A Mayer-Vietoris Spectral Sequence for C*-Algebras and Coarse Geometry

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Abstract

Let A be a C*-algebra that is the norm closure $A = \overline{\sum_{\beta \in \alpha} I_\beta}$ of an arbitrary sum of C*-ideals $I_\beta \subseteq A$. We construct a homological spectral sequence that takes as input the K-theory of $\bigcap_{j \in J} I_j$ for all finite nonempty index sets $J \subseteq \alpha$ and converges strongly to the K-theory of A.

For a coarse space X, the Roe algebra $\mathfrak{C}^*X$ encodes large-scale properties. Given a coarsely excisive cover $\{X_\beta\}_{\beta \in \alpha}$ of X, we reshape $\mathfrak{C}^*X_\beta$ as input for the spectral sequence. From the K-theory of $\mathfrak{C}^*\big(\bigcap_{j \in J} X_j\big)$ for finite nonempty index sets $J \subseteq \alpha$, we compute the K-theory of $\mathfrak{C}^*X$ if α is finite, or of a direct limit C*-ideal of $\mathfrak{C}^*X$ if α is infinite.

Analogous spectral sequences exist for the algebra $\mathfrak{D}^*X$ of pseudocompact finite-propagation operators that contains the Roe algebra as a C*-ideal, and for $\mathfrak{Q}^*X = \mathfrak{D}^*X/\mathfrak{C}^*X$.

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1 Introduction

1.1 Our result for abstract C*-algebras

C*-algebras arise in mathematics and theoretical physics alike. K-theory is a fundamental tool to study and classify these highly-structured algebras.

As a covariant $\mathbb{Z}$-graded functor of C*-algebras over $\mathbb{C}$, K-theory is continuous, is additive, admits suspension isomorphisms $K_s SA \cong K_{s+1}A$, and admits a cyclic 6-term exact sequence induced by C*-ideal inclusions $I \subseteq A$ via boundary maps and Bott periodicity $\beta\colon K_sA \cong K_{s+2}A$. Furthermore, for a sum $A = I_0 + I_1$ of two C*-ideals, there is a Mayer-Vietoris exact sequence,

$$\begin{tikzcd}
K_0(I_0 \cap I_1) \arrow[r] & K_0 I_0 \oplus K_0 I_1 \arrow[r] & K_0 A \arrow[d, "\partial_2^{\mathrm{MV}} \circ \beta"] \\
K_1 A \arrow[u, "\partial_1^{\mathrm{MV}}"] & K_1 I_0 \oplus K_1 I_1 \arrow[l] & K_1(I_0 \cap I_1). \arrow[l]
\end{tikzcd}$$

In algebraic topology, a similar Mayer-Vietoris sequence computes the homology or cohomology of a space X from a cover $X = X_0^\circ \cup X_1^\circ$ by relating the (co)homology of X_0, X_1, and $X_0 \cap X_1$. This topological Mayer-Vietoris sequence generalizes to a spectral sequence: Given a suitable cover $\{X_\beta\}_{\beta\in\alpha}$ of a topological space, the spectral sequence takes as input the (co)homology of $\bigcap_{j\in J} X_j$ for all finite nonempty $J \subseteq \alpha$ and converges to the (co)homology of X, the full space.

It is natural to seek analogous spectral sequences for the K-theory Mayer-Vietoris exact sequence. This is our first main result:

Theorem 6.7.1 (Spectral sequence for arbitrary sums)**.** *Let α be an arbitrary index set: finite, countable, or uncountable. Let $A = \overline{\sum_{\beta\in\alpha} I_\beta}$ be the norm closure of a sum of $|\alpha|$-many C*-ideals $I_\beta \subseteq A$. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} \cong \begin{cases} \displaystyle\bigoplus_{|J|=p+1} K_q\Big(\bigcap_{j\in J} I_j\Big) & \textit{for } p \geq 0, \\ 0 & \textit{for } p < 0, \end{cases}$$

where J ranges over all nonempty finite index subsets $J \subseteq \alpha$. In general, this is a half-page spectral sequence, any term $E^1_{p,q}$ with $p \geq 0$ may be nonzero.

This spectral sequence converges strongly to K_*A. It is functorial with respect to $*$-homomorphisms that preserve α-indexed ideal decompositions.

To prove this, we begin with the finite case $A = I_0 + I_1 + \cdots + I_n$ and construct

C*-algebras of continuous functions from the standard simplex Δ^n to A, interlocking allowed ranges in the $I_j \subseteq A$ on different regions of the simplex.

Sums of these function algebras become a chain of ideals $Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n$. This chain of ideals fits into an already-known spectral sequence that converges strongly to $K_*Q_n \cong K_*(S^nA)$, the K-theory of the n-fold suspension of A. This spectral sequence for ideal inclusions has been described by C. Schochet in [Sch81]; we reprove it to highlight its inner workings, its differentials, and its filtration that guarantees strong convergence. Compared to that spectral sequence, our Theorem 6.7.1 relaxes the input conditions: We do not require that the I_j form a chain of inclusions.

For countable index sets α, we link the spectral sequences for n ideals and $n+1$ ideals – this is only possible on the level of K-theory, not on the level of C*-algebras – and construct a filtration on the spectral sequence via a suitable direct limit. For uncountable sets α, we adapt our direct limit construction to the directed system of finite subsets of α.

1.2 Our application in coarse geometry

Coarse geometry studies the large-scale structure of metric spaces. If two spaces differ only within a compact set, coarse invariants will not detect any difference.

For a coarse space X, i.e., a metric space (X, d), the Roe algebra $\mathfrak{C}^*X$ encodes such large-scale properties. This C*-algebra and the larger algebra $\mathfrak{D}^*X$ are introduced, e.g., by N. Higson and J. Roe in [HR00], or by J. Roe in [Roe96]. Via these algebras, the K-homology of X and further invariants of contemporary research are defined, such as the coarse index when X is a Riemannian manifold. For our work, it suffices to define $\mathfrak{Q}^*X = \mathfrak{D}^*X/\mathfrak{C}^*X$; we will not formulate our results in the language of K-homology.

For certain sets X_0 and X_1 with $X_0 \cup X_1 = X$, there is a coarse Mayer-Vietoris exact sequence: It relates the K-theory of $\mathfrak{C}^*X_0$, $\mathfrak{C}^*X_1$, and $\mathfrak{C}^*(X_0 \cap X_1)$ to the K-theory of $\mathfrak{C}^*X$. Its proof, e.g., in [Roe96], relies on the Mayer-Vietoris sequence for two abstract C*-ideals.

We generalize to arbitrarily many regions. A decomposition $X = \bigcup_{\beta\in\alpha} X_\beta$ is called *coarsely excisive* if, for all nonempty finite subcollections $J \subseteq \alpha$ and $R > 0$, there exists $S > 0$ such that the intersection of the R-neighborhoods is contained in the S-neighborhood of the intersection according to the metric d on X:

$$\bigcap_{j\in J} N_d(X_j, R) \subseteq N_d\Big(\bigcap_{j\in J} X_j, S\Big).$$

This leads to our second main result:

Theorem 7.2.1 (Spectral sequence for coarsely excisive covers)**.** *Let (X, d) be a coarse space and let $\{X_\beta\}_{\beta\in\alpha}$ be a coarsely excisive cover of (X, d). Let $\mathfrak{F}^*$ be either the functor $\mathfrak{C}^*$ from the coarse category to* <u>C*A</u> *or one of the functors $\mathfrak{D}^*$ or $\mathfrak{Q}^*$ from the coarse-continuous category to* <u>C*A</u>*. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} \cong \begin{cases} \bigoplus_{|J|=p+1} K_q\mathfrak{F}^*\Big(\bigcap_{j\in J} X_j\Big) & \text{for } p \geq 0, \\ 0 & \text{for } p < 0, \end{cases}$$

where J ranges over all nonempty finite subcollections of indices in α. For finite α, this spectral sequence converges strongly to $K_*\mathfrak{F}^*X$. In general, the spectral sequence converges strongly to the K-theory of $\overline{\bigcup_J \sum_{j\in J} \mathfrak{F}^*(X_j \subseteq X)}$, a C-ideal of $\mathfrak{F}^*X$, where J ranges over all finite subcollections of indices in α. The spectral sequence is functorial with respect to morphisms (coarse maps for $\mathfrak{C}^*$, or coarse and continuous maps for $\mathfrak{D}^*$ and $\mathfrak{Q}^*$) to other coarse spaces with compatible coarsely excisive covers (Definition 5.1.4).*

As an example, we recompute the known K-theory of $\mathfrak{C}^*\mathbb{R}^n$. Also, we find an infinite coarsely excisive cover of $\mathbb{Z}^\infty$ under many metrics, then show that the K-theory of the direct limit ideal of Roe algebras for this cover in $\mathfrak{C}^*\mathbb{Z}^\infty$ vanishes. Furthermore, we compute the K-theory of the direct limit of Roe algebras for a countable wedge sum $\bigvee_{\mathbb{N}} [0, \infty[$ in a single application of the spectral sequence; this obviates inductive proofs with the Mayer-Vietoris exact sequence.

1.3 Relations to other research

Early motivation for this project was the Partitioned Manifold Index Theorem [Sie12, Proposition 4.9]: Given certain Riemannian manifolds $N \subseteq M$ with G-equivariant covers, the classes of their Dirac operators – elements in K-homology – map to the same element in $K_*C^*_rG$, the K-theory of the group C*-algebra for G, via the coarse index maps. P. Siegel proves this by induction with the coarse Mayer-Vietoris principle for two regions. Our idea was to reprove this theorem by a single application of our Theorem 7.2.1.

Spectral sequences, however, do not construct specific maps on their targets; even strong convergence only leads to isomorphism theorems. Still, if the spectral sequence cannot compute the equality for the Partitioned Manifold Index Theorem, it can classify related C*-algebras for coarse spaces in this setting and decide about the structure of possible morphisms between them.

Sums or inclusion chains of abstract C*-ideals also arise in other settings. In [MM18], D. Mukherjee and R. Meyer construct a gauge-invariant C*-algebra $\mathcal{T}_0$ of the Toeplitz algebra $\mathcal{T}$ for partial product systems. By [MM18, Theorem 4.5], $\mathcal{T}_0$ is a direct limit along $N \in \mathbb{N}$ of images of maps from $\bigoplus_{n<N} K\mathcal{E}_n$ into $\mathcal{T}_0$ where the $\mathcal{E}_n$ are the compact opreators of correspondences. These images are C*-subalgebras of $\mathcal{T}_0$. With our spectral sequences, to compute the K-theory of $\mathcal{T}_0$, we may examine the K-theory of quotients or intersections of these subalgebras.

1.4 Structure of this thesis

Section 2, Fundamentals, establishes the notation and gives an overview of the basic tools. The reader will likely be familiar with several constructions. For a metric space X, we present the coarse algebras $\mathfrak{C}^*X$, $\mathfrak{D}^*X$, and $\mathfrak{Q}^*X = \mathfrak{D}^*X/\mathfrak{C}^*X$.

In Section 3, Ideal inclusions, we show a spectral sequence that takes a chain of C*-ideal inclusions $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p \subseteq \cdots$ across all $p \in \mathbb{N}$. Similar to the material in Section 2, this spectral sequence is known theory. Nonetheless, we reprove it in detail. This will be the basis for the research work in the following sections.

In Section 4, Finite sums of ideals, we construct our spectral sequence that takes intersections $\bigcap_{j\in J} I_j$ of C*-ideals I_j from a finite sum $A = I_0 + I_1 + \cdots + I_n$.

In Section 5, Finite coarse excision, we define coarsely excisive covers and relative coarse algebras. For finite subcollections of a coarsely excisive cover, we show that these relative algebras behave well under intersections and unions. This leads to a version of our spectral sequence for finite coarsely excisive covers. As an example, we recompute the K-theory of $\mathfrak{C}^*\mathbb{R}^n$.

In Sections 6 and 7, Infinite sums of ideals and Infinite coarse excision, we relax the condition that $A = I_0+I_1+\cdots+I_n$ needs to be a finite sum: Now $A = \overline{\sum_{\beta\in\alpha} I_\beta}$ may be a direct limit of sums of arbitrarily many C*-ideals. The spectral sequences from Sections 4 and 5 first generalize to countable ideal decompositions as input instead of only finite decompositions, then to uncountable decompositions. As examples, we compute the K-theory of direct limits of Roe algebras for $\mathbb{Z}^\infty$ and $\bigvee_{\mathbb{N}} [0,\infty[$.

In Section 8, Generalizations, we list ideas for real KO-theory and equivariant spaces.

2 Fundamentals

We will rehearse well-known constructions to establish notation and conventions, beginning with C*-algebras and their K-theory. We introduce the basics of coarse geometry and spectral sequences.

The set $\mathbb{N}$ of natural numbers includes 0.

2.1 C*-algebras

Definition 2.1.1 (Complex Banach algebra)**.** Let $(A, \|-\|)$ be a normed associative algebra over $\mathbb{C}$ that is topologically complete according to its norm. For all x, $y \in A$, the following inequality shall hold: $\|xy\| \leq \|x\| \, \|y\|$. Then we call A a *complex Banach algebra*.

Definition 2.1.2 (C*-algebra, C*-ideal)**.** Let A be a complex Banach algebra. Let A carry an involution $*$, i.e., a map $A \to A$, $x \mapsto x^*$, satisfying $x^{**} = x$, $(\lambda x + y)^* = \overline{\lambda} x^* + y^*$, and $(xy)^* = y^* x^*$ for all x, $y \in A$ and $\lambda \in \mathbb{C}$. Furthermore, $*$ shall satisfy the *C*-identity* $\|xx^*\| = \|x\|^2$. Then $(A, *)$ is a *C*-algebra*.

A closed two-sided ideal in a C*-algebra is called a *C*-ideal*.

Remark 2.1.3. The zero algebra 0 is a C*-algebra. The complex numbers $\mathbb{C}$ themselves are a C*-algebra with $z \mapsto |z|$ as the norm and $z \mapsto \overline{z}$ as the $*$-operation.

For a C*-algebra A with a C*-ideal $I \subseteq A$, the quotient A/I is a well-defined C*-algebra.

Let α be an arbitrary index set and let A_β be a C*-algebra for each $\beta \in \alpha$. The norm completion $\overline{\bigoplus_{\beta \in \alpha} A_\beta}$ of the algebraic direct sum is again a C*-algebra with the norm $\|(a_\beta)_{\beta \in \alpha}\| = \sup \{\|a_\beta\| : \beta \in \alpha\}$ and component-wise addition, multiplication, and involution.

Definition 2.1.4 (Category $\underline{\text{C*A}}$ of C*-algebras)**.** A bounded algebra homomorphism $f\colon A \to B$ between two C*-algebras is called a $*$-*homomorphism* if it preserves the involution: $f(x^*) = f(x)^*$ shall hold for all $x \in A$.

The category $\underline{\text{C*A}}$ encompasses all C*-algebras as objects, together with all $*$-homomorphisms as arrows.

Definition 2.1.5 (Homotopy of C*-algebras)**.** Let f, $g\colon A \to B$ be $*$-homomorphisms between C*-algebras. A *homotopy* in the category of C*-algebras between f and g is a map $H\colon (A \times [0,1]) \to B$ that satisfies:

- For all $a \in A$, $H(a, 0) = f(a)$ and $H(a, 1) = g(a)$.
- For all $a \in A$, the map $[0,1] \to B$, $t \mapsto H(a, t)$ is continuous.

- For all $t \in [0,1]$, the map $A \to B$, $a \mapsto H(a,t)$ is a $*$-homomorphism.

If such a homotopy exists, then f and g are called *homotopic*, denoted by $f \sim g$.

A $*$-homomorphism $f\colon A \to B$ is a *homotopy equivalence* if there exists a $*$-homomorphism $g\colon B \to A$ such that $g \circ f \sim \mathrm{id}(A)$ and $f \circ g \sim \mathrm{id}(B)$. Existence of a homotopy equivalence $A \to B$ is denoted by $A \simeq B$; then A is called *homotopy equivalent* to B.

A C*-algebra A is called *contractible* if $A \simeq 0$.

Homotopy equivalence as C*-algebras is a stronger condition than topological homotopy equivalence.

Lemma 2.1.6. *Let A be a contractible C*-algebra and $p = p^* = p^2$ a projection in A. Then $p = 0$.*

Proof. Let $H\colon A \times [0,1] \to A$ be the contracting homotopy. For all $t \in [0,1]$, we have $\|H(p,t)\| = \|H(pp^*,t)\| = \|H(p,t)\|^2 \in \mathbb{R}_{\geq 0}$ because $x \mapsto H(x,t)$ is a $*$-homomorphism. This implies $\|H(p,t)\| \in \{0,1\}$ and, because of continuity, this norm stays constant across all $t \in [0,1]$. By construction, $H(p,1) = 0$, thus $\|H(p,0)\| = 0$ and $p = 0$. $\square$

Corollary 2.1.7. *The C*-algebra $\mathbb{C}$ is contractible as a topological space, but not contractible as a C*-algebra.*

Proof. One possible topological homotopy is $(z,t) \mapsto tz$. Lemma 2.1.6 precludes a C*-contraction because $1 = \overline{1} = 1 \cdot 1$ is a nonzero projection in $\mathbb{C}$. $\square$

Definition 2.1.8 (Cone)**.** Let A be a C*-algebra. The *cone of A* is the C*-algebra

$$CA = \{f\colon [0,1] \to A : f \text{ is continuous}, f(0) = 0\}.$$

It carries the uniform norm $\|f\| = \sup\{\|f(x)\| : x \in [0,1]\}$; this is well-defined because $[0,1]$ is compact. Algebra multiplication on CA is given by pointwise multiplication of functions. The involution on the cone is defined by $f^*(x) = f(x)^*$.

Let $g\colon A \to B$ be a $*$-homomorphism. The *cone map* $Cg\colon CA \to CB$ is given by $(Cg)(f) = g \circ f\colon [0,1] \to B$. This construction turns $C\colon \underline{\text{C*A}} \to \underline{\text{C*A}}$ into a covariant functor.

This is still basic theory of C*-algebras, but it is reasonable to agree on whether the functions f in the cone must vanish at 0 or vanish at 1. In later sections, we will construct a spectral sequence for C*-algebras; some technical lemmas require cones of algebras.

Proposition 2.1.9. *Let A be a C*-algebra. Then the cone CA is contractible; the zero algebra is a strong deformation retract of CA.*

Proof. Define a homotopy $H\colon CA \times [0,1] \to CA$ by $H(f,t)(x) = f(tx)$. This is continuous because f is continuous. It is the desired deformation retraction because $H(f,0) = 0$ and $H(f,1) = f$ for all $f \in CA$. Since $H(0,t)(x) = 0$ for all t and $x \in [0,1]$, it is even a strong deformation retraction. □

Definition 2.1.10 (Suspension)**.** Let A be a C*-algebra and CA its cone. The *suspension of* A is the subalgebra

$$SA = \{f \in CA : f(1) = 0\}.$$

The suspension SA inherits its C*-algebra structure from CA. Likewise, $*$-homomorphisms $\varphi\colon A \to B$ induce $S\varphi\colon SA \to SB$ via $S\varphi = (C\varphi) \restriction SA$, making $S\colon \underline{\mathrm{C^*A}} \to \underline{\mathrm{C^*A}}$ another covariant functor.

2.2 C*-algebras for spaces

Definition 2.2.1 ($\mathscr{C}(X,A)$, $\mathscr{C}X$)**.** Let X be a compact Hausdorff space and A a C*-algebra. Then $\mathscr{C}(X,A)$ denotes the C*-algebra of A-valued continuous functions on X with the sup-norm $\|f\|_{\mathscr{C}(X,A)} = \sup\{\|f(x)\|_A : x \in X\}$, pointwise addition and multiplication, and $f^*(x) = f(x)^*$. If A is unital, $\mathscr{C}(X,A)$ contains the constant function that maps all points in X to $1 \in A$; this function is then a unit.

Often, $A = \mathbb{C}$; we abbreviate by setting $\mathscr{C}X = \mathscr{C}(X,\mathbb{C})$.

When X fails to be compact, $\mathscr{C}X$ is not a normed algebra because some A-valued functions on X are unbounded. More interesting function algebras impose boundedness:

Definition 2.2.2 ($\mathscr{C}_0(X,A)$, $\mathscr{C}_0X$)**.** Let X be a locally compact Hausdorff space, A a C*-algebra. A continuous function $f\colon X \to A$ *vanishes at infinity* if, for all $\varepsilon > 0$, there exists a compact set $K \subseteq X$ with

$$\{x \in X : \|f(x)\| > \varepsilon\} \subseteq K.$$

The set of all such functions f is denoted $\mathscr{C}_0(X,A)$. In the common case $A = \mathbb{C}$, we shall write $\mathscr{C}_0X = \mathscr{C}_0(X,\mathbb{C})$.

Again, $\mathscr{C}_0(X,A)$ carries a C*-algebra structure under pointwise multiplication and the sup-norm $\|f\| = \sup\{\|f(x)\| : x \in X\}$.

Remark 2.2.3. The sup-norm is well-defined because functions in $\mathscr{C}_0(X,A)$ are necessarily bounded. The C*-algebra $\mathscr{C}_0(X,A)$ has a unit if and only if X is compact and A is unital. For compact X, the algebra $\mathscr{C}_0(X,A)$ coincides with $\mathscr{C}(X,A)$.

Equivalent definitions of $\mathscr{C}_0(X,A)$ embed X into an arbitrary compactification Y, then define $\mathscr{C}_0(X,A)$ as the subset of all continuous functions $f\colon Y \to A$ such that $f \restriction (Y - X) = 0$, then restrict these functions to X.

Taking A-valued functions that vanish at infinity is a contravariant functor from Hausdorff spaces with proper continuous maps into $\underline{\mathrm{C^*A}}$: Let X and Y be Hausdorff spaces and let $f\colon X \to Y$ be a proper continuous map. Then $(-\circ f)\colon \mathscr{C}_0(Y,A) \to \mathscr{C}_0(X,A)$ maps $g\colon Y \to A$ to $g \circ f\colon X \to A$. The composition $g \circ f$ vanishes at infinity because f is proper.

2.3 K-theory of C*-algebras

The exact constructions of the K-theory K_*A for a C*-algebra A are lengthy and shall be omitted; several textbooks, e.g., [WO93] or [RLL00], cover all technical details. The *zeroth K-theory* group K_0A is the Grothendieck group of equivalence classes of projections in a ring of matrices over A modulo homotopy equivalence. The first K-theory group K_1A results from a similar construction with unitary elements of the matrix ring instead of projections.

Both K_0 and K_1 become continuous covariant functors from $\underline{\mathrm{C^*A}}$ to abelian groups: For a morphism $f\colon A \to A'$, the resulting morphism $K_*f\colon K_*A \to K_*A'$ applies f to all matrix entries before taking equivalence classes.

Theorem 2.3.1 (Suspension isomorphism)**.** *For a C*-algebra A, there is an isomorphism $\sigma\colon K_0SA \to K_1A$. This allows $\mathbb{N}$-graded K-theory by defining K_sA as $K_{s-1}SA$ inductively; some authors even define K_1A this way instead of via unitary matrix elements.*

Theorem 2.3.2 (Bott isomorphism)**.** *For all $s \in \mathbb{N}$, there are* Bott isomorphisms *$\beta\colon K_sA \to K_{s+2}A$. This allows $\mathbb{Z}$-graded K-theory by defining $K_sA = K_{s+2}A$ inductively for all $s < 0$.*

Theorem 2.3.3 (Six-term exact sequence)**.** *Let $I \subseteq A$ be a C*-ideal. For all $s \in \mathbb{Z}$, K-theory admits boundary maps $\partial_s\colon K_s(A/I) \to K_{s-1}I$ that make the following six-term sequence exact; the horizontal arrows are induced by ideal inclusion and* *projection:*

$$\begin{tikzcd} K_0 I \arrow[r] & K_0 A \arrow[r] & K_0(A/I) \arrow[d, "\partial_2 \circ \beta"] \\ K_1(A/I) \arrow[u, "\partial_1"] & K_1 A \arrow[l] & K_1 I. \arrow[l] \end{tikzcd}$$

Theorem 2.3.4 (Abstract Mayer-Vietoris exact sequence)**.** *Let A be a C*-algebra such that I_0, $I_1 \subseteq A$ are two C*-ideals with $I_0 + I_1 = A$. There is an exact sequence with Mayer-Vietoris boundary morphisms:*

$$\begin{tikzcd} K_0(I_0 \cap I_1) \arrow[r] & K_0 I_0 \oplus K_0 I_1 \arrow[r] & K_0 A \arrow[d, "\partial_2^{\mathrm{MV}} \circ \beta"] \\ K_1 A \arrow[u, "\partial_1^{\mathrm{MV}}"] & K_1 I_0 \oplus K_1 I_1 \arrow[l] & K_1(I_0 \cap I_1). \arrow[l] \end{tikzcd}$$

2.4 Roe algebras

Definition 2.4.1 (Ample representation)**.** Let A be a separable C*-algebra. Let $\varrho\colon A \to BH$ be a representation of C*-algebras on a separable Hilbert space H, where BH denotes the C*-algebra of all bounded linear operators $H \to H$. Then ϱ is called *ample* if

- ϱ is nondegenerate, and
- $\varrho(0) = 0$ is the only compact operator in $\mathrm{im}(\varrho) \subseteq BH$.

Definition 2.4.2 (Very ample representation)**.** Let A be a separable C*-algebra. A representation $\varrho\colon A \to BH$ of C*-algebras is called *very ample* if it is a countably infinite sum of ample representations.

Remark 2.4.3. To admit ample representations, the Hilbert space H must be both separable and infinite-dimensional. Then suitable ample representations ϱ always exist. According to [HR00], because H is separable, the constructions in Section 2.4 do not depend on the particular choice of H or ϱ up to isomorphy.

Every very ample representation is ample. Most constructions require ample representations. Some isomorphism theorems call for very ample representations, but, because $H \cong \bigoplus_{\mathbb{N}} H$, requiring very ample representations is merely a technical convenience, not a fundamental restriction.

Definition 2.4.4 (Pseudolocal operator)**.** Let X be a locally compact Hausdorff space. For the C*-algebra $\mathscr{C}_0 X$, let $\varrho\colon \mathscr{C}_0 X \to BH$ be an ample representation. Let

$T \in BH$ be an operator such that $\varrho(f)T - T\varrho(f)$ is a compact operator in BH for all $f \in \mathscr{C}_0X$. Then T is called *pseudolocal*.

Definition 2.4.5 (Finite propagation)**.** Let (X, d) be a locally compact metric space and $\varrho\colon \mathscr{C}_0X \to BH$ an ample representation. An operator $T \in BH$ has *finite propagation* if there exists a constant $R > 0$ such that for all f, $g \in \mathscr{C}_0X$ with $d(\operatorname{supp} f, \operatorname{supp} g) \geq R$, the product $\varrho(f)T\varrho(g) \in BH$ is zero.

Definition 2.4.6 ($\mathfrak{D}^*A$)**.** Let (X, d) be a locally compact metric space. Fix an ample representation $\varrho\colon \mathscr{C}_0X \to BH$. The norm closure of the set of all pseudolocal operators in BH with finite propagation forms a C*-algebra, denoted by $\mathfrak{D}^*X$.

Remark 2.4.7. The norm closure turns $\mathfrak{D}^*X$ into a sub-C*-algebra of BH. Without the norm closure, the operators with finite propagation do not form a closed subset. Pseudolocal operators by themselves already form a sub-C*-algebra in BH without additional closure.

Definition 2.4.8 (Locally compact operator)**.** Let (X, d) be a locally compact metric space and $\varrho\colon \mathscr{C}_0X \to BH$ an ample representation. Let $T \in BH$ be an operator such that, for all $f \in \mathscr{C}_0X$, both $\varrho(f)T$ and $T\varrho(f)$ are compact operators in BH. Then T is called *locally compact*.

Remark 2.4.9. Given $\varrho\colon \mathscr{C}_0X \to BH$ ample for a locally compact metric space (X, d), the locally compact operators form a C*-ideal in the algebra of pseudolocal operators.

Definition 2.4.10 ($\mathfrak{C}^*X$, Roe algebra)**.** For a locally compact metric space (X, d) and an ample representation $\varrho\colon \mathscr{C}_0X \to BH$, the *translation algebra* or *Roe algebra* $\mathfrak{C}^*X$ is the norm closure of the operators $T \in BH$ that are both locally compact and have finite propagation.

Remark 2.4.11. The Roe algebra $\mathfrak{C}^*X$ is a C*-ideal in $\mathfrak{D}^*X$.

Remark 2.4.12 (K-homology)**.** Let A be a C*-algebra and A^+ the C*-algebra with a unit adjoined. It is possible to define an abstract dual algebra $\mathfrak{D}^*A^+$ by representing A amply and taking all pseudolocal operators, without defining finite propagation. For $s \in \mathbb{Z}$, we may define the *s-th K-homology group of A* as

$$K^sA = K_{1-s}\mathfrak{D}^*A^+.$$

K-theory of C*-algebras is a covariant functor; K-homology becomes a contravariant functor of C*-algebras. Were we concerned only with abstract C*-algebras, we could consider “K-homology” a bad name for a contravariant functor and to rename it to “K-cotheory”. But K-homology becomes a covariant functor for topological spaces:

Let X be a locally compact metric space and $s \in \mathbb{Z}$. The abelian group

$$K^s X = K_{-s} \mathscr{C}_0 X$$

is the *K-homology of the space* X; this defines a covariant functor from locally compact metric spaces to abelian groups. By [HR00, Lemma 12.3.2], there is an isomorphism $K^s X = K_{s+1}(\mathfrak{D}^* X / \mathfrak{C}^* X)$.

We will not need K-homology and will instead formulate all results in the language of K-theory and Roe algebras. Thus we introduce a notation similar to [Sie12]:

Notation 2.4.13 ($\mathfrak{Q}^* X$)**.** Let X be a locally compact metric space. We write

$$\mathfrak{Q}^* X = \mathfrak{D}^* X / \mathfrak{C}^* X.$$

2.5 Coarse spaces

The most general definition of a coarse space X uses *entourages* or *controlled sets* – collections of subsets of $X \times X$ with axioms to capture a notion of closeness. Following [Roe96, Chapter 2], we will instead work with proper metric spaces, a modest restriction. If our spaces are manifolds, both methods bring the same results.

Definition 2.5.1 (Coarse space)**.** A *coarse space* $X = (X, d)$ is a proper metric space; i.e., a metric space where closed d-bounded sets are compact.

Definition 2.5.2 (Coarse map)**.** Let $f\colon (X, d_X) \to (Y, d_Y)$ be a map between coarse spaces. f is called *coarse* if

- f is uniformly expansive: For $R > 0$, there exists $S > 0$ such that for all x, $x' \in X$ with $d_X(x, x') \leq R$, we have $d_Y(fx, fx') \leq S$.
- f is proper as a map between the metric spaces X and Y: For each bounded set $B \subseteq Y$, the preimage $f^{-1}(B)$ is bounded in X.

Coarse maps are not required to be continuous.

Remark 2.5.3. The identity $\mathrm{id}(X)\colon (X, d) \to (X, d)$ is coarse. Compositions of coarse maps are coarse.

Definition 2.5.4 (Coarse category, coarse-continuous category)**.** The *coarse category* has as objects all coarse spaces and as morphisms all coarse maps.

The *coarse-continuous category* is the subcategory of the coarse category that still comprises all coarse spaces, but that has as morphisms only the coarse maps that are also continuous.

Definition 2.5.5 (Closeness)**.** Let f, $f'\colon (X, d_X) \to (Y, d_Y)$ be two maps between coarse spaces. We call f *close to* f', or *coarsely equivalent to* f', if there exists $S > 0$ such that for all $x \in X$, we have $d_Y(fx, f'x) \leq S$.

Definition 2.5.6 (Coarse equivalence)**.** Let X and Y be coarse spaces with coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is close to $\mathrm{id}(X)$ and $f \circ g$ is close to $\mathrm{id}(Y)$. We call X, Y *coarsely equivalent* and f, g *coarse equivalences.*

Example 2.5.7. Fix $n \in \mathbb{N}$. The lattice $\mathbb{Z}^n$ is coarsely equivalent to Euclidean space $\mathbb{R}^n$ under the metric d_∞ with $d_\infty(x, x') = \sup_{j<n} |x_j - x'_j|$ on both spaces. The inclusion $f\colon \mathbb{Z}^n \to \mathbb{R}^n$ and

$$g\colon \mathbb{R}^n \to \mathbb{Z}^n, \qquad g(x_0, x_1, \ldots, x_{n-1}) = (\lfloor x_0 \rfloor, \lfloor x_1 \rfloor, \ldots, \lfloor x_{n-1} \rfloor)$$

serve as coarse equivalences. For all $z \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$, the distances $d_\infty(z, gfz)$ and $d_\infty(x, fgx)$ are uniformly bounded by the constant 1.

This map g is proper, but it is not continuous.

Coarse equivalences induce isomorphisms on the K-theory of Roe algebras:

Lemma 2.5.8 ([Roe96, Lemma 3.5])**.** *Let X, Y be coarse spaces, $f\colon X \to Y$ a coarse map. Then f induces a functorial homomorphism $f_*\colon K_*\mathfrak{C}^*X \to K_*\mathfrak{C}^*Y$. Coarsely equivalent maps induce the same homomorphism.*

Remark 2.5.9. Similarly, the constructions $\mathfrak{D}^*$ and $\mathfrak{Q}^*$ are functorial, but these functors are merely well-defined on the coarse-continuous category. Only $\mathfrak{C}^*$ is well-defined for coarse non-continuous maps.

All three of $\mathfrak{C}^*$, $\mathfrak{D}^*$, and $\mathfrak{Q}^*$ are covariant functors to C*A: Passing from spaces X and Y to function algebras $\mathscr{C}_0X$ and $\mathscr{C}_0Y$ is contravariant, and passing from function algebras to locally compact or pseudocompact operators with finite propagation is again contravariant.

Corollary 2.5.10. *Let $f\colon X \to Y$ and $g\colon Y \to X$ be coarse equivalences. Then $K_p\mathfrak{C}^*X \cong K_p\mathfrak{C}^*Y$ for all $p \in \mathbb{Z}$.*

Proof. The compositions $g \circ f$ and $f \circ g$ are close to the identities on X and Y. They induce identities in K-theory, thus both $K_p\mathfrak{C}^*f$ and $K_p\mathfrak{C}^*g$ are isomorphisms. $\square$

2.6 Coarsely excisive pairs

We will recapitulate *coarse excision* as defined by J. Roe in [Roe96]. Later, we will define *coarsely excisive covers* to generalize this idea.

Definition 2.6.1 (R-neighborhood)**.** Let (X, d) be a metric space and $Y \subseteq X$ a subspace. For a real number $R > 0$, define the *R-neighborhood of Y* as

$$N_d(Y, R) = \{x \in X : \inf \{d(x, y) : y \in Y\} \leq R\}.$$

When d is a standard metric such as the 1-metric d_1, the Euclidean metric d_2, or the sup-metric d_∞ on $\mathbb{R}^n$ or $\mathbb{Z}^n$, we will also write $N_1 = N_{d_1}$ or, similarly, N_2 or N_∞.

Definition 2.6.2 (Coarsely excisive pair)**.** Let (X, d) be a metric space. Let U and V be subspaces of X with $U \cup V = X$.

The pair (U, V) is called a *coarsely excisive pair for X* if, for every distance $R > 0$, there exists a distance $S > 0$ such that the intersection of the R-neighborhoods is contained in the S-neighborhood of the intersection:

$$N_d(U, R) \cap N_d(V, R) \subseteq N_d(U \cap V, S).$$

Example 2.6.3. For the metric space $\mathbb{R}$ with its standard metric d, the pair of subspaces $(\mathbb{R}_{\leq 0}, \mathbb{R}_{\geq 0})$ is coarsely excisive: The R-neighborhoods are $N_d(\mathbb{R}_{\leq 0}, R) =]\infty, R]$ and $N_d(\mathbb{R}_{\geq 0}, R) = [-R, \infty[$. Their intersection is $[-R, R]$, which, for $S = R$, is the S-neighborhood of $\mathbb{R}_{\leq 0} \cap \mathbb{R}_{\geq 0} = \{0\}$.

In the same vain, $\mathbb{R}^{n+1}$ admits the coarsely excisive pair $(\mathbb{R}^n \times \mathbb{R}_{\leq 0}, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ under d_1, d_2, or d_∞.

Example 2.6.4. For all $S > 0$, the S-neighborhood of $\varnothing$ is again $\varnothing$. This imposes restrictions on eligible coarsely excisive pairs: In any metric space (X, d), disjoint nonempty sets U and V cannot form a coarsely excisive pair. Choose R larger than $\inf \{d(x, y) : x \in U,\, y \in V\}$, then $N(U, R) \cap N(V, R)$ contains points. This is never a subset of $N(U \cap V, S) = N(\varnothing, S) = \varnothing$.

Theorem 2.6.5 ([HRY93, Section 5])**.** *For a coarsely excisive pair (U, V) of (X, d), there is an exact Mayer-Vietoris sequence:*

$$\begin{tikzcd}
K_0\mathfrak{C}^*(U \cap V) \arrow[r] & K_0\mathfrak{C}^*U \oplus K_0\mathfrak{C}^*V \arrow[r] & K_0\mathfrak{C}^*X \arrow[d] \\
K_1\mathfrak{C}^*X \arrow[u] & K_1\mathfrak{C}^*U \oplus K_1\mathfrak{C}^*V \arrow[l] & K_1\mathfrak{C}^*(U \cap V). \arrow[l]
\end{tikzcd}$$

Boths proofs in [HRY93] and [Roe96] reduce this to the Mayer-Vietoris principle for abstract C*-ideals I_0, $I_1 \subseteq A$, Theorem 2.3.4.

The goal of this thesis is to construct a spectral sequence extending that abstract Mayer-Vietoris principle and Theorem 2.6.5 alongside.

2.7 Spectral sequences

A good introduction to spectral sequences is [McC01]. We will give the basic definitions to establish notation. We require no cohomological spectral sequences or ring structures on the pages.

Definition 2.7.1 (Spectral sequence)**.** A *spectral sequence* (of homological type, of abelian groups) is a system of bigraded differential abelian groups $E^r_{p,q}$ for all $0 \neq r \in \mathbb{N}$ and p, $q \in \mathbb{Z}$ with differentials $d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ for all r, p, q such that each $E^{r+1}_{p,q}$ is the homology of d^r at $E^r_{p,q}$.

We define convergence of spectral seuqences with notation similar to [Boa99, Section 5]; that exposition does not assert any common origin of the target group and the $E^r_{*,*}$-terms of the spectral sequence. We re-index to match our spectral sequences of homological type and make explicit the grading of the $\mathbb{Z}$-graded target group.

Definition 2.7.2. Let G be an abelian group with an increasing filtration

$$\cdots \subseteq F^pG \subseteq F^{p+1} \subseteq \cdots \subseteq G$$

for $p \in \mathbb{Z}$. We call the filtration $\{F^pG\}_{p\in\mathbb{Z}}$

- *Hausdorff* if $\bigcap_{p\in\mathbb{Z}} F^pG = 0$,
- *exhaustive* if $\bigcup_{p\in\mathbb{Z}} F^pG = G$, and
- *complete* if the right-derived functor of taking the inverse limit yields the zero group for the inverse system F^pG for $p \to -\infty$.

Definition 2.7.3 (Strong convergence)**.** Let $\{E^r_{p,q}, d^r\}_{r,p,q}$ be a spectral sequence. For $r \geq 1$ and p, $q \in \mathbb{Z}$, write

$$\begin{aligned} Z^r_{p,q} &= \ker d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}, \\ B^r_{p,q} &= \operatorname{im} d^r \colon E^r_{p+r,q-r+1} \to E^r_{p,q}. \end{aligned}$$

Because $E^r_{*,*}$ for $r \geq 2$ is the homology of $E^{r-1}_{*,*}$ under d^{r-1}, an element in $E^r_{p,q}$ may be written as $x + B^{r-1}_{p,q}$ with $x \in E^{r-1}_{p,q}$. Recursively, this allows us to treat $Z^r_{p,q}$ and

$B^r_{p,q}$ as subgroups of $E^1_{p,q}$ and define

$$E^\infty_{p,q} = \Big(\bigcap_{r\geq 1} Z^r_{p,q}\Big) \Big/ \Big(\bigcup_{r\geq 1} B^r_{p,q}\Big).$$

Let $G = \bigoplus_{s\in\mathbb{Z}} G_s$ be a $\mathbb{Z}$-graded abelian group. The spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ *converges strongly* to G if there exist increasing filtrations $\{F^p G_s\}_{p\in\mathbb{Z}}$ of each summand G_s such that these filtrations are Hausdorff, exhaustive, complete, and allow isomorphisms

$$E^\infty_{p,q} \cong F^p G_{p+q} / F^{p-1} G_{p+q}.$$

Definition 2.7.4 (Morphism of spectral sequences)**.** Given two spectral sequences $\{E^r_{p,q}, d^r\}_{r,p,q}$ and $\{\bar{E}^r_{p,q}, \bar{d}^r\}_{r,p,q}$ and $(p',q') \in \mathbb{Z}^2$, a *morphism of spectral sequences of bidegree* (p',q') is a system of morphisms of abelian groups,

$$f = \big\{f^r_{p,q}\colon E^r_{p,q} \to \bar{E}^r_{p+p',q+q'}\big\}_{r,p,q},$$

such that

- the group morphisms commute with the differentials; i.e., $f^r_{*,*} \circ d^r = \bar{d}^r \circ f^r_{*,*}$ for all pages r, and
- each map $f^r_{*,*}$ induces $f^{r+1}_{*,*}$ by passing to homology on $\{E^r_{*,*}, d^r\}_{r,p,q}$ and $\{\bar{E}^r_{*,*}, \bar{d}^r\}_{r,p,q}$.

Remark 2.7.5. Spectral sequences with these morphisms form a category.

By describing a morphism of spectral sequences on the R-th page, all subsequent $f^r_{*,*}$ for $r > R$ and $r = \infty$ are implicitly defined because the $E^r_{*,*}$-terms are iterative homologies of the earlier $E^R_{*,*}$-term. In our setting, we will construct morphisms of spectral sequences only for the $E^1_{*,*}$-terms.

In particular, if $f^R_{*,*}$ is an isomorphism between the differential graded abelian groups $E^R_{*,*}$ and $\bar{E}^R_{*,*}$, then all $f^r_{*,*}$ for $r > R$ and $r = \infty$ become isomorphisms.

3 Ideal inclusions

3.1 Main theorem

Theorem 3.1.1 (Spectral sequence for ideal inclusions)**.** *Let $A = \overline{\bigcup_{p\in\mathbb{N}} I_p}$ be a C*-algebra, where the $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p \subseteq \cdots$ form a chain of closed two-sided ideals. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} = K_{p+q}(I_p/I_{p-1}).$$

This spectral sequence converges strongly to K_*A; i.e., given $s \in \mathbb{Z}$, the groups $E^\infty_{p,q}$ along the diagonal $s = p + q$ pose an extension problem to reconstruct K_sA.

In [Sch81], C. Schochet gave a proof of Theorem 3.1.1.

Nonetheless, we will reprove Theorem 3.1.1 based on very general theory from [CE73]. This extra work reveals the inner mechanisms of the spectral sequence, shows that the convergenge is strong, and highlights naturality of all constructions: The morphisms in K-theory arise from natural inclusions of C*-ideals, quotients of C*-ideals, and boundary maps.

This spectral sequence serves as groundwork for the Mayer-Vietoris results in later sections.

3.2 Abstract H-systems

In [CE73], H. Cartan and S. Eilenberg construct an abstract spectral sequence from a bigraded system of groups, but they omit some details during their proof of convergence. Their construction uses cohomological differentials: On the $E_r^{*,*}$-page, the differential has the degree $(r, 1-r)$. For homological spectral sequences, they suggest the renumbering $E^r_{p,q} = E_r^{-p,-q}$. We will state the main theorem of [CE73] in this renumbered notation, then prove it with all details.

Definition 3.2.1 (Ungraded H-system)**.** Let $H(p, p')$ be abelian groups for $p' \leq p$ from the range $\mathbb{Z} \cup \{\pm\infty\}$. We introduce the shorthand notations

$$H(p) = H(p, -\infty),$$
$$H = H(\infty) = H(\infty, -\infty).$$

For each (p, p') and (q, q') with $-\infty \leq p \leq q \leq \infty$ and $p' \leq p \leq \infty$ and $q' \leq q \leq \infty$, let there be a morphism

$$i \colon H(p, p') \to H(q, q').$$

For each $-\infty \leq p'' \leq p' \leq p \leq \infty$, let

$$\partial\colon H(p,p') \to H(p',p'')$$

be a *connecting homomorphism*. We call this collection of groups together with the above morphisms an *ungraded H-system* if the following axioms are satisfied:

1. $i\colon H(p,p') \to H(p,p')$ is the identity.
2. All triangle and square diagrams built with the morphisms i commute.
3. For all $p'' \leq p' \leq p$, there is an exact sequence
$$\cdots \xrightarrow{\partial} H(p',p'') \xrightarrow{i} H(p,p'') \xrightarrow{i} H(p,p') \xrightarrow{\partial} H(p',p'') \to \cdots. \quad (3.2.1.1)$$
4. For each index $p' \in \mathbb{Z} \cup \{-\infty\}$, the group $H(\infty,p')$ is the direct limit of the morphisms $i\colon H(p,p') \to H(p+1,p')$ along $p' \leq p$.

In [CE73], the indices of these H-systems are denoted by (p,q) instead of (p,p'). To avoid confusion with the bigrading (p,q) of the pages $E^r_{p,q}$ later, we shall use $H(p,p')$.

Definition 3.2.2 (Graded H-system)**.** Let $\{H(p,p'), i, \partial\}_{p,p'}$ be an ungraded H-system as in Definition 3.2.1. We call this a *graded H-system* if it satisfies the following extra axioms:

5. All $H(p,p')$ carry a $\mathbb{Z}$-grading: $H(p,p') = \bigoplus_{s\in\mathbb{Z}} H_s(p,p')$.
6. All morphisms $i\colon H(p,p') \to H(q,q')$ are degree-preserving.
7. All morphisms $\partial\colon H(p,p') \to H(p',p'')$ have degree -1; i.e.,
$$\operatorname{im}\big(\partial \restriction H_s(p,p')\big) \subseteq H_{s-1}(p',p'').$$

Notation 3.2.3. Let $H(p,p')$ for $-\infty \leq p' \leq p \leq \infty$ form a graded H-system. For $r \geq 0$ and $q \in \mathbb{Z}$, write

$$\begin{aligned} Z^r_{p,q} &= \operatorname{im} i\colon H_{p+q}(p,p-r-1) \to H_{p+q}(p,p-1), \\ B^r_{p,q} &= \operatorname{im} \partial\colon H_{p+q+1}(p+r,p) \to H_{p+q}(p,p-1), \\ E^{r+1}_{p,q} &= Z^r_{p,q}/B^r_{p,q}. \end{aligned}$$

In this way, we define $E^r_{p,q}$ only for $r \geq 1$, not for $r \geq 0$. Compared to [CE73], we have shifted the index r in $Z^r_{*,*}$ and $B^r_{*,*}$ by 1 to match our Definition 2.7.3 of these groups as closely as possible; e.g., we write $Z^0_{p,q}$ for what would be denoted by $Z^1_{p,q}$ in [CE73].

Lemma 3.2.4. *We have* $B^0_{p,q} = 0$ *and*

$$E^1_{p,q} \cong Z^0_{p,q} = H_{p+q}(p, p-1).$$

Proof. In the long exact sequence

$$\cdots \to H_{p+q+1}(p,p) \xrightarrow{\partial} H_{p+q}(p,p-1) \xrightarrow{i} H_{p+q}(p,p-1) \xrightarrow{i} H_{p+q}(p,p) \to \cdots,$$

the central map $i\colon H(p,p-1) \to H(p,p-1)$ is the identity by Definiton 3.2.1. Its image is $Z^0_{p,q}$, which is all of $H_{p+q}(p,p-1)$. The exactness of the sequence forces the preceding map ∂ to vanish. The group $B^0_{p,q}$ is the image of ∂, therefore it is the trivial group. □

The main statement in [CE73] becomes:

Theorem 3.2.5. *With Notation 3.2.3, there is a spectral sequence* $\{E^r_{p,q}, d^r_{p,q}\}_{r,p,q}$ *of homological type. Its differentials* $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ *are defined as the composition*

$$\begin{tikzcd} E^r_{p,q} \arrow[rrr, "d^r_{p,q}"] \arrow[d, equal] & & & E^r_{p-r,q+r-1} \arrow[d, equal] \\ Z^{r-1}_{p,q}/B^{r-1}_{p,q} \arrow[r] & Z^{r-1}_{p,q}/Z^r_{p,q} \cong B^r_{p-r,q+r-1}/B^{r-1}_{p-r,q+r-1} \arrow[rr] & & Z^{r-1}_{p-r,q+r-1}/B^{r-1}_{p-r,q+r-1}, \end{tikzcd}$$

where the three maps at the bottom are all constructed in [CE73, Chapter XV, Paragraph 1]: The bottom-left map arises from factoring out the larger group $Z^r_{p,q} \supseteq B^{r-1}_{p,q}$*, the central map is an isomorphism, and the last map arises from the inclusion* $B^r_{p-r,q+r-1} \to Z^{r-1}_{p-r,q+r-1}$*. The homology of* $E^r_{*,*}$ *under* d^r *at* (p,q) *is isomorphic to* $E^{r+1}_{p,q}$*.*

We will first look at our application to K-theory of C*-algebras, then return to the full proof of convergence.

3.3 Application: K-theory

We construct a graded H-system to compute the K-theory of a C*-algebra, taking a chain of ideals as data. This is a $\mathbb{Z}$-graded theory. Bott periodicity forces $K_sA =$

$K_{s+2}A$ for all C*-algebras A over the complex numbers, leaving only two different K-groups to be computed.

Throughout the remainder of Section 3, let

$$0 \subseteq I_0 \subseteq I_1 \subseteq \cdots \subseteq I_p \subseteq \cdots$$

be an increasing chain of C*-ideals for $p \in \mathbb{N}$ and set

$$A = \overline{\bigcup_{p\in\mathbb{N}} I_p}.$$

For convenience, set $I_p = 0$ for $p < 0$, obtaining a $\mathbb{Z}$-graded chain of ideals $(I_p)_{p\in\mathbb{Z}}$.

Commonly, the chain of ideals will stabilize after finitely many steps; i.e., there exists $n \in \mathbb{N}$ with $I_n = I_{n+1} = I_{n+2} = \cdots$. Still, we develop the spectral sequence for the general case without assuming stabilization. The extra work is marginal and the final results of this thesis will rely on that general case.

Definition 3.3.1. For all $\mathbb{Z}$-indices $p' \leq p$ and K-theory degrees $s \in \mathbb{Z}$, define

$$\begin{aligned} H_s(p,p') &= K_s(I_p/I_{p'}), \\ H_s(p) &= K_s I_p, \\ H_s &= K_s A. \end{aligned}$$

The morphisms $i\colon H_s(p,p') \to H_s(p+1,p')$ in K-theory are induced by inclusions of ideals. The morphisms of the form $i\colon H_s(p,p') \to H_s(p,p'+1)$ are induced by the natural projection $I_p/I_{p'} \to I_p/I_{p'+1}$; this projection is well-defined because $I_{p'} \subseteq I_{p'+1}$. All these morphisms commute with each other and preserve the degree in K-theory.

The assignment $H_s(p,p') = K_s(I_p/I_{p'})$ satisfies the direct limit axiom from Definition 3.2.1 regarding $H(p)$ and H:

$$\begin{aligned} K_s(A/I_q) &= \operatorname*{colim}_{p\to\infty} H_s(p,p') = H_s(\infty,p'), \\ K_s A &= \operatorname*{colim}_{p\to\infty} H_s(p,-\infty) = H_s. \end{aligned}$$

For $i\colon H_s(p) \to H_s(p,p')$ and $i\colon H_s(p) \to H_s$, we use the respective limit maps. Because K-theory is a continuous functor, these are induced by inclusions and projections of A. Again, these limit maps preserve the degree s as desired.

Notation 3.3.2. We will deal with two kinds of boundary maps: The K-theoretic boundary map and the connecting homomorphism of the resulting H-system. To distinguish these, throughout Section 3, we shall denote the K-theoretic map by ∂_K and the connecting homomorpism by ∂.

Definition 3.3.3. Let $p'' \leq p' \leq p$ be indices in $\mathbb{Z}$. We implement the connecting homomorphisms ∂ in the diagram

$$\cdots \to H_s(p,p'') \xrightarrow{i} H_s(p,p') \xrightarrow{\partial} H_{s-1}(p',p'') \xrightarrow{i} H_s(p,p'') \to \cdots \tag{3.3.3.1}$$

by a composition of maps in K-theory, making this diagram commutative:

$$\begin{tikzcd}
H_s(p,p') \arrow[rr, "\partial"] \arrow[d, equal] & & H_{s-1}(p',p'') \arrow[d, equal] \\
K_s(I_p/I_{p'}) \arrow[r, "\partial_K"'] & K_{s-1}I_{p'} \arrow[r, "K_s(\mathrm{pr}\colon I_{p'} \to I_{p'}/I_{p''})"'] & K_{s-1}(I_{p'}/I_{p''}).
\end{tikzcd}$$

Lemma 3.3.4. *This choice of connecting homomorphism ∂ in Definition 3.3.3 makes the sequence 3.3.3.1 exact.*

Proof. Let $p'' \leq p' \leq p$ be indices in $\mathbb{Z}$ and $I_{p''} \subseteq I_{p'} \subseteq I_p$ be a chain of ideals in A. According to Definition 3.3.1, we can rewrite the long exact sequence 3.3.3.1 into the top row of the following commutative diagram:

$$\begin{tikzcd}
K_s\left(I_p/I_{p''}\right) \arrow[r, "i"] & K_s\left(I_p/I_{p'}\right) \arrow[r, "\partial"] & K_{s-1}\left(I_{p'}/I_{p''}\right) \arrow[r, "i"] & K_{s-1}\left(I_p/I_{p''}\right) \\
K_sI_p \arrow[r, "\mathrm{pr}_*"'] \arrow[u, "f_*"] & K_s\left(I_p/I_{p'}\right) \arrow[r, "\partial_K"'] \arrow[u, equal] & K_{s-1}I_{p'} \arrow[r, "\mathrm{incl}_*"'] \arrow[u, "g_*"] & K_{s-1}I_p. \arrow[u, "h_*"]
\end{tikzcd}$$

The vertical arrows f_*, g_*, h_* arise from natural projections: They are induced in K-theory from factoring out $I_{p''}$. The identity arrow is also of this type because

$$I_{p''} \subseteq I_{p'} \qquad \Longrightarrow \qquad \frac{I_p/I_{p''}}{I_{p'}/I_{p''}} \cong I_p/I_{p'}.$$

The top row – except possibly at ∂ – matches the long exact sequence in K-theory that corresponds to the short exact sequence $0 \to I_{p'}/I_{p''} \to I_p/I_{p''} \to I_p/I_{p'} \to 0$. We wish to prove that ∂ turns the upper row into a long exact sequence.

We recognize ∂_K as the boundary map in K-theory for the short exact sequence $0 \to I_{p'} \to I_p \to I_p/I_{p'} \to 0$. By naturality of the exact sequence with respect to factoring out $I_{p''}$, the composition $g_* \circ \partial_K$ is the K-theoretic connecting homomorphism to make the upper row exact. By Definition 3.3.3, we have $\partial = g_* \circ \partial_K$. Thus

∂ is the correct arrow to construct the H-system. □

3.4 Filtration

Definition 3.4.1. For $s \in \mathbb{Z}$, the chain of ideals leads to a $\mathbb{Z}$-indexed increasing filtration $\{F^p K_s A\}_{p\in\mathbb{Z}}$ of $K_s A$,

$$F^p H_s = F^p K_s A = \operatorname{im}(i\colon K_s I_p \to K_s A) \subseteq K_s A.$$

To discuss this filtration, we will continue to write $s \in \mathbb{Z}$ for the index in K-theory. Afterwards, the K-theory groups will be indexed by $(p+q)$ to show convergence of the spectral sequence.

Proposition 3.4.2. *For all $s \in \mathbb{Z}$, the filtration $\{F^p K_s A\}_{p\in\mathbb{Z}}$ in Definition 3.4.1 is Hausdorff, exhaustive, and complete according to Definition 2.7.2.*

Proof. The Hausdorff property is immediate because $I_{p'} = 0$ for $p' < 0$, therefore $F^{p'} H_s = \operatorname{im}(i\colon 0 \to K_s A) = 0 \supseteq \bigcap_{p\in\mathbb{Z}} F^p H_s$.

For exhaustion, consider the input of the spectral sequence: An inclusion chain of C*-ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p \subseteq \cdots$ with $\overline{\bigcup_{p\in\mathbb{N}} I_p} = A$. This closure is the direct limit object of the system of ideal inclusions $(I_p \to I_{p+1})_{p\in\mathbb{N}}$ in $\underline{\text{C*A}}$. K-theory is a continuous functor, rendering the K-theory of the limit object A isomorphic to the limit of the system of K-theory groups along the morphisms $i\colon K_s I_p \to K_s I_{p+1}$. By Definition 3.3.1, these are exactly the morphisms that appear in Definition 3.4.1 of the filtration $\{F^p H_s\}_{p\in\mathbb{Z}}$. Finally, universality of the limit object $K_s A$ guarantees that the above system of morphisms i exhausts $K_s A$.

Completeness is trivially satisfied because $F^p H_s = 0$ for all $p < 0$. Both the inverse limit and its right derivative vanish for this system. □

Remark 3.4.3. Even when one C*-ideal $I_0 \subseteq I_1$ is included in another, the K-theory groups need not be connected by a system of injections $K_* I_0 \hookrightarrow K_* I_1 \hookrightarrow \cdots$; for example, let H be an infinite-dimensional Hilbert space, then KH, the compact operators of H, have $K_0 KH = \mathbb{Z}$, but $KH \subseteq BH$ is an inclusion of ideals and $K_0 BH = 0$.

Nonetheless, the filtration $\{F^p H_s\}_{p\in\mathbb{Z}} = \{F^p K_s A\}_{p\in\mathbb{Z}}$ satisfies $F^p H_s \subseteq F^{p+1} H_s$ for all $p \in \mathbb{Z}$: The group $\operatorname{im}(i\colon K_s I_p \to K_s A)$ is a subgroup of $\operatorname{im}(i\colon K_s I_{p+1} \to K_s A)$ because, by definition of an H-system, the morphisms i factor through each other, making this diagram commutative:

$$\begin{tikzcd} K_s I_p \arrow[d, "i"'] \arrow[dr, "i"] & \\ & K_s A = \operatorname{colim}_p K_s I_p = K_s \operatorname{colim}_p I_p. \\ K_s I_{p+1} \arrow[ur, "i"'] & \end{tikzcd}$$

3.5 Convergence

We would like to show that this filtration makes the K-theory spectral sequence converge strongly to K_*A. Here we replace the K-theoretic degree s by $p+q$.

Notation 3.5.1 ($Z^\infty_{p,q}$, $B^\infty_{p,q}$)**.** With $Z^r_{p,q}$ and $B^r_{p,q}$ as in Notation 3.2.5, write

$$Z^\infty_{p,q} = \bigcap_{r\geq 1} Z^r_{p,q}, \qquad B^\infty_{p,q} = \bigcup_{r\geq 1} B^r_{p,q}.$$

For each $q \in \mathbb{Z}$, the filtration $\{F^p K_{p+q} A\}_{p\in\mathbb{Z}}$ from Definition 3.4.1 leads to successive quotients $F^p K_{p+q} A / F^{p-1} K_{p+q} A$ across all $p \in \mathbb{Z}$. We have to show that this (p,q)-indexed collection of quotients coincides with

$$E^\infty_{p,q} = Z^\infty_{p,q} / B^\infty_{p,q} = \Big(\bigcap_{r\geq 1} Z^r_{p,q}\Big) \Big/ \Big(\bigcup_{r\geq 1} B^r_{p,q}\Big).$$

Because the filtration has already been proven Hausdorff, exhaustive, and complete, the convergence will then be strong.

We have shaped our input according to the most general axioms in [CE73]. Even though convergence for more specialized input is proven in that source, convergence is merely claimed for our input. Henceforth, we shall give a full proof.

Lemma 3.5.2. *If there exists $n \in \mathbb{N}$ with $A = I_n = I_{n+1} = I_{n+2} = \cdots$, then the spectral sequence collapses at page $n+1$: We have $E^r_{*,*} = E^{r+1}_{*,*}$ for all $r \geq n+1$, thus $E^\infty_{p,q} = E^{n+1}_{p,q}$.*

Proof. Fix p and $q \in \mathbb{Z}$. If $p < 0$, then $B^r_{p,q}$ and $Z^r_{p,q}$ vanish by definition for all $r \geq 0$ because $I_p = 0$ and we have $E^{r+1}_{p,q} = 0$ here.

Consider the case $p \geq 0$. For all $r \geq n$, we have

$$B^r_{p,q} = \operatorname{im} \partial \colon K_{p+q+1}(I_{p+r}/I_p) \to K_{p+q}(I_p/I_{p-1}),$$

where $I_{p+r} = I_n = A$ for all $r \geq n+1$. Thus all $B^r_{p,q}$ coincide for such high page numbers r. In similar fashion, all $Z^r_{p,q}$ become $\operatorname{im} i \colon K_{p+q} I_p \to K_{p+q}(I_p/I_{p-1})$. The collapse follows from the definition $E^{r+1}_{p,q} = Z^r_{p,q}/B^r_{p,q}$ for all $r \in \mathbb{Z}$. $\square$

This collapsing lemma reveals some structure of the pages $E^r_{*,*}$ when the chain of ideals stabilizes. The following convergence theorem holds with or without stabilization of the ideals.

Theorem 3.5.3. *The E^∞-term admits the desired filtration; i.e.,*

$$E^\infty_{p,q} \cong F^p H_{p+q}/F^{p-1}H_{p+q}.$$

Substituting the definitions lets us rewrite the claim like this, denoting the yet-undefined isomorphism in the middle by f:

$$\begin{aligned} E^\infty_{p,q} = \frac{Z^\infty_{p,q}}{B^\infty_{p,q}} &= \frac{\operatorname{im} i\colon K_{p+q}I_p \to K_{p+q}(I_p/I_{p-1})}{\operatorname{im} \partial\colon K_{p+q+1}(A/I_p) \to K_{p+q}(I_p/I_{p-1})} \\ &\overset{f}{\cong} \frac{\operatorname{im} i\colon K_{p+q}I_p \to K_{p+q}A}{\operatorname{im} i\colon K_{p+q}I_{p-1} \to K_{p+q}A} = \frac{F^p H_{p+q}}{F^{p-1}H_{p+q}}. \end{aligned} \tag{3.5.3.1}$$

Our strategy is to construct the central isomorphism f in 3.5.3.1 explicitly. After giving its construction, we show that f is well-defined, injective, and surjective. That will constitute the proof of Theorem 3.5.3.

Definition 3.5.4. For $x + B^\infty_{p,q}$, an element in $Z^\infty_{p,q}/B^\infty_{p,q}$, we must define $f(x + B^\infty_{p,q})$. Find $y \in K_{p+q}I_p$ with $i(y) = x$. Define

$$f(x + B^\infty_{p,q}) = i^p_A(y) + F^{p-1}K_{p+q}A,$$

where $i^p_A\colon K_{p+q}I_p \to K_{p+q}A$ denotes the standard map from our H-system, induced by the inclusion of algebras.

Lemma 3.5.5. *The morphism f is well-defined: The construction in Definition 3.5.4 is independent of the choice of $y \in K_{p+q}I_p$ with $i(y) = x$.*

Proof. Let y and $y' \in K_{p+q}I_p$ with $i(y - y') \in B^\infty_{p,q}$. We have to show that $f(y) = f(y')$, equivalently, that $f(y - y') \in F^{p-1}K_{p+q}A = \operatorname{im} i\colon K_{p+q}I_{p-1} \to K_{p+q}A$.

Because $i(y - y') \in B^\infty_{p,q} = \operatorname{im} \partial\colon K_{p+q+1}(A/I_p) \to K_{p+q}(I_p/I_{p-1})$, we find $z \in K_{p+q+1}(A/I_p)$ with $\partial(z) = i(y - y')$. This ∂ belongs to the exact sequence

$$\begin{aligned} \cdots \to K_{p+q+1}(A/I_{p-1}) &\xrightarrow{i'} K_{p+q+1}(A/I_p) \\ &\xrightarrow{\partial} K_{p+q}(I_p/I_{p-1}) \xrightarrow{i'} K_{p+q}(A/I_{p-1}) \to \cdots . \end{aligned}$$

Onto this exact sequence, we draw a commutative square, and then extend the right-hand side of the square to a vertical exact sequence in K-theory.

$$\begin{tikzcd}
 & & K_{p+q}I_{p-1} \arrow[d, "i_A^{p-1}"] & \\
 & K_{p+q}I_p \arrow[r, "i_A^p"] \arrow[d, "i"] & K_{p+q}A \arrow[d, "i"] & \\
\cdots \arrow[r, "i'"'] K_{p+q+1}(A/I_p) \arrow[r, "\partial"'] & K_{p+q}(I_p/I_{p-1}) \arrow[r, "i'"'] & K_{p+q}(A/I_{p-1}) \arrow[r, "i'"'] & \cdots .
\end{tikzcd} \tag{3.5.5.1}$$

Chasing $y - y' \in K_{p+q}I_p$ through this diagram, we obtain

$$(i' \circ i)(y - y') = (i' \circ \partial)(z) = 0 \in K_{p+q}(A/I_{p-1})$$

because the bottom row is exact. Then $(i \circ i_A^p)(y - y') = 0$ due to the commutativity of the square. With $i_A^p(y - y') \in \ker i \colon K_{p+q}A \to K_{p+q}(A/I_{p-1})$, we conclude from the exactness of the vertical sequence that $i_A^p(y - y') \in \operatorname{im} i_A^{p-1} = F^{p-1}K_{p+q}A$.

This shows that $f \colon E^\infty_{p,q} \to F^pK_{p+q}A/F^{p-1}K_{p+q}A$ is well-defined. $\square$

Lemma 3.5.6. *The morphism f is injective.*

Proof. Let $x + B^\infty_{p,q}$ be a class in $E^\infty_{p,q}$ that vanishes under f. We have to show that $x \in B^\infty_{p,q}$. This proof looks like the proof of Lemma 3.5.5 in reverse.

Select $y \in K_{p+q}(I_p)$ with $i(y) = x \in \operatorname{im} i \colon K_{p+q}I_p \to K_{p+q}(I_p/I_{p-1})$. From $f\big(i(y) + B^\infty_{p,q}\big) = 0$ and the definition of f, we know $i_A^p(y) \in \operatorname{im} i_A^{p-1} \colon K_{p+q}I_{p-1} \to K_{p+q}A$. Consider diagram 3.5.5.1 again: By exactness of the vertical sequence, $(i \circ i_A^p)(y) = 0$. Commutativity of the square shows that $(i' \circ i)(y) = 0$. Exactness of the bottom row gives $i(y) \in \operatorname{im} \partial \colon K_{p+q+1}(A/I_p) \to K_{p+q}(I_p/I_{p-1})$.

After substituting $i(y) = x$ and $\operatorname{im} \partial = B^\infty_{p,q}$, we have shown $x \in B^\infty_{p,q}$ and therefore the injectivity of f. $\square$

Lemma 3.5.7. *The morphism f is surjective.*

Proof. Let $z + \operatorname{im} i_A^{p-1}$ be in $F^pK_{p+q}A/F^{p-1}K_{p+q}A$. For this $z \in F^pK_{p+q}A = \operatorname{im} i_A^p$, we may find a lift $y \in K_{p+q}I_p$ with $i_A^p(y) = z$. Then $i(y) \in Z^\infty_{p,q}$ already satisfies $f\big(i(y) + B^\infty_{p,q}\big) = z + \operatorname{im} i_A^{p-1}$ by definition of f. Thus f is surjective. $\square$

These lemmas conclude the proof of Theorem 3.5.3.

3.6 Summary

We recall the main theorem of Section 3:

Theorem 3.1.1 (Spectral sequence for ideal inclusions)**.** *Let $A = \overline{\bigcup_{p \in \mathbb{N}} I_p}$ be a C^*-algebra, where the $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p \subseteq \cdots$ form a chain of closed two-sided ideals. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} = K_{p+q}(I_p/I_{p-1}).$$

This spectral sequence converges strongly to K_*A; i.e., given $s \in \mathbb{Z}$, the groups $E^\infty_{p,q}$ along the diagonal $s = p + q$ pose an extension problem to reconstruct K_sA.

The theorem is plausible: Fix $n \in \mathbb{N}$, choose $I_p = 0$ for $p < n$ and $I_n = A$. The spectral sequence begins with $E^1_{p,q} = 0$ for $p \neq n$ and $E^1_{n,q} \cong K_{n+q}A$. In the only nonzero column $E^1_{n,*} \cong E^\infty_{n,*}$, we see the expected $K_sA \cong E^\infty_{n,q}$ for $n + q = s$.

Proof of Theorem 3.1.1. The computation of $E^1_{p,q}$ is straightforward:

$$E^1_{p,q} = \frac{Z^0_{p,q}}{B^0_{p,q}} = \frac{\operatorname{im} \mathrm{id} \colon K_{p+q}(I_p/I_{p-1}) \to K_{p+q}(I_p/I_{p-1})}{\operatorname{im} \partial \colon \underbrace{K_{p+q+1}(I_p/I_p)}_{=0} \to K_{p+q}(I_p/I_{p-1})} \cong K_{p+q}(I_p/I_{p-1}).$$

The differentials $d^r \colon E^r_{*,*} \to E^r_{*,*}$ were defined in Theorem 3.2.5 and have the correct bidegrees $(-r, r-1)$ on page r. The strong convergence of this spectral sequence follows from Proposition 3.4.2 and Theorem 3.5.3. □

Remark 3.6.1. Even though this is a half-plane spectral sequence, its convergence is provable like the convergence of a single-quadrant spectral sequence because we have *exiting differentials*: For each a bidegree $(p, q) \in \mathbb{Z}^2$, all except finitely many differentials $d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ exit the half-plane $E^r_{p',*}$ for $p' \geq 0$ of nonzero groups.

There are more intricate results for half-plane spectral sequences with *entering differentials* or for whole-plane spectral sequences; these will not arise in our setting. Besides the classic reference [McC01], a good resource for convergence theorems is [Boa99].

4 Finite sums of ideals

Let A be a C*-algebra. There is a K-theory spectral sequence for ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = A$. We will postulate a new spectral sequence that weakens the "$\subseteq$" to a mere "+": For ideals I_0, I_1, I_2, …, I_n with $\sum_{j=0}^{n} I_j = A$, there is a spectral sequence that relates the K-theory of their intersections to the K-theory of A.

4.1 Ideal decompositions

Even though Section 4 deals only with the finite case, we will define $*$-homomorphisms that preserve arbitrarily-sized ideal decompositions in light of later sections.

Definition 4.1.1 (Preservation of ideal decompositions)**.** Let α and α' be arbitrary index sets with $\alpha \subseteq \alpha'$. Let A be the norm closure $A = \overline{\sum_{\beta\in\alpha} I_\beta}$ of the sum of $|\alpha|$-many C*-ideals I_β. Let $A' = \overline{\sum_{\beta\in\alpha'} I'_\beta}$ be another C*-algebra written as the sum of $|\alpha'|$-many C*-ideals I'_β.

A $*$-homomorphism $f\colon A \to A'$ *preserves the ideal decomposition* if $f(I_\beta) \subseteq I'_\beta$ for every $\beta \in \alpha$. Both f and the specific decompositions $\{I_\beta\}_{\beta\in\alpha}$ and $\{I'_\beta\}_{\beta\in\alpha'}$ are part of the input data.

Remark 4.1.2 (Naturality w.r.t. ideal decompositions)**.** It is conceivable to define a category of C*-ideal decompositions and decomposition-preserving $*$-homomorphisms, e.g., with cardinal numbers α as index sets to ensure $\alpha \subseteq \alpha'$ wherever $|\alpha| \leq |\alpha'|$. But it will be enough to work in $\underline{\text{C*A}}$, the standard category of C*-algebras, because all natural constructions here will already be natural will w.r.t. ideal decompositions of C*-algebras:

Let $\underline{\text{C}}$ be any category. Let F, $G\colon \underline{\text{C*A}} \to \underline{\text{C}}$ be functors of C*-algebras and let $\eta\colon F \to G$ be a natural transformation. Let $f\colon A \to A'$ be a $*$-homomorphism that preserves an $|\alpha|$-fold ideal decomposition as in Definition 4.1.1. Since $f(I_\beta) \subseteq I'_\beta$ for all $\beta \in \alpha$ and $*$-homomorphisms are compatible with sums, the following diagram commutes in $\underline{\text{C}}$:

$$\begin{tikzcd}
F(A) = F\big(\sum_{\beta\in\alpha} I_\beta\big) \arrow[r, "\eta(A)"] \arrow[d, "F(f) = F(\sum_{\beta\in\alpha} f \restriction I_\beta)"'] & G(A) = G\big(\sum_{\beta\in\alpha} I_\beta\big) \arrow[d, "G(f) = G(\sum_{\beta\in\alpha} f \restriction I_\beta)"] \\
F(A') = F\big(\sum_{\beta\in\alpha} I'_\beta\big) \arrow[r, "\eta(A')"'] & G(A') = G\big(\sum_{\beta\in\alpha} I'_\beta\big).
\end{tikzcd}$$

4.2 Cake pieces

To construct the spectral sequence for finite ideal decompositions, we will define function algebras over certain subsets of the standard simplex.

Definition 4.2.1 (Cake piece)**.** Fix $n \in \mathbb{N}$. The standard *n-simplex* is the topological space

$$\Delta^n = \left\{ (x_0, x_1, \ldots, x_n) \in [0,1]^{n+1} : \sum_{i=0}^{n} x_i = 1 \right\}.$$

Its boundary $\partial\Delta^n$ shall be the subset of points with at least one zero entry. Let $j \in \mathbb{N}$ be an index with $j \leq n$. This index defines the *j-th cake piece*

$$\Delta^n_j = \{(x_0, x_1, \ldots, x_n) \in \Delta^n : x_j \leq x_i \text{ for all } i \leq n\}.$$

Let $J \subseteq \{0, 1, \ldots, n\}$ be a nonempty subset of the $n+1$ indices. This determines an intersection of cake pieces:

$$\Delta^n_J = \bigcap_{j \in J} \Delta^n_j.$$

We will see how Δ^n_J behaves very much like a j-th cake piece, and therefore also call it a *cake piece*.

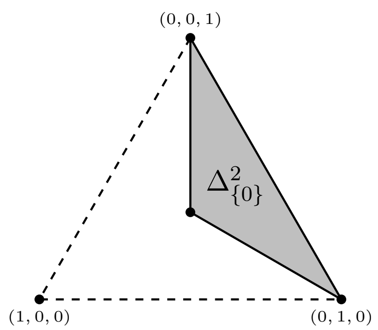

Figure 4.2.2: The simplex Δ^2 with the cake piece $\Delta^2_{\{0\}} = \Delta^2_0 \subseteq \Delta^2$ marked

Cake pieces are closed subsets of Δ^n. The central point $\left(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$ of the n-simplex is part of every cake piece; this point is the only element of Δ^n_J for the full set $J = \{0, 1, \ldots, n\}$.

Arbitrary index subsets $J \neq \varnothing$ make Δ^n_J look like $\Delta^{n+1-|J|}$:

Proposition 4.2.3. *For a nonempty $J \subseteq \{0, 1, \ldots, n\}$, the subset Δ^n_J is the image of $\Delta^{n+1-|J|} \times \{0\}^{|J|-1}$ under a nondegenerate affine transformation in $\mathbb{R}^{n+1}$.*

Proof. We will analyze several cases explicitly by cardinality of J.

Full set. For $J = \{0, 1, \dots, n\}$, the full set of $(n+1)$ elements, we have already argued how Δ^n_J contains only a single point. This is an image of $\Delta^0 \times \{0\}^n \subseteq \mathbb{R}^{n+1}$.

One element. For $n \geq 2$ and $J = \{j\}$, we have $\Delta^n_J = \Delta^n_j$. Without loss of generality, choose $J = \{0\}$. The affine transformation of $\mathbb{R}^{n+1}$ to get Δ^n_0 from the standard n-simplex is

$$f = (f_0, f_1, \dots, f_n)\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1},$$

$$f_0(x) = \frac{x_0}{n+1}, \qquad f_i(x) = x_i + \frac{x_0}{n+1} \text{ for } i \neq 0.$$

The purpose of $f_i(x)$ is to equally distribute among the n other coordinates the value $\frac{nx_0}{n+1}$ that has been taken away from x_0.

The f_i are nontrivial linear maps, and their direct product $f = (f_0, \dots, f_n)$ is an affine automorphism of $\mathbb{R}^{n+1}$. Its inverse is

$$f^{-1} = g = (g_0, g_1, \dots, g_n)\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1},$$

$$g_0(x) = (n+1) \cdot x_0, \qquad g_i(x) = x_i - x_0 \text{ for } i \neq 0.$$

Even though these maps are defined in $\mathbb{R}^{n+1}$, they restrict well to maps on the simplex: For $x = (x_0, \dots, x_n) \in \Delta^n$, we have $\sum_{i=0}^n x_i = 1 = \sum_{i=0}^n f(x_i)$. Furthermore, $f \restriction \Delta^n$ maps into Δ^n_0 because all points $x \in \Delta^n$ satisfy $f_i(x) \geq f_0(x)$. The restricted inverse $f^{-1} \restriction \Delta^n_0$ maps into Δ^n: Positive coordinates stay positive because $x_i \geq x_0$ for all $0 \leq i \leq n$.

Several elements. For $1 < |J| \leq n$, observe how $|J|$ coordinates in Δ^n_J remain equal to each other at all times, and are always the smallest. Without loss of generality, let J be $\{0, 1, \dots, |J| - 1\}$, the $|J|$ first coordinates. We will construct an affine isomorphism

$$h\colon \Delta^{n+1-|J|}_{\{0\}} \to \Delta^n_J$$

by defining h on the $(n+2-|J|)$ corners of $\Delta^{n+1-|J|}_{\{0\}}$, then extending h to the entire cake piece by preserving convex combinations. Thereby, h reduces the case of Δ^n_J to the already-proven case $|J| = 1$.

The central point of $\Delta^{n+1-|J|}$ is a corner of $\Delta^{n+1-|J|}_{\{0\}}$. Have h map this point to the center of Δ^n, this is extremal in Δ^n_J. Biject the remaining $(n+1-|J|)$ corners of $\Delta^{n+1-|J|}_{\{0\}}$ to the corners in Δ^n that belong to the coordinates in $\{0, 1, \dots, n\} - J$; these points remain extremal in Δ^n_J. This bijection can even be chosen to preserve the order of coordinates. $\square$

Corollary 4.2.4. *Let $J \subseteq \{0, 1, \dots, n\}$ be nonempty. Then $\Delta^{n+1-|J|} \cong \Delta^n_J \cong D^{n+1-|J|}$, where $D^{n+1-|J|}$ denotes the $(n+1-|J|)$-dimensional unit disk.*

In particular, if $J = \{j\}$, then $\Delta^n_j \cong D^n$.

Proof. This follows from $\Delta^n \cong D^n$ and Proposition 4.2.3. □

These technical constructions relate various subspaces of simplices to disks. The boundaries of disks are spheres. This will become useful once we consider C*-algebras of functions on these subspaces of simplices: When we force functions to vanish on the boundaries, the C*-algebras can be viewed as suspensions of other algebras.

4.3 Cake algebras

Definition 4.3.1 (Cake algebra)**.** Fix $n \in \mathbb{N}$. Let A be a C*-algebra and $I_j \subseteq A$ be closed two-sided ideals for $j \in \{0, 1, \dots, n\}$ with $A = \sum_{j=0}^n I_j$. This gives rise to a suspension-like C*-algebra, the *cake algebra*

$$\begin{aligned} B &= B(I_0, I_1, \dots, I_n) \\ &= \left\{ f\colon \Delta^n \to A = \sum_{j=0}^n I_j : f \text{ continuous}, f \restriction \partial\Delta^n = 0, f(\Delta^n_j) \subseteq I_j \text{ for all } j \right\}. \end{aligned}$$

For $J \subseteq \{0, 1, \dots, n\}$, define the sub-C*-algebra $B_J \subseteq B$, again called a *cake algebra*, by

$$B_J = \left\{ f \in B : \text{for each } j' \notin J,\ f(\Delta^n_{j'}) = 0 \right\}.$$

Remark 4.3.2. We observe $B_{\{0,1,\dots,n\}} = B$ and $B_\varnothing = 0$. Larger index sets mean larger function algebras because fewer restrictions apply. Whenever $J' \subseteq J$ is a subset, then $B_{J'} \subseteq B_J$ is a subalgebra. For $J = \{j\}$, we can characterize $B_{\{j\}}$:

Proposition 4.3.3. *$B_{\{j\}}$ is isomorphic to the n-fold C*-algebra suspension of I_j.*

Proof. By $\partial\Delta^n_j$, we denote the topological boundary of Δ^n_j as a subset of $\mathbb{R}^{n+1}$. A point $x \in \partial\Delta^n_j$ lies in $\partial\Delta^n$ if $x_j = 0$. Otherwise, we have $x_j = x_{j'}$ for an index $j' \neq j$ and x then lies in $\Delta^n_{j'}$.

Understanding this, we simplify the above definition for $B_J = B_{\{j\}}$:

$$\begin{aligned} B_{\{j\}} &= \left\{ f\colon \Delta^n \to A : f \restriction \partial\Delta^n = 0,\ f(\Delta^n_j) \subseteq I_j,\ f(\Delta^n_{j'}) = 0 \text{ for all } j' \neq j \right\} \\ &= \left\{ f\colon \Delta^n \to A : f \restriction \partial\Delta^n = 0,\ f(\Delta^n_j) \subseteq I_j,\ f(\Delta^n - \Delta^n_j) = 0 \right\} \\ &\cong \left\{ f\colon \Delta^n_j \to I_j : f \restriction \partial\Delta^n_j = 0 \right\}. \end{aligned}$$

Because $\Delta^n_{\{j\}} \cong D^n$, the algebra $B_{\{j\}}$ is isomorphic to the n-fold C*-algebra suspension of I_j; i.e., the A-valued functions on D^n that vanish on the boundary ∂D^n. □

Remark 4.3.4. As subsets of functions that vanish on a given set, the B_J are closed two-sided ideals in B. For $J' \subseteq J$, $B_{J'}$ is a closed two-sided ideal in B_J.

Lemma 4.3.5. *For subsets J and J' of $\{0, 1, \dots, n\}$, we have $B_J \cap B_{J'} = B_{J\cap J'}$.*

Proof. This is immediate from the definition of B_J:

$$\begin{aligned} B_J \cap B_{J'} &= \left\{ f \in B : f(\Delta^n_{j'}) = 0 \text{ for } j' \notin J \right\} \cap \left\{ f \in B : f(\Delta^n_{j'}) = 0 \text{ for } j' \notin J' \right\} \\ &= \left\{ f \in B : f(\Delta^n_{j'}) = 0 \text{ for } j' \text{ with } j' \notin J \text{ or } j' \notin J' \right\} \\ &= B_{J\cap J'}. \end{aligned}$$

□

Definition 4.3.6 (Cake sums Q_p for $p \in \mathbb{Z}$)**.** Let A and $B(I_0, I_1, \dots, I_n)$ be as in Definition 4.3.1. For $p \in \mathbb{Z}$, define the C*-algebra Q_p, called a *cake sum*, by

$$Q_p = \sum_{|J| \le p+1} B_J,$$

where J ranges over all subsets of $\{0, 1, \dots, n\}$ that have cardinality $(p + 1)$ or less.

Remark 4.3.7 (Cake sums for $p < 0$ or $p \ge n$)**.** For $p < 0$, the sum Q_p is either $B_\varnothing$ or an empty sum; both of these are the zero algebra $Q_p = 0$.

For $p \ge n$, the sum Q_p is taken over all B_J for all possible subsets $J \subseteq \{0, 1, \dots, n\}$ including $\{0, 1, \dots, n\}$ itself. By Remark 4.3.2, for all $J \subseteq \{0, 1, \dots, n\}$, the cake algebra B_J is already a subalgebra of $B_{\{0,1,\dots,n\}}$. Thus Q_p is identical to $B_{\{0,1,\dots,n\}} = B$ for $p \ge n$.

Remark 4.3.8 (Inclusions $Q_{p-1} \subseteq Q_p$)**.** We have well-defined inclusions $Q_{p-1} \subseteq Q_p$ for all $p \in \mathbb{Z}$ because Q_p collects at least the cake algebras from Q_{p-1}, possibly more.

If $p \le n$, the relation $B_{J'} \subseteq B_J$ for $J' \subseteq J$ allows another characterization of Q_p by summing over fewer sets:

$$Q_p = \sum_{|J| \le p+1} B_J = \sum_{|J| = p+1} B_J. \tag{4.3.8.1}$$

For $p > n$, this characterization would be false because there are no subsets of cardinality $(n + 2)$ in $\{0, 1, \dots, n\}$.

Lemma 4.3.9. *For all $p' \le p \in \mathbb{Z}$, the cake sum $Q_{p'}$ is a C*-ideal in Q_p. Thus we have well-defined quotients $Q_p/Q_{p'}$, in particular Q_p/Q_{p-1}.*

Proof. For $p' \ge n$, we trivially have $Q_{p'} = Q_p = B$.

We will prove the case $p' < n$. $Q_{p'}$ and Q_p are sub-C*-algebras of the same commutative C*-algebra B and we have $Q_{p'} \subseteq Q_p$. It remains to show that $Q_{p'}$ is an algebraic ideal.

For $f \in Q_{p'}$ and $g \in Q_p$, find $J \subseteq \{0, 1, \ldots, n\}$ such that $f \in B_J$ and $|J| = p' + 1$; such a J exists according to the characterization 4.3.8.1.

By Definition 4.3.1 of B_J, the function f must vanish on at least $(n - p')$ different cake pieces. For any $g \in Q_p$, the pointwise product fg must vanish on the same cake pieces, therefore $fg \in Q_{p'}$. $\square$

Remark 4.3.10. The algebra Q_p is defined by summing over all B_J with $|J| \leq p+1$, not merely over those with $|J| \leq p$. This index shift is deliberate: We are going to define a spectral sequence with the K-theory of quotients of Q_p/Q_{p-1} as input. The index shift will affect the layout of the first page $\{E^1_{p,q}\}_{p,q \in \mathbb{Z}}$.

Consider the trivial input $n = 0$ and $A = I_0$. Here $Q_p = B_{\{0\}}$ for $p \geq 0$ and $Q_p = 0$ for $p < 0$. This leads to a spectral sequence with $E^1_{0,q} \cong K_q A$ and $E^1_{p,q} = 0$ for $p \neq 0$. This is the most desirable layout; the K-theory of the lone ideal $A = I_0$ is not shifted in any way:

$$
\begin{array}{r|cccc}
 & & & & \\
2 & 0 & K_2 A & 0 & 0 \\
1 & 0 & K_1 A & 0 & 0 \\
0 & 0 & K_0 A & 0 & 0 \\
-1 & 0 & K_{-1} A & 0 & 0 \\
\hline
 & -1 & 0 & 1 & 2
\end{array}
\quad p.
$$

4.4 K-theory of cake algebras

We started with a sum of ideals I_0, I_1, …, I_n and have developed a chain of ideals $\cdots \subseteq Q_p \subseteq Q_{p+1} \subseteq \cdots$. The main theorem of Section 4.4 relates the K-theory of this chain to the K-theory of the original ideals:

Theorem 4.4.1. *For $A = I_0 + I_1 + \cdots + I_n$ and the cake sums Q_p defined as before, given $p \in \{0, 1, \ldots, n\}$ and $q \in \mathbb{Z}$, the K-theory of Q_p/Q_{p-1} decomposes as*

$$K_{p+q}\left(Q_p/Q_{p-1}\right) \cong \bigoplus_{|J|=p+1} K_{q+n}\Big(\bigcap_{j \in J} I_j\Big).$$

Remark 4.4.2. For $p \notin \{0, 1, \ldots, n\}$, the K-theory $K_*(Q_p/Q_{p-1})$ vanishes because $Q_p = Q_{p-1}$.

Example 4.4.3. Before we prove Theorem 4.4.1 for all $p \in \{0, 1, \ldots, n\}$, we look at the simplest case, $p = 0$. One-fold intersections of ideals are merely the ideals themselves. The above formula reduces to

$$K_q\left(Q_0/Q_{-1}\right) \cong \bigoplus_{j\leq n} K_{q+n} I_j.$$

Inserting the definitions $Q_{-1} = 0$ and $Q_0 = \sum_{j\leq n} B_{\{j\}}$, we rewrite our claim to

$$K_q\Big(\sum_{j\leq n} B_{\{j\}}\Big) \cong \bigoplus_{j\leq n} K_{q+n} I_j.$$

Lemma 4.3.5 implies that $B_{\{j\}}$ and $B_{\{j'\}}$ overlap trivially as algebras for $j \neq j'$; i.e., $B_{\{j\}} \cap B_{\{j'\}}$ contains only the zero function. The sum $\sum_{j\leq n} B_{\{j\}}$ on the left-hand side is therefore isomorphic to a direct sum $\bigoplus_{j\leq n} B_{\{j\}}$ of the function spaces $B_{\{j\}}$.

In Proposition 4.3.3, we have shown that $B_{\{j\}}$ is isomorphic to the n-fold suspension of I_j, providing the desired shift by n degrees, $K_q B_{\{j\}} \cong K_q S^n I_j \cong K_{q+n} I_j$.

Because taking K-theory commutes with taking direct sums, we have shown the theorem for $p = 0$.

The main ingredient $B_{\{j\}} \cap B_{\{j'\}} = B_\varnothing = 0$ must now be generalized to prove Theorem 4.4.1 for $p > 0$.

Lemma 4.4.4. *Let $J' \neq J''$ be nonempty $(p+1)$-element subsets of $\{0, 1, \ldots, n\}$ and let $f \in Q_p$ lie in the intersection $B_{J'} \cap B_{J''}$. Then f lies already in the next-smaller ideal,*

$$f \in Q_{p-1} = \sum_{|J|=p} B_J.$$

Proof. By Lemma 4.3.5, $B_{J'} \cap B_{J''} = B_{J'\cap J''}$. The algebra $B_{J'\cap J''}$ is a summand of $Q_{|J'\cap J''|}$ This algebra is equal to or a subset of Q_{p-1} because $|J' \cap J''| \leq p$. □

Lemma 4.4.5. *Fix an index subset $J \subseteq \{0, 1, \ldots, n\}$. Let $p \in \mathbb{N}$ be a cardinality. Let $f \in B_J$ vanish on all $(p+1)$-fold intersections of cake pieces: $f \restriction \Delta^n_L = 0$ when $|L| = p + 1$.*

Then f is a finite sum of functions f_L with each $f_L \in B_L$ for $L \subseteq J$ and each occurring set L has cardinality $|L| \leq p$.

Remark 4.4.6. It follows that f is in Q_{p-1}, but the claim is stronger: Only summands B_L with $L \subseteq J$ are required to construct f in Q_{p-1}.

Proof of Lemma 4.4.5. We prove this by induction along p. The base case is $p = 0$: One-fold intersections of cake pieces – where f vanishes by assumption – are the cake pieces themselves, thus $f = 0$, the only function in the zero algebra $B_\varnothing$. This concludes the base case.

For the induction hypothesis, assume that all functions that vanish on p-fold intersections are sums of functions from B_L with $L \subseteq J$ and $|L| \leq p-1$. We will show the claim for p: Let $f \in B_J$ vanish on $(p+1)$-fold intersections of cake pieces.

Consider all subspaces Δ^n_L for $L \subseteq J$ with $|L| = p$. We may treat each as a topological submanifold of $\mathbb{R}^{n+1}$ on its own and consider its boundary $\partial\Delta^n_L$. Each point $x \in \partial\Delta^n_L$ lies on the boundary $\partial\Delta^n$ of the entire simplex Δ^n or in a cake piece Δ^n_j with $j \neq L$, see Figure 4.4.7. If $x \in \partial\Delta^n$, then $f(x) = 0$ by definition of B. If $x \in \Delta^n_j$ with $j \neq J$, then also $f(x) = 0$ because now $x \in \Delta^n_{L\cup\{j\}}$, a $(p+1)$-fold intersection of cake pieces. Thus $f \restriction \partial\Delta^n_L = 0$.

On the interior $\Delta^n_L - \partial\Delta^n_L$, f assumes values in $\bigcap_{j\in L} I_j$ by definition of B_J. From this and the restriction $f \restriction \partial\Delta^n_L = 0$, we can find a function $g_L \in B_L$ such that $f \restriction \Delta^n_L = g_L \restriction \Delta^n_L$. After defining g_L for each $L \subseteq J$ of cardinality p, consider the function

$$f' = f - \sum_{\substack{L\subseteq J\\|L|=p}} g_L.$$

This f' still lies in the C*-algebra B_J because $B_L \subseteq B_J$ for each L. Furthermore, f' vanishes on all p-fold intersections of cake pieces, not merely on the $(p+1)$-fold intersections.

By our induction hypothesis, f' is a finite sum of functions from B_L for $L \subseteq J$ of cardinality $|L| \leq p-1$. Each g_L is in B_L with $L \subseteq J$ and $|L| = p$. Since $f = f' + \sum_L g_L$, we have shown the induction case for cardinality p. $\square$

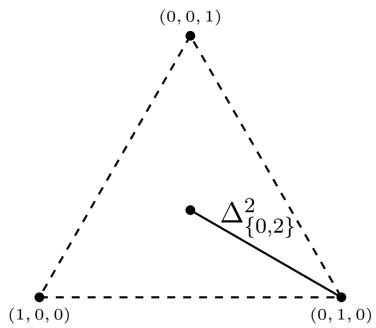

Figure 4.4.7: The two-fold intersection $\Delta^2_{\{0,2\}}$ and its two-point boundary: one point in $\partial\Delta^2$, one in the three-fold intersection $\Delta^2_{\{0,1,2\}}$

Lemma 4.4.8. *Let $J \subseteq \{0, 1, \ldots, n\}$ be a nonempty index set.*

- *Let f be a function in $\sum_{L\subsetneqq J} B_L$. Then $f \restriction \Delta^n_J = 0$.*
- *Conversely, let g be a function in B_J with $g \restriction \Delta^n_J = 0$. Then $g \in \sum_{L\subsetneqq J} B_L$.*

Proof. We have $f \in \sum_{L \subsetneq J} B_L$. Each $L \subsetneq J$ lacks at least one index $j \in J$, therefore $f \restriction \Delta^n_j = 0$ by definition of B_L. Since Δ^n_J is contained in the boundary $\partial\Delta^n_j$, we conclude $f \restriction \Delta^n_J = 0$.

Conversely, let $g \in B_J$ vanish on Δ^n_J. All functions in B_J vanish on $(|J|+1)$-fold intersections of cake pieces. Furthermore, Δ^n_J is the only $|J|$-fold intersection that touches the interior of the support of g. Thus g vanishes on all $|J|$-fold intersections. By Lemma 4.4.5, g is a sum of functions g_L from B_L with $|L| < |J|$ and $L \subseteq J$. □

In the following technical proposition, A and B are general C*-algebras; they need not coincide with $B(I_0, I_1, \dots, I_n)$ that we defined before. Nonetheless, we choose the names A and B here because we will later apply this result to the B_J from Theorem 4.4.1.

Proposition 4.4.9. *Let $X \subseteq \mathbb{R}^n$ be a compact set and let $D \subseteq X$ be a compact subspace of X. Let A be a C*-algebra, $B \subseteq \mathscr{C}(X, A)$ a C*-ideal of functions from X to A, and $\mathrm{Van}_D \subseteq B$ the vanishing ideal of D; i.e., the ideal of functions $f \in B$ with $f \restriction D = 0$.*

Then B/Van_D is isomorphic as a C-algebra to $B' = \{f \restriction D : f \in B\}$.*

This is plausible: When we enlarge D, then functions in Van_D are allowed less variation, thus Van_D becomes smaller and the quotient space B/Van_D becomes larger.

Proof. Define the operator $T\colon B/\mathrm{Van}_D \to B'$ by $T[f] = f \restriction D$. This is a well-defined linear map because for f and $f' \in [f] \in B/\mathrm{Van}_D$, we have $f - f' \in \mathrm{Van}_D$, therefore $f \restriction D = f' \restriction D$.

T is a continuous operator with norm

$$\|T\| = \sup\{\|f \restriction D\| : f \in B \text{ with } \|[f]\| \leq 1\},$$

where $\|[f]\| = \inf\{\|f - g\| : g \in \mathrm{Van}_D\}$. From $\|f \restriction D\| \leq \|[f]\| \leq \|f\|$, we see that T is continuous with norm $\|T\| \leq 1$.

T is bijective: If $T[f] = 0$, then $f \restriction D = 0$, therefore $f \in \mathrm{Van}_D$ and $[f] = 0 \in B/\mathrm{Van}_D$. On the other hand, given $f \restriction D \in B'$ with $f \in B$, surely $[f]$ is a preimage of $f \restriction D$ under T.

As a restriction of functions, T preserves products and the C*-involution. Together with bijectivity of T, we conclude that $\|T\| = 1$ and that T is an isometric *-isomorphism by [Dav96, Theorem I.5.5]. □

We could have obtained $\|T\| = 1$ from an analytical argument, too: Given $f \restriction D$, force $f\colon X \to A$ to decay rapidly outside $D \subseteq X$ by multiplying with bump functions.

B is an ideal in $\mathscr{C}(X, A)$, and A admits an approximate unit.

We shall now return to the setting where $A = I_0 + I_1 + \cdots + I_n$ is a sum of ideals and $B = B(I_0, I_1, \ldots, I_n)$ is the function algebra constructed over the n-simplex.

Lemma 4.4.10. *Let $J \subseteq \{0, 1, \ldots, n\}$ be a nonempty index set of cardinality $|J| = p + 1$. Then*

$$B_J/(B_J \cap Q_{p-1}) \cong S^{n-p}\Big(\bigcap_{j \in J} I_j\Big),$$

where S^{n-p} denotes the $(n-p)$-fold suspension of C^*-algebras.

Recall that Q_p was a sum over all B_L with index sets L of cardinality $|L| = p+1$, thus $B_J \subseteq Q_p$ and dividing by the intersection $(B_J \cap Q_{p-1}) \subseteq Q_p$ is meaningful.

Proof. All functions in B_J are supported in $\bigcup_{j \in J} \Delta^n_j$. Write $D = \Delta^n_J = \bigcap_{j \in J} \Delta^n_j$ for the subset of $\bigcup_{j \in J} \Delta^n_j$ where functions in B_J may take nonzero values in all I_j for $j \in J$ simultaneously. Now

$$B_J \cap Q_{p-1} = \sum_{L \subsetneq J} B_L,$$

and, by Lemma 4.4.8, functions in $B_J \cap Q_{p-1}$ are exactly those functions in B_J that vanish on D. We can apply Proposition 4.4.9 to the function algebra B_J on the base space $X = \bigcup_{j \in J} \Delta^n_j$ and its subset D to get

$$B_J/(B_J \cap Q_{p-1}) \cong \{f \restriction D : \, f \in B_J\}. \tag{4.4.10.1}$$

Considering $D = \Delta^n_J$ an $(n + 1 - |J|)$-dimensional topological manifold on its own, Δ^n_J itself has a boundary $\partial\Delta^n_J$ and a nontrivial interior $(\Delta^n_J - \partial\Delta^n_J)$. In the edge case where $J = \{0, 1, \ldots, n\}$ is the full set, $\Delta^n_{\{0,1,\ldots,n\}}$ is a single point, which is a zero-dimensional manifold with empty boundary $\partial\Delta^n_{\{0,1,\ldots,n\}} = \varnothing$.

The boundary $\partial\Delta^n_J$ is contained in the boundary of the original domain $\bigcup_{j \in J} \Delta^n_j$. Therefore functions in B_J, even when restricted to D as in 4.4.10.1, must still vanish on this new boundary $\partial\Delta^n_J$.

On the interior of $D = \Delta^n_J$, functions in B_J must take values in $\bigcap_{j \in J} I_j$ by definition of B_J, but no further restrictions apply. We can rewrite 4.4.10.1 as

$$B_J/(B_J \cap Q_{p-1}) \cong \left\{ f \colon \Delta^n_J \to \bigcap_{j \in J} I_j : \, f \text{ is continuous and } f \restriction \partial\Delta^n_J = 0 \right\}.$$

Finally, by Lemma 4.2.4, $\Delta^n_J \cong \Delta^{n+1-|J|} = \Delta^{n-p}$ is homeomorphic to the $(n-p)$-dimensional unit disk. This allows us to further rewrite the algebra $B_J/(B_J \cap Q_{p-1})$

as the $(n-p)$-fold suspension in the claim. $\square$

We are now ready to prove the main theorem about the K-theory of the chain of ideals Q_p.

Proof of Theorem 4.4.1. Fix $p \in \{0, 1, \dots, n\}$ and $q \in \mathbb{Z}$. With $Q_p = \sum_{|J|=p+1} B_J$ according to the characterization 4.3.8.1, we have to show:

$$K_{p+q}\left(Q_p/Q_{p-1}\right) \cong \bigoplus_{|J|=p+1} K_{q+n}\Big(\bigcap_{j\in J} I_j\Big).$$

First, we will show that the quotient Q_p/Q_{p-1} decomposes as a direct sum. Let $f \in Q_p$ lie in the images of different inclusions $B_J \to Q_p$ and $B_{J'} \to Q_p$ for $|J| = |J'| = p+1$. By Lemma 4.4.4, we have $f \in Q_{p-1}$ and therefore $[f] = 0 \in Q_p/Q_{p-1}$. This shows that Q_p/Q_{p-1} is a direct sum. Each summand corresponds to one B_J with $|J| = p+1$:

$$Q_p/Q_{p-1} = \bigoplus_{|J|=p+1} B_J/(B_J \cap Q_{p-1}).$$

For each J, we computed $B_J/(B_J \cap Q_{p-1}) \cong S^{n-p}\big(\bigcap_{j\in J}\big)$ in Lemma 4.4.10. Passing to K-theory, we can replace the $(n-p)$-fold suspension with a degree shift by $(n-p)$:

$$K_{p+q}\big(B_J/(B_J \cap Q_{p-1})\big) \cong K_{p+q}\bigg(S^{n-p}\Big(\bigcap_{j\in J} I_j\Big)\bigg) \cong K_{q+n}\Big(\bigcap_{j\in J} I_j\Big).$$

The claim follows because taking K-theory commutes with taking direct sums. $\square$

Theorem 4.4.11. *The inclusion of algebras $B \to \{f\colon \Delta^n \to A :\, f \restriction \partial\Delta^n = 0\}$ induces an isomorphism in K-theory.*

The following Lemmas 4.4.12 to 4.4.15 will prove this theorem. Define the following intermediate algebras:

$$\begin{aligned}
R_0 &= \{f\colon \Delta^n \to A :\, f \restriction \partial\Delta^n = 0\},\\
R_1 &= R_0 \cap \{f :\, f(\Delta^n_0) \subseteq I_0\},\\
R_2 &= R_1 \cap \{f :\, f(\Delta^n_1) \subseteq I_1\},\\
&\;\;\vdots\\
B = R_{n+1} &= R_n \cap \{f :\, f(\Delta^n_n) \subseteq I_n\}.
\end{aligned}$$

To show that $B \to R_0$ is an isomorphism in K-theory, we show that each inclusion

$$\mathrm{incl}\colon R_k \to R_{k-1}$$

induces an isomorphism for $k \in \{n+1, n, \dots, 2, 1\}$.

Lemma 4.4.12. *For $k > k'$, the algebra R_k is a C*-ideal in $R_{k'}$.*

Proof. The additional restrictions to the set of functions in R_k over $R_{k'}$ forces the functions to map points into the given C*-ideals of A instead of anywhere in A. Because all $I_0, \dots, I_n$ are C*-ideals and the multiplication of functions happens pointwise, R_k becomes a C*-ideal in $R_{k'}$. □

Lemma 4.4.13. *The pair of topological spaces $(\Delta^n_k, \partial\Delta^n \cap \Delta^n_k)$ is homeomorphic to $(D^{n-1} \times [0,1], D^{n-1} \times \{1\})$.*

Proof. $\partial\Delta^n \cap \Delta^n_k$ is exactly the k-th face of the n-simplex. In Proposition 4.2.3, we have seen how $\Delta^n_n \cong \Delta^n$. In particular, for one-element sets $J = \{n\}$, the proof shows how Δ^n_J and Δ^n are diffeomorphic via a stretch by the factor $(n+1)$. This stretch has $\partial\Delta^n \cap \Delta^n_n$ as a set of fixed points. Δ^n_n and Δ^n_k are certainly homeomorphic.

The cake piece Δ^n_k is a compact, convex n-dimensional manifold within $\mathbb{R}^{n+1}$ and $\partial\Delta^n \cap \Delta^n_k$ is a convex $(n-1)$-dimensional hypersurface within $\partial\Delta^n_k$. Corollary 4.2.4 relates the simplices to the desired disks. □

Lemma 4.4.14. *For all $k \in \{1, 2, \dots, n+1\}$, the quotient R_{k-1}/R_k has trivial K-theory.*

Proof. The subset $\Delta^n - \Delta^n_{k-1}$ is open in the entire space Δ^n. With the convention that $\{0, 1, \dots, -1\}$ denotes the empty set and that $\{0, 1, \dots, 0\} = \{0\}$, we compute:

$$
\begin{aligned}
R_{k-1}/R_k &= \frac{\Big\{f\colon \Delta^n \to A :\ f(\partial\Delta^n) = 0 \text{ and } f(\Delta^n_j) \subseteq I_j \text{ for } j \in \{0, 1, \dots, k-2\}\Big\}}{\Big\{f\colon \Delta^n \to A :\ f(\partial\Delta^n) = 0 \text{ and } f(\Delta^n_j) \subseteq I_j \text{ for } j \in \{0, 1, \dots, k-1\}\Big\}} \\
&\cong \frac{\big\{f\colon \Delta^n_{k-1} \to A :\ f(\partial\Delta^n \cap \Delta^n_{k-1}) = 0\big\}}{\big\{f\colon \Delta^n_{k-1} \to A :\ f(\partial\Delta^n \cap \Delta^n_{k-1}) = 0 \text{ and } f(\Delta^n_{k-1}) \subseteq I_{k-1}\big\}} \\
&\cong \big\{f\colon \Delta^n_{k-1} \to A/I_{k-1} :\ f(\partial\Delta^n \cap \Delta^n_{k-1}) = 0\big\}.
\end{aligned}
$$

Because of Lemma 4.4.13, the quotient R_{k-1}/R_k is isomorphic to the algebra

$$R' = \big\{f\colon D^{n-1} \times [0,1] \to A/I_{k-1} :\ f\left(D^{n-1} \times \{1\}\right) = 0\big\}.$$

This is a contractible algebra: The homotopy $h\colon R' \times I \to R'$,

$$h(f,t)(x,t') = f(x, t' \cdot t),$$

defines a $*$-homomorphism for each fixed t. This construction is analogous to the proof of Proposition 2.1.9 for the contractibility of cone algebras. Since K-theory is homotopy invariant and $h(f, 0) = 0$ for all f, we conclude that $K_*(R_{k-1}/R_k) = K_*R' = 0$. $\square$

Lemma 4.4.15. incl: $R_k \to R_{k-1}$ *induces an isomorphism in K-theory.*

Proof. We examine the six-term exact sequence associated to the inclusion of the ideal.

$$\begin{tikzcd}
K_0 R_k \arrow[r, "\mathrm{incl}_0"] & K_0 R_{k-1} \arrow[r, "\mathrm{pr}_0"] & K_0(R_{k-1}/R_k) \arrow[d, "\partial_2 \circ \beta"] \\
K_1(R_{k-1}/R_k) \arrow[u, "\partial_1"] & K_1 R_{k-1} \arrow[l, "\mathrm{pr}_1"] & K_1 R_k. \arrow[l, "\mathrm{incl}_1"]
\end{tikzcd}$$

Since $K_p(R_{k-1}/R_k)$ vanishes for both even and odd p as shown in Lemma 4.4.14, incl_p is an isomorphism for all p. This also concludes the proof of Theorem 4.4.11, which is an n-fold application of these lemmas. $\square$

4.5 Review of developed theory

Let A be a C*-algebra that can be written as a finite sum $I_0 + I_1 + \cdots + I_n = A$ of closed two-sided ideals $I_j \subseteq A$. For the *cake pieces* $\Delta^n_j \subseteq \Delta^n$, we have constructed in Definition 4.3.1 a new C*-algebra B of functions into A,

$$\begin{aligned} B &= B(I_0, I_1, \dots, I_n) \\ &= \left\{ f\colon \Delta^n \to A = \sum_{j=0}^{n} I_j \,:\, f \text{ continuous},\ f \restriction \partial\Delta^n = 0,\ f(\Delta^n_j) \subseteq I_j \text{ for all } j \right\}. \end{aligned}$$

We worked with arbitrary index subsets $J \subseteq \{0, 1, \dots, n\}$. For such a J, we have defined

$$B_J = \big\{ f \in B \,:\, \text{for each } j' \notin J,\ f(\Delta^n_{j'}) = 0 \big\}.$$

These *cake algebras* B_J are closed two-sided ideals in $B = B_{\{0,1,\dots,n\}}$. For $p \in \mathbb{Z}$, we defined the *cake sums*

$$Q_p = \sum_{|J| \le p+1} B_J.$$

There are inclusions $Q_{p-1} \to Q_p$ and quotients of C*-ideals, Q_p/Q_{p-1}. This inclusion chain of C*-ideals is the decisive structure: We can later feed these algebras into the spectral sequence for ideal inclusions.

For $p \in \{0, 1, \dots, n\}$, Theorem 4.4.1 computes

$$K_{p+q}\left(Q_p/Q_{p-1}\right) \cong \bigoplus_{|J|=p+1} K_{q+n}\Big(\bigcap_{j \in J} I_j \Big).$$

This expression continues to hold for $p > n$ where $K_{p+q}(Q_p/Q_{p-1}) = 0$: There are no subsets J with $|J| = n+2$. But the expression fails for $p = -1$: To avoid intersections over the empty set, we must explicitly mention $K_{p+q}(Q_p/Q_{p-1}) = 0$ for $p = -1$ whenever we extend Theorem 4.4.1 to all $p \in \mathbb{Z}$.

Theorem 4.4.11 shows that the inclusion $B \to \{f : \Delta^n \to A : f \restriction \partial\Delta^n = 0\}$ induces an isomorphism in K-theory. The C*-algebra on the right-hand side is isomorphic to the n-fold suspension $S^n A$, therefore $K_p B \cong K_{p+n} A$. With this, we can go back to A, the algebra of original interest, even though the quotients Q_p encode information about B.

4.6 Main theorem

Theorem 4.6.1 (Spectral sequence for finite sums of C*-ideals). *Let A be a C*-algebra and $I_0, I_1, \ldots, I_n$ be $(n+1)$ C*-ideals in A with $I_0 + I_1 + \cdots + I_n = A$. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} \cong \begin{cases} \bigoplus\limits_{|J|=p+1} K_q\Big(\bigcap\limits_{j\in J} I_j\Big) & \text{for } 0 \le p \le n, \\ 0 & \text{for } p < 0 \text{ or } p > n. \end{cases} \tag{4.6.1.1}$$

This spectral sequence converges strongly to $K_* A$.

This spectral sequence is functorial for $*$-homomorphisms that preserve ideal decompositions; this will be the next theorem.

Proof of Theorem 4.6.1. For the C*-ideals $I_0, I_1, \ldots, I_n$, define cake algebras $B = B(I_0, I_1, \ldots, I_n)$ and cake sums $Q_p \subseteq B$ for $p \in \mathbb{Z}$ as reviewed in Section 4.5. We have $Q_p = B$ for $p \ge n$ by Remark 4.3.7. For the series of inclusions

$$\cdots = 0 = 0 \subseteq Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n = B = Q_{n+1} = Q_{n+2} = \cdots,$$

Theorem 3.1.1 gives a spectral sequence $\{\bar{E}^r_{p,q}, \bar{d}^r\}_{r,p,q}$ with

$$\bar{E}^1_{p,q} \cong K_{p+q}(Q_p/Q_{p-1}), \tag{4.6.1.2}$$

converging to $K_*(B)$. By Theorem 4.4.1, we can replace the K-theory of these quotients by the K-theory of a more immediate intersection,

$$\bar{E}^1_{p,q} \cong K_{p+q}(Q_p/Q_{p-1}) \cong \begin{cases} \bigoplus\limits_{|J|=p+1} K_{q+n}\Big(\bigcap\limits_{j\in J} I_j\Big) & \text{for } 0 \le p \le n, \\ 0 & \text{for } p < 0 \text{ or } p > n. \end{cases} \tag{4.6.1.3}$$

This spectral sequence converges strongly to $K_*\big(\overline{\bigcup_{p\in\mathbb{Z}} Q_p}\big) = K_*Q_n = K_*B$. By Theorem 4.4.11, $K_qB \cong K_{q+n}A$. To simplify, we will shift down by n all degrees in K-theory, both in 4.6.1.3 and in the expression for the convergence. As a result, we obtain a new spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with

$$E^1_{p,q} \cong \begin{cases} \bigoplus\limits_{|J|=p+1} K_q\Big(\bigcap\limits_{j\in J} I_j\Big) & \text{for } 0 \le p \le n, \\ 0 & \text{for } p < 0 \text{ or } p > n. \end{cases}$$

This spectral sequence converges strongly to K_*A. □

Theorem 4.6.2 (Functoriality of the spectral sequence from Theorem 4.6.1)**.** *The spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ from Theorem 4.6.1 is functorial with respect to $*$-homomorphisms that preserve $(n+1)$-fold ideal decompositions (Definition 4.1.1):*

For $n' \ge n$, let $A' = I'_0 + I'_1 + \cdots + I'_{n'}$ be a C-algebra and let $f\colon A \to A'$ be a $*$-homomorphism such that $f(I_j) \subseteq I'_j$ for all $j \le n$. Let $\{E^r_{p,q}, d^r\}_{r,p,q}$ be the spectral sequence from Theorem 4.6.1 that converges to A and let $\{\bar{E}^r_{p,q}, \bar{d}^r\}_{r,p,q}$ be the spectral sequence that converges to A'.*

Then f induces a morphism $\{f^r_{p,q}\}_{r,p,q}$ of spectral sequences (Definition 2.7.4) of bidegree $(0,0)$ with

$$f^r_{p,q}\colon E^r_{p,q} \to \bar{E}^r_{p,q}$$

for all $r \ge 1$ and p, $q \in \mathbb{Z}$ that commutes with the differentials and, in turn, induces $K_*f\colon K_*A \to K_*A'$ on the convergence targets.

Proof. All constructions since Section 4.3 on the level of C*-algebras have been functorial with respect to ideal decompositions: Cake algebras, sums of cake algebras, cones, suspensions.

Likewise, taking K-theory and constructing direct sums of K-theory groups for each nonempty $J \subseteq \{0, 1, \ldots, n\}$ are functorial in the same way. Thus $\{f^r_{p,q}\}_{r,p,q}$ exists and induces the correct morphism $\{f^\infty_{p,q}\}_{p,q}\colon \{E^\infty_{p,q}\} \to \{\bar{E}^\infty_{p,q}\}$, which induces the desired $K_*f\colon K_*A \to K_*A'$ on the convergence targets. □

5 Finite coarse excision

We generalize coarsely excisive pairs from Definition 2.6.2 to coarsely excisive covers of arbitrary cardinality. Later in this section, we will apply the spectral sequence from Theorem 4.6.1 for finitely many C*-ideals to C*-algebras obtained from finite coarsely excisive covers.

5.1 Coarsely excisive covers

Definition 5.1.1 (Coarsely excisive cover)**.** Let (X, d) be a coarse space. Let $\{X_\beta\}_{\beta\in\alpha}$ be a cover of X of arbitrary cardinality $|\alpha|$ such that each X_β is a closed subset in X.

The cover $\{X_\beta\}_{\beta\in\alpha}$ is called *coarsely excisive* if, for all nonempty finite sets $J \subseteq \alpha$ and for all distances $R > 0$, there exists a distance $S > 0$ such that the intersection of the $|J|$-many R-neighborhoods lies in the S-neighborhood of the $|J|$-fold intersection:

$$\bigcap_{j\in J} N_d(X_j, R) \subseteq N_d\Big(\bigcap_{j\in J} X_j, S\Big). \tag{5.1.1.1}$$

Remark 5.1.2. The distance S may be chosen depending both on R and the particular finite subcollection $\{X_j\}_{j\in J}$ at hand. It is not required that, given R, a single $S > 0$ satisfies 5.1.1.1 uniformly for all subcollections of the cover, or even only for all subcollections of a given cardinality $|J|$.

Remark 5.1.3. This is a straightforward generalization of coarsely excisive pairs. These covers will yield C*-ideals in the Roe algebra of X suitable for our spectral sequence. They behave as expected:

- A coarsely excisive pair of closed sets is a two-set coarsely excisive cover.
- The extra requirement that each X_β be closed in X does not affect any coarse properties. An arbitrary subset $Y \subseteq X$ and its closure $\overline{Y}$ have the same neighborhoods $N_d(Y, R) = N_d(\overline{Y}, R) \subseteq X$ for any given $R > 0$ because d is a metric.
- Provided $\varnothing$ is not a member of a coarsely excisive cover, all intersections of finitely many sets in the cover must contain at least one point. Assume $\{X_j\}_{j\in J}$ are $|J|$ sets in the cover with empty intersection. Then $\{X_j\}_{j\in J}$ does not satisfy 5.1.1.1. This can be seen in a similar way as Example 2.6.4: In a metric space, any two nonempty sets have finite distance from each other, and thus R can be chosen as the maximum of the pairwise distances among the X_j, producing a nonempty $\bigcap_{j\in J} N_d(X_j, R)$ that is not a subset of $N_d(\varnothing, S) = \varnothing$.

Definition 5.1.4 (Compatible coarsely excisive covers)**.** Let (X, d) and (X', d') be coarse spaces and $f\colon X \to X'$ a coarse map. Let $\{X_\beta\}_{\beta\in\alpha}$ be a finite or infinite coarsely excisive cover of X and $\{X'_\beta\}_{\beta\in\alpha'}$ a coarsely excisive cover of X' with $\alpha \subseteq \alpha'$ such that $f(X_\beta) \subseteq X'_\beta$ for every $\beta \in \alpha$.

Then the two covers are called *compatible with* f, or, when f is clear from the context, simply *compatible*.

5.2 Relative Roe algebras

Before we can use our spectral sequence for abstract C*-ideals on coarsely excisive covers, we have to shape our data accordingly – we don't have sums and intersections of C*-ideals, but rather unions and intersections of subspaces X_j. We connect the coarse world to the world of abstract C*-ideal intersections via relative Roe algebras and relative $\mathfrak{D}^*$-algebras.

Notation 5.2.1. In Sections 5.2 through 5.4, let (X, d) be a metric space. Fix a very ample representation $\varrho\colon \mathscr{C}_0X \to BH$ for a separable Hilbert space H to define $\mathfrak{D}^*X$ and $\mathfrak{C}^*X$.

Let $\{X_\beta\}_{\beta\in\alpha}$ be a coarsely excisive cover of (X, d) for an arbitrary index set α. Let $J \subseteq \alpha$ be a nonempty finite subset of indices.

Definition 5.2.2 (Support near a subset)**.** Let $Y \subseteq X$ be a subspace. An operator $T \in \mathfrak{C}^*X$ is *supported near* Y if there exists a constant $R > 0$ (that may depend on T) such that all $f \in \mathscr{C}_0X$ with $d(\operatorname{supp} f, Y) > R$ satisfy $\varrho(f)T = T\varrho(f) = 0 \in BH$.

Definition 5.2.3 (Relative Roe algebra)**.** For $Y \subseteq X$ closed, the *relative Roe algebra of* Y *in* X, denoted $\mathfrak{C}^*(Y \subseteq X)$, is defined as the norm closure of all operators in $\mathfrak{C}^*X$ that are supported near Y.

Definition 5.2.4 (Relative $\mathfrak{D}^*$ algebra)**.** Let $Y \subseteq X$ be a subspace. The C*-algebra $\mathscr{C}_0X$ contains $\mathscr{C}_0(X - Y)$ as a C*-ideal: The inclusion morphism $\mathscr{C}_0(X - Y) \to \mathscr{C}_0X$ extends functions by zero on Y.

Let $T \in \mathfrak{D}^*X$ be an operator such that, for all $f \in \mathscr{C}_0(X - Y) \subseteq \mathscr{C}_0X$, both $\varrho(f)T$ and $T\varrho(f)$ are compact operators. Then T is called *locally compact outside* Y.

For $Y \subseteq X$ closed, the *relative* $\mathfrak{D}^*$ *algebra* $\mathfrak{D}^*(Y \subseteq X)$ is the norm closure of all operators in $\mathfrak{D}^*X$ that are supported near Y and locally compact outside Y.

Remark 5.2.5. The algebra $\mathfrak{C}^*(Y \subseteq X)$ is a C*-ideal in $\mathfrak{C}^*X$; the algebra $\mathfrak{D}^*(Y \subseteq X)$ is a C*-ideal in $\mathfrak{D}^*X$. As subalgebras of $\mathfrak{C}^*X$ and $\mathfrak{D}^*X$, all operators in these ideals are locally compact or pseudocompact, respectively. Each operator has finite propagation or is a norm limit of operators with finite propagation.

For $Y = X$, we have $\mathfrak{C}^*(X \subseteq X) = \mathfrak{C}^*X$ and $\mathfrak{D}^*(X \subseteq X) = \mathfrak{D}^*X$. Operators in $\mathfrak{D}^*(Y \subseteq X)$ act with three strengths on represented functions: pseudocompactly on Y, in a locally compact way near Y, and trivially far away from Y.

Our guideline is [Roe96, Theorem 9.2]: If the coarsely excisive cover $\{X_0, X_1\}$ has only two sets, then there are isomorphisms $\mathfrak{C}^*(X_0 \subseteq X) + \mathfrak{C}^*(X_1 \subseteq X) \cong \mathfrak{C}^*X$ and $\mathfrak{C}^*(X_0 \subseteq X) \cap \mathfrak{C}^*(X_1 \subseteq X) \cong \mathfrak{C}^*(X_0 \cap X_1)$.

Besides for $\mathfrak{C}^*$, we prove a similar result for $\mathfrak{D}^*$ and $\mathfrak{Q}^* = \mathfrak{D}^*/\mathfrak{C}^*$; especially $\mathfrak{D}^*$ requires extra work over the proofs for $\mathfrak{C}^*$. Also, finite subsets $\{X_j\}_{j\in J}$ of arbitary coarsely excisive covers $\{X_\beta\}_{\beta\in\alpha}$ require more care than an inductive application of the result for two regions. Even when $\bigcup_{j\in J} X_j$ happens to be a small subset of X, our C*-algebras still arise from representations of $\mathscr{C}_0X$ in its entirety. Our proofs must safely ignore regions of X far away from $\bigcup_{j\in J} X_j$.

Theorem 5.2.6. *For all closed subsets $Y \subseteq X$ and K-theory degrees $s \in \mathbb{Z}$, there are natural isomorphisms of K-theory groups:*

$$K_s\mathfrak{C}^*(Y \subseteq X) \cong K_s\mathfrak{C}^*Y,$$
$$K_s\mathfrak{D}^*(Y \subseteq X) \cong K_s\mathfrak{D}^*Y.$$

Remark 5.2.7. The proof for $\mathfrak{C}^*$ in [Roe96, Theorem 9.2] passes from $Y \subseteq X$ to the coarsely equivalent $N_d(Y, n) \subseteq X$ for $n \in \mathbb{N}$ and takes the direct limit in K-theory along $n \to \infty$. The isomorphism is induced by the inclusion $Y \to X$. The construction for $\mathfrak{C}^*$ is natural with respect to coarse maps. The construction for $\mathfrak{D}^*$ is natural with respect to maps that are both coarse and continuous.

A proof for $\mathfrak{D}^*$ is in [Sie12, Proposition 3.8]. This construction calls for a very ample representation $\varrho\colon \mathscr{C}_0X \to BH$ as we have required in Notation 5.2.1, not merely for an ample representation.

Lemma 5.2.8. *The representation $\varrho\colon \mathscr{C}_0X \to BH$ can be extended to all Borel functions on X.*

Proof. This follows from [Dav96, Theorem II.1.1] and [Dav96, Proposition II.1.2]: As a nondegenerate representation of $\mathscr{C}_0X$, the given ϱ is equivalent to a direct sum $\bigoplus_{\gamma\in\Gamma} \varrho_\gamma$ of cyclic representations ϱ_γ, each unitarily equivalent to pointwise multiplication with a continuous function f_γ that depends only on the cyclic representation ϱ_γ, on the Hilbert space $L^2(X)$ of H-valued functions using a regular Borel probability measure. Now extend by pointwise multiplication.

The topological space in this construction is compact, but continuous functions on a compact space differ, as an algebra, from $\mathscr{C}_0X$ of a noncompact space X merely by the value at the extra point of the one-point compactification of X. $\square$

Lemma 5.2.9. *Let $\psi\colon X \to [0,1]$ be a Borel function. Extend ϱ to all Borel functions as in Lemma 5.2.8 such that $\varrho(\psi)$ makes sense. Let $Y \subseteq X$ be a closed subset. Let T be an operator in $\mathfrak{D}^*(Y \subseteq X)$. Then the following statments hold.*

- *We have $\varrho(\psi)T \in \mathfrak{D}^*(Y \subseteq X)$.*
- *Let R_{supp} be a distance constant for the support of T near Y; i.e., for functions $f \in \mathscr{C}_0 X$ with $d(\operatorname{supp} f, Y) > R_{\text{supp}}$, the operators $\varrho(f)T$ and $T\varrho(f)$ vanish. Then $\varrho(\psi)T$ is supported near Y with the same distance constant R_{supp}.*
- *If T has finite propagation with distance constant R_{prop}, then $\varrho(\psi)T$ has finite propagation with distance constant R_{prop}. (If T does not admit such an R_{prop}, then T is in the norm completion of operators that do.)*
- *If $T \in \mathfrak{C}^*(Y \subseteq X)$, then also $\varrho(\psi)T \in \mathfrak{C}^*(Y \subseteq X)$.*

The main idea is that the support of a pointwise product of functions $f\psi$ for $f \in \mathscr{C}_0 X$ must be a subset of $\operatorname{supp} f$ within X. Later in this section, ψ will be a function from a Borel partition of unity and $\operatorname{supp}(f\psi)$ may be much smaller than $\operatorname{supp} f$.

Proof of Lemma 5.2.9. We show the claim for all finite-propagation operators. The absolute algebras $\mathfrak{D}^*X$ and $\mathfrak{C}^*Y$ are norm completions of such operators; the general claim follows because the relative algebras $\mathfrak{D}^*(Y \subseteq X)$ and $\mathfrak{C}^*(Y \subseteq X)$ carry the same norm as subalgebras of the absolute algebras.

Let $R_{\text{prop}} > 0$ be a constant of finite propagation for T; i.e., for f, $g \in \mathscr{C}_0 X$ with $d(\operatorname{supp} f, \operatorname{supp} g) \geq R_{\text{prop}}$, the product $\varrho(f)T\varrho(g) \in BH$ is zero. For such f, $g \in \mathscr{C}_0 X$ with $d(\operatorname{supp} f, \operatorname{supp} g) \geq R_{\text{prop}}$, we have $\operatorname{supp}(f\psi) \subseteq \operatorname{supp} f$, therefore

$$d\big(\operatorname{supp}(f\psi), \operatorname{supp} g\big) \geq d(\operatorname{supp} f, \operatorname{supp} g) \geq R_{\text{prop}}$$

and $\varrho(f)\varrho(\psi)T\varrho(g) = \varrho(f\psi)T\varrho(g) = 0$. Thus $\varrho(\psi)T$ has finite propagation with the same constant R_{prop}.

Fix $R_{\text{supp}} > 0$ such that all $f \in \mathscr{C}_0 X$ with $d(\operatorname{supp} f, Y) > R$ satisfy $\varrho(f)T = T\varrho(f) = 0 \in BH$; such an R_{supp} exists because T is supported near Y. Given $f \in \mathscr{C}_0 X$ with $d(\operatorname{supp} f, Y) > R$, we have $\operatorname{supp}(f\psi) \subseteq \operatorname{supp} f$, thus $\varrho(f)\varrho(\psi)T = \varrho(f\psi)T = 0$. Furthermore, $\varrho(\psi)T\varrho(f) = 0$ because $T\varrho(f) = 0$. Thus $\varrho(\psi)T$ is supported near Y with the same distance constant R_{supp}.

For pseudocompactness, given $f \in \mathscr{C}_0 X$, we must show that the following operator is compact:

$$\begin{aligned}\varrho(f)\varrho(\psi)T - \varrho(\psi)T\varrho(f) &= \varrho(f\psi)T - \varrho(\psi)T\varrho(f)\\ &= \varrho(\psi f)T - \varrho(\psi)T\varrho(f)\\ &= \varrho(\psi)\underbrace{\big(\varrho(f)T - T\varrho(f)\big)}_{\text{compact since } T\in\mathfrak{D}^*X},\end{aligned}$$

which is compact as a product with a compact operator. Thus $\varrho(\psi)T$ is pseudolocal.

For local compactness of $\varrho(\psi)T$ outside Y, let f be a function in $\mathscr{C}_0(X-Y)$ Because T is already locally compact outside Y, $\varrho(f)T$ and $T\varrho(f)$ are compact operators. Then $\varrho(f)\varrho(\psi)T = \varrho(\psi)\varrho(f)T$ and $\varrho(\psi)T\varrho(f)$ are also compact. Thus $\varrho(\psi)T$ is locally compact outside Y.

Additionally, if $T \in \mathfrak{C}^*(Y \subseteq X)$, the same argument applied to arbitrary $f \in \mathscr{C}_0X$ shows that $\varrho(f)\varrho(\psi)T$ and $\varrho(\psi)T\varrho(f)$ are compact for all $f \in \mathscr{C}_0X$. Thus $\varrho(\psi)T$ is locally compact if T is. □

5.3 Intersections of relative algebras

Notation 5.3.1. Throughout Section 5.3, in addition to Notation 5.2.1 that defines the finite nonempty subset $\{X_j\}_{j\in J}$ of the coarsely excisive cover $\{X_\beta\}_{\beta\in\alpha}$, write

$$Z = \bigcap_{j\in J} X_j.$$

Lemma 5.3.2. *Let $\mathfrak{F}^*$ denote either the functor $\mathfrak{C}^*$ or $\mathfrak{D}^*$. Then*

$$\mathfrak{F}^*(Z \subseteq X) \subseteq \bigcap_{j\in J} \mathfrak{F}^*(X_j \subseteq X).$$

Proof. For $T \in \mathfrak{F}^*(Z \subseteq X)$, there exists $R > 0$ such that $\varrho(f)T = T\varrho(f) = 0$ for all $f \in \mathscr{C}_0X$ with $d(\operatorname{supp} f, Z) > R$ by definition of T being supported near Z.

Given $j \in J$, let $f_j \in \mathscr{C}_0X$ be a function with $d(\operatorname{supp} f_j, X_j) > R$. Then $d(\operatorname{supp} f_j, Z) > R$ because $Z \subseteq X_j$, therefore T is supported near X_j. This holds for all $j \in J$. For $\mathfrak{F}^* = \mathfrak{C}^*$, this finishes the proof: T is in $\mathfrak{C}^*(X_j \subseteq X)$ for all $j \in J$.

For $\mathfrak{F}^* = \mathfrak{D}^*$, we must show, in addition, that T is locally compact outside X_j for the given $j \in J$; this holds because $Z \subseteq X_j$ and $T \in \mathfrak{D}^*(Z \subseteq X)$ is locally compact outside Z. □

Notation 5.3.3. For a subset $Y \subseteq X$, let

$$\chi(Y)\colon X \to \{0,1\}$$

denote the charateristic function of Y on X; i.e., $\chi(Y)(x) = 1$ if and only if $x \in Y$.

Lemma 5.3.4. *For $\mathfrak{F}^* = \mathfrak{C}^*$ or $\mathfrak{F}^* = \mathfrak{D}^*$, we have*

$$\mathfrak{F}^*(Z \subseteq X) \supseteq \bigcap_{j \in J} \mathfrak{F}^*(X_j \subseteq X);$$

i.e., the inclusion from Lemma 5.3.2 is an equality of sets.

Proof. Fix $T \in \bigcap_{j \in J} \mathfrak{F}^*(X_j \subseteq X)$. We will show that $T \in \mathfrak{F}^*(Z \subseteq X)$.

Support of T near Z. Since T is supported near X_j for all $j \in J$, there are constants $R_j > 0$ such that whenever $f \in \mathscr{C}_0 X$ satisfies $d(\operatorname{supp} f, X_j) > R_j$ for at least one $j \in J$, then $\varrho(f)T = T\varrho(f) = 0$. (It is not necessary that $d(\operatorname{supp} f, X_j) > R_j$ holds for all $j \in J$. It is enough if this holds for one $j \in J$ because "support near a subset" states what happens on the complement of the subset, not on the subset itself.)

The cover $\{X_\beta\}_{\beta \in \alpha}$ is coarsely excisive. In particular, for the constant $R = (\max_{j \in J} R_j) + 1$ and for the chosen finite index set J, there exists $S > 0$ with $\bigcap_{j \in J} N_d(X_j, R) \subseteq N_d(Z, S)$ or, reformulating with complement sets,

$$\big(X - N_d(Z, S)\big) \subseteq \bigcup_{j \in J} \big(X - N_d(X_j, R)\big).$$

This constant S depends on T and the X_j, but not on any function in $\mathscr{C}_0 X$. To finish the proof, choose $f \in \mathscr{C}_0 X$ with $d(\operatorname{supp} f, Z) > S + 1$. We must show that $\varrho(f)T = T\varrho(f) = 0$.

The support of this f lies within $X - N_d(Z, S)$, thus there exists a $j \in J$ with $\operatorname{supp} f \subseteq \big(X - N_d(X_j, R)\big)$ and therefore $d(\operatorname{supp} f, X_j) \geq R > R_j$. Because T is supported near X_j, we conclude that $\varrho(f)T = T\varrho(f) = 0$ as desired. Thus T is supported near Z.

Local compactness of T outside Z. If $\mathfrak{F}^* = \mathfrak{C}^*$, the proof is finished because T is locally compact everywhere in X.

If $\mathfrak{F}^* = \mathfrak{D}^*$, we must show that T is locally compact outside Z. Fix a function $g \in \mathscr{C}_0(X - Z) \subseteq \mathscr{C}_0 X$. We must show that $\varrho(g)T$ and $T\varrho(g)$ are compact.

The support of g may still overlap each X_j. To remedy this, decompose X into $2^{|J|}$ regions: Given $L \subseteq J$, write

$$\begin{aligned} Y_L &= \Big(\bigcap_{j \in L} X_j\Big) - \bigcup_{j \notin L} X_j \\ &= \{x \in X : x \in X_j \text{ if and only if } j \in L\}. \end{aligned}$$

For all $L \subseteq J$, the set Y_L is a Borel set as a finite union, intersection, and set difference of closed sets X_j. For $L \neq L' \subseteq J$, the regions Y_L and $Y_{L'}$ are disjoint. Furthermore,

$$X = \bigcup_{L \subseteq J} Y_L, \qquad Z = Y_J = \bigcap_{j \in J} X_j.$$

Decompose g into $2^{|J|}$ Borel functions on X by multiplying with indicator functions:

$$g = \sum_{L \subseteq J} \chi(Y_L) g.$$

Extend ϱ from $\mathscr{C}_0 X$ to all Borel functions on X. Lemma 5.2.9 shows that the operators $\varrho\big(\chi(Y_L)\big)T$ and $T\varrho\big(\chi(Y_L)\big)$ remain in $\bigcap_{j \in J} \mathfrak{D}^*(X_j \subseteq X)$ because the Y_L are Borel sets.

For each $L \subseteq J$, examine $\chi(Y_L)g$: For $L = J$, we have $\chi(Y_J)g = \chi(Z)g = 0$ since $g \in \mathscr{C}_0(X - Z)$. Both $\varrho\big(\chi(Y_J)g\big)T$ and $T\varrho\big(\chi(Y_J)g\big)$ are the zero operator, thus compact. For $L \neq J$, fix an index $j \in J - L$. Then $\chi(Y_L)g$ vanishes on X_j because Y_L contains no points from X_j by definition. We have $T \in \mathfrak{D}^*(X_j \subseteq X)$, therefore T is locally compact outside X_j, making both $\varrho\big(\chi(Y_L)g\big)T$ and $T\varrho\big(\chi(Y_L)g\big)$ compact.

This shows that the decomposition $g = \sum_{L \subseteq J} \chi(Y_L)g$ allows $\varrho(g)T$ and $T\varrho(g)$ to be written as finite sums of $2^{|J|}$ compact operators each. Such sums are compact. Thus T is locally compact outside Z. □

Proposition 5.3.5. *Let $s \in \mathbb{Z}$ be a degree for K-theory. Then for the nonempty finite index set $J \subseteq \alpha$, there are natural isomorphisms*

$$K_s \mathfrak{C}^*\Big(\bigcap_{j \in J} X_j\Big) \cong K_s\Big(\bigcap_{j \in J} \mathfrak{C}^*(X_j \subseteq X)\Big),$$
$$K_s \mathfrak{D}^*\Big(\bigcap_{j \in J} X_j\Big) \cong K_s\Big(\bigcap_{j \in J} \mathfrak{D}^*(X_j \subseteq X)\Big).$$

Proof. Combine Lemma 5.3.2 with Lemma 5.3.4 for $Z = \bigcap_{j \in J} X_j$:

$$\mathfrak{C}^*(Z \subseteq X) = \bigcap_{j \in J} \mathfrak{C}^*(X_j \subseteq X),$$
$$\mathfrak{D}^*(Z \subseteq X) = \bigcap_{j \in J} \mathfrak{D}^*(X_j \subseteq X).$$

Theorem 5.2.6 relates the K-theory of relative algebras with the K-theory of absolute algebras. This yields the claimed isomorphisms. □

Proposition 5.3.6. *Write $\mathfrak{Q}^* X = \mathfrak{D}^* X / \mathfrak{C}^* X$ as in Notation 2.4.13. Let $s \in \mathbb{Z}$ be a* *degree in K-theory. Then there is a natural isomorphism*

$$K_s\mathfrak{Q}^*\Big(\bigcap_{j\in J} X_j\Big) \cong K_s\Big(\bigcap_{j\in J}\mathfrak{Q}^*(X_j \subseteq X)\Big),$$

Proof. For closed $Y \subseteq X$, consider the following commutative diagram. Its rows are long exact sequences in K-theory. Its vertical isomorphisms are induced by the inclusion of metric spaces $Y \subseteq X$ as in Remark 5.2.7. The third vertical morphism is well-defined as follows because the rows are exact: For an operator $T \in \mathfrak{D}^*(Y \subseteq X)$ whose K-theory class $[T]$ maps to $[T'] \in \mathfrak{D}^*Y$ under the second vertical morphism, the third morphism maps $[T] + K_s\mathfrak{C}^*(Y \subseteq X)$ to $K_s\big([T'] + K_s\mathfrak{C}^*Y\big)$.

$$\begin{tikzcd}
\cdots \arrow[r] & K_s\mathfrak{C}^*(Y \subseteq X) \arrow[r] \arrow[d, "\cong"'] & K_s\mathfrak{D}^*(Y \subseteq X) \arrow[r] \arrow[d, "\cong"'] & \frac{K_s\mathfrak{D}^*(Y\subseteq X)}{K_s\mathfrak{C}^*(Y\subseteq X)} \arrow[r] \arrow[d] & K_{s-1}\mathfrak{C}^*(Y \subseteq X) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & K_s\mathfrak{C}^*Y \arrow[r] & K_s\mathfrak{D}^*Y \arrow[r] & K_s\mathfrak{Q}^*Y \arrow[r] & K_{s-1}\mathfrak{C}^*Y \arrow[r] & \cdots .
\end{tikzcd}$$

All squares commute because $\mathfrak{C}^*Y \to \mathfrak{D}^*Y$ is a C*-ideal inclusion and because the vertical isomorphisms arose from taking natural direct limits.

By the five lemma, the vertical arrow to $K_s\mathfrak{Q}^*Y$ must also be an isomorphism. It is natural by construction. With the isomorphisms from Propositions 5.3.5, we have

$$\begin{aligned}
K_s\mathfrak{Q}^*\Big(\bigcap_{j\in J} X_j\Big) &\cong \frac{K_s\mathfrak{D}^*\big(\bigcap_{j\in J} X_j\big)}{K_s\mathfrak{C}^*\big(\bigcap_{j\in J} X_j\big)} \cong \frac{K_s\big(\bigcap_{j\in J}\mathfrak{D}^*(X_j\subseteq X)\big)}{K_s\big(\bigcap_{j\in J}\mathfrak{C}^*(X_j\subseteq X)\big)} \\
&\cong K_s\Big(\bigcap_{j\in J}\mathfrak{Q}^*(X_j \subseteq X)\Big).
\end{aligned}$$ □

5.4 Sums of relative algebras

Notation 5.4.1. Throughout Section 5.4, in addition to Notation 5.2.1, write

$$Z = \bigcup_{j\in J} X_j.$$

For a subset $Y \subseteq X$, again, denote the indicator function by $\chi(Y)\colon X \to \{0,1\}$.

Lemma 5.4.2. *Let $\mathfrak{F}^*$ be either the functor $\mathfrak{C}^*$ or $\mathfrak{D}^*$. Then*

$$\mathfrak{F}^*(Z \subseteq X) \subseteq \sum_{j\in J}\mathfrak{F}^*(X_j \subseteq X).$$

Proof. By definition, $\mathfrak{F}^*(Y \subseteq X)$ and $\mathfrak{F}^*X$ for closed subsets $Y \subseteq X$ are norm completions of finite-propagation operator algebras. It suffices to check the inclusion for finite-propagation operators in $\mathfrak{F}^*(Z \subseteq X)$; passing to norm completions will then prove the claim.

In this light, let $T \in \mathfrak{C}^*(Z \subseteq X)$ be an operator with finite propagation. We will construct operators T_j for $j \in J$ such that $T_j \in \mathfrak{F}^*(X_j \subseteq X)$ and $\sum_{j \in J} T_j = T$.

Because T has finite propagation, find $R_{\text{prop}} > 0$ such that $\varrho(f) T \varrho(g) = 0 \in BH$ whenever $f,\, g \in \mathscr{C}_0 X$ satisfy $d(\operatorname{supp} f, \operatorname{supp} g) \geq R_{\text{prop}}$. Find $R_{\text{supp}} > 0$ such that T is supported in the R-neighborhood of Z.

Cover $Z = \bigcup_{j \in J} X_j$ by the following sets Y_j for $j \in J$ and their union Y:

$$\begin{aligned} Y_j &= N_d(X_j, R_{\text{prop}} + R_{\text{supp}} + 1) = \{x \in X : d(x, X_j) \leq R_{\text{prop}} + R_{\text{supp}} + 1\}, \\ Y &= \bigcup_{j \in J} Y_j. \end{aligned}$$

Each Y_j is closed in X. Certainly, T is supported in Y, again a closed set.

Choose a linear order $\prec$ on the finite set J. Define a partition $\{P_j\}_{j \in J}$ of Y via

$$\begin{aligned} P_j = &\big\{x \in Z : x \in X_j \text{ and there are no } j' \prec j \text{ with } x \in X_{j'}\big\} \\ &\cup \big\{x \in Y - Z : x \in Y_j \text{ and there are no } j' \prec j \text{ with } x \in Y_{j'}\big\}; \end{aligned}$$

this is a partition of Y because, given either $x \in Z$ or $x \in Y - Z$, exactly one P_j is eligible to contain x. Furthermore, each P_j is a Borel set because it may be written as a finite union, intersection, and difference of Borel sets $X_{j'}$ and $Y_{j'}$.

Extend the representation $\varrho\colon \mathscr{C}_0 X \to BH$ to the Borel functions of X according to Lemma 5.2.8. For all $j \in J$, define operators $T_j \in BH$ by

$$\widetilde{T} = \frac{\varrho\big(\chi(X - Y)\big) T}{|J|}, \qquad T_j = \varrho\big(\chi(P_j)\big) T + \widetilde{T}.$$

By Lemma 5.2.9, $\widetilde{T}$ and all T_j are in $\mathfrak{F}^*(Z \subseteq X)$. The T_j sum to

$$\sum_{j \in J} T_j = \sum_{j \in J} \varrho\big(\chi(P_j)\big) T + \sum_{j \in J} \frac{\varrho\big(\chi(X - Y)\big) T}{|J|} = \varrho\big(\underbrace{\chi(Y) + \chi(X - Y)}_{= 1 \text{ on } X}\big) T = T. \tag{5.4.2.1}$$

The summand $\widetilde{T}$ of T_j merely clarifies $\sum_{j \in J} T_j = T$; it has no deeper meaning. For all functions $g \in \mathscr{C}_0 X$, the products $\varrho(g)\widetilde{T}$ and $\widetilde{T}\varrho(g)$ vanish because $\chi(X - Y)$ is supported further than $R_{\text{prop}} + R_{\text{supp}}$ away from Z, whereas $\widetilde{T} \in \mathfrak{F}^*(Z \subseteq X)$ has the same propagation constant R_{prop} and support distance constant R_{supp} as T by Lemma 5.2.9.

Support of T_j near X_j. For a given $j \in J$, let $f \in \mathscr{C}_0 X$ have support far enough away from X_j:

$$d(\operatorname{supp} f, X_j) > R_{\text{prop}} + R_{\text{supp}} + 1.$$

We will show $\varrho(f)T_j = T_j\varrho(f) = 0$. For $\varrho(f)T_j = 0$, we have

$$\varrho(f)T_j = \varrho\big(f\chi(P_j)\big)T + \underbrace{\varrho(f)\widetilde{T}}_{=0} = 0 \tag{5.4.2.2}$$

because the pointwise product $f\chi(P_j)$ is zero in $\mathscr{C}_0 X$: The function f is supported more than $R_{\text{prop}} + R_{\text{supp}} + 1$ away from X_j, but $P_j \subseteq Y_j = N_d(X_j, R_{\text{prop}} + R_{\text{supp}} + 1)$. For $T_j\varrho(f) = 0$, we similarly show that

$$T_j\varrho(f) = \varrho\big(\chi(P_j)\big)T\varrho(f) + \underbrace{\widetilde{T}\varrho(f)}_{=0} = 0; \tag{5.4.2.3}$$

to see this, observe that $d(X_j, X - P_j) \leq R_{\text{supp}} + 1$ by construction of P_j and $d(X_j, \operatorname{supp} f) > R_{\text{prop}} + R_{\text{supp}} + 1$ by the choice of f. The difference between these two values is more than R_{prop}, the propagation constant of T, thus $\varrho\big(\chi(P_j)\big)T\varrho(f) = 0$.

Local compactness of T_j outside X_j. Let f be a function in $\mathscr{C}_0(X - X_j) \subseteq \mathscr{C}_0 X$. We will show that $\varrho(f)T_j$ and $T_j\varrho(f)$ are compact.

Recall that $\varrho(f)T_j = \varrho\big(f\chi(P_j)\big)T + 0$, thus it suffices to examine the pointwise product $f\chi(P_j)$: It may assume nonzero values only in $(Y_j - Z) \cup X_j$ by definition of P_j. Because $X_j \subseteq Z$, each point from P_j falls either into X_j or into $Y - Z$. We may decompose $f\chi(P_j)$ as

$$f\chi(P_j) = f\chi(P_j \cap X_j) + f\chi(P_j - Z).$$

The left summand is the zero function because f vanishes on X_j. The right summand may be nonzero, but vanishes on Z. Outside Z, the original T is locally compact, therefore $\varrho\big(f\chi(P_j - Z)\big)T$ is compact. Since $f\chi(P_j - Z) = f\chi(P_j)$ and furthermore $\varrho\big(f\chi(P_j)\big)T = \varrho(f)T_j$, the desired operator $\varrho(f)T_j$ is compact.

The difference $\varrho(f)T_j - T_j\varrho(f)$ is compact because $T_j \in \mathfrak{D}^*(Z \subseteq X)$ is pseudocompact. With $\varrho(f)T_j$ already proven compact, $T_j\varrho(f)$ must be compact, too.

Summary. The claim follows from 5.4.2.1, 5.4.2.2, 5.4.2.3, and from the local compactness of T_j outside Z: We have decomposed T into a sum $\sum_{j\in J} T_j$ with each T_j in $\mathfrak{F}^*(X_j \subseteq X)$. $\square$

Lemma 5.4.3. *For $\mathfrak{F}^* = \mathfrak{C}^*$ or $\mathfrak{F}^* = \mathfrak{D}^*$, we have*

$$\mathfrak{F}^*(Z \subseteq X) \supseteq \sum_{j \in J} \mathfrak{F}^*(X_j \subseteq X);$$

i.e., the inclusion from Lemma 5.4.2 is an equality of sets.

Proof. For each $j \in J$, let T_j be an operator in $\mathfrak{F}^*(X_j \subseteq X)$ such that T_j is supported in an R_j-neighborhood of X_j. Define $T = \sum_{j \in J} T_j$. This T is supported in the $(\max_{j \in J} R_j)$-neighborhood of $Z = \bigcup_{j \in J} X_j$.

For $\mathfrak{F}^* = \mathfrak{D}^*$, given $f \in \mathscr{C}_0(X - Z) \subseteq \mathscr{C}_0 X$ and $j \in J$, we know that f is also in $\mathscr{C}_0(X - X_j)$. The operators $\varrho(f)T_j$ and $T_j\varrho(f)$ are compact since $T_j \in \mathfrak{F}^*(X_j \subseteq X)$. The finite sums $\varrho(f)T$ and $T\varrho(f)$ of compact operators are again compact. □

Proposition 5.4.4. *For all $s \in \mathbb{Z}$, there are natural isomorphisms*

$$K_s\mathfrak{C}^*\Big(\bigcup_{j \in J} X_j\Big) \cong K_s\Big(\sum_{j \in J} \mathfrak{C}^*(X_j \subseteq X)\Big),$$
$$K_s\mathfrak{D}^*\Big(\bigcup_{j \in J} X_j\Big) \cong K_s\Big(\sum_{j \in J} \mathfrak{D}^*(X_j \subseteq X)\Big).$$

Proof. Combine Lemma 5.4.2 with Lemma 5.4.3 for $Z = \bigcup_{j \in J} X_j$:

$$\mathfrak{C}^*(Z \subseteq X) = \sum_{j \in J} \mathfrak{C}^*(X_j \subseteq X),$$
$$\mathfrak{D}^*(Z \subseteq X) = \sum_{j \in J} \mathfrak{D}^*(X_j \subseteq X).$$

Theorem 5.2.6 yields the claimed isomorphisms. □

Proposition 5.4.5. *Let $s \in \mathbb{Z}$ be a degree in K-theory. There is a natural isomorphism*

$$K_s\mathfrak{Q}^*\Big(\bigcup_{j \in J} X_j\Big) \cong K_s\Big(\sum_{j \in J} \mathfrak{Q}^*(X_j \subseteq X)\Big).$$

Proof. In the proof of Proposition 5.3.6, we constructed a natural isomorphism for closed subsets $Y \subseteq X$,

$$\frac{K_s\mathfrak{D}^*(Y \subseteq X)}{K_s\mathfrak{C}^*(Y \subseteq X)} \xrightarrow{\cong} \mathfrak{Q}^*Y.$$

Combine this isomorphism with the natural isomorphisms from Proposition 5.4.4:

$$K_s\mathfrak{Q}^*\Big(\bigcup_{j\in J} X_j\Big) \cong \frac{K_s\mathfrak{D}^*\big(\bigcup_{j\in J} X_j\big)}{K_s\mathfrak{C}^*\big(\bigcup_{j\in J} X_j\big)} \cong \frac{K_s\big(\sum_{j\in J}\mathfrak{D}^*(X_j\subseteq X)\big)}{K_s\big(\sum_{j\in J}\mathfrak{C}^*(X_j\subseteq X)\big)}$$
$$\cong K_s\Big(\sum_{j\in J}\mathfrak{Q}^*(X_j\subseteq X)\Big). \qquad \square$$

5.5 Main theorem

We may summarize Propositions 5.3.5, 5.3.6, 5.4.4, and 5.4.5 in a single theorem.

Theorem 5.5.1. *Let (X,d) be a metric space. Let $\{X_\beta\}_{\beta\in\alpha}$ be a finite or infinite coarsely excisive cover of X and let $J\subseteq\alpha$ be a finite nonempty subset.*

Let $\mathfrak{F}^*$ be either the functor $\mathfrak{C}^*$ from the coarse category to $\underline{\text{C*A}}$ or one of the functors $\mathfrak{D}^*$ or $\mathfrak{Q}^*$ from the coarse-continuous category to $\underline{\text{C*A}}$. Let s be a degree in K-theory. Then

$$K_s\mathfrak{F}^*\Big(\bigcap_{j\in J} X_j\Big) \cong K_s\Big(\bigcap_{j\in J}\mathfrak{F}^*(X_j\subseteq X)\Big),$$
$$K_s\mathfrak{F}^*\Big(\bigcup_{j\in J} X_j\Big) \cong K_s\Big(\sum_{j\in J}\mathfrak{F}^*(X_j\subseteq X)\Big).$$

These isomorphisms are natural with respect to morphisms (coarse maps for $\mathfrak{C}^*$, or coarse and continuous maps for $\mathfrak{D}^*$ and $\mathfrak{Q}^*$) to other coarse spaces with compatible coarsely excisive covers (Definition 5.1.4).

When the cover $\{X_\beta\}_{\beta\in\alpha}$ has a finite index set α, the algebras become suitable for our spectral sequence for finite ideal inclusions.

Theorem 5.5.2. *Let (X,d) be a metric space with a finite coarsely excisive cover $\{X_\beta\}_{\beta\in\alpha}$. Let $\mathfrak{F}^*$ be either the functor $\mathfrak{C}^*$ from the coarse category to $\underline{\text{C*A}}$ or one of the functors $\mathfrak{D}^*$ or $\mathfrak{Q}^*$ from the coarse-continuous category to $\underline{\text{C*A}}$. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} \cong \begin{cases} \displaystyle\bigoplus_{|J|=p+1} K_q\mathfrak{F}^*\Big(\bigcap_{j\in J} X_j\Big) & \text{for } 0\le p<|\alpha|,\\ 0 & \text{for } p<0 \text{ or } p\ge|\alpha|,\end{cases}$$

where J ranges over all nonempty subsets of α. This spectral sequence converges strongly to $K_*\mathfrak{F}^*X$ and is functorial with respect to morphisms (coarse maps for $\mathfrak{C}^*$, or coarse and continuous maps for $\mathfrak{D}^*$ and $\mathfrak{Q}^*$) to other coarse spaces with compatible coarsely excisive covers (Definition 5.1.4).

Proof. Apply the spectral sequence from Theorem 4.6.1 about finite sums of abstract C*-algebras to the algebras from Theorem 5.5.1 for coarse spaces.

Both properties from Theorem 5.5.1 are required here: The intersection property guarantees that the E^1-term looks as stated. The sum property of the relative C*-algebras guarantees that the spectral sequence converges to the non-relative C*-algebra of the entire space.

Functoriality of the spectral sequence follows from functoriality of the spectral sequence from Theorem 4.6.1 for finite sums of abstract C*-ideals and from the naturality of the isomorphisms in Theorem 5.5.1. □

5.6 Application: $K_*\mathfrak{C}^*\mathbb{R}^n$

Let d_1 and d_∞ denote the usual 1-metric and sup-metric on $\mathbb{R}^n$: For x, $y \in \mathbb{R}^n$, we have

$$d_1(x,y) = \sum_{j<n} |x_j - y_j|, \qquad d_\infty(x,y) = \max_{j<n} |x_j - y_j|.$$

Theorem 5.6.1. *For the n-dimensional Euclidean space $\mathbb{R}^n$, metrized either with the 1-metric d_1 or the* sup*-metric d_∞, the Roe algebra has the following K-theory:*

$$K_s\mathfrak{C}^*\mathbb{R}^n = \begin{cases} \mathbb{Z} & \text{for } s-n \text{ even,} \\ 0 & \text{for } s-n \text{ odd.} \end{cases}$$

This is known, but we reprove this with our K-theory spectral sequence for finite coarsely excisive covers.

Remark 5.6.2. Towards the end of Section 5.6, some claims and proofs might look like straightforward geometry of $\mathbb{R}^n$. In particular, since d_1 and d_∞ are equivalent metrics on $\mathbb{R}^n$, they must induce equivalent coarse structures; it would suffice to look at only one of them.

Nonetheless, we will conduct these proofs in detail for both d_1 and d_∞ because these results will serve as lemmas for Section 7.3 to compute the K-theory of a C*-ideal of $\mathfrak{C}^*\mathbb{Z}^\infty$ under different metrics.

Definition 5.6.3 (Flasque)**.** Let (X, d) be a metric space. X is *flasque* if there is a coarse map $f\colon X \to X$ satisfying the following conditions:

- The map f is coarsely equivalent to $\mathrm{id}(X)$.
- For all $K \subseteq X$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $f^n(X) \cap K = \varnothing$.
- The powers of f are uniformly coarse: For all $R > 0$, there exists $S > 0$ that, for all $n \in \mathbb{N}$ at once and x, $y \in X$ with $d(x,y) \leq R$, we have $d(f^n x, f^n y) \leq S$.

Lemma 5.6.4. *For a metric space X, the product $X \times \mathbb{R}_{\geq 0}$ is flasque.*

Proof. Consider the self-map f on $X \times \mathbb{R}_{\geq 0}$ with $f(x,t) = (x, t+1)$. Shifting points by a constant distance, f is coarsely equivalent to the identity, yet the powers f^n eventually shift points out of any given bounded set. As an isometry, f is uniformly coarse: Choose $S = R$ for the third condition in Definition 5.6.3. □

Proposition 5.6.5. *Let X be a flasque space. Then $K_*\mathfrak{C}^*X = 0$.*

Proposition 5.6.5 is proven in [Roe88, Proposition 9.4]. To prove Theorem 5.6.1 about $K_*\mathfrak{C}^*\mathbb{R}^n$ with a coarsely excisive cover of $\mathbb{R}^n$, it is helpful to have many flasque intersections.

Definition 5.6.6 (Block decomposition of $\mathbb{R}^n$)**.** Cover $\mathbb{R}^n$ with $n+1$ overlapping subsets, or *blocks*, X_0, X_1, …, X_n, where

$$X_j = \begin{cases}]{-\infty}, 0] \times \mathbb{R}^{n-1} & \text{for } j = 0, \\ [0, \infty[^i \times]{-\infty}, 0] \times \mathbb{R}^{n-j-1} & \text{for } 0 < j < n, \\ [0, \infty[^n & \text{for } j = n. \end{cases}$$

Example 5.6.7. In the simplest case, $\mathbb{R}^1$ is covered with two overlapping rays, one extending into either direction. $\mathbb{R}^2$ is covered with three pieces, a left-hand half-space X_0, a bottom-right-hand quadrant X_1, and a top-right-hand quadrant X_2, as in Figure 5.6.8:

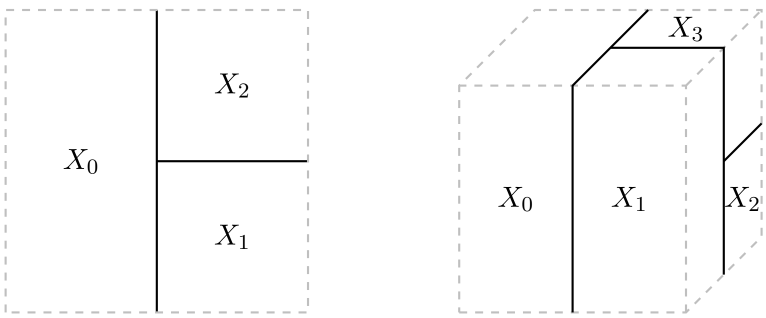

Figure 5.6.8: Decomposition of $\mathbb{R}^2$ into 3 blocks and of $\mathbb{R}^3$ into 4 blocks

Remark 5.6.9. In the block decomposition $\mathbb{R}^n = X_0 \cup X_1 \cup \cdots \cup X_n$, each X_j contains at least one flasque factor, therefore $K_*\mathfrak{C}^*X_j = 0$. Likewise, intersecting fewer than all $n+1$ segments produces a flasque intersection with trivial K-theory of the Roe algebra. Only the $(n+1)$-fold intersection is not flasque; it is the compact one-point set. Its Roe algebra has K-theory $\mathbb{Z}$ in even degrees and zero in odd degrees.

Definition 5.6.10 (Blocky subset of $\mathbb{R}^n$)**.** Let X be a subset of $\mathbb{R}^n$. We call X *blocky* if both of these conditions hold:

- The origin $0 \in \mathbb{R}^n$ is part of X.
- For all $x \in X$, all n coordinates x_j of x, and all $\lambda \geq 0$, varying x_j by λ doesn't leave X; i.e., this point is a part of X:

$$(x_0,\ x_1,\ \ldots,\ x_{j-1},\ \lambda x_j,\ x_{j+1},\ \ldots,\ x_{n-1})$$

Example 5.6.11. Blocky subsets of $\mathbb{R}^n$ are conical, but not all conical subsets are blocky. Consider the upper-right quadrant in $\mathbb{R}^2$: This is blocky. It remains conical, but not blocky, after rotating around the origin by an eighth-turn.

Remark 5.6.12. For a nonempty collection of blocky subsets $\{X_\beta\}_{\beta\in\alpha}$ of $\mathbb{R}^n$, the intersection $\bigcap_{\beta\in\alpha} X_\beta$ is again blocky.

All sets of the block decomposition $\{X_j\}_{j\leq n}$ of $\mathbb{R}^n$ and all their intersections $\bigcap_{j\in J} X_j$ for nonempty $J \subseteq \{0, 1, \ldots, n\}$ satisfy this definition of *blocky*. This is the motivation behind the following Lemma 5.6.13.

Lemma 5.6.13. *Let X_β, $X_\gamma \subseteq \mathbb{R}^n$ be blocky sets. Choose $R > 0$. Then*

$$N_\infty(X_\beta, R) \cap N_\infty(X_\gamma, R) = N_\infty(X_\beta \cap X_\gamma, R), \tag{5.6.13.1}$$

where N_∞ denotes the R-neighborhood under the sup*-metric d_∞.*

Proof. The direction "$\supseteq$" is immediate: If a point $y \in \mathbb{R}^n$ is at most R away from $X_\beta \cap X_\gamma$, then it is at most R away from both X_β and X_γ independently.

To show "$\subseteq$", fix $y \in N_\infty(X_\beta, R) \cap N_\infty(X_\gamma, R)$. We will show that this y is already in $N_\infty(X_\beta \cap X_\gamma, R)$. For each coordinate y_δ, by definition of d_∞, we have $\inf_{x\in X} |y_\delta - x_\delta| \leq R$ for both $X = X_\beta$ and $X = X_\gamma$. In particular, for both $X = X_\beta$ and $X = X_\gamma$,

$$\inf_{x\in X} |y_\delta - x_\delta| = \begin{cases} 0 & \text{if } y_\delta > 0 \text{ and } \{0\}^\delta \times [0, +\infty[\times \{0\}^{n-\delta-1} \subseteq X, \\ 0 & \text{if } y_\delta < 0 \text{ and } \{0\}^\delta \times]-\infty, 0] \times \{0\}^{n-\delta-1} \subseteq X, \\ |y_\delta| \leq R & \text{otherwise.} \end{cases}$$

Certainly, $X_\beta \cap X_\gamma$ is nonempty; at least the origin is part of this intersection. To finish the proof, assume that $d_\infty(y, X_\beta \cap X_\gamma) > R$. Then there exists a δ-th coordinate such that

$$\inf \{|y_\delta - x_\delta| : x \in X_\beta \cap X_\gamma\} > R. \tag{5.6.13.2}$$

Then $|y_\delta| > R$ because $X_\beta \cap X_\gamma$ is blocky. This forbids the "otherwise"-case for both $X = X_\beta$ and $X = X_\gamma$. Both of the remaining two cases force the ray from the origin in the δ-th dimension that contains $(0, \dots, 0, y_\delta, 0, \dots, 0)$ to be a subset of both X_β and X_γ. But now, setting y_δ to zero will not affect the distance to the blocky set $X_\beta \cap X_\gamma$:

$$\begin{aligned} & d_\infty\big((y_0, \dots, y_{\delta-1}, y_\delta, y_{\delta+1}, \dots, y_{n-1}), X_\beta \cap X_\gamma\big) \\ = {} & d_\infty\big((y_0, \dots, y_{\delta-1}, 0, y_{\delta+1}, \dots, y_{n-1}), X_\beta \cap X_\gamma\big). \end{aligned}$$

After setting y_δ to 0, if there are still coordinates remaining that satisfy 5.6.13.2, repeat this argument and set those coordinates to 0, too, without altering the distance to $X_\beta \cap X_\gamma$. Eventually, we find that no coordinates satisfy 5.6.13.2 anymore. Therefore the assumption $d_\infty(y, X_\beta \cap X_\gamma) > R$ is false and $y \in N_\infty(X_\beta \cap X_\gamma, R)$. □

Corollary 5.6.14. *Let $\{X_\beta\}_{\beta\in\alpha}$ be a collection of blocky subsets of $\mathbb{R}^n$ such that $\bigcup_{\beta\in\alpha} X_\beta = \mathbb{R}^n$. Then $\{X_\beta\}_{\beta\in\alpha}$ is coarsely excisive with respect to the* sup*-metric d_∞. In particular, for a nonempty finite index set $J \subseteq \alpha$ and $R > 0$, we have*

$$\bigcap_{j\in J} N_\infty(X_j, R) = N_\infty\Big(\bigcap_{j\in J} X_j, R\Big).$$

Proof. For $|J| = 1$, the claim is trivial. For a $|J|$-fold intersection with $|J| > 1$, use induction along the cardinality of J: Choose $\beta \in J$ and set $J' = J - \{\beta\}$ for which the claim already holds. Then

$$\begin{aligned} \bigcap_{j\in J} N_\infty(X_j, R) &= N_\infty(X_\beta, R) \cap \bigcap_{j\in J'} N_\infty(X_j, R) \\ &= N_\infty(X_\beta, R) \cap N_\infty\Big(\bigcap_{j\in J'} X_j, R\Big) \\ &= N_\infty\Big(\bigcap_{j\in J} X_j, R\Big), \end{aligned}$$

applying Lemma 5.6.13 at the end because $\bigcap_{j\in J'} X_j$ is blocky. □

Proposition 5.6.15. *The block decomposition $\{X_j\}_{j\le n}$ of $\mathbb{R}^n$ from Definiton 5.6.6 is coarsely excisive under the* sup*-metric d_∞: For a nonempty $J \subseteq \{0, 1, \dots, n\}$ and $R > 0$, we have*

$$\bigcap_{j\in J} N_\infty(X_j, R) = N_\infty\Big(\bigcap_{j\in J} X_j, R\Big).$$

Proof. Each X_j is blocky as a cartesian product of copies of $]-\infty, 0]$, $[0, \infty[$, and $\mathbb{R}$. The result follows from Corollary 5.6.14. □

Lemma 5.6.16 (Relating 1-metric and sup-metric)**.** *Let $X \subseteq \mathbb{R}^n$ be arbitrary and choose $R > 0$. Denote by $N_1(X, R)$ the neighborhood of X under the 1-metric d_1 and by $N_\infty(X, R)$ its neighboorhood under the* sup*-metric d_∞. Then*

$$N_1(X, R) \subseteq N_\infty(X, R) \subseteq N_1(X, nR). \tag{5.6.16.1}$$

Proof. For x, $y \in \mathbb{R}^n$ arbitrary, we always have

$$d_1(x, y) = \sum_{j<n} |x_j - y_j| \geq \sup_{j<n} |x_j - y_j| = d_\infty(x, y)$$

and

$$nd_\infty(x, y) = \sum_{j<n} \sup_{j'<n} \left|x_{j'} - y_{j'}\right| \geq \sum_{j<n} |x_j - y_j| = d_1(x, y).$$

These estimates continue to hold when we take $\inf_{x \in X}$ for each term instead of a single point x, yielding the distance to X. Passing to neighborhoods, since larger metrics mean smaller neighborhoods, we get for the inclusion on the right-hand side of 5.6.16.1:

$$\begin{aligned} N_\infty(X, R) &= \{y \in \mathbb{R}^n : d_\infty(X, y) \leq R\} \\ &= \{y \in \mathbb{R}^n : nd_\infty(X, y) \leq nR\} \\ &\subseteq \{y \in \mathbb{R}^n : d_1(X, y) \leq nR\} \\ &= N_1(X, nR). \end{aligned}$$

A similar estimate holds for the inclusion on the left-hand side of 5.6.16.1. □

Proposition 5.6.17. *The block decomposition $\{X_j\}_{j \leq n}$ is coarsely excisive under the 1-metric d_1: Given a nonempty $J \subseteq \{0, 1, \ldots, n\}$ and $R > 0$, we can set $S = nR$ to ensure*

$$\bigcap_{j \in J} N_1(X_j, R) \subseteq N_1\Big(\bigcap_{j \in J} X_j, nR\Big).$$

Proof. Combining Proposition 5.6.15 with Lemma 5.6.16, we get

$$\begin{aligned} \bigcap_{j \in J} N_1(X_j, R) &\subseteq \bigcap_{j \in J} N_\infty(X_j, R) \\ &= N_\infty\Big(\bigcap_{j \in J} X_j, R\Big) \\ &\subseteq N_1\Big(\bigcap_{j \in J} X_j, nR\Big). \end{aligned}$$ □

This finishes the preparations for our proof of Theorem 5.6.1: We would like to show

$$K_s\mathfrak{C}^*\mathbb{R}^n = \begin{cases} \mathbb{Z} & \text{for } s-n \text{ even,} \\ 0 & \text{for } s-n \text{ odd.} \end{cases}$$

Proof of Theorem 5.6.1. The block decomposition $\{X_j\}_{j\leq n}$ of $\mathbb{R}^n$ into $n+1$ pieces from Definition 5.6.6 is coarsely excisive under either the sup-metric by Proposition 5.6.15 or the 1-metric by Proposition 5.6.17.

Intersecting fewer than $n+1$ blocks X_j yields a flasque space Y with trivial $K_*\mathfrak{C}^*Y = 0$. Intersecting all $n+1$ points gives the compact one-point set $\{0\}$ with $K_s\mathfrak{C}^*\{0\} = \mathbb{Z}$ for s even and $K_s\mathfrak{C}^*\{0\} = 0$ for s odd.

These results fit into our spectral sequence from Theorem 5.5.2 for coarsely excisive covers, letting J range over all nonempty subsets of $\{0, 1, \ldots, n\}$: The first page is

$$E^1_{p,q} \cong \begin{cases} \displaystyle\bigoplus_{|J|=p+1} K_q\mathfrak{C}^*\Big(\bigcap_{j\in J} X_j\Big) & \text{for } 0 \leq p \leq n, \\ 0 & \text{for } p < 0 \text{ or } p > n, \end{cases}$$

and the spectral sequence converges to $K_*\mathfrak{C}^*\mathbb{R}^n$. The E^1-term has only one nonzero column $E^1_{n,*}$ from intersecting all $n+1$ pieces:

$$\begin{array}{r|cccccc}
q & & & & & & \\
2 & 0 & \cdots & 0 & \mathbb{Z} & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & \mathbb{Z} & 0 & 0 \\
-1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\hline
 & 0 & \cdots & n-1 & n & n+1 & n+2 \quad p.
\end{array}$$

This spectral sequence collapses on the first page. There is no extension problem to solve. We may read $K_s\mathfrak{C}^*\mathbb{R}^n$ directly from the s-th diagonal $p+q=s$ of $E^1_{*,*}$: If $s-n$ is even, this K-theory is $\mathbb{Z}$; otherwise, it vanishes. $\square$

6 Infinite sums of ideals

For $A = I_0 + I_1 + \cdots + I_n$, we have a spectral sequence. What happens when A is the norm closure of a sum over countably many C*-ideals instead of over finitely many? Now A is a direct limit C*-algebra,

$$A = \overline{\sum_{j \in \mathbb{N}} I_j} = \overline{\bigcup_{n \in \mathbb{N}} \Big(\sum_{j<n} I_j \Big)} = \overline{\bigcup_{\substack{J \subseteq \mathbb{N} \\ |J| \in \mathbb{N}}} \Big(\sum_{j \in J} I_j \Big)}.$$

Can we replace Δ^n by an infinite simplex in the underlying construction? Or can we take the existing spectral sequence for a subalgebra $A(n) = \sum_{j<n} I_j$, then use the inclusion of algebras $A(n) \to A(n+1)$ to link the spectral sequences together? Does the direct limit of these spectral sequences converge to K_*A, or can we at least salvage some information about K_*A from this construction?

After developing a spectral sequence for $A = \overline{\sum_{j \in \mathbb{N}} I_j}$, Section 6.7 will generalize the result to direct limits $A = \overline{\sum_{\beta \in \alpha} I_\beta}$ of sums of uncountably many C*-ideals I_β for arbitrary index sets α.

6.1 Naïve approaches

Let $A = \overline{\sum_{j \in \mathbb{N}} I_j}$ be a C*-algebra with the $I_j \subseteq A$ closed two-sided ideals. For $n \in \mathbb{N}$ and $j \leq n$, we have defined the *cake piece* Δ^n_j in Definition 4.2.1 as a subset of the standard simplex Δ^n. We form C*-algebras $B = B(I_0, \ldots, I_n)$ based on the first $n+1$ ideals like in Definition 4.3.1:

$$\begin{aligned} B &= B(I_0, I_1, \ldots, I_n) \\ &= \left\{ f \colon \Delta^n \to A = \sum_{j=0}^{n} I_j \,:\, f \text{ continuous},\ f \restriction \partial\Delta^n = 0,\ f(\Delta^n_j) \subseteq I_j \text{ for all } j \right\}. \end{aligned}$$

This leads to algebras $B(I_0, \ldots, I_n)_J$ for $J \subseteq \{0, 1, \ldots, n\}$. But everything depends on our initial choice of the simplex Δ^n.

One immediate idea is to generalize the underlying function algebras B_J:

Definition 6.1.1. We define the *infinite simplex*

$$\Delta^{\mathbb{N}} = \left\{ (x_j)_j \in [0,1]^{\mathbb{N}} : \sum_{j=0}^{\infty} x_j \leq 1 \right\}.$$

This becomes a topological space with the usual product topology.

Let $A = \overline{\sum_{j \in \mathbb{N}} I_j}$ be a C*-algebra. Even if we succeed in defining a function space $\Delta^{\mathbb{N}}_J$ for $J \subseteq \mathbb{N}$ and function algebras $B_J \subseteq B$ of certain functions $\Delta^{\mathbb{N}} \to A$, it will be hard interpret the function algebras appropriately. In the finite case with $n+1$ ideals, the algebra

$$\{f \colon \Delta^n \to A : f \restriction \partial\Delta^n = 0\}$$

is isomorphic to the n-fold suspension of A. The suspension isomorphism allowed us to relate the K-theory of ideals to the K-theory of A. When we replace Δ^n with $\Delta^{\mathbb{N}}$, we lose the suspension isomorphism and cannot give a convergence theorem for a spectral sequence.

For another approach, recall the spectral sequence for ideal inclusions, constructed both in Section 3 and by C. Schochet in [Sch81]:

Theorem 3.1.1 (Spectral sequence for ideal inclusions)**.** *Let $A = \overline{\bigcup_{p \in \mathbb{N}} I_p}$ be a C*-algebra, where the $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_p \subseteq \cdots$ form a chain of closed two-sided ideals. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} = K_{p+q}(I_p / I_{p-1}).$$

This spectral sequence converges strongly to K_*A; i.e., given $s \in \mathbb{Z}$, the groups $E^\infty_{p,q}$ along the diagonal $s = p + q$ pose an extension problem to reconstruct K_sA.

In our setting, we do not have $A = \overline{\bigcup_{p \in \mathbb{N}} I_p}$ but merely $A = \overline{\sum_{j=0}^{\infty} I_j}$. A plausible adaption to our setting might be:

- Compute $K_*\big(\sum_{j=0}^{n} I_j\big)$ from $I_0, I_1, I_2, \ldots, I_n$ using our spectral sequence that takes finite sums of ideals.
- For each n, compute $K_*\big(\sum_{j=0}^{n} I_j / \sum_{j=0}^{n-1} I_j\big)$ with the six-term exact sequence.
- Feed these results at once into the spectral sequence from Theorem 3.1.1 to compute K_*A.

The downside is the multilayered computation: We build many spectral sequences, solve an extension problem for every single one, and then fit the results into yet another spectral sequence. This is unlikely to work except in trivial cases where $K_*\big(\overline{\sum_{j=0}^{\infty} I_j}\big)$ would have been straightforward to compute by other means, or when the K-theory would already equal $K_*\big(\sum_{j=0}^{n} I_j\big)$ for an $n \in \mathbb{N}$. Instead, we would like a more robust approach featuring a single spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with terms $E^1_{p,q}$ that are easier to describe and compute.

6.2 Linking two chains of ideals

Notation 6.2.1. Fix a C*-algebra $A = \overline{\sum_{j=0}^{\infty} I_j}$ where the I_j are closed two-sided ideals of A. We denote by $E(n)^r_{p,q}$ the (p,q)-th module in the r-th page of the spectral sequence for the sum of the first n ideals $I_0 + I_1 + \cdots + I_{n-1}$: Each of these spectral sequences $\{E(n)^r_{p,q}, d(n)^r\}_{r,p,q}$ is constructed according to our main Theorem 4.6.1 about finite sums of C*-ideals.

We will construct a morphism of spectral sequences from $E(n)$ to $E(n+1)$. This requires several technical propositions. Each spectral sequence arises from a chain of ideals $Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots$ that strongly depends on a fixed number of ideals I_0, I_1, …, I_j, … chosen in the beginning of the construction – see Section 4.3. To relate $E(n)$ with $E(n+1)$, we first construct morphisms between the two chains of ideals that lead to these two spectral sequences.

Along this way, we diligently track naturality with respect to $*$-homomorphisms that preserve ideal decompositions (Definition 4.1.1 and Remark 4.1.2).

Lemma 6.2.2. *Let $I_0, \ldots, I_n$ be C*-ideals. Fix $k \in \{0, 1, \ldots, n\}$. Define $I'_k = 0$, moreover $I'_j = I_j$ for $j \neq k$. Fix $J \subseteq \{0, 1, \ldots, n\} - \{k\}$. Then*

$$B(I_0, \ldots, I_n)_J = B(I'_0, \ldots, I'_n)_J = B(I'_0, \ldots, I'_n)_{J \cup \{k\}}.$$

Proof. The first equality holds because functions in $B(I_0, \ldots, I_n)_J$ and $B(I'_0, \ldots, I'_n)_J$ are never allowed to take nonzero values in I_k or I'_k, the only ideals that differ in the construction.

The second equality holds because B_J and $B_{J \cup \{k\}}$ differ at most by conditions enforced on the subspace $\Delta^n_k \subseteq \Delta^n$, but on Δ^n_k, functions map to $I'_k = 0$ anyway. $\square$

Proposition 6.2.3. *Let I_0, …, I_n be C*-ideals that sum to A. Construct the cake algebra $B(I_0, \ldots, I_n)$ as in Definition 4.3.1. Now include the zero algebra $0 = I_{n+1}$ as an extra ideal, leading to a different cake algebra $B(I_0, \ldots, I_n, 0)$:*

$$B(I_0, \ldots, I_n) \subseteq \mathscr{C}(\Delta^n, A),$$
$$B(I_0, \ldots, I_n, 0) \subseteq \mathscr{C}(\Delta^{n+1}, A).$$

Let $J \subseteq \{0, 1, \ldots, n\}$ be an index set. There is a suspension isomorphism

$$S\big(B(I_0, \ldots, I_n)_J\big) \cong B(I_0, \ldots, I_n, 0)_{J \cup \{n+1\}}.$$

Figure 6.2.4 gives the geometric idea. Functions on the line vanish on its two end points. Functions on the grey four-sided shape vanish on the continuously drawn boundary lines but not on the dashed line.

We claim that the suspension algebra of the functions on the line is isomorphic to the function algebra on the grey shape.

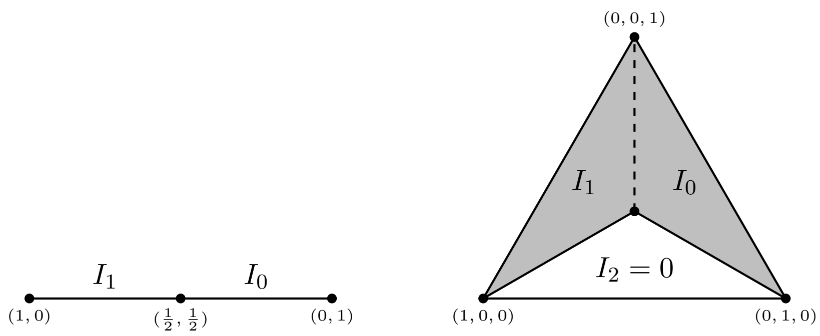

Figure 6.2.4: Geometric idea of Proposition 6.2.3

Proof of Proposition 6.2.3. By Lemma 6.2.2, we can reduce the case of arbitrary J to $J = \{0, 1, \ldots, n\}$ by replacing I_j with 0 for all $j \notin J$, completing the construction in this proof, then re-inserting the original ideals I_j.

Define

$$X = \bigcup_{j=0}^{n} \Delta_j^{n+1}, \qquad Y = X \cap \partial\Delta^{n+1}, \qquad Z = X \cap \Delta_{n+1}^{n+1}.$$

The set X corresponds to the grey area in the example figure above, Y to the grey area's ceiling, Z to its floor. Functions in $B(I_0, \ldots, I_n, 0)$ vanish on Y and Z, but they may assume nontrivial values in I_0, …, I_n on the interior $X - Y - Z$.

The points x in X have $n+2$ barycentric coordinates $(x_0, \ldots x_{n+1})$. For functions in $B(I_0, \ldots, I_n, 0)$, the relative values of the first $n+1$ of these coordinates select ideals among I_0, …, I_n as the function's range. The final coordinate x_{n+2} does not affect that choice, but x_{n+2} is not well-suited to see that $B(I_0, \ldots, I_n, 0)$ is isomorphic to a suspension algebra. Instead, we show this with a reparametrization φ of X. Set

$$y_{\min} = \min\{y_j : y_j \text{ is a barycentric coordinate of } y = (y_0, \ldots, y_n)\},$$

$$\varphi\colon \Delta^n \times \left[\frac{1}{n+2}, 1\right] \to \Delta^{n+1},$$

$$\varphi(y_0, \ldots, y_n, t) = (y_0 - ty_{\min}, \ldots, y_n - ty_{\min}, (n+1)ty_{\min}).$$

This function φ maps continuously to X: φ distributes $(n+1)$ portions of $ty_{\min}$ from the first $(n+1)$ coordinates to the last coordinate. Thus the sum of all coordinates remains 1. Furthermore, φ cannot map to the interior of Δ_{n+1}^{n+1} because the last coordinate cannot be the uniquely smallest: With $t \geq \frac{1}{n+2}$ by definition of φ, we have

$$(n+1)ty_{\min} \geq y_{\min} - ty_{\min}.$$

Equality holds exactly for $t = \frac{1}{n+2}$.

The map φ is surjective onto X: Construct a preimage of $(y_0, \dots, y_n, y_{n+1})$ by distributing $\frac{y_{n+1}}{n+1}$ onto each of the first $n+1$ coordinates, then choose t. Furthermore, φ is injective on the interior of $\Delta^n \times [\frac{1}{n+2}, 1]$.

On the simplex boundaries, we need not check injectivity. Because $B(I_0, \dots, I_n)$ or $B(I_0, \dots, I_n, 0)$ must vanish on the simplex boundaries, it suffices to verify that all points of $\partial\Delta^n \times [\frac{1}{n+2}, 1]$ map to $\partial\Delta^{n+1}$. This holds because $y_{\min} = 0$, thus $y_{\min} - ty_{\min} = 0$ regardless of the value $t \in [\frac{1}{n+2}, 1]$.

On the interior of X, that is, on the interior of $\varphi\big((\Delta^n)^\circ \times \,]\frac{1}{n+2}, 1[\big)$, functions in $B(I_0, \dots, I_n, 0)$ are subject to the restrictions from the relations of the first $n+1$ barycentric coordinates, but not to any restriction from the final zero ideal or from the coordinate $t \in [\frac{1}{n+2}, 1]$. On $\varphi\big((\Delta^n)^\circ \times \big\{\frac{1}{n+2}, 1\big\}\big)$, the functions must be zero. Thus via φ, we see that $B(I_0, \dots, I_n, 0)$ is isomorphic to the suspension of $B(I_0, \dots, I_n)$.

It suffices to parametrize X instead of the entire simplex Δ^{n+1} because every considered C*-function living on X has only one possible extension – by zero – to a function in $B(I_0, \dots, I_n, 0)$. □

Remark 6.2.5. The isomorphism from Proposition 6.2.3 is natural with respect to finite ideal decompositions: A collection of $*$-homomorphisms $i_j \colon I_j \to I'_j$ for $j \in \{0, 1, \dots, n\}$ on the input C*-ideals, together with the constructed isomorphisms on either side, leads to a commutative diagram

$$\begin{tikzcd} S\big(B(I_0, \dots, I_n)_J\big) \arrow[r, "\cong"] \arrow[d, "S(B(i_0, \dots, i_n))"'] & B(I_0, \dots, I_n, 0)_{J \cup \{n+1\}} \arrow[d, "B(i_0, \dots, i_n, 0)"] \\ S\big(B(I'_0, \dots, I'_n)_J\big) \arrow[r, "\cong"'] & B(I'_0, \dots, I'_n, 0)_{J \cup \{n+1\}}. \end{tikzcd}$$

Proposition 6.2.6. *Let $A = I_0 + I_1 + \cdots + I_n$ be a sum of C*-ideals. The function algebras $B_J = B(I_0, \dots, I_n)_J$ for $J \subseteq \{0, 1, \dots, n\}$ give rise to cake sums $Q_p = \sum_{|J| \leq p+1} B_J$ by Definition 4.3.6.*

Let $I_{n+1} = 0$ be an extra zero ideal. The function algebras $\widetilde{B}_J = B(I_0, \dots, I_n, 0)_J$ for this larger set of ideals and $J \subseteq \{0, 1, \dots, n, n+1\}$ give rise to $\widetilde{Q}_p = \sum_{|J| \leq p+1} \widetilde{B}_J$.

Then $\widetilde{Q}_p \cong SQ_p$ for $p \in \mathbb{Z}$ and $\widetilde{Q}_{n+1} = \widetilde{Q}_n$. In other words, the chains of ideal *inclusions* $Q_p \to Q_{p+1}$ *and* $\widetilde{Q}_p \to \widetilde{Q}_{p+1}$ *fit into this commutative diagram:*

$$\begin{tikzcd} SQ_0 \arrow[r] \arrow[d, "\cong"'] & SQ_1 \arrow[r] \arrow[d, "\cong"'] & \cdots \arrow[r] & SQ_n \arrow[d, "\cong"'] \arrow[dr, "\cong"] & \\ \widetilde{Q}_0 \arrow[r] & \widetilde{Q}_1 \arrow[r] & \cdots \arrow[r] & \widetilde{Q}_n \arrow[r, equal] & \widetilde{Q}_{n+1}. \end{tikzcd}$$

Proof. Our index sets will sometimes contain indices in $\{0, 1, \ldots, n\}$, sometimes in $\{0, 1, \ldots, n, n+1\}$. For clarity, given $p \in \mathbb{Z}$, we define these collections of index sets:

$$\mathscr{A}(p) = \{J \subseteq \{0, 1, \ldots, n\} : |J| \le p\},$$
$$\mathscr{B}(p) = \{J \subseteq \{0, 1, \ldots, n+1\} : |J| \le p\}.$$

For $p < 0$, $\mathscr{A}(p)$ and $\mathscr{B}(p)$ are empty and Q_p is the zero algebra.

The following argument will work without modification for all $p \in \mathbb{Z}$. For a given $p \in \mathbb{Z}$, we may write the collection $\mathscr{B}(p+1)$ as the disjoint union of $\mathscr{A}(p+1)$ and $\{J \cup \{n+1\} : J \in \mathscr{A}(p)\}$. Applying this to the definition of $\widetilde{Q}_p$, we obtain

$$\widetilde{Q}_p = \sum_{\mathscr{B}(p+1)} \widetilde{B}_J = \sum_{\mathscr{A}(p+1)} \widetilde{B}_J + \sum_{\mathscr{A}(p)} \widetilde{B}_{J \cup \{n+1\}}.$$

By Lemma 6.2.2, the index $n+1$ can be dropped from $B_{J \cup \{n+1\}}$ with no change because $I_{n+1} = 0$:

$$\widetilde{Q}_p = \sum_{\mathscr{A}(p+1)} \widetilde{B}_J + \sum_{\mathscr{A}(p)} \widetilde{B}_J.$$

By definition of $\mathscr{A}(p)$ as a collection of all index sets up to cardinality p, we have $\mathscr{A}(p) \subseteq \mathscr{A}(p+1)$. The sum simplifies to

$$\widetilde{Q}_p = \sum_{\mathscr{A}(p+1)} \widetilde{B}_J.$$

None of the sets in $\mathscr{A}(p+1)$ contain the index $n+1$. This allows us to rewrite $\widetilde{B}_J$ as SB_J via a natural isomorphism according to Proposition 6.2.3. Taking sums commutes with suspensions:

$$\widetilde{Q}_p = \sum_{\mathscr{A}(p+1)} \widetilde{B}_J \cong \sum_{\mathscr{A}(p+1)} SB_J = SQ_p.$$

This shows the main result. The extra result $\widetilde{Q}_{n+1} = \widetilde{Q}_n$ follows from $\widetilde{Q}_n \cong SQ_n = SQ_{n+1} \cong \widetilde{Q}_{n+1}$. $\square$

Remark 6.2.7. The constructed isomorphism is natural: It is a composition of equalities and the natural isomorphism from Proposition 6.2.3.

6.3 Compatibility of suspensions

Lemma 6.3.1. *Let I be a C*-ideal in A. Then $S(A/I) \cong SA/SI$. Explicitly, there is an isomorphism $\Phi\colon SA/SI \to S(A/I)$ with*

$$\Phi(f+SI)(t) = f(t) + I \qquad \text{for } f\colon [0,1] \to A \text{ with } f(0) = f(1) = 0.$$

Later, we show naturality of Φ in a separate lemma; first, we construct this isomorphism.

Proof of Lemma 6.3.1. The function Φ is well-defined, additive, and multiplicative because I is an ideal; it preserves the involution because I is a C*-ideal. If I contains the range of $\Phi(f)$, then f was already in SI, therefore Φ is injective.

For surjectivity, we will use $A' \otimes \mathscr{C}_0]0,1[\cong SA'$ for arbitrary C*-algebras A' and the nuclearity of $\mathscr{C}_0]0,1[$ as proven, e.g., in [WO93]. As a result, the top two rows of this commutative diagram become exact:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes \mathscr{C}_0]0,1[\arrow[r] \arrow[d, "\cong"] & A \otimes \mathscr{C}_0]0,1[\arrow[r] \arrow[d, "\cong"] & A/I \otimes \mathscr{C}_0]0,1[\arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & SI \arrow[r] \arrow[d, equal] & SA \arrow[r] \arrow[d, equal] & S(A/I) \arrow[r] & 0 \\
0 \arrow[r] & SI \arrow[r] & SA \arrow[r] & SA/SI \arrow[r] \arrow[u, "\Phi"] & 0.
\end{tikzcd}$$

The square in the bottom right commutes by our explicit construction of Φ. Since the bottom row is the standard quotient exact sequence for the inclusion $SI \to SA$, the $*$-homomorphism Φ is an isomorphism by the five lemma. □

Lemma 6.3.2. *The isomorphism $\Phi\colon SA/SI \to S(A/I)$ constructed in Lemma 6.3.1 is natural with respect to $*$-homomorphisms $h\colon A \to A'$ that map I into I', where $I' \subseteq A'$ is a given C*-ideal.*

Proof. Construct $\Phi'\colon SA'/SI' \to S(A'/I')$ for the C*-ideal $I' \subseteq A'$ according to

Lemma 6.3.1. Then $h\colon A \to A'$ gives rise to a diagram:

$$\begin{tikzcd} SA/SI \arrow[r, "\Phi"] \arrow[d, "Sh/S(h \upharpoonright I)"'] & S(A/I) \arrow[d, "S(h/(h \upharpoonright I))"] \\ SA'/SI' \arrow[r, "\Phi'"'] & S(A'/I'). \end{tikzcd}$$

This diagram commutes: Consider $(f + SI) \in SA/SI$ for a given $f\colon [0,1] \to A$ with $f(0) = f(1) = 0$. The upper right path through the diagram maps this to $t \mapsto f(t)+I$ and then to the class containing $t \mapsto (h \circ f)(t) + h(I)$, which is $t \mapsto (h \circ f)(t) + I'$ in $S(A'/I')$.

The lower left path maps $f+SI$ first to $h\circ f+h(SI)+SI'$, which is $h\circ f+SI'$ since $h(I) \subseteq I'$, and then onwards also to $t \mapsto (h \circ f)(t) + I'$ according to the construction of Φ'. □

Proposition 6.3.3. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of C*-algebras. Then for each $s \in \mathbb{Z}$, the following diagram is commutative:*

$$\begin{tikzcd} K_{s+1}(C) \arrow[r, "\partial_{s+1}(C,A)"] \arrow[d, "\sigma_{s+1}(C)"', "\cong"] & K_s(A) \arrow[d, "\cong"', "\sigma_s(A)"] \\ K_s(SC) \arrow[r, "\partial_s(SC,SA)"'] & K_{s-1}(SA). \end{tikzcd}$$

Here σ denotes suspension isomorphisms and ∂ denotes the boundary maps in the long exact K-theory sequences that arise from the original short exact sequence and from its suspension $0 \to SA \to SB \to SC \to 0$.

Even though ∂ is natural with respect to $*$-homomorphisms, the claim does not follow immediately from functoriality because the suspension isomorphisms σ arise only in K-theory, not on the level of C*-algebras.

Proof of Proposition 6.3.3. The *Bott isomorphism*

$$\beta(C)\colon K_0(C) \xrightarrow{\cong} K_1(SC) \xrightarrow{\sigma_2(C)^{-1}} K_2(C),$$

which is a composition of two isomorphisms, and the *exponential map*

$$\delta_0\colon K_0(C) \xrightarrow{\beta(C)} K_2(C) \xrightarrow{\partial_2(C,A)} K_1(A)$$

are constructed in [RLL00, Chapters 11–12] explicitly to make the two outermost paths $\sigma_1(A) \circ \delta_0$ and $\partial_1(SC, SA) \circ \sigma_2(C) \circ \beta(C)$ in the following diagram commute:

$$\begin{tikzcd}
K_0(C) \arrow[r, "\beta(C)", "\cong"'] \arrow[rr, bend left, "\delta_0"] & K_2(C) \arrow[r, "\partial_2(C,A)"] \arrow[d, "\sigma_2(C)"', "\cong"] & K_1(A) \arrow[d, "\cong"', "\sigma_1(A)"] \\
& K_1(SC) \arrow[r, "\partial_1(SC,SA)"'] & K_0(SA).
\end{tikzcd}$$

Because $\beta(C)$ is an isomorphism, commutativity of the square follows from commutativity of the two outermost paths. This commuting square proves the claim for $n = 1$, the lowest n of interest.

For higher s, the claim follows from this base case by composing the entire diagram with an $(s-1)$-fold suspension isomorphism. The higher boundary maps in K-theory are defined precisely to agree with such a suspension. The claim for lower s follows from composing with Bott isomorphisms. □

6.4 Linking spectral sequences

Having linked the chain of ideal inclusions $Q_p \to Q_{p+1}$ with $\widetilde{Q}_p \to \widetilde{Q}_{p+1}$, we can now link the spectral sequences that arise from the first n ideals along increasing cardinalities $n \in \mathbb{N}$.

Proposition 6.4.1. *Let $A = I_0 + I_1 + \cdots + I_{n-1}$ be a sum of C^*-ideals. Construct the spectral sequence $\{E^r_{*,*}, d^r\}_r$ for this n-fold sum as in Theorem 4.6.1.*

Alternatively, add an extra ideal $I_n = 0$ and construct a second spectral sequence $\{\widetilde{E}^r_{*,*}, \widetilde{d}^r\}_r$ for the $(n+1)$-fold sum $I_0 + I_1 + \cdots + I_{n-1} + 0$.

Then there are isomorphisms $E^r_{p,q} \cong \widetilde{E}^r_{p,q-1}$ for all $r \geq 1$ and p, $q \in \mathbb{Z}$ with the following properties:

- *They are natural with respect to $*$-homomorphisms that preserve the ideal decompositions: $*$-homomorphisms $h\colon A \to A'$ for a C^*-algebra $A' = I'_0 + I'_1 + \cdots + I'_{n-1} + 0$ such that, for all $j < n$, we have $h(I_j) \subseteq I'_j$.*

- *They commute with the differentials d^r and $\widetilde{d}^r$.*

Proof. Recall the first page of the spectral sequence from the main statement of

Theorem 4.6.1, adapted to n instead of $n+1$ ideals:

$$E^1_{p,q} \cong \begin{cases} \bigoplus\limits_{|J|=p+1} K_q\Big(\bigcap\limits_{j\in J} I_j\Big) & \text{for } 0 \le p < n, \\ 0 & \text{for } p < 0 \text{ or } p \ge n. \end{cases}$$

Suspend Q_p/Q_{p-1} and compensate for this suspension by a degree shift in K-theory to maintain isomorphy. Apply our isomorphism results from above:

$$\begin{aligned} E^1_{p,q} &\cong K_q(Q_p/Q_{p-1}) \\ &\cong K_{q-1}\big(S(Q_p/Q_{p-1})\big) \\ &\cong K_{q-1}(SQ_p/SQ_{p-1}) \\ &\cong K_{q-1}(\widetilde{Q}_p/\widetilde{Q}_{p-1}) \\ &\cong \widetilde{E}^1_{p,q-1}. \end{aligned}$$

All isomorphisms come from our earlier propositions and are therefore natural. We do not have to show anything new here for naturality with respect to $h\colon A \to A'$ of the isomorphism between the two spectral sequences.

Commutation of isomorphisms and differentials follows from the definition of the differentials according to Theorem 3.2.5 and Definition 3.3.3: The differentials are compositions of maps induced by ideal inclusions, maps induced by natural quotient projections, and boundary maps from long exact sequences in K-theory that arise from ideal inclusions. All these maps commute individually with all isomorphisms applied above; in particular, Proposition 6.3.3 guarantees that the differentials behave well with suspensions. □

Definition 6.4.2 (Link between spectral sequences for increasingly-many ideals)**.** Let $A(n+1) = I_0 + I_1 + \cdots + I_n$ be a sum of $n+1$ C*-ideals and denote by $A(n) \subseteq A(n+1)$ the sum of the first n ideals, $I_0 + I_1 + \cdots + I_{n-1}$.

For all pages $r \ge 1$ and $p, q \in \mathbb{Z}$, let $\{g^r_{p,q}\}\colon \{E(n)^r_{p,q}, d(n)^r\} \to \{\widetilde{E}^r_{p,q-1}, \widetilde{d}^r\}$ be the isomorphism constructed in Proposition 6.4.1 from the spectral sequence for $A(n)$ into the spectral sequence for $I_0 + I_1 + \cdots + I_{n-1} + 0$. The ideal inclusions

$$I_0 = I_0, \qquad I_1 = I_1, \qquad \ldots, \qquad I_{n-1} = I_{n-1}, \qquad 0 \to I_n$$

induce a morphism $\{i^r_{p,q}\}$ that preserves all degrees,

$$\{i^r_{p,q}\}\colon \{\widetilde{E}^r_{p,q}\} \to \{E(n+1)^r_{p,q}\},$$

of spectral sequences between the intermediate $\{\widetilde{E}^r_{p,q}, \widetilde{d}^r\}_{r,p,q}$ and the desired spectral sequence $\{E(n+1)^r_{p,q}, d(n+1)\}_{r,p,q}$ for $A(n+1)$.

The *link* between the spectral sequences for $A(n)$ and $A(n+1)$ is

$$\ell(n) = \{\ell(n)^r_{p,q}\} = \{i^r_{p,q-1} \circ g^r_{p,q}\} \colon \{E(n)^r_{p,q}\} \to \{E(n+1)^r_{p,q-1}\}.$$

Lemma 6.4.3 (Faithfulness of the link $\ell(n)$)**.** *Given $n \in \mathbb{N}$ and $p, q \in \mathbb{Z}$, consider a set of indices $J \subseteq \{0, 1, \dots, n-1\}$ with $|J| = p+1$ and the direct summand $V = K_q\big(\bigcap_{j\in J} I_j\big)$ for J in the group $E(n)^1_{p,q}$.*

Then $\ell(n)^1_{p,q}$ maps V isomorphically onto $V' = K_{q-1}\big(\bigcap_{j\in J} I_j\big)$ in $E(n+1)^1_{p,q-1}$ No other summand of $E(n)^1_{p,q}$ besides V maps onto V' nontrivially. No other summand in $E(n+1)^1_{p,q-1}$ besides V' is a nontrivial image of V.

Proof. We examine the components of $\ell(n)^1_{p,q}$ in the notation of Definition 6.4.2.

The first component of $\ell(n)^1_{p,q}$ is $g^1_{p,q} \colon E(n)^1_{p,q} \to \widetilde{E}^1_{p,q-1}$. As a suspension isomorphism, it agrees with quotients, direct sums, and other suspensions. Because it is natural with respect to $*$-homomorphisms that preserve ideal decompositions, $g^1_{p,q}$ cannot permute V with other direct summands for different choices of ideals than J among the first n ideals. Again by naturality, it cannot map to $K_{q-1}\big(I_n \cap \bigcap_{j\in J} I_j\big)$ within $\widetilde{E}^1_{p,q-1}$ either: This K-theory group must always vanish regardless of A because, by definition, $\{\widetilde{E}^1_{p,q}\}_{r,p,q}$ is the spectral sequence for $I_n = 0$.

On all ideals I_j with $j \neq n$, the second component $i^1_{p,q-1} \colon \widetilde{E}^1_{p,q-1} \to E(n+1)^1_{p,q-1}$ is induced by the identity. It maps $g^1_{p,q}(V)$ necessarily to V' because $n \notin J$ and hits no other summands besides V'.

Thus $\ell(n)^1_{p,q} = i^1_{p,q-1} \circ g^1_{p,q}$ has the claimed isomorphy property and hits no other summands besides V'. It follows that no other summand in $E(n+1)^1_{p,q-1}$ besides V' may be a nontrivial image of V. □

Proposition 6.4.4. *The link $\ell(n)$ between the spectral sequences $\{E(n)^r_{p,q}, d(n)^r\}_{r,p,q}$ and $\{E(n+1)^r_{p,q}, d(n+1)^r\}_{r,p,q}$ is natural with respect to $*$-homomorphisms $h \colon A \to A'$ for another C^*-algebra $A' = I'_0 + I'_1 + \dots + I'_{n'}$ that is a sum of $n'+1 \geq n+1$ C^*-ideals, as long as $h(I_j) \subseteq I'_j$ for all $j \leq n$.*

Proof. We have constructed $\ell(n)$ as a composition of two maps that already satisfy this desired naturality with respect to h as shown in the various earlier lemmas. □

6.5 Main theorem for countably many ideals

Theorem 6.5.1 (Spectral sequence for countable sums). *Let A be the direct limit C*-algebra $\overline{I_0 + I_1 + \cdots + I_j + \cdots}$ of sums of countably many C*-ideals $I_j \subseteq A$. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} \cong \begin{cases} \bigoplus\limits_{|J|=p+1} K_q\Big(\bigcap\limits_{j\in J} I_j\Big) & \textit{for } p \geq 0, \\ 0 & \textit{for } p < 0, \end{cases} \tag{6.5.1.1}$$

where J ranges over all nonempty finite subsets of indices. In general, this is a half-page spectral sequence, any term $E^1_{p,q}$ with $p \geq 0$ may be nonzero.

This spectral sequence converges strongly to K_*A. It is functorial with respect to $*$-homomorphisms that preserve countable ideal decompositions.

We prove Theorem 6.5.1 in two steps: First, in Proposition 6.5.4, we prove existence, that the spectral sequence is well-defined, that it is functorial, and that its $E^1_{*,*}$-term matches the description 6.5.1.1. Later, in Proposition 6.6.12, we prove the strong convergence.

Our strategy is to take the direct limit along a directed system of links $\ell(n)$ for $n \to \infty$, but these links $\ell(n)$ do not connect the spectral sequences $E(n)$ and $E(n+1)$ perfectly. There is an index shift from $E(n)^r_{p,q}$ to $E(n+1)^r_{p,q-1}$. As shown before, the shifted index affects the degree of the K-theory; it has no other effect on equation 6.5.1.1.

Complex K-theory admits Bott isomorphisms β,

$$\cdots \overset{\beta}{\cong} K_{s-2}\Big(\bigcap_{j\in J} I_j\Big) \overset{\beta}{\cong} K_s\Big(\bigcap_{j\in J} I_j\Big) \overset{\beta}{\cong} K_{s+2}\Big(\bigcap_{j\in J} I_j\Big) \overset{\beta}{\cong} \cdots;$$

their naturality allows us to work around the index shift.

Remark 6.5.2. The Bott isomorphisms β are natural with respect to $*$-homomorphisms and commute with the suspension isomorphisms σ for any C*-algebra A:

$$\begin{tikzcd} K_s(A) \arrow[r, "\beta"] \arrow[d, "\sigma_s"'] & K_{s+2}(A) \arrow[d, "\sigma_{s+2}"] \\ K_{s-1}(SA) \arrow[r, "\beta"'] & K_{s+1}(SA). \end{tikzcd}$$

Definition 6.5.3 (Degree-amending link $\lambda(n)$). Compose two links with the Bott isomorphism to create a morphism $\lambda(n)$ of bidegree $(0,0)$, called the *degree-amending* *link*, between two spectral sequences:

$$\lambda(n) = \beta \circ \ell(n+1) \circ \ell(n) \colon \{E(n)^r_{p,q}\}_{r,p,q} \to \{E(n+2)^r_{p,q}\}_{r,p,q}.$$

Proposition 6.5.4. *Let $A = \overline{I_0 + I_1 + \cdots + I_j + \cdots}$ be a sum of C^*-ideals. The spectral sequence postulated in Theorem 6.5.1 with*

$$E^1_{p,q} \cong \begin{cases} \bigoplus\limits_{|J|=p+1} K_q\Big(\bigcap\limits_{j\in J} I_j\Big) & \textit{for } p \geq 0, \\ 0 & \textit{for } p < 0, \end{cases}$$

exists and is functorial with respect to $*$-homomorphisms that preserve countable ideal decompositions.

Proof. Take the direct limit along $\lambda(2n)$ for $n \to \infty$. This yields again a spectral sequence. Functoriality of this spectral sequence with respect to ideal inclusions follows from all earlier constructions that, as we have remarked repeatedly, are natural with respect to $*$-homomorphisms that preserve countable ideal decompositions.

It remains to show that the choice of offset, i.e., $2n$ or $2n+1$, and the position of β among the links ℓ have no effect on the direct limit of degree-amending links; i.e., the following system $\lambda'(2n)$ of morphisms produces the same direct limit:

$$\lambda'(n) = \ell(n+1) \circ \beta \circ \ell(n) \colon \{E(n)^r_{p,q}\} \to \{E(n+2)^r_{p,q}\}.$$

The position of β is irrelevant because the Bott isomorphism is natural with respect to both $*$-homomorphisms and suspension isomorphisms. In Definition 6.4.2, we have defined $\ell(n)$ as a composition of several suspension isomorphisms and several direct morphisms between C*-algebras. Thus for $n \to \infty$, the systems $\lambda(2n)$ and $\lambda'(2n)$ produce naturally isomorphic direct limits, without even requiring index shifts.

To compare the systems $\lambda(2n)$ and $\lambda(2n+1)$ for $n \to \infty$, unroll the compositions λ into their definitions, and take the direct limit – necessarily yielding the same limit as before – across the unrolled system:

$$\begin{aligned} \operatorname*{colim}_{n\to\infty} \lambda(2n+1) &= \operatorname{colim}\big(\cdots \to \bullet \xrightarrow{\ell(n+1)} \bullet \xrightarrow{\ell(n+2)} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\ell(n+3)} \bullet \to \cdots\big) \\ &= \operatorname{colim}\big(\cdots \to \bullet \xrightarrow{\ell(n+1)} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\ell(n+2)} \bullet \xrightarrow{\ell(n+3)} \bullet \to \cdots\big) \\ &= \operatorname{colim}\big(\cdots \to \bullet \xrightarrow{\ell(n+2)} \bullet \xrightarrow{\ell(n+3)} \bullet \xrightarrow{\beta} \bullet \to \cdots\big) \\ &= \operatorname*{colim}_{n\to\infty} \lambda(2n), \end{aligned}$$

because the morphisms ℓ commute with β and because removing the first element of a directed system will not change the limit.

Thus our limit spectral sequence is well-defined as $\operatorname{colim}_n \lambda(2n)$. Finally, because

$$E(n)^r_{p,q} \cong \begin{cases} \bigoplus\limits_{\substack{|J|=p+1\\ \max J<n}} K_q\Big(\bigcap\limits_{j\in J} I_j\Big) & \text{for } 0 \le p < n, \\ 0 & \text{for } p<0 \text{ or } p \ge n, \end{cases}$$

and because the links $\ell(n)$ preserve all structure due to their faithfulness (Lemma 6.4.3), the direct limit $E^r_{p,q}$ along the K-theory morphisms $\lambda(2n)$ for $n \to \infty$ is the desired direct sum of K-theory groups over all nonempty subsets $J \subseteq \mathbb{N}$ with $|J| = p+1$ without restrictions about any maximum of J. □

6.6 Convergence

To prove the convergence of the spectral sequence for countable sums of ideals, we will define a filtration of K_*A via direct limits of existing filtrations.

Even though the following lemma about direct limits is known theory, we reprove it with attention to detail, tracking the naturality of all constructions.

Lemma 6.6.1 (Continuity of inclusion chains of abelian group quotients)**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a directed system of abelian groups along inclusions $i_n\colon G_n \subseteq G_{n+1}$. For all $n \in \mathbb{N}$, let $H_n \subseteq G_n$ be a subgroup such that $H_n \subseteq H_{n+1}$. Write*

$$G = \operatorname*{colim}_{n\to\infty}(G_n, i_n), \qquad H = \operatorname*{colim}_{n\to\infty}(H_n, i_n \restriction H_n).$$

Then there is an isomorphism

$$\Phi\colon G/H \cong \operatorname*{colim}_{n\to\infty}(G_n/H_n)$$

that is natural with respect to morphisms between systems $(G_n)_{n\in\mathbb{N}}$ and $(\widetilde{G}_n)_{n\in\mathbb{N}}$ that preserve their inclusions and their respective systems of subgroups $(H_n \subseteq G_n)_{n\in\mathbb{N}}$ and $(\widetilde{H}_n \subseteq \widetilde{G}_n)_{n\in\mathbb{N}}$.

Proof. For each $n \in \mathbb{N}$, consider the following diagram D_n. All of the horizontal arrows in D_n are natural projections and the square commutes due to naturality of the projections:

$$\begin{tikzcd}
G_n \arrow[r, "\alpha_n"] \arrow[d, "\gamma_n"'] & G_n/H_n \arrow[d, "\gamma_n/H_n"] & \\
G \arrow[r, "\delta_n"'] & G/H_n \arrow[r, "\pi_n"'] & G/H.
\end{tikzcd}$$

To construct a direct limit of the sequence of entire diagrams $(D_n)_{n\in\mathbb{N}}$, link two diagrams D_n and D_{n+1} by the following system of morphisms:

- $i_n \colon G_n \to G_{n+1}$;
- the composition $(\tau_{n+1} \circ i_n/H_n) \colon G_n/H_n \to G_{n+1}/H_n \to G_{n+1}/H_{n+1}$ where the natural projection $\tau_{n+1} \colon G_{n+1}/H_n \to G_{n+1}/H_{n+1}$ is well-defined because $H_n \subseteq H_{n+1}$;
- the natural projection $G/H_n \to G/H_{n+1}$ from dividing by H_{n+1}/H_n; and
- the identities on G and G/H, respectively.

All squares that arise from linking two diagrams D_n and D_{n+1} via these morphisms commute due to naturality of projections; e.g., $\alpha_{n+1} \circ i_n = \tau_{n+1} \circ i_n/H_n \circ \alpha_n$. Take the direct limit along $(D_n)_{n\in\mathbb{N}}$; the limit object is again a commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "\operatorname{colim}_n \alpha_n"] \arrow[d, equal] & \operatorname{colim}_n(G_n/H_n) \arrow[d, "\operatorname{colim}_n(\gamma_n/H_n)"] & \\
G \arrow[r, "\operatorname{colim}_n \delta_n"'] & \operatorname{colim}_n(G/H_n) \arrow[r, "\operatorname{colim}_n \pi_n"'] & G/H.
\end{tikzcd} \tag{6.6.1.1}$$

Furthermore, for each $n \in \mathbb{N}$, there is a short exact sequence

$$0 \to H_n \to G_n \xrightarrow{\alpha_n} G_n/H_n \to 0.$$

Taking direct limits of abelian groups is an exact functor, resulting in a short exact sequence of direct limits which fits into the following diagram as the top row:

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] \arrow[d, equal] & G \arrow[r, "\operatorname{colim}_n \alpha_n"] \arrow[d, equal] & \operatorname{colim}_n(G_n/H_n) \arrow[r] \arrow[d, "\operatorname{colim}_n(\pi_n \circ \gamma_n/H_n)"] & 0 \\
0 \arrow[r] & H \arrow[r] & G \arrow[r, "\operatorname{colim}_n(\pi_n \circ \delta_n)"'] & G/H \arrow[r] & 0.
\end{tikzcd}$$

The right square commutes because its morphisms already commuted in the earlier diagram 6.6.1.1 of direct limits.

Finally, because both rows are short exact sequences, the five lemma guarantees that $\operatorname{colim}_n(\pi_n \circ \gamma_n / H_n)$ is the desired isomorphism Φ. Naturality of Φ follows from the constructions in this proof: Both taking direct limits and taking quotients is natural. $\square$

Notation 6.6.2. For a sum $A = \overline{I_0 + I_1 + \cdots + I_j + \cdots}$ of C*-ideals, write

$$A(n) = I_0 + I_1 + \cdots + I_{n-1}$$

for the sum of the first n ideals and let, for $s \in \mathbb{Z}$,

$$a(n)_s \colon K_s A(n) \to K_s A(n+1)$$

be the map induced in K-theory by the inclusion of C*-algebras $A(n) \subseteq A(n+1)$.

Given $A(n)$, define the cake algebras $B(n)_J = B(I_0, I_1, \ldots, I_{n-1})_J$ for index sets $J \subseteq \{0, 1, \ldots, n-1\}$ as in Definition 4.3.1 and the sums of cake algebras as in Definition 4.3.6,

$$Q(n)_p = \sum_{|J| \leq p+1} B(n)_J$$

for $p \in \mathbb{Z}$ and $J \subseteq \{0, 1, \ldots, n-1\}$.

Remark 6.6.3. By Lemma 4.3.9, we have a chain of ideals $Q(n)_p \subseteq Q(n)_{p+1}$ across all $p \in \mathbb{Z}$. The chain stabilizes with $Q(n)_p = 0$ for $p < 0$ and $Q(n)_p = B(n)_{\{0,1,\ldots,n-1\}}$ for $p \geq n-1$.

Furthermore, $\sum_{p=0}^{n-1} Q(n)_p = B(n)_{\{0,1,\ldots,n-1\}} \cong S^{n-1} A(n)$ by Theorem 4.4.11.

Notation 6.6.4 ($i(n,p)_s$)**.** We recall the filtrations on the spectral sequences for finitely many ideals: The convergence target of $\{E(n)^r_{p,q}, d(n)^r\}$ is $K_* A(n)$, filtered by

$$F^p K_s A(n) \cong \operatorname{im} i \colon K_{s-n+1} Q(n)_p \to K_{s-n+1} S^{n-1} A(n) \tag{6.6.4.1}$$

$$\cong \operatorname{im} i(n,p)_s \colon K_{s-n+1} Q(n)_p \to K_s A(n) \tag{6.6.4.2}$$

as in Definition 3.4.1 for p, $s \in \mathbb{Z}$. In the construction of the spectral sequence in Section 3, the symbol i may also stand for various other maps in K-theory.

Here in Section 6.6, we will write $i(n,p)_s$ for the maps in 6.6.4.2 that define a filtration of $K_s A(n)$, not of $K_{s-n+1} S^{n-1} A(n)$, for a given p. Normally, we would index maps in K-theory with the degree of the domain, but for $i(n,p)_s$, it will be easiest to track the K-theory degree s of the desired convergence target $K_s A(n)$.

We reserve i (without annotation in parentheses) to discuss internals of Section 3.

Remark 6.6.5. Because $Q(n)_p$ is an ideal in $S^{n-1}A(n)$, not in $A(n)$, we have shifted the K-theoretic degree from $K_sQ(n)_p$ to $K_{s-n+1}Q(n)_p$ to compensate. This shift is similar to the shift in the proof of Theorem 4.6.1 about the spectral sequence for sums of finitely many C*-ideals.

Definition 6.6.6 (Link between filtrations)**.** Fix $n \in \mathbb{N}$ and $p \in \mathbb{Z}$. Besides $Q(n)_p$, recall the C*-algebra $\widetilde{Q}(n)_p$ for the $n+1$ ideals I_0, I_1, …, I_{n-1}, 0 as in Proposition 6.2.6. For all $s \in \mathbb{Z}$, define the *link between the filtrations of* $K_*A(n)$ *and* $K_*A(n+1)$,

$$\psi(n,p)_s \colon K_{s-n+1}Q(n)_p \to K_{s-n}Q(n+1)_p,$$

as the composition

$$\begin{tikzcd} K_{s-n+1}Q(n)_p \arrow[r, "\cong"'] \arrow[rrr, bend left, "\psi(n,p)_s"] & K_{s-n}SQ(n)_p \arrow[r, "\cong"'] & K_{s-n}\widetilde{Q}(n)_p, \arrow[r, "a(n)+0"'] & K_{s-n}Q(n+1)_p; \end{tikzcd}$$

here the left arrow is the suspension isomorphism, the middle arrow is the natural isomorphism constructed in Proposition 6.2.6, and the right arrow is induced by the inclusion of the $(n+1)$-fold ideal decomposition $A(n)+0$ into $A(n+1)$.

Lemma 6.6.7. *For all $n \in \mathbb{N}$ and p, $s \in \mathbb{Z}$, the following diagram commutes:*

$$\begin{tikzcd} K_{s-n+1}Q(n)_p \arrow[r, "\psi(n,p)_s"] \arrow[d, "i(n,p)_s"'] & K_{s-n}Q(n+1)_p \arrow[d, "i(n+1,p)_s"] \\ K_sA(n) \arrow[r, "a(n)_s"'] & K_sA(n+1). \end{tikzcd} \tag{6.6.7.1}$$

Proof. Replace $\psi(n,p)_s$ by its definition as a composition of three maps to get the top row of the following diagram where i' denotes the map induced by the ideal inclusion $Q(n)_p \subseteq S^{n-1}A(n)$ with appropriate K-theoretic degree shifts:

$$\begin{tikzcd} K_{s-n+1}Q(n)_p \arrow[r, "\cong"] \arrow[d, "i(n,p)_s"'] & K_{s-n}SQ(n)_p \arrow[r, "\cong"] \arrow[d, "i'"'] & K_{s-n}\widetilde{Q}(n)_p, \arrow[r, "a(n)_s+0"] & K_{s-n}Q(n+1)_p \arrow[d, "i(n+1,p)_s"] \\ K_sA(n) \arrow[r, equal] & K_sA(n) \arrow[rr, "a(n)_s"'] & & K_sA(n+1). \end{tikzcd}$$

The left square commutes by definition of i' because the top arrow is the suspension isomorphism.

The right side commutes because, by Remark 6.2.7, the isomorphism at the top is natural with respect to $*$-homomorphisms that preserve $(n+1)$-fold ideal decompositions: The two $*$-homomorphisms here are the ideal inclusion that induces $a(n)$ and the ideal inclusion $Q(n+1)_p \to S^n A(n+1)$ that induces i' and $i(n+1,p)_s$ and restricts to $\widetilde{Q}(n)_p$; these two ideal inclusions commute with each other already on the level of C*-algebras. $\square$

Definition 6.6.8 (Filtration of K_*A)**.** Fix p, $s \in \mathbb{Z}$ and compose diagram 6.6.7.1 with itself across all $n \geq 1$:

$$\begin{tikzcd}
K_sQ(1)_p \arrow[r, "\psi(1,p)_s"] \arrow[d, "i(1,p)_s"'] & K_{s-1}Q(2)_p \arrow[r, "\psi(2,p)_s"] \arrow[d, "i(2,p)_s"'] & K_{s-2}Q(3)_p \arrow[r, "\psi(3,p)_s"] \arrow[d, "i(3,p)_s"'] & \cdots \\
K_sA(1) \arrow[r, "a(1)_s"'] & K_sA(2) \arrow[r, "a(2)_s"'] & K_sA(3) \arrow[r, "a(3)_s"'] & \cdots .
\end{tikzcd}$$

The direct limit of $K_sA(n)$ for $n \to \infty$ along $a(n)$ is K_sA by continuity of K-theory. Take the direct limit of $K_{s-n+1}Q(n)_p$ for $n \to \infty$ along $\psi(n,p)_s$ and consider, by functoriality of the direct limit, the direct limit of the vertical arrows $i(n,p)_s$,

$$\operatorname*{colim}_{n\to\infty} i(n,p)_s \colon \Big(\operatorname*{colim}_{n\to\infty} K_{s-n+1}Q(n)_p\Big) \to K_sA.$$

With this map, define the *filtration of* K_*A, $\{F^pK_*A\}_{p\in\mathbb{Z}}$, by

$$F^pK_sA = \operatorname{im}\Big(\operatorname*{colim}_{n\to\infty} i(n,p)_s\Big) \subseteq K_sA.$$

Lemma 6.6.9. *The filtration $\{F^pK_sA\}_{p\in\mathbb{Z}}$ of K_sA from Definition 6.6.8 is an increasing filtration.*

Proof. For all $p \leq p'$, we have $Q(n)_p \subseteq Q(n)_{p'}$. Using i as in the notation of Section 3, $i\big(K_{s-n+1}Q(n)_p\big) \subseteq K_{s-n+1}Q(n)_{p'}$. Because the $i(n,p)_s$ arise from the directed system of morphisms i from Remark 3.4.3,

$$K_sQ(n)_p \xrightarrow{i} K_sQ(n)_{p+1} \xrightarrow{i} \cdots \xrightarrow{i} K_sQ(n)_n = K_sQ(n)_{n+1} = \cdots ,$$

merely via isomorphisms that implement a K-theoretic degree shift, we may pull back our direct limit construction for $\operatorname{colim}_n i(n,p)_s$ via these isomorphisms to the system of i. Passing to the direct limit morphism along $n \to \infty$ gives

$$\Big(\operatorname*{colim}_{n\to\infty} i\Big)\Big(\operatorname*{colim}_{n\to\infty} K_{s-n+1}Q(n)_p\Big) \subseteq \operatorname*{colim}_{n\to\infty} K_{s-n+1}Q(n)_{p'}.$$

Thus the filtration is increasing:

$$\begin{aligned} F^p K_s A &= \Big(\operatorname*{colim}_{n\to\infty} i(n,p)_s\Big)\Big(\operatorname*{colim}_{n\to\infty} K_{s-n+1}Q(n)_p\Big) \\ &\subseteq \Big(\operatorname*{colim}_{n\to\infty} i(n,p')_s\Big)\Big(\operatorname*{colim}_{n\to\infty} K_{s-n+1}Q(n)_{p'}\Big) \\ &= F^{p'} K_s A. \end{aligned}$$

□

Lemma 6.6.10. *The p-indexed filtration of K_*A from Definition 6.6.8 is Hausdorff, exhaustive, and complete according to Definition 2.7.2.*

Proof. For $p < 0$, all terms $K_{s-n+1}Q(n)_p$ vanish regardless of s and n, thus their direct limit also vanishes. This renders the filtration Hausdorff and complete.

For any class $[x] \in K_sA$, there is $n \in \mathbb{N}$ such that $K_sA(n)$ contains the preimage of $[x]$. We can choose $p = n$ to make $i(n,n)_s \colon K_{s-n+1}Q(n)_n \to K_sA(n)$ surjective, thereby including that preimage of $[x]$ in the range of $i(n,n)_s$. Commutativity of the diagram in Definition 6.6.8 shows that $[x]$ is in the range of $\operatorname{colim}_n i(n,p)_s$. Thus the filtration is exhaustive. □

Remark 6.6.11. The biggest problem in passing to direct limits for $n \to \infty$ were the iterated suspensions S^{n-1}, but these have already been handled by the degree-amending links $\lambda(2n)$ from Definition 6.5.3 via degree shifts and Bott isomorphisms. Thus throughout Section 6.6, we may rest assured that any direct limits of modules or differentials on $E^r_{p,q}$ for $r \geq 1$ or $r = \infty$ remain well-defined for the spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ for countably many C*-ideals.

Proposition 6.6.12 (Convergence of the limit spectral sequence)**.** *Let A be a C*-algebra with $A = \overline{I_0 + I_1 + \cdots + I_j + \cdots}$, a sum of C*-ideals. The limit spectral sequence constructed in Theorem 6.5.1, defined as the direct limit along the system $\lambda(2n)$ with*

$$E^r_{p,q} = \operatorname*{colim}_{n\to\infty} \Big(\beta \circ \ell(2n+1) \circ \ell(2n) \colon \{E(2n)^r_{p,q}\} \to \{E(2n+2)^r_{p,q}\}\Big),$$

converges strongly to K_*A.

Proof. For all $n \in \mathbb{N}$, the spectral sequence $\{E(n)^r_{p,q}, d(n)^r\}$ converges strongly:

$$E(n)^\infty_{p,q} \cong F^p K_{p+q} A(n) / F^{p-1} K_{p+q} A(n).$$

The terms $E^\infty_{p,q}$ are the direct limits of these $E(n)^\infty_{p,q}$ along the morphisms induced on $E(2n)^\infty_{p,q}$ by the $\lambda(2n)$ because these $\lambda(2n)$ had all desirable properties – naturality with respect to $*$-homomorphisms that preserve countable ideal decompositions and commutativity with the differentials. Furthermore, K-theory is continuous. Along the system of $\lambda(2n)$,

$$\operatorname*{colim}_{n\to\infty}\big(F^pK_{p+q}A(2n)/F^{p-1}K_{p+q}A(2n)\big) \cong \operatorname*{colim}_{n\to\infty}\big(E(2n)^\infty_{p,q}, \lambda(2n)\big) \cong E^\infty_{p,q}.$$

The filtration $\{F^pK_*A\}_{p\in\mathbb{Z}}$ of K_*A is Hausdorff, exhaustive, and complete by Lemma 6.6.9 and $E^\infty_{p,q}$ is isomorphic to $F^pK_{p+q}A/F^{p-1}K_{p+q}A$ by Lemma 6.6.1. Therefore the limit spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ converges strongly to K_*A. $\square$

This concludes the proof of Theorem 6.5.1 about the existence, well-definedness, functoriality, and strong convergence of the limit spectral sequence for countably many C*-ideals.

6.7 Uncountable sums of ideals

In most geometrical applications, if a C*-algebra may be written as a sum of easily computable ideals, this sum will be a countable sum. We have described a spectral sequence for this case. Still, it seems reasonable to generalize the cardinality of the algebra decomposition.

Theorem 6.7.1 (Spectral sequence for arbitrary sums)**.** *Let α be an arbitrary index set: finite, countable, or uncountable. Let $A = \overline{\sum_{\beta\in\alpha} I_\beta}$ be the norm closure of a sum of $|\alpha|$-many C*-ideals $I_\beta \subseteq A$. There is a spectral sequence $\{E^r_{p,q}, d^r\}_{r,p,q}$ with*

$$E^1_{p,q} \cong \begin{cases} \displaystyle\bigoplus_{|J|=p+1} K_q\Big(\bigcap_{j\in J} I_j\Big) & \textit{for } p \geq 0,\\ 0 & \textit{for } p < 0, \end{cases}$$

where J ranges over all nonempty finite index subsets $J \subseteq \alpha$. In general, this is a half-page spectral sequence, any term $E^1_{p,q}$ with $p \geq 0$ may be nonzero.

This spectral sequence converges strongly to K_*A. It is functorial with respect to $*$-homomorphisms that preserve α-indexed ideal decompositions.

This generalizes Theorem 6.5.1 about countable α. To prove Theorem 6.7.1 for uncountable α, we will adapt our construction for countable index sets to suitable direct limits that capture all of α.

Definition 6.7.2 (Directed system of finite sets)**.** For a set α, we define its *directed system of finite sets*,

$$\mathrm{Fin}(\alpha) = \{J \subseteq \alpha : \, |J| \text{ is finite}\};$$

this system is partially ordered by the subset relation $\subseteq$.

Remark 6.7.3. This is indeed a directed system: Given arbitrary J, $J' \in \mathrm{Fin}(\alpha)$, both are equal to or smaller than their union $J \cup J'$, still a finite set.

The direct limit of $\mathrm{Fin}(\alpha)$ in the category of sets is α. All chains, i.e., linear subsystems, in $\mathrm{Fin}(\alpha)$ are either finite or have the order type of $(\mathbb{N}, \leq)$.

We may consider the partially ordered set $\mathrm{Fin}(\alpha)$ itself a thin category: Its elements become objects. Comparable sets $J \subseteq J'$ are linked with a unique morphism $J \to J'$.

Notation 6.7.4 (Algebras for subsets $J \subseteq \alpha$)**.** For the C*-algebra $A = \overline{\sum_{\beta \in \alpha} I_\beta}$ as in the statement of Theorem 6.7.1 and $J \in \mathrm{Fin}(\alpha)$, define a subalgebra $A(J)$ of A by

$$A(J) = \sum_{j \in J} I_j.$$

Let $J' \in \mathrm{Fin}(\alpha)$ be another subset with $J \subseteq J'$. For $s \in \mathbb{Z}$, let

$$a(J, J')_s \colon K_s A(J) \to K_s A(J')$$

be the map induced in K-theory by the inclusion of C*-algebras $A(J) \subseteq A(J')$.

Remark 6.7.5 (Directed system in K-theory)**.** For each $s \in \mathbb{Z}$, consider the functor from $\mathrm{Fin}(\alpha)$ to abelian groups that maps J to $K_s A(J)$ and a comparable pair of sets $J \subseteq J'$ to $a(J, J')_s \colon K_s A(J) \to K_s A(J')$. This turns $\{K_s A(J) : J \in \mathrm{Fin}(\alpha)\}$ with the system of morphisms $\{a(J, J')_s : J \subseteq J' \in \mathrm{Fin}(\alpha)\}$ into a directed system. Because K-theory is continuous,

$$\operatorname*{colim}_{J \in \mathrm{Fin}(\alpha)} K_s A(J) = K_s A.$$

Notation 6.7.6 (Spectral sequence for J)**.** For $J \in \mathrm{Fin}(\alpha)$, the finite sum of ideals $A(J)$ has a spectral sequence $\{E(J)^r_{p,q}, d(J)^r\}_{r,p,q}$ according to Theorem 4.6.1 that converges strongly to $A(J)$ and has first-page terms of the form

$$E(J)^1_{p,q} \cong \begin{cases} \displaystyle\bigoplus_{\substack{|L|=p+1 \\ L \subseteq J}} K_q\Big(\bigcap_{j \in L} I_j\Big) & \text{for } 0 \leq p < |J|, \\ 0 & \text{for } p < 0 \text{ or } p \geq |J|, \end{cases}$$

where L ranges over all nonempty subsets of J.

Remark 6.7.7 (Directed system of spectral sequences)**.** Let $J \subseteq J' \in \mathrm{Fin}(\alpha)$ be two sets such that $|J|$ and $|J'|$ differ by an even number. Then the spectral sequences $\{E(J)^r_{p,q}, d(J)^r\}_{r,p,q}$ and $\{E(J')^r_{p,q}, d(J')^r\}_{r,p,q}$ fit into a directed system of spectral sequences connected by degree-amending links shaped like $\lambda(2n)$ from Definition 6.5.3. These morphisms have bidegree $(0,0)$.

Let $F \subseteq \mathrm{Fin}(\alpha)$ be the subsystem of $\mathrm{Fin}(\alpha)$ of all sets $J \in \mathrm{Fin}(\alpha)$ with even cardinality. Then F and $\mathrm{Fin}(\alpha)$ have the same direct limit α in the category of sets.

Consider the category of spectral sequences of the form $\{E(J)^r_{p,q}, d(J)^r\}_{r,p,q}$ for $J \in F$ with morphisms shaped like $\lambda(2n)$: Passing from $J \in \mathrm{Fin}(\alpha)$ to the spectral sequence $\{E(J)^r_{p,q}, d(J)^r\}_{r,p,q}$ becomes a functor between directed systems.

Remark 6.7.8. It does not matter whether the sets $J \in F$ have even or odd cardinalities. As we have seen in the proof of Proposition 6.5.4, the limit spectral sequence along $(\mathbb{N}, \leq)$ does not depend on whether we consider the subsystem linked by $\lambda(2n)$ or that linked by $\lambda(2n+1)$.

Lemma 6.7.9. *Let $F \subseteq \mathrm{Fin}(\alpha)$ be the subsystem of $\mathrm{Fin}(\alpha)$ of all sets $J \in \mathrm{Fin}(\alpha)$ with even cardinality. Fix $p \in \mathbb{Z}$ with $p \geq 0$ and fix $q \in \mathbb{Z}$.*

Consider all groups $E(J)^1_{p,q}$ for $J \in F$: This is a directed system of abelian groups; the morphisms are restrictions of degree-amending links of the shape of $\lambda(2n)$ from Definition 6.5.3. Then

$$\operatorname*{colim}_{J \in F} E(J)^1_{p,q} = \operatorname*{colim}_{J \in F} \bigoplus_{\substack{|L|=p+1 \\ L \subseteq J}} K_q\Big(\bigcap_{j \in L} I_j\Big) = \bigoplus_{\substack{|L|=p+1 \\ L \subseteq \alpha}} K_q\Big(\bigcap_{j \in L} I_j\Big).$$

Proof. For $J \subseteq J' \in F$, the morphisms $E(J)^1_{p,q} \to E(J')^1_{p,q}$ are well-defined because degree-amending links have bidegree $(0,0)$.

These morphisms were defined as compositions of Bott isomorphisms and links (Definition 6.4.2) between spectral sequences; they preserve all information: Given $L \subseteq J$ of cardinality $p+1$, the direct summand $K_q\big(\bigcap_{j \in L} I_j\big)$ from $E(J)^1_{p,q}$ maps isomorphically onto its copy in the direct sum $E(J')^1_{p,q}$. Given $L \subseteq J'$ such that L is not a subset of J, the direct summand $K_q\big(\bigcap_{j \in L} I_j\big)$ is not in the range.

In the category of abelian groups, the direct limit may be constructed from a large direct sum of all objects, dividing by relations according to the morphisms. Here the links behave like inclusions, enforcing trivial relations.

Finally, F is cofinal in $\mathrm{Fin}(\alpha)$: For any given set $J \subseteq \alpha$, the system F contains a set J' with $J \subseteq J'$. Therefore the desired direct limit is the direct sum taken over all subsets of α that have cardinality $p+1$. $\square$

With these preparations, we may now prove our main theorem.

Proof of Theorem 6.7.1. Let $F \subseteq \mathrm{Fin}(\alpha)$ be the subsystem of $\mathrm{Fin}(\alpha)$ of all sets $J \in \mathrm{Fin}(\alpha)$ with even cardinality.

Consider the directed system of spectral sequences $\{E(J)^r_{p,q}, d(J)^r\}_{r,p,q}$ along all $J \in F$ from Remark 6.7.7. Lemma 6.7.9 guarantees that the page $E^1_{*,*}$ of the limit spectral sequence has the desired structure.

All constructions in earlier sections behaved well with the differentials $d(J)^r$. Here in Section 6.7, we have applied various functorial direct limit constructions. Therefore the limit spectral sequence has the desired differentials.

Likewise, functoriality of the spectral sequence with respect to ideal decompositions follows from all earlier sections and the functoriality of direct limits.

Finally, we must prove the strong convergence. All direct limit results from Section 6.6 about strong convergence along the directed system $(\mathbb{N}, \leq)$ continue to hold for our directed system F because all infinite chains in F have the order type $(\mathbb{N}, \leq)$: Whenever the symbol $n \in \mathbb{N}$ determines a K-theoretic degree shift in Section 6.6, this may be replaced with $n = |J|$ for $J \in F$. Even though the diagram in Definition 6.6.8 of the morphism $\mathrm{colim}_n\, i(n,p)_s$ relies on the linearity of $(\mathbb{N}, \leq)$, the construction itself is worded purely with direct limits: Objects in this category are morphisms of the form $i(n,p)_s$ and morphisms in this category are commutative diagrams. This construction does not require linearity of the underlying system, yet provides the desired filtration on $E^\infty_{*,*}$. □

7 Infinite coarse excision

7.1 Direct limits of coarse algebras

The spectral sequence for infinite sums of C*-ideals allows us to strengthen Theorem 5.5.2, the spectral sequence for finite coarsely excisive covers, to infinite coarsely excisive covers.

Proposition 7.1.1. *Let (X, d) be a coarse space and let $\{X_\beta\}_{\beta\in\alpha}$ be a coarsely excisive cover of (X, d). The index set α may be finite, countably infinite, or uncountable. Let $\mathfrak{F}^*$ be either the functor $\mathfrak{C}^*$ from the coarse category to $\underline{\text{C*A}}$ or one of the functors $\mathfrak{D}^*$ or $\mathfrak{Q}^*$ from the coarse-continuous category to $\underline{\text{C*A}}$. For each $\beta \in \alpha$, consider the C*-ideal $\mathfrak{F}^*X_\beta \cong \mathfrak{F}^*(X_\beta \subseteq X)$ of $\mathfrak{F}^*X$.*

Then the direct limit of finite sums of ideals,

$$\overline{\bigcup_{\substack{J\subseteq\alpha\\ |J|\in\mathbb{N}}} \Big(\sum_{j\in J} \mathfrak{F}^*(X_j \subseteq X)\Big)}, \tag{7.1.1.1}$$

is a C-ideal of $\mathfrak{F}^*X$.*

Proof. Let A denote the C*-algebra in expression 7.1.1.1.

For each $\beta \in \alpha$, the algebra $\mathfrak{F}^*(X_\beta \subseteq X)$ is a C*-ideal in $\mathfrak{F}^*X$. The inclusion $\mathfrak{F}^*(X_\beta \subseteq X) \to \mathfrak{F}^*X$ factors through any finite sum $\sum_{j\in J} \mathfrak{F}^*(X_j \subseteq X)$ when $\beta \in J$, and the inclusion of that finite sum in $\mathfrak{F}^*X$ factors again through A; thus the closed A is a sub-C*-algebra of $\mathfrak{F}^*X$.

To check that A is a C*-ideal in $\mathfrak{F}^*X$, it remains to show that A is an algebraic two-sided ideal. Given $a \in A$ and $b \in \mathfrak{F}^*X$, find a sequence of finite sets $(J_n)_{n\in\mathbb{N}}$ with $J_n \subseteq \alpha$ and a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \sum_{j\in J_n} \mathfrak{F}^*(X_j \subseteq X)$ that converges to a. Finite sums of C*-ideals $\mathfrak{F}^*(X_j \subseteq X)$ are again C*-ideals, thus both $(a_n b)_{n\in\mathbb{N}}$ and $(b a_n)_{n\in\mathbb{N}}$ stay within the closed A. These sequences converge in A to ab and ba respectively because multiplication is continuous. □

Remark 7.1.2. For finite decompositions, the direct limit of finite sums from expression 7.1.1.1 equals $\mathfrak{F}^*X$ by Theorem 5.5.1.

7.2 Corollaries for coarsely excisive covers

Theorem 7.2.1 (Spectral sequence for coarsely excisive covers)**.** *Let (X, d) be a coarse space and let $\{X_\beta\}_{\beta\in\alpha}$ be a coarsely excisive cover of (X, d). Let $\mathfrak{F}^*$ be either the functor $\mathfrak{C}^*$ from the coarse category to $\underline{\text{C*A}}$ or one of the functors $\mathfrak{D}^*$ or $\mathfrak{Q}^*$ from* *the coarse-continuous category to* $\underline{\mathrm{C^*A}}$*. There is a spectral sequence* $\{E^r_{p,q}, d^r\}_{r,p,q}$ *with*

$$E^1_{p,q} \cong \begin{cases} \bigoplus_{|J|=p+1} K_q \mathfrak{F}^* \Big(\bigcap_{j \in J} X_j \Big) & \text{for } p \geq 0, \\ 0 & \text{for } p < 0, \end{cases}$$

where J *ranges over all nonempty finite subcollections of indices in* α*. For finite* α*, this spectral sequence converges strongly to* $K_* \mathfrak{F}^* X$*. In general, the spectral sequence converges strongly to the K-theory of* $\overline{\bigcup_J \sum_{j \in J} \mathfrak{F}^*(X_j \subseteq X)}$*, a C*-ideal of* $\mathfrak{F}^* X$*, where* J *ranges over all finite subcollections of indices in* α*. The spectral sequence is functorial with respect to morphisms (coarse maps for* $\mathfrak{C}^*$*, or coarse and continuous maps for* $\mathfrak{D}^*$ *and* $\mathfrak{Q}^*$*) to other coarse spaces with compatible coarsely excisive covers (Definition 5.1.4).*

Proof. Apply the spectral sequence from Theorem 6.7.1 about arbitrary sums of abstract C*-algebras to the algebras from Theorem 5.5.1 for coarse spaces.

By Theorem 6.7.1, the spectral sequence converges strongly to the K-theory of the norm closure of finite sums of the input ideals. These ideals are $\mathfrak{F}^* X_\beta \cong \mathfrak{F}^*(X_\beta \subseteq X)$ and we have described the norm closure of their finite sums in Proposition 7.1.1.

The special case for finite coarsely excisive covers follows from Remark 7.1.2. □

Remark 7.2.2 (Warning about uncountable decompositions)**.** Let (X, d) be a coarse space. The important C*-algebras $\mathfrak{C}^* X$, $\mathfrak{D}^* X$, and $\mathfrak{Q}^* X$ are defined via very ample representations $\varrho\colon \mathscr{C}_0 X \to BH$ for a separable Hilbert space H. Separability of the Hilbert space is crucial for several isomorphism theorems. Ampleness of the representation guarantees that no two functions $f \neq f' \in \mathscr{C}_0 X$ may be represented on that separable Hilbert space by operators $\varrho(f)$ and $\varrho(f')$ that differ only by a compact operator.

If the coarsely excisive cover $\{X_\beta\}_{\beta \in \alpha}$ of (X, d) has an uncountable index set α, the separability requirement may force $\mathfrak{F}^* X_\beta = \mathfrak{F}^* X_{\beta'}$ for many $\beta \neq \beta'$, or may force outright triviality of ideals. It appears hard to construct interesting examples for uncountable coarse excision.

With Theorem 7.2.1, we can strengthen the following Mayer-Vietoris result. Let (X, d) be a coarse space and X_0, $X_1 \subseteq X$ such that $\{X_0, X_1\}$ is a coarsely excisive cover. All rows and columns in the following diagram are long exact sequences; the index j in $\bigoplus_j$ ranges over $\{0,1\}$:

$$\begin{tikzcd}
\cdots \arrow[r] & K_{s+1}\mathfrak{C}^*X \arrow[r] \arrow[d] & K_s\mathfrak{C}^*(X_0\cap X_1) \arrow[r] \arrow[d] & \bigoplus_j K_s\mathfrak{C}^*X_j \arrow[r] \arrow[d] & K_s\mathfrak{C}^*X \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{s+1}\mathfrak{D}^*X \arrow[r] \arrow[d] & K_s\mathfrak{D}^*(X_0\cap X_1) \arrow[r] \arrow[d] & \bigoplus_j K_s\mathfrak{D}^*X_j \arrow[r] \arrow[d] & K_s\mathfrak{D}^*X \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{s+1}\mathfrak{Q}^*X \arrow[r] \arrow[d] & K_s\mathfrak{Q}^*(X_0\cap X_1) \arrow[r] \arrow[d] & \bigoplus_j K_s\mathfrak{Q}^*X_j \arrow[r] \arrow[d] & K_s\mathfrak{Q}^*X \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_s\mathfrak{C}^*X \arrow[r] & K_{s-1}\mathfrak{C}^*(X_0\cap X_1) \arrow[r] & \bigoplus_j K_{s-1}\mathfrak{C}^*X_j \arrow[r] & K_{s-1}\mathfrak{C}^*X \arrow[r] & \cdots .
\end{tikzcd}$$

The columns are exact by definition of $\mathfrak{C}^*$, $\mathfrak{D}^*$, and $\mathfrak{Q}^*$. Commutativity follows from the naturality of the Mayer-Vietoris sequence.

Proposition 7.2.3. *Let (X,d) be a coarse space and let $\{X_\beta\}_{\beta\in\alpha}$ be a coarsely excisive cover of (X,d). Consider the following diagram with exact columns; horizontal arrows are induced by (direct sums of) inclusions, and unions range over all finite index sets $J\subseteq\alpha$:*

$$\begin{tikzcd}
\bigoplus_{\beta\in\alpha} K_s\mathfrak{C}^*X_\beta \arrow[r] \arrow[d] & K_s\big(\overline{\bigcup_J \sum_{j\in J} \mathfrak{C}^*(X_j\subseteq X)}\big) \arrow[r] \arrow[d] & K_s\mathfrak{C}^*X \arrow[d] \\
\bigoplus_{\beta\in\alpha} K_s\mathfrak{D}^*X_\beta \arrow[r] \arrow[d] & K_s\big(\overline{\bigcup_J \sum_{j\in J} \mathfrak{D}^*(X_j\subseteq X)}\big) \arrow[r] \arrow[d] & K_s\mathfrak{D}^*X \arrow[d] \\
\bigoplus_{\beta\in\alpha} K_s\mathfrak{Q}^*X_\beta \arrow[r] \arrow[d] & K_s\big(\overline{\bigcup_J \sum_{j\in J} \mathfrak{Q}^*(X_j\subseteq X)}\big) \arrow[r] \arrow[d] & K_s\mathfrak{Q}^*X \arrow[d] \\
\bigoplus_{\beta\in\alpha} K_{s-1}\mathfrak{C}^*X_\beta \arrow[r] & K_{s-1}\big(\overline{\bigcup_J \sum_{j\in J} \mathfrak{C}^*(X_j\subseteq X)}\big) \arrow[r] & K_{s-1}\mathfrak{C}^*X.
\end{tikzcd}$$

Then this diagram commutes.

Proof. The columns are exact again by definition of $\mathfrak{C}^*$, $\mathfrak{D}^*$, and $\mathfrak{Q}^*$. Commutativity follows from continuity of K-theory – the algebras in the center column are direct limits – and from functoriality of the spectral sequence in Theorem 6.5.1 with respect to morphisms between C*-algebras; here, direct sums of inclusion morphisms. ☐

7.3 The coarse space $\mathbb{Z}^\infty$

Consider the free $\mathbb{Z}$-module $\mathbb{Z}^\infty = \bigoplus_{\mathbb{N}} \mathbb{Z}$ of $\mathbb{N}$-indexed tuples $(x_0, x_1, \ldots, x_n, \ldots)$ with only finitely many entries different from 0. This space can be metrized in different ways.

Definition 7.3.1 (Weight functions, weighted 1-metric)**.** Let $w\colon \mathbb{N} \to \mathbb{R}_{>0}$ be an arbitrary function, for example

$$w\colon n \mapsto n+1, \qquad w\colon n \mapsto 1, \text{ or } \qquad w\colon n \mapsto \frac{1}{n+1}.$$

We call w a *weight function*. Given a weight function w, define a metric d_w on $\mathbb{Z}^\infty$:

$$d_w\big((x_0, x_1, \ldots x_n, 0, 0, \ldots), (y_0, y_1, \ldots, y_{n'}, 0, 0, \ldots)\big) = \sum_{j=0}^{\infty} w(j)\, |x_j - y_j| \,.$$

Example 7.3.2. For the constant weight function $w\colon n \mapsto 1$, the metric d_w coincides with the usual 1-metric d_1.

Remark 7.3.3 (Topological properties of $\mathbb{Z}^\infty$)**.** For $w\colon n \mapsto n+1$, the metric d_w is proper: With minimum distance $k+1$ between points in the k-th dimension, any ball of finite diameter is finite, thus compact. For $w\colon n \mapsto 1$ or $w\colon n \mapsto \frac{1}{n+1}$, closed d_w-balls with finite radius larger than 1 are not compact anymore. Under $w\colon n \mapsto \frac{1}{n+1}$, the space $(\mathbb{Z}^\infty, d_w)$ is not even locally compact.

More than with the topological properties of these spaces, we are concerned with their coarse properties.

The identity on any coarse space is a coarse map: Choose $S = R$ in Definition 2.5.2. But the identity $\mathbb{Z}^\infty \to \mathbb{Z}^\infty$ fails to be a coarse map when the two spaces are metrized according to two different weight functions among $n \mapsto n+1$, $n \mapsto 1$, and $n \mapsto \frac{1}{n+1}$. The identity ceases to be uniformly expansive: Points with distance 1 in dimension n may have distance $n+1$ or even $(n+1)^2$ in the target space. No constant $S > 0$ can serve as an upper bound across all dimensions n.

If the identity fails as a coarse equivalence, can other maps substitute? The answer is no, for a similar reason:

Proposition 7.3.4 (Coarse properties of $\mathbb{Z}^\infty$)**.** *The weight functions $n \mapsto n+1$ and $n \mapsto 1$ generate different coarse structures on $\mathbb{Z}^\infty$; i.e., there is no coarse equivalence according to Definition 2.5.6.*

Proof. Assume that there is a pair of coarse equivalences

$$f\colon (\mathbb{Z}^\infty, d_{n\mapsto n+1}) \to (\mathbb{Z}^\infty, d_{n\mapsto 1}),$$
$$g\colon (\mathbb{Z}^\infty, d_{n\mapsto 1}) \to (\mathbb{Z}^\infty, d_{n\mapsto n+1}).$$

Then let $S > 0$ satisfy all of these conditions:

- For all $x \in \mathbb{Z}^\infty$, we have $d_{n\mapsto n+1}(x, gfx) \leq S$, this is possible because f and g are a pair of coarse equivalences.
- Whenever x, $x' \in \mathbb{Z}^\infty$ with $d_{n\mapsto 1}(x, x') \leq 1$, then $d_{n\mapsto n+1}(gx, gx') \leq S$, this is possible because g is a coarse map.
- For simplicity, $S \in \mathbb{N}$.

Let x and x' have $d_{n\mapsto 1}(x, x') \leq 1$. Then $d_{n\mapsto n+1}(gx, g0) \leq S$ and in all dimensions S, $(S+1)$, $(S+2)$, ..., the coordinates of $g(x)$ and $g(x')$ must be identical.

Any two points y, y' can be linked by a finite sequence of hops between two points each with $d_{n\mapsto 1}$-distance ≤ 1. By induction, $g(y)$ and $g(y')$ agree in all coordinates from dimension S onwards.

Let $z \in (\mathbb{Z}^\infty, d_{n\mapsto n+1})$ have different coordinates than $g(y)$ in dimension S. Then $(g \circ f)(z)$ and z have distance at least $S+1$. This is not allowed when f and g are coarse equivalences. □

Remark 7.3.5. When we replace the weight function $n \mapsto 1$ with $n \mapsto \frac{1}{n+1}$, the same argument shows that the coarse structure induced by that weight function is not equivalent to $(\mathbb{Z}^\infty, d_{n\mapsto n+1})$ either.

Notation 7.3.6. Let $X \subseteq \mathbb{Z}^\infty$ be a set. For a metric d_w as above and $R > 0$, we write $N_w(X, R)$ instead of $N_{d_w}(X, R)$ for the R-neighborhood of X under the metric d_w according to Definition 2.6.1.

Lemma 7.3.7. *For all three weight functions $w\colon n \mapsto n+1$, $w\colon n \mapsto 1$, and $w\colon n \mapsto \frac{1}{n+1}$, the decomposition $\{X_j\}_{j\in\mathbb{N}}$ with*

$$X_j = \mathbb{Z}^j_{\geq 0} \times \mathbb{Z}_{\leq 0} \times \mathbb{Z}^\infty$$

defines a coarsely excisive cover of $(\mathbb{Z}^\infty, d_w)$.

The block decomposition of $\mathbb{R}^n$ from Definition 5.6.6 was similar, but covered only a finite-dimensional space.

Proof of Lemma 7.3.7. Consider a finite nonempty index set $J \subseteq \mathbb{N}$ and the finite subcollection $\{X_j\}_{j\in J}$ of the decomposition $\{X_j\}_{j\in\mathbb{N}}$. We have to show that, for $R > 0$, there exists $S > 0$ with

$$\bigcap_{j\in J} N_w(X_j, R) \subseteq N_w\Big(\bigcap_{j\in J} X_j, S\Big).$$

Let n denote the highest index in J. We may write

$$\bigcap_{j\in J} X_j = Y_0 \times Y_1 \times \ldots \times Y_{n-1} \times \mathbb{Z}_{\leq 0} \times \mathbb{Z}^\infty,$$

where $Y_j = \{0\}$ for $j \in J$, otherwise $Y_j = \mathbb{Z}_{\geq 0}$. For easier notation, we will write Y_n for $\mathbb{Z}_{\leq 0}$.

We are interested in the R-neighborhood of each X_j for $R > 0$ and in the S-neighborhood of the intersection for a suitable S. Let $x = (x_0, x_1, x_2, \ldots)$ be a point in $\mathbb{Z}^\infty$. The coordinates after the n-th coordinate do not matter anymore: For the intersection $\bigcap_{j\in J} X_j$, the distance of x to $\bigcap_{j\in J} X_j$ depends only on the early coordinates up to the n-th.

Since the metric d_w on $\mathbb{Z}^\infty$ is a weighted 1-metric – distance is the weighted sum of the dimension-wise distances – it makes sense to decompose $(\mathbb{Z}^\infty, d_w)$ into a product of metric spaces, $\mathbb{Z}^{n+1} \times \mathbb{Z}^\infty$, of the first $n+1$ dimensions and the remainder $\mathbb{Z}^\infty$ that is irrelevant for the chosen J. Let p and q be the projections for this product decomposition,

$$\begin{aligned} p&\colon \mathbb{Z}^\infty \to \mathbb{Z}^{n+1}, & p(x_0, x_1, x_2, \ldots) &= (x_0, x_1, \ldots, x_n),\\ q&\colon \mathbb{Z}^\infty \to \mathbb{Z}^{\infty}, & q(x_0, x_1, x_2, \ldots) &= (x_{n+1}, x_{n+2}, \ldots). \end{aligned}$$

Both projections admit one-sided inverse by padding all other coordinates with zeros. We pull back the metric d_w to either factor along these inverses. For $x \in \mathbb{Z}^\infty$ and a space Y that is either $Y = X_j$ for a $j \in J$ or $Y = \bigcap_{j\in J} X_j$, we then have

$$d_w(x, Y) = d_w\big(px, p(Y)\big) + \underbrace{d_w\big(qx, q(Y)\big)}_{=\,0}; \tag{7.3.7.1}$$

the rightmost summand vanishes because both $q(X_j)$ and $q\big(\bigcap_{j\in J} X_j\big)$ are the entire range $q(\mathbb{Z}^\infty)$ by construction. On the finite-dimensional remainder $\mathbb{Z}^{n+1}$, the weight function w admits an upper and a lower bound: There is a constant $M \geq n+1$ such that $\frac{1}{M} \leq w(k) \leq M$ for all $k < n+1$. For x, $y \in \mathbb{Z}^{n+1}$, Lemma 5.6.16 shows

$$d_w(x,y) \leq M d_1(x,y) \leq M d_\infty(x,y) \leq M^2 d_1(x,y) \leq M^3 d_w(x,y). \tag{7.3.7.2}$$

The finite-dimensional space $\mathbb{Z}^{n+1}$ embeds isometrically into $\mathbb{R}^{n+1}$. This embedding is not necessarily the inclusion $\mathbb{Z}^{n+1} \subseteq \mathbb{R}^{n+1}$; rather, depending on w, its image lattice is scaled differently per dimension. Still, we can now apply Proposition 5.6.15 for the sup-metric d_∞ on $\mathbb{R}^{n+1}$:

$$\bigcap_{j\in J} N_\infty\big(p(X_j), R\big) = N_\infty\Big(\bigcap_{j\in J} p(X_j), R\Big).$$

Together with 7.3.7.1 and 7.3.7.2, we conclude that $S = M^3R$ certifies the desired coarse excisiveness in the infinite-dimensional space $\mathbb{Z}^\infty$:

$$\bigcap_{j\in J} N_w(X_j, R) \subseteq N_w\Big(\bigcap_{j\in J} X_j, M^3R\Big).$$ □

Proposition 7.3.8. *Let $\{X_j\}_{j\in\mathbb{N}}$ be the coarsely excisive cover of $\mathbb{Z}^\infty$ as in Lemma 7.3.7 and let A denote the direct limit C^*-algebra*

$$A = \overline{\bigcup_{\substack{J\subseteq\mathbb{N}\\ |J|\in\mathbb{N}}} \sum_{j\in J} \mathfrak{C}^*(X_j \subseteq \mathbb{Z}^\infty)} = \overline{\sum_{j\in\mathbb{N}} \mathfrak{C}^*(X_j \subseteq \mathbb{Z}^\infty)}.$$

Under any of the three considered weight functions, the algebra A then has trivial K-theory: $K_*A = 0$.

Proof. We may use our spectral sequence from Theorem 6.5.1 because the cover $\{X_j\}_{j\in\mathbb{N}}$ is coarsely excisive.

Each X_j is flasque (Definition 5.6.3) because it contains the flasque factor $\mathbb{Z}_{\leq 0}$.

For finite $J \subseteq \mathbb{N}$, the intersection $\bigcap_{j\in J} X_j$ is again flasque: Let n be the largest index in J. We may describe this intersection by listing its one-dimensional factors: First, there are n factors that we may ignore. Then there is the flasque factor $\mathbb{Z}_{\leq 0}$. All further factors form a copy of $\mathbb{Z}^\infty$. Because of the flasque factor, we conclude that $K_*\mathfrak{C}^*\big(\bigcap_{j\in J} X_j\big) = 0$.

Since this holds for all finite $J \subseteq \mathbb{N}$, Theorem 6.5.1 gives a spectral sequence with first page $E^1_{*,*} = 0$, converging to $K_*A = 0$. □

The countable intersection $\bigcap_{j\in\mathbb{N}} X_j$ is a single point, but infinite intersections do not appear in the spectral sequence.

7.4 Wedge sum of rays

Let α be either $\mathbb{N}$ or a finite cardinality with $\alpha > 0$. Let $X_\beta = (\mathbb{R}_{\geq 0}, 0)$ be a ray for all $\beta \in \alpha$, pointed at the origin. Define

$$X = \bigvee_{\beta\in\alpha} X_\beta,$$

a finite or countable wedge sum of the half-open rays glued together at their origins. Define a coarse structure on X by the metric

$$d(x,y) = \begin{cases} |x-y| & \text{if } x \text{ and } y \text{ lie in the same ray,} \\ |x|+|y| & \text{otherwise.} \end{cases}$$

If local compactness were desired, we could remove the common point 0. This would not change any large-scale properties of X. But in this example, local compactness is irrelevant.

Lemma 7.4.1. *Let $Y_\beta = X_0 \cup X_\beta$ for all $\beta \in \alpha$. Cover X by $\{Y_\beta : \beta \in \alpha\}$.*
This is a coarsely excisive cover of the wedge sum X.

Proof. We have to check: For all finite $\{Y_{\beta(0)}, Y_{\beta(1)}, \dots, Y_{\beta(n-1)}\}$ and all $R > 0$, there exists $S > 0$ such that the n-fold intersection of the R-neighborhoods lies in the S-neighborhood of the intersection:

$$\bigcap_{j<n} N_d(Y_{\beta(j)}, R) \subseteq N_d\Big(\bigcap_{j<n} Y_{\beta(j)}, S\Big).$$

For $n = 1$, this is trivial with $S = R$. For $n > 1$, consider n pairwise different $Y_{\beta(j)}$: The intersection $\bigcap_{j<n} Y_{\beta(j)}$ is always $Y_0 = X_0$.

The common point 0 joins all rays X_β, its R-neighborhood $N_d(\{0\}, R)$ is therefore the union of the intervals $[0, R]$ from all rays X_β. Thus:

$$\bigcap_{j<n} N_d(Y_{\beta(j)}, R) = X_0 \cup N_d(\{0\}, R),$$
$$N_d\Big(\bigcap_{j<n} Y_{\beta(j)}, S\Big) = N_d(X_0, S) = X_0 \cup N_d(\{0\}, S).$$

Choose $S = R$ to see that the cover $\{Y_\beta\}_{\beta\in\alpha}$ is coarsely excisive. □

Proposition 7.4.2. *For the wedge sum $X = \bigvee_{\beta\in\alpha} X_\beta$ with each $X_\beta = (\mathbb{R}_{\geq 0}, 0)$ a ray pointed at the origin and the coarsely excisive cover $\{Y_\beta\}_{\beta\in\alpha}$ from Lemma 7.4.1, let A be the direct limit C^*-algebra of sums $\sum_{j\in J} \mathfrak{C}^*(Y_j \subseteq X)$ over finite sets $J \subseteq \alpha$.*

Then

$$K_s A = \begin{cases} 0 & \textit{for } s \textit{ even,} \\ \bigoplus\limits_{\substack{\beta\in\alpha \\ \beta\neq 0}} \mathbb{Z} & \textit{for } s \textit{ odd.} \end{cases}$$

Proof. The set Y_0 remains a flasque ray. Each other Y_β is coarsely equivalent to $\mathbb{R}$ and therefore has K-theory $K_0\mathfrak{C}^*Y_n = 0$ and $K_1\mathfrak{C}^*Y_n = \mathbb{Z}$. This determines the column $E^1_{0,*}$ of the first page.

Each finite intersection of at least two different Y_n is the flasque space $Y_0 = X_0$. The K-theory of its Roe algebra vanishes. Therefore the $E^1_{*,*}$-term looks as follows:

$$\begin{array}{r|cccc}
q & & & & \\
3 & & \vdots & & \\
2 & & 0 & & \\
1 & & \bigoplus\limits_{\substack{\beta\in\alpha \\ \beta\neq 0}} \mathbb{Z} & & \\
0 & & 0 & & \\
-1 & & \bigoplus\limits_{\substack{\beta\in\alpha \\ \beta\neq 0}} \mathbb{Z} & & \\
\hline
 & -1 & 0 & 1 & 2 \quad p.
\end{array}$$

This spectral sequence collapses on the first page. We may read the K-theory of $A \subseteq \mathfrak{C}^*X$ from the only nonzero column: If α is countably infinite, the dimension of the free $\mathbb{Z}$-module in odd degrees is countably infinite; if α is finite, the dimension is $\alpha - 1$. $\square$

For finite α, an alternative proof to compute $A = \mathfrak{C}^*X$ by induction repeats the Mayer-Vietoris principle $\alpha - 1$ times for two-fold coverings: Glue single rays, one after another, to the wedge sum that starts with a single ray.

8 Generalizations

8.1 KO-theory

Instead of K-theory of C*-algebras over $\mathbb{C}$, we may examine *KO-theory* for a C*-algebra A over $\mathbb{R}$, denoted KO_*A. All basic definitions carry over without change, turning KO_* into a $\mathbb{Z}$-graded covariant continuous functor into abelian groups.

The major difference is the degree of the Bott isomorphism: Instead of $K_sA \cong K_{s+2}A$, real Bott periodicity admits a natural isomorphism $\beta\colon KO_sA \cong KO_{s+8}A$ for all $s \in \mathbb{Z}$. As a result, for C*-ideals $I \subseteq A$, the six-term exact sequence

$$\cdots \to K_0I \to K_0A \to K_0(A/I) \xrightarrow{\partial_2\circ\beta} K_1I \to K_1A \to K_1(A/I) \xrightarrow{\partial_1} K_0I \to \cdots$$

becomes a 24-term exact sequence in the real case:

$$\cdots \to KO_0A \to KO_0(A/I) \xrightarrow{\partial_8\circ\beta} KO_7I \to KO_7A \to KO_7(A/I) \xrightarrow{\partial_7} KO_6I \to \cdots .$$

Looking back to the constructions of the various spectral sequences, we relied on the Bott isomorphism merely for the construction of the degree-amending links $\lambda(n)\colon E(n)^r_{p,q} \to E(n+2)^r_{p,q}$ from Definition 6.5.3. These morphisms and Proposition 6.5.4 about the existence of the limit spectral sequence can be adapted to work with KO-theory: Define

$$\lambda_{\mathbb{R}}(n)\colon E(n)^r_{p,q} \to E(n+8)^r_{p,q}$$

by chaining 8 links $\{\ell(n)^r_{p,q}\}_{r,p,q}$ from Definition 6.4.2 – instead of only 2 such links in the case of K-theory – with the real Bott isomorphism. The direct limit along these $\lambda_{\mathbb{R}}(8n)$ for $n \to \infty$ does not depend on the position of the Bott isomorphism within the chain nor on whether we consider the directed system along $\lambda_{\mathbb{R}}(8n)$, $\lambda_{\mathbb{R}}(8n+1)$, …, or $\lambda_{\mathbb{R}}(8n+7)$.

Convergence is proven as in Section 6.6 and generalized to uncountable ideal decompositions as in Section 6.7. This yields a well-defined spectral sequence that computes the KO-theory of C*-ideal sums: The statement from Theorem 6.7.1 holds when we replace K-theory with KO-theory.

8.2 Group actions

Let (X, d) be a metric space. Let G be a countable discrete group that acts on X freely and properly by d-isometries. This action extends to $\mathscr{C}_0X$ via $(gf)(x) = f(g^{-1}x)$ for $g \in G$, $f \in \mathscr{C}_0X$, and $x \in X$. In addition to a very ample representation $\varrho\colon \mathscr{C}_0X \to BH$ for a separable Hilbert space H, let $U\colon G \to H$ be a unitary representation with $U(g)\varrho(f) = \varrho(gf)U(g)$.

This gives rise to C*-algebras $C^*_G X$ and $D^*_G X$ by changing the usual definitions of $\mathfrak{C}^* X$ and $\mathfrak{D}^* X$: The norm closure is taken of only the G-invariant operators in BH that satisfy all other requirements of $\mathfrak{C}^* X$ and $\mathfrak{D}^* X$, respectively. Furthermore, define $Q^*_G X = D^*_G X / C^*_G X$.

In [Sie12, Definition 3.6], for a G-invariant subspace $Y \subseteq X$ that we may assume to be closed, P. Siegel constructs the relative C*-algebras $C^*_G(Y \subseteq X)$ and $D^*_G(Y \subseteq X)$ by imposing on operators in $C^*_G X$ and $D^*_G X$ the conditions from Section 5.2 for support near Y and local compactness outside Y. Finally, define $Q^*_G(Y \subseteq X)$ as the quotient $D^*_G(Y \subseteq X)/C^*_G(Y \subseteq X)$. There is a long exact sequence for $s \in \mathbb{Z}$,

$$\cdots \to K_s C^*_G X \to K_s D^*_G X \to K_s Q^*_G X \to K_{s-1} C^*_G X \to \cdots .$$

Furthermore, $Q^*_G X \cong \mathfrak{Q}^* X_G$. The sequence may thus be rewritten with the K-homology of X_G instead of $Q^*_G X$ according to Remark 2.4.12.

Let the functor F^*_G stand for either C^*_G, D^*_G, or Q^*_G. According to [Sie12, Propositions 3.8, 3.9], for closed G-invariant subspaces $Y \subseteq X$ and $s \in \mathbb{Z}$, we have

$$K_s F^*_G(Y \subseteq X) \cong K_s F^*_G Y.$$

For G-invariant coarsely excisive covers $\{X_0, X_1\}$ of X, we have

$$\begin{aligned} F^*_G(X_0 \subseteq X) + F^*_G(X_1 \subseteq X) &= F^*_G X, \\ F^*_G(X_0 \subseteq X) \cap F^*_G(X_1 \subseteq X) &= F^*_G(X_0 \cap X_1 \subseteq X). \end{aligned}$$

This leads to a Mayer-Vietoris exact sequence. We may expect a generalization of our spectral sequence to arbitrary G-invariant coarsely excisive covers $\{X_\beta\}_{\beta\in\alpha}$ of X. The definitions of $C^*_G(Y \subseteq X)$ and $D^*_G(Y \subseteq X)$ treat the G-invariance in the least intrusive way possible. There should be no difficulty in adapting the equations to finite selections $J \subseteq \alpha$ of the coarsely excisive cover:

$$\begin{aligned} K_s\Big(\sum_{j\in J} F^*_G(X_j \subseteq X)\Big) &\cong K_s F^*_G\Big(\bigcup_{j\in J} X_j\Big), \\ K_s\Big(\bigcap_{j\in J} F^*_G(X_j \subseteq X)\Big) &\cong K_s F^*_G\Big(\bigcap_{j\in J} X_j\Big), \end{aligned}$$

with F^*_G standing for C^*_G, D^*_G, or Q^*_G. The constructions will be similar to those leading to our Theorem 5.5.1 for these equations where G is trivial.

This provides a G-invariant version of our spectral sequence from Theorem 7.2.1 for coarsely excisive covers.

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