicity we will state the precise formula just for $n = 1$, but the same idea works exactly for the higher n:

$$h(re^{\pi is}, (0)^{crs}) := (re^{\frac{\pi}{4}i(2s+1)}, (0)^{crs}) \qquad 0 \leq s \leq 1$$

$$h(re^{\pi is}, re^{\frac{\pi}{2}it}) := \begin{cases} (re^{\frac{\pi}{4}i(1-t)}, (0)^{crs}), & s = 0, 0 \le t \le 1 \\ (re^{\frac{\pi}{4}i(3+t)}, (0)^{crs}), & s = 1, 0 \le t \le 1 \end{cases}$$

$$h(re^{\pi is}, (1)^{crs}) := \begin{cases} (re^{\pi i0}, re^{\frac{\pi}{2}i4s}), & 0 \le s \le \frac{1}{4} \\ (re^{\pi i1}, re^{\frac{\pi}{2}i(4-4s)}), & \frac{3}{4} \le s \le 1 \end{cases}$$

$$h(re^{\pi is}, (1)^{crs}) := (re^{\frac{\pi}{2}i(4s-1)}, (1)^{crs}) \qquad \frac{1}{4} \le s \le \frac{3}{4}$$

extending the definition inside $D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$ by regarding the inside as the join of $(D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} (0)^{crs}) \cup (S^{n-1}_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}) \cup (D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} (1)^{crs})$ to $(re^{\frac{\pi}{2}i}, re^{0i})$ with the equal r. The map h is clearly a homeomorphism, therefore we denote its inverse which is again a coarse map by h^{-1}. The point of this definition is that the map $f \circ (\Phi_{e^{crs}} \times_{\mathbb{R}_+} 1_{I^{crs}_{\mathbb{R}_+}}) \circ h^{-1}$ is a coarse map of $(D^n_{\mathbb{R}_+}, S^{n-1}_{\mathbb{R}_+})$ to (X, Y), which therefore represents an element of $\pi^{crs}_n(X, Y)$, with some basepoint. But by the exact coarse homotopy sequence $\pi^{crs}_n(X, Y) = 0$; thus $f \circ (\Phi_{e^{crs}} \times_{\mathbb{R}_+} 1_{I^{crs}_{\mathbb{R}_+}}) \circ h^{-1}$ can be extended to a coarse map of $D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$ that sends $D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} (1)^{crs}$ and $S^{n-1}_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$ to Y, i.e., there exists a coarse homotopy $F : D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to X$ (rel $S^{n-1}_{\mathbb{R}_+}$) such that

$$F \circ i_0^* \sim_{cl} f \circ (\Phi_{e^{crs}} \times_{\mathbb{R}_+} 1_{I^{crs}_{\mathbb{R}_+}}) \circ h^{-1},$$

$$F((D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} (1)^{crs}) \cup (S^{n-1}_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+})) \overset{cl}{\subseteq} Y.$$

Hence, by applying h again, $H := F \circ h$ extends $f \circ (\Phi_{e^{crs}} \times_{\mathbb{R}_+} 1_{I^{crs}_{\mathbb{R}_+}})$ to a coarse map of $D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$ that sends $D^n_{\mathbb{R}_+} \times_{\mathbb{R}_+} (1)^{crs}$ to Y. Since f has already been defined on $\Phi_{e^{crs}}(S^{n-1}_{\mathbb{R}_+})$ and because $\Phi_{e^{crs}}$ is a coarse equivalence on $D^n_{\mathbb{R}_+} \setminus S^{n-1}_{\mathbb{R}_+}$, hence f can be extended by $H \circ \Psi_{e^{crs}}$ to a coarse map of $(Z \times_{\mathbb{R}_+} (0)^{crs}) \cup ((e^{crs} \cup M^{n-1}) \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+})$ such that $f((e^{crs} \cup M^{n-1}) \times_{\mathbb{R}_+} (1)^{crs}) \subseteq Y$, where $\Psi_{e^{crs}}$ is coarse inverse of $\Phi_{e^{crs}}$ on e^{crs}. These process defines an extension of f to a function $f : (Z \times_{\mathbb{R}_+} (0)^{crs}) \cup (M^n \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}) \to X$ such that $f(M^n \times_{\mathbb{R}_+} (1)^{crs}) \subseteq Y$. Moreover this extension is a coarse map: for $(Z \times_{\mathbb{R}_+} (0)^{crs}) \cup (M^n \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+})$ is a coarse CW-complex, and the composite of each of its characteristic maps with f is coarse; hence f is coarse by Lemma 2.1.10. The inductive step is now complete, hence f can be extended to a function $f : Z \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to X$ such that $f(Z \times_{\mathbb{R}_+} (1)^{crs}) \subseteq Y$, that is, $i_* : [Z, Y]^{crs}_* \to [Z, X]^{crs}_*$ is onto.

It is easy to deduce that i_* is also injective. For suppose $f, g : Z \to Y$ are pointed coarse maps such that $i \circ f \simeq_{crs} i \circ g$ by a pointed coarse homotpy $F : Z \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to X$. Since $Z \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$ is a coarsely path connected coarse CW-complex and $K := (Z \times_{\mathbb{R}_+} (0)^{crs}) \cup (z^{crs} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}) \cup (Z \times_{\mathbb{R}_+} (1)^{crs})$ is a subcomplex with $F(K) \subseteq Y$, so F can be coarsely deformed to a coarse map $G : Z \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to Y$ such that G coincides with F on K. That is, G is a pointed coarse homotopy between f and g. □

It is easy to extend Theorem 5.2.2 to a weak coarse homotopy equivalence which is strongly compatible with the basepoint projections, by using the coarse mapping cylinder defined as follows:

Definition 5.2.3. Let X and Y be pointed coarse topological R-spaces and let $f : X \to Y$ be a pointed coarse map which is strongly compatible with the basepoint projections. If the inverse image of bounded subsets under f are bounded (for example, this is the case if X and Y are pointed proper coarse topological R-spaces with X unital), then we define the *coarse mapping cylinder* of f, M_f^{crs}, to be the pointed coarse topological R-space obtained from Y by weakly coarse attaching $(X \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs})/(*_{crs} \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs})$ via f, where X has been identified with the subspace $\{[x,(1)^{crs}] |\ x \in X\}$.

Let $g : X \to M_f^{crs}$ be the inclusion of X in $X \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}$ (as $X \times_{\mathbb{R}_+} (0)^{crs}$), followed by the natural map, and let $h : M_f^{crs} \to Y$ be the map induced by the identity map of Y and the map from $X \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}$ to Y that sends each $(x, p_X(x)e^{\frac{\pi}{2}is})$ to $f(x)$. Clearly, $f = h \circ g$. Moreover,

Theorem 5.2.4. *Let X and Y be pointed proper coarse topological $\mathbb{R}_+$-spaces with X unital, and let $f : X \to Y$ be a pointed coarse map which is strongly compatible with the basepoint projections. The map $h : M_f^{crs} \to Y$ defined as above, is indeed a coarse homotopy equivalence.*

Proof. To show that h is a coarse homotopy equivalence, define $j : Y \to M_f^{crs}$ to be the inclusion of Y in the disjoint union $Y \amalg ((X \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs})/(*_{crs} \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs}))$ followed by the natural map. Then $h \circ j = 1_Y$, and $j \circ h : M_f^{crs} \to M_f^{crs}$ is given by

$$j \circ h(y) = y,$$
$$j \circ h([x, p_X(x)e^{\frac{\pi}{2}is}]) = f(x).$$

A coarse homotopy $F : M_f^{crs} \times_{\mathbb{R}_+} I_{\mathbb{R}_+}^{crs} \to M_f^{crs}$ between $1_{M_f^{crs}}$ and $j \circ h$ can be defined by

$$F(y, p_Y(y)e^{\frac{\pi}{2}it}) = y,$$
$$F([x, p_X(x)e^{\frac{\pi}{2}is}], p_X(x)e^{\frac{\pi}{2}it}) = [x, p_X(x)e^{\frac{\pi}{2}i(s+t(1-s))}].$$

That the map F is coarse follows from Theorem 2.2.10 by an argument similar to what we have seen in the proof of Theorem 3.2.9. □

Corollary 5.2.5. *Let X and Y be coarse CW-complexes with X unital. Given a weak coarse homotopy equivalence $f : X \to Y$ strongly compatible with the basepoint projections, and a coarsely path connected coarse CW-complex Z, $f_* : [Z; X]_*^{crs} \to [Z; Y]_*^{crs}$ is an one-one correspondence (where Z has a coarse 0-cell as basepoint, and X, Y have any basepoints that correspond under f).*

Proof. As we have seen before, we can assume, without lose of generality, that our coarse CW-complexes are full. By Theorem 5.2.4, f is the composite

$$X \xrightarrow{g} M_f^{crs} \xrightarrow{h} Y,$$

where M_f^{crs} is the coarse mapping cylinder of f, the map g is the inclusion map, and h is a coarse homotopy equivalence. The critical point here is again that the proof of Theorem 4.0.5 guarantees that each element of $\pi_k^{crs}(M_f^{crs})$ (resp. each element of $[Z; M_f^{crs}]_*^{crs}$) can be represented by a pointed coarse map $S^k_{\mathbb{R}_+} \to M_f^{crs}$ (resp. by a pointed coarse map $Z \to M_f^{crs}$) which is compatible with the basepoint projections, that is, h is indeed a weak coarse homotopy equivalence (resp. h_* is indeed an one-one correspondence). Now, since both f and h are weak coarse homotopy equivalences, so is g; hence $g_* : [Z; X]_*^{crs} \to [Z; M_f]_*^{crs}$ is an one-one correspondence by Theorem 5.2.2. Therefore, $f_* = h_* \circ g_*$ is an one-one correspondence. □

Note that in the proof of Corollary 5.2.5, we do not need to show that M_f^{crs} is a coarse CW-complex, because looking at the details of the proof of Theorem 5.2.2, one can see that we did not actually use the assumption that X is a coarse CW-complex. Now, the coarse whitehead's theorem follows immediately.

Theorem 5.2.6. *If $f : X \to Y$ is a weak coarse homotopy equivalence of coarsely path connected coarse CW-complexes, then f is a coarse homotopy equivalence.*

Proof. By corollary 5.2.5, $f_* : [Y; X]_*^{crs} \to [Y; Y]_*^{crs}$ is a one-one correspondence. Now, by looking at the details of the proof of Theorem 5.2.2 and Corollary 5.2.5, one can see that since the identity map 1_Y is compatible with the basepoint projections, the coarse map $\gamma : Y \to X$ for which $f_*([\gamma]^{crs}) = [1_Y]^{crs}$ is indeed compatible with the basepoint projections. On the other hand, $f \circ \gamma \simeq_{crs} 1_Y$ immediately implies that γ is also a weak homotopy equivalence. So by a similar argument there exists $\delta : X \to Y$ compatible with the basepoint projections such that $\gamma \circ \delta \simeq_{crs} 1_X$. From $f \circ \gamma \simeq_{crs} 1_Y$ follows that $(f \circ \gamma) \circ \delta \simeq_{crs} \delta$, since δ is compatible with the basepoint projections. Therefore,

$$f \simeq_{crs} f \circ (\gamma \circ \delta) \simeq_{crs} (f \circ \gamma) \circ \delta \simeq_{crs} \delta,$$

so that $\gamma \circ f \simeq_{crs} 1_X$ as well, and so γ is a coarse homotopy inverse to f. □

The next important theorem in this chapter is the Coarse Cellular Approximation Theorem, which in a sense is the analogue for coarse CW-complexes of the Cellular Approximation Theorem for CW-complexes in the classical case. The following is our understanding of a coarse cellular map:

Definition 5.2.7. Let X and Y be coarse CW-complexes. A coarse map $f : X \to Y$ is called *cellular* if $f(X^n) \subseteq Y^n$ for each $n \geq 0$.

Theorem 5.2.8. *If X and Y are full coarse CW-complexes, Y is coarsely path connected, and $f : X \to Y$ is a coarse map such that $f|_K$ is coarse cellular for some subcomplex K of X (possibly empty), then there exists a coarse cellular map $g : X \to Y$ such that $g|_K = f|_K$ and $g \simeq_{crs} f$ rel M.*

Proof. This is very similar to Theorem 5.2.2: by induction on the skeletons of X, we define a coarse homotopy $F : X \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to Y$ that starts with f ends with a coarse cellular map, and is the coarsely constant homotopy on $K \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$. Since, for each coarse 0-cell e^{crs}_0 of $X \setminus K$, there is a coarse path in Y from $f(e^{crs}_0)$ to a coarse 0-cell of Y, we can certainly define F on $X^0 \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \cup K \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$. Suppose, then, that F has been extended to $X^{n-1} \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+}$, and that $F(K^{n-1} \times_{\mathbb{R}_+} (1)^{crs}) \subseteq Y^{n-1}$. Just as in Theorem 5.2.2, F can be extended to each coarse n-cell of $X \setminus K$, since $\pi^{crs}_n(Y, Y^n) = 0$ by Theorem 5.1.3; and the result is a coarse map such that $F(X^n \times_{\mathbb{R}_+} (1)^{crs}) \subseteq Y^n$. This completes the inductive step, and so gives the required coarse homotopy $F : X \times_{\mathbb{R}_+} I^{crs}_{\mathbb{R}_+} \to Y$. □

Theorem 5.2.9. *Given a coarsely path connected full coarse CW-complex X and an integer $n \geq 0$, there exists a full coarse CW-complex Y, having X as a coarse subcomplex, such that, if $i : X \to Y$ is the inclusion map*

(i) $i_* : \pi^{crs}_k(X) \to \pi^{crs}_k(Y)$ *is isomorphic for* $k < n$;

(ii) $\pi^{crs}_n(Y) = 0$.

Proof. Let A be a set of generators for the group $\pi^{crs}_n(X)$ (for example, the set of all elements of $\pi^{crs}_n(X)$). For each $\alpha \in A$, take a representative pointed coarse map $\Phi^n_\alpha : S^n_{\mathbb{R}_+} \to X$ which by Theorems 4.5 and Theorem 5.2.8 may be assumed to be a homeomorphism onto its image, strongly compatible with the basepoint projections and cellular. Let Y be the coarse topological $\mathbb{R}_+$-space obtained from X by coarse attaching coarse cells $D^{n+1}_{\mathbb{R}_+}$ by the coarse maps Φ^n_α, one for each $\alpha \in A$. Obviously, Y is a full coarse CW-complex having X as its subcomplex. Moreover, by Theorem 5.1.3(i) $i_* : \pi^{crs}_k(X) = \pi^{crs}_k(Y^n \cup X) \to \pi^{crs}_k(Y)$ is isomorphic for $k < n$, and onto for $k = n$. But for each $\alpha \in A$, $i_*(\alpha) \in \pi^{crs}_n(Y)$ is represented by the coarse map $i \circ \Phi^n_\alpha : S^n_{\mathbb{R}_+} \to Y$; and this is clearly coarsely homotopic to the coarse constant map to the basepoint of Y, since Y has an coarse $(n+1)$-cell attached by Φ^n_α. Hence $\pi^{crs}_n(Y) = 0$. □

This process can be iterated, so as to "kill off" $\pi^{crs}_k(Y) = 0$ for all $k \geq n$.

Theorem 5.2.10. *Given a coarsely path connected full coarse CW-complex X and an integer $n \geq 0$, there exists a coarse CW-complex Y, having X as a coarse subcomplex, such that, if $i : X \to Y$ is the inclusion map* *(i)* $i_* : \pi_k^{crs}(X) \to \pi_k^{crs}(Y)$ *is isomorphic for* $k < n$*;*

(ii) $\pi_k^{crs}(Y) = 0$ *for* $k \geq n$*.*

Proof. By repeated applications of Theorem 5.2.9, there is a sequence of full coarse CW-complexes $X \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$, each a coarse subcomplex of the next, such that for each $m \geq 1$, if $i : X \to Y_m$ is the inclusion map,

(i) $i_* : \pi_k^{crs}(X) \to \pi_k^{crs}(Y_m)$ is isomorphic for $k < n$, and

(ii) $\pi_k^{crs}(Y_m) = 0$ for $n \leq k < n + m$.

Let $Y = \bigcup_{m=1}^{\infty} Y_m$ equipped with the weak topology and with the weak coarse structure. Since we could choose the representative pointed maps $\Phi_\alpha^n : S_{\mathbb{R}_+}^n \to X$ strongly compatible with the basepoint projections, therefore we can define a basepoint projection $p_Y : Y \to \mathbb{R}_+$ with desired properties, that is, Y is indeed a coarse CW-complex having each Y_m, and X as a coarse subcomplex.

To prove (i) and (ii), note that, given any k, $i_* : \pi_k^{crs}(Y^{k+1}) \to \pi_k^{crs}(Y)$ is an isomorphism. But Y^{k+1} is the $(k+1)$-skeleton of each Y_m for which $n + m > k$, so that $i_* : \pi_k^{crs}(Y^{k+1}) \to \pi_k^{crs}(Y_m)$ is also an isomorphism for such m. Hence $i_* : \pi_k^{crs}(Y_m) \to \pi_k^{crs}(Y)$ is an isomorphism, and (i) and (ii) are now immediate. □

We finish this chapter by introducing coarse Eilenberg-Maclane spaces:

Definition 5.2.11. We have proved in Theorem 4.0.5 that

$$\pi_k^{crs}(S_{\mathbb{R}_+}^n) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k < n. \end{cases}$$

It follows from Theorem 5.2.10 that there exists a coarse CW-complex K such that $\pi_k^{crs}(K) = 0$ for $k \neq n$ and $\pi_n^{crs}(K) = \mathbb{Z}$. Such a coarse CW-complex is called an *coarse Eilenberg-Maclane space* $K^{crs}(\mathbb{Z}, n)$.

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