Constructive aspects of string-localized quantum field theory

Dissertation

for the award of the degree
"Doctor rerum naturalium" (Dr. rer. nat.)
of the Georg-August-Universität Göttingen

within the doctoral program ProPhys
of the Georg-August University School of Science (GAUSS)

submitted by

Christian Gaß

from Kirchheimbolanden

Göttingen, 2022

Thesis committee

Prof. Dr. Karl-Henning Rehren,
Institut für theoretische Physik, Georg-August-Universität Göttingen

Prof. Dr. Dorothea Bahns,
Mathematisches Institut, Georg-August-Universität Göttingen

Prof. Dr. Steffen Schumann,
Institut für theoretische Physik, Georg-August-Universität Göttingen

Members of the examination board

Reviewer:
Prof. Dr. Karl-Henning Rehren,
Institut für theoretische Physik, Georg-August-Universität Göttingen

Second Reviewer:
Prof. Dr. Dorothea Bahns,
Mathematisches Institut, Georg-August-Universität Göttingen

Further members of the examination board:

Prof. Dr. Steffen Schumann,
Institut für theoretische Physik, Georg-August-Universität Göttingen

Prof. Laura Covi, PhD,
Institut für theoretische Physik, Georg-August-Universität Göttingen

Prof. Dr. Stefan Kehrein,
Institut für theoretische Physik, Georg-August-Universität Göttingen

Prof. Dr. Jens Niemeyer,
Institut für Astrophysik, Georg-August-Universität Göttingen

Date of the oral examination: 26.04.2022

Abstract

String-localized quantum field theory (SLFT) as an alternative to gauge theory has been under investigation for the last decade because of its conceptual benefits. This thesis is dedicated to the development and investigation of constructive tools for both perturbative and non-perturbative aspects of SLFT.

On the non-perturbative side, we provide explicit formulas for the two-point function and the propagator of the so-called "escort field" of the string-localized photon potential. These are needed to describe the dressing of Dirac fields in quantum electrodynamics (QED) with photon clouds and have a potential application in the scattering theory of these dressed Dirac fields. We continue to derive the two-point functions and propagators of string-localized potentials starting from the two-point function of the photon escort field. We also prove no-go results for similar non-perturbative dressing constructions in models with massless self-interacting fields. In QED, these constructions have been implemented by Mund, Rehren and Schroer and work well. We show that such a Mund-Rehren-Schroer construction already fails at an early stage for massless Yang-Mills theory and the graviton self-coupling.

The main focus of this thesis is on perturbation theory with string-localized fields. We propose a setup for string-localized perturbation theory and outline a new method to define the time-ordering operation if string-localized fields are involved. We determine the wavefront set of string-localized propagators and show that the divergences in loop graph contributions stay pure ultraviolet divergences: the new singularities arising from the string-localization are harmless in the sense of distributions. Furthermore, we investigate and compare different methods to reduce the renormalization freedom in SLFT. These considerations can be seen as a first step towards a full axiomatic construction of the string-localized S-matrix in a Bogoliubov-Epstein-Glaser spirit. Finally, we discuss examples of perturbative constructions in SLFT in low orders of perturbation theory.

Author's note

Parts of this dissertation have been published in peer-reviewed journals or have recently been accepted for publication. The respective manuscripts (see also [35, 37, 38] in the bibliography) are the following:

(1) C. Gaß, J. M. Gracia-Bondía, and J. Mund. Revisiting the Okubo-Marshak argument. *Symmetry*, 13(9), 2021. *Preprint: arXiv:2108.01792* . Author's contribution: Implementation of the string independence principle at first and second order (at tree level) for the case that each potential depends on a different string variable, which is necessary to get well-defined propagators. In particular, realization that string independence can only be achieved when one symmetrizes in all string variables.

(2) C. Gaß. Renormalization in string-localized field theories: a microlocal analysis. Accepted for publication in *Annales Henri Poincaré*. *Preprint: arXiv:2107.12834*.

(3) C. Gaß, K.-H. Rehren, and F. Tippner. On the spacetime structure of infrared divergencies in QED. Accepted for publication in *Letters in Mathematical Physics*. *Preprint: arXiv:2109.10148*. Author's contribution: Original derivation of the explicit formulas for the simply and doubly string-integrated two-point function (the correct $i\varepsilon$-prescription is due to F. Tippner). Original derivation of the "derivative formula" in Prop. 3.14. Derivation of the statements about Minkowski space Gram determinants (with improvements made by K.-H. Rehren). The author only made minor contributions to the sections concerned with vertex operators.

Throughout the thesis, we mark the chapters and sections that are based on (or sometimes verbatim included) parts of these papers.

Acknowledgements

I want to thank Karl-Henning Rehren for his dedicated supervision of my PhD project and for the many valuable discussions that we had in the last three and a half years.

I also thank Dorothea Bahns and Steffen Schumann for being part of the thesis committee and being available whenever I posed mathematical, physical or organizational questions. Moreover, I want to express my gratitude to the remaining members of the examination board.

Furthermore, I owe thanks to Jens Mund and José M. Gracia-Bondía, with whom I worked on a paper and who proofread another one that entered in this dissertation. Their comments, explanations and suggestions were appreciated and very welcome.

I thank the Studienstiftung des deutschen Volkes e.V. for both the financial support and their additional offers, which allowed me to interact with PhD students working in other fields.

Last but not least, I thank Karim Shedid Attifa and Christian T. Preuß for proofreading this dissertation and giving me an external feedback on some of the presented concepts.

Contents

Chapter 1

Background

Quantum field theory (QFT) as the fundamental physical theory describing interactions between elementary particles has been a tremendous success. Over the last decades, the predictions of QFT, especially of quantum electrodynamics (QED), have been tested and validated on a wide range of energy scales and turned out to be outstandingly accurate. From a mathematical perspective, the precision and accuracy of those predictions is somewhat surprising, for QFT is typically formulated perturbatively in terms of formal power series, which come with several obstacles. First, these series need to be renormalized in order to obtain (ultraviolet-)finite results at each order of the expansion. Second, the question of convergence of the series after renormalization [22] and the existence of the adiabatic limit [32] have not yet been answered in a satisfactory way, despite recent progress [20] on the existence of the weak adiabatic limit. Furthermore, the perturbative description of the strong interaction breaks down at low energy scales and so far, the formation of bound particles like hadrons has to be modelled by effective or stochastic approaches [30]. Despite constant efforts for almost a century, no interacting quantum field theory in four spacetime dimensions has yet been constructed in a mathematically rigorous way and the task to put QFT on a safe mathematical ground is still ongoing.

Mathematical approaches to QFT are usually based on a set of axioms like the axioms of Gårding and Wightman [78] or Haag and Kastler (see for example [40]). From these axioms, one typically tries to rigorously construct an interacting QFT, or at least derive central properties that a QFT, which is subject to the axioms, must possess. In Wightman's approach, the central objects of axiomatization are the quantum fields. One of the Wightman axioms states that quantum fields are operator-valued distributions, which means that after smearing with test function f, a quantum field $\phi(x)$ becomes a – typically unbounded – operator $\phi(f)$ on a domain that is dense in some Hilbert space. The distributional nature of quantum fields is necessary but gives rise to certain subtleties: In order to make quantitative predictions, one needs to rely on perturbation theory, and in perturbative constructions, quantum fields need to be multiplied with each other. However, the product of distributions makes no sense in general and thus, naively taking powers of quantum fields results in divergences, which need to be removed by renormalization. The implementation of renormalization schemes in certain quantum field theoretic setups is a central topic of this thesis.

In contrast to Gårding and Wightman, Haag and Kastler do not axiomatize properties of quantum fields but properties of algebras of observables (therefore, it is usually referred to as algebraic QFT, or AQFT). In the Haag-Kastler approach, one can go back to talking about quantum fields by using an algebra which is generated by polynomials of quantum fields. However, one then loses the power of the theory of bounded operators. An advantage of the Haag-Kastler axioms is that they are easily generalizable to curved spacetimes (see for example [15, 73] for introductions).

AQFT has been considerably boosted by modular theory. This powerful abstract tool in the theory of von-Neumann-algebras turned out to be particularly useful for addressing fundamental issues in QFT, such as locality, covariance and thermal states. From the connection between modular localization and Wigner particles observed by Brunetti, Guido and Longo [10], the modern theory of string-localized quantum fields was born [52, 53]. String-localized field theory (SLFT) is one attempt to better understand quantum field theory by weakening the locality properties of certain auxiliary and non-observable quantum fields. In some cases, it presents an alternative to the well-established gauge theoretic approaches to QFT that suffer from a number of well-known drawbacks like the appearance of ghost fields or negative norm states in intermediate steps, which must be removed by means of complicated procedures such as the BRST procedure [4, 69] (see also [62, 75] for introductions). String-localized fields have been considered since the early days of QFT [19, 42, 46], although with quite different motivation than the cited works of Mund, Schroer and Yngvason [52, 53].

The aim of this thesis is to contribute to the advancement of string-localized quantum field theory, in particular to analytic aspects relevant for perturbation theory. We strive to be as mathematically rigorous as possible. Therefore, the basis for the derivations in this thesis is the most rigorous approach to implement perturbation theory in QFT that is currently available. This is the causal method introduced by Bogoliubov [6], which was later worked out in full detail in a seminal work by Epstein and Glaser [32]. In contrast to most phenomenological approaches, which are typically set up in momentum space, the Bogoliubov-Epstein-Glaser (BEG) approach is formulated in configuration space (position space).

Since both SLFT and BEG perturbation theory are not well-known to many, we give a concise overview of SLFT's history and achievements in the following Section 1.1 and a brief introduction to the BEG scheme in Section 1.2. A more concrete and technical introduction to string-localized fields is then given in Chapter 2. Chapter 3 is dedicated to non-perturbative constructions in SLFT. We perform explicit computations of massless string-localized two-point functions and also prove no-go results for certain constructions involving self-interactions of massless string-localized fields. In Chapter 4, we outline a possible general setup of string-localized perturbation theory, including a prescription for the time-ordering operation if string-localized fields are involved. We then discuss steps towards a BEG scheme in SLFT in Chapter 5, where we in particular investigate the nature of divergences in loop graphs and adjust several renormalization prescriptions, which are known from gauge theory, to the SLFT framework. In Chapter 6, we work out examples in low orders of perturbation theory. Finally, the results are discussed and interpreted in Chapter 7.

1.1 String-localized quantum field theory

In 1935, Jordan introduced finite line integrals over the potential of the Maxwell field strength in his work on a gauge invariant formulation of QED [42]. Exponentials of these integrals are used to dress the fermion field. Two decades after Jordan, Dirac [19] considered similar constructions where the line integrals in the exponentials now extend to (spacelike) infinity, and a few years after Dirac, Mandelstam [46] proposed to formulate QED only in terms of two gauge invariant quantities: the Maxwell field strength $F_{\mu\nu}(x)$ and the modified Dirac field

$$\Psi(x, P) := \psi(x)\ \exp\left\{-ie \int_{-\infty}^{x} d\xi^{\mu}\, A_{\mu}(\xi)\right\}, \tag{1.1}$$

where $\psi(x)$ is the standard Dirac field, P is the spacelike path along which the line integral in Eq. (1.1) is to be taken and A_μ is a potential of $F_{\mu\nu}$, i.e., $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The above mentioned works [19,42,46] can be regarded as the first appearances of string-localization in QFT. The dependence of $\Psi(x, P)$ on the path P becomes a dependence on additional spacelike variables in modern formulations involving string-like quantities, which is described in Section 2.1.

In the 1980's, Steinmann came back to Mandelstam's description, proving a Jost-Schroer theorem [65] and considering perturbative constructions [66] involving string-localized fields. In the first of the cited works, Steinmann sketches two problems of gauge theories: first, the difficulty in interpreting the whole gauge theoretic formalism and in particular in explaining the physical significance of the fields involved; second, the non-applicability of (rigorous) results from axiomatic QFT to formulations that work in physical gauges. He then relates to the formulation of Mandelstam, which can be seen as a solution to the first of the sketched problems. At the time of Steinmann's works, the second problem essentially was unattended in a string-like formulation, with the exceptions of his own Jost-Schroer theorem and considerations by Buchholz and Fredenhagen [14]. The latter had shown that particles are always localized but that in certain cases the corresponding fields, which are only mathematical auxiliary quantities, might be localized in stringlike regions.

It took another two decades until Mund, Schroer and Yngvason [52, 53] gave the modern formulation of string-localized fields, based on mathematical considerations on modular localization by Brunetti, Guido and Longo [10]. Since then, many rigorous statements on string-localized fields have been derived, diminishing Steinmann's second problem. As a motivation, we will briefly sketch the conceptual results that have been derived by now.[1]

- Since the modern formulation of SLFT arose from modular theory, it is not surprising that string-localized fields satisfy the Bisognano-Wichmann property [53], which relates the modular group and conjugation associated with a wedge (and the vacuum state) to Lorentz boosts and reflections associated to this wedge [5].

[1] The following list and the remaining part of this section is essentially an extension of parts of the introduction of the author's paper [35].

The same authors also showed that the wedge algebras satisfy the Reeh-Schlieder property [58].

- An important benefit of SLFT is that the string-localized potential for the massless field strength of arbitrary helicity $s \in \mathbb{N}$ is a Lorentz (or Poincaré) covariant rank-s tensor field that lives on Hilbert space and not on an indefinite Krein space like its point-localized gauge field equivalents [49, 53].

- Closely related to the previous point, SLFT can be formulated without the appearance of unphysical degrees of freedom. The string-localized potentials of the field strength tensors of arbitrary mass $m \geq 0$ and spin respectively helicity $s \in \mathbb{N}$ have the number of degrees of freedom that is expected from fields transforming under the respective Wigner representation. For the massive string-localized potential $A^{(s)}$ of spin s, there is a hierarchy of couplings/relations to string-localized fields of all lower integer tensor ranks $0 \leq r < s$. These fields are usually called "escort fields". In the limit $m \longrightarrow 0$, the escort fields decouple from the original field and from each other, carrying away the excess number of $2s + 1$ spin states over 2 helicity states [49].

- The string-localized potential of the field strength tensor associated with the Wigner (m, s) representation for mass $m > 0$ and spin $s \in \mathbb{N}$ has an improved short distance (SD) scaling behavior. It was conjectured by Mund, Schroer and Yngvason that this improved scaling behavior has a positive effect on the renormalizability of string-localized models compared to their gauge theoretic analogues [53]. In a recent paper, the author of this thesis was able to put their claim on safer mathematical ground [35].

- Besides the representations of finite spin respectively helicity, Wigner's classification of representations of the Poincaré group contains so-called infinite spin representations, which are known to be in conflict with point-locality [79]. String-localization, however allows for the construction of such infinite spin fields [52].

- Stress-energy tensors that yield the correct Poincaré generators can be constructed for massless fields of arbitrary finite and infinite spin/helicity [49, 50, 59], circumventing the Weinberg-Witten theorem [76], which excludes the existence of such stress energy tensors for massless point-localized fields of helicity $s \geq 2$.

- The DVZ discontinuity [70, 81] in the massless limit of massive gravitons, which is present in point-localized theories, can be removed in SLFT [49, 50].

- A possible resolution of the Velo-Zwanziger problem of non-causal propagation [71] within the framework of SLFT has been outlined [61]. However, more work is needed on that matter to clarify all details.

- As mentioned before, the language of string-localized field theory allows to reformulate gauge theories on Hilbert space without the appearance of unphysical degrees of freedom. However, Buchholz et al. [13] pointed out the importance of unphysical degrees of freedom in QED. This criticism has led to a more thorough analysis of the implementation of Gauss' law within the framework of SLFT [48] and to the realization how the unphysical degrees of freedom, i.e., the photon escort field, can have the necessary effect on the interacting theory without being separately present in the field content of the theory. This observation is closely related to the next point in our list and is addressed in more detail in Section 3.2.

- A derivation of the dressing factor of the Dirac field in the spirit of Eq. (1.1) as well as an investigation of its consequences on QED has been worked out recently [51] with the emphasis on the infrared problems of QED.

- The concept of gauge symmetry as a fundamental principle absent in SLFT. The string-localized potentials of the field strength tensors of arbitrary mass $m \geq 0$ and spin respectively helicity $s \in \mathbb{N}$ can be expressed explicitly as line integrals of the latter. From this fact, one can deduce the non-existence of the strong CP problem in string-localized quantum chromodynamics (QCD) [37]. This statement is discussed in more detail in Section 7.1, where we also shed light on a remaining caveat in the argument.

To summarize, extensive research on conceptual aspects of SLFT has revealed many benefits. On the other hand, the implementation of string-localized perturbation theory is only in its beginnings. Besides some conceptual considerations [16, 47], only low-order computations at tree level have been performed. The Lie algebra structure of pure massless Yang-Mills theory and of the weak interaction as well as the chirality of the latter have been derived at tree-level to second order of perturbation theory in a bottom-up approach – the structure of these interactions is constrained by the requirement that the scattering matrix be string independent [37, 39]. Parts of these results are due to the author of this dissertation and are included in full detail in Section 6.1.

Calculations at higher orders of perturbation theory as well as computations of loop graphs involving internal string-localized potentials have not yet been attacked. The main reason for this is the most evident disadvantage of SLFT: The analytic structure of propagators of string-localized potentials is highly complicated. Consequently, an extension of the causal renormalization procedure as described by Epstein and Glaser [32] naively seems very involved and is currently not at hand. In a recent article [35], the author of this thesis proved some statements about the nature of renormalization in SLFT and discussed constraints on the setup of string-localized perturbation theory. These results clear the way for an BEG renormalization scheme in SLFT and are displayed in full detail in Chapters 4 and 5.

1.2 Causal perturbation theory in point-localized QFT

The central object to describe the scattering of quantum fields is the scattering operator, or S-matrix $\mathbb{S}[g]$, which is usually defined as a functional of a test function $g \in \mathcal{S}(\mathbb{R}^{1+3})$

in terms of a formal power series (Dyson series) in $g(x)$ [6],

$$\mathbb{S}[g] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int S_n(x_1, \ldots, x_n)\, g(x_1) \ldots g(x_n)\, dx_1 \ldots dx_n. \tag{1.2}$$

More general, $\mathbb{S}$ can be considered as a functional of a multiplet of test functions $\underline{g} \in \mathcal{S}(\mathbb{R}^{1+3})^k$ but within the framework of this thesis, it is sufficient to consider a single g. A generalization to a multiplet is comparably straight-forward and can be found in [32]. The S_n from Eq. (1.2) are operator-valued distributions that need to be constructed recursively. The distributional nature of the S_n and the fact that $g \in \mathcal{S}(\mathbb{R}^{1+3})$ is a test function are crucial for the construction. But for a physical interpretation of $\mathbb{S}[g]$ as a formal power series in a coupling *constant*, the test function $g(x)$ must eventually be sent to a true constant in the so-called adiabatic limit $g(x) \to g$. Especially in models with massless particles, the existence and performance of a suitable adiabatic limit is an intricate question [20, 25, 32], which will not be addressed in this thesis.

In the following, we sketch the construction of $\mathbb{S}[g]$ in point-localized QFT before the adiabatic limit is taken. As a first step for this construction, one notices that Eq. (1.2) implies that the operator-valued distributions S_n can be assumed to be symmetric under exchange of any two arguments $x_i \leftrightarrow x_j$ without loss of generality because each argument is smeared with the same test function.

One then axiomatizes properties, which $\mathbb{S}[g]$ should possess. It should be unitary, Poincaré covariant and respect causality. Using these properties (and an additional enhanced symmetry property), one can relate S_1 to the interaction Lagrangian L describing a physical model by

$$S_1(x) = iL(x), \tag{1.3}$$

where the Wick-polynomial L must be Hermitian, Poincaré covariant and local [6, 32] in the sense that

$$[L(x), L(y)] = 0 \quad \text{if } x \text{ and } y \text{ are spacelike separated.} \tag{1.4}$$

In a next step, one derives that the S_n are time-ordered products of S_1, i.e.,

$$S_n(x_1, \ldots, x_n) = i^n\, T(L(x_1) \ldots L(x_n)), \tag{1.5}$$

where the time-ordered product on the right-hand side coincides with the usual product if the arguments can be meaningfully ordered in time, that is

$$T(L(x_1) \ldots L(x_n)) = L(x_{j_1})L(x_{j_2}) \ldots L(x_{j_n}) \quad \text{if} \quad x^0_{j_1} > x^0_{j_2} > \cdots > x^0_{j_n}. \tag{1.6}$$

With this at hand, the Dyson series (1.2) for the S-matrix can be written as

$$\mathbb{S}[g] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int T(L(x_1) \ldots L(x_n))\, g(x_1) \ldots g(x_n)\, dx_1 \ldots dx_n. \tag{1.7}$$

There is a snag to the right-hand side of Eq. (1.7). It is not evident how to define the time-ordered product if $x_i^0 = x_j^0$ for some $i \neq j$. Naively, one can try to define T by use of Heaviside step functions via

$$T(L(x_1)\dots L(x_n)) \stackrel{?}{=} \sum_{\pi\in\mathfrak{S}_n} \theta(x^0_{\pi(1)} - x^0_{\pi(2)})\dots\theta(x^0_{\pi(n-1)} - x^0_{\pi(n)})L(x_{\pi(1)})\dots L(x_{\pi(n)}), \tag{1.8}$$

where $\mathfrak{S}_n$ denotes the symmetric group of order n. However, both the Heaviside functions and the interaction Lagrangians are of distributional nature and their product as in Eq. (1.8) will in general not be well-defined. Indeed, using the expression on the right-hand side of Eq. (1.8) for the time-ordered products yields unrenormalized scattering amplitudes that exhibit the well-known ultraviolet divergences in loop graph contributions to the S-matrix within the common momentum space approaches.

A solution to the issue of defining the time-ordered products that appear in the Dyson series of the S-matrix was given in a seminal paper by Epstein and Glaser in terms of distribution splitting [32]. A more modern formulation, which also applies to curved spacetimes, would be in terms of the extension of products of distributions [9]. One proves that the right-hand side of Eq. (1.8) is well-defined outside a diagonal set where some of the arguments coincide and defines $T(L(x_1)\dots L(x_n))$ as an extension of this right-hand side to the full space. In this way, the time-ordered products $T(L(x_1)\dots L(x_n))$ stay finite (in the sense of operator-valued distributions) at every step of the construction of $\mathbb{S}[g]$ and no divergent loop integrals will appear. Therefore, this Bogoliubov-Epstein-Glaser (BEG) approach is sometimes referred to as renormalization without regularization. The extension procedure might be ambiguous and the infinite counterterms in the usual momentum space approaches to renormalization correspond to finite constants that reflect the freedom in choosing an extension of the right-hand side of Eq. (1.8). These renormalization constants are a priori free and need to be absorbed into physical quantities or constrained by physical (and mathematical) considerations such as power counting. Making sense of all free parameters is the BEG equivalent of examining the renormalizability of a model.

The construction of the time-ordered products $T(L(x_1)\dots L(x_n))$ is a very non-trivial inductive procedure. In the language of Feynman graphs, the graphs contributing to $T(L(x_1)\dots L(x_n))$ might contain divergent subgraphs. Due to the constraints coming from the described axioms for $\mathbb{S}[g]$, in particular causality, the extensions $T(L(x_1)\dots L(x_n))$ must be consistent with all extensions of time-ordered products $T(L(x_1)\dots L(x_k))$ of lower order $k < n$. In an inductive procedure, one can show that the time-ordered product $T(L(x_1)\dots L(x_n))$ is well-defined and uniquely determined outside the "small diagonal"

$$D_n := \{\, (x_1,\dots,x_n) \in (\mathbb{R}^{1+3})^n \mid x_1 = \dots = x_n \,\} \tag{1.9}$$

provided that all extensions $T(L(x_1)\dots L(x_k))$ of lower order $k < n$ have already been constructed [9]. In this way, one obtains a consistent inductive construction and sees that the extension $T(L(x_1)\dots L(x_n))$ is only ambiguous at D_n.

The construction of the S-matrix and time-ordered products on the level of operators is somewhat abstract and difficult to grasp. Using Wick's theorem [77], operator products $L(x_1)\dots L(x_n)$ can be written as a sum of products of expectation values multiplied by normal-ordered operators. Similarly, the time-ordered products $T(L(x_1)\dots L(x_n))$ can be written as a sum of products of time-ordered expectation values multiplied normal-ordered operators, *wherever they are defined*. The extension of the operator-valued time-ordered products across D_n then reduces to the extension of time-ordered expectation values across D_n. As long as the extensions are translation invariant – or satisfy a spectral condition in curved spacetimes [9] – this procedure yields well-defined expressions by Theorem 0 of Epstein and Glaser [32].

Remark 1.1. The extensions of time-ordered distributions in the BEG approach and the corresponding renormalization freedom can be classified in two categories. First, the BEG scheme provides a configuration space analogue of the renormalization of divergent loop integrals and second, it accounts for possible finite ambiguities in the definition of time-ordering. This is discussed in more detail in Chapter 4.

By now, the BEG procedure is well-established for point-localized QFT over both Minkowski space [32] and curved spacetimes [9]. A transition to SLFT is a non-trivial task but a step towards such a transition over Minkowski space is introduced in Chapter 4 of this thesis. There we also propose a method to define the time-ordering of string-localized expressions, which goes beyond the currently available methods.

1.3 Conventions

Let us fix the conventions, which are used in this thesis. The convention for the Minkowski metric is with mostly negative signs, $\eta = \mathrm{diag}(1,-1,-1,-1)$. The Minkowski product of two four-vectors x and y is generically denoted by $(xy) := \eta_{\mu\nu}x^\mu y^\nu$ and x^2 is used for the Minkowski square of x. The Fourier transform $\hat{f}(p) = \mathcal{F}f(p)$ of a function $f(x)$ over Euclidean space $\mathbb{R}^n$ is defined with negative sign in the exponent, the back transform has a positive sign. All factors of 2π are absorbed in the back transform. The Fourier transform over Minkowski space $\mathbb{R}^{1+3}$ has an overall minus sign in the exponent due to our convention of a "mostly negative" Minkowski metric:

$$\hat{f}(p) := \int d^4x\; e^{i(px)} f(x), \quad f(x) := \int \frac{d^4p}{(2\pi)^4}\; e^{-i(px)}\hat{f}(p) \tag{1.10}$$

for a generic f living on $\mathbb{R}^{1+3}$. This sign convention has effects on the signs of the wavefront sets computed in Section 5.5 because statements about wavefront sets are usually formulated over Euclidean space (see for example [41] or Section 5.1, where we give a concise overview of important statements). When relevant, it is always specified whether statements pertain to Euclidean space $\mathbb{R}^n$ or Minkowski space $\mathbb{R}^{1+3}$.

Chapter 2

Introduction: string-localized fields

String-localized fields for arbitrary mass $m \geq 0$ and spin/helicity s, including string-localized fields of infinite spin, can be constructed from the irreducible (m, s) Wigner representations of the Poincaré group [52,53]. String-localized fields have a distributional dependence on an additional spacelike variable e, which indicates the direction of their localization. String-localization is reflected in the commutation relations of string-localized fields. If e and e' are spacelike directions and $\phi(x, e)$ and $\varphi(x', e')$ are string-localized fields, then

$$[\phi(x, e), \varphi(x', e')]_{\pm} = 0 \quad \text{if} \quad (x + \mathbb{R}_{\geq 0}e - x' - \mathbb{R}_{\geq 0}e')^2 < 0, \tag{2.1}$$

where the commutator is to be taken for bosonic and the anti-commutator for fermionic fields, as usual [52, 65]. It is evident from Eq. (2.1) that string-localized fields do not satisfy the local commutativity axiom in the Wightman setting, which needs to be adjusted to the string-locality of the fields. This was already observed by Steinmann [65] and is not in conflict with Einstein causality, for quantum fields are in general no observable quantities.

In this thesis, we will not consider general string-localized fields but only string-localized potentials for the field-strength tensors of arbitrary mass $m \geq 0$ and positive integer spin respectively helicity $s \in \mathbb{N}$ as well as fields that are closely related to these potentials, which are usually referred to as "escort fields" [49]. String-localized fields of half-integer or infinite spin will not appear in this thesis.

2.1 String-localized potentials of integer spin or helicity

We start by defining string-localized potentials for arbitrary masses and spins respectively helicities.

Definition 2.1. For arbitrary mass $m \geq 0$ and for spin respectively helicity $s \in \mathbb{N}$, let $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}(x)$ denote the field strength tensor associated with the (m, s) Wigner representation [74], where the brackets around neighboring indices indicate an antisymmetrization. Let further $H \subset \mathbb{R}^{1+3}$ denote the open subset of spacelike vectors. The string-localized potential $A_{\mu_1\dots\mu_s}(x,e)$ of $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}(x)$ is defined as an s-fold line integral over $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}(x)$ in direction $e \in H$,

$$A_{\mu_1\dots\mu_s}(x,e) := I_e^s F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}(x) e^{\nu_1}\dots e^{\nu_s}, \tag{2.2}$$

where $I_e^s f(x) := \int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_s\ f(x+(\lambda_1+\cdots+\lambda_s)e)$ for some generic f [49,53,56].

Remark 2.2. In the literature (see for example [37,49,53]), the string variables are usually taken to be elements of the closed subset $H_{-1} := \{ e \in \mathbb{R}^{1+3} \mid e^2 = -1 \} \subset H \subset \mathbb{R}^{1+3}$. We prove in Section 5.6.2 that it does not severely affect the singularity structures in SLFT whether one chooses $e \in H$ or $e \in H_{-1}$. The restriction to the *open* set H, however, has the advantages that one can easily take derivatives with respect to the string variables and that restrictions of distributions to open subsets are much easier to handle than restrictions to closed subsets.

Using the Bianchi identity for the field strength tensors,

$$\partial_\kappa F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]} + \partial_{\mu_1} F_{[\nu_1\kappa]\dots[\mu_s\nu_s]} + \partial_{\nu_1} F_{[\kappa\mu_1]\dots[\mu_s\nu_s]} = 0, \tag{2.3}$$

and similar for the other index pairs $[\mu_i\nu_i]$, as well as the relation $(e\partial)I_e = -1$, it is easy to verify that $A_{\mu_1\dots\mu_s}(x,e)$ is indeed a potential for $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}(x)$. The string-localized potentials from Eq. (2.2) are symmetric under exchange of any two Lorentz indices $\mu_i \leftrightarrow \mu_j$ and axial with respect to the string variable e,

$$e^{\mu_1} A_{\mu_1\dots\mu_s}(x,e) = 0, \tag{2.4}$$

which follows from the skewsymmetry of the field strength tensor under exchange of any pair of indices $\mu_i \leftrightarrow \nu_i$ (as indicated by the brackets). Moreover, by its definition as integral over the field strength, the string-localized potential $A_{\mu_1\dots\mu_s}(x,e)$ lives on the same Hilbert space as $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}$. This remains true even in the massless case, where the gauge potentials in standard approaches are usually only defined on an indefinite Krein space that contains the Hilbert space of the field strength [60]. Moreover, the defining Eq. (2.2) tells us that the string-localized potentials transform covariantly under representations $U(a,\Lambda)$ of the Poincaré group, where $a \in \mathbb{R}^{1+3}$ is a translation vector and Λ is a Lorentz matrix,

$$U(a,\Lambda) A_{\mu_1\dots\mu_s}(x,e) U(a,\Lambda)^* = \Lambda^{\nu_1}_{\ \mu_1} \dots \Lambda^{\nu_s}_{\ \mu_s} A_{\nu_1\dots\nu_s}(\Lambda x + a, \Lambda e). \tag{2.5}$$

The good short distance behavior of string-localized potentials advertised in the previous section can be read off Eq. (2.2). The field strength of spin or helicity s scales as $|x|^{-(s+1)}$ with respect to $x = 0$ and the s-fold string integration implies that the string-localized potentials scale as $|x|^{-1}$ for all masses and spins respectively helicities with respect to $x = 0$. However, due to the delocalization of the string-localized potentials it is not clear that this improved scaling behavior has a positive effect on renormalizability. New singularities might appear in loop graph contributions to the S-matrix, which depend on the string variables and have the effect that renormalization is no longer a short distance (or in momentum space: an ultraviolet) problem. In such a case, the short distance behavior alone would not imply improved renormalizability of string-localized models.

In Section 5.5, we present a proof that the renormalization of loop graph contributions to the S-matrix stays a short distance problem and thus put the conjecture of improved renormalizability on safer mathematical ground.

Remark 2.3. Recently, a more general alternative to the Definition 2.1 of string-localized potentials has been proposed. This alternative leads to a further delocalization – the result is less than string-localized – but yields better analytic properties. Namely, one can define

$$\tilde{A}_{\mu_1\dots\mu_s}(x, e_1, \dots, e_s) := I_{e_1} \cdots I_{e_s} \frac{1}{s!} \sum_{\pi \in \mathfrak{S}_s} F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}(x) e^{\nu_1}_{\pi(1)} \dots e^{\nu_s}_{\pi(s)}, \tag{2.6}$$

where $\mathfrak{S}_s$ is the symmetric group of order s. Similarly to the case of string-localized potentials, one can easily show that the field $\tilde{A}_{\mu_1\dots\mu_s}$ from Eq. (2.6) is indeed a potential for the field strength. For $s = 2$, the potential $\tilde{A}_{\mu_1\mu_2}$ has been used to investigate the coupling of gravitons to the stress-energy tensor (SET) of a scalar field [11].[1] In this thesis, we use the term "multi-string-localized" for potentials of the form as in Eq. (2.6).

2.2 Two-point functions of string-localized potentials

In this section, we introduce the two-point functions of string-localized potentials for arbitrary $m \geq 0$ and $s \in \mathbb{N}$ as described by Mund, Rehren and Schroer [49].[2] These authors did not consider questions of well-definedness, and in the current section, we shall also not address this matter. Questions concerning well-definedness and the analytical and singularity structure are discussed in Chapter 5.

We employ the notation $\langle\!\langle \bullet \rangle\!\rangle := (\Omega, \bullet\, \Omega)$ for vacuum expectation values and for the two-point function of two arbitrary point-localized fields $X(x)$ and $X'(x')$ of mass $m \geq 0$, we always use the notation

$$\langle\!\langle X(x)X'(x') \rangle\!\rangle := \int d\mu_m(p)\, e^{-i(p(x-x'))}\, {}_mM^{X,X'}(p), \tag{2.7}$$

$$\text{with} \qquad d\mu_m(p) := \frac{d^4p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0) \tag{2.8}$$

being the measure on the mass shell and where ${}_mM^{X,X'}(p)$ is a polynomial in p. In momentum space, the string integrations from Eq. (2.2) become multiplication with

[1] To be precise, only the coupling of the escort fields to the SET of a scalar field has been considered in the cited work. Escort fields for string-localized potentials will be defined in Section 2.3. Their coupling to the SET yields a trivial S-matrix but has non-perturbative effects. We briefly sketch non-perturbative effects related to the escort fields in Section 3.2.

[2] The present section is based on the author's work [35].

string-dependent factors $\pm i(pe)_\pm^{-1} := \pm i \lim_{\varepsilon\downarrow 0} [(pe) \pm i\varepsilon]^{-1}$. For example, we have

$$\begin{aligned}\widehat{I_e f}(p) &= \int d^4x \int_0^\infty ds\, e^{i(px)} f(x+se) = \int d^4x \int_0^\infty ds\, e^{i(p[x-se])} f(x) \\ &= \hat{f}(p) \int_0^\infty ds\, e^{-is(pe)} := \frac{-i\hat{f}(p)}{(pe)_-}.\end{aligned} \tag{2.9}$$

Therefore, the string dependence of the two-point functions of string-localized potentials can be absorbed into the integral kernel ${}_mM(p)$ in Eq. (2.7), i.e.,

$$\langle\!\langle X(x,e)X'(x',e')\rangle\!\rangle = \int d\mu_m(p)\, e^{-i(p(x-x'))}\, {}_mM^{X,X'}(p,e,e'), \tag{2.10}$$

where now ${}_mM^{X,X'}(p,e,e')$ is a polynomial in the variables p, e and e' with possible denominators $-i(pe)_-^{-1}$ and $i(pe')_+^{-1}$. ${}_mM^{X,X'}(p,e,e')$ is separately homogeneous of degree $\omega = 0$ in both string variables since each string integration in Eq. (2.2) is accompanied by a factor e^{ν_i}. Of course, one of the fields in Eq. (2.10) might be point-localized, and in this case the dependence on the respective string variable is trivial.

The expressions Eq. (2.9) and Eq. (2.10) are to be understood as distributions in all their variables, including the string variables. Both the string-localized and point-localized two-point functions from Eq. (2.10) and Eq. (2.7), respectively, are translation invariant. We therefore often redefine $x - x' \to x$ and perform the transition from distributions over $(\mathbb{R}^{1+3})^2 \times H^2$ to distributions over $\mathbb{R}^{1+3} \times H^2$,

$$\langle\!\langle X(x,e)X'(x',e')\rangle\!\rangle \quad \xrightarrow{x-x'\to x} \quad \langle\!\langle X(e)X'(e')\rangle\!\rangle(x). \tag{2.11}$$

The two-point functions of the string-localized potentials arise from the two-point function of the field strength tensors by Eq. (2.2). For arbitrary mass $m \geq 0$ and spin respectively helicity $s = 1$, the kernel of the field strength two-point function is given by [74]

$${}_mM^{F,F}_{\mu\nu\kappa\lambda}(p) = -\eta_{\mu\kappa}p_\nu p_\lambda + \eta_{\mu\lambda}p_\nu p_\kappa + \eta_{\nu\kappa}p_\mu p_\lambda - \eta_{\nu\lambda}p_\mu p_\kappa, \tag{2.12}$$

and thus, by Eq.s (2.2) and (2.9), the kernel of the associated string-localized potential is given by

$$\begin{aligned}{}_mM^{A,A}_{\mu\kappa}(p,e,e') &= -\eta_{\mu\kappa} + \frac{e_\kappa p_\mu}{(pe)_-} + \frac{e'_\mu p_\kappa}{(pe')_+} - \frac{(ee')p_\mu p_\kappa}{(pe)_-(pe')_+} \\ &=: -E_{\mu\kappa}(p,e,e').\end{aligned} \tag{2.13}$$

The quantity $E_{\mu\kappa}(p,e,e')$ from Eq. (2.13) turns out to be the central building block of the two-point functions of string-localized potentials of arbitrary spin or helicity $s \in \mathbb{N}$. To construct these two-point functions, one makes a useful change of notation. Because the string-localized potentials are totally symmetric under exchange of Lorentz indices, no information is lost if all indices are contracted with the same dummy four-vector f^μ. Therefore, we can define

$$A^{(s)}_f(x,e) := f^{\mu_1}\cdots f^{\mu_s} A_{\mu_1\cdots\mu_s}(x,e) \tag{2.14}$$

and also

$$E_{ff} := f^\mu E_{\mu\nu}(p,e,-e) f^\nu, \quad E_{f'f'} := f'^\mu E_{\mu\nu}(p,-e',e') f'^\nu, \quad \text{and}$$
$$E_{ff'} := f^\mu E_{\mu\nu}(p,e,e') f'^\nu, \tag{2.15}$$

where the signs in the arguments of E_{ff}, $E_{f'f'}$ and $E_{ff'}$ ensure that only denominators $(pe)_-^{-1}$ and $(pe')_+^{-1}$ with the correct imaginary shift appear.

With this useful notation, Mund, Rehren and Schroer [49] were able to write down the kernel of the two-point function of the string-localized potentials for arbitrary $m \geq 0$ and $s \in \mathbb{N}$ in a concise way. It reads

$${}_mM^{A_f^{(s)},A_{f'}^{(s)}}(p,e,e') = (-1)^s \sum_{2n \leq s} \beta_n^s \, (E_{ff})^n (E_{f'f'})^n (E_{ff'})^{s-2n} \tag{2.16}$$

with numerical coefficients β_n^s that are of no particular interest in this thesis. The derivation of the β_n^s for the different cases of massive and massless potentials can be found in [49]. The features of Eq. (2.16), which are important for this thesis because they are relevant to prove well-definedness and to determine the singularity structure of the two-point functions (see Chapter 5), are the following:

1. The kernel from Eq. (2.16) is homogeneous of degree $\omega = 0$ separately in all variables p, e and e'.

2. The kernel from Eq. (2.16) only contains denominators $(pe)_-^{-k}(pe')_+^{-l}$, but no "mixed denominators" of the form $(pe)_-^{-1}(pe)_+^{-1}$, where the same string variable has different imaginary shifts.

Remark 2.4. Similar to the case of the string-localized potentials, we do not lose information about the multi-string-localized potentials $\tilde{A}_{\mu_1\ldots\mu_s}(x,e_1,\ldots,e_s)$ from Remark 2.3 if all the indices are contracted with the same dummy four-vector f^μ by the symmetry of the potentials,

$$\tilde{A}_f^{(s)}(x,e_1,\ldots,e_s) := f^{\mu_1}\ldots f^{\mu_s}\tilde{A}_{\mu_1\ldots\mu_s}(x,e_1,\ldots,e_s). \tag{2.17}$$

The average over all permutations of the string variables in the definition of $\tilde{A}_{\mu_1\ldots\mu_s}$ can be dropped because symmetry is now encoded in the contraction with f^μ. One might therefore hope that there is a concise formula of similar nature as Eq. (2.16) for the potentials $\tilde{A}_{\mu_1\ldots\mu_s}$. Although this is unknown at present, the important features for the content of this thesis are clearly there. The kernel of the two-point function of the string-localized potential arises from contraction of the kernel of the associated field strength with

$$q_-^{\nu_1\ldots\nu_s}(p,e)\; q_+^{\lambda_1\ldots\lambda_s}(p,e') := \frac{(-i)^s e^{\nu_1}\ldots e^{\nu_s}}{(pe)_-^s} \times \frac{i^s e'^{\lambda_1}\ldots e'^{\lambda_s}}{(pe')_+^s} \tag{2.18}$$

and similarly, the kernel of the multi-string-localized potential arises from contraction with a symmetrization of

$$q_-^{\nu_1}(e_1)\dots q_-^{\nu_s}(e_s) \times q_+^{\lambda_1}(e_1')\dots q_+^{\lambda_s}(e_s'). \tag{2.19}$$

Hence, we already know that the kernel of the multi-string-localized potential $\tilde{A}_f^{(s)}$ satisfies similar properties as the kernel of the string-localized potential, which are that

$\tilde{1}$. ${}_mM^{\tilde{A}_f^{(s)},\tilde{A}_{f'}^{(s)}}(p, e_1, \dots, e_s')$ is homogeneous of degree $\omega = 0$ in all variables separately, and moreover,

$\tilde{2}$. ${}_mM^{\tilde{A}_f^{(s)},\tilde{A}_{f'}^{(s)}}(p, e_1, \dots, e_s')$ contains no mixed denominators, simply because each string variable can appear in at most one denominator.

Remark 2.5 (Relation to axial gauges)**.** The new denominators $(pe)_\pm^{-1}$ are similar to the denominators that appear in gauge theories with axial or lightcone gauges [45] but there are important differences.[3] First, the two-point functions of string-localized potentials are treated as distributions in all variables, including the string variables, whereas the preferred direction in axial gauges is typically assumed to be fixed. The interpretation of two-point functions as distributions also in the string variables is of great importance for renormalization, as we shall see in Sections 5.2 to 5.5. The second important difference is that the string-localized potentials are covariant by Eq. (2.5) while the axial gauge potentials are not.

2.3 Escort fields

In the case of massive string-localized potentials, there is a hierarchy of fields associated with the potential of spin s, which describes the latter's coupling to string-localized fields $a^{(r)}_{\mu_1\dots\mu_r}$ of all smaller spins $r = 0, 1, \dots, s-1$. Writing $A_{\mu_1\cdots\mu_s} \equiv a^{(s)}_{\mu_1\dots\mu_s}$, these relations are given by [49]

$$\partial^{\mu_1} a^{(r)}_{\mu_1\dots\mu_r} = -m a^{(r-1)}_{\mu_2\dots\mu_r}, \quad \eta^{\mu_1\mu_2} a^{(r)}_{\mu_1\dots\mu_r} = -a^{(r-2)}_{\mu_3\dots\mu_r} \tag{2.20}$$

for all $r \in \{1, \dots, s\}$ and $r \in \{2, \dots, s\}$, respectively. The fields from Eq. (2.20) coupled to the string-localized field of spin s are called "escort fields". One can relate the massive point-localized potential $A^{\mathrm{P}}_{\mu_1\dots\mu_s}(x)$, where P stands for Proca, to the string-localized potential and its escort fields via

$$A^{\mathrm{P}}_{\mu_1\dots\mu_s}(x) = A_{\mu_1\dots\mu_s}(x,e) + \sum_{r<s} (-m)^{-(s-r)}\, P^{(r)\nu_1\dots\nu_r}_{\mu_1\dots\mu_s}(\partial)\, a^{(r)}_{\nu_1\dots\nu_r}(x,e), \tag{2.21}$$

where the $P^{(r)\nu_1\dots\nu_r}_{\mu_1\dots\mu_s}(\partial)$ are differential operators of order $s - r$ [49].

[3] See also [44] and Section 25.4.3 of [62] for further literature on axial and light-cone gauges.

It is clear from the first relation in Eq. (2.20) that the escort fields of uneven degree – even and uneven understood in the sense of counting down from s – decouple in the massless limit. On the other hand, the even escort fields stay coupled even in the massless limit due to the trace relation in Eq. (2.20). Mund, Rehren and Schroer showed that there is a unique linear combination of the massive string-localized potential and its even escort fields, which is traceless in the massless limit, and the massless limit of this combination coincides with the massless string-localized potentials from Eq. (2.2) [49]. These massless string-localized potentials live on the Hilbert space of their field strengths and thus inherit properties from the latter. They are traceless with respect to any index pair and have vanishing divergence – remember that they are also axial with respect to the string variable and symmetric under exchange of any pair of Lorentz indices, as described in Section 2.1. To summarize,

$$e^{\mu_1} A_{\mu_1\dots\mu_s}(x,e) = 0, \quad \partial^{\mu_1} A_{\mu_1\dots\mu_s}(x,e) = 0, \quad \eta^{\mu_1\mu_2} A_{\mu_1\dots\mu_s}(x,e) = 0 \quad \text{at } m = 0. \tag{2.22}$$

The second and third property from Eq. (2.22) can now also be explained by the decoupling of the massless string-localized potentials from their escort fields. Doing the simple combinatorics, one sees that all conditions together imply that the massless string-localized potentials carry 2 degrees of freedom, as is expected from fields transforming under the massless $\pm s$ Wigner representations. The unphysical degrees of freedom have been absorbed into the decoupled escort fields. To see this explicitly, note that a symmetric tensor of rank $s \geq 1$ in $d = 4$ spacetime dimensions that is also subject to the axiality condition from Eq. (2.22) has

$$\binom{s+3}{s} - \binom{s+2}{s-1} = \frac{s(s+3)}{2} + 1 \tag{2.23}$$

degrees of freedom. The trace condition from Eq. (2.22) removes another $\frac{(s-2)(s+1)}{2} + 1$ degrees of freedom, i.e., the number or degrees of freedom of a symmetric and axial tensor of rank $s - 2$ in four spacetime dimensions,

$$\frac{s(s+3)}{2} + 1 - \left[\frac{(s-2)(s+1)}{2} + 1\right] = 2s + 1. \tag{2.24}$$

Finally, the divergence condition from Eq. (2.22) removes another $2s - 1$ degrees of freedom, which is the number of degrees of freedom of a symmetric, traceless and axial tensor of rank $s - 1$. We are thus left with the desired

$$2s + 1 - (2s - 1) = 2 \tag{2.25}$$

degrees of freedom.

In the massive case, the even escort fields are related to both the trace and the divergence,

$$\partial^{\mu_1}\partial^{\mu_2} a^{(r)}_{\mu_1\dots\mu_r} = m^2 a^{(r-2)}_{\mu_3\dots\mu_r} = -m^2 \eta^{\mu_1\mu_2} a^{(r)}_{\mu_1\dots\mu_r}. \tag{2.26}$$

Therefore, the trace degrees of freedom are not independent and the massive string-localized potential $A_{\mu_1 \dots \mu_s}$ possesses as many degrees of freedom as a symmetric, axial *and* traceless tensor field of rank s. That is, it has the desired

$$\frac{s(s+3)}{2} + 1 - \left[\frac{(s-2)(s+1)}{2} + 1 \right] = 2s + 1 \tag{2.27}$$

degrees of freedom. In particular, the massive string-localized potentials and their mentioned traceless combinations, which yield the correct massless limit, carry the same number of degrees of freedom.

One can attempt to set up physical models that are formulated on Hilbert space even in the massless case by replacing the usual gauge potentials with the string-localized potentials from Definition 2.1. Programmes of this type have been attacked but are mostly unfinished in the sense that they have only been verified in low orders and/or tree level of perturbation theory. Examples are massless Yang-Mills theory [37], whose current state is presented in Section 6.1 and scalar QED [67], which is not addressed in this thesis.

However, the decoupling of the escort fields and the possibility to formulate perturbation theory with massless particles directly on Hilbert space does not mean that the escort fields are unimportant in the massless case. It has been pointed out by Buchholz and collaborators [13] that unphysical degrees of freedom are important in QFT and indeed, the escort fields reappear in a non-perturbative guise [48, 51]. This topic is discussed in more detail in Chapter 3.

2.4 The string independence principle

A string dependence of measurable quantities is not observed in experiment and therefore it is a reasonable requirement that observables must be independent of the string variables. This *principle of string independence* (SI) is the SLFT analogue of the gauge invariance principle in gauge theories.

The requirement that observables, and also quantities that are closely related to them, are string independent turns out to give strong constraints on string-localized models. In this thesis, the central quantity subjected to the string independence principle is the string-localized S-matrix. To be precise, the string-localized analogue of the Dyson series Eq. (1.7) is required to be string independent in the adiabatic limit at each fixed order in g. This is achieved if the string variation of each time-ordered product in the Dyson series is a total divergence. The details of such a requirement depend on the formulation of perturbation theory in SLFT and are discussed in Chapter 4, after such a formulation has been worked out. We then encounter in Chapters 5 and 6 that this form of the SI principle has many facets:

- At first order, it gives rise to strong constraints on the form of the interaction Lagrangian of string-localized models.
- At second order and tree level, it constrains the ambiguities of string-localized propagators. These ambiguities belong to the second category of the renormalization freedom inherent to the BEG approach described in Remark 1.1. For example,

in string-localized QED and massless Yang-Mills theory, the string independence principle uniquely fixes the propagator of the respective string-localized potentials, see Sections 5.7.2 and 6.1. The effect of the SI principle on propagators in other models has not yet been investigated.

- More general, the SI principle exhibits features of a renormalization condition. That means, it gives constraints on the allowed extensions of time-ordered products in a string-localized BEG scheme. These constraints resemble Ward identities, for example, the master Ward identity [25, 26].

The described constraints coming from SI are of similar nature as constraints in the perturbative gauge theoretic framework of Scharf and collaborators, which originate from the gauge invariance principle [1, 29, 60]. However, it remains unproven whether the SI principle in SLFT and the gauge invariance principle in the gauge theoretic framework are formally equivalent.

To implement the string independence principle in perturbation theory, we will later make use of the fact that the string derivative of the string-localized potentials Eq. (2.2) is a symmetrized gradient of certain auxiliary fields. To verify this, note that by Eq. (2.9), we have

$$\partial_{e^\mu} \frac{-i}{(pe)_-} = -ip_\mu \frac{(-i)^2}{(pe)_-^2} \quad \Rightarrow \quad \partial_{e^\mu} I_e f(x) = I_e^2 \partial_\mu f(x) \tag{2.28}$$

for a generic f. In Section 5.2, we prove that the distributional product $\frac{1}{(pe)_-^2}$, which appears in Eq. (2.28), is indeed well-defined if e is spacelike (see Corollary 5.13). In the case $s = 1$, the Bianchi identity $\partial_\kappa F_{\mu\nu} + \text{cyclic} = 0$ gives

$$\begin{aligned} \partial_{e^\kappa} A_\mu(x,e) &= \partial_{e^\kappa} I_e F_{\mu\nu}(x) e^\nu \\ &= I_e^2 \partial_\kappa F_{\mu\nu}(x) e^\nu + I_e F_{\mu\kappa}(x) \\ &= I_e^2 [\partial_\mu F_{\kappa\nu} + \partial_\nu F_{\mu\kappa}] e^\nu + I_e F_{\mu\kappa}(x) \\ &= \partial_\mu I_e A_\kappa(x,e), \end{aligned} \tag{2.29}$$

where we have used that $I_e^2 (e\partial) F_{\mu\kappa} = -I_e F_{\mu\kappa}$. Thus, introducing the auxiliary field $w_\mu(x,e) := I_e A_\mu(x,e)$, we get that

$$\partial_{e^\kappa} A_\mu(x,e) = \partial_\mu w_\kappa(x,e). \tag{2.30}$$

The Bianchi identity holds for any index pair $[\mu_i \nu_i]$, $i = 1, \ldots, s$ of the field strengths $F_{[\mu_1\nu_2]\ldots[\mu_s\nu_s]}(x)$ of arbitrary spin respectively helicity s and moreover, the chain rule implies $\partial_{e^\mu} I_e^s f(x) = s I_e^{s+1} \partial_\mu f(x)$. Consequently, Eq. (2.2) gives

$$\partial_{e^\kappa} A_{\mu_1\ldots\mu_s}(x,e) = \sum_{i=1}^{s} \partial_{\mu_i} w_{\mu_1\ldots\mu_{i-1}\kappa\mu_{i+1}\ldots\mu_s}(x,e) \tag{2.31}$$

for the string derivative of the string-localized potentials for arbitrary $s \in \mathbb{N}$, with the symmetric auxiliary fields $w_{\mu_1\ldots\mu_s}(x,e) = I_e A_{\mu_1\ldots\mu_s}(x,e)$. By their definition, the auxiliary fields inherit certain properties of the string-localized potentials: the auxiliary fields $w_{\mu_1 \ldots \mu_s}$ are axial with respect to e and satisfy the Klein-Gordon equation. In the massless case, they additionally are divergence-free and traceless, while there is a hierarchy of auxiliary fields of the type Eq. (2.20) in the massive case.

Chapter 3

The hybrid approach to SLFT

The string-localized potential lives on the Hilbert space of its field strength even in the massless case. Therefore, a pure string-localized QFT, where all (massless) gauge bosons are replaced by string-localized potentials, is also formulated on Hilbert space. However, Buchholz and collaborators have pointed out the importance of unphysical degrees of freedom, which seem to be absent in SLFT, for certain non-perturbative constructions in QED [13]: in the gauge theoretic framework, the existence of gauge bridges that are needed to ensure the validity of Gauss' law in QFT relies on the presence of unphysical degrees of freedom. Buchholz and collaborators point out that their construction cannot be formulated solely in terms of the string-localized photon potential, which carries only physical degrees of freedom.

However, it is not true that there are no unphysical degrees of freedom *at all* in SLFT. In Section 2.3, we have encountered the escort fields associated with string-localized potentials, which carry away the unphysical degrees of freedom in the massless limit of massive fields [49, 50]. In that sense, the distinction between physical and unphysical degrees of freedom is very clear in SLFT. The massless string-localized potentials carry the physical degrees of freedom, the escort fields the unphysical ones. One can lift the string-localized potentials from Hilbert space to Krein space in the same way as one lifts the field strength tensors in gauge theories. In this way, one gets relations between the gauge potential and the string-localized potential [50]. They differ from each other by gradients of the escort fields.

We constrain our considerations in this section to helicities $s = 1$ and $s = 2$ (because of their physical relevance in describing photons, gluons and gravitons). The massless string-localized potential for helicity $s = 1$ will be denoted by $A_\mu(x, e)$, the one for helicity $s = 2$ by $h_{\mu\nu}(x, e)$. The corresponding gauge potentials, which we choose to be in Feynman gauge, get an index "K" standing for "Krein field". Then, on Krein space, the string-localized potentials and the gauge potentials differ by operator-valued gauge transformations,

$$A_\mu(x, e) = A^{\mathrm{K}}_\mu(x) + \partial_\mu \varphi(x, e), \tag{3.1a}$$

$$h_{\mu\nu}(x, e) = h^{\mathrm{K}}_{\mu\nu}(x) + \partial_\mu \Phi_\nu(x, e) + \partial_\nu \Phi_\mu(x, e) + \partial_\mu \partial_\nu \phi(x, e), \tag{3.1b}$$

where $\varphi(x, e)$, $\Phi_\mu(x, e)$ and $\phi(x, e)$ are the associated escort fields [50, 51]. If $A_\mu(x, e)$

is coupled to a conserved current $j^\mu(x)$ (for example to the current of QED or scalar QED), or if the graviton potential $h_{\mu\nu}(x,e)$ is coupled to a symmetric and conserved stress energy tensor $T^{\mu\nu}(x)$, then the string-localized and Krein Lagrangian differ by a total divergence,

$$\begin{aligned} L(x,e) &= A_\mu(x,e) j^\mu(x) \overset{(3.1a)}{=} L^{\mathrm{K}}(x) + \partial_\mu V^\mu(x,e), \\ \tilde{L}(x,e) &= h_{\mu\nu}(x,e) T^{\mu\nu}(x) \overset{(3.1b)}{=} \tilde{L}^{\mathrm{K}}(x) + \partial_\mu \tilde{V}^\mu(x,e), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} L^{\mathrm{K}}(x) &= A^{\mathrm{K}}_\mu(x) j^\mu(x), \qquad V^\mu(x,e) = \varphi(x,e) j^\mu(x), \\ \tilde{L}^{\mathrm{K}}(x) &= h^{\mathrm{K}}_{\mu\nu}(x) T^{\mu\nu}(x), \quad \tilde{V}^\mu(x,e) = \left(2\Phi_\nu(x,e) + \partial_\nu \phi(x,e)\right) T^{\mu\nu}(x). \end{aligned} \tag{3.3}$$

More general, we make the following definition.

Definition 3.1. We say that a string-localized Lagrangian $L(x,e)$ and a point-localized Lagrangian $L^{\mathrm{K}}(x)$ are part of an L-V pair if they differ by the total divergence of some Wick polynomial $V^\mu(x,e)$, i.e.,

$$L(x,e) - L^{\mathrm{K}}(x) = \partial_\mu V^\mu(x,e). \tag{3.4}$$

If $L(x,e)$ and $L^{\mathrm{K}}(x)$ are part of an L-V pair, their difference gives rise to a trivial S-matrix in the adiabatic limit.[1] Therefore, the unphysical degrees of freedom in the examples from Eq. (3.3), do not enter the perturbative expansion of the S-matrix. The problem of constructing physical models in SLFT then splits into two parts – the construction of the scattering matrix *on Hilbert space* and the investigation of non-perturbative effects arising from the contribution of $\partial_\mu V^\mu(x,e)$ *on a detour through Krein space*. This double-tracked approach is usually called the "hybrid approach to SLFT" [48, 51]. Alternatively, one could also say that the hybrid approach allows for the construction of string-localized models on Hilbert space by splitting the construction into a point-localized perturbative part and a string-localized non-perturbative part.

Remark 3.2. We say "detour through Krein space" because one can hope to descend back to Hilbert space after the construction. For QED, this is indeed the case: positivity is restored by restricting the string variables to be purely spatial, i.e., $e = (0, \vec{e})$, thereby breaking Lorentz covariance [48, 51]. So far, it is not clear that positivity can be restored in general models.

3.1 String-localized fields on Krein and Hilbert space

Lifted to Krein space, the massless string-localized potentials for helicities $s = 1, 2$ can be related to the respective point-localized Krein potentials by Eq.s (3.1a) and (3.1b).

[1] It is a non-trivial requirement, or rather a renormalization condition in the BEG sense, that a total divergence gives rise to a trivial S-matrix, see for example [11].

On Hilbert space, they satisfy the conditions from Eq. (2.22) but it is not clear whether these conditions are satisfied also on Krein space. In the following, we investigate what properties of the string-localized potentials (and the field strengths) for helicities $s = 1, 2$ survive the lifting process. For that, we choose the point-localized potentials to be in Feynman gauge, that is (see for example [60]),

$$\begin{aligned} {}_0M^{A^{\mathrm{K}},A^{\mathrm{K}}}_{\mu\nu}(p) &= -\eta_{\mu\nu}, \\ {}_0M^{h^{\mathrm{K}},h^{\mathrm{K}}}_{\mu\nu\kappa\lambda}(p) &= \frac{1}{2}(\eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu\kappa} - \eta_{\mu\nu}\eta_{\kappa\lambda}). \end{aligned} \tag{3.5}$$

Before we split our considerations into the separate cases of $s = 1$ and $s = 2$, we formulate a general property of all fields. The Fourier transform of the two-point functions of all fields in Eq.s (3.1a) and (3.1b) and of the respective field strengths is supported on the mass-zero shell. Therefore, they all satisfy the wave equation also on the larger Krein space,

$$\Box X = 0 \quad \text{for} \quad X \in \{\, A_\mu, A^{\mathrm{K}}_\mu, \varphi, F_{\mu\nu}; h_{\mu\nu}, h^{\mathrm{K}}_{\mu\nu}, \Phi_\mu, \phi, F_{\mu\kappa\nu\lambda} \,\}. \tag{3.6}$$

3.1.1 Helicity one

On Krein space, the two-point function of the string-localized potential can not only be obtained from the two-point function of the field strength, but also from the two-point function of the Krein potential in Feynman gauge [49]. The index of the first Krein field A^{K}_α in Eq. (3.5) is hit by a factor $J^{-\mu}_\kappa(p, e)$, the index of the second Krein field by a factor $J^{+\nu}_\lambda(p, e')$, where

$$J^{\pm\beta}_\alpha(p, e) := \delta^\beta_\alpha - \frac{p_\alpha e^\beta}{(pe)_\pm}, \tag{3.7}$$

so that

$${}_0M^{A,A}_{\kappa\lambda}(p, e, e') = J^{-\mu}_\kappa(p, e) J^{+\nu}_\lambda(p, e') {}_0M^{A^{\mathrm{K}},A^{\mathrm{K}}}_{\mu\nu}(p). \tag{3.8}$$

This is in agreement with the result coming from the fact that A_μ and A^{K}_μ are potentials for the same field strength,

$$A_\mu = I_e F_{\mu\nu} e^\nu = I_e \partial_\mu(eA^{\mathrm{K}}) - I_e(e\partial)A^{\mathrm{K}}_\mu = A^{\mathrm{K}}_\mu + \partial_\mu I_e(eA^{\mathrm{K}}) = \check{J}^{-\kappa}_\mu A^{\mathrm{K}}_\kappa, \tag{3.9}$$

where $\check{J}^{-\beta}_\alpha = \delta^\beta_\alpha + I_e \partial_\alpha e^\beta$ is the inverse Fourier transform of $J^{-\beta}_\alpha$. Comparing Eq. (3.9) to Eq. (3.1a), we can read off a relation between the Krein potential and the escort field, namely

$$\varphi(x, e) = I_e(eA^{\mathrm{K}}). \tag{3.10}$$

Because $e^\alpha J^{\pm\beta}_\alpha = 0$, the axiality condition from Eq. (2.22) is satisfied also on Krein space. All fields satisfy the wave equation, from which we can infer that the divergence of the string-localized potential $A_\mu(x, e)$ is non-vanishing on Krein space,

$$(\partial A) = (\partial A^{\mathrm{K}}) + \Box\varphi = (\partial A^{\mathrm{K}}) \neq 0, \tag{3.11}$$

since

$$\partial^\mu \langle\!\langle A^{\mathrm{K}}_\mu A^{\mathrm{K}}_\nu \rangle\!\rangle = \partial^\mu \int d\mu_0(p)\, e^{-i(px)}\, [-\eta_{\mu\nu}] = -\partial_\nu W_0(x) \neq 0, \tag{3.12}$$

where $W_0(x) = -(2\pi)^{-2} \lim_{\varepsilon\downarrow 0}[x^2 - i\varepsilon x^0]^{-1}$ is the two-point function of a massless scalar Klein-Gordon field. Also the field strength $F_{\mu\nu}$ has non-vanishing divergence on Krein space,

$$\partial^\mu F_{\mu\nu} = \Box A^{\mathrm{K}}_\nu - \partial_\nu(\partial A^{\mathrm{K}}) = -\partial_\nu(\partial A^{\mathrm{K}}) \neq 0 \tag{3.13}$$

by virtue of Eq.s (3.11) and (3.12). But the field strength satisfies the Bianchi identity also on Krein space,

$$\partial_\kappa F_{\mu\nu} + \text{ cyclic } = \partial_\kappa\partial_\mu A^{\mathrm{K}}_\nu - \partial_\kappa\partial_\nu A^{\mathrm{K}}_\mu + \partial_\mu\partial_\nu A^{\mathrm{K}}_\kappa - \partial_\mu\partial_\kappa A^{\mathrm{K}}_\nu + \partial_\nu\partial_\kappa A^{\mathrm{K}}_\mu - \partial_\nu\partial_\mu A^{\mathrm{K}}_\kappa = 0. \tag{3.14}$$

A summary comparing the properties of the fields on Hilbert and Krein space can be found in Table 3.1.

Property	Hilbert space	Krein space
Wave equation for all fields	satisfied	satisfied
(eA)	$= 0$	$= 0$
(∂A)	$= 0$	$= (\partial A^{\mathrm{K}}) \neq 0$
$\partial^\mu F_{\mu\nu}$	$= 0$	$= -\partial_\nu(\partial A^{\mathrm{K}}) \neq 0$
$\partial_\kappa F_{\mu\nu} +$ cyclic	$= 0$	$= 0$

Table 3.1: Properties of helicity one fields on Krein and Hilbert space.

3.1.2 Helicity two

There are more escort fields and correspondingly also more unphysical degrees of freedom for helicity $s = 2$ than for $s = 1$ but the analysis will be similar to the one presented in the last section. Resembling the case $s = 1$, the escort fields from Eq. (3.1b) can be read off from the fact that $h_{\mu\nu}$ and $h^{\mathrm{K}}_{\mu\nu}$ are potentials for the same field strength,

$$\begin{aligned} h_{\mu\nu} = I_e^2 e^\kappa e^\lambda F_{\mu\kappa\nu\lambda} &= I_e^2 e^\kappa e^\lambda \left(\partial_\kappa\partial_\lambda h^{\mathrm{K}}_{\mu\nu} + \partial_\mu\partial_\nu h^{\mathrm{K}}_{\kappa\lambda} - \partial_\mu\partial_\lambda h^{\mathrm{K}}_{\kappa\nu} - \partial_\kappa\partial_\nu h^{\mathrm{K}}_{\mu\lambda}\right) \\ &= h^{\mathrm{K}}_{\mu\nu} + \partial_\mu\partial_\nu I_e^2(eh^{\mathrm{K}}e) + \partial_\mu I_e(h^{\mathrm{K}}e)_\nu + \partial_\nu I_e(h^{\mathrm{K}}e)_\mu \\ &= \check{J}_\mu^{-\varrho} \check{J}_\nu^{-\sigma} h^{\mathrm{K}}_{\varrho\sigma}, \end{aligned} \tag{3.15}$$

giving $\Phi_\mu = I_e(h^{\mathrm{K}}e)_\mu$ and $\phi = I_e^2(eh^{\mathrm{K}}e)$. Also, the two-point function (2.16) of the string-localized potential $h_{\mu\nu}$ arises from the action of $J_\alpha^{\pm\beta}$ on the two-point function (3.5) of $h^{\mathrm{K}}_{\mu\nu}$ [49, 50], giving

$$\begin{aligned}{}_0M^{h,h}_{\mu\nu\kappa\lambda}(p,e,e') &= J_\mu^{-\alpha}(p,e)J_\nu^{-\beta}(p,e)J_\kappa^{+\varrho}(p,e')J_\lambda^{+\sigma}(p,e')\,{}_0M^{h^{\mathrm{K}},h^{\mathrm{K}}}_{\alpha\beta\varrho\sigma}(p) \\ &= \frac{1}{2}\big[E_{\mu\kappa}(e,e')E_{\nu\lambda}(e,e') + E_{\nu\kappa}(e,e')E_{\mu\lambda}(e,e') \\ &\qquad -E_{\mu\nu}(e,-e)E_{\kappa\lambda}(-e',e')\big]\,. \end{aligned} \tag{3.16}$$

Since $e^\alpha J_\alpha^{\pm\beta} = 0$, $h_{\mu\nu}(x,e)$ remains axial with respect to e^μ on Krein space. Writing the field strength as the (double) curl of the Krein potential, one easily derives that the Bianchi identity

$$\partial_\varrho F_{\mu\kappa\nu\lambda} + \partial_\mu F_{\kappa\varrho\nu\lambda} + \partial_\kappa F_{\varrho\mu\nu\lambda} = 0 = \partial_\varrho F_{\mu\kappa\nu\lambda} + \partial_\nu F_{\mu\kappa\lambda\varrho} + \partial_\lambda F_{\mu\kappa\varrho\nu} \tag{3.17}$$

is valid on Krein space. Let us introduce the notation $h \equiv \mathrm{tr}(h) := \eta^{\mu\nu}h_{\mu\nu}$ and $h^{\mathrm{K}} \equiv \mathrm{tr}(h^{\mathrm{K}}) := \eta^{\mu\nu}h^{\mathrm{K}}_{\mu\nu}$ for the traces of the string-localized and Krein potential, respectively. By Eq. (3.5), we have

$${}_0M^{\mathrm{tr}(h^{\mathrm{K}}),h^{\mathrm{K}}}_{\kappa\lambda}(p) = \eta^{\mu\nu}\,{}_0M^{h^{\mathrm{K}},h^{\mathrm{K}}}_{\mu\nu\kappa\lambda}(p) = -\eta_{\kappa\lambda}, \tag{3.18}$$

so ${}_0M^{\mathrm{tr}(h^{\mathrm{K}}),h^{\mathrm{K}}}_{\kappa\lambda}$ does not vanish on the mass-zero shell and thus $\langle\!\langle h^{\mathrm{K}} h^{\mathrm{K}}_{\mu\nu}\rangle\!\rangle \neq 0$, implying that the trace h^{K} is non-zero. One also finds that

$$\begin{aligned}{}_0M^{\mathrm{tr}(h),h^{\mathrm{K}}}_{\kappa\lambda}(p,e) &= \eta^{\mu\nu}J_\mu^{-\alpha}(p,e)\;J_\nu^{-\beta}(p,e)\;{}_0M^{h^{\mathrm{K}},h^{\mathrm{K}}}_{\alpha\beta\varrho\sigma}(p) \\ &= -\frac{p_\kappa e_\lambda + p_\lambda e_\kappa}{(pe)_-} + O(p^2)\end{aligned} \tag{3.19}$$

does not vanish on the mass-shell, meaning that $\langle\!\langle h(e)h^{\mathrm{K}}_{\mu\nu}\rangle\!\rangle \neq 0$ and thus $h = \eta^{\mu\nu}h_{\mu\nu} \neq 0$ on Krein space. With the same reasoning, one also finds that $\partial^\mu h_{\mu\nu} \equiv (\partial h)_\nu \neq 0$ on Krein space and trivially, that $(\partial h^{\mathrm{K}})_\nu \neq 0$. From Eq. (3.1b), we can infer

$$\begin{aligned} h = h^{\mathrm{K}} + 2(\partial\Phi), \quad (\partial h)_\nu = (\partial h^{\mathrm{K}})_\nu + \partial_\nu(\partial\Phi), \\ \Rightarrow \quad \partial^\mu\left(h_{\mu\nu} - h^{\mathrm{K}}_{\mu\nu}\right) = \frac{1}{2}\partial_\nu\left(h - h^{\mathrm{K}}\right). \end{aligned} \tag{3.20}$$

Finally, expressing the field strength as (double) curl of the Krein potential, we find that neither the Ricci trace $\eta^{\mu\nu}F_{\mu\kappa\nu\lambda}$ nor the divergence $\partial^\mu F_{\mu\kappa\nu\lambda}$ vanishes on Krein space because

$$\begin{aligned} \eta^{\mu\nu}F_{\mu\kappa\nu\lambda} &= \partial_\kappa\partial_\lambda h^{\mathrm{K}} - \partial_\lambda(\partial h^{\mathrm{K}})_\kappa - \partial_\lambda(\partial h^{\mathrm{K}})_\kappa \neq 0, \\ \partial^\mu F_{\mu\kappa\nu\lambda} &= \partial_\kappa\left(\partial_\lambda(\partial h^{\mathrm{K}})_\nu - \partial_\nu(\partial h^{\mathrm{K}})_\lambda\right) \neq 0. \end{aligned} \tag{3.21}$$

A summary comparing the properties of the fields on Hilbert and Krein space can be found in Table 3.2.

Remark 3.3. For the multi-string-localized potentials from Remark 2.3, one can do similar considerations that yield relations as displayed in Table 3.2. We do not go into detail here but the reader should be aware that the proofs in the subsequent Sections 3.4.1 and 3.4.2 apply also to the multi-string-localized potentials.

Property	Hilbert space	Krein space
Wave equation for all fields	satisfied	satisfied
$(eh)_\nu$	$= 0$	$= 0$
$(\partial h)_\nu$	$= 0$	$= (\partial h^{\mathrm{K}})_\nu + \partial_\nu(\partial\Phi) \neq 0$
$h = \eta^{\mu\nu} h_{\mu\nu}$	$= 0$	$= h^{\mathrm{K}} + 2(\partial\Phi) \neq 0$
$\partial^\mu F_{\mu\kappa\nu\lambda}$	$= 0$	$= \partial_\kappa \left(\partial_\lambda(\partial h^{\mathrm{K}})_\nu - \partial_\nu(\partial h^{\mathrm{K}})_\lambda\right) \neq 0$
$\eta^{\mu\nu} F_{\mu\kappa\nu\lambda}$	$= 0$	$= \partial_\kappa\partial_\lambda h^{\mathrm{K}} - \partial_\lambda(\partial h^{\mathrm{K}})_\kappa - \partial_\lambda(\partial h^{\mathrm{K}})_\kappa \neq 0$
Bianchi identity for $F_{\mu\kappa\nu\lambda}$	satisfied	satisfied

Table 3.2: Properties of helicity two fields on Krein and Hilbert space.

3.2 Non-perturbative constructions in SLFT

The interaction Lagrangian of string-localized QED, obtained by inserting the conserved fermion current $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ into Eq.s (3.2) and (3.3), is part of an L-V pair,

$$A_\mu(x,e) j^\mu(x) = A^{\mathrm{K}}_\mu(x) j^\mu + \partial_\mu \left[\varphi(x,e) j^\mu(x)\right]. \tag{3.22}$$

The trivial interaction $q\partial_\mu \left[\varphi(x,e) j^\mu(x)\right] = q\partial_\mu\varphi(x,e) j^\mu(x)$ yields

$$i\gamma^\mu \left[\partial_\mu - iq\partial_\mu\varphi(x,e)\right]\psi(x) = m\psi(x) \tag{3.23}$$

as equations of motion for the Dirac field $\psi(x)$, with the classical solution

$$\psi_e(x) = e^{iq\varphi(x,e)}\psi_0(x), \tag{3.24}$$

where $\psi_0(x)$ is the free Dirac field satisfying $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ [51]. Remembering that the escort field $\varphi(x,e)$ is given by an integral over the Krein potential by Eq. (3.10), we recognize that $\psi_e(x)$ from Eq. (3.24) formally is the same as Mandelstam's gauge invariant $\Psi(x,P)$ from Eq. (1.1). However, $\psi_e(x)$ is a classical solution but not a quantum field. Moreover, the escort field $\varphi(x,e)$ is logarithmically divergent in the infrared and needs to be regularized, for formally

$$\langle\!\langle \varphi(e)\varphi(e')\rangle\!\rangle(x) = I_e I_{-e'} e^\mu e'^\nu \langle\!\langle A^{\mathrm{K}}_\mu A^{\mathrm{K}}_\nu \rangle\!\rangle(x) = -(ee') \int d\mu_0(p)\, \frac{e^{-ipx}}{(pe)_-(pe')_+}, \tag{3.25}$$

and the right-hand side scales as $|p|^{-4}$ with respect to $p = 0$. As expected from the fact that the escort field lives on Krein space, its two-point function (3.25) is not positive definite. Mund, Rehren and Schroer [51] showed that positivity can be restored if one chooses the string variables to be purely spacelike, that is, $e^0 = e'^0 = 0$, thereby breaking Lorentz invariance. In a next step, they split the two-point function in a singular and a regularized part by introducing an infrared cutoff function $v(p)$ with $v(0) = 1$ and defining it as a massless limit,

$$\begin{aligned}\langle\!\langle \varphi(e)\varphi(e')\rangle\!\rangle(x) &= -(ee') \lim_{m\to 0}\left[\int d\mu_m(p)\,\frac{e^{-ipx} - v(p)}{(pe)_-(pe')_+} + \int d\mu_m(p)\,\frac{v(p)}{(pe)_-(pe')_+}\right]\\ &=: \langle\!\langle \varphi(e)\varphi(e')\rangle\!\rangle_v(x) + \lim_{m\to 0} d_{m,v}(e,e'),\end{aligned} \tag{3.26}$$

where $\langle\!\langle \varphi(e)\varphi(e')\rangle\!\rangle_v(x)$ is the regularized part and where the x-independent expression $d_{m,v}(e,e')$ diverges to $+\infty$ in the limit $m \to 0$. Starting with this splitting, Mund, Rehren and Schroer [51] could construct the renormalized exponential of the smeared escort field, $\varphi(g,c)$, where the test function c, which smears out the string variables, needs to be of total weight $\int d^3\vec{e}\, c(\vec{e}) = 1$:

$$:e^{i\varphi(g,c)}:_v \;:=\; \lim_{m\to 0} e^{-\frac{1}{2}\hat{g}(0)^2\, d_{m,v}(c,c)}\; :e^{i\varphi(g,c)}: \,. \tag{3.27}$$

With the regularized exponential from Eq. (3.27) constructed, these authors derived an uncountable superselection structure corresponding to (then superselected) photon clouds accompanying the Dirac field, which arises from the fact that $d_{m,v}(c,c)$ diverges to $+\infty$ in the massless limit unless $c \equiv 0$. Finally, defining $g = q\delta_x = q\delta(x-\cdot)$ as smearing function, they define the vertex operators $V_{q,c}(x) := \,:e^{i\varphi(q\delta_x,c)}:_v$ and complete the construction of the dressed Dirac field

$$\psi_{q,c}(x) := V_{q,c}(x)\,\psi_0(x). \tag{3.28}$$

The breaking of Lorentz invariance in their construction is unproblematic since it is known that photon clouds break Lorentz invariance [34, 51].

Similar constructions to the ones of Mund, Rehren and Schroer in the framework of QED have been studied by Brüers for the coupling of a string-localized graviton potential to the SET of a scalar Klein-Gordon field $\chi(x)$ but the situation there is less clear [11]. The escort fields of the string-localized graviton field can be combined into a single field $\beta_\mu(x,e)$ and an analogue of the dressed Dirac field is the shifted field

$$\chi_{q,c}(x) = \chi_0(x - q\beta(x,c)), \tag{3.29}$$

where χ_0 is the free field. The shift of χ_0 by an operator-valued coordinate transformation $q\beta(x,c)$ as depicted in Eq. (3.29) seems natural from the perspective of general relativity – forgetting for the moment that the shift is an operator-valued distribution that is in addition infrared divergent if naively written down as in Eq. (3.29). However, the shifted field is not the only solution to the problem and work is needed to clarify the ambiguities.

3.3 Singularity structure of infrared divergences in QED

Together with K.-H. Rehren and F. Tippner, the author of this thesis was able to derive an analytic representation of the two-point function (3.25) of the escort field $\varphi(x, e)$, identifying the finite and the divergent part explicitly [38]. This makes the undetermined regularization function $v(p)$ obsolete and offers the opportunity to directly compute correlations of the vertex operator and the dressed Dirac field. In this section, we derive the analytic representation of $\langle\!\langle \varphi(e)\varphi(e')\rangle\!\rangle(x)$. We start with an investigation of Minkowski space Gram determinants, which are central building blocks of this two-point function, in Section 3.3.1. We then perform the actual derivation of the two-point functions of interest in Section 3.3.2. In Section 3.3.3, the results are related to the vertex operators $V_{q,c}$, which we encountered in Section 3.2.

3.3.1 Gram determinants in Minkowski space

Gram determinants appear at various places in the two-point functions of string-localized fields to be derived in Section 3.3.2. Their properties over Euclidean space are well-established and understood. However, their properties and a geometrical interpretation over Minkowski space are not – and hence are interesting from a purely mathematical perspective. Therefore, this section, in which we prove statements about Gram determinants over Minkowski space, should not be seen as a mere introduction to tools needed for physical application but also as a list of mathematical results that stand on their own feet.

For a collection of vectors $y_1, \dots, y_n$, the associated Gram determinant is defined by

$$\det_{y_1\dots y_n} := \det \begin{pmatrix} y_1^2 & (y_2 y_1) & \cdots & (y_n y_1) \\ (y_1 y_2) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ (y_1 y_n) & (y_2 y_n) & \cdots & y_n^2 \end{pmatrix}. \tag{3.30}$$

Over Euclidean space, Gram determinants vanish if and only if the appearing vectors are linearly dependent. In Minkowski space however, Gram determinants vanish on a bigger set of configurations. A vanishing Gram determinant in Minkowski space implies that "something" is lightlike but does not always mean that the vectors are linearly dependent. Of course, the Gram determinant vanishes if the system $\{ y_1, \dots, y_n \}$ is linearly dependent but in contrast to the Euclidean case, there is no "only if" in the Minkowskian case.

Remark 3.4. If the number n of vectors exceeds the dimension of the space (or spacetime), then the system $\{ y_1, \dots, y_n \}$ must be linearly dependent and consequently, the Gram determinant $\det_{y_1\dots y_n}$ must vanish. This simple observation gives a relation between scalar products of the involved vectors and proves useful later in Section 5.7.1.

In the following, we restrict our considerations to properties of Minkowski space

Gram determinants of two or three Minkowski vectors,

$$\begin{aligned}\det_{y_1 y_2} &= y_1^2 y_2^2 - (y_1 y_2)^2, \quad \text{and} \\ \det_{y_1 y_2 y_3} &= y_1^2 y_2^2 y_3^2 - y_1^2 (y_2 y_3)^2 - y_2^2 (y_3 y_1)^2 - y_3^2 (y_1 y_2)^2 + 2(y_1 y_2)(y_2 y_3)(y_3 y_1)\end{aligned} \tag{3.31}$$

and to the cofactors (signed subdeterminants) of the 3×3-determinant,

$$\Lambda_{y_i} := (y_i y_j)(y_i y_k) - y_i^2 (y_j y_k), \quad \text{for } i, j, k \text{ pairwise distinct.} \tag{3.32}$$

Let us first prove a relation between the 3×3-determinant and some of its cofactors, which has two interesting corollaries.

Lemma 3.5. *We have* $y_1^2 \det_{y_1 y_2 y_3} = \det_{y_1 y_2} \det_{y_1 y_3} - \Lambda_{y_1}^2$.

Proof. The proof is just a short computation,

$$\begin{aligned}&y_1^2 \det_{y_1 y_2 y_3} \\ &= (y_1^2)^2 y_2^2 y_3^2 - (y_1^2)^2 (y_2 y_3)^2 - y_1^2 y_2^2 (y_3 y_1)^2 - y_1^2 y_3^2 (y_1 y_2)^2 + 2 y_1^2 (y_1 y_2)(y_2 y_3)(y_3 y_1) \\ &= \left[y_1^2 y_2^2 - (y_1 y_2)^2\right] \left[y_1^2 y_3^2 - (y_1 y_3)^2\right] \\ &\qquad - (y_1^2)^2 (y_2 y_3)^2 + 2 y_1^2 (y_1 y_2)(y_2 y_3)(y_3 y_1) - (y_1 y_2)^2 (y_1 y_3)^2 \\ &= \left[y_1^2 y_2^2 - (y_1 y_2)^2\right] \left[y_1^2 y_3^2 - (y_1 y_3)^2\right] - \left[(y_1 y_2)(y_1 y_3) - y_1^2 (y_2 y_3)\right]^2 \\ &= \det_{y_1 y_2} \det_{y_1 y_3} - \Lambda_{y_1}^2.\end{aligned}$$

□

Corollary 3.6 (of Lemma 3.5)**.** *Suppose that* $\det_{y_1 y_2 y_3} = 0$*. Then all* 2×2*-determinants of the form* $\det_{y_i y_j}$ *have the same sign or vanish.*

Proof. By Lemma 3.5, we have $y_1^2 \det_{y_1 y_2 y_3} = \det_{y_1 y_2} \det_{y_1 y_3} - \Lambda_{y_1}^2$, so if $\det_{y_1 y_2 y_3} = 0$, we have

$$\det_{y_1 y_2} \det_{y_1 y_3} = \Lambda_{y_1}^2 \geq 0,$$

which means that either $\operatorname{sgn} \det_{y_1 y_2} = \operatorname{sgn} \det_{y_1 y_3}$ or that at least one of them is zero. By symmetry of the Gram determinant under exchanging the y_i, we get the claim also for the other combinations of indices. □

Corollary 3.7 (of Lemma 3.5)**.** *Suppose that* $\det_{y_1 y_2 y_3} = 0 = \det_{y_i y_j}$ *for some* $i \neq j \in \{1, 2, 3\}$*. then* $\Lambda_{y_i} = \Lambda_{y_j} = 0$.

Proof. Let $i, j, k \in \{1, 2, 3\}$ be pairwise distinct. Using Lemma 3.5, we obtain

$$\begin{aligned}\Lambda_{y_i}^2 &= \det_{y_i y_j} \det_{y_i y_k} - y_i^2 \det_{y_1 y_2 y_3} = 0 - 0, \\ \Lambda_{y_j}^2 &= \det_{y_i y_j} \det_{y_j y_k} - y_i^2 \det_{y_1 y_2 y_3} = 0 - 0,\end{aligned}$$

which proves the claim. □

The 3×3-determinant and its gradients can also be expressed via its minors in the following way, as can be seen by a short computation.

Lemma 3.8. *Let $i, j, k \in \{1, 2, 3\}$ be pairwise distinct. We have*

$$\begin{aligned} \det{}_{y_1y_2y_3} &= \det{}_{y_iy_j} {y_k}^2 + \Lambda_{y_j}(y_k y_i) + \Lambda_{y_i}(y_k y_j) \quad \textit{and} \\ \partial_{y_k} \det{}_{y_1y_2y_3} &= 2\left\{\det{}_{y_iy_j} y_k + \Lambda_{y_j} y_i + \Lambda_{y_i} y_j\right\}. \end{aligned} \tag{3.33}$$

Later, we will also need the inverse Gram matrix, which can easily be computed by use of Lemma 3.8:

Lemma 3.9. *Let G be the Gram matrix associated with the vectors y_1, y_2 and y_3 and let* $\det G = \det_{y_1y_2y_3} \neq 0$. *Then*

$$G^{-1} = \frac{1}{\det_{y_1y_2y_3}} \begin{pmatrix} \det_{y_2y_3} & \Lambda_{y_3} & \Lambda_{y_2} \\ \Lambda_{y_3} & \det_{y_1y_3} & \Lambda_{y_1} \\ \Lambda_{y_2} & \Lambda_{y_1} & \det_{y_1y_2} \end{pmatrix}. \tag{3.34}$$

The vanishing of Gram determinants over Minkowski space is related to the gradients $\partial_{y_k} \det_{y_1y_2y_3}$ by the following lemma.

Lemma 3.10. *Let* $\det_{y_1y_2y_3} = 0$*, then* $\partial_{y_k} \det_{y_1y_2y_3}$ *is lightlike for all* $k \in \{1, 2, 3\}$*. Similarly, let* $\det_{y_1y_2} = 0$*, then* $\partial_{y_k} \det_{y_1y_2}$ *is lightlike for* $k = 1, 2$.

Proof. Let $w_k := \partial_{y_k} \det_{y_1y_2y_3}$. Since

$$(y_i \partial_{y_k}) \det{}_{y_1y_2y_3} = 2\delta_{ik} \det{}_{y_1y_2y_3} = 0 \tag{3.35}$$

if $\det_{y_1y_2y_3} = 0$, Lemma 3.8 gives ${w_k}^2 = 2\det_{y_iy_j} \det_{y_1y_2y_3} = 0$, where i, j, k are pairwise distinct. The proof is similar for the 2×2 determinant. □

Most probably, Lemma 3.10 can easily be generalized to $\det_{y_1 \ldots y_n}$ but only properties of $\det_{y_1y_2y_3}$ and $\det_{y_1y_2}$ are of physical interest in this thesis. Lemma 3.10 is in some sense the Minkowski space generalization of the "if and only if" between the vanishing of the Euclidean Gram determinant and the linear dependency of the appearing Euclidean vectors.

Of particular interest for SLFT is the setup where some of the vectors are spacelike. For spacelike string variables $e_1, e_2 \in H$, we will encounter the determinants $\det_{xe_i}$, $\det_{e_1e_2}$ and $\det_{xe_1e_2}$. Such determinants have more properties than general ones, which we investigate in the following. We start by proving the following statement about a non-zero and two non-lightlike vectors.

Lemma 3.11. *Let $y_1 \neq 0$ and let y_2 and y_3 have non-zero Minkowski square. Suppose further that* $\det_{y_1y_2y_3} = 0$, $\det_{y_1y_2} = 0$ *and* $\det_{y_2y_3} = 0$. *Then the set $\{y_1, y_2, y_3\}$ is linearly dependent.*

Proof. Since $y_2^2 \neq 0$ and $y_3^2 \neq 0$ by assumption, we can define the vectors

$$v := y_1 - \frac{(y_1 y_2)}{y_2^2} y_2, \quad w := y_2 - \frac{(y_2 y_3)}{y_3^2} y_3. \tag{3.36}$$

It is easy to check that $v^2 = \frac{\det_{y_1 y_2}}{y_2^2} = 0$ and $w^2 = \frac{\det_{y_2 y_3}}{y_3^2} = 0$. Moreover,

$$(vw) = \frac{(y_2 y_3)}{y_2^2 y_3^2} \Big((y_2 y_3)(y_1 y_2) - y_2^2 (y_1 y_3) \Big) = \frac{(y_2 y_3)}{y_2^2 y_3^2} \Lambda_{y_2} = 0 \tag{3.37}$$

by Corollary 3.7. Hence, v and w are both lightlike and Minkowski-orthogonal to each other. This means that either one of them is zero or they are both non-zero but linearly dependent. If $v = 0$, then y_1 and y_2 are linearly dependent, if $w = 0$, then y_2 and y_3 are linearly dependent. If v and w are non-zero but linearly dependent, $\lambda v + \mu w = 0$ for $\lambda, \mu \neq 0$, then

$$\mu y_1 + \left(\lambda - \mu \frac{(y_1 y_2)}{y_2^2} \right) y_2 - \lambda \frac{(y_2 y_3)}{y_3^2} y_3 = 0, \tag{3.38}$$

which means that y_1, y_2 and y_3 are linearly dependent. □

While $\det_{y_1 y_2}$ can be exchanged with $\det_{y_1 y_3}$ in the previous Lemma 3.11 by symmetry of the assumptions under the exchange $y_2 \leftrightarrow y_3$, it is crucial that we require $\det_{y_2 y_3} = 0$, i.e., it is crucial that the 2×2 sub-determinant of the two non-lightlike vectors vanishes. To see that the statement is in general not true if we require $\det_{y_1 y_2} = \det_{y_1 y_3} = 0$ instead of $\det_{y_1 y_2} = \det_{y_2 y_3} = 0$, we can choose a lightlike y_1 and find y_2, y_3 that form a counterexample. For example, take

$$y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad y_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We have $y_1^2 = 0 = (y_1 y_2) = (y_1 y_3) = (y_2 y_3)$, which yields

$$\det_{y_1 y_2 y_3} = \det_{y_1 y_2} = \det_{y_1 y_3} = 0, \quad \det_{y_2 y_3} = 1,$$

but clearly, the set $\{y_1, y_2, y_3\}$ is not linearly dependent.

Lemma 3.12. *Let $y_1 \neq 0$, let y_2 and y_3 have non-zero Minkowski square and suppose that the set $\{y_1, y_2, y_3\}$ is not linearly dependent. Then*

$$\begin{pmatrix} \partial_{y_1} \det_{y_1 y_2 y_3} \\ \partial_{y_2} \det_{y_1 y_2 y_3} \\ \partial_{y_3} \det_{y_1 y_2 y_3} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.39}$$

Proof. By Lemma 3.8, we can write

$$\begin{pmatrix} \partial_{y_1}\det_{y_1y_2y_3} \\ \partial_{y_2}\det_{y_1y_2y_3} \\ \partial_{y_3}\det_{y_1y_2y_3} \end{pmatrix} = 2\begin{pmatrix} \det_{y_2y_3} & \Lambda_{y_3} & \Lambda_{y_2} \\ \Lambda_{y_3} & \det_{y_1y_3} & \Lambda_{y_1} \\ \Lambda_{y_2} & \Lambda_{y_1} & \det_{y_1y_2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{3.40}$$

Since y_1, y_2 and y_3 are linearly independent, it follows that all 2×2 sub-determinants must vanish. But then Lemma 3.11 implies that y_1, y_2 and y_3 are linearly dependent, which contradicts our assumption. □

We have now proven two statements about Gram determinants if two of the appearing vectors are non-lightlike. Assuming that these vectors are spacelike, we can make a statement about the sign of the 2×2 subdeterminants of $\det_{y_1y_2y_3}$, provided the system $\{ y_1, y_2, y_3 \}$ is not linearly dependent.

Lemma 3.13. *Let* $y_1 \neq 0$*,* $y_2^2 < 0$ *and* $y_3^2 < 0$*,* $\det_{y_1y_2y_3} = 0$ *and suppose that the set* $\{y_1, y_2, y_3\}$ *is not linearly dependent. Then* $\det_{y_iy_j} \geq 0$ *for all* i, j*.*

Proof. Since $\det_{y_1y_2y_3} = 0$, all vectors $w_i := \partial_{y_i}\det_{y_1y_2y_3}$ are lightlike by Lemma 3.10. We define the vectors

$$v_1 := y_2 - \frac{(y_2y_3)}{y_3^2}y_3, \quad v_2 := y_1 - \frac{(y_1y_3)}{y_3^2}y_3, \quad v_3 := y_1 - \frac{(y_1y_2)}{y_2^2}y_2 \tag{3.41}$$

with ${v_1}^2 = \frac{1}{y_3^2}\det_{y_2y_3}$, ${v_2}^2 = \frac{1}{y_3^2}\det_{y_1y_3}$ and ${v_3}^2 = \frac{1}{y_2^2}\det_{y_1y_2}$. Moreover, since

$$(y_i\partial_{y_j})\det_{y_1y_2y_3} = 2\delta_{ij}\det_{y_1y_2y_3} = 0 \tag{3.42}$$

if $\det_{y_1y_2y_3} = 0$, we have that $(v_iw_j) = 0$ for all i, j. Suppose now that one of the determinants $\det_{y_iy_j}$ is negative. Then the vector v_k, where $k \notin \{i, j\}$ is timelike since y_2 and y_3 are spacelike. But then $(v_kw_l) = 0$ for all $l = 1, 2, 3$ implies that all w_l are actually the zero-vector because a timelike vector cannot be Minkowski-orthogonal to a non-zero lightlike vector. Hence the set $\{y_1, y_2, y_3\}$ would be linearly dependent since w_l are linear combinations of y_1, y_2 and y_3, which is a contradiction to the assumption. □

The statement from Lemma 3.13 that the determinants $\det_{y_iy_j}$ are non-negative in the case when $\det_{y_1y_2y_3} = 0$ but when the vectors y_1, y_2 and y_3 are not linearly dependent (and two of them are spacelike) is relevant for the $i\varepsilon$-prescription of the string-localized photon propagator. This will become clear in Section 3.3.4.

3.3.2 The escort field two-point function in configuration space

In this section, we have to deal with logarithms and square roots of complex variables. We choose their branch cuts along the negative real axis. Let $W_0(x)$ be the two-point function of a massless scalar Klein-Gordon field, that is

$$W_0(x) := \int d\mu_0(p)\, e^{-ipx} = -\frac{1}{(2\pi)^2}\lim_{\varepsilon\downarrow 0}\frac{1}{(x^0 - i\varepsilon)^2 - |\vec{x}|^2} = -\frac{1}{(2\pi)^2}\lim_{\varepsilon\downarrow 0}\frac{1}{x^2 - i\varepsilon x^0}. \tag{3.43}$$

The two-point function of the massless Krein potential of helicity $s = 1$ in Feynman gauge is $\langle\!\langle A^{\mathrm{K}}_\mu A^{\mathrm{K}}_\nu \rangle\!\rangle(x) = -\eta_{\mu\nu} W_0(x)$, giving the formal and infrared divergent two-point function

$$\langle\!\langle \varphi(e)\varphi(e') \rangle\!\rangle(x) = -(ee') I_e I_{-e'} W_0(x) \tag{3.44}$$

of the escort field $\varphi(e) = I_e(A^{\mathrm{K}} e)$. To derive an explicit representation of Eq. (3.44), we start by computing the single string integral $I_e W_0(x)$.

Lemma 3.14. *Let $e \in H$. The single string integral $I_e W_0(x)$ over $W_0(x)$ is given by*

$$\begin{aligned}(2\pi)^2 I_e W_0(x) &= \frac{1}{2} \lim_{\varepsilon \downarrow 0} \frac{\ln\left(-(xe) + \sqrt{-\det_{x_\varepsilon e}}\right) - \ln\left(-(xe) - \sqrt{-\det_{x_\varepsilon e}}\right)}{\sqrt{-\det_{x_\varepsilon e}}} \\ &=: f(x,e),\end{aligned} \tag{3.45}$$

where the Gram determinant $\det_{x_\varepsilon e} = (x^2 - i\varepsilon x^0)e^2 - (xe)^2$ *contains an imaginary shift. Moreover, $f(x,e)$ is independent of the branch of the complex square root.*

Remark 3.15. Note the formal symmetry of $f(x,e)$ under the exchange $x \leftrightarrow e$, which can be understood by performing a change of variables $s \longrightarrow \frac{1}{s}$ in the integral

$$\int_0^\infty ds \, \frac{1}{(x+se)^2 - i\varepsilon(x^0 + se^0)}. \tag{3.46}$$

Proof (see also [38] for an alternative proof). The spacelike string variable $e \in H$ can be boosted to a purely spatial vector $(0, \vec{e})^T$ and because the integrated two-point function $I_e W_0(x)$ is Lorentz invariant by invariance of the measure $d\mu_0(p)$,

$$\begin{aligned} I_{\Lambda e} W_0(\Lambda x) &= \int_0^\infty ds \int d\mu_0(p)\; e^{-ip\Lambda(x+se)} = \int_0^\infty ds \int d\mu_0(\Lambda p)\; e^{-ip(x+se)} \\ &= I_e W_0(x), \end{aligned}$$

it is sufficient to compute $I_e W_0(x)$ in a Lorentz frame where $e^0 = 0$. Let $u = (1, \vec{0})^T$ so that $W_0(x) = \lim_{\varepsilon\downarrow 0}[(x - i\varepsilon u)^2]^{-1}$, then in a frame with $e^0 = 0$, we have

$$\begin{aligned}(2\pi)^2 I_e W_0(x) &= -\lim_{\varepsilon\downarrow 0} \int_0^\infty ds \; \frac{1}{(x - i\varepsilon u)^2 + 2s(xe) + s^2 e^2} \\ &= -e^2 \lim_{\varepsilon\downarrow 0} \int_0^\infty ds \; \frac{1}{(e^2 s + (xe))^2 + \det_{x_\varepsilon e}}.\end{aligned} \tag{3.47}$$

To perform the integral, we decompose the denominator in Eq. (3.47). This means taking the square root of the complex determinant $-\det_{x_\varepsilon e}$. We proceed without specifying the branch of the square root and after the computation, we will see that the result is independent of the branch. We then have

$$
\begin{aligned}
(2\pi)^2 I_e W_0(x) &= \lim_{\varepsilon\downarrow 0} \frac{1}{2\sqrt{-\det_{x_\varepsilon e}}} \int_0^\infty ds\, \frac{1}{s + \frac{(xe)+\sqrt{-\det_{x_\varepsilon e}}}{e^2}} - \frac{1}{s + \frac{(xe)-\sqrt{-\det_{x_\varepsilon e}}}{e^2}} \\
&= -\lim_{\varepsilon\downarrow 0} \frac{\ln\left(\frac{(xe)+\sqrt{-\det_{x_\varepsilon e}}}{e^2}\right) - \ln\left(\frac{(xe)-\sqrt{-\det_{x_\varepsilon e}}}{e^2}\right)}{2\sqrt{-\det_{x_\varepsilon e}}} \\
&= \lim_{\varepsilon\downarrow 0} \frac{\ln\left(-(xe) + \sqrt{-\det_{x_\varepsilon e}}\right) - \ln\left(-(xe) - \sqrt{-\det_{x_\varepsilon e}}\right)}{2\sqrt{-\det_{x_\varepsilon e}}},
\end{aligned}
\tag{3.48}
$$

where the last transformation follows because $e^2 = -|e^2|$ and the positive factor $|e^2|$ cancels in the difference of the two logarithms. The independence of the branch of the square root follows because numerator and denominator of the last version of Eq. (3.48) change sign simultaneously under a change of the branch. □

Remark 3.16. Note that the difference of two logarithms in the distribution $f(x, e)$ is not necessarily the logarithm of the quotient of the arguments since we deal with complex logarithms. To make the arguments of the logarithms dimensionless, one can for example insert a positive denominator

$$
\sqrt{|(x^2 - i\varepsilon x^0)e^2|} = \sqrt{|x^2 e^2 - i\varepsilon e^2 x^0|} \tag{3.49}
$$

into both logarithms.

The well-definedness of $f(x, e) = (2\pi)^2 I_e W_0(x)$ as a distribution over $\mathbb{R}^{1+3} \times H$ is easier to prove in momentum space. We postpone this proof to Section 5.3. The double integral Eq. (3.44), however, is infrared divergent and not well-defined as it stands. To give meaning to it, we compute the cutoff integral $I^a_{e_2} I_{e_1} W_0(x)$, with $e_1 = e$, $e_2 = -e'$ and where $I^a_{e_2}$ stands for a string integration with the finite upper integral border $a > 0$. Now, the computation for general spacelike string variables is quite cumbersome and two spacelike vectors can in general not be boosted into a purely spatial plane. However, for the application to the Mund-Rehren-Schroer construction, only purely spatial string variables with $e^0_1 = e^0_2 = 0$ are needed. Therefore, we restrict our discussion to that case.

The double integral contains dilogarithm functions of complex arguments. Since the dilogarithm is usually not considered a standard function, we display its definition,

$$
\mathrm{Li}_2(z) := \int_0^z dt\, \frac{-\ln(1-t)}{t}, \tag{3.50}
$$

with a branch cut along $[1, \infty)$. Inside the circle $|z| < 1$, it is given by a power series

$$
\mathrm{Li}_2(z) = \sum_{n=1}^\infty \frac{z^n}{n^2}, \quad \text{for} \quad |z| < 1. \tag{3.51}
$$

The dilogarithm satisfies several functional equations, some of which we employ during the computation for the cutoff integral $I^a_{e_2} I_{e_1} W_0(x)$. We do not give an introduction to properties of the dilogarithm at this point and refer to the literature, e.g. [80], for further details. We are now ready to prove the following statement.

Theorem 3.17. *Let e_1 and e_2 be purely spatial, that is, $e_1^0 = e_2^0 = 0$, linearly independent with $e_1^2 = e_2^2 = -1$ and denote the angle between $\vec{e}_1$ and $\vec{e}_2$ by $\gamma \in (0, \pi)$, so that $\det_{e_1 e_2} = \sin^2 \gamma \neq 0$. Let further $u = (1, \vec{0})^T$. Then*

$$I^a_{e_2} I_{e_1} W_0(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{(2\pi)^2} \left[\frac{1}{2} f(e_1, e_2) \ln\left(\frac{4(ae_2)^2}{(x - i\varepsilon u)^2} \right) + \frac{H(x, e_1, e_2)}{(e_1 e_2)} \right] + O(a^{-1}), \tag{3.52}$$

where $f(e_1, e_2)$ is the distribution f from Lemma 3.14 restricted to the set $(\mathbb{S}^2)^2$ of two purely spatial unit vectors, given by $f(e_1, e_2) = \frac{\gamma}{\sin\gamma}$, and where $H(x, e_1, e_2)$ is the distribution

$$H = -\frac{\cos\gamma}{2\sin\gamma} \left[\gamma \ln\left(\frac{\sin^4\gamma}{D} \right) + \pi(\zeta_1 + \zeta_2) - \frac{i}{2} \left\{ \mathrm{Li}_2\left(e^{i\gamma} e^{\zeta_1} e^{\zeta_2} \right) \begin{array}{l} + (e^{\zeta_1} \leftrightarrow -e^{-\zeta_1}) \\ + (e^{\zeta_2} \leftrightarrow -e^{-\zeta_2}) \\ - (e^{i\gamma} \leftrightarrow e^{-i\gamma}) \end{array} \right\} \right],$$

which is homogeneous of degree $\omega = 0$ in all three variables x, e_1 and e_2 and where

$$D := \frac{\det_{x_\varepsilon e_1 e_2}}{(x - i\varepsilon u)^2 e_1^2 e_2^2}, \qquad \pm e^{\pm\zeta_i} := \frac{\Lambda_{e_i} \pm \sqrt{\det_{e_1 e_2} \det_{x_\varepsilon e_i}}}{\sqrt{e_i^2 \det_{x_\varepsilon e_1 e_2}}} \tag{3.53}$$

with Λ_{e_i} as in Eq. (3.32) and with the subscript ε in the determinants indicating that x^2 is to be understood as $(x - i\varepsilon u)^2$.

The proof is similar to the one given in our paper [38]. However, in the framework of this thesis, we display all details of the computation, which have partially been omitted in the paper, where conciseness is more important.

Proof. The cutoff double integral is given by a finite line integral over the distribution $f(x, e_1)$ from Lemma 3.14,

$$\begin{aligned} I^a_{e_2} I_{e_1} W_0(x) = \frac{1}{2} \int_0^a ds \, & \frac{\ln\left(-(xe_1) - s(e_1 e_2) + \sqrt{-\det_{(x+se_2)_\varepsilon e_1}} \right)}{\sqrt{-\det_{(x+se_2)_\varepsilon e_1}}} \\ & - \frac{\ln\left(-(xe_1) - s(e_1 e_2) - \sqrt{-\det_{(x+se_2)_\varepsilon e_1}} \right)}{\sqrt{-\det_{(x+se_2)_\varepsilon e_1}}}. \end{aligned} \tag{3.54}$$

To compute this integral, we first assume that the component $x^\perp$ of x perpendicular to e_1 and e_2 is spacelike. In such a case, there is a Lorentz boost Λ such that $(\Lambda x)^0 = 0$ and

$\Lambda e_i = e_i$ for $i = 1, 2$, i.e., we can assume x to be purely spatial as well. Then, because all appearing vectors are purely spatial, we have

$$e_i^2 \det_{xe_1e_2} \geq 0 \quad \text{and} \quad \det_{y_iy_j} \geq 0 \quad \text{for} \quad y_i, y_j \in \{x, e_1, e_2\}, \tag{3.55}$$

and consequently, Lemma 3.5 gives $|\det_{xe_i}\det_{e_1e_2}| \geq |\Lambda_{e_i}|$, so that the variables ζ_i defined in Eq. (3.53) are real. A short computation shows that

$$\det_{(x+se_2)e_1} = \frac{[\det_{e_1e_2} s - \Lambda_{e_1}]^2 + e_1^2 \det_{xe_1e_2}}{\det_{e_1e_2}} =: \frac{\det_{e_1e_2}}{\Gamma_1^2} \left[1 + (s\Gamma_1 + \Gamma_2)^2\right], \tag{3.56}$$

where we have defined

$$\Gamma_1 := \frac{\det_{e_1e_2}}{\sqrt{e_1^2 \det_{xe_1e_2}}} \quad \text{and} \quad \Gamma_2 := -\frac{\Lambda_{e_1}}{\sqrt{e_1^2 \det_{xe_1e_2}}}, \tag{3.57}$$

with Γ_1 being non-negative due to Eq. (3.55). Thus, we can perform a change of the integration variable according to

$$\begin{aligned} C(s) &= s\Gamma_1 + \Gamma_2 + \sqrt{1 + (s\Gamma_1 + \Gamma_2)^2} \\ \text{with} \quad C(s)^{-1} &= -(s\Gamma_1 + \Gamma_2) + \sqrt{1 + (s\Gamma_1 + \Gamma_2)^2}. \end{aligned} \tag{3.58}$$

Note that both $C(s)$ and $C(s)^{-1}$ are positive for real Γ_i. Using that Γ_1 is non-negative and that the ζ_i are real, we find

$$\begin{aligned} f(x + se_2, e_1) &= \frac{\Gamma_1}{\sqrt{\det_{e_1e_2}}} \left\{ \frac{\ln\left[\frac{e^{-\zeta_2}}{2\Gamma_1}(1 + e^{i\gamma}e^{\zeta_2}C(s))(1 - e^{-i\gamma}e^{\zeta_2}C(s)^{-1})\right]}{i[C(s) + C(s)^{-1}]} \right. \\ &\qquad \left. - \frac{\ln\left[\frac{e^{-\zeta_2}}{2\Gamma_1}(1 + e^{-i\gamma}e^{\zeta_2}C(s))(1 - e^{i\gamma}e^{\zeta_2}C(s)^{-1})\right]}{i[C(s) + C(s)^{-1}]} \right\} \\ &= \frac{\Gamma_1}{\sqrt{\det_{e_1e_2}}} \left\{ \frac{\ln\left[(1 + e^{i\gamma}e^{\zeta_2}C(s))(1 - e^{-i\gamma}e^{\zeta_2}C(s)^{-1})\right]}{i[C(s) + C(s)^{-1}]} \right. \\ &\qquad \left. - \frac{\ln\left[(1 + e^{-i\gamma}e^{\zeta_2}C(s))(1 - e^{i\gamma}e^{\zeta_2}C(s)^{-1})\right]}{i[C(s) + C(s)^{-1}]} \right\}. \end{aligned} \tag{3.59}$$

Writing the double integral in the new integration variable yields

$$\begin{aligned} \int_0^a ds\, f(x + se_2, e_1) = \frac{1}{2i\sqrt{\det_{e_1e_2}}} \int_{C(0)}^{C(a)} \frac{dC}{C} &\left\{\ln\left[(1 + e^{i\gamma}e^{\zeta_2}C)(1 - e^{-i\gamma}e^{\zeta_2}C^{-1})\right]\right. \\ &\left. - \ln\left[(1 + e^{-i\gamma}e^{\zeta_2}C)(1 - e^{i\gamma}e^{\zeta_2}C^{-1})\right]\right\}. \end{aligned} \tag{3.60}$$

Using the definition of the dilogarithm Eq. (3.50), we obtain

$$\int_0^a ds\, f(x+se_2,e_1) = \left.\frac{\mathrm{Li}_2(e^{-i\gamma}e^{\zeta_2}C^{-1}) - \mathrm{Li}_2(-e^{i\gamma}e^{\zeta_2}C) - (e^{i\gamma}\leftrightarrow e^{-i\gamma})}{2i\sqrt{\det_{e_1e_2}}}\right|_{C(0)}^{C(a)}, \tag{3.61}$$

where we are allowed to apply the logarithm law to relate the sum of derivatives of the dilogarithm to logarithms of products because we do not cross any branch cuts. Inserting $C(0)^{\pm 1} = e^{\mp\zeta_1}$, the contribution of the lower boundary of the integral (3.61) reads

$$\frac{i\left[\mathrm{Li}_2(e^{-i\gamma}e^{\zeta_2}e^{\zeta_1}) - \mathrm{Li}_2(-e^{i\gamma}e^{\zeta_2}e^{-\zeta_1}) - \mathrm{Li}_2(e^{i\gamma}e^{\zeta_2}e^{\zeta_1}) + \mathrm{Li}_2(-e^{-i\gamma}e^{\zeta_2}e^{-\zeta_1})\right]}{2\sqrt{\det_{e_1e_2}}}. \tag{3.62}$$

To relate Eq. (3.62) to the form presented in the theorem, we need the functional identity [80]

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z) \tag{3.63}$$

of the dilogarithm. With the help of this identity and splitting the sum of dilogarithms in Eq. (3.62) in a symmetric and and antisymmetric part with respect to the simultaneous exchange of $e^{\pm\zeta_1} \leftrightarrow -e^{\mp\zeta_1}$ and $e^{\pm\zeta_2} \leftrightarrow -e^{\mp\zeta_2}$, we obtain

$$\begin{aligned}
&\mathrm{Li}_2(e^{-i\gamma}e^{\zeta_2}e^{\zeta_1}) - \mathrm{Li}_2(-e^{i\gamma}e^{\zeta_2}e^{-\zeta_1}) - \mathrm{Li}_2(e^{i\gamma}e^{\zeta_2}e^{\zeta_1}) + \mathrm{Li}_2(-e^{-i\gamma}e^{\zeta_2}e^{-\zeta_1})\\
&= -\frac{1}{2}\left\{\mathrm{Li}_2\left(e^{i\gamma}e^{\zeta_1}e^{\zeta_2}\right) \begin{array}{l} +\,(e^{\zeta_1}\leftrightarrow -e^{-\zeta_1})\\ +\,(e^{\zeta_2}\leftrightarrow -e^{-\zeta_2})\\ -\,(e^{i\gamma}\leftrightarrow e^{-i\gamma})\end{array}\right\}\\
&\quad + \frac{1}{4}\left\{\ln^2(-e^{i\gamma}e^{\zeta_2}e^{\zeta_1}) - \ln^2(e^{i\gamma}e^{-\zeta_2}e^{\zeta_1}) - \ln^2(-e^{i\gamma}e^{-\zeta_2}e^{-\zeta_1}) + \ln^2(e^{i\gamma}e^{\zeta_2}e^{-\zeta_1})\right\}
\end{aligned} \tag{3.64}$$

and the last line in Eq. (3.64) is nothing but

$$\begin{aligned}
&\frac{1}{4}\left\{\left[\ln(-e^{i\gamma}) + (\zeta_1+\zeta_2)\right]^2 - \left[\ln(e^{i\gamma}) + (\zeta_1-\zeta_2)\right]^2\right.\\
&\qquad \left. - \left[\ln(-e^{i\gamma}) - (\zeta_1+\zeta_2)\right]^2 + \left[\ln(e^{i\gamma}) - (\zeta_1-\zeta_2)\right]^2\right\}\\
&= (\zeta_1+\zeta_2)\ln(-e^{i\gamma}) - \ln(e^{i\gamma})(\zeta_1-\zeta_2)\\
&= -i(\pi-\gamma)(\zeta_1+\zeta_2) - i\gamma(\zeta_1-\zeta_2).
\end{aligned} \tag{3.65}$$

Thus, the full contribution of the lower boundary is

$$-\frac{i}{4\sin\gamma}\left\{\mathrm{Li}_2\left(e^{i\gamma}e^{\zeta_1}e^{\zeta_2}\right) \begin{array}{l} +\,(e^{\zeta_1}\leftrightarrow -e^{-\zeta_1})\\ +\,(e^{\zeta_2}\leftrightarrow -e^{-\zeta_2})\\ -\,(e^{i\gamma}\leftrightarrow e^{-i\gamma})\end{array}\right\} + \frac{(\pi-\gamma)(\zeta_1+\zeta_2) + \gamma(\zeta_1-\zeta_2)}{2\sin\gamma}. \tag{3.66}$$

For large values of a, we have

$$C(a) = 2\Gamma_1 a + 2\Gamma_2 + O(a^{-1}) \quad \text{and} \quad C(a)^{-1} = O(a^{-1}) \tag{3.67}$$

and by the functional identity Eq. (3.63) and the power series representation (3.51), the dilogarithm satisfies

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = O(|z|^{-1}) \quad \text{and} \quad \mathrm{Li}_2(z) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z) - O(|z|^{-1}) \quad \text{for} \quad |z| > 1. \tag{3.68}$$

Hence, the upper boundary of the double integral Eq. (3.61) is given by

$$\begin{aligned} &\frac{1}{4i}\frac{\ln^2(e^{i\gamma}e^{\zeta_2}C(a)) - \ln^2(e^{-i\gamma}e^{\zeta_2}C(a))}{\sqrt{\det_{e_1e_2}}} + O(a^{-1}) \\ &= \frac{\gamma}{\sin\gamma}[\zeta_2 + \ln(C(a))] + O(a^{-1}) \end{aligned} \tag{3.69}$$

so that

$$\begin{aligned} (2\pi)^2 I^a_{e_2} I_{e_1} W_0 = \frac{1}{2\sin\gamma}&\left[\pi(\zeta_1+\zeta_2) - \frac{i}{2}\left\{\mathrm{Li}_2\left(e^{i\gamma}e^{\zeta_1}e^{\zeta_2}\right) \begin{array}{l} + (e^{\zeta_1} \leftrightarrow -e^{-\zeta_1}) \\ + (e^{\zeta_2} \leftrightarrow -e^{-\zeta_2}) \\ - (e^{i\gamma} \leftrightarrow e^{-i\gamma}) \end{array}\right\}\right] \\ &+ \frac{\gamma}{\sin\gamma}\ln(C(a)) + O(a^{-1}). \end{aligned} \tag{3.70}$$

Because $(e_1e_2) = -\cos\gamma$, we have established that

$$\begin{aligned} (2\pi)^2 I^a_{e_2} I_{e_1} W_0 &= \frac{H(x,e_1,e_2)}{(e_1e_2)} - \frac{\gamma}{2\sin\gamma}\ln\left(\frac{\sin^4\gamma}{D}\right) + \frac{\gamma}{\sin\gamma}\ln(C(a)) + O(a^{-1}) \\ &= \frac{H(x,e_1,e_2)}{(e_1e_2)} + \frac{1}{2}f(e_1,e_2)\ln\left(\frac{4\Gamma_1^2a^2D}{\sin^4\gamma}\right) + O(a^{-1}) \\ &= \frac{H(x,e_1,e_2)}{(e_1e_2)} + \frac{1}{2}f(e_1,e_2)\ln\left(\frac{4(ae_2)^2}{x^2}\right) + O(a^{-1}), \end{aligned} \tag{3.71}$$

which is the desired result for spacelike $x^\perp$. If $x^\perp$ is not spacelike, $I^a_{e_2} I_{e_1} W_0$ is defined as the distributional boundary value of the analytic continuation of Eq. (3.71) to the backward tube $x - i\varepsilon u$. □

The cutoff integral $I^a_{e_2} I_{e_1} W_0$ from Theorem 3.17 can be related to the v-regularized version $(I_{e_2} I_{e_1} W_0)_v = \frac{\langle\langle \varphi(e_1)\varphi(-e_2)\rangle\rangle_v}{(e_1e_2)}$ from Section 3.2 because the difference between the two regularizations is independent of x, for

$$\partial_\mu (I_{e_2} I_{e_1} W_0)_v = I_{e_2} I_{e_1} \partial_\mu W_0 = \lim_{a\to\infty} I^a_{e_2} I_{e_1} \partial_\mu W_0 = \lim_{a\to\infty} \partial_\mu (I^a_{e_2} I_{e_1} W_0). \tag{3.72}$$

Thus, we can write

$$(I_{e_2} I_{e_1} W_0)_\nu = \lim_{\varepsilon\downarrow 0} \frac{1}{(2\pi)^2} \left[-\frac{1}{2} f(e_1, e_2) \ln\left(-\mu_\nu^2 (x - i\varepsilon u)^2\right) + \frac{H(x, e_1, e_2)}{(e_1 e_2)} \right], \quad (3.73)$$

where μ_ν depends on the regularizing function ν, and possibly also on the string variables e_1 and e_2.

Even after regularization, the two-point function of the escort field is fairly complicated due to the appearance of dilogarithms. The two-point function of the string-localized potential itself, however, has a much simpler structure. This is due to the fact that only derivatives of the doubly integrated W_0 appear, as can be seen by the form of $E_{\mu\kappa}(p, e, e')$ in Eq. (2.13). It is less difficult than one might naively expect to compute $\lim_{a\to\infty} \partial_\mu (I_{e_2}^a I_{e_1} W_0)$.

Theorem 3.18. *For* $e_1^0 = e_2^0 = 0$*, we have*

$$I_{e_2} I_{e_1} \partial_\mu W_0 = -\frac{1}{(2\pi)^2} \frac{1}{2} \left[f(e_1, e_2) \partial_\mu + f(x, e_2) \partial_{e_1 \mu} + f(x, e_1) \partial_{e_2 \mu} \right] \ln(\det{}_{x_\varepsilon e_1 e_2}). \quad (3.74)$$

Remark 3.19. Similar to the case of $I_e W_0$, there is a formal symmetry of $I_{e_2} I_{e_1} \partial_\mu W_0$ under the exchange of any pair of variables. This symmetry can be understood by performing a change of variables $(s_1, s_2) \to (\frac{1}{s_1}, \frac{s_2}{s_1})$ in

$$\int_0^\infty ds_1 \int_0^\infty ds_2 \frac{x_\mu + s_1 e_{1\mu} + s_2 e_{2\mu}}{[(x + s_1 e_1 + s_2 e_2 - i\varepsilon u)^2]^2}. \quad (3.75)$$

Proof (see also [38])[2]. We compute the derivative only for regular configurations of the variables x, e_1 and e_2. The $i\varepsilon$-prescription $(x - i\varepsilon u)$ then gives the generalization for arbitrary configurations. For regular configurations, Eq. (3.73) together with the homogeneity of $H(x, e_1, e_2)$ of degree $\omega = 0$ in x imply $x^\mu I_{e_2} I_{e_1} \partial_\mu W_0 = -f(e_1, e_2)$. Because $(e_i \partial) I_{e_i} = -1$, Lemma 3.14 gives $e_1^\mu I_{e_2} I_{e_1} \partial_\mu W_0 = -f(x, e_2)$ and $e_2^\mu I_{e_2} I_{e_1} \partial_\mu W_0 = -f(x, e_1)$. Furthermore, $I_{e_2} I_{e_1} \partial_\mu W_0$ must be a linear combination $b_1 e_1 + b_2 e_2 + b_3 x$ because it is defined as a convergent integral over

$$\partial_\mu \frac{1}{(x + s_1 e_1 + s_2 e_2)^2} = -2 \frac{x_\mu + s_1 e_{1\mu} + s_2 e_{2\mu}}{[(x + s_1 e_1 + s_2 e_2)^2]^2}. \quad (3.76)$$

Therefore, we have

$$\begin{pmatrix} x^\mu I_{e_2} I_{e_1} \partial_\mu W_0 \\ e_1^\mu I_{e_2} I_{e_1} \partial_\mu W_0 \\ e_2^\mu I_{e_2} I_{e_1} \partial_\mu W_0 \end{pmatrix} = -\begin{pmatrix} f(e_1, e_2) \\ f(x, e_2) \\ f(x, e_1) \end{pmatrix} = \begin{pmatrix} x^2 & (xe_1) & (xe_2) \\ (xe_1) & e_1^2 & (e_1 e_2) \\ (xe_2) & (e_1 e_2) & e_2^2 \end{pmatrix} \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix} \quad (3.77)$$

and because we have restricted ourselves to regular configurations, the coefficients b_1, b_2 and b_3 can be obtained by using Lemma 3.9. A rearrangement of the result with the help of Lemma 3.8 yields Eq. (3.74). □

[2]The following proof of Theorem 3.18 from our paper [38] is mainly due to K.-H. Rehren. The original proof, which was given by the author of this thesis, is a rather complicated direct computation of the derivative.

Both the regularized double integral $(I_{e_2} I_{e_1} W_0)_v$ and its derivative Eq. (3.74) contain a factor

$$f(e_1, e_2) = \frac{\gamma}{\sin\gamma}, \tag{3.78}$$

which is regular at $\gamma = 0$ but singular at $\gamma = \pi$, reflecting the fact that the integral of a distribution over a full line is (in general) no longer a distribution and implying that it is not allowed to make the identification $e = e'$ in $\langle\langle A_\mu(e)A_\nu(e')\rangle\rangle$ or $\langle\langle \varphi(e)\varphi(e')\rangle\rangle_v$. Still, the singularity of $f(e_1, e_2)$ at $\gamma = \pi$ is integrable with respect to the invariant measure on $\mathbb{S}^2 \times \mathbb{S}^2$ so that $f(e_1, e_2)$ is a well-defined distribution. However, if there are further constraints on the string variables beyond $e_i^2 = -1$ and $e_i^0 = 0$, this singularity is no longer integrable so that the result of such a further restriction does not yield a well-defined distribution.

3.3.3 Application: vertex operators

The results of Section 3.3.2 and the explicit form of $\langle\langle \varphi(e)\varphi(e')\rangle\rangle_v = -(ee')(I_{-e'} I_e W_0)_v$, in particular the knowledge of the infrared finite part $H(x, e, -e')$, can be used to investigate scattering amplitudes of the dressed Dirac field $\psi_{q,c}(x)$ from the Mund-Rehren-Schroer construction outlined in Section 3.2. Using our explicit representation (3.73), Mund, Rehren and Schroer [51] were able to derive the correlation functions of the vertex operators $V_{q,c}(x) = :e^{i\varphi(q\delta_x, c)}:$ introduced in Section 3.2:

$$\begin{aligned} &\langle\langle V_{q_1,c_1}(x_1)\dots V_{q_n,c_n}(x_n)\rangle\rangle \\ &= \delta_{\sum_i q_i c_i, 0} \prod_{i<j} \left\{ \left(\frac{-1}{(x_i - x_j)^2 - i\varepsilon(x_i^0 - x_j^0)} \right)^{-\frac{q_i q_j}{8\pi^2}\langle c_i, c_j\rangle} e^{-\frac{q_i q_j}{4\pi^2}\tilde{H}(x_i - x_j, c_i, c_j)} \right\}, \end{aligned} \tag{3.79}$$

with the Kronecker delta $\delta_{\sum_i q_i c_i, 0}$ giving rise to an uncountable superselection rule, the partially smeared distribution $\tilde{H}(x, c_i, c_j) := \int d\mu_{\mathbb{S}^2}(e_i) \int d\mu_{\mathbb{S}^2}(e_j)\ H(x, e_i, -e_j)$ and with

$$\langle c_i, c_j\rangle := \int d\mu_{\mathbb{S}^2}(e_i) \int d\mu_{\mathbb{S}^2}(e_j)\ c(e_i) c_(e_j)\ \frac{\pi - \theta(\vec{e}_i, \vec{e}_j)}{\tan\theta(\vec{e}_i, \vec{e}_j)}, \tag{3.80}$$

$\theta(\vec{e}_i, \vec{e}_j)$ being the angle between $\vec{e}_i$ and $\vec{e}_j$.

In order to have a chance to explicitly compute Eq. (3.79), one can choose $c_0(e) = \frac{1}{4\pi}$ to be the constant test function on $\mathbb{S}^2$. Then, the partially smeared and regularized escort field two-point function is [38, 51]

$$\begin{aligned} \langle\langle \varphi(c_0)\varphi(c_0)\rangle\rangle_v(x) &= -\frac{1}{(2\pi)^2}\frac{1}{2} \ln\left(-\tilde{\mu}_v^2(x^2 - i\varepsilon x^0)\right) + \tilde{H}_{c_0}(x), \\ \text{with} \quad \tilde{H}_{c_0}(x) &= \frac{1}{(2\pi)^2}\frac{x^0}{2|\vec{x}|} \ln\left(\frac{x^0 - i\varepsilon - |\vec{x}|}{x^0 - i\varepsilon + |\vec{x}|}\right) \end{aligned} \tag{3.81}$$

and the vertex operator two-point function, obtained from Eq. (3.79), is [38]

$$\langle\!\langle V^*_{q,c_0} V_{q,c_0} \rangle\!\rangle(x) = \left[\frac{\left(\frac{x^0 - i\varepsilon - |\vec{x}|}{x^0 - i\varepsilon + |\vec{x}|} \right)^{\frac{x^0}{|\vec{x}|}}}{-(x^2 - i\varepsilon x^0)} \right]^{\frac{\alpha}{2\pi}} \tag{3.82}$$

These results can be used to investigate the toy model of a scattering theory of vertex operators without the Dirac field, which works in $1+1$ dimensions [12, 21]. If the method from these references is naively applied to the present case, however, the function $\tilde{H}$ can produce a result of modulus greater than one, so it cannot directly be interpreted as a scattering amplitude [38, 68]. Hence, more work is needed on the topic. Especially, one has to properly understand how the presence of $\tilde{H}$ affects the scattering theory of vertex operators and also of the full dressed Dirac field. Such considerations are not part of this thesis but are currently studied as part of a master's thesis [68].

3.3.4 The string-localized photon propagator in configuration space

The non-perturbative constructions described in Sections 3.2 and 3.3.3 involve the *two-point function* of the escort field. Time-ordering is not required. In perturbation theory, however, the S-matrix is a time-ordered exponential and consequently contains *time-ordered two-point functions*, i.e., propagators, instead of ordinary two-point functions. We shall see in Section 5.4.1 that one choice of a propagator of the string-localized photon potential is given by the *kinematic* propagator

$$\begin{aligned} &\langle\!\langle T_0 A_\mu(e) A_\nu(e') \rangle\!\rangle(x) \\ \equiv\ &-\frac{i}{(2\pi)^2} \lim_{\varepsilon \downarrow 0} \left(\eta_{\mu\nu} + e_\nu I_e \partial_\mu + e'_\mu I_{-e'} \partial_\nu + (ee') I_e I_{-e'} \partial_\mu \partial_\nu \right) \frac{1}{x^2 - i\varepsilon}, \end{aligned} \tag{3.83}$$

and all other possible choices of a propagator can at most differ from Eq. (3.83) by certain linear combinations of derivatives of string-integrated Dirac deltas, as we shall see in Section 5.4.2.

Due to the multiplication with Heaviside functions, the propagator has a different $i\varepsilon$-prescription than the two-point function. Introducing again the notation $e_1 := e$ and $e_2 := -e'$, we find

$$I_{e_1} I_{e_2} \partial_\mu \frac{1}{x^2 - i\varepsilon} = \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2\, \partial_\mu \frac{1}{(x + \lambda_1 e_1 + \lambda_2 e_2)^2 - i\varepsilon} \tag{3.84}$$

and since $\lambda_i \geq 0$, we can absorb the infinitesimal imaginary shift into all Minkowski squares,

$$x^2 \to x^2 - i\varepsilon \quad \text{and} \quad e_j^2 \to e_j^2 - i\varepsilon, \quad j = 1, 2. \tag{3.85}$$

Therefore, in contrast to the case of the two-point function, the imaginary shift does not change its sign on the line (or wedge) of integration even if we consider general spacelike string variables $e_1, e_2 \in H$. We can hence use the formulas derived in Section 3.3.2 to describe the string-localized photon propagator for general spacelike e_j after adjusting the $i\varepsilon$-prescription accordingly. For the 2×2 Gram determinants, the prescription Eq. (3.85) yields

$$\begin{aligned} \det_{e_1e_2} &\to \det_{e_1e_2} - i(e_1^2+e_2^2)\varepsilon = \det_{e_1e_2} + i\varepsilon \quad \text{since} \quad e_j^2 < 0, \\ \det_{xe_j} &\to \det_{xe_j} - i(x^2+e_j^2)\varepsilon, \end{aligned} \tag{3.86}$$

and for the 3×3 determinant, we obtain

$$\det_{xe_1e_2} \to \det_{xe_1e_2} - i\varepsilon[\det_{e_1e_2} + \det_{xe_1} + \det_{xe_2}]. \tag{3.87}$$

The 2×2 determinants only appear in the distributions $f(e_1,e_2)$ and $f(x,e_j)$. The latter are only singular if $x^2 \le 0$, and hence we can, without loss of generality, rewrite Eq. (3.86) to

$$\det_{e_1e_2} \to \det_{e_1e_2} + i\varepsilon, \quad \det_{xe_j} \to \det_{xe_j} + i\varepsilon. \tag{3.88}$$

The 3×3 determinant only appears in the doubly string-integrated part of the propagator Eq. (3.83) as a denominator

$$\frac{1}{\det_{xe_1e_2} - i\varepsilon[\det_{e_1e_2} + \det_{xe_1} + \det_{xe_2}]}, \tag{3.89}$$

which is of course only singular if $\det_{xe_1e_2} = 0$ and arises from the derivatives of the logarithm in Eq. (3.74). Indeed, if $\det_{xe_1e_2} = 0$ but the set $\{x, e_1, e_2\}$ is not linearly dependent, then Lemmas 3.11 and 3.13 imply

$$\det_{e_1e_2} + \det_{xe_1} + \det_{xe_2} > 0 \tag{3.90}$$

so that the imaginary shift in Eq. (3.89) is non-zero. Thus, as long as the set $\{x, e_1, e_2\}$ is not linearly dependent, we have

$$\begin{aligned} &(2\pi)^2 i\langle\langle T_0 A_\mu(e) A_\nu(e')\rangle\rangle(x) \\ &= \lim_{\varepsilon\downarrow 0}\Bigg\{\frac{\eta_{\mu\nu}}{x^2 - i\varepsilon} + e_\nu\partial_\mu f(x,e) + e'_\mu\partial_\nu f(x,-e') \\ &\quad + \frac{(ee')}{2}\partial_\nu\left[\left(f(e,-e')\partial_\mu + f(x,-e')\partial_{e\mu} + f(x,e)\partial_{-e'\mu}\right)\ln(\det_{xee'} - i\varepsilon)\right]\Bigg\}, \end{aligned} \tag{3.91}$$

where the 2×2 determinants appearing in $f(\cdot,\cdot)$ are defined as in Eq. (3.88). The $i\varepsilon$-prescription in Eq. (3.91) is only valid if the set $\{x, e_1, e_2\}$ is not linear dependent and by now, it is unclear whether one can find suitable coordinates in which the string-localized photon propagator is locally integrable with respect to this set of linear dependency.

However, we prove the well-definedness of the kinematic string-localized photon propagator in a momentum space consideration in Section 5.4.1. This well-definedness suggests that such coordinates do indeed exist.

3.3.5 Higher helicities

The string-localized potential $A_{\mu_1\ldots\mu_s}(x,e)$ of the massless field strength $F_{[\mu_1\nu_1]\ldots[\mu_s\nu_s]}(x)$ of helicity $s \in \mathbb{N}$ is defined as an s-fold integral

$$A_{\mu_1\ldots\mu_s}(x,e) = I_e^s F_{[\mu_1\nu_1]\ldots[\mu_s\nu_s]}(x)e^{\nu_1}\ldots e^{\nu_s}. \tag{3.92}$$

Consequently, the two-point function of the helicity s potential is an $2s$-fold string integral over a $2s$-fold x-derivative of the two-point function $W_0(x)$ of a massless scalar Klein-Gordon field. Naively, one might hence expect that one needs to compute an increasing number of evermore complex integrals in order to explicitly determine the two-point functions of massless string-localized potentials of higher helicities. However, it turns out that no further integrals need to be computed. We can instead determine the two-point functions for higher helicities by *taking derivatives* of the expressions that we have already derived in Sections 3.3.2 and 3.3.4.

At the heart of this idea is the observation Eq. (2.28), which tells us that derivatives with respect to the string variables of string integrals are the same as higher string integrals over x-derivatives,

$$\partial_{e^\mu} I_e f(x) = I_e^2 \partial_\mu f(x) \tag{3.93}$$

for some generic f. Thus, we have

$$\begin{aligned} I_e^s I_{-e'}^s \partial_{\mu_1}\ldots\partial_{\mu_{2s}} W_0(x) &\sim \partial_{e_{\mu_1}}\ldots\partial_{e_{\mu_{s-1}}}\partial_{e'_{\mu_s}}\ldots\partial_{e'_{\mu_{2s-2}}} I_e I_{-e'} \partial_{\mu_{2s-1}}\partial_{\mu_{2s}} W_0(x) \\ &= \partial_{e_{\mu_1}}\ldots\partial_{e_{\mu_{s-1}}}\partial_{e'_{\mu_s}}\ldots\partial_{e'_{\mu_{2s-2}}}\partial_{\mu_{2s-1}} I_e I_{-e'} \partial_{\mu_{2s}} W_0(x), \end{aligned} \tag{3.94}$$

where the double integral over the gradient of $W_0(x)$ in the last line is given by the derivative formula from Theorem 3.18. Thus, the two-point functions of massless string-localized potentials of arbitrary helicities are contractions of derivatives of the distribution from Theorem 3.18, which only contains the distributions $f(x,e)$, $f(x,-e')$, $f(e,-e')$ and $\ln(\det_{x_\varepsilon ee'})$. In particular, no dilogarithms appear.

Remark 3.20. Note that the observation from this section is a rare example of a consideration that cannot be applied to the multi-string-localized potentials from Remark 2.3: in order to be able to rewrite repeated string-integrals as derivatives, the repeated integrals must be in the same direction.

3.4 Massless self-interactions and the non-existence of L-V pairs

The success of the non-perturbative constructions by Mund, Rehren and Schroer in QED [51] and the results of Brüers on similar constructions for the coupling of gravitons to the stress energy tensor of a scalar field [11] outlined in Section 3.2 automatically raise the question whether recipes of their type are applicable in general models. The answer turns out to be "no". There is no photon self-interaction and consequently the photon itself is not dressed by a photon cloud. Similarly, a graviton self-interaction is not present if one only considers the coupling of a graviton to the SET of other fields. If self-interactions of string-localized (or gauge) potentials are involved, the matter becomes more complicated. One must expect that self-interacting bosons do not only dress the fields, which they accompany, but also themselves.

In the following, we prove that constructions of the type described in Section 3.2 already fail in the first step: the interaction Lagrangian of massless string-localized Yang-Mills theory and the cubic part of a string-localized graviton self-interaction are not part of an L-V pair.

Remark 3.21. We shall derive in Section 4.2 that perturbation theory in SLFT should be set up in such a way that each string-localized potential in the Dyson series for the S-matrix depends on its own string variable but the fixed-order terms in the S-matrix must be symmetric under exchange of any pair of string variables. To keep notation simple, we do not explicitly write the e-dependence of the string-localized potentials and the escort fields in the following Sections 3.4.1 and 3.4.2. The concrete nature of the number of string variables appearing in a Lagrangian is irrelevant for our proofs. Similarly, with slight adjustments for the escort fields, the derivations in Section 3.4.2 also apply for the multi-string-localized potential

$$\tilde{h}_{\mu\nu}(x, e_1, e_2) = \frac{1}{2} I_{e_1} I_{e_2} F_{\mu\kappa\nu\lambda}(x) \left[e_1^\kappa e_2^\lambda + e_2^\kappa e_1^\lambda \right]. \tag{3.95}$$

3.4.1 Massless Yang-Mills theory

We will derive in Section 6.1 (see also [37]) that string independence at second order of perturbation theory constrains the interaction Lagrangian describing a self-interaction of string-localized potentials of mass $m = 0$ and helicity $s = 1$ to be of the Yang-Mills form

$$\begin{aligned} L_{\mathrm{YM}} = {} & \frac{g}{2} f_{abc} :A_{a\mu}(x) A_{b\nu}(x) F_c^{\mu\nu}(x): \\ & + \frac{g^2}{2} f_{abc} f_{cde} :A_{a\mu}(x) A_{b\nu}(x) A_d^\mu(x') A_e^\nu(x'): \delta(x - x'), \end{aligned} \tag{3.96}$$

where $g \in \mathcal{S}(\mathbb{R}^{1+3})$ is a coupling to be sent to a constant in the adiabatic limit and the f_{abc} are the structure constants of a Lie algebra of compact type. Summation over repeated Latin indices is understood.

Theorem 3.22. *The string-localized Lagrangian L_{YM} from Eq. (3.96) is not part of an L-V pair.*

Proof. It is sufficient to prove that the cubic part $L^{(3)}$ of the Lagrangian L_{YM} is not part of an L-V pair. We lift the field strengths $F_a^{\mu\nu}(x)$ and the string-localized potentials $A_a^\mu(x, e)$ to Krein space in order to connect the $A_{a\mu}(x, e)$ with the Krein potentials $A_{a\mu}^{\mathrm{K}}(x)$ in Feynman gauge via the escort fields $\varphi_a(x, e)$, as prescribed in Section 3.1,

$$A_a^\mu(x, e) = A_{a\mu}^{\mathrm{K}}(x) + \partial_\mu \varphi_a(x, e), \quad \text{with} \quad \varphi_a(x, e) = I_e(A_a^{\mathrm{K}} e). \tag{3.97}$$

Then, the cubic part of the Lagrangian (3.96) can be written as

$$\begin{aligned} L^{(3)} &= \frac{g}{2} f_{abc} :(A^{\mathrm{K}}_{a\mu} + \partial_\mu \varphi_a)(A^{\mathrm{K}}_{b\nu} + \partial_\nu \varphi_b) F^{\mu\nu}_c: \\ &= L^{\mathrm{K},(3)} + g f_{abc} :A^{\mathrm{K}}_{a\mu} \partial_\nu \varphi_b F^{\mu\nu}_c: + \frac{g}{2} f_{abc} :\partial_\mu \varphi_a \partial_\nu \varphi_b F^{\mu\nu}_c:, \end{aligned} \tag{3.98}$$

where $L^{\mathrm{K},(3)} = \frac{g}{2} f_{abc} :A^{\mathrm{K}}_{a\mu} A^{\mathrm{K}}_{b\nu} F^{\mu\nu}_c:$ is the cubic part of the point-localized massless Yang-Mills Lagrangian on Krein space. The last term in Eq. (3.98) can be expressed in terms of the escort field and the Krein potential and becomes a total divergence in the adiabatic limit by virtue of the wave equation,

$$\begin{aligned} &\frac{g}{2} f_{abc} :\partial_\mu \varphi_a \partial_\nu \varphi_b F^{\mu\nu}_c: \\ &= g f_{abc} :\partial_\mu \varphi_a \partial_\nu \varphi_b \partial^\mu A^{\mathrm{K}\nu}_c: \\ &= \frac{g}{2} f_{abc} \partial_\mu \Big(:\varphi_a \partial_\nu \varphi_b \partial^\mu A^{\mathrm{K}\nu}_c: + :\partial^\mu \varphi_a \partial_\nu \varphi_b A^{\mathrm{K}\nu}_c: - :\varphi_a \partial^\mu \partial_\nu \varphi_b A^{\mathrm{K}\nu}_c: \Big). \end{aligned} \tag{3.99}$$

However, the second term in the last line of Eq. (3.98) does not form a divergence in the adiabatic limit,

$$g f_{abc} :A^{\mathrm{K}}_{a\mu} \partial_\nu \varphi_b F^{\mu\nu}_c: = g f_{abc} \partial_\nu \Big(:A^{\mathrm{K}}_{a\mu} \varphi_b F^{\mu\nu}_c: \Big) - g f_{abc} :A^{\mathrm{K}}_{a\mu} \varphi_b \partial_\nu F^{\mu\nu}_c: \tag{3.100}$$

because on Krein space, we have $\partial_\nu F^{\mu\nu}_c = \partial_\mu (\partial A^{\mathrm{K}}_c) \neq 0$, see Table 3.1. Still, the obstructing term in Eq. (3.100) could be compensated by a term that vanishes identically on Hilbert space but gives a non-zero contribution on Krein space. Up to a total divergence, the only possible such term is

$$\Delta L^{(3)} := g \tilde{f}_{abc} :(\partial A_a)(A_b A_c):, \tag{3.101}$$

where the coefficients $\tilde{f}_{abc}$ can be assumed to satisfy $\tilde{f}_{abc} = \tilde{f}_{acb}$ without loss of generality. Then $\Delta L^{(3)}$ from Eq. (3.101) equals

$$\begin{aligned} \Delta L^{(3)} &= g \tilde{f}_{abc} :(\partial A^{\mathrm{K}}_a) \left[(A^{\mathrm{K}}_b A^{\mathrm{K}}_c) + \partial_\mu \varphi_b \partial^\mu \varphi_c + 2 \partial^\mu \varphi_b A^{\mathrm{K}}_{c\mu} \right]: \\ &= \tilde{L}^{\mathrm{K},(3)} + \frac{g}{2} \tilde{f}_{abc} \partial^\mu \Big(2 :(\partial A^{\mathrm{K}}_a) \partial_\mu \varphi_b \varphi_c: - :\partial_\mu (\partial A^{\mathrm{K}}_a) \varphi_b \varphi_c: \Big) \\ &\quad + 2 g \tilde{f}_{abc} :(\partial A^{\mathrm{K}}_a) \partial^\mu \varphi_b A^{\mathrm{K}}_{c\mu}:, \end{aligned} \tag{3.102}$$

where we defined $\tilde{L}^{\mathrm{K},(3)} := g \tilde{f}_{abc} :(\partial A^{\mathrm{K}}_a)(A^{\mathrm{K}}_b A^{\mathrm{K}}_c):$. The last term in Eq. (3.102) equals

$$\begin{aligned} 2 g \tilde{f}_{abc} :(\partial A^{\mathrm{K}}_a) \partial^\mu \varphi_b A^{\mathrm{K}}_{c\mu}: = {}& 2 g \tilde{f}_{abc} \partial^\mu \Big(:(\partial A^{\mathrm{K}}_a) \varphi_b A^{\mathrm{K}}_{c\mu}: \Big) \\ & - 2 g \tilde{f}_{abc} :(\partial A^{\mathrm{K}}_a) \varphi_b (\partial A^{\mathrm{K}}_c): \\ & - 2 g \tilde{f}_{abc} :\partial^\mu (\partial A^{\mathrm{K}}_a) \varphi_b A^{\mathrm{K}}_{c\mu}:. \end{aligned} \tag{3.103}$$

Thus, combining Eq.s (3.98), (3.100), (3.102) and (3.103), we obtain

$$\begin{aligned} L^{(3)} + \Delta L^{(3)} \overset{\mathrm{div}}{=} {}& L^{\mathrm{K},(3)} + \tilde{L}^{\mathrm{K},(3)} - 2 g \tilde{f}_{abc} :(\partial A^{\mathrm{K}}_a) \varphi_b (\partial A^{\mathrm{K}}_c): \\ & - g (f_{cba} + 2 \tilde{f}_{abc}) :\partial^\mu (\partial A^{\mathrm{K}}_a) \varphi_b A^{\mathrm{K}}_{c\mu}:, \end{aligned} \tag{3.104}$$

where $\overset{\text{div}}{=}$ again means an equality up to a total divergence. The requirement that $L^{(3)} + \Delta L^{(3)}$ be part of an L-V pair hence translates to the requirements

$$\tilde{f}_{abc} :(\partial A^{\mathrm{K}}_a)\varphi_b(\partial A^{\mathrm{K}}_c): \overset{!}{=} 0, \tag{3.105a}$$

$$(2\tilde{f}_{abc} - f_{abc}) :\partial^\mu(\partial A^{\mathrm{K}}_a)\varphi_b A^{\mathrm{K}}_{c\mu}: \overset{!}{=} 0. \tag{3.105b}$$

While the requirement Eq. (3.105a) can be satisfied if $\tilde{f}_{abc} = -\tilde{f}_{cba}$, the requirement Eq. (3.105b) can only be satisfied if $f_{abc} = \tilde{f}_{abc} = 0$ since the fields are independent and $f_{abc} = -f_{acb}$ while $\tilde{f}_{abc} = \tilde{f}_{acb}$. This proves the claim. □

3.4.2 Graviton self-interaction

A possible interaction Lagrangian describing a graviton self-interaction is expected to have infinitely many terms of arbitrarily high power in the linearized metric tensor, which is interpreted as the graviton field. Furthermore, it seems reasonable that all these terms should sum up to the expansion of the Einstein-Hilbert Lagrangian as a power series in the gravitational coupling constant κ [60]. Indeed, one can prove in the gauge theoretic framework that the expansion of the Einstein-Hilbert Lagrangian is perturbatively gauge invariant, i.e., gauge invariant to all orders of perturbation theory, at tree level [23]. But it is not known whether (and if so, in which way) the Einstein-Hilbert Lagrangian is the unique gauge invariant solution.

At present, no string-localized analogue of such a statement has been proven and it is unclear if the expansion of the Einstein-Hilbert Lagrangian, where now the linearized metric is interpreted as the string-localized potential $h_{\mu\nu}(x, e)$ of the linearized curvature tensor, is string independent at all orders of perturbation theory. However, there is a string independent cubic self-coupling of $h_{\mu\nu}(x, e)$, which is unique up to total divergences and overall prefactors. This cubic self-coupling Lagrangian coincides with the cubic part of the expansion of the Einstein-Hilbert Lagrangian [36],

$$L^{(3)}_G = :h^{\mu\nu}\left[\partial_\mu h_{\kappa\lambda}\partial_\nu h^{\kappa\lambda} + 2\partial^\kappa h_{\mu\lambda}\partial^\lambda h_{\nu\kappa}\right]: . \tag{3.106}$$

Although the full interaction Lagrangian describing the self-coupling of massless string-localized potentials of helicity $s = 2$ is not yet known, we can show that it cannot be part of an L-V pair by proving that the cubic part alone is not part of one. Indeed, this is the case:

Theorem 3.23. *The cubic part $L^{(3)}_G$ of the string-localized graviton self-coupling from Eq. (3.106) is not part of an L-V pair.*

The proof of Theorem 3.23 is much more involved than the proof for massless Yang-Mills theory. Thus, we do not display it here but transfer it to Appendix A. One needs to heavily employ a useful lemma for cubic polynomials of solutions of the wave equation (see also [60]), stating that

$$\partial_\mu f_1 \partial^\mu f_2 f_3 = \frac{1}{2}\partial^\mu(\partial_\mu f_1 f_2 f_3 + f_1 \partial_\mu f_2 f_3 - f_1 f_2 \partial_\mu f_3) =: \mathfrak{div}(\partial_\mu f_1 \partial^\mu f_2 f_3) \tag{3.107}$$

if $\Box f_i = 0$ for $i = 1, 2, 3$, which can easily be verified by direct computation.

Chapter 4

Perturbation theory with string-localized fields

We now turn to perturbation theory in the framework of SLFT. In particular, we explore the construction of the string-localized version of the Dyson series Eq. (1.2) for the S-matrix. There is no straightforward transition of the Bogoliubov-Epstein-Glaser (BEG) construction of the S-matrix to string-localized field theories. The first and main obstacle for such a transition is the construction of time-ordered products of string-localized fields: how can one order two or more infinitely extended strings in time and what are the ambiguities in the construction? A partial answer to these questions has been given by Cardoso, Mund and Várilly with the method of string chopping [16] but it is unclear whether this method works in general. In Section 4.1, we sketch the method of string chopping and then propose another method for time-ordering string-localized fields, which goes beyond string chopping.

Another obstacle for the construction of a string-localized Dyson series for the S-matrix is that very little is known on renormalization in SLFT. In a recent paper [35], the author was able to shed some light on this question. The results of this paper concerning renormalization are the central topic of Sections 5.1 to 5.6. However, the nature of renormalization crucially depends on the setup of string-localized perturbation theory and in particular, before addressing renormalization, one must declare the nature of the string-localization of the S-matrix. These issues are described in Section 4.2, which is based on the mentioned paper [35].

The implementation of an axiomatic and comprehensive BEG scheme to construct the string-localized S-matrix is beyond the scope of this thesis. In particular, we do not give a rigorous classification of all possible ambiguities of time-ordered products in SLFT. However, the considerations in this and the following chapter contribute to several aspects that are important to establish a BEG scheme.

Remark 4.1. Another central question in perturbation theory is the existence of the adiabatic limit or at least a weak version of it. However, as explained in the introductory Section 1.2, that question is not addressed in this thesis. We focus on aspects of a BEG construction in SLFT before taking the adiabatic limit.

4.1 Time-ordering in string-localized field theory

Already in point-localized QFT, time-ordering and the construction of time-ordered products are somewhat subtle operations. The BEG description outlined in the introductory Section 1.2 is the most rigorous method available to perform such a construction. It is well-defined at all steps but gives rise to ambiguities. The BEG scheme is a geometric approach to time-ordering in relativistic theories that takes into account the causal structure of quantum fields, i.e., commutativity at spacelike distance.

One approach to generalize such a geometric description of time-ordering is string chopping, which has been implemented first for linear string-localized fields [16], with a recent generalization to very specific models where the coupling between string-localized fields is skewsymmetric [37]. We briefly sketch the method of string chopping in Section 4.1.1. String chopping is fairly abstract, it is so far unclear whether it can be implemented for general string-localized models and it seems difficult to use string chopping for practical applications. In Section 4.1.2, we therefore introduce a different method to define time-ordering in SLFT, which is an attempt to go beyond string chopping. In this introduction, we only outline the proposed method. A full and rigorous description is not yet available. It is a so far unproven conjecture of the author of this thesis that this new method is a generalization of string chopping to arbitrary models. Relations between string chopping and the new method are discussed at the end of this thesis in Section 7.2.

4.1.1 State of the art: string chopping

The earliest description of string chopping [16] only describes the time-ordering for linear string-localized fields. Recently, this method has been generalized to models that contain skewsymmetric self-interactions of string-localized fields [37]. A generalization to arbitrary string-localized models has not yet been given. Indeed, the author expects that string chopping is, without major adjustments, not suited to describe arbitrary models involving string-localized fields. This expectation comes from the following observation. The generalization of string chopping, which was done in the mentioned work [37], was only possible because the cubic part of the massless Yang-Mills Lagrangian (cf. Eq. (3.96)) has totally skewsymmetric constants of proportionality f_{abc}. In the paper [37], we performed computations only up to second order in the coupling constant and thus, time-ordering the quartic part of the massless Yang-Mills Lagrangian was not necessary. Going beyond second order is not possible so far, because the constants of proportionality $\tilde{f}_{abcd} := f_{abx} f_{xcd}$ in front of the quartic term of the interaction Lagrangian are not totally skewsymmetric and it is unclear how to implement the string chopping prescription in that case.[1]

This does not mean that one cannot compute anything. One can still make the usual

[1]One possibility might be via "color ordering" (see for example [31]). There, one notices that the four-gluon-vertex can be described by a sum of cubic vertices. However, this procedure would have to be adjusted to a position space consideration. Even if it turns out that color ordering can help to implement string chopping in massless Yang-Mills theory, one is still miles away from a general string chopping description.

ansatz and expand the Dyson series as a sum of divergent products of propagators by means of Wick's theorem and renormalize these products in retrospective to get a finite result. This is one solution of the problem but it is then unclear what the ambiguities in the construction are.

Now, let us briefly outline the string chopping method for the case of linear string-localized quantum fields, as described by Cardoso, Mund and Várilly [16]. These authors define the time-ordered products

$$T_n \, \varphi_1(x_1, e_1) \dots \varphi_n(x_n, e_n) \tag{4.1}$$

of n string-localized fields φ_i depending on different x- and e-variables. They first show that the n strings $x_i + \mathbb{R}_{\geq 0} e_i$ can be chopped into a finite number of compact segments plus infinite tails, which can be mutually compared in time – unless two or more of the strings intersect. This gives a comprehensive classification of the ambiguities of a BEG-like construction in SLFT, but *only for linear string-localized fields*.

The authors of [16] proceed by subjecting the T_n to certain assumptions:

P1 Initial condition: $T_1 \varphi(x, e) \stackrel{!}{=} \varphi(x, e)$.

P2 Linearity: The T_n are n-linear mappings from the space of linear fields into operator-valued distributions acting on a specified domain.

P3 Symmetry: T_n is symmetric under exchange of any pair of arguments.

P4 Causality: If φ_i is localized on S_i, where S_i is either the string $x_i + \mathbb{R}_{\geq 0} e_i$ or a segment of it, and if furthermore all S_j, $j = 1, \dots, k$, are later than all strings S_l, $l = k + 1, \dots, n$, then

$$\begin{aligned} &T_n \, \varphi_1(x_1, e_1) \dots \varphi_n(x_n, e_n) \\ &= T_k \, \varphi_1(x_1, e_1) \dots \varphi_k(x_k, e_k) \; T_{n-k} \, \varphi_{k+1}(x_{k+1}, e_{k+1}) \dots \varphi_n(x_n, e_n). \end{aligned} \tag{4.2}$$

P1 to P3 are the same requirements as in point-localized QFT (see for example [25]), while P4 is adjusted according to the string localization.

Cardoso, Mund and Várilly then show that under these assumptions the T_n are uniquely fixed outside the large string diagonal Δ_n, which is the set where two or more of the string intersect, and that they can be expanded according to Wick's theorem [16].

4.1.2 Beyond string chopping

We propose an alternative method to implement time-ordering in SLFT. No general set of axioms to define time-ordered products involving string-localized fields is available, and we propose to reconnect to the axioms of point-localized time-ordered products even if string-localized fields are involved.

To do so, recall that the string-localized potentials $A_{\mu_1\dots\mu_s}(x,e)$ from Eq. (2.2) are defined as integrals over the field strength tensors $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}(x)$,

$$A_{\mu_1\dots\mu_s}(x,e) = \int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_s \, F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}\Big(x + \sum_{i=1}^{s} \lambda_i e_i\Big) e^{\nu_1} \dots e^{\nu_s}. \tag{4.3}$$

The field strengths are point-localized. Motivated by Eq. (4.3), we define the time-ordered product of expressions involving string-localized potentials as the corresponding integral over the respective time-ordered products expressed in terms of the corresponding field strengths. For example, we define the time-ordered product of two string-localized photon potentials $A_\mu(x,e)$ and $A_\nu(x',e')$ as

$$T[A_\mu(x,e)A_\nu(x',e')] := \int_0^\infty d\lambda \int_0^\infty d\lambda' \, T[(Fe)_\mu(x+\lambda e)(Fe)_\nu(x'+\lambda' e')], \tag{4.4}$$

with

$$\begin{aligned} &T[(Fe)_\mu(x+\lambda e)(Fe)_\nu(x'+\lambda' e')] \\ &= \begin{cases} (Fe)_\mu(x+\lambda e)(Fe)_\nu(x'+\lambda' e') & \text{if } (x-x'+\lambda e-\lambda' e')^0 > 0, \\ (Fe)_\nu(x'+\lambda' e')(Fe)_\mu(x+\lambda e) & \text{if } (x-x'+\lambda e-\lambda' e')^0 < 0. \end{cases} \end{aligned} \tag{4.5}$$

By local commutativity of the field strength, this time-ordering recipe is unique whenever $x + \lambda e \neq x' + \lambda' e'$, which guarantees the Lorentz invariance of Eq. (4.5). The obvious choice for the corresponding time-ordered two-point function is then

$$\begin{aligned} &\langle\langle T A_\mu(x,e) A_\nu(x',e')\rangle\rangle \\ &:= \int_0^\infty d\lambda \int_0^\infty d\lambda' \, \Big[\theta\big((x-x'+\lambda e-\lambda' e')^0\big)\langle\langle (Fe)_\mu(x+\lambda e)(Fe)_\nu(x'+\lambda' e')\rangle\rangle \\ &\qquad + \theta\big(-(x-x'+\lambda e-\lambda' e')^0\big)\langle\langle (Fe)_\nu(x'+\lambda' e') Fe)_\mu(x+\lambda e)\rangle\rangle\Big], \end{aligned} \tag{4.6}$$

and by the axioms of the *point-localized* BEG construction, the integrand of Eq. (4.6) is unique up to a distribution supported on $\{x - x' + \lambda e - \lambda' e'\} = 0$. That is, any other time-ordered two-point function of the string-localized A_μ can differ from Eq. (4.6) only by a sum of derivatives of a string-integrated Dirac delta. We shed more light on the well-definedness and singularity structure of propagators of the form Eq. (4.6) and their ambiguities in Chapter 5.

With the method we propose, an automatic string chopping is achieved because the time-ordering of string integrals is defined as integral over the time-ordered integrands. Instead of chopping a full string into (extended) segments, which are comparable in time, we compare each pair of points on the strings separately.

Without mentioning, we have assumed that the two string-localized photon potentials in the time-ordered product Eq. (4.4) and the propagator Eq. (4.6) depend on different string variables e and e'. This dependence on *different* string variables is necessary for the following reason. By our considerations from Sections 2.2 and 3.3, the two-point function of the string-localized photon potential is

$$\langle\!\langle A_\mu(e)A_\nu(e')\rangle\!\rangle(x) = -\Big(\eta_{\mu\nu} + e_\nu\partial_\mu I_e + e'_\mu\partial_\nu I_{-e'} + (ee')I_e I_{-e'}\partial_\mu\partial_\nu\Big)\, W_0(x), \qquad (4.7)$$

with the ordinary massless scalar two-point function $W_0(x)$ as in Eq. (3.43). By Lemma 3.14 and Theorem 3.18, Eq. (4.7) is built from $W_0(x)$, the distributions $f(x,e)$, $f(x,-e')$ and $f(e,-e')$ from Section 3.3, the inverse Gram determinant $\det^{-1}_{x_\varepsilon ee'}$ and smooth functions. In particular, $\langle\!\langle A_\mu(e)A_\nu(e')\rangle\!\rangle$ contains a term proportional to $f(e,-e')$, which diverges in the limit $e' \to e$.[2] Thus, an identification $e' = e$ in the two-point function and hence also in the time-ordered two-point function is not possible.

To define the string-localized S-matrix, one must declare how the string-localized equivalent of its Dyson series depends on string variables. We discuss this topic in full detail in the following section. For now, we consider the simple example of string-localized QED,where the interaction Lagrangian $L = {:}A_\mu(x,e)j^\mu(x){:}$ only depends on a single string variable. To avoid divergences in the string-localized two-point functions, each Lagrangian shall depend on its own string variable. Our approach then yields

$$\begin{aligned} &T[L(x_1,e_1)\dots L(x_n,e_n)] \qquad\qquad (4.8)\\ &:= \int_0^\infty d^n\underline{\lambda}\, T[\,{:}(Fe_1)_{\mu_1}(x_1+\lambda_1 e_1)j^{\mu_1}(x_1){:}\,\dots\,{:}(Fe_n)_{\mu_n}(x_n+\lambda_n e_n)j^{\mu_n}(x_n){:}\,] \end{aligned}$$

so that the time-ordering of string-localized objects becomes an integral over time-ordered point-localized objects and for the latter, we know how to define time-ordering. In the example of QED, we can introduce $y_i = x_i + \lambda_i e_i$ so that the time-ordered product Eq. (4.8), which has n arguments in point-localized QED, becomes a time-ordered product in $2n$ variables of the fields $(Fe_1)_{\mu_i}(y_i)$ and $j^{\mu_i}(x_i) = {:}\overline{\psi}(x_i)\gamma^{\mu_i}\psi(x_i){:}$ because

$${:}(Fe_1)_{\mu_i}(y_i)\; j^{\mu_i}(x_i){:} = (Fe_1)_{\mu_i}(y_i)\; j^{\mu_i}(x_i). \qquad (4.9)$$

The time-ordering with respect to the $2n$ variables $\{x_1, y_1, \dots, x_n, y_n\}$ then works as it does in the point-localized case. In particular, the time-ordered product

$$T[(Fe_1)_{\mu_1}(y_1)j^{\mu_1}(x_1)(Fe_2)_{\mu_2}(y_2)j^{\mu_2}(x_2)] \qquad (4.10)$$

of the four variables $\{x_1, y_1, x_2, y_2\}$ is unique outside the thin diagonal

$$\begin{aligned} \{x_1 = x_2 = y_1 = y_2\} &= \{x_1 = x_2 = x_1 + \lambda_1 e_1 = x_2 + \lambda_2 e_2\} \\ &= \{x_1 = x_2 \,\wedge\, \lambda_1 = \lambda_2 = 0\}, \end{aligned} \qquad (4.11)$$

by the reasoning of Epstein-Glaser, *provided that all lower time-ordered products have been constructed.* Let us for the moment assume that these lower time-ordered products have been constructed. Then the requirement $\lambda_1 = \lambda_2 = 0$ from Eq. (4.11) cancels the string integrations and the time-ordered product after string integration is ambiguous

[2]This is the limit $\gamma \to \pi$ in $f(e_1,e_2) = \frac{\gamma}{\sin\gamma}$ discussed in the end of Section 3.3.

only at the x-diagonal. That is, two time-ordering recipes T_1 and T_2 can – after string integration – only differ by a term supported on the x diagonal $\{ x_1 = x_2 \}$,

$$\Big(T_1[L(x_1,e_1)L(x_2,e_2)] - T_2[L(x_1,e_1)L(x_2,e_2)]\Big)\Big|_{x_1 \neq x_2} = 0, \tag{4.12}$$

again: provided that the lower time-ordered products have been constructed. Eq. (4.12) has the following physical consequence. The difference $T_1 - T_2$ gives rise to an induced term in the Lagrangian and the fact that this difference is supported on the x-diagonal ensures that such an induced term is local in the sense that all strings appearing in it emanate from the same x-variable.

Remember that we are still working within the example of QED. Let us write the second order tree graph contribution explicitly. We can carry out the string integrations to obtain

$$\begin{aligned} &T[L(x_1,e_1)L(x_2,e_2)]|_{\text{tree}} \\ &= \langle\!\langle TA_\mu(x_1,e_1)A_\nu(x_2,e_2)\rangle\!\rangle :j^\mu(x_1)j^\nu(x_2): \\ &+ \sum_{\chi,\phi} \langle\!\langle T\chi(x_1)\phi(x_2)\rangle\!\rangle :A_\mu(x_1,e_1)A_\nu(x_2,e_2)\frac{\partial j^\mu(x_1)}{\partial \chi}\frac{\partial j^\nu(x_2)}{\partial \phi}: \\ &+ \text{ ambiguities supported on } \{ x_1 = x_2 \}. \end{aligned} \tag{4.13}$$

But in Eq. (4.13), the propagators have not yet been fixed and according to the Epstein-Glaser reasoning, they can also have ambiguities. In contrast to the full time-ordered product, which is an integral over time-ordered products of four arguments, the propagator of the string-localized photon potential appearing in Eq. (4.13) is (a vacuum expectation value) of an integral over a time-ordered product of the two arguments $y_i = x_i + \lambda_i e_i$, $i = 1, 2$. Its ambiguity is thus a linear combination of derivatives of a doubly string-integrated Dirac delta.

We postpone a generalization of these considerations in QED until after we have clarified the string dependence of the S-matrix in a generic model in the next section.

4.2 The string-localized scattering matrix

We now outline the construction of the string-localized analogue of the Dyson series Eq. (1.2) for the S-matrix with time-ordering as defined in the previous section. These considerations are partially taken from the author's paper [35] with adjustments made to better fit into the context of this thesis.

As a first step towards a perturbative construction of the string-localized S-matrix, one must declare the nature of the string-localization. Is string-localization a feature of the potentials, the Lagrangian or the S-matrix? That is to say: Does each field come with its own string variable, do the fields in the interaction Lagrangian L depend on the same string variable or do *all* appearing fields depend on the same string variable:

$$L = L(x, e_1, \cdots, e_k) \qquad \text{(each SL field has its own string variable),} \qquad (4.14a)$$

$$L = L(x, e) \qquad \text{(all SL fields in } L \text{ depend on the same } e\text{),} \qquad (4.14b)$$

$$\mathbb{S} = \mathbb{S}[g; e] \qquad \text{(there is only a single string variable).} \qquad (4.14c)$$

In a generic model, the three alternatives (4.14a), (4.14b) and (4.14c) result in completely different analytic properties of the corresponding perturbation theory. Note, however, that the alternatives (4.14a) and (4.14b) are equivalent if L is at most linear in the string-localized potentials, as is the case in QED, which was our example in the previous Section 4.1.2.

Alternative (4.14c) is desirable if one wants to keep the delocalization as small as possible and has been employed in models with lightlike string variables and massive string-localized potentials [39]. However, it is in general not realizable due to the divergence of $f(e, -e')$ at $e' = e$ that we have encountered in Section 3.3 and described in the context of time ordering in the previous Section 4.1.2. We will see another facet of that divergence in a Hörmander analysis in Section 5.2, see Corollary 5.13 below. The divergence of $f(e, -e')$ at $e' = e$ rules out alternative (4.14c) for spacelike strings, which is the only case that we consider in this thesis.

The interaction Lagrangian L depends only on a single x-variable. One can therefore argue that alternative (4.14b) is a natural choice to set up perturbation theory in SLFT. In this case, loop graph contributions would consist of products of propagators in x and e and one must expect renormalization to become very complicated. Recent observations by the author of this thesis, José M. Gracia-Bondía and Jens Mund in the string-localized equivalent of massless Yang-Mills theory suggest that alternative (4.14b) does not reproduce the standard model of particle physics [37]. In the cited work, we consider the alternative (4.14a) from the beginning without mentioning other alternatives but one can verify without too much effort that the Lie algebra structure of gluon self-interactions is not compatible with alternative (4.14b) by adjusting Section 2.3 in [37] according to $L(x, e_1, e_2) \to L(x, e)$. The complete reasoning is displayed explicitly in Section 6.1.3 of this thesis. This observation rules out alternative (4.14b) for phenomenological reasons.

We are thus left with the alternative (4.14a), i.e., $L = L(x, e_1, \cdots, e_k) =: L(x, \boldsymbol{e})$, which is also employed in the cited work [37]. The analyses therein additionally require a symmetry under exchange of all string variables that appear at a fixed order of perturbation theory. This symmetry can be achieved by smearing all string variables with the same averaging function $c \in \mathcal{D}(H)$ with $\int d^4e\, c(e) = 1$.[3] With this at hand, we are finally able to write down a candidate for the string-localized S-matrix,

$$\mathbb{S}[g; c] := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \prod_{l=1}^{k} \int d^4x_j \int d^4e_{j,l}\, g(x_j) c(e_{j,l}) S_n(x_1, \boldsymbol{e}_1; \ldots; x_n, \boldsymbol{e}_n), \tag{4.15}$$

[3]The test function c needs to have integral equal to 1 if the string-localized potential is to remain a potential for the field strength after smearing out the e-variable.

where the first-order coupling $S_1(x, \boldsymbol{e}) = {:}L(x, \boldsymbol{e}){:}$ is the Wick-ordered interaction Lagrangian. The property that c integrates to unity ensures consistency if L is a sum of terms where different powers of string-localized potentials appear. The higher-order couplings S_n are time-ordered products of the first-order coupling, which need to be constructed recursively. For the construction of these time-ordered products, we stick to the procedure outlined in Section 4.1.2.

The time-ordered products of operator-valued distributions (i.e., of the interaction Lagrangian L) that appear in the Dyson series for $\mathbb{S}$ are usually reduced to products of numerical distributions by taking expectation values and employing Wick's theorem. In the point-localized case, that means [9, 32]

$$T[L(x_1)\dots L(x_n)] \\ = \sum_{j_1,\dots,j_n} \frac{1}{j_1!\dots j_n!} \langle\!\langle T\,{:}\varphi^{j_1}(x_1){:}\,\dots\,{:}\varphi^{j_n}(x_n){:}\,\rangle\!\rangle \, {:}\frac{\partial^{|j_1|}L(x_1)}{\partial^{j_1}\varphi}\dots\frac{\partial^{|j_n|}L(x_n)}{\partial^{j_n}\varphi}{:}\,, \tag{4.16}$$

where φ is an array of all quantum fields appearing in L and the j_i are multi-indices, and where we have employed formal derivatives within Wick polynomials. If the time-ordering operation was not involved in Eq. (4.16), the expectation value

$$\langle\!\langle\, {:}\varphi^{j_1}(x_1){:}\,\dots\,{:}\varphi^{j_n}(x_n){:}\,\rangle\!\rangle \tag{4.17}$$

would factorize into a sum of products of two-point functions, which is well-defined as a product of distributions [9, 32, 41, 57]. But a factorization of the time-ordered expectation values $\langle\!\langle T\,{:}\varphi^{j_1}(x_1){:}\,\dots\,{:}\varphi^{j_n}(x_n){:}\,\rangle\!\rangle$ into a product of propagators does not hold everywhere but only outside the diagonal set described in Section 1.2, provided that the lower time-ordered products have already been fixed. In the BEG-scheme, one constructs a well-defined distribution $\langle\!\langle T\,{:}\varphi^{j_1}(x_1){:}\,\dots\,{:}\varphi^{j_n}(x_n){:}\,\rangle\!\rangle$ by extension across that diagonal set [9, 32].

For example, if ϕ is a point-localized scalar field, then $\langle\!\langle\,{:}\phi(x_1)^2{:}\;{:}\phi(x_2)^2{:}\,\rangle\!\rangle = 2\langle\!\langle\phi(x_1)\phi(x_2)\rangle\!\rangle^2$, where the product of distributions on the right-hand side is well-defined. However, the square of the Feynman propagator is ill-defined and for the time-ordered product, one instead has

$$\langle\!\langle T\,{:}\phi(x_1)^2{:}\;{:}\phi(x_2)^2{:}\,\rangle\!\rangle := \big[2\langle\!\langle T\phi(x_1)\phi(x_2)\rangle\!\rangle^2\big]_{\text{extended across }\{x_1=x_2\}}\,. \tag{4.18}$$

The ambiguities of choosing an extension as in the right-hand side of Eq. (4.18) are precisely the BEG ambiguities. We shall discuss the details in Section 5.1.

Let us go back to the string-localized case and outline our programme. In Chapter 5, we prove the well-definedness of string-localized two-point functions and propagators and also that products of string-localized propagators exist whenever they exist in point-localized QFT. Consequently, products involving string-localized propagators of the type (4.18) make sense whenever they make sense in point-localized QFT and standard renormalization techniques can be employed to perform renormalization in SLFT. The fact that we were forced to choose alternative (4.14a) for the string dependence of the S-matrix has profound impact on renormalization. It is of great importance for the proofs in Sections 5.3 and 5.5 that all appearing products of distributions are products only in the spacetime variables but not in the string variables.

In the second part of Chapter 5, we investigate the "finite renormalization freedom" of choosing a propagator for string-localized fields and discuss methods that reduce this freedom. One such method is the implementation of the string independence principle that we have already encountered in Section 2.4. We shall see that there are more methods to reduce the extension freedom, some of which are not compatible. One therefore has to take care, which methods for the reduction of the renormalization freedom one chooses. However, since the string independence principle is at the heart of SLFT, it should be given supremacy over the other methods.

Now that we are clear about how the string-localized version of the Dyson series should look like, we can give an explicit description of what string independence in string-localized perturbation theory means. Noting that all string variables of S_n in Eq. (4.15) are smeared in the same test function, we see that only the symmetric part S_n^{symm} of S_n enters the Dyson series, where

$$S_n^{\mathrm{symm}} = \frac{1}{(nk)!} \sum_{\pi \in \mathfrak{S}_{nk}} S_n(x_1, e_{\pi(1,1)}, \ldots, e_{\pi(1,k)}; \ldots; x_n, e_{\pi(n,1)}, \ldots, e_{\pi(n,k)}), \tag{4.19}$$

where $\mathfrak{S}_{nk}$ is the symmetric group of order nk. Introducing $d_{e_{j,l}} := \partial_{e_{j,l}\kappa} de^{\kappa}_{j,l}$ as notation for the variation with respect to $e_{j,l}$, we require that

$$d_{e_{j,l}} S_n^{\mathrm{symm}} = \partial_\mu Q^\mu_{n,j,l}, \tag{4.20}$$

so that the total divergence on the right-hand side of Eq. (4.20) gives a trivial contribution to the S-matrix in the adiabatic limit $g \to const$. The requirement (4.20) has many important consequences on the construction of the Dyson series, which will appear at many places in the remaining parts of this thesis. For some implications of the string independence principle at low orders of perturbation theory, see [37, 39].

Besides the string independence requirement, which is the string-localized analogue of perturbative gauge invariance [1, 29, 60], the string-localized S-matrix should of course be subject to similar constraints as in point-localized QFT. It should be unitary, Poincaré invariant and causal [6, 32]. The S-matrix can also be subjected to model-dependent discrete symmetries [74].

Causality takes a special role among these requirements because the causal structure, or the commutation relations, are different in SLFT than in point-localized QFT. In the string chopping description for linear string-localized fields [16], causality is realized by a causal factorization adjusted to the string locality, see Property P4 from Section 4.1.1. For an implementation of a causal factorization in our time-ordering prescription, presented in Section 4.1.2, we aim at a causal factorization, which is closely related to the standard one from point-localized QFT, only formulated in terms of all variables x_j and $y_{j,l} \equiv x_j + \lambda_{j,l} e_{j,l}$. However, to implement such a factorization rule, some obstacles need to be overcome.

Using our definition that time-ordered products involving string-localized fields are given by the respective string-integrals over time-ordered products involving the field strength tensors, we can rewrite the Dyson series for the string-localized S-matrix Eq. (4.15) as

$$\mathbb{S}[G] := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \int d^4 x_j \, d^{4k} \boldsymbol{y}_j \, G^{\underline{\nu}_j}(x_j, \boldsymbol{y}_j) \, S_{n\underline{\nu}_1 \cdots \underline{\nu}_n}(x_1, \boldsymbol{y}_1; \ldots; x_n, \boldsymbol{y}_1), \quad (4.21)$$

with the tensor-valued auxiliary "test function"

$$\begin{aligned} G^{\underline{\nu}}(x, \boldsymbol{y}) = g(x) \prod_{l=1}^{k} \int d^4 e_l \, c(e_l) \, e_l^{\nu_{l,1}} \ldots e_l^{\nu_{l,s_l}} \\ \times \int_0^{\infty} d^{s_l} \boldsymbol{\lambda}_l \, \delta\Big(y_l - x - \Big(\sum_{i=1}^{s_l} \lambda_{l,i}\Big) e_l\Big), \end{aligned} \quad (4.22)$$

where $s_l \in \mathbb{N}$ is the spin resp. helicity of the string-localized potential that depends on the variable e_l. The adjusted n-th order coupling in Eq. (4.21) is then

$$S_{n\underline{\nu}_1 \cdots \underline{\nu}_n}(x_1, \boldsymbol{y}_1; \ldots; x_n, \boldsymbol{y}_1) = S_n(x_1, \boldsymbol{e}_1; \ldots; x_n, \boldsymbol{e}_n)\big|_{A_{\underline{\mu}}(x_j, e_{j,l}) \to F_{\underline{\mu}\underline{\nu}}(y_{j,l})}. \quad (4.23)$$

It is clear that $G^{\underline{\nu}}(x, \boldsymbol{y})$ is not a true test function. Caused by the string integrations, which it contains, it is neither compactly supported nor does it fall off in negative string direction.

In a first attempt, one can try to implement a causal factorization of the string-localized S-matrix in two steps. First, require for true (tensor-valued) test functions $G, H \in \mathcal{D}\big((\mathbb{R}^{1+3})^{k+1}, (\mathbb{R}^{1+3})^{|\nu|}\big)$ that

$$\mathbb{S}[G + H] = \mathbb{S}[G]\mathbb{S}[H] \quad (4.24)$$

if none of the points x and y_l in the support of H is in the causal future of any of the points x' or y'_l in the support of G. In a second step, check if the transition from test functions to the stringy function from Eq. (4.22) is well-defined. It is however foreseeable that such a factorization rule would imply that the time-ordering prescription is ambiguous on a huge set because we are blind to what happens in situations of the type

> "Some points of $(x, \boldsymbol{y})$ are later than all points $(x', \boldsymbol{y}')$ and all of these are later than the remaining points of $(x, \boldsymbol{y})$".

From a physical perspective, it is clear that the time-ordering prescription should be unique also in this type of situation. Thus, an improvement of the described causal factorization property is highly desirable. What blocks our way to an improved factorization rule is the normal ordering of the interaction Lagrangian.

In the example of QED, which we discussed in Section 4.1.2, we had the advantage that the normal-ordered product $:F_{\mu\nu}(y) j^{\mu}(x):$ is the same as the product $F_{\mu\nu}(y) :j^{\mu}(x):$. We were then able to define the time-ordered product of n copies of the QED interaction Lagrangian as a time-ordered product of $2n$ variables, which satisfies a causal factorization rule with respect to all these variables. This is no longer possible in generic models, where we have

$$:\phi(x)\chi(x'):\ = \phi(x)\chi(x') - \langle\!\langle \phi(x)\chi(x') \rangle\!\rangle \tag{4.25}$$

for generic (linear) quantum fields $\phi(x)$ and $\chi(x')$ with a generally non-vanishing vacuum expectation value on the right-hand side. Thus, if one were to drop the normal-ordering of the interaction Lagrangian, the resulting Wick expansion of the n-th order coupling would in general contain terms that could be interpreted as "self-contractions" in the standard formulations.

Remark 4.2. The fact that the vacuum expectation values of the gluon potential are color diagonal while the constants of proportionality f_{abc} are totally skewsymmetric is precisely the reason why the string chopping procedure could be generalized to the cubic part of the massless Yang-Mills Lagrangian [37]: the vacuum expectation values corresponding to "self-contractions" add up to zero.

We propose the following ad hoc improvement of the previously described causal factorization rule.

1. Drop the normal-ordering of the string-localized interaction Lagrangian.

2. Formally construct the n-th order coupling $S_{n\underline{\nu}_1\cdots\underline{\nu}_n}$ as point-localized time-ordered product of the $n(k+1)$ variables $(x_j, \boldsymbol{y}_j)$, $j = 1, \ldots, n$ with a point-localized causal factorization rule for *all* variables.[4]

3. Since the result is a point-localized time-ordered product, it has a Wick expansion. Thus, subtract all terms in the Wick expansion that would correspond to self-contractions before dropping the normal-ordering of the interaction Lagrangian. This effectively restores the normal-ordering of the interaction Lagrangian.

4. Perform the string integrations.

A proper and axiomatic formulation of this ad hoc procedure is still missing. However, the outlined rules allow us to perform computations.

By similar reasoning as in the example of QED discussed in Section 4.1.2, the outlined ansatz yields that the full time-ordered products

$$T[L(x_1, e_{1,1}, \ldots, e_{1,k}) \ldots L(x_n, e_{n,1}, \ldots, e_{n,k})] \tag{4.26}$$

of nk or $n(k+1)$ arguments (that is, after dropping the normal-ordering of L) are unique outside the thin x-diagonal $\{ x_1 = \cdots = x_n \ \wedge \ \lambda_{j,l} = 0 \}$ provided that all lower time-ordered products have been constructed. The ambiguities of the lower time-ordered products might possess string-localized features, whose concrete nature still needs to be clarified.

[4]Or as time-ordered products of nk arguments if L contains only string-localized potentials so that the x-dependence disappears and only the dependence on the y-variables remains.

Remark 4.3. We have argued in this chapter that we are left with alternative (4.14a) for a proper setup of perturbation theory in string-localized quantum field theory. The reader should be aware that the alternatives (4.14a) to (4.14c) have to be adjusted and that our arguments need to be reevaluated if one employs the multi-string-localized potentials from Remark 2.3 instead of string-localized potentials. The most obvious consequence is that alternative (4.14c) drops out automatically if multi-string-localized higher-spin potentials are involved because there is more than one string variable from the start. A second aspect is that one needs to investigate an appropriate symmetrization in the string variables.

Chapter 5

Renormalization in SLFT

In this chapter, we investigate BEG renormalization in string-localized field theory. For these investigations, we employ mathematical tools of microlocal analysis. These tools might not be well-known to many physicists, so we give an introduction in Section 5.1. We start the actual investigations in Section 5.2, where we describe the additional singularities that arise from string integration with microlocal methods. We proceed to give proofs of the well-definedness and singularity structure of the two-point functions and propagators of string-localized potentials in Sections 5.3 and 5.4. In the latter section, we also investigate ambiguities in the definition of the propagators, which correspond to a (re-)normalization freedom in the BEG sense. In contrast to the previous considerations in Section 3.3, we start by working in momentum space and then transfer the results to configuration space. A major result of the chapter is the existence of products of string-localized propagators outside an x-diagonal, which we derive in Section 5.5 based on the formulation of string-localized perturbation theory outlined in Section 4.2. Section 5.6 is an excursion, where we check if the presented results also work for other choices of string variables. In Section 5.7, we give a list of different approaches how to remove ambiguities in string-localized propagators and comment on their non-trivial interplay.

Sections 5.1 to 5.6 are mainly based on the author's paper [35]. Parts of them, especially the biggest part of the introductory Section 5.1, are incorporated verbatim from this reference.[1]

5.1 Introduction: wavefront sets and renormalization

As described in Sections 1.2, 4.1 and 4.2, perturbation theory in point-localized QFT is typically formulated by writing matrix elements of the scattering operator as products of numerical distributions – the propagators of the quantum fields involved in a certain model – with the help of Wick's theorem [32, 77]. However, products or higher powers of distributions make no sense in general and also the products of propagators in the Wick expansion for the scattering operator are divergent. At n-th order of perturbation

[1]The corresponding section in the reference is in turn based on parts of Hörmander's book [41]. See also [33] for a concise but detailed introduction to distribution theory and wavefront sets.

theory, they only make sense outside the thin diagonal $\{ x_1 = \cdots = x_n \} \subset (\mathbb{R}^{1+3})^n$, or after exploiting translation invariance, outside the origin $\{ z = 0 \} \subset (\mathbb{R}^{1+3})^{n-1}$, where $z = (x_1 - x_n, \cdots, x_{n-1} - x_n)$. In momentum space, the non-existence of these products manifests itself in the well-known ultraviolet (UV) divergences of loop integrals contributing to scattering amplitudes. Renormalization in a mathematically rigorous sense is the extension of non-existent products of distributions in configuration space across the origin $\{ z = 0 \}$ [9, 32].

Once the existence of *some* extension across the origin has been established, one must address the question of uniqueness. Two extensions can only differ by a distribution supported on the origin, i.e., by a linear combination of derivatives of the Dirac delta, since both extensions must be equal to the original distribution outside the origin. Vice versa, adding an arbitrary linear combination of derivatives of the Dirac delta to a particular extension gives another extension. These ambiguities are precisely the BEG renormalization freedom. They can be controlled via constraints on the short-distance scaling behavior of the extensions, i.e., the scaling behavior with respect to $z = 0$ [9] (cf. also [64]), by requiring that the extension does not scale worse than the original distribution. This type of constraint is often referred to as *power counting*.

Example 5.1. Consider the distribution $D := [x^2 - i0]^{-1} \in \mathcal{S}'(\mathbb{R}^{1+3})$, which is a multiple of the massless Feynman propagator. We will see in Example 5.8 that the square of D is defined on $\mathbb{R}^{1+3} \setminus 0$ but not on the full space $\mathbb{R}^{1+3}$. For now, we are only interested in constructing an extension. First, note that D is homogeneous, $D(\lambda x) = \lambda^{-2} D(x)$ for all $\lambda > 0$. Correspondingly, the square $(D|_{\mathbb{R}^{1+3}\setminus 0})^2$ scales as λ^{-4}. Power counting is the requirement that any admissible extension does not scale worse than the non-extended distribution, i.e., one requires that $\lim_{\lambda \downarrow 0} \lambda^{4+\omega} w(\lambda x) = 0$ for any admissible extension w of $(D|_{\mathbb{R}^{1+3}\setminus 0})^2$ and for all $\omega > 0$.

It is a simple task to verify that on $\mathbb{R}^{1+3} \setminus 0$, the square of D coincides with the divergence of the vector-valued distribution

$$v^\mu := \frac{1}{2} \frac{x^\mu \ln(x^2 - i0)}{(x^2 - i0)^2}. \tag{5.1}$$

Since v^μ is locally integrable with respect to x at $x = 0$, it is a well-defined distribution[2] on the full space $\mathbb{R}^{1+3}$ and thus, the divergence $\overline{D^2} := \partial_\mu v^\mu$ defines an extension of $(D|_{\mathbb{R}^{1+3}\setminus 0})^2$. It is also admissible by power counting since $\lim_{\lambda \downarrow 0} \lambda^\omega \ln(\lambda^2) = 0$ for all $\omega > 0$.

An arbitrary extension w of $(D|_{\mathbb{R}^{1+3}\setminus 0})^2$ can only differ from $\overline{D^2}$ by a linear combination of derivatives of the Dirac delta. Power counting introduces an upper bound on the number of derivatives appearing in said linear combination. In the case at hand,

$$w - \overline{D^2} = c_0 \delta(x) \tag{5.2}$$

[2] The reader may try to verify that the logarithm does not cause any trouble by using the tools that we present in the remaining part of the section.

for some constant c_0 and any admissible extension w since the Dirac delta already scales like λ^{-4}. The free parameter c_0 in Eq. (5.2) introduces a renormalization freedom to the model under consideration. It usually needs to be fixed by physical reasoning.

The method used to obtain the special extension $\overline{D^2}$ is called *differential renormalization* but there are also other well-established methods (see for example [25] for an introduction or [9, 17] for more abstract considerations).

Remark 5.2. The massless Feynman propagator D from Example 5.1 is homogeneous. Therefore, it is obvious how to define the scaling behavior with respect to the origin. A more general definition can for example be found in [9].

A definition of a product of two distributions, which satisfies the known rules of calculus, as well as a criterion for its existence was found by Hörmander [41, Theorem 8.2.10.] (as a special case of Lemma 5.4 below). If u and v are distributions over an open subset $X \subset \mathbb{R}^n$, their product uv can be defined as the pullback of the tensor product $u \otimes v$ by the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x, x)$ if

$$(x; p) \in \mathrm{WF}\, u \quad \text{implies} \quad (x; -p) \notin \mathrm{WF}\, v, \tag{5.3}$$

where the wavefront set $\mathrm{WF}\, u$ of a distribution u is a subset of the cotangent bundle $\dot{T}^*(X)$ over X deprived of the elements $(x; 0)$ (as indicated by the dot). $\mathrm{WF}\, u$ gives a refined characterization of the singularities of u:

Definition 5.3 (see Ch. 8 in [41])**.** Let $u \in \mathcal{D}'(X)$ for $X \subset \mathbb{R}^n$ open. Then the *singular support* $\operatorname{singsupp} u$ of u is the set of points in X that have no open neighborhood where u is smooth. The *frequency set* $\Sigma_x(u)$ of u over a point $x \in X$ is defined as an intersection

$$\Sigma_x(u) := \bigcap_{\substack{\phi \in C_c^\infty(X) \\ \phi(x) \neq 0}} \Sigma(\phi u), \tag{5.4}$$

where $\Sigma(\phi u)$ is the cone of directions in $\mathbb{R}^n \setminus 0$ having no conic neighborhood in which the Fourier transform of the compactly supported distribution ϕu is rapidly decaying. Finally, the *wavefront set* $\mathrm{WF}\, u$ of u is the closed subset of $\dot{T}^*(X)$ defined by

$$\mathrm{WF}\, u := \{\, (x; p) \in \dot{T}^*(X) \mid p \in \Sigma_x(u) \,\} \tag{5.5}$$

so that the projection of $\mathrm{WF}\, u$ onto the first component yields the singular support.

Thus, the wavefront set does not only encode the information about the singularities of a distribution but also about the high frequencies that are responsible for their appearance. It is easy to verify that the wavefront set is a closed and conic subset of $\dot{T}^*(X)$, where *conic* means that the wavefront set is invariant under scaling the second variable with positive scalars.

The proofs in the following parts of this section are based on several standard statements about properties of the wavefront set. For convenience of the reader, we now concisely list the statements on which we rely later.

The Hörmander product of two distributions u and v is defined as a pullback of their tensor product, provided that the criterion Eq. (5.3) is satisfied. One can then also give a bound on the wavefront set of the product [41, Theorem 8.2.10.], namely

$$\mathrm{WF}(uv) \subset \{\, (x; p+k) \mid (x;p) \in \mathrm{WF}\, u \text{ or } p = 0,\ (x;k) \in \mathrm{WF}\, v \text{ or } k = 0 \,\}. \tag{5.6}$$

The Hörmander product of two distributions is an important special case of the pullback of distributions but we will also need to consider other pullbacks in order to examine the wavefront set of string-localized propagators.

Lemma 5.4 (Thm. 8.2.4. in [41])**.** *The pullback f^*u of a distribution $u \in \mathcal{D}'(Y)$ by a smooth map $f : X \to Y$, where $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are open, can be defined such that it coincides with the pullback of smooth maps if $u \in C^\infty(Y)$, provided that $N_f \cap \mathrm{WF}\, u = \emptyset$, where*

$$N_f := \{\, (f(x); p) \in Y \times \mathbb{R}^n \mid {}^t f'(x) p = 0 \,\}. \tag{5.7}$$

is the set of normals of the map f. Moreover, we have

$$\mathrm{WF}(f^*u) \subset f^*\, \mathrm{WF}\, u := \{\, (x; {}^t f'(x) p) \mid (f(x); p) \in \mathrm{WF}\, u \,\}. \tag{5.8}$$

The distributions that appear in quantum field theory are often solutions of partial differential equations. For such distributions, one can give bounds on their wavefront set:

Lemma 5.5 (Eq. (8.1.11) and Thm. 8.3.1. in [41])**.** *Let $u \in \mathcal{D}'(X)$ for $X \subset \mathbb{R}^n$ open and let $P = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha$ be a differential operator of order m on X with smooth coefficients. Then*

$$\mathrm{WF}(Pu) \subset \mathrm{WF}\, u \subset \mathrm{WF}(Pu) \cup \operatorname{char} P, \tag{5.9}$$

where the characteristic set char P *is defined in terms of the principal symbol $P_m(x,p) := \sum_{|\alpha|=m} a_\alpha(x) p^\alpha$ of P via*

$$\operatorname{char} P := \{\, (x;p) \in \dot{T}^*(X) \mid P_m(x,p) = 0 \,\}. \tag{5.10}$$

In particular, if u solves $Pu = 0$, then $\mathrm{WF}\, u \subset \operatorname{char} P$.

We will also deal with several homogeneous distributions. These are automatically tempered [41, Theorem 7.1.18.] and the wavefront set of a homogeneous distribution is closely related to the wavefront set of its Fourier transform:

Lemma 5.6 (Thm. 8.1.8. in [41])**.** *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be homogeneous in $\mathbb{R}^n \setminus 0$. Then*

$$\begin{aligned}
(x;p) \in \mathrm{WF}\, u &\Leftrightarrow (p; -x) \in \mathrm{WF}\, \hat{u} && \text{if} \quad x \neq 0 \quad \text{and} \quad p \neq 0,\\
x \in \operatorname{supp} u &\Leftrightarrow (0; -x) \in \mathrm{WF}\, \hat{u} && \text{if} \quad x \neq 0,\\
p \in \operatorname{supp} \hat{u} &\Leftrightarrow (0; p) \in \mathrm{WF}\, u && \text{if} \quad p \neq 0.
\end{aligned}$$

Remark 5.7. The statements from [41] displayed in this section are formulated over Euclidean space with the sign convention of the Fourier transform described in the end of the introduction. The mentioned change of the sign convention due to physical reasons when working over Minkowski space implies that the covector components of wavefront sets over Minkowski space get an additional sign.

Example 5.8. We show that the wavefront set of the massless Feynman propagator D from Example 5.1 is given by

$$\mathrm{WF}\, D = \{ (x; \lambda x) \mid x^2 = 0,\ x \neq 0,\ \lambda > 0 \} \cup \dot{T}_0^*. \tag{5.11}$$

First, we have $\dot{T}_0^* \subset \mathrm{WF}\, D$ by Lemma 5.5 since D is a fundamental solution of the wave equation and since $\mathrm{WF}\, \delta(x) = \dot{T}_0^*$. The latter wavefront set can be computed by using that $\widehat{\varphi\delta}(p) = \varphi(0)$ for $\varphi \in C_c^\infty(\mathbb{R}^{1+3})$. When $x \neq 0$, D is the pullback of the homogeneous distribution $[t - i0]^{-1} \in \mathcal{S}'(\mathbb{R})$ by the map $f : \mathbb{R}^{1+3} \setminus 0 \to \mathbb{R}$ with $f(x) = x^2$. To verify this, note that the Fourier transform of $[t \pm i0]^{-1}$ is a multiple of the Heaviside distribution $\theta(\pm\lambda)$ and thus, by Lemma 5.6,

$$\mathrm{WF}[t \pm i0]^{-1} = \{ (0; \lambda) \mid \lambda \gtrless 0 \} \tag{5.12}$$

and $N_f \cap \mathrm{WF}[t - i0]^{-1} = \emptyset$. Hence, the pullback is defined by Lemma 5.4. The wavefront set of the pullback is thus contained in the right-hand side of Eq. (5.11) by Lemma 5.4, where the inverted sign of λ comes from the fact that we work over Minkowski space, as explained in Remark 5.7. Since the wavefront set is conic and the projection onto the first component yields the singular support, $\mathrm{WF}\, D$ cannot be smaller than the right-hand side of Eq. (5.11).

Since λ has a fixed sign, the Hörmander square of D exists when $x \neq 0$ but because the wavefront set over $x = 0$ contains any direction, the square is not defined at $x = 0$.

Examples 5.1 and 5.8 are prototypical for an extension problem in point-localized gauge theories. The situation becomes much more complex in string-localized field theories. There, the propagators are not only distributions in the variables x and x' but also in spacelike string directions e and e'. The string-localization can induce new singularities to the propagator and moreover, the structure of these singularities depends on the formulation of a string-localized perturbation theory, as we shall investigate in Section 4.2. We will then prove in Section 5.5 that in a proper setup of string-localized perturbation theory, the wavefront sets of string-localized propagators are actually contained in the wavefront sets of certain point-localized propagators, that is to say that the renormalization problem does not get worse in SLFT despite the delocalization.

To prove the latter statement, another standard theorem from microlocal analysis about partially smeared distributions will play a central role:

Lemma 5.9 (Thm. 8.2.12. in [41])**.** *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open and let $K \in \mathcal{D}'(X \times Y)$ with the corresponding linear transformation $\mathcal{K}$ from $\mathcal{D}(Y)$ to $\mathcal{D}'(X)$, i.e.,*

$$[\mathcal{K}\varphi](\phi) = K(\phi \otimes \varphi). \tag{5.13}$$

Then

$$\mathrm{WF}(\mathcal{K}\varphi) \subset \{\, (x;p) \mid (x,y;p,0) \in \mathrm{WF}\, K \text{ for some } y \in \operatorname{supp}\varphi \,\}. \tag{5.14}$$

The two-point functions and propagators of string-localized potentials are tensor-valued distributions. To investigate their singularity structure and the existence of products, we need the definition of the wavefront set of a a distribution with values in $\mathbb{C}^n$.

Definition 5.10 (see [18])**.** Let $X \subset \mathbb{R}^m$ and let $u = (u_j)_j \in \mathcal{D}'(X, \mathbb{C}^n)$ be a vector-valued distribution. Then $\mathrm{WF}\, u = \bigcup_{j=1}^n \mathrm{WF}\, u_j$.

5.2 Singularities arising from string integration

The following section is partially taken from the author's paper [35], with adjustments made to better fit into the context of this thesis. In Section 2.2, we have seen that string integration leads to multiplication with the distributions

$$(pe)_\pm^{-1} := \lim_{\varepsilon\downarrow 0} \frac{1}{(pe) \pm i\varepsilon} \tag{5.15}$$

in momentum space. These distributions become singular when $(pe) = 0$ and thus introduce additional singularities to string-localized two-point functions and propagators. Let us characterize the new singularities in detail for general directions $e \in \mathbb{R}^{1+3}$.

Lemma 5.11. *The expressions $U_\pm(p,e) := (pe)_\pm^{-1}$ are tempered distributions on $(\mathbb{R}^{1+3})^2$ with*

$$\mathrm{WF}\, U_\pm = \{\, (p,e;\lambda e,\lambda p) \mid \lambda \lessgtr 0,\ (pe) = 0,\ (p,e) \neq (0,0) \,\} \cup \dot{T}^*_{(0,0)}, \tag{5.16}$$

where $\dot{T}^*_{(0,0)}$ is the cotangent space at $(p,e) = (0,0)$ deprived of the zero-covector.

Proof. First note that if $U_\pm$ are well-defined distributions, they are also tempered because they are homogeneous. When $(p,e) \neq (0,0)$, $U_\pm$ are the pullbacks of the distributions $[t \pm i0]^{-1} \in \mathcal{S}'(\mathbb{R})$ by the map $f : (\mathbb{R}^{1+3})^2 \setminus (0,0) \to \mathbb{R}$, $f(p,e) = (pe)$ with set of normals

$$N_f = \{\, ((pe);\lambda) \in \mathbb{R}^2 \mid \lambda e = \lambda p = 0,\ (p,e) \neq (0,0) \,\} = \{\, (t;0) \mid t \in \mathbb{R} \,\} \tag{5.17}$$

so that $N_f \cap \mathrm{WF}[t \pm i0]^{-1} = \emptyset$. Thus, by Lemma 5.4, Remark 5.7 and the form of $\mathrm{WF}[t \pm i0]^{-1}$ given in Eq. (5.12), we have

$$\mathrm{WF}\, U_\pm|_{(p,e)\neq(0,0)} \subset f^*\, \mathrm{WF}[t \pm i0]^{-1} = \{\, (p,e;\lambda e,\lambda p) \mid (pe) = 0,\ \lambda \lessgtr 0 \,\}. \tag{5.18}$$

Eq. (5.18) must actually be an equality since the wavefront set is conic and the projection onto the first component must yield the singular support. Since $U_\pm$ are locally integrable at $(p,e) = (0,0)$, we have established their existence as tempered distributions.

It remains to show that the wavefront set over $(p, e) = (0, 0)$ is the whole cotangent space (deprived of the zero-covector). To do so, we introduce the bilinear form

$$A := \frac{1}{2}\begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} \tag{5.19}$$

on $(\mathbb{R}^{1+3})^2$ such that $A(p, e) = (pe)$ and $4A^2 = \mathbb{I}$. By [41, Theorem 6.2.1.],

$$(\partial_p \partial_e)\,[(pe) \pm i0]^{-3} = a_\pm \delta(p, e), \tag{5.20}$$

where $a_\pm$ are non-vanishing constants that are unimportant for the following arguments. Moreover, we have

$$(\partial_p \partial_e)^2 U_\pm = 4\,[(pe) \pm i0]^{-3} \quad \Rightarrow \quad (\partial_p \partial_e)^3 U_\pm = 4a_\pm \delta(p, e). \tag{5.21}$$

Consequently $\mathrm{WF}\,\delta(p, e) = \dot{T}^*_{(0,0)} \subset \mathrm{WF}\,U_\pm$ by Lemma 5.5 and the proof is completed. □

The distributions $U_\pm$ in Lemma 5.11 depend on a general string direction $e \in \mathbb{R}^{1+3}$. In SLFT, however, the string directions are usually restricted to a set of spacelike (or lightlike, see [39]) directions. Within our derivations, they are elements of the *open* subset $H \subset \mathbb{R}^{1+3}$ of spacelike directions, as explained in Section 2.1. The restriction of a distribution to an open subset always exists and it follows immediately from Definition 5.3 that the wavefront set of the restricted distribution is the restriction of the wavefront set. We therefore define:

Definition 5.12. Let $u_\pm(p, e) := U_\pm(p, e)|_{\mathbb{R}^{1+3}\times H}$ denote the restriction of the distributions $U_\pm$ over $(\mathbb{R}^{1+3})^2$ from Lemma 5.11 to the open subset $\mathbb{R}^{1+3}\times H$ of spacelike string directions with

$$\mathrm{WF}\,u_\pm = \{\,(p, e; x, \xi) \mid (p, e; x, \xi) \in \mathrm{WF}\,U_\pm,\ e \in H\,\} \tag{5.22}$$

by definition of the wavefront set.

Lemma 5.11 has the following important consequence for the restricted distributions $u_\pm$.

Corollary 5.13. *Hörmander products $(u_+)^k$ and $(u_-)^k$ of the restrictions to spacelike string variables do exist for arbitrary $k \in \mathbb{N}$, but the Hörmander product $u_+ \cdot u_-$ with opposite imaginary shift does not exist. Moreover,*

$$\mathrm{WF}\left[(u_\pm)^k\right] = \mathrm{WF}\,u_\pm. \tag{5.23}$$

Proof. If $e \in H$, then $(p, e) \neq (0, 0)$ and

$$\mathrm{WF}\,u_\pm = \{\,(p, e; \lambda e, \lambda p) \mid e \in H,\ \lambda \lessgtr 0,\ (pe) = 0\,\}. \tag{5.24}$$

The Hörmander product of two distributions exists if Eq. (5.3) is satisfied. Since the sign of λ in Eq. (5.24) is fixed by the sign of the imaginary shift, $(u_\pm)^2$ are defined but $u_+ \cdot u_-$ is not. It also follows immediately from the shape of $\operatorname{WF} u_\pm$ and Eq. (5.6) that

$$\operatorname{WF}\left[(u_\pm)^2\right] \subset \operatorname{WF} u_\pm \tag{5.25}$$

and both sides must be equal since the wavefront set is conic and the projection onto the first component must yield the singular support. By induction, we get the statement for arbitrary powers. □

Remark 5.14. In the literature, the string variables are usually considered as elements of the *closed* subset $H_{-1} \subset \mathbb{R}^{1+3}$ of spacelike vectors with Minkowski square $e^2 = -1$ (as for example in [37, 49]). The restriction of a distribution to a closed subset is much more involved than the restriction to an open subset. It does not always exist and even if it does, it may affect the form of the wavefront set [41]. We will briefly sketch in Section 5.6.2 why the restriction to H_{-1} is indeed unproblematic. For our purposes, however, the simpler case of the restriction to the open subset H is sufficient.

By our considerations in Section 4.2, each string-localized potential appearing in the Dyson series for the string-localized S-matrix depends on a different string variable. The products of propagators are products only in the x-variables but not in the e-variables. We can thus smear out the string variables and investigate the existence of the distributional product in x *after smearing*. Let us determine the effect of this smearing on the new singularities coming from $u_\pm$. More detailed, each denominator $u_\pm(p, e)$ in the two-point function (2.16) is accompanied by a numerator e^μ and this numerator, although unproblematic, must be included in our considerations. We have:

Lemma 5.15. *For $c \in \mathcal{D}(H)$, we define*

$$q_{c,\pm}^{\mu_1\cdots\mu_s}(p) := \int d^4e\, c(e) \frac{(\pm i)^s e^{\mu_1} \cdots e^{\mu_s}}{[(pe) \pm i0]^s}. \tag{5.26}$$

Then $q_{c,\pm}^{\mu_1\cdots\mu_s}(p) \in \mathcal{S}'(\mathbb{R}^{1+3})$ with $\operatorname{WF} q_{c,\pm}^{\mu_1\cdots\mu_s}(p) = \{\, (0; \lambda e) \mid \lambda \lessgtr 0, e \in \operatorname{supp} c \,\}$.

Proof. The expressions $q_{c,\pm}^{\mu_1\cdots\mu_s}(p)$ are the results of smearing distributions of the form appearing in Corollary 5.13 times a smooth (tensor-valued) function in the string variable. Therefore, they are well-defined distributions. By homogeneity, they are also tempered.

Since $e \in H$ is spacelike and hence non-zero, the wavefront set of $q_{c,\pm}^{\mu_1\cdots\mu_s}(p)$ must be contained in $\{\, (p; x) \mid (p, e; x, 0) \in \operatorname{WF} u_\pm \,\}$ by Lemma 5.9, with $u_\pm$ as in Corollary 5.13 and $\operatorname{WF} u_\pm$ as in Eq. (5.24). This yields

$$(p; x) \in \operatorname{WF} q_{c,\pm}^{\mu_1\cdots\mu_s}(p) \quad \Rightarrow \quad p = 0 \text{ and } x = \lambda e \text{ for some } \lambda \lessgtr 0 \text{ and } e \in \operatorname{supp} c. \tag{5.27}$$

To show that any such element $(0; \lambda e)$ is in the wavefront set, note that the Fourier transform of $q_{c,\pm}^{\mu_1\cdots\mu_s}(p)$ is

$$\int d^4p\, e^{i(px)} q_{c,\pm}^{\mu_1\cdots\mu_s}(p) \sim \int d^4e\, c(e)\, e^{\mu_1} \cdots e^{\mu_s} I_{\pm e}^s \delta(x) \tag{5.28}$$

with support $\{ x = \lambda e \mid \lambda \leqslant 0,\ e \in \operatorname{supp} c \}$. By homogeneity and Lemma 5.6, $(0; x)$ is an element of $\operatorname{WF} q_{c,\pm}^{\mu_1\cdots\mu_s}(p)$ if and only if x is in the support of the Fourier transform. This proves the claim. □

Note that Lemma 5.15 implies that the new singularities coming from string integration are pure infrared singularities after smearing out the string variables.

Remark 5.16. Let $\boldsymbol{e} = (e_1, \dots, e_k) \in H^k$ and $\boldsymbol{e'} = (e'_1, \dots, e'_l) \in H^l$. Note that the existence criterion for "full" x-products after smearing out the string variables $(\boldsymbol{e}, \boldsymbol{e'})$ is the same existence criterion as for "partial" x-products of distributions in $\mathbb{R}^{1+3} \times H^{k+l}$. Let us have a closer look at this, starting with the x-product of smeared distributions and assuming that all string variables are smeared with the same test function $c \in \mathcal{D}(H)$.[3]

Let $u \in \mathcal{D}'(\mathbb{R}^{1+3} \times H^k)$ and $v \in \mathcal{D}'(\mathbb{R}^{1+3} \times H^l)$. Lemma 5.9 gives

$$\begin{aligned} \operatorname{WF} u(x, c^{\otimes k}) &\subset \{ (x; p) \mid (x, \boldsymbol{e}; p, 0) \in \operatorname{WF} u(x, \boldsymbol{e}) \text{ for some } \boldsymbol{e} \in \operatorname{supp} c^{\otimes k} \}, \\ \operatorname{WF} v(x, c^{\otimes l}) &\subset \{ (x; p) \mid (x, \boldsymbol{e'}; p, 0) \in \operatorname{WF} v(x, \boldsymbol{e'}) \text{ for some } \boldsymbol{e'} \in \operatorname{supp} c^{\otimes l} \} \end{aligned} \tag{5.29}$$

so that the product $u(x, c^{\otimes k}) v(x, c^{\otimes l})$ of the partially smeared distributions exists if

$$(x, \boldsymbol{e}; p, 0) \in \operatorname{WF} u(x, \boldsymbol{e}) \quad \text{implies} \quad (x, \boldsymbol{e'}; -p, 0) \notin \operatorname{WF} v(x, \boldsymbol{e'}), \tag{5.30}$$

for any $e_i, e'_j \in \operatorname{supp} c$. On the other hand, the partial x-product before smearing is nothing but the pull-back to the x-diagonal by the map

$$\Delta_x : \mathbb{R}^{1+3} \times H^{k+l} \to (\mathbb{R}^{1+3})^2 \times H^{k+l}, (x, \boldsymbol{e}, \boldsymbol{e'}) \mapsto (x, x, \boldsymbol{e}, \boldsymbol{e'}) \tag{5.31}$$

with

$${}^t\Delta'_x(x, e, e') \begin{pmatrix} p \\ k \\ \xi \\ \xi' \end{pmatrix} = \begin{pmatrix} p + k \\ \xi \\ \xi' \end{pmatrix} = 0 \quad \text{if and only if} \quad \xi = \xi' = 0 \quad \text{and} \quad p + k = 0. \tag{5.32}$$

Therefore, by Lemma 5.4, the criteria for the existence of the partial product and the product after smearing are the same. In the following, we will work with the product after smearing because there, it is easier to perform a transition from momentum to configuration space. However, for a BEG construction in SLFT, the time-ordered products should be defined *before* smearing so that one does not have to extend across an open set.

Remark 5.17. Indeed, the singularities of the multi-string-localized potentials (2.6) from Remark 2.3 are also pure infrared singularities after smearing out the string variables. To see this, note that in momentum space the additional singularities of the multi-string-localized potentials are

$$\int d^4(e_1) \dots d^4 e_s\, c(e_1) \dots c(e_s)\, \frac{(\pm i)^s e_1^\mu \dots e_s^\mu}{(pe_1)_\pm \dots (pe_s)_\pm}, \tag{5.33}$$

which is nothing but the s-fold p-product of $q_{c,\pm}^\mu(p)$ with itself, whose wavefront set is contained in $\dot{T}_0^* \mathbb{R}^{1+3}$. This can be seen in a similar manner to the proof of Lemma 5.15.

[3]This assumption is not necessary for the subsequent reasoning but since it is satisfied anyway in our considerations, we use it here to simplify the notation.

Lemma 5.11, Corollary 5.13 and Lemma 5.15 are the starting point for the full analysis of the singularities of string-localized propagators that we subsequently perform. We proceed to show that the singularities induced by string-integration are completely harmless if string-localized perturbation theory is set up as described in Section 4.2.

5.3 Well-definedness of string-localized two-point functions

With the microlocal tools listed in Section 5.1 and the wavefront set of the denominators $(pe)_\pm^{-1}$ derived in Section 5.2 at hand, we are now ready to analyze the well-definedness and singularity structure of the string-localized two-point functions introduced in Section 2.2. We have:

Theorem 5.18. *For all masses $m \geq 0$ and all spins respectively helicities $s \in \mathbb{N}$, the two-point function of the string-localized potential, given by*

$$\langle\!\langle A_f^{(s)}(e) A_{f'}^{(s)}(e') \rangle\!\rangle(x) = \int d\mu_m(p)\, e^{-ipx}\, {}_mM^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e'), \tag{5.34}$$

with the kernel ${}_mM^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e')$ as defined in Eq. (2.16), is a well-defined distribution over $\mathbb{R}^{1+3} \times H^2$.

Proof. At $m = 0$, the point-localized scalar two-point function

$$W_0(x) = \int d\mu_0(p)\, e^{-ipx} \tag{5.35}$$

is homogeneous, as is its Fourier transform, the measure $d\mu_0(p)$. Since [57, Theorem IX.48]

$$\mathrm{WF}\, W_0(x) = \{\, (0; p) \mid p^2 = 0,\ p^0 < 0 \,\} \cup \{\, (\pm|\vec{x}|, \vec{x}; \lambda|\vec{x}|, \pm\lambda\vec{x}) \mid \vec{x} \in \mathbb{R}^3, \lambda < 0 \,\}. \tag{5.36}$$

and since $W_0(x)$ is supported everywhere, Lemma 5.6 on the wavefront set of homogeneous distributions gives[4]

$$\mathrm{WF}\, d\mu_0(p) = \{\, (p; \lambda p) \mid p^2 = 0, p^0 > 0, \lambda \neq 0 \,\} \cup \dot{T}_0^*\mathbb{R}^{1+3}. \tag{5.37}$$

In the massive case, homogeneity is lost but the mass hyperboloid $H_m^+ := \{\, p^2 = m^2, p^0 > 0 \,\}$ is a smooth and closed submanifold and the measure $d\mu_m(p)$ is a smooth density on H_m^+, so that [7, 41] (cf. also [36])

$$\mathrm{WF}\, d\mu_m(p) = \{\, (p; \lambda p) \mid p^2 = m^2, p^0 > 0, \lambda \neq 0 \,\}. \tag{5.38}$$

[4]Note again the inverted sign convention for the Fourier transform over Minkowski space, which is important at this point. See also Section 1.3.

The kernel ${}_mM^{A_f^{(s)},A_{f'}^{(s)}}(p,e,e')$ is homogeneous of degree $\omega = 0$ in p and contains only denominators $u_-(p,e)^k$ and $u_+(p,e')^l$. Tensoring $d\mu_m(p)$, $u_-(p,e)^k$ and $u_+(p,e')^l$ with the constant distribution in the missing string variables, we see that the Hörmander criterion for the triple product

$$d\mu_m(p) \cdot u_-(p,e)^k \cdot u_+(p,e')^l \tag{5.39}$$

is satisfied for all masses $m \geq 0$ whenever $p \neq 0$ by Eq. (5.37), Eq. (5.38) and Corollary 5.13. Since $d\mu_m(p)$ is smooth at $p = 0$ for $m > 0$ and since $d\mu_0(p)$ is homogeneous of degree $\tilde{\omega} = -2$,

$$d\mu_m(p)\, {}_mM^{A_f^{(s)},A_{f'}^{(s)}}(p,e,e') \tag{5.40}$$

is locally integrable with respect to $p = 0$ and hence a well-defined distributions. The vacuum expectation values are then the Fourier back transform with respect to p of (5.40). □

Next, we determine the wavefront set of the partially smeared two-point functions.

Theorem 5.19. *The wavefront set of the partially smeared two-point function*

$$\langle\!\langle A_f^{(s)}(c)A_{f'}^{(s)}(c)\rangle\!\rangle(x) := \int d^4e \int d^4e'\, c(e)\, c(e')\, \langle\!\langle A_f^{(s)}(c)A_{f'}^{(s)}(c)\rangle\!\rangle(x) \in \mathcal{S}'(\mathbb{R}^{1+3}) \tag{5.41}$$

is contained in the wavefront set of the two-point function $W_m(x)$ of a scalar and point-localized Klein-Gordon field, i.e.,

$$\mathrm{WF}\left[\langle\!\langle A_f^{(s)}(c)A_{f'}^{(s)}(c)\rangle\!\rangle(x)\right] \subset \mathrm{WF}\, W_m(x), \tag{5.42}$$

where the wavefront set on the righthand side of Eq. (5.42) is given by Eq. (5.36) also for $m > 0$ [57].

To prove Theorem 5.19, we can exploit homogeneity for the massless case. For the massive case, we first prove an auxiliary lemma.

Lemma 5.20. *Let $u, v \in \mathcal{S}'(\mathbb{R}^{1+3})$. Suppose further that $\hat{u}$ is polynomially bounded, that $\mathrm{WF}\,\hat{u} \subset \dot{T}_0^*$, that $\hat{v}$ is smooth on a neighborhood of $p = 0$ and that the Hörmander product $\hat{u}\hat{v}$ is an element of $\mathcal{S}'(\mathbb{R}^{1+3})$. Then $\mathrm{WF}\left[\mathcal{F}^{-1}(\hat{u}\hat{v})\right] \subset \mathrm{WF}\, v$.*

Proof (see also [35]). We have to investigate the decay properties of the Fourier transforms of $\phi\mathcal{F}^{-1}(\hat{u}\hat{v})$ for $\phi \in C_c^\infty(\mathbb{R}^{1+3})$. Since ϕ is compactly supported and smooth, we know that its Fourier transform is a Schwartz function, $\hat{\phi} \in \mathcal{S}(\mathbb{R}^{1+3})$. Moreover, $\mathcal{F}^{-1}(\hat{u}\hat{v})$ is a tempered distribution since by assumption also $\hat{u}\hat{v}$ is tempered. Then

$$\left[\mathcal{F}\left(\phi\mathcal{F}^{-1}(\hat{u}\hat{v})\right)\right](p) = \hat{\phi} * \hat{u}\hat{v}(p) = \hat{u}\hat{v}(\hat{\phi}(p - \cdot)) \tag{5.43}$$

is smooth and polynomially bounded [57, Thm. IX.4].

To investigate the decay of $\hat{u}\hat{v}(\hat{\phi}(p-\cdot))$, we introduce a second cutoff function $\chi \in C_c^\infty(\mathbb{R}^{1+3})$ with $0 \le \chi \le 1$, $\chi \equiv 1$ on $B_r(0)$ and $\operatorname{supp}\chi \subset B_R(0)$, where $B_\varrho(p)$ is the closed ball of *Euclidean* radius ϱ and center $p \in \mathbb{R}^{1+3}$, and where $0 < r < R$ is chosen such that $B_R(0) \cap \operatorname{singsupp} \hat{v} = \emptyset$.

Then $\hat{u}$ is smooth on $\operatorname{supp}(1-\chi)$, $\hat{v}$ is smooth on $\operatorname{supp}\chi$ and $\hat{u}\hat{v} = \chi\hat{u}\hat{v} + (1-\chi)\hat{u}\hat{v}$. The first term is unproblematic, for there are constants N, C and C' such that

$$\begin{aligned}|\chi\hat{u}\hat{v}(\hat{\phi}(p-\cdot))| &\le C \sum_{|\alpha+\beta|\le N} \sup_{k\in B_R(0)} \left|k^\alpha \partial_k^\beta \hat{\phi}(p-k)\right| \\ &\le C'(1+|p|)^N \sum_{|\alpha+\beta|\le N} \sup_{k\in B_R(p)} \left|k^\alpha \partial_k^\beta \hat{\phi}(k)\right|, \end{aligned} \tag{5.44}$$

where $|p|$ is the Euclidean norm of p. The right-hand side of Eq. (5.44) is rapidly decaying since $\hat{\phi} \in \mathcal{S}(\mathbb{R}^{1+3})$ and since the supremum is taken over a compact set around p.

To estimate the second term $(1-\chi)\hat{u}\hat{v}$, note that the smooth function $(1-\chi)\hat{u}$ is polynomially bounded and thus

$$[(1-\chi)\hat{u}\hat{v}]\left(\hat{\phi}(p-\cdot)\right) = \hat{v}\left((1-\chi)\hat{u}(\cdot)\,\hat{\phi}(p-\cdot)\right) \tag{5.45}$$

falls off rapidly if $\hat{v}\left(\hat{\phi}(p-\cdot)\right) = \widehat{\phi v}$ falls off rapidly. Thus, also the full expression falls off rapidly if $\widehat{\phi v}$ does, which proves the lemma. □

With Lemma 5.20 at hand, we are ready to prove Theorem 5.19.

Proof of Theorem 5.19. To prove the massive case, we define the distributions

$$\hat{u}(p) = \int d^4e \int d^4e'\, c(e)c(e')\, {}_mM^{A_f^{(s)},A_{f'}^{(s)}}(p,e,e') \tag{5.46}$$

and $\hat{v}(p) = d\mu_m(p)$ for $m > 0$. $\hat{u}(p)$ is homogeneous of degree 0 in p and arises from contraction of the distributions $q_{c,\pm}^{\mu_1\cdots\mu_s}$ from Lemma 5.15 with a polynomial in p. Local integrability of $\hat{u}(p)$ at $p = 0$ ensures its existence as a tempered distribution and by Lemma 5.15, $\mathrm{WF}\,\hat{u} \subset \dot{T}_0^*\mathbb{R}^{1+3}$. $\hat{v}$ is smooth at $p = 0$ because $m > 0$ and hence $\hat{u}$ and $\hat{v}$ satisfy the assumptions of Lemma 5.20. Therefore, the part of Theorem 5.32 concerning the massive case is a special case of Lemma 5.20 and

$$\mathrm{WF}\,\langle\!\langle A_f^{(s)}(c) A_{f'}^{(s)}(c)\rangle\!\rangle \subset \mathrm{WF}\,v = \mathrm{WF}\,W_m(x). \tag{5.47}$$

In the massless case, the partially smeared string-localized two-point function is

$$\langle\!\langle A_f^{(s)}(c) A_{f'}^{(s)}(c)\rangle\!\rangle = \int d\mu_0(p)\,\hat{u}(p)\,e^{-ipx} \tag{5.48}$$

with $\hat{u}(p)$ as in Eq. (5.46). Here we have $\dot{T}_0^*\mathbb{R}^{1+3} \subset \mathrm{WF}\,d\mu_0(p)$, so Lemma 5.20 does not apply. However, by local integrability with respect to $p = 0$, $d\mu_0(p)\,\hat{u}(p)$ is a well-defined distribution because $\mathrm{WF}\,\hat{u} \subset \dot{T}_0^*\mathbb{R}^{1+3}$ and since $\hat{u}(p)$ can only introduce additional singularities at $p = 0$, Lemma 5.6 gives the claim for the vacuum expectation value (5.48). □

Corollary 5.21. *All Hörmander powers*

$$v_{f_1 \dots f_{2n}}(x) := \prod_{i=1}^{n} \langle\!\langle A^{(s)}_{f_{2i-1}}(c) A^{(s)}_{f_{2i}}(c) \rangle\!\rangle(x) \tag{5.49}$$

are well-defined elements of $\mathcal{S}'(\mathbb{R}^{1+3})$ *with*

$$\mathrm{WF}\, v_{f_1 \dots f_{2n}} \subset \mathrm{WF}\, W_m(x) \cup \{\, (0; p) \mid p^2 \geq 0,\ p^0 < 0 \,\}. \tag{5.50}$$

Proof. The wavefront set of $W_m(x)$ is directed and from its form given by Eq. (5.36) we see that the Hörmander square of $W_m(x)$ exists. Using the estimate Eq. (5.6) on the wavefront set of the Hörmander product, we also get that

$$\mathrm{WF}\left[(W_m(x))^2 \right] \subset \mathrm{WF}\, W_m(x) \cup \{\, (0; p) \mid p^2 \geq 0,\ p^0 < 0 \,\}, \tag{5.51}$$

where the second component comes from the fact that the sum of two backwards directed lightlike vectors can also be a backwards directed timelike vector. The statement for arbitrary powers follows by induction because any linear combination of backwards directed timelike and lightlike vectors remains a backwards directed timelike or lightlike vector. Finally, Theorem 5.19 gives the same statement for the powers of partially smeared two-point functions of string-localized potentials. □

We have thus shown that the singularity structure of the partially smeared two-point functions of string-localized potentials of arbitrary mass and spin respectively helicity is not worse than in the point-localized case.

Remark 5.22. Note again that all proofs in this section work analogously for the multi-string-localized potentials from Remark 2.3.

5.4 Propagators and their ambiguities

In the preceding section, we have shown that the two-point functions of string-localized potentials are well-defined and that the wavefront set of the partially smeared two-point functions is contained in the wavefront set of the two-point function of the scalar Klein-Gordon field. In this and the following section, we prove similar statements about the respective propagators. Both sections are again partially taken from the author's paper [35].

In accordance with the BEG scheme, there is no unique choice of a propagator in general. To visualize that, note that any (smooth) term proportional to $p^2 - m^2$ can be added to the kernel of a two-point function without changing the latter because $(p^2 - m^2) d\mu_m(p) = 0$. However, adding such a term to the propagator yields a contribution $\frac{p^2 - m^2}{p^2 - m^2 + i0} = 1$, which becomes a Dirac delta in configuration space. This freedom in choosing a propagator is well-known and of course, we have come across it already in Section 1.2 and Chapter 4 when discussing the ambiguities in the BEG approach to perturbation theory. As described in Section 5.1, the freedom of choosing an extension is usually restricted by the power counting principle.

In Section 5.4.1, we introduce a special choice of (string-localized) propagators – the "kinematic" propagators – and prove their well-definedness. In Section 5.4.2, we discuss a reasonable class of different choices of propagators before we perform a similar analysis to the one from Section 5.3 for all propagators in that class in Section 5.5.

5.4.1 The kinematic propagator

A special choice of a propagator, the "kinematic" propagator $\langle\langle T_0\varphi\chi\rangle\rangle$, is obtained by replacing the measure $d\mu_m(p)$ in the two-point function by $\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i0}$. In the case of the point-localized Klein-Gordon field $\phi(x)$, the kinematic propagator is simply the Feynman propagator,

$$\langle\langle T_0\phi\phi\rangle\rangle(x) = \theta(x^0)W_m(x) + \theta(-x^0)W_m(-x) = D_F(x), \tag{5.52}$$

where θ is the Heaviside step function. The derivation of Eq. (5.52) is a standard task (see for example the textbooks [55, 62]), while the proof of the existence as a distributional product in the sense of Hörmander is less known but clear from the shape of the wavefront set of $W_m(x)$ given by Eq. (5.36) – see also [57]. In the general case of two arbitrary point-localized fields X and X' given by Eq. (2.7), the kinematic propagator is a derivative of $D_F(x)$,

$$\langle\langle T_0XX'\rangle\rangle(x) = {}_mM^{X,X'}(i\partial)D_F(x) \tag{5.53}$$

with the kernel ${}_mM^{X,X'}$ of the two-point function of X and X'. To avoid ambiguities in the definition of the kinematic propagator, we need to fix some conventions:

1. All terms in ${}_mM^{X,X'}(p)$, which are proportional to $p^2 - m^2$ and hence vanish on the mass-shell and yield no contribution to the two-point function, are set to zero.
2. In the massive case, all factors p^2 that might appear in ${}_mM^{X,X'}(p)$ are replaced by factors m^2.

Note that the kinematic propagator Eq. (5.53) should be seen as a so-called offshell propagator in the sense of Brouder-Dütsch-Fredenhagen [8, 27, 28].[5] Consequently, the kinematic propagator does not necessarily respect the equations of motions, which on-shell fields satisfy.

For the string-localized potentials from Eq. (2.1), the integral kernel ${}_mM^{A_f^{(s)},A_{f'}^{(s)}}$ of the two-point function actually only depends on p, $\frac{-ie}{(pe_-)}$ and $\frac{ie'}{(pe')_+}$, so that we can write

$$\begin{aligned}\langle\langle T_0A_f^{(s)}(e)A_{f'}^{(s)}(e')\rangle\rangle(x) &= i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ipx}}{p^2-m^2+i0}\,{}_mM^{A_f^{(s)},A_{f'}^{(s)}}\left(p, \frac{-ie}{(pe)_-}, \frac{ie'}{(pe')_+}\right)\\ &= {}_mM^{A_f^{(s)},A_{f'}^{(s)}}(i\partial, eI_e, e'I_{-e'})D_F(x).\end{aligned} \tag{5.54}$$

[5] Details on offshell propagators are given in Section 5.7.4.

We prove the well-definedness of the kinematic propagator Eq. (5.54).

Theorem 5.23. *The kinematic propagators of the string-localized potentials for arbitrary mass $m \geq 0$ and spin/helicity $s \in \mathbb{N}$ defined by*

$$\langle\!\langle T_0 A_f^{(s)}(e) A_{f'}^{(s)}(e') \rangle\!\rangle(x) := i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i(px)}}{p^2 - m^2 + i0} \, {}_mM^{A_f^{(s)}, A_{f'}^{(s)}}(p, e, e') \tag{5.55}$$

are well-defined distributions on $\mathbb{R}^{1+3} \times H^2$.

Proof. In the massive case, the distribution $[p^2 - m^2 + i0]^{-1}$ can be seen as the pullback of $[t + i0]^{-1} \in \mathcal{S}'(\mathbb{R})$ by the map $f : \mathbb{R}^{1+3} \to \mathbb{R}$ with $f(p) = p^2 - m^2$ with set of normals

$$N_f = \{ (p^2 - m^2; \lambda) \in \mathbb{R}^2 \mid \lambda p = 0 \} \tag{5.56}$$

because $N_f \cap \mathrm{WF}[t + i0]^{-1} = \emptyset$. Then

$$\mathrm{WF}[p^2 - m^2 + i0]^{-1} \subset f^* \, \mathrm{WF}[t + i0]^{-1} = \{ (p; \lambda p) \mid p^2 = m^2, \ \lambda < 0 \}, \tag{5.57}$$

where the inverted sign of λ again comes from the fact that we work over Minkowski space, as explained in Remark 5.7. In the massless case, we can use Lemma 5.6 on the wavefront set of homogeneous distributions and describe $\mathrm{WF}[p^2 + i0]^{-1}$ in terms of the wavefront set of the massless Feynman propagator from Example 5.8, giving

$$\mathrm{WF}[p^2 + i0]^{-1} = \{ (p; \lambda p) \mid p^2 = 0, \ p \neq 0, \ \lambda < 0 \} \cup \dot{T}_0^* \mathbb{R}^{1+3}. \tag{5.58}$$

Comparing $\mathrm{WF}[p^2 + i0]^{-1}$ and $\mathrm{WF}[p^2 - m^2 + i0]^{-1}$ to the wavefront sets of $d\mu_0(p)$ and $d\mu_m(p)$ from the proof of Theorem 5.18 on the well-definedness of string-localized two-point functions, we see that the proof of the well-definedness of the string-localized kinematic propagators works completely analogous to the proof for the two-point functions because the only difference is that now p^0 has arbitrary sign. The sign of p^0, however, was irrelevant for the proof of Theorem 5.18. □

5.4.2 Ambiguities of string-localized propagators

We next investigate the ambiguities of propagators of string-localized potentials in detail. The following considerations are again partially taken from the author's paper [35].

We start by considering the ambiguities in the point-localized case. As outlined in the beginning of the current Section 5.4, adding a term of the form

$$(p^2 - m^2) \, {}_m\tilde{M}^{X,X'}(p) \quad \text{e.g., with} \quad {}_m\tilde{M}^{X,X'}(p) \quad \text{a polynomial} \tag{5.59}$$

to the kernel ${}_mM^{X,X'}(p)$ will have no effect on the two-point function but yield a different propagator. If T denotes a generic time-ordering recipe and T_0 the kinematic time-ordering, then this ambiguity is a linear combination of derivatives of the Dirac delta,

$$\langle\!\langle TXX' \rangle\!\rangle(x) - \langle\!\langle T_0XX' \rangle\!\rangle(x) = \sum_a C_a \partial^a \delta(x) \tag{5.60}$$

with multi-indices a.

Remark 5.24. To work with multi-indices over Minkowski space and to be able to use all rules like the Einstein summation convention, Epstein and Glaser used quadri-indices, superquadri-indices and multi-superquadri-indices in their original work [32]. For simplicity, we stick to the slightly misleading but established term "multi-index", hoping that the reader understands this abuse of notation. For example, if both fields in Eq. (5.60) are Minkowski scalars, then the right-hand side is to be interpreted as

$$c_0\delta(x) + c_{1\kappa}\partial^\kappa\delta(x) + c_{2\kappa\lambda}\partial^\kappa\partial^\lambda\delta(x) + \ldots; \tag{5.61}$$

if both fields are vectors, then the right-hand side should be interpreted as

$$c_{0\mu\nu}\delta(x) + c_{1\mu\nu\kappa}\partial^\kappa\delta(x) + c_{2\mu\nu\kappa\lambda}\partial^\kappa\partial^\lambda\delta(x) + \ldots \tag{5.62}$$

and so on.

The ambiguities on the right-hand side of Eq. (5.60) lie within the freedom of the BEG construction. The freedom can be reduced by the power counting argument described in the beginning of Section 5.1. In the present form, power counting can be formulated as follows: The propagator must not have a worse scaling behavior with respect to $x = 0$ than the two-point function. In the massless case, where the two-point functions and propagators are homogeneous, it is clear what that means (see also Example 5.1). For a generic distribution, one can use the scaling degree, sometimes also called "Steinmann" scaling degree:

Definition 5.25 (see [9, 64])**.** For any test function $\phi \in \mathcal{D}(\mathbb{R}^n)$, we define a family $(\phi^\lambda)_\lambda$ of scaled test functions indexed by a positive real number λ,

$$\phi^\lambda(\cdot) := \lambda^{-n}\phi(\lambda^{-1}\cdot) \tag{5.63}$$

and for a distribution $t \in \mathcal{D}'(\mathbb{R}^n)$, we consider the family of scaled distributions $t_\lambda(\phi) := t(\phi^\lambda)$. t is then said to have scaling degree $\mathrm{sd}\, t = \omega$ with respect to the origin $x = 0$ if

$$\omega = \inf\left\{\omega' \in \mathbb{R} \mid \lim_{\lambda\downarrow 0}\lambda^{\omega'}t_\lambda = 0 \text{ in the sense of distributions}\right\}. \tag{5.64}$$

Remark 5.26. If t is a locally integrable function, then $t_\lambda(\phi) = \int d^nx\; t(\lambda x)\phi(x)$.

The two-point functions of the field strength tensors of spin or helicity s have scaling degree $\omega_F = 2s + 2$ and the Dirac delta has scaling degree $\omega_\delta = 4$ (in four spacetime dimensions), so if the generic fields X and X' in Eq. (5.60) are replaced by the field strengths and the power counting requirement is implemented, we obtain

$$\langle\!\langle TF^{(s)}_\times F^{(s)}_\times\rangle\!\rangle - \langle\!\langle T_0F^{(s)}_\times F^{(s)}_\times\rangle\!\rangle = \sum_{|a|\le 2(s-1)} C_{\times,a}\partial^a\delta(x), \tag{5.65}$$

where $\times$ is a placeholder for the Lorentz indices of the field strengths.

The requirement that the time-ordering for the string-localized potentials arises from the time-ordering of the corresponding field strength by appropriate string integration and contraction is central to our approach. Consequently, the freedom of choosing a propagator for the string-localized potentials arises from Eq. (5.65) as

$$\langle\!\langle T A_\times^{(s)}(e) A_\times^{(s)}(e') \rangle\!\rangle - \langle\!\langle T_0 A_\times^{(s)}(e) A_\times^{(s)}(e') \rangle\!\rangle = \sum_{|a|\le 2(s-1)} C_{\times,a}(e,e') I_e^s I_{-e'}^s \partial^a \delta(x) \tag{5.66}$$

with string integrals over derivatives of the Dirac delta appearing on the right-hand side and where $C_{\times,a}(e,e')$ is the appropriate contraction of $C_{\times,a}$ from Eq. (5.65) with the string variables.

Lemma 5.27. *Let $j,k,l \in \mathbb{N}_0$. The expression*

$$I_e^j I_{-e'}^k \partial^{\mu_1} \dots \partial^{\mu_l} \delta(x) \tag{5.67}$$

is a well-defined distribution over $\mathbb{R}^{1+3} \times H^2$ if $j+k-l<4$.

Proof. $I_e^j I_{-e'}^k \partial^{\mu_1} \dots \partial^{\mu_l} \delta(x)$ is the Fourier back transform with respect to the x-variable of

$$(-1)^{j+l} i^{j+k+l} \frac{p_{\mu_1} \cdots p_{\mu_l}}{(pe)_-^j (pe')_+^k}, \tag{5.68}$$

which exists as a Hörmander product if $p \neq 0$ by Lemma 5.11 and Corollary 5.13, and is locally integrable with respect to $p=0$ if $j+k-l<4$. □

The integrability condition $j+k-l<4$ from Lemma 5.27 introduces a lower bound on the number of derivatives that can appear on the right-hand side of Eq. (5.66); it implies $|a| > 2s-4$.

Thus, the scaling behavior of the field strengths gives an upper bound on $|a|$, while the requirement for local integrability in momentum space gives a lower bound. In the massless case, the two-point functions of field strength and string-localized potential are homogeneous of degree $\varrho = -\omega_F = -(2s+2)$. The requirement that the propagators are homogeneous of the same degree then restricts the freedom to $|a| = 2s-2$ at $m=0$. In summary, we demand

$$\langle\!\langle T A_\times^{(s)} A_\times^{(s)} \rangle\!\rangle - \langle\!\langle T_0 A_\times^{(s)} A_\times^{(s)} \rangle\!\rangle = \sum_{|a|=2s-2} C_{\times,a}(e,e') I_e^s I_{-e'}^s \partial^a \delta \quad \text{at} \quad m=0, \tag{5.69a}$$

$$\langle\!\langle T A_\times^{(s)} A_\times^{(s)} \rangle\!\rangle - \langle\!\langle T_0 A_\times^{(s)} A_\times^{(s)} \rangle\!\rangle = \sum_{|a|=2s-3}^{2s-2} C_{\times,a}(e,e') I_e^s I_{-e'}^s \partial^a \delta \quad \text{at} \quad m>0 \tag{5.69b}$$

for a general time-ordering recipe T ordering string-localized potentials.

Remark 5.28. We want to stress that the requirements $|a| = 2s-2$ for $m=0$ and $2s-3 \le |a| \le 2s-2$ for $m>0$ are stronger than the power counting constraints. The latter would only imply $|a| \le 2s-2$ but the integrability condition – an *infrared effect* – forbids all $|a| \le 2s-4$. At $m=0$, homogeneity gives an even stronger constraint and excludes all $|a| < 2s-2$.

5.5 Renormalization of divergent amplitudes

We now turn to the renormalization of divergent amplitudes in SLFT. Each order in the Dyson series for string-localized field theories Eq. (4.15) can be written as a sum of time-ordered n-point functions times a normal-ordered product of quantum fields. One particular way of constructing the time-ordered n-point functions is by extensions of sums of products of propagators. In this section, we prove that the extension of such products of propagators is not worse than it is in the point-localized case. We start by proving this statement for kinematic propagators in Section 5.5.1 and generalize it in Section 5.5.2 to all propagators described in Section 5.4.2.

Remark 5.29. It is important to define the time-ordering of string-localized fields *before* smearing out the string variables so that one does not need to extend products of distributions across *open* sets. However, to prove the existence of x-products of propagators – only such products appear in the string-localized Dyson series – outside the x-diagonal, it proves useful to first smear out the string variables, similar to what we did for the two-point functions in Section 5.3. By Remark 5.16, proving the existence of partially smeared products is nothing but proving the existence of x-products before smearing out the string variables. Hence there is no conflict between "time-ordering before smearing" and "proving the existence of time-ordered x-products after smearing".

The results in this section are again taken from the author's paper [35]; some paragraphs are verbatim quotes from the paper.

5.5.1 Products of kinematic propagators

To investigate the existence of products of kinematic string-localized propagators after the string variables have been smeared out, recall Lemma 5.15, which says that the $q_{c,\pm}^{\mu_1\cdots\mu_s}(p)$ are smooth when $p \neq 0$ and therefore, they can at most contribute to the wavefront set over $p = 0$. Since the $q_{c,\pm}^{\mu_1\cdots\mu_s}(p)$ are the only singular objects appearing in the smeared kernel

$$\int d^4e \int d^4e'\, c(e)c(e')\, {}_mM^{A_f^{(s)},A_{f'}^{(s)}}(p,e,e') \tag{5.70}$$

of the kinematic propagator of the string-localized potentials, we arrive at the following statement.

Lemma 5.30. *The Fourier transform*

$$\mathcal{F}\left[\langle\!\langle T_0 A_{c,f}^{(s)} A_{c,f'}^{(s)} \rangle\!\rangle(x)\right](p) = i \int d^4e \int d^4e'\, c(e)c(e')\, \frac{{}_mM^{A_f^{(s)},A_{f'}^{(s)}}(p,e,e')}{p^2 - m^2 + i0} \tag{5.71}$$

of the smeared kinematic string-localized propagators for all masses $m \geq 0$ and all spins respectively helicities $s \in \mathbb{N}$ are tempered distributions with

$$\mathrm{WF}\left(\mathcal{F}\left[\langle\!\langle T_0 A_{c,f}^{(s)} A_{c,f'}^{(s)} \rangle\!\rangle(x)\right](p)\right) \subset \mathrm{WF}\, \frac{1}{p^2 - m^2 + i0} \cup \dot{T}_0^*. \tag{5.72}$$

In the massless case, the Fourier transform (5.71) is homogeneous. Therefore, the wavefront set of the massless kinematic propagator in configuration space can be determined easily from Eq. (5.72) by use of Lemma 5.6. We obtain our first result concerning the existence of products of propagators.

Theorem 5.31 (massless case)**.** *At $m = 0$, the wavefront set of the smeared string-localized kinematic propagator*

$$\langle\langle T_0 A^{(s)}_{c,f} A^{(s)}_{c,f'} \rangle\rangle(x) = i \int \frac{d^4p}{(2\pi)^4} e^{-i(px)} \int d^4e \int d^4e'\, c(e)c(e') \frac{{}_mM^{A^{(s)}_f, A^{(s)}_{f'}}(p,e,e')}{p^2 + i0} \tag{5.73}$$

is contained in the wavefront set of the massless point-localized Feynman propagator from Eq. (5.11). In particular, products of massless string-localized kinematic propagators and their product with the propagators of point-localized fields are well-defined on $\mathbb{R}^{1+3} \setminus 0$.

In the massive case, homogeneity is lost and a transition from momentum to configuration space needs more effort. We encountered the same situation when investigating powers of two-point functions in Section 5.3. There, we proved the auxiliary Lemma 5.20, which made it possible to also consider the massive case. We use this lemma again to prove the following statement.

Theorem 5.32 (massive case)**.** *At $m > 0$, the wavefront set of the smeared string-localized kinematic propagator*

$$\langle\langle T_0 A^{(s)}_{c,f} A^{(s)}_{c,f'} \rangle\rangle(x) = i \int \frac{d^4p}{(2\pi)^4} e^{-i(px)} \int d^4e \int d^4e'\, c(e)c(e') \frac{{}_mM^{A^{(s)}_f, A^{(s)}_{f'}}(p,e,e')}{p^2 - m^2 + i0} \tag{5.74}$$

is contained in the wavefront set of the massive point-localized Feynman propagator. In particular, products of massive string-localized kinematic propagators, their product with massless string-localized kinematic propagators and with the propagators of point-localized fields are well-defined on $\mathbb{R}^{1+3} \setminus 0$.

Proof. We define the distributions

$$\hat{u}(p) = \int d^4e \int d^4e'\, c(e)c(e')\, {}_mM^{A^{(s)}_f, A^{(s)}_{f'}}(p,e,e') \tag{5.75}$$

and $\hat{v}(p) = [p^2 - m^2 + i0]^{-1}$ with $m > 0$. $\hat{u}(p)$ is homogeneous of degree 0 in p and arises from contraction of the distributions $q^{\mu_1\cdots\mu_s}_{c,\pm}$ from Lemma 5.15 with a polynomial in p. Local integrability of $\hat{u}(p)$ at $p = 0$ ensures its existence as a tempered distribution and by Lemma 5.15, $\mathrm{WF}\,\hat{u} \subset \dot{T}^*_0$. $\hat{v}$ is smooth at $p = 0$ because $m > 0$ and hence $\hat{u}$ and $\hat{v}$ satisfy the assumptions of Lemma 5.20. Therefore, Theorem 5.32 is a special case of Lemma 5.20 and

$$\mathrm{WF}\,\langle\langle T_0 A^{(s)}_{c,f} A^{(s)}_{c,f'} \rangle\rangle \subset \mathrm{WF}\,v = \mathrm{WF}\left[\int \frac{d^4p}{(2\pi)^4} \frac{e^{-i(px)}}{p^2 - m^2 + i0}\right], \tag{5.76}$$

where v is the massive scalar Feynman propagator, whose wavefront set is the same as for the massless scalar Feynman propagator given by Eq. (5.11) [7]. □

Remark 5.33. Note that Lemma 5.20 is only helpful at $m > 0$ since the wavefront set of the massless kernel $[p^2 + i0]^{-1}$ contains $\dot{T}_0^*$, so that the Hörmander product $\hat{u}\hat{v}$ of the respective $\hat{u}$ and $\hat{v}$ does not exist at $p = 0$. For the same reason, there is no straightforward generalization of Lemma 5.20 to the massless case.

Theorems 5.31 and 5.32 show that renormalization of divergent amplitudes stays an extension problem across the x-diagonal despite the delocalization in SLFT if the *kinematic* propagators from Theorem 5.23 are employed. However, the transition from the two-point functions to the propagators might not be unique as we have argued in Section 5.4.2. In the following section, we prove that Theorems 5.31 and 5.32 are generalizable to all propagators described in Section 5.4.2.

5.5.2 Products of non-kinematic propagators

The propagators described in Section 5.4.2 differ from the kinematic propagators of the string-localized potentials of spin or helicity s by a linear combination of derivatives of a $2s$-fold string-integrated Dirac delta with constraints on the number of appearing derivatives coming from power counting and the requirement that the x-Fourier transform be locally integrable with respect to $p = 0$. In the massless case, homogeneity gives an even stronger restriction on the number of appearing derivatives. These constraints led us to the ambiguities Eq.s (5.69a) and (5.69b). We determine the wavefront sets of the relevant partially smeared string integrated Dirac deltas, which appear in Eq.s (5.69a) and (5.69b) and form the difference between $\langle\!\langle T A^{(s)}_{c,\times} A^{(s)}_{c,\times}\rangle\!\rangle$ and $\langle\!\langle T_0 A^{(s)}_{c,\times} A^{(s)}_{c,\times}\rangle\!\rangle$.

Lemma 5.34. *Let $s \in \mathbb{N}$, $c \in \mathcal{D}(H)$ and let a be a multi-index with $|a| \geq 2s - 3$ (and $|a| \geq 0$ if $s = 1$). Then the partially smeared derivative*

$$\delta^{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}_{a,s,c}(x) := \int d^4e \int d^4e'\, c(e)c(e')\, e^{\mu_1}e'^{\nu_1}\ldots e^{\mu_s}e'^{\nu_s} I^s_e I^s_{-e'}\partial^a\delta(x) \tag{5.77}$$

of the Dirac delta is a tempered distribution over $\mathbb{R}^{1+3}$ with

$$\mathrm{WF}\,\delta^{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}_{a,s,c}(x) \subset \mathrm{WF}\,\delta(x). \tag{5.78}$$

Proof. The expression $\delta^{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}_{a,s,c}(x)$ is the Fourier back transform of the homogeneous product of distributions

$$(-ip)^a q^{\mu_1\cdots\mu_s}_{c,-}(p) q^{\nu_1\cdots\nu_s}_{c,+}(p), \tag{5.79}$$

which exists by Lemma 5.15 and the integrability constraint $|a| \geq 2s - 3$. Hence, $\delta^{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}_{a,s,c}$ is a well defined tempered distribution. Note that $q_{c,\pm}$ is smooth when $p \neq 0$ by Lemma 5.15 so that there are no issues with well-definedness of the expression (5.79). Moreover, the same lemma gives

$$\mathrm{WF}\left[(-ip)^{|a|}_\times q^{\mu_1\cdots\mu_s}_{c,+}(p) q^{\kappa_1\cdots\kappa_s}_{c,-}(p)\right] \subset \dot{T}_0^* \tag{5.80}$$

so that homogeneity implies

$$\mathrm{WF}\left[\delta_{a,s,c}^{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(x)\right] = \dot{T}_0^* = \mathrm{WF}\,\delta(x) \tag{5.81}$$

by Lemma 5.6, where the equality comes from the fact that the Fourier transform (5.79) is supported everywhere. □

Since the wavefront set of the sum of two distributions is contained in the union of their wavefront sets, adding a linear combination of derivatives of smeared string-deltas subject to Eq.s (5.69a) or (5.69b), respectively, to any distribution does not affect the wavefront set over $\mathbb{R}^{1+3} \setminus 0$ by Lemma 5.34. Moreover, because the kinematic propagators of the field strengths are derivatives of a fundamental solution of the wave equation, $\dot{T}_0^*$ is already contained in their wavefront set and hence also in the wavefront set of the kinematic propagators of the string-localized potentials since the wavefront set of a string-integrated smeared Dirac delta is given by Eq. (5.78). For time-ordering recipes T that are subject to Eq.s (5.69a) and (5.69b), we have thus shown that the renormalization problem is the same as for T_0. This statement is formalized in the following theorem.

Theorem 5.35. *Let T denote a time-ordering recipe that is subject to Eq. (5.69a) if $m = 0$ and Eq. (5.69b) if $m > 0$. Then*

$$\mathrm{WF}\langle\!\langle T A_{c,\times}^{(s)} A_{c,\times}^{(s)} \rangle\!\rangle \subset \mathrm{WF}\langle\!\langle T_0 A_{c,\times}^{(s)} A_{c,\times}^{(s)} \rangle\!\rangle \tag{5.82}$$

and consequently, products of $\langle\!\langle T A_{c,\times}^{(s)} A_{c,\times}^{(s)} \rangle\!\rangle$ as well as their products with propagators of point-localized fields exist on $\mathbb{R}^{1+3} \setminus 0$.

We have thus shown that renormalization of divergent amplitudes in SLFT stays a pure ultraviolet problem not only if one uses the kinematic propagators discussed in Section 5.4.1 but also for a large class of propagators satisfying Eq.s (5.69a) or (5.69b), respectively. That is, we have generalized Theorems 5.31 and 5.32 to all propagators that

1. arise from one of the field strength propagators displayed in Eq. (5.65) by appropriate string-integration, and
2. are subject to the constraints of power counting, integrability in momentum space and, at $m = 0$, homogeneity of the same degree as the two-point function.

Note that only the lower bounds on $|a|$ are needed in the proof of Theorem 5.35, but not the constraints coming from power counting. The latter are only an additional requirement in order to reduce the (finite) renormalization freedom.

Remark 5.36. In all our considerations in Sections 5.5.1 and 5.5.2, we have only considered pure string-localized propagators $\langle\!\langle T A_{c,\times}^{(s)} A_{c,\times}^{(s)} \rangle\!\rangle$, whereas also mixed propagators like $\langle\!\langle T F_{\times}^{(s)} A_{c,\times}^{(s)} \rangle\!\rangle$ are non-vanishing in SLFT. However, it should be obvious that such propagators are subject to similar statements as Theorems 5.31, 5.32 and 5.35.

5.6 Other choices of string variables

In this thesis, we always work with spacelike string variables e living in the open subset $H = \{ e^2 < 0 \} \subset \mathbb{R}^{1+3}$. In the literature, one also finds other choices: lightlike string variables [39], normalized spacelike string variables with Minkowski square $e^2 = -1$ [37,49,50,52,53] or purely spacelike string variables $e = (0, \vec{e})$ with $\vec{e}^2 = 1$ [48,51], all of which correspond to restrictions of the string variables to *closed* subsets (or more precisely, closed submanifolds). Such restrictions are much more subtle than the restriction to H used in this thesis. We briefly examine the described options in the following Sections 5.6.1 to 5.6.3, which are taken from the author's paper [35].

5.6.1 Lightlike strings

Lightlike string directions have been employed in [39] when dealing with massive string-localized potentials, where they promise a computational advantage. The authors of [39] were able to set equal all string variables appearing in the Dyson series for the scattering operator describing the weak interaction by exploiting that the problematic denominator $([(pe) + i0][(pe) - i0])^{-1}$ in $E_{\mu\kappa}(p, e, e')|_{e=e'}$ from Eq. (2.13) drops out when $e^2 = 0$. This simplification of $E_{\mu\kappa}$ yields an essential reduction of the complexity of tree-graph calculations. Similarly, one can check that also the problematic terms in the kernel for $s = 2$ given by Eq. (2.16) drop out, resulting in an even bigger computational simplification than for $s = 1$.[6]

However, the authors of [39] restricted their considerations to tree graph contributions, where no products (or convolution products in momentum space) of several $E_{\mu\kappa}(p, e, e')|_{e=e'}$ appear. It is very likely that this changes when treating loop graph contributions and therefore, the divergent denominators will pop up again in loop amplitudes, resulting in complex renormalization schemes and spoiling the computational advantage that was achieved at tree level.

One can also think of SLFT with lightlike strings where not all string variables are set equal. However, an analysis similar to the one presented in this thesis cannot be performed in that case. This is due to the fact that the restriction to the closed set of lightlike string directions causes trouble. Without loss of generality, we can investigate the restriction to lightlike string variables with zero-component equal to 1, which is given by the pullback of the respective inclusion map [41, Corollary 8.2.7.], provided that this pullback exists. Thus, consider the map

$$\begin{aligned} &\iota : \mathbb{R}^{1+3} \times (0, 2\pi) \times (0, \pi) \to (\mathbb{R}^{1+3})^2, (p, \varphi, \vartheta) \mapsto (p, e), \\ &\text{where} \quad e = (1, \sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)^t, \end{aligned} \tag{5.83}$$

so that the desired restriction is the pullback $\iota^* U_\pm$ with $U_\pm(p, e) = (pe)_\pm^{-1} \in \mathcal{D}'(\mathbb{R}^{1+3} \times \mathbb{R}^{1+3})$ as in Lemma 5.11.

[6] It is a conjecture of the author that the problematic denominators drop out for any helicity but whether that turns out to be true or not is of no interest for our current considerations.

Remark 5.37. The submanifold of elements (p, e), where e is lightlike and has 0-component equal to one is $\mathbb{R}^{1+3} \times \mathbb{S}^2$. To avoid confusion with coordinate-related singularities, one needs several charts. The map ι corresponds to only a single chart but is enough to demonstrate the issues that come with lightlike strings.

Having a look at Lemma 5.11 and using

$$ {}^t\iota' \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \sin\vartheta(\eta_1 \sin\varphi - \eta_2 \cos\varphi) \\ \eta_3 \sin\vartheta - \cos\vartheta(\eta_1 \cos\varphi + \eta_2 \sin\varphi) \end{pmatrix} \tag{5.84} $$

for $(\xi, \eta) \in (\mathbb{R}^{1+3})^2$, one can easily verify that $\iota^* U_\pm$ is well-defined but that the wavefront set of the pullback contains elements $(p, \varphi, \vartheta; \lambda e, 0, 0)$ when p becomes proportional to e, the latter being defined as in Eq. (5.83). Note that the singular-support-criterion $(pe) = 0$ is met when p is proportional to e only if e is lightlike (or $p = 0$).

Hence, there is no immediate analogue of Theorems 5.18 and 5.19, Corollary 5.21 and Theorems 5.23, 5.31 and 5.32 for the case of lightlike strings and in particular, analyses as performed in Section 5.5, which led to a simple renormalization description, are not feasible for lightlike strings because lightlike strings produce additional singularities also when $0 \neq p = \lambda e$. This problem is worse in the massless case than in the massive case, for $p^2 + i0$ is singular when $p = \lambda e$, but $p^2 - m^2 + i0$ with $m > 0$ is not.

In conclusion, spacelike strings seem preferable over lightlike strings from analytic and heuristic viewpoints. Nevertheless, lightlike strings cannot be fully excluded at the present time.

5.6.2 Closed subsets of spacelike strings

In Remark 5.14, we claimed that the restriction to the closed submanifold H_{-1} of normalized spacelike string directions with Minkowski square -1 is harmless. In principle, this restriction can cause similar issues as the restriction to the lightlike string directions, but a brief analysis shows that it is indeed much better behaved than the latter. Similar to the case of lightlike strings, we consider an inclusion map

$$ \begin{aligned} &\tilde\iota : \mathbb{R}^{1+3} \times \mathbb{R} \times (0, 2\pi) \times (0, \pi) \to (\mathbb{R}^{1+3})^2, \\ &(p, \tau, \varphi, \vartheta) \mapsto (p, e), \text{ where } e = \begin{pmatrix} \sinh\tau \\ \cosh\tau \sin\vartheta \cos\varphi \\ \cosh\tau \sin\vartheta \sin\varphi \\ \cosh\tau \cos\vartheta \end{pmatrix}, \end{aligned} \tag{5.85} $$

which is again only a single chart but a generalization to cover the full submanifold is straightforward. For $(\xi, \eta) \in (\mathbb{R}^{1+3})^2$, we have

$$ {}^t\tilde\iota' \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta^0 \cosh\tau - \sinh\tau \, [\eta_1 \sin\vartheta \cos\varphi + \eta_2 \sin\vartheta \sin\varphi + \eta_3 \cos\vartheta] \\ \cosh\tau \sin\vartheta(\eta_1 \sin\varphi - \eta_2 \cos\varphi) \\ \cosh\tau \, [\eta_3 \sin\vartheta - \cos\vartheta(\eta_1 \cos\varphi + \eta_2 \sin\varphi)] \end{pmatrix} \tag{5.86} $$

and thus, the pullback $\tilde{\iota}^* U_\pm$ is well-defined by Lemmas 5.4 and 5.11. In contrast to the case of lightlike string variables, the wavefront set of the pullback does not contain elements $(p, \tau, \varphi, \vartheta; \lambda e, 0, 0, 0)$, provided that $p \neq 0$. This can be seen by inserting $\eta = \lambda p$, $\lambda \neq 0$, into Eq. (5.86) and noting that the pullback is only singular when

$$(pe) = p^0 \sinh\tau - \cosh\tau\, [p_1 \sin\vartheta \cos\varphi + p_2 \sin\vartheta \sin\varphi + p_3 \cos\vartheta] = 0. \qquad (5.87)$$

Consequently, the results in Sections 5.4 and 5.5 remain valid also if one restricts to H_{-1}. We nevertheless chose to consider the restriction to the open set H in the main part of the thesis because it is much simpler and also exhibits the practical advantage that one can easily derive with respect to the string variables.

Remark 5.38. A qualitative and simpler argument that the restriction to H_{-1} is unproblematic is the homogeneity in the string variables of all string-localized propagators of degree $\omega = 0$: When one interprets H as $H_{-1} \times \mathbb{R}_{\geq 0}$, the "radial" part is constant and can simply be integrated out with the radial part of the test function.

5.6.3 Purely spacelike strings

Another case appearing in the literature [48, 51] is the case of purely spacelike string variables $e = (0, \vec{e})$, for example with $|\vec{e}| = 1$. It is motivated by the fact that the inner product $-(ee')$ becomes positive definite, which is not the case in H or H_{-1}. Because the set $\{\, e \in \mathbb{R}^{1+3} \mid e^0 = 0,\ |\vec{e}| = 1 \,\}$ is just the two-sphere $\mathbb{S}^2$, this case can be investigated by adjusting the inclusion map Eq. (5.83) from the lightlike case by setting the zero-component of e to 0 instead of 1. Then the only – but very important – difference in the wavefront set analysis is the criterion for the singular support, which becomes

$$\vec{p} \cdot \vec{e} = 0 \quad \text{instead of} \quad \vec{p} \cdot \vec{e} = p^0 \quad \text{in the lightlike case.} \qquad (5.88)$$

The wavefront set of the restriction of $U_\pm(p, e)$ to purely spacelike strings can then only contain elements $(p, \varphi, \vartheta; \lambda e, 0, 0)$ if $\vec{p} = 0$ but not when $p = \kappa e$ for some $\kappa \neq 0$, in contrast to the lightlike case. Also in contrast to the lightlike case, the proofs of well-definedness from Theorems 5.18 and 5.23 can easily be transferred to the case of purely spacelike string variables with norm $|\vec{e}| = 1$.

However, the appearance of the critical elements $(p, \varphi, \vartheta; \lambda e, 0, 0)$ in the wavefront set of $U_\pm(p, e)|_{e^0=0, |\vec{e}|=1}$ can happen for arbitrary p^0 and hence, the wavefront sets of the string-localized propagators are non-trivially affected by that restriction, so that Theorems 5.19, 5.31 and 5.32 as well as Corollary 5.21 cannot be directly transferred to a restriction to purely spacelike strings.

We want to emphasize that in the mentioned applications [48, 51], which are the non-perturbative constructions that we have outlined in Section 3.2, a time-ordering of the string-localized expression is not required because there, the string-localized part is perturbed with a point-localized Lagrangian.

Remark 5.39. In our derivation of the configuration space representation of string-localized two-point functions in Section 3.3.2, in particular in Theorem 3.17, we encountered a singular contribution $f(e_1, e_2) = \frac{\gamma}{\sin\gamma}$ for purely spacelike string variables $e_i^0 = 0$ with $|\vec{e}_i| = 1$, where γ is the angle between $\vec{e}_1$ and $\vec{e}_2$. For the singularity at $e_1 = e_2$ of this term to be integrable, we must not further restrict the string variables. If we nevertheless did, not only the wavefront set estimates, but also the proofs of well-definedness from Sections 5.3, 5.4 and 5.5 would not work. In this sense, the results from Section 3.3.2 also show how much smearing in the string variables is needed in order to obtain well-defined string-localized two-point functions.

5.7 Methods to reduce ambiguities and their interplay

A particular part of fixing the BEG renormalization freedom is the removal of the ambiguities of propagators. In our considerations in Section 5.4.2, we have classified the ambiguities in the choice of a propagator for field strength tensors and their string-localized potentials, constraining them by two requirements:

1. Scaling conditions like power counting or homogeneity of a fixed degree, and for the potentials also integrability conditions.
2. The BEG renormalization freedom of the string-localized potentials arises from the freedom of the field strength by appropriate string integration (and contraction).

The scaling requirements reduce an infinite number of free parameters to a finite number, the second requirement connects the freedom for $A_{\mu_1\ldots\mu_s}(x, e)$ to the freedom of $F_{[\mu_1\nu_1]\ldots[\mu_s\nu_s]}(x)$. Still, free parameters remain and need to be fixed. In the literature [2, 8, 25, 27, 54, 60], one finds many advanced methods to fix the BEG freedom, which are partially related to each other and might have a non-trivial interplay.

In the case of vector or tensor fields, one basic requirement is that the propagators should inherit certain symmetry properties of the fields and that other algebraic conditions should be satisfied. We start our considerations on the reduction of the BEG freedom with the implementation of such conditions in Section 5.7.1 .

In reference [60], perturbative gauge invariance gives strong constraints on the free parameters. The gauge concept is absent in SLFT but as a substitute, we have the string independence principle, whose effect on the free constants we investigate in Section 5.7.2. The methods from the other references [2, 8, 25, 27, 54] are described in Sections 5.7.3 and 5.7.4, where we also give examples of applications to SLFT. The methods turn out to be partially incompatible and the implications of that observation are discussed in Section 5.7.5.

Remark 5.40. The methods that we describe in Sections 5.7.3 and 5.7.4 are formulated for the renormalization of general amplitudes in the original works that deal with point-localized QFT [8, 27, 28, 54, 72] However, to display their application to string-localized fields and to discuss the issues coming from the partial incompatibility, it is enough to consider their application to the BEG ambiguities of propagators. Therefore, we restrict our considerations to propagators but one may try to generalize our derivations to arbitrary amplitudes.

5.7.1 Algebraic conditions

The simplest way to constrain the free parameters in propagators is the implementation of algebraic conditions. Such conditions (along with the string independence principle discussed in the next section) might be regarded as the most natural type of constraints on the free parameters of propagators. A prime example is that propagators should inherit symmetry properties of the involved fields. For example, the propagator of the Maxwell field strength $F_{\mu\nu}(x)$ is homogeneous of degree $\omega = -4$ and thus has an ambiguity

$$\langle\!\langle T F_{\mu\nu} F_{\kappa\lambda} \rangle\!\rangle(x) = \langle\!\langle T_0 F_{\mu\nu} F_{\kappa\lambda} \rangle\!\rangle(x) + c_{\mu\nu\kappa\lambda}\delta(x). \tag{5.89}$$

By Lorentz covariance, the tensor-valued constant $c_{\mu\nu\kappa\lambda}$ can be expressed by a combination of three scalar constants times factors of the Minkowski metric,

$$c_{\mu\nu\kappa\lambda} = c_1\eta_{\mu\nu}\eta_{\kappa\lambda} + c_2\eta_{\mu\kappa}\eta_{\nu\lambda} + c_3\eta_{\mu\lambda}\eta_{\nu\kappa}. \tag{5.90}$$

The requirement that the propagator should inherit the symmetry properties of the field strength then means that we demand $c_{\mu\nu\kappa\lambda} = -c_{\nu\mu\kappa\lambda} = -c_{\mu\nu\lambda\kappa} = c_{\nu\mu\lambda\kappa}$.[7] This condition yields a reduction from a space of three real parameters to a single one, because it implies $c_1 = 0$ and $c := c_2 = -c_3$, so that we are left with

$$\langle\!\langle T F_{\mu\nu} F_{\kappa\lambda} \rangle\!\rangle(x) = \langle\!\langle T_0 F_{\mu\nu} F_{\kappa\lambda} \rangle\!\rangle(x) + c(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\kappa})\delta(x). \tag{5.91}$$

Because time-ordered products are required to be symmetric [32], one should also impose a symmetry condition under exchange of $(\mu, \nu) \leftrightarrow (\kappa, \lambda)$ but this condition is already implemented in Eq. (5.91). The requirements are similar for higher spins respectively helicities, with the additional condition of symmetry under exchange of any pair of neighboring indices $[\mu_i\nu_i] \leftrightarrow [\mu_j\nu_j]$ in $F_{[\mu_1\nu_1]\dots[\mu_s\nu_s]}$.

Remark 5.41. The requirement that the propagator of the Maxwell field strength has the correct symmetry properties implies that the propagator of the corresponding string-localized potential is axial. This can easily verified by inserting

$$\begin{aligned} \langle\!\langle T A_\mu(e) F_{\kappa\lambda} \rangle\!\rangle(x) &= I_e \langle\!\langle T F_{\mu\nu} F_{\kappa\lambda} \rangle\!\rangle(x) e^\nu \\ \text{and} \quad \langle\!\langle T A_\mu(e) A_\kappa(e') \rangle\!\rangle(x) &= I_e I_{-e'} \langle\!\langle T F_{\mu\nu} F_{\kappa\lambda} \rangle\!\rangle(x) e^\nu e'^\lambda, \end{aligned} \tag{5.92}$$

into Eq. (5.91).

The requirement that the propagator must respect the symmetries of the involved fields might seem trivial at first. However, there are other algebraic constraints that are less trivial. We discuss such constraints at the example of the requirement that the propagator of the linearized Riemann tensor $F_{[\mu\nu][\kappa\lambda]}$, i.e., the graviton field strength, should have vanishing Ricci trace because

$$\eta^{\mu_1\mu_2} F_{[\mu_1\nu_1][\mu_2\nu_2]} = 0. \tag{5.93}$$

[7] The kinematic propagator trivially inherits these properties.

Recall that the kinematic propagator of the massless $s = 2$ field strength is given by

$$\langle\!\langle T_0 F_{[\mu_1\nu_1][\mu_2\nu_2]} F_{[\kappa_1\lambda_1][\kappa_2\lambda_2]} \rangle\!\rangle(x) = M^{FF}_{\mu_1\nu_1\mu_2\nu_2\kappa_1\lambda_1\kappa_2\lambda_2}(i\partial) D_F(x), \tag{5.94}$$

where the polynomial M^{FF} can be obtained from the kernel Eq. (3.5) of the massless $s = 2$ Krein potential by letting the curl operator $J^{\gamma}_{\alpha\beta}(\partial) := \delta^{\gamma}_{\alpha}\partial_\beta - \delta^{\gamma}_{\beta}\partial_\alpha$ act on each index of the latter,

$$M^{FF}_{\mu_1\nu_1\mu_2\nu_2\kappa_1\lambda_1\kappa_2\lambda_2}(i\partial) = J^{\varrho_1}_{\mu_1\nu_1}(i\partial) J^{\varrho_2}_{\mu_2\nu_2}(i\partial) J^{\sigma_1}_{\kappa_1\lambda_1}(i\partial) J^{\sigma_2}_{\kappa_2\lambda_2}(i\partial) M^{h^{\mathrm{K}}h^{\mathrm{K}}}_{\varrho_1\varrho_2\sigma_1\sigma_2}, \tag{5.95}$$

giving M^{FF} as a sum of 48 terms. In order to not have to deal with all these terms explicitly, it proves useful to introduce a shorthand notation. This will also help to write the allowed ambiguities of the propagator in a concise way. Moreover, it is easier to work in momentum space, so we replace all $i\partial$ in Eq. (5.95) by p. The field strength $F_{[\mu_1\nu_1][\mu_2\nu_2]}$ is skewsymmetric under the exchange $(\mu_i \leftrightarrow \nu_i)$, $i = 1, 2$, and symmetric under the exchange $((\mu_1, \nu_1) \leftrightarrow (\mu_2, \nu_2))$. Without losing any information, we can thus contract the μ_i with the same dummy vector f and the ν_i with the same dummy vector g, giving

$$F_{f,g} := f^{\mu_1} f^{\mu_2} g^{\nu_1} g^{\nu_2} F_{[\mu_1\nu_1][\mu_2\nu_2]}. \tag{5.96}$$

With this notation, Eq. (5.95) becomes

$$M^{F_{f,g}F_{f',g'}}(p) = (p[fg][f'g']p)^2 - \frac{1}{2}(p[fg][fg]p)\,(p[f'g'][f'g']p), \tag{5.97}$$

where we have introduced the shorthand notation $[xy]$ for the skewsymmetric tensor built from the vectors x and y,

$${}_{\mu}[xy]_{\nu} := x_\mu y_\nu - x_\nu y_\mu, \quad \text{or} \quad [xy] = xy^t - yx^t \tag{5.98}$$

so that for example

$$\begin{aligned}(p[fg][f'g']p) = (pf)(gf')(g'p) - (pg)(ff')(g'p) \\ - (pf)(gg')(f'p) + (pg)(fg')(f'p).\end{aligned} \tag{5.99}$$

The momentum space kernel Eq. (5.97) of the kinematic propagator only depends on the skewsymmetric tensors $[fg]$ and $[f'g']$ and is symmetric under the exchange $[fg] \leftrightarrow [f'g']$. It thus inherits the basic symmetry properties that we require the propagator of the field strength to satisfy and all admissible corrections to this propagator must possess the same symmetry properties.

We now turn to the Ricci trace, which can in our shorthand notation be computed by applying $\Box_f$ to Eq. (5.97) because

$$\Box_f f^{\mu_1} f^{\mu_2} g^{\nu_1} g^{\nu_1} F_{[\mu_1\nu_1][\mu_2\nu_2]} = 2\eta^{\mu_1\mu_2} g^{\nu_1} g^{\nu_1} F_{[\mu_1\nu_1][\mu_2\nu_2]} \tag{5.100}$$

Let us first show that the kinematic propagator does *not* satisfy this tracelessness condition.

Lemma 5.42. *The kinematic propagator of the (massless) graviton field strength tensor has non-vanishing Ricci trace,*

$$\eta^{\mu_1\mu_2}\langle\!\langle T_0 F_{[\mu_1\nu_1][\mu_2\nu_2]}F_{[\kappa_1\lambda_1][\kappa_2\lambda_2]}\rangle\!\rangle(x) = C_{\alpha\beta\nu_1\nu_2\kappa_1\lambda_1\kappa_2\lambda_2}\partial^\alpha\partial^\beta\delta(x) \neq 0 \tag{5.101}$$

for some non-vanishing tensor of constants $C_{\alpha\beta\nu_1\nu_2\kappa_1\lambda_1\kappa_2\lambda_2}$.

Proof. We compute $\Box_f M^{F_{f,g}F_{f',g'}}(p)$,

$$\begin{aligned}
&\Box_f M^{F_{f,g}F_{f',g'}}(p)\\
&= \partial_{f^\mu}\{2\,[p^\mu(g[f'g']p) - (pg)(pg')f'^\mu + (pg)(pf')g'^\mu]\ (p[fg][f'g']p)\\
&\quad - \left[p^\mu(g[fg]p) - (pg)^2 f^\mu + (pg)(pf)g^\mu\right]\ (p[f'g'][f'g']p)\}\\
&= 2\,[p^\mu(g[f'g']p) - (pg)(pg')f'^\mu + (pg)(pf')g'^\mu]\\
&\qquad \times \left[p_\mu(g[f'g']p) - (pg)(pg')f'_\mu + (pg)(pf')g'_\mu\right]\\
&\quad + \left[p^2g^2 + 2(pg)^2\right]\ (p[f'g'][f'g']p)\\
&= p^2\left[g^2(p[f'g'][f'g']p) + 2(g[f'g']p)^2\right]\\
&\quad + 2(pg)^2\left[(p[f'g'][f'g']p) + (pg')^2f'^2 + (pf')^2g'^2 - 2(pf')(pg')(f'g')\right]\\
&= p^2\left[g^2(p[f'g'][f'g']p) + 2(g[f'g']p)^2\right].
\end{aligned}$$

The overall prefactor p^2 in $\Box_f M^{F_{f,g}F_{f',g'}}(p)$ ensures that the two-point function is traceless but gives rise to a Dirac delta term for the kinematic propagator. Besides the p^2, there remain two more factors of p^μ, implying that the trace of the kinematic propagator is a second derivative of a Dirac delta, as claimed in the Lemma. □

In the following, we make the propagator traceless. A generic propagator of the graviton field strength, which has the same degree of homogeneity $\omega = -6$ as the two-point function, can differ from the kinetic propagator by a linear combination of second order derivatives of the Dirac delta,

$$\begin{aligned}
\langle\!\langle TF_{[\mu_1\nu_1][\mu_2\nu_2]}F_{[\kappa_1\lambda_1][\kappa_2\lambda_2]}\rangle\!\rangle = {}& \langle\!\langle T_0F_{[\mu_1\nu_1][\mu_2\nu_2]}F_{[\kappa_1\lambda_1][\kappa_2\lambda_2]}\rangle\!\rangle\\
&+ c_{\alpha\beta\mu_1\nu_1\mu_2\nu_2\kappa_1\lambda_1\kappa_2\lambda_2}\partial^\alpha\partial^\beta\delta
\end{aligned} \tag{5.102}$$

Lemma 5.43. *Requiring that the propagator of the field strength respects the symmetries of the fields, the ambiguity on the right-hand side of Eq. (5.102) can be expressed in terms of six scalar parameters.*

Proof. We again employ the previously used shorthand notation and work in momentum space. The kinematic propagator respects the desired symmetries, so the allowed correction terms (or ambiguities) must respect them as well. That is, they can be expressed solely in terms of p and the skewsymmetric tensors $[fg]$ and $[f'g']$, and are symmetric under the exchange $[fg] \leftrightarrow [f'g']$. Moreover, they must contain at least one factor of p^2 and be homogeneous of degree 4 in p. Naively, we find seven linearly independent admissible correction terms, which can be chosen as

$$p^2 N_1 := p^2 \, (p[fg][fg][f'g'][f'g']p), \tag{5.103a}$$

$$p^2 N_2 := p^2 \, (p[fg][f'g'][fg][f'g']p), \tag{5.103b}$$

$$p^2 N_3 := p^2 \, \{(p[fg][f'g'][f'g'][fg]p) + (p[f'g'][fg][fg][f'g']p)\}, \tag{5.103c}$$

$$p^2 N_4 := p^2 \, \{(p[fg][fg]p) \, \mathrm{Tr}([f'g'][f'g']) + (p[f'g'][f'g']p) \, \mathrm{Tr}([fg][fg])\} \tag{5.103d}$$

and

$$(p^2)^2 K_1 := (p^2)^2 \, \mathrm{Tr}([f'g'][f'g'][fg][fg]), \tag{5.104a}$$

$$(p^2)^2 K_2 := (p^2)^2 \, \mathrm{Tr}([f'g'][fg][f'g'][fg]), \tag{5.104b}$$

$$(p^2)^2 K_3 := (p^2)^2 \, \mathrm{Tr}([fg][fg]) \, \mathrm{Tr}([f'g'][f'g']). \tag{5.104c}$$

However, there is another, non-trivial, linear dependence coming from the fact that the vectors p, f, g, f' and g' are necessarily linearly dependent because they are five vectors in a four-dimensional space. This implies

$$\det{}_{pff'gg'} = \det \begin{pmatrix} p^2 & (pf) & (pf') & (pg) & (pg') \\ (pf) & f^2 & (ff') & (fg) & (fg') \\ (pf') & (ff') & f'^2 & (f'g) & (f'g') \\ (pg) & (fg) & (f'g) & g^2 & (gg') \\ (pg') & (fg') & (f'g') & (gg') & g'^2 \end{pmatrix} = 0, \tag{5.105}$$

i.e., the vanishing of the Gram determinant yields a relation between the scalar products of the five vectors. Working out all scalar products, we find

$$\det{}_{pff'gg'} = 2N_1 - 2N_2 + N_3 - \frac{1}{2} N_4 + p^2 \left[-K_1 + \frac{1}{2} K_2 + \frac{1}{4} K_3 \right], \tag{5.106}$$

so that we can eliminate for example N_4 in favor of the other terms and are thus left with a six-parameter space. □

By Lemma 5.43, the kernel of a generic propagator of the massless $s = 2$ field strength is given by

$$\mathcal{P}^{F_{f,g} F_{f',g'}}(p) := M^{F_{f,g} F_{f',g'}}(p) + p^2 \sum_{i=1}^{3} a_i N_i + (p^2)^2 \sum_{i=1}^{3} b_i K_i \tag{5.107}$$

with six free parameters a_i and b_i, $i = 1, 2, 3$. We investigate the implications of the tracelessness requirement.

Proposition 5.44. *The condition of vanishing trace, $\square_f \mathcal{P}^{F_{f,g}F_{f',g'}}(p) = 0$, reduces the six-parameter-space* $\mathrm{span}(p^2 N_i, (p^2)^2 K_i)$ *of ambiguities in Eq. (5.107) to a one-parameter-space. Correspondingly, the kernel of any traceless propagator of the massless $s = 2$ field strength, which respects the symmetries of the fields, must be of the form*

$$\mathcal{P}^{F_{f,g}F_{f',g'}}_{\text{traceless}}(p) := M^{F_{f,g}F_{f',g'}}(p) + p^2 \left[\frac{1}{2}N_1 - N_2\right] + b_3 (p^2)^2 \left[-6K_1 + 6K_2 + K_3\right]. \tag{5.108}$$

Proof. We have to find linear combinations

$$\square_f \left\{ p^2 \sum_{i=1}^{3} a_i N_i + (p^2)^2 \sum_{i=1}^{3} b_i K_i \right\} \tag{5.109}$$

that compensate the non-vanishing trace

$$\frac{1}{2}\square_f M^{F_{f,g}F_{f',g'}}(p) = p^2 \left[\frac{1}{2} g^2 (p[f'g'][f'g']p) + (g[f'g']p)^2\right] \tag{5.110}$$

of the kinematic propagator. To find all possible such combinations, we compute the trace of each term,

$$\begin{aligned}
\frac{1}{2}\square_f N_1 &= -2(pg)(g[f'g'][f'g']p) - g^2 (p[f'g'][f'g']p),\\
\frac{1}{2}\square_f N_2 &= (g[f'g']p)^2 - (pg)(g[f'g'][f'g']p),\\
\frac{1}{2}\square_f N_3 &= 2(pg)(g[f'g'][f'g']p) - (pg)^2 \operatorname{Tr}([f'g'][f'g'])\\
&\quad - p^2 (g[f'g'][f'g']g) + 2(g[f'g']p)^2 - g^2 (p[f'g'][f'g']p),\\
\frac{1}{2}\square_f K_1 &= -g^2 \operatorname{Tr}([f'g'][f'g']) - 2(g[f'g'][f'g']g),\\
\frac{1}{2}\square_f K_2 &= -2(g[f'g'][f'g']g),\\
\frac{1}{2}\square_f K_3 &= -6g^2 \operatorname{Tr}([f'g'][f'g']).
\end{aligned} \tag{5.111}$$

Comparing the terms in Eq. (5.110) with the respective ones in Eq. (5.111), the tracelessness condition translates to the requirement

$$\begin{aligned}
a_1 + a_3 = \frac{1}{2} &\quad \text{from} \quad g^2 (p[f'g'][f'g']p),\\
-2a_1 - a_2 + 2a_3 = 0 &\quad \text{from} \quad (pg)(g[f'g'][f'g']p),\\
a_2 + 2a_3 = -1 &\quad \text{from} \quad (g[f'g']p)^2,\\
a_3 = 0 &\quad \text{from} \quad (pg)^2 \operatorname{Tr}([f'g'][f'g']),\\
a_3 + 2b_1 + 2b_2 = 0 &\quad \text{from} \quad (g[f'g'][f'g']g),\\
b_1 + 6b_3 = 0 &\quad \text{from} \quad g^2 \operatorname{Tr}([f'g'][f'g']).
\end{aligned} \tag{5.112}$$

Eq. (5.112) yields the constants from the proposition. In particular, the only remaining unspecified parameter is b_3. □

We have shown that the requirement that the propagator of a quantum field inherits certain algebraic properties of the underlying fields, for example (skew)symmetry under exchange of Lorentz indices or vanishing Ricci trace, drastically reduces the ambiguities in the choice of a propagator. In the subsequent sections, we investigate more involved methods to remove the ambiguities.

5.7.2 The effect of the string independence principle

We discuss the effect of the string independence principle on the freedom of choosing a propagator at the example of a coupling of a string-localized photon potential to a point-localized conserved current $j^\mu(x)$,

$$S_1(x,e) = :L(x,e): = g(x) :A_\mu(x,e) j^\mu(x): \tag{5.113}$$

with a coupling "constant" $g \in \mathcal{S}(\mathbb{R}^{1+3})$. The propagator first appears at second order of perturbation theory, at which the tree graph contribution to the Dyson series can be written as

$$\begin{aligned} &T[S_1(x,e)S_1(x',e')]|_{\text{tree}} \\ &= g(x)g(x') \Bigg\{ \langle\langle T A_\mu(x,e) A_\nu(x',e')\rangle\rangle :j^\mu(x) j^\nu(x'): \qquad (5.114\text{a}) \\ &\quad + \sum_{\phi,\chi} \langle\langle T\phi(x)\chi(x')\rangle\rangle :A_\mu(x,e)A_\nu(x',e') \frac{\partial j^\mu(x)}{\partial\phi(x)} \frac{\partial j^\nu(x')}{\partial\chi(x')}: \Bigg\}. \qquad (5.114\text{b}) \end{aligned}$$

For this to be string independent, the string variations of each of the lines (5.114a) and (5.114b) with respect to one of the string variables must form a divergence separately.[8] Let us thus compute the string variation of the first line (5.114a) with respect to e. Employing that the propagator of the string-localized potential arises by appropriate string integration and contraction of the field strength's propagator, which we in turn require to possess the correct symmetry properties as displayed in Section 5.7.1, the generic string-localized propagator appearing there has one free real parameter c,

$$\langle\langle T A_\mu(x,e) A_\nu(x',e')\rangle\rangle = \langle\langle T_0 A_\mu(x,e) A_\nu(x',e')\rangle\rangle + c(\eta_{\mu\nu}(ee') - e'_\mu e_\nu) I_e I_{-e'} \delta(x-x'). \tag{5.115}$$

The kinematic part of the propagator (5.115) commutes with the string variation d_e and with partial derivatives if no field equations are involved. Employing the auxiliary field

[8] A symmetrization in the string variables is not necessary since this corresponds to an exchange $(x \leftrightarrow x')$ under which $T[S_1(x,e)S_1(x',e')]|_{\text{tree}}$ is symmetric.

$w(x,e) := w_\mu(x,e)de^\mu \equiv I_e A_\mu(x,e)de^\mu$ from Eq. (2.30), we thus obtain

$$\begin{aligned} &d_e \langle\langle T A_\mu(x,e) A_\nu(x',e') \rangle\rangle \\ &= \partial_\mu \langle\langle T_0 w(x,e) A_\nu(x',e') \rangle\rangle \\ &\quad + c \left[(\eta_{\mu\nu}(ee') - e'_\mu e_\nu)(\partial_\kappa de^\kappa) + (\eta_{\mu\nu}(e'de) - e'_\mu de_\nu) \right] I_e I_{-e'} \delta(x-x'), \end{aligned} \tag{5.116}$$

yielding

$$d_e\ (5.114a) = g(x)g(x')\partial_\mu \langle\langle T_0 w(x,e) A_\nu(x',e') \rangle\rangle :j^\mu(x) j^\nu(x'): \tag{5.117a}$$

$$\begin{aligned} &+ c g(x) g(x') :j^\mu(x) j^\nu(x'): \\ &\times \left[(\eta_{\mu\nu}(ee') - e'_\mu e_\nu)(\partial_\kappa de^\kappa) + (\eta_{\mu\nu}(e'de) - e'_\mu de_\nu) \right] I_e I_{-e'} \delta(x-x'). \end{aligned} \tag{5.117b}$$

The kinetic part (5.117a) becomes a total divergence in the adiabatic limit and thus satisfies the string independence requirement Eq. (4.20), while the non-kinetic part (5.117b) is an obstruction to string independence. Thus, the string independence principle fixes $c = 0$, resolving the ambiguity of the photon propagator completely.

Similar derivations can be done in generic models but the result may be model dependent. For example, we shall derive an analogue statement for string-localized massless Yang-Mills theory in Section 6.1, where kinematic propagators must be employed as well. In the present case of the photon propagator, that is, for helicity $s = 1$, the only algebraic condition on the propagator is that it has the correct symmetry properties associated with the field strengths, resulting in the axiality requirement for the string-localized propagator as discussed in the previous section.

For $s = 2$, we have derived in Section 5.7.1 that the kinematic propagator is not traceless. Explicitly implementing the "vanishing trace" condition in the propagator thus means that we cannot employ the kinematic propagators. However, if the $s = 2$ potential is coupled to a traceless quantity, the trace component of the propagator does not contribute. In such a case, the kinematic propagator can still be used as an ansatz. One example is the coupling of a massless string-localized $s = 2$ potential to the Maxwell stress energy tensor, which we discuss in Section 6.2. For generic models, however, it remains a task of future research to find out whether a traceless choice of a propagator is consistent with the string independence principle.

5.7.3 NST renormalization of massless amplitudes

In this section, we describe a subtle method to remove ambiguities from propagators of massless fields, which is based on representation theory of the Lorentz group. It is formulated in terms of symmetric tensor representations of the Lorentz group on harmonic and homogeneous polynomials – see for example [3], where however the case of symmetric tensor representations of $SO(n)$ instead of $SO(1, n-1)$ is discussed. We thus emphasize that this method is naively only applicable to massless amplitudes because massive amplitudes are not homogeneous. One may, however, attempt to generalize it to massive amplitudes using the scaling mass expansion [24], as outlined by Várilly and Gracia-Bondía [72].

The original description of the method presented in the following is due to Nikolov, Stora and Todorov (NST) and applies to both tree and loop graphs [54]. We will briefly sketch their method, which is formulated in the point-localized case, before turning to applications in SLFT. Such applications to SLFT were first proposed by Várilly and Gracia-Bondía [72], who mention a transition of the NST concept to SLFT but do not work out their proposal or give any details. After introducing the method, we apply it to string-localized QED and to the propagator of the massless $s = 2$ field strength, restricting our considerations to tree level. We also compare the results to the implications of the string independence principle derived in Section 5.7.2.

The NST notion of renormalization of divergent amplitudes [54] is based on irreducible representations of the Lorentz group on harmonic (in the Minkowski sense) and homogeneous polynomials. We introduce their method in a different guise. While Nikolov, Stora and Todorov talk about the transition from general unrenormalized to renormalized amplitudes, we focus on the ambiguities of propagators and therefore consider the transition from two-point functions to propagators. Furthermore, we reformulate their ideas in terms of the Casimir operator C of the Lorentz group, which is, in the notation of Section 5.7.1, given by

$$C := (x_\mu \partial_\nu - x_\nu \partial_\mu)^2 = 2x^2\Box - 4(x\partial) - 2(x\partial)^2 \tag{5.118}$$

and hence only consists of powers of the Euler operator $(x\partial)$ and $x^2\Box$. Thus, if $H_l(x)$ is a Minkowski-harmonic polynomial, i.e., $\Box H_l(x) = 0$, which is homogeneous of degree l, then

$$CH_l(x) = -2l(l+2)H_l(x). \tag{5.119}$$

One can easily compute that $C\,(x^2)^k = 0$ for all $k \in \mathbb{N}$, which just reflects the Lorentz invariance of x^2. Consider now the two-point function of a massless scalar Klein-Gordon field

$$W_0(x) = -\frac{1}{(2\pi)^2} \lim_{\varepsilon \downarrow 0} \frac{1}{(x - i\varepsilon u)^2} =: -\frac{1}{(2\pi)^2} \frac{1}{x_-^2} \tag{5.120}$$

with a forward timelike vector u of which the distributional limit $\varepsilon \downarrow 0$ is independent.[9] Recall Eq. (5.36), which tells us that the wavefront set of $W_0(x)$ is directed and which implies that Hörmander powers $[W_0(x)]^k$ exist with

$$\mathrm{WF}\, W_0^k \subset \mathrm{WF}\, W_0 \cup \{\, (0; p) \mid p^2 \geq 0,\ p^0 < 0 \,\}. \tag{5.121}$$

Since the Hörmander product satisfies the Leibniz rule, one easily finds that the Casimir operator C acts trivially on powers of W_0,

$$C \frac{1}{(x_-^2)^k} = 0 \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.122}$$

[9] We replaced $u = (1, 0, 0, 0)^t$ from Eq. (3.43) by a generic forward timelike u to display the Lorentz invariance of W_0 more explicitly.

We next derive the two-point function of the Maxwell field strength by taking derivatives of W_0. We have

$$\partial_\mu \partial_\kappa W_0(x) = \frac{2}{(2\pi)^2} \frac{\eta_{\mu\kappa} x^2 - 4 x_\mu x_\kappa}{(x_-^2)^3}, \tag{5.123}$$

and thus, acting with the kernel ${}_0M^{FF}_{\mu\nu\kappa\lambda}(i\partial)$ of the Maxwell field strength from Eq. (2.12) on the scalar two-point function W_0 yields the two point function of the Maxwell field strength

$${}_0M^{FF}_{\mu\nu\kappa\lambda}(i\partial) W_0(x) = -\frac{1}{(2\pi)^2} \frac{H_{\mu\nu\kappa\lambda}(x)}{(x_-^2)^3}, \tag{5.124}$$

with the harmonic and homogeneous polynomial

$$H_{\mu\nu\kappa\lambda}(x) = -4 \left[(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\nu\kappa}\eta_{\mu\lambda}) x^2 - 2(\eta_{\nu\lambda} x_\mu x_\kappa - \eta_{\mu\lambda} x_\nu x_\kappa - \eta_{\nu\kappa} x_\mu x_\lambda + \eta_{\mu\kappa} x_\nu x_\lambda) \right]. \tag{5.125}$$

Then, having Eq. (5.119) in mind, we obtain

$$C\, {}_0M^{FF}_{\mu\nu\kappa\lambda}(i\partial) W_0(x) = -16\, {}_0M^{FF}_{\mu\nu\kappa\lambda}(i\partial) W_0(x), \tag{5.126}$$

where we have used that the terms with one derivative of C acting on $H_{\mu\nu\kappa\lambda}$ and the other on $(x_-^2)^{-3}$ cancel each other,

$$4 \left[x^2 \partial_\alpha H_{\mu\nu\kappa\lambda} \partial^\alpha (x_-^2)^{-3} - (x\partial) H_{\mu\nu\kappa\lambda} (x\partial)(x_-^2)^{-3} \right] = 0. \tag{5.127}$$

Thus, the two-point function of the Maxwell field strength is an eigenvalue of the Casimir operator and hence transforms under an irreducible representation of the Lorentz group, which is usually labeled as $(1, 1)$ representation [74]. The situation is similar for higher helicities.

However, when dealing with propagators, complications arise. The wavefront set of the massless Feynman propagator contains the whole cotangent space over $x = 0$ and hence its Hörmander powers are ill-defined over $x = 0$, which means that the above computations for the two-point functions can only be transferred to the propagator away from the origin,

$$\left({}_0M^{FF}_{\mu\nu\kappa\lambda}(i\partial) D_F(x) \right)\Big|_{x\neq 0} = -\frac{i}{(2\pi)^2} \left(\frac{H_{\mu\nu\kappa\lambda}(x)}{(x^2 - i0)^3} \right)\Big|_{x\neq 0}. \tag{5.128}$$

The NST renormalization condition [54] then is the requirement that the extension of the propagator across the origin should transform under the same irreducible representation of the Lorentz group, which can be rephrased to the requirement that the propagator should transform under the same irreducible representation of the Lorentz group as the two-point function.

Remark 5.45. The authors of [72] transfer their off-origin computation to the extended propagator and claim that the kinematic propagator is transforming under the correct representation. However, such a computation overlooks the delta-contributions at $x = 0$. Working in momentum space, we correct that mistake, and our result is in agreement with the original paper [54, last equation of Section 5 therein], where no derivation of the result is given.

To circumvent the issues at $x = 0$, it proves useful to work in momentum space. There, the propagators are given by a single denominator $p^2 + i0$ times a polynomial in p, so no issues with ill-defined powers $[p^2 - i0]^{-k}$ arise. This transition can easily be performed, for the Casimir operator of the Lorentz group has the same form in momentum and configuration space. If $\tilde{H}^l(p)$ is a homogeneous polynomial of degree $l \in \mathbb{N}$, then

$$[2p^2\Box_p - 4(p\partial_p) - 2(p\partial_p)^2]\frac{\tilde{H}^l(p)}{p^2 + i0} = -2l(l+2)\frac{\tilde{H}^l(p)}{p^2 + i0} + 2p^2\frac{\Box_p\tilde{H}^l(p)}{p^2 + i0} \tag{5.129}$$

because all terms proportional to $\delta(p)$ drop out since $p_\mu\delta(p) = 0$. Thus, in order for the propagator to transform under an irreducible representation of the Lorentz group, its momentum space kernel must be Minkowski harmonic. This is obviously not true for the kinematic propagator. We explicitly derive the NST propagator for the Maxwell field strength in momentum space.

Proposition 5.46. *There is a unique propagator of the Maxwell field strength transforming under the* $(1, 1)$ *representation of the Lorentz group, whose kernel is given by*

$$\begin{aligned} P^{FF}_{\frac{1}{2},\mu\nu\kappa\lambda} &= -\frac{H_{\mu\nu\kappa\lambda}(p)}{8} \\ &= -\left(\eta_{\nu\lambda}p_\mu p_\kappa - \eta_{\mu\lambda}p_\nu p_\kappa - \eta_{\nu\kappa}p_\mu p_\lambda + \eta_{\mu\kappa}p_\nu p_\lambda\right) + \frac{1}{2}p^2(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\kappa}). \end{aligned} \tag{5.130}$$

Proof. By Eq. (5.91), a generic kernel of the propagator of the Maxwell field strength, which respects the latter's symmetries, must be of the form

$$P^{FF}_{c,\mu\kappa\nu\lambda} = -\left(\eta_{\nu\lambda}p_\mu p_\kappa - \eta_{\mu\lambda}p_\nu p_\kappa - \eta_{\nu\kappa}p_\mu p_\lambda + \eta_{\mu\kappa}p_\nu p_\lambda\right) + cp^2(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\kappa}). \tag{5.131}$$

For the propagator to transform under the $(1, 1)$ representation of the Lorentz group, $P^{FF}_{c,\mu\kappa\nu\lambda}$ must be Minkowski harmonic, that is,

$$0 \stackrel{!}{=} \Box_p P^{FF}_{c,\mu\nu\kappa\lambda} = -4(1 - 2c)(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\kappa}), \tag{5.132}$$

implying $c = \frac{1}{2}$ and giving

$$P^{FF}_{\frac{1}{2},\mu\nu\kappa\lambda} = -\frac{H_{\mu\nu\kappa\lambda}(p)}{8}, \tag{5.133}$$

which proves the claim. □

Reformulating the statement of Proposition 5.46 to configuration space, we find that the propagator of the Maxwell field strength, which transforms under the correct irreducible representation of the Lorentz group, is given by

$$\frac{1}{8}H_{\mu\nu\kappa\lambda}(\partial)D_F(x), \tag{5.134}$$

which is precisely the one that Nikolov-Stora-Todorov write down [54]. Várilly and Gracia-Bondía [72], however, have missed a Dirac delta contribution.

We proved in Section 5.7.2 that the string independence principle at second order and tree level of string-localized QED implies that the kinematic propagator of the string-localized photon potential must be employed. However, Proposition 5.46 says that the propagator transforming under the $(1,1)$ representation of the Lorentz group differs from the kinematic one by a string-integrated Dirac delta. Thus:

Corollary 5.47. *The propagator from Proposition 5.46, which transforms under the* $(1,1)$ *representation of the Lorentz group, is in conflict with the string independence principle.*

We discuss the issues coming from Corollary 5.47 in more detail in Section 5.7.5. First, we perform a similar analysis to the one from Proposition 5.46 for the propagator of the massless $s=2$ field strength.

Proposition 5.48. *The requirement that the propagator of the massless field strength of helicity* $s=2$ *transforms under the (irreducible)* $(2,2)$ *representation of the Lorentz group fixes the free parameter* b_3 *from Eq. (5.108) to* $b_3=\frac{1}{48}$. *Thus, there is a unique propagator of the massless* $s=2$ *field strength, which respects the latter's symmetries, is traceless and transforms under the* $(2,2)$*-representation of the Lorentz group, and whose kernel is given by*

$$\mathcal{P}^{F_{f,g}F_{f',g'}}_{\text{traceless},(2,2)}(p) := M^{F_{f,g}F_{f',g'}}(p) + p^2\left[\frac{1}{2}N_1 - N_2\right] + \frac{(p^2)^2}{48}\left[-6K_1+6K_2+K_3\right], \tag{5.135}$$

with the N_i *and* K_i *as in Eq.s (5.103) and (5.104), respectively.*

Proof. The kernel of a traceless propagator respecting the symmetries of the massless $s=2$ field strength must be of the form Eq. (5.108). In order to transform under the $(2,2)$ representation of the Lorentz group, the kernel must also be Minkowski harmonic in p. Thus, we need to check if there is a choice of the constant b_3 such that

$$\square_p\mathcal{P}^{F_{f,g}F_{f',g'}}_{\text{traceless}} = \square_p\left\{M^{F_{f,g}F_{f',g'}} + p^2\left[\frac{1}{2}N_1 - N_2\right] + b_3(p^2)^2\left[-6K_1+6K_2+K_3\right]\right\} \overset{!}{=} 0.$$

The K_i, given by Eq. (5.104), are independent of p and therefore

$$b_3\square_p(p^2)^2\left[-6K_1+6K_2+K_3\right] = 24b_3p^2\left[-6K_1+6K_2+K_3\right]. \tag{5.136}$$

The N_i, given by Eq. (5.103), are homogeneous in p of degree 2, and thus

$$\Box_p p^2 N_i = 16 N_i + p^2 \Box_p N_i, \tag{5.137}$$

with

$$\begin{aligned} \Box_p N_1 &= 2\,\mathrm{Tr}([fg][fg][f'g'][f'g']) = 2K_1, \\ \Box_p N_2 &= 2\,\mathrm{Tr}([fg][f'g'][fg][f'g']) = 2K_2. \end{aligned} \tag{5.138}$$

It is easy to verify that

$$(p[fg][f'g']p)\,\mathrm{Tr}([fg][f'g']) = 2N_2, \tag{5.139}$$

and with that identity in mind, we compute

$$\begin{aligned} \Box_p M^{F_{f,g}F_{f',g'}} = {} & 4(p[fg][f'g']p)\,\mathrm{Tr}([fg][f'g']) + 4(p[fg][f'g'][fg][f'g']p) \\ & + 2(p[fg][f'g'][f'g'][fg]p) + 2(p[f'g'][fg][fg][f'g']p) \\ & - (p[f'g'][f'g']p)\ \mathrm{Tr}([fg][fg]) - (p[fg][fg]p)\ \mathrm{Tr}([f'g'][f'g']) \\ & - 4(p[fg][fg][f'g'][f'g']p) \\ = {} & 12N_2 + 2N_3 - N_4 - 4N_1. \end{aligned} \tag{5.140}$$

N_3 and N_4 do not appear in the traceless propagator but fortunately, these terms appear only as combination $2(N_3 - \frac{1}{2}N_4)$ in Eq. (5.140), which is the same combination that appears in the vanishing Gram determinant Eq. (5.106). Replacing N_3 and N_4 in Eq. (5.140) in favor of the other terms by said Gram identity yields

$$\Box_p M^{F_{f,g}F_{f',g'}} = 16N_2 - 8N_1 - p^2\left(-2K_1 + K_2 + \frac{1}{2}K_3\right), \tag{5.141}$$

so that

$$\Box_p \mathcal{P}^{F_{f,g}F_{f',g'}}_{\text{traceless}} = \frac{1-48b_3}{2} p^2 (6K_1 - 6K_2 - K_3) \stackrel{!}{=} 0, \tag{5.142}$$

implying $b_3 = \frac{1}{48}$. □

The kernel from Proposition 5.48 can be used to write down the configuration space representation of the propagator of the massless $s = 2$ field strength, which is traceless, has the correct symmetry properties and transforms under the $(2, 2)$ representation of the Lorentz group. Namely, we have

$$\langle\!\langle T_{\text{traceless},(2,2)} F_{f,g} F_{f',g'} \rangle\!\rangle(x) = \mathcal{P}^{F_{f,g}F_{f',g'}}_{\text{traceless},(2,2)}(i\partial) D_F(x). \tag{5.143}$$

It is as of yet unclear whether the $s = 2$ propagator (5.143) exhibits a similar incompatibility with the string independence principle as the Maxwell propagator. An investigation of this question remains for future research.

5.7.4 The BDF construction

Yet another approach to reduce the BEG renormalization freedom is due to Brouder, Dütsch and Fredenhagen (BDF) [8,25,27,28]. It is based on defining onshell time-ordered products in terms of offshell fields. In the usual physics literature, fields are considered as onshell, i.e., they satisfy an equation of motion. Moreover, one typically employs kinematic time-ordering without caring much about the BEG ambiguities. In this standard case, however, multilinearity of the time-ordering operation and the onshell property of the field are incompatible.

For example, consider a free real scalar Klein-Gordon field ϕ of mass m, which satisfies $(\square + m^2)\phi = 0$. Then the kinematic propagator commutes with derivatives,

$$\langle\!\langle T_0 \partial^\mu \partial^\nu \phi(x)\phi(y)\rangle\!\rangle = \partial_x^\mu \partial_x^\nu \langle\!\langle T_0 \phi(x)\phi(y)\rangle\!\rangle \tag{5.144}$$

Contracting Eq. (5.144) with $\eta_{\mu\nu}$, using the Klein-Gordon equation and assuming multilinearity of T_0, we run into trouble, for we obtain

$$-m^2\langle\!\langle T_0\phi(x)\phi(y)\rangle\!\rangle = -m^2\langle\!\langle T_0\phi(x)\phi(y)\rangle\!\rangle - i\delta(x-y) \quad \Leftrightarrow \quad \delta(x-y) = 0, \tag{5.145}$$

which is absurd. To resolve issues of this type, Dütsch and Fredenhagen introduced a construction of time-ordered products in terms of offshell fields, which are not subject to any differential equation (of motion) [27,28] and this work was continued later by Brouder and Dütsch [8]. In his book [25], Dütsch presents the construction in a smooth and complete way. Let us sketch the BDF construction in the point-localized case.

One starts with an algebra $\mathcal{A}_{\text{off}}$ generated by offshell fields and their derivatives, which are not subject to any equation of motion, and defines a two-sided ideal $\mathcal{I}$ in $\mathcal{A}_{\text{off}}$, which is generated by the equation of motion. The quotient algebra $\mathcal{A}_{\text{on}} = \frac{\mathcal{A}_{\text{off}}}{\mathcal{I}}$ is then the algebra of "local onshell field polynomials", and the canonical surjection

$$\pi : \mathcal{A}_{\text{off}} \to \mathcal{A}_{\text{on}}, \ A \mapsto \pi A := A + \mathcal{I} = [A] \tag{5.146}$$

is an algebra homomorphism [25]. Derivatives in $\mathcal{A}_{\text{on}}$ are defined in terms of derivatives in $\mathcal{A}_{\text{off}}$. For $\pi A \in \mathcal{A}_{\text{on}}$, choose any $B \in \mathcal{A}_{\text{off}}$ such that $\pi A = \pi B$ and define

$$\partial^\mu(\pi A) := \pi(\partial^\mu B) \tag{5.147}$$

To define onshell time-ordered products in terms of offshell fields, BDF then define an algebra homomorphism

$$\xi : \mathcal{A}_{\text{on}} \to \mathcal{A}_{\text{off}} \tag{5.148}$$

picking a representative of $[A]$, implying that $\pi \circ \xi = \text{id}$. BDF then subject ξ to a set of axioms so that the onshell time-ordered product satisfies certain desirable properties. Beyond the homomorphism property and $\pi \circ \xi = \text{id}$, BDF propose the following axioms for the real scalar Klein-Gordon field ϕ [25]:

- Lorentz transformations commute with $\xi\pi$ $(=: \xi \circ \pi)$.

- If $\mathcal{A}_1 \subset \mathcal{A}_{\text{off}}$ is the subspace of $\mathcal{A}_{\text{off}}$ spanned by the linear field ϕ and its partial derivatives, then $\xi\pi(\mathcal{A}_1) \subset \mathcal{A}_1$.
- $\xi\pi$ does not increase the mass dimension of the fields.

These axioms must be adjusted to models containing other fields. For example, having the homogeneity of massless two-point functions in mind, one should require that the mass dimension *stays the same* after application of $\xi\pi$ to (polynomials of) massless fields. The homomorphism property of ξ implies that it is enough to determine the map $\xi\pi$ on all partial derivatives of linear fields in order to know everything we need to know about the map ξ. Moreover, $\pi\xi = \text{id}$ imply that $(\xi\pi)^2 = (\xi\pi)$, i.e., $\xi\pi$ is a projection and finally, we know that $\xi\pi(A) - A \in \mathcal{I}$ by definition. We thus see that the axioms give strong constraints on the form of ξ (or of $\xi\pi$).

With π and ξ at hand, BDF define the onshell time-ordered product of offshell fields by

$$T_{\text{on}}[\pi(A_1)(x_1)\dots\pi(A_n)(x_n)] := T_{\text{off}}[\xi\pi(A_1)(x_1)\dots\xi\pi(A_n)(x_n)], \tag{5.149}$$

where T_{off} commutes with derivatives and there is no conflict of the type Eq. (5.145) because the $\xi\pi(A_i) \in \mathcal{A}_{\text{off}}$ are offshell fields. Before turning to more complicated fields and a possible transition to SLFT, we shed more light on the BDF approach by displaying how the axioms constrain $\xi\pi$ for low order derivatives of the real scalar Klein-Gordon field (see also [25] for this example and [8] for further examples in point-localized QFT).

For the real scalar Klein-Gordon field ϕ, the ideal $\mathcal{I}$ is generated by the Klein-Gordon equation $(\Box + m^2)\phi = 0$. The properties $\xi\pi(\mathcal{A}_1) \subset \mathcal{A}_1$ and $\xi\pi(\phi) - \phi \in \mathcal{I}$ imply that

$$\xi\pi(\phi) = \phi + \sum_{n=1}^{\infty} c_{0,n}(\Box + m^2)^n\phi, \tag{5.150}$$

and since $\xi\pi$ must not increase the mass dimension, we obtain $c_{0,n} = 0$ for all $n \in \mathbb{N}$ and thus

$$\xi\pi(\phi) = \phi. \tag{5.151}$$

The same reasoning yields $\xi\pi(\partial^\mu\phi) = \partial^\mu\phi$. For the second derivative, the situation is more interesting. Already including the Lorentz covariance axiom and that the mass dimension must not increase, we have

$$\xi\pi(\partial^\mu\partial^\nu\phi) = \partial^\mu\partial^\nu\phi + c\eta^{\mu\nu}(\Box + m^2)\phi \tag{5.152}$$

with a free parameter c. By definition of π and of derivatives on $\mathcal{A}_{\text{on}}$ given by Eq. (5.147), we have

$$(\Box + m^2)\pi\phi = \pi\big((\Box + m^2)\phi\big) = 0, \tag{5.153}$$

and therefore, contracting Eq. (5.152) with $\eta_{\mu\nu}$ yields

$$-m^2\xi\pi\phi = -m^2\phi = \Box\phi + 4c\Box\phi + 4cm^2\phi, \tag{5.154}$$

from which we can read off that $c = -\frac{1}{4}$. For the real scalar field, T_{off} is nothing but the kinematic time-ordering T_0 (the propagator is unique for the field itself and the off shell time-ordered product commutes with derivatives by definition). Let us summarize what we have found:

$$\begin{aligned}
\langle\!\langle T_{\text{on}}\pi(\phi(x))\pi(\phi(y))\rangle\!\rangle &= \langle\!\langle T_0\phi(x)\phi(y)\rangle\!\rangle \\
\langle\!\langle T_{\text{on}}\pi(\partial^\mu\phi(x))\pi(\phi(y))\rangle\!\rangle &= \partial_x^\mu\langle\!\langle T_0\phi(x)\phi(y)\rangle\!\rangle \\
\langle\!\langle T_{\text{on}}\pi(\partial^\mu\partial^\nu\phi(x))\pi(\phi(y))\rangle\!\rangle &= \partial_x^\mu\partial_x^\nu\langle\!\langle T_0\phi(x)\phi(y)\rangle\!\rangle - \frac{1}{4}\eta^{\mu\nu}(\square + m^2)\langle\!\langle T_0\phi(x)\phi(y)\rangle\!\rangle \\
&= \partial_x^\mu\partial_x^\nu\langle\!\langle T_0\phi(x)\phi(y)\rangle\!\rangle + \frac{i}{4}\eta^{\mu\nu}\delta(x-y)
\end{aligned} \tag{5.155}$$

and similar for the other propagators including derivatives up to second order of ϕ.

For general fields, there might arise additional ambiguities coming from the possible non-uniqueness of T_{off}. For example, we have seen in Section 5.7.1 that the kinematic time-ordering does not respect the vanishing Ricci trace of the massless $s = 2$ field strength, which is a linear relation between field components, and that there is a one-parameter space of traceless propagators. To the best knowledge of the author, such issues have not yet been addressed in the BDF formalism. One solution might be to include the tracelessness condition (or more general: algebraic conditions between field components) into the ideal $\mathcal{I}$ but more research is due in that direction.

A first approach to adjust the BDF framework to string-localized field theory has been studied by K. Shedid Attifa's [63], who worked on the topic for certain cases of massive string-localized fields while at the same time, the author of this thesis was doing computations for massless string-localized potentials. At the time, the author was frequently discussing the matter with K. Shedid Attifa. While the latter's approach includes the string integration operator I_e in the construction of the map $\xi\pi$ and also relates string-localized and point-localized potentials via the BDF procedure, the approach that we present in the following is solely formulated in the SLFT framework and includes only derivatives in the construction of the map $\xi\pi$. We exemplify the BDF method in SLFT at the example of the string-localized photon potential $A_\mu(x, e)$ (of mass $m = 0$), where the ideal $\mathcal{I}$ is generated by the equations $\partial^\mu A_\mu = 0$ and $\square A = 0$. Let us write down the axioms to which we subject the BDF framework, in particular the map ξ, in that case.

(a) ξ is an algebra homomorphism.

(b) $\pi\xi = \text{id}$, and hence $\xi\pi$ is a projection.

(c) Lorentz transformations commute with $\xi\pi$.

(d) If $\mathcal{A}_1 \subset \mathcal{A}_{\text{off}}$ is the subspace of $\mathcal{A}_{\text{off}}$ spanned by the linear field ϕ and its partial derivatives, then $\xi\pi(\mathcal{A}_1) \subset \mathcal{A}_1$.

(e) Since A_μ is massless, $\xi\pi$ does not change the mass dimension.

(f) The string-localized potential and the Maxwell field strength are not considered as independent fields. We rather define $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and require that $\xi\pi(F_{\mu\nu})$ and $\xi\pi(\partial^{\alpha_1} \dots \partial^{\alpha_n} F_{\mu\nu})$ are independent of e for all $n \in \mathbb{N}$.

(g) $\xi\pi(A_\mu)$ and all partial derivatives $\xi\pi(\partial^{\alpha_1} \dots \partial^{\alpha_n} A_\mu)$ are homogeneous of degree 0 in the string variable e.

Note that the axioms (a)-(e) already imply that $\xi\pi(A_\mu) = A_\mu$. The offshell time-ordered product of the string-localized potential $A_\mu(x, e)$ is ambiguous but our findings in Section 5.7.2 imply that the string independence principle is only consistent with kinematic time-ordering. We axiomatize this as well.

(h) The offshell time-ordering T_{off} is given by the kinematic time-ordering T_0, as dictated by the string independence principle.

The axiality of the string-localized potential is an algebraic condition, which we impose already on the offshell field. That is $e^\mu A_\mu = 0$ as a relation in $\mathcal{A}_{\text{off}}$ and hence, the homomorphism property of ξ implies that

$$e^\mu \xi\pi(\partial^{\alpha_1} \dots \partial^{\alpha_n} A_\mu) = 0. \tag{5.156}$$

In contrast to the case of the real point-localized scalar field that we discussed earlier, the first non-trivial adjustments arising from the action of $\xi\pi$ appear already for $\partial_\kappa A_\mu$ because the divergence (∂A) is a generator for the ideal. We make the general ansatz

$$\xi\pi(\partial_\kappa A_\mu) = \partial_\kappa A_\mu + cB_{\kappa\mu}(e)\,(\partial A). \tag{5.157}$$

with a constant c and a tensor

$$B_{\mu\kappa}(e) := \eta_{\mu\kappa} - \frac{e_\mu e_\kappa}{e^2}, \tag{5.158}$$

whose shape is determined by the axioms (c) and (g) as well as the axiality condition Eq. (5.156). Contracting Eq. (5.157) with $\eta^{\mu\kappa}$ yields

$$0 \overset{!}{=} (\partial A)[1 + 3c] \quad \Rightarrow \quad c = -\frac{1}{3} \quad \text{and} \quad \xi\pi(\partial_\kappa A_\mu) = \partial_\kappa A_\mu - \frac{1}{3} B_{\kappa\mu}(e)\,(\partial A). \tag{5.159}$$

Due to the symmetry of $B_{\mu\kappa}$, we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \xi\pi(\partial_\mu A_\nu - \partial_\nu A_\mu) = \xi\pi(F_{\mu\nu}) \tag{5.160}$$

and since $T_{\text{off}} = T_0$ by axiom (h), this is independent of e and point-localized. Moreover, the axiality of A_μ and of $B_{\mu\kappa}$ with respect to both indices together with Eq.s (5.159) and (5.160) imply that

$$(e\partial)A_\mu = \xi\pi((e\partial)A_\mu) = \xi\pi((e\partial)A_\mu - \partial_\mu(eA)) = -F_{\mu\nu}e^\nu, \tag{5.161}$$

so that $A_\mu = I_e F_{\mu\nu} e^\nu$ in $\mathcal{A}_{\rm off}$. Let us now turn to the second derivative. Without changing the mass dimension, respecting Lorentz covariance, the axiality of A_μ, homogeneity in e of degree 0 and symmetry under exchange of the Lorentz indices of the derivatives, we have the general ansatz

$$\begin{aligned}\xi\pi(\partial_\kappa\partial_\lambda A_\mu) = \partial_\kappa\partial_\lambda A_\mu + \Big[& c_1(B_{\mu\kappa}\partial_\lambda + B_{\mu\lambda}\partial_\kappa) + c_2\eta_{\kappa\lambda}D_\mu \\ & + c_3\frac{e_\kappa e_\lambda}{e^2}D_\mu + c_4(B_{\mu\kappa}e_\lambda + B_{\mu\lambda}e_\kappa)\frac{(e\partial)}{e^2}\Big](\partial A) \\ & + \Big[d_1\eta_{\kappa\lambda}\eta_{\varrho\mu} + d_2\frac{e_\kappa e_\lambda}{e^2}\eta_{\varrho\mu} + d_3(B_{\mu\kappa}\eta_{\varrho\lambda} + B_{\mu\lambda}\eta_{\varrho\kappa})\Big]\Box A^\varrho,\end{aligned} \tag{5.162}$$

where we have introduced the “axial derivative” $D_\mu := \partial_\mu - \frac{e_\mu(e\partial)}{e^2}$. We first implement the condition of vanishing divergence,

$$\begin{aligned}0 &\overset{!}{=} \eta^{\mu\lambda}\xi\pi(\partial_\kappa\partial_\lambda A_\mu) \\ &= \partial_\kappa(\partial A)\,[1 + 4c_1 + c_2] + \frac{e_\kappa(e\partial)}{e^2}\,[-c_1 - c_2 + 3c_4] + \Box A_\kappa\,[d_1 + 4d_3]\,.\end{aligned} \tag{5.163}$$

Second, we implement the wave equation

$$0 \overset{!}{=} \eta^{\kappa\lambda}\xi\pi(\partial_\kappa\partial_\lambda A_\mu) = \Box A_\mu\,[1 + 4d_1 + d_2 + 2d_3] + D_\mu(\partial A)\,[2c_1 + 4c_2 + c_3]\,. \tag{5.164}$$

Finally, we implement the axiom (f), which states that $\xi\pi(\partial_\kappa\partial_\lambda A_\mu) - \xi\pi(\partial_\kappa\partial_\mu A_\lambda) = \xi\pi(\partial_\kappa F_{\lambda\mu})$ must be independent of e. We have

$$\begin{aligned}&\xi\pi(\partial_\kappa\partial_\lambda A_\mu) - \xi\pi(\partial_\kappa\partial_\mu A_\lambda) \\ &= \partial_\kappa F_{\lambda\mu} + \Big[c_1(B_{\mu\kappa}\partial_\lambda - B_{\kappa\lambda}\partial_\mu) + c_2(\eta_{\kappa\lambda}D_\mu - \eta_{\kappa\mu}D_\lambda) \\ &\qquad + c_3\frac{e_\kappa}{e^2}(e_\lambda D_\mu - e_\mu D_\lambda) + c_4(B_{\mu\kappa}e_\lambda - B_{\kappa\lambda}e_\mu)\frac{(e\partial)}{e^2}\Big](\partial A) \\ &\quad + \Big[d_1(\eta_{\kappa\lambda}\eta_{\varrho\mu} - \eta_{\kappa\mu}\eta_{\varrho\lambda}) + d_2\frac{e_\kappa}{e^2}(e_\lambda\eta_{\varrho\mu} - e_\mu\eta_{\varrho\lambda}) + d_3(B_{\mu\kappa}\eta_{\varrho\lambda} - B_{\kappa\lambda}\eta_{\varrho\mu})\Big]\Box A^\varrho \\ &= \partial_\kappa F_{\lambda\mu} + \Big[(\eta_{\mu\kappa}\partial_\lambda - \eta_{\kappa\lambda}\partial_\mu)(c_1 - c_2) - \frac{e_\kappa}{e^2}(e_\mu\partial_\lambda - e_\lambda\partial_\mu)(c_1 + c_3) \\ &\qquad - (\eta_{\kappa\lambda}e_\mu - \eta_{\kappa\mu}e_\lambda)\frac{(e\partial)}{e^2}(c_2 + c_4)\Big](\partial A) \\ &\quad + \Big[(d_1 - d_3)(\eta_{\kappa\lambda}\eta_{\varrho\mu} - \eta_{\kappa\mu}\eta_{\varrho\lambda}) + (d_2 + d_3)\frac{e_\kappa}{e^2}(e_\lambda\eta_{\varrho\mu} - e_\mu\eta_{\varrho\lambda})\Big]\Box A^\varrho.\end{aligned} \tag{5.165}$$

To remove the e-dependence of Eq. (5.165), we must require

$$c_1 + c_3 = 0, \quad c_2 + c_4 = 0, \quad d_2 + d_3 = 0 \quad \text{and} \quad c_1 - c_2 = d_1 - d_3, \tag{5.166}$$

where the last condition comes from the fact that

$$\partial^\alpha F_{\alpha\mu} = \Box A_\mu - \partial_\mu(\partial A) \tag{5.167}$$

is point-localized. The conditions from Eq.s (5.163) and (5.164) yield more constraints on the parameters,

$$\begin{gathered} 1+4c_1+c_2=0, \quad -c_1-c_2+3c_4=0, \quad d_1+4d_3=0, \\ 1+4d_1+d_2+2d_3=0 \quad \text{and} \quad 2c_1+4c_2+c_3=0. \end{gathered} \tag{5.168}$$

Together, the conditions (5.166) and (5.168) form a linear system of nine equations for seven free parameters. However, this system is consistent and has the unique solution

$$c_1=-\frac{4}{15},\ \ c_2=\frac{1}{15},\ \ c_3=\frac{4}{15},\ \ c_4=-\frac{1}{15},\ \ d_1=-\frac{4}{15},\ \ d_2=-\frac{1}{15},\ \ d_3=\frac{1}{15}. \tag{5.169}$$

We thus end up with

$$\begin{aligned} \xi\pi(\partial_\kappa\partial_\lambda A_\mu) = \partial_\kappa\partial_\lambda A_\mu + \frac{1}{15}\Big[&-4(B_{\mu\kappa}\partial_\lambda + B_{\mu\lambda}\partial_\kappa) + \eta_{\kappa\lambda}D_\mu \\ &+4\frac{e_\kappa e_\lambda}{e^2}D_\mu - (B_{\mu\kappa}e_\lambda + B_{\mu\lambda}e_\kappa)\frac{(e\partial)}{e^2}\Big](\partial A) \\ &+\frac{1}{15}\Big[-4\eta_{\kappa\lambda}\eta_{\varrho\mu} - \frac{e_\kappa e_\lambda}{e^2}\eta_{\varrho\mu} + B_{\mu\kappa}\eta_{\varrho\lambda} + B_{\mu\lambda}\eta_{\varrho\kappa}\Big]\Box A^\varrho \end{aligned} \tag{5.170}$$

and

$$\begin{aligned} \xi\pi(\partial_\kappa F_{\lambda\mu}) &= \partial_\kappa F_{\lambda\mu} - \frac{1}{3}\left[\eta_{\mu\kappa}\eta_{\varrho\lambda} - \eta_{\lambda\kappa}\eta_{\varrho\mu}\right]\ (\partial^\varrho(\partial A) - \Box A^\varrho) \\ &= \partial_\kappa F_{\lambda\mu} - \frac{1}{3}\left[\eta_{\kappa\lambda}\partial^\varrho F_{\varrho\mu} - \eta_{\mu\kappa}\partial^\varrho F_{\varrho\lambda}\right] \end{aligned} \tag{5.171}$$

with $\eta^{\kappa\lambda}\xi\pi(\partial_\kappa F_{\lambda\mu}) = 0$. Note that the Bianchi identity is satisfied simply because of the definition of $F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda$,

$$\xi\pi(\partial_\kappa F_{\lambda\mu} + \text{cyclic}) = 0. \tag{5.172}$$

Our findings translate as follows to the propagators of the string-localized photon potential and the Maxwell field strength.

$$\begin{aligned} \langle\langle T_{\text{on}}\pi(A_\mu(e))\pi(A_\nu(e'))\rangle\rangle(x) &= \langle\langle T_0 A_\mu(e)A_\nu(e')\rangle\rangle(x), \\ \langle\langle T_{\text{on}}\pi(\partial_\kappa A_\mu(e))\pi(A_\nu(e'))\rangle\rangle(x) &= \partial_\kappa\langle\langle T_0 A_\mu(e)A_\nu(e')\rangle\rangle(x) \\ &\quad + \frac{i}{3}B_{\mu\kappa}(e)\left[(ee')I_e I_{-e'}\partial_\nu - e_\nu I_e\right]\delta(x), \\ \langle\langle T_{\text{on}}\pi(F_{\kappa\mu})\pi(A_\nu(e'))\rangle\rangle(x) &= \langle\langle T_0 F_{\kappa\mu}A_\nu(e')\rangle\rangle(x), \\ \langle\langle T_{\text{on}}\pi(F_{\kappa\mu})\pi(\partial_\lambda A_\nu(e'))\rangle\rangle(x) &= -\partial_\lambda\langle\langle T_0 F_{\kappa\mu}A_\nu(e')\rangle\rangle(x) \\ &\quad - \frac{i}{3}B_{\nu\lambda}(e')\left[e'_\mu\partial_\kappa - e'_\kappa\partial_\mu\right]I_{-e'}\delta(x), \\ \langle\langle T_{\text{on}}\pi(F_{\kappa\mu})\pi(F_{\lambda\nu})\rangle\rangle(x) &= \langle\langle T_0 F_{\kappa\mu}F_{\lambda\nu}\rangle\rangle(x), \end{aligned} \tag{5.173}$$

and so on.

Remark 5.49. In the previous derivations, we have axiomatized the map ξ for the string-localized potential A_μ and *defined* the Maxwell field strength as the curl of A_μ, axiomatizing that this curl and its partial derivatives must be point-localized and independent of the string variables. One can attempt to start from the opposite end, axiomatizing ξ for the Maxwell field strength $F_{\mu\nu}(x)$ and defining $A_\mu = I_e F_{\mu\nu} e^\nu$. However, it is then unclear how to incorporate the string integration operator I_e into the construction of the map ξ. Assuming that string integration commutes with ξ, one runs into trouble. To illustrate the issue, note that

$$I_e e^\mu \langle\langle T_{\text{on}} \pi(\partial_\kappa F_{\lambda\mu})\pi(A_\nu(e'))\rangle\rangle(x) \neq \langle\langle T_{\text{on}} \pi(\partial_\kappa A_\lambda(e))\pi(A_\nu(e'))\rangle\rangle(x) \tag{5.174}$$

in our previous construction. More research would be needed to clarify this matter but since the construction of ξ from axioms in terms of the potential works just fine – as we have seen – we leave it at that.

5.7.5 On the interference of different methods

In Sections 5.7.1 to 5.7.4, we have discussed different methods to reduce the BEG renormalization freedom beyond power counting. In particular, we have given examples of the implementation of

- algebraic conditions,
- the consequences of the string independence principle,
- the NST renormalization prescription, and
- the BDF onshell formalism

in time-ordering within the setting of string-localized field theory. One might think of more such construction, such as the implementation of a Master Ward Identity [26] in SLFT, of which the string independence principle as formulated in Eq. (4.20) is probably a special case. But in the framework of this thesis, we constrain ourselves – due to time constraints during the work on a thesis – to the investigation of the listed examples.

The NST prescription and the BDF onshell formalism are both special cases of onshell extensions of distributions as described by Bahns and Wrochna [2]. They describe the onshell extension of distributions across the origin by means of so-called "operators of essential order $n \in \mathbb{N}_0$" of which partial differential operators with smooth coefficients are a special case.[10] Roughly speaking, Bahns and Wrochna introduce a machinery to construct extensions $\overline{u} \in \mathcal{D}'(\mathbb{R}^n)$ of distributions $u \in \mathcal{D}'(\mathbb{R}^n \setminus 0)$, which are subject to a certain set of differential equations,

$$P_i u = 0 \text{ on } \mathbb{R}^n \setminus 0 \text{ for a set of partial differential operators } P_i,\ i = 1, \dots, k, \tag{5.175}$$

[10] The reason why we have introduced the NST prescription by means of the Casimir operator was so that one can easily see that the NST prescription is a special case of onshell extensions in the sense of Bahns and Wrochna.

such that $P_i\overline{u} = 0$ on $\mathbb{R}^n$ for all $i = 1, \ldots, k$.

Now, our findings in Section 5.7.4 suggest that the string independence principle and the BDF construction go hand in hand. They imply the use of kinematic time-ordered propagators

$$\langle\langle T_0 A_\mu(e) A_\nu(e') \rangle\rangle \quad \text{and} \quad \langle\langle T_0 F_{\mu\nu} F_{\kappa\lambda} \rangle\rangle \tag{5.176}$$

for the string-localized photon potential and the Maxwell field strength, respectively. In Section 6.1, we shall find that the same is true for string-localized massless Yang-Mills theory. However, the NST prescription, which relies on the refined notion of Lorentz covariance in the sense of a transformation behavior of massless propagators under irreducible representations of the Lorentz group, is not compatible with the implications of the string independence principle and the BDF construction. To see this, recall Corollary 5.47.

A string dependence of cross sections is not observed in experiment, and hence the string independence principle should be treated as paramount. Consequently, it invalidates the NST prescription. A physical interpretation of this result is due to K.-H. Rehren: the refined notion of Lorentz covariance that applies to the two-point functions, is destroyed by the interaction.

For helicity $s = 2$, the situation is not as clear as for the Maxwell case. We have found in Section 5.7.1 that the one-parameter space of propagators respecting the vanishing of the Ricci trace does not contain the kinematic propagator for the massless $s = 2$ field strength. To validate that one of the traceless propagators is consistent with the BDF construction, which might even fix the free parameter, one needs to apply the results to physical models. This task remains for future research.

In summary, we have introduced two different realizations of onshell extensions of distributions, which are mutually incompatible. From this, we learn that one should be careful when implementing differential relations in time-ordering and that it is not a priori clear, which recipes for the construction of onshell extensions are physically justified and which are not.

Chapter 6

Examples

In this section, we have a closer look at examples for the concepts presented and derived in Chapters 4 and 5. In particular, the examples display the power of the string independence principle. In Section 6.1, we show that the latter constrains the form of the self-coupling of massless string-localized potentials of helicity $s = 1$ to be of Yang-Mills type, while we show in Section 6.2 that the string independence principle forbids a coupling of a string-localized graviton potential to the Maxwell stress energy tensor. In Section 6.3, we outline some obstacles that still block the way to a perturbative description of the self-coupling of string-localized graviton potentials.

The consequences of the no-go result for the graviton coupling to the Maxwell SET are unclear at the present time. The result seems to imply that a construction of a string-localized graviton-photon coupling *on Hilbert space* does not work. One still can attempt to construct the model in the hybrid approach (i.e., on Krein space), which was outlined in the introductory paragraphs of Chapter 3. However, this means that one loses one of SLFT's main advantages. The no-go result also raises the question if there is an underlying principle that decides whether a string-localized model can be formulated on Hilbert space or not.

6.1 String-localized massless Yang-Mills theory

In this section, we show that the string independence principle – as formulated in Eq. (4.20) – implies at second order and tree level of perturbation theory that a generic renormalizable self-coupling of massless string-localized potentials of helicity $s = 1$ must be of Yang-Mills type. The derivations are based on joint work with J. M. Gracia Bondía and J. Mund [37]. The contribution of the author to that work is the derivation of the described result with several independent string variables in the S_n of the Dyson series, that is, a realization of alternative (4.14a) described in Section 4.2. The original derivation by Gracia-Bondía and Mund was within alternative (4.14c), which gives rise to ill-defined two-point functions and propagators. In addition to the content of the mentioned work [37], we here prove that the alternative (4.14b) for the setup of string-localized perturbation theory is in conflict with the string independence principle.

6.1.1 String independence at first order of perturbation theory

To derive the Yang-Mills structure, we start with a generic ansatz for a cubic coupling of n string-localized potentials $A_{\mu a}(x,e)$, $a = 1,\dots,n$, and show that the string independence principle strongly constrains its form at first order of perturbation theory. Our derivations are similar in spirit to the machinery used by Scharf and collaborators [1, 29, 60] in the gauge theoretic framework.

Theorem 6.1 (see Prop. 1 in [37])**.** *Suppose that we are given n massless string-localized potentials $A_{\mu a}(x,e)$, $a = 1,\dots,n$, of helicity $s = 1$. Then their most general cubic coupling $S_1 = i\,{:}L{:}$, which is renormalizable by power counting and satisfies the string independence principle Eq. (4.20), must be of the form*

$$S_1(x,e_1,e_2) = i\frac{g}{2} f_{abc}\,{:}A_{\mu a}(x,e_1)A_{\nu b}(x,e_2)F_c^{\mu\nu}(x){:} \tag{6.1}$$

with completely skewsymmetric constants f_{abc}, and where summation over repeated Latin indices understood.

Proof (similar to the proof in [37] but with details added). For the sake of readability, we drop the colons that indicate normal ordering and introduce the notation $A^i_{\mu a} \equiv A_{\mu a}(x,e_i)$ for the string-localized potentials. To begin with, we write down the most general ansatz for a cubic coupling of the fields $A_{\mu a}(x,e)$, which is renormalizable by power counting,

$$S_1'(x,e_1,e_2,e_3) = igf^1_{abc}A^1_{\mu a}A^2_{\nu b}\partial^\mu A^{3\nu}_c, \tag{6.2}$$

where the coefficients f^1_{abc} are a priori unspecified. We then split $f^1_{abc} \equiv d_{abc} + f^2_{abc}$ into a symmetric and skewsymmetric part under exchange of the second and third indices,

$$d_{abc} = d_{acb} \quad \text{and} \quad f^2_{abc} = -f^2_{acb}. \tag{6.3}$$

Only the part of S_1', which is symmetric under exchange of any string variable, gives a non-trivial contribution to the S-matrix. In particular, we can symmetrize S_1' in $e_2 \leftrightarrow e_3$ without loss of generality to see that the d_{abc}-contribution forms a total divergence in the adiabatic limit and can thus be neglected,

$$d_{abc}A^1_{\mu a}\left(A^2_{\nu b}\partial^\mu A^{3\nu}_c + A^3_{\nu b}\partial^\mu A^{2\nu}_c\right) = \partial^\mu\left(d_{abc}A^1_{\mu a}A^2_{\nu b}A^{3\nu}_c\right) \tag{6.4}$$

because $\partial^\mu A^1_{\mu a} = 0$ on Hilbert space. The most general ansatz Eq. (6.2) hence reduces to

$$S_1''(x,e_1,e_2,e_3) = igf^2_{abc}A^1_{\mu a}A^2_{\nu b}\partial^\mu A^{3\nu}_c \tag{6.5}$$

with the constants f^2_{abc} from Eq. (6.3). We can repeat the above procedure for the new coupling S_1'': Split $f^2_{abc} := f^+_{abc} + f^-_{abc}$ with $f^+_{abc} = f^+_{bac}$ and totally skewsymmetric f^-_{abc}. The totally skewsymmetric part can then be rewritten as

$$S_1''^{-}(x,e_1,e_2,e_3) := igf^-_{abc}A^1_{\mu a}A^2_{\nu b}\partial^\mu A^{3\nu}_c = i\frac{g}{2}f^-_{abc}A^1_{\mu a}A^2_{\nu b}F^{\mu\nu}_c \equiv S_1''^{-}(x,e_1,e_2) \tag{6.6}$$

and depends trivially on e_3. Since the test function $c \in \mathcal{D}(H)$ averaging over the string variables has integral equal to unity, it is consistent to simply ignore e_3 in $S_1''^-$. Furthermore, $S_1''^-(x, e_1, e_2)$ is intrinsically symmetric under the exchange $e_1 \leftrightarrow e_2$, i.e., $S_1''^- = S_1''^{-,\text{symm}}$. To compute the string variation of $S_1''^-$, we remember that the string-derivative of string-localized potentials is a symmetric gradient of auxiliary fields. For helicity $s = 1$, Eq. (2.30) yields

$$d_e A_{\mu a}(x,e) = \partial_{e^\kappa} A_{\mu a}(x,e) de^\kappa = \partial_\mu w_{a\kappa}(x,e) de^\kappa =: \partial_\mu w_a(x,e), \tag{6.7}$$

with $w_a(x,e) = I_e A_\kappa(x,e) de^\kappa$. Thus,

$$\begin{aligned} d_{e_1} S_1''^-(x,e_1,e_2) &= i\frac{g}{2} f^-_{abc} \partial_\mu w^1_a A^2_{\nu b} F^{\mu\nu}_c \\ &= i\frac{g}{2} f^-_{abc} \left\{ \partial_\mu \left[w^1_a A^2_{\nu b} F^{\mu\nu}_c \right] - \frac{1}{2} w^1_a F_{\mu\nu b} F^{\mu\nu}_c \right\} \\ &= i\frac{g}{2} f^-_{abc} \partial_\mu \left[w^1_a A^2_{\nu b} F^{\mu\nu}_c \right] \\ &=: \partial_\mu Q^\mu(x,e_1,e_2), \end{aligned} \tag{6.8}$$

where the term with the two field strength tensors vanishes by the skewsymmetry of f^-_{abc}. Therefore, the f^-_{abc} part is string independent in the adiabatic limit. Next, consider the f^+_{abc}-part of S_1''. We have

$$\begin{aligned} d_{e_1} \frac{1}{3!} \sum_{\pi \in \mathfrak{S}_3} f^+_{abc} A^{\pi(1)}_{\mu a} A^{\pi(2)}_{\nu b} \partial^\mu A^{\pi(3)\nu}_c = \frac{f^+_{abc}}{6} \Big\{ & \partial_\mu w^1_a \left(A^2_{\nu b} \partial^\mu A^{3\nu}_c + A^3_{\nu b} \partial^\mu A^{2\nu}_c \right) \\ & + \partial_\nu w^1_b \left(A^2_{\mu a} \partial^\mu A^{3\nu}_c + A^3_{\mu a} \partial^\mu A^{2\nu}_c \right) \\ & + \partial^\mu \partial^\nu w^1_c \left(A^2_{\mu a} A^3_{\nu b} + A^3_{\mu a} A^2_{\nu b} \right) \Big\}. \end{aligned} \tag{6.9}$$

The first line of the right-hand side of Eq. (6.9) has two contracted derivatives and forms a total divergence because all fields satisfy the wave equation,[1]

$$\partial_\mu w^1_a A^2_{\nu b} \partial^\mu A^{3\nu}_c = \frac{1}{2} \partial_\mu \left[w^1_a A^2_{\nu b} \partial^\mu A^{3\nu}_c + \partial^\mu w^1_a A^2_{\nu b} A^{3\nu}_c - w^1_a \partial^\mu A^2_{\nu b} A^{3\nu}_c \right], \tag{6.10}$$

and the other two lines can be integrated by parts. Additionally using the symmetry properties $f^+_{abc} = f^+_{bac} = -f^+_{acb}$, we obtain

$$d_{e_1} \frac{1}{3!} \sum_{\pi \in \sigma_3} f^+_{abc} A^{\pi(1)}_{\mu a} A^{\pi(2)}_{\nu b} \partial^\mu A^{\pi(3)\nu}_c = \text{div} + \frac{2}{3} f^+_{abc} \partial_\mu A^2_{\nu a} \partial^\nu A^{3\mu}_b w^1_c. \tag{6.11}$$

The operators $\partial_\mu A^2_{\nu a}$, $\partial^\nu A^{3\mu}_b$ and w^1_c are linearly independent and therefore, the right-hand side of Eq. (6.11) only forms a total divergence if the f^+_{abc} vanish identically. Thus, string independence implies that $f^2_{abc} \equiv f^-_{abc} =: f_{abc}$, which finishes the proof. □

[1] We have encountered such divergences earlier, recall Eq. (3.107).

Remark 6.2. The advantage that SLFT is formulated on Hilbert space and that correspondingly no unphysical ghost fields appear fully comes to bear in the proof of Theorem 6.1. In the gauge theoretic framework, where ghost field contributions have to be considered, similar derivations are far more involved, see for example [60]. The same holds for the second order considerations in the next section.

The coupling S_1 from Eq. (6.1) with totally skewsymmetric coefficients f_{abc} is the most general cubic self-coupling of massless string-localized potentials of helicity $s = 1$, which is renormalizable by power counting and string independent at first order of perturbation theory. However, string independence at first order does not yet imply that the coupling is of Yang-Mills type. But this should not be expected since the quartic term $\sim g^2 A^4$, which appears in Yang-Mills theory, is quadratic in the coupling constant and hence a second order contribution. Furthermore, we know by now that the f_{abc} must be totally skewsymmetric but it is unclear whether they satisfy the Jacobi identity, which would make them structure constants of a Lie algebra (of compact type – due to the *total* skewsymmetry). We thus need to go to the next order of perturbation theory to get a better picture.

6.1.2 String independence at second order and tree level

In our paper [37], we treat string independence at second order and tree level in the string chopping framework, which has been adjusted to the purpose by Jens Mund in an appendix of that paper. Moreover, we consider kinematic propagators right from the start in the mentioned work, not considering possible ambiguities of the propagators. The following derivation of the implications of second order string independence are slightly different from the derivations in the paper because we use the method described in Section 4.1.2 and because we *derive* that kinematic propagators must be employed to ensure string independence. The computations still are partially the same as in our work [37], only with more details added. In the end, the result turns out to be the same as in the paper.

By our considerations in Chapter 4, the second order tree graph contribution to the S-matrix is given by a four-fold string integral

$$T[S_1(x,e_1,e_2)S_1(x',e_1',e_2')]\big|_{\text{tree}} = \int_0^\infty d^4s\ \ T[S_1(x,y_1,y_2)S_1(x',y_1',y_2')]\big|_{\text{tree}}\,, \tag{6.12}$$

with $y_i^{(\prime)} = x^{(\prime)} + s_i^{(\prime)} e_i^{(\prime)}$. We can then use Wick's theorem to expand the integrand, which is a point-localized time-ordered product of $n(k+1) = 6$ variables, with the possible appearance of "self-contractions" that would have to be removed manually (as outlined in Section 4.2). In the case at hand, no self-contractions appear, for the gluon propagators are diagonal in the color indices, $\langle\!\langle TA_{\mu a}A_{\nu b}\rangle\!\rangle \sim \delta_{ab}$, with the Kronecker delta δ_{ab}, while the coefficients f_{abc} are totally skewsymmetric. Thus, Wick's theorem applies without any adjustments and we have

$$T[S_1(x,y_1,y_2)S_1(x',y'_1,y'_2)]\big|_{\text{tree}} = \sum_{\varphi,\chi} \langle\!\langle T\varphi(\xi)\chi(\xi')\rangle\!\rangle :\frac{\partial S_1(x,y_1,y_2)}{\partial\varphi(\xi)}\frac{\partial S_1(x',y'_1,y'_2)}{\chi(\xi')}: . \tag{6.13}$$

Inserting the explicit form of S_1 derived in the previous section, carrying out the string integrals over Eq. (6.13), indicating a dependence on x' by a prime at the field and a dependence on e'_i by an upper index i', and dropping again the colons indicating normal ordering for the sake of readability, we obtain

$$\begin{aligned} &T[S_1(x,e_1,e_2)S_1(x',e'_1,e'_2)]\big|_{\text{tree}} \\ &= i^2\frac{g(x)g(x')}{4} f_{abc}f_{xyz}\Big\{\langle\!\langle TF^{\mu\nu}_c F'^{\kappa\lambda}_z\rangle\!\rangle A^1_{\mu a}A^2_{\nu b}A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} \\ &\qquad + \Big[\langle\!\langle TA^1_{\mu a}F'^{\kappa\lambda}_z\rangle\!\rangle A^2_{\nu b}F^{\mu\nu}_c A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} + (e_1 \leftrightarrow e_2)\Big] \\ &\qquad + \Big[\langle\!\langle TF^{\mu\nu}_c A'^{1'}_{\kappa x}\rangle\!\rangle A^1_{\mu a}A^2_{\nu b}A'^{2'}_{\lambda y}F'^{\kappa\lambda}_z + (e'_1 \leftrightarrow e'_2)\Big] \\ &\qquad + \Big[\Big(\langle\!\langle TA^1_{\mu a}A'^{1'}_{\kappa x}\rangle\!\rangle A^2_{\nu b}F^{\mu\nu}_c A'^{2'}_{\lambda y}F'^{\kappa\lambda}_z + (e_1 \leftrightarrow e_2)\Big) + (e'_1 \leftrightarrow e'_2)\Big]\Big\} . \end{aligned} \tag{6.14}$$

By our reasoning in Sections 4.1.2 and 4.2, the left-hand side of Eq. (6.14) is uniquely fixed outside the x-diagonal $\{\, x = x' \,\}$, provided that all lower time-ordered products have already been fixed. But they have not: The propagators on the right-hand side are so far unspecified. The propagator of the field strength, which is homogeneous of degree $\omega = -4$, has a freedom

$$\langle\!\langle TF^{\mu\nu}_c F'^{\kappa\lambda}_z\rangle\!\rangle = \delta_{cz}\Big[\langle\!\langle T_0F^{\mu\nu}F'^{\kappa\lambda}\rangle\!\rangle + c(\eta^{\mu\kappa}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\kappa})\delta(x-x')\Big] \tag{6.15}$$

with a constant c and the kinematic propagator T_0, and consequently,

$$\langle\!\langle TA^{1\mu}_a F'^{\kappa\lambda}_z\rangle\!\rangle = \delta_{az}\Big[\langle\!\langle T_0A^{1\mu}F'^{\kappa\lambda}\rangle\!\rangle + c(\eta^{\mu\kappa}e^\lambda_1 - \eta^{\mu\lambda}e^\kappa_1)I_{e_1}\delta(x-x')\Big], \tag{6.16a}$$

$$\langle\!\langle TF^{\mu\nu}_c A'^{1'}_{\kappa x}\rangle\!\rangle = \delta_{cx}\Big[\langle\!\langle T_0F^{\mu\nu}A'^{1'\kappa}\rangle\!\rangle + c(\eta^{\mu\kappa}e'^{\nu}_1 - \eta^{\nu\kappa}e'^{\mu}_1)I_{-e'_1}\delta(x-x')\Big], \tag{6.16b}$$

$$\langle\!\langle TA^{1\mu}_a A'^{1'\kappa}_x\rangle\!\rangle = \delta_{ax}\Big[\langle\!\langle T_0A^{1\mu}A'^{1'\kappa}\rangle\!\rangle + c(\eta^{\mu\kappa}(e_1e'_1) - e'^{\mu}_1e^\kappa_1)I_{e_1}I_{-e'_1}\delta(x-x')\Big] \tag{6.16c}$$

with the same constant c.

Our strategy to investigate the effect of second order string independence at tree level is the following. We start by inserting the kinematic propagators into Eq. (6.14) and then compute the variation with respect to e_1 of the symmetrized version of Eq. (6.14) to see whether string independence can be achieved if we employ kinematic propagators. In the next step, we investigate the contribution of the c-terms and how they interfere with the previous results.

Inserting the kinematic propagators into Eq. (6.14) and symmetrizing the expression in all string variables yields

$$
\begin{aligned}
&T_0[S_1(x,e_1,e_2)S_1(x',e_1',e_2')]\Big|_{\text{tree}}^{\text{symm}} \\
&= i^2\frac{g(x)g(x')}{4} f_{abc}f_{xyz}\Big\{\langle\!\langle T_0 F_c^{\mu\nu} F'^{\kappa\lambda}_z\rangle\!\rangle A^1_{\mu a}A^2_{\nu b}A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} \\
&\qquad + 2\langle\!\langle T_0 A^1_{\mu a} F'^{\kappa\lambda}_z\rangle\!\rangle A^2_{\nu b}F_c^{\mu\nu}A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} \\
&\qquad + 2\langle\!\langle T_0 F_c^{\mu\nu} A'^{1'}_{\kappa x}\rangle\!\rangle A^1_{\mu a}A^2_{\nu b}A'^{2'}_{\lambda y}F'^{\kappa\lambda}_z \\
&\qquad + 4\langle\!\langle T_0 A^1_{\mu a} A'^{1'}_{\kappa x}\rangle\!\rangle A^2_{\nu b}F_c^{\mu\nu}A'^{2'}_{\lambda y}F'^{\kappa\lambda}_z\Big\}^{\text{symm}}.
\end{aligned}
\tag{6.17}
$$

Let us write the symmetrized braces in Eq. (6.17) explicitly, exploiting the skewsymmetry of $F^{\mu\nu}$ and f_{abc} and the fact that the propagators are "color diagonal". For readability, we drop the overall prefactor $i^2\frac{g(x)g(x')f_{abc}f_{xyc}}{4!}$ and obtain

$$
\begin{aligned}
&\langle\!\langle T_0 F^{\mu\nu}F'^{\kappa\lambda}\rangle\!\rangle\Big[A^1_{\mu a}A^2_{\nu b}A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} + A^1_{\mu a}A^{1'}_{\nu b}A'^{2}_{\kappa x}A'^{2'}_{\lambda y} + A^1_{\mu a}A^{2'}_{\nu b}A'^{1'}_{\kappa x}A'^{2}_{\lambda y} \\
&\qquad + A^{1'}_{\mu a}A^{2'}_{\nu b}A'^{1}_{\kappa x}A'^{2}_{\lambda y} + A^2_{\mu a}A^{2'}_{\nu b}A'^{1}_{\kappa x}A'^{1'}_{\lambda y} + A^{1'}_{\mu a}A^{2}_{\nu b}A'^{1}_{\kappa x}A'^{2'}_{\lambda y}\Big] \\
&+ \langle\!\langle T_0 A^1_\mu F'^{\kappa\lambda}\rangle\!\rangle\Big[A^2_{\nu a}F_b^{\mu\nu}A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} + A^{1'}_{\nu a}F_b^{\mu\nu}A'^{2}_{\kappa x}A'^{2'}_{\lambda y} + A^{2'}_{\nu a}F_b^{\mu\nu}A'^{1'}_{\kappa x}A'^{2}_{\lambda y}\Big] \\
&+ \langle\!\langle T_0 A^2_\mu F'^{\kappa\lambda}\rangle\!\rangle\Big[A^1_{\nu a}F_b^{\mu\nu}A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} + A^{1'}_{\nu a}F_b^{\mu\nu}A'^{1}_{\kappa x}A'^{2'}_{\lambda y} + A^{2'}_{\nu a}F_b^{\mu\nu}A'^{1'}_{\kappa x}A'^{1}_{\lambda y}\Big] \\
&+ \langle\!\langle T_0 A^{1'}_\mu F'^{\kappa\lambda}\rangle\!\rangle\Big[A^2_{\nu a}F_b^{\mu\nu}A'^{1}_{\kappa x}A'^{2'}_{\lambda y} + A^{1}_{\nu a}F_b^{\mu\nu}A'^{2}_{\kappa x}A'^{2'}_{\lambda y} + A^{2'}_{\nu a}F_b^{\mu\nu}A'^{1}_{\kappa x}A'^{2}_{\lambda y}\Big] \\
&+ \langle\!\langle T_0 A^{2'}_\mu F'^{\kappa\lambda}\rangle\!\rangle\Big[A^2_{\nu a}F_b^{\mu\nu}A'^{1'}_{\kappa x}A'^{1}_{\lambda y} + A^{1'}_{\nu a}F_b^{\mu\nu}A'^{2}_{\kappa x}A'^{1}_{\lambda y} + A^{1}_{\nu a}F_b^{\mu\nu}A'^{1'}_{\kappa x}A'^{2}_{\lambda y}\Big] \\
&+ \langle\!\langle T_0 F^{\mu\nu}A'^{1'}_\kappa\rangle\!\rangle\Big[A^1_{\mu a}A^2_{\nu b}A'^{2'}_{\lambda x}F'^{\kappa\lambda}_y + A^1_{\mu a}A^{2'}_{\nu b}A'^{2}_{\lambda x}F'^{\kappa\lambda}_y + A^{2'}_{\mu a}A^{2}_{\nu b}A'^{1}_{\lambda x}F'^{\kappa\lambda}_y\Big] \\
&+ \langle\!\langle T_0 F^{\mu\nu}A'^{2'}_\kappa\rangle\!\rangle\Big[A^1_{\mu a}A^2_{\nu b}A'^{1'}_{\lambda x}F'^{\kappa\lambda}_y + A^1_{\mu a}A^{1'}_{\nu b}A'^{2}_{\lambda x}F'^{\kappa\lambda}_y + A^{1'}_{\mu a}A^{2}_{\nu b}A'^{1}_{\lambda x}F'^{\kappa\lambda}_y\Big] \\
&+ \langle\!\langle T_0 F^{\mu\nu}A'^{1}_\kappa\rangle\!\rangle\Big[A^{1'}_{\mu a}A^2_{\nu b}A'^{2'}_{\lambda x}F'^{\kappa\lambda}_y + A^{1'}_{\mu a}A^{2'}_{\nu b}A'^{2}_{\lambda x}F'^{\kappa\lambda}_y + A^{2'}_{\mu a}A^{2}_{\nu b}A'^{1'}_{\lambda x}F'^{\kappa\lambda}_y\Big] \\
&+ \langle\!\langle T_0 F^{\mu\nu}A'^{2}_\kappa\rangle\!\rangle\Big[A^1_{\mu a}A^{1'}_{\nu b}A'^{2'}_{\lambda x}F'^{\kappa\lambda}_y + A^1_{\mu a}A^{2'}_{\nu b}A'^{1'}_{\lambda x}F'^{\kappa\lambda}_y + A^{2'}_{\mu a}A^{1'}_{\nu b}A'^{1}_{\lambda x}F'^{\kappa\lambda}_y\Big] \\
&+ 4!\Big[\langle\!\langle T_0 A^1_\mu A'^{1'}_\kappa\rangle\!\rangle A^2_{\nu a}F_b^{\mu\nu}A'^{2'}_{\lambda x}F'^{\kappa\lambda}_y\Big]^{\text{symm}},
\end{aligned}
$$

where we have canceled a factor 4 in the numerator against the same factor in the denominator and not expanded the last line in Eq. (6.17) because this line cannot be further simplified. To compute the string variation, we note that

$$
d_{e_1}\langle\!\langle T_0 A^1_{\mu a}\bullet\rangle\!\rangle = \partial_\mu\langle\!\langle T_0 w^1_a\bullet\rangle\!\rangle, \tag{6.18}
$$

which implies that the variation with respect to e_1 of the lines containing $\langle\!\langle T_0 A^1_\mu F'^{\kappa\lambda}\rangle\!\rangle$,

$\langle\!\langle T_0 F^{\mu\nu} A'^1_\kappa \rangle\!\rangle$, $\langle\!\langle T_0 A^1_\mu A'^{i^{(\prime)}}_\kappa \rangle\!\rangle$ or $\langle\!\langle T_0 A^{i^{(\prime)}}_\mu A'^1_\kappa \rangle\!\rangle$ are total divergences. For example,

$$\begin{aligned} d_{e_1} \langle\!\langle T_0 A^1_\mu F'^{\kappa\lambda} \rangle\!\rangle A^2_{\nu a} F^{\mu\nu}_b A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} &= \partial_\mu \langle\!\langle T_0 w^1 F'^{\kappa\lambda} \rangle\!\rangle A^2_{\nu a} F^{\mu\nu}_b A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} \\ &= \mathrm{div} - \frac{1}{2} \langle\!\langle T_0 w^1 F'^{\kappa\lambda} \rangle\!\rangle F_{\mu\nu a} F^{\mu\nu}_b A'^{1'}_{\kappa x} A'^{2'}_{\lambda y}, \end{aligned} \tag{6.19}$$

where div is a shorthand notation for a total divergence. The last term in Eq. (6.19) vanishes in the sum because $f_{abc} = -f_{bac}$. Thus, only terms where e_1 appears outside the propagators can cause obstructions to string independence (if we vary with respect to e_1). Furthermore, by symmetry under exchange of $x \leftrightarrow x'$, it is enough to compute the string variation of the sum of terms where e_1 appears in an x-dependent field. Let us write down all these terms, dropping again the common prefactor $i^2 \frac{g(x)g(x') f_{abc} f_{xyc}}{4!}$:

$$\begin{aligned} &\langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda} \rangle\!\rangle A^1_{\mu a} \left[A^2_{\nu b} A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} + A^{1'}_{\nu b} A'^2_{\kappa x} A'^{2'}_{\lambda y} + A^{2'}_{\nu b} A'^{1'}_{\kappa x} A'^2_{\lambda y} \right] \\ &+ A^1_{\nu a} F^{\mu\nu}_b \left[\langle\!\langle T_0 A^2_\mu F'^{\kappa\lambda} \rangle\!\rangle A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} + \langle\!\langle T_0 A^{1'}_\mu F'^{\kappa\lambda} \rangle\!\rangle A'^2_{\kappa x} A'^{2'}_{\lambda y} + \langle\!\langle T_0 A^{2'}_\mu F'^{\kappa\lambda} \rangle\!\rangle A'^{1'}_{\kappa x} A'^2_{\lambda y} \right] \\ &+ A^1_{\mu a} F'^{\kappa\lambda}_y \left\{ \langle\!\langle T_0 F^{\mu\nu} A'^{1'}_\kappa \rangle\!\rangle A^2_{\nu b} A'^{2'}_{\lambda x} + \text{ 5 terms symmetrizing in } (2, 1', 2') \right\} \\ &+ A^1_{\nu a} F'^{\kappa\lambda}_y \left\{ \langle\!\langle T_0 A^2_\mu A'^{1'}_\kappa \rangle\!\rangle F^{\mu\nu}_b A'^{2'}_{\lambda x} + \text{ 5 terms symmetrizing in } (2, 1', 2') \right\}, \end{aligned} \tag{6.20}$$

The kinematic propagator (5.53) is defined as a derivative of the Feynman propagator and thus $\langle\!\langle T_0 \partial^x_\mu X(x) X'(x') \rangle\!\rangle \equiv \partial^x_\mu \langle\!\langle T_0 X(x) X'(x') \rangle\!\rangle$ as long as no field equation is involved. Using this fact and also that $\partial_\mu A^i_\nu - \partial_\nu A^i_\mu = F_{\mu\nu}$, we can combine the first two lines and the last two lines to obtain

$$\begin{aligned} &\langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda} \rangle\!\rangle \partial_\mu w^1_a A^2_{\nu b} A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} + \partial_\nu w^1_a F^{\mu\nu}_b \langle\!\langle T_0 A^2_\mu F'^{\kappa\lambda} \rangle\!\rangle A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} \\ &= \mathrm{div} - w^1_a A^2_{\nu b} A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} \partial_\mu \langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda} \rangle\!\rangle \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} &\partial_\mu w^1_a F'^{\kappa\lambda}_y \langle\!\langle T_0 F^{\mu\nu} A'^{1'}_\kappa \rangle\!\rangle A^2_{\nu b} A'^{2'}_{\lambda x} + \partial_\nu w^1_a F'^{\kappa\lambda}_y \langle\!\langle T_0 A^2_\mu A'^{1'}_\kappa \rangle\!\rangle F^{\mu\nu}_b A'^{2'}_{\lambda x} \\ &= \mathrm{div} - w^1_a A^2_{\nu b} A'^{2'}_{\lambda x} F'^{\kappa\lambda}_y \partial_\mu \langle\!\langle T_0 F^{\mu\nu} A'^{1'}_\kappa \rangle\!\rangle \end{aligned} \tag{6.22}$$

after application of d_{e_1} to the first term of each line in Eq. (6.20). Thus, the string variation of Eq. (6.20) with respect to e_1 is given by

$$\begin{aligned} &\mathrm{div} - w^1_a \left[A^2_{\nu b} A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} + A^{1'}_{\nu b} A'^2_{\kappa x} A'^{2'}_{\lambda y} + A^{2'}_{\nu b} A'^{1'}_{\kappa x} A'^2_{\lambda y} \right] \partial_\mu \langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda} \rangle\!\rangle \\ &\quad - w^1_a F'^{\kappa\lambda}_y \left[A^2_{\nu b} A'^{2'}_{\lambda x} \partial_\mu \langle\!\langle T_0 F^{\mu\nu} A'^{1'}_\kappa \rangle\!\rangle + \text{ 5 terms symmetrizing in } (2, 1', 2') \right] \end{aligned} \tag{6.23}$$

Recall that the pertinent kinematic propagators are

$$\begin{aligned}\langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda}\rangle\!\rangle &= \left(\eta^{\nu\lambda}\partial^\mu\partial^\kappa - \eta^{\mu\lambda}\partial^\nu\partial^\kappa - \eta^{\nu\kappa}\partial^\mu\partial^\lambda + \eta^{\mu\kappa}\partial^\nu\partial^\lambda\right) D_F(x-x'),\\ \langle\!\langle T_0 F^{\mu\nu} A'^{1'}_{\kappa}\rangle\!\rangle &= \left(e'^{\nu}_1\partial^\mu\partial_\kappa - e'^{\mu}_1\partial^\nu\partial_\kappa - \delta^\nu_\kappa\partial^\mu(e'_1\partial) + \delta^\mu_\kappa\partial^\nu(e'_1\partial)\right) I_{-e'_1} D_F(x-x')\\ &= \left(e'^{\nu}_1\partial^\mu - e'^{\mu}_1\partial^\nu\right)\partial_\kappa I_{-e'_1} D_F(x-x') + \left(\delta^\mu_\kappa\partial^\nu - \delta^\nu_\kappa\partial^\mu\right) D_F(x-x'),\end{aligned} \tag{6.24}$$

and thus, we have

$$\begin{aligned}\partial_\mu\langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda}\rangle\!\rangle &= \left(\eta^{\nu\lambda}\partial^\kappa - \eta^{\nu\kappa}\partial^\lambda\right)\Box D_F(x-x'),\\ \partial_\mu\langle\!\langle T_0 F^{\mu\nu} A'^{1'}_{\kappa}\rangle\!\rangle &= e'^{\nu}_1\partial_\kappa I_{-e'_1}\Box D_F(x-x') - \delta^\nu_\kappa\Box D_F(x-x').\end{aligned} \tag{6.25}$$

Inserting $\Box D_F(x-x') = -i\delta(x-x')$, we obtain

$$\begin{aligned}\partial_\mu\langle\!\langle T_0 F^{\mu\nu} F'^{\kappa\lambda}\rangle\!\rangle &= i\left(\eta^{\nu\kappa}\partial^\lambda - \eta^{\nu\lambda}\partial^\kappa\right)\delta(x-x'),\\ \partial_\mu\langle\!\langle T_0 F^{\mu\nu} A'^{1'}_{\kappa}\rangle\!\rangle &= i\left(\delta^\nu_\kappa\delta(x-x') - e'^{\nu}_1\partial_\kappa I_{-e'_1}\delta(x-x')\right).\end{aligned} \tag{6.26}$$

Inserting Eq. (6.26) into Eq. (6.23) gives

$$\begin{aligned}\operatorname{div} &-iw^1_a\left[A^2_{\nu b}A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} + A^{1'}_{\nu b}A'^{2}_{\kappa x}A'^{2'}_{\lambda y} + A^{2'}_{\nu b}A'^{1'}_{\kappa x}A'^{2}_{\lambda y}\right]\left(\eta^{\nu\kappa}\partial^\lambda - \eta^{\nu\lambda}\partial^\kappa\right)\delta(x-x')\\ &-iw^1_a F'^{\kappa\lambda}_{\ \ y}\left[A^2_{\nu b}A'^{2'}_{\lambda x}\left(\delta^\nu_\kappa\delta(x-x') - e'^{\nu}_1\partial_\kappa I_{-e'_1}\delta(x-x')\right)\right.\\ &\qquad\left. +\ 5 \text{ terms symmetrizing in } (2,1',2')\right].\end{aligned} \tag{6.27}$$

Using $\partial_\kappa I_{-e'_1}\delta(x-x') = -\partial'_\kappa I_{-e'_1}\delta(x-x')$ we see that the term with the string-integrated Dirac delta forms a total divergence,

$$-iw^1_a(e'_1A^2_b)F'^{\kappa\lambda}_{\ \ y}A'^{2'}_{\lambda x}\partial'_\kappa I_{-e'_1}\delta(x-x') = \operatorname{div} + \frac{i}{2}w^1_a(e'_1A^2_b)F'^{\kappa\lambda}_{\ \ y}F'_{\kappa\lambda x}I_{-e'_1}\delta(x-x'), \tag{6.28}$$

where the last term vanishes in the sum over all color indices due to the skewsymmetry of f_{xyc}. Thus, only the point-localized Dirac deltas remain and Eq. (6.27) becomes

$$\begin{aligned}\operatorname{div} &-iw^1_a\left[A^{2\kappa}_b A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} + A^{1'\kappa}_b A'^{2}_{\kappa x}A'^{2'}_{\lambda y} + A^{2'\kappa}_b A'^{1'}_{\kappa x}A'^{2}_{\lambda y}\right]\partial^\lambda\delta\\ &+iw^1_a\left[A^{2\lambda}_b A'^{1'}_{\kappa x}A'^{2'}_{\lambda y} + A^{1'\lambda}_b A'^{2}_{\kappa x}A'^{2'}_{\lambda y} + A^{2'\lambda}_b A'^{1'}_{\kappa x}A'^{2}_{\lambda y}\right]\partial^\kappa\delta\\ &-iw^1_a F'^{\kappa\lambda}_{\ \ y}\left[A^2_{\kappa b}A'^{2'}_{\lambda x} + A^2_{\kappa b}A'^{1'}_{\lambda x} + A^{2'}_{\kappa b}A'^{2}_{\lambda x} + A^{2'}_{\kappa b}A'^{1'}_{\lambda x} + A^{1'}_{\kappa b}A'^{2}_{\lambda x} + A^{1'}_{\kappa b}A'^{2'}_{\lambda x}\right]\delta.\end{aligned} \tag{6.29}$$

Integrating the first two lines by parts gives

$$\begin{aligned}
\text{div} + & i\partial^\lambda w_a^1 \left[A_b^{2\kappa} A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} + A_b^{1'\kappa} A'^{2}_{\kappa x} A'^{2'}_{\lambda y} + A_b^{2'\kappa} A'^{1'}_{\kappa x} A'^{2}_{\lambda y}\right] \delta \\
& -i\partial^\kappa w_a^1 \left[A_b^{2\lambda} A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} + A_b^{1'\lambda} A'^{2}_{\kappa x} A'^{2'}_{\lambda y} + A_b^{2'\lambda} A'^{1'}_{\kappa x} A'^{2}_{\lambda y}\right] \delta \\
& -i w_a^1 \left[F_b^{\kappa\lambda} A'^{1'}_{\kappa x} A'^{2'}_{\lambda y} + F_b^{\kappa\lambda} A'^{2}_{\kappa x} A'^{2'}_{\lambda y} + F_b^{\kappa\lambda} A'^{1'}_{\kappa x} A'^{2}_{\lambda y}\right] \delta \\
& -i w_a^1 F'^{\kappa\lambda}_y \left[A^2_{\kappa b} A'^{2'}_{\lambda x} + A^2_{\kappa b} A'^{1'}_{\lambda x} + A^{2'}_{\kappa b} A'^{2}_{\lambda x} + A^{2'}_{\kappa b} A'^{1'}_{\lambda x} + A^{1'}_{\kappa b} A'^{2}_{\lambda x} + A^{1'}_{\kappa b} A'^{2'}_{\lambda x}\right] \delta.
\end{aligned} \tag{6.30}$$

Eq. (6.30) only contains terms proportional to $\delta(x - x')$. Restoring the global prefactor $i^2 \frac{g(x)g(x')f_{abc}f_{xyc}}{4!}$, multiplying by a factor 2 for the omitted terms where e_1 appears in x'-dependent fields and integrating out the Dirac delta, Eq. (6.30) becomes

$$\begin{aligned}
& d_{e_1} T_0[S_1(x, e_1, e_2) S_1(x', e_1', e_2')]\big|^{\text{symm}}_{\text{tree}} \\
&= \text{div} + i^3 \frac{2g^2(x) f_{abc} f_{xyc}}{4!} \Big\{\partial^\lambda w_a^1 \left[A_b^{2\kappa} A^{1'}_{\kappa x} A^{2'}_{\lambda y} + A_b^{1'\kappa} A^{2}_{\kappa x} A^{2'}_{\lambda y} + A_b^{2'\kappa} A^{1'}_{\kappa x} A^{2}_{\lambda y}\right] \\
&\quad - \partial^\kappa w_a^1 \left[A_b^{2\lambda} A^{1'}_{\kappa x} A'^{2'}_{\lambda y} + A_b^{1'\lambda} A^{2}_{\kappa x} A^{2'}_{\lambda y} + A_b^{2'\lambda} A^{1'}_{\kappa x} A^{2}_{\lambda y}\right] \\
&\quad - w_a^1 \left[F_b^{\kappa\lambda} A^{1'}_{\kappa x} A^{2'}_{\lambda y} + F_y^{\kappa\lambda} A^{1'}_{\kappa b} A^{2'}_{\lambda x} + F_y^{\kappa\lambda} A^{2'}_{\kappa b} A^{1'}_{\lambda x} + \ (2 \times 3 = 6) \text{ similar terms}\right]\Big\}.
\end{aligned} \tag{6.31}$$

The lines containing a derivative of the auxiliary field can be rewritten as a string variation with respect to e_1 and the linear combination of fields in the last line can be rewritten as a linear combination of the constants $f_{abc}f_{xyc}$, so that we have

$$\begin{aligned}
& d_{e_1} T_0[S_1(x, e_1, e_2) S_1(x', e_1', e_2')]\big|^{\text{symm}}_{\text{tree}} \\
&= \text{div} + i^3 \frac{2g^2(x) f_{abc} f_{xyc}}{4!} d_{e_1} \Big\{(A_a^1 A_y^{2'})(A_b^2 A_x^{1'}) + (A_a^1 A_y^{2'})(A_b^{1'} A_x^{2}) + (A_a^1 A_y^{2})(A_b^{2'} A_{\kappa x}^{1'}) \\
&\qquad - (A_a^1 A_x^{1'})(A_b^2 A'^{2'}_y) - (A_a^1 A_x^{2})(A_b^{1'} A_y^{2'}) - (A_a^1 A_x^{1'})(A_b^{2'} A_y^{2})\Big\} \\
&\quad - \frac{2i^3 g^2(x) \left[f_{abc}f_{xyc} + f_{axc}f_{ybc} + f_{ayc}f_{bxc}\right]}{4!} \left(w_a^1 F_b^{\kappa\lambda} A^{1'}_{\kappa x} A^{2'}_{\lambda y} + \ 2 \text{ similar terms}\right) \\
&= \text{div} + d_{e_1} \left\{-i^3 \frac{g^2(x)}{2} f_{abc} f_{xyc} (A_a^1 A_x^{2'})(A_b^2 A_y^{1'})\right\}^{\text{symm}} \\
&\quad - \frac{2i^3 g^2(x) \left[f_{abc}f_{xyc} + f_{axc}f_{ybc} + f_{ayc}f_{bxc}\right]}{4!} \left(w_a^1 F_b^{\kappa\lambda} A^{1'}_{\kappa x} A^{2'}_{\lambda y} + \ 2 \text{ similar terms}\right).
\end{aligned} \tag{6.32}$$

The two terms in Eq. (6.32) that obstruct string independence at second order and tree level are of different nature. The first one, which is a string variation, lies within the BEG freedom of defining

$$\begin{aligned} S_2(x,e_1,e_2,x',e_1',e_2') &\equiv T[S_1(x,e_1,e_2)S_1(x',e_1',e_2')] \\ &:= i^2\Big\{T_0[L(x,e_1,e_2)L(x',e_1',e_2')] \\ &+ i\frac{g^2(x)}{2}f_{abc}f_{xyc}(A_a^1A_x^{2'})(A_b^2A_y^{1'})\delta(x-x')\Big\}. \end{aligned} \tag{6.33}$$

In fact, the correction term in Eq. (6.33) is the quartic term of the Yang-Mills Lagrangian. The second obstructing term in Eq. (6.32), which contains an auxiliary field without any derivative, is no string variation of anything. To achieve string independence by employing kinematic propagators, it needs to vanish. This can only be achieved if

$$f_{abc}f_{xyc} + f_{axc}f_{ybc} + f_{ayc}f_{bxc} = 0, \tag{6.34}$$

which is the Jacobi identity.

There is one task left. Namely, we need to investigate what happens if we employ non-kinematic propagators. That means, we must determine the effect of the c-terms in Eq.s (6.15) and (6.16a-6.16c). To see that the string independence principle implies $c = 0$, it is enough to consider the contribution of the c-term from Eq. (6.16c). The symmetrization of this contribution is

$$\begin{aligned} &\Big\{T[S_1(x,e_1,e_2)S_1(x',e_1',e_2')]\big|_{\text{tree},\langle\langle TAA'\rangle\rangle,c}\Big\}^{\text{symm}} \\ &= i^2c\frac{g(x)g(x')}{4!}f_{abc}f_{ayz} \\ &\times\Big\{\Big[F_{\kappa c}^{\ \nu}F'^{\kappa\lambda}_{\ z}(e_1e_1') - e'_{1\mu}F_{\kappa c}^{\mu\nu}e_{1\kappa}F'^{\kappa\lambda}_{\ z}\Big]A_{\nu b}^2A'^{2'}_{\lambda y}I_{e_1}I_{-e_1'}\delta(x-x')\Big\}^{\text{symm}}. \end{aligned} \tag{6.35}$$

This term, containing a doubly string-integrated Dirac delta, cannot be compensated by a BEG renormalization of S_2, which must be supported on the x-diagonal by our considerations in the end of Section 4.2. The string variation with respect to e_1 of Eq. (6.35) hence must be a total divergence to fulfill the string independence principle. To see that this is not the case, it is enough to consider all terms proportional to $e'_{1\alpha}de_1^\alpha$ after application of d_{e_1} to Eq. (6.35). These terms are

$$\begin{aligned} &d_{e_1}\Big\{T[S_1(x,e_1,e_2)S_1(x',e_1',e_2')]\big|_{\text{tree},\langle\langle TAA'\rangle\rangle,c}\Big\}^{\text{symm}}\Big|_{e'_{1\alpha}de_1^\alpha} \\ &= i^2c\frac{g(x)g(x')\,e'_{1\alpha}de_1^\alpha}{4!}f_{abc}f_{ayz}F_{\kappa c}^{\ \nu}F'^{\kappa\lambda}_{\ z}\Big[A_{\nu b}^2A'^{2'}_{\lambda y} + A_{\nu b}^{2'}A'^{2}_{\lambda y}\Big] \\ &\times\Big[I_{e_1}I_{-e_1'} + I_{e_1'}I_{-e_1}\Big]\delta(x-x'), \end{aligned} \tag{6.36}$$

which does clearly not form a total divergence: the only derivatives are the ones hidden in the field strength tensors, and due to the skewsymmetry of the f_{abc}, for example the term $\partial_\kappa A_c^{3\nu}A_{\nu b}^2 \times \ldots$ reproduces itself upon integration by parts. We therefore conclude that $c = 0$ and arrive at the desired result.

Theorem 6.3. *Renormalizability by power counting and string independence up to second order and tree level imply*

1. *kinematic propagators of F and A must be employed,*
2. $S_1(x, e_1, e_1) = i\frac{g}{2} f_{abc} :A_{\mu a}(x, e_1) A_{\nu b}(x, e_2) F_c^{\mu\nu}(x):$*, and*
3. $S_2 = i^2 \left\{ T_0[LL] + i\frac{g^2(x)}{2} f_{abc} f_{xyc} (A_a^1 A_x^{2'})(A_b^2 A_y^{1'}) \delta(x - x') \right\}$

with totally skewsymmetric constants f_{abc} that satisfy the Jacobi identity (6.34) and are therefore structure constants of a Lie algebra of compact type. That is to say: self-interactions of massless string-localized potentials of helicity $s = 1$ must be of Yang-Mills type.

Theorem 6.3 tells us about the implications of the string independence principle at second order and tree level. Namely, it says that a renormalizable (by power counting) self-interaction of massless string-localized potentials of helicity $s = 1$ must be exactly what we expect it to be – *to second order of perturbation theory and at tree level*. A proof of perturbative string independence, that is, that Eq. (4.20) is satisfied to all orders of perturbation theory and also at loop level, and that additionally no new induced term needs to be added to $T_0[S_1 \dots S_1]$ at higher orders of perturbation theory, has not yet been given. This remains for future work.

Remark 6.4. Two remarks, which are of similar spirit but lead in opposite directions, are due on the derivations in this section.

- First, note that the derivation of Theorem 6.3 works completely analogous in the BDF onshell formalism described in Section 5.7.4. The string independence principle requires (for identically colored gluons) $\langle\langle T A_\mu A_\nu \rangle\rangle = \langle\langle T_0 A_\mu A_\nu \rangle\rangle$ and even if we had not postulated the connection between the propagators involving $F_{\mu\nu a}$ and $A_{\mu a}$, the BDF construction yields that all propagators appearing in Eq. (6.14) must be kinematic if $\langle\langle T A_\mu A_\nu \rangle\rangle = \langle\langle T_0 A_\mu A_\nu \rangle\rangle$, see Eq. (5.173).

- On the other hand, one can also think of dropping the requirement that the propagators of the string-localized potentials arise from the propagators of the field strength by appropriate string integration. Similarly, the BDF construction is not a God-given tool.

  If we assume that the propagators of the potential and the field strength possess an independent renormalization freedom, the free constants in Eq. (6.15) and (6.16a) to (6.16c) are potentially different. In such a case, the contribution of the c-term of $\langle\langle T F_{\mu\nu} F'_{\kappa\lambda} \rangle\rangle$ can absorb the quartic term that must be added to S_2. In that sense, inducing a term to S_2 and adding a non-kinematic term to $\langle\langle T_0 F_{\mu\nu} F'_{\kappa\lambda} \rangle\rangle$ can be treated interchangeably (at least to second order of perturbation theory) if the requirement that the string-localized propagators must arise from the field strength propagator by appropriate integration is dropped.

Let us relate these observations to the appearance of "self-contractions" as described in Section 4.2. The cubic part of the interaction $L^{(3)}_{\mathrm{YM}} = \frac{g}{2} f_{abc} A_{\mu a} A_{\nu b} F^{\mu\nu}_c$ does not give rise to self-contractions because all two-point functions and propagators are "color-diagonal" while the structure constants f_{abc} are totally skewsymmetric. However, there will in general appear self-contractions within the induced quartic term. From that point of view it thus seems preferable to work without an induced term and hence with a non-kinematic field strength propagator.

From our derivations up to second order and tree level, it is not clear whether one of the prescribed approaches is preferable. String independence at higher orders or loop level might shed light on the subject but such computations have not yet been performed and remain for future research.

6.1.3 Why every potential needs its own string variable

In this section we present the argument why the alternative (4.14b) for the setup of string-localized perturbation theory, i.e., the setup where all fields in the interaction Lagrangian depend on the same string variable, is in conflict with the string independence principle. By adjusting the proof of Theorem 6.1, it is easy to verify that string independence at first order,

$$d_e S(x,e) \stackrel{!}{=} \partial_\mu Q^\mu(x,e) \tag{6.37}$$

enables the same coupling $S_1(x,e)$ as Eq. (6.1) with the only difference being the dependence on a single string variable. However, second order string independence as derived in Section 6.1.2 cannot be achieved in the present setup. To prove this, we repeat the analysis from the previous section in the new setup, starting with the tree level expansion of the kinematic time-ordering,

$$\begin{aligned} &T_0[S_1(x,e_1)S_1(x',e_2)]|_{\mathrm{tree}} \qquad (6.38)\\ &= \frac{g(x)g(x')}{4} f_{abc} f_{xyz} \Big\{ \langle\!\langle T_0 F^{\mu\nu}_c F'^{\kappa\lambda}_z \rangle\!\rangle A^1_{\mu a} A^1_{\nu b} A'^2_{\kappa x} A'^2_{\lambda y} + 2 \langle\!\langle T_0 A^1_{\mu a} F'^{\kappa\lambda}_z \rangle\!\rangle A^1_{\nu b} F^{\mu\nu}_c A'^2_{\kappa x} A'^2_{\lambda y} \\ &\qquad + 2 \langle\!\langle T_0 F^{\mu\nu}_c A'^2_{\kappa x} \rangle\!\rangle A^1_{\mu a} A^1_{\nu b} A'^2_{\lambda y} F'^{\kappa\lambda}_z + 4 \langle\!\langle T_0 A^1_{\mu a} A'^2_{\kappa x} \rangle\!\rangle A^1_{\nu b} F^{\mu\nu}_c A'^2_{\lambda y} F'^{\kappa\lambda}_z \Big\}, \end{aligned}$$

a symmetrization of which in $(e_1 \leftrightarrow e_2)$ just corresponds to an exchange $x \leftrightarrow x'$ and is therefore not necessary. A similar analysis to the one performed in the previous section yields

$$\begin{aligned} &d_{e_1} T_0[S_1(x,e_1)S_1(x',e_2)]|_{\mathrm{tree}} \qquad (6.39)\\ &= \operatorname{div} - \frac{g(x)g(x')}{2} f_{abc} f_{xyz} \Big\{ \partial_\mu \langle\!\langle T_0 F^{\mu\nu}_c F'^{\kappa\lambda}_z \rangle\!\rangle A'^2_{\kappa x} + 2 \partial_\mu \langle\!\langle T_0 F^{\mu\nu}_c A'^2_{\kappa x} \rangle\!\rangle F'^{\kappa\lambda}_z \Big\} w^1_a A^1_{\nu b} A'^2_{\lambda y}, \end{aligned}$$

where div means a term which becomes a total divergence in the adiabatic limit. Inserting the Dirac deltas (6.26) into (6.39), we realize that the term with the string-integrated delta forms a total divergence, as it did in the previous section, so that the obstruction to string independence in Eq. (6.39) reads

$$\begin{aligned} & ig(x)g(x')\left[f_{abc}f_{xyc}w_a^1(A_b^1 A'^2_y)A'^2_{x\kappa}\partial^\kappa\delta(x-x') + f_{abc}f_{cyz}w_a^1 A^1_{\kappa b}F'^{\kappa\lambda}_z A^2_{y\lambda}\delta(x-x')\right] \\ =\ & \operatorname{div} -i\frac{g^2(x)}{2}f_{abc}f_{xyc}d_{e_1}\left[(A_b^1A_y^2)(A_a^1A_x^2)\right] \\ & + ig^2(x)\left[f_{abc}f_{cyz}w_a^1A^1_{\kappa b}F_z^{\kappa\lambda}A^2_{\lambda y} - \frac{1}{2}f_{abc}f_{xyc}w_a^1F_b^{\kappa\lambda}A^2_{\lambda y}A^2_{\kappa x}\right]. \end{aligned} \tag{6.40}$$

Thus, the first obstruction can be cured by inducing a term to S_2, as it was the case in the previous section. The last line of Eq. (6.40), however, cannot be related to the Jacobi identity – or more general, to any identity that the f_{abc} must satisfy – because it is a sum of terms with different dependence on the string variables, which cannot be resolved even by symmetrizing in $(e_1 \leftrightarrow e_2)$. Therefore, this obstruction to string independence is not resolvable and alternative (4.14b) does not work for massless Yang-Mills theory.

6.2 Graviton coupling to the Maxwell SET

We turn to an example where string independence at second order and tree level excludes a model, which at first sight seems physically reasonable: the coupling of the Maxwell stress energy tensor

$$T^{FF}_{\mu\nu} = \frac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F_{\mu\alpha}F_\nu{}^\alpha \tag{6.41}$$

to a string-localized graviton potential $h^{\mu\nu}$. We want to stress that the following derivations are formulated entirely over Hilbert space and that they do not exclude non-perturbative constructions formulated on Krein space similar to the ones described in Section 3.2. We start by deriving how the string independence principle at first order constrains the interaction Lagrangian.

Theorem 6.5. *The most general coupling of a string-localized graviton potential $h_{\mu\nu}(x,e)$ with two string-localized Maxwell potentials $A_\mu(x,e)$, which is of smallest possible UV dimension and whose string variation is a total divergence is given by*

$$\begin{aligned} S_1(x,e_1,e_2,e_3) &= {:}h_{\mu\nu}(e_1)\,[\partial^\mu A^\kappa(e_2) - \partial^\kappa A^\mu(e_2)]\,[\partial^\nu A_\kappa(e_3) - \partial_\kappa A^\nu(e_3)]{:} \\ &= {:}h_{\mu\nu}(e_1)F^{\mu\kappa}F^\nu_\kappa{:} \equiv S_1(x,e_1) \end{aligned} \tag{6.42}$$

with the point-localized Maxwell field strength $F^{\mu\nu}(x)$.

Before we prove the theorem, recall from Subsection 3.1.2 that the string-localized graviton potential is traceless on Hilbert space, see also Table 3.2. Therefore, the coupling S_1 from Eq. (6.42) is identically equal to

$$\tilde{S}_1(x,e_1) := -{:}h^{\mu\nu}(e_1)T^{FF}_{\mu\nu}{:} \tag{6.43}$$

on Hilbert space, where $T^{FF}_{\mu\nu}$ is the Maxwell stress energy tensor from Eq. (6.41).

Proof (of Theorem 6.5). For readability, we drop the colons indicating normal-ordering. Similar to the treatment of massless Yang-Mills theory in the preceding section, we introduce the auxiliary fields

$$\begin{aligned} & u(x,e) := I_e A_\mu(x,e) de^\mu \quad \text{and} \quad w_\mu(x,e) := I_e h_{\mu\nu}(x,e) de^\mu \\ \Rightarrow \quad & d_e A_\mu(x,e) = \partial_\mu u(x,e) \quad \text{and} \quad d_e h_{\mu\nu}(x,e) = \partial_\mu w_\nu(x,e) + \partial_\nu w_\mu(x,e). \end{aligned} \tag{6.44}$$

The unique candidate for a coupling of two photon potentials with one graviton potential of smallest possible UV dimension $d_{UV} = 3$ is

$$L^3(x, e_1, e_2, e_3) := A^{1\mu} A^{2\nu} h^3_{\mu\nu}, \tag{6.45}$$

where we have again used the notation $A^{i\mu} = A^\mu(x, e_i)$ and similar for $h_{\mu\nu}$. Using the symmetry of $h_{\mu\nu}$, the string variation with respect to e_1 of the symmetrization of L^3 in the string variables is given by

$$\begin{aligned} d_{e_1} L^{3,\text{symm}} &= \frac{1}{3} d_{e_1} \left[A^{1\mu} A^{2\nu} h^3_{\mu\nu} + A^{3\mu} A^{1\nu} h^2_{\mu\nu} + A^{2\mu} A^{3\nu} h^1_{\mu\nu} \right] \\ &= \frac{1}{3} \left[\partial^\mu u^1 A^{2\nu} h^3_{\mu\nu} + A^{3\mu} \partial^\nu u^1 h^2_{\mu\nu} + A^{2\mu} A^{3\nu} (\partial_\mu w^1_\nu + \partial_\nu w^1_\mu) \right] \\ &\overset{\text{div}}{=} -\frac{1}{3} \left[u^1 \partial^\mu A^{2\nu} h^3_{\mu\nu} + w^1_\mu \partial^\nu A^{2\mu} A^3_\nu + (e_2 \leftrightarrow e_3) \right] \end{aligned} \tag{6.46}$$

where we have exploited that both $h_{\mu\nu}$ and A_μ are divergence-free on Hilbert space. Since Eq. (6.46) is not a total divergence, the string independence principle excludes the candidate L^3 already at first order of perturbation theory.

The candidates of the next higher UV dimension contain two derivatives and are thus of UV dimension $d_{UV} = 5$. As described by Eq. (3.107), candidate Lagrangians where these derivatives are contracted form a total divergence because both A_μ and $h_{\mu\nu}$ satisfy the wave equation. In the end, we are only interested in the symmetrization of the candidate Lagrangians with respect to the exchange of any pair of string variables. Modulo such an exchange of string variables, we find six terms that do not form a total divergence on their own,

$$\begin{aligned} & L^{5,1} := \partial^\mu A^{1\kappa} \partial^\nu A^2_\kappa h^3_{\mu\nu}, \quad L^{5,2} := A^{1\kappa} \partial^\mu \partial^\nu A^2_\kappa h^3_{\mu\nu}, \quad L^{5,3} := \partial^\nu \partial_\kappa A^{1\mu} A^{2\kappa} h^3_{\mu\nu}, \\ & L^{5,4} := \partial^\nu A^{1\mu} A^{2\kappa} \partial_\kappa h^3_{\mu\nu}, \quad L^{5,5} := \partial_\kappa A^{1\mu} \partial^\nu A^{2\kappa} h^3_{\mu\nu}, \quad L^{5,6} := A^{1\mu} \partial^\nu A^{2\kappa} \partial_\kappa h^3_{\mu\nu}. \end{aligned} \tag{6.47}$$

However, some of these candidates add up to a total divergence. Namely, we have

$$\begin{aligned} L^{5,1} + L^{5,2} &= \partial^\mu (A^{1\kappa} \partial^\nu A^2_\kappa h^3_{\mu\nu}), \quad L^{5,3} + L^{5,4} = \partial_\kappa (\partial^\nu A^{1\mu} A^{2\kappa} h^3_{\mu\nu}), \\ L^{5,4} + L^{5,6} &= \partial^\nu (A^{1\mu} A^{2\kappa} \partial_\kappa h^3_{\mu\nu}) \quad \text{and} \quad L^{5,5} + L^{5,6} = \partial_\kappa (A^{1\mu} \partial^\nu A^{2\kappa} h^3_{\mu\nu}). \end{aligned} \tag{6.48}$$

Consequently, the most general ansatz for of UV dimension $d_{UV} = 5$ has two free parameters and is given by

$$L^5(x, e_1, e_2, e_3) := c_1 \partial^\mu A^{1\kappa} \partial^\nu A^2_\kappa h^3_{\mu\nu} + c_2 \partial^\mu A^{1\kappa} \partial_\kappa A^{2\nu} h^3_{\mu\nu}. \tag{6.49}$$

We have

$$d_{e_1} L^{5,\text{symm}} = \frac{c_1}{3}\left[\partial^\mu \partial^\kappa u^1 \partial^\nu A^2_\kappa h^3_{\mu\nu} + \partial^\mu A^{2\kappa}\partial^\nu A^3_\kappa \partial_\mu w^1_\nu + (e_2 \leftrightarrow e_3)\right] + \frac{c_2}{6}\left[\partial^\mu \partial^\kappa u^1 \partial_\kappa A^{2\nu} h^3_{\mu\nu} + \partial^\mu A^{2\kappa}\partial_\kappa \partial^\nu u^1 h^3_{\mu\nu} + \partial^\mu A^{2\kappa}\partial_\kappa A^{3\nu}(\partial_\mu w^1_\nu + \partial_\nu w^1_\mu) + (e_2 \leftrightarrow e_3)\right]. \tag{6.50}$$

Again, the terms with two contracted derivatives form a total divergence by Eq. (3.107), so that

$$d_{e_1} L^{5,\text{symm}} \overset{\text{div}}{=} \frac{c_1}{3}\partial^\mu \partial^\kappa u^1 \partial^\nu A^2_\kappa h^3_{\mu\nu} + \frac{c_2}{6}\left[\partial^\mu A^{2\kappa}\partial_\kappa \partial^\nu u^1 h^3_{\mu\nu} + \partial^\mu A^{2\kappa}\partial_\kappa A^{3\nu}\partial_\nu w^1_\mu\right] + (e_2 \leftrightarrow e_3). \tag{6.51}$$

Writing the symmetrization in e_2 and e_3 explicitly, we see that the term containing a $\partial_\nu w^1_\mu$ is a total divergence,

$$\left[\partial^\mu A^{2\kappa}\partial_\kappa A^{3\nu} + \partial^\mu A^{3\kappa}\partial_\kappa A^{2\nu}\right]\partial_\nu w^1_\mu = \partial_\nu\left\{\left[\partial^\mu A^{2\kappa}\partial_\kappa A^{3\nu} + \partial^\mu A^{3\kappa}\partial_\kappa A^{2\nu}\right] w^1_\mu - \partial^\mu A^{3\kappa}\partial_\kappa A^2_\mu w^{1\nu}\right\}, \tag{6.52}$$

because all fields are divergence-free on Hilbert space. Integrating the remaining terms in Eq. (6.51) by parts, we have

$$d_{e_1} L^{5,\text{symm}} \overset{\text{div}}{=} \frac{2c_1 + c_2}{6} u^1 \partial^\nu \partial^\mu A^2_\kappa \partial_\kappa h^3_{\mu\nu} + (e_2 \leftrightarrow e_3). \tag{6.53}$$

The right-hand side of Eq. (6.53) does not form a total divergence and hence, the string independence principle implies $2c_1 + c_2 = 0$. Thus,

$$\begin{aligned} L^{5,\text{symm}} &= c_1\left\{(\partial^\mu A^{1\kappa}\partial^\nu A^2_\kappa - 2\partial^\mu A^{1\kappa}\partial_\kappa A^{2\nu})h^3_{\mu\nu}\right\}^{\text{symm}} \\ &\overset{\text{div}}{=} c_1\left\{(\partial^\mu A^{1\kappa}\partial^\nu A^2_\kappa - 2\partial^\mu A^{1\kappa}\partial_\kappa A^{2\nu} + \partial^\kappa A^{1\mu}\partial_\kappa A^{2\nu})h^3_{\mu\nu}\right\}^{\text{symm}} \\ &= c_1\left\{F^{\mu\kappa}F^\nu{}_\kappa h^3_{\mu\nu}\right\}^{\text{symm}}, \end{aligned} \tag{6.54}$$

where we have again used Eq. (3.107) to insert the term with two contracted derivatives and where the symmetrization in the last line is trivial because the dependence on e_1 and e_2 becomes trivial. This concludes the proof. □

Writing the first order coupling in the form given by Eq. (6.43) is of advantage because the stress energy tensor $T^{FF}_{\mu\nu}$ from Eq. (6.41) is traceless and conserved. We hence can write

$$T[\tilde{S}_1 \tilde{S}'_1]\big|_{\text{tree}} = \langle\!\langle T h^{\mu\nu} h'^{\varrho\sigma}\rangle\!\rangle :T^{FF}_{\mu\nu} T'^{FF}_{\varrho\sigma}: + \langle\!\langle T F_{\alpha\beta} F'_{\xi\eta}\rangle\!\rangle :\frac{\partial T^{FF}_{\mu\nu}}{\partial F_{\alpha\beta}} h^{\mu\nu} \frac{\partial T'^{FF}_{\varrho\sigma}}{\partial F'_{\xi\eta}} h'^{\varrho\sigma}:, \tag{6.55}$$

where the prime indicates a dependence on the primed variables. Because the Maxwell stress energy tensor is traceless, the contribution of the trace of the graviton propagator $\langle\!\langle T h^{\mu\nu} h'^{\varrho\sigma}\rangle\!\rangle$ does not contribute to Eq. (6.55). Therefore, it is a valid ansatz to use kinematic propagators.

Remark 6.6. We will see that the obstruction to second order string independence comes from the contraction $\langle\!\langle T F_{\mu\nu} F'_{\kappa\lambda} \rangle\!\rangle$ and hence is not related to the choice of propagator for the string-localized graviton potential.

It is clear that Eq. (6.55) is already symmetric under the exchange $(e \leftrightarrow e')$, so there is no need for another symmetrization in the string variables. We compute the string variation of the kinematic time-ordered product $T_0[\tilde{S}_1 \tilde{S}'_1]\big|_{\text{tree}}$:

$$\begin{aligned} d_e T_0[\tilde{S}_1 \tilde{S}'_1]\big|_{\text{tree}} &= 2(\partial^\mu \langle\!\langle T_0 w^\nu h'^{\varrho\sigma} \rangle\!\rangle) :T^{FF}_{\mu\nu} T'^{FF}_{\varrho\sigma}: \\ &+ 2\langle\!\langle T_0 F_{\alpha\beta} F'_{\xi\eta} \rangle\!\rangle : \frac{\partial T^{FF}_{\mu\nu}}{\partial F_{\alpha\beta}} (\partial^\mu w^\nu) \frac{\partial T'^{FF}_{\varrho\sigma}}{\partial F'_{\xi\eta}} h'^{\varrho\sigma}: . \end{aligned} \tag{6.56}$$

Because the Maxwell stress energy tensor is conserved, the first line of Eq. (6.56) is a total divergence. Using that $\partial^\mu w_\mu = \eta_{\varrho\sigma} h'^{\varrho\sigma} = 0$ on Hilbert space, we find

$$\begin{aligned} d_e T_0[\tilde{S}_1 \tilde{S}'_1]\big|_{\text{tree}} &\overset{\text{div}}{=} 4\langle\!\langle T_0 F_{\mu\kappa} F'_{\varrho\lambda} \rangle\!\rangle :F_\nu{}^\kappa (\partial^\mu w^\nu + \partial^\nu w^\mu) F'_\sigma{}^\lambda h'^{\varrho\sigma}: \\ &\overset{\text{div}}{=} -4(\partial^\mu \langle\!\langle T_0 F_{\mu\kappa} F'_{\varrho\lambda} \rangle\!\rangle) :F_\nu{}^\kappa w^\nu F'_\sigma{}^\lambda h'^{\varrho\sigma}: \\ &= -4i :F_\nu{}^\kappa w^\nu F'_\sigma{}^\lambda h'^{\varrho\sigma}: \big(\eta_{\kappa\varrho}\partial_\lambda - \eta_{\kappa\lambda}\partial_\varrho\big)\, \delta(x-x'), \end{aligned} \tag{6.57}$$

where we have used the fact that $\langle\!\langle T_0 \partial^x_\mu X(x) X'(x') \rangle\!\rangle = \partial^x_\mu \langle\!\langle T_0 X(x) X'(x') \rangle\!\rangle$ for the kinematic time-ordering as long as no field equation is involved and also the Bianchi identity to obtain the second line and then inserted Eq. (6.26) for the contracted derivative of the kinematic propagator of the Maxwell field strength to obtain the last line. Eq. (6.57) implies that there is a point-localized obstruction to second order string independence at tree level. Integrating the last line of Eq. (6.57) by parts, dropping the colons for the sake of readability and integrating out the Dirac delta to remove the x'-dependence, we obtain

$$\begin{aligned} d_e T_0[\tilde{S}_1 \tilde{S}'_1]\big|_{\text{tree}} &\overset{\text{div}}{=} 4i\left[\partial_\lambda F_{\nu\varrho} w^\nu + F_{\nu\varrho}\partial_\lambda w^\nu - \partial_\varrho F_{\nu\lambda} w^\nu - F_{\nu\lambda}\partial_\varrho w^\nu\right] F_\sigma{}^\lambda h'^{\varrho\sigma} \\ &= 4i\left[F_{\nu\varrho}\partial_\lambda w^\nu - \partial_\nu F_{\varrho\lambda} w^\nu - F_{\nu\lambda}\partial_\varrho w^\nu\right] F_\sigma{}^\lambda h'^{\varrho\sigma}, \end{aligned} \tag{6.58}$$

where we have in the second step used the Bianchi identity to combine the first and third term of the first line and where the prime in $h'^{\varrho\sigma}$ now only indicates a dependence on e' but not on x'. Since $h'^{\varrho\sigma} = h'^{\sigma\varrho}$, the first term in Eq. (6.58) is symmetric under the exchange and is the string variation of an induced term,

$$\begin{aligned} 4iF_{\nu\varrho}\partial_\lambda w^\nu F_\sigma{}^\lambda h'^{\varrho\sigma} &= 2iF_{\nu\varrho}(\partial_\lambda w^\nu + \partial^\nu w_\lambda) F_\sigma{}^\lambda h'^{\varrho\sigma} \\ &= 2i d_e\left[F_{\nu\varrho} F_{\sigma\lambda} h^{\nu\lambda}(e) h^{\varrho\sigma}(e')\right]. \end{aligned} \tag{6.59}$$

The second term in Eq. (6.58) can be rewritten in a similar manner but is no string variation,

$$-\partial_\nu F_{\varrho\lambda} w^\nu F_\sigma{}^\lambda h'^{\varrho\sigma} \overset{\text{div}}{=} \frac{1}{2} F_{\varrho\lambda} F_\sigma{}^\lambda w_\nu \partial^\nu h'^{\varrho\sigma}. \tag{6.60}$$

Finally, using the Bianchi identity and integrating by parts repeatedly, the third term in Eq. (6.58) can be written as

$$\begin{aligned} -F_{\nu\lambda}\partial_\varrho w^\nu F_\sigma{}^\lambda h'^{\varrho\sigma} =& -d_e\left[F_{\nu\lambda}h^\nu{}_\varrho(e)F_\sigma{}^\lambda h'^{\varrho\sigma}(e')\right] + F_{\nu\lambda}\partial^\nu w_\varrho F_\sigma{}^\lambda h'^{\varrho\sigma} \\ \overset{\text{div}}{=}& -d_e\left[F_{\nu\lambda}h^\nu{}_\varrho(e)F_\sigma{}^\lambda h'^{\varrho\sigma}(e')\right] \\ & -\frac{1}{4}\partial_\sigma\left[F_{\nu\lambda}F^{\nu\lambda}\right]w_\varrho h'^{\varrho\sigma} - F_{\nu\lambda}F_\sigma{}^\lambda w_\varrho \partial^\nu h'^{\varrho\sigma} \\ \overset{\text{div}}{=}& \, d_e\left[-F_{\nu\lambda}h^\nu{}_\varrho(e)F_\sigma{}^\lambda h'^{\varrho\sigma}(e') + \frac{1}{8}F_{\nu\lambda}F^{\nu\lambda}h_{\varrho\sigma}(e)h^{\varrho\sigma}(e')\right] \\ & - F_{\nu\lambda}F_\sigma{}^\lambda w_\varrho\partial^\nu h'^{\varrho\sigma}, \end{aligned} \tag{6.61}$$

so that the right-hand side of Eq. (6.58) sums up to

$$d_e T_0[\tilde{S}_1\tilde{S}'_1]\big|_{\text{tree}} \overset{\text{div}}{=} 4id_e L_{\text{ind}}(x,e,e') + 2iF_{\varrho\lambda}F_\sigma{}^\lambda w_\nu\left[\partial^\nu h'^{\varrho\sigma} - 2\partial^\varrho h'^{\nu\sigma}\right] \tag{6.62}$$

with a candidate induced Lagrangian

$$\begin{aligned} L_{\text{ind}}(x,e,e') =& -F_{\nu\lambda}F_\sigma{}^\lambda h^\nu{}_\varrho(e)h'^{\varrho\sigma}(e') - \frac{1}{2}F_{\nu\varrho}F_{\lambda\sigma}h^{\nu\lambda}(e)h^{\varrho\sigma}(e') \\ & + \frac{1}{8}F_{\nu\lambda}F^{\nu\lambda}h_{\varrho\sigma}(e)h^{\varrho\sigma}(e'). \end{aligned} \tag{6.63}$$

In Appendix B, we show that any admissible induced Lagrangian can only be a linear combination

$$\Delta L = c_1\, F^{\kappa\lambda}F_{\kappa\lambda}h^{\mu\nu}h_{\mu\nu} + c_2\, F^{\mu\kappa}F_\mu{}^\lambda h^{\kappa\varrho}h^\lambda{}_\varrho + c_3\, F_{\mu\varrho}F_{\nu\sigma}h^{\mu\nu}h^{\varrho\sigma}, \tag{6.64}$$

so L_{ind} is a special case of ΔL. However, the last term on the right-hand side of Eq. (6.62) is not the string variation of any such term and hence a non-removable obstruction to string independence at second order. We have thus shown:

Theorem 6.7. *A perturbative description on Hilbert space of the coupling of a string-localized graviton potential to the Maxwell stress energy tensor is in conflict with the string independence principle at second order of perturbation theory (and tree level).*

We emphasize that Theorem 6.7 applies only to the formulation of the graviton-Maxwell coupling *on Hilbert space*. A formulation on Krein space, where the fields satisfy less relations and where the escort fields need to be taken into account, is not excluded by our derivations. The considerations of Brüers [11] for the coupling of a graviton escort field to the SET of a massive scalar field indeed suggest that pure Krein space constructions work fine. Still, the exclusion of a Hilbert space construction of the model at hand demonstrates the restrictive power of the string independence principle.

6.3 Graviton self-coupling: current state and obstacles

An investigation of a self-coupling of string-localized graviton potentials has been started by the author in his Master's thesis but as of yet, it is unclear how a perturbative description of such a self-coupling can be implemented. In the following, we describe several obstacles that currently block the way.

To start with, note that the cubic self-coupling $L_G^{(3)}$ from Eq. (3.106) differs by a total divergence from the coupling

$$\tilde{L}_G^{(3)} = :h^{\mu\nu} G_{\mu\alpha,\beta} G_\nu{}^{\alpha,\beta}:, \tag{6.65}$$

where $G_{\mu\alpha,\beta} = \partial_\mu h_{\alpha\beta} - \partial_\alpha h_{\mu\beta}$ are "partial" field strengths. On Hilbert space, the partial field strength satisfies the Bianchi identity

$$\partial_\kappa G_{\mu\nu,\alpha} + \partial_\mu G_{\nu\kappa,\alpha} + \partial_\nu G_{\kappa\mu,\alpha} = 0 \tag{6.66}$$

and the non-trivial conditions

$$\partial^\mu G_{\mu\nu,\alpha} = \partial^\nu G_{\mu\nu,\alpha} = \partial^\alpha G_{\mu\nu,\alpha} = \eta^{\mu\alpha} G_{\mu\nu,\alpha} = \eta^{\nu\alpha} G_{\mu\nu,\alpha} = 0. \tag{6.67}$$

Moreover, its string variation is given by

$$d_e G_{\mu\nu,\alpha} = \partial_\alpha \left[\partial_\mu w_\nu - \partial_\nu w_\mu\right]. \tag{6.68}$$

The cubic part of the graviton self-coupling Eq. (6.65) is thus of similar shape as the graviton coupling to the Maxwell SET, which we have discussed in the previous section. There are two main differences: First, the Maxwell field strength is string independent while the non-trivial string variation of the partial field strength $G_{\mu\nu,\alpha}$ is given by Eq. (6.68). Second, the Maxwell field strength and the graviton potential are independent fields, whose mixed two-point function vanishes, while the mixed two-point function of the gravitation potential and the partial field strength $G_{\mu\nu,\alpha}$ is non-zero. Thus, there are more terms in the Wick expansion of the graviton self-coupling.

To consider the second order tree graph contribution, we employ again our notation $X(x, e_i) \to X^i$, $X(x', e_i) \to X'^i$ and drop the colons indicating normal-ordering for the sake of readability. For a generic and so far unspecified time-ordering prescription, we have

$$\begin{aligned} T\left[\tilde{L}_G^{(3)} \tilde{L}_G^{(3)}\right]\Big|_{\text{tree}}^{\text{symm}} = \Big\{ & \langle\!\langle T h^{1\mu\nu} h'^{4\varrho\sigma} \rangle\!\rangle G^2_{\mu\kappa,\lambda} G^{3\kappa,\lambda}_\nu G'^5_{\varrho\alpha,\beta} G'^{6\alpha,\beta}_\sigma \\ & + 2\langle\!\langle T G^2_{\mu\kappa,\lambda} h'^{4\varrho\sigma} \rangle\!\rangle h^{1\mu\nu} G^{3\kappa,\lambda}_\nu G'^5_{\varrho\alpha,\beta} G'^{6\alpha,\beta}_\sigma \\ & + 2\langle\!\langle T h^{1\mu\nu} G'^5_{\varrho\alpha,\beta} \rangle\!\rangle G^2_{\mu\kappa,\lambda} G^{3\kappa,\lambda}_\nu h'^{4\varrho\sigma} G'^{6\alpha,\beta}_\sigma \\ & + 4\langle\!\langle T G^2_{\mu\kappa,\lambda} G'^5_{\varrho\alpha,\beta} \rangle\!\rangle h^{1\mu\nu} G^{3\kappa,\lambda}_\nu h'^{4\varrho\sigma} G'^{6\alpha,\beta}_\sigma \Big\}^{\text{symm}}. \end{aligned} \tag{6.69}$$

The investigation of the tree graph contribution from Eq. (6.69) is far more involved than for the previous case of the graviton coupling to the Maxwell SET.

The most intricate issue is the choice of propagators. In the case of the graviton coupling to the Maxwell SET, we employed the kinematic propagator for the graviton potential and effectively removed the trace component by coupling $h^{\mu\nu}$ to the traceless SET $T^{FF}_{\mu\nu}$ instead of $F_{\mu\kappa}F_\nu{}^\kappa$. Indeed, our (tree level) derivations in the previous section are equivalent to considering the coupling $h^{\mu\nu}F_{\mu\kappa}F_\nu{}^\kappa$ and using the traceless propagator

$$\langle\!\langle T_\eta h^{\mu\nu} h'^{\varrho\sigma}\rangle\!\rangle := \left(\eta^{\mu\kappa}\eta^{\nu\lambda} - \frac{1}{4}\eta^{\mu\nu}\eta^{\kappa\lambda}\right)\left(\eta^{\varrho\alpha}\eta^{\sigma\beta} - \frac{1}{4}\eta^{\varrho\sigma}\eta^{\alpha\beta}\right)\langle\!\langle T_0 h_{\kappa\lambda} h'_{\alpha\beta}\rangle\!\rangle. \tag{6.70}$$

The propagator $\langle\!\langle T_\eta h^{\mu\nu} h'^{\varrho\sigma}\rangle\!\rangle$ does not respect the axiality of the potential and is thus a similarly poor choice as the kinematic propagator. In the previous section, T_η respectively T_0 served our purpose: the sector with a graviton contraction is string independent, while the obstruction came from the sector with a contraction of Maxwell field strengths. Due to the scaling degree of the graviton propagator and the field content in the problematic term in Eq. (6.62), the obstruction to second order string independence in the graviton-Maxwell coupling cannot be resolved by changing the graviton propagator. Hence, we did not raise this issue in the previous section.

In the present case, however, the situation is different. Here, there is only a graviton sector, so if there is an obstruction to second order string independence at tree level, it can only come from this sector. For that reason, a careful treatment of the choice of propagators is due. But as of yet, the details of such a proper treatment are unclear:

Employing T_0 or T_η (the latter does act non-trivial on $G_{\mu\nu,\alpha}$ as well), it seems that one runs into similar trouble as in the case of the graviton-Maxwell coupling. However, due to the dependence on six instead of two string variables and the non-trivial string variation of $G_{\mu\nu,\alpha}$, the computations are far more involved than they were in the previous case.[2]

Another option is to use one of the traceless propagators derived in Section 5.7.1, which have the benefit that they respect the axiality of the potential $h_{\mu\nu}$. One can then for example use the BDF construction to determine the propagator of the partial field strength $G_{\mu\nu,\alpha}$ from the traceless propagator of $h_{\mu\nu}$. However, also this treatment comes with certain flaws.

All admissible traceless propagators differ from the kinematic propagator by a linear combination of derivatives of string-integrated Dirac deltas. In momentum space, the coefficients of this linear combination are determined by the allowed correction terms N_1, N_2 and K_i, $i = 1, 2, 3$ (in the notation of Section 5.7.1; see also Proposition 5.44) with the dummy vectors g and g' replaced by

$$q^\mu := -i\frac{e^\mu}{(pe)_-} \quad \text{and} \quad q'^\mu := i\frac{e'^\mu}{(pe')_+}, \quad \text{respectively.} \tag{6.71}$$

We have

$$d_e q^\mu = \frac{-i}{(pe)_-}\left[de^\mu - \frac{e^\mu(pde)}{(pe)_-}\right], \tag{6.72}$$

[2]As of yet, it seems that not all obstructions to second order string independence can be resolved but due to the complexity of the terms, some caveats may remain.

and using the distributional identity $(pq) = -i = const.$, we obtain

$$\begin{aligned} d_e(p[fq]X) = d_e(pf)(qX) &= \frac{(pf)}{(pe)_-}\left[(Xde)(pq) - (pde)(Xq)\right] \\ &= -\frac{(pf)}{(pe)_-}(p[deq]X) \end{aligned} \tag{6.73}$$

for some dummy vector X. Eq. (6.73) implies that a gradient can be pulled out after performing the string variation if a skewsymmetric tensor $[fq]$ is contracted with p. This is always the case for the kernel of the kinematic propagator (see Eq. (5.97)) and forms the foundation of the identity

$$d_e\langle\langle T_0 h_{\mu\nu}(e) h'_{\varrho\sigma}(e')\rangle\rangle = \partial_\mu\langle\langle T_0 w_\nu(e) h'_{\varrho\sigma}(e')\rangle\rangle + \partial_\nu\langle\langle T_0 w_\mu(e) h'_{\varrho\sigma}(e')\rangle\rangle. \tag{6.74}$$

Partial derivatives are needed in order for possible obstructions to sum up to a total divergence. However, the action of d_e on the correction terms N_i and K_j, which are needed to make the kinematic propagator traceless, does not always produce derivatives. Recall for example that

$$N_1 = (p[fq][fq][f'q'][f'q']p) \tag{6.75}$$

so that

$$\begin{aligned} d_e N_1 = &-\frac{(pf)}{(pe)_-}(p[deq][fq][f'q'][f'q']p) \\ &+ (p[fq][f(d_e q)][f'q'][f'q']p). \end{aligned} \tag{6.76}$$

The first term on the right-hand side of Eq. (6.76) is obviously a gradient, while the second term contains a contribution

$$\begin{aligned} -(pq)(pq')(f[f(d_e q)][f'q']f') &= -(f[f(d_e q)][f'q']f') \\ &\supset \frac{i}{(pe)_-}(f[fde][f'q']f'), \end{aligned} \tag{6.77}$$

which contains no derivative at all and gives rise to a string-integrated Dirac delta and a possibly highly non-local obstruction to string independence. It is an open (and presumably very tedious) task to check whether all obstructions of this type cancel each other.

Also the BDF construction comes with issues. Recall from Section 5.7.4 that this construction contains the axial tensor $B_{\mu\nu} = \eta_{\mu\nu} - \frac{e_\mu e_\nu}{e^2}$, which has a non-trivial string variation. Thus, the time-ordered products from the BDF construction do in general not commute with string variation. The consequences of this observation are not known as of yet.

To sum up, there remain many obstacles to the perturbative construction of a self-coupling of string-localized graviton potentials. At present, we do not know whether these obstacles are just technical difficulties or whether they indicate a physical (no-go) feature of the model.

Chapter 7

Discussion and outlook

In the main body of this dissertation, we have to a large extent dispensed with extensive interpretations of our results, an exception being Section 5.7.5, where we discussed the interference of different methods to reduce renormalization ambiguities. In the following, we catch up on this and give more detailed interpretations of our different results. We also outline loose ends and open tasks for future research.

7.1 Self-interactions and non-perturbative effects

We have shown in Sections 3.4.1 and 3.4.2 that the cubic part of the interaction Lagrangians of massless string-localized Yang-Mills theory and of the string-localized graviton self-interaction are not part of an L-V pair. These findings suggest that one must not expect the existence of an L-V pair in massless self-interacting models. On the other hand, the non-perturbative constructions in QED by Mund, Rehren and Schroer [48, 51] as well as the ones for the coupling of the graviton escort fields to the stress energy tensor of a point-localized scalar field by Brüers [11], which we have outlined in Section 3.2, rely on the existence of such an L-V pair. Hence, constructions of that type cannot be performed for massless Yang-Mills theory and graviton self-interactions.

Consequently, there is no straightforward way to adjust the hybrid approach to self-interacting models. The relations between point-localized and string-localized models become less clear than for theories without self-interactions of string-localized fields. There is, however, a possible way to circumvent our more or less expected no-go results by a mix of perturbative and non-perturbative constructions. Such constructions, which take into account the self-interactions in a perturbative way, have been described in a more phenomenological setting [43], which uses the original Dirac string fields that we introduced in Section 1.1. It remains an interesting question whether the constructions made in [43] can be reformulated in terms of escort fields. Indeed, there is work in progress by Rehren and collaborators that indicates that the answer is in the affirmative (private conversation with K.H. Rehren).

An observation due to J.-M. Gracia-Bondía, which is part of our joint work [37] together with J. Mund, is related to the θ-term in QCD. In our perturbative derivations (up to second order) for string-localized Yang-Mills theory in Section 6.1, we have disregarded any term that is a total divergence and hence gives rise to a trivial S-matrix in the adiabatic limit. One particular term that is a total divergence in the adiabatic limit is the θ-term[1]

$$L_\theta := \theta\varepsilon^{\mu\nu\kappa\lambda}\left[F_{\mu\nu a}F_{\kappa\lambda a} + 2g f_{abc} f_{adf} F_{\mu\nu a} A_{\kappa d} A_{\lambda f}\right] \tag{7.1}$$

with the totally skewsymmetric Levi-Civita tensor $\varepsilon^{\mu\nu\kappa\lambda}$, and where $L_\theta = 2\theta\partial_\mu K^\mu$ with the Chern-Simons current (see for example [62])

$$K^\mu = \varepsilon^{\mu\nu\kappa\lambda}\left[A_{\nu a}F_{\kappa\lambda a} + \frac{2}{3} g f_{abc} f_{adf} F_{\mu\nu a} A_{\kappa d} A_{\lambda f}\right]. \tag{7.2}$$

Despite not contributing to perturbation theory, the θ-term needs to be considered in gauge theories, for it can produce non-perturbative effects in non-trivial topologies [75].[2] There, one can achieve a finite action even if the gauge field is topologically non-trivial, provided that the gauge field approaches a pure gauge in the long distance limit $x \to \infty$. However, there is no concept of gauge in string-localized field theory and consequently, a pure gauge configuration of the string-localized potential does not exist. This suggests the conclusion that there is no θ-term in string-localized massless Yang-Mills theory.

There is a small caveat in this reasoning. It is true that there is no pure gauge in string-localized QFT; but there is the escort field, which is logarithmically divergent at large distances and thus can give rise to boundary effects. To fix the loose ends, one needs to take into account possibly existent formulations of the mentioned perturbative-plus-non-perturbative constructions [43] in terms of the escort fields.

7.2 Time-ordering methods in SLFT

The only fully rigorous method to define time-ordered products involving string-localized fields that is available at the present time is string chopping, which was introduced by Cardoso, Mund and Várilly [16]. However, string chopping only works for expressions that are linear in the string-localized potentials (and the respective field strength tensors) and for a small set of other models that can be reduced to the linear case [37]. In Sections 4.1.2 and 4.2, we outlined an ad hoc alternative to string chopping. Indeed, the outlined prescription is expected to be an ad hoc *generalization* of string chopping, for it realizes an automatic chopping.

To see that, remember that we have proposed to define time-ordered products involving string-localized potentials as the respective string integrals over point-localized products that depend on nk or $n(k+1)$ instead of n variables at n-th order of perturbation

[1]Remember that in our notation, the field strength tensors are always given by $F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}$, and not by $F'_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g f_{abc} A_{\mu b} A_{\nu c}$, which is typically employed in non-Abelian gauge theories.

[2]Introducing all details is beyond the scope of this thesis. We refer the reader to Weinberg's book [75] for a thorough introduction. Note also that topological issues are usually treated in Euclidean geometry.

theory, where k is the number of string-localized potentials that appear in the interaction Lagrangian L. The time-ordered products depend on nk arguments if L only contains string-localized potentials and on $n(k+1)$ arguments if L additionally contains point-localized fields.

The nk arguments $y_{j,l} = x_j + \lambda_{j,l} e_{j,l}$ for $j = 1, \ldots, n$, $l = 1, \ldots, k$ and $\lambda_{j,l} \geq 0$ correspond to points on the strings, so that defining time-ordering with respect to these arguments automatically chops the strings.

However, our ad hoc generalization comes with the drawback that we have to drop the Wick ordering of the interaction Lagrangian in an intermediate step , in order to have a causal factorization rule with respect to all variables of the new time-ordered products. This gives rise to "self-contractions" in the Wick expansion, which have to be removed again in retrospective. At the present time, it seems that a "good" causal factorization rule – meaning one where the set of ambiguities in a BEG construction is not tremendously large – only holds for the time-ordered products with the normal ordering of the interaction Lagrangian dropped.

In order to set up a comprehensive BEG construction in SLFT, which is soundly based on a set of reasonable axioms, one needs to lift the fog surrounding our proposed method, in particular the appearance and removal of "self-contractions".

7.3 Renormalization of divergent amplitudes

Despite the fact that a full BEG scheme of the string-localized S-matrix is still missing, we were able to characterize the structure of (true) divergences in string-localized perturbation theory.

The BEG freedom of time-ordered products of two arguments that arises from our method can easily be determined. It is a linear combination of derivatives of string-integrated Dirac deltas. We classified the corresponding ambiguities of string-localized propagators in Section 5.4.2 and formulated reasonable constraints on these ambiguities coming from power counting and IR constraints (local integrability at $p = 0$ in momentum space). We determined that the wavefront set of all admissible choices of string-localized propagators in Section 5.5 are contained in the wavefront set of the ordinary Feynman propagator after smearing out the string variables. As a consequence, products of string-localized propagators exist whenever the respective products in the point-localized case exist. That is to say, the regularization of divergent loop graph contributions stays a pure UV problem in SLFT. In particular, the new singularities that are introduced by the string integration operation become harmless after smearing out the string variables. In combination with the good UV behavior of string-localized potentials, it thus remains an interesting question whether one can formulate renormalizable models in SLFT where the point-localized counterpart is non-renormalizable.

This result crucially relies on the fact that each string-localized potential in the Dyson series Eq. (4.15) comes with its own string variable. Only then are the products of distributions appearing in said Dyson series pure x-products and a pullback to the e-diagonal is not necessary.

The proofs in Sections 5.2 to 5.5 show that one can regularize the Wick expansion of the Dyson series whenever one can do so in the point-localized case. However, we have not given a full classification of all possible ambiguities in the construction.

Our results on renormalization can also be applied to models with axial gauges because the analytic structure of propagators in axial or lightcone gauges is similar to the one of string-localized propagators (see for example [45] for an introduction to axial gauges).[3] Axial gauge propagators, however, are usually not treated as distributions in the variable n, which represents the preferred direction and is the analogue of the string variable in SLFT. Hence, the singularities that arise when the Minkowski product of n with the momentum p vanishes are of a different nature than the ones discussed in this thesis. The singularities at $(pn) = 0$ were an important reason for the decreasing interest in axial gauges over the past decades.

Adjusting the framework of axial gauges by treating the respective propagators as distributions in n and letting each appearing axial gauge field depend on its own n, our results can be transferred with benefit to axial gauge theories (with n spacelike). Thus, the singularities at $(pn) = 0$ in axial gauges do not cause additional problems for renormalization if they are treated as described in this dissertation.

Axial gauges suffer from analytic complexity but also offer advantages, in particular if each axial gauge field comes with its own n: They prove useful in the so-called spinor-helicity formalism that drastically reduces the computational effort to determine gluon scattering matrix elements [62, Chapters 25.4.3 and 27]. Due to the close formal connection between axial gauge and string-localized potentials, it is worthwhile to investigate whether the spinor-helicity formalism can be adjusted to the string-localized setup of perturbation theory presented in this thesis.

7.4 Reduction of the renormalization freedom

The different methods to reduce the renormalization freedom beyond power counting described in Section 5.7 are partially incompatible. We have illustrated this incompatibility by applying several such methods to string-localized QED in Sections 5.7.2 to 5.7.4. We found that the NST prescription, which is based on requiring a certain transformation behavior of massless propagators (or general massless amplitudes) under the action of the Lorentz group, yields a different propagator of the Maxwell field strength than the BDF construction, in which onshell time-ordered products are defined as a particular choice of offshell time-ordered products. The BDF construction goes hand in hand with the string independence principle, while the NST prescription contradicts string independence.

Since both the NST prescription and the BDF construction are special cases of onshell extensions of distributions in the sense of Bahns and Wrochna [2], we find that such onshell extensions must in general be taken with a grain of salt. In the case at hand, we have resolved the issue as follows (see also Section 5.7.5):

- A string dependence of observable quantities, say, cross sections, is not observed

[3] The following paragraphs on the connection to axial gauges are taken from the author's paper [35].

in experiment and hence the string independence principle should be treated as paramount. While the BDF construction is consistent with string independence and even benefits from it (by fixing the choice for the offshell time-ordering, see Section 5.7.4), the NST prescription is in conflict with string independence. Thus, the NST prescription should not be implemented.

- In retrospective, the failure of the NST prescription can be made comprehensible by the interpretation (due to K.-H. Rehren) that the interaction destroys the refined NST notion of Lorentz covariance.

When Várilly and Gracia-Bondía first outlined an application of the NST prescription to string-localized fields [72], they already referred to the importance of the string independence principle, and at the same time raised the question whether an additional reduction of the renormalization freedom beyond power counting and string independence is necessary, suggesting the NST prescription as solution:

> *"Of course, the string 'ought not to be seen', and the program becomes to demonstrate whether, and how, this simple criterion is enough to determine interaction vertices and govern perturbative renormalization* [. . .].
> [. . .]
> *What we realize is that the construction of string-local fields* [. . .] *rests on the bedrock of a never-ambiguous time-ordered product of the field strengths."*

At this point, we cannot answer the question whether power counting and string independence are enough to fix the BEG freedom completely in a generic model but we have seen that the NST prescription is not the solution to fixing possibly remaining free parameters in perturbative renormalization, at least not in QED.

7.5 The construction of models in SLFT

The perturbative construction of models involving string-localized potentials remains an important task for future research. We have seen in Section 6.1 that the construction of massless Yang-Mills theory to second order (and at tree level) works fine but a proof of perturbative string independence to all orders (also at least at tree level) is still pending. On the other hand, our derivations in Section 6.2 show that the string independence principle at second order forbids a perturbative construction on Hilbert space of the coupling of a string-localized graviton to the Maxwell stress energy tensor. Since this is a model of physical interest, the implications of this no-go result need to be clarified. At present, the consequences are not clear. It could simply be that one needs to resort to a Krein space construction, where there are also unphysical degrees of freedom and where the fields satisfy less relations. It is also conceivable that the inclusion of the trace of the graviton potential (or more general, an additional scalar field) could help resolve the issue. On the other hand, the no-go result could have a profound physical meaning. For example, it could be that only the coupling of the escort field to the Maxwell SET, which is a total divergence and hence should only produce non-perturbative effects, can be implemented. Such a program was investigated by Brüers for the SET of a scalar field [11].

It is an interesting fact that perturbative constructions on Hilbert space in SLFT seem to work fine for the standard model interactions. We have discussed (pure) massless Yang-Mills theory in Section 6.1, while the weak interaction has been investigated in [39], where it is derived that the string independence principle implies the appearance of a scalar field ("Higgs") as well as the chirality of the interaction. There are also unpublished notes on constructions in QED. To the best knowledge of the author, only a gluon-matter interaction has not yet been attacked in an SLFT framework.

Beyond the standard model, very little is known. There are positive results in scalar QED [67], while we derived the mentioned no-go result in Section 6.2. There is work in progress by Mund, Rehren and Schroer on the Abelian Higgs model. The obstacles to a graviton self-coupling, which we have summarized in Section 6.3, need to be investigated. Having both the positive and negative results in mind, it remains an interesting open question, which Hilbert space models are allowed and which are excluded by the string independence principle.

7.6 Loop graphs in SLFT

Another interesting question for future research is the treatment of loop graphs in SLFT. In particular, one must investigate how the string independence principle can be formulated at loop level. We proved in Chapter 5 that the true divergences corresponding to ill-defined loop integrals stay pure ultraviolet divergences in SLFT. This implies that the well-known tools from standard QFT for UV/short-distance regularization can be employed in SLFT. To be able to treat loop graph contributions, one needs to find out how the string independence principle interacts with the extension procedure of products of propagators. Formally, introducing some generic distribution $v(x, \underline{\tilde{e}})$, a prototypical example to consider would be

$$
\begin{aligned}
&d_e \left[\langle\!\langle T_0 A_\mu(e) A_\nu(e')\rangle\!\rangle(x) v(x, \underline{\tilde{e}})\right]_{\text{ext}} \\
&= \left[d_e \langle\!\langle T_0 A_\mu(e) A_\nu(e')\rangle\!\rangle(x) v(x, \underline{\tilde{e}})\right]_{\text{ext}} \\
&= \left[(\partial_\mu \langle\!\langle T_0 w(e) A_\nu(e')\rangle\!\rangle(x)) v(x, \underline{\tilde{e}})\right]_{\text{ext}} \\
&\overset{?}{=} \partial_\mu \left[\langle\!\langle T_0 w(e) A_\nu(e')\rangle\!\rangle(x) v(x, \underline{\tilde{e}})\right]_{\text{ext}} - \left[\langle\!\langle T_0 w(e) A_\nu(e')\rangle\!\rangle(x) \partial_\mu v(x, \underline{\tilde{e}})\right]_{\text{ext}} \\
&\quad + \text{ambiguities supported on } \{\, x = 0 \,\},
\end{aligned}
\tag{7.3}
$$

where "ext" stands for "extended". Because the regularization is only x-dependent, we can pull d_e into the extended product but one has to investigate the effect of pulling out the gradient in the last step. Note in particular that the products to be extended in the last and second last line of Eq. (7.3) are not of the same UV dimension. Only when the interaction of the string variation with the extension procedure is clarified, one can hope to implement perturbative string independence also at loop level.

7.7 Final comments

We have investigated both perturbative and non-perturbative constructive aspects of string-localized quantum field theory, finding many affirmative answers to our questions but also certain no-go results.

Non-perturbative aspects On the positive side, the explicit computation of the two-point function of the photon escort field allowed us to characterize the structure of infrared divergences in QED. On the negative side, we showed that the Lagrangians of massless Yang-Mills theory and the graviton self-coupling are not part of an L-V pair, indicating that there is no immediate analogue for the non-perturbative constructions of Mund, Rehren and Schroer in QED [51] if self-interactions of massless fields are involved. It remains a task to investigate whether one can do perturbative-plus-non-perturbative constructions similar to the ones that have been proposed in the point-localized case [43]. As previously mentioned, the work in this direction has already been started by Rehren and collaborators.

Perturbative aspects We have shown that the regularization of divergent loop integrals stays a pure ultraviolet problem in SLFT, indicating that the good UV behavior of string-localized fields has positive effects on renormalizability because power counting remains a meaningful indicator for renormalizability. We also have proposed a method to define the time-ordering operation if string-localized fields are involved but a full axiomatic implementation of this method is still pending. Despite the fact that no fully constructed BEG scheme is available for SLFT at the present time, we investigated different methods to reduce the BEG ambiguities in SLFT, including the NST and BDF constructions as well as effects of the string independence principle. However, these methods turned out to be partially incompatible, and thus one needs to be careful when applying them. We have also found that a perturbative formulation of string-localized massless Yang-Mills theory on Hilbert space works well in low orders of perturbation theory (and tree level), while the same construction fails for the coupling of a string-localized graviton potential to the Maxwell SET.

Implications We have found some negative — or no-go — results, which indicate that further research is needed to clarify certain issues that are present in SLFT. It is possible that there are ways within SLFT to circumvent these issues, for example a formulation of certain models on Krein instead of Hilbert space within the hybrid approach or a generalization of the L-V connection to point-localized theories. Research in these directions has already been started.

At the same time, our positive results clearly speak in favor of string-localized quantum field theory. The construction of string-localized models is advancing and sheds new light on SLFT in general. The explicit computation of the two-point functions of the photon escort field and the corresponding vertex operators opens the door for explicit computations in the scattering theory of dressed Dirac fields. The progress on renormalization that we have made paves the way towards a treatment of loop graphs in SLFT. In summary, we can say that we have found many new pieces of the puzzle but at present, we cannot clearly see how the full picture looks like.

Appendix A

Proof of Theorem 3.23

In this appendix, we give a detailed proof of Theorem 3.23 about then non-existence of an L-V pair for the string-localized graviton self-interaction. For the convenience of the reader, we display the theorem again:

Theorem A.1 (Theorem 3.23)**.** *The cubic part*

$$L_G^{(3)} = :h^{\mu\nu}\left[\partial_\mu h_{\kappa\lambda}\partial_\nu h^{\kappa\lambda} + 2\partial^\kappa h_{\mu\lambda}\partial^\lambda h_{\nu\kappa}\right]: \tag{A.1}$$

of the string-localized graviton self-coupling from Eq. (3.106) is not part of an L-V pair.

In the following proof, we make excessive use of the notation defined on the right-hand side of Eq. (3.107).

Proof. Remember from Eq. (3.1b) the relation

$$h_{\mu\nu} = h^{\mathrm{K}}_{\mu\nu} + \partial_\mu\Phi_\nu + \partial_\nu\Phi_\mu + \partial_\mu\partial_\nu\phi \tag{A.2}$$

between the string-localized potential $h_{\mu\nu}$ and the Krein potential $h^{\mathrm{K}}_{\mu\nu}$, with the escort fields $\Phi_\mu = I_e(h^{\mathrm{K}}e)_\mu$ and $\phi = I_e^2(eh^{\mathrm{K}}e)$. Moreover, remember that $h_{\mu\nu}$ and the field strength $F_{\mu\kappa\nu\lambda}$ satisfy less constraints on Krein space than on Hilbert space. By our findings from Section 3.1.2 (see Table 3.2), the string-localized potential, the Krein potential and the escort fields only satisfy the following conditions on Krein space:

$$h_{\mu\nu} = h_{\nu\mu}, \quad h^{\mathrm{K}}_{\mu\nu} = h^{\mathrm{K}}_{\nu\mu}, \quad e^\mu h_{\mu\nu} = \Box h_{\mu\nu} = \Box h^{\mathrm{K}}_{\mu\nu} = \Box\Phi_\mu = \Box\phi = 0. \tag{A.3}$$

In particular, the trace $h := \eta^{\mu\nu}h_{\mu\nu}$ and the divergence $(\partial h)_\nu$ do not vanish on Krein space. Denoting by h^{K} the trace of the Krein field, we found the relations

$$\begin{aligned} & h = h^{\mathrm{K}} + 2(\partial\Phi), \quad \partial^\mu h_{\mu\nu} = \partial^\mu h^{\mathrm{K}}_{\mu\nu} + \partial_\nu(\partial\Phi) \\ \Rightarrow \quad & \partial^\mu\left(h_{\mu\nu} - h^{\mathrm{K}}_{\mu\nu}\right) = \frac{1}{2}\partial_\nu(h - h^{\mathrm{K}}). \end{aligned} \tag{A.4}$$

Similar to the simpler case of Yang-Mills theory, we split the proof in two parts. First, we insert Eq. (A.2) into the Lagrangian $L_G^{(3)}$ from Eq. (3.106) to show that $L_G^{(3)}$ is a sum of a point-localized Lagrangian, a total divergence and certain obstructing terms. In the second step, we show that the obstructing terms cannot be removed by adding terms to $L_G^{(3)}$ that vanish on Hilbert space but not on Krein space. For readability, we drop the colons indicating normal ordering but all terms are to be understood as normally ordered.

Part 1: Using Eq. (A.2), let us first insert the Krein and escort fields into the string-localized potential outside the brackets in the Lagrangian Eq. (3.106) to get the expression

$$L_G^{(3)} = \left(h^{K\mu\nu} + \partial^\mu\Phi^\nu + \partial^\nu\Phi^\mu + \partial^\mu\partial^\nu\phi\right)\left(\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + 2\partial^\alpha h_{\mu\beta}\partial^\beta h_{\nu\alpha}\right) \tag{A.5a}$$
$$= h^{K\mu\nu}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + 2\left(h^{K\mu\nu} + 2\partial^\mu\Phi^\nu + \partial^\mu\partial^\nu\phi\right)\partial^\alpha h_{\mu\beta}\partial^\beta h_{\nu\alpha} + \partial^\mu V_\mu^{(1)} \tag{A.5b}$$

where $V_\mu^{(1)}$ on the right-hand side of (A.5b) arises from

$$\partial^\mu V_\mu^{(1)} := \mathfrak{div}\left[\left(\partial^\mu\Phi^\nu + \partial^\nu\Phi^\mu + \partial^\mu\partial^\nu\phi\right)\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right], \tag{A.6}$$

where $\mathfrak{div}$ is the notation for the special total divergence introduced in Eq. (3.107). A further investigation of the first term in Eq. (A.5b) yields

$$\begin{aligned}
&h^{K\mu\nu}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\\
&= h^{K\mu\nu}\partial_\mu\left[h^K_{\alpha\beta} + 2\partial_\alpha\Phi_\beta + \partial_\alpha\partial_\beta\phi\right]\partial_\nu\left[h^{K\alpha\beta} + \partial^\alpha\Phi^\beta + \partial^\beta\Phi^\alpha + \partial^\alpha\partial^\beta\phi\right]\\
&= h^{K\mu\nu}\partial_\mu h^K_{\alpha\beta}\partial_\nu\left[h^{K\alpha\beta} + \partial^\alpha\Phi^\beta + \partial^\beta\Phi^\alpha + \partial^\alpha\partial^\beta\phi\right]\\
&\quad + 2h^{K\mu\nu}\partial_\mu\partial_\alpha\Phi_\beta\partial_\nu\left[h^{K\alpha\beta} + \partial^\beta\Phi^\alpha\right] + h^{K\mu\nu}\partial_\mu\partial_\alpha\partial_\beta\phi\partial_\nu h^{K\alpha\beta} + \partial^\mu V_\mu^{(2)},
\end{aligned}$$

where $V_\mu^{(2)}$ arises from

$$\begin{aligned}
\partial^\alpha V_\alpha^{(2)} := \mathfrak{div}\,\{&h^{K\mu\nu}\partial_\mu\partial_\alpha\partial_\beta\phi\partial_\nu\left[2\partial^\alpha\Phi^\beta + \partial^\alpha\partial^\beta\phi\right]\\
&+2h^{K\mu\nu}\partial_\mu\partial_\alpha\Phi_\beta\partial_\nu\left[\partial^\alpha\Phi^\beta + \partial^\alpha\partial^\beta\phi\right]\}.
\end{aligned} \tag{A.7}$$

We next consider the second term in Eq. (A.5b), which can be expanded to

$$\begin{aligned}
&2\left(h^{K\mu\nu} + 2\partial^\mu\Phi^\nu + \partial^\mu\partial^\nu\phi\right)\partial^\alpha\left[h^K_{\mu\beta} + \partial_\mu\Phi_\beta + \partial_\beta\Phi_\mu + \partial_\mu\partial_\beta\phi\right]\\
&\quad\times\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha + \partial_\alpha\Phi_\nu + \partial_\nu\partial_\alpha\phi\right]\\
&= 2\left(h^{K\mu\nu} + 2\partial^\mu\Phi^\nu + \partial^\mu\partial^\nu\phi\right)\partial^\alpha\left[h^K_{\mu\beta} + \partial_\mu\Phi_\beta\right]\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha\right] + \partial^\mu V_\mu^{(3)}\\
&= 2h^{K\mu\nu}\partial^\alpha\left[h^K_{\mu\beta} + \partial_\mu\Phi_\beta\right]\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha\right] + 4\partial^\mu\Phi^\nu\partial^\alpha h^K_{\mu\beta}\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha\right]\\
&\quad 2\partial^\mu\partial^\nu\phi\partial^\alpha h^K_{\mu\beta}\partial^\beta h^K_{\nu\alpha} + \partial^\mu V_\mu^{(3)} + \partial^\mu V_\mu^{(4)},
\end{aligned}$$

where $V_\mu^{(3)}$ and $V_\mu^{(4)}$ are defined by

$$\begin{aligned}
\partial^\mu V_\mu^{(3)} = 2\,\mathfrak{div}\,\Big\{&\left(h^{K\mu\nu} + 2\partial^\mu\Phi^\nu + \partial^\mu\partial^\nu\phi\right)\partial^\alpha\partial_\beta\left[\,\Phi_\mu + \partial_\mu\phi\right]\\
&\times\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha + \partial_\alpha\Phi_\nu + \partial_\nu\partial_\alpha\phi\right]\\
&+\left(h^{K\mu\nu} + 2\partial^\mu\Phi^\nu + \partial^\mu\partial^\nu\phi\right)\partial^\alpha\left[h^K_{\mu\beta} + \partial_\mu\Phi_\beta\right]\partial^\beta\partial_\alpha\left[\Phi_\nu + \partial_\nu\phi\right]\Big\},\\
\partial^\mu V_\mu^{(4)} = 2\,\mathfrak{div}\,\Big\{&\left(2\partial^\mu\Phi^\nu + \partial^\mu\partial^\nu\phi\right)\partial^\alpha\partial_\mu\Phi_\beta\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha\right] + \partial^\mu\partial^\nu\phi\partial^\alpha h^K_{\mu\beta}\partial^\beta\partial_\nu\Phi_\alpha\Big\}.
\end{aligned}$$

So far, we have

$$\begin{aligned} L_G^{(3)} &= h^{K\mu\nu}\partial_\mu h^K_{\alpha\beta}\partial_\nu\left[h^{K\alpha\beta} + 2\partial^\alpha\Phi^\beta + \partial^\alpha\partial^\beta\phi\right] + 2h^{K\mu\nu}\partial_\mu\partial_\alpha\Phi_\beta\partial_\nu\left[h^{K\alpha\beta} + \partial^\beta\Phi^\alpha\right] \\ &\quad + h^{K\mu\nu}\partial_\mu\partial_\alpha\partial_\beta\phi\partial_\nu h^{K\alpha\beta} + 2h^{K\mu\nu}\partial^\alpha\left[h^K_{\mu\beta} + \partial_\mu\Phi_\beta\right]\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha\right] \\ &\quad + 4\partial^\mu\Phi^\nu\partial^\alpha h^K_{\mu\beta}\partial^\beta\left[h^K_{\nu\alpha} + \partial_\nu\Phi_\alpha\right] + 2\partial^\mu\partial^\nu\phi\partial^\alpha h^K_{\mu\beta}\partial^\beta h^K_{\nu\alpha} + \partial^\mu\sum_{i=1}^4 V_\mu^{(i)}. \end{aligned}$$

The combination of terms that contain a scalar field can be transformed to

$$\begin{aligned} L_G^{(3)} &\supset 2\left[h^{K\mu\nu}\partial_\mu h^K_{\alpha\beta}\partial_\nu\partial^\alpha\partial^\beta\phi + \partial^\mu\partial^\nu\phi\partial^\alpha h^K_{\mu\beta}\partial^\beta h^K_{\nu\alpha}\right] \\ &= \partial^\beta V_\beta^{(5)} - 2\partial_\alpha\partial^\beta h^K_{\mu\beta}h^{K\alpha\nu}\partial_\nu\partial^\mu\phi, \end{aligned}$$

with $V_\beta^{(5)}$ defined by $\partial^\beta V_\beta^{(5)} = 2\partial^\beta\left[\partial_\alpha h^K_{\mu\beta}h^{K\alpha\nu}\partial_\nu\partial^\mu\phi\right]$. Next, we collect the terms with two Φ^μ's and one $h^K_{\alpha\beta}$,

$$\begin{aligned} L_G^{(3)} &\supset 2h^{K\mu\nu}\partial_\mu\partial_\alpha\Phi_\beta\partial_\nu\partial^\beta\Phi^\alpha + 2h^{K\mu\nu}\partial^\alpha\partial_\mu\Phi_\beta\partial^\beta\partial_\nu\Phi_\alpha + 4\partial^\mu\Phi^\nu\partial^\alpha h^K_{\mu\beta}\partial^\beta\partial_\nu\Phi_\alpha \\ &= \partial^\alpha V_\alpha^{(6)} - 4\partial^\beta\partial^\nu(\partial\Phi)\partial^\mu\Phi_\beta h^K_{\mu\nu}, \end{aligned}$$

where $V_\alpha^{(6)}$ is defined by $\partial^\alpha V_\alpha^{(6)} = 4\partial^\alpha\left[\partial^\beta\partial^\nu\Phi_\alpha\partial^\mu\Phi_\beta h^K_{\mu\nu}\right]$. Finally, we collect the terms with one Φ^μ and two $h^K_{\alpha\beta}$'s,

$$\begin{aligned} L_G^{(3)} &\supset 2h^{K\mu\nu}\partial_\mu h^K_{\alpha\beta}\partial_\nu\partial^\alpha\Phi^\beta + 2h^{K\mu\nu}\partial_\mu\partial_\alpha\Phi_\beta\partial_\nu h^{K\alpha\beta} + 2h^{K\mu\nu}\partial^\alpha h^K_{\mu\beta}\partial^\beta\partial_\nu\Phi_\alpha \\ &\quad + 2h^{K\mu\nu}\partial^\alpha\partial_\mu\Phi_\beta\partial^\beta h^K_{\nu\alpha} + 4\partial^\mu\Phi^\nu\partial^\alpha h^K_{\mu\beta}\partial^\beta h^K_{\nu\alpha} \\ &= \partial^\alpha V_\alpha^{(7)} - 2h^{K\mu\nu}h^K_{\mu\beta}\partial^\beta\partial_\nu(\partial\Phi) - 4\partial^\mu\partial^\alpha h^K_{\alpha\beta}\partial^\nu\Phi^\beta h^K_{\mu\nu}, \end{aligned}$$

where $V_\alpha^{(7)}$ is defined by $\partial^\alpha V_\alpha^{(7)} = 2\partial^\alpha\left[h^{K\mu\nu}h^K_{\mu\beta}\partial^\beta\partial_\nu\Phi_\alpha + 2\partial^\mu h^K_{\alpha\beta}\partial^\nu\Phi^\beta h^K_{\mu\nu}\right]$. Combining all transformations that we have performed, we obtain

$$\begin{aligned} L_G^{(3)} &= L_G^{K,(3)} + \partial^\mu\sum_{i=1}^7 V_\mu^{(i)} - 2h^{K\mu\nu}h^K_{\mu\beta}\partial^\beta\partial_\nu(\partial\Phi) \\ &\quad - 4\partial^\mu\left[\left(\partial h^K\right)_\beta + \partial_\beta(\partial\Phi)\right]\partial^\nu\Phi^\beta h^K_{\mu\nu} - 2\partial_\alpha\left(\partial h^K\right)_\mu h^{K\alpha\nu}\partial_\nu\partial^\mu\phi, \end{aligned} \tag{A.8}$$

where $L_G^{K,(3)} = h^{K\mu\nu}\partial_\mu h^K_{\alpha\beta}\partial_\nu h^{K\alpha\beta} + 2h^{K\mu\nu}\partial^\alpha h^K_{\mu\beta}\partial^\beta h^K_{\nu\alpha}$ is point-localized.

Part 2: Naively, the obstructing terms in Eq. (A.8) imply that $L_G^{(3)}$ is not given by the sum of a point-localized Lagrangian and a total divergence. However, they could be canceled by terms that vanish exactly on Hilbert space but are non-zero on Krein space. In Yang-Mills theory, there was only a single term that could possibly have such an effect. In the present case, there are many more; namely, all possible terms that contain a trace h or a divergence $\partial^\mu h_{\mu\nu}$, modulo total divergences. We give a full list of these terms and all total divergences that they can form.

Sector $hh_{\mu\nu}h_{\alpha\beta}$: In the sector with exactly one trace field, we find the terms

$$h\partial_\mu h_{\alpha\beta}\partial^\alpha h^{\mu\beta}, \tag{A.9a}$$
$$\partial_\mu h h_{\alpha\beta}\partial^\alpha h^{\mu\beta}, \tag{A.9b}$$
$$\partial_\mu\partial^\alpha h h_{\alpha\beta}h^{\mu\beta}, \tag{A.9c}$$
$$hh_{\alpha\beta}\partial^\alpha(\partial h)^\beta, \tag{A.9d}$$
$$h(\partial h)_\beta(\partial h)^\beta, \tag{A.9e}$$
$$\partial_\mu h h^{\mu\beta}(\partial h)_\beta, \tag{A.9f}$$

From the terms (A.9), we can form three different total divergences:

$$\begin{aligned}
D_1 &:= \partial_\mu\left[hh_{\alpha\beta}\partial^\alpha h^{\mu\beta}\right] = (\text{A.9a}) + (\text{A.9b}) + (\text{A.9d}),\\
D_2 &:= \partial_\mu\left[\partial^\alpha hh_{\alpha\beta}h^{\mu\beta}\right] = (\text{A.9b}) + (\text{A.9c}) + (\text{A.9f}),\\
D_3 &:= \partial^\mu\left[hh_{\mu\beta}(\partial h)^\beta\right] = (\text{A.9d}) + (\text{A.9e}) + (\text{A.9f}).
\end{aligned}$$

Sector $hhh_{\mu\nu}$: In the sector with exactly two trace fields, we find the terms

$$\partial_\mu h\partial_\nu hh^{\mu\nu}, \tag{A.10a}$$
$$\partial_\mu\partial_\nu hhh^{\mu\nu}, \tag{A.10b}$$
$$\partial_\mu hh\partial_\nu h^{\mu\nu}, \tag{A.10c}$$
$$hh\partial_\mu\partial_\nu h^{\mu\nu}, \tag{A.10d}$$

from which we can form the two independent total divergences

$$\begin{aligned}
D_4 &:= \partial_\mu\left[h\partial_\nu hh^{\mu\nu}\right] = (\text{A.10a}) + (\text{A.10b}) + (\text{A.10c}),\\
D_5 &:= \partial_\mu\left[hh\partial_\nu h^{\mu\nu}\right] = 2\,(\text{A.10c}) + (\text{A.10d}).
\end{aligned}$$

Sector $h_{\mu\nu}h_{\kappa\lambda}h_{\alpha\beta}$: In the sector that contains no trace fields but only divergences $(\partial h)_\mu$, we find the terms

$$\partial_\mu h^{\mu\nu}h^{\alpha\beta}\partial_\nu h_{\alpha\beta}, \tag{A.11a}$$
$$\partial_\mu\partial_\nu h^{\mu\nu}h^{\alpha\beta}h_{\alpha\beta}, \tag{A.11b}$$
$$(\partial h)^\mu(\partial h)^\beta h_{\mu\beta}, \tag{A.11c}$$
$$(\partial h)^\mu h^{\alpha\beta}\partial_\alpha h_{\mu\beta}, \tag{A.11d}$$
$$\partial_\alpha(\partial h)^\mu h^{\alpha\beta}h_{\mu\beta}, \tag{A.11e}$$

from which we can built the two independent total divergences

$$\begin{aligned}
D_6 &:= \partial^\mu\left[(\partial h)_\mu h^{\alpha\beta}h_{\alpha\beta}\right] = 2\,(\text{A.11a}) + (\text{A.11b})\,,\\
D_7 &:= \partial_\alpha\left[(\partial h)^\mu h^{\alpha\beta}h_{\mu\beta}\right] = (\text{A.11a}) + (\text{A.11b}) + (\text{A.11c}).
\end{aligned}$$

Thus, there are 15 possible terms that vanish on Hilbert space but not on Krein space and seven total divergences that these terms can form. Therefore, we are left with eight independent terms that we can choose. The most general term, which can possibly remove the obstructing terms in Eq. (A.8) can thus be chosen to be of the form

$$\begin{aligned}\Delta L := {} & c_1 h\partial_\mu h_{\alpha\beta}\partial^\alpha h^{\mu\beta} + c_2\partial_\mu h h_{\alpha\beta}\partial^\alpha h^{\mu\beta} + c_3 h(\partial h)_\beta(\partial h)^\beta \\ & + c_4\partial_\mu h\partial_\nu h h^{\mu\nu} + c_5\partial_\mu h h(\partial h)^\mu + c_6(\partial h)^\nu h^{\alpha\beta}\partial_\nu h_{\alpha\beta} \\ & + c_7(\partial h)^\mu(\partial h)^\beta h_{\mu\beta} + c_8(\partial h)^\mu h^{\alpha\beta}\partial_\alpha h_{\mu\beta}.\end{aligned} \tag{A.12}$$

All terms in Eq. (A.12) can be expressed in terms of the Krein field and the escort fields by Eq. (A.2). Ignoring all total divergences (as indicated by the div above the equality sign), this gives

$$\begin{aligned}h\partial_\mu h_{\alpha\beta}\partial^\alpha h^{\mu\beta} \overset{\text{div}}{=} {} & h^{\mathrm{K}}\partial_\mu h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha h^{\mathrm{K}\mu\beta} + 2h^{\mathrm{K}}\partial_\mu h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha\partial^\beta\Phi^\mu \\ & + 2(\partial\Phi)\partial_\mu h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha h^{\mathrm{K}\mu\beta} + 4(\partial\Phi)\partial_\mu h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha\partial^\beta\Phi^\mu,\end{aligned} \tag{A.13a}$$

$$\begin{aligned}\partial_\mu h h_{\alpha\beta}\partial^\alpha h^{\mu\beta} \overset{\text{div}}{=} {} & \partial_\mu h^{\mathrm{K}} h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha h^{\mathrm{K}\mu\beta} + \partial_\mu h^{\mathrm{K}} h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha\partial^\beta\Phi^\mu \\ & + \partial_\mu h^{\mathrm{K}}\partial_\beta\Phi_\alpha\partial^\alpha h^{\mathrm{K}\mu\beta} + 2\partial_\mu(\partial\Phi)h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha h^{\mathrm{K}\mu\beta} \\ & + 2\partial_\mu(\partial\Phi)h^{\mathrm{K}}_{\alpha\beta}\partial^\alpha\partial^\beta\Phi^\mu + 2\partial_\mu(\partial\Phi)\partial_\beta\Phi_\alpha\partial^\alpha h^{\mathrm{K}\mu\beta},\end{aligned} \tag{A.13b}$$

$$\begin{aligned}h(\partial h)_\beta(\partial h)^\beta \overset{\text{div}}{=} {} & h^{\mathrm{K}}(\partial h^{\mathrm{K}})_\beta(\partial h^{\mathrm{K}})^\beta + 2h^{\mathrm{K}}(\partial h^{\mathrm{K}})_\beta\partial^\beta(\partial\Phi) \\ & + 2(\partial\Phi)(\partial h^{\mathrm{K}})_\beta(\partial h^{\mathrm{K}})^\beta + 4(\partial\Phi)(\partial h^{\mathrm{K}})_\beta\partial^\beta(\partial\Phi),\end{aligned} \tag{A.13c}$$

$$\partial_\mu h\partial_\nu h h^{\mu\nu} \overset{\text{div}}{=} \partial_\mu h^{\mathrm{K}}\partial_\nu h^{\mathrm{K}} h^{\mathrm{K}\mu\nu} + 4\partial_\mu h^{\mathrm{K}}\partial_\nu(\partial\Phi)h^{\mathrm{K}\mu\nu} + 4\partial_\mu(\partial\Phi)\partial_\nu(\partial\Phi)h^{\mathrm{K}\mu\nu}, \tag{A.13d}$$

$$\begin{aligned}\partial_\mu h h(\partial h)^\mu \overset{\text{div}}{=} {} & \partial_\mu h^{\mathrm{K}} h^{\mathrm{K}}(\partial h^{\mathrm{K}})^\mu + 2\partial_\mu h^{\mathrm{K}}(\partial\Phi)(\partial h^{\mathrm{K}})^\mu \\ & + 2\partial_\mu(\partial\Phi)h^{\mathrm{K}}(\partial h^{\mathrm{K}})^\mu + 4\partial_\mu(\partial\Phi)(\partial\Phi)(\partial h^{\mathrm{K}})^\mu,\end{aligned} \tag{A.13e}$$

$$\begin{aligned}(\partial h)^\nu((h\partial_\nu h)) \overset{\text{div}}{=} {} & (\partial h^{\mathrm{K}})^\nu h^{\mathrm{K}}_{\alpha\beta}\partial_\nu h^{\mathrm{K}\alpha\beta} + 2(\partial h^{\mathrm{K}})^\nu h^{\mathrm{K}}_{\alpha\beta}\partial_\nu\partial^\beta\Phi^\alpha \\ & + 2(\partial h^{\mathrm{K}})^\nu\partial_\alpha\Phi_\beta\partial_\nu h^{\mathrm{K}\alpha\beta} + 2(\partial h^{\mathrm{K}})^\nu\partial_\alpha\Phi_\beta\partial_\nu\partial^\beta\Phi^\alpha \\ & + (\partial h^{\mathrm{K}})^\nu h^{\mathrm{K}}_{\alpha\beta}\partial_\nu\partial^\beta\partial^\alpha\phi + (\partial h^{\mathrm{K}})^\nu\partial_\alpha\partial_\beta\phi\partial_\nu h^{\mathrm{K}\alpha\beta},\end{aligned} \tag{A.13f}$$

$$\begin{aligned}(\partial h)^\mu(\partial h)^\beta h_{\mu\beta} \overset{\text{div}}{=} {} & (\partial h^{\mathrm{K}})^\mu(\partial h^{\mathrm{K}})^\beta h^{\mathrm{K}}_{\mu\beta} + 2(\partial h^{\mathrm{K}})^\mu(\partial h^{\mathrm{K}})^\beta\partial_\mu\Phi_\beta \\ & + (\partial h^{\mathrm{K}})^\mu(\partial h^{\mathrm{K}})^\beta\partial_\mu\partial_\beta\phi + 2(\partial h^{\mathrm{K}})^\mu\partial^\beta(\partial\Phi)h^{\mathrm{K}}_{\mu\beta} \\ & + 2(\partial h^{\mathrm{K}})^\mu\partial^\beta(\partial\Phi)\partial_\mu\Phi_\beta + \partial^\mu(\partial\Phi)\partial^\beta(\partial\Phi)h^{\mathrm{K}}_{\mu\beta},\end{aligned} \tag{A.13g}$$

$$\begin{aligned}(\partial h)^\mu h^{\alpha\beta}\partial_\alpha h_{\mu\beta} \stackrel{\text{div}}{=}\, & (\partial h^{\mathrm{K}})^\mu h^{\mathrm{K}\alpha\beta}\partial_\alpha h^{\mathrm{K}}_{\mu\beta} + (\partial h^{\mathrm{K}})^\mu h^{\mathrm{K}\alpha\beta}\partial_\alpha\partial_\mu\Phi_\beta \\ & + (\partial h^{\mathrm{K}})^\mu h^{\mathrm{K}\alpha\beta}\partial_\alpha\partial_\beta\Phi_\mu + (\partial h^{\mathrm{K}})^\mu h^{\mathrm{K}\alpha\beta}\partial_\mu\partial_\alpha\partial_\beta\phi \\ & + (\partial h^{\mathrm{K}})^\mu \partial^\beta\Phi^\alpha\partial_\alpha h^{\mathrm{K}}_{\mu\beta} + (\partial h^{\mathrm{K}})^\mu\partial^\beta\Phi^\alpha\partial_\alpha\partial_\mu\Phi_\beta \\ & + \partial^\mu(\partial\Phi)h^{\mathrm{K}\alpha\beta}\partial_\alpha h^{\mathrm{K}}_{\mu\beta} + \partial^\mu(\partial\Phi)h^{\mathrm{K}\alpha\beta}\partial_\alpha\partial_\beta\Phi_\mu \\ & + \partial^\mu(\partial\Phi)\partial^\beta\Phi^\alpha\partial_\alpha h^{\mathrm{K}}_{\mu\beta}.\end{aligned} \tag{A.13h}$$

To show that no linear combination of the correction terms (A.13a)-(A.13h) can remove the obstructing term in Eq. (A.8), it is enough to consider the sector $h^{\mathrm{K}}_{\mu\nu}h^{\mathrm{K}}_{\kappa\lambda}\Phi_\alpha$. We have

$$L_G^{(3)} + \Delta L\Big|_{h^{\mathrm{K}}_{\mu\nu}h^{\mathrm{K}}_{\kappa\lambda}\Phi_\alpha} \stackrel{\text{div}}{=} \Phi^\alpha\Big\{\partial_\alpha(\partial h^{\mathrm{K}})^\mu(\partial h^{\mathrm{K}})_\mu(4-4c_3+4c_7-c_8) \tag{A.14a}$$

$$+\,\partial_\alpha\partial^\nu(\partial h^{\mathrm{K}})^\mu h^{\mathrm{K}}_{\mu\nu}(4+2c_2+2c_7+c_8) \tag{A.14b}$$

$$+\,\partial^\nu(\partial h^{\mathrm{K}})^\mu\partial_\alpha h^{\mathrm{K}}_{\mu\nu}(4+2c_2+2c_7) \tag{A.14c}$$

$$+\,\partial_\alpha\partial^\nu h^{\mathrm{K}\mu\beta}\partial_\beta h^{\mathrm{K}}_{\mu\nu}(4-4c_1+4c_2+2c_8) \tag{A.14d}$$

$$+\,\partial^\mu(\partial h^{\mathrm{K}})_\alpha(\partial h^{\mathrm{K}})_\mu(4-2c_7+3c_8) \tag{A.14e}$$

$$+\,\partial^\mu\partial^\nu(\partial h^{\mathrm{K}})_\alpha h^{\mathrm{K}}_{\mu\nu}(4+c_8) \tag{A.14f}$$

$$+\,\partial^\mu\partial^\nu h^{\mathrm{K}}_{\mu\nu}(\partial h)_\alpha(2c_6-2c_7+2c_8) \tag{A.14g}$$

$$+\,\partial^\beta\partial^\mu\partial^\nu h^{\mathrm{K}}_{\mu\nu}h_{\alpha\beta}(2c_6+c_8) \tag{A.14h}$$

$$+c_8\partial^\beta(\partial h^{\mathrm{K}})^\nu\partial_\nu h^{\mathrm{K}}_{\alpha\beta}\Big\}\,. \tag{A.14i}$$

Line (A.14f) implies $c_8 = -4$, while line (A.14i) implies $c_8 = 0$. Both conditions cannot be satisfied simultaneously, which implies that the obstructions in Eq. (A.8) cannot be fully resolved. Thus, $L_G^{(3)}$ is not part of an L-V pair, which concludes the proof of Theorem 3.23 □

Appendix B

Admissible counterterms for the graviton-photon coupling

This appendix has two aims: First, we give a full list of possible independent quartic counterterms for the graviton-photon coupling described in Section 6.2, where independent means that the terms can not be combined to a total divergence. Second, we show that only three of these possible terms are admissible because they are the only choices whose string variation with respect to the string variables of the photon potentials form a total divergence. These terms turn out to be the terms where the string variation with respect to the photon string variables is identically zero,

$$F^{\mu\nu}F_{\mu\nu}h^{\kappa\lambda}h_{\kappa\lambda}, \quad F^{\mu\lambda}F^{\nu}{}_{\lambda}h_{\mu\kappa}h_{\nu}{}^{\kappa}, \quad \text{and} \quad F^{\mu\nu}F^{\kappa\lambda}h_{\mu\kappa}h_{\nu\lambda}. \tag{B.1}$$

As a consequence, the obstruction to second order string independence in Eq. (6.62) cannot be compensated by an induced Lagrangian.

B.1 A list of independent counterterms

The possible quartic counterterms for the graviton photon coupling must all consist of two string-localized graviton potentials, two string-localized Maxwell potentials and two derivatives. We group them into classes with different contraction schemes and for the sake of readability drop the colons, which indicate Wick-ordering.

Class I: This class has the contraction scheme $[\partial^\lambda, \partial_\lambda, A^\kappa, A_\kappa, h^{\mu\nu}, h_{\mu\nu}]$ from which we can form the three terms

$$\partial_\lambda A_\kappa \partial^\lambda A^\kappa h^{\mu\nu} h_{\mu\nu}, \tag{B.2a}$$

$$A_\kappa A^\kappa \partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu}, \tag{B.2b}$$

$$A_\kappa \partial^\lambda A^\kappa h^{\mu\nu} \partial_\lambda h_{\mu\nu}. \tag{B.2c}$$

From these terms, we can form the two total divergences

$$D_{\mathrm{I}1} := \text{(B.2a)} + 2\,\text{(B.2c)} \quad \text{and} \quad D_{\mathrm{I}2} := \text{(B.2b)} + 2\,\text{(B.2c)}, \tag{B.3}$$

so that there is one independent possible counterterm left. We choose the first term Eq. (B.2a).

Class II: This class has the contraction scheme $[\partial^\mu, \partial_\mu, A_\kappa, A_\lambda, h^{\kappa\varrho}, h^\lambda{}_\varrho]$ from which we can form the four terms

$$\partial_\mu A_\lambda \partial^\mu A_\kappa h^{\kappa\varrho} h^\lambda{}_\varrho, \tag{B.4a}$$
$$\partial_\mu A_\lambda A_\kappa \partial^\mu h^{\kappa\varrho} h^\lambda{}_\varrho, \tag{B.4b}$$
$$\partial_\mu A_\lambda A_\kappa h^{\kappa\varrho} \partial^\mu h^\lambda{}_\varrho, \tag{B.4c}$$
$$A_\lambda A_\kappa \partial^\mu h^{\kappa\varrho} \partial_\mu h^\lambda{}_\varrho. \tag{B.4d}$$

From these terms, we can form the two total divergences

$$D_{\mathrm{II}1} := (\mathrm{B.4a}) + (\mathrm{B.4b}) + (\mathrm{B.4c}) \quad \text{and} \quad D_{\mathrm{II}2} := (\mathrm{B.4b}) + (\mathrm{B.4c}) + (\mathrm{B.4d}), \tag{B.5}$$

so that there are two independent possible counterterms left. We choose the first two terms Eq. (B.4a) and Eq. (B.4b).

Class III: This class has the contraction scheme $[\partial_\mu, \partial_\nu, A_\kappa, A^\kappa, h^{\mu\varrho}, h^\nu{}_\varrho]$ from which we can form the four terms

$$\partial_\mu A_\kappa \partial_\nu A^\kappa h^{\mu\varrho} h^\nu{}_\varrho, \tag{B.6a}$$
$$A_\kappa A^\kappa \partial_\nu h^{\mu\varrho} \partial_\mu h^\nu{}_\varrho, \tag{B.6b}$$
$$A_\kappa \partial_\mu A^\kappa \partial_\nu h^{\mu\varrho} h^\nu{}_\varrho, \tag{B.6c}$$
$$A_\kappa \partial_\mu \partial_\nu A^\kappa h^{\mu\varrho} h^\nu{}_\varrho. \tag{B.6d}$$

From these terms, we can form the two total divergences

$$D_{\mathrm{III}1} := (\mathrm{B.6a}) + (\mathrm{B.6c}) + (\mathrm{B.6d}) \quad \text{and} \quad D_{\mathrm{III}2} := (\mathrm{B.6b}) + 2\,(\mathrm{B.6c}), \tag{B.7}$$

so that there are two independent possible counterterms left. We choose the first two terms Eq. (B.6a) and Eq. (B.6b).

Class IV: This class has the contraction scheme $[\partial_\mu, \partial_\nu, A^\mu, A^\nu, h^{\kappa\lambda}, h_{\kappa\lambda}]$ from which we can form the four terms

$$\partial_\nu A_\mu \partial^\mu A^\nu h^{\kappa\lambda} h_{\kappa\lambda}, \tag{B.8a}$$
$$A_\mu \partial^\mu A^\nu h^{\kappa\lambda} \partial_\nu h_{\kappa\lambda}, \tag{B.8b}$$
$$A_\mu A^\nu \partial^\mu h^{\kappa\lambda} \partial_\nu h_{\kappa\lambda}, \tag{B.8c}$$
$$A^\mu A^\nu h^{\kappa\lambda} \partial_\mu \partial_\nu h_{\kappa\lambda}. \tag{B.8d}$$

From these terms, we can form the two total divergences

$$D_{\mathrm{IV}1} := (\mathrm{B.8b}) + (\mathrm{B.8c}) + (\mathrm{B.8d}) \quad \text{and} \quad D_{\mathrm{IV}2} := (\mathrm{B.8a}) + 2\,(\mathrm{B.8b}), \tag{B.9}$$

so that there are two independent possible counterterms left. We choose the first two terms Eq. (B.8a) and Eq. (B.8c).

Class V: This class has the contraction scheme $[\partial_\mu, \partial_\nu, A_\varrho, A_\sigma, h^{\mu\nu}, h^{\varrho\sigma}]$ from which we can form the four terms

$$\partial_\mu A_\varrho \partial_\nu A_\sigma h^{\mu\nu} h^{\varrho\sigma}, \tag{B.10a}$$
$$\partial_\mu \partial_\nu A_\varrho A_\sigma h^{\mu\nu} h^{\varrho\sigma}, \tag{B.10b}$$
$$\partial_\mu A_\varrho A_\sigma h^{\mu\nu} \partial_\nu h^{\varrho\sigma}, \tag{B.10c}$$
$$A_\varrho A_\sigma h^{\mu\nu} \partial_\mu \partial_\nu h^{\varrho\sigma}. \tag{B.10d}$$

From these terms, we can form the two total divergences

$$D_{\mathrm{V1}} := (\mathrm{B.10a}) + (\mathrm{B.10b}) + (\mathrm{B.10c}) \quad \text{and} \quad D_{\mathrm{V2}} := (\mathrm{B.10d}) + 2\,(\mathrm{B.10c}), \tag{B.11}$$

so that there are two independent possible counterterms left. We choose the first two terms Eq. (B.10a) and Eq. (B.10d).

Class VI: This class has the contraction scheme $[\partial_\mu, \partial_\nu, A_\varrho, A_\sigma, h^{\mu\varrho}, h^{\nu\sigma}]$ from which we can form the six terms

$$\partial_\mu A_\varrho \partial_\nu A_\sigma h^{\mu\varrho} h^{\nu\sigma}, \tag{B.12a}$$
$$\partial_\mu \partial_\nu A_\varrho A_\sigma h^{\mu\varrho} h^{\nu\sigma}, \tag{B.12b}$$
$$\partial_\mu A_\varrho A_\sigma \partial_\nu h^{\mu\varrho} h^{\nu\sigma}, \tag{B.12c}$$
$$A_\varrho A_\sigma \partial_\nu h^{\mu\varrho} \partial_\mu h^{\nu\sigma}, \tag{B.12d}$$
$$A_\varrho \partial_\mu A_\sigma \partial_\nu h^{\mu\varrho} h^{\nu\sigma}, \tag{B.12e}$$
$$\partial_\nu A_\varrho \partial_\mu A_\sigma h^{\mu\varrho} h^{\nu\sigma}. \tag{B.12f}$$

From these terms, we can form the three total divergences

$$\begin{aligned} D_{\mathrm{VI1}} &:= (\mathrm{B.12a}) + (\mathrm{B.12b}) + (\mathrm{B.12c}), \\ D_{\mathrm{VI2}} &:= (\mathrm{B.12b}) + (\mathrm{B.12e}) + (\mathrm{B.12f}), \\ D_{\mathrm{VI3}} &:= (\mathrm{B.12c}) + (\mathrm{B.12d}) + (\mathrm{B.12e}), \end{aligned} \tag{B.13}$$

so that there are three independent possible counterterms left. We choose the terms Eq. (B.12a), Eq. (B.12d) and Eq. (B.12f).

Class VII: The last class has the contraction scheme $[\partial_\mu, \partial_\nu, A^\nu, A_\kappa, h^{\mu\lambda}, h^\kappa{}_\lambda]$ from which we can form the nine terms

$$\partial_\mu A^\nu A_\kappa \partial_\nu h^{\mu\lambda} h^\kappa{}_\lambda, \tag{B.14a}$$
$$\partial_\mu A^\nu \partial_\nu A_\kappa h^{\mu\lambda} h^\kappa{}_\lambda, \tag{B.14b}$$
$$\partial_\mu A^\nu A_\kappa h^{\mu\lambda} \partial_\nu h^\kappa{}_\lambda, \tag{B.14c}$$
$$A^\nu \partial_\mu A_\kappa \partial_\nu h^{\mu\lambda} h^\kappa{}_\lambda, \tag{B.14d}$$
$$A^\nu \partial_\mu A_\kappa h^{\mu\lambda} \partial_\nu h^\kappa{}_\lambda, \tag{B.14e}$$
$$A^\nu \partial_\mu \partial_\nu A_\kappa h^{\mu\lambda} h^\kappa{}_\lambda, \tag{B.14f}$$

$$A^\nu A_\kappa \partial_\nu h^{\mu\lambda} \partial_\mu h^\kappa{}_\lambda, \tag{B.14g}$$

$$A^\nu A_\kappa h^{\mu\lambda} \partial_\mu \partial_\nu h^\kappa{}_\lambda, \tag{B.14h}$$

$$A^\nu \partial_\nu A_\kappa h^{\mu\lambda} \partial_\mu h^\kappa{}_\lambda. \tag{B.14i}$$

From these terms, we can form the six total divergences

$$\begin{aligned} D_{\mathrm{VII1}} &:= (\mathrm{B.14a}) + (\mathrm{B.14b}) + (\mathrm{B.14c}), \\ D_{\mathrm{VII2}} &:= (\mathrm{B.14d}) + (\mathrm{B.14e}) + (\mathrm{B.14f}), \\ D_{\mathrm{VII3}} &:= (\mathrm{B.14c}) + (\mathrm{B.14e}) + (\mathrm{B.14h}), \\ D_{\mathrm{VII4}} &:= (\mathrm{B.14a}) + (\mathrm{B.14d}) + (\mathrm{B.14g}), \\ D_{\mathrm{VII5}} &:= (\mathrm{B.14b}) + (\mathrm{B.14f}) + (\mathrm{B.14i}), \\ D_{\mathrm{VII6}} &:= (\mathrm{B.14g}) + (\mathrm{B.14h}) + (\mathrm{B.14i}). \end{aligned} \tag{B.15}$$

However, not all of the divergences D_{VII1} to D_{VII6} are independent. To see this, note that Eq. (B.15) can be interpreted as the linear system of equations (after identifying any total divergence with 0)

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} (\mathrm{B.14a}) \\ (\mathrm{B.14b}) \\ (\mathrm{B.14c}) \\ (\mathrm{B.14d}) \\ (\mathrm{B.14e}) \\ (\mathrm{B.14f}) \\ (\mathrm{B.14g}) \\ (\mathrm{B.14h}) \\ (\mathrm{B.14i}) \end{pmatrix} = 0. \tag{B.16}$$

It is clear that the matrix in Eq. (B.16) has rank 5 because the sum of the first, second and sixth row equals the sum of the remaining rows. Thus, there are four independent possible counterterms in this class. We choose Eq. (B.14a), Eq. (B.14b), Eq. (B.14e) and Eq. (B.14g).

Summing everything up, we found that the most general quartic counterterm is of the form

$$\begin{aligned} \Delta L = {} & c_{\mathrm{I}}\, \partial_\lambda A_\kappa \partial^\lambda A^\kappa h^{\mu\nu} h_{\mu\nu} + c_{\mathrm{II1}}\, \partial_\mu A_\lambda \partial^\mu A_\kappa h^{\kappa\varrho} h^\lambda{}_\varrho \\ & + c_{\mathrm{II2}}\, \partial_\mu A_\lambda A_\kappa \partial^\mu h^{\kappa\varrho} h^\lambda{}_\varrho + c_{\mathrm{III1}}\, \partial_\mu A_\kappa \partial_\nu A^\kappa h^{\mu\varrho} h^\nu{}_\varrho \\ & + c_{\mathrm{III2}}\, A_\kappa A^\kappa \partial_\nu h^{\mu\varrho} \partial_\mu h^\nu{}_\varrho + c_{\mathrm{IV1}}\, \partial_\nu A_\mu \partial^\mu A^\nu h^{\kappa\lambda} h_{\kappa\lambda} \\ & + c_{\mathrm{IV2}}\, A_\mu A^\nu \partial^\mu h^{\kappa\lambda} \partial_\nu h_{\kappa\lambda} + c_{\mathrm{V1}}\, \partial_\mu A_\varrho \partial_\nu A_\sigma h^{\mu\nu} h^{\varrho\sigma} \\ & + c_{\mathrm{V2}}\, A_\varrho A_\sigma h^{\mu\nu} \partial_\mu \partial_\nu h^{\varrho\sigma} + c_{\mathrm{VI1}}\, \partial_\mu A_\varrho \partial_\nu A_\sigma h^{\mu\varrho} h^{\nu\sigma} \\ & + c_{\mathrm{VI2}}\, A_\varrho A_\sigma \partial_\nu h^{\mu\varrho} \partial_\mu h^{\nu\sigma} + c_{\mathrm{VI3}}\, \partial_\nu A_\varrho \partial_\mu A_\sigma h^{\mu\varrho} h^{\nu\sigma} \\ & + c_{\mathrm{VII1}}\, \partial_\mu A^\nu A_\kappa \partial_\nu h^{\mu\lambda} h^\kappa{}_\lambda + c_{\mathrm{VII2}}\, \partial_\mu A^\nu \partial_\nu A_\kappa h^{\mu\lambda} h^\kappa{}_\lambda \\ & + c_{\mathrm{VII3}}\, A^\nu \partial_\mu A_\kappa h^{\mu\lambda} \partial_\nu h^\kappa{}_\lambda + c_{\mathrm{VII4}}\, A^\nu A_\kappa \partial_\nu h^{\mu\lambda} \partial_\mu h^\kappa{}_\lambda \end{aligned} \tag{B.17}$$

and hence has sixteen free parameters.

B.2 Constraints due to string independence

The only string dependence of cubic coupling $:h^{\mu\nu}T^{FF}_{\mu\nu}:$ appears in the graviton potential $h_{\mu\nu}$ because the Maxwell stress energy tensor is entirely built from the field strength and not from the photon potential itself. Consequently, the term

$$2iF_{\varrho\lambda}F_{\sigma}{}^{\lambda}w_{\nu}\left[\partial^{\nu}h'^{\varrho\sigma}-2\partial^{\varrho}h'^{\nu\sigma}\right], \tag{B.18}$$

which obstructs second order string independence, only consists of the Maxwell field strength, the graviton potential and the auxiliary field w_ν. In order to achieve string independence at second order and tree level, the term Eq. (B.18) must be canceled by the string variation of an induced Lagrangian. We have derived in the last Section B.1 that the most general such induced Lagrangian is of the form as given by Eq. (B.17). However, the induced Lagrangian is further constrained. Its string variation must cancel the obstruction (B.18) but otherwise form a total divergence so that it does not cause new obstructions to string independence. In particular, all terms where the auxiliary field u, which is related to the photon potential, appears must form a total divergence separately. In the following, we thus compute the part of the string variation of ΔL from Eq. (B.17) that contains an auxiliary field u in order to further constrain the form of an admissible induced Lagrangian. This analysis can be done sectorwise by considering the different contractions of the graviton potentials separately.

Remark B.1. Before starting the computations, we give a short explanation. To implement the string independence principle, one should make each potential dependent on its own string variable and then compute the string variation with respect to one string variable of the symmetrized expression – as we have done for the examples discussed in Chapter 6. To reduce the complexity of the computations, we slightly simplify this procedure. Formally, we make all potentials depend on the same string variable and compute the string variation with respect to this single variable. In the present context, this procedure leads to the same result.[1] Qualitatively, this can be understood by realizing that we neither have to deal with contractions $e_\mu A^\mu(x,e)=0$ while $e_{1\mu}A^\mu(x,e_2)\neq 0$ nor with propagators or string-integrated Dirac deltas at the present level. In general, the result might not be the same.

Additionally, we remark that we continue to drop the colons that indicate normal-ordering.

Sector (i): This is the sector where the graviton potentials are fully contracted among each other. The respective terms in the induced Lagrangian Eq. (B.17) are

$$\Delta L_{(\mathrm{i})}=c_{\mathrm{I}}\,\partial_\lambda A_\kappa\partial^\lambda A^\kappa h^{\mu\nu}h_{\mu\nu}+c_{\mathrm{IV}1}\,\partial_\nu A_\mu\partial^\mu A^\nu h^{\kappa\lambda}h_{\kappa\lambda}+c_{\mathrm{IV}2}\,A_\mu A^\nu\partial^\mu h^{\kappa\lambda}\partial_\nu h_{\kappa\lambda}. \tag{B.19}$$

[1]The reader, who is not convinced, may repeat our derivations in the remaining part of this section in the proper way. This, however, is a long and cumbersome task that fills too much space to be included in this dissertation.

We have

$$
\begin{aligned}
d_e\, \Delta L_{\text{(i)}}\big|_u =&\; 2c_{\text{I}}\, \partial_\lambda \partial_\kappa u \partial^\lambda A^\kappa h^{\mu\nu} h_{\mu\nu} + 2c_{\text{IV1}}\, \partial_\nu \partial_\mu u \partial^\mu A^\nu h^{\kappa\lambda} h_{\kappa\lambda} \\
&+ 2c_{\text{IV2}}\, \partial_\mu u A^\nu \partial^\mu h^{\kappa\lambda} \partial_\nu h_{\kappa\lambda} \\
\overset{\text{div}}{=}&\, 2u \left\{ 2(c_{\text{I}} + c_{\text{IV1}})\, \partial^\lambda A^\kappa \left[\partial_\lambda h^{\mu\nu} \partial_\kappa h_{\mu\nu} + h^{\mu\nu} \partial_\lambda \partial_\kappa h_{\mu\nu} \right] \right. \\
&\left. - c_{\text{IV2}} \left[\partial_\mu A^\nu \partial^\mu h^{\kappa\lambda} \partial_\nu h_{\kappa\lambda} + A^\nu \partial^\mu h^{\kappa\lambda} \partial_\nu \partial_\mu h_{\kappa\lambda} \right] \right\},
\end{aligned}
\tag{B.20}
$$

which implies

$$
c_{\text{IV1}} = -c_{\text{I}} \quad \text{and} \quad c_{\text{IV2}} = 0. \tag{B.21}
$$

Thus, the string independence principle implies that

$$
\Delta L_{\text{(i)}} = c_{\text{I}} \left[\partial_\lambda A_\kappa \partial^\lambda A^\kappa h^{\mu\nu} h_{\mu\nu} - \partial_\nu A_\mu \partial^\mu A^\nu h^{\kappa\lambda} h_{\kappa\lambda} \right] = \frac{1}{2} c_{\text{I}} F^{\kappa\lambda} F_{\kappa\lambda} h^{\mu\nu} h_{\mu\nu}. \tag{B.22}
$$

Sector (ii): This sector comprises of all terms in ΔL where there are no contractions between the graviton potentials. These terms are

$$
\begin{aligned}
\Delta L_{\text{(ii)}} =&\; c_{\text{V1}}\, \partial_\mu A_\varrho \partial_\nu A_\sigma h^{\mu\nu} h^{\varrho\sigma} + c_{\text{V2}}\, A_\varrho A_\sigma h^{\mu\nu} \partial_\mu \partial_\nu h^{\varrho\sigma} \\
&+ c_{\text{VI1}}\, \partial_\mu A_\varrho \partial_\nu A_\sigma h^{\mu\varrho} h^{\nu\sigma} + c_{\text{VI2}}\, A_\varrho A_\sigma \partial_\nu h^{\mu\varrho} \partial_\mu h^{\nu\sigma} \\
&+ c_{\text{VI3}}\, \partial_\nu A_\varrho \partial_\mu A_\sigma h^{\mu\varrho} h^{\nu\sigma}
\end{aligned}
\tag{B.23}
$$

with

$$
\begin{aligned}
d_e\, \Delta L_{\text{(ii)}}\big|_u =&\; 2c_{\text{V1}}\, \partial_\mu \partial_\varrho u \partial_\nu A_\sigma h^{\mu\nu} h^{\varrho\sigma} + 2c_{\text{V2}}\, \partial_\varrho u A_\sigma h^{\mu\nu} \partial_\mu \partial_\nu h^{\varrho\sigma} \\
&+ 2c_{\text{VI1}}\, \partial_\mu \partial_\varrho u \partial_\nu A_\sigma h^{\mu\varrho} h^{\nu\sigma} + 2c_{\text{VI2}}\, \partial_\varrho u A_\sigma \partial_\nu h^{\mu\varrho} \partial_\mu h^{\nu\sigma} \\
&+ 2c_{\text{VI3}}\, \partial_\nu \partial_\varrho u \partial_\mu A_\sigma h^{\mu\varrho} h^{\nu\sigma} \\
\overset{\text{div}}{=}&\, 2u \left\{ \partial_\mu \partial_\nu \partial_\varrho A_\sigma h^{\mu\nu} h^{\varrho\sigma}\ (c_{\text{V1}} + c_{\text{VI1}} + c_{\text{VI3}}) \right. \\
&+ \partial_\mu \partial_\nu A_\sigma \partial_\varrho h^{\mu\nu} h^{\varrho\sigma}\ (c_{\text{V1}} + c_{\text{VI3}}) \\
&+ \partial_\nu \partial_\varrho A_\sigma h^{\mu\nu} \partial_\mu h^{\varrho\sigma}\ (c_{\text{V1}} + 2c_{\text{VI1}} + c_{\text{VI3}}) \\
&+ \partial_\nu A_\sigma \partial_\varrho h^{\mu\nu} \partial_\mu h^{\varrho\sigma}\ (c_{\text{V1}} - c_{\text{VI2}} + c_{\text{VI3}}) \\
&+ \partial_\varrho A_\sigma h^{\mu\nu} \partial_\mu \partial_\nu h^{\varrho\sigma}\ (-c_{\text{V2}} + c_{\text{VI1}}) \\
&\left. + A_\sigma \partial_\varrho h^{\mu\nu} \partial_\mu \partial_\nu h^{\varrho\sigma}\ (-c_{\text{V2}} - c_{\text{VI2}}) \right\}.
\end{aligned}
\tag{B.24}
$$

This yields

$$
c_{\text{V2}} = c_{\text{VI1}} = c_{\text{VI2}} = 0 \quad \text{and} \quad c_{\text{VI3}} = -c_{\text{V1}} \tag{B.25}
$$

so that the string independence principle implies

$$
\Delta L_{\text{(ii)}} = c_{\text{V1}} \left[\partial_\mu A_\varrho \partial_\nu A_\sigma - \partial_\varrho A_\mu \partial_\nu A_\sigma \right] h^{\mu\nu} h^{\varrho\sigma} = \frac{1}{2} c_{\text{V1}}\, F_{\mu\varrho} F_{\nu\sigma} h^{\mu\nu} h^{\varrho\sigma}. \tag{B.26}
$$

Sector (iii): In this last sector, one index of one graviton potential is contracted with one index of the second graviton potential. The corresponding terms in ΔL are given by

$$\begin{aligned}\Delta L_{(\mathrm{iii})} = {} & c_{\mathrm{II1}}\, \partial_\mu A_\lambda \partial^\mu A_\kappa h^{\kappa\varrho} h^\lambda{}_\varrho + c_{\mathrm{II2}}\, \partial_\mu A_\lambda A_\kappa \partial^\mu h^{\kappa\varrho} h^\lambda{}_\varrho \\ & + c_{\mathrm{III1}}\, \partial_\mu A_\kappa \partial_\nu A^\kappa h^{\mu\varrho} h^\nu{}_\varrho + c_{\mathrm{III2}}\, A_\kappa A^\kappa \partial_\nu h^{\mu\varrho} \partial_\mu h^\nu{}_\varrho \\ & + c_{\mathrm{VII1}}\, \partial_\mu A^\nu A_\kappa \partial_\nu h^{\mu\lambda} h^\kappa{}_\lambda + c_{\mathrm{VII2}}\, \partial_\mu A^\nu \partial_\nu A_\kappa h^{\mu\lambda} h^\kappa{}_\lambda \\ & + c_{\mathrm{VII3}}\, A^\nu \partial_\mu A_\kappa h^{\mu\lambda} \partial_\nu h^\kappa{}_\lambda + c_{\mathrm{VII4}}\, A^\nu A_\kappa \partial_\nu h^{\mu\lambda} \partial_\mu h^\kappa{}_\lambda .\end{aligned} \tag{B.27}$$

We have

$$\begin{aligned} d_e\, \Delta L_{(\mathrm{iii})}\big|_u = {} & 2c_{\mathrm{II1}}\, \partial_\mu \partial_\lambda u \partial^\mu A_\kappa h^{\kappa\varrho} h^\lambda{}_\varrho + c_{\mathrm{II2}}\, \partial_\mu \partial_\lambda u A_\kappa \partial^\mu h^{\kappa\varrho} h^\lambda{}_\varrho \\ & + c_{\mathrm{II2}}\, \partial_\mu A_\lambda \partial_\kappa u \partial^\mu h^{\kappa\varrho} h^\lambda{}_\varrho + 2c_{\mathrm{III1}}\, \partial_\mu \partial_\kappa u \partial_\nu A^\kappa h^{\mu\varrho} h^\nu{}_\varrho \\ & + 2c_{\mathrm{III2}}\, \partial_\kappa u A^\kappa \partial_\nu h^{\mu\varrho} \partial_\mu h^\nu{}_\varrho + c_{\mathrm{VII1}}\, \partial_\mu \partial^\nu u A_\kappa \partial_\nu h^{\mu\lambda} h^\kappa{}_\lambda \\ & + c_{\mathrm{VII1}}\, \partial_\mu A^\nu \partial_\kappa u \partial_\nu h^{\mu\lambda} h^\kappa{}_\lambda + c_{\mathrm{VII2}}\, \partial_\mu \partial^\nu u \partial_\nu A_\kappa h^{\mu\lambda} h^\kappa{}_\lambda \\ & + c_{\mathrm{VII2}}\, \partial_\mu A^\nu \partial_\nu \partial_\kappa u h^{\mu\lambda} h^\kappa{}_\lambda + c_{\mathrm{VII3}}\, \partial^\nu u \partial_\mu A_\kappa h^{\mu\lambda} \partial_\nu h^\kappa{}_\lambda \\ & + c_{\mathrm{VII3}}\, A^\nu \partial_\mu \partial_\kappa u h^{\mu\lambda} \partial_\nu h^\kappa{}_\lambda + c_{\mathrm{VII4}}\, \partial^\nu u A_\kappa \partial_\nu h^{\mu\lambda} \partial_\mu h^\kappa{}_\lambda \\ & + c_{\mathrm{VII4}}\, A^\nu \partial_\kappa u \partial_\nu h^{\mu\lambda} \partial_\mu h^\kappa{}_\lambda \\ \overset{\mathrm{div}}{=} {} & u\, \big\{ \partial_\mu \partial^\kappa A_\nu \partial_\kappa h^{\mu\lambda} h^\nu{}_\lambda (2c_{\mathrm{II1}} - c_{\mathrm{II2}} + c_{\mathrm{VII1}} + c_{\mathrm{VII2}}) \\ & + \partial_\mu \partial^\kappa A_\nu h^{\mu\lambda} \partial_\kappa h^\nu{}_\lambda (2c_{\mathrm{II1}} + c_{\mathrm{II2}} + c_{\mathrm{VII2}} - c_{\mathrm{VII3}}) \\ & + \partial^\kappa A_\nu \partial_\kappa h^{\mu\lambda} \partial_\mu h^\nu{}_\lambda (2c_{\mathrm{II1}} - c_{\mathrm{II2}} + c_{\mathrm{VII1}} + c_{\mathrm{VII2}} - c_{\mathrm{VII4}}) \\ & + \partial^\kappa A_\nu h^{\mu\lambda} \partial_\kappa \partial_\mu h^\nu{}_\lambda (2c_{\mathrm{II1}} + c_{\mathrm{II2}} + c_{\mathrm{VII2}}) \\ & + \partial_\lambda A_\kappa \partial^\mu h^{\kappa\varrho} \partial_\mu h^\lambda{}_\varrho (c_{\mathrm{II2}} + c_{\mathrm{VII1}} - c_{\mathrm{VII3}}) \\ & + A_\kappa \partial^\mu \partial_\lambda h^{\kappa\varrho} \partial_\mu h^\lambda{}_\varrho (c_{\mathrm{II2}} + c_{\mathrm{VII1}} - c_{\mathrm{VII4}}) \\ & + \partial_\mu \partial_\nu A_\kappa \partial^\kappa h^{\mu\lambda} h^\nu{}_\lambda (4c_{\mathrm{III1}} - c_{\mathrm{VII1}} + 2c_{\mathrm{VII2}} + c_{\mathrm{VII3}}) \\ & + \partial_\nu A_\kappa \partial^\kappa h^{\mu\lambda} \partial_\mu h^\nu{}_\lambda (2c_{\mathrm{III1}} + c_{\mathrm{VII2}} + c_{\mathrm{VII3}} - c_{\mathrm{VII4}}) \\ & + \partial_\nu A_\kappa h^{\mu\lambda} \partial^\kappa \partial_\mu h^\nu{}_\lambda (2c_{\mathrm{III1}} + c_{\mathrm{VII2}}) \\ & + A_\varrho \partial^\varrho \partial^\mu h^{\nu\kappa} \partial_\nu h_{\mu\kappa} (-4c_{\mathrm{III2}} + c_{\mathrm{VII3}} - c_{\mathrm{VII4}}) \\ & + \partial_\mu A_\nu \partial^\nu \partial^\kappa h^{\mu\lambda} h_{\kappa\lambda} (-c_{\mathrm{VII1}} + c_{\mathrm{VII3}}) \big\} .\end{aligned} \tag{B.28}$$

The resulting linear system of equations for the coefficients has the one-parameter space of solutions given by

$$c_{\mathrm{VII2}} = -2c_{\mathrm{II1}}, \quad c_{\mathrm{III1}} = c_{\mathrm{II1}} \quad \text{and} \quad c_{\mathrm{II2}} = c_{\mathrm{III2}} = c_{\mathrm{VII1}} = c_{\mathrm{VII3}} = c_{\mathrm{VII4}} = 0. \tag{B.29}$$

Hence, the string independence principle implies

$$\begin{aligned} L_{(\mathrm{iii})} &= c_{\mathrm{II1}} \left[\partial_\mu A_\lambda \partial^\mu A_\kappa - 2\partial_\kappa A^\mu \partial_\mu A_\lambda + \partial_\kappa A_\mu \partial_\lambda A^\mu h^{\mu\varrho} \right] h^{\kappa\varrho} h^\lambda{}_\varrho \\ &= c_{\mathrm{II1}} F^{\mu\kappa} F_\mu{}^\lambda h^{\kappa\varrho} h^\lambda{}_\varrho .\end{aligned} \tag{B.30}$$

Combining the results from all three sectors, we have thus shown that the string independence principle constrains the form of any possible quartic induced Lagrangian to

$$\Delta L = c_1\, F^{\kappa\lambda} F_{\kappa\lambda} h^{\mu\nu} h_{\mu\nu} + c_2\, F^{\mu\kappa} F_{\mu}{}^{\lambda} h^{\kappa\varrho} h^{\lambda}{}_{\varrho} + c_3\, F_{\mu\varrho} F_{\nu\sigma} h^{\mu\nu} h^{\varrho\sigma} \tag{B.31}$$

with three free parameters c_1, c_2 and c_3. In particular, the string variation with respect to the helicity one field is trivial for any admissible induced term.

Glossary

Abbreviations

AQFT	Algebraic quantum field theory
BEG	Bogoliubov-Epstein-Glaser prescription [6, 32]
BDF	Brouder-Dütsch-Fredenhagen approach, see Section 5.7.4
BRST	Becchi-Rouet-Stora-Tyutin method [4, 69]
CP	Charge (and) Parity
DVZ	van-Dam-Veltman-Zakharov discontinuity [70, 81]
K (index)	Krein (indicating that something is defined on Krein space)
NST	Nikolov-Stora-Todorov prescription, see Section 5.7.3
QCD	Quantum chromodynamics
QED	Quantum electrodynamics
QFT	Quantum field theory
SD	Short distance
SET	Stress energy tensor
SI	String independence
SLFT	String-localized field theory
UV	ultraviolet

Mathematical symbols

$\det_{y_1 \ldots y_m}$	The Gram determinant associated with $y_1, \ldots, y_m$ in $\mathbb{R}^n$.
$\overset{\text{div}}{=}$	Equality up to a total divergence.
$\mathfrak{div}$	A special total divergence, see Eq. (3.107).
$\mathcal{D}(X)$	The space of compactly supported smooth test functions over an open set $X \subset \mathbb{R}^n$.
$\mathcal{D}'(X)$	The space of distributions over the open set $X \subset \mathbb{R}^n$ and topological dual of $\mathcal{D}(X)$.
H	The open subset of spacelike vectors in $\mathbb{R}^{1+3}$ (not to be confused with the homogeneous distribution from Theorem 3.17).
H_{-1}	The hyperboloid of spacelike directions e with $e^2 = -1$.
I_e	The string integration operator $I_e f(x) = \int_0^\infty ds\, f(x + se)$.
$\mathbb{R}_{\geq 0}$	The set of real numbers greater or equal to 0.
$\mathbb{R}^{1+3}$	Minkowski space.
$\mathcal{S}(\mathbb{R}^n)$	The Schwartz space of smooth functions, which together with all their derivatives decay faster than any polynomial.
$\mathcal{S}'(\mathbb{R}^n)$	The space of tempered distributions over $\mathbb{R}^n$ and topological dual of $\mathcal{S}(\mathbb{R}^n)$.
$U(a, \Lambda)$	A representation of the Poincaré group belonging to the translation vector $a \in \mathbb{R}^{1+3}$ and the Lorentz matrix Λ.
$W_m(x)$	The scalar Klein-Gordon two-point function associated to mass $m \geq 0$.
$\mathrm{WF}\, u$	The wavefront set of the (tempered) distribution u.
$\langle\langle \chi(x)\phi(x') \rangle\rangle$	The vacuum expectation value of the quantum fields $\chi(x)$ and $\phi(x')$.
$\langle\langle T\chi(x)\phi(x') \rangle\rangle$	A generic propagator of the quantum fields $\chi(x)$ and $\phi(x')$.
$\langle\langle T_0\chi(x)\phi(x') \rangle\rangle$	The kinematic propagator of the quantum fields $\chi(x)$ and $\phi(x')$.
$[fg]$	The antisymmetric tensor built from the vectors f and g.

Bibliography

[1] A. Aste and G. Scharf. Non-abelian gauge theories as a consequence of perturbative quantum gauge invariance. *Int. J. Mod. Phys. A*, **14**:3421–3434, 1999.

[2] D. Bahns and M. Wrochna. On-shell extension of distributions. *Annales Henri Poincaré*, **15**:2045–2067, 2014.

[3] V. Bargmann and I. T. Todorov. Spaces of Analytic Functions on a Complex Cone as Carriers for the Symmetric Tensor Representations of SO(N). *J. Math. Phys.*, **18**:1141–1148, 1977.

[4] C. Becchi, A. Rouet, and R. Stora. Renormalization of the Abelian Higgs-Kibble Model. *Commun. Math. Phys.*, **42**:127–162, 1975.

[5] J. J. Bisognano and E. H. Wichmann. On the duality condition for a Hermitian scalar field. *J. Math. Phys.*, **16**:985–1007, 1975.

[6] N. N. Bogoliubov and D. V. Shirkov. *Introduction to the Theory of Quantized Fields*. Interscience, 1959.

[7] C. Brouder, N. V. Dang, and F. Hélein. A smooth introduction to the wavefront set. *Journal of Physics A: Mathematical and Theoretical*, **47**:443001, 2014.

[8] C. Brouder and M. Dütsch. Relating on-shell and off-shell formalisms in perturbative quantum field theory. *J. Math. Phys.*, **49**:052303, 2008.

[9] R. Brunetti and K. Fredenhagen. Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds. *Commun. Math. Phys.*, **208**:623–661, 2000.

[10] R. Brunetti, D. Guido, and R. Longo. Modular localization and Wigner particles. *Rev. Math. Phys.*, **14**:759–786, 2002.

[11] T. Brüers. Master's Thesis. Universität Göttingen, 2021.

[12] D. Buchholz. Collision theory for waves in two dimensions and a characterization of models with trivial S-matrix. *Commun. Math. Phys.*, **45**:1–8, 1975.

[13] D. Buchholz, F. Ciolli, G. Ruzzi, and E. Vasselli. On string-localized potentials and gauge fields. *Lett Math Phys*, **109**:2601–2610, 2019.

[14] D. Buchholz and K. Fredenhagen. Locality and the structure of particle states. *Commun. Math. Phys.*, **84**:1–54, 1982.

[15] C. Bär and K. Fredenhagen (Eds.). *Quantum Field Theory on Curved Spacetimes*. Springer, Berlin-Heidelberg, 2009.

[16] L. T. Cardoso, J. Mund, and J. C. Várilly. String chopping and time-ordered products of linear string-localized quantum fields. *Math. Phys. Anal. Geom.*, **21**:3, 2018.

[17] N.V. Dang. The Extension of Distributions on Manifolds, a Microlocal Approach. *Ann. Henri Poincaré*, **17**:819–859, 2016.

[18] Nils Dencker. On the propagation of polarization sets for systems of real principal type. *Journal of Functional Analysis*, **46**(3):351–372, 1982.

[19] P. A. M. Dirac. Gauge-invariant formulation of quantum electrodynamics. *Canadian Journal of Physics*, **33**:650–660, 1955.

[20] P. Duch. Weak Adiabatic Limit in Quantum Field Theories with Massless Particles. *Ann. Henri Poincaré*, **19**:875–935, 2018.

[21] W. Dybalski and J. Mund. Interacting massless infraparticles in 1+1 dimensions. *arXiv:2109.02128*.

[22] F. J. Dyson. Divergence of Perturbation Theory in Quantum Electrodynamics. *Phys. Rev.*, **85**:631–632, 1952.

[23] M. Dütsch. Proof of perturbative gauge invariance for tree diagrams to all orders. *Annalen der Physik*, **14**:438–461, 2005.

[24] M. Dütsch. The scaling and mass expansion. *Ann. Henri Poincaré*, **16**:163, 2015.

[25] M. Dütsch. *From classical field theory to perturbative quantum field theory*. Birkhäuser, Basel, 2018.

[26] M. Dütsch and F. M. Boas. The Master Ward Identity. *Reviews in Mathematical Physics*, **14**(9):977–1049, 2002.

[27] M. Dütsch and K. Fredenhagen. The Master Ward Identity and Generalized Schwinger-Dyson Equation in Classical Field Theory. *Commun. Math. Phys.*, **243**:275–314, 2003.

[28] M. Dütsch and K. Fredenhagen. Causal perturbation theory in terms of retarded products, and a proof of the Action Ward Identity. *Rev. Math. Phys.*, **16**:1291–1348, 2004.

[29] M. Dütsch and G. Scharf. Perturbative gauge invariance: the electroweak theory. *Ann. Phys. (Leipzig)*, **8**:359–387, 1999.

[30] R. K. Ellis, Stirling W. J., and B. R. Webber. *QCD and Collider Physics*. Cambridge University Press, Cambridge, 1996.

[31] H. Elvang and Y. Huang. *Scattering amplitudes in gauge theory and gravity*. Cambridge University Press, 2015.

[32] H. Epstein and V. J. Glaser. The role of locality in perturbation theory. *Ann. Inst. Henri Poincaré A*, **19**:211–295, 1973.

[33] G. Friedlander and M. Joshi. *Introduction to the theory of distributions*. Cambridge University Press, Cambridge, 2nd edition, 1998.

[34] J. Fröhlich, G. Morchio, and F. Strocchi. Charged sectors and scattering states in quantum electrodynamics. *Ann. Phys.*, **119**:241–284, 1979.

[35] C. Gaß. Renormalization in string-localized field theories: a microlocal analysis. *arXiv:2107.12834,* accepted for publication in *Annales Henri Poincaré*.

[36] C. Gaß. Master's Thesis. Universität Göttingen, 2018.

[37] C. Gaß, J. M. Gracia-Bondía, and J. Mund. Revisiting the Okubo-Marshak argument. *Symmetry*, 13(9), 2021.

[38] C. Gaß, K.-H. Rehren, and F. Tippner. On the spacetime structure of infrared divergencies in QED. *Letters in Mathematical Physics*, **112**:37, 2022.

[39] J. M. Gracia-Bondía, J. Mund, and J. C. Várilly. The chirality theorem. *Ann. Henri Poincaré*, **19**:843–874, 2018.

[40] R. Haag. *Local Quantum Physics*. Springer, Berlin-Heidelberg, 2nd edition, 1996.

[41] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer, Berlin, 2nd edition, 1990.

[42] P. Jordan. Zur Quantenelektrodynamik. III. Eichinvariante Quantelung und Diracsche Magnetpole. *Zeitschrift fur Physik*, **97**:535–537, 1935.

[43] M. Lavelle and D. McMullan. Constituent quarks from QCD. *Physics Reports*, **279**(1):1–65, 1997.

[44] G. Leibbrandt. Light-cone gauge in Yang-Mills theory. *Phys. Rev. D*, **29**:1699–1708, 1984.

[45] G. Leibbrandt. Introduction to noncovariant gauges. *Rev. Mod. Phys.*, **59**:1067–1119, 1987.

[46] S. Mandelstam. Quantum electrodynamics without potentials. *Ann. Phys. (NY)*, **19**:1–24, 1962.

[47] J. Mund and E.T. de Oliveira. String-Localized Free Vector and Tensor Potentials for Massive Particles with Any Spin: I. Bosons. *Commun. Math. Phys.*, **355**:1243–1282, 2017.

[48] J. Mund, K.-H. Rehren, and B. Schroer. Gauss' Law and string-localized quantum field theory. *JHEP*, **01**:2020, 001.

[49] J. Mund, K.-H. Rehren, and B. Schroer. Helicity decoupling in the massless limit of massive tensor fields. *Nucl. Phys. B*, **924**:699–727, 2017.

[50] J. Mund, K.-H. Rehren, and B. Schroer. Relations between positivity, localization and degrees of freedom: the Weinberg–Witten theorem and the van Dam–Veltman–Zakharov discontinuity. *Phys. Lett. B*, **773**:625–631, 2017.

[51] J. Mund, K.-H. Rehren, and B. Schroer. Infraparticle fields and the formation of photon clouds. *Journal of High Energy Physics*, **4**:83, 2022.

[52] J. Mund, B. Schroer, and J. Yngvason. String-localized quantum fields from Wigner representations. *Phys. Lett. B*, **596**:156–162, 2004.

[53] J. Mund, B. Schroer, and J. Yngvason. String-localized quantum fields and modular localization. *Commun. Math. Phys.*, **268**:621–672, 2006.

[54] N. M. Nikolov, R. Stora, and I. Todorov. Renormalization of massless Feynman amplitudes in configuration space. *Rev. Math. Phys.*, **26**:1430002, 2014.

[55] M. E. Peskin and D. V. Schroeder. *An introduction to quantum field theory*. Westview, Boulder, CO, 1995.

[56] M. Plaschke and J. Yngvason. Massless, string localized quantum fields for any helicity. *J. Math. Phys.*, **53**:042301, 2012.

[57] M. Reed and B. Simon. *Methods of modern mathematical physics II*. Academic Press, San Diego, 1975.

[58] H. Reeh and S. Schlieder. Bemerkungen zur Unitäräquivalenz von Lorentzinvarianten Feldern. *Nuovo Cimento*, **22**:1051–1068, 1961.

[59] K.-H. Rehren. Pauli–Lubański limit and stress-energy tensor for infinite-spin fields. *JHEP*, **11**:130, 2017.

[60] G. Scharf. *Gauge Field Theories: Spin One and Spin Two*. Dover, New York, 2016.

[61] B. Schroer. The role of positivity and causality in interactions involving higher spin. *Nucl. Phys. B*, **941**:91–144, 2019.

[62] M. D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, Cambridge, 2014.

[63] K. Shedid Attifa. Master's Thesis. Universität Göttingen, 2020.

[64] O. Steinmann. *Perturbation Expansions in Axiomatic Field Theory*. Lect. Notes in Phys. **11**, Springer, Berlin, 1971.

[65] O. Steinmann. A Jost–Schroer theorem for string fields. *Commun. Math. Phys.*, **87**:259–264, 1982.

[66] O. Steinmann. Perturbative QED in terms of gauge invariant fields. *Ann. Phys. (NY)*, **157**:232–254, 1984.

[67] F. C. Tippner. Bachelor's Thesis. Universität Göttingen, 2020.

[68] F. C. Tippner. Master's Thesis. Universität Göttingen, ongoing.

[69] I. V. Tyutin. Lebedev Institute preprint. N39, 1975.

[70] T. van Dam and M. Veltman. Massive and massless Yang–Mills and gravitational fields. *Nucl. Phys. B*, **22**:397–411, 1970.

[71] G. Velo and D. Zwanziger. Noncausality and other defects of interaction Lagrangians for particles with spin one and higher. *Phys. Rev.*, **188**:2218, 1969.

[72] J. C. Várilly and J. M. Gracia-Bondía. Stora's fine notion of divergent amplitudes. *Nuclear Physics B*, **912**:28–37, 2016.

[73] R. M. Wald. *Quantum field theory in curved spacetime and black hole thermodynamics*. University of Chicago press, 1994.

[74] S. Weinberg. *The Quantum Theory of Fields I.* Cambridge University Press, Cambridge, 1995.

[75] S. Weinberg. *The Quantum Theory of Fields II.* Cambridge University Press, Cambridge, 1996.

[76] S. Weinberg and E. Witten. Limits on massless particles. *Phys. Lett. B*, **96**:59–62, 1980.

[77] G. C. Wick. The evaluation of the collision matrix. *Phys. Rev.*, **80**:268–272, 1950.

[78] A. S. Wightman and L. Gårding. Fields as operator-valued distributions in relativistic quantum theory. *Ark. Fys.*, **28**(13), 1964.

[79] J. Yngvason. Zero-mass infinite spin representations of the Poincaré group and quantum field theory. *Commun. Math. Phys.*, **18**:195–203, 1970.

[80] D. Zagier. *The dilogarithm function. In: Frontiers in number theory, physics, and geometry II.* Springer, Berlin, Heidelberg, 2007.

[81] V. I. Zakharov. Linearized graviton theory and the graviton mass. *JETP Lett.*, **12**:312–313, 1970.

Christian Gaß

CV

Personal Data

Date of Birth	**22 October 1992**, *Kirchheimbolanden*.
Nationality	**German**.

Education

2012	**Abitur**, *Gymnasium Weierhof am Donnersberg*. Overall grade: 1,0
2012–2015	**Bachelor studies physics**, *Universität Göttingen*, Thesis "Monte-Carlo techniques for the integration of high-dimensional phase-spaces" (Supervisor: Prof. Dr. Steffen Schumann). Overall grade: 1,6
2015–2016	**Bachelor studies mathematics**, *Universität Göttingen*, Thesis "Extension of distributions" (Supervisor: Prof. Dr. Dorothea Bahns). Overall grade: 1,7
2015–2018	**Master studies physics**, *Universität Göttingen*, Thesis "Self-interactions of string-local fields of spin two" (Supervisor: Prof. Dr. Karl-Henning Rehren). Overall grade: 1,1
since 10/2018	**PhD studies**, *Universität Göttingen*, Thesis "Constructive aspects of string-localized quantum field theory" (Supervisor: Prof. Dr. Karl-Henning Rehren).

Languages

German (native), English, Basic French

Employments

2016–2018	**Student assistant**, *Universität Göttingen*, part-time.
since 10/2018	**Scientific assistant**, *Universität Göttingen*, part-time.

Scholarships and awards

2010	**Certamen Rheno Palatinum**, 2nd main price.
since 05/2019	**PhD scholarship**, *Studienstiftung des deutschen Volkes*.

Scientific publications

C. Gaß, J. M. Gracia-Bondía and J. Mund, *“Revisiting the Okubo-Marshak argument”*, Symmetry 13(9), 2021.

Preprints

C. Gaß, *“Renormalization in string-localized field theories: a microlocal analysis”*, arXiv:2107.12834, (accepted for publication in “Annales Henri Poincaré”).

C. Gaß, K.-H. Rehren and F. Tippner, *“On the spacetime structure of infrared divergencies in QED”*, arXiv:2109.10148, (accepted for publication in “Letters in Mathematical Physics”).

Talks

02/2019 **43rd LQP workshop**, *“String-local field theories and the method of Epstein and Glaser”*, Florence, Italy.

07/2019 **Mathematics of interacting QFT models**, *“Perturbative construction of interactions involving string-local bosonic potentials”*, York, UK.

08/2021 **DPG Conference of the SMuK section**, *“Renormalization in string-localized field theory”*, virtual conference.

09/2021 **Operator Algebras Seminar, University of Tor Vergata**, *“Renormalization in string-localized field theories: a microlocal analysis”*, Rome, Italy.

Teaching experience

2016–2020 **Universität Göttingen**, tutor for Analytical Mechanics, Classical Field Theory, Quantum Mechanics, Advanced Quantum Mechanics, Mathematical Methods of Physics III.