Local invariants of four-dimensional Riemannian manifolds and their application to the Ricci flow

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This thesis is dedicated
to the memory of my grandparents
Giorgos and Efthalia.

Abstract

In this thesis, we study the four-dimensional Ricci flow with the help of local invariants. If $(M^4, g(t))$ is a solution to the Ricci flow and $x \in M$, we can associate to the point x a one-parameter family of curves, which lie on a smooth quadric in $\mathbb{P}(T_x M \otimes \mathbb{C})$. This allows us to reformulate the Cheeger-Gromov-Hamilton Compactness Theorem in the context of these curves. Furthermore we study Type I singularities in dimension four and give a characterization of the corresponding singularity models in the context of these curves as well.

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Contents

1. Introduction

The Ricci flow is a geometric evolution equation which deforms Riemannian metrics on manifolds by their Ricci tensor, an equation which turns out to exhibit many similarities with the heat equation. It was introduced by Richard Hamilton in 1982 in his seminal paper [9]. Hamilton's program was to use Ricci flow in order to approach Thurston's Geometrizaton Conjecture. His first result towards this direction was accomplished in this first paper, where Hamilton classified closed 3-manifolds with positive Ricci curvature using Ricci flow. Hamilton's theorem states that under the normalized (volume-preserving) Ricci flow on a closed 3-manifold with positive Ricci curvature, the metric converges exponentially fast in every C^k-norm to a constant positive sectional curvature metric. Four years later Hamilton managed to classify in [10] 4-manifolds with positive curvature operator as well. In 2002 and 2003, Grisha Perelman posted three papers on arXiv [24], [25] , [26] and completed Hamiltons work towards proving the Geometrization Conjecture.

The Ricci flow is a type of nonliner heat equation for the metric and it is expected, that it develops singularities. The most basic examples of Ricci flow singularities are the shrinking round sphere and the neckpinch singularity discussed in Chapter 2 of [2]. Understanding the formation of singularities is a very crucial step. This step was done by Hamilton in [12]. This paper discusses (among other topics) singularity formation, the classification of singularities, applications of estimates and singularity analysis to the Ricci flow with surgery. To study singularities one should take dilations about sequences of points and times where the time tends to the singularity time T. The limit solutions of such sequences, if they exist, are ancient solutions. One distinguishes singularities in two types: those formed at $T < \infty$ and those formed at $T = \infty$. One can show that in the first case the curvature blows up in finite time. There is further categorization of finite time singularities in Type I and Type IIa. Type I singularities blow up in finite time at the rate of the standard shrinking sphere. Type IIa singularites are formed slowly in the sense, that in terms of the curvature scale, the time to blow up is longer than that of the Type I. The prototype of a Type IIa singularity is the degenerate neckpinch. Given a singularity type, the way of picking suitable sequences of points and times about which we dilate, lead to ancient limits, called singularity models. In this text we will only focus on Type I singularities.

We would like to make the notion of the convergence mentioned above a little bit more precise. It has its roots in the converge theory developed by Cheeger and Gromov. Hamilton proved in [13, Theorem 1.2] a compactness theorem, which is today known as the Cheeger-Gromov-Hamilton Compactness Theorem. It roughly states, that any sequence of complete solutions to the Ricci flow having curvatures uniformly bounded from above and injectivity radii uniformly bounded from below, contains a convergent subsequence and the limit exists in an ancient time interval. Showing the bound on the injectivity radius has been a huge obstacle for Hamilton, but the problem was solved by Perelman in [24]. Perelman showed that if the solution becomes singular in finite time his No Local Collapsing Theorem provides such an estimate. In other words, he showed that if $T < \infty$, then there exists $\kappa > 0$ such that the singularity model is κ-noncollapsed at all scales. A nice exposition on Perelman's arguments can be found in [4].

There is some special class of solution to the Ricci flow called Ricci solitons. Ricci solitons correspond to self-similar solution to the Ricci flow and change only by scaling and pullback by diffeomorphisms. They are a natural extension of Einstein metrics, are possible singularity models of the Ricci flow and are critical points of Perelman's λ-entropy and μ-entropy. There exists a special kind of Ricci solitons, which are called gradient Ricci solitons. Ricci solitons can be categorized by their behaviour in steady, shrinking or expanding. Hamilton and Ivey proved in [12] and [17] respectively, that on a compact manifold, a gradient steady or expanding Ricci soliton is necessarily an Einstein metric. More generally, any compact, steady or expanding Ricci soliton must be Einstein. This follows from Perelman's result in [24], that any compact Ricci soliton is necessarily a gradient Ricci soliton. Furthermore by the results of Hamilton and Ivey in [11] and [17] respectively, in dimension $n \leq 3$, there are no compact shrinking Ricci solitons other than the sphere and its quotients. The classification of 3-dimensional gradient shrinking Ricci solitons was done by the works of Perelman [25], Ni-Wallach [22] and Cao-Chen-Zhu [5]. They showed that a 3-dimensional gradient shrinking Ricci soliton is a quotient of either $\mathbb{S}^3$ or $\mathbb{R}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. This means that the only noncompact nonflat 3-dimensional gradient shrinking Ricci solitons are the round cylinder and its quotients. In this text we will focus on the 4-dimensional gradient shrinking Ricci solitons. In dimension 4 there is no full classification of the gradient shrinking Ricci solitons. There is some classification done under curvature assumptions by Ni and Wallach [23] and Naber [20]. A conjecture, normally attributed to Hamilton, is that a suitable blow up sequence for a Type I singularity converges to a nontrivial gradient shrinking Ricci soliton [12]. In the case where the blow up limit is compact, the conjecture was confirmed by Sesum [28, Theorem 1.1]. In the general case, blow up to a gradient shrinking soliton was proved by Naber [20, Theorem 1.4]. However, it remained an open question whether the limit soliton Naber constructed is non trivial (i.e. flat). Enders, Müller and Topping eliminated in [6, Theorem 1.4] this possibility.

In this thesis we try to contribute in the direction of understanding the 4-dimensional gradient shrinking Ricci solitons, which can appear as singularity models for Type I singularities. This is done by considering local invariants for a 4-dimensional Riemannian manifold and trying to interpret the limiting solitons in the language of these local invariants. Let's be more precise.

In Chapter 2 we describe a construction of A. N. Tyurin. Tyurin showed in [29], that for any 4-dimensional Riemannian manifold (M^4, g) and fixed point $x \in M$, one can define in natural way three quadratic forms in $\Lambda^2 T_x M$. These are given by the exterior power evaluated at a volume form, the second exterior power of the Riemannian metric g and the curvature tensor of the Riemannian connection respectively. After complexifying, their projectivization defines three quadrics in $\mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C})$. For any point $x \in M$ at which the quadratic forms are linearly independent, the intersection of these three quadrics defines a singular $K3$ surface. After performing a resolution of the singular points the resolved $K3$ is a double branched cover of a smooth quadric in $\mathbb{P}(T_x M \otimes \mathbb{C})$. In many cases the branching locus corresponds to a curve of bidegree $(4, 4)$ in the product of two projective lines. The branching curve denoted by Γ_x will be our local invariant for the 4-dimensional manifold M. Its coefficients will be determined by the components of the Riemann curvature tensor. Note that four years later, V. V. Nikulin in [21] extended the result to the case of pseudo-Riemannian manifolds with a Lorentz metric.

In Chapter 3 we demonstrate an introduction to the basic theory of the Ricci flow and give some examples. Furthermore we introduce Ricci solitons and their canonical form.

In Chapter 4 we do some explicit calculations, compute examples of local invariants for some 4-dimensional gradient shrinking Ricci solitons.

In Chapter 5 we prove Proposition 5.2.1, which gives the evolution of the coefficients of the branching curve under the Ricci flow.

In Chapter 6 we prove Theorem 6.2.8, which states, that convergence of manifolds in the Cheeger-Gromov sence implies convergence for branching curves. This is the main theorem of our text. We use this result and combine it with the result of Enders, Müller and Topping mentioned above, in order to obtain a characterization of the gradient shrinking Ricci solitons, which can appear as singularity models for Type I singularities. We call this result Corollary 6.2.9. The fact that we only deal with Type I singularities can be explained by the following facts. In the Type I case Perelman's No Local Collpasing Theorem holds and thus by performing a blow up analysis we can pass to the limit. As a result, we can use the branching curves construction, in order to characterize the limiting curve. One should also have in mind, that the blow up analysis for Type II and Type III singularities is very limited, especially in the four dimensional case. The interested reader could take a look for example at John Lott's paper [18], where he gives an extension of Hamiltons Compactness Theorem, that does not assume a lower injectivity radius bound, in terms of Riemannian groupoids .

2. A local invariant of a four-dimensional Riemannian manifold

In this chapter we explain Tyurin's arguments, reproduce the construction and demonstrate it explicitely in a way that suits our needs. This chapter is organized as follows: In the first section we define the quadrics and describe the intersection of the first two. In the next section we intersect with the third quadric and decribe the branching curve. In the last section we make the argument about the surface of type $K3$ more precise and describe it in terms of its exceptional divisors.

2.1. The geometry of three quadrics in $\mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C})$

Let (M, g) be a four-dimensional Riemannian manifold. We denote by $T_x M$ the tangent space at the point $x \in M$. We are going to define three quadratics forms on $\Lambda^2 T_x M$.

The quadratic form v_x :

We define the map

$$\begin{aligned} \Lambda^2 T_x M \times \Lambda^2 T_x M &\rightarrow \Lambda^4 T_x M \\ (u, h) &\mapsto u \wedge h. \end{aligned}$$

Recall, that the volume form vol_M on M is a nowhere vanishing section of $\Lambda^4 T_x^* M$. We identify $\Lambda^4 T_x M$ with $\mathbb{R}$ by evaluating $u \wedge v$ on the volume form, i.e. $\mathrm{vol}_M(u \wedge v)$. So we obtain a bilinear form $v_x : \Lambda^2 T_x M \times \Lambda^2 T_x M \to \mathbb{R}$. This is a well defined bilinear form and does not depent on the choice of basis on $\Lambda^2 T_x M$.

Let now $\{x^i\}_{i=1}^4$ denote local coordinates around x, such that $\{\frac{\partial}{\partial x^i}\}_{i=1}^4$ is a basis for $T_x M$ and $\{dx^i\}_{i=1}^4$ is the dual to it. Then $\{\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\}_{1 \le i < j \le 4}$ and $\{dx^i \wedge dx^j\}_{1 \le i < j \le 4}$, are bases for

$\Lambda^2 T_x M$ and $(\Lambda^2 T_x M)^* \simeq \Lambda^2 T_x^* M$ respectively. Let $u, h \in \Lambda^2 T_x M$ be given by

$$u = \sum_{1 \leq i < j \leq 4} u^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{2.1}$$

and

$$h = \sum_{1 \leq i < j \leq 4} h^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{2.2}$$

with respect to this basis. Then

$$\begin{aligned} \Lambda^2 T_x M \times \Lambda^2 T_x M &\rightarrow \Lambda^4 T_x M \\ (u, h) &\mapsto (u^{12}h^{34} - u^{13}h^{24} + u^{14}h^{23} \\ &\quad + u^{23}h^{14} - u^{24}h^{13} + u^{34}h^{12}) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}. \end{aligned} \tag{2.3}$$

Recall, that the Riemannian volume form is given by $\sqrt{|\det(g)|} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. Then, the bilinear form v_x is now given by

$$\begin{aligned} v_x : \Lambda^2 T_x M \times \Lambda^2 T_x M &\rightarrow \mathbb{R} \\ (u, h) &\mapsto \sqrt{|\det(g)|}(u^{12}h^{34} - u^{13}h^{24} + u^{14}h^{23} + u^{23}h^{14} - u^{24}h^{13} + u^{34}h^{12}). \end{aligned}$$

The associated quadratic form $v_x : \Lambda^2 T_x M \rightarrow \mathbb{R}$ is now given by

$$v_x(u) = 2\sqrt{|\det(g)|}(u^{12}u^{34} - u^{13}u^{24} + u^{14}u^{23}). \tag{2.4}$$

The quadratic form $\Lambda^2 g_x$:

We need at this point the notion of the Kulkarni-Nomizu product. This product is defined for two symmetric $(2,0)$-tensors and gives as a result a $(4,0)$-tensor. Specifically, if k and l are symmetric $(2,0)$-tensors, then the product is defined by

$$\begin{aligned} (k \owedge l)(u_1, u_2, u_3, u_4) \quad := \quad & k(u_1, u_3) l(u_2, u_4) + k(u_2, u_4) l(u_1, u_3) \\ & - k(u_1, u_4) l(u_2, u_3) - k(u_2, u_3) l(u_1, u_4). \end{aligned}$$

Consider now the Riemannian metric g_x and let $u = u_1 \wedge u_2$ and $h = h_1 \wedge h_2$. We define a symmetric bilinear form $\Lambda^2 g_x$ on $\Lambda^2 T_x M$ by defining it on totally decomposable vectors as follows

$$\begin{array}{rcl} \Lambda^2 g_x : \Lambda^2 T_xM \times \Lambda^2 T_xM & \to & \mathbb{R} \\ (u,h) & \mapsto & \frac{1}{2}(g_x \owedge g_x)(u_1,u_2,h_1,h_2) \\ & & = g_x(u_1,h_1)g_x(u_2,h_2) - g_x(u_1,h_2)g_x(u_2,h_1). \end{array}$$

and extending it bilinearly to a bilinear form on the whole $\Lambda^2 T_xM$. This is a well defined bilinear form and does not depent on the choice of basis on $\Lambda^2 T_xM$. This can be also found in the book [27].

For u and h like in (2.1) and (2.2) we obtain, that in components

$$\Lambda^2 g_x(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l}) = \det \begin{bmatrix} g_{ik} & g_{jk} \\ g_{il} & g_{jl} \end{bmatrix} = \frac{1}{2}(g_x \owedge g_x)_{ijkl}.$$

So we obtain a quadratic form

$$\Lambda^2 g_x(u) = \frac{1}{2} \sum_{1 \leq i,k<j,l \leq 4} (g_x \owedge g_x)_{ijkl} u^{ij} u^{kl}. \tag{2.5}$$

The quadratic form $\mathbf{R}_x$:

Let now Rm_x denote the $(4,0)$-Riemann curvature tensor at $x \in M$. We define a symmetric bilinear form R_x on $\Lambda^2 T_xM$ by defining it on totally decomposable vectors as follows

$$\begin{array}{rcl} \mathrm{R}_x : \Lambda^2 T_xM \times \Lambda^2 T_xM & \to & \mathbb{R} \\ (u_1 \wedge u_2, h_1 \wedge h_2) & \mapsto & \mathrm{Rm}_x(u_1,u_2,h_2,h_1). \end{array}$$

and extending it bilinearly to a bilinear form on the whole $\Lambda^2 T_xM$. This is a well defined bilinear form and does not depent on the choice of basis on $\Lambda^2 T_xM$. This can be also found in the book [27].

In the basis $\{\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\}_{1 \leq i<j \leq 4}$ (compare (2.1) and (2.2)) we obtain,

$$\begin{array}{rcl} \mathrm{R}_x : \Lambda^2 T_xM \times \Lambda^2 T_xM & \to & \mathbb{R} \\ (\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^l}) & \mapsto & R_{(ij)(kl)} = R_{ijlk}, \end{array}$$

where $R_{ijlk} = \mathrm{Rm}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k})$. Notice the convention $R_{(ij)(kl)} = R_{ijlk}$, which is used in the whole text. The associated quadratic form is given by

$$\mathrm{R}_x(u) = \sum_{1\leq i,k<j,l\leq 4} R_{ijlk} u^{ij} u^{kl}. \tag{2.6}$$

2.1.1 Remark. There is a reason behind choosing to introduce the quadratic forms with respect to this special basis coming from local coordinates. It is a very common fact when working with the Ricci flow, that the evolution equations of the various geometric quantities are writen with respect to local coordinates.

From now on vector spaces are turned into complexified ones. The quadratic forms (2.4), (2.5) and (2.6) define three quadrics in $\mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C}) \cong \mathbb{P}^5$, given by

$$\mathbb{P}(v_x) = \{[u] \in \mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C}) : u^{12}u^{34} - u^{13}u^{24} + u^{14}u^{23} = 0\}, \tag{2.7}$$

$$\mathbb{P}(\Lambda^2 g_x) = \{[u] \in \mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C}) : \frac{1}{2} \sum_{1\leq i,k<j,l\leq 4} (g_x \owedge g_x)_{ijkl} u^{ij} u^{kl} = 0\} \tag{2.8}$$

and

$$\mathbb{P}(\mathrm{R}_x) = \{[u] \in \mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C}) : \sum_{1\leq i,k<j,l\leq 4} R_{ijlk} u^{ij} u^{kl} = 0\}. \tag{2.9}$$

We would like to take now a closer look at the Grassmannian $\mathrm{Gr}_2(T_x M \otimes \mathbb{C})$ of two-dimensional linear subspaces of $T_x M \otimes \mathbb{C}$. We prefer to look at it as the variety $\mathrm{Gr}_1(\mathbb{P}(T_x M \otimes \mathbb{C}))$ of lines in $\mathbb{P}(T_x M \otimes \mathbb{C})$, where $\mathbb{P}(T_x M \otimes \mathbb{C}) \cong \mathbb{P}^3$. Let

$$w = \sum_{i=1}^{4} w^i \frac{\partial}{\partial x^i}$$

and

$$\tilde{w} = \sum_{j=1}^{4} \tilde{w}^j \frac{\partial}{\partial x^j},$$

where $w, \tilde{w} \in T_x M \otimes \mathbb{C}$. Then $[w] = [w_1, w_2, w_3, w_4]$ and $[\tilde{w}] = [\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4]$ correspond to points in $\mathbb{P}(T_x M \otimes \mathbb{C})$. Their projective span, denoted by $\mathbb{P}$-span$([w], [\tilde{w}])$ represents a line in $\mathbb{P}(T_x M \otimes \mathbb{C})$.

Let pl denote the Plücker embedding

$$\begin{aligned} \mathrm{pl}: \mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C})) &\to \mathbb{P}(\Lambda^2(T_xM \otimes \mathbb{C})) \\ \mathbb{P}\text{-span}([w],[\tilde{w}]) &\mapsto [w \wedge \tilde{w}]. \end{aligned} \tag{2.10}$$

In other words, the Plücker embedding maps a line in $\mathbb{P}(T_xM \otimes \mathbb{C})$ to a point in $\mathbb{P}(\Lambda^2(T_xM \otimes \mathbb{C}))$. Since

$$\begin{aligned} w \wedge \tilde{w} = {} & (w^1\tilde{w}^2 - w^2\tilde{w}^1)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + (w^1\tilde{w}^3 - w^3\tilde{w}^1)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + \\ & + (w^1\tilde{w}^4 - w^4\tilde{w}^1)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} + (w^2\tilde{w}^3 - w^3\tilde{w}^2)\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + \\ & + (w^2\tilde{w}^4 - w^4\tilde{w}^2)\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4} + (w^3\tilde{w}^4 - w^4\tilde{w}^3)\frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}, \end{aligned}$$

the coordinates of $[w \wedge \tilde{w}]$ in the basis $\{\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\}_{1 \le i < j \le 4}$ are given by

$$[w^1\tilde{w}^2 - w^2\tilde{w}^1, w^1\tilde{w}^3 - w^3\tilde{w}^1, w^1\tilde{w}^4 - w^4\tilde{w}^1, w^2\tilde{w}^3 - w^3\tilde{w}^2, w^2\tilde{w}^4 - w^4\tilde{w}^2, w^3\tilde{w}^4 - w^4\tilde{w}^3].$$

We will denote these coordinates by $[u^{12}, u^{13}, u^{14}, u^{23}, u^{24}, u^{34}]$. Oberve that they correspond to the 2×2 minors of the matrix

$$\begin{bmatrix} w^1 & \tilde{w}^1 \\ w^2 & \tilde{w}^2 \\ w^3 & \tilde{w}^3 \\ w^4 & \tilde{w}^4 \end{bmatrix}.$$

We will show now, that $\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C}))$ can be naturally realized as a quadric hypersurface in $\mathbb{P}(\Lambda^2(T_xM \otimes \mathbb{C}))$. Recall that a vector $u \in \Lambda^2(T_xM \otimes \mathbb{C})$ is called totally decomposable if there exist linear independent vectors $w, \tilde{w} \in T_xM \otimes \mathbb{C}$, such that $u = w \wedge \tilde{w}$. Observe that

$$\mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C})) = \{[u] \in \mathbb{P}(\Lambda^2(T_xM \otimes \mathbb{C})) : u \in \Lambda^2(T_xM \otimes \mathbb{C}) \text{ is totally decomposable}\}.$$

2.1.2 Lemma. *The vector $u \in \Lambda^2(T_xM \otimes \mathbb{C})$ is totally decomposable if and only if $u \wedge u = 0$, in coordinates*

$$u^{12}u^{34} - u^{13}u^{24} + u^{14}u^{23} = 0.$$

Proof. Let $u \in \Lambda^2(T_xM \otimes \mathbb{C})$ be totally decomposable, i.e. $u = w \wedge \tilde{w}$. Then

$$u \wedge u = w \wedge \tilde{w} \wedge w \wedge \tilde{w} = 0.$$

We can write u as

$$\begin{aligned} u &= u^{12}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + u^{13}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + u^{14}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} + \\ &\quad + u^{23}\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + u^{24}\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4} + u^{34}\frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}. \end{aligned}$$

Then by a simple computation we obtain, that

$$u \wedge u = 2(u^{12}u^{34} - u^{13}u^{24} + u^{14}u^{23})\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4}.$$

Thus $u \wedge u = 0$ imples that $u^{12}u^{34} - u^{13}u^{24} + u^{14}u^{23} = 0$. Therefore, if u is totally decomposable, then it satisfies that $u^{12}u^{34} - u^{13}u^{24} + u^{14}u^{23} = 0$.

Conversely, let

$$\begin{aligned} u &= u^{12}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + u^{13}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + u^{14}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^4} + \\ &\quad + u^{23}\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + u^{24}\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4} + u^{34}\frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4} \end{aligned}$$

be a vector satisfying

$$u^{12}u^{34} - u^{13}u^{24} + u^{14}u^{23} = 0. \tag{2.11}$$

Then $u \wedge u = 0$. Now we want to show, that u is totally decomposable. For this we consider the following cases.

(i) Suppose first, that $u^{12}, u^{13} \neq 0$. Then using equation (2.11) we can show that

$$u = \Big(u^{12}\frac{\partial}{\partial x^1} + \frac{u^{23}u^{12}}{u^{13}}\frac{\partial}{\partial x^2} + \frac{u^{23}u^{14} - u^{13}u^{24}}{u^{13}}\frac{\partial}{\partial x^4}\Big) \wedge \Big(\frac{\partial}{\partial x^2} + \frac{u^{13}}{u^{12}}\frac{\partial}{\partial x^3} + \frac{u^{14}}{u^{12}}\frac{\partial}{\partial x^4}\Big).$$

(ii) Let $u^{12} = u^{13} = 0$. Then equation (2.11) yields $u^{14}u^{23} = 0$. So we have $u^{14} = 0$ or $u^{23} = 0$ or both are zero. If in this case $u^{14} = u^{23} = 0$ we can write u a

$$u = u^{24}\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^4} + u^{34}\frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4} = (u^{24}\frac{\partial}{\partial x^2} + u^{34}\frac{\partial}{\partial x^3}) \wedge \frac{\partial}{\partial x^4}.$$

If $u^{14} = 0$, $u^{23} \neq 0$, then we decompose u as

$$u = (u^{23}\frac{\partial}{\partial x^2} - u^{34}\frac{\partial}{\partial x^4}) \wedge \Big(\frac{\partial}{\partial x^3} + \frac{u^{24}}{u^{23}}\frac{\partial}{\partial x^4}\Big).$$

If $u^{14} \neq 0$, $u^{23} = 0$ we can write u as

$$u = (u^{14}\frac{\partial}{\partial x^1} + u^{24}\frac{\partial}{\partial x^2} + u^{34}\frac{\partial}{\partial x^3}) \wedge \frac{\partial}{\partial x^4}.$$

So u is totally decomposable.

(iii) If $u^{12} = 0$. $u^{13} \neq 0$, equation (2.11) gives us $u^{13}u^{24} = u^{14}u^{23}$ and u can be decomposed as

$$u = (u^{13}\frac{\partial}{\partial x^1} + u^{23}\frac{\partial}{\partial x^2} - u^{34}\frac{\partial}{\partial x^4}) \wedge \Big(\frac{\partial}{\partial x^3} + \frac{u^{14}}{u^{13}}\frac{\partial}{\partial x^4}\Big).$$

(iv) If $u^{13} = 0$, $u^{12} \neq 0$, equation (2.11) gives us that $u^{12}u^{34} = -u^{14}u^{23}$ and u can be decomposed as

$$u = (u^{12}\frac{\partial}{\partial x^1} - u^{23}\frac{\partial}{\partial x^3} - u^{24}\frac{\partial}{\partial x^4}) \wedge \Big(\frac{\partial}{\partial x^2} + \frac{u^{14}}{u^{12}}\frac{\partial}{\partial x^4}\Big).$$

Thus we see that in all the cases u is totally decomposable. □

So indeed $\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C}))$ is embedded as a quadric hypersurface in $\mathbb{P}(\Lambda^2(T_xM \otimes \mathbb{C}))$. Taking now into account (2.7) we observe that we can identify $\mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C})))$ with the quadric $\mathbb{P}(v_x)$.

The quadric surface $\mathbb{P}(g_x)$:

Now the metric g_x defines a quadratic form $T_xM \otimes \mathbb{C} \to \mathbb{C}$ by

$$g_x(w) = \sum_{i,j=1}^{4} g_{ij}w^i w^j,$$

where $g_{ij} = g_{ji}$. It defines a quadric surface

$$\mathbb{P}(g_x) = \{[w] \in \mathbb{P}(T_xM \otimes \mathbb{C}) : \sum_{i,j=1}^{4} g_{ij}w^i w^j = 0\}.$$

This quadric is non-degenerate, since the quadratic form g_x is non-degenerate. So its rank equals four and it corresponds to a smooth quadric in $\mathbb{P}(T_xM \otimes \mathbb{C})$.

2.1.3 Remark. Recall, that if a quadric is mapped to a quadric under a projective trasformation, then the rank of the coefficient matrix is not changed. Thus one can classify quadrics in complex projective spaces up to their rank. Precisely, in $\mathbb{P}^3$ there are four of them: rank 4 corresponds to a smooth quadric, rank 3 to a quadric cone, rank 2 to a pair of planes and rank 1 to a double plane.

We need at this point some theory on spinor bundles. We will recall some facts on *spin* and $spin^{\mathbb{C}}$ structures on 4-manifolds. Heuristically, one can see *spin* and $spin^{\mathbb{C}}$ structures as generalizations of orientantions. The tangent bundle TM gives rise to a principal $O(4)$-bundle of frames denoted by $P_{O(4)}$. The manifold is said to be orientable if this bundle can be reduced to a $SO(4)$-bundle denoted by $P_{SO(4)}$. We define the group $Spin(4) = SU(2) \times SU(2)$ to be the double cover of $SO(4)$. This is the universal cover. If we make a further reduction, we obtain a principal $Spin(4)$-bundle denoted by $P_{Spin(4)}$. We have then, that the map

$$\xi : P_{Spin(4)} \to P_{SO(4)}$$

is a double covering and say that the manifold is *spin*. To find the complex analogue we replace $SO(4)$ by the group $SO(4) \times \mathbb{S}^1$ and consider its double cover. We define the group

$$Spin^{\mathbb{C}}(4) = (Spin(4) \times \mathbb{S}^1)/\{\pm 1\} = Spin(4) \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

This is the desired double cover of $SO(4) \times \mathbb{S}^1$. Finally we define M to be $spin^{\mathbb{C}}$, if given the bundle $P_{SO(4)}$, there are principal bundles $P_{\mathbb{S}^1}$ and $P_{Spin^{\mathbb{C}}(4)}$, with a $Spin^{\mathbb{C}}(4)$ equivariant bundle map, a double cover

$$\xi' : P_{Spin^{\mathbb{C}}(4)} \to P_{SO(4)} \times P_{\mathbb{S}^1}.$$

It is a known fact, that in dimension four any orientable manifold has a (non-unique) $spin^{\mathbb{C}}$ structure. The $spin^{\mathbb{C}}$ representation now allows us to consider the associated vector bundle S, called the spinor bundle for a given $spin^{\mathbb{C}}$ structure. This is a complex vector bundle. In the four-dimensional case this vector bundle splits into the sum of two subbundles S^+, S^-, such that

$$S = S^+ \oplus S^-.$$

Further details on this theory can be found in the book [8].

Let $\mathbb{P}(S_x^+) \cong \mathbb{P}^1$ and $\mathbb{P}(S_x^-) \cong \mathbb{P}^1$ denote the projectivizations of the fibers of the spinor bundles S^+ and S^- over x respectively. Consider now the Segre embedding

$$\begin{aligned} \mathbb{P}(S_x^-) \times \mathbb{P}(S_x^+) &\rightarrow \mathbb{P}(S_x^- \otimes S_x^+) \\ [\rho^-] \times [\rho^+] &\mapsto [\rho^- \otimes \rho^+]. \end{aligned}$$

One can show, that $S_x^- \otimes S_x^+ \cong T_xM \otimes \mathbb{C}$.

Let now $\{e_i\}_{i=1}^4$ be a local orthonormal frame for $T_xM \otimes \mathbb{C}$. We will be working with this frame from now on, because it is more convient for computational reasons. The Segre embedding with respect to the basis $\{e_i\}_{i=1}^4$ is given by

$$\begin{aligned} \sigma : \mathbb{P}(S_x^-) \times \mathbb{P}(S_x^+) &\rightarrow \mathbb{P}(T_xM \otimes \mathbb{C}) \\ ([a^1, a^2], [b^1, b^2]) &\mapsto [a^1b^1 + a^2b^2, i(a^2b^2 - a^1b^1), -i(a^1b^2 + a^2b^1), a^2b^1 - a^1b^2] \\ &=: [w^1, w^2, w^3, w^4]. \end{aligned} \tag{2.12}$$

This is a well defined map. In order to pick coordinates on $\mathbb{P}(S_x^-)$ and $\mathbb{P}(S_x^+)$ one should observe the projection of ξ' onto the first factor:

$$P_{Spin^{\mathbb{C}}(4)} \rightarrow P_{SO(4)}.$$

A point in the fiber of $P_{SO(4)}$ over x is a basis for T_xM and a point in the fiber of $P_{Spin^{\mathbb{C}}(4)}$ over x is a basis for the spinor $S_x = S_x^+ \oplus S_x^-$.

2.1.4 Remark. Recall that the "classical" Segre embedding is given by

$$\begin{aligned} \Sigma : \mathbb{P}^1 \times \mathbb{P}^1 &\rightarrow \mathbb{P}^3 \\ ([a^1, a^2], [b^1, b^2]) &\mapsto [a^1b^1, a^2b^2, a^1b^2, a^2b^1] =: [W^1, W^2, W^3, W^4]. \end{aligned}$$

The image is just the quadric surface $W^1W^2 - W^3W^4 = 0$ and the rank of the quadric is four, i.e. it's a smooth quadric. The associated symmetric matrix is

$$\Sigma = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & -1/2 & 0 \end{bmatrix}$$

Let now

$$B = \begin{bmatrix} 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & i & 1 \end{bmatrix},$$

so that $B^t \Sigma B = I_4$. Then

$$B^{-1} \begin{bmatrix} W^1 \\ W^2 \\ W^3 \\ W^4 \end{bmatrix} = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \\ w^4 \end{bmatrix}.$$

One we can easily observe that the image of the Segre embedding is just the quadric surface $(w^1)^2 + (w^2)^2 + (w^3)^2 + (w^4)^2 = 0$ and the rank of the quadric is four, i.e. it s a smooth quadric. Thus $\mathbb{P}(g_x)$ can be written with respect to the orthonormal basis $\{e_i\}_{i=1}^4$ for $T_xM \otimes \mathbb{C}$ as

$$\mathbb{P}(g_x) = \{[w] \in \mathbb{P}(T_xM \otimes \mathbb{C}) : (w^1)^2 + (w^2)^2 + (w^3)^2 + (w^4)^2 = 0\}. \tag{2.13}$$

The quadric $\mathbb{P}(g_x)$ has two rulings by lines and a unique line of each ruling passes through each point of the quadric. More precisely: fix a point $[a^1, a^2] \in \mathbb{P}(S_x^-)$. Then

$$t_+ := \sigma(\{[a^1, a^2]\} \times \mathbb{P}(S_x^+))$$

is a line in $\mathbb{P}(T_xM \otimes \mathbb{C})$. Similarly for fixed $[b^1, b^2] \in \mathbb{P}(S_x^+)$,

$$t_- := \sigma(\mathbb{P}(S_x^-) \times \{[b^1, b^2]\})$$

is also a line in $\mathbb{P}(T_xM \otimes \mathbb{C})$. So the quadric contains two families of lines denoted by $\mathcal{F}_-$ and $\mathcal{F}_+$ respectively such that,

$$\mathcal{F}_- = \bigcup_{[b^1,b^2]\in\mathbb{P}(S_x^+)} \{t_-\}, \;\; \mathcal{F}_+ = \bigcup_{[a^1,a^2]\in\mathbb{P}(S_x^-)} \{t_+\}.$$

If we choose any point of t_+, we can find a unique line of the family $\mathcal{F}_-$ passing through it. Analogously for every point of t_- we can find a unique line of $\mathcal{F}_+$ passing through it. Furthermore it holds that no two lines from the same family intersect and that any two lines belonging to different families intersect in a unique point of the quadric. The lines $\mathbb{P}(S_x^{\pm})$ are called the rectilinear generators of the quadric and

$$\mathbb{P}(g_x) = \sigma(\mathbb{P}(S_x^-) \times \mathbb{P}(S_x^+)). \tag{2.14}$$

The Plücker Embedding:

Every t_- or t_+ is a line in $\mathbb{P}(T_xM \otimes \mathbb{C})$. We will compute their images under the Plücker embedding. By setting first $[b^1, b^2] = [1, 0]$ and then $[b^1, b^2] = [0, 1]$ in (2.12) we can easily see, that

$$t_+ = \mathbb{P}\text{-span}([a^1, -ia^1, -ia^2, a^2], [a^2, ia^2, -ia^1, -a^1]).$$

We compute, that $(a^1e_1 - ia^1e_2 - ia^2e_3 + a^2e^4) \wedge (a^2e_1 + ia^2e_2 - ia^1e_3 - a^1e^4)$ equals

$$\begin{gathered} 2ia^1a^2e_1 \wedge e_2 + i\{(a^2)^2 - (a^1)^2\}e_1 \wedge e_3 - (a^1)^2 - (a^2)^2e_1 \wedge e_4 \\ -(a^1)^2 - (a^2)^2e_2 \wedge e_3 + i\{(a^1)^2 - (a^2)^2\}e_2 \wedge e_4 + 2ia^1a^2e_3 \wedge e_4. \end{gathered}$$

Thus we obtain, that the coordinates of $\mathrm{pl}(t_+)$ in the basis $\{e_i \wedge e_j\}_{1\le i<j\le 4}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ are

$$[2ia^1a^2, i\{(a^2)^2 - (a^1)^2\}, -(a^1)^2 - (a^2)^2, -(a^1)^2 - (a^2)^2, i\{(a^1)^2 - (a^2)^2\}, 2ia^1a^2].$$

On the other hand by setting first $[a^1, a^2] = [1, 0]$ and then $[a^1, a^2] = [0, 1]$ in (2.12), we have that

$$t_- = \mathbb{P}\text{-span}([b^1, -ib^1, -ib^2, -b^2], [b^2, ib^2, -ib^1, b^1]).$$

In this case the coordinates of $\mathrm{pl}(t_-)$ in the basis $\{e_i \wedge e_j\}_{1 \leq i < j \leq 4}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ are

$$[2ib^1b^2, i\{(b^2)^2 - (b^1)^2\}, (b^1)^2 + (b^2)^2, -(b^1)^2 - (b^2)^2, i\{(b^2)^2 - (b^1)^2\}, -2ib^1b^2].$$

It is well known, that in dimension 4 the Hodge $*$-operator induces a natural decomposition of $\Lambda^2 T_xM$ on an oriented manifold M given by

$$\Lambda^2 T_xM = \Lambda^2_+ T_xM \oplus \Lambda^2_- T_xM,$$

where $\Lambda^2_+ T_xM$ and $\Lambda^2_- T_xM$ correspond to the eigenspaces $+1$ and -1 respectively. Furthermore elements of $\Lambda^2_+ T_xM$ and $\Lambda^2_- T_xM$ are called self-dual and anti-self-dual respectively. We will perform a change of basis for $\Lambda^2 T_xM \otimes \mathbb{C}$. We would like to express the coordinates of $\mathrm{pl}(t_+)$ and $\mathrm{pl}(t_-)$ in the basis $\mathfrak{B}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ given by

$$\begin{aligned} f_1^{\pm} &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4) \\ f_2^{\pm} &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \mp e_2 \wedge e_4) \\ f_3^{\pm} &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3). \end{aligned}$$

One can observe, that $\{f_i^+\}_{i=1}^3$ is basis for $\Lambda^2_+ T_xM \otimes \mathbb{C}$, where $*f_i^+ = f_i$, $i = 1,2,3$ and that $\{f_i^-\}_{i=1}^3$ is basis for $\Lambda^2_- T_xM \otimes \mathbb{C}$, where $*f_i^- = -f_i$, $i = 1,2,3$. By using the change of basis matrix

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}$$

we compute, that the coordinates $[u^{12}, u^{13}, u^{14}, u^{23}, u^{24}, u^{34}]$ in the basis $\mathfrak{B}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ are given by

$$[u^1, u^2, u^3, u^4, u^5, u^6] := [u^{12} + u^{34}, u^{13} - u^{24}, u^{14} + u^{23}, u^{12} - u^{34}, u^{13} + u^{24}, u^{14} - u^{23}].$$

Thus the coordinates of $\mathrm{pl}(t_+)$ in the basis $\mathfrak{B}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ are

$$[2ia^1a^2, i\{(a^2)^2 - (a^1)^2\}, -(a^1)^2 - (a^2)^2, 0, 0, 0] \tag{2.15}$$

and the coordinates of $\mathrm{pl}(t_-)$ in the basis $\mathfrak{B}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ are

$$[0, 0, 0, 2ib^1b^2, i\{(b^2)^2 - (b^1)^2\}, (b^1)^2 + (b^2)^2]. \tag{2.16}$$

By (2.15) and (2.16) one can easily see, that $\mathcal{F}_+$ and $\mathcal{F}_-$ are embedded conics in $\mathbb{P}(\Lambda^2 T_xM \otimes \mathbb{C})$ given by the equations

$$\begin{cases} u^4 = u^5 = u^6 = 0 \\ (u^1)^2 + (u^2)^2 + (u^3)^2 = 0 \end{cases} \tag{2.17}$$

and

$$\begin{cases} u^1 = u^2 = u^3 = 0 \\ (u^4)^2 + (u^5)^2 + (u^6)^2 = 0 \end{cases} \tag{2.18}$$

respectively. We will denote these conics by $\mathcal{C}_+$ and $\mathcal{C}_-$. Obviously each of the two conics is sitting in a plane in $\mathbb{P}(\Lambda^2 T_xM \otimes \mathbb{C})$. The first plane is $\mathbb{P}(\Lambda^2_+ T_xM \otimes \mathbb{C})$ and the second is $\mathbb{P}(\Lambda^2_- T_xM \otimes \mathbb{C})$. They are given by the equations

$$u^4 = u^5 = u^6 = 0 \tag{2.19}$$

and

$$u^1 = u^2 = u^3 = 0 \tag{2.20}$$

respectively. Obviously $\mathbb{P}(\Lambda^2_+ T_xM \otimes \mathbb{C}) \cap \mathbb{P}(\Lambda^2_- T_xM \otimes \mathbb{C}) = \emptyset$.

The projectivized tangent bundle:

Let now $\mathcal{T} := T\mathbb{P}(g_x)$ denote the tangent bundle of the quadric $\mathbb{P}(g_x)$ and $\mathbb{P}(\mathcal{T})$ its projectivization. Then one can write

$$\mathbb{P}(\mathcal{T}) = \{(t_+ \cap t_-, l) : l \subset \mathbb{P}(T_xM \otimes \mathbb{C}) \text{ is a line tangent to } \mathbb{P}(g_x) \text{ at the point } t_+ \cap t_-\},$$

which is an algebraic subvariety of $\mathbb{P}(g_x) \times \mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C})) \subset \mathbb{P}(T_xM \otimes \mathbb{C}) \times \mathrm{Gr}_1(\mathbb{P}(T_xM \otimes$

$\mathbb{C})$). We will now apply the Plücker embedding on the second factor. We define the map

$$\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl} : \mathbb{P}(g_x) \times \mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C})) \to \mathbb{P}(g_x) \times \mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C}))).$$

Then

$$(\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big) := \{(t_+ \cap t_-, \mathrm{pl}(l)) : \, l \subset \mathbb{P}(T_xM \otimes \mathbb{C}) \text{ is a line tangent to } \mathbb{P}(g_x) \text{ at the point } t_+ \cap t_-\}.$$

Thus $(\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)$ is naturally an algebraic subvariety

$$(\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big) \subset \mathbb{P}(g_x) \times \mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C}))) \subset \mathbb{P}(T_xM \otimes \mathbb{C}) \times \mathbb{P}(\Lambda^2 T_xM \otimes \mathbb{C}) \cong \mathbb{P}^3 \times \mathbb{P}^5.$$

If we now denote by

$$\begin{array}{rcl} \pi : (\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big) & \to & \mathrm{pl}(Gr_1(\mathbb{P}(T_xM \otimes \mathbb{C}))) \\ (t_+ \cap t_-, \mathrm{pl}(l)) & \mapsto & \mathrm{pl}(l) \end{array}$$

and

$$\begin{array}{rcl} \tau : (\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big) & \to & \mathbb{P}(g_x) \\ (t_+ \cap t_-, \mathrm{pl}(l)) & \mapsto & t_+ \cap t_- \end{array}$$

the natural projections, we are interested in the geometry of $\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)$. We would like to give a description in $\mathbb{P}(\Lambda^2 T_xM \otimes \mathbb{C})$ of the image of the set of lines tangent to the quadric $\mathbb{P}(g_x)$ at the point $t_+ \cap t_-$ under the Plücker embedding. All these lines lie on one plane and pass through one point, so in $\mathbb{P}(\Lambda^2 T_xM \otimes \mathbb{C})$ they form a line given by $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-))$. Thus

$$\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big) = \{\mathrm{pl}(l) \in \mathbb{P}\text{-span}(\mathrm{pl}(t_+), \mathrm{pl}(t_-)) : \, \mathrm{pl}(t_+) \in \mathcal{C}_+, \, \mathrm{pl}(t_-) \in \mathcal{C}_-\} \tag{2.21}$$

and

$$\dim\Big[\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)\Big] = \dim(\mathcal{C}_+) + \dim(\mathcal{C}_-) + 1 = 3,$$

because $\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)$ is the join of the varieties $\mathcal{C}_+$ and $\mathcal{C}_-$. We can now describe $(\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)$ by

$$(\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big) = \{(t_+ \cap t_-, \mathrm{pl}(l)) : \mathrm{pl}(l) \in \mathbb{P}\text{-span}(\mathrm{pl}(t_+), \mathrm{pl}(t_-))\}. \tag{2.22}$$

We are going to show now that the variety $\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)$ is singular and we will determine its singular locus. By (2.17), (2.18) and (2.21) we see that the variety $\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)$ is defined by the equations

$$\begin{cases} (u^1)^2 + (u^2)^2 + (u^3)^2 = 0 \\ (u^4)^2 + (u^5)^2 + (u^6)^2 = 0. \end{cases} \tag{2.23}$$

The system of equations (2.23) shows that the singular points of $\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)$ are given by

$$\mathrm{Sing}\Big(\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)\Big) = \mathcal{C}_+ \cup \mathcal{C}_-.$$

Let's explain why. We will fix a coordinate system on $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-))$. Let T_+ and T_- denote the vector space representations of $\mathrm{pl}(t_+)$ and $\mathrm{pl}(t_-)$ in the basis $\mathcal{B}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ respectively. Then

$$T_+ = 2ia^1a^2 f_1^+ + i\{(a^2)^2 - (a^1)^2\} f_2^+ + [-(a^1)^2 - (a^2)^2] f_3^+ + 0f_1^- + 0f_2^- + 0f_3^-$$

and

$$T_- = 0f_1^+ + 0f_2^+ + 0f_3^+ + 2ib^1b^2 f_1^- + i\{(b^2)^2 - (b^1)^2\} f_2^- + [(b^1)^2 + (b^2)^2] f_3^-.$$

We have then, that

$$\mathrm{span}(T_+, T_-) = \{\lambda T_+ + \mu T_- : \lambda, \mu \in \mathbb{C}\}$$

is a plane in $\Lambda^2 T_xM \otimes \mathbb{C}$. So we obtain a projective coordinate system on $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-))$.

A point on this line has coordinates in the basis $\mathfrak{B}$ of $\Lambda^2(T_xM \otimes \mathbb{C})$ given by

$$[2\lambda i a^1 a^2, \lambda i\{(a^2)^2 - (a^1)^2\}, \lambda\{-(a^1)^2 - (a^2)^2\}, 2\mu i b^1 b^2, \mu i\{(b^2)^2 - (b^1)^2\}, \mu\{(b^1)^2 + (b^2)^2\}] \tag{2.24}$$

for scalars λ and μ. Obviously, by (2.23) the Jacobian matrix of the polynomials defining the variety is

$$\begin{bmatrix} 2u^1 & 2u^2 & 2u^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2u^4 & 2u^5 & 2u^6 \end{bmatrix}$$

and its rank at the point $\mathrm{pl}(t_+)$ or $\mathrm{pl}(t_-)$ is equal to one, i.e. lower than on any other point of $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-))$.

2.2. The intersection of three quadrics

Consider the intersection

$$\begin{aligned} S_x &= \mathbb{P}(v_x) \cap \mathbb{P}(\Lambda^2 g_x) \cap \mathbb{P}(\mathrm{R}_x) \\ &= \mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C}))) \cap \mathbb{P}(\Lambda^2 g_x) \cap \mathbb{P}(\mathrm{R}_x) \end{aligned}$$

of the three quadrics in $\mathbb{P}(\Lambda^2 T_xM \otimes \mathbb{C})$. We consider a line l tangent to the quadric $\mathbb{P}(g_x)$. By the discussion in the previous section it corresponds to a point in $\mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C})))$. The condition that the line l is tagent to the quadric $\mathbb{P}(g_x)$ is equivalent to the condition that $\mathrm{pl}(l) \in \mathbb{P}(\Lambda^2 g_x)$. So

$$\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})(\mathbb{P}(\mathcal{T}))\Big) = \mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C}))) \cap \mathbb{P}(\Lambda^2 g_x).$$

This means that,

$$S_x = \pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})(\mathbb{P}(\mathcal{T}))\Big) \cap \mathbb{P}(\mathrm{R}_x).$$

Therefore S_x must have singularities

$$\mathrm{Sing}(S_x) \supset \mathrm{Sing}\Big(\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})(\mathbb{P}(\mathcal{T}))\Big)\Big) \cap \mathbb{P}(\mathrm{R}_x) = (\mathcal{C}_+ \cap \mathbb{P}(\mathrm{R}_x)) \cup (\mathcal{C}_- \cap \mathbb{P}(\mathrm{R}_x)).$$

2.2.1 Definition. The variety S_x is called the local invariant of the Riemannian manifold (M, g) at the point x.

2.2.2 Remark. Notice, that if $\mathrm{R}_x = \kappa\Lambda^2 g_x$, $\kappa \in \mathbb{C}^*$, the manifold at the point x is a manifold of constant curvature in any two dimensional direction. In such a case, S_x is not defined and we shall not consider such points on M.

In the folowing we assume that the quadric $\mathbb{P}(\mathrm{R}_x)$ intersects the non-singular points of $\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big)$ transversally and intersects the singular locus $\mathcal{C}_+ \cup \mathcal{C}_-$ transversally as well. It follows by [14], Proposition 17.18 that S_x is the complete intersection of the quadrics $\mathbb{P}(v_x)$, $\mathbb{P}(\Lambda^2 g_x)$, $\mathbb{P}(\mathrm{R}_x)$.

2.2.3 Remark. Recall that two varieties intersect transversally if they intersect transversally at each point of their intersection, i.e. they are smooth at this point and their separate tangent spaces at that point span the tangent space of the ambient variety at that point. In other words if X and Y are projective subvarieties of $\mathbb{P}^n$, then X and Y intersect transversally if at every point $u \in X \cap Y$, $T_u X \oplus T_u Y = T_u \mathbb{P}^n$. Thus transversality depends on the choice of the ambient variety. In particular, transversality always fails whenever two subvarieties are tangent.

Recall that the complete intersection of three quadrics in $\mathbb{P}^5$ is a $K3$ surface (more details on that can be found in the Appendix A). Thus S_x is a (singular) $K3$ surface. The quadric $\mathbb{P}(\mathrm{R}_x)$ interesects the singular locus $\mathcal{C}_+ \cup \mathcal{C}_-$ transversally and each intersection $\mathbb{P}(\mathrm{R}_x) \cap \mathcal{C}_+$, $\mathbb{P}(\mathrm{R}_x) \cap \mathcal{C}_-$, consists of four ordinary double points (the transversal intersection of quadric and conic gives a 0-dimensional variety of degree 4). We wil denote the set of these points by $\mathrm{Sing}(S_x) = \{\mathrm{pl}(t_+^1), \mathrm{pl}(t_+^2), \mathrm{pl}(t_+^3), \mathrm{pl}(t_+^4), \mathrm{pl}(t_-^1), \mathrm{pl}(t_-^2), \mathrm{pl}(t_-^3), \mathrm{pl}(t_-^4)\}$.

Consider now the the algebraic subvariety

$$\tilde{S}_x = \pi^{-1}(\mathbb{P}(\mathrm{R}_x)) \subset \mathbb{P}(T_x M \otimes \mathbb{C}) \times \mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C}).$$

The next step is to show, that $\tilde{S}_x$ is the resolution of the singular points of S_x. We consider the map

$$\tilde{\pi} : \tilde{S}_x \to S_x.$$

Then $\tilde{S}_x$ is the resolution of the singular points of S_x, if and only if

$$\tilde{S}_x \setminus \tilde{\pi}^{-1}(\mathrm{Sing}(S_x)) \cong S_x \setminus \mathrm{Sing}(S_x).$$

By the definiton of $\tilde{\pi}^{-1}$ this is indeed an isomorphism.

We would like to compute now $\tilde{\pi}^{-1}(\mathrm{Sing}(S_x))$, or in other words to find the blow ups of the singular points $\mathrm{pl}(t_+^i)$, $\mathrm{pl}(t_-^j)$, $1 \leq i, j \leq 4$.

2.2.4 Remark. Let's recall the notion of the blow up of a complex surface at a point. Let $q \in U \subset X$ be an open neighborhood and (x, y) local coordinates such that $q = (0, 0)$ in this coordinate system. Define

$$\tilde{U} := \{((x, y), [z, w]) \in U \times \mathbb{P}^1 : xw = yz\}.$$

We have then the projection onto the first factor

$$\begin{array}{rcl} p_U : \tilde{U} & \to & U \\ ((x, y), [z, w]) & \mapsto & (x, y). \end{array}$$

If $(x, y) \neq (0, 0)$, then $p_U^{-1}((x, y)) = (((x, y), [z, w]))$. Furthermore we have $p_U^{-1}(q) = \{q\} \times \mathbb{P}^1$. This implies that the restriction

$$p_U : p_U^{-1}(U \setminus \{q\}) \to U \setminus \{q\}$$

is an isomorphism and $p_U^{-1}(q) \cong \mathbb{P}^1$ is a curve contracted by p_U to a point. Now let us take the gluing of X and $\tilde{U}$ along $X \setminus \{q\}$ and $\tilde{U} \setminus \{q\} \cong U \setminus \{q\}$. In this way we obtain a surface $\tilde{X}$ together with a morphism $p : \tilde{X} \to X$. Notice that p gives an isomoprhism between $X \setminus \{q\}$ and $\tilde{X} \setminus p^{-1}(q)$ and contracts the curve $\mathbb{P}^1 \cong p^{-1}(q)$ to the point q. The morphism $p : \tilde{X} \to X$ is called the blow-up of X along q. The curve $p^{-1}(q) \cong \mathbb{P}^1$ is called exceptional curve or exceptional divisor of the blow-up.

We obtain that

$$E_i := \tilde{\pi}^{-1}(\mathrm{pl}(t_+^i)) = \{(t_+^i \cap t_-, \mathrm{pl}(t_+^i)) : t_- \subset \mathcal{F}_-\} \cong \mathbb{P}^1, \tag{2.25}$$

$$F_j := \tilde{\pi}^{-1}(\mathrm{pl}(t_-^j)) = \{(t_+ \cap t_-^j, \mathrm{pl}(t_-^j)) : t_+ \subset \mathcal{F}_+\} \cong \mathbb{P}^1, \tag{2.26}$$

for $1 \leq i, j \leq 4$. Observe that this means, that $\tilde{\pi}$ is the blow-up of S_x along $\mathrm{pl}(t_+^i), \mathrm{pl}(t_-^j)$ for $1 \leq i, j \leq 4$ and the curves E_i, F_j, for $1 \leq i, j \leq 4$ are the exceptional divisor of the blow-up. In other words, $\tilde{\pi}$ contracts the curves E_i to the points $\mathrm{pl}(t_+^i)$ and the curves F_j to the points $\mathrm{pl}(t_-^j)$ for $1 \leq i, j \leq 4$.

The branching curve Γ_x :

We will show that the map

$$\tilde{\tau} : \tilde{S}_x \to \mathbb{P}(g_x)$$

is a double branched cover at a generic point, where $\tilde{\tau}$ is the restriction of τ to $\tilde{S}_x$. The term "double branched cover" means, that there exists a closed subset Br of $\mathbb{P}(g_x)$, such that $\tilde{\tau}$ restricted to $\tilde{S}_x \setminus \mathrm{Ram}$, where $\mathrm{Ram} := \tilde{\tau}^{-1}(\mathrm{Br})$ is a topological double cover of $\mathbb{P}(g_x) \setminus \mathrm{Br}$. Points in Br and Ram are called branching points and ramification points respectively. The term "generic" stands for the fact that as we will see sometimes $\tilde{\tau}$ represents a branched double cover followed by a blow-up. Before describing the preimage $\tilde{\tau}^{-1}(t_+ \cap t_-)$ we would like to be more precise.

The block decomposition of the Riemann curvature operator in dimension four is given by

$$\mathfrak{Rm} = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix},$$

where A and C correspond to the operators associated to

$$W_+ + \frac{\mathrm{scal}}{24} g \owedge g$$

and

$$W_- + \frac{\mathrm{scal}}{24} g \owedge g$$

respectively and B is the operator associated to the curvature-like tensor

$$\frac{1}{2}\overset{\circ}{\mathrm{Ric}} \owedge g = \frac{1}{2}\Big(\mathrm{Ric} - \frac{\mathrm{scal}}{4} g\Big) \owedge g.$$

Recall, that in dimension four

$$\mathrm{Rm} = W_+ + W_- + \frac{1}{2}\overset{\circ}{\mathrm{Ric}} \owedge g + \frac{\mathrm{scal}}{24} g \owedge g,$$

where W_+, W_- denote the Weyl parts of the curvature and $\overset{\circ}{\mathrm{Ric}}$ the traceless Ricci tensor.

Consider now the block decomposition above and let $u = u_1 + u_2 \in \Lambda^2_+(T_xM \otimes \mathbb{C}) \oplus \Lambda^2_-(T_xM \otimes \mathbb{C})$. Then

$$\begin{aligned}
\mathrm{R}_x(u) &= \Lambda^2 g_x(\mathfrak{Rm}(u), u) \\
&= \Lambda^2 g_x(A(u_1), u_1) + \Lambda^2 g_x(B(u_2), u_1) + \Lambda^2 g_x(B^t(u_1), u_2) + \Lambda^2 g_x(C(u_2), u_2) \\
&= \Lambda^2 g_x(A(u_1), u_1) + 2\Lambda^2 g_x(B(u_2), u_1) + \Lambda^2 g_x(C(u_2), u_2).
\end{aligned}$$

Now the quadric $\mathbb{P}(\mathrm{R}_x)$ is given by

$$\begin{aligned}
\mathbb{P}(\mathrm{R}_x) &= \{[u] = [u_1 + u_2] \in \mathbb{P}(\Lambda^2 T_xM \otimes \mathbb{C}) : \Lambda^2 g_x\Big(A(u_1), u_1\Big) + \\
&\quad +2\Lambda^2 g_x\Big(B(u_2), u_1\Big) + \Lambda^2 g_x\Big(C(u_2), u_2\Big) = 0\}.
\end{aligned}$$

We would like to describe the intersection of $\mathbb{P}(\mathrm{R}_x)$ with $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-))$. As explained previously, a point on the line $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-))$ is expressed as $[\lambda T_+ + \mu T_-]$. Let us set $u_1 = \lambda T_+ \in \Lambda^2_+(T_xM \otimes \mathbb{C})$ and $u_2 = \mu T_- \in \Lambda^2_-(T_xM \otimes \mathbb{C})$. We obtain, that

$$\begin{aligned}
& \Lambda^2 g_x\Big(A(\lambda T_+), \lambda T_+\Big) + 2\Lambda^2 g_x\Big(B(\mu T_-), \lambda T_+\Big) + \Lambda^2 g_x\Big(C(\mu T_-), \mu T_-\Big) = 0 \\
\Rightarrow\ & \lambda^2 \Lambda^2 g_x\Big(A(T_+), T_+\Big) + 2\lambda\mu \Lambda^2 g_x\Big(B(T_-), T_+\Big) + \mu^2 \Lambda^2 g_x\Big(C(T_-), T_-\Big) = 0 \\
\Rightarrow\ & \lambda^2 \Lambda^2 g_x\Big((\mathfrak{W}_+ + \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_+})(T_+), T_+\Big) + 2\lambda\mu \Lambda^2 g_x\Big(B(T_-), T_+\Big) + \\
& +\mu^2 \Lambda^2 g_x\Big((\mathfrak{W}_- + \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_-})(T_-), T_-\Big) = 0 \\
\Rightarrow\ & \lambda^2 \Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big) + \lambda^2 \frac{\mathrm{scal}}{12} \Lambda^2 g_x\Big(T_+, T_+\Big) + 2\lambda\mu \Lambda^2 g_x\Big(B(T_-), T_+\Big) + \\
& +\mu^2 \Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) + \mu^2 \frac{\mathrm{scal}}{12} \Lambda^2 g_x\Big(T_-, T_-\Big) = 0 \\
\Rightarrow\ & \lambda^2 \Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big) + 2\lambda\mu \Lambda^2 g_x\Big(B(T_-), T_+\Big) + \mu^2 \Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = 0,
\end{aligned}$$

where $\mathfrak{W}_+$ and $\mathfrak{W}_-$ correspond to the operators associated to W_+ and W_- respectively. Notice, that in the last implication we are using the fact, that $\pi\Big((\mathrm{id}_{\mathbb{P}(g_x)} \times \mathrm{pl})\big(\mathbb{P}(\mathcal{T})\big)\Big) = \mathrm{pl}(\mathrm{Gr}_1(\mathbb{P}(T_xM \otimes \mathbb{C}))) \cap \mathbb{P}(\Lambda^2 g_x)$.

By assuming that $\mu \neq 0$ and setting $s = \frac{\lambda}{\mu}$ we obtain a quadratic equation in the variable s given by

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big)s^2 + 2\Lambda^2 g_x\Big(B(T_-), T_+\Big)s + \Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = 0. \tag{2.27}$$

We can consider the previous equation naturally, as an equation that determines S_x. The discriminant of the equation is given by

$$\Delta = 4\Big(\Lambda^2 g_x(B(T_-), T_+)\Big)^2 - 4\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+)\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big).$$

Thus there are three possible cases for the intersection of the quadric and the line.

(i) If $\Delta \neq 0$, then the intersection consists of exactly two distinct points:

- $\mathbb{P}\text{-span}(\mathrm{pl}(t_+), \mathrm{pl}(t_-)) \cap \mathbb{P}(\mathrm{R}_x) = \{\mathrm{pl}(l), \mathrm{pl}(l')\}$, where $\mathrm{pl}(l), \mathrm{pl}(l') \neq \mathrm{pl}(t_+), \mathrm{pl}(t_-)$. Then

  $$\tilde{\pi}^{-1}(\mathrm{pl}(l)) = (t_+ \cap t_-, \mathrm{pl}(l)),\ \tilde{\pi}^{-1}(\mathrm{pl}(l')) = (t_+ \cap t_-, \mathrm{pl}(l'))$$

  and

  $$\tilde{\tau}^{-1}(t_+ \cap t_-) = \{\tilde{\pi}^{-1}(\mathrm{pl}(l)), \tilde{\pi}^{-1}(\mathrm{pl}(l'))\}$$

  are two distinct points. Both these points are nonsingular points of $\tilde{S}_x$.

- $\mathbb{P}\text{-span}(\mathrm{pl}(t_+^i), \mathrm{pl}(t_-)) \cap \mathbb{P}(\mathrm{R}_x) = \{\mathrm{pl}(t_+^i), \mathrm{pl}(l)\}$, for some $i = 1, ..., 4$, where $\mathrm{pl}(l) \neq \mathrm{pl}(t_+^i), \mathrm{pl}(t_-)$. Then

  $$\tilde{\tau}^{-1}(t_+^i \cap t_-) = \{(t_+ \cap t_-, \mathrm{pl}(t_+^i)), \tilde{\pi}^{-1}(\mathrm{pl}(l))\},$$

  are two distinct points. Both these points are nonsingular points of $\tilde{S}_x$.

- $\mathbb{P}\text{-span}(\mathrm{pl}(t_+), \mathrm{pl}(t_-^j)) \cap \mathbb{P}(\mathrm{R}_x) = \{\mathrm{pl}(t_-^j), \mathrm{pl}(l)\}$, for some $j = 1, ..., 4$, where $\mathrm{pl}(l) \neq \mathrm{pl}(t_+), \mathrm{pl}(t_-^j)$. Then

  $$\tilde{\tau}^{-1}(t_+ \cap t_-^j) = \{(t_+ \cap t_-, \mathrm{pl}(t_-^j)), \tilde{\pi}^{-1}(\mathrm{pl}(l))\},$$

are two distinct points. Both these points are nonsingular points of $\tilde{S}_x$.

- $\mathbb{P}$-span$(\mathrm{pl}(t^i_+), \mathrm{pl}(t^j_-)) \cap \mathbb{P}(\mathrm{R}_x) = \{\mathrm{pl}(t^i_+), \mathrm{pl}(t^j_-)\}$, for some $i, j = 1, ..., 4$. Then

$$\tilde{\tau}^{-1}(t^i_+ \cap t^j_-) = \{(t^i_+ \cap t^j_-, \mathrm{pl}(t^i_+)), (t^i_+ \cap t^j_-, \mathrm{pl}(t^j_-))\},$$

are two distinct points. Both these points are nonsingular points of $\tilde{S}_x$.

(ii) If $\Delta = 0$, but not all coefficients are equal to zero, then the line has exactly one double point in common with the quadric $\mathbb{P}(\mathrm{R}_x)$, which is possible if and only if the line is tangent to the quadric at that point:

- $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-)) \cap \mathbb{P}(\mathrm{R}_x) = \{\mathrm{pl}(l)\}$, where $\mathrm{pl}(l) \neq \mathrm{pl}(t_+), \mathrm{pl}(t_-)$. This is the case that $\mathbb{P}$-span$(\mathrm{pl}(t_+), \mathrm{pl}(t_-))$ is tangent to the quadric $\mathbb{P}(R_x)$ at the point $\mathrm{pl}(l)$. Then

$$\tilde{\tau}^{-1}(t_+ \cap t_-) = \{\tilde{\pi}^{-1}(\mathrm{pl}(l))\}.$$

Obviously in this case $t_+ \cap t_-$ corresponds to a branching point and $\tilde{\pi}^{-1}(\mathrm{pl}(l))$ is a ramification point.

(iii) If $\Delta = 0$ and all coefficients are simultaneously equal to zero, then the line lies entirely in $\mathbb{P}(\mathrm{R}_x)$:

- $\mathbb{P}$-span$(\mathrm{pl}(t^i_+), \mathrm{pl}(t^j_-)) \subset \mathbb{P}(\mathrm{R}_x)$, for some $i, j = 1, ..., 4$. Then

$$\begin{aligned} \tilde{\tau}^{-1}(t^i_+ \cap t^j_-) \quad = \quad & \{\tilde{\pi}^{-1}(\mathrm{pl}(l)) : \ \mathrm{pl}(l) \in \mathbb{P}\text{-span}(\mathrm{pl}(t^i_+), \mathrm{pl}(t^j_-)) \setminus \{\mathrm{pl}(t^i_+), \mathrm{pl}(t^j_-)\}\} \cup \\ & \cup\{(t^i_+ \cap t^j_-, \mathrm{pl}(t^i_+))\} \cup \{(t^i_+ \cap t^j_-, \mathrm{pl}(t^j_-))\} =: \mathbb{P}^1_{t^i_+ \cap t^j_-}, \end{aligned}$$

where $\mathbb{P}^1_{t^i_+ \cap t^j_-} \cong \mathbb{P}$-span$(\mathrm{pl}(t^i_+), \mathrm{pl}(t^j_-)) \cong \mathbb{P}^1$, since $\tilde{\pi}$ maps the curve $\tilde{\tau}^{-1}(t^i_+ \cap t^j_-)$ one to one onto the singular line $\mathbb{P}$-span$(\mathrm{pl}(t^i_+), \mathrm{pl}(t^j_-))$. Here $t^i_+ \cap t^j_-$ corresponds again to a branching point and in this special case the branching curve $\Gamma_x \subset \mathbb{P}(g_x)$ at the point $t^i_+ \cap t^j_-$ is singular.

Thus the branching curve is described by

$$\begin{aligned} \Gamma_x \quad = \quad & \{([a^1, a^2], [b^1, b^2]) \in \mathbb{P}(S^-_x) \times \mathbb{P}(S^+_x) : \\ & \Big(\Lambda^2 g_x(B(T_-), T_+)\Big)^2 - \Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big)\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = 0\}. \end{aligned} \tag{2.28}$$

The next propositions can be found in Nikulin's paper [21].

2.2.5 Remark. The branching curve will serve as our local invariant in this text. Precisely, we will use this local invariant in oder to obtain a characterization for the singularity models for Type I singularities for four dimensional Ricci flows. The type of the curve is invariant under the choice of basis for $T_xM \otimes \mathbb{C}$. For example, as we will see in Chapter 4, the branching curve associated to a point of $\mathbb{S}^3 \times \mathbb{R}$, is a 4-typle diagonal and that of $\mathbb{S}^2 \times \mathbb{S}^2$, is a double rectangle.

2.2.6 Proposition. *Assume that the branching curve Γ_x has only finite number of singular points. Then $\tilde{\tau} : \tilde{S}_x \to \mathbb{P}(g_x)$ is a branched double cover for all points $t_+ \cap t_- \in \Gamma_x$, except for the singular points, at which $\tilde{\tau}$ is a branched double cover followed by a blow-up.*

Recall that for a covering map $\tilde{\tau} : \tilde{S}_x \to \mathbb{P}(g_x)$, there exists a homeomorpish $\hat{\sigma} : \tilde{S}_x \to \tilde{S}_x$, such that $\tilde{\tau} \circ \hat{\sigma} = \tilde{\tau}$, that is to say $\hat{\sigma}$ is a lift of $\tilde{\tau}$. The map $\hat{\sigma}$ is called a deck transformation.

2.2.7 Proposition. *Assume that the branching curve Γ_x has only finite number of singular points. Then the deck transformation $\hat{\sigma}$ of the branched double cover is everywhere defined on $\tilde{S}_x$.*

Then there are given on $\tilde{S}_x$ nonsingular rational curves (exceptional curves) $\bar{E}_i := \hat{\sigma}(E_i) \cong \mathbb{P}^1$ and $\bar{F}_j := \hat{\sigma}(F_j) \cong \mathbb{P}^1$, where $1 \leq i, j \leq 4$.

2.3. The Picard sublattice

2.3.1 Remark. We should recall some basic knowledge from algebraic geometry. A Weil divisor D on a nonsingular surface $\tilde{S}_x$ is a formal linear combination

$$D = \sum_i d_i C_i$$

of irreducible curves. A curve itself can be regarded as a divisor, seen as a formal linear sum of its irreducible components. $\mathrm{Div}(\tilde{S}_x)$ denotes the set of all divisors and has the structure of a free, abelian group. The divisor is called effective and we denote it by $D \geq 0$ if all coefficients d_i are non-negative and not all zero. On the other hand the tensor product and the dual endow the set of all isomorphism classes of holomorphic line bundles on $\tilde{S}_x$ with the structure of an abelian group. This group is called the Picard group and is denoted by

$\mathrm{Pic}(\tilde{S}_x)$. There is a natural group homomorphism

$$\begin{array}{rcl} \mathrm{Div}(\tilde{S}_x) & \to & \mathrm{Pix}(\tilde{S}_x) \\ D & \mapsto & \mathcal{O}_{\tilde{S}_x}(D), \end{array}$$

where $\mathcal{O}_{\tilde{S}_x}(D)$ denotes the line bundle associated to the divisor D. More details on how this association is constructed can be found in the book [1] on page 27. Two divisors D and F are called linearly equivalent and we write $D \sim F$ if $\mathcal{O}_{\tilde{S}_x}(D) \cong \mathcal{O}_{\tilde{S}_x}(F)$. It can be shown that $\mathrm{Pic}(\tilde{S}_x)$ can be equivalently defined as

$$\mathrm{Pic}(\tilde{S}_x) = \mathrm{Div}(\tilde{S}_x)/\sim .$$

The next notion we will need, is the notion of a linear system. The complete linear system of $D \in \mathrm{Div}(\tilde{S}_x)$ is defined by

$$|D| = \{F \in \mathrm{Div}(\tilde{S}_x) : F \geq 0, D \sim F\},$$

which means that $|D|$ is the set of all effective divisors on $\tilde{S}_x$, which are linearly equivalent to D. We call a linear subspace of $|D|$ a linear system on $\tilde{S}_x$. The base locus of $|D|$ is given by

$$\mathrm{Bs}(|D|) = \bigcap_{F \in |D|} F.$$

Finally, there exists a one-to-one correspondence between regular maps $\Phi : \tilde{S}_x \to \mathbb{P}^n$ to projective space and linear systems on $\tilde{S}_x$. To the regular map Φ we can associate the linear system $\Phi^*|\mathcal{H}|$, where $\mathcal{H} = \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is the hyperplane divisor.

Consider the maps $\Phi_5 : \tilde{S}_x \to \mathbb{P}(\Lambda^2 T_x M \otimes \mathbb{C})$, $\Phi_3 : \tilde{S}_x \to \mathbb{P}(T_x M \otimes \mathbb{C})$, $p_+ \circ \Phi_3 : \tilde{S}_x \to \mathcal{F}_+$, $p_- \circ \Phi_3 : \tilde{S}_x \to \mathcal{F}_-$, where $p_+ : \mathbb{P}(g_x) \to \mathcal{F}_+$ and $p_- : \mathbb{P}(g_x) \to \mathcal{F}_-$ are projections. By the Remark 2.3.1 we obtain four linear systems H_5, H_3, H_+ and H_- respectively.

We denote by $h_5, h_3, h_+, h_-, p_{t_+^k \cap t_-^m} \in \mathrm{Pic}(\tilde{S}_x)$ and e_i, f_j, $\bar{e}_i$, $\bar{f}_j \in \mathrm{Pic}(\tilde{S}_x)$ the classes of divisors of H_5, H_3, H_+, H_-, $\mathbb{P}^1_{t_+^k \cap t_-^m}$ and E_i, F_j, $\bar{E}_i$, $\bar{F}_i$ respectively, where $1 \leq i, j \leq 4$ and some $k, m \in \{1, ..., 4\}$. An introduction to the basic theory of $K3$ surfaces can be found in the Appendix A.

The following Theorem can be found in [21].

2.3.2 Theorem. *For the nonsingular K3 surface $\tilde{S}_x$ hold:*

(i) The elements h_5, e_i, f_j for $1 \leq i, j \leq 4$, generate in $Pic(\tilde{S}_x)$ a primitive sublattice

$$\Lambda = \Big\{h_5,\ \{e_i : 1 \leq i \leq 4\},\ \{f_j : 1 \leq j \leq 4\},\ h_+ = \frac{1}{2}\Big(h_5 - \sum_{i=1}^{4} e_i\Big),\ h_- = \frac{1}{2}\Big(h_5 - \sum_{j=1}^{4} f_j\Big)\Big\}.$$

Furthermore

$$\begin{array}{lll} h_5^2 = 8, & e_i^2 = -2, & f_j^2 = -2, \\ h_5 \cdot e_i = 0,\ \forall i, & h_5 \cdot f_j = 0,\ \forall j, & \\ e_i \cdot e_l = 0,\ \forall i \neq l, & f_j \cdot f_l = 0,\ \forall j \neq l, & \\ e_i \cdot f_j = 0,\ \forall i, j & & \end{array}.$$

The elements h_3, $\bar{e}_i$, $\bar{f}_j \in \Lambda$ and

$$h_3 = h_+ + h_-,$$

$$\bar{e}_i = f - e_i - \sum_j p_{t_+^i \cap t_-^j},$$

$$\bar{f}_j = e - f_j - \sum_i p_{t_+^i \cap t_-^j}.$$

(ii) If the elements $p_{t_+^k \cap t_-^m}$ exist, then it holds:

$$\begin{array}{ll} p_{t_+^k \cap t_-^m} \cdot h_5 = 1, & p_{t_+^k \cap t_-^m} \cdot e_i = \delta_{ki}, \\ p_{t_+^k \cap t_-^m} \cdot f_j = \delta_{mj}, & p^2_{t_+^k \cap t_-^m} = -2 \end{array}.$$

The sublattice spanned by Λ and the elements $p_{t_+ \cap t_-}$ has rank equal to $\operatorname{rank}(\Lambda) + 1$. *If distinct elements $p_{t_+ \cap t_-}$ exist, then they are mutually orthogonal.*

(iii) The complete linear systems $|h_5|$, $|h_3|$, $|h_+|$, $|h_-|$, $|p_{t_+ \cap t_-}|$, $|e_i|$, $|f_j|$, $|\bar{e}_i|$, $|\bar{f}_j|$ coincide with the original ones H_5, H_3, H_+, H_-, $P_{t_+ \cap t_-}$ and E_i, F_j, $\bar{E}_i$, $\bar{F}_j$ and the linear systems $|h_5|$, $|h_3|$, $|h_+|$, $|h_-|$ have no basis points.

2.3.3 Remark. Notice that h_5 plays the role of polarization, so the $K3$ surfaces we are dealing with are always lattice polarized $K3$ surfaces with a polarization of degree 8. The interested reader can found details on lattice polarized $K3$ surface and their coarse moduli space in the last section of the Appendix.

3. Ricci flow basics

In this chapter we introduce the Ricci flow equation and give some basic examples of Ricci flows. The Ricci flow was introduced by Hamilton in 1982 in his seminal paper [9]. After the work of Perelman it was proven to be a very powerful tool towards the classification of 3-dimensional Riemannian manifolds and was used in order to prove Thurston's Geometrization Conjecture. In the second section of this chapter we introduce Ricci solitons, which are special solutions to the Ricci flow and often arise as singularity models of the Ricci flow after performing some rescaling arguments, which will become clear in Chapter 6.

3.1. The Ricci flow equation and examples

Let (M, g_0) be a smooth Riemannian manifold . The Ricci flow is a PDE that describes the evolution of the Riemannian metric tensor:

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2\mathrm{Ric} \\ g(0) &= g_0, \end{aligned}$$

where $g(t)$ is a one-parameter family of metrics on M and $\mathrm{Ric} := \mathrm{Ric}_{g(t)}$ denotes the Ricci curvature with respect to $g(t)$. The minus sign makes the Ricci flow a heat-type equation, so it is expected to "average out" the curvature. In order to get a feeling of the evolution equation, we will look at some simple examples.

3.1.1 Example (Einstein metrics). Suppose that the initial metric g_0 is Ricci flat, i.e. $\mathrm{Ric}_{g_0} = 0$. In this case the metric will remain stationary for all subsequent times. Concrete examples are the Euclidean space $\mathbb{R}^n$ and the flat torus $\mathbb{T}^n = \mathbb{S}^1 \times ... \times \mathbb{S}^1$. Suppose now that the initial metric is an Einstein metric, i.e. $\mathrm{Ric}_{g_0} = \kappa g_0$, $\kappa \in \mathbb{R}$. A solution $g(t)$ with $g(0) = g_0$ is given by

$$g(t) = (1 - 2\kappa t)g_0.$$

If $\kappa > 0$, $\kappa = 0$ or $\kappa < 0$ we call the solution shrinking, steady or expanding respectively.

The simplest shrinking solution is that of the unit sphere $(\mathbb{S}^n, g_0)$ endowed with the round metric. It holds that $\mathrm{Ric}_{g_0} = (n-1)g_0$, so

$$g(t) = (1 - 2(n-1)t)g_0$$

is a maximal solution to the Ricci flow defined on the time interval $(-\infty, T)$, where $T = \frac{1}{2(n-1)}$. That is, under the Ricci flow $\mathbb{S}^n$ stays round and shrinks homothetically at a steady rate. Observe that at time T (called the singularity time) the sphere shrinks to a point. By contrast, the simplest expanding solution is that of the hyperbolic space $\mathbb{H}^n$ endowed with the hyperbolic metric of a constant sectional curvature -1. In this case $\mathrm{Ric}_{g_0} = -(n-1)g_0$, so

$$g(t) = (1 + 2(n-1)t)g_0$$

is a solution to the Ricci flow and the manifold expands homothetically for all times.

3.1.2 Example (Quotient metrics). Let $M = N/G$ be a quotient of a Riemannian manifold N by a discrete group of isometries G. Then it will remain so under the Ricci flow, as the Ricci flow on N preserves the isometry group. For example $\mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2$ shrinks to a point in finite time, as does its cover $\mathbb{S}^n$.

3.1.3 Example (Product Metrics). Let $M \times N$ be a product manifolds endowed with the product metric. Under the Ricci flow the metric will remain a product metric and each factor evolves independently. For example for $\mathbb{S}^n \times \mathbb{S}^1$, the first factor shrinks to a point in finite time, while the second factor stays stationary.

3.2. Ricci solitons

Before we discuss Ricci solitons we list at first the types of long-existing solutions of the Ricci flow.

3.2.1 Definition. An ancient solution to the Ricci flow is a solution that exists on a past time interval $(-\infty, \omega)$. An immortal solution is a solution that exists for a future time interval (α, ∞). An eternal solution is a solution that exists for all time $(-\infty, \infty)$.

From taking limits of dilations of singularities we obtain long-existing solutions. This will become clear in Chapter 6.

3.2.2 Definition. A triple (M^n, g, X) is called a Ricci soliton if there exists a complete vector field X on M and $\kappa \in \mathbb{R}$, such that

$$\operatorname{Ric} + \frac{1}{2}\mathcal{L}_X g = \kappa g. \tag{3.1}$$

We distinguish the following cases. If $\kappa = 0$, then it is called a steady Ricci soliton, if $\kappa < 0$ an expanding Ricci soliton and if $\kappa > 0$ a shrinking Ricci soliton. In the case where X vanishes identically, we just have the case of Einstein metrics.

3.2.3 Definition. A triple (M^n, g, f) is called a gradient Ricci soliton, if there exists a gradient vector field $X = \nabla^g f = \operatorname{grad} f$ for some $f \in C^\infty(M)$ (called the potential function) and $\kappa \in \mathbb{R}$, such that

$$\operatorname{Ric} + \nabla^g \nabla^g f = \kappa g. \tag{3.2}$$

3.2.4 Remark. Recall, that $\nabla^g \nabla^g f = \operatorname{Hess}(f) = \frac{1}{2}\mathcal{L}_{\nabla^g f} g$.

We distiguish the following cases. If $\kappa = 0$, then it is called a gradient steady Ricci soliton, if $\kappa < 0$ a gradient expanding Ricci soliton and if $\kappa > 0$ a gradient shrinking Ricci soliton. In the case where $f = \text{const}$, we just have the case of Einstein metrics.

Similar to Einstein metrics, Ricci solitons give rise to special solutions to the Ricci flow. Suppose that $(M, g(t))$ is a solution of the Ricci flow. One says $g(t)$ is a self-similar solution of the Ricci flow if there exist scalars $\sigma(t)$ and diffeomorphisms $\phi(t)$ on M, such that

$$g(t) = \sigma(t)\phi(t)^*(g_0). \tag{3.3}$$

A metric of this form changes only by diffeomorphisms and rescaling. The following Lemma can be found in [2] on page 23.

3.2.5 Lemma. *If $(M^n, g(t))$ is a solution to the Ricci flow having the special form*

$$g(t) = \sigma(t)\phi(t)^*(g_0),$$

then there exists a vector field X on M^n, such that (M^n, g_0, X) solves

$$\mathrm{Ric}_{g_0} + \frac{1}{2}\mathcal{L}_X g_0 = \kappa g_0.$$

Conversely given any solution (M^n, g_0, X) of the Ricci soliton equation, there exists one-parameter families of scalars $\sigma(t)$ and diffeomorhisms $\phi(t)$ such that $(M^n, g(t))$ becomes a solution of the Ricci flow when $g(t)$ is defined by

$$g(t) = \sigma(t)\phi(t)^*(g_0).$$

3.2.6 Remark. A gradient Ricci soliton satisfying the equation

$$\mathrm{Ric}_{g_0} + \nabla^{g_0}\nabla^{g_0} f_0 = \kappa g_0$$

corresponds to the self similar solution

$$g(t) = (1 - 2\kappa t)\phi(t)^*(g_0),$$

where $\phi(t)$ is the one-parameter family of diffeomorphisms generated by $X(t) = \frac{\nabla f_0}{1-2\kappa t}$.

Hamilton showed in [12] the following result, which can also be found in [3] on page 156. We present an outline of the proof.

3.2.7 Lemma. *Let (M^n, g_0, f_0) be a complete, gradient Ricci soliton. Then*

$$scal_{g_0} + |\nabla^{g_0} f_0|^2_{g_0} - 2\kappa f_0 = C_0, \tag{3.4}$$

for some constant C_0.

Proof. A computation shows, that

$$\nabla_i \mathrm{scal}_{g_0} = 2\mathrm{R}_{ij}\nabla_j f_0.$$

This computation can be found in the book [3] on page 156. We will verify that the covariant derivative of the left hand side of (3.4) equals zero:

$$\begin{aligned}\nabla_i(\mathrm{scal}_{g_0} + |\nabla f_0|^2_{g_0} - 2\kappa f_0) &= \nabla_i \mathrm{scal}_{g_0} + 2\nabla_i\nabla_j f_0 \nabla_j f_0 - 2\kappa\nabla_i f \\ &= 2\mathrm{R}_{ij}\nabla_j f_0 + 2\nabla_i\nabla_j f_0 - 2\kappa\nabla_i f_0 \\ &= 2(R_{ij} + \nabla_i\nabla_j f_0 - \kappa g_{ij})\nabla_j f_0 \\ &= 0.\end{aligned}$$

□

We will restrict ourselves to the case of gradient shrinking Ricci solitons. For a gradient, shrinking Ricci soliton it is always possible to rescale the metric by 2κ and shift the function f_0 by the constant $-C_0$, so that the soliton equation becomes

$$\mathrm{Ric}_{g_0} + \nabla^{g_0}\nabla^{g_0} f_0 = \frac{1}{2} g_0$$

and the identity (3.4) takes the form

$$\mathrm{scal}_{g_0} + |\nabla^{g_0} f_0|^2_{g_0} - f_0 = 0.$$

We call such a soliton a normalized gradient shrinking Ricci soliton.

We say that the gradient soliton is complete if (M^n, g_0) is complete and the vector field $\nabla^{g_0} f_0$ is complete.

The following result gives the canonical form for the associated time-dependent version of a normalized gradient shrinking Ricci soliton. We demonstrate the proof, which can be found in [3] on page 154 as well.

3.2.8 Theorem. *Let (M^n, g_0, f_0) be a complete normalized gradient shrinking Ricci soliton. Then there exists a solution $g(t)$ of the Ricci flow with $g(0) = g_0$, diffeomorphisms $\phi(t)$ with $\phi(0) = id_M$, functions $f(t)$ with $f(0) = f_0$ defined for all t with*

$$\sigma(t) = 1 - t > 0,$$

such that the following hold:

(i) $\phi(t) : M^n \to M^n$ *is the one-parameter family of diffeomorphisms generated by* $X(t) = \frac{\nabla^{g_0} f_0}{1-t}$,

(ii) $g(t) = (1-t)\phi(t)^*(g_0)$ *on* $(-\infty, 1)$,

(iii) $f(t) = f_0 \circ \phi(t) = \phi(t)^*(f_0)$.

Futhermore,

$$\begin{aligned} \mathrm{Ric}_{g(t)} + \nabla^{g(t)}\nabla^{g(t)} f(t) &= \frac{1}{2(1-t)} g(t), \\ \frac{\partial}{\partial t} f(t) &= |\nabla^{g(t)} f(t)|^2_{g(t)}. \end{aligned}$$

Proof. We define $\sigma(t) = 1 - t$. Since the vector field $\nabla^{g_0} f_0$ is complete. there exists a 1-parameter family of diffeomorphisms $\phi(t) : M^n \to M^n$ generated by the vector fields

$$X(t) = \frac{\nabla^{g_0} f_0}{1-t},$$

defined for all $\sigma(t) > 0$. Furthermore define $f(t) = f_0 \circ \phi(t)$ and $g(t) = \sigma(t)\phi(t)^* g_0$. Then

$$\frac{\partial}{\partial \mathrm{t}}\Big|_{t=t_0} g(t) = -\frac{1}{\sigma(t_0)} g(t_0) + \sigma(t_0) \frac{\partial}{\partial \mathrm{t}}\Big|_{t=t_0} (\phi(t)^*(g_0)).$$

We compute, that

$$\begin{aligned} \sigma(t_0) \frac{\partial}{\partial \mathrm{t}}\Big|_{t=t_0} (\phi(t)^*(g_0)) &= \sigma(t_0) \mathcal{L}_{(\phi(t_0)^{-1})_* \frac{\partial}{\partial \mathrm{t}}\big|_{t=t_0} \phi(t)} \phi(t_0)^*(g_0) \\ &= \mathcal{L}_{\nabla^{g(t_0)} f(t_0)} g(t_0). \end{aligned}$$

This follows from the fact, that

$$\frac{\partial}{\partial \mathrm{t}}\Big|_{t=t_0} \phi(t) = \frac{\nabla^{g_0} f_0}{\sigma(t_0)} = \phi(t_0)^*(\nabla^{g(t_0)} f(t_0)).$$

We now evaluate at t instead of t_0 and obtain

$$\frac{\partial}{\partial \mathrm{t}} g(t) = -\frac{1}{\sigma(t)} g(t) + \mathcal{L}_{\nabla^{g(t)} f(t)} f(t).$$

Now

$$\begin{aligned}
-2\mathrm{Ric}_{g(t)} &= \phi(t)^*(-2\mathrm{Ric}_{g_0}) = \phi(t)^*(-g_0 + \mathcal{L}_{\nabla^{g_0} f_0} g_0) \\
&= -\frac{1}{\sigma(t)} g(t) + \mathcal{L}_{\nabla^{g(t)} f(t)} g(t).
\end{aligned}$$

So we obtain , that

$$\frac{\partial}{\partial t} g(t) = -\frac{1}{\sigma(t)} g(t) + \mathcal{L}_{\nabla^{g(t)} f(t)} g(t) = -2\mathrm{Ric}_{g(t)}.$$

Finally we calculate

$$\begin{aligned}
\frac{\partial f}{\partial t}(x,t) &= \Big(\frac{\partial}{\partial t}\phi(t)\Big)(f_0)(x) = \frac{1}{\sigma(t)} |\nabla_{g_0} f_0|^2(\phi(t)(x)) \\
&= |\nabla^{g(t)} f(t)|^2_{g(t)}.
\end{aligned}$$

□

3.2.9 Remark. It really doesn't matter what the end time is. One is allowed to shift the time, so that $g(-1) = g_0$, $\phi(-1) = \mathrm{id}_M$ and $f(-1) = f_0$. Then one would obtain $g(t)$ defined in the time interval $(-\infty, 0)$. This remark should make clear any ambiguity, when we discuss the result of Enders, Müller and Topping [6] in Chapter 6.

More details on Ricci solitons can be found in the books [2], [3], [4].

4. Examples of local invariants

In this chapter we will compute the local invariants for solutions $(M, g(t))$. The first local invariant is the branching curve and the second one is the $K3$ surface. We look at the examples of $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}^2$, $\mathbb{P}^2$ and $\mathbb{S}^4$.

4.1. The example of $(\mathbb{S}^3 \times \mathbb{R}, g(t))$

4.1.1. The branching curve

The initial metric g_0 (with respect to spherical coordinates on the $\mathbb{S}^3$ factor) is given by

$$g_0 = d\phi_1^2 + \sin^2\phi_1 d\phi_2^2 + \sin^2\phi_1 \sin^2\phi_2 d\phi_3^2 + dx^2.$$

Recall that the Ricci flow evolves each factor of a product metric seperately and if we use the formula for the evolution of the round metric on the sphere, we obtain that a solution to the Ricci flow is given by

$$g(t) = (1-4t)d\phi_1^2 + (1-4t)\sin^2\phi_1 d\phi_2^2 + (1-4t)\sin^2\phi_1 \sin^2\phi_2 d\phi_3^2 + dx^2.$$

The set $\left\{\frac{\partial}{\partial\phi_1}, \frac{\partial}{\partial\phi_2}, \frac{\partial}{\partial\phi_3}, \frac{\partial}{\partial x}\right\}$ constitutes a basis for T_xM. We obtain a time-dependent orthonormal frame, with respect to which the metric becomes diagonal by setting

$$\left\{e_a = \frac{1}{\sqrt{1-4t}}\frac{\partial}{\partial\phi_1}, e_b = \frac{1}{\sqrt{1-4t}\sin\phi_1}\frac{\partial}{\partial\phi_2}, e_c = \frac{1}{\sqrt{1-4t}\sin\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_3}, e_d = \frac{\partial}{\partial x}\right\},$$

where $e_\alpha := e_\alpha(x,t)$, $\alpha = a,b,c,d$. Then

$$\begin{aligned}
\Big\{e_a \wedge e_b &= \frac{1}{(1-4t)\sin\phi_1}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_2}, e_a\wedge e_c = \frac{1}{(1-4t)\sin\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_3},\\
e_a\wedge e_d &= \frac{1}{\sqrt{1-4t}}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial x}, e_b\wedge e_c = \frac{1}{(1-4t)\sin^2\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial\phi_3},\\
e_b\wedge e_d &= \frac{1}{\sqrt{1-4t}\sin\phi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial x}, e_c\wedge e_d = \frac{1}{\sqrt{1-4t}\sin\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_3}\wedge\frac{\partial}{\partial x}\Big\}
\end{aligned}$$

is an orthonormal frame for $\Lambda^2 T_x M$ and

$$\begin{aligned}
\Big\{f_1^\pm &= \frac{1}{\sqrt{2}}(e_a\wedge e_b \pm e_c\wedge e_d) = \frac{1}{\sqrt{2}(1-4t)\sin\phi_1}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_2} \pm \frac{1}{\sqrt{2}\sqrt{1-4t}\sin\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_3}\wedge\frac{\partial}{\partial x},\\
f_2^\pm &= \frac{1}{\sqrt{2}}(e_a\wedge e_c \mp e_b\wedge e_d) = \frac{1}{\sqrt{2}(1-4t)\sin\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_3} \mp \frac{1}{\sqrt{2}\sqrt{1-4t}\sin\phi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial x},\\
f_3^\pm &= \frac{1}{\sqrt{2}}(e_a\wedge e_d \pm e_b\wedge e_c) = \frac{1}{\sqrt{2}\sqrt{1-4t}}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial x} \pm \frac{1}{\sqrt{2}(1-4t)\sin^2\phi_1\sin\phi_2}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial\phi_3}\Big\}.
\end{aligned}$$

The Christoffel symbols are given by

$$\begin{array}{lll}
{\Gamma_{22}}^1 = -\cos\phi_1\sin\phi_1, & {\Gamma_{12}}^2 = \dfrac{\cos\phi_1}{\sin\phi_1}, & {\Gamma_{13}}^3 = \dfrac{\cos\phi_1}{\sin\phi_1}\\
{\Gamma_{33}}^1 = -\cos\phi_1\sin\phi_1\sin^2\phi_2, & {\Gamma_{21}}^2 = \dfrac{\cos\phi_1}{\sin\phi_1}, & {\Gamma_{23}}^3 = \dfrac{\cos\phi_2}{\sin\phi_2}\\
 & {\Gamma_{33}}^2 = -\cos\phi_2\sin\phi_2, & {\Gamma_{31}}^3 = \dfrac{\cos\phi_1}{\sin\phi_1},\\
 & & {\Gamma_{32}}^3 = \dfrac{\cos\phi_2}{\sin\phi_2}.
\end{array}$$

The components of the $(3,1)$-Riemann curvature tensor are given by

$$\begin{array}{lll}
{R_{122}}^1 = \sin^2\phi_1, & {R_{121}}^2 = \dfrac{\cos^2\phi_1 - 1}{\sin\phi_1}, & {R_{131}}^3 = \dfrac{\cos^2\phi_1-1}{\sin\phi_1},\\
{R_{212}}^1 = -\sin^2\phi_1, & {R_{211}}^2 = 1, & {R_{311}}^3 = 1,\\
{R_{133}}^1 = \sin^2\phi_1\sin^2\phi_2, & {R_{233}}^2 = \sin^2\phi_1\sin^2\phi_2, & {R_{232}}^3 = -\sin^2\phi_1,\\
{R_{313}}^1 = -\sin^2\phi_1\sin^2\phi_2, & {R_{323}}^2 = -\sin^2\phi_1\sin^2\phi_2, & {R_{322}}^3 = \sin^2\phi_1.
\end{array}$$

The components of the $(4,0)$-Riemann curvature tensor are given by

$$\begin{aligned}
R_{1221} &= (1-4t)\sin^2\phi_1, & R_{1212} &= -(1-4t)\sin^2\phi_1,\\
R_{2121} &= -(1-4t)\sin^2\phi_1, & R_{2112} &= (1-4t)\sin^2\phi_1,\\
R_{1331} &= (1-4t)\sin^2\phi_1\sin^2\phi_2, & R_{2332} &= (1-4t)\sin^4\phi_1\sin^2\phi_2,\\
R_{3131} &= -(1-4t)\sin^2\phi_1\sin^2\phi_2, & R_{3232} &= -(1-4t)\sin^4\phi_1\sin^2\phi_2
\end{aligned},$$

and

$$\begin{aligned}
R_{1313} &= -(1-4t)\sin^2\phi_1\sin^2\phi_2,\\
R_{3113} &= (1-4t)\sin^2\phi_1\sin^2\phi_2,\\
R_{2323} &= -(1-4t)\sin^4\phi_1\sin^2\phi_2,\\
R_{3223} &= (1-4t)\sin^4\phi_1\sin^2\phi_2.
\end{aligned}$$

Then

$$\begin{aligned}
R_{abba} &= \frac{1}{1-4t}, & R_{abab} &= -\frac{1}{1-4t}, & R_{acac} &= -\frac{1}{1-4t},\\
R_{baba} &= -\frac{1}{1-4t}, & R_{baab} &= \frac{1}{1-4t}, & R_{caac} &= \frac{1}{1-4t},\\
R_{acca} &= \frac{1}{1-4t}, & R_{bccb} &= \frac{1}{1-4t}, & R_{bcbc} &= -\frac{1}{1-4t},\\
R_{caca} &= -\frac{1}{1-4t}, & R_{cbcb} &= -\frac{1}{1-4t} & R_{cbbc} &= \frac{1}{1-4t}.
\end{aligned}$$

We are now in position to compute the scalar curvature.

$$\begin{aligned} R_{aa} &= R_{baa}{}^{b} + R_{caa}{}^{c} = \frac{2}{1-4t} \\ R_{bb} &= R_{abb}{}^{a} + R_{cbb}{}^{c} = \frac{2}{1-4t} \\ R_{cc} &= R_{acc}{}^{a} + R_{bcc}{}^{b} = \frac{2}{1-4t}. \end{aligned}$$

Thus $\mathrm{scal} = \frac{6}{1-4t}$ and $\frac{\mathrm{scal}}{12} = \frac{1}{2(1-4t)}$. Furthermore

$$\begin{aligned} \Lambda^2 g_x(\mathfrak{Rm}(f_1^+), f_1^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)) \\ &= \frac{1}{2}(R_{abba} + R_{abdc} + R_{cdba} + R_{cddc}) \\ &= \frac{1}{2(1-4t)}. \end{aligned}$$

$$\begin{aligned} \Lambda^2 g_x(\mathfrak{Rm}(f_2^+), f_2^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_c - e_b \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_c - e_b \wedge e_d)) \\ &= \frac{1}{2}(R_{acca} - R_{acdb} + R_{bdca} + R_{bddb}) \\ &= \frac{1}{2(1-4t)}. \end{aligned}$$

$$\begin{aligned} \Lambda^2 g_x(\mathfrak{Rm}(f_3^+), f_3^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_d + e_b \wedge e_c)), \frac{1}{\sqrt{2}}(e_a \wedge e_d + e_b \wedge e_c)) \\ &= \frac{1}{2}(R_{adda} - R_{adcb} + R_{bcda} + R_{bccb}) \\ &= \frac{1}{2(1-4t)}. \end{aligned}$$

$$\begin{aligned}\Lambda^2 g_x(\mathfrak{Rm}(f_1^-), f_1^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)) \\ &= \frac{1}{2}(R_{abba} + R_{abdc} - R_{cdba} - R_{cddc}) \\ &= \frac{1}{2(1-4t)}.\end{aligned}$$

$$\begin{aligned}\Lambda^2 g_x(\mathfrak{Rm}(f_2^-), f_2^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_c + e_b \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_c - e_b \wedge e_d)) \\ &= \frac{1}{2}(R_{acca} - R_{acdb} + R_{bdca} - R_{bddb}) \\ &= \frac{1}{2(1-4t)}.\end{aligned}$$

$$\begin{aligned}\Lambda^2 g_x(\mathfrak{Rm}(f_3^-), f_3^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_d - e_b \wedge e_c)), \frac{1}{\sqrt{2}}(e_a \wedge e_d + e_b \wedge e_c)) \\ &= \frac{1}{2}(R_{adda} + R_{adcb} - R_{bcda} - R_{bccb}) \\ &= -\frac{1}{2(1-4t)}.\end{aligned}$$

$$\begin{aligned}\Lambda^2 g_x(\mathfrak{Rm}(f_1^-), f_1^-) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)) \\ &= \frac{1}{2}(R_{abba} - R_{abdc} - R_{cdba} + R_{cddc}) \\ &= \frac{1}{2(1-4t)}.\end{aligned}$$

$$\begin{aligned}\Lambda^2 g_x(\mathfrak{Rm}(f_2^-), f_2^-) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_c + e_b \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_c + e_b \wedge e_d)) \\ &= \frac{1}{2}(R_{acca} + R_{acdb} + R_{bdca} + R_{bddb}) \\ &= \frac{1}{2(1-4t)}.\end{aligned}$$

$$\begin{aligned}\Lambda^2 g_x(\mathfrak{Rm}(f_3^-), f_3^-) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_d - e_b \wedge e_c)), \frac{1}{\sqrt{2}}(e_a \wedge e_d - e_b \wedge e_c)) \\ &= \frac{1}{2}(R_{adda} - R_{adcb} - R_{bcda} + R_{bccb}) \\ &= \frac{1}{2(1-4t)}.\end{aligned}$$

This means that the matrices of the bilinear forms

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(\cdot), \cdot\Big) = \Lambda^2 g_x\Big((A - \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_+})(\cdot), \cdot\Big)$$

and

$$\Lambda^2 g_x\Big(\mathfrak{W}_-(\cdot), \cdot\Big) = \Lambda^2 g_x\Big((C - \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_-})(\cdot), \cdot\Big)$$

are given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the matrix of the bilinear form $\Lambda^2 g_x\Big(B(\cdot), \cdot\Big)$ by

$$\begin{bmatrix} \frac{1}{2(1-4t)} & 0 & 0 \\ 0 & \frac{1}{2(1-4t)} & 0 \\ 0 & 0 & -\frac{1}{2(1-4t)} \end{bmatrix}.$$

We are now going to compute the branching curve. Obviously

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big) = \Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = 0$$

and

$$\Lambda^2 g_x\Big(B(T_-), T_+\Big) = -\frac{2}{1-4t} a^1 a^2 b^1 b^2 + \frac{1}{1-4t}(a^1)^2 (b^2)^2 + \frac{1}{1-4t}(a^2)^2 (b^1)^2,$$

where

$$T_- = 2ib^1b^2f_1^- + i\{(b^2)^2 - (b^1)^2\}f_2^- + [(b^1)^2 + (b^2)^2]f_3^-$$

and

$$T_+ = 2ia^1a^2f_1^+ + i\{(a^2)^2 - (a^1)^2\}f_2^+ + [-(a^1)^2 - (a^2)^2)]f_3^+.$$

We compute that

$$\left[\Lambda^2 g_x\Big(B(T_-), T_+\Big)\right]^2 = \frac{1}{(1-4t)^2}(a^1b^2 - a^2b^1)^4.$$

Thus

$$\Gamma_x = \{([a^1, a^2], [b^1, b^2]) \in \mathbb{P}(S_x^-) \times \mathbb{P}(S_x^+) : (a^1b^2 - a^2b^1)^4 = 0\}.$$

This curve is never smooth and has multiplicity four. Notice, that in this the branching curve represents geometrically a quadruple diagonal.

4.1.2. The K3 surface

In the notation of (2.27), the $K3$ surface is described by the polynomial

$$\begin{aligned} \Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big)s^2 + 2\Lambda^2 g_x\Big(B(T_-), T_+\Big)s + \Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) &= 0 \\ \Big(-2a^1a^2b^1b^2 + (a^1)^2(b^2)^2 + (a^2)^2(b^1)^2\Big)s &= 0. \end{aligned}$$

4.2. The example of $(\mathbb{S}^2 \times \mathbb{S}^2, g(t))$

4.2.1. The branching curve

The initial metric g_0 (with respect to spherical coordinates on both $\mathbb{S}^2$ factors) is given by

$$g_0 = d\phi_1^2 + \sin^2\phi_1 d\phi_2^2 + d\psi_1^2 + \sin^2\psi_1 d\psi_2^2.$$

Now a solution to the Ricci flow is given by

$$g(t) = (1-2t)d\phi_1^2 + (1-2t)\sin^2\phi_1 d\phi_2^2 + (1-2t)d\psi_1^2 + (1-2t)\sin^2\psi_1 d\psi_2^2.$$

The set $\left\{\frac{\partial}{\partial\phi_1}, \frac{\partial}{\partial\phi_2} \frac{\partial}{\partial\psi_1}, \frac{\partial}{\partial\psi_2}\right\}$ constitutes a basis for T_xM. We obtain an orthonormal frame, with respect to which the metric becomes diagonal by setting

$$\left\{e_a = \frac{1}{\sqrt{1-2t}}\frac{\partial}{\partial\phi_1}, e_b = \frac{1}{\sqrt{1-2t}\sin\phi_1}\frac{\partial}{\partial\phi_2}, e_c = \frac{1}{\sqrt{1-2t}}\frac{\partial}{\partial\psi_1}, e_d = \frac{1}{\sqrt{1-2t}\sin\psi_1}\frac{\partial}{\partial\psi_2}\right\}$$

Then

$$\begin{aligned}\Big\{e_a \wedge e_b &= \frac{1}{(1-2t)\sin\phi_1}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_2}, e_a\wedge e_c = \frac{1}{(1-2t)}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\psi_1},\\ e_a\wedge e_d &= \frac{1}{(1-2t)\sin\psi_1}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\psi_2}, e_b\wedge e_c = \frac{1}{(1-2t)\sin\phi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial\psi_1},\\ e_b\wedge e_d &= \frac{1}{\sqrt{1-2t}\sin\phi_1\sin\psi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial\psi_2}, e_c\wedge e_d = \frac{1}{\sqrt{1-2t}\sin\psi_1}\frac{\partial}{\partial\psi_1}\wedge\frac{\partial}{\partial\psi_2}\Big\}\end{aligned}$$

is an orthonormal frame for $\Lambda^2 T_xM$ and

$$\left\{f_1^\pm = \frac{1}{\sqrt{2}}(e_a\wedge e_b \pm e_c\wedge e_d), f_2^\pm = \frac{1}{\sqrt{2}}(e_a\wedge e_c \mp e_b\wedge e_d), f_3^\pm = \frac{1}{\sqrt{2}}(e_a\wedge e_d \pm e_b\wedge e_c)\right\}.$$

The Christoffel symbols are given by

$$\begin{aligned}&{\Gamma_{22}}^1 = -\cos\phi_1\sin\phi_1, \quad {\Gamma_{12}}^2 = \frac{\cos\phi_1}{\sin\phi_1}, \quad {\Gamma_{44}}^3 = -\frac{\cos\psi_1}{\sin\psi_1} \quad {\Gamma_{34}}^4 = \frac{\cos\psi_1}{\sin\psi_1}\\ &{\Gamma_{21}}^2 = \frac{\cos\phi_1}{\sin\phi_1}, \quad {\Gamma_{43}}^4 = \frac{\cos\psi_1}{\sin\psi_1}\end{aligned}$$

The components of the $(3,1)$-Riemann curvature tensor are given by

$$\begin{aligned}&{R_{122}}^1 = \sin^2\phi_1, \quad {R_{121}}^2 = \frac{\cos^2\phi_1 - 1}{\sin\phi_1}, \quad {R_{344}}^3 = \sin^2\psi_1, \quad {R_{343}}^4 = \frac{\cos^2\psi_1 - 1}{\sin\psi_1},\\ &{R_{212}}^1 = -\sin^2\phi_1, \quad {R_{211}}^2 = 1, \quad {R_{434}}^3 = -\sin^2\psi_1, \quad {R_{433}}^4 = 1.\end{aligned}$$

The components of the $(4,0)$-Riemann curvature tensor are given by

$$\begin{aligned}R_{1221} &= (1-2t)\sin^2\phi_1, & R_{1212} &= -(1-2t)\sin^2\phi_1,\\ R_{2121} &= -(1-2t)\sin^2\phi_1, & R_{2112} &= (1-2t)\sin^2\phi_1, \quad ,\end{aligned}$$

and

$$\begin{aligned} R_{3443} &= (1-2t)\sin^2\psi_1, & R_{3434} &= -(1-2t)\sin^2\psi_1, \\ R_{4343} &= -(1-2t)\sin^2\psi_1, & R_{4334} &= (1-2t)\sin^2\psi_1, \end{aligned} \quad ,$$

Then

$$\begin{aligned} R_{abba} &= \frac{1}{1-2t}, & R_{abab} &= -\frac{1}{1-2t}, \\ R_{baba} &= -\frac{1}{1-2t}, & R_{baab} &= \frac{1}{1-2t}, \end{aligned} \quad ,$$

and

$$\begin{aligned} R_{cddc} &= \frac{1}{1-2t}, & R_{cdcd} &= -\frac{1}{1-2t}, \\ R_{cdcd} &= -\frac{1}{1-2t}, & R_{cddc} &= \frac{1}{1-2t}, \end{aligned} \quad ,$$

We are now in position to compute the scalar curvature.

$$\begin{aligned} R_{aa} &= R_{baa}{}^{b} = \frac{1}{1-2t} \\ R_{bb} &= R_{abb}{}^{a} = \frac{1}{1-2t} \\ R_{cc} &= R_{acc}{}^{a} = \frac{1}{1-2t} \\ R_{dd} &= R_{cdd}{}^{c} = \frac{1}{1-2t}. \end{aligned}$$

Thus scal $= \frac{4}{1-2t}$ and $\frac{\text{scal}}{12} = \frac{1}{3(1-2t)}$. Furthermore

$$\begin{aligned}
\Lambda^2 g_x(\mathfrak{Rm}(f_1^+), f_1^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)) \\
&= \frac{1}{2}(R_{abba} + R_{abdc} + R_{cdba} + R_{cddc}) \\
&= \frac{1}{2(1-2t)}.
\end{aligned}$$

$$\begin{aligned}
\Lambda^2 g_x(\mathfrak{Rm}(f_1^-), f_1^-) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)) \\
&= \frac{1}{2}(R_{abba} - R_{abdc} - R_{cdba} + R_{cddc}) \\
&= \frac{1}{2(1-2t)}.
\end{aligned}$$

This means that the matrices of the bilinear forms

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(\cdot), \cdot\Big) = \Lambda^2 g_x\Big(\big(A - \frac{\text{scal}}{12}\text{Id}_{\Lambda_+}\big)(\cdot), \cdot\Big)$$

and

$$\Lambda^2 g_x\Big(\mathfrak{W}_-(\cdot), \cdot\Big) = \Lambda^2 g_x\Big(\big(C - \frac{\text{scal}}{12}\text{Id}_{\Lambda_-}\big)(\cdot), \cdot\Big)$$

are given by

$$\begin{bmatrix} \frac{1}{6(1-2t)} & 0 & 0 \\ 0 & -\frac{1}{3(1-2t)} & 0 \\ 0 & 0 & -\frac{1}{3(1-2t)} \end{bmatrix}$$

and the matrix of the bilinear form $\Lambda^2 g_x\Big(B(\cdot), \cdot\Big)$ by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We are now going to compute the branching curve.

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big) = -\frac{2}{3(1-2t)}(a^1)^2(a^2)^2,$$
$$\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = -\frac{2}{3(1-2t)}(b^1)^2(b^2)^2.$$

where

$$T_- = 2ib^1b^2f_1^- + i\{(b^2)^2 - (b^1)^2\}f_2^- + [(b^1)^2 + (b^2)^2]f_3^-$$

and

$$T_+ = 2ia^1a^2f_1^+ + i\{(a^2)^2 - (a^1)^2\}f_2^+ + [-(a^1)^2 - (a^2)^2)]f_3^+.$$

Thus

$$\Gamma_x = \{([a^1, a^2], [b^1, b^2]) \in \mathbb{P}(S_x^-) \times \mathbb{P}(S_x^+) : \ (a^1a^2b^1b^2)^2 = 0\}.$$

Notice, that in this the branching curve represents geometrically a double rectangle.

4.2.2. The $K3$ surface

In the notation of (2.27), the $K3$ surface is described by the polynomial

$$\begin{aligned} \Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big)s^2 + 2\Lambda^2 g_x\Big(B(T_-), T_+\Big)s + \Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) &= 0 \\ (a^1)^2(a^2)^2s^2 + (b^1)^2(b^2)^2 &= 0. \end{aligned}$$

4.3. The example of $(\mathbb{S}^2 \times \mathbb{R}^2, g(t))$

4.3.1. The branching curve

The initial metric g_0 (with respect to spherical coordinates on the $\mathbb{S}^2$ factor) is given by

$$g_0 = d\phi_1^2 + \sin^2\phi_1 d\phi_2^2 + dx^2 + dy^2.$$

In this case a solution to the Ricci flow is given by

$$g(t) = (1-2t)d\phi_1^2 + (1-2t)\sin^2\phi_1 d\phi_2^2 + dx^2 + dy^2.$$

The set $\left\{\frac{\partial}{\partial\phi_1}, \frac{\partial}{\partial\phi_2}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ constitutes a basis for T_xM. We obtain an orthonormal frame, with respect to which the metric becomes diagonal by setting

$$\left\{e_a = \frac{1}{\sqrt{1-2t}}\frac{\partial}{\partial\phi_1}, e_b = \frac{1}{\sqrt{1-2t}\sin\phi_1}\frac{\partial}{\partial\phi_2}, e_c = \frac{\partial}{\partial x}, e_d = \frac{\partial}{\partial y}\right\}$$

Then

$$\begin{aligned}\Big\{e_a\wedge e_b &= \frac{1}{(1-2t)\sin\phi_1}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_2}, e_a\wedge e_c = \frac{1}{\sqrt{1-2t}}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial x},\\ e_a\wedge e_d &= \frac{1}{\sqrt{1-2t}}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial y}, e_b\wedge e_c = \frac{1}{\sqrt{1-2t}\sin\phi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial x},\\ e_b\wedge e_d &= \frac{1}{\sqrt{1-2t}\sin\phi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial y}, e_c\wedge e_d = \frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\Big\}\end{aligned}$$

is an orthonormal frame for $\Lambda^2 T_xM$ and

$$\begin{aligned}\Big\{f_1^{\pm} &= \frac{1}{\sqrt{2}}(e_a\wedge e_b \pm e_c\wedge e_d) = \frac{1}{(1-2t)\sin\phi_1}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_2} \pm \frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y},\\ f_2^{\pm} &= \frac{1}{\sqrt{2}}(e_a\wedge e_c \mp e_b\wedge e_d) = \frac{1}{\sqrt{1-2t}}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial x} \mp \frac{1}{\sqrt{1-2t}\sin\phi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial y}\\ f_3^{\pm} &= \frac{1}{\sqrt{2}}(e_a\wedge e_d \pm e_b\wedge e_c) = \frac{1}{\sqrt{1-2t}}\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial y} \pm \frac{1}{\sqrt{1-2t}\sin\phi_1}\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial x}\Big\}.\end{aligned}$$

The Christoffel symbols in this case are given by

$$\Gamma_{22}{}^1 = -\cos\phi_1\sin\phi_1, \qquad \Gamma_{12}{}^2 = \frac{\cos\phi_1}{\sin\phi_1}, \qquad \Gamma_{21}{}^2 = \frac{\cos\phi_1}{\sin\phi_1}.$$

The components of the $(3,1)$-Riemann curvature tensor are given by

$$R_{122}{}^1 = \sin^2\phi_1, \qquad R_{121}{}^2 = \frac{\cos^2\phi_1 - 1}{\sin\phi_1},$$

$$R_{212}{}^1 = -\sin^2\phi_1, \qquad R_{211}{}^2 = 1.$$

The components of the $(4, 0)$-Riemann curvature tensor are given by

$$R_{1221} = (1-2t)\sin^2\phi_1, \qquad R_{1212} = -(1-2t)\sin^2\phi_1,$$
$$R_{2121} = -(1-2t)\sin^2\phi_1, \qquad R_{2112} = (1-2t)\sin^2\phi_1.$$

Then

$$R_{abba} = \frac{1}{1-2t}, \qquad R_{abab} = -\frac{1}{1-2t},$$
$$R_{baba} = -\frac{1}{1-2t}, \qquad R_{baab} = \frac{1}{1-2t}.$$

The Ricci curvature is

$$\begin{aligned} R_{aa} = R_{baa}{}^{b} &= \frac{1}{1-2t} \\ R_{bb} = R_{abb}{}^{a} &= \frac{1}{1-2t}. \end{aligned}$$

Thus the scalar curvature is given by $\mathrm{scal} = \frac{2}{1-2t}$ and $\frac{\mathrm{scal}}{12} = \frac{1}{6(1-2t)}$. Furthermore

$$\begin{aligned} \Lambda^2 g_x(\mathfrak{Rm}(f_1^+), f_1^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)) \\ &= \frac{1}{2}(R_{abba} + R_{abdc} + R_{cdba} + R_{cddc}) \\ &= \frac{1}{2(1-2t)}. \end{aligned}$$

$$\begin{aligned} \Lambda^2 g_x(\mathfrak{Rm}(f_1^-), f_1^+) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b + e_c \wedge e_d)) \\ &= \frac{1}{2}(R_{abba} + R_{abdc} - R_{cdba} - R_{cddc}) \\ &= \frac{1}{2(1-2t)}. \end{aligned}$$

$$\begin{aligned}\Lambda^2 g_x(\mathfrak{Rm}(f_1^-), f_1^-) &= \Lambda^2 g_x(\mathfrak{Rm}(\frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)), \frac{1}{\sqrt{2}}(e_a \wedge e_b - e_c \wedge e_d)) \\ &= \frac{1}{2}(R_{abba} - R_{abdc} - R_{cdba} + R_{cddc}) \\ &= \frac{1}{2(1-2t)}.\end{aligned}$$

This means that the matrices of the bilinear forms

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(\cdot), \cdot\Big) = \Lambda^2 g_x\Big((A - \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_+})(\cdot), \cdot\Big)$$

and

$$\Lambda^2 g_x\Big(\mathfrak{W}_-(\cdot), \cdot\Big) = \Lambda^2 g_x\Big((C - \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_-})(\cdot), \cdot\Big)$$

are given by

$$\begin{bmatrix} \frac{1}{3(1-2t)} & 0 & 0 \\ 0 & -\frac{1}{6(1-2t)} & 0 \\ 0 & 0 & -\frac{1}{6(1-2t)} \end{bmatrix}$$

and the matrix of the bilinear form $\Lambda^2 g_x\Big(B(\cdot), \cdot\Big)$ by

$$\begin{bmatrix} \frac{1}{2(1-2t)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We are now going to compute the branching curve.

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big) = -\frac{2}{(1-2t)}(a^1)^2(a^2)^2,$$

$$\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = -\frac{2}{(1-2t)}(b^1)^2(b^2)^2,$$

$$\Lambda^2 g_x\Big(B(T_-), T_+\Big) = -\frac{2}{1-2t}a^1a^2b^1b^2.$$

where

$$T_- = 2ib^1b^2f_1^- + i\{(b^2)^2 - (b^1)^2\}f_2^- + [(b^1)^2 + (b^2)^2]f_3^-$$

and

$$T_+ = 2ia^1a^2f_1^+ + i\{(a^2)^2 - (a^1)^2\}f_2^+ + [-(a^1)^2 - (a^2)^2)]f_3^+.$$

We compute that

$$\left[\Lambda^2 g_x\Big(B(T_-), T_+\Big)\right]^2 = \frac{4}{(1-2t)^2}(a^1a^2b^1b^2)^2 = \Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big)\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big).$$

One observes, that in this case the branching curve doesn't exist.

4.3.2. The $K3$ surface

In the notation of (2.27), the $K3$ surface is described by the polynomial

$$\begin{aligned}\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+)T_+\Big)s^2 + 2\Lambda^2 g_x\Big(B(T_-), T_+\Big)s + \Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) &= 0\\ (a^1)^2(a^2)^2s^2 + 2a^1a^2b^1b^2s + (b^1)^2(b^2)^2 &= 0.\end{aligned}$$

4.4. The example of $(\mathbb{P}^2, g(t))$

4.4.1. The branching curve

The initial metric g_{FS} is the Fubini-Study metric. A solution to the Ricci flow is given by

$$g(t) = (1-2\kappa t)g_{FS},$$

where $\kappa > 0$. By working exactly in the same way as is the previous examples one can obtain that the matrix of the bilinear form

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(\cdot), \cdot\Big) = \Lambda^2 g_x\Big((A - \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_+})(\cdot), \cdot\Big)$$

is given by

$$\begin{bmatrix} \frac{1}{2(1-2\kappa t)} - \frac{\mathrm{scal}}{12} & 0 & 0 \\ 0 & \frac{1}{2(1-2\kappa t)} - \frac{\mathrm{scal}}{12} & 0 \\ 0 & 0 & \frac{1}{2(1-2\kappa t)} - \frac{\mathrm{scal}}{12} \end{bmatrix},$$

that of

$$\Lambda^2 g_x\Big(\mathfrak{W}_-(\cdot),\cdot\Big) = \Lambda^2 g_x\Big((C - \frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_-})(\cdot),\cdot\Big)$$

by

$$\begin{bmatrix} \frac{3}{2(1-2\kappa t)} - \frac{\mathrm{scal}}{12} & 0 & 0 \\ 0 & -\frac{\mathrm{scal}}{12} & 0 \\ 0 & 0 & -\frac{\mathrm{scal}}{12} \end{bmatrix}$$

and finally the matrix of the bilinear form $\Lambda^2 g_x\Big(B(\cdot),\cdot\Big)$ by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We are now going to compute the branching curve.

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+),T_+\Big) = 0,$$

$$\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-),T_-\Big) = -\Big(\frac{6}{1-2\kappa t} - \frac{\mathrm{scal}}{3}\Big)(b^1)^2(b^2)^2.$$

where

$$T_- = 2ib^1b^2f_1^- + i\{(b^2)^2 - (b^1)^2\}f_2^- + [(b^1)^2 + (b^2)^2]f_3^-$$

and

$$T_+ = 2ia^1a^2f_1^+ + i\{(a^2)^2 - (a^1)^2\}f_2^+ + [-(a^1)^2 - (a^2)^2)]f_3^+.$$

Thus there is no curve and the branching locus is the whole quadric $\mathbb{P}(g_x)$.

4.4.2. The $K3$ surface

In the notation of (2.27), the $K3$ surface is described by the polynomial

$$\begin{aligned}\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+),T_+\Big)s^2+2\Lambda^2 g_x\Big(B(T_-),T_+\Big)s+\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-),T_-\Big) &= 0\\ (b^1b^2\mu)^2 &= 0.\end{aligned}$$

4.5. The example of $(\mathbb{S}^4, g(t))$

In this section we show that the local invarants for the solution $(\mathbb{S}^4, g(t))$ do not exist. The initial metric g_0 (with respect to spherical coordinates on $\mathbb{S}^4$) is given by

$$g_0 = d\phi_1^2+\sin^2\phi_1 d\phi_2^2+sin^2\phi_1\sin^2\phi_2 d\phi_3^2+\sin^2\sin^2\phi_2\sin^2\phi_3 d\phi_4^2.$$

A solution to the Ricci flow is given by

$$g(t)=(1-6t)d\phi_1^2+(1-6t)\sin^2\phi_1 d\phi_2^2+(1-6t)sin^2\phi_1\sin^2\phi_2 d\phi_3^2+(1-6t)\sin^2\sin^2\phi_2\sin^2\phi_3 d\phi_4^2.$$

Working exactly as in the previous examples ones can compute that matrices of the bilinear forms $\Lambda^2 g_x\Big(\mathfrak{W}_+(\cdot),\cdot\Big)$ and $\Lambda^2 g_x\Big(\mathfrak{W}_-(\cdot),\cdot\Big)$ are given by

$$\begin{bmatrix}\frac{1}{3(1-6t)} & 0 & 0\\ 0 & \frac{1}{3(1-6t)} & 0\\ 0 & 0 & \frac{1}{3(1-6t)}\end{bmatrix}$$

and the matrix of the bilinear form $\Lambda^2 g_x\Big(B(\cdot),\cdot\Big)$ by

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

which implies that

$$\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big) = 0,$$
$$\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = 0,$$

with T_- and T_+ as in the previous examples. Thus in this case both the branching curve and the $K3$ surface do not exist.

5. Evolving the branching curve under the Ricci flow

5.1. The evolution of the curvature operator

5.1.1. Uhlenbeck's Trick

In this section we will demonstrate Uhlenbeck's trick in order to derive the evolution equation for the curvature operator with respect to an evolving orthonormal frame. This technique was introduced in Hamilton's paper [10] and can be found also in both books [2] and [3]. We will use this evolution equation in the next section in order to compute the evolution equation for the coefficients of the branching curve. First we recall the evolution equation of the components Riemann curvature tensor with respect to some local coordinate system.

5.1.1 Proposition.

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &- \sum_{p=1}^{n}(R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}),\end{aligned}$$

where

$$B_{ijkl} := -\sum_{p,q,r,s=1}^{n} g^{pr}g^{qs}R_{ipjq}R_{krls} = -\sum_{p,q=1}^{n} R_{pij}{}^{q}R_{qlk}{}^{p}.$$

Furthermore, note that B is quadratic in the Riemann curvature tensor and satisfies the following algebraic identity

$$B_{ijkl} = B_{jilk} = B_{klij}.$$

We assume that $(M^n, g(t))$ is a solution of the Ricci flow for $t \in [0, T)$ and $\{e_\alpha\}$ is a local orthonormal frame in an open subset $U \subset M$, with respect to the initial metric g_0. We will evolve this frame, so that it remains orthonormal with respect to $g(t)$. For each $x \in U$ we define:

$$\begin{aligned} \frac{\partial}{\partial t} e_\alpha(x,t) &= \mathfrak{Ric}(e_\alpha(x,t)) \\ e_\alpha(x,0) &= e_\alpha^0(x). \end{aligned} \tag{5.1}$$

In this case $\mathfrak{Ric} \in \mathrm{End}(TM)$, i.e. is seen as a $(1,1)$-tensor. For notational simplicity we will just write e_α instead of $e_\alpha(x,t)$ from now on.

5.1.2 Proposition. *If $(M^n, g(t))$ is a solution of the Ricci flow and $\{e_\alpha\}$ is a local frame satisfying (5.1), then*

$$\frac{\partial}{\partial t} g(e_\alpha, e_\beta) = 0.$$

This means that if $\{e_\alpha^0\}$ is orthonormal with respect to g_0, then $\{e_\alpha\}$ remains orthonormal with respect to $g(t)$.

Proof.

$$\begin{aligned} \frac{\partial}{\partial t}(g(e_\alpha, e_\beta)) &= (\frac{\partial}{\partial \mathrm{t}} g)(e_\alpha, e_\beta) + g(\frac{\partial}{\partial \mathrm{t}} e_\alpha, e_\beta) + g(e_\alpha, \frac{\partial}{\partial \mathrm{t}} e_\beta) \\ &= -2\mathrm{Ric}(e_\alpha, e_\beta) + g(\mathfrak{Ric}(e_\alpha, e_\beta)) + g(e_\alpha, \mathfrak{Ric}(e_\beta)) = 0. \end{aligned}$$

$\square$

Let $\{x^i\}_{i=1}^n$ denote local coordinates on U. Then

$$e_\alpha = \sum_{i=1}^n e_\alpha^i \frac{\partial}{\partial x^i}.$$

As a result

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \mathrm{Rm}(e_\alpha, e_\beta, e_\gamma, e_\delta) \\ &= \mathrm{Rm}\,(\sum_{i=1}^n e_\alpha^i \frac{\partial}{\partial x^i}, \sum_{j=1}^n e_\beta^j \frac{\partial}{\partial x^j}, \sum_{k=1}^n e_\gamma^k \frac{\partial}{\partial x^k}, \sum_{l=1}^n e_\delta^l \frac{\partial}{\partial x^l}) \\ &= \sum_{i,j,k,l=1}^n e_\alpha^i e_\beta^j e_\gamma^k e_\delta^l R_{ijkl}. \end{aligned}$$

5.1.3 Proposition. *Let $(M^n, g(t))$ be a solution to the Ricci flow then $R_{\alpha\beta\gamma\delta}$ evolves by*

$$\frac{\partial}{\partial t} R_{\alpha\beta\gamma\delta} = \Delta R_{\alpha\beta\gamma\delta} + 2(B_{\alpha\beta\gamma\delta} - B_{\alpha\beta\delta\gamma} + B_{\alpha\gamma\beta\delta} - B_{\alpha\delta\beta\gamma}),$$

where

$$B_{\alpha\beta\gamma\delta} = -\sum_{\varepsilon,\zeta,\eta,\theta=1}^{n} g^{\varepsilon\eta} g^{\zeta\theta} R_{\alpha\varepsilon\beta\zeta} R_{\gamma\eta\delta\theta}.$$

Proof. Observing that

$$\frac{\partial}{\partial t} e^i_\alpha = \sum_{l=1}^{n} R^i_l e^l_\alpha,$$

we compute

$$\begin{aligned}
\frac{\partial}{\partial t} R_{\alpha\beta\gamma\delta} &= \sum_{i,j,k,l=1}^{n} \frac{\partial}{\partial t}(e^i_\alpha e^j_\beta e^k_\gamma e^l_\delta R_{ijkl}) \\
&= \sum_{i,j,k,l=1}^{n} \Big[\Big(\frac{\partial}{\partial t} e^i_\alpha \Big) e^j_\beta e^k_\gamma e^l_\delta R_{ijkl} + e^i_\alpha \Big(\frac{\partial}{\partial t} e^j_\beta \Big) e^k_\gamma e^l_\delta R_{ijkl} + e^i_\alpha e^j_\beta \Big(\frac{\partial}{\partial t} e^k_\gamma \Big) e^l_\delta R_{ijkl} \\
&+ e^i_\alpha e^j_\beta e^k_\gamma \Big(\frac{\partial}{\partial t} e^l_\delta \Big) R_{ijkl} + e^i_\alpha e^j_\beta e^k_\gamma e^l_\delta \Big(\frac{\partial}{\partial t} R_{ijkl} \Big) \Big] \\
&= \sum_{i,j,k,l=1}^{n} \Big[\Big(\sum_{m=1}^{n} R^i_m e^m_\alpha \Big) e^j_\beta e^k_\gamma e^l_\delta R_{ijkl} + e^i_\alpha \Big(\sum_{m=1}^{n} R^j_m e^m_\beta \Big) e^k_\gamma e^l_\delta R_{ijkl} \\
&+ e^i_\alpha e^j_\beta \Big(\sum_{m=1}^{n} R^k_m e^m_\gamma \Big) e^l_\delta R_{ijkl} + e^i_\alpha e^j_\beta e^k_\gamma \Big(\sum_{m=1}^{n} R^l_m e^m_\delta \Big) R_{ijkl} \\
&+ e^i_\alpha e^j_\beta e^k_\gamma e^l_\delta \Big(\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\
&- \sum_{p=1}^{n} (R^p_i R_{pjkl} + R^p_j R_{ipkl} + R^p_k R_{ijpl} + R^p_l R_{ijkp}) \Big) \Big] \\
&= e^i_\alpha e^j_\beta e^k_\gamma e^l_\delta \left[\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \right].
\end{aligned}$$

□

One observes that now, the last four terms from the evolution equation of Rm in Proposition 5.1.1 have been eliminated.

5.1.2. The structure of the evolution equation

Consider the curvature operator $\mathfrak{Rm} \in \mathrm{End}(\Lambda^2 T_x M) \cong \Lambda^2 T^*_x M \otimes \Lambda^2 T_x M$. In components we have

$$\mathfrak{Rm}(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}, dx^q \wedge dx^r) = R_{(ij)}{}^{(qr)} = \sum_{q,r=1}^{n} g^{qp} g^{rs} R_{(ij)(ps)}.$$

Then the components of the square of the curvature operator, denoted by $\mathfrak{Rm}^2 \in \mathrm{End}(\Lambda^2 T_x M)$ are given by

$$\begin{aligned}(R^2)_{(ij)}{}^{(kl)} &= \sum_{q,r=1}^{n} R_{(qr)}{}^{(kl)} R_{(ij)}{}^{(qr)} \\ &= \sum_{p,q,r,s=1}^{n} R_{(qr)}{}^{(kl)} g^{qp} g^{rs} R_{(ij)(ps)}.\end{aligned}$$

The associated bilinear form is determined by its components

$$(R^2)_{(ij)(kl)} = \sum_{p,q,r,s=1}^{n} R_{(qr)(kl)} g^{qp} g^{rs} R_{(ij)(ps)}.$$

Thus the components of the associated $(4,0)$-tensor are

$$(R^2)_{ijkl} = -\sum_{p,q,r,s=1}^{n} R_{qrkl} g^{qp} g^{rs} R_{ijps}.$$

By using the symmetries of the curvature tensor, the first Bianchi identity $R_{ijkl} + R_{iklj} + R_{iljk}$ and the definiton of B_{ijkl} one can show, that

$$(R^2)_{ijkl} = 2(B_{ijkl} - B_{ijlk}).$$

We are now going to introduce the Lie algebra square. Let $\mathfrak{g}$ be any Lie algebra endowed with a scalar product $\langle \cdot, \cdot \rangle$ and let $\{F^\alpha\}$ be a basis of $\mathfrak{g}$. The structure constants of $\mathfrak{g}$ are defined by

$$[F^\alpha, F^\beta] = \sum c_\gamma{}^{\alpha\beta} F^\gamma,$$

where $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$. Let $\{F_\alpha\}$ denote the basis algebraically dual to $\{F^\alpha\}$ such that $F_\beta(F^\alpha) = \delta^\alpha_\beta$.

Given a symmetric bilinear form L on $\mathfrak{g}^*$, we may regards L as an element of $S^2(\mathfrak{g})$ whose components are given by

$$L_{\alpha\beta} = L(F_\alpha, F_\beta).$$

There is a commutative bilinear operation $\# : S^2(\mathfrak{g}) \times S^2(\mathfrak{g}) \to S^2(\mathfrak{g})$ given in the following way. Let

$$L = \sum L_{\alpha\beta} F^\alpha \otimes F^\beta$$

and

$$M = \sum M_{\gamma\delta} F^\gamma \otimes F^\delta.$$

Then

$$\begin{aligned} L\#M &= \sum L_{\alpha\beta} M_{\gamma\delta} (F^\alpha \otimes F^\beta)\#(F^\gamma \otimes F^\delta) \\ &:= \sum L_{\alpha\beta} M_{\gamma\delta} [F^\alpha, F^\gamma] \otimes [F^\beta, F^\delta] \\ &= \sum L_{\alpha\beta} M_{\gamma\delta} c_\varepsilon{}^{\alpha\gamma} c_\zeta{}^{\beta\delta} F^\varepsilon \otimes F^\zeta. \end{aligned}$$

Then

$$(L^\#)_{\varepsilon\zeta} := (L\#L)_{\varepsilon\zeta} = \sum L_{\alpha\beta} L_{\gamma\delta} c_\varepsilon{}^{\alpha\gamma} c_\zeta{}^{\beta\delta}.$$

The associated operator is given in coordinates by

$$(L^\#)_\varepsilon{}^\zeta = \sum \langle F^\zeta, F^\theta \rangle L_{\alpha\beta} L_{\gamma\delta} c_\varepsilon{}^{\alpha\gamma} c_\theta{}^{\beta\delta} = \sum L_{\alpha\beta} L_{\gamma\delta} c_\varepsilon{}^{\alpha\gamma} c^{\zeta\beta\delta}.$$

For each $x \in M$ we can give $\Lambda^2 T_x^* M$ the structure of a Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{so}(n)$. Thus we can view R as an element of $S^2(\mathfrak{so}(n))$.

We can define the Lie algebra square $\mathfrak{Rm}^\#$ of the curvature operator in components by

$$(R^\#)_{(ij)}{}^{(kl)} = \sum R_{(pq)(uv)} R_{(rs)(wx)} c_{(ij)}{}^{(pq)(rs)} c^{(kl)(uv)(wx)}.$$

The components of the associated bilinear form are given by

$$(R^\#)_{(ij)(kl)} = \sum R_{(pq)(uv)} R_{(rs)(wx)} c_{(ij)}{}^{(pq)(rs)} c_{(kl)}{}^{(uv)(wx)}$$

and those of the associated $(4, 0)$-tensor by

$$(R^{\#})_{ijkl} = \sum R_{pquv} R_{rswx} c_{(ij)}{}^{(pq)(rs)} c_{(lk)}{}^{(uv)(wx)}.$$

Then by using the definition of B_{ijkl} and the fact that the structure constants are those for $\mathfrak{so}(n)$ one can show, that

$$(R^{\#})_{ijkl} = 2(B_{iklj} - B_{iljk}).$$

An explicit argument for that can be found in the book [2] on page 185.

As a result by the construction above we can write down an evolution equation for the curvature operator with respect to an orthonormal frame which evolves to stay orthonormal. This is given in the next theorem.

5.1.4 Theorem. *If $g(t)$ is a solution of the Ricci flow, the curvature operator evolves by*

$$\frac{\partial}{\partial t} \mathfrak{Rm} = \Delta \mathfrak{Rm} + \mathfrak{Rm}^2 + \mathfrak{Rm}^{\#}.$$

5.1.5 Remark. One has exactly the same evolution equation for the bilinear form R and the $(4, 0)$-tensor Rm as well. This follows from the fact that we are working with an orthonormal frame.

Hamilton showed in his paper [10], that in dimension four we have the following evolution equations for the various parts of the curvature operator, with respect to the block decomposition of the curvature operator $\mathfrak{Rm} = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$ mentioned in Chapter 2.

5.1.6 Corollary.

Let $(M^4, g(t))$ be a solution to the Ricci flow. The evolution equation for the curvature operator breaks up in to three equations:

$$\frac{\partial}{\partial t} A = \Delta A + A^2 + 2A^{\#} + BB^t,$$

$$\frac{\partial}{\partial t} B = \Delta B + AB + BC + 2B^{\#},$$

$$\frac{\partial}{\partial t} C = \Delta C + C^2 + 2C^{\#} + B^t B.$$

5.1.7 Remark. Here the Lie algebra square is given by the adjoint matrix. Precisely $A^\# = \det A \cdot (A^t)^{-1}$, $B^\# = \det B \cdot (B^t)^{-1}$, $C^\# = \det C \cdot (C^t)^{-1}$.

5.2. The evolution equation for the coefficients of the branching curve

We would like to compute the evolution of the coefficients of the branching curve (2.28) under the Ricci flow. We polarize and define the following $(4,0)$-tensor

$$\begin{aligned}\eta(u_+,u_-,h_+,h_-) &= \Lambda^2 g_x\Big(B(u_-),h_+\Big)\Lambda^2 g_x\Big(B(h_-),u_+\Big)\\ &\quad -\Lambda^2 g_x\Big((A-\frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_+})(u_+),h_+\Big)\Lambda^2 g_x\Big((C-\frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_-})(u_-),h_-\Big),\end{aligned}$$

where $u_+ + u_-,\ h_+ + h_- \in (\Lambda^2_+ T_xM \otimes \mathbb{C}) \oplus (\Lambda^2_- T_xM \otimes \mathbb{C})$ and $\eta := \eta(x,t)$ $\Lambda^2 g_x := \Lambda^2 g(x,t)$, $A := A(x,t)$, $B := B(x,t)$, $C := C(x,t)$.

5.2.1 Proposition. *Then tensor η evolves under the Ricci flow as follows*

$$\begin{aligned}&\Big[\Big(\frac{\partial}{\partial t}-\Delta\Big)\eta\Big](u_+,u_-,h_+,h_-) =\\ &\Lambda^2 g_x\Big((AB+BC+2B^\#)(u_-),h_+\Big)\Lambda^2 g_x\Big(B(h_-),u_+\Big)+\\ &+\Lambda^2 g_x\Big(B(u_-),h_+\Big)\Lambda^2 g_x\Big((AB+BC+2B^\#)(h_-),u_+\Big)-\\ &-\Lambda^2 g_x\Big((A^2+2A^\#+BB^t-\frac{1}{6}|\mathrm{Ric}|^2\mathrm{Id}_{\Lambda_+})(u_+),h_+\Big)\Lambda^2 g_x\Big((C-\frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_-})(u_-),h_-\Big)-\\ &-\Lambda^2 g_x\Big((A-\frac{\mathrm{scal}}{12}\mathrm{Id}_{\Lambda_+})(u_+),h_+\Big)\Lambda^2 g_x\Big((C^2+2C^\#+B^tB-\frac{1}{6}|\mathrm{Ric}|^2\mathrm{Id}_{\Lambda_-})(u_-),h_-\Big)-\\ &-2\Lambda^2 g_x\Big(\nabla B(u_-),h_+\Big)\Lambda^2 g_x\Big(\nabla B(h_-),u_+\Big)+2\Lambda^2 g_x\Big(\nabla A(u_+),h_+\Big)\Lambda^2 g_x\Big(\nabla C(u_-),h_-\Big).\end{aligned}$$

Proof. By using Corollary 5.1.2 we obtain that $(\frac{\partial}{\partial t}\eta)(u_+,u_-,h_+,h_-) =$

$$\begin{aligned}
= \quad & \Lambda^2 g_x\Big((\frac{\partial}{\partial t} B)(u_-), h_+\Big)\Lambda^2 g\Big(B(h_-),u_+\Big) + \Lambda^2 g_x\Big(B(u_-),h_+\Big)\Lambda^2 g_x\Big((\frac{\partial}{\partial t} B)(h_-),u_+\Big) \\
& -\Lambda^2 g_x\Big((\frac{\partial}{\partial t}(A - \frac{\text{scal}}{12}\text{Id}_{\Lambda_+}))(u_+),h_+\Big)\Lambda^2 g_x\Big((C - \frac{\text{scal}}{12}\text{Id}_{\Lambda_-})(u_-),h_-\Big) \\
& -\Lambda^2 g_x\Big((A - \frac{\text{scal}}{12}\text{Id}_{\Lambda_+})(u_+),h_+\Big)\Lambda^2 g_x\Big((\frac{\partial}{\partial t}(C - \frac{\text{scal}}{12}\text{Id}_{\Lambda_-}))(u_-),h_-\Big) \\
= \quad & \Lambda^2 g_x\Big((\Delta B + AB + BC + 2B^\#)(u_-),h_+\Big)\Lambda^2 g_x\Big(B(h_-),u_+\Big) \\
& +\Lambda^2 g_x\Big(B(u_-),h_+\Big)\Lambda^2 g_x\Big((\Delta B + AB + BC + 2B^\#)(h_-),u_+\Big) \\
& -\Lambda^2 g_x\Big((\Delta A + A^2 + 2A^\# + BB^t - (\frac{1}{12}\Delta\text{scal} + \frac{1}{6}|\text{Ric}|^2)\text{Id}_{\Lambda_+}))(u_+),h_+\Big) \\
& \Lambda^2 g_x\Big((C - \frac{\text{scal}}{12}\text{Id}_{\Lambda_-})(u_-),h_-\Big) - \Lambda^2 g_x\Big((A - \frac{\text{scal}}{12}\text{Id}_{\Lambda_+})(u_+),h_+\Big) \\
& \Lambda^2 g_x\Big((\Delta C + C^2 + 2C^\# + B^t B - (\frac{1}{12}\Delta\text{scal} + \frac{1}{6}|\text{Ric}|^2)\text{Id}_{\Lambda_-}))(u_-),h_-\Big).
\end{aligned}$$

The result follows immediately. Notice that the ∇ terms come from the Leibniz rule for the Laplacian of the product of two tensors. $\square$

6. Type I singularities on 4-dimensional manifolds

By the results of Naber [20] and Enders, Müller, Topping [6] on Type I singularities for the Ricci flow, it follows that along any sequence of times converging to the finite extinction time T, parabolic rescalings will subconverge to a normalized nonflat gradient shrinking Ricci soliton. In this chapter we use the construction of Chapter 2 and apply it to this result, in order to obtain a characterization of the nonflat gradient shrinking solitons in the language of local invariants.

6.1. Cheeger-Gromov-Hamilton Compactness Theorem

In this section we will introduce the Compactness Theorem of Cheeger-Gromov-Hamilton. This theorem is crucial when we try to analyze singularities in the Ricci flow. We will start with some definitions. Further details and the proof of the Theorem can be found in [4].

6.1.1 Definition. A marked solution to the Ricci flow is a 4-tuple $(M^n, g(t), x, F)$, where M^n is a Riemannian manifold , $x \in M^n$ is a choice of point, called the origin , $t \in (\alpha, \omega)$ with $-\infty \leq \alpha < 0 < \omega \leq +\infty$ and F is a frame at x which is orthonormal with respect to the initial metric g_0.

6.1.2 Definition. Let $K \subset M^n$ be a compact set and $\{g_i\}_{i\in\mathbb{N}}$, g_∞ Riemannian metrics on M^n. For $k \in \{0\} \cup \mathbb{N}$ we say that g_i converges in C^k to g_∞ uniformly on K if, for all $\varepsilon > 0$ there exists $\iota_0 = \iota_0(\varepsilon)$ such that for $\iota \geq \iota_0$

$$\sup_{0\leq m\leq k} \sup_{x\in K} |\nabla^m (g_i - g_\infty)|_{g_\infty} < \varepsilon,$$

where the covariant derivative ∇ is with respect to g_∞.

6.1.3 Definition. We say that a sequence of open sets $\{U_i\}_{i\in\mathbb{N}}$ in a manifold M is an exhaustion of M^n by open sets if for every compact set $K \subset M^n$ there exists $\iota_0 \in \mathbb{N}$ such that $K \subset U_i$ for all $\iota \geq \iota_0$.

6.1.4 Definition. Let $\{U_i\}_{i\in\mathbb{N}}$ be an exhaustion of M^n by open sets and let g_i be Riemannian metrics on U_i. We say that (U_i, g_i) converges in C^∞ to (M^n, g_∞) uniformly on compact sets in M^n if for any compact set $K \subset M^n$ and any $k > 0$ there exists $\iota_0 = \iota_0(K, k)$ such that $\{g_i\}_{i\geq\iota_0}$ converges in C^k to g_∞ uniformly on K.

A solution $g(t)$of the Ricci flow, $t \in I$ for some interval I, is said to be complete, if for each $t \in I$ the Riemannian metric $g(t)$ is complete.

6.1.5 Definition. A sequence $\{(M_i^n, g_i(t), x_i, F_i)\}_{i\in\mathbb{N}}$, $t \in (\alpha, \omega)$ of smooth, complete, marked solutions to the Ricci flow converges to a complete, marked solution to the Ricci flow $(M_\infty^n, g_\infty(t), x_\infty, F_\infty)$, $t \in (\alpha, \omega)$, if

(i) there exists an exhaustion $\{U_i\}_{i\in\mathbb{N}}$ of M_{∞^n} by open sets with $x_\infty \in U_i$ for all $i \in \mathbb{N}$,

(ii) there exists a sequence of diffeomoprhisms

$$\phi_i : U_i \to \phi_i(U_i) \subset M_i^n,$$

with $\phi_i(x_\infty) = x_i$ and $(\phi_i)_* F_\infty = F_i$ for all $i \in \mathbb{N}$ and

(iii) $\left(U_i, \phi_i^*\left[g_i(t)|_{\phi(U_i)}\right]\right)$ converges in C^∞ to $(M_\infty^n, g_\infty(t))$ uniformly on compact sets in M_∞^n.

The next theorem is known in the literature as the Cheeger-Gromov-Hamilton Compactness Theorem.

6.1.6 Theorem. *Let $\{(M_i^n, g_i(t), x_i, F_i)\}_{i\in\mathbb{N}}$, $t \in (\alpha, \omega) \ni 0$ be a sequence of smooth, complete, marked solutions to the Ricci flow. If*

(i) $|\mathrm{Rm}_{g_i(t)}|_{g_i(t)} \leq C_0$ on $M_i \times (\alpha, \omega)$, for some constant $C_0 < \infty$ idependent of i,

(ii) $\mathrm{inj}_{g_i(0)}(x_i) \geq \delta$, for some constant $\delta > 0$,

then there exists a subsequence $\{j_i\}_{i\in\mathbb{N}}$ such that $\{(M_{j_i}^n, g_{j_i}(t), x_{j_i}, F_{j_i})\}_{i\in\mathbb{N}}$ converges to a complete marked solution to the Ricci flow $(M_\infty^n, g_\infty(t), x_\infty, F_\infty)$, $t \in (\alpha, \omega)$, as $i \to \infty$.

6.2. Type I singularities and the branching curve

6.2.1 Definition. A complete solution $(M^n, g(t))$ to the Ricci flow defined on a finite time interval $[0, T)$, $T < \infty$ is called a Type I Ricci flow if there exists some constant $C > 0$ such that for all $t \in [0, T)$

$$\sup_M |\operatorname{Rm}_{g(t)}|_{g(t)} \leq \frac{C}{T-t}.$$

Furthermore we say that the solution $g(t)$ develops a Type I singularity at time T.

We know that if the singularity time T is finite, then the curvature becomes unbounded

$$\lim_{t\to T}\left(\sup_M |\operatorname{Rm}_{g(t)}|_{g(t)}\right) = \infty.$$

A complete proof for this argument can be found in [2]. The most well known examples of Type I singularities are the neckpinch singularity modelled on a shrinking cylinder and those modelled on flows starting at a positive Einstein metric or more general at a gradient shrinking Ricci soliton with bounded curvature.

One can show that in the Type I case if we apply the parabolic maximum principle to the evolution equation of $|\operatorname{Rm}|^2$ we obtain that

$$\sup_M |\operatorname{Rm}_{g(t)}|_{g(t)} \geq \frac{1}{8(T-t)},$$

for all $t \in [0, T)$. The proof of this result can be found in [3] on page 295. A detailed exposition on the maximum principles can be found in Chapter 4 of [2]. In the proof of the statement above one uses a version of the parabolic maximum principle for scalars, which applies to complete solutions of the Ricci flow and can be found on page 276 of [3].

6.2.2 Definition. A quantity $A(t)$ is said to blow up at the Type I rate as $t \to T$ is there exist cosntants $C \geq c > 0$ such that

$$\frac{c}{T-t} \leq A(t) \leq \frac{C}{T-t},$$

for all $t \in [0, T)$.

6.2.3 Definition. A sequence of points and times $\{(x_i, t_i)\}$ with $x_i \in M^n$ and $t_i \to T$ is called an essential blow up sequence if there exists a constant $c > 0$ such that

$$|\operatorname{Rm}_{g(t_i)}|_{g(t_i)}(x_i) \geq \frac{c}{T - t_i}.$$

6.2.4 Definition. A point $x \in M^n$ in a Type I Ricci flow is called a Type I singular point if there exists an essential blow-up sequence with $x_i \to x$ on M^n. The set of all Type I singular points is denoted by Σ_I.

We will need the following lemmas in order to prove our result for Type I singularities.

6.2.5 Lemma. *Let (M^4, g) be a 4-dimensional Riemannian manifold and $x \in M$, such that the branching curve Γ_x exists. Then Γ_x remains invariant under scalings of the metric by a constant factor.*

Proof. Let κ be some constant factor and and let $\tilde{g} = \kappa g$. Then we know that $\Lambda^2\tilde{g} = \kappa^2\Lambda^2 g$, and $\widetilde{\mathfrak{Rm}} = \frac{1}{\kappa}\mathfrak{Rm}$. Then the branching curve is given by

$$\begin{aligned}
\Gamma_x^{\tilde{g}} &= \{([a^1, a^2], [b^1, b^2]) \in \mathbb{P}(S_x^-) \times \mathbb{P}(S_x^+) : \Big(\kappa^2\Lambda^2 g_x\Big(\frac{1}{\kappa}B(T_-), T_+\Big)\Big)^2 - \\
&\quad \kappa^2\Lambda^2 g_x\Big(\frac{1}{\kappa}\mathfrak{W}_+(T_+), T_+\Big)\kappa^2\Lambda^2 g_x\Big(\frac{1}{\kappa}\mathfrak{W}_-(T_-), T_-\Big) = 0\} \\
&= \{([x^0, x^1], [y^0, y^1]) \in \mathbb{P}(S_x^-) \times \mathbb{P}(S_x^+) : \kappa^2\Big(\Lambda^2 g_x(B(T_-), T_+)\Big)^2 - \\
&\quad \kappa^2\Lambda^2 g_x\Big(\mathfrak{W}_+(T_+), T_+\Big)\Lambda^2 g_x\Big(\mathfrak{W}_-(T_-), T_-\Big) = 0\} \\
&= \Gamma_x^g.
\end{aligned}$$

□

6.2.6 Lemma. *Let $\{(M^n, g_i(t), x, F_i(t))\}_{i\in\mathbb{N}}$, $t \in (\alpha, \omega) \ni 0$ be a sequence of smooth, complete, marked solutions to the Ricci flow, where the time-dependent frame $F_i(t)$ evolves to stay orthonormal. If the sequence converges to a complete marked solution to the Ricci flow $(M_\infty, g_\infty(t), x_\infty, F_\infty(t))$, $t \in (\alpha, \omega)$ as $i \to \infty$, where $F_\infty(t)$ evolves to stay orthonormal, then the sequence $\{(M, \operatorname{Rm}_{g_i(t)}, x, F_i(t))\}_{i\in\mathbb{N}}$, $t \in (\alpha, \omega) \ni 0$ converges to $(M_\infty, \operatorname{Rm}_{g_\infty(t)}, x_\infty, F_\infty(t))$, $t \in (\alpha, \omega)$ as $i \to \infty$.*

Proof. Let $\{U_i\}_{i\in\mathbb{N}}$ be an exhaustion of M_∞ by open sets with $x_\infty \in U_i$ for all $i \in \mathbb{N}$. Furthermore let $\phi_i : U_i \to \phi_i(U_i) \subset M$ be a sequence of diffeomorphisms with $\phi_i(x_\infty) = x$ and $(\phi_i)_* F_\infty(t) = F_i(t)$ for all $i \in \mathbb{N}$ and $t \in (\alpha, \omega)$. We know that $(U_i, \phi_i^*\Big[g_i(t)|_{\phi(U_i)}\Big])$ converges in C^∞ to $(M_\infty, g_\infty(t))$ uniformly on compact sets in M_∞. But uniform convergence of $\phi_i^*\left[g_i(t)|_{\phi(U_i)}\right]$ to $g_\infty(t)$ in C^k for any $k \geq 2$ implies immediately uniform convergence of $\phi_i^*\left[\mathrm{Rm}_{g_i(t)}|_{\phi(U_i)}\right]$ to $\mathrm{Rm}_{g_\infty(t)}$ in C^{k-2}. This comes from the fact, that the components of the Riemann curvature tensor are determined by the second (spatial) derivatives of the components of the Riemannian metric tensor. Thus one can deduce that $(U_i, \phi_i^*\left[\mathrm{Rm}_{g_i(t)}|_{\phi(U_i)}\right])$ converges in C^∞ to $(M_\infty, \mathrm{Rm}_{g_\infty(t)})$ uniformly on compact sets in M_∞. □

6.2.7 *Remark.* There is a reason behind the fact that we choose to work with an evolving orthonormal frame, which evolves to stay orthonormal. We want to prove the next Theorem, which states, that convergence of metrics implies convergence of curves. This extra assumption guarantees us the desired extra control over the convergence of curves. One should also observe, that the Theorem 6.1.6 can be reformulated in the context of evolving orthonormal frames directly. This will be a crucial step in the proof of Corollary 6.2.9.

6.2.8 Theorem. *Let $\{(M^4, g_i(t), x, F_i(t))\}_{i\in\mathbb{N}}$, $t \in (\alpha, \omega) \ni 0$ be a sequence of smooth, complete, marked solutions to the Ricci flow, where the time-dependent frame $F_i(t)$ evolves to stay orthonormal and assume, that the sequence converges to a complete marked solution to the Ricci flow $(M^n_\infty, g_\infty(t), x_\infty, F_\infty(t))$, $t \in (\alpha, \omega)$ as $i \to \infty$, where $F_\infty(t)$ evolves to stay orthonormal as well. Let $\{\Gamma_x^{g_i(t)}\}_{i\in\mathbb{N}}$ be the sequence of one-paramenter families of branching curves associated to $x \in M$ and $\Gamma_{x_\infty}^{g_\infty(t)}$ the one-parameter family of branching curves associated to $x_\infty \in M_\infty$ (if this exists). Then $\Gamma_x^{g_i(t)}$ converges to $\Gamma_{x_\infty}^{g_\infty(t)}$ as $i \to \infty$, in the sense that the coefficients of the curves converge.*

Proof. By the Lemma 6.2.6 we know that the Cheeger-Gromov convergence can be extended to the case of Riemann curvature tensors as well. The coefficients of the branching curve are given by polynomials of components of Rm. By the elemantary fact that a polynomial is a continuous function the result follows. □

We will need at this point the notion of the parabolic rescaling (or parabolic dilation) of Ricci flows. The Ricci flow has scaling properties, that are essential for blow up analysis for singularities. Let $(M^n, g(t))$ be a Ricci flow on $[0, T)$. Given a scaling factor $\lambda > 0$, if one defines a new flow $\hat{g}(t) = \lambda^{-1} g(T + \lambda t)$, for $t \in [-\lambda^{-1}T, 0)$, then

$$\frac{\partial}{\partial t}\hat{g}(t) = -2\mathrm{Ric}_{\hat{g}(t)}$$

and so $\hat{g}(t)$ is also a Ricci flow. Under this scaling, the Ricci tensor is invariant, but sectional curvatures and scalar curvature are scaled by the factor λ. The connection also remains invariant. The main use of this rescaling will be to analyse singularities that develop under the Ricci flow. In such a case the curvature tends to infinity, so we perform a rescaling of the flow where the curvature is becoming large, in such a way that we can pass to a limit, which will be a new Ricci flow encoding some of the information contained in the singularity. This is a very successful strategy in many branches if geometric analysis.

6.2.9 Corollary. *Let $(M^4, g(t))$ be a Type I Ricci flow on $[0, T)$ and $x \in \Sigma_I$. Furthermore let $\Gamma_x^{g(t)}$ be the one-paramater family of branching curves associated to x. Let us choose a sequence of scaling factors λ_i, such that $\lambda_i \to 0$. We define the rescaled Ricci flows $(M^4, g_i(t), x, F_i(t))$ by*

$$g_i(t) = \lambda_i^{-1} g(T + \lambda_i t),\ t \in [-\lambda_i^{-1} T, 0),$$

where the time-dependent frame $F_i(t)$ evolves to stay orthonormal. Then the one-parameter family of curves $\Gamma_x^{g_i(t)}$ is $\Gamma_x^{g(T+\lambda_i t)}$ and subconverges to the one-parameter family of curves $\Gamma_{x_\infty}^{g_\infty(t)}$ (if this exists) of a nontrivial normalized gradient shrinking Ricci soliton $(M_\infty^n, g_\infty(t), x_\infty, F_\infty(t))$, $t \in (-\infty, 0)$ in canonical form, where $F_\infty(t)$ evolves to stay orthonormal.

Proof. By the Lemma 6.2.5 the branching curves are invariant under scalings of the metric by a constant factor. Thus $\Gamma_x^{g_i(t)} = \Gamma_x^{g(T+\lambda_i t)}$. By the Compactness Theorem of Cheeger-Gromov-Hamilton 6.1.6, there exists a subsequence $\{j_i\}$ such that $(M^4, g_{j_i}(t), x, F_{j_i}(t))$ converges to a complete, pointed ancient solution to the Ricci flow $(M_\infty^4, g_\infty(t), x_\infty, F_\infty(t))$ on $(-\infty, 0)$. By the result of Enders-Müller-Topping ([6], Theorem 1.4) this singularity model is given by a nontrivial normalized gradient shrinking Ricci soliton in canonical form. The result follows immediately from Theorem 6.2.8. □

6.2.10 Remark. As the Corollary shows, the (limiting) branching curve can serve as an invariant of the singularity model for Type I singularities in dimension four. It still remains open which normalized nontrivial gradient shrinking Ricci solitons can occur as singularity models in the four dimensional case. Thus by studying limiting curves of Type I singularities, we get indications for the possible singularity models for Type I singularities in the four dimensional case. Precisely, we hope that this approach (or even more the sophisticated $K3$ surfaces approach, which will be discussed in a forthcoming paper) will contribute in the direction of determing the generic singularity models in dimension four. In the next, we will discuss what we understand under the notion of stability.

Following the discussion in the Introduction, one of the keys to understand the nature of singularities, that develop in the Ricci flow is to adequately classify the set of singularitiy models that may arrise. The signularity formation in the three dimensional case has been fairly well understood. It follows by the Hamilton-Ivey pinching estimate ([12] and [17]), that the only possible three dimensional singularity models have nonnegative sectional curvature. This is a highly restrictive condition. On the other hand, Máximo showed in [19], that in dimension four, the singularity models for finite singularities can have Ricci curvature of mixed sign. As a result the only restriction on the curvature remaining for $n \geq 4$ is nonnegative scalar curvature, which is unfortunately a too week restriction to be useful.

Thus in dimension greater that three, a full classification of the possible singularity models is rather impractical. A more promising alterative would be to classify the generic or at least the stable singularity models. A singularity model developing certain original data is labeled stable, if flows starting from all sufficient small perturbations of that data develop singularities with the same singularity model. Furthermore, a singularity model is labeled generic, if flows that start from an open dense subset of all possible initial data develop singularities having the same singularity model. Clearly, a singularity model can be generic only if it is stable. More details can be found in [16].

It is conjectured by experts, that the only candidates for generic singularity models in dimension four are $\mathbb{S}^4$, $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}^2$. These singularity models are known to be generic. There is another soliton, which is now known yet if it is generic or not. This the $(\mathcal{L}^2_{-1}, h)$, which is the blow down soliton constructed by Feldman, Ilmanen ann Knopf in [7]. If the blow down soliton is generic, then it will be also in the list.

A possible application of our construction could be to contribute in the direction of determining the generic sigularity models in dimension four. The idea would be find to a four dimensional manifold, such that its singularity model is the blow down soliton. Following our Corollary, this would imply a convergence result for branching curves. We could make a small perturbation of the initial data and compute the associated family of branching curves at the singular point. If now the new family of branching curves doesn't converge to the family of branching curves associated to the blow down soliton, then we could deduce, that the blow down soliton is not a generic singularity model.

We believe strongly, that by choosing the $K3$ surface as an invariant instead of the branching curve, we can obtain even better results. The reason is, that the $K3$ surfaces approach is more a sophisticated tool and their moduli space is well understood. Recall, that the interested reader can find more details on the coarse moduli space for lattice polarized $K3$ surfaces in the Appendix. This will be part of our forthcoming work. The hope is, that these invariants will provide us with a better understanding of the generic singularity models for Type I singularities for the four dimensional Ricci flow.

A. $K3$ surfaces

This is a short introduction to the theory of $K3$ surfaces. We list the results without giving any proofs. The interested reader can look up the proofs of the statements below in the books [1] or [15]. The first book is a very good reference for the general theory of algebraic surfaces and the second one is a comprehensive reference for the theory $K3$ surfaces.

A.1. Definition and examples

A.1.1 Definition. A complex algebraic $K3$ surface is a projective, connected complex surface X with $\mathcal{K}_X \cong \mathcal{O}_X$ (trivial canonical bundle $\mathcal{K}_X$) and $H^1(X, \mathcal{O}_X) = 0$.

Recall that the canonical line bundle is given by $\mathcal{K}_X = \Lambda^2 T^* X^{(1,0)}$. In other words it is the highest exterior power of the holomorphic cotangent bundle. Holomorphic sections of $\mathcal{K}_X$ correspond to holomorphic top-degree forms.

A.1.2 Remark. From now on by a $K3$ surface we will mean a complex algebraic $K3$ surface.

Let's take a look at some examples of $K3$ surfaces.

A.1.3 Example. (K3 surfaces of degree 4,6 and 8) Let X be a smooth complete intersection of type $(d_1, .., d_{n-2})$ in $\mathbb{P}^n$, i.e. X is a surface which is the transversal intersection of $n-2$ hypersurfaces of degree $d_1, ..., d_{n-2}$ respectively. Without loss of generality we may assume that $d_i \geq 2$ for every i. Then X becomes a $K3$ surface only when

(i) $n = 3$ and $d_1 = 4$, i.e. X is a quartic surface in $\mathbb{P}^3$

(ii) $n = 4$ and $(d_1, d_2) = (2, 3)$, i.e. X is the complete intersection of a quadric and a cubic in $\mathbb{P}^4$

(iii) $n = 5$ and $(d_1, d_2, d_3) = (2, 2, 2)$, i.e. X is the complete intersection of three quadrics in $\mathbb{P}^5$.

A.2. Lattices

A.2.1 Definition. A finitely generated, free $\mathbb{Z}$-module L of rank n together with a symmetric, non-degenerate bilinear form

$$\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z},$$

is called a lattice of rank n. With respect to a choice of basis for the $\mathbb{Z}$-module, the symmetric bilinear form may be represented by a matrix denoted again with L . The lattice L is called unimodular if $\det(L) = \pm 1$ and it is called even, if

$$\langle x, x \rangle \equiv 0 (\mathrm{mod} 2),$$

i.e. the associated quadratic form takes only even values. If $\langle x, x \rangle > 0$ for all $x \neq 0$ it is called positive-definite. A lattice is definite if it is either positive or negative definite, otherwise it is called indefinite. The signature of L is that of the quadratic form $L \otimes \mathbb{R}$ over $\mathbb{R}$.

A.2.2 Example. (The hyperbolic plane U) As a $\mathbb{Z}$-module it is $\mathbb{Z}^2$ and if e_1, e_2 is the standard basis, the Gram matrix $[\langle e_i, e_j \rangle]_{i,j}$ is just

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is an even, unimodular rank 2 lattice of signature $(1, 1)$.

A.2.3 Example. (The root lattice E_8) As a $\mathbb{Z}$-module, $E_8 = \mathbb{Z}^8$ and on the canonical basis the Gram matrix $[B(e_i, e_j)]_{i,j}$ is just the Cartan matrix of the root system E_8 and is explicitely given by

$$E_8 = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The lattice E_8 is even, unimodular and positive definite. Changing all signs yields $-E_8$, a negative definite lattice.

A.3. Topological and analytical invariants

We would like to compute the Euler number $e(X)$. For a $K3$ surface X holds by definition $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$. By Serre duality we have that $h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$. So the Euler characteristic is

$$\chi(X, \mathcal{O}_X) = \sum_{i=0}^{2} (-1)^i h^i(X, \mathcal{O}_X) = 2.$$

Now by Noether's formula

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(K_X^2 + e(X))$$

and by using the fact that $K_X^2 = 0$ we obtain that the Euler number is

$$e(X) = 24.$$

The next Proposition gives us the singular cohomologies for a $K3$ surface. In particular we have the following result.

A.3.1 Proposition. *Let X be a $K3$ surface. Then*

- $H^0(X, \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \cong \mathbb{Z}$

- $H^1(X,\mathbb{Z}) \cong H^3(X,\mathbb{Z}) \cong 0$
- $H^2(X,\mathbb{Z})$ *is torsion free and is a free abelian group of rank* 22*. Furthermore if we equip it with the cup product pairing it becomes an even, indefinite, unimodular lattice of signature* $(3,19)$.

We automaticaly obtain that $H^0(X,\mathbb{Z}) \cong \mathbb{Z}$ because X is connected and $H^4(X,\mathbb{Z}) \cong \mathbb{Z}$ because X is oriented. The proof for the rest of the statements can be found in [1].

From the previous Propositon we have that

$$b_i(X) = \operatorname{rank}(H^i(X,\mathbb{Z})),$$

so that $b_0 = b_4 = 1$ and $b_1 = b_3 = 0$. We showed before that $e(X) = 24$ and from the standard theory for algebraic surfaces one has

$$e(X) = \sum (-1)^i b_i(X).$$

So one obtains that $b_2(X) = \operatorname{rank}(H^2(X,\mathbb{Z})) = 22$.

A.3.2 Theorem. *Every K3 surface is Kähler.*

A.3.3 Proposition. *For a K3 surface the Hodge diamond is given by*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

A.3.4 Proposition. *Let X be a K3 surface. Then* $H^2(X,\mathbb{Z})$ *endowed with the cup product pairing forms a lattice, isometric to the K3 lattice*

$$\Lambda_{K3} := (-E_8) \oplus (-E_8) \oplus U \oplus U \oplus U.$$

A.3.5 Remark. So the lattice Λ_{K3} has rank 22 and signature $(3,19)$.

A.3.6 Theorem. *Every $K3$ surface is simply connected.*

A.4. Moduli of $K3$ surfaces

The isomorphism $H^2(X,\mathbb{Z}) \cong \Lambda_{K3}$ is not unique.

A.4.1 Definition. A marking on X is a choice of isometry $\phi : H^2(X,\mathbb{Z}) \to \Lambda_{K3}$.

Let $\omega \in H^{2,0}(X) = H^0(X,\Omega_X^2)$ be any class. Then $(\omega,\omega) = 0$ and $(\omega,\bar{\omega}) > 0$. So, if ϕ is a marking for X and $\phi_{\mathbb{C}}$ its complexification, then $\phi_{\mathbb{C}}(H^{2,0}(X))$ defines a point in

$$\Omega := \{[\omega] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) : \langle \omega,\omega \rangle = 0, \langle \omega,\bar{\omega} \rangle > 0\}.$$

Ω is a 20-dimensional quasi-projective variety called the period space of $K3$ surfaces. The point defined by $\phi_{\mathbb{C}}(H^{2,0}(X))$ is the period point of the marked $K3$ surface (X,ϕ).

A.4.2 Theorem (Surjectivity of the period map)**.** *For each $[\omega] \in \Omega$, there is some marked $K3$ surface (X,ϕ) such that $[\omega] = \phi_{\mathbb{C}}(H^{2,0}(X))$.*

In other words, every point of Ω occurs as the period point of some marked $K3$ surface. Also, as a corollary of the so called weak Torelli theorem for $K3$ surfaces, we have an injectivity statement.

A.4.3 Theorem. *If (X,ϕ) and (X',ϕ') are marked $K3$ surfaces with*

$$\phi_{\mathbb{C}}(H^{2,0}(X)) = \phi'_{\mathbb{C}}(H^{2,0}(X')) \in \Omega,$$

then X and X' are isomoprhic.

Indeed, the period space Ω is a fine moduli space for marked $K3$ surfaces and it is observed that $\dim_{\mathbb{C}}(\Omega) = 20$.

We will now introduce lattice polarized $K3$ surfaces.

A.4.4 Proposition. *For a $K3$ surface holds*

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X).$$

Here $\mathrm{NS}(X)$ denotes the Neron-Severi group.

It is well known that that for a projective surface holds that

$$\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X) \subset H^2(X,\mathbb{Z}).$$

So $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$ becomes a sublattice of $H^2(X,\mathbb{Z})$, called the Picard lattice (or Néron-Severi lattice). The Picard number $\rho(X) = \mathrm{rank}(\mathrm{NS}(X)) = \mathrm{rank}(\mathrm{Pic}(X))$ is at most the dimension of $H^{1,1}(X)$.

A.4.5 Proposition. *Let X be a $K3$ surface. Then*

$$0 \le \rho(X) \le 20$$

and the signature of the intersection form on $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ *is* $(1, \rho(X) - 1)$.

Let Λ be a lattice of signature $(1, r-1)$, that can be primitively embedded in the $K3$ lattice Λ_{K3}. A lattice polarized $K3$ surface of degree $2k$, for $k > 0$ is a $K3$ surface X together with a primitive embedding

$$\iota : \Lambda \hookrightarrow \mathrm{Pic}(X),$$

such that $\iota(\Lambda)$ contains a psudo-ample element $h \in \mathrm{Pic}(X)$, such that $h^2 = 2k$.

There is coarse moduli space of lattice polarized $K3$ surfaces with is constructed as follows. Fix an embedding $\Lambda \hookrightarrow \Lambda_{K3}$ and define

$$\Omega_\Lambda = \{[\omega] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) : \langle \omega, \omega \rangle = 0,\ \langle \omega, \bar{\omega} \rangle > 0,\ \langle \omega, m \rangle = 0,\ \forall m \in \Lambda\}.$$

Ω_Λ is called the period space of lattice polarized $K3$ surfaces of degree $2k$. As before, we have surjectivity and injectivity results and this time, the coarse moduli space for lattice polarized $K3$ surfaces (forgetting the marking) is constructed as the quotient

$$\mathcal{M}_{K3,\Lambda} := \Omega_\Lambda / \{\gamma \in \mathrm{Aut}(\Lambda_{K3}) : \gamma(\Lambda) = \Lambda\}.$$

Notice that $\dim_{\mathbb{C}}(\mathcal{M}_{K3,\Lambda}) = 20 - \mathrm{rank}(\Lambda)$.

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