Nahms Equations and Nilpotent Orbits

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With that dissertation my almost ten years long period in Göttingen reaches its end. I think this is the perfect place and time to express my sentimentality and to say thank you:

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To not overload this paragraph I reduce myself to the formulation of a general thank you to all employees, members and students of the institute for all the opportunities to grow.

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Contents

1 Introduction

The general project of the gauge theory group in Göttingen is to use nilpotent orbits as target manifolds in the theory of generalised Seiberg-Witten equations. Such a target manifold has to admit rich structures[Pid04]: It needs to be a hyperkähler manifold with a rotational Sp(1)-action and an additional tri-hamiltonian action of some Lie group G. Later on we shall need a suitable compactification of the moduli space based on the Weitzenböck formula with $\rho_2 = 0$

$$\|\mathscr{D}_A u\|^2 - \|\nabla^A u\|^2 = \frac{1}{2}\langle s, \rho_0 \circ u\rangle + 2\langle \mu \circ u, F_A^+\rangle \quad \text{for any spinor } u$$

in which the concrete shape of the moment map μ becomes relevant.
The hyperkähler structure on a nilpotent orbit in a complex semi-simple Lie algebra can be given explicitely via Nahms equation, this goes back to P. Kronheimers work in [Kro90b]. In his description it is also not difficult to find the desired group actions, however, the concrete shape of the moment map is mystical since the construction as well as the identification of solutions to Nahms equations with the orbit become very clear when the hyperkähler structure is seen as a complex symplectic structure with a holomorphic symplectic form. Unfortunately, this determines the real moment map μ only partly and so it is the purpose of this dissertation to contribute to the understanding of the missing piece.

Since the mentioned compactification is motivated by the geometry we intent to visualise some geometric perspectives on Nahms equations in the first chapter. The considerations here are based on basic knowledge on Nahms equations which is e.g. described in [Kro90b], [Kro90a], [Kov96] and [Swa99].
With that in mind we add some details to the way P. Kronheimer defined in [Kro90b] the moduli space of solutions to Nahms equations. We then switch to the pictures of A. Kovalev [Kov96] and O. Biquard [Biq96] to identify the orbit using their methods. With this done, we close the chapter with an explanation of the problem. Actually, we are not the first mathematicians working on the moment map: A. Dancer, F. Kirwan and A. Swann suggest in their paper [AS14] a point of view that leads into a explicit formula, e.g. for $\mathfrak{sl}_3\,\mathbb{C}$, which is unfortunately not explicit enough for our purpose. Also the construction was generalised to any adjoint orbit ([Kro90a], [Kov96], [Biq96]) and even further ([Bie97]), however, we focus here on the nilpotent orbits since these carry in contrast to general orbits the desired rotational action.

Initiated by V. Pidstrygach, the actual work begins thereafter: We shall interpret Nahms equations as flow equations on a homomorphism space and that way reformulate them as recurrence relation in linear algebraic terms only. Here, we closely follow the discussion of E. Cattani, A. Kaplan and W. Schmid in [ES86]. Our contribution is a tool to deal with the representation theory: the Moyal product we know from quantisation theory.
However, it turns out that with all the collected knowledge we are not able to finalize computations and so will probably need additional algebraic formulations and tools. Nevertheless, the first steps in the iteration can be made precise so that we can give accurate estimates on the decay of the solutions and, that way, estimates on the moment map. With a priori estimates as in [Sch10] we may already be in the position to deduce some information.

We will indicate how the computations are linked to the universal enveloping algebra of $\mathfrak{sl}_2\,\mathbb{C}$ which is very well understood and so a promising algebraic structure in which we can work. Again, it turns out that this description is not general enough so that we have to look further.
And that is the reason why this doctoral thesis has an open end. It can be seen as an invitation for discussions and suggestions that lead sometime to an explicit description of the moment map.

2 Nahms Equations from Different Perspectives

Different perspectives on the equations makes us more flexible in notation and proofs. In this chaper, we will introduce Nahms equations from geometric point of views that are present in the standard literature, e.g. [Kro90b] or [Kro90a].

2.1 Unperturbed Nahms Equations as Gradientflow

On the Lie algebra $\mathfrak{g}$ of a compact semi-simple Lie group G we can consider the cubic form $\varphi : \mathfrak{g} \otimes \mathbb{R}^3 \to \mathbb{R}$

$$\varphi(\xi_1, \xi_2, \xi_3) = \langle \xi_1, [\xi_2, \xi_3] \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the metric induced by the Killing form. And so φ is not only invariant with respect to the adjoint action of G but also with respect to cyclic permutations as well as the SO(3)-action on $\mathbb{R}^3$. The linearisiation is

$$D_{(\xi_1,\xi_2,\xi_3)}\varphi(\zeta_1, \zeta_2, \zeta_3) = \langle \zeta_1, [\xi_2, \xi_3] \rangle + \langle \zeta_2, [\xi_3, \xi_1] \rangle + \langle \zeta_3, [\xi_1, \xi_2] \rangle$$

so that we find the gradient of φ to be

$$\nabla\varphi(\xi_1, \xi_2, \xi_3) = \begin{pmatrix} [\xi_2, \xi_3] \\ [\xi_3, \xi_1] \\ [\xi_1, \xi_2] \end{pmatrix}.$$

The gradient flow is the first shape of Nahms equations we shall see in this text.

$$\dot{T} = -\nabla\phi(T) \quad \text{i.e.} \quad \begin{cases} \dot{T}_1 + [T_2, T_3] = 0 \\ \dot{T}_2 + [T_3, T_1] = 0 \\ \dot{T}_3 + [T_1, T_2] = 0 \end{cases},$$

We find its critical points being exactly the commuting triples (τ_1, τ_2, τ_3).

2.2 Perturbed Nahms Equations as Gradient Flow

The perturbed cubic form $\psi : \mathfrak{g}^3 \to \mathbb{R}$ is given by

$$\psi(\xi_1, \xi_2, \xi_3) = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + \langle \xi_1, [\xi_2, \xi_3] \rangle.$$

As φ, this functional is invariant also under adjoint action of G, cyclic permutation and SO(3) action onf $\mathbb{R}^3$. Its linearisation

$$D_{(\xi_1,\xi_2,\xi_3)}\psi(\zeta_1, \zeta_2, \zeta_3) = \langle \zeta_1, 2\xi_1 + [\xi_2, \xi_3] \rangle + \langle \zeta_2, 2\xi_2 + [\xi_3, \xi_1] \rangle + \langle \zeta_3, 2\xi_3 + [\xi_1, \xi_2] \rangle$$

leads to the gradient flow

$$\dot{A} = -\nabla\psi(A) \quad \text{that is} \quad \begin{cases} \dot{A}_1 + 2A_1 + [A_2, A_3] = 0 \\ \dot{A}_2 + 2A_2 + [A_3, A_1] = 0 \\ \dot{A}_3 + 2A_3 + [A_1, A_2] = 0 \end{cases}.$$

We can see that the critical points are given by triples $(\tilde{\sigma}_i)$ such that $[\tilde{\sigma}_i, \tilde{\sigma}_j] = -2\tilde{\sigma}_k$ for even permutation (ijk) of (123).
These two gradient flows are actually equivalent: The transform $\mathbb{R} \to \mathbb{R}_{\leq 0}$ with $t \mapsto s(t)$

$$\begin{cases} s(t) = -\frac{1}{2}\, e^{-2t} \\ -2sT(s) = A(t) \end{cases} \quad \text{inverted by} \quad \begin{cases} t(s) = -\frac{1}{2}\, \log(-2s) \\ e^{2t}A(t) = T(s) \end{cases}$$

transfers the gradient flow equation of φ into the gradient flow equation of ψ.

2.3 Grassmannian Perspective

The functional φ is SO(3)-invariant and so $\varphi(\xi_1, \xi_2, \xi_3)$ does actually not depent the concrete triple (ξ_1, ξ_2, ξ_3), it rather depends on the three-plane which is spanned by them. This motived A. Swann in [Swa99] to consider

$$\varphi : \widetilde{\mathrm{Gr}_3\,\mathfrak{g}} \to \mathbb{R} \quad \text{via} \quad \mathrm{span}\{e_1, e_2, e_3\} \mapsto \varphi(e_1, e_2, e_3)$$

as a functional on the space of oriented three-planes in $\mathfrak{g}$.
With relatively simple computations like

$$\frac{\mathrm{d}}{\mathrm{d}s}|T_i|^2 = 2\varphi(T_1, T_2, T_3) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\langle T_i, T_j\rangle = 0.$$

Swann has proven in [Swa99] that the components T_i of a solutions T to Nahms equations corresponding to a nilpotent orbit remain linearly independed and so define a conformal basis of some three plane.

2.4 Extended Formulations

We shall extend the above equations by a forth variable T_0 respectively A_0 to the set of equations

$$\dot{T}_i + [T_0, T_i] + [T_j, T_k] = 0 \quad \text{and} \quad \dot{A}_i + 2A_i + [A_0, A_i] + [A_j, A_k] = 0.$$

In the first part of the dissertation, the construction of the moduli spaces, we shall work with the extended equations and then later go back to the non-extended form.
Those equations are invariant under group $\mathcal{G} = \mathscr{C}^2(\mathbb{R}; G)$ acting by

$$g.(T_0, T_1, T_2, T_3) = (\mathrm{Ad}_g\, T_0 - \dot{g}g^{-1}, \mathrm{Ad}_g\, T_1, \mathrm{Ad}_g\, T_2, \mathrm{Ad}_g\, T_3) \quad \text{and in the same way } g.(A_0, A_1, A_2, A_3).$$

The short computation shows that this type of action preserves the equations

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{Ad}_g\, T_i) + \left[\mathrm{Ad}_g\, T_0 - \dot{g}g^{-1}, T_i\right] + \left[\mathrm{Ad}_g\, T_j, \mathrm{Ad}_g\, T_k\right] \\
&= \mathrm{Ad}_g\, \dot{T}_i + \left[\dot{g}g^{-1}, \mathrm{Ad}_g\, T_i\right] - \left[\dot{g}g^{-1}, T_i\right] + \mathrm{Ad}_g([T_0, T_1] + [T_j, T_k]) \\
&= \mathrm{Ad}_g(\dot{T}_i + [T_0, T_1] + [T_j, T_k])
\end{aligned}$$

and so we can reduce the system from four variables back to three variables via the action of $\mathcal{G}$:
For any T_0, we can solve

$$\mathrm{Ad}_u\, T_0 - \dot{u}u^{-1} = 0, \text{or alternatively in matrix notation: } \dot{u} = uT_0$$

which is a linear first-order differential equation in u. And the solution is unique if we impose $u(0) = 1$. That is, in any $\mathcal{G}$-orbit there is a unique solution with $T_0 = 0$, i.e. we work in the slice in which $T_0 = 0$. The same computations can be done for the perturbed formulation.
We shall show in later stages of this text that Nahms equations are actually the moment map of the $\mathcal{G}$-action and so postpone this discussion to later since we have to equip the path space with additional structures.
The $\mathcal{G}$-action looks like a gauge action and so motivates the next two perspectives.

2.5 Extended Unperturbed Solutions as Anti-Self-Dual Instantons

Consider $\mathbb{R}^4$ with coordinates $(x_0 = s, x_1, x_2, x_3)$ with standard inner product as Riemannian metric and induced hodge star operator $\star$. A connection T on the trivial principal G-bundle P over $\mathbb{R}^4$ is called anti-self dual if its curvature $F_T \in \Omega^2(P; \mathfrak{g})^G_{\mathfrak{g}} = \Omega^2(\mathbb{R}^4; \mathrm{Ad}\, P)$ satisfies

$$F_T + \star F_T = 0.$$

Such connections are exaclty the Yang-Mills connections and, hence, are instantons.
In the trivialisation of the bundle induced by the coordinates on $\mathbb{R}^4$ we can say that T has components $T_0, T_1, T_2, T_3 : \mathbb{R}^4 \to \mathfrak{g}$

$$T = T_0\, \mathrm{d}s + \sum_{i=1}^{3} T_i\, \mathrm{d}x_i = T_0\, \mathrm{d}s + T_1\, \mathrm{d}x_1 + T_2\, \mathrm{d}x_2 + T_3\, \mathrm{d}x_3.$$

The curvature F_T can be computed to be

$$F_T = \mathrm{d}T + \frac{1}{2}[T,T] = \sum_{i,j=0}^{4} (\partial_i T_j - \partial_j T_k + [T_i, T_j])\, \mathrm{d}x_i \wedge \mathrm{d}x_j.$$

so that the anti-self-duality equation reads as

$$\partial_i T_j - \partial_j T_k + \partial_k T_l - \partial_l T_k + [T_i, T_j] + [T_k, T_l] = 0 \quad \text{for even permutations } (ijkl) \text{ of } (0123).$$

If we reduce the dimension by requiring translation invariance T with respect to $\mathbb{R}^3$ in the x_1, x_2, x_3-components we can simplify the anti-self-duality equations to

$$\frac{\mathrm{d}}{\mathrm{d}s} T_i + [T_0, T_i] + [T_j, T_k] = 0.$$

This means that the solutions to Nahms equations can be understood as $\mathbb{R}^3$-translation invariant instantons of the trivial G-bundle over $\mathbb{R}^4$. We have the same perspective for the perturbed equations:

2.6 Extended Perturbed Solutions as Anti-Self-Dual Instantons

Starting now with a trivial principal G-bundle over $\mathbb{R}^4 \setminus \{0\}$ and requirering that the connections are invariant under the rotational action of $\mathrm{SO}(4)$ a connection A on that bundle is of the form

$$A = A_0(r)\frac{\mathrm{d}r}{r} + A_1(r)\theta_1 + A_2(r)\theta_2 + A_3(r)\theta_3$$

where θ_l denote the rotation invariant forms on S^3 and $r > 0$ so that $\frac{\mathrm{d}r}{r}$ together with the θ_l forms a orthonormal basis. The anti-self-duality equations for A, as in the previous section, are

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r} A_i\right) + [A_0, A_i] + [A_j, A_k] = 0$$

so that, after the change $r = e^t$, we get the extended pertubed Nahm's equations

$$\dot{A}_i + 2A_i + [A_0, A_i] + [A_j, A_k] = 0.$$

2.7 Complex Formulation of the Extended Non-Perturbed Equations

We complexify the formulation: Instead of four paths $T = (T_0, T_1, T_2, T_3)$ in $\mathfrak{g}$ we remains with two paths (α, β) in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. This can be done via

$$\alpha = T_0 + iT_1 \quad \text{and} \quad \beta = T_2 + iT_3.$$

The complexification $\mathfrak{g}_{\mathbb{C}}$ admits an involution

$$\alpha^* = -T_0 + iT_1 \quad \text{and} \quad \beta^* = -T_2 + iT_3,$$

which enables us to pass from (α, β) back to T with

$$T_0 = \frac{1}{2}(\alpha - \alpha^*) \quad T_1 = \frac{1}{2i}(\alpha + \alpha^*)$$
$$T_2 = \frac{1}{2}(\beta - \beta^*) \quad T_3 = \frac{1}{2i}(\beta + \beta^*)$$

An easy computation shows that the extended Nahm's equations are equivalent to the two equations

$$\text{the complex equation} \quad \dot{\beta} + [\alpha, \beta] = 0$$
$$\text{the real equation} \quad \frac{\mathrm{d}}{\mathrm{d}s}(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0$$

But only the complex equation is invariant under complexified gauge actions: $\mathcal{G}_\mathbb{C} = \mathscr{C}^2(\mathbb{R}; \mathfrak{g}_\mathbb{C})$ acting by

$$g.(\alpha, \beta) = (g.\alpha, g.\beta) = (\mathrm{Ad}_g\, \alpha - \dot{g}g^{-1}, \mathrm{Ad}_g\, \beta).$$

The same procedure does apply to the extended perturbed equations, here we get

$$\begin{aligned} &\text{the complex equation} && \dot{\beta} + 2\beta + [\alpha, \beta] = 0 \\ &\text{the real equation} && \frac{\mathrm{d}}{\mathrm{d}s}(\alpha + \alpha^*) + 2(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0. \end{aligned}$$

2.8 Model Solutions

Let us give a few examples of solutions and compare their asymptotic behaviour: Here we use a $\mathfrak{su}(2)$-triple (σ_i) in $\mathfrak{g}$ with commutation relations $[\sigma_1, \sigma_2] = 2\sigma_3$ etc, the triple of (τ_i) is assume to commute pairwise and with all the σ's. Also, $s_0 < 0$. We shall here translate solutions of the non-perturbed equations into solutions

Unperturbed Equations			Perturbed Equations		
$T_i(s)$	$s \to -\infty$	$s \to 0$	$A(t)$	$t \to -\infty$	$t \to \infty$
τ_i	τ_i	τ_i	$e^{-2t}\tau_i$	∞	0
$\sigma_i/2s$	0	pole, $\mathrm{res}_0 = \sigma_i/2$	$-\sigma_i$	$-\sigma_i$	$-\sigma_i$
$\sigma_i/2(s - s_0)$	0	$-\sigma_i/2s_0$	$-\sigma_i/(1 + 2s_0e^{2t})$	$-\sigma_i$	0
$\tau_i + \sigma_i/2s$	τ_i	pole, $\mathrm{res}_0 = \sigma_i/2$	$e^{-2t}\tau_i - \sigma_i$	∞	$-\sigma_i$
$\tau_i + \sigma_i/2(s - s_0)$	τ_i	$\tau_i - \sigma_i/2s_0$	$e^{-2t}\tau_i - \sigma_i/(1 + 2s_0e^{2t})$	∞	0

of the perturbed equations via the reparametrisation

$$\begin{cases} s(t) = -\frac{1}{2}\, e^{-2t} \\ -2sT(s) = A(t) \end{cases} \qquad \text{inverted by} \qquad \begin{cases} t(s) = -\frac{1}{2}\, \log(-2s) \\ e^{2t}A(t) = T(s) \end{cases}$$

and that way also compare their asymptotics: A closer look at the reparametrisation tells us that poles or polynomials decay like s^{-1} on $\mathbb{R}_{\leq 0}$ is translated into well-defined limits on $\mathbb{R}$ while well-defined values and limits are substituted by exponential growth respectively decay.

3 Conventions and Notations

3.1 In $\mathfrak{su}(2)$ and $\mathfrak{sl}_2\,\mathbb{C}$

The Lie algebra $\mathfrak{su}(2)$ of skew-hermitian matrices $\mathfrak{su}(2) = \{\sigma \in \mathrm{Mat}_{2\times 2}\,\mathbb{C} \mid \sigma + \bar{\sigma}^t = 0\}$ comes naturally with the Killing form which in this thesis shall be rescaled to $\langle\sigma,\sigma'\rangle := \frac{1}{8}\,\mathrm{tr}(\mathrm{ad}_\sigma\,\mathrm{ad}_{\sigma'}) = \frac{1}{2}\,\mathrm{tr}(\sigma\sigma')$. The Pauli matrices

$$\sigma_1 = \begin{pmatrix} -i & \\ & i \end{pmatrix} \quad\text{and}\quad \sigma_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \quad\text{and}\quad \sigma_3 = \begin{pmatrix} & -i \\ -i & \end{pmatrix}$$

follow the well-known commutation relations

$$[\sigma_i,\sigma_j] = 2\sum_{k=1}^{3}\varepsilon_{ijk}\sigma_k \quad \text{where } \varepsilon_{ijk} \text{ is the full-antisymmetric Levi-Civita tensor.}$$

Furthermore, $\{\sigma_i\}$ is a basis of $\mathfrak{su}(2)$ with $|\sigma_i|^2 = \langle\sigma_i,\sigma_i\rangle = -1$ and $\langle\sigma_i,\sigma_j\rangle = 0$ for $i \neq j$, so that the Killing form is not only non-degenerate which makes $\mathfrak{su}(2)$ semi-simple, it is even negative-definite.
In its complexification $\mathfrak{sl}(2) = (\mathfrak{su}(2))_{\mathbb{C}} = \{a \in \mathrm{Mat}_{2\times 2}\,\mathbb{C} \mid \mathrm{tr}\,a = 0\}$ the elements $\{i\sigma_i\}$ build an orthonormal basis. However it is common to use

$$h = i\sigma_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad\text{and}\quad e = \frac{1}{2}(\sigma_2 + i\sigma_3) = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \quad\text{and}\quad f = \frac{1}{2}(-\sigma_2 + i\sigma_3) = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix},$$

with well-known relations

$$[h,e] = 2e \quad\text{and}\quad [h,f] = -2f \quad\text{and}\quad [e,f] = h$$

and $|h|^2 = 1$, $|e|^2 = |f|^2 = 0$ and $\langle h,e\rangle = \langle h,f\rangle = 0$ as well as $\langle e,f\rangle = \frac{1}{2}$.
As a non-degenerate bilinear form the Killing form is a natural way to identify $\mathfrak{sl}_2\,\mathbb{C}$ with its dual $(\mathfrak{sl}_2\,\mathbb{C})^\vee$, respectively $\mathfrak{su}(2)$ with $(\mathfrak{su}(2))^\vee$.

3.2 Hyperkähler Structure on $\mathbb{H}^n$

Let $\mathbb{H}$ be the space of quaternions, that is the non-commutative algebra $\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k \mid a,b,c,d \in \mathbb{R}\}$ with $i^2 = j^2 = k^2 = ijk = -1$ and the reals being the center. The quaternionic conjugation is the $\mathbb{R}$-linear anti-automorphism of $\mathbb{H}$ defined on the generators by $1 \mapsto 1$, $i \mapsto -i$, $j \mapsto -j$ and $k \mapsto -k$. Like for the complex numbers, it defines the real and the imaginary part of a quaternion $q \in \mathbb{H}$ as $2\,\mathrm{Re}\,q = q + \bar{q}$ and $2\,\mathrm{Im}\,q = q - \bar{q}$ and the absolute value $|q|^2 = q\bar{q}$.

There are two common ways to consider $\mathbb{H}$ and its powers $\mathbb{H}^n$ as a left quaternionic vector space: A quaternion $q \in \mathbb{H}$ can act on $h \in \mathbb{H}^n$ either by left $q.h = qh$ or by the right conjugated multiplication $q.h = h\bar{q}$. We shall use the multiplication from the left so that the $\mathbb{H}$-linear endomorphisms $\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$ act via matrices $Q \in \mathrm{Mat}_{n\times n}\,\mathbb{H}$ by $Q.h = h\bar{Q}^t$. This choice defines us the three complex structures just by multiplying with i, j or k from the left

$$\begin{aligned} I(x_0 + x_1 i + x_2 j + x_3 k) &= -x_1 + x_0 i - x_3 j + x_2 k \\ J(x_0 + x_1 i + x_2 j + x_3 k) &= -x_2 + x_3 i + x_0 j - x_1 k \\ K(x_0 + x_1 i + x_2 j + x_3 k) &= -x_3 - x_2 i + x_1 j + x_0 k. \end{aligned}$$

Combining these with the hermitian structure

$$V(h,h') = \sum_{l=1}^{n} h_l \bar{h}'_l$$

we additionally find a Riemannian metric $\langle\cdot,\cdot\rangle$ and three symplectic structures ω_S for $S = I,J,K$ being

$$\langle h,h'\rangle = \mathrm{Re}\,V(h,h') \quad\text{and}\quad \omega_S(h,h') = \mathrm{Re}\,S\,V(h,h') = \langle Sh,h'\rangle.$$

Using $\{x_0, x_1, x_2, x_3\}$ as coordinates on $\mathbb{H} \to \mathbb{R}^4$ we can write all these structures in terms of forms

$$\begin{aligned}
\langle h, h' \rangle &= \sum_{l=1}^{n} (\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)(h_l, h_l') \\
\omega_I(h, h') &= -\sum_{l=1}^{n} (\mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3)(h_l, h_l') \\
\omega_J(h, h') &= -\sum_{l=1}^{n} (\mathrm{d}x_0 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_1)(h_l, h_l') \\
\omega_K(h, h') &= -\sum_{l=1}^{n} (\mathrm{d}x_0 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_2)(h_l, h_l').
\end{aligned}$$

This is nothing else but saying that $\mathbb{H}^n$ is actually a flat hyperkähler manifold.

There is an important group action on $\mathbb{H}$: An element $q \in \mathrm{Sp}(1) = \{q \in \mathbb{H} \mid |q| = 1\}$ acts on an $h \in \mathbb{H}$ via conjugation $q.h = qh\bar{q}$. This action fixes the real part but acts in a norm-perserving way on $\mathrm{Im}\,\mathbb{H} = \{ix_1 + jx_2 + kx_3\}$ and so actually factorises through $\mathrm{SO}(3)$. This map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ has kernel $\{\pm 1\}$ and is nothing else but the universal (two-fold) covering $\mathrm{Sp}(1) \to \mathrm{SO}(3)$. If we fix an arbitrary complex structure on $\mathbb{H}$, say I, the $\mathrm{Sp}(1)$-action $(\mathbb{H}, I) \to (\mathbb{H}, I)$ is not holomorphic. But it is bi-holomorphic when we regard $q : h \mapsto qh\bar{q}$ as a map $(\mathbb{H}, I) \to (\mathbb{H}, qI\bar{q})$. In other words, via the $\mathrm{SO}(3)$-rotation in the sphere $\{aI + bJ + cK \mid (a, b, c) \in S^2\}$ of complex structures any two choices of complex structures on $\mathbb{H}$ are equivalent.

Restricting ourselves to the complex manifold $(\mathbb{H}, I)$ we can give the hyperkähler structure in terms of a holomorphic symplectic form. Via the choice of I, we can identify $\mathbb{H}^n = \mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}$

$$h_0 + ih_1 + jh_2 + kh_3 \mapsto z + wj = (z, w) \quad \text{where} \quad z = h_0 + ih_1,\ w = h_2 + ih_3, \qquad h_0, h_1, h_2, h_3 \in \mathbb{R}^n.$$

Here, the complex structure I acts on each summand in $\mathbb{C}^n \oplus \mathbb{C}^n$ separately as $I(z, w) = (iz, iw)$. The interpretation as tensor product motivates the choice $J(z, w) = (-\bar{w}, \bar{z})$ being related to the complex structure on the second factor $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}$. The conjugation on $\mathbb{H}$ now reads as $\overline{(z, w)} = (\bar{z}, -w)$. Using $jw = \bar{w}j$ for $w \in \mathbb{C}$, the hermitan form $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{C}$ becomes

$$\tilde{V}(z + wj, z' + w'j) = \sum_{l=1}^{n} (z_l + w_l j)(\bar{z}_l' - w_l' j) = \sum_{l=1}^{n} z_l \bar{z}_l' + w_l \bar{w}_l' - (z_l w_l' - w_l z_l')j$$

and decomposes

$$\langle z + wj, z'w'j \rangle = \mathrm{Re}\ z_l \bar{z}_l' + w_l \bar{w}_l' \quad \text{and} \quad \omega(z + wj, z' + w'j) = -(z_l w_l' - w_l z_l').$$

These two structures again written as Riemannian metric and complex valued I-holomorphic 2-form are

$$\langle \cdot, \cdot \rangle = \mathrm{Re} \sum_{l=1}^{n} \mathrm{d}z_l \otimes \mathrm{d}\bar{z}_l + \mathrm{d}w_l \otimes \mathrm{d}\bar{w}_l \quad \text{and} \quad \omega = -\sum_{l=1}^{n} \mathrm{d}z_l \wedge \mathrm{d}w_l.$$

This way $\mathbb{H}$ became a kähler manifold with holomorphic symplectic form.

4 Nilpotent Orbits as Moduli Spaces of Nahms Equations

In this chapter we will introduce the moduli space of solutions to Nahms equations which corresponds to a nilpotent coadjoint orbit in the semi-simple, complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. More precisely we construct this space several times: once as a quotient of paths $\mathfrak{g}\otimes\mathbb{H}$ defined on $\mathbb{R}$, once with paths on $\mathbb{R}_{\leq 0}$ and one last time in the complex world with paths in $\mathfrak{g}_{\mathbb{C}}\otimes\mathbb{C}^2$ on $\mathbb{R}_{\leq 0}$. All those perspectives are present in the literature and so we shall go through all of them here.

The critical points of the perturbed Nahms equations, i.e. the time-indepedent solutions, are exactly the negative of $\mathfrak{su}(2)$-triples in $\mathfrak{g}$. In contrast to [Kro90b] we shall not consider solutions to perturbed Nahms equations converging to a critical point but all paths with such a convergence and find the space of solutions as zero-level of the momentum map of the gauge action. This way the quotient inherits its structure more natural from a hyperkähler reduction. This is done e.g. by O. Biquard in [Biq96] in the non-pertubed setup. When we have justified that the performed quotient is a manifold we close this section with group actions of G, $\mathrm{Sp}(1)$ and symmetries of the domain $\mathbb{R}$.
We aim to give more detailed proofs when the Nahm-background becomes relevant and so we have decided to not give any details concering the theory of Banach-manifolds, the relevant material can be found e.g. in Serge Langs book *Differential Manifolds*[Lan72].

Since the formulations of Nahms equations on $\mathbb{R}$ and $\mathbb{R}_{\leq 0}$ are linked by a reparametrisation $\mathbb{R}\to\mathbb{R}_{\leq 0}$ it is not necesessary to give proofs in the presentation of the moduli spaces as in A. Kovalevs[Kov96] and O. Biquards[Biq96] description. Here we do need to take care of only one and not two asymptotics: The asymptotics at $t=\infty$ is now translated into an initial value at $s=0$ - that would have simplified most proofs since the boundary condition at $t=\infty$ is translated into a proper inital value at $s=0$ now, however we still have to ensure the correct asymptotic at $s=-\infty$.

Just as Donaldson in [Don84] we fix an identification between $\mathbb{H}$ and $\mathbb{C}^2$. Replacing $\mathfrak{g}\otimes\mathbb{H}$ with $\mathfrak{g}_{\mathbb{C}}\otimes\mathbb{C}^2$ brings us into the complex formulation: The three Nahms equations are replaced by one complex and one real equation. The complex symplectic reduction leads to the moduli space of solutions to the complex equation modulo complex gauge elements. We shall interpret the smooth structure on this space via the real formulation presented before, that is why we have to spend some time with the real equation.
Now, when both spaces are identified with each other we pass to our goal of this chapter: the identification of the moduli space and the nilpotent orbit via the $G_{\mathbb{C}}$-action and the formulation of the moment map.

Let G be a compact, semi-simple Lie group and a homomorphism $\rho:\mathfrak{su}(2)\to\mathfrak{g}$ of Lie algebras. Schurs Lemma tells us that ρ is either an injection or the zero-map:

Proposition 4.1. *A representation $\rho:\mathfrak{su}(2)\to\mathfrak{g}$ is either the zero map or an injection.*

So let us fix a non-trivial $\rho:\mathfrak{su}(2)\to\mathfrak{g}$ and that way identify $\mathfrak{su}(2)$ with its image. This allows us to abuse the notation with writing $\sigma_l\in\mathfrak{g}$ for $l=1,2,3$.
By the assumptions on G the Killing form on its Lie algebra $\mathfrak{g}$ induces an G-invariant inner product $\langle\cdot,\cdot\rangle$.

4.1 On the Real Line $\mathbb{R}$

Given an $\mathfrak{su}(2)$-triple $\{\sigma_l\}_{l=1}^3$ in the Lie algebra $\mathfrak{g}$ and some small enough $\delta > 0$ which is fixed later. The base space $\mathcal{A}$ is the subspace of $\mathscr{C}^1(\mathbb{R};\mathfrak{g}) \otimes \mathbb{R}^4 = \mathscr{C}^1(\mathbb{R};\mathfrak{g}) \otimes \mathbb{H}$ such that its elements $A = (A_0, A_1, A_2, A_3) = A_0 + iA_1 + jA_2 + kA_3$ have the following asymptotics

- $\|A_l\|_+ < \infty$ for $l = 0,1,2,3$ where $\|w\|_+ \coloneqq \sup_{t\geq 0} e^{2t}(|w(t)| + |\dot{w}(t)|)$
- $\|A_0\|_- < \infty$ and $\|A_l + \sigma_l\|_- < \infty$ for $l = 1,2,3$ where $\|w\|_- \coloneqq \sup_{t\leq 0} e^{-\delta t}(|w(t)| + |\dot{w}(t)|)$

The space $\mathcal{A}$ is seen to be a quaternionic space affine over the Banach space $\{a \in \mathscr{C}^1(\mathbb{R},\mathfrak{g}) \mid \|a\|_\pm < \infty\} \otimes \mathbb{H}$. Hence $\mathcal{A}$ has the structure of a flat Banach manifold whose tangent space at a point $A \in \mathcal{A}$ is given by $T_A\mathcal{A} = \{a \in \mathscr{C}^1(\mathbb{R},\mathfrak{g}) \mid \|a\|_\pm < \infty\} \otimes \mathbb{H}$. We can use the L^2-inner product on the first factor to define a Riemannian metric on $\mathcal{A}$ by

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle = \int_{-\infty}^{\infty} e^{2t} |\mathrm{d}A|^2 \quad \text{i.e.} \quad \langle\!\langle a, a' \rangle\!\rangle = \int_{-\infty}^{\infty} e^{2t} \langle a(t), a'(t) \rangle \,\mathrm{d}t.$$

The asymptotic conditions ensure that the integrals are well-defined. The so-defined metric is compatible with the complex structurs coming from $\mathbb{H}$. Consequently, we can define to each of it a symplectic form

$$\begin{aligned} I(a_0,a_1,a_2,a_3) &= (-a_1, a_0, -a_3, a_2) \quad \text{so that} \quad \omega_I = -\int_{-\infty}^{\infty} \mathrm{d}A_0 \wedge \mathrm{d}A_1 + \mathrm{d}A_2 \wedge \mathrm{d}A_3 \\ J(a_0,a_1,a_2,a_3) &= (-a_2, a_3, a_0, -a_1) \quad \text{so that} \quad \omega_J = -\int_{-\infty}^{\infty} \mathrm{d}A_0 \wedge \mathrm{d}A_2 + \mathrm{d}A_3 \wedge \mathrm{d}A_1 \\ K(a_0,a_1,a_2,a_3) &= (-a_3, -a_2, a_1, a_0) \quad \text{so that} \quad \omega_K = -\int_{-\infty}^{\infty} \mathrm{d}A_0 \wedge \mathrm{d}A_3 + \mathrm{d}A_1 \wedge \mathrm{d}A_2. \end{aligned}$$

To summarise: This way $\mathcal{A}$ is a flat hyperkähler manifold.
Geometrically, we work with $\mathcal{A}$ as it were the space of connections on the trivial G-bundle $\mathbb{R}^4 \times G \to \mathbb{R}^4$ being $\mathbb{R}^3$-translation invariant with certain decay: Chosen coordinates (x_l) on $\mathbb{R}^4$ we can write $A \in \mathcal{A}$ as a connection via

$$A = (A_0, A_1, A_2, A_3) \longleftrightarrow A = \mathrm{d} + \sum_{l=0}^{3} A_l \,\mathrm{d}x_l \longleftrightarrow \nabla_A \xi = (\dot{\xi} - \mathrm{ad}_{A_0} \xi, \mathrm{ad}_{A_1} \xi, \mathrm{ad}_{A_2} \xi, \mathrm{ad}_{A_3} \xi).$$

With that said it is natural to consider the action of the gauge group

$$\begin{aligned} &\mathcal{G} = \{g \in \mathscr{C}^2(\mathbb{R}; G) \mid \|\dot{g}g^{-1}\|_\pm < \infty,\ \|\mathrm{Ad}_g \sigma_i - \sigma_i\|_- < \infty\} \\ &\text{via} \quad g.A \coloneqq (g.A_0, g.A_1, g.A_2, g.A_3) = (\mathrm{Ad}_g A_0 - \dot{g}g^{-1}, \mathrm{Ad}_g A_1, \mathrm{Ad}_g A_2, \mathrm{Ad}_g A_3). \end{aligned}$$

It is worth to point out that any path $g \in \mathcal{G}$ actually converges for $t \to \pm\infty$ since the group G is supposed to be compact. And so we shall restrict ourselves already here to the pointed gauge group

$$\mathcal{G}^0 = \{g \in \mathcal{G} \mid g(\infty) = 1\} \subset \mathcal{G}$$

being a normal subgroup in $\mathcal{G}$ with $\mathcal{G}/\mathcal{G}^0 = G$. With that choice we can find the Lie algebra

$$\mathcal{G}^0 \to \mathrm{Lie}\,\mathcal{G}^0 = \{\xi \in \mathscr{C}^2(\mathbb{R};\mathfrak{g}) \mid \|\dot{\xi}\|_\pm < \infty,\ \|[\xi,\sigma_i]\|_\pm < \infty,\ \xi(\infty) = 0\} \quad \text{via} \quad g \mapsto \dot{g}g^{-1}.$$

For any such ξ the ordinary differential equation $\dot{g}g^{-1} = \xi$ has a unique solution given $g(0) = 1$ - this becomes clear when written in matrix notation as the linear equation $\dot{g} = \xi g$. And so the above map modells $\mathcal{G}^0$ as a Banach manifold over the Banach space $(\mathrm{Lie}\,\mathcal{G}^0, \|\cdot\|_+ + \|\cdot\|_-)$. The condition $\xi(\infty) = 0$ is relevant for the completeness of $\mathrm{Lie}\,\mathcal{G}^0$, for $\mathrm{Lie}\,\mathcal{G}$ we would have to identify

$$\mathrm{Lie}\,\mathcal{G} = \{\xi \in \mathscr{C}^2(\mathbb{R};\mathfrak{g}) \mid \|\dot{\xi}\|_\pm < \infty,\ \|[\xi,\sigma_i]\|_\pm < \infty\}$$

with $\mathrm{Lie}\,\mathcal{G}^0 \oplus \mathfrak{g}$. The exponential map $\mathrm{Lie}\,\mathcal{G}^0 \to \mathcal{G}^0$ is given by the pointwise exponential map $\mathfrak{g} \to G$.
The $\mathcal{G}$-invariance of the hyperkähler structure follows directly from the Ad-invariance of the metric and the shape of the induced $\mathcal{G}$-action on the tangent spaces: For $g \in \mathcal{G}$ it is found to be $T_A\mathcal{A} \to T_{g.A}\mathcal{A}$

$$g.a = (\mathrm{Ad}_g a_0, \mathrm{Ad}_g a_1, \mathrm{Ad}_g a_2, \mathrm{Ad}_g a_3).$$

It turns out that the $\mathcal{G}$-action is an hyperkähler action, we will prove the $\mathcal{G}^0$-action to be tri-hamiltonian. In order to do so we compute the fundamental vector fields on $\mathcal{A}$:

$$K^\xi_{\mathcal{G}}(A) = (\mathrm{ad}_\xi A_0 - \dot{\xi}, \mathrm{ad}_\xi A_1, \mathrm{ad}_\xi A_2, \mathrm{ad}_\xi A_3) = -\nabla_A \xi$$

where we have interpreted $\xi \in \mathrm{Lie}\,\mathcal{G}$ as a $\mathbb{R}^3$-translation invariant section of the vector bundle $\mathrm{Ad}\,P = \mathbb{R}^4 \times \mathfrak{g}$. The next step is a computational one: For $a \in T_A\mathcal{A}$ and $\xi \in \mathrm{Lie}\,\mathcal{G}^0$ we have

$$\begin{aligned}
\left\langle\!\!\left\langle K^\xi_{\mathcal{G}^0}, a \right\rangle\!\!\right\rangle &= \int_{-\infty}^{\infty} e^{2t}(\langle \mathrm{ad}_\xi A_0 - \dot{\xi}, a_0\rangle + \langle \mathrm{ad}_\xi A_1, a_1\rangle + \langle \mathrm{ad}_\xi A_2, a_2\rangle + \langle \mathrm{ad}_\xi A_3, a_3\rangle)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{2t}(-\langle \dot{\xi}, a_0\rangle + \sum_{l=0}^{3}\langle \xi, [A_l, a_l]\rangle)\,\mathrm{d}t = \int_{-\infty}^{\infty} -\frac{\mathrm{d}}{\mathrm{d}t} e^{2t}\langle \xi, a_0\rangle + e^{2t}\langle \xi, \dot{a}_0 + 2a_0\rangle + e^{2t}\sum_{l=0}^{3}\langle \xi, [A_l, a_l]\rangle\,\mathrm{d}t \\
&= -\lim_{t\to\infty}\langle \xi, e^{2t}a_0\rangle + \lim_{t\to-\infty}\langle \xi, e^{2t}a_0\rangle + \int_{-\infty}^{\infty} e^{2t}\left\langle \xi, \dot{a}_0 + 2a_0 + \sum_{l=0}^{3}[A_l, a_l]\right\rangle \mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{2t}\left\langle \xi, \dot{a}_0 + 2a_0 + \sum_{l=0}^{3}[A_l, a_l]\right\rangle \mathrm{d}t.
\end{aligned}$$

Here it was helpful that we have already restricted ourselves to the action of $\mathcal{G}^0$ since this means that we have to consider only $\xi \in \mathrm{Lie}\,\mathcal{G}^0$. The point of this being that the exponential decay of $\dot{\xi}$ implies that also ξ reaches its limit $\xi(\infty) = 0$ exponentially so that after the partial integration the integral does converge and both limit-terms in the second last line vanish.
The same computation yields to the formal adjoint of ∇_A applied on $\eta = (\eta_0, \eta_1, \eta_2, \eta_3)$

$$\nabla_A^*\eta = -(\dot{\eta}_0 + 2\eta_0 + \sum_{l=0}^{3}\mathrm{ad}_{A_l}\xi_l).$$

Coming back to the moment map, we use $\mathrm{d}\mu^S_{\mathcal{G}^0}(\xi) = \omega_S(K^\xi_{\mathcal{G}^0}, a) = -\left\langle\!\!\left\langle K^\xi_{\mathcal{G}^0}, Sa\right\rangle\!\!\right\rangle$ for $S = I, J, K$ to finally find

$$\begin{aligned}
\mu^I_{\mathcal{G}^0}(A) &= \dot{A}_1 + 2A_1 + [A_0, A_1] + [A_2, A_3] \\
\mu^J_{\mathcal{G}^0}(A) &= \dot{A}_2 + 2A_2 + [A_0, A_2] + [A_3, A_1] \\
\mu^K_{\mathcal{G}^0}(A) &= \dot{A}_3 + 2A_3 + [A_0, A_3] + [A_1, A_2]
\end{aligned}$$

which we combine into one map[1] $\mu_{\mathcal{G}^0} = i\mu^I_{\mathcal{G}^0} + j\mu^J_{\mathcal{G}^0} + k\mu^K_{\mathcal{G}^0}$. Clearly, each component of μ has values in

$$\mathscr{C}_{\exp}(\mathbb{R};\mathfrak{g}) = \{f \in \mathscr{C}^0(\mathbb{R};\mathfrak{g}) \mid \sup_{t\geq 0} e^{2t}|f(t)| + \sup_{t\leq 0} e^{-\delta t}|f(t)| < \infty\}$$

which is a closed subspace of $(\mathrm{Lie}\,\mathcal{G}^0)^\vee$ via

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle : f \mapsto \left(\xi \mapsto \int_{-\infty}^{\infty} e^{2t}\langle f, \xi\rangle\,\mathrm{d}t\right).$$

The defining equations of $\mathcal{N}$

$$\begin{cases}
\dot{A}_1 + 2A_1 + [A_0, A_1] + [A_2, A_3] = 0 \\
\dot{A}_2 + 2A_2 + [A_0, A_2] + [A_3, A_1] = 0 \\
\dot{A}_3 + 2A_3 + [A_0, A_3] + [A_1, A_2] = 0.
\end{cases}$$

are called Nahm equations or sometimes Nahm-Schmidt equation (with our boundary conditions).
It is straight-forward to check that $\mu_{\mathcal{G}^0}$ is $\mathcal{G}^0$-equivariant, and so $\mathcal{G}^0$ acts on $\mathcal{N} = \mu^{-1}_{\mathcal{G}^0}(0)$. Before we can perform the hyperkähler quotient $\mathcal{N}/\mathcal{G}^0$ we have to answer several questions positively: $\mathcal{N}$ needs to be a submanifold and the $\mathcal{G}^0$-action needs to be free and proper.

Lemma 4.2. *The space $\mathcal{N}$ of solutions to Nahms equations in $\mathcal{A}$ is a submanifold of $\mathcal{A}$.*

[1] The right multiplication for the complex structure would have given us $\dot{A}_i + 2A_i + [A_0, A_i] - [A_j, A_k] = 0$ as Nahms Equations.

Proof. All we need to check is that 0 is actually a regular value of μ. As a quadratic map μ is continuously differentiable with linearisation $D_A\mu : \mathcal{T}_A\mathcal{A} \to \mathscr{C}_{\exp}(\mathbb{R};\mathfrak{g})\otimes\mathfrak{sp}(1)$. We have to check if for any $\xi = (\xi_1,\xi_2,\xi_3) \in \mathscr{C}_{\exp}(\mathbb{R};\mathfrak{g})\otimes\mathfrak{sp}(1)$ there is an $a \in T_A\mathcal{A}$ such that $D_A\mu(a) = \xi$. This is nothing else but solving the following set of linear differential equations

$$\begin{cases} \dot a_1 + 2a_1 + [A_0,a_1] + [a_0,A_1] + [A_2,a_3] + [a_2,A_3] = \xi_1 \\ \dot a_2 + 2a_2 + [A_0,a_2] + [a_0,A_2] + [A_3,a_1] + [a_3,A_1] = \xi_2 \\ \dot a_3 + 2a_3 + [A_0,a_3] + [a_0,A_3] + [A_1,a_2] + [a_1,A_2] = \xi_3 \end{cases}$$

for $a = (a_0,a_1,a_2,a_3)$. We have three equations and four variables, so we can choose $a_0 = 0$ and only solve for (a_1,a_2,a_3). For the model solution $S = (0,-\sigma_1,-\sigma_2,-\sigma_3) \in \mathcal{A}$ we can write $A = S + b$ and use this to find a linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} + 2\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} + \underbrace{\begin{pmatrix} & \mathrm{ad}_{\sigma_3} & -\mathrm{ad}_{\sigma_2}\\ -\mathrm{ad}_{\sigma_3} & & \mathrm{ad}_{\sigma_1}\\ \mathrm{ad}_{\sigma_2} & -\mathrm{ad}_{\sigma_1} & \end{pmatrix}}_{\tilde S}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} + \underbrace{\begin{pmatrix} \mathrm{ad}_{b_0} & -\mathrm{ad}_{b_3} & \mathrm{ad}_{b_2}\\ \mathrm{ad}_{b_3} & \mathrm{ad}_{b_0} & -\mathrm{ad}_{b_1}\\ -\mathrm{ad}_{b_2} & \mathrm{ad}_{b_1} & \mathrm{ad}_{b_0} \end{pmatrix}}_{B}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}\xi_1\\\xi_2\\\xi_3\end{pmatrix}$$

At this point it becomes clear that the problem is more subtile compared to the considerations in [Kro04] on the compact interval in stead of $\mathbb{R}$ since we have to take care of the boundary conditions and have no intival values here. However, we can tackle this problem with similar functional analytic methods used in this article. We shall show three following statements:

1. The differential operator $D_A\mu = \frac{\mathrm{d}}{\mathrm{d}t} + 2 + \tilde S$ is an isomorphism for $A = S$, i.e. $B = 0$.
2. The operator B is a compact.
3. The differential operator $\frac{\mathrm{d}}{\mathrm{d}t} + 2 + \tilde S + B$ injects, i.e. the only solution with $\xi = 0$ is $a = 0$.

That means that $D_A\mu$ defines an injective Fredholm operator of index 0 which means it is an isomorphism and so the claim is true.
The advantage of this functional analytic approach is that the operator theory automatically cares about the asymptotic. Of course the standard existence and uniqueness results for differential equation apply here as well but with those we would have to take of the limiting behaviour separately.

1. For any $\xi \in \mathscr{C}_{\exp}(\mathbb{R};\mathfrak{g})$ the solution with initial value $a(t_0) = \tilde a_0$ is given by

   $$a(t) = \exp(-(2+\tilde S)t)\int_{t_0}^{t}\exp((2+\tilde S)t')\,\xi(t')\,\mathrm{d}t' + \exp(-(2+\tilde S)t)\,\tilde a_0.$$

   We have to choose $\tilde a_0 = 0$ to ensure the correct asymptotics of the second summand.
   The eigenvalues of $\tilde S$ are 0 and $\pm i\sqrt{\mathrm{ad}_{\mathrm{cas}}}$ so that $\exp(t'\tilde S)$ is unitary on the orthogonal complement of its kernel and so can be inverted there. Consequently, the integral of $\exp(\tilde S t')$ can be computed and is seen to be bounded due to $e^{2t}|\xi(t)| \le M$. This implies with

   $$e^{2t}a(t) = \exp(-\tilde S t)\int_{t_0}^{t}\exp(\tilde S t')\,e^{2t'}\xi(t')\,\mathrm{d}t'$$

   that $|e^{2t}a(t)|$ is bounded for large t. The same argument works for $|e^{-\delta t}a(t)|$ for $t \to -\infty$.
   This shows that a has the correct asymptotic behaviour. From $\dot a = -2a - \tilde S a + \xi$ it follows that also $\dot a$ has the correct asymptotic behaviour and so $\|a\|_\pm < \infty$.

2. We can see B more or less as a multiplication operator with an exponentially decaying function and so a bounded operator. We show that for any $\|\cdot\|_\pm$-bounded sequence (a_n) the sequence (Ba_n) has a convergent subsequence. On any compact set in $\mathbb{R}$ we can apply the Arzela-Ascoli theorem and thus only need to show that (Ba_n) is pointwise bounded and equicontinuous. Due to the exponential decay we can then find a convergent subsequence for all of $\mathbb{R}$ with a diagonal argument:
   The family (a_n) is bounded, and so automatically pointwise bounded. Since $\|a\|_\pm < \infty$ implies that the derivative is bounded, all a_n are Lipschitz continuous with a common Lipschitz constant which is

sufficient for equicontinuity of the family. Both properties are preserved by the bounded operator B. Let us recursively construct a convergent subsequence of (a_n) by a diagonal argument: Define the inital sequence $a^{(0)}$ to be (a_n). For any $N \in \mathbb{N}$ there is a subsequence $a^{(N)}$ in $a^{(N-1)}$ which converges uniformly on the interval $[-N;N]$ and thus with respect to $\|\cdot\|_\pm$ on $[-N;N]$ - this is a consequence of the Arzela-Ascoli theorem. We shall now show that the diagonal sequence $\tilde{a} = (\tilde{a}_N)$ given by $\tilde{a}_N = a_N^{(N)}$ converges with respect to $\|\cdot\|_\pm$:
Let $\varepsilon > 0$. There is some integer $R > 0$ such that $|a_n| < \varepsilon/2$ outside of $[-R;R]$ for any $n \in \mathbb{N}$. By construction $\tilde{a}$ is a subsequence of the convergent sequence $a^{(R)}$ and so there is some $R' > R$ such that $\|\tilde{a}_n - \tilde{a}_m\| < \varepsilon$ holds on $[-R;R]$ and so on all of $\mathbb{R}$ for any $n,m > R'$. And so it follows that $\tilde{a}$ is a $\|\cdot\|_\pm$-Cauchy sequence and, hence, converges.

3. The injectivity is a consequence of the uniqueness of solutions to ordinary differential equations: As $a = 0$ is a solution to $\dot{a} + (2 + \tilde{S} + B)a = 0$ it is the only one.

These arguments show that the differential μ is a submersion and so they complete the proof. □

So $\mathcal{N}$ is a manifold on which $\mathcal{G}^0$ acts, to ensure that the quotient is reasonable defined we show that the action is free and proper. Having the methods already from [RR18] in mind, we adopt to our situation by structuring the claims into two separate statements each which some more explanations:

Lemma 4.3. *For any $A_0 \in \mathscr{C}^1(\mathbb{R};\mathfrak{g})$ with $\|A_0\|_\pm < \infty$ there is an element $g \in \mathscr{C}^2(\mathbb{R};G)$ with $\|\dot{g}g^{-1}\|_\pm < \infty$ such that $g.A_0 = \operatorname{Ad}_g A_0 - \dot{g}g^{-1} = 0$. Two such g and g' differ by a constant factor from the left: $g'g^{-1}$ is a constant path in G. And so there is a unique such g with initial value $g(\infty) = 1$.*

Proof. We firstly have to solve the differential equation $h.A_0 = \operatorname{Ad}_h A_0 - \dot{h}h^{-1} = 0$. Passing to the notation as G were a matrix group we can write this to be

$$\dot{h} = hA_0.$$

So h is determined by a linear ordinary differential equation. The correct asymptotics is a consequence of $|\dot{h}h^{-1}| = |\operatorname{Ad}_h A_0| = |A_0|$ so that $\|\dot{h}h^{-1}\|_\pm = \|A_0\|_\pm < \infty$. By the definition of an action from $g.A_0 = 0$ if follows $g^{-1}.0 = A_0$. For two such g and g' we have $g'g^{-1}.0 = 0$ which is equivalent to $g'g^{-1}$ being constant.
From $\|\dot{g}g^{-1}\|_\pm < \infty$ and the compactness of G it follows that g converges for both $t \to \infty$ and $t \to -\infty$. And so $g(\infty)^{-1}g$ is the desired path in G with value 1 at $t = \infty$. □

This is the right moment to point out that if A_0 were a component of some $A \in \mathcal{A}$ as above we cannot conclude g being a gauge element in general as such a g has the additional requirement $\|\operatorname{Ad}_g \sigma_i - \sigma_i\|_- < \infty$. From $\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Ad}_g \sigma_i = [\dot{g}g^{-1}, \operatorname{Ad}_g \sigma] = [A_0, \operatorname{Ad}_g \sigma]$ we can deduce that $\operatorname{Ad}_g \sigma_i$ reaches its limit exponentially but it is in general only a representation conjugate to (σ_i) and not (σ_i) itself.

However this information is enough to deduce the freeness of the action as follows: Let $g \in \mathcal{G}^0$ have any fixpoint $A \in \mathcal{A}$ and $u \in \mathscr{C}^2(\mathbb{R};G)$ be the unique path such that $u.A_0 = 0$ and $u(\infty) = 1$. From the fact that ugu^{-1} gauges 0 back to 0 we can deduce that it is the constant path and, comparing with the value at $t = \infty$, it follows that g was already the constant path $g = 1$. Here again it became relevant to work with $\mathcal{G}^0$ instead of $\mathcal{G}$.
What is left is the properness:

Lemma 4.4. *The gauge group $\mathcal{G}^0$ acts properly on $\mathcal{N}$ and also on $\mathcal{A}$.*

Proof. Let $(g_n) \subset \mathcal{G}^0$ be an arbitrary sequence. We have to show that if there is a converging sequence $(A^n) \subset \mathcal{A}$ for which also $(g_n.A^n) \subset \mathcal{A}$ converges there is a converging subsequence in (g_n). As $\mathcal{G}^0$ is modelled over its Lie algebra we have to show that the family (ζ_n), $\zeta_n = \dot{g}_n g_n^{-1}$, has a convergent subsequence.
We aim to apply the Arzela-Ascoli theorem to the family (ζ_n) on any compact interval in $\mathbb{R}$ and deduce the existence of a convergent subsequence as before. The boundedness of (ζ_n) is direct consequence of the convergence of (A_n) and $(g_n.A_n)$

$$\|\zeta_n\| = \|\dot{g}_n g_n^{-1}\| = \|\operatorname{Ad}_{g_n} A_0^n - g_n.A_0^n\| \le \|\operatorname{Ad}_{g_n} A_0^n\| + \|g_n.A_0^n\| = \|A_0^n\| + \|g_n.A_0^n\|.$$

The equicontinuity follows from the boundedness with respect to $\|\cdot\|_\pm$: The first derivatives are bounded by the same number which means the family shares the same Lipschitz constant.
Using the same diagonal argument as the second step of the proof of Lemma 4.2. we deduce that (ζ_n) contains a convergent subsequence. ☐

We can now pass to the quotient

$$\mathcal{M} = \frac{\{\text{Solutions to Nahms Equations in } \mathcal{A}\}}{\mathcal{G}^0} = \frac{\mathcal{N}}{\mathcal{G}^0}.$$

The infinite dimensional setup requires some additional work to invest for $\mathcal{M}$ being a manifold:

Theorem 4.5. *The quotient $\mathcal{M} = \frac{\mathcal{N}}{\mathcal{G}^0}$ is a hyperkähler manifold.*

Proof. The main point in the proof is to show that $\mathcal{M}$ is a manifold. If that is done we can just induce all the necessary structures from $\mathcal{N}$ to $\mathcal{M}$ via the $\mathcal{G}^0$-equivariance, the integrability of the hyperkähler structure follows as usual in the hyperkähler setting already from the closedness of the symplectic forms.
The smoothness of the quotient is proven using the slice technique, similar to Kronheimers proofs in [Kro04] and [Kro90a]: To any $A \in \mathcal{A}$ we regard the slice

$$\mathcal{S}_A = \{A + a \mid a \in T_A\mathcal{A},\ \|a\|_\pm < \varepsilon,\ \nabla_A^* a = \dot{a}_0 + 2a_0 + \sum_{l=0}^{3}[A_l, a_l] = 0\}$$

which is, speaking geometrically, orthogonal to the $\mathcal{G}^0$-orbit, i.e. $T_A\mathcal{S}_A \perp T_A\mathcal{G}^0$, as we have seen in the previous computation of the $\mathcal{G}^0$-momentum map, here $\varepsilon > 0$ is later chosen small. We have to show that for any $A + b \in \mathcal{S}_A$ the only $g \in \mathcal{G}^0$ such that $g.(A+b) \in \mathcal{S}_A$ is exactly $g = 1$. In the setup of an $A + b \in \mathcal{S}_A$ and $g.(A+b) \in \mathcal{S}_A$ we put $b' = g.(A+b) - A$ and since $A + b' \in S_A$ we know

$$-\nabla_A^* b' = \dot{b}_0' + 2b_0' + \sum_{l=0}^{3}[A_l, b_l'] = 0.$$

Considering this as a differential equation of order 2 for g with initial value $g(\infty) = 1$ our aim is to show that $g = 1$ is the only solution when ε is chosen small enough. The usual existence and uniqueness statement of Picard-Lindelöf will not suffice here because it would be tricky to handle the initial value being a limit and not a value at finite time. And so again we apply analysis in the infinite dimensional setting: Local invertibility comes as a consequence of the invertibility of the differential, if the statement is modelled over the correct spaces we have dealed with the asymptotics as well.
We shall pass to the Lie algebra $\operatorname{Lie}\mathcal{G}^0$: Since $g(\infty) = 1$ the gauge g is necessarily in the image of the exponential map with $g = \exp\xi$ around $t = \infty$ and so we can ask for the linearised equation in ξ: If the linearisation is an isomorphism the uniqueness of the solution is garanteed. It reads as

$$\nabla_A^*\nabla_{A+b}\xi = \nabla_A^*(\nabla_A\xi + [b, \xi]) = 0.$$

The isomorphy of the operator $\nabla_A^*\nabla_A + \nabla_A^* \operatorname{ad}_b : \operatorname{Lie}\mathcal{G}^0 \to \mathscr{C}_{\exp}(\mathbb{R}; \mathfrak{g})$ is achieved by the following line of arguments:

1. $\nabla_S^*\nabla_S$ is an isomorphism where $S = (0, -\sigma_1, -\sigma_2, -\sigma_3)$ is again the model solution
2. $\nabla_A^*\nabla_A$ is an injective compact perturbation of $\nabla_S^*\nabla_S$
3. $\nabla_A^*\nabla_A + \nabla_A^* \operatorname{ad}_b$ remains for $\|b\|_\pm < \varepsilon$ small enough still invertible

The first two points show that $\nabla_A^*\nabla_A$ is an injective operator of Fredholm index 0 and so an isomorphism to which we add in point 3 an operator of controlled norm. If ε is small enough the added operator has norm so small that we remain with an isomorphism because the space of invertible operators is an open subset of the space of operators.

1. For the model solution $S = (0, -\sigma_1, -\sigma_2, -\sigma_3)$ we can compute the operator $\nabla_S^* \nabla_S$ directly to be

$$\nabla_S^* \nabla_S \xi = \ddot{\xi} + 2\dot{\xi} + \sum_{j=1}^{3} \mathrm{ad}_{\sigma_j}^2 \xi.$$

We can decompose $\mathfrak{g} = \oplus_{k=0}^{\infty} \mathrm{Eig}(\mathrm{cas}, \nu_k)$ into eigenspaces of the Casimir operator $\mathrm{cas} = \mathrm{ad}_{\sigma_1}^2 + \mathrm{ad}_{\sigma_2}^2 + \mathrm{ad}_{\sigma_3}^2$ with eigenvalues $\nu_k = k(k-1)$, $k \in \mathbb{N}_0$. In this decomposition the above equation $\ddot{\xi}_k + 2\dot{\xi}_k + \nu_k \xi_k = \zeta_k$ can easily be solved with the shorter notation $\tilde{\nu}_k = \sqrt{\nu_k - 1}$, $k \geq 2$, to be

$$\xi_k(t) = -\frac{1}{\tilde{\nu}_k} \int_{-\infty}^{t} \sin(\tilde{\nu}_k(\tau - t)) e^{\tau - t} \zeta_k(\tau) \mathrm{d}\tau + C \cos(\tilde{\nu}_k t) e^{-t} + C' \sin(\tilde{\nu}_k t) e^{-t}$$

where C and C' are constants of the integration. From $\|\zeta\|_{\pm} < \infty$ we can read off that $|\zeta_k| \leq e^{\delta t} \|\zeta\|_-$ for $t \to -\infty$ respectively $|\zeta_k| \leq e^{-2t} \|\zeta\|_-$ for $t \to \infty$. And so the intergration from $-\infty$ to t is well-defined, since we can estimate ζ_k and the trigonometric terms against 1. The same elementary estimates show that the integral converges like e^{-2t} as $t \to \infty$. For the convergence of the whole ξ, we have to choose $C = C' = 0$ which makes the solution also unique. A very similar calculation also works for $k = 0, 1$, i.e. $\nu_k = 0$.

2. The computation of $\ker \nabla_A^* \nabla_A$ reduces due to the equality $\langle\!\langle \nabla_A^* \nabla_A \xi, \xi \rangle\!\rangle = \langle\!\langle \nabla_A \xi, \nabla_A \xi \rangle\!\rangle$ to the computation of the kernel $\ker \nabla_A$ which turns out to be trivial: $\ker \nabla_A^* \nabla_A = \ker \nabla_A = \{0\}$.
Now using that $A = S + a$ we can write

$$(\nabla_A^* \nabla_A - \nabla_S^* \nabla_S)\xi = 2\big[a_0, \dot{\xi}\big] + [\dot{a}_0, \xi] + \sum_{l=0}^{3} \mathrm{ad}_{a_l} \mathrm{ad}_{\sigma_l} \xi + \sum_{l=0}^{3} \mathrm{ad}_{A_l} \mathrm{ad}_{a_l} \xi$$

As in the computation of the properness of the $\mathcal{G}^0$-action on $\mathcal{A}$ the right hand side is seen to define a compact operator in ξ, this time also using that a and A are uniformly continuous.

3. The norm of the composition of the two bounded operators ∇_A^* and $\xi \mapsto (\mathrm{ad}_{b_0} \xi, \mathrm{ad}_{b_1} \xi, \mathrm{ad}_{b_2} \xi, \mathrm{ad}_{b_3} \xi)$ can be bounded by $\|\nabla_A\| \|b\|$ and so the operator $\nabla_A^* \nabla_{A+b}$ is close enough to the invertible operator $\nabla_A^* \nabla_A$ and hence invertible itself for appropriately small ε.

And so it follows that $\xi = 0$, i.e. $g = 1$, in a neighbourhood of $t = \infty$ and so everywhere. □

In the proof that $\nabla_S^* \nabla_S$ is an isomorphism we have seen that δ needs to be chosen such that it is smaller than the smallest positive eigenvalue of the Casimir operator: This is why $\delta < 1$ is sufficient in our setup to deduce that the quotient is a manifold.
It is also a direct consequence that we can identify the tangent space at some $[A] \in \mathcal{M}$ with a subspace of $T_A \mathcal{N}$ respectively the subspace of such $a \in T_A \mathcal{A}$ that fulfill

$$\begin{cases} \dot{a}_0 + 2a_0 + [A_0, a_0] + [A_1, a_1] + [A_2, a_2] + [A_3, a_3] = 0 \\ \dot{a}_1 + 2a_1 + [A_0, a_1] + [a_0, A_1] + [A_2, a_3] + [a_2, A_3] = 0 \\ \dot{a}_2 + 2a_2 + [A_0, a_2] + [a_0, A_2] + [A_3, a_1] + [a_3, A_1] = 0 \\ \dot{a}_3 + 2a_3 + [A_0, a_3] + [a_0, A_3] + [A_1, a_2] + [a_1, A_2] = 0 \end{cases}$$

With that we are almost done with presenting the construction of the moduli space. But there are a few symmetries on $\mathcal{N}$ that pass down to $\mathcal{M}$ and some that do not:

- From the $\mathcal{G}$-action on $\mathcal{N}$ it remains an action of $\mathcal{G}/\mathcal{G}^0 = G$ on $\mathcal{M}$ simply by choosing another limit point at $t = \infty$. Comparing to the $\mathcal{G}^0$-moment map computation above, it is automatic to see this action being tri-hamiltonian with moment map $\mu_G : \mathcal{M} \to \mathfrak{g}^\vee \otimes \mathfrak{sp}(1)$ given by

$$\mu_G(A) = \lim_{t \to \infty} e^{2t} (A_1(t), A_2(t), A_3(t)) = \lim_{t \to \infty} e^{2t} \, \mathrm{Im} \, A(t).$$

- The idea of an $\mathrm{Sp}(1)$-action on the quotient $\mathcal{M}$ is to act just on the $\mathbb{H}$ component by the adjoint action. However in general Ad_q, $q \in \mathrm{Sp}(1)$, does not preserve the boundary conditions at $t = -\infty$. To deal with that problem we integrate the representation $\rho : \mathfrak{sp}(1) = \mathfrak{su}(2) \to \mathfrak{g}$ to the groups $\rho : \mathrm{Sp}(1) \to G$ and correct by any $g \in \mathscr{C}^2(\mathbb{R}; G)$ with $g(-\infty) = \rho(q^{-1})$ such that $g \circ \mathrm{Ad}_q \in \mathcal{G}^0$, e.g. one which is constant in some neighborhood of $t = -\infty$. Since

$$q.(A_0, A_1, A_2, A_3) = g.(A_0 + \mathrm{Ad}_q(iA_1 + jA_2 + kA_3)), \quad A \in \mathcal{A}.$$

maps $\mathcal{G}^0$-orbit into $\mathcal{G}^0$-orbit it induces an Sp(1)-action on the quotient $\mathcal{M}$.
With the usual map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, $q \mapsto (q_{kl})$ we can rewrite this to be

$$q.(A_0, A_1, A_2, A_3) = g.(A_0 + i\sum_{l=1}^{3} q_{1l}A_l + j\sum_{l=1}^{3} q_{2l}A_l + k\sum_{l=1}^{3} q_{3l}A_l).$$

In other words Sp(1) acts on both factors of $\mathfrak{g} \otimes \mathbb{H}$ in $\mathscr{C}^1(\mathbb{R}; \mathfrak{g} \otimes \mathbb{H})$. But the correction by $g \in \mathcal{G}$ does not affect the rotational action of Sp(1) in the sphere of complex structures and so this action is also a rotational one.

- Nahms equations are invariant with respect to the time shift: The additive real numbers act on $\mathcal{N}$ by $(\lambda.A)(t) = A(t + \lambda)$. This action descents to the quotient $\mathcal{M}$. To check this we firstly have to observe that $\lambda.(g.A) = (\lambda.g).(\lambda.A)$ with $(\lambda.g)(t) = g(t + \lambda)$ for any $g \in \mathcal{G}^0$. Since for any gauge element g also $\lambda.g$ is a gauge element it turns out that the shifting action preserves the $\mathcal{G}^0$-orbits and so passes to $\mathcal{M}$.

- It is also possible to act via scaling of the semi-group $\mathbb{R}_{>0}$. A $\lambda \in \mathbb{R}_{>0}$ acts on an $A \in \mathcal{N}$ via

$$(\lambda.A)(t) = \frac{e^{-2t}}{e^{-2t} + 2\lambda} A(-\frac{1}{2}\log(e^{-2t} + 2\lambda)).$$

This action does preserve the $\mathcal{G}$-orbits in $\mathcal{A}$ but not the smaller $\mathcal{G}^0$-orbits. That is the reason why it is not induced to the quotient $\mathcal{M}$.
It is worth pointing out that the action of λ essentially rescales $s_\lambda : \mathbb{R} \to \mathbb{R}_{-\log(2\lambda)/2}$ via $s_\lambda(t) = -\frac{1}{2}\log(e^{-2t} + 2\lambda)$ so that $(\lambda.A)(\infty) = A(-\log(2\lambda)/2)$. The prefactor corrects the chain rule $\dot{s}_\lambda(t) = \frac{e^{-2t}}{e^{-2t}+2\lambda}$, checking whether this really defines an action reduces to $s_\lambda \circ s_{\lambda'} = s_{\lambda+\lambda'}$.

Although this latter action looks a bit unnatural it is not. That becomes clear when passing to the Nahms picture on $\mathbb{R}_{\leq 0}$. Following the transformation

$$\begin{cases} s(t) = -\frac{1}{2}e^{-2t} \\ -2sT(s) = A(t) \end{cases} \quad \text{inverted by} \quad \begin{cases} t(s) = -\frac{1}{2}\log(-2s) \\ e^{2t}A(t) = T(s) \end{cases}$$

we transform the equation

$$\dot{A}_i + 2A_i + [A_0, A_i] + [A_j, A_k] = 0 \quad \text{into} \quad \dot{T}_i + [T_0, T_i] + [T_j, T_k] = 0$$

and the model solution

$$(0, -\sigma_1, -\sigma_2, -\sigma_3) \quad \text{into} \quad (0, \sigma_1/2s, \sigma_2/2s, \sigma_3/2s).$$

With $s(s_\lambda(t)) = s(t) - \lambda$ we just map

$$(\lambda.A)(t) = \frac{e^{-2t}}{e^{-2t} + 2\lambda} A(-\frac{1}{2}\log(e^{-2t} + 2\lambda)) \quad \text{into} \quad (\lambda.T)(s) = T(s - \lambda)$$

to the shift along the real line.

Solutions $A \in \mathcal{A}$ were supposed to asymptotically approach $(0, -\sigma_1, -\sigma_2, -\sigma_3)$ and so for our model on $\mathbb{R}_{\leq 0}$ we demand the T's to decay like $(0, \sigma_1/2s, \sigma_2/2s, \sigma_3/2s)$. In order to start with a well-defined norm on $\mathscr{C}^2(\mathbb{R}_{\leq 0}; \mathfrak{g})$ and we choose the model solution to be the non-singular $(0, \sigma_1/2(s-1), \sigma_2/2(s-1), \sigma_3/2(s-1))$ on $\mathbb{R}_{\leq 0}$ and that way have well-defined values at $s = 0$ but preserve the asymptotics.

4.2 On the Negative Half-Line $\mathbb{R}_{\leq 0}$

Let $\{\sigma_l\}_{l=1}^3$ be a $\mathfrak{su}(2)$-triple in $\mathfrak{g}$. For some small $\delta > 0$ we choose the base space $\mathcal{T}$ to be the subspace of $\mathscr{C}^1(\mathbb{R}_{\leq 0};\mathfrak{g}) \otimes \mathbb{H}$ consisting of quadrupels $T = (T_0, T_1, T_2, T_3) = T_0 + iT_1 + jT_2 + kT_3$ such that

- $\|T_0\| < \infty$ and $\|T_l - \sigma_l/2(s-1)\| < \infty$ for $l = 1, 2, 3$ where $\|w\| = \sup_{s\leq 0}(|s-1|^{1+\delta}|w(s)| + |s-1|^{2+\delta}|\dot{w}(s)|)$

As in [Biq96] we regard $\mathcal{T}$ as Banach manifold modelled over the Banach space $\{t \in \mathscr{C}^1(\mathbb{R}_{\leq 0};\mathfrak{g}) \otimes \mathbb{H} \mid \|t\| < \infty\}$. Consequently, this space happens to be the tangent space at any point and it is seen to be a quaternionic vector space if we write it as $\{f \in \mathscr{C}^1(\mathbb{R}_{\leq 0};\mathfrak{g}) \mid \|f\| < \infty\} \otimes \mathbb{H}$, so we can write down the hyperkähler structure on $\mathcal{T}$ to be given by

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle = \int_{-\infty}^0 |\mathrm{d}T|^2 \quad \text{i.e.} \quad \langle\!\langle t, t' \rangle\!\rangle = \int_{-\infty}^0 \langle t(s), t'(s)\rangle \,\mathrm{d}s$$

$$I(t_0, t_1, t_2, t_3) = (-t_1, t_0, -t_3, t_2) \quad \text{so that} \quad \omega_I = -\int_{-\infty}^0 \mathrm{d}T_0 \wedge \mathrm{d}T_1 + \mathrm{d}T_2 \wedge \mathrm{d}T_3$$

$$J(t_0, t_1, t_2, t_3) = (-t_2, t_3, t_0, -t_1) \quad \text{so that} \quad \omega_J = -\int_{-\infty}^0 \mathrm{d}T_0 \wedge \mathrm{d}T_2 + \mathrm{d}T_3 \wedge \mathrm{d}T_1$$

$$K(t_0, t_1, t_2, t_3) = (-t_3, -t_2, t_1, t_0) \quad \text{so that} \quad \omega_K = -\int_{-\infty}^0 \mathrm{d}T_0 \wedge \mathrm{d}T_3 + \mathrm{d}T_1 \wedge \mathrm{d}T_2.$$

The well-definedness of all those integrals is a consequence of the decay condition.
We shall now define the group of gauge elements to be

$$\mathcal{G} = \{g \in \mathscr{C}^2(\mathbb{R}_{\leq 0};\mathfrak{g}) \mid \left\|\dot{g}g^{-1}\right\| < \infty,\ \|(\mathrm{Ad}_g \sigma_i - \sigma_i)/2(s-1)\| < \infty\}$$

and the pointed gauge group being a normal subgroup in $\mathcal{G}$

$$\mathcal{G}^0 = \{g \in \mathscr{C}^2(\mathbb{R}_{\leq 0};\mathfrak{g}) \mid \left\|\dot{g}g^{-1}\right\| < \infty,\ \|(\mathrm{Ad}_g \sigma_i - \sigma_i)/2(s-1)\| < \infty,\ g(0) = 1\}$$

both acting via

$$g.(T_0, T_1, T_2, T_3) = (g.T_0, g.T_1, g.T_2, g.T_3) = (\mathrm{Ad}_g T_0 - \dot{g}g^{-1}, \mathrm{Ad}_g T_1, \mathrm{Ad}_g T_2, \mathrm{Ad}_g T_3).$$

On a tangent vector $t \in T_T\mathcal{T}$ it induces

$$g.t = g.(t_0, t_1, t_2, t_3) = (\mathrm{Ad}_g t_0, \mathrm{Ad}_g t_1, \mathrm{Ad}_g t_2, \mathrm{Ad}_g t_3).$$

Taking the Ad-invariance of the metric on $\mathfrak{g}$ into account we can follow now directly that $\mathcal{G}$ respects the hyperkähler structure of $\mathcal{T}$. We intend to find the moment map of the $\mathcal{G}^0$- action and firstly have to figure out that the Lie algebra of $\mathcal{G}$ is given by

$$\mathrm{Lie}\,\mathcal{G}^0 = \{\xi \in \mathscr{C}^2(\mathbb{R}_{\leq 0};\mathfrak{g}) \mid \left\|\dot{\xi}\right\| < 0,\ \|\mathrm{ad}_\xi \sigma_i/2(s-1)\| < \infty,\ \xi(0) = 0\}$$

and the corresponding fundamental vector fields of the action turns out to be

$$K^\xi_{\mathcal{G}^0}(T) = (\mathrm{ad}_\xi T_0 - \dot{\xi}, \mathrm{ad}_\xi T_1, \mathrm{ad}_\xi T_2, \mathrm{ad}_\xi T_3).$$

The computation

$$\begin{aligned}
\left\langle\!\!\left\langle K^\xi_{\mathcal{G}^0}, t \right\rangle\!\!\right\rangle &= \int_{-\infty}^0 \left\langle \mathrm{ad}_\xi T_0 - \dot{\xi}, t_0 \right\rangle + \langle \mathrm{ad}_\xi T_1, t_1\rangle + \langle \mathrm{ad}_\xi T_2, t_2\rangle + \langle \mathrm{ad}_\xi T_3, t_3\rangle)\,\mathrm{d}s \\
&= \int_{-\infty}^0 (-\left\langle \dot{\xi}, t_0\right\rangle + \sum_{l=0}^3 \langle \xi, [T_l, t_l]\rangle)\,\mathrm{d}s = \int_{-\infty}^0 -\frac{\mathrm{d}}{\mathrm{d}s}\langle \xi, t_0\rangle + \left\langle \xi, \dot{t}_0\right\rangle + \sum_{l=0}^3 \langle \xi, [T_l, t_l]\rangle\,\mathrm{d}s \\
&= \int_{-\infty}^0 \left\langle \xi, \dot{t}_0 + \sum_{l=0}^3 [T_l, t_l] \right\rangle(s)\,\mathrm{d}s.
\end{aligned}$$

leads us as in Kronheimers picture on $\mathbb{R}$ to the $\mathcal{G}$-equivariant hyperkähler momentum map $\mu : \mathcal{T} \to \mathscr{C}_{\exp}(\mathbb{R}_{\leq 0}; \mathfrak{g}) \otimes \mathfrak{sp}(1) \subset (\operatorname{Lie}\mathcal{G})^\vee \otimes \mathfrak{sp}(1)$ with components

$$\begin{aligned}
\mu^I_{\mathcal{G}^0}(T) &= \dot{T}_1 + [T_0, T_1] + [T_2, T_3] \\
\mu^J_{\mathcal{G}^0}(T) &= \dot{T}_2 + [T_0, T_2] + [T_3, T_1] \\
\mu^K_{\mathcal{G}^0}(T) &= \dot{T}_3 + [T_0, T_3] + [T_1, T_2]
\end{aligned}$$

We say that $T \in \mathcal{T}$ solves Nahms equations if $\mu(T) = 0$ and call $\mathcal{N} = \mu^{-1}(0)$ the solution space. There are a few things to check here: $\mathcal{N}$ needs to be a submanifold of $\mathcal{T}$ on which $\mathcal{G}^0$ acts freely and properly. When this is done we can pass down to the quotient

$$\mathcal{M} = \frac{\{\text{solutions to Nahms equations in } \mathcal{T}\}}{\mathcal{G}^0} = \frac{\mathcal{N}}{\mathcal{G}^0}$$

using the scheme of a hyperkähler reduction and thus inherit the presented $\mathcal{G}^0$-preserved structure. The tangent space $T_{[T]}\mathcal{M}$ at $[T] \in \mathcal{M}$ can be seen as the subspace of $T_T\mathcal{T}$ such that the tangent vectors fulfill

$$\begin{cases}
\dot{t}_0 + [T_0, t_0] + [T_1, t_1] + [T_2, t_2] + [T_3, t_3] = 0 \\
\dot{t}_1 + [T_0, t_1] + [t_0, T_1] + [T_2, t_3] + [t_2, T_3] = 0 \\
\dot{t}_2 + [T_0, t_2] + [t_0, T_2] + [T_3, t_1] + [t_3, T_1] = 0 \\
\dot{t}_3 + [T_0, t_3] + [t_0, T_3] + [T_1, t_2] + [t_1, T_2] = 0
\end{cases} .$$

Just as in the previous case, the model on $\mathbb{R}$, we can give several group actions which we shall define now

- From the $\mathcal{G}$-action on $\mathcal{N}$ it remains an hyperkähler action of $\mathcal{G}/\mathcal{G}^0 = G$. Similary to the previous $\mathcal{G}^0$-moment map computation the moment map $\mu_G : \mathcal{M} \to \mathfrak{g}^\vee \otimes \mathfrak{sp}(1)$ turns out to be equivariant with

  $$\mu_G(T) = (T_1(0), T_2(0), T_3(0)).$$

- The $\operatorname{Sp}(1)$-action on $\mathcal{T}$ has to consists of two ingrediences since a simple Ad_q, $q \in \operatorname{Sp}(1)$, would not respect the boundary conditions. And so we choose a path in $\mathscr{C}^2(\mathbb{R}_{\leq 0}; G)$ such that $g \circ \operatorname{Ad}_q \in \mathcal{G}^0$. This means we have to have $g(-\infty) = \rho(q^{-1})$, it can be constant outside of some compact set. The action on a $[T] \in \mathcal{M}$ is now given on representatives

  $$q.(T_0, T_1, T_2, T_3) = g.(T_0 + \operatorname{Ad}_q(iT_1 + jT_2 + kT_3)).$$

  With the usual map $\operatorname{Sp}(1) \to \operatorname{SO}(3)$, $q \mapsto (q_{kl})$ we can rewrite this to be

  $$q.(T_0, T_1, T_2, T_3) = g.(T_0 + i\sum_{l=1}^3 q_{1l}T_l + j\sum_{l=1}^3 q_{2l}T_l + k\sum_{l=1}^3 q_{3l}T_l)$$

  and it also follows that $\operatorname{SO}(3)$ is a rotational action.

- Nahms equations are an autonomous system so that negative shifts in time defines a symmetry

  $$(\lambda.T)(s) = T(s - \lambda).$$

  This action does only preserve the $\mathcal{G}$-orbits in $\mathcal{T}$ but not the smaller $\mathcal{G}^0$ orbits and so does not descent to the quotient.

- In Kromheimers picture we also had the shift action along the real line. This one is translated into the scaling action: For some $h \in \mathbb{R}_{>0}$ we can consider $(h.T)(s) = hT(hs)$. The quotient $\mathcal{M}$ inherits this action since $h.(g.T)$ and $g.(h.T)$ are linked by the gauge transformation $(h.g)(s) = g(hs)$ which in turn implies that the $\mathcal{G}^0$-orbits are invariant under this way of acting.

4.3 The Complex Formulation on $\mathbb{R}_{\leq 0}$

The next three sections deal with the complex picture, its relation to the real world and the correspondence to the nilpotent orbits. We shall now choose the complex structure I and that way identify $\mathbb{H}$ with $\mathbb{C}^2$. This means that we have to adjust all the above objects: In stead of paths in $\mathfrak{g} \otimes \mathbb{H}$ we now work with paths in $\mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}$, i.e. with pairs (α, β) of which each component itself is a path in the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$: We write $T = T_0 + iT_1 + jT_2 + kT_3 = \alpha + j\beta = (\alpha, \beta)$, i.e.

$$\begin{cases} \alpha = T_0 + iT_1 \\ \beta = T_2 + iT_3. \end{cases}$$

The representation $\rho : \mathfrak{su}(2) \to \mathfrak{g}$ becomes complexified to an $\mathfrak{sl}_2\,\mathbb{C}$-representation on $\mathfrak{g}_{\mathbb{C}}$

$$\begin{cases} h = \quad i\sigma_1 \\ 2e = \sigma_2 + i\sigma_3. \end{cases}$$

The gauge transformations now take values in the universal complexified group $G_{\mathbb{C}}$ of G. We shall need here the Cartan composition which is given by the isomorphism

$$G \times \mathfrak{g} \to G_{\mathbb{C}} \quad \text{via} \quad (g, \xi) \mapsto g \exp(i\xi).$$

For matrices, this is just the polar decomposition and so we call elements from G unitary and elements of the shape $\exp(i\xi)$ self-adjoint. This also corresponds to involution on $\mathcal{G}_{\mathbb{C}}$ given by

$$(g\, \exp(i\xi))^* = \exp(i\xi)\, g^{-1} \quad \text{and} \quad (\xi_1 + i\xi_2)^* = -\xi_1 + i\xi_2 = -\overline{\xi_1 + i\xi_2}$$

on the Lie algebra level, here $\xi_1, \xi_2 \in \mathfrak{g}$ were real elements. And so $(\mathrm{Ad}_g\, \eta)^* = \mathrm{Ad}_{(g^*)^{-1}}\, \eta^*$.
With the involution we can reconstruct the T's from (α, β) using

$$\alpha^* = -T_0 + iT_1 \quad \text{and} \quad \beta^* = -T_2 + iT_3$$

and find

$$T_0 = \frac{1}{2}(\alpha - \alpha^*) \qquad T_1 = -\frac{i}{2}(\alpha + \alpha^*)$$
$$T_2 = \frac{1}{2}(\beta - \beta^*) \qquad T_3 = \frac{i}{2}(\beta + \beta^*).$$

With these relations we can easily check Nahms equations in the (α, β)-formulation to be

$$\begin{cases} \dot{\beta} + [\alpha, \beta] = 0 \\ \frac{\mathrm{d}}{\mathrm{d}s}(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0. \end{cases}$$

The next discussion devides into three parts: The complex moment map leads to the first equation, the so-called complex equation, and the quotient modulo the complexified gauge group. Afterwards the correspondence to the real picture is discussed. While the quotient does contain equivalence classes of solutions to the complex equation we have to show that any $\mathcal{G}^0_{\mathbb{C}}$ orbit also contains a unique $\mathcal{G}^0$-orbit of solutions to the second equation to which we refer as the real equation. In the third part we shall prove that the moment map of the $G_{\mathbb{C}}$ on the moduli space is an isomorphism with the coadjoint orbit $\mathrm{Ad}_{G_{\mathbb{C}}}\, e$.
Most of the definitions are motivated by the way we interpret $\mathbb{H}$ as $\mathbb{C}^2$ just as presented in the conventional chapter. This procedure is linear and does not have any effect on the analysis behind. That is why most of the presented objects behave similar to the objects in the real model on $\mathbb{R}_{\leq 0}$.

4.3.1 The Moduli Spaces in the Complex Picture

For a given $\mathfrak{sl}_2\,\mathbb{C}$-triple $\{h,e,f\}$ in a complexified semi-simple Lie algebra $\mathfrak{g}_\mathbb{C}$ we consider $\mathcal{T}_\mathbb{C}$ to be the subspace of $\mathscr{C}^1(\mathbb{R}_{\le 0};\mathfrak{g}_\mathbb{C}\oplus\mathfrak{g}_\mathbb{C})$ consisting of pairs (α,β) such that

- $\|\alpha - h/2(s-1)\| < \infty$, $\|\beta - 2e/2(s-1)\| < \infty$ where $\|w\| = \sup_{s\le 0}(|s-1|^{1+\delta}|w(s)| + |s-1|^{2+\delta}|\dot{w}(s)|)$

This space $\mathcal{T}_\mathbb{C}$ is seen to be a complex space affine over $\{(a,b)\in\mathscr{C}^1(\mathbb{R}_{\le 0};\mathfrak{g}_\mathbb{C}\oplus\mathfrak{g}_\mathbb{C}) \mid \|(a,b)\|<\infty\}$. Regarding $\mathcal{T}_\mathbb{C}$ as a complex Banach manifold modelled over $\{(a,b)\in\mathscr{C}^1(\mathbb{R}_{<0};\mathfrak{g}_\mathbb{C}\oplus\mathfrak{g}_\mathbb{C}) \mid \|(a,b)\|<\infty\}\otimes\mathbb{C}$. This complex vector space inherits the complex structure I and comes equipped with the I-holomorphic symplectic form

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle = \mathrm{Re}\int_{-\infty}^{0} \mathrm{d}\alpha\otimes\mathrm{d}\bar{\alpha} + \mathrm{d}\beta\otimes\mathrm{d}\bar{\beta}$$

$$I(a,b) = (ia, ib) \quad \text{and} \quad \omega_\mathbb{C} = -\int_{-\infty}^{0}\mathrm{d}\alpha\wedge\mathrm{d}\beta.$$

The well-definedness of all those integrals is a consequence of the decay condition.
We shall now define the group of gauge elements to be

$$\mathcal{G}_\mathbb{C} = \{g\in\mathscr{C}^2(\mathbb{R}_{\le 0};G_\mathbb{C}) \mid \|\dot{g}g^{-1}\|<\infty,\ \|(\mathrm{Ad}_g h - h)/2(s-1)\|<\infty,\ \|(\mathrm{Ad}_g e - e)/2(s-1)\|<\infty\}$$

and the pointed gauge group being a normal subgroup in $\mathcal{G}_\mathbb{C}$

$$\mathcal{G}^0_\mathbb{C} = \{g\in\mathscr{C}^2(\mathbb{R}_{\le 0};G_\mathbb{C}) \mid \|\dot{g}g^{-1}\|<\infty,\ \|(\mathrm{Ad}_g h - h)/2(s-1)\|<\infty,\ \|(\mathrm{Ad}_g e - e)/2(s-1)\|<\infty,\ g(0)=1\}$$

both acting via

$$g.(\alpha,\beta) = (g.\alpha, g.\beta) = (\mathrm{Ad}_g\,\alpha - \dot{g}g^{-1}, \mathrm{Ad}_g\,\beta).$$

On a tangent vector $(a,b)\in T_{(\alpha,\beta)}\mathcal{T}_\mathbb{C}$ it induces

$$g.(a,b) = (\mathrm{Ad}_g\,a, \mathrm{Ad}_g\,b).$$

And so it follows that the $\mathcal{G}_\mathbb{C}$-action is holomorphic and preserves the structures on $\mathcal{T}_\mathbb{C}$. We intent to compute the moment map of the $\mathcal{G}^0_\mathbb{C}$- action and firstly have to figure out that the Lie algebra of $\mathcal{G}$ is given by

$$\mathrm{Lie}\,\mathcal{G}^0 = \{\xi\in\mathscr{C}^2(\mathbb{R}_{\le 0};\mathfrak{g}_\mathbb{C}) \mid \|\dot{\xi}\|<0,\ \|(\mathrm{ad}_\xi\,h/2(s-1)\|<\infty,\ \|(\mathrm{ad}_\xi\,f/2(s-1)\|<\infty,\ \xi(0)=0\}$$

and the corresponding fundamental vector fields of the action turn out to be

$$K^\xi_\mathcal{G}(\alpha,\beta) = (\mathrm{ad}_\xi\,\alpha - \dot{\xi}, \mathrm{ad}_\xi\,\beta).$$

From the computation

$$\begin{aligned}\left\langle\!\!\left\langle K^\xi_{\mathcal{G}^0_\mathbb{C}}, (a,b)\right\rangle\!\!\right\rangle &= \int_{-\infty}^{0}\left(\left\langle \mathrm{ad}_\xi\,\alpha - \dot{\xi}, a\right\rangle + \langle \mathrm{ad}_\xi\,\beta, b\rangle\right)\mathrm{d}s \\ &= \int_{-\infty}^{0}\left(-\left\langle\dot{\xi},a\right\rangle + \langle\xi,[\alpha,a]+[\beta,b]\rangle\right)\mathrm{d}s = \int_{-\infty}^{0} -\frac{\mathrm{d}}{\mathrm{d}s}\langle\xi,a\rangle + \langle\xi,\dot{a}\rangle + \langle\xi,[\alpha,a]+[\beta,b]\rangle\,\mathrm{d}s \\ &= \int_{-\infty}^{0}\langle\xi,\dot{a}+[\alpha,a]+[\beta,b]\rangle\,\mathrm{d}s\end{aligned}$$

we can find the complex valued moment map

$$\mu_{\mathcal{G}^0_\mathbb{C}}(\alpha,\beta) = \dot{\beta} + [\alpha,\beta]$$

and to pass as before to the quotient of the complex symplectic manifold modulo a structure preserving gauge action

$$\mathcal{M}_\mathbb{C} = \frac{\{\text{solutions to the complex equation in } \mathcal{T}_\mathbb{C}\}}{\mathcal{G}^0_\mathbb{C}} = \frac{\mathcal{N}_\mathbb{C}}{\mathcal{G}^0_\mathbb{C}}.$$

If the space were a manifold we could just inherit the structure via the invariance. But the step of the complexification basically caused to the problem that $\mathcal{G}^0_\mathbb{C}$ is too huge. In general the properness will fail so that the quotient is not necessarily a hausdorff space. However, the next chapter will take care of this problem because it will just identify $\mathcal{M}$ and $\mathcal{M}_\mathbb{C}$.

4.3.2 The Real Equation

Due to the equivalence of the complex equation $\mu_{\mathcal{G}^0_{\mathbb{C}}}(\alpha,\beta) = 0$ and the two equations $\mu^J_{\mathcal{G}^0}(T) = \mu^K_{\mathcal{G}^0}(T) = 0$ there is additional information hidden in the real equation $\mu^I_{\mathcal{G}^0}(T) = 0$. Ignoring this for the moment the identification between T and (α,β) includes $\mathcal{N}$ in $\mathcal{N}_{\mathbb{C}}$. Since $\mathcal{G}^0$ is a subgroup of $\mathcal{G}^0_{\mathbb{C}}$ the map

$$\mathcal{M} \to \mathcal{M}_{\mathbb{C}} \quad \text{via} \quad T \mapsto (\alpha,\beta) = (T_0 + iT_1, T_2 + iT_3)$$

is well-defined. We want to show that this map actually is an bijection which is equivalent to say

- Injectivity: If (α,β) and (α',β') are $\mathcal{G}^0_{\mathbb{C}}$-equivalent in $\mathcal{N} \subseteq \mathcal{N}_{\mathbb{C}}$ then they are already $\mathcal{G}^0$-equivalent.
- Surjectivity: For any $(\alpha,\beta) \in \mathcal{N}_{\mathbb{C}}$ there is some $g \in \mathcal{G}^0_{\mathbb{C}}$ such that $g.(\alpha,\beta)$ solves the real equation.

And so it is all about finding a helpful form of $\mu^I_{\mathcal{G}^0}(\alpha,\beta) = 0$ to work with gauge elements: Donaldson gives in [Don84] a useful formulation in terms of operators on $\mathscr{C}^2(\mathbb{R}_{\leq 0};\mathfrak{g}_{\mathbb{C}})$ and, relying on that, proves the bijectivity. Here is what he did:

$$\mathrm{ad}_{\mu^I_{\mathcal{G}^0}(\alpha,\beta)} = \left[\partial_\alpha, \bar{\partial}_\alpha\right] - \left[\partial_\beta, \bar{\partial}_\beta\right] \quad \text{where} \quad \begin{array}{ll} \partial_\alpha = \frac{\mathrm{d}}{\mathrm{d}s} - \mathrm{ad}_{\alpha^*} & \partial_\beta = -\,\mathrm{ad}_{\beta^*} \\ \bar{\partial}_\alpha = \frac{\mathrm{d}}{\mathrm{d}s} + \mathrm{ad}_\alpha & \bar{\partial}_\beta = \mathrm{ad}_\beta \end{array}$$

which transform under the gauge element $g \in \mathcal{G}_{\mathbb{C}}$ as

$$\begin{array}{ll} \partial_{g.\alpha} = \mathrm{Ad}_{(g^*)^{-1}} \circ \partial_{\alpha^*} \circ \mathrm{Ad}_{g^*} & \partial_{g.\beta} = -\,\mathrm{Ad}_{(g^*)^{-1}} \circ \partial_{\beta^*} \circ \mathrm{Ad}_{g^*} \\ \bar{\partial}_{g.\alpha} = \mathrm{Ad}_g \circ \partial_\alpha \circ \mathrm{Ad}_{g^{-1}} & \bar{\partial}_{g.\beta} = \mathrm{Ad}_g \circ \partial_\beta \circ \mathrm{Ad}_{g^{-1}} \end{array}$$

We can here read off that $\mu^I_{\mathcal{G}^0}$ is transformed only under the selfajoint part $p = g^*g$ of $g \in \mathcal{G}_{\mathbb{C}}$ as

$$\mathrm{Ad}_{g^{-1}}\, \mu_i(g.(\alpha,\beta)) = \mu_i(\alpha,\beta) - \bar{\partial}_\alpha p^{-1} \partial_\alpha p + \bar{\partial}\beta p^{-1} \partial_\beta p.$$

And so if for some $g \in \mathcal{G}_{\mathbb{C}}$ with (α,β) also $g.(\alpha,\beta)$ fulfill the real equation, $p = g^*g$ satisfies

$$-\bar{\partial}_\alpha p^{-1} \partial_\alpha p + \bar{\partial}_\beta p^{-1} \partial_\beta p = 0 \quad \text{with} \quad p(0) = 1$$

from which already $p = 1$ follows. But this is nothing else to say that g was unitary and so $g \in \mathcal{G} \subseteq \mathcal{G}_{\mathbb{C}}$. This proves the injectivity.

Let us now come to the existence of such a gauge: The idea is to see the equation for g as a Euler-Lagrange equation, the corresponding functional is given by

$$\mathscr{L}(g) = \frac{1}{2} \int_{-\infty}^0 |g.\alpha + (g.\alpha)^*|^2 + |g.\beta|^2.$$

Lemma 4.6. *The functional $\mathscr{L}$ has a stationary point if and only if $\mu^I_{\mathcal{G}^0}(g.(\alpha,\beta)) = 0$.*

Proof. Donaldsons arguments also work in our setup and so we mainly add some details on top:
Let us firstly compute the variations of $g.\alpha$ and $g.\beta$ in terms of δg:

$$\delta(\mathrm{Ad}_g\, \alpha - \dot{g}g^{-1}) = \left[\delta g g^{-1}, \mathrm{Ad}_g\, \alpha\right] - \dot{\delta g} g^{-1} + \dot{g}g^{-1}\delta g g^{-1} \quad \text{and} \quad \delta(\mathrm{Ad}_g\, \beta) = \left[\delta g g^{-1}, \mathrm{Ad}_g\, \beta\right]$$

We shall make two assumptions:

- If suffices to find the Euler-Lagrange equation at $g = 1$: An arbitrary g is a stationary point of $\mathscr{L}$ with respect to (α,β) if and only if the constant path $g = 1$ is stationary of $\mathscr{L}$ with respect to $g.(\alpha,\beta)$. But both (α,β) and $g.(\alpha,\beta)$ are equivalent in the quotient and so it suffices to consider $g = 1$.
  This assumption simplifies $\delta(\mathrm{Ad}_g\, \alpha - \dot{g}g^{-1}) = [\delta g, \alpha] - \dot{\delta g}$ and $\delta(\mathrm{Ad}_g\, \beta) = [\delta g, \beta]$.
- Due to the polar decomposition we can divide variations in $\mathcal{G}^0_{\mathbb{C}}$ in two directions: We can vary along the self-dual part and along the unitary part. But being a solution to the real equation or not is a property being preserved under unitary gauge transformations. That is why only the self-adjoint direct that is relevant for us: we assume that δg is self-adjoint, i.e. $\delta g^* = \delta g$.
  The two assumptions simplify $\delta(\mathrm{Ad}_g\, \alpha - \dot{g}g^{-1})^* = -[\delta g, \alpha^*] - \dot{\delta g}$.

The rest of the proof is purely computational:

$$\begin{aligned}
\delta\mathscr{L}(1) &= \frac{1}{2}\,\delta\,\mathrm{Re}\int \langle \alpha+\alpha^*, \alpha+\alpha^*\rangle + \langle \beta, \beta^*\rangle \\
&= \mathrm{Re}\,\frac{1}{2}\int 2\langle \alpha+\alpha^*, \delta(\alpha+\alpha^*)\rangle + 2\big\langle \alpha+\alpha^*, -\dot{\delta g}\big\rangle + \langle \delta\beta, \beta^*\rangle + \langle \beta, \delta\beta^*\rangle \\
&= \mathrm{Re}\,\frac{1}{2}\int 2\langle \alpha+\alpha^*, [\delta g, \alpha-\alpha^*]\rangle - 2\frac{\mathrm{d}}{\mathrm{d}s}\langle \alpha+\alpha^*, \delta g\rangle + 2\Big\langle \frac{\mathrm{d}}{\mathrm{d}s}(\alpha+\alpha^*), \delta g\Big\rangle + \langle [\delta g, \beta], \beta^*\rangle - \langle \beta, [\delta g, \beta^*]\rangle \\
&= \mathrm{Re}\int \langle \delta g, [\alpha-\alpha^*, \alpha+\alpha^*]\rangle + \Big\langle \delta g, \frac{\mathrm{d}}{\mathrm{d}s}(\alpha+\alpha^*)\Big\rangle + \langle \delta g, [\beta, \beta^*]\rangle \\
&= \mathrm{Re}\int \Big\langle \delta g, \frac{\mathrm{d}}{\mathrm{d}s}(\alpha+\alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*]\Big\rangle
\end{aligned}$$

This computation shows that $g = 1$ was already a stationary point if (α, β) solved the real equation. In other words, g is a stationary point of $\mathscr{L}$ with respect to (α, β) if and only if $g.(\alpha, \beta)$ solves the real equation. □

The existence of a minimizing path p follows now from the Direct method of Calculus of Variations, compare again against [Don84]. It follows the following train of arguments:

1. The functional is bounded from below and so there is a sequence reaching the infimum.
2. The minimizing sequence contains a convergent subsequence.
3. The functional is lower semi-continuous.

Now the existence of a minimizing p follows. Putting $g = \sqrt{p}$ we have found a gauge element minimizing $\mathscr{L}$ and thus $g.(\alpha, \beta)$ solves the real equation.
We shall leave this leck of details and directly go into the idenfication with the complex coadjoint orbit.

4.3.3 Identification as Coadjoint Orbits

To identify the moduli space $\mathcal{M}_{\mathbb{C}}$ with an adjoint orbit let us go through computation of the $G_{\mathbb{C}}$-moment map again: At a point we couple a tangent vector $T_{(\alpha,\beta)}\mathcal{M}_{\mathbb{C}}$ which we shall represent by a pair $(a, b) \in T_{(\alpha,\beta)}\mathcal{T}_{\mathbb{C}}$ with a fundamental vector field of the $\mathcal{G}_{\mathbb{C}}$-action. Both satisfy their own relation:

$$\dot{b} + [\alpha, b] + [a, \beta] = 0 \quad \text{and} \quad K^{\xi}_{\mathcal{G}_{\mathbb{C}}}(\alpha, \beta) = (\mathrm{ad}_\xi\,\alpha - \dot{\xi}, \mathrm{ad}_\xi\,\beta).$$

Their pairing via $\omega_{\mathbb{C}}$ is

$$\begin{aligned}
\omega_{\mathbb{C}}(K^{\xi}_{\mathcal{G}_{\mathbb{C}}}, (a, b)) &= -\int_{-\infty}^{0} \big(\big\langle \mathrm{ad}_\xi\,\alpha - \dot{\xi}, b\big\rangle - \langle \mathrm{ad}_\xi\,\beta, a\rangle\big)\,\mathrm{d}s \\
&= -\int_{-\infty}^{0} \big(-\big\langle \dot{\xi}, b\big\rangle + \langle \xi, [\alpha, b] - [\beta, a]\rangle\big)\,\mathrm{d}s = -\int_{-\infty}^{0} -\frac{\mathrm{d}}{\mathrm{d}s}\langle \xi, b\rangle + \big\langle \xi, \dot{b}\big\rangle + \langle \xi, [\alpha, b] + [a, \beta]\rangle\,\mathrm{d}s \\
&= -\lim_{s\to-\infty}\langle \xi, a\rangle + \lim_{s\to 0}\langle \xi, a\rangle + \int_{-\infty}^{0} \big\langle \xi, \dot{b} + [\alpha, b] + [a, \beta]\big\rangle\,\mathrm{d}s = \langle \xi(0), b(0)\rangle
\end{aligned}$$

so that the moment map of the $G_{\mathbb{C}}$-action is given by

$$\mu_{\mathbb{C}} : \mathcal{M}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} \quad \text{with} \quad \mu_{\mathbb{C}}(\alpha, \beta) = \beta(0).$$

In the remaining text of this chapter we shall investigate that this moment map has values the adjoint orbit of $e \in \mathfrak{g}_{\mathbb{C}}$ and actually defines an isomorphism of kähler manifolds equipped with a holomorphic symplectic form if the orbit comes with the canonical Kirillov-Kostant-Souriau two-form $\omega \in \Omega^2(\mathfrak{g}_{\mathbb{C}}; \mathbb{C})$

$$\omega_\lambda(\mathrm{ad}_\xi, \mathrm{ad}_{\xi'}) = \langle \lambda, [\xi, \xi']\rangle.$$

Both, source and target of $\mu_{\mathbb{C}}$, inherit their complex structure from the complexification process of $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$ and so μ is holomorphic.

The equivalence of the symplectic forms, i.e.

$$\omega_{\mathbb{C}} = \mu_{\mathbb{C}}^* \omega \quad \text{or alternatively} \quad \omega_{\mathbb{C}}(K^{\xi}_{\mathcal{G}_{\mathbb{C}}}(\alpha,\beta), K^{\xi'}_{\mathcal{G}_{\mathbb{C}}}(\alpha,\beta)) = \langle \beta(0), [\xi(0), \xi'(0)] \rangle$$

for any $\xi, \xi' \in \operatorname{Lie} \mathcal{G}_{\mathbb{C}}$, follows from a direct computation:

$$\begin{aligned}
\omega_{\mathbb{C}}(K^{\xi}_{\mathcal{G}_{\mathbb{C}}}(\alpha,\beta), K^{\xi'}_{\mathcal{G}_{\mathbb{C}}}(\alpha,\beta)) &= -\int_{-\infty}^{0} \left\langle \operatorname{ad}_\xi \alpha - \dot{\xi}, \operatorname{ad}_{\xi'} \beta \right\rangle - \left\langle \operatorname{ad}_{\xi'} \alpha - \dot{\xi}', \operatorname{ad}_\xi \beta \right\rangle \mathrm{d}s \\
&= -\int_{-\infty}^{0} \langle \operatorname{ad}_\xi \alpha, \operatorname{ad}_{\xi'} \beta \rangle - \langle \operatorname{ad}_{\xi'} \alpha, \operatorname{ad}_\xi \beta \rangle - \left\langle \dot{\xi}, \operatorname{ad}_{\xi'} \beta \right\rangle + \left\langle \dot{\xi}', \operatorname{ad}_\xi \beta \right\rangle \mathrm{d}s \\
&= -\int_{-\infty}^{0} \langle (-\operatorname{ad}_{\xi'} \operatorname{ad}_\xi + \operatorname{ad}_\xi \operatorname{ad}_{\xi'})\alpha, \beta \rangle + \left\langle \operatorname{ad}_{\xi'} \dot{\xi} - \operatorname{ad}_\xi \dot{\xi}', \beta \right\rangle \mathrm{d}s \\
&= -\int_{-\infty}^{0} \left\langle \operatorname{ad}_{[\xi,\xi']} \alpha, \beta \right\rangle - \left\langle \frac{\mathrm{d}}{\mathrm{d}s} [\xi, \xi'], \beta \right\rangle \mathrm{d}s \\
&= -\int_{-\infty}^{0} \langle [\xi,\xi'], \operatorname{ad}_\alpha \beta \rangle + \left\langle [\xi,\xi'], \dot{\beta} \right\rangle - \frac{\mathrm{d}}{\mathrm{d}s} \langle [\xi,\xi'], \beta \rangle \mathrm{d}s \\
&= \langle [\xi,\xi'], \beta \rangle \Big|_{-\infty}^{0} - \int_{-\infty}^{0} \left\langle [\xi,\xi'], \dot{\beta} + [\alpha,\beta] \right\rangle \mathrm{d}s = \langle \beta(0), [\xi(0), \xi'(0)] \rangle.
\end{aligned}$$

It remains to show the bijectivity of μ:

Theorem 4.7. *The moment map $\mu_{\mathbb{C}} : \mathcal{M}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ of the $G_{\mathbb{C}}$ action*

$$\mu(\alpha,\beta) = \beta(0)$$

is an isomorphism between the moduli space $\mathcal{M}_{\mathbb{C}}$ and the coadjoint orbit $\operatorname{Ad}_{G_{\mathbb{C}}} e$.

In more elementary words the theorem just states that for any $(\alpha,\beta) \in \mathcal{N}_{\mathbb{C}}$ there is a $g \in \mathcal{G}_{\mathbb{C}}$ such that

$$g.(\alpha,\beta) = (\alpha_0, \beta_0) \quad \text{where} \quad \alpha_0(s) = \frac{1}{2(s-1)} h, \quad \beta_0(s) = \frac{2}{2(s-1)} e.$$

As before, finding such a gauge takes care about the asymptotic conditions. We will follow Biquards strategy from [Biq96] with some slight modifications: when we have found such a gauge transformation on some interval $]-\infty; s^*]$ we can follow the claim on the whole half-line just by existence and uniqueness of a solution on the compact interval $[s^*, 0]$. For us it will be easier to turn the problem around and show that for any $(\alpha,\beta) \in \mathcal{N}_{\mathbb{C}}$ there is some $g \in \mathcal{G}_{\mathbb{C}}$ such that $g.(\alpha_0,\beta_0) = (\alpha,\beta)$ and deduce the asymptotics from the first equation $g.\alpha_0 = \alpha$ as in previous proofs of comparable statements. This differential equation $g.\alpha_0 = \alpha$ should fix g in such a way that the second equation $\operatorname{Ad}_g \beta_0 = \beta$ will follow. This is almost the case as the next lemma illustrates:

Lemma 4.8. *Let $(\alpha,\beta), (\alpha',\beta') \in \mathcal{N}_{\mathbb{C}}$. If there is a gauge transformation $g \in \mathcal{G}_{\mathbb{C}}$ such that $g.\alpha = \alpha'$ and $\operatorname{Ad}_g \beta(s^*) = \beta'(s^*)$ at some point s^*, then $g.(\alpha,\beta) = (\alpha',\beta')$.*

Proof. We only have to check that $\operatorname{Ad}_g \beta = \beta'$. Differentiating $B = \operatorname{Ad}_g \beta - \beta'$ to

$$\begin{aligned}
\dot{B} &= \left[\dot{g}g^{-1}, \operatorname{Ad}_g \beta\right] + \operatorname{Ad}_g \dot{\beta} - \dot{\beta}' \\
&= [\operatorname{Ad}_g \alpha - \alpha', \operatorname{Ad}_g \beta] - \operatorname{Ad}_g[\alpha,\beta] + [\alpha',\beta'] = -[\alpha', \operatorname{Ad}_g \beta - \beta'] \\
&= -[\alpha', B].
\end{aligned}$$

leads us to a differential equation $\dot{B} = -[\alpha', B]$. Due to $B(s^*) = 0$ we know that the only solution is $B = 0$ which implies the claim. ☐

We can now focus on the first equation outside of some compact set. The idea will be to use the implicit function theorem again and that way ensure the existence of a gauge $g.\alpha_0 = \alpha$. In the second step we deal with the second equation $\operatorname{Ad}_g \beta_0 = \beta$.

Proof of the theorem. We shall show as in [Biq96] that there is a gauge $g = \exp\xi$ such that $\mathrm{Ad}_g\,\alpha_0 - \dot g g^{-1} = \alpha$. For some $(a,b) \in T_{(\alpha,\beta)}\mathcal{N}_{\mathbb{C}}$ the linearised equation

$$\dot\xi - \frac{1}{2(s-1)}\,\mathrm{ad}_h\,\xi = a$$

can be solved directly to be

$$\xi(s) = \exp\!\left(\frac{\log|s-1|}{2}\,\mathrm{ad}_h\right)\int_s^0 \exp\!\left(\frac{-\log|s-1|}{2}\,\mathrm{ad}_h\right)\alpha(s')\,\mathrm{d}s'.$$

The asymptotics of a garantuees the asymptotics of $\dot\xi$ and $\ddot\xi$, this can be checked e.g. via decomposing $\mathfrak{g}_{\mathbb{C}}$ into its weight spaces with respect to the given $\mathfrak{sl}_2\,\mathbb{C}$-representation. This implies the existence of some $g \in \mathcal{G}_{\mathbb{C}}$ with $g.\alpha_0 = \alpha$ on some interval $]-\infty; s_0]$ and so we have $g^{-1}.(\alpha,\beta) = (\alpha_0,\beta')$. We are left to show that $\beta' = \beta_0$. The real equation gives us already that

$$\dot\beta' + \frac{1}{2(s-1)}\,\mathrm{ad}_h\,\beta' = 0 \quad \text{so that} \quad \beta'(s) = \exp\!\left(-\frac{\log|s-1|}{2}\,\mathrm{ad}_h\right)\beta'_0 \quad \text{for some} \quad \beta'_0 \in \mathfrak{g}_{\mathbb{C}}\,.$$

It is actually so that β'_0 is restricted by the asymptotics of β to be in $\beta'_0 \in e + \bigoplus_{\mu>2}\mathrm{Eig}(\mathrm{ad}_h,\mu)$. This unwelcome summand in $\bigoplus_{\mu>2}\mathrm{Eig}(\mathrm{ad}_h,\mu)$ is canceled by the correct choice of g which was not unique, yet. It can be precomposed with any gauge g' such that $g'.\alpha_0 = \alpha_0$. Such gauges are of the shape

$$g'(s) = \exp\!\left(-\frac{\log|s-1|}{2}\,\mathrm{ad}_h\right)\exp(\zeta) \quad \text{where} \quad \zeta \in \bigoplus_{\mu>0}\mathrm{Eig}(\mathrm{ad}_h,\mu).$$

When we have shown that there is such ζ with the property that $\mathrm{Ad}_{\exp(\zeta)}\,\beta'_0 = e$ we are done.
Following the argument from [Kro90b] let us write $\beta'_0 = e+\zeta'$ and consider $\exp(\mathrm{ad}_\zeta)(e+\zeta') = e$ as an equation for ζ: If ζ' is small we can parametrise ζ through ζ' in a neighbourhood of $\zeta' = 0$ if the differential of the function $Z(\zeta) = \exp(\mathrm{ad}_\zeta)(e+\zeta') - e$ surjects at $\zeta' = 0$. It is known from $\mathfrak{sl}_2\,\mathbb{C}$-representation theory that

$$D_0 Z(\zeta) = -\,\mathrm{ad}_e\,\zeta \quad \text{in other words} \quad D_0 Z = -\,\mathrm{ad}_e : \bigoplus_{\mu>0}\mathrm{Eig}(\mathrm{ad}_h,\mu) \to \bigoplus_{\mu>2}\mathrm{Eig}(\mathrm{ad}_h,\mu)$$

is a surjective map and so the claim follows in a neighbourhood. For general ζ', we use the homogeneity coming from the conjugation with h: We have that $e = \exp(2r)\exp(-r\,\mathrm{ad}_h)e$ and to we can apply the operator $\exp(2r)\exp(-r\,\mathrm{ad}_h)$ to the equation. Since $\zeta' \in \bigoplus_{\mu>2}\mathrm{Eig}(\mathrm{ad}_h,\mu)$ there is some huge r such that ζ' is mapped into the neighbourhood of 0 where we can find a corresponding ζ, gauging back yields a solution and hence proves the existence. This completes the proof of the theorem. □

The following diagramm does not only summarize the introduced spaces and relevant maps it also clarifies the problem: The two vertical arrows, the moment maps, differ in the number of components. While the

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{T\mapsto(\alpha,\beta)} & \mathcal{M}_{\mathbb{C}} \\ \Big\downarrow{\scriptstyle T\mapsto T_i(0)} & & \Big\downarrow{\scriptstyle (\alpha,\beta)\mapsto\beta(0)} \\ \mathfrak{g}\otimes\mathbb{R}^3 & & \mathrm{Ad}_{G_{\mathbb{C}}}\,e \end{array}$$

moment map of the G-action on $\mathcal{M}$ has values in $\mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$ the complex moment map only takes values in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\oplus\mathfrak{g}$. Even more, it is the isomorphism between $\mathcal{M}_{\mathbb{C}}$ and the orbit. We have explained that the orbit is a suitable hyperkähler manifold for the theory of the generalised Seiberg-Witten equations and so we only have to give $\mathrm{Ad}_{G_{\mathbb{C}}}\,e \to \mathfrak{g}\oplus\mathfrak{g}\oplus\mathfrak{g}$. Again, we know that this map is just the identity in the last two components when writing $\mathfrak{g}^3 = \mathfrak{g}\oplus\mathfrak{g}_{\mathbb{C}}$. But what is the first component? The direct approach following the arrows leads nowhere: Let ξ be in the orbit of e, i.e. $\xi = \operatorname{Ad}_{g_0} e$ for some $g_0 \in G_{\mathbb{C}}$. For $g \in \mathcal{G}_{\mathbb{C}}$ such that $g(0) = g_0$ the solution $g.(\alpha_0, \beta_0)$ of the complex equation corresponds via $\mu_{\mathbb{C}}$ to ξ. There is a second gauge $u \in \mathcal{G}^0$ such that $u.g.(\alpha_0, \beta_0)$ solves the real equation. Hence we have found with $T_0 = \operatorname{Re} u.g.\alpha_0$ and $T_1 = \operatorname{Im} u.g.\alpha_0$, $T_2 = \operatorname{Re} u.g.\beta_0$ and $T_3 = \operatorname{Im} u.g.\beta_0$ solutions to the real setup. It is not surprising that we find with $u(0) = 1$ and $g(0) = g_0$

$$\mu^J_{\mathcal{G}^0}(T) = \operatorname{ev}_{s=0} \operatorname{Re}(\operatorname{Ad}_u \operatorname{Ad}_g e) = \operatorname{Re}\xi \quad \text{and} \quad \mu^K_{\mathcal{G}^0}(T) = \operatorname{ev}_{s=0} \operatorname{Im}(\operatorname{Ad}_u \operatorname{Ad}_g e) = \operatorname{Im}\xi$$

and so we can concentrate on the third part:

$$\mu^I_{\mathcal{G}^0}(T) = \operatorname{ev}_{s=0} \operatorname{Re} u.g.\alpha_0 = \operatorname{Re}(\operatorname{Ad}_{g_0} h - \dot{u}(0) - \dot{g}(0)g_0^{-1}).$$

It could have been so easy if there were not the depence on g and u. Of course, there is much freedom in the choice of g, e.g. we can take it to be constant around $s = 0$ so that $\dot{g}(0) = 0$. However there is less freedom in the choice of u: In general u will not be in the real gauge group $\mathcal{G}^0$ since $g.(\alpha, \beta)$ does not solve the real equation, this means that $\dot{u}(0)$ has a non-trivial imagary part and so can not be omitted - its real part could have been chosen to be zero.
The idea of this clarification is two-fold: Firstly, the easiest approach how to compute $\mu^I_{\mathcal{G}^0}$ through $\mu_{\mathbb{C}}$ does not lead anywhere. And secondly, finding the moment map is nothing else but finding $\dot{u}(0)$ and so a big step into the direction of a computation for u which enables us to give solutions T. That might indicate why we are heading towards solving Nahms equations in the next chapter.

5 Nahms Equations as Flow Equation of the Bilinear Form

In this section we shall give an approach to actually solve the real equation for nilpotent orbits going back to section 6 in the article *Degeneration of Hodge Structures* of E. Cattani, A. Kaplan and W. Schmid, referenced as [ES86]. Here they have introduced Nahms equation as flow equations to a quadratic form $Q : S^2 \operatorname{Hom}(\mathfrak{su}(2);\mathfrak{g}) \to \operatorname{Hom}(\mathfrak{su}(2);\mathfrak{g})$. Combining this with the power series in which we can expand the nilpotent orbits around $s = -\infty$ we can find a recurrence relation on the coefficients in the series. This reduces the problem of solving to methods of representation theory.

Recall from the introduction that we can see Nahms equations as gradient flow

$$\begin{cases} \dot{T}_1 + [T_2, T_3] = 0 \\ \dot{T}_2 + [T_3, T_1] = 0 \\ \dot{T}_3 + [T_1, T_2] = 0 \end{cases}$$

to the functional $\varphi(\xi_1, \xi_2, \xi_3) = \langle \xi_1, [\xi_2, \xi_3] \rangle$ and that the nilpotent orbit of the $\mathfrak{su}(2)$-triple decays like $(\sigma_1, \sigma_2, \sigma_3)/2s$ and so is nothing but $\rho/2s$ for the $\mathfrak{su}(2)$-representation ρ on $\mathfrak{g}$. This way of thinking means to not see T as three paths in $\mathfrak{g}$ linked by Nahms equation but rather as a flow equation on $\operatorname{Hom}(\mathfrak{su}(2);\mathfrak{g})$ which asymptotically is $\rho/2s$.
With the identification $\operatorname{Hom}(\mathfrak{su}(2);\mathfrak{g}) = (\mathfrak{su}(2))^\vee \otimes \mathfrak{g} = \mathfrak{su}(2) \otimes \mathfrak{g}$ we can define a bilinear symmetric form Q on $\operatorname{Hom}(\mathfrak{su}(2);\mathfrak{g})$ by

$$S^2(\mathfrak{su}(2) \otimes \mathfrak{g}) = S^2\,\mathfrak{su}(2) \otimes S^2\,\mathfrak{g} \;\oplus\; \Lambda^2\,\mathfrak{su}(2) \otimes \Lambda^2\,\mathfrak{g} \xrightarrow{\mathrm{pr}_2} \Lambda^2\,\mathfrak{su}(2) \otimes \Lambda^2\,\mathfrak{g} \xrightarrow{[\cdot,\cdot]\otimes[\cdot,\cdot]} \mathfrak{su}(2) \otimes \mathfrak{g},$$

or explicitely

- on $\mathfrak{su}(2) \otimes \mathfrak{g}$ as $Q(\sigma \otimes \varphi, \sigma' \otimes \varphi') = \frac{1}{2}[\sigma, \sigma'] \otimes [\varphi, \varphi']$
- on $\operatorname{Hom}(\mathfrak{su}(2);\mathfrak{g})$ as $Q(\Phi, \Phi')([\sigma, \sigma']) = \frac{1}{2}\left([\Phi(\sigma), \Phi'(\sigma')] - [\Phi(\sigma'), \Phi'(\sigma)]\right)$

The first noticable property of Q is that we can characterise Lie algebra homomorphisms $\mathfrak{su}(2) \to \mathfrak{g}$, i.e. representations, as follows:

Lemma 5.1. *An element $\Phi \in \operatorname{Hom}(\mathfrak{su}(2);\mathfrak{g})$ defines a representation of $\mathfrak{su}(2)$ on $\mathfrak{g}$ if and only if $\Phi = Q(\Phi, \Phi)$.*

Proof. From the computation

$$Q(\Phi, \Phi)([\sigma, \sigma']) = \frac{1}{2}\left([\Phi(\sigma), \Phi(\sigma')] - [\Phi(\sigma'), \Phi(\sigma)]\right) = [\Phi(\sigma), \Phi(\sigma')]$$

the claim follows directly. □

Nahms equations are now given in terms of the flow equation

$$\dot{\Phi}(s) = -Q(\Phi(s), \Phi(s)).$$

Indeed, for $T_l(s) = \Phi(s)(\frac{\sigma_l}{2})$ we find

$$\begin{cases} \dot{T}_1 + [T_2, T_3] = 0 \\ \dot{T}_2 + [T_3, T_1] = 0 \\ \dot{T}_3 + [T_1, T_2] = 0 \end{cases}.$$

5.1 Recurrence Relation for Nilpotent Orbits

Referring to [ES86] around $s = -\infty$ we can expand the path Φ as a half-integer power series

$$\Phi(-s) = \rho\, s^{-1} + \sum_{n=2}^{\infty} \Phi_n s^{-(1+\frac{n}{2})}$$

with ρ corresponding to the nilpotent orbit. We shall talk later about the convergence of this series, let us firstly focus of the interplay of the expansion with the flow equation

$$\begin{aligned}\dot\Phi(-s) &= -\rho s^{-2} - \sum_{k=2}^{\infty}(1+\frac{k}{2})\,\Phi_k\, s^{-(2+\frac{k}{2})}\\ Q(\Phi(-s),\Phi(-s)) &= Q(\rho,\rho)s^{-2} + 2\sum_{k=2}^{\infty} Q(\rho,\Phi_k)\, s^{-(2+\frac{k}{2})} + \sum_{k,l=2}^{\infty} Q(\Phi_k,\Phi_l)s^{-(2+\frac{k+l}{2})}\\ &= Q(\rho,\rho)s^{-2} + 2\sum_{k=2}^{\infty} Q(\rho,\Phi_k)\, s^{-(2+\frac{k}{2})} + \sum_{k=2}^{\infty}\sum_{l=2}^{k-2} Q(\Phi_l,\Phi_{k-l})s^{-(2+\frac{k}{2})}\end{aligned}$$

and compare the coefficents

$$(1+\frac{k}{2})\Phi_k - 2Q(\rho,\Phi_k) = \sum_{l=2}^{k-2} Q(\Phi_l,\Phi_{k-l}).$$

As in [ES86], let us determine the operator $Q(\rho,\cdot)$ in the basis of Pauli matrices $[\sigma_i,\sigma_j] = 2\sum_{k=1}^{3}\varepsilon_{ijk}\,\sigma_k$:

$$\begin{aligned}Q(\rho,\Phi)(\sigma_3) &= \frac{1}{2}\,Q(\rho,\Phi)([\sigma_1,\sigma_2]) = \frac{1}{4}\,([\rho(\sigma_1),\Phi(\sigma_2)] - [\rho(\sigma_2),\Phi(\sigma_1)])\\ &= -\frac{1}{8}\,([\rho(\sigma_1),\Phi([\sigma_1,\sigma_3])] + [\rho(\sigma_2),\Phi([\sigma_2,\sigma_3])]) = -\frac{1}{8}\sum_{l=1}^{2}[\rho(\sigma_l),\Phi([\sigma_l,\sigma_3])]\end{aligned}$$

which also works for σ_1 and σ_2. Since the adjoint actions $\mathrm{ad}_{\rho(\sigma)}$ and $\mathrm{ad}^\vee_\sigma\,\xi = -\xi\circ\mathrm{ad}_\sigma$ act on different components in $\mathfrak{su}(2)\otimes\mathfrak{g}$ they commute and so we can rewrite the previous result as

$$Q(\rho,\cdot) = \frac{1}{8}\sum_{j=1}^{3}\mathrm{ad}_{\rho(\sigma_j)}\,\mathrm{ad}^\vee_{\sigma_j} = -\frac{1}{16}\left(\sum_{j=1}^{3}(\mathrm{ad}_{\rho(\sigma_j)} + \mathrm{ad}^\vee_{\sigma_j})^2 - \mathrm{ad}^2_{\rho(\sigma_j)} - (\mathrm{ad}^\vee_{\sigma_j})^2\right).$$

In general, $\mathrm{Hom}(\mathfrak{su}(2);\mathfrak{g})$ is an $\mathfrak{su}(2)$-representation in three different ways:

i) Composing the adjoint action of $\mathfrak{g}$ with ρ gives a representation of $\mathfrak{su}(2)$ on $\mathrm{Hom}(\mathfrak{su}(2);\mathfrak{g}) = \mathfrak{su}(2)\otimes\mathfrak{g}$ which acts on the $\mathfrak{g}$-component.

ii) The coadjoint action of $\mathfrak{su}(2)$ on $\mathrm{Hom}(\mathfrak{su}(2);\mathfrak{g})$ acts on the $\mathfrak{su}(2)$-component in the tensor product: $\mathrm{ad}^\vee_a\,\Phi = -\Phi\circ\mathrm{ad}_a$.

iii) The previous two actions commute and so can be combined into one $a\mapsto \mathrm{ad}_{\rho(a)} + \mathrm{ad}^\vee_a$ which corresponds to the induced action on the homomorphism space rather than a component.

Each of the above sums of squares is related to the action of the Casimir element in $\mathfrak{su}(2)$ on the relevant representation which acts as a multiplication operator on the irreducible subrepresentation. With respect to our basis $\{\sigma_j\}$ we have that $\mathrm{cas} = -(\sigma_1^2+\sigma_2^2+\sigma_3^2)$.

Instead of dealing with the $\mathfrak{su}(2)$-representation theory here we shall pass to the complexification, i.e. substitute $\mathfrak{g}$ by $\mathfrak{g}_\mathbb{C} = \mathfrak{g}\otimes\mathbb{C}$, $\mathfrak{su}(2)$ by $\mathfrak{sl}_2\,\mathbb{C}$ and extend to above maps complex linearly. This has two reasons: on the one hand, we do not need to worry in diagonalisation arguments and, on the other hand, we would have passed to the complexification later on anyway. There is much material about $\mathfrak{sl}_2\,\mathbb{C}$-representations covered in later chapters, [Hum97] is also a good reference. We shall go on with the decomposition into $\mathfrak{sl}_2\,\mathbb{C}$-subrepresentations: $\mathfrak{sl}_2\,\mathbb{C} = S^2$ and $\mathfrak{g}_\mathbb{C} = \bigoplus A_r\otimes S^r$ where S^d is the usual irreducible $\mathfrak{sl}_2\,\mathbb{C}$-representation of dimension $d+1$. For the homomorphisms we can write

$$\mathrm{Hom}(\mathfrak{sl}_2\,\mathbb{C};\mathfrak{g}_\mathbb{C}) = S^2\otimes\left(\bigoplus_{r=0}^{N} A_r\otimes S^r\right) = \bigoplus_{r=0}^{N}\left(A_r\otimes\bigoplus_{\varepsilon=-1}^{1} S^{r+2\varepsilon}\right) \quad\text{and}\quad \Phi = \sum_{r=0}^{N}\sum_{\varepsilon=-1}^{1}\Phi^{(r,\varepsilon)}\ \text{accordingly}$$

where we put $S^{-2} = S^{-1} = \{0\}$, i.e. $\Phi^{(0,-1)} = \Phi^{(0,-1)} = 0$. This leads the above Casimir-terms to be

$$\sum_{l=1}^{3}(\mathrm{ad}_{\rho(\sigma_l)} + \mathrm{ad}^{\vee}_{\sigma_l})^2\,\Phi^{(r,\varepsilon)} = -(r+2\varepsilon)(r+2\varepsilon+2)\,\Phi^{(r,\varepsilon)}$$
$$\sum_{l=1}^{3}\mathrm{ad}^2_{\rho(\sigma_l)}\;\Phi^{(r,\varepsilon)} = -r(r+2)\,\Phi^{(r,\varepsilon)}$$
$$\sum_{l=1}^{3}(\mathrm{ad}^{\vee}_{\sigma_l})^2\,\Phi^{(r,\varepsilon)} = -8\,\Phi^{(r,\varepsilon)}$$

and consequently

$$\left(1+\frac{k}{2}\right)\Phi_k^{(r,\varepsilon)} - 2Q(\rho,\Phi_k^{(r,\varepsilon)}) = \left(1+\frac{k}{2}+\frac{1}{8}\left((r+2\varepsilon)(r+2\varepsilon+2)-r(r+2)-8\right)\right)\Phi_k^{(r,\varepsilon)} = \frac{1}{2}\left(k+\varepsilon^2+\varepsilon(r+1)\right)\Phi_k^{(r,\varepsilon)}.$$

And so $\Phi^{(r,\varepsilon)}$ are determined recursively by

$$(k+\varepsilon^2+\varepsilon(r+1))\Phi_k^{(r,\varepsilon)} = 2\sum_{l=2}^{k-2} Q(\Phi_l,\Phi_{k-l})^{(r,\varepsilon)}$$

unless the prefactor $k+\varepsilon^2+\varepsilon(r+1)$ vanishes which is exactly the case for $k=r$ and $\varepsilon=-1$. In other words, the recurrence relation determines all components of the Φ_n except $\xi_k = \Phi_k^{(k,-1)}$, $0 \le k \le N$ where $N+1$ was the dimension of the highest-weight module in the decomposition of $\mathrm{Hom}(\mathfrak{sl}_2\,\mathbb{C};\mathfrak{g}_{\mathbb{C}})$. We shall consider these $\Phi_k^{(k,-1)}$ as initial data for the recurrence and so compute the other components of the Φ_k. If all the input terms are real also the output is real and so if we restrict ourselves later to real variables to remain in $\mathfrak{g}$. We summarize the material from [ES86] in the next theorem:

Theorem 5.2. *For any given $\xi_1,\ldots,\xi_n \in \mathfrak{g}_{\mathbb{C}}$ there is a solution Φ to Nahms equations such that $\Phi_k^{(k,-1)} = \xi_k$ for $0 \le k \le N$. If the $\xi_j \in \mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$ were real the resulting $\Phi \in \mathrm{Hom}(\mathfrak{sl}_2\,\mathbb{C};\mathfrak{g}_{\mathbb{C}})$ restricts to a real homomorphism $\Phi \in \mathrm{Hom}(\mathfrak{su}(2);\mathfrak{g})$.*

Let us loose some words about the convergence of the series $\Phi(-s) = \rho\, s^{-1} + \sum_{n=2}^{\infty}\Phi_n s^{-(1+\frac{n}{2})}$ again following [ES86]: The norms on $\mathfrak{sl}_2\,\mathbb{C}$ and $\mathfrak{g}_{\mathbb{C}}$ induce a norm on $\mathrm{Hom}(\mathfrak{sl}_2;\mathfrak{g}_{\mathbb{C}})$ and so also on the space of bilinear forms on the homomorphisms with respect to which Q is bounded. Consequently we can find an index n_0 and a constant $q \ge 1$ depending on the precise shape the decomposition of $\mathfrak{g}_{\mathbb{C}}$ and Q such that for any $n \ge n_0$

$$|\Phi_n| \le \frac{q}{n}\sum_{l=2}^{n-2}|\Phi_l|\,|\Phi_{n-l}|.$$

Let us put $p = \max\{|\Phi_1|,\ldots,|\Phi_{n_0}|,1\}$ so that $|\Phi_k| \le q^k p^{k+1}$ for any $k \le n_0$. Assuming this estimate also for all $k \le K$ for some K, we find

$$|\Phi_{K+1}| \le \frac{q}{n}\sum_{l=2}^{n}|\Phi_l|\,|\Phi_{n-l}| \le \frac{q}{n}\sum_{l=2}^{n} q^l p^{l+1}\, q^{K-l}p^{K-l+1} \le q^{K+1}p^{K+2}.$$

And so it follow from induction that the series

$$\Phi(-s) = \rho\, s^{-1} + \sum_{n=2}^{\infty}\Phi_n s^{-(1+\frac{n}{2})}$$

convergences for all $-s > \sqrt{pq}$. That is the reason why such a sequence is in general only defined on some interval $]-\infty; s_0]$ and not on all of $\mathbb{R}_{\le 0}$. However, we can make use of the time-shift symmetry of the equations to construct solutions on $\mathbb{R}_{\le 0}$. In other words, out of any solution we can explicitely construct this way a solution that represents an element of the previously introduced moduli spaces.

Let us come back to the recurrence relation: With the input data $\{\xi_k\}$ we intend to find all Φ_n. This does not only require an explicit decomposition of $\mathfrak{g}_{\mathbb{C}}$ but also an explicit decomposition of tensor products into irreducible $\mathfrak{sl}_2\,\mathbb{C}$-representations, e.g. to determine $Q(\Phi_l, \Phi_{k-l})^{(r,\varepsilon)}$ concretely. And when this is done we have to find the way back to homomorphisms. And so we shall focus on the concrete example

$$\mathfrak{g}_{\mathbb{C}} = \mathrm{End}_0(S^d) = \mathfrak{sl}_{d+1}\,\mathbb{C} \quad \text{with the standard } \mathfrak{sl}_2\,\mathbb{C}\text{-module structure}$$

which means nothing else but to consider the adjoint orbit of maximal rank.
This has two reasons: Firstly, the nilpotent orbits are computed up to $d = 3$ so that there is hope to produce new examples. Secondly, from the representational point of view, $\mathrm{End}(S^d)$ is relatively simple: Interpreting S^d here as polynomials we can work relatively concrete with well-known objects:

- We already know that $\mathrm{End}\, S^d = \bigoplus_{m=0}^{d} S^{2m}$, i.e. each irreducible representation occurs only once.
- With the previous point in mind we can compute an explicit isomorphism $S^m \otimes S^n = \bigoplus_{l=0}^{\min\{m,n\}} S^{m+n-2l}$.
- Polynomials will be easy enough to handle so that we can write Q in a simple and condensed form.

This choice becomes very reasonable when taking our main tool, the Moyal product, into account. The inspiration for this approach had V. Pidstrygach. So that the main reference for the general material concering the representation theory we present here was covered in his seminar talks while concreter computations are joint work. Nevertheless, the Moyal product is known from quantisation theory so that there is much material covered in physics papers. A brief introduction with some proofs can also be found in *Classical Invariant Theory* by P. Olver [Olv99].

5.2 The Moyal Product as Computational Tool

For two $f, g \in \mathbb{C}[x, y]$ we write

$$\{f, g\}_m = \frac{1}{m!}\Pi\ (\partial_x \wedge \partial_y)^m f \otimes g = \sum_{j=0}^{m} \frac{(-1)^{m-j}}{j!\,(m-j)!}\ \partial_x^j \partial_y^{m-j} f \cdot \partial_x^{m-j} \partial_y^j g.$$

where $\Pi(f \otimes g) = fg$ is just the standard product of polynomials. The Moyal product $f * g$ of f and g now is set to be

$$f * g = \sum_{m=0}^{\infty} \{f, g\}_m = \Pi \circ \exp(\partial_x \wedge \partial_y) f \otimes g.$$

We shall justify the terminus product and collect some fundamental properties of the Moyal product in the next theorem:

Theorem 5.3. *The space of polynomials $(\mathbb{C}[x, y], *)$ equipped with the Moyal product is an associative, unital but non-commutative Algebra. Moreover, all Moyal brackets and so also the Moyal product are* $\mathrm{Sl}_2\,\mathbb{C}$-*equivariant or, formulated for the Lie algebra, $\mathfrak{sl}_2\,\mathbb{C}$ acts via derivations.*
We find with $\bigoplus_{m=0}^{n} S^m$, $\bigoplus_{m=0}^{n} S^{2m}$ and $\mathbb{C}[x, y]_{\mathrm{ev}} = \bigoplus_{m=0}^{\infty} S^{2m}$ some $\mathrm{Sl}_2\,\mathbb{C}$-*invariant subalgebras.*

Proof. Let us firstly check the associativity, i.e. we have to check that $(f * g) * h = f * (g * h)$ for any three polynomials f, g, h. The claim can be rewritten as

$$\Pi\ \exp(\partial_x \wedge \partial_y)\ \Pi \otimes \mathrm{id}\ \exp((\partial_x \wedge \partial_y) \otimes \mathrm{id}) = \Pi\ \exp(\partial_x \wedge \partial_y)\ \mathrm{id} \otimes \Pi\ \exp(\mathrm{id} \otimes (\partial_x \wedge \partial_y)).$$

The main idea is to permute the differential operators through the mulitplication operators. We have

$$\begin{aligned}(\partial_x \otimes \partial_y - \partial_y \otimes \partial_x)\ \Pi \otimes \mathrm{id} &= \partial_x \Pi \otimes \partial_y - \partial_y \Pi \otimes \partial_x = \Pi(\partial_x \otimes \mathrm{id} + \mathrm{id} \otimes \partial_x) \otimes \partial_y - \Pi(\partial_y \otimes \mathrm{id} + \mathrm{id} \otimes \partial_y) \otimes \partial_x \\ &= \Pi \otimes \mathrm{id}\ (\partial_x \otimes \mathrm{id} \otimes \partial_y + \mathrm{id} \otimes \partial_x \otimes \partial_y - \partial_y \otimes \mathrm{id} \otimes \partial_x - \mathrm{id} \otimes \partial_y \otimes \partial_x)\end{aligned}$$

and so by iterated application

$$\begin{aligned}\exp(\partial_x \wedge \partial_y)\ \Pi \otimes \mathrm{id} &= \Pi \otimes \mathrm{id}\ \exp(\partial_x \otimes \mathrm{id} \otimes \partial_y + \mathrm{id} \otimes \partial_x \otimes \partial_y - \partial_y \otimes \mathrm{id} \otimes \partial_x - \mathrm{id} \otimes \partial_y \otimes \partial_x) \\ \exp(\partial_x \wedge \partial_y)\ \mathrm{id} \otimes \Pi &= \mathrm{id} \otimes \Pi\ \exp(\partial_x \otimes \partial_y \otimes \mathrm{id} + \partial_x \otimes \mathrm{id} \otimes \partial_y - \partial_y \otimes \partial_x \otimes \mathrm{id} - \partial_y \otimes \mathrm{id} \otimes \partial_x).\end{aligned}$$

Tha associativity is now a direct consequence of

$$\begin{aligned}&\exp(\partial_x\otimes\mathrm{id}\otimes\partial_y+\mathrm{id}\otimes\partial_x\otimes\partial_y-\partial_y\otimes\mathrm{id}\otimes\partial_x-\mathrm{id}\otimes\partial_y\otimes\partial_x)\exp(\partial_x\otimes\partial_y\otimes\mathrm{id}-\partial_y\otimes\partial_x\otimes\mathrm{id})\\ &=\exp(\partial_x\otimes\partial_y\otimes\mathrm{id}+\partial_x\otimes\mathrm{id}\otimes\partial_y-\partial_y\otimes\partial_x\otimes\mathrm{id}-\partial_y\otimes\mathrm{id}\otimes\partial_x)\exp(\mathrm{id}\otimes\partial_x\otimes\partial_y-\mathrm{id}\otimes\partial_y\otimes\partial_x).\end{aligned}$$

Let's proceed with the equivariance. The $\mathrm{Sl}_2\,\mathbb{C}$-action on $\mathbb{C}[x,y]$ is defined by the precomposition $A.f = f\circ A^{-1}$ where $A\in\mathrm{Sl}_2\,\mathbb{C}$ and $f\in\mathbb{C}[x,y]$. And so it follows that

$$\partial_x\wedge\partial_y\ A.(f\otimes g)=\partial_x\wedge\partial_y\ (f\circ A^{-1}\otimes g\circ A^{-1})=\det A^{-1}\ (\partial_x\wedge\partial_y\, f\otimes g)\circ A^{-1}=A.(\partial_x\wedge\partial_y\, f\otimes g)$$

where $\det A = 1$ became relevant. Combining this with $\Pi A.(f\otimes g) = A.\Pi(f\otimes g)$ we obtain that

$$\{A.f, A.g\}_m = A.\{f,g\}_m \quad\text{and}\quad A.(f*g)=(A.f)*(A.g).$$

This completes the proof. □

We will use the Moyal bracket $\{\cdot,\cdot\}_N$ to define equivariant homomorphisms via

$$\lambda^{(N)}: S^n\to\mathrm{Hom}(S^n;\mathrm{Hom}(S^m;S^{m+n-2N}))\quad\text{via}\quad f\mapsto(\lambda_f^{(N)}=\{f,\cdot\}_N : g\mapsto\{f,g\}_N).$$

This defines in particular an equivariant embedding of $S^{2m}\to\mathrm{End}(S^d)$ via $f\mapsto\lambda_f^{(m)}$ and thus the decomposition

$$\oplus\lambda^{(m)}:\bigoplus_{m=0}^{d}S^{2m}\to\mathrm{End}(S^d).$$

That is the way we identify polynomials with endomorphisms.

5.3 The Recurrence Relation for $\mathrm{End}(S^d)$

Let us now explain firstly how the Moyal product decomposes tensor products and then afterwards secondly how we can bring Q into a handy shape. Concering the first point, for any two functions $f\in S^n$ and $g\in S^m$ with $n\le m$ we have that

$$f*g=\sum_{l=0}^{n}\{f,g\}_k=\underbrace{\{f,g\}_0}_{\in S^{m+n}}+\underbrace{\{f,g\}_1}_{\in S^{m+n-2}}+\ldots+\underbrace{\{f,g\}_n}_{\in S^{m-n}}$$

and so the $f*g$ gives the components of $f\otimes g$ in the irreducible representations $S^{m+n-l}\subseteq S^m\otimes S^n$ as each bracket is $\mathrm{Sl}_2\,\mathbb{C}$ equivariant and non-trivial: The Moyal product

$$*: S^m\otimes S^n\to\bigoplus_{l=0}^{\min\{m,n\}}S^{m+n-2l}$$

defines an isomorphism of $\mathrm{Sl}_2\,\mathbb{C}$-representations.
Let us now concretise the decomposition $S^2\otimes\mathrm{End}(S^d)=\oplus_{l=0}^{d}\oplus_{\varepsilon=-1}^{1}S^{2l+2\varepsilon}$. We already have found

$$S^2\otimes S^{2l}\to S^{2l+2\varepsilon}\quad\text{via}\quad\sigma\otimes f\mapsto\{\sigma,f\}_{1-\varepsilon}=\lambda_\sigma^{(1-\varepsilon)}f$$

and now need to find its equivariant inverse. We know from Schurs Lemma that any two equivariant maps $S^{2l+2\varepsilon}\to S^2\otimes S^{2l}$ differ by a multiplicative constant. So the idea is to firstly give any equivariant, non-trivial map $S^2\otimes S^{2l+2\varepsilon}\to S^{2l}$ and then compose with $\{\cdot,\cdot\}_{1-\varepsilon}$, the composition is just a multiplication operator which we can use the determine the correct normalisation.
We begin with

$$S^{2l+2\varepsilon}\to S^2\otimes S^{2l}\ \text{ via }\ g\mapsto\sum_{l=1}^{3}\sigma_l\otimes\{\sigma_l,g\}_{1+\varepsilon}=xy\otimes\{xy,g\}_{1+\varepsilon}-(x^2\otimes\{\frac{y^2}{2},g\}_{1+\varepsilon}+y^2\otimes\{\frac{x^2}{2},g\}_{1+\varepsilon})$$

where we have used $\sigma_1 = -ixy$, $\sigma_2 = (y^2 + x^2)/2$ and $\sigma_3 = -i(y^2 - x^2)/2$. The equivalence is seen by writing this map as

$$\operatorname{id} \otimes \lambda^{(1+\varepsilon)}(\sum_{l=1}^{3} \sigma_l \otimes \sigma_l) \quad \text{where} \quad \operatorname{id} \otimes \lambda^{(1+\varepsilon)} : S^2 \otimes S^2 \to S^2 \otimes \operatorname{Hom}(S^{2l+2\varepsilon}; S^{2l})$$

and use the invariance of the sum $-2\sum \sigma_l \otimes \sigma_l = \sum(\sigma_l \otimes 1 + 1 \otimes \sigma_l)^2 - 1 \otimes \sum \sigma_l^2 - \sum \sigma_l^2 \otimes 1$ being a multiple of the Casimir elements and the equivariance of all the other maps. And we only have to renormalize by computing the precomposition $S^2 \otimes S^{2l} \to S^{2l+2\varepsilon} \to S^2 \otimes S^{2l}$:

- $\varepsilon = 1$: $x^2 \otimes x^{2l} \mapsto x^{2l+2} \mapsto -\frac{1}{2}(2l+2)(2l+1)\, x^2 \otimes x^l$
- $\varepsilon = 0$: $x^2 \otimes x^{l-1}y^{l+1} - y^2 \otimes x^{l+1}y^{l-1} \mapsto 4(l+1)x^l y^l \mapsto -4l(l+1)(-x^2 \otimes x^{l+1}y^{l-1} + y^2 \otimes x^{l-1}y^{l+1})$
- $\varepsilon = -1$: $x^2 \otimes x^{l-1}y^{l+1} - 2xy \otimes x^l y^l + y^2 \otimes x^{l+1}y^{l-1} \mapsto 2l(2l+1)x^{l-1}y^{l-1} \mapsto -\frac{1}{2} 2l(2l+1)\,(x^2 \otimes x^{l-1}y^{l+1} - 2xy \otimes x^l y^l + y^2 \otimes x^{l+1}y^{l-1})$

To summarise: The inverse map $\{\cdot,\cdot\}_{1-\varepsilon} : S^2 \otimes S^{2l} \to S^{2l+2\varepsilon}$ is

$$S^{2l+2\varepsilon} \to S^2 \otimes S^{2l} \quad \text{via} \quad g \mapsto \alpha_{2l,\varepsilon}\left(xy \otimes \{xy, g\}_{1+\varepsilon} - (x^2 \otimes \{\frac{y^2}{2}, g\}_{1+\varepsilon} + y^2 \otimes \{\frac{x^2}{2}, g\}_{1+\varepsilon})\right)$$

$$\text{with constants} \quad \alpha_{2l,1} = -\frac{2}{(2l+1)(2l+2)}, \quad \alpha_{2l,0} = -\frac{1}{2l(2l+2)} \quad \text{and} \quad \alpha_{2l,-1} = -\frac{2}{2l(2l+1)}.$$

Concering the second point, since our bilinear form Q computes commutators we need to express these in terms of λ's. The next Lemma does even more: it computes compositions.

Lemma 5.4. *For $f \in S^n$ and $N \leq n$ we consider $\lambda_f^{(N)} \in \operatorname{Hom}(S^k; S^{k+n-2N})$ as well as $\lambda_g^{(M)} \in \operatorname{Hom}(S^k; S^{k+m-2M})$ for some $g \in S^m$ and $M \leq m$. According to the decomposition*

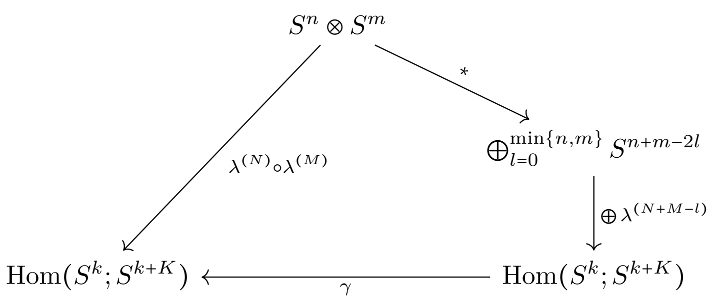
$$\operatorname{Hom}(S^k; S^{k+K}) = \bigoplus_{l=0}^{\min\{k,k+K\}} S^{2k+K-2l} \quad \text{where} \quad K = n - 2N + m - 2M$$

the composition $\lambda_f^{(N)}\lambda_g^{(M)}\big|_{S^k}$ has components

$$\lambda_f^{(N)}\lambda_g^{(M)}\big|_{S^k} = \sum_{l=0}^{\min\{k,k+K\}} \gamma_{N,M}^l(n,m,k)\, \lambda_{\{f,g\}_l}^{(N+M-l)}\big|_{S^k}.$$

In particular the coefficients depend on the degree of the polynomials it is applied to.

Proof. This is basically an application of Schurs Lemma: On the right hand side, $f \otimes g \in S^n \otimes S^m$ is considered via the Moyal product as a sum of $\{f,g\}_l$ mapped via λ into the homomorphisms. The left hand side is another way to map $f \otimes g$ into the homomorphism space.



As both maps are $\operatorname{Sl}_2 \mathbb{C}$-equivariant these two sides differ componentwise by a factor. ☐

We have computed some of these coefficients using the python programm from the appendix:

- $\lambda_f^{(0)}\lambda_g^{(0)} = \lambda_{fg}^{(0)}$
- $\lambda_f^{(0)}\lambda_g^{(1)} = \frac{m}{n+m}\,\lambda_{fg}^{(1)} - \frac{k}{n+m}\,\lambda_{\{f,g\}_1}^{(0)}$
- $\lambda_f^{(1)}\lambda_g^{(0)} = \frac{n}{n+m}\,\lambda_{fg}^{(1)} + \frac{k+n+m}{n+m}\,\lambda_{\{f,g\}_1}^{(0)}$
- $\lambda_f^{(0)}\lambda_g^{(2)} = \frac{m(m-1)}{(n+m)(n+m-1)}\,\lambda_{fg}^{(2)} + \frac{(m-1)(k-1)}{(n+m)(n+m-2)}\,\lambda_{\{f,g\}_1}^{(1)} + \frac{k(k-1)}{(n+m-1)(n+m-2)}\,\lambda_{\{f,g\}_2}^{(0)}$
- $\lambda_f^{(2)}\lambda_g^{(0)} = \frac{n(n-1)}{(n+m)(n+m-1)}\,\lambda_{fg}^{(2)} + \frac{(n-1)(k+m+n-1)}{(n+m)(n+m-2)}\,\lambda_{\{f,g\}_1}^{(1)} + \frac{(k+n+m-1)(k+n+m-2)}{(n+m-1)(n+m-2)}\,\lambda_{\{f,g\}_2}^{(0)}$
- $\lambda_f^{(1)}\lambda_g^{(1)} = \frac{2mn}{(n+m)(n+m-1)}\,\lambda_{fg}^{(2)} + \frac{-kn+m(k+m+n-2)}{(n+m)(n+m-2)}\,\lambda_{\{f,g\}_1}^{(1)} - \frac{2k(k+n+m-2)}{(n+m-1)(n+m-2)}\,\lambda_{\{f,g\}_2}^{(0)}$

Let $\Phi = \Phi^{(2n,\varepsilon)} \in S^{2(n+\varepsilon)}$ with corresponding homomorphism $\alpha_{2n,\varepsilon}\,\sum_{i=1}^3 \sigma_i \otimes \lambda_{\sigma_i}^{(1+\varepsilon)}\Phi \in S^2 \otimes S^{2n}$ and similarly for $\Psi \in S^{2(m+\delta)}$. We can now compute

$$\begin{aligned}
Q(\Phi,\Psi) &= Q(\alpha_{2n,\varepsilon}\sum_{i=1}^3 \sigma_i \otimes \lambda_{\sigma_i}^{(1+\varepsilon)}\Phi, \alpha_{2m,\varepsilon}\sum_{j=1}^3 \sigma_j \otimes \lambda_{\sigma_j}^{(1+\delta)}\Psi) \\
&= \alpha_{2n,\varepsilon}\alpha_{2m,\delta}\sum_{i,j=1}^3 \{\sigma_i,\sigma_j\}_1 \otimes \left[\lambda^{(n)}_{\lambda_{\sigma_i}^{(1+\varepsilon)}\Phi}, \lambda^{(m)}_{\lambda_{\sigma_j}^{(1+\delta)}\Psi}\right].
\end{aligned}$$

Following the general formula we have

$$\left[\lambda^{(n)}_{\lambda_{\sigma_i}^{(1+\varepsilon)}\Phi}, \lambda^{(m)}_{\lambda_{\sigma_j}^{(1+\delta)}\Psi}\right] = \sum_{r=0}^{n+m} (\gamma^r_{n,m}(2n,2m,d) - (-1)^r \gamma^r_{m,n}(2m,2n,d))\,\lambda^{(n+m-r)}_{\{\lambda_{\sigma_i}^{(1+\varepsilon)}\Phi, \lambda_{\sigma_j}^{(1+\delta)}\Psi\}_r}$$

and so find

$$Q(\Phi,\Psi)^{(2r)} = \alpha_{2n,\varepsilon}\alpha_{2m,\delta}\,(\gamma^r_{n,m}(2n,2m,d) - (-1)^r\gamma^r_{m,n}(2m,2n,d))\sum_{i,j=1}^3 \{\sigma_i,\sigma_j\}_1 \otimes \{\lambda_{\sigma_i}^{(1+\varepsilon)}\Phi, \lambda_{\sigma_j}^{(1+\delta)}\Psi\}_{n+m-r}.$$

We are going to express $\{\lambda_{\sigma_i}^{(1+\varepsilon)}\Phi, \lambda_{\sigma_j}^{(1+\delta)}\Psi\}_{n+m-r}$ as a sum of symmetric and anti-symmetric terms in i and j and then deduce that, due to the anti-symmetry of $\{\sigma_i,\sigma_j\}_1$ in i and j, only the anti-symmetric part contributes to the summation. And so our aim is to simplify these nested moyal brackets by iterated application of the composition formular. To keep the notation a bit slimmer we notate: $\varepsilon' = 1+\varepsilon$, $\delta' = \delta+1$, $r' = n+m-r$. This brings us to

$$\begin{aligned}
&\{\lambda_{\sigma_i}^{(\varepsilon')}\Phi, \lambda_{\sigma_j}^{(\delta')}\Psi\}_{r'} = \lambda^{(r')}_{\lambda_{\sigma_i}^{(\varepsilon')}\Phi}\lambda_{\sigma_j}^{(\delta')}\Psi \\
&= \sum_{a=0}^{\min\{r'+\delta',2\}} (-1)^a \gamma^a_{r',\delta'}(2n,2,2(m+\delta))\,\lambda^{(r'+\delta'-a)}_{\lambda^{(a)}_{\sigma_j}\lambda^{(\varepsilon')}_{\sigma_i}\Phi}\Psi \\
&= \sum_{a=0}^{\min\{r'+\delta',2\}} (-1)^a \gamma^a_{r',\delta'}(2n,2,2(m+\delta)) \sum_{b=0}^{\min\{a+\varepsilon',2\}} (-1)^b \gamma^b_{a,\varepsilon'}(2,2,2(n+\varepsilon))\lambda^{(r'+\delta'-a)}_{\lambda^{(\varepsilon'+a-b)}_{\{\sigma_i,\sigma_j\}_b}\Phi}\Psi \\
&= \sum_{a=0}^{\min\{r'+\delta',2\}} (-1)^a \gamma^a_{r',\delta'}(2n,2,2(m+\delta)) \sum_{b=0}^{\min\{a+\varepsilon',2\}} (-1)^{r'+\delta'+\varepsilon'} \gamma^b_{a,\varepsilon'}(2,2,2(n+\varepsilon))\,\lambda^{(r'+\delta'-a)}_{\Psi}\lambda^{(\varepsilon'+a-b)}_{\Phi}\{\sigma_i,\sigma_j\}_b \\
&= \sum_{a=0}^{\min\{r'+\delta',2\}} (-1)^a \gamma^a_{r',\delta'}(2n,2,2(m+\delta)) \sum_{b=0}^{\min\{a+\varepsilon',2\}} (-1)^{r'+\delta'+\varepsilon'} \gamma^b_{a,\varepsilon'}(2,2,2(n+\varepsilon))\times \\
&\quad \times \sum_{c=0}^{\min\{r'+\delta'+\varepsilon'-b,4-2b\}} \gamma^c_{r'+\delta'-a,\varepsilon'+a-b}(2(m+\delta),2(n+\varepsilon),4-2b)(-1)^{r'+\delta'+\varepsilon'-b}\lambda^{(r'+\delta'+\varepsilon'-b-c)}_{\{\sigma_i,\sigma_j\}_b}\{\Phi,\Psi\}_c.
\end{aligned}$$

And so we have found $Q(\Phi,\Psi)^{(2r)}$ being a linear combination of $\lambda^{(r'+\delta'+\varepsilon'-b-c)}_{\{\sigma_i,\sigma_j\}_b}\{\Phi,\Psi\}_c$. Ignoring the prefactor for the moment the calculation goes on with

$$Q(\Phi,\Psi)^{(2r)} = \sum_{b,c}\mathrm{const}(b,c)\sum_{i,j=1}^3 \{\sigma_i,\sigma_j\}_1 \otimes \lambda^{(r'+\delta'+\varepsilon'-b-c)}_{\{\sigma_i,\sigma_j\}_b}\{\Phi,\Psi\}_c.$$

By the mentioned antisymmetry of the first component the only non-vanishing summand is $b = 1$ so that we are left with the homomorphism

$$\begin{aligned} Q(\Phi,\Psi)^{(2r)} &= \sum_c \mathrm{const}(c) \sum_{i,j=1}^{3} \{\sigma_i,\sigma_j\}_1 \otimes \lambda^{(r'+\delta'+\varepsilon'-1-c)}_{\{\sigma_i,\sigma_j\}_1}\{\Phi,\Psi\}_c \\ &= 2\sum_c \mathrm{const}(c) \sum_{k=1}^{3} \sigma_k \otimes \lambda^{(r'+\delta'+\varepsilon'-1-c)}_{\sigma_k}\{\Phi,\Psi\}_c \end{aligned}$$

whose compontents are computed by replacing the tensor product with the Moyal product. And so with the notation $\omega' = \omega + 1$

$$Q(\Phi,\Psi)^{(2r,\omega)} = 2\sum_c \mathrm{const}(c) \sum_{k=1}^{3} \lambda^{(2-\omega')}_{\sigma_k}\lambda^{(r'+\delta'+\varepsilon'-1-c)}_{\sigma_k}\{\Phi,\Psi\}_c$$

$$= 2\sum_c \mathrm{const}(c) \sum_{k=1}^{3} \sum_{p=0}^{\min\{2,r'+\delta'+\varepsilon'-\omega'+1-c\}} \gamma^p_{2-\omega',r'+\delta'+\varepsilon'-1-c}(2,2,2(n+\varepsilon)+2(m+\varepsilon)-2c)\lambda^{(r'+\delta'+\varepsilon'-\omega'+1-c-p)}_{\{\sigma_k,\sigma_k\}_p}\{\Phi,\Psi\}_c.$$

Since $\sigma_1 = -ixy$, $\sigma_2 = (y^2+x^2)/2$ and $\sigma_3 = -i(y^2-x^2)/2$ we can compute

$$\sum_{k=1}^{3}\{\sigma_k,\sigma_k\}_p = \begin{cases} -3 & p=2 \\ 0 & p=1 \\ 0 & p=0 \end{cases}.$$

so that the above $\lambda^{(r'+\delta'+\varepsilon'+1-c-\omega'-p)}_{\{\sigma_k,\sigma_k\}_p}$ is a non-zero map if and only if $p = 2$ and $r'+\delta'+\varepsilon'+1-c-\omega'-p \le 4-2p$. This fixes $p = 2$ and $c = r'+\delta'+\varepsilon'-\omega'-1$.
Let us now discuss whether $b = 1$, $p = 2$ and $c = r'+\delta'+\varepsilon'-\omega'-1$ are in the summation domain, if not the sum is vanishing already: We have

- $a \in [0;\min\{r'+\delta',2\}]$
- $b \in [0;\min\{a+\varepsilon',2\}]$
- $c \in [0;\min\{r'+\delta'+\varepsilon'-b,4-2b\}]$
- $p \in [0;\min\{2,r'+\delta'+\varepsilon'-\omega'+1-c\}]$

This leads us to

- With this c the upper bound of p is 2 and so there is the p-summand if we have the c-summand.
- $b = 1$ is in the summation domain $[0;\min\{a+\varepsilon',2\}]$ unless $a+\varepsilon' \neq 0$ which is equivalent to $a = 0$ and $\varepsilon' = 0$. There is only the $a = 0$ summand if and only if $r' = 0$ and $\varepsilon' = 0$, i.e. if $\varepsilon = -1$ and $r = n+m$. By the symmetry of Q we can run the same argument for $\delta' = 0$.
  And so there is no $b = 1$ term if and only if $r = n+m$, $\varepsilon = \delta = -1$. We would have then just have the multiplication
  $$Q(\Phi^{(2n-2)},\Psi^{(2m-2)})^{(2n+2m)} = \left(\sum_{i,j=1}^{3}\sigma_i\sigma_j\right)\Phi\Psi.$$
- The condition $c \ge 0$ says $\omega'+1 \le r'+\varepsilon'+\delta'$ and so excludes some values for ω already. Taking $c \le 2$ into account yields to $r'+\varepsilon'+\delta' \le 3+\omega'$.

And so the above result is probably non-zero as long as

$$1+\omega' \le r'+\varepsilon'+\delta' \le 3+\omega' \quad \text{alternatively} \quad \omega+r \le n+m+\varepsilon+\delta \le 2+\omega+r$$

and in all other cases we have $Q(\Phi,\Psi) = 0$. With $c = n+m+\varepsilon+\delta-r-\omega = \frac{|\Phi|+|\Psi|}{2}-(r+\omega)$ we finally find

$$\begin{aligned} Q(\Phi,\Psi)^{(2r,\omega)} &= R(n,\varepsilon,m,\delta;r,\omega)\,\{\Phi,\Psi\}_{\frac{|\Phi|+|\Psi|}{2}-(r+\omega)} \\ R(n,\varepsilon,m,\delta;r,\omega) &= \alpha_{2n,\varepsilon}\alpha_{2m,\delta}\left(\gamma^r_{n,m}(2n,2m,d)-(-1)^r\gamma^r_{m,n}(2m,2n,d)\right)\times \\ &\times \sum_{a=0}^{\min\{r'+\delta',2\}} (-1)^{a+1}\gamma^a_{r',\delta'}(2n,2,2(m+\delta))\,\gamma^1_{a,\varepsilon'}(2,2,2(n+\varepsilon))\times \\ &\times \gamma^{n+\varepsilon+m+\delta-r-\omega}_{r'+\delta'-a,\varepsilon'+a-1}(2(m+\delta),2(n+\varepsilon),2)\,2\,(-3)\,\gamma^2_{1-\omega,\omega+1}(2,2,2(r+\omega)). \end{aligned}$$

This expression for Q is quite a simple one: It computes Moyal brackets of the input. And so the recurrence relation boils down to compute nestes Moyal brackets of the initial data. This is, e.g., illustrated when iterating the first few steps for endomorphisms up to $n \le 6$: The relation

$$(n+\omega^2+\omega(2r+1))\Phi_n^{(2r,\omega)} = 2\sum_{0<l<n} Q(\Phi_l,\Phi_{n-l})^{(2r,\omega)} \quad \text{with inital values} \quad \Phi_l^{(2l,-1)} = \xi_l \in S^{2(l-1)}$$

yields to

- The recurrence relation implies that all components of Φ_2 vanish except $\Phi_2^{(2,-1)} = \xi_2 \in S^0$
- Next one is Φ_4 with $(4+\omega^2+\omega(2r+1))\Phi_4^{(2r,\omega)} = 2Q(\Phi_2,\Phi_2)^{(2r,\omega)} = 2Q(\xi_2,\xi_2)^{(2r,\omega)}$. As $Q(\xi_2,\xi_2)^{(2r,\omega)} = R(1,1,1,1;r,\omega)\,\{\xi_2,\xi_2\}_{-r-\omega}$ which is non-vanishing if and only if $r+\omega=0$, i.e. if $r=\omega=0$ or $r=-\omega=1$. We consequently find
  - $\Phi_4^{(0,0)} = \frac{1}{4}\,2R(1,1,1,1;0,0)\,\xi_2^2 = \frac{R(1,1,1,1;0,0)}{2}\,\xi_2^2$
  - $\Phi_4^{(2,-1)} = \frac{1}{2}\,2R(1,1,1,1;1,-1)\,\xi_2^2 = R(1,1,1,1;1,-1)\,\xi_2^2$
  - $\Phi_4^{(4,-1)} = \xi_4 \in S^2$ as initial value
  - all other components vanish
- For Φ_6 we have to consider $(6+\omega^2+\omega(2r+1))\,\Phi_6^{(2r,\omega)} = 4Q(\Phi_2,\Phi_4)^{(2r,\omega)} = 4Q(\Phi_2^{(2,-1)},\Phi_4^{(0,0)})^{(2r,\omega)} + 4Q(\Phi_2^{(2,-1)},\Phi_4^{(2,-1)})^{(2r,\omega)} + 4Q(\Phi_2^{(2,-1)},\Phi_4^{(4,-1)})^{(2r,\omega)}$. The first two terms are only non-vanishing if $r+\omega=0$, so as before, while the third term is $Q(\Phi_2^{(2,-1)},\Phi_4^{(4,-1)})^{(2r,\omega)} = R(1,-1,2,-1;r,\omega)\,\{\xi_2,\xi_4\}_{1-(r+\omega)}$ which only fives contributions if $r+\omega=1$ as ξ_2 is a scalar.
  - $\Phi_6^{(0,0)} = \frac{1}{6}\,4\big(R(1,1,0,0;0,0)\,\frac{1}{4}\,2R(1,1,1,1;0,0) + R(1,1,1,-1;0,0)\,\frac{1}{2}\,2R(1,1,1,1;1,-1)\big)\,\xi_2^3 =$ $= \frac{R(1,1,0,0;0,0)\,R(1,1,1,1;0,0)+2R(1,1,1,-1;0,0)\,R(1,1,1,1;1,-1)}{3}\,\xi_2^3$
  - $\Phi_6^{(2,-1)} = \frac{1}{4}\,4\big(R(1,1,0,0;1,-1)\,\frac{1}{4}\,2R(1,1,1,1;0,0) + R(1,1,1,-1;0,0)\,\frac{1}{2}\,2R(1,1,1,1;1,-1)\big)\,\xi_2^3 =$ $= \frac{R(1,1,0,0;1,-1)\,R(1,1,1,1;0,0)+2R(1,1,1,-1;0,0)\,R(1,1,1,1;1,-1)}{2}\,\xi_2^3$
  - $\Phi_6^{(0,1)} = \frac{1}{8}\,4\,R(1,1,2,-1;0,1)\,\xi_2\xi_4 = \frac{R(1,1,2,-1;0,1)}{2}\,\xi_2\xi_4$
  - $\Phi_6^{(2,0)} = \frac{1}{6}\,4\,R(1,1,2,-1;1,0)\,\xi_2\xi_4\,\frac{2R(1,1,2,-1;0,1)}{3}\,\xi_2\xi_4$
  - $\Phi_6^{(4,-1)} = \frac{1}{2}\,4\,R(1,1,2,-1;2,-1)\,\xi_2\xi_4 = 2R(1,1,2,-1;2,-1)\,\xi_2\xi_4$
  - $\Phi_6^{(6,-1)} = \xi_6 \in S^4$ as inital value
  - all other components vanish
- there are no odd components, i.e. $\Phi_1 = \Phi_3 = \Phi_5 = 0$.

Let us now focus on the coefficients R and so also on γ. Using the composition formula again we are actually able to find relations for the γs that might help later to simplify R or products of R.

5.4 About the γ-Coefficients

With the composition formula

$$\lambda_f^{(D-j)}\lambda_g^{(j)} = \sum_{l=0}^{D} \gamma_{D-j,j}^{l}(n,m,k)\,\lambda_{\{f,g\}_l}^{(D-l)}$$

we can use the symmetries of $\{\cdot,\cdot\}$ to find further information on the γ that may help to simplify the prefactor R. To get the first symmetry, we just apply the composition formula twice

$$
\begin{aligned}
&\{f,\{g,h\}_j\}_{D-j} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,\{\{f,g\}_a,h\}_{D-a} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^D\,\{h,\{g,f\}_a\}_{D-a} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^D \sum_{b=0}^{D} \gamma^b_{D-a,a}(|h|,|g|,|f|)\,\{\{h,g\}_b,f\}_{D-b} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^D \sum_{b=0}^{D} \gamma^b_{D-a,a}(|h|,|g|,|f|)\,(-1)^D\,\{f,\{g,h\}_b\}_{D-b}
\end{aligned}
$$

which leads with the Kronecker-delta δ to

$$\sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|) \sum_{b=0}^{D} \gamma^b_{D-a,a}(|h|,|g|,|f|) = \delta_j(b).$$

Including another step yields

$$
\begin{aligned}
&\{f,\{g,h\}_j\}_{D-j} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,\{\{f,g\}_a,h\}_{D-a} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^{D-a}\,\{h,\{f,g\}_a\}_{D-a} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^{D-a} \sum_{b=0}^{D} \gamma^b_{D-a,a}(|h|,|f|,|g|)\,\{\{h,f\}_b,g\}_{D-b} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^{D-a} \sum_{b=0}^{D} \gamma^b_{D-a,a}(|h|,|f|,|g|)\,(-1)^{D-b}\,\{g,\{h,f\}_b\}_{D-b} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^{D-a} \sum_{b=0}^{D} \gamma^b_{D-a,a}(|h|,|f|,|g|)\,(-1)^{D-b} \sum_{c=0}^{D} \gamma^c_{D-b,b}(|g|,|f|,|h|)\,\{\{g,h\}_c,f\}_{D-c} \\
&= \sum_{a=0}^{D} \gamma^a_{D-j,j}(|f|,|g|,|h|)\,(-1)^{D-a} \sum_{b=0}^{D} \gamma^b_{D-a,a}(|h|,|f|,|g|)\,(-1)^{D-b} \sum_{c=0}^{D} \gamma^c_{D-b,b}(|g|,|f|,|h|)\,(-1)^{D-c}\,\{f,\{g,h\}_c\}_{D-c}
\end{aligned}
$$

which gives

$$\sum_{a=0}^{D} \gamma_{D-j,j}(|f|,|g|,|h|)\,(-1)^a \sum_{b=0}^{D} \gamma_{D-a,a}(|h|,|f|,|g|)\,(-1)^b \sum_{c=0}^{D} \gamma_{D-b,b}(|g|,|f|,|h|)\,(-1)^c = (-1)^D\,\delta_j(c)$$

These are just matrix multiplications: Let us write $\Gamma_D(|f|,|g|,|h|)$ and the sign-twisted $\tilde{\Gamma}_D(|f|,|g|,|h|)$ for the matrix with entries

$$\Gamma_D(|f|,|g|,|h|)_{a,b} = \gamma^a_{D-b,b}(|f|,|g|,|h|) \quad \text{and} \quad \tilde{\Gamma}_D(|f|,|g|,|h|)_{a,b} = (-1)^a \gamma^a_{D-b,b}(|f|,|g|,|h|).$$

In that style we can rewrite the above relations as

$$\Gamma_D(|f|,|g|,|h|)\,\Gamma_D(|h|,|g|,|f|) = 1 \quad \text{as well as} \quad \tilde{\Gamma}_D(|f|,|g|,|h|)\,\tilde{\Gamma}_D(|h|,|f|,|g|)\,\tilde{\Gamma}_D(|g|,|h|,|f|) = (-1)^D.$$

This is the (open) end of the discussion. These relations are a nice-to-have but so far have not lead to any deep insight, the prefactors R in Q are not of the above shape to use the above relations. However the shape of Q makes it clear what to expect at the end: A series of nested Moyal brackets applied to combinations of the initial data. Again, such nested Moyal brackets can be reorganised using the composition formula and so we are able to express the result in terms of easily computable derivatives of the inital data. The main idea is to look for another tool which can handle this nesting procedure more easily and abstractly.

Before we have actually found the composition formula we were working in the universal enveloping algebra of $\mathfrak{sl}_2\mathbb{C}$ as both, the space $\mathrm{End}(S^d)$ as well as the polynomials with the Moyal bracket, can be formulated very nicely there. Even an intrinsic description for some of the γ's and their relation is possible. This is what we present now. During our work on the project we have stopped going into this direction and focussed on the computation of concrete solutions. The next chapter indicates again V. Pidstrygachs idea where to look for the overall picture.

6 Algebraic Backround in the Universal Enveloping Algebra

The consideration of Q in the previous chapter enabled us to formulate the problem of solving Nahms equations for nilpotent orbits in an algebraic language of represenation theory or, in general, linear algebra. We consider this as a key step towards an explicit solution. The next problem is to understand the algebraic structure behind the computations in particular behind the composition formula, ideally in a formulation that is well developped. In this chapter we shall relate previous considerations to the universal enveloping algebra of $\mathfrak{sl}_2\,\mathbb{C}$. However, this language is not fully developped, yet, and not sufficient since it can only deal with endomorphisms: Since any representation is a Lie algebra homomorphism it extends naturally to the universal enveloping algebra:

$$\mathfrak{sl}_2\,\mathbb{C} \to \mathrm{End}(S^d) \quad \text{extends to} \quad \mathscr{U}(\mathfrak{sl}_2\,\mathbb{C}) \to \mathrm{End}(S^d).$$

With this map we will find a way to describe $\mathrm{End}(S^d)$ inside the universal enveloping algebra.
We can also see the polynomial in $\mathscr{U}$: The Moyal-product induces a Lie algebra structure on $S^2 = \mathbb{C}[x,y]_2$ that coincides with the algebra $\mathfrak{sl}_2\,\mathbb{C}$ and that way defines a map of algebras

$$\mathfrak{sl}_2\,\mathbb{C} \to (\mathbb{C}[x,y]_{\mathrm{ev}}, *) \quad \text{which extends to} \quad \mathscr{U}(\mathfrak{sl}_2\,\mathbb{C}) \to (\mathbb{C}[x,y]_{\mathrm{ev}}, *).$$

In other words, everything comes together in $\mathscr{U} = \mathscr{U}(\mathfrak{sl}_2\,\mathbb{C})$. However, in the computation of Q we also had terms like $\lambda^{(1+\varepsilon)}_{\sigma_i}\Phi^{(2n,\varepsilon)}$ in which $\lambda^{(1+\varepsilon)}_{\sigma_i}$ is not acting as an endomorphism of $S^{2n+2\varepsilon}$ but as a homomorphism $S^{2n+2\varepsilon} \to S^{2n}$. It is the task of future work to include these homomorphisms into the presented picture or to find another suitable language. We shall present in this chapter that our knowledge about $\mathscr{U}$ covers the picture partly and thus may be a fruitful approach for necessary generalisations.

This chapter constists of two parts: The first section sets up the framework and the notation of $\mathfrak{sl}_2\,\mathbb{C}$-representations, explains how to relate $\mathrm{End}(S^d)$ and $(\mathbb{C}[x,y]_{\mathrm{ev}}, *)$ to the univeral enveloping algebra and so tries to develop the underlying picture. Even if we give some simple proofs this text is not intended to be an introduction to the representation theory of $\mathfrak{sl}_2\,\mathbb{C}$. Theorems, missing proofs and arguments can be found in J. Humphreys book *Introduction to Lie Algebras and Representation Theory*([Hum97]).
On our way towards the decomposition of $\mathscr{U}$ into its irreducible subrepresentations the dicussion seems to be interrupted by a short discussion of symmetric tensors as $\mathrm{SO}(d)$-represenation. These will be used later in terms of symmetric polynomials again so that we have decided to cover this part already there. As before our explanations do not follows some particular article or book, some inspirations for computations and overviews concerning the tensors and polynomials are taken and given in the articles [Bru18] and [MD17].
It was the work of F. Bayen and C. Fronsdal in [BF80] that motivated the point of view from the universal enveloping algebra. And so we close this chapter with an $\mathfrak{sl}_2\,\mathbb{C}$-intrinsic description of some γ-coefficients following their ideas.

6.1 The Universal Enveloping Algebra and $\mathfrak{sl}_2\,\mathbb{C}$-Representation Theory

The universal enveloping algebra is a fundamental object in the study of Lie algebra representations and so it is to us. We shall start with its definition and then later pass to the universal property:

Definition 6.1. The universal enveloping algebra of $\mathfrak{sl}_2\,\mathbb{C}$ is the full tensor algebra $T\,\mathfrak{sl}_2\,\mathbb{C} = \bigoplus_{k=0}^{\infty}(\mathfrak{sl}_2\,\mathbb{C})^{\otimes k}$ modulo the two-sided ideal $\mathscr{I}$ generated by the elements $\mathscr{I} = \langle a\otimes b - b\otimes a - [a,b] \mid a,b \in \mathfrak{sl}_2\,\mathbb{C}\rangle$:

$$\mathscr{U} \coloneqq \mathscr{U}(\mathfrak{sl}_2\,\mathbb{C}) \coloneqq T\,\mathfrak{sl}_2\,\mathbb{C}/\mathscr{I}.$$

The algebra structure is inherited from the tensor algebra, notationally, we shall not write a particular symbol for the product and simply write $ef, e^2 = ee \in \mathscr{U}$. As both $\mathbb{C}$, $\mathfrak{sl}_2\,\mathbb{C} \subseteq T\,\mathfrak{sl}_2\,\mathbb{C}$ we also find them included in $\mathscr{U}$. Moreover, the $\mathfrak{sl}_2\,\mathbb{C}$-module structure of $T\,\mathfrak{sl}_2\,\mathbb{C}$ given by the adjoint action descends to $\mathscr{U}$ as the ideal $\mathscr{I}$ is preserved: It remains a derivation so that $\mathrm{ad}_a(bc) = (\mathrm{ad}_a\,b)c + b(\mathrm{ad}_a\,c) = (ab-ba)c + b(ac-ca) = abc - bca$ for any $a,b,c \in \mathfrak{sl}_2\,\mathbb{C}$.

On can now show

Theorem 6.2 (Universal Property of the Universal Enveloping Algebra). *Let* $(\mathcal{A},\cdot)$ *be any complex algebra with an unit element. Any map* $\varphi : \mathfrak{sl}_2\,\mathbb{C} \to \mathcal{A}$ *with* $\varphi(a)\cdot\varphi(b) - \varphi(b)\cdot\varphi(a) = \varphi([a,b])$ *extends uniquely to an algebra homomorphism* $\varphi : \mathscr{U} \to \mathcal{A}$ *with* $\varphi|_{\mathfrak{sl}_2\,\mathbb{C}} = \varphi$.
Additionally, if $\mathcal{A}$ *is a* $\mathfrak{sl}_2\,\mathbb{C}$*-represenation space such that* φ *is equivariant its extension is also equivariant.*

We intend to not overload the notation so that the extension φ according to the universal property is denoted by the same letter as the extended map φ.
Clearly, any representation $\mathfrak{sl}_2\,\mathbb{C} \to \mathrm{End}(V)$ on a vector space V extends to some $\mathscr{U} \to \mathrm{End}(V)$ as the assumtion is fulfilled already by the definition, so does the $\mathfrak{sl}_2\,\mathbb{C}$-action on $\mathscr{U}$ extend to an algebra homomorphism $\mathrm{ad} : \mathscr{U} \to \mathrm{End}(\mathscr{U})$, e.g. $\mathrm{ad}_{e^2} = \mathrm{ad}_e^2$ so that $\mathrm{ad}_e^2\, a = \mathrm{ad}_e(ea - ae) = e^2 a - 2eae + ae^2$.
This extension is a simple way to compare different representations or to compare representations with other algebra maps. Another way is of course given by concrete computations and expressions in a basis. Clearly, any basis of $\mathfrak{sl}_2\,\mathbb{C}$ generates the algebra $\mathscr{U}$. For a vector space basis we recall the theorem by H. Poincaré, G. Birkhoff and E. Witt.

Theorem 6.3. *For any basis* $\{x_1, x_2, x_3\}$ *of* $\mathfrak{sl}_2\,\mathbb{C}$ *the linear span of* $\{x_1^a x_2^b x_3^c \mid a,b,c \in \mathbb{N}_0\}$ *is all of* $\mathscr{U}$.

As usually the study of the algebra and its homomorphisms leads to the question about its center which is generated by one element, the so-called Casimir element

$$\mathrm{cas} = h^2 + 2(ef + fe) = -(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

in terms of the standard basis $\{e, f, h\}$ or the Pauli-basis $\{\sigma_1, \sigma_2, \sigma_3\}$. A proof of this theorem follows e.g. from the theorem of Harish-Chandra:

Theorem 6.4. *The center of* $\mathscr{U}$ *is generated by the Casimir element, i.e. the center is* $\mathbb{C}[\mathrm{cas}]$.

Consequently, for any algebra homomorphism $\varphi : \mathscr{U} \to \mathcal{A}$ we find $\varphi(\mathrm{cas}) = \varphi(h)^2 + 2(\varphi(e)\varphi(f) + \varphi(f)\varphi(e))$ being central in the image of φ, i.e. it commutes with any $\varphi(a)$ where $a \in \mathfrak{sl}_{\mathbb{C}}$. In our future examples $\varphi(\mathrm{cas})$ acts as multiplication with a scalar on all of $\mathcal{A}$, say $\varphi(\mathrm{cas}) = c \in \mathbb{C}$. Consequently, φ factorises through the quotient by the two-sided ideal $\mathscr{I}_c = \langle \mathrm{cas} - c \rangle$. This leads to the idea of a Verma module:

6.1.1 Verma Modules

Let $c \in \mathbb{C}$ be any number and $\mathscr{I}_c = \langle \mathrm{cas} - c \rangle$ the two-sided ideal generated by $\mathrm{cas} - c \in \mathscr{U}$. The Verma module is defined to be the algebra given by the quotient

$$\pi_c : \mathscr{U} \to \mathscr{U}_c := \mathscr{U} / \mathscr{I}_c.$$

Notice that $\mathscr{I}_c$ is central in $\mathscr{U}$ and hence $\mathscr{U}_c$ is again a $\mathfrak{sl}_2\,\mathbb{C}$-representation. Clearly, as $\{e^a f^b h^c \mid a,b,c \in \mathbb{N}_0\}$ spans $\mathscr{U}$ their image under π_c spans $\mathscr{U}_c$. But here we can simplify further:

- Due to the relation $ef - fe = h$ and $\mathrm{cas} = h^2 + 2(ef - fe)$ we can rewrite

  $$\mathrm{cas} = h^2 + 2h + 4fe = h^2 - 2h + 4ef$$

  and so in $\mathscr{U}_c$

  $$fe = \frac{1}{4}(c - h^2 - 2h) \quad \text{and} \quad ef = \frac{1}{4}(c - h^2 + 2h).$$

  In other words, products ef and fe are polynomials in h.

- For a general polynomial $q(h)$ in h we can deduce from $hf = fh - 2f = f(h-2)$ that $q(h)f = fq(h-2)$. The same consideration for $he = (h+2)e$ leads to

  $$f^n q(h) = q(h + 2n) f^n \quad \text{and} \quad e^n q(h) = q(h - 2n) e^n.$$

- This brings us to a recursive simplification

$$e^n f^m = e^{n-1}(ef)f^{m-1} = e^{n-1}\,\frac{c-h^2+2h}{4}\,f^{m-1} = e^{n-1}f^{m-1}\,\frac{c-(h-2(m-1))^2+2(h-2(m-1))}{4} = \ldots$$

  Consequently, $e^n f^m$ is either $e^{n-m}q(h)$ if $m \le n$ or $f^{m-n}q(h)$ if $n \le m$ for some polynomial q.

And so $\mathscr{U}_c$ is already generated by $\{e^a h^b, f^a h^b \mid a, b \in \mathbb{N}_0\}$. Since these elements are also linearly independent we have proven

Proposition 6.5. *Each of the two families* $\{e^a h^b, f^a h^b \mid a, b \in \mathbb{N}_0\}$ *and* $\{h^a e^b, h^a f^b \mid a, b \in \mathbb{N}_0\}$ *spans* $\mathscr{U}_c$.

In a similary fashion we shall prove the next proposition:

Proposition 6.6. *The family* $\{\mathrm{ad}_f^l\, e^k \mid k \in \mathbb{N}_0,\ 0 \le l \le 2k\}$ *is a vector space basis of* $\mathscr{U}_c$.

Proof. We show inductively that $\mathrm{span}\{\mathrm{ad}_f^l\, e^m \mid m \le k, 0 \le l \le 2m\} = \mathrm{span}\{e^a h^b, f^a h^b \mid a+b \le k\}$. The claim follows then from the fact that the number of generators on the left concides with the number of basis elements on the right.
Let $l < k$. As a first step we will show by induction that there is a polynomial $p_{k,l}$ of degree l such that

$$\mathrm{ad}_f^l\, e^k = p_{k,l}(h)e^{k-l} \in h^l e^{k-l} + \mathrm{span}\{h^a e^b \mid a+b \le k-1\}$$

A direct computation gives

$$\begin{aligned}\mathrm{ad}_f^{l+1}\, e^k &= \mathrm{ad}_f\, p_{k,l}(h)e^{k-l} = f p_{k,l}(h)\, e^{k-l} - p_{k,l}(h)e^{k-l-1}\, ef = p_{k,l}(h+2)\, fe\, e^{k-l-1} - p_{k,l}(h)e^{k-l-1}\, ef \\ &= \frac{1}{4}\left(p_{k,l}(h+2)\,(c-h^2-2h)\, e^{k-l-1} - p_{k,l}(h)e^{k-l-1}\,(c-h^2+2h)\right) \\ &= \frac{1}{4}\left(p_{k,l}(h+2)\,(c-h^2-2h) - p_{k,l}(h)\,(c-(h-2(k-l-1))^2+2(h-2(k-l-1)))\right) e^{k-l-1}.\end{aligned}$$

In this shape it is easy to see that the h^{l+2}-terms cancel so that $p_{k,l+1}$ is a polynomial of degree $l+1$ in h.
If $k \ge l$, we do the same: There is a polynomial $p_{k,l}$ of degree $2k-l$ such that

$$\mathrm{ad}_f^l\, e^k = p_{k,l}(h)f^{k-l}$$

which simply follows from the fact that $\mathrm{ad}_f^k\, e^k = p_{k,k}(h)$ is just a polynomial in h so that the precise form $\mathrm{ad}_f^l\, e^k = \mathrm{ad}_f^{l-k}\, p_{k,k}(h)$ follows from permuting f- and h-terms only. Thus the claim is proven. ☐

Even though the first basis $\{e^n h^m, f^n h^m\}$ comes more natural from the Poincaré-Birkhoff-Witt-Theorem, the latter basis $\{\mathrm{ad}_f^l\, e^k\}$ turns out to be more handy when decomposing $\mathscr{U}_c$ further to finally understand this $\mathfrak{sl}_2\,\mathbb{C}$-module:

- From $\mathrm{ad}_{\mathrm{cas}}\, e^k = 2k(2k-1)\, e^k$ and the $\mathrm{Sl}_2\,\mathbb{C}$-invariance of the Casimir element it follows

$$\mathrm{ad}_{\mathrm{cas}}\, \mathrm{ad}_f^l\, e^k = 2k(2k-1)\ \mathrm{ad}_f^l\, e^k,$$

  i.e. with $\{\mathrm{ad}_f^l\, e^k\}$ we found an eigenbasis to $\mathrm{ad}_{\mathrm{cas}}$, each $\{\mathrm{ad}_f^l\, e^k \mid l = 0, \ldots, 2k\}$ being a $\mathfrak{sl}_2\,\mathbb{C}$-submodule.

- Those $\mathrm{ad}_{\mathrm{cas}}$-eigenspaces decompose further into one-dimensional weight spaces: $\mathrm{ad}_h\, e^k = 2k\, e^k$ and hence $\mathrm{ad}_h\, \mathrm{ad}_f\, e^k = (\mathrm{ad}_f\, \mathrm{ad}_h - \mathrm{ad}_{[h,f]})e^k = 2(k-1)\, \mathrm{ad}_f\, e^k$ and correspondingly

$$\mathrm{ad}_h\, \mathrm{ad}_f^l\, e^k = 2(k-l)\, \mathrm{ad}_f\, e^k \quad \text{for any} \quad l = 0, \ldots, 2k.$$

This way we have decomposed $\mathscr{U}_c$ into its irreducible subpresentation via a suitable choice of a basis which firstly decomposes $\mathscr{U}_c$ into eigenspaces of $\mathrm{ad}_{\mathrm{cas}}$ and these secondly into weight spaces.

6.1.2 Classification of Irreducible Representations

Let, for the moment, V be a finite dimensional and irreducible representation $\rho : \mathfrak{sl}_2\,\mathbb{C} \to \operatorname{End}(V)$ that means that there is no non-trivial subspace in V being preserved under the action of $\mathfrak{sl}_2\,\mathbb{C}$. Our situation is special since we have $[\operatorname{ad}_h, \operatorname{ad}_e] = 2\operatorname{ad}_e$ and $[\operatorname{ad}_h, \operatorname{ad}_f] = -2\operatorname{ad}_f$. The following general result from linear applies to our situation and so can be used to study the structure of V:

Lemma 6.7. *For $A, B \in \operatorname{End}(V)$ on a complex vector space V with $[A, B] = bB$ the following is true:*

- *$B(V_a) \subseteq V_{a+b}$ where $V_a = \ker(A - a)$*
- *B is nilpotent, in particular there is a eigenvector $w \in V$ of A such that $Bw = 0$*

Proof. The equation $(A - (a + b))B = AB - (a + b)B = BA + bB - (a + b)B = B(A - a)$ implies the first claim. This also implies that all possible eigenvalues of A are $a + bk$, $k \in \mathbb{N}_0$. But as W is finite dimensional there is necessarily a k for which $V_{a+(k+1)b} = \{0\}$ but $V_{a+kb} \neq \{0\}$ - correspondingly, this generalised eigenspace is contained in the kernel of B. □

This Lemma applies to gerneral representations $\rho : \mathfrak{sl}_2\,\mathbb{C} \to \operatorname{End}(V)$ and not only to the adjoint represenation. Since $[\rho(h), \rho(e)] = 2\rho(e)$ there is an eigenvector $v \in V$ of hightest weight, i.e. with maximal eigenvalue $\rho(h)v = \nu\, v$ satisfying $\rho(e)v = 0$. We define the subspace $W = \bigoplus_{k=0}^{\dim V} \rho(f)^k v$ in V. Similar to previous computations we duduce that $\rho(h)\rho(f)^k = \rho(f)^k(\rho(h) - 2)^k$ and $\rho(e)\rho(f)^{k-1} = \rho(f)^k q_k(\rho(h))$ for some polynomial q_k. Consequently, W is fixed by the $\mathfrak{sl}_2\,\mathbb{C}$ action and hence it is a non-trivial subrepresentation and so all of V:

$$V = \bigoplus_{k=0}^{\dim V} \rho(f)^k v.$$

Let now have V dimension d, then

- $\operatorname{tr}\rho(h) = \operatorname{tr}[\rho(e), \rho(f)] = 0$ and $\operatorname{tr}\rho(h) = \nu + (\nu - 2) + \ldots + (\nu - (d - 1)) = d \cdot \nu - d \cdot (d - 1)$ and so
$$\nu = d - 1 \quad \text{is the highest weight.}$$
- The fact that $v \in \operatorname{Eig}(\rho(h), d - 1)$ implies $\operatorname{Eig}(\rho(h), d - 1 - 2k) = \mathbb{C} \cdot \rho(f)^k v$ so that
$$V = \bigoplus_{k=0}^{d} \operatorname{Eig}(\rho(h), d - 1 - 2k).$$
- As $[\text{cas}, a] = 0$ for any $a \in \mathfrak{sl}_2\,\mathbb{C}$ we have $[\rho(\text{cas}), \rho(a)] = 0$. Hence $\rho(\text{cas})|_V = \text{const}$ so that a direct computation shows
$$\begin{aligned}\rho(\text{cas})v &= (\rho(h)^2 + 2(\rho(e)\rho(f) + \rho(f)\rho(e))v = (\rho(h)^2 + \rho(h) + 4\rho(f)\rho(e))v \\ &= d(d - 1)\, v\end{aligned}$$
as $\rho(h)v = (d - 1)\, v$ and $\rho(e)v = 0$ on the highest weight vector v.

The decomposition of any $\mathfrak{sl}_2\,\mathbb{C}$-module into its irreducible subrepresentations now goes as follows: If we are lucky, the operator $\operatorname{ad}_{\text{cas}}$ determines a decomposition into finite dimensional eigenspaces as summands. Here, the eigenvalue $d(d-2)$ of $\rho(\text{cas})$ already determines the dimension and so also their multiplicity. To determine irreducibles in $\operatorname{Eig}(\operatorname{ad}_{\text{cas}}, d(d-1))$ concretely we can choose a basis of the subspace of highest weight vectors $\operatorname{Eig}(\rho(h), d - 1)$ and use the action of ad_f to transport this basis to the other weight Eigenspaces.
It is not difficult to show that all n-dimensional $\mathfrak{sl}_2\,\mathbb{C}$-representations are isomorphic: simply map one heighest weight vector to another and extend by equivariance. This so defined map is equivariant and non-trivial and so is necessarily an isomorphism. This argument relies on Schurs Lemma:

Theorem 6.8. *The only non-trivial and equivariant endomorphisms of an irreducible representation are homotheties, in particular, they are autmorphisms.*
More generally, equivariant homomorphisms between two irreducible representations are either the zero map or isomorphisms.

A simple consequence is the following: Let V and W be not necessarily irreducible and $f \in \operatorname{Hom}(V;W)$ equivariant. Then the restriction $f|_{\tilde{V}}$ to any irreducible component $\tilde{V}$ in V is still equivariant and can only map non-trivially to irreducible components $\tilde{W} \subseteq W$ that have the same dimension as $\tilde{V}$. Even better, $f|_{\tilde{V}} : \tilde{V} \to \tilde{W}$ is an isomorphism.

We finish this section by giving the standard irreducible representations $S^d = \mathbb{C}[x,y]_d$ being the space of complex, homogenuous polynomials in two variables x and y of degree d, i.e. $S^d = \operatorname{span}_{\mathbb{C}}\{y^d, xy^{d-1}, \dots x^{d-1}y, y^d\}$. The action of $\mathfrak{sl}_2\,\mathbb{C}$ is defined as follows:

$$\rho(h) = y\partial_y - x\partial_y = \begin{pmatrix} d & & & & \\ & d-2 & & & \\ & & \ddots & & \\ & & & -d+2 & \\ & & & & -d \end{pmatrix} = \operatorname{diag}(d, d-2, \dots, -d+2, -d),$$

$$\rho(e) = -y\partial_x = \begin{pmatrix} 0 & -1 & & & \\ & 0 & -2 & & \\ & & \ddots & \ddots & \\ & & & 0 & -d \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad \rho(f) = -x\partial_y = \begin{pmatrix} 0 & & & & \\ d & 0 & & & \\ & \ddots & 0 & & \\ & & 2 & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

where matrices are given in those coordinates.

It follows from the above discussion that any finite dimensional $\mathfrak{sl}_2\,\mathbb{C}$-representation V can be written as

$$V = \bigoplus_{k=0}^{N} V_k \otimes S^k \quad \text{with} \quad V_k \otimes S^k = \operatorname{Eig}(\operatorname{ad}_{\mathrm{cas}}; k(k-1))$$

where $V_k = \mathbb{C}^{n_k}$ is the multiplicity, i.e. contains the information how many S^k-representations occur. The corresponding projectors onto the finite dimensional $\operatorname{ad}_{\mathrm{cas}}$-eigenspaces are given in the following lemma:

Lemma 6.9. *Let $A \in \operatorname{End}(\mathbb{C}^n)$ be a diagonalized operator $A = \operatorname{diag}(a_1, \dots a_n)$ with $a_1 \leq a_2 \leq \dots \leq a_n$. The polynomial*

$$p_k(x) = \prod_{\substack{j=1 \\ a_j \neq a_k}}^{n} \frac{x - a_j}{a_k - a_j}$$

takes values $p_k(a_k) = 1$ and $p_k(a_j) = 0$ for all $a_j \neq a_k$, $j = 1, \dots, n$.
Consequently, the diagonal operator $p_k(A)$ is the identity operator on $\operatorname{Eig}(A, a_k)$ *and vanishes on* $\operatorname{Eig}(A, a_j)$ *for any j such that $a_j \neq a_k$, in particular, $p_k(A)$ is the projector onto* $\operatorname{Eig}(A, a_k)$.
Similarly, $a_k p_k(A)$ coincides with A on $\operatorname{Eig}(A, a_k)$ *and vanishes on any other* $\operatorname{Eig}(A, a_j)$.

6.1.3 The Decomposition of $\mathscr{U}$ into Irreducible Submoduls

The decomposition of $\mathscr{U}$ into its $\mathfrak{sl}_2\,\mathbb{C}$-subrepresentations is linked to the decomposition of symmetric tensors $S^m V$ on a d-dimensional vector space V into $\mathrm{SO}(V)$-subrepresentations. Since we come back later to the tensors we shall collect some knowledge here already and transfer it to $\mathscr{U}$ via complexification for $V = \mathfrak{sl}_2\,\mathbb{C}$. Further and more concrete computations in physics language related to the symmetric powers can be found in [Bru18] and [MD17], here motivated by multi-pole expansion and quantum physics.

Let V be a d-dimensional vector space equipped with a non-degenerate and positive-definite bilinear form b and a chosen orthonormal basis. We shall consider the symmetric powers $S^m V$ as a $\mathrm{SO}(V)$-module. For us there are two maps relevant: The trace, on the one hand, as a $\mathrm{SO}(d) = \mathrm{SO}(V,b)$-equivariant self-contraction

$$\operatorname{tr} : S^m V \to S^{m-2} V \quad \text{given by} \quad (\operatorname{tr} T)_{i_3 \dots i_m} = \sum_{j=1}^{d} T_{jji_3\dots i_m}$$

in the orthonormal basis. Since the trace map surjects and is $\mathrm{SO}(d)$-equivariant we find the decomposition of S^mV into the subrepresentations given by the kernel and the image of the trace. Let us denote the first space, the kernel, the space of traceless symmetric tensors of rank m by S_0^mV. And so that we have justified the following decomposition of S^mV into subrepresentations $S^mV = S_0^mV \oplus S^{m-2}V$ and by iteration

$$S^mV = S_0^mV \oplus S_0^{m-2}V \oplus S_0^{m-4}V \oplus \dots$$

where the last summand is $S_0^2V \oplus V^{\otimes 0}$ if m was even or $S_0^1V = S^1V = V$ if m was odd.
Let us analyse how each of the summands actually sits inside S^mV. This leads us to the second map $S^{m-2}V \to S^mV$: It comes from the product in the symmetric algebra which is in the basis

$$(ST)_{i_1\dots i_{m+n}} = \frac{1}{(m+n)!}\sum_{\sigma\in P_{m+n}} S_{i_{\sigma(1)}\dots i_{\sigma(m)}}T_{i_{\sigma(m+1)}\dots i_{\sigma(m+n)}} \quad \text{where} \quad S\in S^mV,\ T\in S^nV,$$

as the multiplication with b: $T\mapsto bT$. By definition of $\mathrm{SO}(V,b)$, b is invariant and so is the multiplication with it equivariant. Let us compute the components

$$\mathrm{tr}(bT)_{i_1\dots i_m} = \sum_{j=1}^d (bT)_{jji_1\dots i_m} = \frac{1}{(m+2)!}\sum_{j=1}^d\sum_{\sigma\in P_{m+2}} b_{i_{\sigma(1)}i_{\sigma(2)}}T_{i_{\sigma(3)}\dots i_{\sigma(m+2)}} \quad \text{where} \quad i_{m+1}=i_{m+2}=j.$$

Here we need to count specific permutations:

- There are $2m!$ permutations fixing both j's in the first places, namely all elements in P_m and $(12)P_m$.
- There are $2((m+1)!-m!) = 2mm!$ permutations that fix either the first or the second position but not both, and the same amount of permutation mapping the first position to the second or the second to the first but not interchanging both. And so we have $4mm!$ permutation keeping exactly one j in the two first positions.
- There are $m(m-1)m!$ permutations that map boths j's away from the first positions.

Those numbers sum indeed up to $2m! + 4mm! + m(m-1)m! = (m+2)! = |P_{m+2}|$. We find

$$\begin{aligned}\mathrm{tr}(bT)_{i_1\dots i_m} &= \frac{1}{(m+2)!}\left(2m!\sum_{j=1}^d b_{jj}T_{i_1\dots i_m} + 4mm!\sum_{j=1}^d\sum_{l=1}^d b_{ji_l}T_{ji_1\dots\hat{i}_h\dots i_m} + m(m-1)m!\sum_{l,l'=1,l\neq l}^d b_{i_li_{l'}}T_{jji_1\dots\hat{i}_l\dots\hat{i}_{l'}\dots i_m}\right)\\ &= \frac{m!}{(m+2)!}\left((2d+4m)\,T + m(m-1)\,b\,\mathrm{tr}\,T\right)_{i_1\dots i_m}\end{aligned}$$

A simple consequence is the following: if the tensor T is traceless, we have $\mathrm{tr}(bT) = \frac{2(d+2m)}{(m+2)(m+1)}T$ so that $b: S_0^{m-2}V \to S^mV$ in particular injects. Consequently, $S^2V = S_0^2V \oplus bV^{\otimes 0}$ and inductively it follows

$$S^mV = S_0^mV \oplus bS_0^{m-2}V + b^2S_0^{m-4}V \oplus \dots$$

It is actually true that each of S_0^mV is an irreducible $\mathrm{SO}(d)$-representation but we will not need this fact and so directly transfer this to the $\mathfrak{sl}_2\mathbb{C}$-representation $\mathscr{U}$:
The basis $\{\sigma_1,\sigma_2,\sigma_3\}$ is an orthonormal basis of $\mathfrak{su}(2)$ with respect inner product which is the negative of Killing form. And so we can apply the above theory to $\mathrm{SU}(2)$ as the covering of $\mathrm{SO}(3)$. The complexification then leads to $\mathfrak{sl}_2\mathbb{C}$-representations, e.g. we can substitute $S^m\mathbb{C}^3$ by $S^m\,\mathfrak{sl}_2\mathbb{C} \subseteq \mathscr{U}$ via

$$T \mapsto \sum_{i_1,i_2,i_3} T_{i_1i_2i_3}\sigma_{i_1}\sigma_{i_2}\sigma_{i_3} \quad \text{and in particular} \quad b\mapsto \mathrm{cas}.$$

Under this map, we shall denote by $\mathcal{S}^m$ the image of $S^m\mathbb{C}^3$ and by $\mathcal{R}^m$ the image of $S_0^m\mathbb{C}^3$ each of them being a $\mathfrak{sl}_2\mathbb{C}$-subrepresentation in $\mathscr{U}$. Since $\mathscr{U}$ is spanned by all $\mathcal{S}^m$ we can summarise the above explanation for $\mathrm{SO}(d)$ in the next theorem [BF80]

Theorem 6.10. *We can decompose*

$$\mathscr{U} = \bigoplus_{m=0}^{\infty} \mathcal{S}^m \quad \text{and} \quad \mathcal{S}^m = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} \text{cas}^k \mathcal{R}^{m-2k}$$

into irreducible subrepresentations. Here $\mathcal{R}^m \subseteq \mathcal{S}^m$ can be characterised by

i) the image of the symmetric and traceless tensor $S_0^m \mathbb{C}^3$

ii) $\mathcal{R}^m = \text{span}\{\text{ad}_f^j\, e^m \mid j = 0, \ldots, 2m\}$

iii) $\mathcal{R}^m = \text{Eig}(\text{ad}_{\text{cas}}, 2m(2m+1)) \cap \mathcal{S}^m$

Proof. Again, $\mathscr{U} = \bigoplus \mathcal{S}^m$, $\mathcal{S}^m = \bigoplus \text{cas}^k \mathcal{R}^{m-2k}$ since $b \mapsto$ cas and the first characterisation are already explained. The third characterisation follows from the second using the material of $\mathfrak{sl}_2\,\mathbb{C}$-representations covered before and so it remains to check the second characterisation. The observation $e^m = (\sigma_2 + i\sigma_3)^m$ motivates to consider the following tensor: We define the tensor E by its components as follows: For any index set $I \in \{1,2,3\}^m$

- $E_I = 0$ when ever I contains the index 3
- $E_I = i^{m-k}$ when ever I contains k times the index 1 and $m-k$ times the index 2

First of all, E is symmetric with $\sum_{i_1,\ldots,i_m} E_{i_1\ldots i_m} \ldots \sigma_{i_1}\sigma_{i_m} = (\sigma_2 + i\sigma_3)^m = e^m$. The trace of E is computed as follows: Let $I \in \{1,2,3\}^{m-2}$ be any index set, we find

$$(\text{tr}\, E)_I = E_{11I} + E_{22I} + E_{33I}.$$

If I contains at least once the index 3 it is directly clear that $(\text{tr}\, E)_I = 0$. Otherwise if k denotes the number of index 1 in I we have $E_{11I} = i^{m-k-2} = -i^{m-k}$ and $E_{22I} = i^{m-k}$. This shows that $\text{tr}\, E = 0$ and so $e^m \in \mathcal{R}^m$. Since $\mathcal{R}^m$ is $\mathfrak{sl}_2\,\mathbb{C}$-invariant of dimension not bigger than $2m+1$ we can deduce that $\mathcal{R}^m = \text{span}\{\text{ad}_f^j\, e^m \mid j = 0, \ldots, 2m\}$. This completes the proof. □

We also have seen in the previous material about $\mathfrak{sl}_2\,\mathbb{C}$-representations that $\mathcal{R}^m = \text{span}\{\text{ad}_f^j\, e^m \mid j = 0, \ldots, 2m\}$ is actually irreducible and so we have found the decomposition of $\mathscr{U}$ into irreducible $\mathfrak{sl}_2\,\mathbb{C}$-subrepresentations.

6.2 Formulation of the Moyal Product in $\mathscr{U}$

From the representation point of view $\mathcal{R}^m$ is $2m+1$ dimensional and so isomorphic to S^{2m}. On the other hand the map $\mu : \mathfrak{sl}_2\,\mathbb{C} \to S^2 = \mathbb{C}[x,y]_2$

$$h \mapsto xy\ , \quad e \mapsto \frac{y^2}{2}\ , \quad f \mapsto -\frac{x^2}{2}$$

identifies the two irreducible representations $\mathfrak{sl}_2\,\mathbb{C}$ and S^2. Unfortunately, this map $\mathfrak{sl}_2\,\mathbb{C} \to S^2 \subseteq \mathbb{C}[x,y]$ does not extend to the universal enveloping algebra when $\mathbb{C}[x,y]$ is being equipped with the standard product on polynomials. If perturbed to the Moyal product

$$f * g = \sum_{m=0}^{\infty} \{f,g\}_m \quad \text{where} \quad \{f,g\}_m = \sum_{j=0}^{m} \frac{(-1)^{m-j}}{j!\,(m-j)!}\, \partial_x^j \partial_y^{m-j} f \cdot \partial_x^{m-j} \partial_y^j g$$

we can check that

$$\mu(a) * \mu(b) - \mu(b) * \mu(a) = \{\mu(a), \mu(b)\}_1 = [a,b]$$

on these basis elements and that way extent μ to an equivariant algebra homomorphism $\mu : \mathscr{U} \to \mathbb{C}[x,y]_{\text{ev}}$, the non-commutative moment map from the $\text{Sl}_2\,\mathbb{C}$ action on $\mathbb{C}^2$. Due to

$$\begin{aligned}\mu(\text{cas}) &= \mu(h) * \mu(h) + 2(\mu(e) * \mu(f) + \mu(f) * \mu(e)) = xy * xy + 2\left(\frac{y^2}{2} * (-\frac{x^2}{2}) + (-\frac{x^2}{2}) * \frac{y^2}{2}\right) \\ &= (x^2y^2 - 1) - \frac{1}{2}(x^2y^2 - 4xy + 2) - \frac{1}{2}(x^2y^2 + 4xy + 2) = -3\end{aligned}$$

the homomorphism μ factorises through the Verma module $\mathscr{U}_{-3}$. From the representation point of view now both $\mathscr{U}_{-3} = \bigoplus_{m=0}^{\infty} S^{2m}$ and $\mathbb{C}[x,y]_{\mathrm{ev}} = \bigoplus_{m=0}^{\infty} S^{2m}$ coincide and hence $\mu \circ \pi_{-3} : \mathscr{U}_{-3} \to \mathbb{C}[x,y]_{\mathrm{ev}}$ is an equivariant algebra isomorphism if and only if it is non-trivial on any irreducible component. But that is clear if we simply evaluate at e^k: $\mu(e^k) = (\mu(e))^k = 2^{-k} y^{2k}$ and Schurs Lemma implies now the isomorphy between $\mathcal{R}^m$ and $\mathbb{C}[x,y]_{2m}$ and so between $\mathscr{U}_{-3}$ and $\mathbb{C}[x,y]_{\mathrm{ev}}$.
The Moyal bracket $\{\cdot,\cdot\}_m : S^m \otimes S^m \to \mathbb{C}$ defines a symmetric invariant bilinear form if m is even and antisymmetric if m is odd. This can be extended to an invariant inner product on $\mathbb{C}[x,y]_{\mathrm{ev}}$ by $\langle f, g\rangle = \mathrm{pr}_{\mathbb{C}}(f * g)$. The invariance has two consequences:

- The operator $\mathrm{ad}_{\mathrm{cas}}$ is symmetric. This implies $S^k \perp S^l$ for $k \neq l$ since two different $\mathrm{ad}_{\mathrm{cas}}$-eigenspaces have to be orthogonal.
- The operator ad_h is antisymmetric. This gives that $\mathrm{Eig}(\mathrm{ad}_h, \nu) \perp \mathrm{Eig}(\mathrm{ad}_h, \nu')$ unless $\nu = -\nu'$.

And so for some $\mathrm{Eig}(\mathrm{ad}_{\mathrm{cas}}, 2m(2m+1)) \cap \mathrm{Eig}(\mathrm{ad}_h, 2(m-k)) = \mathbb{C}\, x^k y^{2m-k}$ the only non-orthogonal elements are multiples of $x^{2m-k} y^k$. This consideration also gives that the so defined bilinear form is non-degenerate.

6.3 Composition Formula for $\mathrm{End}\, S^d$ in $\mathscr{U}$

The standard representation ρ on S^d is nothing but the map $\rho : \mathfrak{sl}_2\,\mathbb{C} \to \mathrm{End}(S^d)$ given by

$$h \mapsto \lambda^{(1)}_{xy} = y\partial_y - x\partial_x \ , \quad e \mapsto \frac{1}{2}\lambda^{(1)}_{y^2} = -y\partial_x \ , \quad f \mapsto -\frac{1}{2}\lambda^{(1)}_{x^2} = -x\partial_y.$$

The induced representation on $\mathrm{End}(S^d)$ itself decomposes it as $\mathrm{End}\, S^d = \bigoplus_{k=0}^{d} S^{2k}$. This can also be seen using the homomorphism $\lambda^{(k)} : S^{2k} \to \mathrm{End}(S^d)$ mapping $f \mapsto \{f,\cdot\}_k$, i.e. $\lambda^{(k)}_f(g) = \{f,g\}_k$. Just as the Moyal product also the map

$$\lambda : \bigoplus_{k=0}^{d} S^{2k} \to \mathrm{End}(S^d)$$

is equivariant. The pullback of the Killing metric on $\mathrm{End}(S^d)$ via λ differs on each component by a conformal factor, this is a consequence of the invariance of both metrics.

The endomorphism space is an $\mathfrak{sl}_2\,\mathbb{C}$-representation via ρ which lifts to an homomorphism of algebras $\mathscr{U} \to \mathrm{End}(S^d)$. Here, we can compute

$$\rho(\mathrm{cas}) = \rho(h)^2 + 2(\rho(e)\rho(f) + \rho(f)\rho(e)) = (y\partial_y - x\partial_x)^2 + 2(y\partial_x x\partial_y + x\partial_y y\partial_x) = d(d+2)$$

on S^d which becomes clear when applied to an element, e.g. x^d. And so ρ factorises through the Verma module $\mathscr{U}_{d(d+2)}$. We now have found two maps with co-domain $\mathrm{End}(S^d)$: the algebra homomorphism $\rho : \mathscr{U} \to \mathrm{End}(S^d)$ and $\lambda : \mathbb{C}[x,y]_{\mathrm{ev},\leq 2d} = \bigoplus S^{2m} \to \mathrm{End}(S^d)$. We want to define a map $R : \oplus S^{2m} \to \mathscr{U}$ such that $\lambda = \rho \circ R$. We have

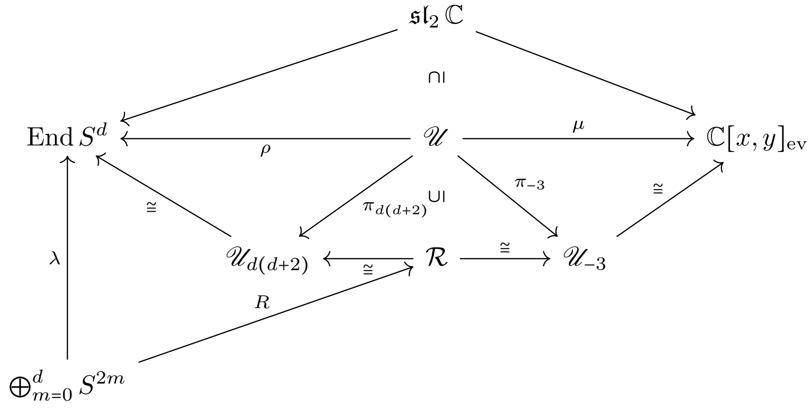
$$\lambda^{(m)}_{(y^2/2)^m} = \frac{(-1)^m}{2^m\, m!}\frac{(2m)!}{m!}\, y^m \partial_x^m \quad \text{and} \quad \rho(e^m) = \rho(e)^m = \left(\lambda^{(1)}_{y^2/2}\right)^m = \left(-\frac{1}{2}\, y\partial_x\right)^m = \left(-\frac{1}{2}\right)^m y^m \partial_x^m$$

and so demand $R(y^{2m}/2^m) = \frac{(2m)!}{(m!)^2}\, e^m$ which we extend equivariantly to

$$R(x^k y^{2m-k}) = (-1)^k \frac{(2m-k)!}{(m!)^2}\, \mathrm{ad}_f^k\, e^m.$$

In particular, R takes values in $\mathcal{R} = \bigoplus \mathcal{R}^m \subseteq \mathscr{U}$ which itself is isomorphic to any $\mathscr{U}_c$ by the projection π_c. Although $\mathcal{R}$ is only a subspace and not a subalgebra of $\mathscr{U}$ the map ρ remains multiplicative on $\mathcal{R}$.

All the presented maps are again collected in the following diagram



The map $\mu \circ \pi_{-3} \circ R : \mathbb{C}[x,y]_{\mathrm{ev},\leq 2d} = \bigoplus S^{2m} \to \mathbb{C}[x,y]_{\mathrm{ev}}$ is for representational reasons a rescaling on each summand which we can compute by

$$\left(\frac{y^2}{2}\right)^m \overset{R}{\longmapsto} \frac{(2m)!}{(m!)^2}\, e^m \overset{\mu\circ\pi_{-3}}{\longmapsto} \frac{(2m)!}{(m!)^2}\left(\frac{y^2}{2}\right)^m.$$

The idea is now to relate the formula

$$\lambda_f^{(n)}\lambda_g^{(m)} = \sum_{l=0}^{d} \gamma_{n,m}^{l}(2n,2m,d)\,\lambda_{\{f,g\}_l}^{(n+m-l)}\;, \quad f \in S^{2n},\; f \in S^{2m}$$

with $\mathscr{U}$. We should point out here that we necessarily have to restrict ourselves to the endomorphism $\lambda_f^{(n)}$ with $f \in S^{2n}$, i.e. $\deg f = 2n$, and so can only to relate the coefficients $\gamma_{n,m}^{l}(2n,2m,d)$ for all n, m, d to $\mathscr{U}$. The above formula is translated to $\mathcal{R}$ as

$$R(f)R(g) = \sum_{l=0}^{d} c_{n,m}^{l} R(\{f,g\}_l), \quad c_{n,m}^{l} = \gamma_{n,m}^{l}(2n,2m,d) = (-1)^{n+m-l} c_{m,n}^{l}.$$

We can be a litte more explicit considering $f = x^n y^n$, $g = x^m y^m$ so that

$$\begin{aligned}\{x^n y^n, x^m y^m\}_l &= \sum_{j=0}^{l} \frac{(-1)^{l-j}}{j!(l-j)!}\,\partial_x^j x^n\,\partial_y^{l-j} y^n\,\partial_x^{l-j} x^m\,\partial_y^j y^m\\ &= \sum_{j=0}^{l} \frac{(-1)^{l-j}}{j!(l-j)!}\,\frac{n!}{(n-j)!}\,\frac{n!}{(n-(l-j))!}\,\frac{m!}{(m-(l-j))!}\,\frac{m!}{(m-j)!}\,x^{n+m-l}y^{n+m-l} = a_{n,m}^{l}\,x^{n+m-l}y^{n+m-l}.\end{aligned}$$

With the notation $R(x^l y^l) = \frac{(-1)^l}{l!}\,\mathrm{ad}_f^l\, e^l = \frac{(-1)^l}{l!}\,p_l(h)$ we intend to emphazise that it is just a polynomial in h over $\mathbb{C}[\mathrm{cas}]$. We have found

$$p_n(h)p_m(h) = \sum_{l=0}^{n+m} c_{n,m}^{l} a_{n,m}^{l}\, p_l(h) \quad \text{in } \mathscr{U}_{-3} = \mathcal{R}, \text{ with coefficients } a_{n,m}^{l} \text{ from above.}$$

More general we have in $\mathcal{R}$

$$p_n(h)p_m(h) = \sum_{l=0}^{n+m} c_{n,m}^{l} A_{n,m}^{l}\, p_l(h) \quad \text{in } \mathscr{U} \text{ with } A_{n,m}^{l} \in \mathbb{C}[\mathrm{cas}].$$

This translates information about the coefficients $\gamma_{n,m}^{l}(2n,2m,d)$ to information in $\mathscr{U}$ about p_l. Since $\{p_l\}$ is a basis of the space of polynomials in h over $\mathbb{C}[\mathrm{cas}]$ we are interested in the decomposition of two-fold product $p_n p_m$ in terms of that basis which can easily generalised to many-fold products of the ps. This is the reason why we have to invest some time in the understanding of the p which lead us to [BF80] and description in terms of the Legendre polynomials.

Firstly we shall extend it to an equivariant polynomial $p_l : \mathfrak{sl}_2\,\mathbb{C} \to \mathscr{U}$:

1. We extend homogeneuously to $\mathbb{C}\,h \subseteq \mathfrak{sl}_2\,\mathbb{C}$: For any $\alpha \in \mathbb{C}$, we define $p_k(\alpha h) \coloneqq \alpha^k p_k(h)$.
2. We extent equivariantly to the orbit $\mathbb{C}\ \mathrm{Sl}_2\,\mathbb{C} \cdot h$: For any $g \in \mathrm{Sl}_2\,\mathbb{C}$, we demand $p_k(\alpha h^g) = \alpha^k (p_k(h))^g$. If $g \in \mathrm{Stab}\,h$ fixes h it also fixes $p_k(h)$ so that this step is valid.
   This polynomial map is uniformly continuous over compacta and so we can proceed the next step:
3. We extend continuously to all of $\mathfrak{sl}_2\,\mathbb{C}$ as $\mathbb{C}\ \mathrm{Sl}_2\,\mathbb{C} \cdot h$ is dense in there.

This way we have obtained an $\mathrm{Sl}_2\,\mathbb{C}$-equivariant and $\mathbb{C}$-homogeneous polynomial expression $\mathfrak{sl}_2\,\mathbb{C} \to \mathscr{U}$ of degree m which actually has values in the subrepresentation $\mathcal{R}^m$. It is a polynomial in the following sense:

Definition 6.11. Let $\sigma_1, \sigma_2, \sigma_3$ be any orthonormal basis on $\mathfrak{sl}_2\,\mathbb{C}$ and let a_1, a_2, a_3 be the associated coordinates on $\mathfrak{sl}_2\,\mathbb{C}$, i.e. any $a = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$ for any $a \in \mathfrak{sl}_2\,\mathbb{C}$. A monomial $\mathfrak{sl}_2\,\mathbb{C} \to \mathscr{U}$ of degree k is the contraction of a rank k tensor $P = (P_{i_1 \ldots i_k})$ with the basis

$$\sum_{i_1,\ldots,i_k=1}^{3} P_{i_1\ldots i_k} a_{i_1}\sigma_{i_1} \cdot \ldots a_{i_k}\sigma_{i_k}.$$

By the equivariance of the expression this is indepenend of the choice of basis. By a polynomial we mean a sum of such monomials.

The constructed polynomials p_m are indeed of this shape: Since $p_m(h) \in \mathcal{R}^m = \mathrm{im}(S^m\mathbb{C}^3 \to \mathscr{U})$ there is a symmetric tensor L of rank m such that $p_m(h) = \sum_{i_1,\ldots,i_k=1}^{3} L_{i_1\ldots i_m}\sigma_{i_1}\ldots\sigma_{i_m}$. Homogenisation as well as extension by equivariance preserve this form so that the polynomial on $\mathfrak{sl}_2\,\mathbb{C}$ corresponds to the tensor L. The construction does not only prove the existence of such polynomial p_m is also shows uniqueness:

Lemma 6.12. *For any $m \in \mathbb{N}_0$ there is, up to a constant factor, a unique equivariant, homogeneuous polynomial $\mathfrak{sl}_2\,\mathbb{C} \to \mathcal{R}^m$.*

Proof. The previous construction built p_m up only from its value $p_m(h)$ which was in $\mathbb{C}[\mathrm{cas}, h]$. Consequently, we only have to show that such a polynomial fulfills $q(h) \in \mathbb{C}[h, \mathrm{cas}]$. But this means nothing else but $q(h) \in \mathcal{R}^m \cap \mathrm{Eig}(\mathrm{ad}_h, 0)$ which follows directly from the equivariance: $\exp(h)$ necessarily acts trivially on $p_m(h)$ whose differentiation shows $\mathrm{ad}_h\, p_m(h) = 0$. □

6.4 Relation between the γ-Coefficients and the Legendre Polynomials

We can use the correspondence between the polynomials and the traceless symmetric tensors to give these polynomials more or less explicit: To any symmetric tensor we can associate a polynomial by

$$\vartheta(T)(x_1,\ldots,x_d) = \sum_{i_1,\ldots,i_m=1}^{d} T_{i_1\ldots i_m} x_{i_1}\cdot\ldots\cdot x_{i_m}$$

as we have done already on our way from S^mV to $\mathscr{U}$. Under this identification a traceless tensor corresponds to a harmonic polynomial:

$$\begin{aligned}
\Delta\vartheta(T)(x_1,\ldots,x_d) &= \sum_{j=1}^{d}\partial_j^2 \sum_{i_1,\ldots,i_m=1}^{d} T_{i_1\ldots i_m}\, x_{i_1}\cdot\ldots\cdot x_{i_m} \\
&= \sum_{j=1}^{d}\sum_{i_1,\ldots,i_m=1}^{d} T_{i_1\ldots i_m}\left(\delta_j(i_1)\delta_j(i_2)x_{i_3}\cdot\ldots\cdot x_{i_m} + \text{symm. in } (i_1\ldots i_m)\right) \\
&= m(m-1)\sum_{j=1}^{d}\sum_{i_1,\ldots,i_m=1}^{d} T_{jji_3\ldots i_m}\, x_{i_3}\cdot\ldots\cdot x_{i_m} \\
&= m(m-1)\sum_{i_3,\ldots,i_m=1}^{d} (\mathrm{tr}\,T)_{i_3\ldots i_m}\, x_{i_3}\cdot\ldots\cdot x_{i_m} \\
&= m(m-1)\,\vartheta(\mathrm{tr}\,T)(x_1,\ldots,x_d).
\end{aligned}$$

Consequently our aim is to find a harmonic, symmetric and homogeneuous polynomial of degree m. Such polynomials are known as spherical harmonics of which we shall consider only the solid Legendre polynomials: We define via

$$\ell_m(x) = \sqrt{n}^m \, |x|^m \, l_m\left(\frac{x_1 + \ldots + x_n}{\sqrt{n}\,|x|}\right)$$

a symmetric, harmonic and homogeneuous polynomial of degree m on $\mathbb{R}^n$. Here, l_m is the mth Legendre polynomial which are equivalently defined by one of the following prescribtions

- $(m+1)l_{m+1}(t) = (2m+1)t\,l_m(t) - m\,l_{m-1}(t)$ with $l_0(t) = 1$ and $l_1(t) = t$
- $l_m(t) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} l_{mk}\, t^{m-2k}$ where $l_{mk} = \frac{(-1)^k}{2^k} \frac{(2n-2k)!}{(n-k)!(n-2k)!k!} = \frac{(-1)^k}{2^k} \binom{2n-2k}{n-k, n-2k, k}$
- $t^m = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} t_{mk}\, l_{m-2k}(t)$ where $t_{mk} = \frac{(2m-4k+1)m!}{2^k k!(2m-2k+1)!!}$

Correspondingly, the tensor L_m associated to ℓ_m

$$\ell_m(x) = \sqrt{n}^m \, |x|^m \, l_m\left(\frac{x_1 + x_2 + \ldots + x_n}{\sqrt{n}\,|x|}\right) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} l_{mk} n^k \, (x_1^2 + \ldots + x_n^2)^k (x_1 + x_2 + \ldots + x_n)^{m-2k}$$

is symmetric and traceless and thus pushed to $\mathscr{U}$ defines a multiple of p_m. Unfortunately, we have not been able to give an explicit formala for the p's and so used a computer programm as in the appendix to find $p_m(h) = \mathrm{ad}_f^m \, e^m$

- $p_0(h) = 1$
- $p_1(h) = -h$
- $p_2(h) = 3h^2 - \mathrm{cas}$
- $p_3(h) = -15h^3 + 3(-4 + 3\mathrm{cas})h$
- $p_4(h) = 105h^4 + 30(10 - 3\mathrm{cas})h^2 + 9(\mathrm{cas}^2 - 8\mathrm{cas})$
- $p_5(h) = -945h^5 - 1050(6 + \mathrm{cas})h^3 + 15(-15\mathrm{cas}^2 + 200\mathrm{cas} - 192)h$

In the paper [BF80] by F. Bayen and C. Fronsdal the considered polynomials are $R(x^m y^m)$ which is in our convention $p_m(h) = (-1)^m m!\, R(x^m y^m)$, the adopted recurrence relation for p_m combined with our conventions for the Killing form on $\mathfrak{sl}_2\,\mathbb{C}$ is

$$p_{m+1}(a) = (2m+1)\, a\, p_m(a) + m^2(m^2 - 1 - \mathrm{cas})|a|^2\, p_{m-1}(a), \quad a \in \mathfrak{sl}_2\,\mathbb{C}.$$

There is clearly some relation to the recurrence relation for l_m. A lot of work has been invested in the Legrende polynomials and so it is no surprise that there are explicit formulas for

$$l_m(t) l_n(t) = \sum_{j=0}^{m+n} z_{n,m}^j l_j(t), \quad \text{where} \quad z_{mn}^j = \frac{2(m+n) - 4j + 1}{2(m+n) - 2j + 1} \frac{\frac{(j-\frac{1}{2})!}{j!} \frac{(m-j-\frac{1}{2})!}{(m-j)!} \frac{(n-j-\frac{1}{2})!}{(n-j)!}}{\frac{(m+n-j-\frac{1}{2})!}{(m+n-j)!}}$$

with some complicated but combinatorical expression for the coefficients $z_{n,m}^l$. There are several articles in that direction, e.g. [AlS56]. The discussion ends here since we have then focussed on the recurrence formula for Q. The knowledge about the Legendre polynomials and the universal enveloping algebra might bring some light in the dark about the coefficients and to the bilinear form Q and that way motivate the bigger picture behind in which we can solve Nahms equations for nilpotent orbits of maximal rank in $\mathrm{End}(S^d)$ with already developed algebraic terminology.

7 Outlook

It was mentioned already that the presented approach is not fully developped yet, and so it is not difficult to list gaps and ToDo's. There are two main points here to discuss: How the open end could be closed and how this method may be generalised to other orbits or other Lie algebras.

From our point of view it does not make too much sense to start brute-force computations of the recurrence relation of Q. Of course, a computer can do that but we will not be able to interpret the results, yet. We have to develop the algebraic language further to understand what is going on with the nilpotent orbits. And that is why it looks most promising to search for a well-developped algebraic structure generalising the universal enveloping algebra in such that way that we can not only formulate relations between λ's as endomorphisms but also between λ's as homomorphisms. We are optimisitic that in this language the problem of solving the Q-recurrence is easier, controlable and interpretable.
Reducing our high standards for the moment we would already be happy with a closed formula for the γ's so that we maybe can go on with the computation. Here we have two possibilies to start with: On the one hand, a python program can compute these coefficients and we try to guess formulas. This is doable, e.g. for $\lambda^{(0)}\lambda^{(n)}$, but tricky if mix-terms appear as in $\lambda^{(1)}\lambda^{(n)}$. Ideally this path is combined with the knowledge we can collect from the Legendre polynomial-approach: When we fully understand how the symmetrisation, i.e. the map from the polynomials to the enveloping algebra, works we can use the relations between the Legendre polynomials and translate them for our problem. Even though this will just give some γ's it might indicate in which direction we have to look.

So far we have restricted ourselves to the standard representation of $\mathfrak{sl}_2\mathbb{C}$ on $\mathrm{End}(S^d)$ which corresponds to the nilpotent element given by the Jordan block of maximal rank. We can approach the other orbits in the following way: Due to complex linear algebra we know that we can choose e to be in Jordan normal form. The Jacobson-Morozov theorem states that we can complete e to an $\mathfrak{sl}_2\mathbb{C}$-triple $\{e, f, h\}$ and that way compute the decomposition of $\mathrm{End}(S^d)$ with respect to this $\mathfrak{sl}_2\mathbb{C}$-trepresentation. We can even be more precise here: If we denote by J_k the nilpotent Jordan block of rank $k-1$, we decompose $\mathrm{End}(S^4)$, e.g. as

- For $e = \mathrm{diag}(J_3, J_2)$ we find $\mathrm{End}(S^d) = \mathbb{C}^2 \otimes S^0 \oplus \mathbb{C}^2 \otimes S^1 \oplus \mathbb{C}^2 \otimes S^2 \oplus \mathbb{C}^2 \otimes S^3 \oplus S^4$
- For $e = \mathrm{diag}(J_3, J_1, J_1)$ we find $\mathrm{End}(S^d) = S^0 \oplus S^2 \oplus \mathbb{C}^2 \otimes S^3 \oplus S^4 \oplus S^6$

And so the problem we have to deal with becomes clear: The different components do not only occur once as for the standard representation but several times. However we are convinced this can be done with some linear algebra tricks about which we think later.

A Gauge Transformations and the Baker–Campbell–Hausdorff formula

So far we have used quite often the existence of some gauge element with addition properties. If it is about gauging a compontent to the standard solution or to zero, to solution of the real equation. In this section we shall introduce a method how such a gauge can be computed systematically - for a limited number of very special cases.

We consider a two-dimensional Lie algebra of some Lie group spanned by ξ_1 and ξ_2, this means in particular that $[\xi_1, \xi_2] = r_1\xi_1 + r_2\xi_2$. The unique gauge element $g : \mathbb{R} \to G$ which eliminates $\xi = a_1\xi_1 + a_2\xi_2$, i.e. $g.\xi = \mathrm{Ad}_g\,\xi - \dot{g}g^{-1} = 0$, with initial value $g(0) = 1$ is given by

$$g(t) = \exp(t(a_1\xi_1 + a_2\xi_2)).$$

This relies on the fact that ξ is time-independed so that we can easily compute the derivative of g.
Another approach could be the following: We try to eliminate the ξ_1-component in ξ and the ξ_2-component after each other. This can be by firstly gauging ξ with $g_1(t) = \exp(\kappa_1(t)\xi_1)$ and then with $g_2(t) = \exp(\kappa_2(t)\xi_2)$ both with $\kappa_i(0) = 0$. The uniqueness tell us then that $g = g_2g_1$. In more concrete terms:

$$\exp(\kappa_1\,\xi_1).(a_1\xi_1 + a_2\xi_2) = \exp(\kappa_1\,\mathrm{ad}_{\xi_1})(a_1\xi_1 + a_2\xi_2) - \dot{\kappa}_1\xi_1 = a_2\,\exp(\kappa_1\,\mathrm{ad}_{\xi_1})\xi_2 + (a_1 - \dot{\kappa}_1)\xi_1$$

All the nested commutators of ξ_1 and ξ_2 remain in the Lie algebra so that there are functions $w_1 = w_1(t)$ and $w_2 = w_2(t)$ such that $\exp(\kappa_1(t)\,\mathrm{ad}_{\xi_1})\xi_2 = w_1(t)\xi_1 + w_2(t)\kappa_2(t)$, here w_1 and w_2 can be computed in terms of the commutation relations and κ_1. The above equation becomes now

$$\exp(\kappa_1(t)\,\xi_1).(a_1\xi_1 + a_2\xi_2) = a_2w_2(t)\,\xi_2 + (a_1 + a_2w_1(t) - \dot{\kappa}_1(t))\xi_1$$

so that the choice of $\dot{\kappa}_1 = a_1 + a_2w_1$ with $\kappa_1(0) = 0$ determines κ_1 uniquely.
By that step we have decreased the dimension of the considered Lie algebra by 1 so that it remains to find $g_2 = \exp(\kappa_2\,\xi_2)$. Due to

$$\exp(\kappa_2(t)\,\xi_2).\exp(\kappa_1(t)\,\xi_1).(a_1\xi_1 + a_2\xi_2) = \exp(\kappa_2\,\xi_2).(a_2w_2(t)\,\xi_2) = (a_2w_2(t) - \dot{\kappa}_2(t))\,\xi_2$$

the choice $\dot{\kappa}_2 = a_2w_2$ with $\kappa_2(0) = 0$ is necessary.
We have now found two gauges, g and g_2g_1, that solve the same differential equation $g.\xi = 0$ and $g(0) = 1$ and so they coincide

$$\exp(t(a_1\xi_1 + a_2\xi_2)) = \exp(\kappa_2(t)\,\xi_2)\exp(\kappa_1(t)\,\xi_1).$$

This is nothing but a Baker-Campell-Hausdorff problem. Furthermore, we may be lucky and find the map $(a_1, a_2) \mapsto (\kappa_1(1), \kappa_2(t))$ to be invertible. This then would correspond to a formula for

$$\exp(\xi_2)\exp(\xi_1) = \exp(b_1\xi_1 + b_2\xi_2).$$

Although the complexity of such problems increases rapidly, here we would have to solve differential equations explicitely and hope for controlable solutions, this approach generalises to computations in higher-dimensional Lie algbras. But not to all of them, we have to have a linear basis $\{\xi_1, \dots, \xi_n\}$ such that

$$\mathrm{span}\{\xi_1, \dots, \xi_k\} \subseteq \mathfrak{g} \quad \text{is a } k\text{-dimensional Lie subalgebra of } \mathfrak{g}$$

for any $k \leq n$, i.e. $\mathfrak{g}$ is solvable. The proof just follows the above presented arguments recursively.

Let us compute some (known) examples. The first two examples are just as in the example, X and Y span a two-dimensional Lie algebra. This was also computed by A. Van-Brunt and M. Visser presented in [VV15], which looks from our point of view more complicated to our approach. The last and third example will deal with the Lie algebra $\mathfrak{sl}_2\,\mathbb{C}$, explained M. Matone in [Mat16]. He reduces the problem to the two-dimensional case and uses results from there. With our restrictions on the Lie algebras the example of $\mathfrak{sl}_2\,\mathbb{C}$ is already pretty general for the three-dimensional case: The Jacobi-equality together with the existence of the chain of Lie subalgebras gives determing dependences between possible coefficients in the commutation relations.

Example 1: X, Y with $[X, Y] = \lambda X$

Let's start with the situation of a two dimensional Lie algebra spanned by X and Y where $\mathrm{ad}_Y X = \lambda X$ and λ is a non-zero real number. In this case it does not matter which component is firstly gauged away and so we just start with the computation: It is our aim to decompose

$$\exp(aX + bY)$$

into factors. In order to do so we compute the action of $g_Y(t) = \exp(\kappa_Y(t)Y)$ on $aX + bY$:

$$\exp(\kappa_Y \,\mathrm{ad}_Y)(aX + bY) = a \exp(\kappa_Y \,\mathrm{ad}_Y)X + bY = ae^{\lambda\kappa_Y} X + bY$$

and so the choice $\kappa_Y(t) = bt$ gives $g_Y(t) = \exp(-btY)$ and it remains to compute $\dot{\kappa}_X(t) = ae^{-\lambda b t}$ which is solved by $\kappa_X(t) = -\frac{a}{bt}(e^{-\lambda b t} - 1)$ in the right way. Evaluating at $t = 1$ we have found

$$\exp(aX + bY) = \exp(-a\,\frac{e^{-\lambda b} - 1}{b}\,X)\ \exp(bY).$$

Now, let's turn the question around and produce the coefficients such that $\exp(\alpha X)\exp(\beta Y) = \exp(aX+bY)$. This means nothing but solving the system of equations

$$\begin{cases} -a\,\frac{e^{-\lambda b}-1}{b} & = \alpha \\ b & = \beta \end{cases}$$

for (a, b). That's not too difficult and we obtain

$$\exp(\alpha X)\exp(\beta Y) = \exp(\frac{\alpha\beta}{1 - e^{-\lambda\beta}} + \beta Y).$$

Example 2: X, Y with $[X, Y] = xX + yY$

First things first, we can assume that both x and y are non-zero since this case was handled earlier alread. Now we take care about $\exp(\mathrm{ad}_Y)(aX + bY)$. Let $\mathrm{ad}_Y^k(aX + bY) = a_k X + b_k Y$ so that

$$a_{k+1}X + b_{k+1}Y = \mathrm{ad}_Y^{k+1}(aX + bY) = \mathrm{ad}_Y(a_k X + b_k Y) = a_k(xX + yY) = a_k xX + a_k yY$$

and we can read off

$$\begin{cases} a_{k+1} = xa_k, & a_0 = a \\ b_{k+1} = ya_k, & b_0 = b, b_1 = y \end{cases} \quad \text{to find} \quad \begin{cases} a_k = ax^k \\ b_k = a\frac{y}{x}\,x^k + \delta_0(k)\,\frac{bx-ay}{x} \end{cases}$$

where δ represents the Delta-Kronecker-Tensor given by $\delta_k(n) = 1$ if and only if $k = n$ and $\delta_k(n) = 0$ otherwise. This leads us to

$$\exp(\kappa_Y \,\mathrm{ad}_Y)(aX + bY) = ae^{\kappa_Y x}X + \frac{1}{x}\,(ay\,e^{\kappa_Y x} + (bx - ay))Y$$

what requires us to solve

$$x\dot{\kappa}_Y = ay\,e^{\kappa_Y x} + (bx - ay) \quad \Longleftrightarrow \quad 1 = -\frac{-x\dot{\kappa}_Y e^{-x\kappa_Y}}{ay + (bx - ay)e^{-x\kappa_Y}} = -\frac{1}{bx - ay}\frac{\mathrm{d}}{\mathrm{d}t}\log(ay + (bx - ay)e^{-x\kappa_Y}).$$

And so we find κ_Y to be given implicitly by

$$\log\frac{ay + (bx - ay)e^{-x\kappa_Y(t)}}{bx} = -(bx - ay)t$$

which his in turn equivalent to the explicit form

$$\kappa_Y(t) = -\frac{1}{x}\log\left(\frac{bxe^{-(bx-ay)t} - ay}{bx - ay}\right)$$

If $bx - ay = 0$ we have not inhomogeneity in the differential equation to that $\kappa_Y(t) = -\frac{1}{x}\log(1 - ayt)$. Now for κ_X, it has to be chosen in such a way that

$$\dot\kappa_X = ae^{x\kappa_Y} = a\,\frac{x\dot\kappa_Y - (bx - ay)}{ay} = \frac{1}{y}\left(x\dot\kappa_Y - (bx - ay)\right)$$

where we have linked $e^{x\kappa_Y}$ with $\dot\kappa_Y$ via the differential equation. And so we have determined κ_X to be

$$\kappa_X(t) = \frac{x}{y}\,\kappa_Y(t) - \frac{bx - ay}{y}\,t = -\frac{1}{y}\log\left(\frac{bxe^{-(bx-ay)t} - ay}{bx - ay}\right) - \frac{bx - ay}{y}\,t = -\frac{1}{y}\log\left(\frac{bx - aye^{(bx-ay)t}}{bx - ay}\right)$$

And so we have finally computed

$$\exp(atX + btY) = \exp(\kappa_X(t)\,X)\exp(\kappa_Y(t)\,Y).$$

Let's proceed by inverting the picture and solve

$$\begin{cases} \alpha = \kappa_Y(1) \\ \beta = \kappa_X(1) \end{cases}$$

When writing $r = bx - ay$ we can simplify the second equation by using $\beta = \kappa_X = \frac{x}{y}\,\alpha - \frac{r}{y}$. And so we find $r = x\alpha - \beta y$. This empowers uns to rewrite the equation for κ_Y and solve for a

$$re^{-x\alpha} = bxe^r - ay = (r + ay)e^r - ay = ay(e^r - 1) + re^r \quad\Longleftrightarrow\quad a = \frac{r}{y}\,\frac{e^{-x\alpha} - e^r}{e^r - 1}$$

and then with $x\alpha - y\beta = r = bx - ay$ for b

$$b = \frac{1}{x}\,(r + ay) = \frac{r}{x}\,(1 + \frac{e^{-x\alpha} - e^r}{e^r - 1}) = \frac{r}{x}\,\frac{e^{-x\alpha} - 1}{e^r - 1}.$$

Of course we could have changed this basis here with $[X, Y] = xX + yY$ to the basis of Example 1 or even to $[X, Y] = X$. But this would lead to a Baker-Campell-Hausdorff decomposition and so to exponentials in the new basis vectors. So again, we would have exponentials in linear combinations of X and Y which does not help us at all. That is why we intended to consider this Example 2 really as an example not as a special case. And this is also the reason why we do not change the basis in $\mathfrak{sl}_2\,\mathbb{C}$:

Example 3: Standard Basis of $\mathfrak{sl}_2\,\mathbb{C}$

Any triple X, Y, Z with real and non-trivial relations

$$[Y, X] = uX \quad [Z, X] = yY \quad [Z, Y] = uZ$$

span the Lie algebra $\mathfrak{sl}_2\,\mathbb{C}$. We can compute already all the necessary formulas

$$\exp(\kappa_Z\,\mathrm{ad}_Z)Y = Y + \kappa_Z\,\mathrm{ad}_Z\,Y + \frac{\kappa_Z^2}{2}\,\mathrm{ad}_Z^2\,Y + \ldots = Y + u\kappa_Z Z$$
$$\exp(\kappa_Z\,\mathrm{ad}_Z)X = X + \kappa_Z\,\mathrm{ad}_Z\,X + \frac{\kappa_Z^2}{2}\,\mathrm{ad}_Z^2\,X + \ldots = X + y\kappa_Z Y + \frac{yu}{2}\,\kappa_Z^2 Z$$

so that

$$\exp(\kappa_Z\,\mathrm{ad}_Z)(aX + bY + cZ) = aX + (ay\kappa_Z + b)\,Y + (a\,\frac{yu}{2}\,\kappa_Z^2 + bu\kappa_Z + c)Z.$$

In the next step we use

$$\exp(\kappa_Y\,\mathrm{ad}_Y)X = e^{u\kappa_Y}\,X$$
$$\exp(\kappa_Y\,\mathrm{ad}_Y)Z = e^{-u\kappa_Y}\,Z.$$

which leads us to the differential equations for the κ:

$$\begin{aligned}\dot\kappa_Z &= a\,\frac{uy}{2}\,\kappa_Z^2 + bu\kappa_Z + c = \frac{ayu}{2}\left(\kappa_Z + \frac{b}{ay}\right)^2 + \left(c - \frac{ub^2}{2ay}\right)\\ \dot\kappa_Y &= ay\kappa_Z + b\\ \dot\kappa_X &= ae^{u\kappa_Y}\,.\end{aligned}$$

We abbreviate $p = \frac{ayu}{2}$ and $q = c - \frac{ub^2}{2ay}$ to find

$$\begin{aligned}&\kappa_Z(t) = \sqrt{\frac{q}{p}}\,\tan(\sqrt{pq}\,(t-\tau)) - \frac{b}{ay}\,, \quad \tau = \frac{1}{\sqrt{pq}}\,\arctan\left(\sqrt{\frac{p}{q}}\,\frac{b}{ay}\right)\\ &\dot\kappa_Y(t) = ay\sqrt{\frac{q}{p}}\,\tan(\sqrt{pq}\,(t-\tau)) \quad\Longrightarrow\quad \kappa_Y(t) = -\frac{ay}{p}\,\log|\cos(\sqrt{pq}\,(t-\tau))| = -\frac{2}{u}\,\log|\cos(\sqrt{pq}\,(t-\tau))|\\ &\dot\kappa_X(t) = \frac{a}{\cos^2(\sqrt{pq}\,(t-\tau))} \quad\Longrightarrow\quad \kappa_X(t) = \frac{a}{\sqrt{pq}}\,\tan(\sqrt{pq}\,(t-\tau))\end{aligned}$$

This shape relies on $pq > 0$, if $pq < 0$ the the roots are complex so that the trigonometric functions are replaced by hyperbolic functions.
The situation is a little different for $q = 0$. Here we find

$$\kappa_Z(t) = \frac{b}{ay}\left(\frac{1}{1 - \frac{bu}{2}\,t} - 1\right) \quad\Longrightarrow\quad \kappa_Y(t) = -\frac{2}{u}\,\log\left(1 - \frac{bu}{2}\,t\right) \quad\Longrightarrow\quad \kappa_X(t) = \frac{abu}{2}\left(\frac{1}{1 - \frac{bu}{2}\,t} - 1\right)$$

For the case $q \neq 0$ let us solve the system

$$\begin{cases}\gamma = \kappa_Z(1)\\ \beta = \kappa_Y(1)\\ \alpha = \kappa_X(1)\end{cases}$$

for a, b and c. From $\alpha = \kappa_X(1)$ we can directly read off that

$$\tan(\sqrt{pq}\,(t-\tau)) = \frac{a\alpha}{\sqrt{pq}} = \frac{\alpha}{\sqrt{\frac{p}{a}\frac{q}{a}}} = \frac{\alpha}{\sqrt{\frac{p}{a}\frac{q}{a}}}$$

which leads when combined with γ to

$$\gamma = \frac{a\alpha}{p} - \frac{b}{ay} = \frac{2\alpha}{uy} - \frac{1}{y}\,\frac{b}{a} \quad\Longleftrightarrow\quad \frac{b}{a} = \frac{2\alpha}{u} - y\gamma$$

The β-equation can be transformed to

$$e^{u\beta} = \frac{1}{\cos^2(\sqrt{pq}\,(t-\tau))} = 1 + \tan^2(\sqrt{pq}\,(t-\tau)) = \frac{a^2\alpha^2}{pq} = \frac{\alpha^2}{\frac{p}{a}\frac{q}{a}} \quad\Longleftrightarrow\quad \frac{e^{u\beta}}{\alpha^2}\,\frac{yu}{2} = \frac{q}{a} = \frac{c}{a} - \frac{u}{2y}\left(\frac{b}{a}\right)^2$$

And so we have determined the quotients $\frac{b}{a}$ and $\frac{c}{a}$. This is the best we can hope for as long we do not consider the tan-term explicitely since the prefactors in the κ's are invariant under rescaling a, b and c simultaneously. Consequently, we have to fix a by

$$\tan\left(a\sqrt{\frac{p}{a}\,\frac{q}{a}} - \arctan\left(\sqrt{\frac{p}{q}}\,\frac{1}{y}\,\frac{b}{a}\right)\right) = \frac{\alpha}{\sqrt{\frac{p}{a}\frac{q}{a}}}$$

The solution is messy, the discussion when the solutions for κ as well as the inverse problem is left out anyway and so with presenting a solution strategy we close this example here.

B Python-Codes

The Code for the Components in the Composition Formula

```
import sympy as sp
from sympy import *
from sympy import simplify, expand, reduced, S
import math

sp.init_printing()

a,n,b,m,p,k = sp.symbols('a n b m p k')
x,y = sp.symbols('x y')

f = x**a * y**(n-a)
g = x**b * y**(m-b)
h = x**p * y**(k-p)

d=4

#computes the coefficient of the moyal bracket of {x^ay^b,x^sy^t}_k
#wrt x^(a+s-k)y^(b+t-k) and returns it
def moyalbracket(f=0,g=0,m=0):
    result = 0
    for j in range(0,m+1):
        result += Rational ((-1)**(m-j) , factorial(j)*factorial(m-j) ) * sp.diff(f,x,j,y,m-j) * sp.diff(g,x,m-j,y,j)
    return result

coeffs={}
lambdacoeff={}
for N in range (0,d+1):
    for M in range(0,d-N+1):
#compute \lamnda ^{(N)}_f \lambda^{(M)}_g (h) = = \sum _{j=0} ^{N+M} alpha_{N+M-j} \lamnda ^{(N+M-j)} _{ {f,g}_j }
        poly = cancel(moyalbracket( f, moyalbracket(g,h,M) , N) * x**(-a-b-p+N+M) * y**(-n-m-k+a+b+p+N+M))
        for j in range (0,N+M+1):
            #computes the coeffiencent of Lambda ^{ (N+M-j)} _{ {f,g}_j } (h) = { {f,g}_j , h}_{N+M-j}
            lambdacoeff.update({(N,M,N+M-j):cancel(moyalbracket(moyalbracket(f,g,j),h,N+M-j)*x**(-a-b-p+N+M)*y**(-n-m-k+a+b+p+N
#We compute the iterations: We have already compute the first r
```

```
#coeffs={3: m*(m-1)*(m-2)/((n+m)*(n+m-1)*(n+m-2))}
        for j in range(0,N+M+1):
            i = N+M-j
#           print(j,i)
            helper=poly.as_poly(p).coeff_monomial(p**i)
#           print(helper)
            for l in range(i+1,N+M+1):
                helper = helper - coeffs[(N,M,l)] * lambdacoeff[(N,M,l)].as_poly(p).coeff_monomial(p**i)
#           print(helper)
            helper = cancel (helper / lambdacoeff[(N,M,i)].as_poly(p).coeff_monomial(p**i))
            coeffs.update({(N,M,i): helper})
```

The Code for the Computations of the Polynomials p_m

```
import sympy
from sympy import *
from sympy.abc import c,h
from sympy import eye, degree, print_latex

#c=h**2+2(ef+fe) is Casimir
pp=Rational(1,4)*(c-h**2+2*h)# ef=pp(h)
pm=Rational(1,4)*(c-h**2-2*h)# fe=pm(h)=pp(h-2)
q0=1+h-h #initial polynomial, somewhy just 1 is not accepted
dim=5
d=dim-1

#we compute all polynomials up to degree r*d
polycoeff={}
poly={}
for n in range(0,2*d+1): #runs through S**2n
    q=q0
    polycoeff.update({(n,0):q})
    for l in range(0,n):
        q=q.subs({h:(h+2)})*pm-q*pm.subs({h:h-2*(n-l)})
        q=q.simplify()
        polycoeff.update({(n,l):q})
    poly.update({ n: q })
```

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