Aperiodic dynamical inclusions of C^*-algebras

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Jonathan Peter Taylor
from New Lambton Heights, Australia

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Thesis Committee

Prof. Dr. Ralf Meyer	Mathematisches Institut Göttingen
Prof. Dr. Thomas Schick	Mathematisches Institut Göttingen
Prof. Dr. Chenchang Zhu	Mathematisches Institut Göttingen

Members of the Examination Board

Reviewer	Prof. Dr. Ralf Meyer	Mathematisches Institut Göttingen
Second Reviewer	Prof. Dr. Thomas Schick	Mathematisches Institut Göttingen

Further members of the Examination Board

Prof. Dr. Dorothea Bahns	Mathematisches Institut Göttingen
Prof. Dr. Damaris Schindler	Mathematisches Institut Göttingen
Prof. Dr. Frank Gounelas	Mathematisches Institut Göttingen
Prof. Dr. Thorsten Hohage	Institut für Numerische und Angewandte Mathematik Göttingen

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Chapter 1

Introduction

One of the most common ways to construct interesting C^*-algebras is to build crossed products from dynamical systems. Given a pair of C^*-algebras $A \subseteq B$ one can ask whether B 'acts' in a meaningful way on A, and if so can B be explicitly described in terms of these dynamics. With no further conditions on the C^*-algebras $A \subseteq B$, the answer is probably not. For this reason, many authors have explored the question by adding several helper-conditions to the inclusion $A \subseteq B$ and in various capacities have answered the question positively.

Two of these helper conditions have been so far inescapable. The first is that A should contain an approximate identity for B (*non-degeneracy*), and the second is that B should be generated as a C^*-algebra by normalisers of A. A *normaliser* of the inclusion $A \subseteq B$ is an element $m \in B$ such that m^*Am and mAm^* are contained in A. This ensures that one can consider closed A-submodules of normalisers, called *slices*, without losing information about B not encoded by these. There are typically two more ingredients in the definition of these pairs, although these are the conditions which are modified most often in papers on the topic. One usually requires that A is maximal in some way in B, and that there is a conditional expectation of some kind $P : B \to A$. The flavour of maximality changes from author to author, many papers building stronger results from the same conditions or showing similar results hold under weaker ones. The expectation is sometimes removed in favour of some other condition at surface level, but often one can build kinds of conditional expectations using the replaced condition.

One of the first bids to describe these pairs of C^*-algebras was by Kumjian [13] and Renault [22] to study commutative Cartan subalgebras. Renault considered pairs $A \subseteq B$ where A is a maximal commutative subalgebra (*masa*) in B, and there is a faithful conditional expectation $P : B \to A$. In combination with results from [13], Renault was able to describe these pairs in terms of twisted groupoid C^*-algebras for étale, locally compact, Hausdorff, second countable, effective groupoids. This was later expanded upon by Exel [7] to allow for noncommutative Cartan subalgebras by replacing (masa) with a virtual maximality condition and to describe them as reduced section algebras of Fell bundles over inverse semigroups. This was later expanded upon by Kwaśniewski and Meyer in [15], who showed that the Cartan pairs of Exel could be described as crossed products by closed and purely outer actions, as well as showing that Exel's virtual maximality condition is equivalent to a number of other conditions on the inclusion.

Another condition one may put on the pair $A \subseteq B$ include *aperiodicity*, which is explored in more detail in [16], [14], and [17]. This condition guarantees that there is at most one pseudo-expectation $E : B \to I(A)$, that is, a conditional expectation taking values in Hamana's injective hull of A. One particular subset of these expectations are expectations that take values in the local multiplier algebra $M_{\text{loc}}(A)$ of A, which embeds in the injective hull of A (cf. [9]).

In this thesis we aim to mimic results of the previously mentioned authors and papers while adjusting some of the conditions. We shall consider inclusions $A \subseteq B$ that are aperiodic and have a faithful conditional expectation taking values in the local multiplier algebra of A. One of the challenges that arises is that the conditional expectation $E : B \to M_{\text{loc}}(A)$ maps part of B isomorphically onto a subalgebra of $M_{\text{loc}}(A)$. This is undesireable since the inclusion $A \subseteq M_{\text{loc}}(A)$ is quite badly behaved, as $M_{\text{loc}}(A)$ does not have interesting dynamics on A. Thus we shall insist on a technical condition limiting the multiplicative domain of E as described by Choi [6]. This ensures that the 'intersection' between B and $M_{\text{loc}}(A)$ is as small as possible, namely is exactly A itself.

One of the main tools we use throughout this article is an analogous construction of the local multiplier algebra applied to Hilbert bimodules. Given a Hilbert A–B-bimodule X, one can construct a Hilbert $M_{\text{loc}}(A)$–$M_{\text{loc}}(B)$-bimodule, called the *local multiplier module of X*, analogously to the construction of the local multiplier algebra. We prove some useful properties of the local multiplier module of a Hilbert bimodule, following Ara and Matthiew [1]. We also show how some bimodule-specific properties interact with that local multiplier module construction.

Our main application of the local multiplier module construction is to take an aperiodic action of an inverse semigroup by Hilbert bimodules on a C^*-algebra A, and gain a corresponding Fell bundle over an inverse semigroup with unit fibre $M_{\text{loc}}(A)$. To then gain an inverse semigroup action on $M_{\text{loc}}(A)$ is not immediate, but does follow from an application of the main theorem of [3]. We can then show that this induced action on $M_{\text{loc}}(A)$ gives rise to an Exel-Cartan inclusion, and so we then have the results of [7], [15] which we can apply to this induced inclusion. We then show that with another technical condition (that the canonical conditional expectation has minimal multiplicative domain), much of the structure of the induced Cartan inclusion descends to the original action of interest.

In the second part of this thesis we consider pairs $A \subseteq B$ of C^*-algebras where the subalgebra A is commutative. We consider a generalised case of Renault's commutative Cartan subalgebras [22], which we call *essential commutative Cartan subalgebras*, where we allow for conditional expectations taking values in the local multiplier algebra of A. The construction of the Weyl groupoid and twist by Renault [21],[22] is applicable to regular and non-degenerate inclusions of commutative C^*-algebras, so there is no need to modify this. Allowing for local multiplier algebra-valued conditional expectations in these pairs gives rise to an extended class of Weyl groupoids, which by construction are always effective, but need not be Hausdorff. To analyse such groupoids, Kwaśniewski and Meyer [16] introduce the *essential groupoid C^*-algebra* with a construction that works for étale groupoids with locally compact Hausdorff unit space. We are then able to show that any essential commutative Cartan pair is isomorphic to the canonical pair arising from the Weyl twist and its essential groupoid C^*-algebra, comparing very closely with Renault's classification of commutative Cartan pairs. Moreover, for Hausdorff groupoids this construction agrees with the reduced groupoid C^*-algebra. Hence we recover the main results of Renault in [22].

We also show that automorphisms of the Weyl twist of an essential Cartan pair $A \subseteq B$ are in natural correspondence with automorphisms of B that preserve the subalgebra A. Thus dynamics on the Weyl twist translate perfectly to dynamics on the Cartan pair and vice versa.

Chapter 2

Local multiplier modules and actions

Analogous to the construction of the local multiplier algebra of a C^*-algebra (cf. [1, Definition 2.3.1]), we define the local multiplier module for Hilbert C^*-modules. We recall some key properties of the local multiplier algebra both from [1] and others that follow from short arguments. We also briefly study the relationship between a C^*-algebra and its local multiplier algebra, and show how some of these properties translate into the Hilbert module setting. Additionally, we discuss some properties specific to Hilbert (bi)modules and how these affect properties of local multiplier modules.

2.1 Local multiplier algebras

Throughout, A will denote a fixed C^*-algebra, and $\mathcal{I}_e(A)$ will be the lattice of essential ideals of A ordered by containment. Note that $\mathcal{I}_e(A)$ is directed, as the intersection of two essential ideals is itself an essential ideal.

Definition 2.1.1 ([1, Definition 2.3.1])**.** Let $I, J \in \mathcal{I}_e(A)$ with $J \subseteq I$. The restriction map $M(I) \to M(J)$ is injective and the collection of these restrictions over the net of essential ideals gives rise to the inductive limit

$$M_{\mathrm{loc}}(A) := \varinjlim_{I \in \mathcal{I}_e(A)} M(I),$$

which we call the *local multiplier algebra of A*.

By definition, the multiplier algebras $M(I)$ canonically embed into $M_{\mathrm{loc}}(A)$ for all essential ideals $I \lhd A$. Moreover, if $J \lhd A$ is not essential, the ideal $J \oplus J^\perp$ is essential and we have $M(J) \subseteq M(J) \oplus M(J^\perp) = M(J \oplus J^\perp)$, which then includes into $M_{\mathrm{loc}}(A)$. In this fashion, one may think of $M_{\mathrm{loc}}(A)$ as the C^*-algebra generated by multipliers on ideals of A. Considering that ideals of A are in bijective correspondence with open subsets of the spectrum $\hat{A}$ of A, we see that $M_{\mathrm{loc}}(A)$ is densely spanned by multipliers defined on these open subsets, hence the name *local*.

Lemma 2.1.2 ([1, Lemma 2.3.2])**.** *Let $\tilde{I} \lhd M_{\mathrm{loc}}(A)$ be an ideal. Then $\tilde{I} \cap A = \{0\}$ if and only if $\tilde{I} = 0$. Moreover, if $\tilde{I}$ is an essential ideal of $M_{\mathrm{loc}}(A)$, then $\tilde{I} \cap A$ is an essential ideal of A.*

Lemma 2.1.3 ([1, Lemma 2.3.6])**.** *For each $I \in \mathcal{I}_e(A)$ we have $M_{\mathrm{loc}}(I) = M_{\mathrm{loc}}(A)$. Let (A_i) be a family of C^*-algebras. Then*

$$M_{\mathrm{loc}}\left(\bigoplus_i A_i\right) = \prod_i M_{\mathrm{loc}}(A_i).$$

Here the product $\prod_{i \in I} A_i$ of C^*-algebras A_i consists of bounded families of elements $(a_i)_{i \in I}$, that is, each a_i is an element of the C^*-algebra A_i, and there is a constant $C \in \mathbb{R}$ such that $||a_i|| \leqslant C$ for all $i \in I$. This differs from the Cartesian product of the A_i as sets, as that product may contain unbounded families. This boundedness criterium is needed to ensure that the product carries a norm, namely the supremum norm $||(a_i)_{i \in I}|| := \sup_{i \in I} ||a_i||$.

In the setting where $A = C_0(U)$ is a commutative C^*-algebra, essential ideals are of the form $C_0(V)$ for dense open subsets $V \subseteq U$, whereby the local multiplier algebra of A loses the information of ∂V for every dense open $V \subseteq U$. In the noncommutative setting, one may think of the quotient A/I by an essential ideal $I \in \mathcal{I}_e(A)$ as the 'boundary' of I. Lemma 2.1.3 implies that even if A/I is non-trivial, the quotient $M_{\mathrm{loc}}(A)/M_{\mathrm{loc}}(I)$ is always zero.

Lemma 2.1.4. *Let $I \lhd A$ be an ideal. Then $A \cap M_{\mathrm{loc}}(I) = A \cap M(I)$, where the intersection is taken in $M_{\mathrm{loc}}(A)$.*

Proof. Since $I \oplus I^\perp$ is an essential ideal in A we have the inclusion $A \subseteq M(I \oplus I^\perp) = M(I) \oplus M(I)^\perp$ in $M_{\mathrm{loc}}(A)$. The left and right summands each embed respectively into the orthogonal summands $M_{\mathrm{loc}}(I)$ and $M_{\mathrm{loc}}(I^\perp)$, and so $A \cap M_{\mathrm{loc}}(I) \subseteq (M(I) \oplus M(I^\perp)) \cap M_{\mathrm{loc}}(I) = M(I)$. Thus $A \cap M_{\mathrm{loc}}(I) = A \cap A \cap M_{\mathrm{loc}}(I) \subseteq A \cap M(I)$. The reverse inclusion holds as $M(I) \subseteq M_{\mathrm{loc}}(I)$. □

Lemma 2.1.5. *Let $\tilde{I} \lhd M_{\mathrm{loc}}(A)$ be an ideal. Then $\tilde{I} \subseteq M_{\mathrm{loc}}(A \cap \tilde{I})$.*

Proof. Lemma 2.1.3 implies $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(\tilde{I} \cap A) \oplus M_{\mathrm{loc}}((\tilde{I} \cap A)^\perp)$. Set $K := \tilde{I} \cap M_{\mathrm{loc}}((\tilde{I} \cap A)^\perp)$. This is an ideal in $M_{\mathrm{loc}}(A)$ and so Lemma 2.1.2 gives $K \cap A = \{0\}$ if and only if $K = \{0\}$. Lemma 2.1.4 then implies $K \cap A = \tilde{I} \cap M_{\mathrm{loc}}((\tilde{I} \cap A)^\perp) \cap A = (\tilde{I} \cap A) \cap M((\tilde{I} \cap A)^\perp) = \{0\}$. Thus $\tilde{I}$ must be contained in $M_{\mathrm{loc}}((\tilde{I} \cap A))$. □

2.2 Local multiplier modules

Let A and B be C^*-algebras. If X is a Hilbert A–B-bimodule we denote the right inner product by single angle brackets $\langle \cdot, \cdot \rangle$ and the left inner product by double angle brackets $\langle\langle \cdot, \cdot \rangle\rangle$. We denote the source and range ideals $s(X) = \overline{\mathrm{span}}\{\langle x, y \rangle : x, y \in X\}$ and $r(X) = \overline{\mathrm{span}}\{\langle\langle x, y \rangle\rangle : x, y \in X\}$. By definition of these ideals, the bimodule X always induces a Morita equivalence between $s(X)$ and $r(X)$.

The bidmodule structure also allows us to classify all subbimodules in terms of their range and source ideals. If $Y \subseteq X$ is a Hilbert A–B-subbimodule, one sees readily that $Y \subseteq r(Y) \cdot X$ and $Y \subseteq X \cdot s(Y)$. This is in fact equality, since for $x \in X$ and $y, z \in Y$ we have $x \cdot \langle y, z \rangle = \langle\langle x, y \rangle\rangle \cdot z \in A \cdot Y \subseteq Y$, so $X \cdot s(Y) \subseteq Y$ (and similarly $r(Y) \cdot X \subseteq X$).

For a Hilbert A–B-bimodule X, we denote by $X^* = \{x^* : x \in X\}$ its *opposite module.* The opposite module X^* has a B–A-bimodule structure. The addition in X^* is given by $x^* + y^* = (x + y)^*$ for $x, y \in X$, the left B-action is given by $b \cdot x^* = (x \cdot b^*)^*$ for $b \in B$ and $x \in X$, and the right A-action is given by $x^* \cdot a = (a^* \cdot x)^*$ for $a \in A$ and $x \in X$. The left and right inner products are given by $\langle\langle x^*, y^* \rangle\rangle = \langle x, y \rangle$ and $\langle x^*, y^* \rangle = \langle\langle x, y \rangle\rangle$. The opposite bimodule satisfies $X \otimes_B X^* \cong r(X)$ and $X^* \otimes_A X \cong s(X)$, via maps defined on elementary tensors by $x \otimes y^* \mapsto \langle\langle x, y\rangle\rangle$ and $x^* \otimes y \mapsto \langle x, y\rangle$ respectively for $x, y \in X$.

One may always consider A equipped with the canonical bimodule structure given by left and right multiplication, and inner products $\langle a, b\rangle = a^*b$ and $\langle\langle a, b\rangle\rangle = ab^*$. The multiplier algebra $M(A)$ of A can then be identified with the operators $A \to A$ that are adjointable with respect to the right inner product, when acting from the left (see [18, Theorem 2.4]). For a more in depth introduction to Hilbert modules we recommend [18].

Notation 2.2.1. Let X and Y be right Hilbert B-modules. The rank-one operators $X \to Y$ are of the form $z \mapsto y \cdot \langle x, z\rangle$ for $x, z \in X$ and $y \in Y$. The space $\mathcal{K}(X, Y)$ of compact operators $X \to Y$ is the completion of the span of rank-one operators $X \to Y$, in the operator norm $||T|| = \sup_{||x||=1} ||Tx||$. We denote the space of adjointable operators $X \to Y$ by $\mathcal{L}(X, Y)$. If X and Y are Hilbert A–B-bimodules, and we wish to distinguish between operators that are left-adjointable and right-adjointable, we shall write

$$\begin{aligned}
\mathcal{K}_R(X, Y) &= \{f : X \to Y | f \text{ is compact in the right Hilbert module structure}\} \\
\mathcal{L}_R(X, Y) &= \{f : X \to Y | f \text{ is adjointable in the right Hilbert module structure}\} \\
\mathcal{K}_L(X, Y) &= \{f : X \to Y | f \text{ is compact in the left Hilbert module structure}\} \\
\mathcal{L}_L(X, Y) &= \{f : X \to Y | f \text{ is adjointable in the left Hilbert module structure}\}.
\end{aligned}$$

In the case where X and Y are bimodules, and no subscript of L or R is specified as in $\mathcal{K}(X, Y)$ and $\mathcal{L}(X, Y)$, we assume these refer to the spaces of maps compatible with the right Hilbert module structure of X and Y. If $X = Y$, the spaces $\mathcal{K}_*(X)$ and $\mathcal{L}_*(X)$ for $* = L, R$ are C^*-algebras with multiplication given by composition, *-operation given by taking adjoints, and operator norm. In this scenario we have $\mathcal{L}_L(X) \cong M(\mathcal{K}_L(X)) \cong M(s(X))$ and $\mathcal{L}_R(X) \cong M(\mathcal{K}_R(X)) \cong M(r(X))$ by [18, Theorem 2.4].

Let X be a Hilbert A–B-bimodule. The space $\mathcal{L}_R(s(X), X)$ carries a $M(r(X))$–$M(s(X))$-bimodule structure by defining

$$(\tau \cdot T \cdot \sigma)a = \tau(T(\sigma a)),$$

for $\tau \in M(r(X))$, $\sigma \in M(s(X))$, and $a \in s(X)$. For $S, T \in \mathcal{L}_R(s(X), X)$ we see that S^*T is an adjointable operator on $s(X)$, so uniquely defines a multiplier $\langle S, T\rangle$ of $s(X)$. Similarly, ST^* defines an adjointable operator on X which in turn uniquely defines a multiplier on $r(X) \cong X \otimes_B X^*$, which we label $\langle\langle S, T\rangle\rangle$. The assignments $(S, T) \mapsto \langle S, T\rangle$ and $(S, T) \mapsto \langle\langle S, T\rangle\rangle$ define a Hilbert $M(r(X))$–$M(s(X))$-bimodule structure on $\mathcal{L}_R(s(X), X)$.

We define a Hilbert $M(r(X))$–$M(s(X))$-bimodule structure on $\mathcal{L}_L(r(X), X)$ similarly, and this is eloquently denoted by expressing adjointable maps in $\mathcal{L}_L(r(X), X)$ in post-fix notation: for $P \in \mathcal{L}_L(r(X), X)$ and $a \in r(X)$, the map $P : r(X) \to X$ evaluated at a is written aP.

For $\tau \in M(r(X))$, $\sigma \in M(s(X))$, and $P \in \mathcal{L}_L(r(X), X)$ we define $\tau \cdot P \cdot \sigma$ as the map $a \mapsto ((a\tau)P)\sigma$. This gives $\mathcal{L}_L(r(X), X)$ a $M(r(X))$–$M(s(X))$-bimodule structure. For $P, Q \in \mathcal{L}_L(r(X), X)$ the compositions P^*Q and PQ^* define unique multipliers $\langle P, Q\rangle \in M(s(X))$ and $\langle\langle P, Q\rangle\rangle \in r(X)$ respectively, which give rise to a Hilbert $M(r(X))$–$M(s(X))$-bimodule structure on $\mathcal{L}_L(r(X), X)$.

There are two isomorphisms $X \cong \mathcal{K}_R(s(X), X)$ and $X \cong \mathcal{K}_L(r(X), X)$ given by mapping $x \in X$ to the respective left and right creation operators of x. That is, $x \mapsto |x\rangle \in \mathcal{K}_R(s(X), X)$, where $|x\rangle$ is the function mapping $b \in s(X)$ to $|x\rangle b = x \cdot b$. Similarly $\langle\langle x| \in \mathcal{K}_L(r(X), X)$ is the operator mapping $a \in r(X)$ to $a\langle\langle x| := a \cdot x$. Denote the inverses of these isomorphisms by $\Xi_R : \mathcal{K}_R(s(X), X) \to X$ and $\Xi_L : \mathcal{K}_L(r(X), X) \to X$ respectively.

Lemma 2.2.2. *Let X be a Hilbert A–B-bimodule. Then $\mathcal{L}_R(s(X), X)$ and $\mathcal{L}_L(r(X), X)$ are isometrically isomorphic as Hilbert $M(s(X))$–$M(r(X))$-bimodules via the map taking $T \in \mathcal{L}_R(s(X), X)$ to the operator $\check{T} : r(X) \to X$ given by*

$$r(X) \ni a \mapsto \Xi_R(aT) \in X.$$

The inverse of this map is then given by mapping $P \in \mathcal{L}_L(r(X), X)$ to the map $\hat{P} \in \mathcal{L}_R(s(X), X)$ given by

$$s(X) \ni b \mapsto \Xi_L(Pb) \in X.$$

Proof. Fix $T \in \mathcal{L}_R(s(X), X)$. Since X and $\mathcal{K}_R(s(X), X)$ are isomorphic, they have equal range ideals. Thus the map $a \mapsto \Xi_R(aT)$ is well defined since $aT \in r(X) \cdot \mathcal{L}_R(s(X), X) = \mathcal{K}_R(s(X), X)$. To see that $\check{T}$ is adjointable, fix $x \in X$ and $a \in r(X)$. Then

$$\langle\langle a\check{T}, x\rangle\rangle = \langle\langle \Xi_R(aT), x\rangle\rangle = \langle\langle aT, |x\rangle\rangle\rangle,$$

which is the unique multiplier on $r(X)$ specified by $aT|x\rangle^* = a(|x\rangle \circ T^*)^*$ (note that this is indeed a multiplier, since we can express it as a composition of adjointable maps). Thus $\check{T}$ has adjoint $X \to r(X)$ given by mapping x to the unique multiplier of $r(X)$ associated to $|x\rangle \circ T^*$. This belongs to $r(X) = r(X)^* r(X) = \mathcal{K}(r(X))$ and not solely $M(r(X))$ since $|x\rangle$ is a compact operator, so the composition is also.

For $S, T \in \mathcal{L}_R(s(X), X)$ and $x \in X$ we have $x\langle S^*, T\rangle = x(S^* \circ T)$ by definition, which is exactly

$$\Xi_R((|x\rangle \circ S^*) \circ T) = \Xi_R(|x\rangle S^* T) = \Xi_R(|x\rangle)\langle S, T\rangle = x\langle S, T\rangle.$$

Thus $S^* \circ T = \check{S}^* \circ \check{T}$ whereby $\langle S, T\rangle = \langle \check{S}, \check{T}\rangle$ and the assignment $T \mapsto \check{T}$ is isometric.

Lastly we show that $\check{\hat{P}} = P$ for all $P \in \mathcal{L}_L(r(X), X)$. This shall show that the map $T \mapsto \check{T}$ is a surjective map, hence an isomorphism, and the map $P \mapsto \hat{P}$ is its inverse. Fix $P \in \mathcal{L}_L(r(X), X)$. For all $a \in r(X)$ we have $a\check{\hat{P}} = \Xi_R(a\hat{P})$. This is the unique element of X such that $\langle\langle \Xi_R(a\hat{P})| = a\check{\hat{P}}$. Hence $\Xi_R(a\hat{P})b$ is the unique element of X satisfying $\Xi_R(a\hat{P})b = a\hat{P}b = a\Xi_L(Pb) = \Xi_L(aPb)$ for all $b \in s(X)$. Thus $a\check{\hat{P}} = \Xi_R(a\hat{P}) = aP$ whereby $\check{\hat{P}} = P$. ☐

Lemma 2.2.2 allows us to consider $\mathcal{L}_R(s(X), X)$ and $\mathcal{L}_L(r(X), X)$ as an analogue of the multiplier algebra for Hilbert modules.

If $K \subseteq J \lhd B$ are both essential ideals, then any adjointable operator $T \in \mathcal{L}(J, X \cdot J)$ restricts to a map $T|_K : K \to X \cdot J$. Moreover, $T|_K$ has range contained in $X \cdot K$ since each $a \in K$ can be written as $a = a_1 a_2$ for some $a_1, a_2 \in K$ and we have $T|_K a = (Ta_1)a_2 \in X \cdot J \cdot K = X \cdot K$. The restriction $T|_K$ is adjointable from $K \to X \cdot K$, with adjoint $T^*|_{X \cdot K}$, which takes values in K since $T^*(X \cdot K) = T^*(X \cdot J) \cdot K \subseteq K$. Thus, restriction of operators gives rise to a module homomorphism $\mathcal{L}(J, X \cdot J) \to \mathcal{L}(K, X \cdot K)$. If $T \in \mathcal{L}(J, X \cdot J)$ restricts to the zero operator on K, then $\{0\} = T^*(X \cdot K) = T^*(X \cdot J) \cdot K$, whereby $T^*(X \cdot J)$ is an ideal that annihilates K. But K has zero annihilator in J as K is essential in J, whereby $T^* = 0$. In particular, $T = 0$ if and only if $T|_K = 0$, and the map $\mathcal{L}(J, X \cdot J) \to \mathcal{L}(K, X \cdot K)$ is injective. These restriction maps give rise to an inductive system over essential ideals in B, ordered by reverse inclusion: $I \leqslant J$ if and only if $J \subseteq I$.

Definition 2.2.3. Let X be a right Hilbert B-module. We define the *(right-)local multiplier module* of X as the inductive limit

$$X_{\mathrm{loc}} := \varinjlim_{J \in \mathcal{I}_e(B)} \mathcal{L}_R(J, X \cdot J)$$

taken over essential ideals $J \lhd B$.

Analogously, if X is a left Hilbert A-module we define the *(left-)local multiplier module* of X as

$$_{\mathrm{loc}}X := \varinjlim_{I \in \mathcal{I}_e(A)} \mathcal{L}_L(I, I \cdot X).$$

Taking the case $X = B$, with the canonical B-bimodule structure inherited from multiplication, we have that $\mathcal{L}_R(I, B \cdot I) = \mathcal{L}_R(I, I) = M(I) = \mathcal{L}_L(I, I) = \mathcal{L}_L(I, I \cdot B)$, thus $B_{\mathrm{loc}} = {}_{\mathrm{loc}}B = M_{\mathrm{loc}}(B)$, and these constructions generalise the local multiplier algebra.

Lemma 2.2.4. *Let X be a right Hilbert B-module. For each ideal $J \lhd B$ there is an isomorphism $\mathcal{L}_R(J, X \cdot J)^* \cong \mathcal{L}_L(J, J \cdot X^*)$ of left Hilbert B-modules. In particular $(X_{\mathrm{loc}})^* \cong {}_{\mathrm{loc}}(X^*)$.*

Proof. For $T \in \mathcal{L}_R(J, X \cdot J)$, define $S_T : J \to J \cdot X^* = (X \cdot J)^*$ by $S_T a := (T a^*)^*$. Then the map $T \mapsto S_T$ gives an anti-isomorphism between $\mathcal{L}_R(J, X \cdot J)$ and $\mathcal{L}_L(J, J \cdot X^*)$. Moreover, these anti-isomorphisms entwine the restriction maps in the inductive limit, and so the inductive limits are isomorphic. □

Many of the following results may be generalised to left Hilbert modules using either Lemma 2.2.4 or symmetry arguments.

Lemma 2.2.5. *Let X be a right Hilbert B-module. For any $J \in \mathcal{I}_e(B)$ we have $(X \cdot J)_{\mathrm{loc}} = X_{\mathrm{loc}}$.*

Proof. This follows because the collection of essential ideals of J is cofinal in $\mathcal{I}_e(A)$, over which the inductive limit is taken. □

In the case where X is a Hilbert bimodule, all subbimodules are of the form $X \cdot J$ for some ideal J. In particular, a subbimodule $Y \subseteq X$ has zero orthogonal complement in X if and only if the source ideal of Y is an essential ideal of the source of X. In the bimodule case, this then gives that the local multiplier module of any subbimodule with zero orthogonal complement is equal to the local multiplier module of the whole original module.

Corollary 2.2.6. *For any ideal $J \lhd B$ we have $X_{\mathrm{loc}} = (X \cdot (J \oplus J^{\perp}))_{\mathrm{loc}}$.*

Lemma 2.2.7. *Let X and Y be right Hilbert B-modules. Then $(X \oplus Y)_{\mathrm{loc}} = X_{\mathrm{loc}} \oplus Y_{\mathrm{loc}}$.*

Proof. This follows because the map $\mathcal{L}(J, X \cdot J) \oplus \mathcal{L}(J, Y \cdot J) \to \mathcal{L}(J, (X \oplus Y) \cdot J)$, $(T, S) \mapsto T + S$ is an isomorphism entwining the restriction maps for all ideals $J \lhd B$. □

Lemma 2.2.8. *The module X_{loc} is a right Hilbert $M_{\mathrm{loc}}(B)$-module.*

Proof. The argument at the start of this section shows that $\mathcal{L}_R(J, X \cdot J)$ is a right Hilbert $M(J)$-module for each essential ideal $J \lhd B$. The inductive limit structure is preserved since all the inductive limit maps are restrictions of operators, which clearly preserve the right actions and inner products. Hence there is an induced right Hilbert $M_{\mathrm{loc}}(B)$-module structure on X_{loc}. □

In a Hilbert bimodule there are, at first glance, two ways to take an orthgonal complement of a subbimodule (one for each inner product). If $Y \subseteq X$ is a Hilbert subbimodule of a Hilbert bimodule X, and if $x \in X$ satisfies $\langle x, y \rangle = 0$ for all $y \in Y$, then we also see that $\langle\langle y, x \rangle\rangle \cdot z = y \langle x, z \rangle = 0$ for all $y, z \in Y$. Thus $\langle\langle y, x \rangle\rangle$ belongs to the annihilator of the range ideal $r(Y)$ for all $y \in Y$. However, the inner product $\langle\langle y, x\rangle\rangle$ must also lie in $r(Y)$, as one may always write $y = ay'$ for some $a \in r(Y)$ and $y' \in Y$ by the Cohen-Hewitt factorisation theorem (see, for example, [18, Lemma 4.4]). Thus if $x \in X$ annihilates Y in the right inner product, it also does so in the left. Symmetrically, we see that orthogonal complements with respect to both inner products agree. We shall denote both by $Y^\perp$.

Lemma 2.2.9. *Let X be a Hilbert A–B-bimodule and let $Y \subseteq X$ be a closed Hilbert subbimodule. Then $(Y^\perp)_{\text{loc}} = (Y_{\text{loc}})^\perp$ and $X_{\text{loc}} = Y_{\text{loc}} \oplus Y^\perp_{\text{loc}}$.*

Proof. We have $Y = X \cdot s(Y)$ and so by Lemma 2.2.7 we gain

$$X_{\text{loc}} = (X \cdot (s(Y) \oplus s(Y)^\perp))_{\text{loc}} = Y_{\text{loc}} \oplus (X \cdot s(Y)^\perp)_{\text{loc}}.$$

The right summand must then be equal to both $(Y^\perp)_{\text{loc}}$ and $(Y_{\text{loc}})^\perp$. □

Lemma 2.2.10. *Let X be a Hilbert A–B-bimodule and let $J \lhd B$ be an ideal. Then $(X \cdot J)_{\text{loc}} = X_{\text{loc}} \cdot M_{\text{loc}}(J)$.*

Proof. First we show that $(X \cdot J)_{\text{loc}} \subseteq X_{\text{loc}} \cdot M_{\text{loc}}(J)$. Fix an essential ideal $J' \lhd B$, and define $J'_{\text{ess}} := JJ' \oplus (JJ')^\perp$, and note J'_{ess} is an essential ideal of A. Fix $\xi \in \mathcal{L}(J'_{\text{ess}}, (X \cdot J) \cdot J'_{\text{ess}})$. Then $\xi = \xi|_{JJ'} \oplus \xi|_{(JJ')^\perp}$ and since $JJ'_{\text{ess}} = JJ'$ we have $\xi|_{(JJ')^\perp} = 0$, giving $\xi = \xi \cdot 1_J$. Thus

$$\xi \in \mathcal{L}(J'_{\text{ess}}, (X \cdot J)J'_{\text{ess}}) \cdot M_{\text{loc}}(J) \subseteq (X \cdot J)_{\text{loc}} \cdot M_{\text{loc}}(J).$$

This gives $(X \cdot J)_{\text{loc}} \subseteq X_{\text{loc}} \cdot M_{\text{loc}}(J)$.

Lemmas 2.2.7 and 2.2.9 together imply that $X_{\text{loc}} \cdot M_{\text{loc}}(J) \oplus X_{\text{loc}} \cdot M_{\text{loc}}(J^\perp) = X_{\text{loc}} = (X \cdot J)_{\text{loc}} \oplus (X \cdot J^\perp)_{\text{loc}}$. For $\xi \in X_{\text{loc}}$, $\tau \in M_{\text{loc}}(J)$, and $\eta \in (X \cdot J^\perp)_{\text{loc}}$ we have

$$\langle \eta, \xi \cdot \tau\rangle = \langle \eta, \xi\rangle\tau \in \langle (X \cdot J^\perp)_{\text{loc}}, X_{\text{loc}}\rangle \cdot M_{\text{loc}}(J).$$

The set $\langle (X \cdot J^\perp)_{\text{loc}}, X_{\text{loc}}\rangle$ is contained in $M_{\text{loc}}(J^\perp)$ by the above argument, and so the inner product $\langle \eta, \xi \cdot \tau\rangle$ is contained in $M_{\text{loc}}(J^\perp) \cdot M_{\text{loc}}(J)$, which is zero by Lemma 2.1.3. Thus $X_{\text{loc}} \cdot M_{\text{loc}}(J) \subseteq (X \cdot J)_{\text{loc}}$ must hold. □

Lemma 2.2.11. *Let X be a Hilbert A–B-bimodule. Then X_{loc} and ${}_{\text{loc}}X$ are Hilbert $M_{\text{loc}}(A)$–$M_{\text{loc}}(B)$-bimodules.*

Proof. We show that X_{loc} is a left Hilbert $M_{\text{loc}}(A)$-module. The case for ${}_{\text{loc}}X$ then follows by a symmetric argument, or from Lemma 2.2.4. For each essential ideal $J \lhd B$, let $I_J := r(X \cdot J) \oplus r(X \cdot J)^\perp$. Then I_J is an essential ideal in A and we claim $I_J \cdot X = X \cdot J$. To see this, note that I_J satisfies $I_J \cdot X = r(X \cdot J) \cdot X \oplus r(X \cdot J)^\perp \cdot X$. The range ideal of $r(X \cdot J)^\perp \cdot X \cdot J$ is $r(X \cdot J)^\perp \cdot r(X \cdot J) = \{0\}$, so $r(X \cdot J)^\perp \cdot X \subseteq (X \cdot J)^\perp$. Since J is essential in $s(X)$, we see $s((X \cdot J)^\perp) \cap J = \{0\}$ implies that $s((X \cdot J)^\perp) = \{0\}$, whereby $(X \cdot J)^\perp = \{0\}$ and $I_J \cdot X = X \cdot J$.

For $\tau \in M(I_J)$ and $T \in \mathcal{L}_R(J, X \cdot J)$ we define $(\tau T)a := \tau(Ta)$. This bilinear map commutes with the maps in the inductive system, giving the left action of $M_{\text{loc}}(A)$ on X_{loc}. For $T, S \in \mathcal{L}_R(J, X \cdot J)$, we define the left inner product $\langle\langle T, S\rangle\rangle$ as the element of $M(r(X \cdot J))$ corresponding to TS^* under the isomorphism $\mathcal{L}(X \cdot J) \cong M(r(X \cdot J))$ arising from $X \cdot J \cong \mathcal{K}_R(J, X \cdot J)$.

We also see clearly by construction that $\langle\langle T, S\rangle\rangle P = TS^*P = T\langle S, P\rangle$ for all $T, S, P \in X_{\text{loc}}$, giving the necessary compatibility of left and right inner products. □

We now show that if X is a Hilbert bimodule, then the left and right local multiplier modules agree.

Proposition 2.2.12. *Let X be a Hilbert A–B-bimodule. For each ideal $J \lhd s(X)$ and the corresponding ideal $I_J := r(X \cdot J)$, the isomorphisms $\mathcal{L}_R(J, X \cdot J) \cong \mathcal{L}_L(I_J, I_J \cdot X)$ of Hilbert $M(I_J)$–$M(J)$-bimodules from Lemma 2.2.2 preserve the inductive limit structures of X_{loc} and ${}_{\text{loc}}X$. In particular, ${}_{\text{loc}}X \cong X_{\text{loc}}$.*

Proof. Let $\mathcal{L}_R(J, X \cdot J) \ni T \mapsto \check{T} \in \mathcal{L}_L(I_J, I_J \cdot X)$ denote the isomorphism from Lemma 2.2.2.

Fix an essential ideal $J' \lhd J$ and note $I' = r(X \cdot J')$ is essential in I. If $T \in \mathcal{L}_R(J, X \cdot J)$ then $T|_{J'}$ gives the operator $\widecheck{T|_{J'}} \in \mathcal{L}_L(I', I' \cdot X)$. For any $a \in I'$ and $b \in J'$ we have $[a\widecheck{T|_{J'}}]b = aT|_{J'}b = aTb = [a\check{T}]b$, so $\widecheck{T|_{J'}} = \check{T}|_{I'}$. Thus the isomorphisms $\mathcal{L}_R(J, X \cdot J) \cong \mathcal{L}_L(I_J, I_J \cdot X)$ arising from Lemma 2.2.2 preserve the inductive limit structures and induce an isomorphism of ${}_{\text{loc}}X$ and X_{loc}. □

Proposition 2.2.12 allows us to suppress the left- and right- prefixes of the local multiplier module of a bimodule. In principle these differ as sets, but we shall identify both under the isomorphism specified above. This is a problem that does not arise in the case of local multiplier algebras, as these (when considered as Hilbert bimodules) are symmetric, that is, $M_{\text{loc}}(A)^* = M_{\text{loc}}(A)$ explicitly.

Lemma 2.2.13. *Let X be a Hilbert A–B-bimodule and let $Y \subseteq X_{\text{loc}}$ be a Hilbert $M_{\text{loc}}(A)$–$M_{\text{loc}}(B)$-subbimodule of X_{loc}. Then $Y = \{0\}$ if and only if $Y \cap X = \{0\}$.*

Proof. Suppose $X \cap Y = \{0\}$. We have $Y = X_{\text{loc}} \cdot s(Y)$ by the classification of Hilbert subbimodules, and so

$$\begin{aligned} \{0\} &= X \cap Y \\ &= X \cap (X_{\text{loc}} \cdot s(Y)) \\ &\supseteq (X \cdot (s(Y) \cap B)) \cap (X_{\text{loc}} \cdot s(Y)) \\ &= X \cdot (s(Y) \cap B). \end{aligned}$$

Thus $s(Y) \cap B \subseteq s(X)^{\perp}$, and so by Lemma 2.1.5 we have $s(Y) \subseteq M_{\text{loc}}(s(X)^{\perp})$. Hence $Y = X_{\text{loc}} \cdot s(Y) \subseteq X_{\text{loc}} M_{\text{loc}}(s(X)^{\perp}) = (X \cdot s(X)^{\perp})_{\text{loc}} = \{0\}$ by Lemma 2.2.10. □

Lemma 2.2.13 is the bimodule analogue of detection of ideals for an inclusion $A \subseteq M_{\text{loc}}(A)$. It is worth noting that this argument requires X to be a Hilbert bimodule, since without this condition it is no longer true in general that $Y = X \cdot s(Y)$ (for example, all closed subspaces of a Hilbert space have the same source ideal $\mathbb{C}$). This shall not be a problem for us, as in later sections we shall exclusively examine Hilbert bimodules.

Although the construction of the local multiplier module commutes with right multiplication by an ideal (as in Lemma 2.2.10), this is not true for balanced tensor products of Hilbert bimodules. One particular consequence of this is that if X is a Morita equivalence A–B-bimodule, it is not always true that X_{loc} induces a Morita equivalence between $M_{\text{loc}}(A)$ and $M_{\text{loc}}(B)$. This failure occurs because X_{loc} need not be full, even when X is.

Example 2.2.14. Let $X = \mathcal{H}$ be an infinite-dimensional Hilbert space. Then $\mathcal{H}$ induces a Morita equivalence between $\mathbb{C}$ and the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$, with $s(\mathcal{H}) = \mathbb{C}$ and $r(\mathcal{H}) = \mathcal{K}(\mathcal{H})$. Clearly, $\mathcal{H}_{\text{loc}} = \mathcal{L}(\mathbb{C}, \mathcal{H}) \cong \mathcal{H}$ as $\mathbb{C}$ is simple and unital. Hence $\mathcal{H}_{\text{loc}} \otimes_{\mathbb{C}} \mathcal{H}^*_{\text{loc}} \cong \mathcal{K}(\mathcal{H})$. However, $(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}^*)_{\text{loc}} \cong \mathcal{K}(\mathcal{H})_{\text{loc}} = \mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$ (considered as a Hilbert bimodule over $M_{\text{loc}}(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$), which is strictly larger than $\mathcal{K}(\mathcal{H})$ since $\mathcal{H}$ is infinite-dimensional.

Proposition 2.2.15. *Let X be a Hilbert A–B-bimodule and let Y be a Hilbert B–C-bimodule. There is an isometric bimodule map $\Phi : X_{\text{loc}} \otimes_{M_{\text{loc}}(B)} Y_{\text{loc}} \to (X \otimes_B Y)_{\text{loc}}$ extending the canonical embedding $X \otimes_B Y \subseteq X_{\text{loc}} \otimes_{M_{\text{loc}}(B)} Y_{\text{loc}}$.*

Proof. Fix $J \in \mathcal{I}_e(B)$ and $K \in \mathcal{I}_e(C)$. Then, there is $J' \in \mathcal{I}_e(C)$ such that $J \cdot Y = Y \cdot J'$, and consequently $X \cdot J \otimes_B Y \cdot K = X \otimes_B J \cdot Y \cdot K = X \otimes_B Y \cdot J'K$. Fix $\xi \in \mathcal{L}(J, X \cdot J)$ and $\eta \in \mathcal{L}(K, Y \cdot K)$. Denote by $\mathrm{mult}_{J'K,Y} : J'K \otimes_B Y \to J'K \cdot Y$ the unitary map implementing the left multiplication of B on Y. Identify η with $\eta|_{J'K} \in \mathcal{L}(J'K, Y \cdot J'K)$ and define

$$\varphi_{\xi,\eta} := (\xi \otimes 1) \circ \mathrm{mult}_{J'K,Y}^{-1} \circ \eta : J'K \to (X \otimes_B Y)J'K.$$

This is an adjointable map since $\xi \otimes 1, \mathrm{mult}_{J'K,Y}$, and η each are. We claim there is a bimodule homomorphism $\Phi : X_{\mathrm{loc}} \otimes_{M_{\mathrm{loc}}(B)} Y_{\mathrm{loc}} \to (X \otimes_B Y)_{\mathrm{loc}}$ that maps the elementary tensor $\xi \otimes \eta$ to $\varphi_{\xi,\eta}$ for $\xi \in \mathcal{L}(J, X \cdot J)$ and $\eta \in \mathcal{L}(Y, Y \cdot K)$. First, note from the definition that for $\tau \in M(r(X \cdot J))$, $\omega \in M(J \cap r(Y \cdot J'K))$ and $\sigma \in M(J'K)$, we have $\varphi_{\tau\xi\omega,\eta\sigma} = \tau\varphi_{\xi,\omega\eta}\sigma$. The assignment $(\xi, \eta) \mapsto \varphi_{\xi,\eta}$ thus respects the balanced tensor product structure and Φ will be a bimodule homomorphism. To see that the assignment is isometric, consider J' as above and fix $\xi_1, \xi_2 \in \mathcal{L}(J, X \cdot J)$ and $\eta_1, \eta_2 \in \mathcal{L}(J'K, Y \cdot J'K)$. For $b \in J'K$ we compute

$$\begin{aligned}
\langle \varphi_{\xi_1,\eta_1}, \varphi_{\xi_2,\eta_2} \rangle(b) &= \varphi^*_{\xi_1,\eta_1}\varphi_{\xi_2,\eta_2}(b) \\
&= (\eta_1^* \circ \mathrm{mult}_{J'K,Y} \circ \xi_1^* \otimes 1)(\xi_2(a) \otimes y), \qquad \text{where } a \cdot y = \eta_2(b), \\
&= \eta_1^*(\xi_1^*\xi_2(a)y) \\
&= \eta_1^*\xi_1^*\xi_2\eta_2(b) \\
&= \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle(b).
\end{aligned}$$

Lastly, for $x \in X$, $y \in Y$, and $c \in C$ we have $\varphi_{x,y}(c) = x \otimes y \cdot c$, so Φ restricts to the canonical embedding of $X \otimes_B Y$ into $(X \otimes_B Y)_{\mathrm{loc}}$. □

The map given in Proposition 2.2.15 will in many cases not be adjointable. To see this, recall Example 2.2.14 and note that $\mathcal{K}(\mathcal{H})$ embeds identically into $\mathcal{B}(\mathcal{H})$, but since $\mathcal{K}(\mathcal{H})$ is an essential ideal this map cannot have an adjoint.

The image of $X_{\mathrm{loc}} \otimes_{M_{\mathrm{loc}}(B)} Y_{\mathrm{loc}}$ in the localisation of the tensor product does however 'detect' all Hilbert $M_{\mathrm{loc}}(A)$–$M_{\mathrm{loc}}(C)$-subbimodules of $(X \otimes_B Y)_{\mathrm{loc}}$. This follows as $X_{\mathrm{loc}} \otimes_{M_{\mathrm{loc}}(B)} Y_{\mathrm{loc}}$ contains $X \otimes_B Y$, which detects such subbimodules by Lemma 2.2.13. This then implies that $(X_{\mathrm{loc}} \otimes_{M_{\mathrm{loc}}(B)} Y_{\mathrm{loc}})^\perp = \{0\}$, giving that $s(X_{\mathrm{loc}} \otimes_{M_{\mathrm{loc}}(B)} Y_{\mathrm{loc}})$ is an essential ideal of $s((X \otimes_B Y)_{\mathrm{loc}})$.

2.3 Non-triviality conditions for bimodules

One may also ask how the construction of the local multiplier module process alters non-triviality conditions for bimodules. We are interested in two particular conditions for Hilbert bimodules: pure outerness and aperiodicity. Pure outerness is of interest to us for the classification of noncommutative Cartan pairs (cf. [7], [15]). Aperiodicity is explored in [16] and grants us two properties of interest: uniqueness of conditional expectations taking value in the injective hull of a C^*-algebra, in particular, the local multiplier algebra ([9, Theorem 1], [17]), and pure outerness of the local multiplier module of a bimodule (see Proposition 2.3.4 ahead). In later sections we shall use this to construct Cartan pairs by applying the local multiplier module construction to a special class of actions that do not initially give rise to Cartan inclusions.

Definition 2.3.1 ([14, Definition 4.3]). Let X be a Banach A-bimodule. We say that X is *purely outer* if the only ideal of $J \lhd A$ such that $X \cdot J \cong J$ is $J = \{0\}$.

Definition 2.3.2 ([14])**.** Let X be a normed A-bimodule. We say that $x \in X$ satisfies *Kishimoto's condition* if for any $\varepsilon > 0$ and any non-zero hereditary subalgebras $D \subseteq A$, there exists $a \in D$ with $a \geqslant 0$ and $||a|| = 1$ such that $||axa|| < \varepsilon$. We say that X is *aperiodic* if all $x \in X$ satisfy Kishimoto's condition.

Lemma 2.3.3 ([16, Lemma 5.12])**.** *Subbimodules, quotient bimodules, extensions, finite direct sums, and inductive limits of aperiodic normed A-bimodules remain aperiodic. If $f : X \to Y$ is a bounded A-bimodule homomorphism with dense range and X is aperiodic, then so is Y. If $D \subseteq A$ is hereditary, then an aperiodic A-bimodule is also aperiodic as a D-bimodule. If $J \in \mathcal{I}_e(A)$ and X an A-bimodule, then JXJ is aperiodic as a J-bimodule if and only if X is aperiodic as an A-bimodule.*

By [16, Lemma 5.10] no positive non-zero element of A satisfies Kishimoto's condition when considering A as an A-bimodule. In particular, no ideals of A are aperiodic other than the zero ideal. Thus, if X is an aperiodic Banach bimodule and $J \lhd A$ is an ideal such that $X \cdot J \cong J$, then J is also aperiodic, giving $J = 0$. This shows that aperiodic Banach bimodules are purely outer. The converse is in general false. However if X is a Hilbert A-bimodule and A contains an essential ideal that is simple or of type I, then they are equivalent by [14, Theorem 8.1].

Proposition 2.3.4. *Let X be an aperiodic Hilbert A-bimodule. Then X_{loc} is a purely outer $M_{\mathrm{loc}}(A)$-bimodule.*

Proof. Let $\tilde{I} \lhd M_{\mathrm{loc}}(A)$ be an ideal such that $X_{\mathrm{loc}} \cdot \tilde{I} \cong \tilde{I}$. Let $\Phi : X_{\mathrm{loc}} \cdot \tilde{I} \to \tilde{I}$ be such an isomorphism. Then Φ restricts to an injective A-bimodule map $\phi : X \cap (X_{\mathrm{loc}} \cdot \tilde{I}) \to M_{\mathrm{loc}}(A)$. The image of ϕ is an aperiodic A-bimodule by [16, Lemma 5.12]. By [17, Proposition 3.16], the algebra $M_{\mathrm{loc}}(A)$ contains no non-zero aperiodic A-bimodule. Hence the image of ϕ is zero, and so $X \cap (X_{\mathrm{loc}} \cdot \tilde{I}) = \{0\}$. Lemma 2.2.13 then gives $\tilde{I} \cong X_{\mathrm{loc}} \cdot \tilde{I} = \{0\}$. ☐

2.4 A Galois connection for Hilbert modules

Let $A \subseteq B$ be C^*-algebras and let $\mathcal{I}(A)$ and $\mathcal{I}(B)$ be their respective ideal lattices. There are maps $i : \mathcal{I}(A) \to \mathcal{I}(B)$ and $r : \mathcal{I}(B) \to \mathcal{I}(A)$ given by

$$r(J) = A \cap J, \qquad i(I) = BIB,$$

for $I \in \mathcal{I}(A)$ and $J \in \mathcal{I}(B)$, where $BIB := \overline{\operatorname{span}} BIB$ is the ideal generated by I in B. Green [10] observed that these maps form a *monotone Galois connection.* That is, the maps r and i have the property that for any $I \in \mathcal{I}(A)$ and $J \in \mathcal{I}(B)$, the containment $I \subseteq r(J)$ holds if and only if $i(I) \subseteq J$. It follows then that the maps i and r also satisfy $i \circ r \circ i = i$ and $r \circ i \circ r = r$. The map i preserves joins (sum closure of ideals) and the map r preserves meets (intersections of ideals), and these maps further restrict to mutually inverse isomorphisms

$$i(\mathcal{I}(A)) \cong r(\mathcal{I}(B)).$$

This argument extends to Hilbert modules. Let X be a Hilbert A-module and let Y be a Hilbert B-module. Suppose there is an isometric embedding $X \subseteq Y$. Denote by $\mathrm{Mod}_A(X)$ and $\mathrm{Mod}_B(Y)$ the collections of A-submodules of X and B-submodules of Y respectively. There are analogously defined restriction and induction maps $r : \mathrm{Mod}_B(Y) \to \mathrm{Mod}_A(X)$ and $i : \mathrm{Mod}_A(X) \to \mathrm{Mod}_B(Y)$ given by

$$r(Z) = Z \cap X, \qquad i(W) = W \cdot B,$$

where $W \cdot B = \overline{\operatorname{span}}\, W \cdot B$ is the Hilbert B-module generated by W in Y. One then readily checks

$$\begin{aligned} W \subseteq r(Z) &\implies i(W) = W \cdot B \subseteq r(Z) \cdot B \subseteq Z, \text{ and} \\ i(W) \subseteq Z &\implies i(W) \cap X \subseteq Z \cap X \\ &\implies W \subseteq r(W), \end{aligned}$$

so these maps also induce a monotone Galois correspondence. A similar argument works for inclusions $A_i \subseteq B_i$ for $i = 1, 2$ and Hilbert bimodules $X \subseteq Y$, where X is a Hilbert A_1–A_2-bimodule and Y is a Hilbert B_1–B_2-bimodule. Thus we have shown that r is a right adjoint to i_-.

In the setting of the inclusion $A \subseteq M_{\text{loc}}(A)$, have an alternative way to induce ideals of $M_{\text{loc}}(A)$ and Hilbert $M_{\text{loc}}(A)$-(bi)modules. If $I \lhd A$ is an ideal and if X is a Hilbert A-module, we gain an ideal $i_+(I) := M_{\text{loc}}(I)$ of $M_{\text{loc}}(A)$ and may also induce the Hilbert $M_{\text{loc}}(A)$-(bi)module $i_+(X) := X_{\text{loc}}$ from X. These maps i_+ do not however induce a Galois correspondence with restriction r. One counterexample is to consider the compact operators $\mathcal{K}(\mathcal{H})$ as in Example 2.2.14. Since the local multiplier algebra of $\mathcal{K}(\mathcal{H})$ is the bounded operators $\mathcal{B}(\mathcal{H})$, and $\mathcal{K}(\mathcal{H})$ is also an ideal in $\mathcal{B}(\mathcal{H})$, we see that $\mathcal{K}(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H}) = r(\mathcal{K}(\mathcal{H}))$ but $i_+(\mathcal{K}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$ is not contained in $\mathcal{K}(\mathcal{H})$ if $\mathcal{H}$ has infinite dimension, violating the Galois correspondence condition.

If we restrict to the Hilbert subbimodules $W \subseteq X$, $Z \subseteq Y$ satisfying $W = r(i_+(W))$ and $Z = i_+(r(Z))$, we do however gain

$$r(Z) \subseteq W \iff Z \subseteq i_+(W).$$

Restricting to this class of ideals, we gain a Galois correspondence and so i_+ is then the right adjoint to r for this restricted class.

Chapter 3

Inducing inverse semigroup actions on the local multiplier algebra

We shall give the definition of an inverse semigroup action on a C^*-algebra A by Hilbert bimodules as defined and used in [5], [15], [16], and then apply our local multiplier module construction. This gives rise to a Fell bundel over the acting inverse semigroup, which after some refinement, gives an action of an inverse semigroup on $M_{\mathrm{loc}}(A)$.

3.1 Inverse semigroup actions and crossed products

Throughout the rest of this thesis, A shall denote a C^*-algebra and S shall denote a unital inverse semigroup. There is a canonical partial order on S given by $t \leqslant u$ if and only if $t = ut^*t$ for $u, t \in S$.

Definition 3.1.1 (cf. [5, Definition 4.7]). An *action* $\mathcal{E} = (\mathcal{E}_t, \mu_{t,u})_{t,u\in S}$ of S on A by Hilbert bimodules consists of

- Hilbert A-bimodules $\mathcal{E}_t$ for each $t \in S$; and
- bimodule isomorphisms $\mu_{t,u} : \mathcal{E}_t \otimes_A \mathcal{E}_u \to \mathcal{E}_{tu}$;

satisfying

(i) $\mathcal{E}_1 = A$, with the canonical A-bimodule structure;

(ii) the maps $\mu_{1,t} : A \otimes_A \mathcal{E}_t \to \mathcal{E}_t$ and $\mu_{t,1} : \mathcal{E}_t \otimes_A A \to \mathcal{E}_t$ are the canonical isomorphisms coming from the respective left and right actions of A on $\mathcal{E}_t$; and

(iii) associativity: for all $t, u, v \in S$, the following diagram commutes:

$$\begin{array}{ccccc} (\mathcal{E}_t \otimes_A \mathcal{E}_u) \otimes_A \mathcal{E}_v & \xrightarrow{\mu_{t,u} \otimes_A \mathrm{id}_{\mathcal{E}_v}} & \mathcal{E}_{tu} \otimes_A \mathcal{E}_v & \overset{\mu_{tu,v}}{\searrow} & \\ \mathrm{ass}\updownarrow & & & & \mathcal{E}_{tuv} \\ \mathcal{E}_t \otimes_A (\mathcal{E}_u \otimes_A \mathcal{E}_v) & \xrightarrow[\mathrm{id}_{\mathcal{E}_t} \otimes_A \mu_{u,v}]{} & \mathcal{E}_t \otimes_A \mathcal{E}_{uv} & \underset{\mu_{t,uv}}{\nearrow} & \end{array}$$

We shall now give the necessary concepts to build the full and essential crossed products for inverse semigroup actions. These are the same definitions as in [15, Section 2.2] and [16, Section 4], and we refer the reader there for a more in depth explanation.

If $t \leqslant u$ for $t, u \in S$ and $\mathcal{E}$ is an action of S on A, then there is an inclusion map $\mathcal{E}_t \hookrightarrow \mathcal{E}_u$ gained from the multiplication maps. These inclusions then restrict to isomorphisms

$j_{u,t} : \mathcal{E}_t \to \mathcal{E}_u \cdot s(\mathcal{E}_t) = r(\mathcal{E}_t) \cdot \mathcal{E}_u$. For each $v \leqslant t, u$, the maps $j_{t,v}$ and $j_{u,v}$ induce an isomorphism $\vartheta^v_{t,u} : \mathcal{E}_u \cdot s(\mathcal{E}_v) \to \mathcal{E}_t \cdot s(\mathcal{E}_v)$ by defining $\vartheta^v_{t,u} := j_{t,v} \circ j^{-1}_{u,v}$. Define

$$I_{t,u} = \overline{\sum_{v \leqslant t,u} s(\mathcal{E}_v)},$$

the closed ideal generated by $s(\mathcal{E}_v)$ for all $v \leqslant t, u$. We call the ideal $I_{t,u}$ the *intersection ideal for* t, u. This is contained in $s(\mathcal{E}_u) \cap s(\mathcal{E}_t)$ and the inclusion may be strict. There is a unique Hilbert bimodule isomorphism $\vartheta_{t,u} : \mathcal{E}_u \cdot I_{t,u} \to \mathcal{E}_v \cdot I_{t,u}$ which for each $v \leqslant t, u$ restricts to $\vartheta^v_{t,u}$ on $\mathcal{E}_t \cdot s(\mathcal{E}_v)$ by [5, Lemma 2.4]. The *algebraic crossed product* $A \rtimes_{\text{alg}} S$ is defined as the quotient of $\bigoplus_{t \in S} \mathcal{E}_t$ by the linear span of $\vartheta_{u,t}(\xi)\delta_u - \xi\delta_t$ for $t, u \in S$ and $\xi \in \mathcal{E}_t \cdot I_{t,u}$. There is a *-algebra structure on $A \rtimes_{\text{alg}} S$ with multiplication and involution induced by the maps $\mu_{t,u}$ and the involutions $\mathcal{E}_t^* \to \mathcal{E}_{t^*}$. There is then a maximal C^*-norm on $A \rtimes_{\text{alg}} S$.

Definition 3.1.2 ([15, Definition 2.7])**.** The *full crossed product* $A \rtimes S$ of the action $\mathcal{E}$ is defined as the maximal C^*-completion of the *-algebra $A \rtimes_{\text{alg}} S$.

We also define some non-triviality conditions for inverse semigroup actions. Kwaśniewski and Meyer ([15], [16]) show that actions satisfying these conditions give rise to inclusions of C^*-algebras satisfying certain maximality conditions that are useful in the analysis of Cartan-like pairs.

Definition 3.1.3 ([16, Definitions 6.1, 6.9])**.** Let $\mathcal{E}$ be an action of an inverse semigroup S on A. We say that the action is *purely outer* if $\mathcal{E}_t \cdot I_{1,t}^{\perp}$ is a purely outer A-bimodule for each $t \in S$. We say that the action is *aperiodic* if $\mathcal{E}_t \cdot I_{1,t}^{\perp}$ is an aperiodic A-bimodule for each $t \in S$.

A definition of the reduced crossed product for such an action can be found in [15, Section 2.2]. This involves the construction of a weak conditional expectation $E : A \rtimes S \to A''$ taking values in the enveloping von Neumann algebra of A. One may also consider *essentially defined conditional expectations* or *local expectations* that instead take values in the local multiplier algebra $M_{\text{loc}}(A)$ of A.

Recall that for a C^*-algebra A we denote Hamana's injective hull by $I(A)$ [12].

Definition 3.1.4 ([16, Definitions 3.1,3.7])**.** Let $A \subseteq B$ be a C^*-inclusion. A *generalised expectation* for $A \subseteq B$ consists of another C^*-inclusion $A \subseteq \tilde{A}$ and a completely positive contractive linear map $E : B \to \tilde{A}$ such that E restricts to the identity on A. We say that E is *faithful* if $E(b^*b) = 0$ implies $b = 0$ for all $b \in B$, and E is *almost faithful* if $E((ba)^*ba) = 0$ for all $a \in B$ implies $b = 0$ for all $b \in B$. If $\tilde{A} = M_{\text{loc}}(A)$ we call E an *essentially defined conditional expectation* or a *local expectation.* If $\tilde{A} = I(A)$ then we call E a *pseudo-expectation.*

Since $M_{\text{loc}}(A)$ canonically embeds in $I(A)$ by a result of Frank [9] we can consider $M_{\text{loc}}(A) \subseteq I(A)$. With this one may consider local expectations as a special case of pseudo-expectations.

The dense subalgebra $A \rtimes_{\text{alg}} S$ of $A \rtimes S$ is exactly the span of the bimodules $\mathcal{E}_t$ under the canonical inclusions $\mathcal{E}_t \to A \rtimes_{\text{alg}} S$, $\xi \mapsto [\xi\delta_t]$. Each of these maps is injective, so we identify each $\mathcal{E}_t$ with its image in $A \rtimes_{\text{alg}} S$. We define a local expectation $EL : A \rtimes S \to M_{\text{loc}}(A)$ as follows Let $t \in S$. Recalling the isomorphism $\vartheta_{1,t} : \mathcal{E}_t \cdot I_{1,t} \to I_{1,t}$, each $\xi \in \mathcal{E}_t$ defines an element of $\mathcal{M}(I_{1,t}) \subseteq M_{\text{loc}}(A)$ via $EL(\xi) : a \mapsto \vartheta_{1,t}(\xi \cdot a)$. By [16, Proposition 4.4] this extends to a local expectation $EL : A \rtimes S \to M_{\text{loc}}(A)$, which we call the *canonical local expectation.* We denote by $\mathcal{N}_{EL}$ the largest ideal contained in $\ker(EL)$.

Definition 3.1.5. The *essential crossed product* is defined as the quotient

$$A \rtimes_{\text{ess}} S := (A \rtimes S)/\mathcal{N}_{EL}.$$

The local expectation EL descends to a local expectation $A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A)$, which we also denote by EL.

Theorem 3.1.6 ([16, Theorem 4.12])**.** *The canonical local expectation $EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A)$ is faithful.*

Renault [22], and earlier Kumjian [13] considered inclusions of C^*-algebras $A \subseteq B$ such that the larger algebra B is densely spanned by elements that normalise A in the following sense: $n \in B$ is a normaliser if $n^*An, nAn^* \subseteq A$. This has since been taken as a standard assumption for Cartan pair-like objects in [7], [15], and many others. For an action $\mathcal{E}$ of a unital inverse semigroup S on a C^*-algebra A (or more generally a Fell bundle over S with unit fibre A), the inclusion $A \subseteq A \rtimes S$ has this property. This is because each $\mathcal{E}_t$ carries an A-bimodule structure, and together they span a dense subspace of $A \rtimes S$.

Definition 3.1.7. Let $A \subseteq B$ be an inclusion of C^*-algebras. We call the inclusion *regular* if the set of normalisers $N(A, B) = \{n \in B : n^*An, nAn^* \subseteq A\}$ spans a dense subspace of B. Closed subspaces $M \subseteq N(A, B)$ such that $AM, MA \subseteq M$ are called *slices*, and the collection of slices for the inclusion $A \subseteq B$ is denoted $\mathcal{S}(A, B)$. A *subslice* N of $M \in \mathcal{S}(A, B)$ is a slice N contained in M.

If $A \subseteq B$ is a non-degenerate inclusion then $M^*M, MM^* \subseteq A$ for any slice $M \in \mathcal{S}(A, B)$, giving each slice a Hilbert A-bimodule structure with inner products induced from the multiplication in B. The set $\mathcal{S}(A, B)$ then becomes an inverse semigroup with operation $M \cdot N := \overline{\text{span}}\, MN$ and *-operation given by the adjoint in B. For brevity of notation we write MN to denote the closed linear span of products of elements in M and N. This gives rise to the tautological action of $\mathcal{S}(A, B)$ on A, where the bimodules for the action are the slices, and the multiplication isomorphisms are induced by the multiplication in B. In the case where one has a closed and purely outer action $\mathcal{E}$ of S on A, Kwaśniewski and Meyer [15] showed one can reconstruct slices for the inclusion $A \subseteq A \rtimes_{\text{r}} S$ from the bimodules $\mathcal{E}_t$.

In general if $A \subseteq B$ is a regular non-degenerate inclusion then the slice inverse semigroup $\mathcal{S}(A, B)$ acts tautologically on A via the action $\mathcal{E}_X = X$ for slices $X \in \mathcal{S}(A, B)$, and multiplication maps given by the multiplication in B.

We then gain the following classification of certain C^*-inclusions in terms of actions by inverse semigroups.

Theorem 3.1.8. *Let $A \subseteq B$ be a non-degenerate inclusion of C^*-algebras, and let $E : B \to I(A)$ be a faithful pseudo-expectation. Suppose there is a densely spanning inverse subsemigroup $S \subseteq \mathcal{S}(A, B)$ that acts aperiodically on A. Then there is an isomorphism $\varphi : A \rtimes_{\text{ess}} S \to B$ that restricts to the identity on A and entwines E with the canonical local expectation EL. In particular, the expectation E takes values in $M_{\text{loc}}(A)$ and the inclusion $A \subseteq B$ is aperiodic.*

Proof. The inclusion $A \subseteq A \rtimes S$ is aperiodic by [16, Proposition 6.3], so the local expectation $EL : A \rtimes S \to M_{\text{loc}}(A) \subseteq I(A)$ is the unique pseudo-expectation for the inclusion by [17, Theorem 3.6]. Let $\Phi : A \rtimes S \to B$ be the canonical *-homomorphism. This is surjective as it spans each of the slices in S. Then $E \circ \Phi : A \rtimes S \to I(A)$ is a pseudo-expectation, and so is equal to EL by uniqueness.

The map Φ then descends to an isomorphism $\varphi : A \rtimes S/\ker(\Phi) \to B$, so it suffices to show that the kernel of Φ is $\mathcal{N}_{EL}$, the largest ideal contained in the kernel of EL, as taking the quotient by this ideal gives the essential crossed product. Fix $x \in \ker(\Phi)$. Then $EL(x^*x) = E(\Phi(x^*x)) = 0$ so $x \in \mathcal{N}_{EL}$. Conversely, $x \in \mathcal{N}_{EL}$ if and only if $0 = EL((xy)^*xy)$ by [16, Proposition 3.5], so $0 = EL((xy)^*xy) = E(\Phi(xy)^*\Phi(xy))$. Since E is faithful, it is almost faithful by [16, Corollary 3.7], and since Φ is surjective we see that $\Phi(x) = 0$. Thus Φ descends to an isomorphism $\varphi : A \rtimes_{\text{ess}} S \to B$. The inclusion $A \subseteq A \rtimes_{\text{ess}} S$ is then aperiodic since the quotient map $q : A \rtimes S \to A \rtimes_{\text{ess}} S$ descends to a bounded surjective bimodule map $(A \rtimes S)/A \to (A \rtimes_{\text{ess}} S)/A$, and the image of an aperiodic bimodule is aperiodic by Lemma 2.3.3. □

The universal property of $I(A)$ ensures that for any C^*-inclusion $A \subseteq B$ there is always a pseudo-expectation $E : B \to I(A)$ extending the identity map on A. Under the conditions of Theorem 3.1.8 we then gain that E must in fact take values in $M_{\text{loc}}(A)$, as the canonical pseudo-expectation associated to the essential crossed product does.

3.2 The dual groupoid to an inverse semigroup action

If $\mathcal{E}$ is an action of S on A, there is an induced action $\hat{\mathcal{E}} = (\hat{\mathcal{E}}_t)_{t\in S}$ of S on $\hat{A}$ such that $\hat{\mathcal{E}}_t : \widehat{s(\mathcal{E}_t)} \to \widehat{r(\mathcal{E}_t)}$ for each $t \in S$. The construction of the transformation groupoid $\hat{A} \rtimes S$ associated to $\mathcal{E}$ can be found in [15, Section 2.3].

Definition 3.2.1 ([16]). We call $\hat{\mathcal{E}}$ the *dual action* to the action $\mathcal{E}$ of S on A. The transformation groupoid $\hat{A} \rtimes S$ is called the *dual groupoid*.

The unit space of the dual groupoid $\hat{A} \rtimes S$ is homeomorphic to $\hat{A}$ via the map $\hat{A} \to (\hat{A} \rtimes S)^{(0)}$, $[\pi] \mapsto [1, [\pi]]$. We often identify $\hat{A}$ and $(\hat{A} \rtimes S)^{(0)}$ under this map. By [15, Proposition 2.17], an action $\mathcal{E} : S \rightharpoonup A$ is closed if and only if the unit space $\hat{A}$ of $\hat{A} \rtimes S$ is a closed subset.

By construction of the essential crossed product, there are injective A-bimodule maps $\mathcal{E}_t \to A \rtimes_{\text{ess}} S$ where the image of each $\mathcal{E}_t$ is a slice for this inclusion. Let $B := A \rtimes_{\text{ess}} S$. We gain a canonical homomorphism $S \to \mathcal{S}(A, B)$ by mapping $t \in S$ to the corresponding image of $\mathcal{E}_t$ in B. We then gain a canonical groupoid homomorphism $\phi : \hat{A} \rtimes S \to \hat{A} \rtimes \mathcal{S}(A, B)$ via $\phi[t, [\pi]] = [\mathcal{E}_t, [\pi]]$. The inverse semigroup S also canonically maps into the bisection inverse semigroup $\text{Bis}(\hat{A} \rtimes \mathcal{S}(A, B))$ by mapping $t \in S$ to the bisection $\{[\mathcal{E}_t, [\pi]] : [\pi] \in \widehat{s(\mathcal{E}_t)}\}$. In general these homomorphisms are not injective. For example, if A_1 and A_2 are C^*-algebras, consider the action generated by a self-inverse automorphism on A_1, and extend this to an automorphism α on $A := A_1 \oplus A_2$ by acting trivially on A_2. Then the germ relation for a group action is trivial since groups only have one idempotent: the unit. Thus over every irreducible representation of A_2 there is non-trivial isotropy, but if we consider $\hat{A} \rtimes \mathcal{S}(A, A \rtimes_{\text{ess}} \mathbb{Z}_2)$, we see that $[\alpha, [\pi]] = [\text{id}, [\pi]]$ for all $[\pi] \in A_2$ as $\alpha|_C = \text{id}|_C$. These two germs differ in $\hat{A} \rtimes \mathbb{Z}_2$, since the only germ relations we may take there are trivial, so $[\alpha, [\pi]] \neq [\text{id}, [\pi]]$ in $\hat{A} \rtimes \mathbb{Z}_2$ for all $[\pi] \in \hat{A}$.

This can be remedied by assuming that our action has enough idempotents.

Lemma 3.2.2. *Let $\mathcal{E}$ be a purely outer action of S on A such that for each the open sets $\hat{\mathcal{E}}_e \subseteq \hat{A}$ for idempotents $e \in S$ form a basis for the topology on $\hat{A}$. Write $B := A \rtimes_{\text{ess}} S$. Then the canonical homomorphism $\phi : \hat{A} \rtimes S \to \hat{A} \rtimes \mathcal{S}(A, B)$ is injective.*

Proof. Fix $[t, [\pi]] \in \hat{A} \rtimes S$ with $[\hat{\mathcal{E}}_t, [\pi]] = [A, [\pi]]$, that is, $\phi[t, [\pi]]$ is a unit in $\hat{A} \rtimes \mathcal{S}(A, B)$. Then there is an open subset $U \subseteq \hat{A}$ such that $\hat{\mathcal{E}}_t|_U = \text{id}_U$. Let $e \in S$ be an idempotent such that $\hat{\mathcal{E}}_e \subseteq U$ and $[\pi] \in \mathcal{E}_e$. Then $\mathcal{E}_t \cdot \mathcal{E}_e = \mathcal{E}_e$ giving $\mathcal{E}_e = \mathcal{E}_t \cdot s(\mathcal{E}_e)$, and we see that $e = te$ since the action $\mathcal{E}$ is purely outer. Thus $[t, [\pi]] = [e, [\pi]]$ is a unit since e is an idempotent. □

We note that by [2, Theorem 7.2] we can without loss of generality assume our inverse semigroup has enough idempotents, so that the criteria of Lemma 3.2.2 are always satisfied.

3.3 Extending actions to local multiplier algebras

The results of Exel in [7] and Kwaśniewski–Meyer in [15] require that the action of S on A is *closed*, that is, the conditional expectation EL takes values in A. Since for our purposes the canonical expectation EL need not take values in A, we are unable to immediately use these results. To circumvent this, we construct a closed action on $M_{\text{loc}}(A)$ from the action on A, then show that the inclusion $A \subseteq A \rtimes_{\text{ess}} S$ embeds into the crossed product associated to this extended action.

Unfortunately, one cannot simply take an action $\mathcal{E}$ of S on A, replace the modules with the local multiplier counterparts, and then gain an action on $M_{\text{loc}}(A)$. The obstruction to this is that the map in Proposition 2.2.15 is not always an isomorphism, so we do not always have $(\mathcal{E}_t)_{\text{loc}} \otimes_{M_{\text{loc}}(A)} (\mathcal{E}_u)_{\text{loc}} \cong (\mathcal{E}_t \otimes_A \mathcal{E}_u)_{\text{loc}}$. Our solution is to instead create a non-saturated Fell bundle with these local multiplier modules, and then gain a saturated Fell bundle using [3, Theorem 7.2], which is then equivalent to an inverse semigroup action on $M_{\text{loc}}(A)$.

Definition 3.3.1 ([2, Definition 2.10])**.** Let S be an inverse semigroup. A *Fell bundle* over S is a collection $\mathcal{A} = (\mathcal{A}_t)_{t\in S}$ of Banach spaces $\mathcal{A}_t$ together with a multiplication $\cdot : \mathcal{A}_t \times \mathcal{A}_u \to \mathcal{A}_{tu}$ for all $t, u \in S$, an involution $^* : \mathcal{A}_t \to \mathcal{A}_{t^*}$ for each $t \in S$, linear maps $j_{t,u} : \mathcal{A}_u \to \mathcal{A}_t$ for $t, u \in S$ with $u \leqslant t$ satisfying the following:

(i) the multiplication is bilinear $\mathcal{A}_t \times \mathcal{A}_u \to \mathcal{A}_{tu}$ for all $t, u \in S$;

(ii) the multiplication is associative;

(iii) $||a \cdot b|| \leqslant ||a|| \cdot ||b||$ for all $a, b \in \bigcup_{t\in S} \mathcal{A}_t$;

(iv) * is conjugate linear on each $\mathcal{A}_t$;

(v) $(a^*)^* = a$, $||a^*|| = ||a||$, and $(a \cdot b)^* = b^* \cdot a^*$ for all $a, b \in \bigcup_{t\in S} \mathcal{A}_t$;

(vi) $||a^*a|| = ||a||^2$ and a^*a is a positive element in the C^*-algebra $\mathcal{A}_{t^*t}$ for all $t \in S$ and $a \in \mathcal{A}_t$;

(vii) $j_{t,u}$ is an isometric linear map for all $t, u \in S$ with $u \leqslant t$;

(viii) if $v \leqslant u \leqslant t$ then $j_{t,v} = j_{t,u} \circ j_{u,v}$;

(ix) if $s \leqslant t$ and $u \leqslant v$ in S then $j_{t,s}(a) \cdot j_{v,u}(b) = j_{tv,su}(a \cdot b)$ for all $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_v$;

(x) if $s \leqslant t$ then $j_{t,s}(a)^* = j_{t^*,s^*}(a^*)$ for all $a \in \mathcal{A}_s$.

If $\mathcal{A}_s \cdot \mathcal{A}_t$ spans a dense subspace of $\mathcal{A}_{st}$ for all $s, t \in S$, we say that the Fell bundle $\mathcal{A}$ is *saturated*. If S is unital we call $\mathcal{A}_1$ the *unit fibre* of the Fell bundle $\mathcal{A}$.

One can build a C^*-algebra out of the sections of a Fell bundle.

Definition 3.3.2 ([7, Definition 3.4], [15, Definition 2.7])**.** Let $\mathcal{A}$ be a Fell bundle over a unital inverse semigroup S with unit fibre $\mathcal{A}_1 = A$. Let $\mathcal{L}(\mathcal{A}) := \bigoplus_{t\in S} \mathcal{A}_t$ and $N := \operatorname{span}\{a_s\delta_s - j_{t,s}(a_s)\delta_t : s,t \in S,\ s \leqslant t,\ a_s \in \mathcal{A}_s\}$. The *full cross sectional C^*-algebra* is $C^*(\mathcal{A})$ is defined as the maximal C^*-completion of $\mathcal{L}(\mathcal{A})/N$ with multiplication and involution inherited from the Fell bundle.

Each fibre $\mathcal{A}_t$ of the Fell bundle $\mathcal{A}$ embeds canonically in $\mathcal{L}(\mathcal{A})/N$ via $\mathcal{A}_t \ni x \mapsto [x\delta_t] \in \mathcal{L}(\mathcal{A})/N$, and so embeds in the full C^*-algebra $C^*(\mathcal{A})$. Identifying each fibre with its image in $C^*(\mathcal{A})$ we see that $\mathcal{E}_t \cdot I_{t,u} = \mathcal{E}_u \cdot I_{t,u} = \mathcal{E}_t \cap \mathcal{E}_u$ for each $t,u \in S$. In particular, we have $I_{1,t} = \mathcal{E}_t \cap A$.

Proposition 3.3.3. *Let A be a C^*-algebra, S an inverse semigroup, and $\mathcal{E} = (\mathcal{E}_t, \mu_{t,u})_{t,u\in S}$ an action of S on A by Hilbert bimodules. There exists a Fell bundle $\mathcal{A}$ over S with unit fibre $M_{\mathrm{loc}}(A)$ such that $\mathcal{A}_t = (\mathcal{E}_t)_{\mathrm{loc}}$ for each $t \in S$.*

Proof. Define $\mathcal{A}_t := (\mathcal{E}_t)_{\mathrm{loc}}$. For $t,u \in S$ we define the multiplication by

$$\xi_t \cdot \eta_u := \tilde{\mu}_{t,u}(\xi_t \otimes_{M_{\mathrm{loc}}(A)} \eta_u), \qquad \xi_t \in \mathcal{A}_t, \eta_u \in \mathcal{A}_u,$$

where $\tilde{\mu}_{t,u}$ is the map induced by $\mu_{t,u}$ using Proposition 2.2.15. By Lemma 2.2.4 and Proposition 2.2.12 we canonically identify $(\mathcal{E}_t)^*_{\mathrm{loc}}$ and $(\mathcal{E}_{t^*})_{\mathrm{loc}}$, and can define the $^* : \mathcal{A}_t \to \mathcal{A}_{t^*}$ by mapping each $\xi_t \in (\mathcal{E}_t)_{\mathrm{loc}}$ to its corresponding adjoint $\xi_t^* \in (\mathcal{E}_t)^*_{\mathrm{loc}} = (\mathcal{E}_{t^*})_{\mathrm{loc}}$.

One readily checks that this structure satisfies the axioms of a Fell bundle. □

Definition 3.3.4. We call the Fell bundle $\mathcal{A}$ defined in Proposition 3.3.3 the *induced local Fell bundle* of the action $\mathcal{E}$.

The main use of the induced local Fell bundle will be that it 'closes' the action. By this, we mean the associated conditional expectation for this Fell bundle shall be a genuine conditional expectation $C^*(\mathcal{A}) \to M_{\mathrm{loc}}(A)$ rather than a generalised local expectation taking values in $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ (note here that the term 'genuine' is in reference to the inclusion $M_{\mathrm{loc}}(A) \subseteq C^*(\mathcal{A})$). This is however not immediately true, but it can be done without loss of generality. To show this, we recall that any such inverse semigroup action can be refined to one with "enough idempotents".

Proposition 3.3.5 ([15, Proposition 5.2])**.** *Let $\mathcal{E}$ be an action of a unital inverse semigroup S on a C^*-algebra A. Let $\iota : S \to \mathrm{Bis}(\hat{A} \rtimes S)$ be the canonical inclusion $\iota(t) = U_t := \{[\hat{\mathcal{E}}_t, [\pi]] : [\pi] \in \widehat{s(\mathcal{E}_t)}\}$. There is an action $\bar{\mathcal{E}}$ of $\bar{S} := \mathrm{Bis}(\hat{A} \rtimes S)$ on A such that $\bar{\mathcal{E}}_{\iota(t)} = \mathcal{E}_t$ for all $t \in S$ and a canonical isomorphism $A \rtimes S \cong A \rtimes \mathrm{Bis}(\hat{A} \rtimes S)$ that restricts to the identity on each $\mathcal{E}_t = \bar{\mathcal{E}}_{\iota(t)}$. If the map ι is surjective, we say that $\mathcal{E}$ is* fine.

The action $\bar{\mathcal{E}}$ in Proposition 3.3.5 is called the *refinement* of $\mathcal{E}$. The proof of Proposition 3.3.5 in [15] shows that each bimodule $\bar{\mathcal{E}}_U$ for $U \in \mathrm{Bis}(\hat{A} \rtimes S)$ is of the form

$$\bar{\mathcal{E}}_U = \overline{\sum_{i\in C} \mathcal{E}_{t_i} J_i},$$

for some $t_i \in S$ and $J_i \lhd A$, where C is some indexing set.

Lemma 3.3.6. *The action $\mathcal{E}$ is aperiodic if and only if the refinement $\bar{\mathcal{E}}$ is aperiodic.*

Proof. If the refined action $\bar{\mathcal{E}}$ is aperiodic then $\mathcal{E}_t \cdot I_{1,t}^{\perp} = \bar{\mathcal{E}}_{U_t} \cdot I_{1,U_t}^{\perp}$ is aperiodic for all $t \in S$. Thus $\mathcal{E}$ is aperiodic.

Suppose the action $\mathcal{E}$ is aperiodic. Fix $U \in \operatorname{Bis}(\hat{A} \rtimes S)$ and write $\bar{\mathcal{E}}_U = \overline{\sum_{i\in C} \mathcal{E}_{t_i} \cdot J_i}$ for some $t_i \in S$ and $J_i \lhd A$, where $i \in C$ and C is some indexing set. For each $U \in \operatorname{Bis}(\hat{A} \rtimes S)$ the supremum over $V \leqslant 1, U$ in $\operatorname{Bis}(\hat{A} \rtimes S)$ is the bisection $U \cap \hat{A}$. Thus the intersection ideal $I_{1,U}$ is given by $\bar{\mathcal{E}}_{U\cap\hat{A}} = \overline{\sum_i \mathcal{E}_{t_i} \cdot J_i} \cap A$. For each $i \in C$ we have

$$\begin{aligned} J_i \cdot I_{1,U}^{\perp} &= J_i \cdot \left(\overline{\sum_{k\in C} \mathcal{E}_{t_k} \cdot J_k} \cap A\right)^{\perp} \\ &\subseteq J_i \cdot (\mathcal{E}_{t_i} \cdot J_i \cap A)^{\perp} \\ &= J_i \cdot (\mathcal{E}_{t_i} \cap J_i)^{\perp}. \end{aligned}$$

Thus for each $i \in C$ we have

$$\begin{aligned} \mathcal{E}_{t_i} \cdot J_i \cdot I_{1,U}^{\perp} &\subseteq \mathcal{E}_{t_i} \cdot J_i \cdot (\mathcal{E}_{t_i} \cap J_i)^{\perp} \\ &= \mathcal{E}_{t_i} \cdot J_i \cdot I_{1,t}^{\perp} \\ &\subseteq \mathcal{E}_{t_i} \cdot I_{1,t_i}^{\perp}. \end{aligned}$$

The bidmodule $\mathcal{E}_{t_i} \cdot I_{1,t_i}^{\perp}$ is aperiodic as the action $\mathcal{E}$ is aperiodic, and so we see that each $\mathcal{E}_{t_i} \cdot J_i \cdot I_{1,U}^{\perp}$ is an aperiodic bimodule. The span of these bimodules over $i \in C$ is a dense subspace of $\bar{\mathcal{E}}_U \cdot I_{1,U}^{\perp}$. Thus the bimodule $\bar{\mathcal{E}}_U \cdot I_{1,U}^{\perp}$ is aperiodic by [14, Lemma 4.2], and so the action $\bar{\mathcal{E}}$ is aperiodic. ☐

Corollary 3.3.7. *The canonical isomorphism $A \rtimes_{\mathcal{E}} S \cong A \rtimes_{\bar{\mathcal{E}}} \bar{S}$ entwines the canonical local expectations associated to each action, and hence the isomorphism descends to an isomorphism $A \rtimes_{\mathcal{E},\mathrm{ess}} S \cong A \rtimes_{\bar{\mathcal{E}},\mathrm{ess}} \bar{S}$ of the essential crossed products.*

Proof. The canonical local conditional expectations agree on each $\mathcal{E}_t = \bar{\mathcal{E}}_{U_t}$ for each $t \in S$, and these span dense subspaces of both $A \rtimes_{\mathcal{E}} S$ and $A \rtimes_{\bar{\mathcal{E}}} \bar{S}$. Thus the isomorphism descends to the essential crossed products. ☐

Proposition 3.3.5, Lemma 3.3.6, and Corollary 3.3.7 together allow us to, without loss of generality, assume that any action we wish to consider is refined. In particular, we gain the intersection property that allows us to write any intersection ideal associated to $t, u \in S$ as $I_{t,u} = \mathcal{E}_{t\cap u}$ for some $t \cap u \in S$. From this point on we identify an action $\mathcal{E}$ with its image in the refinement in this way, or we may assume the action $\mathcal{E}$ is fine.

Lemma 3.3.8. *Let $\mathcal{E} : S \rightharpoonup A$ be a fine action, so that $S \cong \operatorname{Bis}(\hat{A} \rtimes S)$. For each $t \in S$ let $\mathcal{I}_{1,t} := \overline{\sum_{v\leqslant 1,t} s(\mathcal{A}_v)}$ be the intersection ideal for $1, t \in S$ for the induced local Fell bundle $\mathcal{A}$. Assume that the action $\mathcal{E}$ is fine. Then $\mathcal{I}_{1,t} = M_{\mathrm{loc}}(I_{1,t})$.*

Proof. Identifying S with $\operatorname{Bis}(\hat{A} \rtimes S)$ we see that $\mathcal{I}_{1,t} = \tilde{\mathcal{E}}_{t\cap\hat{A}} = (\mathcal{E}_{t\cap\hat{A}})_{\mathrm{loc}} = (I_{1,t})_{\mathrm{loc}} = M_{\mathrm{loc}}(I_{1,t})$. ☐

Corollary 3.3.9. *The canonical weak conditional expectation $\tilde{E} : C^*(\mathcal{A}) \to M_{\mathrm{loc}}(A)''$ and local expectation $\tilde{EL} : C^*(\mathcal{A}) \to M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ are genuine (that is, take values in $M_{\mathrm{loc}}(A)$, and agree).*

Proof. Using the characterisation of $\tilde{EL}$ in [4, Lemma 4.5] and the definition of $\tilde{EL}$, we see that for $t \in S$ and $\xi \in \mathcal{A}_t$ we have

$$\tilde{E}(\xi) = \tilde{EL}(\xi) = \xi \cdot 1_{I_{1,t}},$$

where $1_{I_{1,t}}$ is the unit in $M_{\mathrm{loc}}(I_{1,t})$, which is equal to $\mathcal{I}_{1,t}$ by Lemma 3.3.8. □

The canonical weak expectation mentioned in Corollary 3.3.9 is that defined in [4, Lemma 4.5]. The definition of this expectation is not given in here, since under the conditions we impose this expectation is equal to the canonical local expectation.

We shall now use [3, Theorem 7.2] to gain a saturated Fell bundle $\tilde{\mathcal{E}}$ over an inverse semigroup $\tilde{S}$, such that any constructed C^*-algebras from $\mathcal{A}$ and $\tilde{\mathcal{E}}$ are isomorphic via an isomorphism that entwines conditional expectations. If the original action is aperiodic, the induced local action we construct will be closed and purely outer.

Proposition 3.3.10. *Let $\mathcal{E}$ be an action of S on A. There exists an inverse semigroup $\tilde{S}$ and a closed inverse semigroup action $\tilde{\mathcal{E}}$ of $\tilde{S}$ on $M_{\mathrm{loc}}(A)$ such that $M_{\mathrm{loc}}(A) \rtimes \tilde{S} \cong C^*(\mathcal{A})$ via an isomorphism that maps $M_{\mathrm{loc}}(A)$ identically to itself and entwines the canonical local conditional expectations. Moreover, if $\mathcal{E}$ is an aperiodic action, then the action $\tilde{\mathcal{E}}$ is purely outer.*

Proof. The saturated Fell bundle exists by [3, Theorem 7.2] and the bimodules associated to the action are of the form $\tilde{\mathcal{E}}_x = r(\mathcal{A}_{t_1}) \ldots r(\mathcal{A}_{t_n})\mathcal{A}_t$ for some $t_1, \ldots, t_n, t \in S$, and the isomorphism described in [3, Theorem 7.2] maps each $\mathcal{A}_t$ identically to itself. We see that the isomorphism $C^*(\tilde{\mathcal{E}}) \to M_{\mathrm{loc}}(A) \rtimes \tilde{S}$ must preserve the conditional expectations, since on each fibre $\mathcal{A}_x$ we have that the conditional expectation is the restriction of $\tilde{E}$ on $\mathcal{A}_t$ to a subbimodule. The conditional expectation is genuine on $C^*(\mathcal{A})$ by Corollary 3.3.9, so the action $\tilde{\mathcal{E}}$ is closed. If $\mathcal{E}$ is an aperiodic action, then each of these bimodules acts purely outerly since each $\mathcal{A}_t \cdot \mathcal{I}_{1,t}^{\perp} = (\mathcal{E}_t \cdot I_{1,t}^{\perp})_{\mathrm{loc}}$ is purely outer by Lemma 2.3.4. □

Remark 3.3.11. The inverse semigroup $\tilde{S}$ in Proposition 3.3.10 is called the *prefix expansion* of S and its construction can be found in [3]. The prefix expansion of S is the inverse semigroup generated by S under alternative relations. These relations give rise to more idempotents in $\tilde{S}$, as well as a canonical injective *partial homomorphism* $\pi : S \to \tilde{S}$, which is an injective map satisfying $\pi(t)^* = \pi(t^*)$, $\pi(tu) \leqslant \pi(t)\pi(u)$ for all $t, u \in S$, and if $t \leqslant u$ then $\pi(t) \leqslant \pi(u)$. In the context of this induced local Fell bundle, we have shown that $\tilde{\mathcal{E}}_t^* = \tilde{\mathcal{E}}_{t^*}$, $\tilde{\mathcal{E}}_t \otimes_{M_{\mathrm{loc}}(A)} \tilde{\mathcal{E}}_u \subseteq \tilde{\mathcal{E}}_{tu}$ for all $t, u \in S$, and $\tilde{\mathcal{E}}_t \subseteq \tilde{\mathcal{E}}_u$ in $C^*(\tilde{\mathcal{E}})$ whenever $t \leqslant u$, which exactly describes a partial homomorphism.

The prefix expansion inverse semigroup (denoted $\mathrm{Pr}(S)$) in general differs from the refinement $\mathrm{Bis}(\hat{A} \rtimes S)$ in Proposition 3.3.5 for two main reasons. The construction of the prefix inverse semigroup depends only on the inverse semigroup S and is therefore independent of the C^*-algebra A and the action $S \curvearrowright A$, whereas the refinement $\mathrm{Bis}(\hat{A} \rtimes S)$ can differ based on both these choices. For example, the idempotent lattice of $\mathrm{Bis}(\hat{A} \rtimes S)$ is canonically isomorphic to the ideal lattice of A. The second place where the construction of $\mathrm{Pr}(S)$ differs is a consequence of its universal property. Since the prefix inverse semigroup is universal for submultiplicative maps from S, the canonical map $S \to \mathrm{Pr}(S)$ may not be a semigroup homomorphism, but rather a submultiplicative map. The map $S \to \mathrm{Bis}(\hat{A} \rtimes S)$ however is always a semigroup homomorphism.

Definition 3.3.12. Let $\mathcal{E}$ be an action of S on A. We call the action $\tilde{\mathcal{E}}$ of $\tilde{S}$ on $M_{\mathrm{loc}}(A)$ the *induced local action.*

The induced local action coming from an aperiodic action $\mathcal{E}$ gives rise to a Cartan inclusion $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} S$ in the sense of Exel [7] by [15, Theorem 4.3]. The inclusion we are interested in however is $A \subseteq A \rtimes_{\mathrm{ess}} S$, so we require a way to descend the structure of the induced inclusion to the original inclusion.

Theorem 3.3.13. *Let $\mathcal{E}$ be an action of S on A. There is an injective homomorphism $A \rtimes_{\mathrm{ess}} S \hookrightarrow M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ that commutes with the inclusions $\mathcal{E}_t \hookrightarrow \tilde{\mathcal{E}}_t \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ for each $t \in S$. That is, for each $t \in S$ the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E}_t & \longrightarrow & A \rtimes_{\mathrm{ess}} S \\ \big\downarrow & & \big\downarrow \\ \tilde{\mathcal{E}}_t & \longrightarrow & M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S} \end{array}$$

Proof. By Proposition 3.3.5 we may assume the action $\mathcal{E}$ is fine. Recall that $A \rtimes S$ is defined as the maximal C^*-completion of $A \rtimes_{\mathrm{alg}} S$, which is in turn the quotient of $\bigoplus_{t \in S} \mathcal{E}_t$ by $\mathcal{N} := \operatorname{span}\{\xi_t\delta_t - \vartheta_{s,t}(\xi_t)\delta_s : t, s \in S, \xi_t \in \mathcal{E}_t \cdot I_{t,s}\}$. Similarly, $M_{\mathrm{loc}}(A) \rtimes \tilde{S}$ is the quotient of $\bigoplus_{t \in \tilde{S}} \tilde{\mathcal{E}}_t$ by the ideal $\tilde{\mathcal{N}} = \operatorname{span}\{\xi_t\delta_t - \tilde{\vartheta}_{s,t}(\xi_t)\delta_s : t, s \in \tilde{S}, \xi_t \in \tilde{\mathcal{E}}_t \cdot M_{\mathrm{loc}}(I_{t,s})\}$. The inclusion $\bigoplus_{t \in S} \mathcal{E}_t \to \bigoplus_{t \in \tilde{S}} \tilde{\mathcal{E}}_t$ descends to both quotients since under this inclusion as we have $\mathcal{N} \subseteq \tilde{\mathcal{N}}$ whereby we gain a map $A \rtimes_{\mathrm{alg}} S \to M_{\mathrm{loc}}(A) \rtimes_{\mathrm{alg}} \tilde{S}$. This map then extends to a homomorphism $i : A \rtimes S \to M_{\mathrm{loc}}(A) \rtimes S$.

To see this map descends further to the essential and reduced crossed products, we first show that the map entwines conditional expectations. For $\xi_t \in \mathcal{E}_t$, the element $EL(\xi_t) \in M_{\mathrm{loc}}(A)$ is defined as the multiplier in $M(I_{1,t})$ mapping $a \mapsto \xi_t a$, where $\mathcal{E}_t \cdot I_{1,t}$ and $I_{1,t}$ are identified via $\vartheta_{1,t}$. By the proof of Corollary 3.3.9 the element $\tilde{E}(\xi_t) \in M_{\mathrm{loc}}(A)$ is given as $\tilde{E}(\xi_t) = \xi_t \cdot 1_{I_{1,t}}$. Thus for $a \in I_{1,t}$ we have $\tilde{E}(i(\xi_t))a = \xi_t \cdot a$, where $\tilde{\mathcal{E}}_t \cdot \mathcal{I}_{1,t}$ is identified with $\mathcal{I}_{1,t}$ by applying Proposition 2.2.15 to the isomorphism $\vartheta_{1,t}$. Since the bimodules $\mathcal{E}_t$ span a dense subspace of $A \rtimes_{\mathrm{ess}} S$, we see that $EL = \tilde{E} \circ i$. Since $A \rtimes_{\mathrm{ess}} S$ is the quotient of $A \rtimes S$ by the ideal $N_{EL} = \{a \in A \rtimes S : EL(a^*a) = 0\} = \{a \in A \rtimes S : \tilde{E} \circ i(a^*a) = 0\}$ we have $i(N_{EL}) \subseteq N_{\tilde{E}}$, so i descends to a map on $A \rtimes_{\mathrm{ess}} S$.

Lastly we show that i is injective on $A \rtimes_{\mathrm{ess}} S$. To see this, we note that an element $a \in A \rtimes_{\mathrm{ess}} S$ is mapped to zero under i if and only if it satisfies $\tilde{E}(i(a^*a)) = EL(a^*a) = 0$, giving $a = 0$. ☐

Theorem 3.3.13 allows the dynamic and algebraic structures of the actions $\mathcal{E}$ and $\tilde{\mathcal{E}}$ to be encoded in the one algebra $M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$, and provides a setting useful for computations. Throughout the rest of this article we identify the modules $\mathcal{E}_t, \tilde{\mathcal{E}}_t$, and the algebras $A, M_{\mathrm{loc}}(A)$ and $A \rtimes_{\mathrm{ess}} S$ with their images in $M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ via Theorem 3.3.13.

Remark 3.3.14. The refinement from S to $\operatorname{Bis}(\hat{A} \rtimes S)$ is required since taking local multiplier algebras does not commute with taking inductive limits. Given a countable collection $(A_n)_{n \in \mathbb{N}}$ of non-zero C^*-algebras, one may consider the inductive system given by algebras $B_n = \bigoplus_{j=1}^{n} A_j$ with maps $\iota_n = (\mathrm{id}_{B_n}, 0) : B_n \to B_n \oplus A_{n+1} = B_{n+1}$. The inductive limit of this system is the direct sum $\bigoplus_{n \in \mathbb{N}} A_n$. Finite direct sums agree with finite direct products, so Lemma 2.1.3 gives that $M_{\mathrm{loc}}(\cdot)$ preserves finite direct sums. By Lemma 2.1.3, the local multipiler algebra of the inductive limit of the system B_n is then

$$M_{\mathrm{loc}}\left(\varinjlim B_n\right) = M_{\mathrm{loc}}\left(\bigoplus_{n \in \mathbb{N}} A_n\right) = \prod_{n \in \mathbb{N}} M_{\mathrm{loc}}(A_n),$$

which is a unital C^*-algebra. The inductive limit of local multiplier algebras is however

$$\varinjlim M_{\mathrm{loc}}(B_n) = \varinjlim \bigoplus_{j=1}^{n} M_{\mathrm{loc}}(A_j) = \bigoplus_{n \in \mathbb{N}} M_{\mathrm{loc}}(A_n).$$

This cannot be isomorphic to the local multiplier algebra of the inductive limit, since direct sums of infinitely many non-zero C^*-algebras are never unital.

Chapter 4

Aperiodic dynamical inclusions

With Theorem 3.3.13 we see that $A \rtimes_{\text{ess}} S$ embeds into $M_{\text{loc}}(A) \rtimes_{\text{r}} \tilde{S}$ in a way that preserves the inclusion $A \subseteq M_{\text{loc}}(A)$. As briefly mentioned, the inclusion $A \subseteq M_{\text{loc}}(A)$ is problematic, and so we wish to ensure that the intersection of $A \rtimes_{\text{ess}} S$ and $M_{\text{loc}}(A)$ in $M_{\text{loc}}(A) \rtimes_{\text{r}} \tilde{S}$ is as small as possible. The local conditional expectation $EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A)$ restricts to a *-homomorphism on at least A, since $EL|_A = \text{id}_A$, and so to minimise the intersection between $M_{\text{loc}}(A)$ and $A \rtimes_{\text{ess}} S$ we investigate the circumstances under which EL restricts to a *-homomorphism only on A.

4.1 Inclusions with minimal multiplicative domain

Definition 4.1.1. Let $A \subseteq B$ be an inclusion of C^*-algebras and let $E : B \to \tilde{A}$ be a generalised expectation. We say that E has *minimal multiplicative domain* (MMD) if the following subset is equal to A:

$$\mu(E) := \{b \in B : E(b^*b) = E(b^*)E(b), E(bb^*) = E(b)E(b^*)\}.$$

The set $\mu(E)$ for an expectation E is the (two-sided) multiplicative domain of E; the largest C^*-subalgebra of B, on which E restricts to a *-homomorphism. This characterisation is adapted from Choi [6, Theorem 3.1] in which the multiplicative domain for 2-positive maps between C^*-algebras is defined. This formulation applies here as generalised expectations are completely positive by definition, and the two-sided condition ensures that $\mu(E)$ is *-closed. The 'minimal' descriptor of the definition of (MMD) stems from the fact that we always have $A \subseteq \mu(E)$, as E restricts to the identity on A.

If E is faithful, then E restricts to an injective *-homomorphism $\mu(E) \hookrightarrow \tilde{A}$. Thus, in the case of a faithful genuine conditional expectation $E : B \to A$, the multiplicative domain of E is always minimal.

Lemma 4.1.2. *Let $\mathcal{E}$ be an action of S on A and let $\tilde{\mathcal{E}}$ be its induced local action. Consider $A \rtimes_{\text{ess}} S$ as a subalgebra of $M_{\text{loc}}(A) \rtimes_{\text{r}} \tilde{S}$ as in Theorem 3.3.13. Then the multiplicative domain of $EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A)$ is equal to the intersection $M_{\text{loc}}(A) \cap A \rtimes_{\text{ess}} S$, where the intersection is taken in $M_{\text{loc}}(A) \rtimes_{\text{r}} \tilde{S}$.*

Proof. Let $\tilde{E} : M_{\text{loc}}(A) \rtimes_{\text{r}} \tilde{S} \to M_{\text{loc}}(A)$ be the canonical conditional expectation for the action $\tilde{\mathcal{E}}$. By Theorem 3.3.13 the inclusion $A \rtimes_{\text{ess}} S \subseteq M_{\text{loc}}(A) \rtimes_{\text{r}} \tilde{S}$ entwines conditional expectations, so we have $\tilde{E}|_{A \rtimes_{\text{ess}} S} = EL$. Since $\tilde{E}$ is a faithful genuine expectation we have $\mu(\tilde{E}) = M_{\text{loc}}(A)$. Thus

$$\mu(EL) = \mu(\tilde{E}|_{A \rtimes_{\text{ess}} S}) = \mu(\tilde{E}) \cap A \rtimes_{\text{ess}} S = M_{\text{loc}}(A) \cap A \rtimes_{\text{ess}} S. \qquad \square$$

Corollary 4.1.3. *The local expectation EL for the inclusion $A \subseteq A \rtimes_{\text{ess}} S$ has minimal multiplicative domain if and only if $M_{\text{loc}}(A) \cap A \rtimes_{\text{ess}} S = A$.*

Corollary 4.1.3 does not work generally for the modules $\mathcal{E}_t \subseteq A \rtimes_{\text{ess}} S$: it fails even for non-unital ideals $I \lhd A$ since Lemma 2.1.4 gives $M_{\text{loc}}(I) \cap A \rtimes_{\text{ess}} S = M(I) \cap A \supseteq I$. However, we always will have containment $\mathcal{E}_t \subseteq \tilde{\mathcal{E}}_t \cap A \rtimes_{\text{ess}} S$.

Lemma 4.1.4. *Let $\mathcal{E}$ be an action such that $EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A)$ has minimal multiplicative domain. For $t \in S$ we have $\mathcal{E}_t = (\tilde{\mathcal{E}}_t \cap A \rtimes_{\text{ess}} S) \cdot s(\mathcal{E}_t)$.*

Proof. The inclusion $\mathcal{E}_t \subseteq (\tilde{\mathcal{E}}_t \cap A \rtimes_{\text{ess}} S) \cdot s(\mathcal{E}_t)$ follows since $\mathcal{E}_t$ is contained in both $\tilde{\mathcal{E}}_t$ and $A \rtimes_{\text{ess}} S$, and $\mathcal{E}_t = \mathcal{E}_t \cdot s(\mathcal{E}_t)$. We shall show that $\tilde{\mathcal{E}}_t \cap A \rtimes_{\text{ess}} S$ is a Hilbert A-bimodule, and then the reverse inclusion follows as the source ideals will be equal.

For $\xi, \eta \in \tilde{\mathcal{E}}_t \cap A \rtimes_{\text{ess}} S$, both $\langle \xi, \eta \rangle = \xi^* \eta$ and $\langle\langle \xi, \eta \rangle\rangle = \xi \eta^*$ belong to $M_{\text{loc}}(A)$ as $\tilde{\mathcal{E}}_t$ is a Hilbert $M_{\text{loc}}(A)$-bimodule, and belong to $A \rtimes_{\text{ess}} S$ as $\xi, \eta \in A \rtimes_{\text{ess}} S$ which is closed under its own multiplication. Thus the inner products of $\tilde{\mathcal{E}}_t \cap A \rtimes_{\text{ess}} S$ take values in $M_{\text{loc}}(A) \cap A \rtimes_{\text{ess}} S$, which is A by Corollary 4.1.3. Then $\tilde{\mathcal{E}}_t \cap A \rtimes_{\text{ess}} S$ is closed under the left and right A-multiplications as both $\tilde{\mathcal{E}}_t$ and $A \rtimes_{\text{ess}} S$ are, and is norm-closed as an intersection of closed subsets of $M_{\text{loc}}(A) \rtimes_{\text{r}} \tilde{S}$. □

4.2 Slice reconstruction

With these stronger results for inclusions with minimal multiplicative domain we can now define the class of inclusions we wish to study.

Definition 4.2.1. Let $A \subseteq B$ be an inclusion of C^*-algebras. We say the inclusion is an *aperiodic dynamical inclusion* if the following conditions hold:

1. $A \subseteq B$ is non-degenerate;
2. there exists an inverse subsemigroup $S \subseteq \mathcal{S}(A, B)$ such that the tautological action of S on A is aperiodic; and
3. there exists a faithful pseudo-expectation $E : B \to I(A)$ with minimal multiplicative domain.

Proposition 4.2.2. *Let $A \subseteq B$ be an aperiodic dynamical inclusion. Let $E : B \to I(A)$ be the associated faithful pseudo-expectation with minimal multiplicative domain and let $S \subseteq \mathcal{S}(A, B)$ be the distinguished inverse subsemigroup acting aperiodically. Then there is an isomorphism $\varphi : A \rtimes_{\text{ess}} S \to B$ that entwines conditional expectations and hence restricts to the identity map on A.*

Proof. Aperiodic dynamical inclusions satisfy the hypotheses of Theorem 3.1.8, which then gives the desired results. □

Corollary 4.2.3. *Let $\mathcal{E}$ be an aperiodic action such that the local expectation $EL : A \rtimes_{\text{ess}} S \to M_{\text{loc}}(A)$ has minimal multiplicative domain. Then $A \subseteq A \rtimes_{\text{ess}} S$ is an aperiodic dynamical inclusion, and up to isomorphism every aperiodic dynamical inclusion is of this form.*

One of the statements in [15, Theorem 5.6] is that the bisection inverse semigroup $\text{Bis}(\hat{A} \rtimes S)$ for a closed and purely outer action is isomorphic to the slice inverse semigroup $\mathcal{S}(A, A \rtimes_{\text{r}} S)$. Slices for the inclusion $A \subseteq A \rtimes_{\text{r}} S$ can then be recovered from their intersections with the bimodules $\mathcal{E}_t$. Thus all non-trivial slices for the inclusion $A \subseteq A \rtimes_{\mathrm{r}} S$ inherit pure-outerness, and there can be no wayward slices outside of those coming from the action. We shall apply this to the inclusion $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$, and then use (MMD) to descend properties to the inclusion $A \subseteq A \rtimes_{\mathrm{ess}} S$.

Lemma 4.2.4. *Let $\mathcal{E} : S \rightharpoonup A$ be a closed and purely outer action. Let $X \subseteq A \rtimes_{\mathrm{r}} S$ be a slice. Then $X = \overline{\sum_{t \in S} X \cap \mathcal{E}_t}$.*

Proof. The inclusion $\overline{\sum_{t \in S} X \cap \mathcal{E}_t} \subseteq X$ is clear. The proof of [15, Theorem 5.6] shows that X is the closure of the span of slices $\mathcal{E}_t \cdot J_t$ for some $t \in T \subseteq S$ and ideals $J_t \lhd A$. Each $\mathcal{E}_t \cdot J_t$ is contained in $\mathcal{E}_t$ and X, so we gain the desired result. □

In the remainder of this chapter we shall fix an action $\mathcal{E}$ of S on A such that the canonical local expectation $EL : A \rtimes_{\mathrm{ess}} S \to M_{\mathrm{loc}}(A)$ has minimal multiplicative domain. We shall denote by $\tilde{\mathcal{E}}$ the induced local action of $\tilde{S}$ on $M_{\mathrm{loc}}(A)$ arising from $\mathcal{E}$. By Proposition 3.3.5 we may and will without loss of generality assume the action $\mathcal{E}$ to be fine.

Corollary 4.2.5. *Let $\tilde{Y} \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ be a slice for the induced inclusion $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$. Then $\tilde{Y} = \{0\}$ if and only if $\tilde{Y} \cap A \rtimes_{\mathrm{ess}} S = \{0\}$, where the intersection is taken in $M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ using Theorem* 3.3.13.

Proof. If $\tilde{Y} = \{0\}$ then clearly $\tilde{Y} \cap A \rtimes_{\mathrm{ess}} S = \{0\}$. Conversely if $\tilde{Y} \neq \{0\}$ then Lemma 4.2.4 implies that for some $t \in S$ we have $\tilde{Y} \cap \tilde{\mathcal{E}}_t \neq \{0\}$, which is a subbimodule of $\tilde{\mathcal{E}}_t$. Lemma 2.2.13 then gives $\{0\} \neq \tilde{Y} \cap \tilde{\mathcal{E}}_t \cap \mathcal{E}_t \subseteq \tilde{Y} \cap A \rtimes_{\mathrm{ess}} S$. □

If $X \subseteq A \rtimes_{\mathrm{ess}} S$ is a slice, one would want to analyse a corresponding slice $\check{X} \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ generated by X.

Lemma 4.2.6. *Let $X \subseteq A \rtimes_{\mathrm{ess}} S$ be a slice for the inclusion $A \subseteq A \rtimes_{\mathrm{ess}} S$. Considering $A \rtimes_{\mathrm{ess}} S \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ as in Theorem* 3.3.13, *$\check{X} := \overline{\operatorname{span}} M_{\mathrm{loc}}(A) \cdot X \cdot M_{\mathrm{loc}}(A)$ is a slice for $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ and satisfies $X = (\check{X} \cap A \rtimes_{\mathrm{ess}} S) \cdot s(X)$.*

Proof. We first show that $\check{X}$ is a slice for $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$. Fix $x, y \in X$ and $a \in M(J)$ for some $J \in \mathcal{I}_e(A)$. Let $I := s(J \cdot X) \oplus s(J \cdot X)^{\perp}$. This ideal is essential in A by construction, and satisfies $X \cdot I = J \cdot X$. By the Cohen-Hewitt Factorisation Theorem for all $b \in I$ there exists $z \in X$ and $c \in J$ such that $yb = cz$. We then have $(x^*ay)b = x^*(ac)z \in X^*M(J)JX = X^*JX = s(J \cdot X) \subseteq I$. We also note that $b(x^*ay) = (xb^*)^*ay$ belongs to $(X \cdot I)^*M(J)X = (J \cdot X)^*M(J)X = X^*JM(J)X \subseteq I$. Thus x^*ay is a multiplier on I, so belongs to the local multiplier algebra. Since $||x^*ay|| \leqslant ||x|| \cdot ||y|| \cdot ||a||$, it follows that $x^*M_{\mathrm{loc}}(A)y \subseteq M_{\mathrm{loc}}(A)$ for all $x, y \in X$. Thus $\check{X}^*M_{\mathrm{loc}}(A)\check{X} = \overline{\operatorname{span}}\, M_{\mathrm{loc}}(A) \cdot X^* \cdot M_{\mathrm{loc}}(A) \cdot X \cdot M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A)^3 = M_{\mathrm{loc}}(A)$. Symmetrically $\check{X}M_{\mathrm{loc}}(A)\check{X}^* \subseteq M_{\mathrm{loc}}(A)$, so $\check{X}$ is a slice.

Similarly to Lemma 4.1.4, the inclusion $X \subseteq (\check{X} \cap A \rtimes_{\mathrm{ess}} S) \cdot s(X)$ is clear. For the reverse inclusion, we see that $\check{X} \cap A \rtimes_{\mathrm{ess}} S$ has inner products taking value in $M_{\mathrm{loc}}(A) \cap A \rtimes_{\mathrm{ess}} S$, which is equal to A by Corollary 4.1.3. By cutting down with the source ideal $s(X)$, we then gain the desired equality. □

Corollary 4.2.7. *Let $X \subseteq A \rtimes_{\mathrm{ess}} S$ be a slice and let $\check{X} = \overline{\operatorname{span}}\, M_{\mathrm{loc}}(A) \cdot X \cdot M_{\mathrm{loc}}(A)$ be the slice in $M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ generated by X. Then for any Hilbert A-subbimodule $Y \subseteq \check{X} \cap A \rtimes_{\mathrm{ess}} S$ we have $Y = \{0\}$ if and only if $Y \cap X = \{0\}$.*

Proof. The source ideal of $\check{X}$ annihilates $M_{\mathrm{loc}}(s(X)^{\perp})$ in $M_{\mathrm{loc}}(A)$ since

$$\check{X}s(X)^{\perp} \subseteq \overline{\operatorname{span}}\, M_{\mathrm{loc}}(A)XM_{\mathrm{loc}}(s(X)^{\perp}) = \{0\}.$$

Thus $s(\check{X}) \subseteq M_{\mathrm{loc}}(s(X)^{\perp})^{\perp} = M_{\mathrm{loc}}(s(X))$. We then have $s(\check{X} \cap A \rtimes_{\mathrm{ess}} S) \subseteq M_{\mathrm{loc}}(s(X)) \cap A \rtimes_{\mathrm{ess}} S$, which is equal to $M(s(X)) \cap A$ by Lemma 2.1.4 and Corollary 4.1.3.

We claim that $s(X)$ is an essential ideal of $s(\check{X} \cap A \rtimes_{\mathrm{ess}} S)$. Note that the only element of $M_{\mathrm{loc}}(s(X))$ that annihilates $s(X)$ is 0, and so the annihilator of $s(X)$ in $M_{\mathrm{loc}}(A)$ is $M_{\mathrm{loc}}(s(X)^{\perp})$ by Lemma 2.1.3. We then see that $X \cdot M_{\mathrm{loc}}(A) \cdot s(X)^{\perp} \subseteq X \cdot M_{\mathrm{loc}}(s(X)^{\perp}) = \{0\}$, giving $\check{X} \cdot s(X)^{\perp} = \{0\}$. Thus $s(X)$ must be an essential ideal of $s(\check{X} \cap A \rtimes_{\mathrm{ess}} S)$. Hence if $Y \subseteq \check{X} \cap A \rtimes_{\mathrm{ess}} S$ is a non-zero Hilbert A-subbimodule, then $Y \cdot s(X) \neq \{0\}$ as $s(X)$ is essential in $s(\check{X} \cap A \rtimes_{\mathrm{ess}} S)$.

Thus $Y \cdot s(X) \subseteq (\check{X} \cap A \rtimes_{\mathrm{ess}} S) \cdot s(X) = X$, so $\{0\} \neq Y \cdot s(X) \subseteq X \cap Y$. Contrapositively, $Y \cap X = \{0\}$ gives $Y = \{0\}$. □

The conditional expectation having minimal multiplicative domain allows us to analyse slices of $A \subseteq A \rtimes_{\mathrm{ess}} S$ by analysing slices of $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ and descending down to the smaller inclusion. If the action $\mathcal{E}$ is aperiodic, then the induced action $\tilde{\mathcal{E}}$ is closed and purely outer by Proposition 3.3.10, and so the inclusion $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ satisfies the hypotheses of Lemma 4.2.4. We use this now to show that all slices in an aperiodic dynamical inclusion act aperiodically, and all choices of inverse subsemigroup $S \subseteq \mathcal{S}(A, B)$ that densely span B give rise to the same essential crossed product. From this point on we assume that the action $\mathcal{E}$ is aperiodic, so that the induced local action $\tilde{\mathcal{E}}$ is purely outer.

Lemma 4.2.8. *Suppose that the action $\mathcal{E}$ is aperiodic. Let $X \subseteq A \rtimes_{\mathrm{ess}} S$ be a slice for the inclusion $A \subseteq A \rtimes_{\mathrm{ess}} S$. Then $X = \{0\}$ if and only if $X \cap \mathcal{E}_t = \{0\}$ for all $t \in S$.*

Proof. The 'only if' direction is clear.

Suppose that $X \cap \mathcal{E}_t = \{0\}$ for all $t \in S$. Let $\check{X} := \overline{\operatorname{span}}\, M_{\mathrm{loc}}(A) \cdot X \cdot M_{\mathrm{loc}}(A)$ be the slice for the larger inclusion $M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(A) \rtimes_{\mathrm{r}} \tilde{S}$ generated by X. The equality $X \cap \mathcal{E}_t = \{0\}$ implies that $X \cap (\mathcal{E}_t \cap \check{X} \cap A \rtimes_{\mathrm{ess}} S) = \{0\}$ for each $t \in S$, and so by Corollary 4.2.7 we have $\{0\} = \mathcal{E}_t \cap \check{X} \cap A \rtimes_{\mathrm{ess}} S = \mathcal{E}_t \cap \check{X} \supseteq \mathcal{E}_t \cap \check{X} \cap \tilde{\mathcal{E}}_t$. The module $\tilde{\mathcal{E}}_t \cap \check{X}$ is a subbimodule of $\tilde{\mathcal{E}}_t$ and $\mathcal{E}_t$ detects subbimodules of its local multiplier module by Lemma 2.2.13. Thus $\mathcal{E}_t \cap \check{X} = \{0\}$ implies that $\tilde{\mathcal{E}}_t \cap \check{X} = \{0\}$. Since the action $\mathcal{E}$ is aperiodic, the induced local action is closed and purely outer by Proposition 3.3.10, and so satisfies the criteria of Lemma 4.2.4. Hence $\check{X} \cap \tilde{\mathcal{E}}_t = \{0\}$ for all $t \in S$ if and only if $\check{X} = \{0\}$. □

Corollary 4.2.9. *Suppose that the action $\mathcal{E}$ is aperiodic. For $X \in \mathcal{S}(A, A \rtimes_{\mathrm{ess}} S)$, the subslice $X_{\mathrm{ess}} := \overline{\sum_{t\in S} X \cap \mathcal{E}_t}$ has zero orthogonal complement in X, and so has non-zero intersection with all non-zero subslices of X. Equivalently, $s(X_{\mathrm{ess}})$ is an essential ideal of $s(X)$.*

Proof. We see $X_{\mathrm{ess}}^{\perp} \cap X$ satisfies $X_{\mathrm{ess}}^{\perp} \cap X \cap \mathcal{E}_t = \bigcap_{u\in S}(X \cap \mathcal{E}_u)^{\perp} \cap (X \cap \mathcal{E}_t) \subseteq (X \cap \mathcal{E}_t)^{\perp} \cap (X \cap \mathcal{E}_t) = \{0\}$. Hence $X_{\mathrm{ess}}^{\perp} \cap X = \{0\}$ by Lemma 4.2.8. If $Y \subseteq X$ is a non-zero subslice, then $Y \cap X_{\mathrm{ess}} = Y \cap \overline{\sum_{t\in S} X \cap \mathcal{E}_t} \supseteq \overline{\sum_{t\in S} Y \cap \mathcal{E}_t}$, which is zero if and only if $Y = \{0\}$ by Lemma 4.2.8. That this is equivalent to $s(X_{\mathrm{ess}}) \lhd s(X)$ being an essential ideal follows from the fact that any subbimodule $Y \subseteq X$ satisfies $Y = X \cdot s(Y)$. □

Theorem 4.2.10. *Let $A \subseteq B$ be an aperiodic dynamical inclusion. Then the tautological action of $\mathcal{S}(A, B)$ on A is aperiodic. If $T \subseteq \mathcal{S}(A, B)$ is an inverse subsemigroup that spans a dense subalgebra of B, then there are isomorphisms*

$$B \cong A \rtimes_{\mathrm{ess}} T \cong A \rtimes_{\mathrm{ess}} \mathcal{S}(A, B).$$

Moreover these isomorphisms map A identically to itself and entwine conditional expectations.

Proof. Let $S \subseteq \mathcal{S}(A, B)$ be an inverse subsemigroup that acts aperiodically and spans a dense subalgebra of B (note that such a semigroup exists since $A \subseteq B$ is an aperiodic dynamical inclusion). By Theorem 3.1.8 there is an isomorphism $B \cong A \rtimes_{\mathrm{ess}} S$ that maps A identically to itself and entwines conditional expectations.

Let $X \subseteq A \rtimes_{\mathrm{ess}} S \cong B$ be a slice and let $Y := X \cdot (X \cap A)^{\perp}$. By Corollary 4.2.9 the subslice $Y_{\mathrm{ess}} := \overline{\sum_{t \in S} Y \cap \mathcal{E}_t}$ has zero orthogonal complement in Y and so $s(Y_{\mathrm{ess}})$ is an essential ideal of $s(Y)$. For each $t \in S$ the ideal $s(Y \cap \mathcal{E}_t \cdot I_{1,t}^{\perp})$ is an essential ideal of $s(Y \cap \mathcal{E}_t)$ since $Y \cap A = \{0\}$. Applying [16, Lemma 5.12] shows that $Y \cap \mathcal{E}_t$ is an aperiodic A-bimodule since $Y \cap \mathcal{E}_t \cdot I_{1,t}^{\perp}$ is. Thus Y is aperiodic as the closed linear span of aperiodic subbimodules $Y \cap \mathcal{E}_t$, and so X acts aperiodically on A. Thus any choice of $T \subseteq \mathcal{S}(A, B)$ makes the inclusion $A \subseteq B$ an aperiodic dynamical inclusion, and so $B \cong A \rtimes_{\mathrm{ess}} T$ by Theorem 3.1.8. Particularly the choice $T = \mathcal{S}(A, B)$ gives $B \cong A \rtimes_{\mathrm{ess}} \mathcal{S}(A, B)$. □

The topology on the dual groupoid $\hat{A} \rtimes \mathcal{S}(A, B)$ has a basis given by slices of the inclusion $A \subseteq B$. Lemma 4.2.8 shows that any non-zero slice $X \in \mathcal{S}(A, B)$ intersects at least one $\mathcal{E}_t$ for some $t \in S$. In the topology of the groupoid, we see then that any open subset of $\hat{A} \rtimes \mathcal{S}(A, B)$ must intersect the open bisection defined by $\mathcal{E}_t$. Thus the bisections defined by $\mathcal{E}_t$ for $t \in S$ cover a dense subset of the dual groupoid for the full slice action.

Corollary 4.2.11. *Suppose the action $\mathcal{E}$ is aperiodic. Let $B = A \rtimes_{\mathrm{ess}} S$. The canonical groupoid homomorphism $\phi : \hat{A} \rtimes S \to \hat{A} \rtimes \mathcal{S}(A, B)$, $\phi[t, [\pi]] = [\mathcal{E}_t, [\pi]]$ has open range. If for each $t \in S$ and each ideal $I \lhd A$ there is an idempotent $e \in S$ with $\mathcal{E}_t \cdot I = \mathcal{E}_e$, then ϕ is injective.*

Proof. Each $\phi[t, [\pi]]$ belongs to $\hat{\mathcal{E}}_t$ for each $[t, [\pi]] \in A \rtimes S$ and $\bigcup_{t \in S} \hat{\mathcal{E}}_t$ is open in $A \rtimes \mathcal{S}(A, B)$ as each $\hat{\mathcal{E}}_t$ is an open bisection, and so the image of ϕ is open.

The last part giving that ϕ is injective follows from Lemma 3.2.2. □

Remark 4.2.12. The map ϕ in Corollary 4.2.11 is not necessarily injective, since the semigroup $\mathcal{S}(A, B)$ may have a more refined lattice of idempotents than S. For example if S is a group (viewed as an inverse semigroup) acting on a C^*-algebra A, then the germ relation is trivial since the only idempotent in a group is the identity. The inverse semigroup $\mathcal{S}(A, B)$ will always contain the ideal lattice of A as idempotents, and so the quotient may identify more germs. The condition of Lemma 3.2.2 ensures there are enough idempotents, which holds if in particular the action $\mathcal{E}$ is fine.

We do not know if the map ϕ is automatically injective for aperiodic inverse semigroup actions. If the action is topologically non-trivial, that is, the bimodules $\mathcal{E}_t \cdot I_{1,t}$ induce partial homeorphisms that do not fix any open subset of $\hat{A}$, then ϕ is injective. Every topologically non-trivial action is aperiodic by [14, Theorem 8.1], but the converse is not known (unless the algebra A contains an essential ideal that is simple or of Type I). If there exists a C^*-algebra A and a non-zero Hilbert A-bimodule X that is aperiodic but not topologically non-trivial, then X would generate such an action on A where the map ϕ would fail to be injective.

Chapter 5

Groupoid C^*-algebras and aperiodicity

For the rest of this thesis we consider inclusions $A \subseteq B$ where the subalgebra A is commutative. By the Gelfand-Naimark theorem, A is isomorphic to the algebra of continuous functions over some locally compact Hausdorff space, and this isomorphism comes from a contravariant equivalence between the categories of commutative C^*-algebras and locally compact Hausdorff spaces. Such pairs of C^*-algebras were studied by Renault [22], who considers regular non-degenerate maximal abelian (masa) inclusions into separable C^*-algebras with a faithful genuine conditional expectation. Renault showed that, up to isomorphism, these *commutative Cartan pairs* are given by C^*-algebras of twists over effective étale locally compact Hausdorff second countable groupoids. This is achieved by constructing a groupoid and twist to each such Cartan pair, called the *Weyl groupoid* and *Weyl twist*.

In this chapter we recall the construction of the Weyl groupoid and twist, and make comparisons between maximal abelian inclusions of C^*-algebras and certain properties of aperiodicity associated to the inclusion. We shall also give the construction of the essential (twisted) groupoid C^*-algebra of Kwaśniewski and Meyer [16], as this will allow us to further generalise the results of Renault in the following sections of the thesis.

5.1 The Weyl groupoid and Weyl twist

Let G be an étale groupoid. A *twist* over G is a central extension Σ of G by $G^{(0)} \times \mathbb{T}$; the unit space times the circle group

$$G^{(0)} \times \mathbb{T} \hookrightarrow \Sigma \twoheadrightarrow G.$$

By definition, Σ carries a central action of $\mathbb{T}$. From this one may construct a canonical line bundle associated to the twist (G, Σ) as $L := \frac{\Sigma \times \mathbb{C}}{\mathbb{T}}$, where the quotient is by the action $z(\sigma, \lambda) = (z\sigma, \bar{z}\lambda)$ for $z \in \mathbb{T}$, $\sigma \in \Sigma$, and $\lambda \in \mathbb{C}$. Sections of this bundle are then equivalent to functions $\Sigma \to \mathbb{C}$ satisfying $f(z\sigma) = \bar{z}f(\sigma)$ for all $z \in \mathbb{T}$, and we may often identify the two.

Renault defines the Weyl groupoid and Weyl twist in [22] for commutative and non-degenerate inclusions of C^*-algebras. In particular, we need not alter the definition to accommodate for our more general definition of essential Cartan pairs.

Definition 5.1.1. Let $A = C_0(X)$ be a commutative C^*-subalgebra of B containing an approximate unit for B. For each normaliser $n \in N(A, B)$ we have $n^*n, nn^* \in A$ by [22, Lemma 4.5]. Define $\mathrm{dom}(n) := \{x \in X : n^*n(x) > 0\}$ and $\mathrm{ran}(n) := \{x \in X : nn^*(x) > 0\}$.

Kumjian [13] described how normalisers of an inclusion $A \subseteq B$ of a commutative C^*-algebra act as partial homeomorphisms on the Gelfand dual X of $A = C_0(X)$.

Proposition 5.1.2 ([13, 1.6])**.** *Let $A = C_0(X)$ be a commutative C^*-subalgebra of B containing an approximate unit for B. Let $n \in N(A, B)$ be a normaliser. There exists a unique homeomorphism $\alpha_n : \mathrm{dom}(n) \to \mathrm{ran}(n)$ satisfying*

$$n^* a n(x) = a(\alpha_n(x)) n^* n(x),$$

for all $a \in C_0(X)$ and $x \in \mathrm{dom}(n)$.

Lemma 5.1.3 ([13, 1.7])**.** *Let A be a commutative C^*-subalgebra of B containing an approximate unit for B. Then*

1. *if $a \in A$ then $\alpha_a = \mathrm{id}_{\mathrm{dom}(a)}$;*
2. *for $m, n \in N(A, B)$ we have $\alpha_{mn} = \alpha_m \circ \alpha_n$ and $\alpha_n^{-1} = \alpha_{n^*}$.*

Corollary 5.1.4. *Let A be a commutative C^*-subalgebra of B containing an approximate unit for B. Let $M \subseteq B$ be a slice for the inclusion. For $m, n \in M$, let $U := \mathrm{dom}(m) \cap \mathrm{dom}(n)$. Then $\alpha_m|_U = \alpha_n|_U$*

Proof. Since M is a slice, the element m^*n belongs to A, and so the associated partial homeomorphism α_{m^*n} is the identity map on $\mathrm{dom}(m^*n)$ by Lemma 5.1.3. The same lemma implies that $\mathrm{id}_{\mathrm{dom}(m^*n)} = \alpha_{m^*n} = \alpha_m^{-1} \circ \alpha_n$, whereby $\mathrm{id}_{\mathrm{dom}(m^*n)}$ and $\alpha_m^{-1} \circ \alpha_n$ have the same domain, namely U. Thus, on U we have

$$\alpha_m|_U = \alpha_m|_U \circ \mathrm{id}_{\mathrm{dom}(m^*n)} = \alpha_m|_U \alpha_m^{-1} \circ \alpha_n|_U = \alpha_n|_U.$$

□

The collection of partial homeomorphism $\mathcal{G}(A) := \{\alpha_n : n \in N(A, B)\}$ forms a pseudogroup acting on $X = \hat{A}$.

Definition 5.1.5 ([22, Definition 4.2])**.** We call $\mathcal{G}(A)$ the *Weyl pseudogroup* of the pair (A, B). Define the *Weyl groupoid* $G(A, B)$ of (A, B) as the groupoid of germs of $\mathcal{G}(A)$.

Concretely, the Weyl groupoid consists of equivalence classes of pairs (α_n, x) where $n \in N(A, B)$ and $x \in \mathrm{dom}(n) \subseteq X = \hat{A}$. Two pairs (α_n, x) and (α_m, y) are equivalent if $x = y$ and there is an open neighbourhood $U \subseteq X$ of x with $\alpha_n|_U = \alpha_m|_U$. The composition of two equivalence classes $[\alpha_n, x]$ and $[\alpha_m, y]$ in $G(A, B)$ is defined if $x = \alpha_m(y)$, and the product is given by $[\alpha_n, x] \cdot [\alpha_m, y] = [\alpha_n \circ \alpha_m, y]$, which by Lemma 5.1.3 is equal to $[\alpha_{nm}, y]$. The inverse is given by $[\alpha_n, x]^{-1} = [\alpha_{n^*}, \alpha_n(x)]$.

Renault [22] also defines the Weyl twist $\Sigma(A, B)$ by considering pairs (n, x) where $n \in N(A, B)$ and $x \in \mathrm{dom}(n)$. Two pairs (n, x) and (m, y) are equivalent in the twist if $x = y$ and there are functions $a, b \in C_0(X) = A$ with $a(x), b(x) > 0$ and $an = bm$. The product of equivalence classes $[n, x]$ and $[m, y]$ in the twist is defined whenever the product $[\alpha_n, x]$ and $[\alpha_m, y]$ is defined in the Weyl groupoid, and is given by $[m, x] \cdot [n, y] = [mn, y]$. Similarly to the Weyl groupoid, the inverse of $[n, x]$ in the Weyl twist is $[n^*, \alpha_n(x)]$. The canonical surjection $\Sigma(A, B) \to G(A, B)$ is given by $[n, x] \mapsto [\alpha_n, x]$, which Renault shows to indeed be a twist over $G(A, B)$ if $A \subseteq B$ is a maximal abelian and non-degenerate inclusion (cf. [22, Proposition 4.12]).

Recall that an étale groupoid G is *effective* if the interior of the isotropy of G is the unit space. By construction, the Weyl groupoid is effective.

Lemma 5.1.6 ([23, Corollary 3.2.7])**.** *The Weyl groupoid is effective.*

Proof. Let G be the Weyl groupoid for a pair (A, B). We must show that the interior of the isotropy of G is the unit space $G^{(0)}$. Let $n \in N(A, B)$ be a normaliser such that the bisection $U_n := \{[\alpha_n, x] : x \in \mathrm{dom}(n)\}$ is contained in the isotropy of G. Then for all $[\alpha_n, x] \in U_n$ we have $\alpha_n(x) = r[\alpha_n, x] = s[\alpha_n, x] = x$, thus for all $x \in \mathrm{dom}(n)$ we have $\alpha_n(x) = x$. Hence $[\alpha_n, x] = [\mathrm{id}, x] \in G^{(0)}$ for all $x \in \mathrm{dom}(n)$, thus $U_n \subseteq G^{(0)}$. Since bisections of the form U_n form a basis for the topology of $G^{(0)}$, we see that G is effective. □

Let $D \subseteq X$ be a subset of a topological space X. We say D is *nowhere dense* if its closure has empty interior. We say D is *meagre* if it is a countable union of nowhere dense sets. In a Baire space, an open subset of X is never meagre unless it is empty. From this point we shall only be considering Baire spaces, this includes the class of locally Hausdorff spaces, which encompasses the groupoids we wish to consider.

Kwaśniewski and Meyer define the essential (twisted) groupoid C^*-algebra as follows (cf. [16, Section 7.2]). Let (G, Σ) be a twist over an étale groupoid with locally compact Hausdorff unit space. For each open bisection $U \subseteq G$ let $C_c(U, \Sigma)$ be the compactly supported continuous sections of the canonical line bundle $L := \frac{\mathbb{C}\times\Sigma}{\mathbb{T}}$. Let $\mathcal{C}_c(U, \Sigma)$ be the collection of sections in $C_c(U, \Sigma)$, extended by zero to sections on G. Let $\mathcal{C}_c(G, \Sigma) = \overline{\mathrm{span}}_{U\subseteq G}\, \mathcal{C}_c(U, \Sigma)$, where U ranges over open bisections of G. The space $\mathcal{C}_c(G, \Sigma)$ carries a convolution product and involution given by

$$f * g(\gamma) := \sum_{r(\gamma)=r(\eta)} f(\eta) \cdot g(\eta^{-1}\gamma), \quad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

for all $f, g \in \mathcal{C}_c(G, \Sigma)$ and $\gamma \in G$. The *full twisted groupoid C^*-algebra* $C^*(G, \Sigma)$ is defined as the maximal C^*-completion of the *-algebra $\mathcal{C}_c(G, \Sigma)$. If G is Hausdorff then all functions in $\mathcal{C}_c(G, \Sigma)$ are continuous, and so $\mathcal{C}_c(G, \Sigma) = C_c(G, \Sigma)$. Thus this construction agrees with the construction of Renault [21] if the groupoid G is Hausdorff.

The reduced twisted groupoid C^*-algebra $C^*_{\mathrm{r}}(G, \Sigma)$ is defined by constructing a family of regular representations and taking the norm induced from them. Importantly, by [16, Proposition 7.10] one may embed $C^*_{\mathrm{r}}(G, \Sigma)$ into the space of Borel sections of the line bundle L (denoted $\mathfrak{B}(G, \Sigma)$) via an injective and contractive linear map, and there is a generalised expectation $C^*_{\mathrm{r}}(G, \Sigma) \to \mathfrak{B}(G^{(0)})$ given by restricting these sections to the unit space. We will often identify $C^*_{\mathrm{r}}(G, \Sigma)$ with its image in $\mathfrak{B}(G, \Sigma)$, so that we may consider elements of $C^*_{\mathrm{r}}(G, \Sigma)$ as functions $\Sigma \to \mathbb{C}$.

Definition 5.1.7 (cf. [16, Proposition 7.18])**.** The *essential twisted groupoid C^*-algebra* $C^*_{\mathrm{ess}}(G, \Sigma)$ is defined as the quotient of $C^*_{\mathrm{r}}(G, \Sigma)$ by the ideal

$$J_{\mathrm{sing}} := \{f \in C^*_{\mathrm{r}}(G, \Sigma) : (f^*f)|_{G^{(0)}} \text{ has meagre support}\}.$$

5.1.1 Aperiodic vs Masa inclusions

Since we are now considering commutative C^*-algebras, aperiodicity is equivalent to both topological non-triviality and pure outerness for Hilbert bimodules (cf. [17, Theorem 8.1]). The partial homeomorphisms induced by normalisers in Proposition 5.1.2 detect whether a normaliser commutes with the subalgebra A. If A is maximal abelian, then these induced partial homeomorphisms must be non-trivial for non-trivial normalisers, and we can use this to show that masa inclusions are aperiodic.

Lemma 5.1.8. *Let $A \subseteq B$ be a regular masa inclusion. Let $n \in N(A,B)$ be a normaliser. Then the slice $M_n := \overline{\operatorname{span}}\, AnA$ generated by n acts aperiodically on A. That is, the bimodule $M_n \cdot (M_n \cap A)^{\perp}$ is an aperiodic A-bimodule.*

Proof. Note first that the inclusion is non-degenerate by [20, Theorem 2.6]. We must show that $M_n \cdot (M_n \cap A)^{\perp}$ is an aperiodic A-bimodule. Let $V \subseteq \operatorname{dom}(n)$ be the interior of the set of points $x \in \operatorname{dom}(n)$ with $\alpha_n(x) = x$. For $m \in M_n \cdot C_0(V)$, we shall show that $m^*m = mm^*$. Corollary 5.1.4 implies that $\alpha_m = \alpha_n|_{\operatorname{dom}(m)}$, which is the identity $\operatorname{id}_{\operatorname{dom}(m)}$ as $\operatorname{dom}(m) \subseteq V$. Thus for $x \in \operatorname{dom}(m) \subseteq V$ we have

$$m^*m(x)^2 = m^*mm^*m(x) = m^*m(x)mm^*(\alpha_m(x)) = m^*m(x)mm^*(x),$$

whereby $m^*m(x) = mm^*(x)$. Otherwise, $x \notin \operatorname{dom}(m)$ occurs when $m^*m(x) = 0$ by definition, and we have

$$m^*m(x)^2 = mm^*mm^*(x) = \begin{cases} 0, & \text{if } mm^*(x) = 0, \\ mm^*(x)m^*m(\alpha_m^{-1}(x)), & \text{otherwise.} \end{cases}$$

In the first case we achieve the desired result. Otherwise we have $\alpha_m^{-1}(x) = \alpha_n^{-1}(x) \neq x$, since $x \notin V$, whereby $\alpha_m^{-1}(x)$ also does not belong to V. It follows that $mm^*(x) = mm^*(x)m^*m(\alpha_m^{-1}(x)) = 0$ since m^*m has support contained in V. Thus $mm^*(x) = m^*m(x) = 0$ for $x \notin V$, whereby $m^*m = mm^*$. It follows then that $M_n \cdot C_0(V)$ commutes with $C_0(X)$.

Since A is maximal abelian in B, we see that $M_n \cdot C_0(V) \subseteq A$, whereby $C_0(V) \subseteq M_n \cap A$ as $C_0(V) \subseteq s(M_n)$. Conversely for any $a \in M_n \cap A$ we have $\alpha_a = \operatorname{id}_{\operatorname{dom}(a)}$, whereby $\operatorname{dom}(a) \subseteq V$ and $a \in C_0(V)$. Thus $C_0(V) = M_n \cap A$ and so $C_0(V)^{\perp} = C_0(X \backslash \overline{V})$. Let $W := X \backslash \overline{V}$, so that $(M_n \cap A)^{\perp} = C_0(W)$. Recall that hereditary subalgebras of commutative C^*-algebras are exactly ideals, so fix an open subset $U \subseteq X$. If $U \cap W = \varnothing$ then we have $C_0(U)(M_n \cdot (M_n \cap A)^{\perp})C_0(U) = C_0(U)M_nC_0(W)C_0(U) = \{0\}$. Else if $U \cap W \neq \varnothing$, fix an element $x \in U \cap W$ with $\alpha_n(x) \neq x$, which exists since $U \cap W$ is open and disjoint from V. Fix neighbourhoods $U_1, U_2 \subseteq X$ of x and $\alpha_n(x)$ with $U_1 \cap U_2 = \varnothing$. By restricting U_1 to $U_1 \cap \alpha_n^{-1}(U_2)$, we can assume that $U_1 \cap \alpha_n(U_1)$ is empty. Then for any $a \in C_0(U_1) \subseteq C_0(W \cap V)$ we have

$$(ana)^*(ana)(x) = a^*(x)(n^*a^*an)(x)a(x) = (a^*a)(x)(a^*a)(\alpha_n(x))(n^*n)(x) = 0.$$

If $b \in A$ and $c \in C_0(W)$ then $ab, ca \in C_0(U_1)$ and so $||a(bnc)a|| = 0$. By [14, Lemma 4.2] the closed linear span of such elements bnc for $b \in A$, $c \in C_0(W)$ is aperiodic, and so the bimodule $\overline{\operatorname{span}}\, A \cdot n \cdot C_0(W) = M_n \cdot (M_n \cap A)^{\perp}$ is aperiodic. □

The proof of Lemma 5.1.8 shows that a slice M with trivial intersection with the subalgebra is *topologically non-trivial.* That is, the induced partial homeomorphism $\widehat{s(M)} \to \widehat{r(M)}$ of the spectrum of A does not restrict to the identity map on any non-empty open subset. In general this may be a stronger condition than the bimodule M being aperiodic, but for bimodules over Type I C^*-algebras (in particular, commutative C^*-algebras) these conditions are equivalent by [17, Theorem 8.1].

Corollary 5.1.9. *Let $A \subseteq B$ be a regular masa inclusion. Then all slices $M \in \mathcal{S}(A,B)$ act aperiodically on A.*

Proof. We have that $M \cdot (M \cap A)^\perp$ consists only of normalisers. For any $x \in M \cdot (M \cap A)^\perp$ we have $M_x := \overline{\operatorname{span}}\, AxA \subseteq M \cap (M \cap A)^\perp$. Since $M_x \cap A \subseteq M \cdot (M \cap A)^\perp \cap A = \{0\}$, we see that $M_x = M_x \cdot (M_x \cap A)^\perp$, which is an aperiodic A-bimodule by Lemma 5.1.8. Thus x satisfies Kishimoto's condition for all $x \in M \cdot (M \cap A)^\perp$, whereby M acts aperiodically on A (that is, the action of $\mathcal{S}(A, B)$ on A by slices is aperiodic). □

Chapter 6

Essential commutative Cartan subalgebras of C^*-algebras

We now define an essential commutative Cartan pair in analogy to Renault's definition [22, Definition 5.1].

Definition 6.0.1. Let $A \subseteq B$ be an inclusion of C^*-algebras with A commutative. We say the pair (A, B) is an *essential commutative Cartan pair* if the following conditions hold:

(ECC1) A is a regular subalgebra of B;

(ECC2) A is a maximal abelian subalgebra of B (masa);

(ECC3) there exists a faithful local expectation $E : B \to M_{\mathrm{loc}}(A)$.

There are some notable differences in this definition to the definition of Renault in [22]. Firstly, the condition that A is a non-degenerate subalgebra of B has been shown to be redundant by Pitts [20, Theorem 2.7], as all regular masa inclusions have this property. Secondly, the conditional expectation need not take values in A, but may take values in $M_{\mathrm{loc}}(A)$. Since $A \subseteq B$ is assumed to be regular and masa, Corollary 6.2.2 implies that $A \subseteq B$ is an aperiodic inclusion, and so E is the unique pseudo-expectation for the inclusion by [14, Proposition 8.2]. Lastly, the condition that B be a separable C^*-algebra was shown to be unnecessary by Kwaśniewski and Meyer in [15], and also by Raad in [23].

We will prove analogues of many of Renault's results in [22] replacing the reduced twisted groupoid C^*-algebra with Kwaśniewski and Meyer's essential twisted groupoid C^*-algebra. The construction of the essential groupoid C^*-algebra allows us to better detect the underlying topology of the associated Weyl groupoid, as this groupoid may be non-Hausdorff when the associated conditional expectation takes values in $M_{\mathrm{loc}}(C_0(G^{(0)}))$ instead of $C_0(G^{(0)})$. If the Weyl groupoid is Hausdorff, then the reduced and essential groupoid C^*-algebra constructions agree and the results we prove here will recover those of Renault in [22].

6.1 Comparison to aperiodic dynamical inclusions

We show now that essential commutative Cartan pairs are exactly aperiodic dynamical inclusions when the subalgebra is commutative. To show this we need the following technical results.

Lemma 6.1.1. *Let $A \subseteq B$ be a non-degenerate inclusion with $A \subseteq B$ commutative. Then $M(A)$ canonically embeds into $M(B)$, and A is masa in B if and only if $M(A)$ is masa in $M(B)$.*

Proof. Since $A \subseteq B$ is non-degenerate, a multiplier $\tau \in M(A)$ extends to a multiplier of B by defining $\tau(ab) = (\tau a)b$ for all $ab \in AB = B$.

Suppose that $A \subseteq B$ is masa and fix $\tau \in M(B)$ that commutes with $M(A)$. Then for all $a \in A$ the elements τa and $a\tau$ commute with A, and so belong to A since A is masa. Thus $\tau A, A\tau \subseteq A$ whereby $\tau \in M(A)$ and $M(A)$ is masa.

Now suppose $M(A) \subseteq M(B)$ is masa and fix $b \in B$ that commutes with A. Then b commutes with $M(A)$ by [15, Lemma 3.4] and so $b \in M(A)$. Let $(e_\lambda) \subseteq A$ be an approximate unit for A. Then (e_λ) is an approximate unit for B as the inclusion is non-degenerate, and so we have $b = \lim_\lambda be_\lambda \in M(A) \cdot A = A$. Thus $A \subseteq B$ is masa. □

Lemma 6.1.2. *Let $A \subseteq B$ be a regular non-degenerate inclusion with faithful local expectation $E : B \to M_{\mathrm{loc}}(A)$. If $M \subseteq B$ is a slice with $M \cdot I \subseteq \mu(E)$ for an essential ideal $I \lhd A$, then $M \subseteq \mu(E)$.*

Proof. We must show that $E(m^*m) = E(m)^*E(m)$ and $E(mm^*) = E(m)E(m)^*$ for all $m \in M$. For $a \in I$ we have $ma \in M \cdot I \subseteq \mu(E)$, whereby $E(m^*m)a = E(m^*ma) = E(m)^*E(m)a$. Similarly $aE(m^*m) = aE(m)^*E(m)$. Since elements of $M_{\mathrm{loc}}(A)$ are determined by how they act on essential ideals, we see that $E(m^*m) = E(m)^*E(m)$. A similar argument shows $E(mm^*) = E(m)E(m)^*$, and we see that m belongs to $\mu(E)$. □

Proposition 6.1.3. *Let $A \subseteq B$ be an inclusion of C^*-algebras with A commutative. Then $A \subseteq B$ is an essential commutative Cartan pair if and only if it is an aperiodic dynamical inclusion.*

Proof. For an essential commutative Cartan pair $A \subseteq B$, Corollary 5.1.9 then gives an aperiodic action of slices on the subalgebra A by a spanning inverse subsemigroup of $\mathcal{S}(A, B)$, namely all of $\mathcal{S}(A, B)$. Since A is commutative, the local multiplier algebra $M_{\mathrm{loc}}(A)$ is also commutative. Thus the multiplicative domain of the faithful pseudo-expectation E is commutative, as it is isomorphic to a subalgebra of the local multiplier algebra. In particular, since $\mu(E)$ contains A and A is maximal abelian, we see that $\mu(E) = A$, so the conditional expectation has minimal multiplicative domain.

Now suppose $A \subseteq B$ is an aperiodic dynamical inclusion. The inclusion is non-degenerate by [20, Theorem 2.7], and the pseudo-expectation takes values in the local multiplier algebra by Theorem 3.1.8. We must show that $A \subseteq B$ is masa. For contradiction, suppose that $A \subseteq B$ is not masa. Lemma 6.1.1 then implies that the inclusion $M(A) \subseteq M(B)$ is not masa. Thus $M(A)' \cap M(B)$, the algebra of multipliers in $M(B)$ that commute with $M(A)$ is a commutative subalgebra of $M(B)$ containing $M(A)$. Since unital C^*-algebras are spanned by their unitaries, there exists a unitary $u \in M(A)' \cap M(B)$ that does not belong to $M(A)$. The space $M_u := u \cdot A$ is then a slice for the inclusion $A \subseteq B$ with $M_u \cong A$ but M_u is not contained in A. By Theorem 4.2.10 the bimodule $M_u \cdot (M_u \cap A)^\perp$ is aperiodic, hence purely outer, and so must be zero as it is isomorphic to an ideal of A. However $M_u \cap A \subseteq M_u \cdot ((M_u \cap A) \oplus (M_u \cap A)^\perp) \subseteq A = \mu(E)$ giving $M_u \subseteq A = \mu(E)$ by Lemma 6.1.2. Hence $u \in M(A)$, a contradiction, thus $A \subseteq B$ is masa. □

Throughout let $A \subseteq B$ be an essential commutative Cartan pair with faithful local expectation $E : B \to M_{\mathrm{loc}}(A)$.

6.2 The evaluation map

Renault [22] defines an evaluation map $B \to C^*_{\mathrm{r}}(G, \Sigma)$ directly taking elements of B to sections of the canonical line bundle associated to Σ. An integral part to this construction is that the conditional expectation takes values in the subalgebra $A = C_0(X)$, and so elements may be considered as functions on X. This approach somewhat breaks down in our setting: Gonshor [11, Theorem 1] showed that the local multiplier algebra of $C_0(X)$ is isomorphic to the algebra $\mathfrak{B}(X)/\mathfrak{M}(X)$ of Borel measurable functions quotient by meagrely supported functions. Thus elements of $M_{\mathrm{loc}}(C_0(X))$ are not functions on X per se, but are expressed as equivalence classes of functions up to meagre support. Since singletons are (often) meagre in X, elements of $M_{\mathrm{loc}}(A)$ cannot be evaluated at points. We circumvent this problem by showing that normalisers for the inclusion $A \subseteq B$ can be represented by functions defined on a dense open subset, and so give a well defined class in the Borel-mod-meagre setting since X is a Baire space.

Lemma 6.2.1. *Let $A \subseteq B$ be a regular non-degenerate inclusion of C^*-algebras. If B is densely spanned by slices $W \subseteq B$ with the property that $W \cdot (W \cap A)^{\perp}$ is an aperiodic A-bimodule, then the inclusion $A \subseteq B$ is aperiodic.*

Proof. This is equivalent to [16, Proposition 6.3]. □

Corollary 6.2.2. *Let $A \subseteq B$ be a regular masa inclusion. Then $A \subseteq B$ is an aperiodic inclusion.*

Proof. Combine Lemmas 5.1.8 and 6.2.1. □

Lemma 6.2.3. *Let $A \subseteq B$ be a regular masa inclusion. Then the inclusion $A \subseteq B$ is aperiodic, and hence there is at most one local conditional expectation $E : B \to M_{\mathrm{loc}}(A)$, and $E(W) \subseteq M(W \cap A)$ for any slice $W \subseteq B$ of the inclusion.*

Proof. Let $E : B \to M_{\mathrm{loc}}(A)$ be a local conditional expectation. By Corollary 6.2.2 the inclusion $A \subseteq B$ is aperiodic, so has at most one pseudo-expectation by [14, Propositions 3.9,3.16]. Lemma 5.1.8 states that for any slice $W \subseteq B$, the subslice $W \cdot (W \cap A)^{\perp}$ is an aperiodic A-bimodule. Then [16, Lemma 5.10] gives that $M_{\mathrm{loc}}(A)$ contains no non-zero aperiodic bimodules, and the image of an aperiodic bimodule under a bounded bimodule map is aperiodic by [16, Lemma 5.12]. Thus $E(W \cdot (W \cap A)^{\perp}) = \{0\}$.

For a normaliser $n \in N(A, B)$ and a slice $W \subseteq B$ with $n \in W$, the local multiplier $E(n)$ is determined by how it acts on the essential ideal $(W \cap A) \oplus (W \cap A)^{\perp}$. Fix $a \in W \cap A$ and $a^{\perp} \in (W \cap A)^{\perp}$. Then $E(n)(a + a^{\perp}) = E(na) + E(na^{\perp})$. Since $na \in W \cdot (W \cap A) = W \cap A \subseteq A$, we have $E(na) = na$, and since $na^{\perp} \in W \cdot (W \cap A)^{\perp}$, we have $E(na^{\perp}) = 0$. Thus $E(n)$ is a multiplier on $W \cap A$, identifying $M(W \cap A) \subseteq M_{\mathrm{loc}}(W \cap A) \subseteq M_{\mathrm{loc}}(A)$ via the canonical inclusions. □

Corollary 6.2.4. *For each normaliser $n \in N(A, B)$ the element $E(n) \in M_{\mathrm{loc}}(A)$ is represented by a continuous bounded function on the dense open subset V_n^{ess} of $X = \hat{A}$. Moreover, the images of finite sums of normalisers under E are represented by continuous bounded functions on dense open subsets of X.*

Proof. By Lemma 6.2.3 we have that $E(n)$ belongs to $M(I)$ for some essential ideal $I \lhd A$. Essential ideals of $A = C_0(X)$ are of the form $I = C_0(U)$ for dense open subsets $U \subseteq X$. This, coupled with the identification of $M(C_0(U))$ with the algebra $C_{\mathrm{b}}(U)$ of bounded continuous functions on U allows us to express $E(n)$ as a bounded continuous function on U.

For a finite linear collection of normalisers $n_i \in N(A,B)$, we have $E(\sum_i n_i) = \sum_i E(n_i)$, where each $E(n_i)$ is a function on a dense open subset $U_i \subseteq X$. The sum of these functions defines a continuous and bounded function on the intersection $\bigcap_i U_i$, which is again dense and open since the collection $(U_i)_i$ is finite. □

Corollary 6.2.5. *Let $n \in N(A,B)$ be a normaliser, and let $V_n := \{x \in X : \alpha_n(x) = x\}^\circ$; the interior of the set of fixed points for α_n. Then $E(n)$ is represented in $M_{\text{loc}}(A)$ by a bounded continuous function on $V_n^{\text{ess}} := V_n \cup X\backslash\overline{V_n}$ which we denote by $n|_{V_n^{\text{ess}}}$, and for any $g \in C_0(V_n)$ we have $n|_{V_n^{\text{ess}}} g = E(ng) = ng$, and for any $g^\perp \in C_0(V_n)^\perp$ we have $n|_{V_n^{\text{ess}}} g = E(ng) = 0$. If $b = \sum_i n_i$ is a sum of finitely many normalisers $n_i \in N(A,B)$, then $E(b)$ is represented in $M_{\text{loc}}(A)$ by a function defined on a dense open subset $V_b^{\text{ess}} := \bigcap_i V_{n_i}^{\text{ess}}$.*

Proof. Let $W_n := \overline{\operatorname{span}}\, AnA$ be the smallest slice containing n. Then $s(W_n)$ is exactly $\overline{\operatorname{span}}\, An^*AnA = C_0(\operatorname{dom}(n))$, in which $C_0(V_n) \oplus C_0(X\backslash\overline{V_n} \cap \operatorname{dom}(n))$ is an essential ideal. Since $\alpha_n|_{V_n} = \operatorname{id}_{V_n}$ we have that n commutes with $C_0(V_n)$. The element n restricts to a multiplier of $C_0(V_n)$ since $A \subseteq B$ is masa, and so n is represented by a continuous and bounded function on V_n. Conversely for all $x \in X\backslash\overline{V_n}$, we have that $\alpha_n(x) \neq x$, whereby $W_n \cdot C_0(X\backslash\overline{V_n})$ is a topologically non-trivial Hilbert A-bimodule. The subbimodule $W_n \cdot C_0(X\backslash\overline{V_n})$ is then purely outer by [17, Theorem 8.1], whereby $W_n \cdot C_0(X\backslash\overline{V_n}) \cap A = \{0\}$. Thus $E(n)$ gives a multiplier of $C_0(V_n)$ by Lemma 6.2.3, which extends by zero to a multiplier of $C_0(V_n) \oplus C_0(X\backslash\overline{V_n}) = C_0(V_n^{\text{ess}})$, which is in turn given by a bounded function $E(n) = n|_{V_n^{\text{ess}}}$ on V_n^{ess}. Then for any $g \in C_0(V_n)$ we have $n|_{V_n^{\text{ess}}} g = E(ng) = ng$ and for any $g^\perp \in C_0(V_n)^\perp$ we have $n|_{V_n^{\text{ess}}} g^\perp = E(ng^\perp) = 0$.

The result about finite sums of normalisers follows since the intersection of finitely many dense open subsets of a topological space is again dense and open. □

In Corollary 6.2.5 a different choice of normalisers summing to $b \in \operatorname{span} N(A,B)$ may give a different dense open subset $V_{\text{ess}}^b \subseteq X$. However such a differing choice can only change the resulting set V_{ess}^b on a meagre set, since V_{ess}^b is always dense and open. For our purposes we only need that at least one such V_{ess}^b exists.

Lemma 6.2.6. *Let $n \in N(A,B)$ and let $U_n := \{[\alpha_n, x] : x \in \operatorname{dom}(n)\}$ be the open bisection of $G(A,B)$ determined by n. There is a section $\hat{n} \in \mathcal{C}_0(U_n, \Sigma(A,B))$ defined by*

$$\hat{n}[m,x] = \begin{cases} \frac{(m^*n)|_{V_{m^*n}}(x)}{\sqrt{m^*m(x)}}, & x \in V_{m^*n}^{\text{ess}}, \\ 0, & \textit{otherwise.} \end{cases}$$

If $b = \sum_i n_i$ is a sum of finitely many normalisers then we define $\hat{b} \in \mathcal{C}_0(G(A,B), \Sigma(A,B))$ by

$$\hat{b}[m,x] = \begin{cases} \frac{(m^*b)|_{V_{m^*b}}(x)}{\sqrt{m^*m(x)}}, & x \in V_{m^*b}^{\text{ess}} \\ 0, & \textit{otherwise.} \end{cases}$$

The maps $\hat{b}$ and $\sum_i \hat{n_i}$ agree on $U_m \cdot V_{m^*b}^{\text{ess}}$ for each normaliser $m \in N(A,B)$, and so agree on a dense open subset of $G(A,B)$.

Proof. If $[m,x] = [m',x]$ for some $m, m' \in N(A,B)$ then either both $\hat{n}[m,x] = \hat{n}[m',x] = 0$ or $x \in V_{m^*n} \cup V_{m'^*n}$ and there exist positive functions $a, a' \in A = C_0(X)$ with

$a(x), a'(x) > 0$ and $ma = m'a'$. Then

$$\frac{(m'^*n)|_{V_{m'^*n}}(x)}{\sqrt{m'^*m'(x)}} = \frac{a'(x)}{a'(x)}\frac{(m'^*n)|_{V_{m'^*n}}(x)}{\sqrt{m'^*m'(x)}} = \frac{a(x)}{a(x)}\frac{(m^*n)|_{V_{m^*n}}(x)}{\sqrt{m^*m(x)}} = \frac{(m^*n)|_{V_{m^*n}}(x)}{\sqrt{m^*m(x)}}.$$

Thus the function $\hat{n}$ gives a well defined section of the canonical line bundle associated to the twist $(G(A,B), \Sigma(A,B))$. One notes also that the support of $\hat{n}$ is contained in U_n, as outside this set either x does not belong to $V^{\text{ess}}_{m^*n}$ or $(m^*n)|_{V_{m^*n}}(x) = 0$.

To see the section $\hat{n}$ is quasi-continuous, that is, its restriction to U_n is continuous, we note that for $[\alpha_m, x] \in U_n$ we have $[m, x] = [zn, x]$ for some $z \in \mathbb{T}$. Thus, on U_n the section $\hat{n}$ reduces to $\hat{n}[zn, x] = \frac{\bar{z}(n^*n)|_{V_{n^*n}}(x)}{\sqrt{n^*n(x)}} = \bar{z}\sqrt{n^*n}(x)$, which is continuous and vanishing at topological infinity of U_n since n^*n vanishes at topological infinity of $\text{dom}(n) = s(U_n)$.

The proof that $\hat{b}$ is well defined follows since $(m^*n_i)|_{V^{\text{ess}}_{m^*b}} + (m^*n_j)|_{V^{\text{ess}}_{m^*b}}$ is exactly $(m^*(n_i + n_j))|_{V^{\text{ess}}_{m^*b}}$ for all $m \in N(A,B)$ and for any summands n_i, n_j describing b. Since $V^{\text{ess}}_{m^*b} \subseteq V^{\text{ess}}_{m^*n_i}$ for each i we see that $(m^*b)|_{V^{\text{ess}}_{m^*b}}(x) = \sum_i (m^*n_i)|_{V^{\text{ess}}_{m^*n_i}}(x)$ for all $x \in V^{\text{ess}}_{m^*b}$, whereby $\hat{b}$ and $\sum_i \hat{n_i}$ agree on $V^{\text{ess}}_{m^*b}$. The union of $U_m \cdot V^{\text{ess}}_{m^*b}$ over all normalisers $m \in N(A,B)$ is then a dense open subset of $G(A,B)$ since the sets U_m form a basis for the topology on $G(A,B)$, and each $U_m \cdot V^{\text{ess}}_{m^*b}$ is open. □

Lemma 6.2.7. *Let $n \in N(A,B)$ be a normaliser. Any section $f \in \mathcal{C}_c(U_n, \Sigma(A,B))$ is of the form $\hat{n} \cdot h_f$ for a function $h_f \in C_0(X)$. Moreover, the assignment $\phi_n : f = \hat{n} \cdot h_f \mapsto nh_f$ is a well defined $C_0(X)$-bimodule map.*

Proof. There is $h_f \in C_0(X)$ with $f = \hat{n} \cdot h_f$ because $\hat{n}$ is a non-vanishing section over U_n, and so since the support of f is contained in U_n, there exists $h_f \in C_0(X)$ with compact support contained in $s(U_n)$ satisfying $\hat{n}h = f$.

To see that the map $\hat{n} \cdot h_f \mapsto nh_f$ is well defined, suppose $f = \hat{n}h_f = \hat{n}g_f$. Then $\hat{n}(h_f - g_f) = 0$, so for all $[m, x] \in \Sigma(A,B)$ with $[\alpha_m, x] \in U_n$ we have

$$0 = \hat{n}[m, x](h_f - g_f)(x),$$

which implies that $h_f(x) = g_f(x)$ for all $x \in s(U_n) = \text{dom}(n)$ as $\hat{n}$ is a non-vanishing section on U_n. Then

$$E((n(h_f - g_f))^* n(h_f - g_f)) = \overline{(h_f - g_f)} E(n^*n)(h_f - g_f) = \overline{(h_f - g_f)} n^*n(h_f - g_f)$$

and for all $x \in X$ we have

$$\overline{(h_f - g_f)} n^*n(h_f - g_f)(x) = |h_f(x) - g_f(x)| n^*n(x) = 0,$$

as either $x \in \text{dom}(n) = \text{supp}(n^*n)$ where $h_f(x) - g_f(x) = 0$, or $x \notin \text{dom}(n)$ so $n^*n(x) = 0$.

For $f, g \in C_0(X)$, $\hat{n}h \in \mathcal{C}_c(U_n, \Sigma(A,B))$, $m \in N(A,B)$, and $x \in \text{dom}(m) \cap s(U_n)$ we have

$$(f\hat{n}hg)[m, x] = f(\alpha_n(x)) \frac{(m^*n)|_{V_{m^*n}}(x)}{\sqrt{m^*m(x)}} h(x)g(x).$$

Since h has compact support $K \subseteq X$, the function $(f \circ \alpha_n)|_{\text{dom}(n) \cap K}$ extends to some $F \in C_0(X)$ by Tietze's extension theorem, and we have $f\hat{n}h = \hat{n}hF$. We claim that

$fnh = nhF$. To see this, we note that for $x \in \mathrm{dom}(n) \cap K$ we have

$$\begin{aligned}
&(fnh - nhF)^*(fnh - nhF)(x) \\
&\quad = (h^*n^*f^*fnh - F^*h^*n^*fnh - h^*n^*f^*nhF + F^*h^*n^*nhF)(x) \\
&\quad = 2|f(\alpha_n(x))|^2(n^*n)(x)|h(x)|^2 - 2|f(\alpha_n(x))|^2(n^*n)(x)|h(x)|^2 \\
&\quad = 0.
\end{aligned}$$

Alternatively if $x \notin \mathrm{dom}(n) \cap K$ then all the terms in the above calculation evaluate to zero as $(n^*n)(x)|h(x)|^2$ is a common factor in each term. Thus

$$\phi_n(f\hat{n}hg) = \phi_n(\hat{n}hFg) = nhFg = fnhg = f\phi_n(\hat{n}h)g,$$

so ϕ_n is a $C_0(X)$-bimodule homomorphism. □

Lemma 6.2.8. *For $n \in N(A,B)$ let ϕ_n be the $C_0(X)$-bimodule map in Lemma 6.2.7. Let $D := \bigoplus^{\mathrm{alg}}_{n\in N(A,B)} \mathcal{C}_c(U_n, \Sigma(A,B))$ and define $\Phi : D \to B$ by*

$$\Phi((f_n)_{n\in N(A,B)}) = \sum_{n\in N(A,B)} \phi_n(f_n).$$

*Define $c : D \to \mathcal{C}_c(G(A,B), \Sigma(A,B))$ by $c((f_n)_{n\in N(A,B)}) = \sum_{n\in N(A,B)} f_n$. There is a *-algebra structure on D such that both Φ and c are *-homomorphisms. Moreover, c is surjective and Φ has dense range in B.*

Proof. For $f \in \mathcal{C}_c(U_n, \Sigma(A,B))$ and $g \in \mathcal{C}_c(U_m, \Sigma(A,B))$ in D, write $f = \hat{n}j_f$ and $g = \hat{m}k_g$ with $j_f, k_g \in C_0(X)$ as in Lemma 6.2.7. Define the product $fg = \widehat{nh_f m}k_g \in \mathcal{C}_c(U_{nh_fm}, \Sigma(A,B))$, extending linearly to all of D, and define the involution $(\hat{n}j_f)^* = j_f^*\widehat{n^*} \in \mathcal{C}_c(U_{n^*}, \Sigma(A,B))$, extending anti-linearly. A brief computation shows that this gives D a *-algebra structure.

For $f = \hat{n}h_f \in \mathcal{C}_c(U_n, \Sigma(A,B))$ and $g = \hat{m}k_g \in \mathcal{C}_c(U_m, \Sigma(A,B))$, we have

$$\Phi(\hat{n}h_f\hat{m}k_g) = \Phi(\widehat{nh_fm}k_g) = \phi_{nh_fm}(\widehat{mh_fn}k_g) = mh_fnk_g = \phi_n(\hat{n}h_f)\phi_m(\hat{m}k_g) = \Phi(f)\Phi(g).$$

Since f has compact support there is $k \in C_0(X)$ with compact support such that $f = k\hat{n}h_f$, and so $\Phi(f^*) = \phi_{n^*}(\overline{h_f}\widehat{n^*}\bar{k}) = \overline{h_f}\phi_{n^*}(\widehat{n^*}\bar{k}) = (k\hat{n}h_f)^* = \Phi(f)^*$. Thus Φ is a *-homomorphism.

To see that c is a *-homomorphism, let $\hat{n}h \in \mathcal{C}_c(U_n, \Sigma(A,B))$ and $\hat{m}k \in \mathcal{C}_c(U_m, \Sigma(A,B))$. Considering $\hat{n}h$ and $\hat{m}k$ as sections of the canonical line bundle associated to the twist $(G(A,B), \Sigma(A,B))$, for $[\alpha_p, x] \in G(A,B)$ we have

$$\begin{aligned}
(c(\hat{n}h)\cdot c(\hat{m}k))[\alpha_p, x] &= \sum_{\alpha_q(y)=\alpha_p(x)} \hat{n}h[\alpha_q, y]\hat{m}k[\alpha_{q^*p}, x] \\
&= \hat{n}[\alpha_n, y]h(y)\hat{m}[\alpha_{n^*p}, x]k(x) \\
&= \begin{cases} \hat{n}[\alpha_n, \alpha_m(x)]h(\alpha_m(x))\hat{m}[\alpha_m, x]k(x), & [\alpha_p, x] = [\alpha_{nm}, x] \\ 0, & \text{otherwise,} \end{cases} \\
&= c(\widehat{nhm}k)[\alpha_p, x].
\end{aligned}$$

The second and third equalities hold since $\hat{n}$ and $\hat{m}$ have support contained in U_n and U_m, respectively. For $f = \hat{n}h \in \mathcal{C}_c(U_n, \Sigma(A, B))$ we have

$$\begin{aligned} c(f^*)[\alpha_p, x] &= \bar{h}\widehat{n^*}[\alpha_p, x] \\ &= \overline{(h\hat{n})[\alpha_{p^*}, \alpha_p(x)]} \\ &= \begin{cases} \overline{h(x)\hat{n}[\alpha_n, \alpha_{n^*}(x)]}, & \alpha_p = \alpha_{n^*}, \\ 0, & \text{otherwise}, \end{cases} \\ &= (\hat{n}h)^*[\alpha_p, x] \\ &= c(f)^*[\alpha_p, x], \end{aligned}$$

thus c is a *-homomorphism.

That c is surjective is clear, since $\mathcal{C}_c(G(A, B), \Sigma(A, B))$ is defined as the span of spaces $\mathcal{C}_c(U, \Sigma(A, B))$ over all bisections $U \subseteq G$, and bisections of the form U_n for normalisers $n \in N(A, B)$ form a basis for the topology on G.

To see that Φ has dense range, it suffices to show that any normaliser $n \in N(A, B)$ can be approximated by elements in the image of Φ since $A \subseteq B$ is a regular subalgebra. For $n \in N(A, B)$ we can write $n = mh$ for $m \in N(A, B)$ and $h \in C_0(X) = A$ by the Cohen-Hewitt factorisation theorem. Since $C_c(X)$ is dense in $C_0(X)$, we have compactly supported functions h_k with $h_k \to h$ uniformly. Since n^*n has support contained in $\mathrm{dom}(n) = s(U_n)$, we can pick functions h_k with support contained in $s(U_n)$. Then $n = mh = \lim_{k\to\infty} mh_k$, and each mh_k is exactly the image of $\hat{m}h_k \in \mathcal{C}_c(U_m, \Sigma(A, B))$. Thus the image of Φ is dense in B. □

Proposition 6.2.9. *Let D, Φ, and c be as in Lemma 6.2.8, and identify A with $C_0(G^{(0)})$. There is a linear map $R : \mathcal{C}_c(G(A, B), \Sigma(A, B)) \to M_{\mathrm{loc}}(C_0(G^{(0)}))$ that restricts functions to the unit space $G^{(0)}$ on G, such that the following diagram commutes:*

$$\begin{array}{ccc} D & \xrightarrow{\quad\Phi\quad} & B \\ \big\downarrow c & & \big\downarrow E \\ \mathcal{C}_c(G(A, B), \Sigma(A, B)) & \xrightarrow{\quad R\quad} & M_{\mathrm{loc}}(C_0(G^{(0)}) = M_{\mathrm{loc}}(A) \end{array}$$

Proof. Let $f \in \mathcal{C}_c(U_n, \Sigma(A, B))$. Then f has support contained in U_n, so $f|_{G^{(0)}}$ has support contained in $U_n \cap G^{(0)}$. Since f is continuous and compactly supported on U_n, $f|_{G^{(0)}}$ is continuous and bounded on $U_n \cap G^{(0)}$, so defines a multiplier of $C_0(U_n \cap G^{(0)})$. Since $f|_{G^{(0)}}$ is zero outside of $U_n \cap G^{(0)}$, we see that $f|_{G^{(0)}}$ extends (by zero) to a multiplier on $C_0((U_n \cap G^{(0)}) \cup (G^{(0)}\backslash\overline{(U_n \cap G^{(0)})})) = C_0(U_n \cap G^{(0)}) \oplus C_0(U_n \cap G^{(0)})^{\perp}$, which is an essential ideal in $C_0(G^{(0)})$. Thus $f|_{G^{(0)}}$ defines an element of $M_{\mathrm{loc}}(C_0(G^{(0)})$, and the map R is well defined.

Write $f = \hat{n}h$ for some $h \in C_0(X)$ using Lemma 6.2.7. To show that $E(\Phi(f)) = c(f)|_{G^{(0)}}$ it suffices to show that $E(\Phi(f))g = c(f)|_{G^{(0)}}g$ for $g \in C_0(X)$ supported on a dense open subset. Let $V_n = \{x \in \mathrm{dom}(n) : \alpha_n(x) = x\}^{\circ}$ be as in Corollary 6.2.5. Then $U_n \cap G^{(0)}$ is contained in V_n since $\alpha_n(x) = x$ for all $x \in U_n \cap G^{(0)}$ and $U_n \cap G^{(0)}$ is an open set. Conversely, if $x \in V_n$ then there is a neighbourhood of x contained in $\mathrm{dom}(n)$ which is fixed by α_n, and so $[\alpha_n, x] = [\mathrm{id}, x] \in U_n \cap G^{(0)}$, hence $V_n = U_n \cap G^{(0)}$. For $g \in C_0(V_n)$ we have $E(\Phi(f))g = E(nh)g = nhg = (nh)|_{V_n}g = (nh)|_{U_n \cap G^{(0)}}g$ by Corollary 6.2.5. For $g^{\perp} \in C_0(V_n)^{\perp}$ we have $E(n)g^{\perp} = 0$ by Corollary 6.2.5, and $(nh)|_{G^{(0)}}g^{\perp} = 0$ since the supports of $n|_{G^{(0)}}$ and $g^{\perp}$ have zero intersection. Thus $E(\Phi(f)) = c(f)|_{G^{(0)}}$ as both agree on an essential ideal of $C_0(G^{(0)})$. ☐

It is worth noting that the restriction map R in Proposition 6.2.9 is the same as the conditional expectation EL defined in [16, Proposition 4.3] on the suspace $\mathcal{C}_c(G, \Sigma)$, and so is the canonical local multiplier algebra-valued expectation giving rise to the essential groupoid C^*-algebra.

Corollary 6.2.10. *The map Φ descends via c to a *-homomorphism*

$$\Psi_0 : \mathcal{C}_c(G(A, B), \Sigma(A, B)) \to B$$

satisfying $\Psi_0(c(f_n)) = \Phi(f_n)$ for all $(f_n) \in D$, and $R = P \circ \Psi_0$.

Proof. It suffices to show that the kernel of c is contained in the kernel of Φ. Contrapositively, we must show that if $\Phi((f_n)) \neq 0$ for some $(f_n) \in D$ then $c(f_n) \neq 0$. Fix such $d \in D$ with $\Phi(d) \neq 0$. Since c and Φ are *-homomorphisms and P is faithful, we have

$$R(c(d)^*c(d)) = R(c(d^*d)) = E(\Phi(d^*d)) = E(\Phi(d)^*\Phi(d)) \neq 0.$$

In particular, $c(d)^*c(d)$ cannot be zero since R is linear, whereby $c(d) \neq 0$. ☐

Theorem 6.2.11. *Let $A \subseteq B$ be an essential Cartan pair with faithful local conditional expectation $E : B \to M_{\text{loc}}(A)$. Let $(G(A, B), \Sigma(A, B))$ by the Weyl twist associated to $A \subseteq B$. There is a *-isomorphism $C^*_{\text{ess}}(G(A, B), \Sigma(A, B)) \cong B$ that restricts to an isomorphism $C_0(G^{(0)}) \cong A$ and entwines E with the canonical local expectation $C^*_{\text{ess}}(G(A, B), \Sigma(A, B)) \to M_{\text{loc}}(C_0(G^{(0)})$, where $M_{\text{loc}}(A)$ is identified with $M_{\text{loc}}(C_0(G^{(0)})$.*

Proof. Corollary 6.2.10 gives a *-homomorphism $\Psi_0 : \mathcal{C}_c(G(A, B), \Sigma(A, B)) \to B$, and so this extends to a *-homomorphism $\Psi : C^*(G(A, B), \Sigma(A, B)) \to B$ by the universal property of the full twisted groupoid C^*-algebra. Since the canonical local conditional expectation $EL : C^*(G(A, B), \Sigma(A, B)) \to M_{\text{loc}}(A)$ is the continuous extension of the restriction map $R : \mathcal{C}_c(G(A, B), \Sigma(A, B)) \to M_{\text{loc}}(C_0(G^{(0)})$, Proposition 6.2.9 implies that Ψ entwines EL with E. Thus for $a \in C^*(G(A, B), \Sigma(A, B))$ we have $EL(a^*a) = E(\Psi(a)^*\Psi(a)) = 0$ if and only if $a \in \ker(\Psi)$ as E is faithful. In particular, the kernel of Ψ is contained in the kernel of EL, so Ψ descends to a homomorphism $\psi : C^*_{\text{ess}}(G(A, B), \Sigma(A, B)) \to B$. Moreover, ψ is injective since it entwines expectations, and both EL and E are faithful. The map ψ is surjective since Φ has dense image in B by Lemma 6.2.8. ☐

6.3 Uniqueness of the Weyl groupoid and twist

The Weyl groupoid and twist associated to an essential Cartan inclusion $A \subseteq B$ are not unique among the class of all étale groupoids giving rise to an isomorphic essential Cartan pair. The Weyl groupoid is however 'minimal' in a certain sense, as we can show that any other twist over a groupoid giving the same Cartan pair will have the Weyl groupoid and twist as a quotient. We also show that the Weyl pair is unique among twists over effective groupoids.

Theorem 6.3.1. *Let $A \subseteq B$ be an essential Cartan pair and let (G, Σ) be the associated Weyl twist. Let (H, Ω) be a twist over an étale groupoid with locally compact Hausdorff unit space. Suppose there is an isomorphism $\varphi : C^*_{\text{ess}}(H, \Omega) \to C^*_{\text{ess}}(G, \Sigma)$ that restricts to an isomorphism $\varphi|_{C_0(H^{(0)})} : C_0(H^{(0)}) \to C_0(G^{(0)})$ and entwines conditional expectations.*

Then there is a surjective twisted groupoid homomorphism $\beta_\varphi : (H, \Omega) \to (G, \Sigma)$ that restricts to a homeomorphism of unit spaces, and the twist (H, Ω) is an extension of (G, Σ) by a normal isotropy group bundle in which the unit space $H^{(0)}$ is dense. Moreover β_φ is a local homeomorphism.

Lemma 6.3.2. *Let $U, V \subseteq H$ be open bisections in a twisted groupoid (H, Ω). Suppose that $U \cap V = \varnothing$. Then for $f \in \mathcal{C}_c(U, \Omega)$ and $g \in \mathcal{C}_c(V, \Omega)$ we have $||f|| \leqslant ||f + g||$ in the essential groupoid C^*-algebra.*

Proof. The map $P : \mathfrak{B}(H, \Omega) \to \mathfrak{B}(H^{(0)}, \Omega)$ taking sections and restricting them to the unit space is a generalised expectation by [16, Proposition 7.10], hence P is contractive. The functions f^*g and g^*f have supports contained in $U^{-1}V$ and $V^{-1}U$ respectively, which do not intersect the unit space $H^{(0)}$ since $U \cap V = \varnothing$. Since f and g are supported on bisections, f^*f and g^*g are functions on the unit space, so $P(f^*f) = f^*f$ and $P(g^*g) = g^*g$. Moreover, $C_0(H^{(0)})$ is a commutative C^*-algebra so $||f^*f + g^*g|| \geqslant ||f^*f||$. Thus we have

$$\begin{aligned} ||f + g||^2 &= ||(f + g)^*(f + g)|| \\ &= ||f^*f + f^*g + g^*f + g^*g|| \\ &\geqslant ||P(f^*f + f^*g + g^*f + g^*g)|| \\ &= ||P(f^*f) + P(g^*g)|| \\ &= ||f^*f + g^*g|| \\ &\geqslant ||f^*f|| \\ &= ||f||^2. \end{aligned}$$ □

Lemma 6.3.3. *Let $f \in C^*_{\mathrm{ess}}(H, \Omega)$ and suppose that f is represented by a function $g \in C^*_{\mathrm{r}}(H, \Omega)$, which we consider as a subspace of $\mathfrak{B}(H, \Omega)$ by* [16, Proposition 7.10]. *If g has continuous restriction to some open bisection $U \subseteq H$, then* $\sup_{\gamma \in U} |g(\gamma)| \leqslant ||f||$.

Proof. Since g is continuous on U the supremum $\sup_{\gamma \in U} |g(\gamma)|$ is equal to $\sup_{\gamma \in C} |g(\gamma)|$ for any comeagre subset $C \subseteq U$. Thus taking an infimum over comeagre subsets of U changes nothing, and we see

$$\sup_{\gamma \in U} |g(\gamma)|^2 = \inf_{C \subseteq U} \sup_{\gamma \in C} |g(\gamma)|^2 \leqslant \inf_{D \subseteq H^{(0)}} \sup_{x \in D} |g^*g(x)| = ||EL(g^*g)|| = ||EL(f^*f) \leqslant ||f||^2,$$

where $D \subseteq H^{(0)}$ ranges over all comeagre subsets of $H^{(0)}$. □

Lemma 6.3.4. *Let G be an effective groupoid with locally compact Hausdorff unit space. Let $U \subseteq G$ be a bisection, and let $\overline{U}^\circ$ be the interior of the closure of U. Then $U = \overline{U}^\circ \cdot s(U) = r(U) \cdot \overline{U}^\circ$.*

Proof. The inclusion $U \subseteq \overline{U}^\circ \cdot s(U)$ is clear. Fix $\gamma \in \overline{U}^\circ \cdot s(U)$. Then there exists $\eta \in U$ with $s(\eta) = s(\gamma)$, and we have $\gamma\eta^{-1} \in \overline{U}^\circ \cdot s(U) \cdot U^{-1} \subseteq \overline{G^{(0)}}^\circ$. Since G is effective and the closure of the unit space consists of isotropy, we see that $\overline{G^{(0)}}^\circ = G^{(0)}$ and so $\gamma\eta^{-1} \in G^{(0)}$. Thus $\gamma = \eta \in U$, and so $\overline{U}^\circ \cdot s(U) = U$. A similar argument shows $r(U) \cdot \overline{U}^\circ = U$. □

Lemma 6.3.5. *Let X be a Baire space. Let $x \in X$ be a point and let $U \subseteq X$ be a neighbourhood of x. Let $V_1, \ldots, V_\ell \subseteq X$ be open sets such that $x \notin \overline{V_k}^\circ$ for $1 \leqslant k \leqslant \ell$. That is, x does not lie in the interior of the closure of any of the V_k. Then $Z := U \setminus \bigcup_{k=1}^{\ell} \overline{V_k}$ is a non-empty open set, and x lies in the closure of Z.*

Proof. Since Z is the intersection of U with the complements of finitely many open sets, we see that Z is open. To see that Z is non-empty, we first note that Z can be expressed as

$$Z = U \backslash \bigcup_{k=1}^{\ell} \overline{V_k} = \left(U \backslash \bigcup_{k=1}^{\ell} \partial(\overline{V_k}^{\circ}) \right) \cap \left(U \backslash \bigcup_{k=1}^{\ell} \overline{V_k}^{\circ} \right).$$

Since each $\partial(\overline{V_k}^{\circ})$ is the boundary of an open set, it is meagre. Thus Z is a comeagre subset of $U \backslash \bigcup_{k=1}^{\ell} \overline{V_k}^{\circ}$, and hence is dense since X is a Baire space. The set $U \backslash \bigcup_{k=1}^{\ell} \overline{V_k}^{\circ}$ is non-empty since x is not contained in any $\overline{V_k}^{\circ}$ by hypothesis, so x lies in the closure of Z and in particular Z is non-empty. □

Proof of Theorem 6.3.1. First we shall establish that a map $\beta_\varphi : (H, \Omega) \to (G, \Sigma)$ exists. Let $\sigma \in \Omega$ and let $U \subseteq H$ be a bisection with $\sigma \in \Omega|_U$. Let θ_φ be the inverse to $\varphi|_{H^{(0)}}^* : G^{(0)} \to H^{(0)}$; the homeomorphism induced by considering the restricted isomorphism of commutative C^*-algebras $\varphi|_{H^{(0)}} : C_0(H^{(0)}) \to C_0(G^{(0)})$. Viewing elements of $\mathcal{C}_c(U, \Omega)$ as continuous functions $\Omega|_U \to \mathbb{C}$ satisfying $f(\tau z) = f(\tau)\bar{z}$ for $z \in \mathbb{T}$, let $f_\sigma : \Omega|_U \to \mathbb{C}$ be such a continuous compactly supported function with $f(\sigma) = 1$. Then $\varphi(f_\sigma)$ is a normaliser of $C_0(G^{(0)}) \subseteq C^*_{\text{ess}}(G, \Sigma)$, and $\varphi(f_\sigma^* f_\sigma)(\theta_\varphi(s(\sigma))) = (f_\sigma^* f_\sigma)(s(\sigma)) = |f_\sigma(\sigma)|^2 = 1$, so $\theta_\varphi(s(\sigma)) \in \text{dom}(\varphi(f_\sigma))$. Thus $\beta_\varphi(\sigma, U, f_\sigma) := [\varphi(f_\sigma), \theta_\varphi(s(\sigma))]$ is an element of the Weyl twist Σ.

We claim that $\beta_\varphi(\sigma, U, f_\sigma)$ does not depend on the choice of $U \subseteq H$ or $f_\sigma \in \mathcal{C}_c(U, \Sigma)$. To see this, fix another bisection $U' \subseteq H$ with $\sigma \in \Omega|_{U'}$ and $g_\sigma \in \mathcal{C}_c(U', \Omega)$ with $g_\sigma(\sigma) = 1$.

Then $U \cap U'$ is non-empty since it contains $q(\sigma)$, and there exists $h \in C_0(X)$ with $\text{supp}(h) \subseteq s(U \cap U')$ and $h|_K = 1$ on a compact neighbourhood K of $s(\sigma)$. The functions $f_\sigma \cdot h$ and $g_\sigma \cdot h$ are both compactly supported sections with support contained in $U \cap U'$. Thus there are functions $a, a' \in C_0(X)$ with $a(s(\sigma)) = 1 = a'(s(\sigma))$ and $f_\sigma \cdot ha = g_\sigma \cdot ha'$. We then have $\varphi(f_\sigma)\varphi(ha) = \varphi(g_\sigma)\varphi(ha')$ and $\varphi(ha)(\theta_\varphi(s(\sigma))) = ha(s(\sigma)) = 1 = ha'(s(\sigma)) = \varphi(ha')(\theta_\varphi(s(\sigma)))$, so $[\varphi(f_\sigma), \theta_\varphi(s(\sigma))] = [\varphi(g_\sigma), \theta_\varphi(s(\sigma))]$ and β_φ depends only on $\sigma \in \Omega$. By overloading notation we write $\beta_\varphi : \Omega \to \Sigma$ for the map $\sigma \mapsto \beta_\varphi(\sigma) := \beta_\varphi(\sigma, U, f_\sigma)$.

To see that β_φ is a groupoid homomorphism, fix a composable pair $(\sigma, \tau) \in \Omega^{(2)}$. For all $a \in C_0(G^{(0)})$ we have

$$\begin{aligned}
a(r(\beta_\varphi(\tau))) &= a(\alpha_{\varphi(f_\tau)}(\theta_\varphi(s(\tau)))) \\
&= a(\alpha_{\varphi(f_\tau)}(\theta_\varphi(s(\tau))))\varphi(f_\tau^* f_\tau)(\theta_\varphi(s(\tau))) \\
&= (\varphi(f_\tau)^* a \varphi(f_\tau))(\theta_\varphi(s(\tau))) \\
&= f_\tau^* \varphi^{-1}(a) f_\tau(s(\tau)) \\
&= \varphi^{-1}(a)(r(\tau)) f_\tau^* f_\tau(s(\tau)) \\
&= a(\theta_\varphi(r(\tau))) \\
&= a(\theta_\varphi(s(\sigma))) \\
&= a(s(\beta_\varphi(\sigma))).
\end{aligned}$$

Since this holds for all $a \in C_0(G^{(0)})$, we see that $r(\beta_\varphi(\tau)) = s(\beta_\varphi(\sigma))$, so the pair $(\beta_\varphi(\sigma), \beta_\varphi(\tau))$ is composable.

Let $U, V \subseteq H$ be bisections with $\sigma \in \Omega|_U$ and $\tau \in \Omega|_V$, and let $f_\sigma \in \mathcal{C}_c(U, \Omega)$ and $f_\tau \in \mathcal{C}_c(V, \Omega)$ be functions with $f_\tau(\sigma) = f_\tau(\tau) = 1$. The product $f_\sigma \cdot f_\tau$ belongs to $\mathcal{C}_c(UV, \Omega)$, and since both functions are supported on bisections we have $(f_\sigma \cdot f_\tau)(\sigma\tau) =$

$f_\sigma(\sigma)f_\tau(\tau) = 1$, whereby

$$\begin{aligned}\beta_\varphi(\sigma\tau) &= [\varphi(f_\sigma f_\tau), \theta_\varphi(s(\sigma\tau))]\\ &= [\varphi(f_\sigma), \theta_\varphi(s(\sigma))] \cdot [\varphi(f_\tau), \theta_\varphi(s(\tau))]\\ &= \beta_\varphi(\sigma)\beta_\varphi(\tau).\end{aligned}$$

To see that β_φ is a homomorphism of twists, fix an element $\sigma \in H^{(0)} \times \mathbb{T} \subseteq \Omega$. Since H is étale, the unit space $H^{(0)}$ is open and it follows that $H^{(0)} \times \mathbb{T}$ is open as the preimage of $H^{(0)}$ under the quotient map $\Omega \to H$. Fix an open subset $U \subseteq H^{(0)}$ with $\sigma \in U \times \mathbb{T}$. Then $\mathcal{C}_c(U, \Omega)$ embeds into $C_0(H^{(0)})$ as the compactly supported functions on U, and so $\varphi(f) \in C_0(G^{(0)})$ for any $f \in \mathcal{C}_c(U, \Omega)$. In particular, if $f_\sigma \in \mathcal{C}_c(U, \Omega)$ satisfies $f_\sigma(\sigma) = 1$ then $\beta_\varphi(\sigma) = [\varphi(f_\sigma), \theta_\varphi(s(\sigma))]$ descends to a unit in Ω, and so β_φ induces a homomorphism of twists.

We now show that β_φ is a local homeomorphism. The topology on Σ is generated by open sets specified by normalisers $n \in N(A, B)$ and the homeomorphisms

$$G_n : \mathbb{T} \times \operatorname{dom}(n) \to \Sigma|_{U_n}, \qquad G_n(t, x) = [tn, x].$$

For $\sigma \in \Omega$ and an open bisection $U \subseteq H$ with $\sigma \in \Omega|_U$, pick $f_\sigma \in \mathcal{C}_c(U, \Omega)$ with $f_\sigma(\sigma) = 1$. Let $U_\sigma \subseteq \Omega$ be the open support of f_σ. Define the map $H_\sigma : \mathbb{T} \times s(U_\sigma) \to U_\sigma$ by $H_\sigma(z, x) = z\tau_x$, where $\tau_x \in U_\sigma$ is the unique element of U_σ with $s(\tau_x) = x$ and $f_\sigma(\tau_x) > 0$. Then H_σ is a homeomorphism since f_σ is continuous and non-zero on U_σ. Note now that $f_\sigma(z\tau_x) = \bar{z}f_\sigma(\tau_x)$, so $zf_\sigma(z\tau_x) > 0$ for any $\tau_x \in U_\sigma$ with $f_\sigma(\tau_x) > 0$, giving $\beta(z\tau_x) = [z\varphi(f_\sigma), \theta_\varphi(s(\tau_x))]$. Letting $n_\sigma := \varphi(f_\sigma)$, for $(z, x) \in \mathbb{T} \times s(U_\sigma)$ we compute

$$\begin{aligned}G_{n_\sigma}^{-1} \circ \beta_\varphi \circ H_\sigma(z, x) &= G_{n_\sigma}^{-1} \circ \beta_\varphi(z\tau_x)\\ &= G_{n_\sigma}^{-1}[z\varphi(f_\sigma), \theta_\varphi(s(\tau_x))]\\ &= G_{n_\sigma}^{-1}[zn_\sigma, \theta_\varphi(s(\tau_x))]\\ &= (z, \theta_\varphi(s(\tau_x))).\end{aligned}$$

Thus the composition $G_{n_\sigma}^{-1} \circ \beta_\varphi \circ H_\sigma$ agrees with the homeomorphism $\mathrm{id}_\mathbb{T} \times \theta_\varphi$ on its domain, implying that β_φ is a homeomorphism on U_σ. Hence β_φ is a local homeomorphism. This argument also shows that β_φ descends to a local homeomorphism $\tilde{\beta}_\varphi : H \to G$, as one need only ignore the $\mathbb{T}$ component in the homeomorphisms H_σ and G_{n_σ}.

For contradiction, suppose that β_φ is not surjective. Fix $[n, x] \in \Sigma$ with $[n, x] \neq \beta_\varphi(\sigma)$ for any $\sigma \in \Omega$. Then for any open bisection $U \subseteq H$ and $f \in \mathcal{C}_c(U, \Omega)$, the element $\varphi(f)$ is a normaliser in $C^*_{\mathrm{ess}}(G, \Sigma)$ and so we may consider $\varphi(f)$ as a function $\Sigma \to \mathbb{C}$ via Lemma 6.2.6. We then have $\operatorname{supp}(\varphi(f)) = \{[\varphi(f), y] : y \in \operatorname{dom}(\varphi(f))\} = \{[\varphi(f), \theta_\varphi(x)] : x \in \operatorname{supp}(f^*f)\}$, which is exactly the image of the support of f under β_φ. Thus $\varphi(f)[n, x] = 0$ for all $f \in \mathcal{C}_c(V, \Omega)$, and since each f has compact support, the function $\varphi(f)$ must vanish on an open bisection neighbourhood of $[n, x]$.

For $f \in \mathcal{C}_c(U, \Omega)$ let $U_f := \operatorname{supp}(f) \subseteq H$ be the open support of f in H. Note that since β_φ restricts to a homeomorphism of unit spaces and is a local homeomorphism, the image of a bisection is a bisection under this map, so $\beta_\varphi(U_f)$ is an open bisection in G. We claim that $[\alpha_n, x]$ does not belong to the interior of the closure of $\beta_\varphi(U_f)$. To see this, suppose that $[\alpha_n, x] \in \overline{\beta_\varphi(U_f)}^\circ$. Then there exists an open neighbourhood $V_{[\alpha_n, x]} \subseteq \overline{\beta_\varphi(U_f)}$ with $[\alpha_n, x] \in V_{[\alpha_n, x]}$. Then the set

$$V_{[\alpha_n, x]}^{-1} \cdot \overline{\beta_\varphi(U_f)}^\circ = \{v^{-1}u : v \in V_{[\alpha_n, x]}, u \in \overline{\beta_\varphi(U_f)}^\circ, r(v) = r(u)\},$$

is an open set contained in $\overline{\beta_\varphi(U_f)}^{-1}\overline{\beta_\varphi(U_f)} \subseteq \overline{\varphi(U_f^{-1}U_f)} \subseteq \overline{G^{(0)}}$. Thus $V_{[\alpha_n,x]}^{-1} \cdot \overline{\beta_\varphi(U_f)}^\circ$ is contained in $G^{(0)}$ since G is effective by Lemma 5.1.6. This implies

$$V_{[\alpha_n,x]} = V_{[\alpha_n,x]} \cdot V_{[\alpha_n,x]}^{-1} \cdot \overline{\beta_\varphi(U_f)}^\circ = r(V_{[\alpha_n,x]}) \cdot \overline{\beta_\varphi(U_f)}^\circ \subseteq r(\beta_\varphi(U_f)) \cdot \overline{\beta_\varphi(U_f)}^\circ,$$

and Lemma 6.3.4 then shows that $V_{[\alpha_n,x]} \subseteq \beta_\varphi(U_f)$, contradicting that $[n,x]$ does not belong to the image of β_φ.

Let $f_k \in \mathcal{C}_c(U_k,\Omega)$ for $k = 1,\ldots,\ell$ be a finite collection of such normalisers, so that their sum is a generic element of $\mathcal{C}_c(H,\Omega)$. For each $k = 1,\ldots,\ell$ let U_k be the open support of f_k in H. Let $\hat{n}$ be the function from Lemma 6.2.6 associated to the normaliser n. Let W be an open bisection neighbourhood of $[\alpha_n,x]$ such that the function $\hat{n} : \Sigma \to \mathbb{C}$ satisfies $|\hat{n}[m,y]| \geqslant \hat{n}[n,x]/2$ for all $[m,y] \in \Sigma|_W$. Note that we can do this since $\hat{n}$ restricts to a continuous function on a bisection of G. Define $Z \subseteq G$ by

$$Z := W \backslash \left(\bigcup_{k=1}^{n} \overline{\beta_\varphi(U_k)} \right).$$

Lemma 6.3.5 asserts that Z is non-empty and open. By Lemma 6.3.3 we then have

$$\begin{aligned} \left\| \hat{n} - \sum_{k=1}^{\ell} \varphi(f_k) \right\| &\geqslant \sup_{[m,x]\in\Omega|_Z} \left| \hat{n}[m,y] - \sum_{k=1}^{\ell} \varphi(f_k)[m,y] \right| \\ &= \sup_{[m,y]\in\Omega|_W} |\hat{n}[m,y]| \\ &\geqslant |\hat{n}[n,x]|/2 \\ &= \sqrt{n^*n(x)}/2. \end{aligned}$$

Since $\sqrt{n^*n(x)}$ is positive, we see that n does not lie in the closure of the image of $\mathcal{C}_c(H,\Omega)$, which is $C^*_{\mathrm{ess}}(G,\Sigma)$, rendering a contradiction. Hence β_φ is surjective.

Finally we shall show that the (twist over the) unit space $H^{(0)}$ is dense in the kernel of β_φ, showing that (H,Ω) is an extension of (G,Σ) by a normal isotropy subbundle inside of which $H^{(0)}$ is dense. Let $\tilde{\beta}_\varphi : H \to G$ be the map descending β_φ to the quotient H. The kernel of $\tilde{\beta}_\varphi$, that is, the preimage of the unit space $G^{(0)}$ under $\tilde{\beta}_\varphi$ is open since G is étale. Let $U \subseteq \ker(\tilde{\beta}_\varphi)$ be an open bisection and suppose that $U \cap H^{(0)} = \varnothing$. Then for any $f \in \mathcal{C}_c(U,\Omega)$ we have $[\varphi(f), \theta_\varphi(x)] \in G^{(0)} \times \mathbb{T}$ for any $x \in \mathrm{dom}(f)$, so $\varphi(f) \in C_0(G^{(0)})$. Since φ restricts to an isomorphism of subalgebras $C_0(H^{(0)}) \cong C_0(G^{(0)})$, the function f must represent an element of $C_0(H^{(0)})$ in $C^*_{\mathrm{ess}}(H,\Omega)$. So there is a function $g \in C_0(H^{(0)})$ such that $f = g$ in $C^*_{\mathrm{ess}}(G)$. Consider f and g as sections of the line bundle associated to (H,Ω). Since $U \cap H^{(0)} = \varnothing$, Lemma 6.3.2 gives $||f|| \leqslant ||f-g|| = 0$, implying that $f = 0$. Thus U must be empty, so every non-empty open subset of $\ker(\tilde{\beta}_\varphi)$ has non-empty intersection with $H^{(0)}$; equivalently $H^{(0)}$ is dense in $\ker(\tilde{\beta}_\varphi)$. □

Corollary 6.3.6. *Let $A \subseteq B$ be an essential Cartan pair and let (G,Σ) be the associated Weyl twist. Let (H,Ω) be a twist over an étale groupoid with locally compact Hausdorff unit space. Suppose there is an isomorphism $\varphi : C^*_{\mathrm{ess}}(H,\Omega) \to C^*_{\mathrm{ess}}(G,\Sigma)$ that entwines conditional expectations and restricts to an isomorphism $\varphi|_{C_0(H^{(0)})} : C_0(H^{(0)}) \to C_0(G^{(0)})$. If H is effective, then the map $\beta_\varphi : (H,\Omega) \to (G,\Sigma)$ in Theorem* 6.3.1 *is an isomorphism of twists.*

Proof. Let $\tilde{\beta}_\varphi : H \to G$ be induced by β_φ. The kernel of $\tilde{\beta}_\varphi$ is an open normal isotropy group bundle in H by Theorem 6.3.1, and so is contained in the interior of the isotropy of H. Since H is effective, the interior of the isotropy is exactly the unit space of H and so the map $\tilde{\beta}_\varphi$ is injective, hence an isomorphism between H and G.

Suppose $\beta_\varphi(\omega) \in \Omega^{(0)}$ for some $\omega \in \Omega$. Let $q_\Omega : \Omega \to H$ and $q_\Sigma : \Sigma \to G$ be the canonical quotient maps associated to the twists. Since β_φ is a homomorphism of twists, we have $\tilde{\beta}_\varphi(q_\Omega(\omega)) = q_\Sigma(\beta_\varphi(\omega)) \in G^{(0)}$, so $q_\Omega(\omega) \in H^{(0)}$ as $\tilde{\beta}_\varphi$ is an isomorphism. Hence $\omega \in H^{(0)} \times \mathbb{T}$, so $\omega = (s(\omega), z)$ for some $z \in \mathbb{T}$. Since $\beta_\varphi(\omega) \in \Sigma^{(0)}$ we have $\beta_\varphi(\omega) = \beta_\varphi(s(\omega))$. Thus for any function $f \in \mathcal{C}_c(H^{(0)}, \Omega)$ with $f(\omega) = 1$ we have $f(s(\omega)) = 1$. Identifying $\Omega^{(0)} = H^{(0)} \times \{1\}$ with $H^{(0)}$ we see $1 = f(\omega) = f(s(\omega), z) = zf(s(\omega), 1) = zf(s(\omega)) = z$. Thus $\omega = (s(\omega), 1) = s(\omega)$, so $\omega \in \Omega^{(0)}$. Hence β_φ is injective, so is an isomorphism of twists. ☐

6.3.1 Non-uniqueness for non-effective groupoids

One may then ask the question of when two étale groupoids give rise to isomorphic essential groupoid C^*-algebras. We are not able to answer this question in full generality, but we can provide this for the condition when one groupoid is contained in another as an open subgroupoid. We answer this question in the case where the twist over the groupoid is trivial.

Proposition 6.3.7. *Let $H \subseteq G$ be an open subgroupoid of an étale groupoid with locally compact Hausdorff unit space. Consider $C^*_{\mathrm{ess}}(H) \subseteq C^*_{\mathrm{ess}}(G)$. Then $C^*_{\mathrm{ess}}(H) = C^*_{\mathrm{ess}}(G)$ if and only if for every $\gamma \in G$ there exists a bisection $V \subseteq H$ with $\gamma \in \overline{V}^\circ \cdot s(V)$. In particular, H is dense in G and $H^{(0)} = G^{(0)}$.*

Proof. Suppose first that for every $\gamma \in G$ there is some bisection $V \subseteq H$ with $\gamma \in \overline{V}^\circ \cdot s(V)$. Let $U \subseteq G$ be a bisection neighbourhood of γ contained in $\overline{V}^\circ \cdot s(V)$; such a bisection exists since $\overline{V}^\circ \cdot s(V)$ is open. We claim that $\mathcal{C}_c(U)$ and $\mathcal{C}_c(V \cdot s(U))$ are equal in the essential groupoid C^*-algebra. To show this, fix $f \in \mathcal{C}_c(U)$. Note that $s|_V^{-1} \circ s : U \to V \cdot s(U)$ is a homeomorphism, so we can define $\tilde{f} := f \circ s|_V^{-1} \circ s$ on V, and extend by zero to a function on H. Then $\tilde{f}$ is a continuous and compactly supported function on V precisely since f is, and $s|_V^{-1} \circ s$ is a homeomorphism between U and $V \cdot s(U)$.

We claim $f - \tilde{f}$ has meagre support, which by [16, Proposition 7.18] shows that f and $\tilde{f}$ represent the same element of $C^*_{\mathrm{ess}}(G)$. Since $s|_V^{-1} \circ s$ restricts to the identity on $U \cap V$, the open support of $f - \tilde{f}$ must be contained in $(\mathrm{supp}^\circ(f) \backslash (U \cap V)) \cup (\mathrm{supp}^\circ(\tilde{f}) \backslash (U \cap V))$. Since the open supports of f and $\tilde{f}$ are contained in U and V respectively, this reduces to $\mathrm{supp}^\circ(f) \backslash V \cup \mathrm{supp}^\circ(\tilde{f}) \backslash U$. We shall show that each of these components of the union is meagre.

Note that $\mathrm{supp}^\circ(f) \subseteq U \subseteq \overline{V}$, and $\overline{V} \backslash V$ is meagre as it is closed and contains no open subsets. Hence $\mathrm{supp}^\circ(f) \backslash V$ is meagre.

We claim that U is dense in the support of $\tilde{f}$. Then by a similar argument as above the set $\mathrm{supp}^\circ(\tilde{f}) \subseteq \overline{U} \backslash U$ is meagre. Fix an open subset $W \subseteq \mathrm{supp}^\circ(\tilde{f})$. Note that W is open in G since $\mathrm{supp}^\circ(\tilde{f})$ is. Suppose $W \cap U = \varnothing$. We shall show that W is then empty, whereby $\mathrm{supp}^\circ(\tilde{f})$ is contained in the closure of U. Note that since $s(W) \subseteq s(\mathrm{supp}^\circ(\tilde{f})) \subseteq s(U)$, it suffices to show that $U \cdot s(W)$ is empty. We then note that $(U \cdot s(W)) \cap V = U \cap V \cdot s(W) = U \cap W = \varnothing$, whereby $U \cdot s(W)$ is empty since U lies in the closure of V, so every non-empty open subset of U has non-empty intersection with V. Thus $W \cap U \neq \varnothing$ for all non-empty open subsets $W \subseteq \mathrm{supp}^\circ(\tilde{f})$, whereby $\mathrm{supp}^\circ(\tilde{f})$ is contained in the closure of U so $\mathrm{supp}^\circ(\tilde{f}) \backslash U$ is meagre. It follows that $\mathrm{supp}^\circ(f - \tilde{f})$ is a meagre set and the homeomorphism $s|_V^{-1} \circ s$ induces an identification of $\mathcal{C}_c(U)$ and $\mathcal{C}_c(V)$ in $C^*_{\text{ess}}(G)$. We can do this on neighbourhoods around any point in G, so a partition of unity argument shows that $C^*_{\text{ess}}(H)$ is dense in $C^*_{\text{ess}}(G)$, whereby they are equal.

We shall show the converse via contrapositive. Suppose there exists $\gamma \in G \backslash H$ with the property that, for any bisection $V \subseteq H$ we have $\gamma \notin \overline{V}^\circ \cdot s(V)$. That is, for any bisection $V \subseteq H$ either no open neighbourhood of γ is contained in $\overline{V}$, or $s(\gamma) \notin s(V)$. Let U be a bisection containing γ and let $K \subseteq U$ be a compact neighbourhood also containing γ, and denote the interior of K by K°. Let $f \in \mathcal{C}_0(U)$ be a function with $f|_K = 1$. We claim that f does not belong to $C^*_{\text{ess}}(H)$. Recall that the subspaces $\mathcal{C}_c(V)$ for bisections $V \subseteq H$ span a dense subalgebra of $C^*_{\text{ess}}(H)$, so it suffices to show that f cannot be approximated by any finite sum of elements belonging to subspaces of this form. Fix bisections $V_i \subseteq H$ and functions $g_i \in \mathcal{C}_c(V_i)$ for $i = 1, \ldots, n$. By assumption, for each $i \leqslant 1$, at least one of the following two statements is true: that $s(\gamma) \notin s(V_i)$ or $\gamma \notin \overline{V_i}^\circ$.

Let I denote the set of values of i for which $s(\gamma) \notin s(V_i)$. For $i \in I$, g_i has compact support $K_i \subseteq V_i$, and so $s(K_i)$ is closed in $G^{(0)}$. Since $G^{(0)}$ is regular, there are disjoint open neighbourhoods V_i' and W_i' separating $s(K_i)$ and $\{s(\gamma)\}$. Let $W_i = s|_U^{-1}(W_i' \cap s(U))$, that is, the lift of W_i' to U under the source map. Then $g_i|_{W_i} = 0$, since g_i is only non-zero on its support, and the source of g_i's support does not intersect the source of W_i. Let $W := \bigcap_{i \in I} W_i$, and note that this is an open neighbourhood of γ.

For the second case, let J denote the set of values of i such that $s(\gamma) \in s(V_i)$ but $\gamma \notin \overline{V_i}^\circ$. The set $U_J' := K^\circ \backslash \left(\bigcup_{i \in J} \overline{V_i}\right)$ is then open, non-empty, and has γ as a limit by Lemma 6.3.5.

The set $Z := U_J' \cap W$ is non-empty since $\gamma \in W \cap \overline{U_J'}$, and W is open so intersects U_J'. Moreover, Z is open as both U_J' and W are, and contained in K° since U_J' is. Thus each g_i is zero on Z as g_i is zero on either W_i (if $i \in I$) or is zero on U_J' (if $i \in J$), since U_J' lies in the complement of each V_i, which in turn contains the support of g_i. Thus the sum of the g_i is again zero on Z, and so

$$\left. \left(f - \sum_{i=1}^n g_i \right) \right|_Z = f|_Z = 1,$$

as $Z \subseteq K$. By Lemma 6.3.3 we see that $||f - \sum_i g_i|| \geqslant 1$ and so f does not lie in the closure of the span of $\mathcal{C}_c(V)$ for bisections $V \subseteq H$. □

Remark 6.3.8. If (G, Σ) is a twist over an effective étale Hausdorff locally compact second countable groupoid, then [22, Proposition 4.11] states that the Weyl groupoid associated to the pair $C_0(G^{(0)}) \subseteq C^*_{\text{r}}(G, \Sigma)$ is canonically isomorphic to G. When G is globally Hausdorff and second countable, effectivity is equivalent to the condition that G is topologically principal: that there is a dense set of points in the unit space of G that have trivial isotropy. The conditions of Proposition 6.3.7 can only occur trivially if G is Hausdorff, so this is only an interesting result for non-Hausdorff groupoids. Notably, if one has a groupoid G satisfying the conditions of Proposition 6.3.7 such that the unit space $H := G^{(0)}$ is dense in G, then $C^*_{\text{ess}}(G) = C^*_{\text{ess}}(H) = C_0(G^{(0)})$. However, G need not consist only of units since in a non-Hausdorff groupoid (with Hausdorff unit space) the unit space is not closed.

Example 6.3.9. Consider the line with two origins $G = (0, 1] \cup \{0_1, 0_0\}$. Define the range and source maps $r, s : G \to (0, 1] \cup \{0_0\}$ by $r(t) = s(t) = t$ for $t > 0$ and $r(0_i) = s(0_i) = 0_0$

for $i = 0, 1$. Define a multiplication $G_s \times_r G \to G$ by $t \cdot t = t$ for $t > 0$ and $0_i \cdot 0_j = 0_{i+j \mod 2}$. Then the unit space of G is homeomorphic to the interval $[0, 1]$, and the groupoid G satisfies the conditions of Proposition 6.3.7 with the dense open subgroupoid $H = G^{(0)}$. Thus $C^*_{\text{ess}}(G) \cong C[0, 1]$, but clearly G is not isomorphic to $[0, 1]$ as a groupoid, since it is both non-Hausdorff and contains isotropy (although, very little isotropy).

Example 6.3.9 illustrates that sometimes 'small' amounts of isotropy do not change the resulting essential groupoid C^*-algebra. The next result states that when G is effective, then removing any more isotropy will change the resulting essential groupoid C^*-algebra, so essential groupoids are minimal in a certain sense.

Corollary 6.3.10. *Let G be an effective étale groupoid with locally compact Hausdorff unit space. Then the only open subgroupoid $H \subseteq G$ with $C^*_{\text{ess}}(H) = C^*_{\text{ess}}(G)$ is $H = G$.*

Proof. Fix $\gamma \in G$. By Proposition 6.3.7 there exists a bisection $U \subseteq H$ with $\gamma \in \overline{U}^\circ \cdot s(U)$. Let $\eta \in U$ be the unique element of U with $s(\eta) = s(\gamma)$. Then $\gamma\eta^{-1}$ belongs to $\overline{U}^\circ \cdot s(U) \cdot U^{-1}$ which is an open set contained in the closure of the unit space of G, so is contained in $\overline{G^{(0)}}^\circ$. Since G has Hausdorff unit space, the closure of the unit space consists only of isotropy, so $\overline{G^{(0)}}^\circ$ is contained in the interior of the isotropy of G. Since G is effective, the interior of the isotropy is again the unit space, and so we have $\gamma\eta^{-1} \in G^{(0)}$ implying $\gamma = \eta \in U \subseteq H$. This holds for all $\gamma \in G$ so we see $G = H$. $\square$

Remark 6.3.11. As mentioned earlier, twists over effective groupoids are not the only twists that give rise to essential Cartan inclusions. Theorem 6.3.1 implies that such groupoids have effective twisted quotients giving rise to the same Cartan pair, and in particular twists descend to the effective quotients of these groupoids.

Unfortunately it is not exactly known what properties of a groupoid H and a twist Ω over H ensure that the inclusion $C_0(H^{(0)}) \subseteq C^*_{\text{ess}}(H, \Omega)$ is maximal abelian. As shown, the Weyl groupoid associated to an essential commutative Cartan pair is effective, but Example 6.3.9 shows that these are not the only such groupoids.

6.4 Twisted groupoid automorphisms and Cartan automorphisms

Having established the link between twists over certain étale groupoids and essential Cartan pairs, one may ask whether automorphisms of one object induce automorphisms of the other. We provide constructions in both directions: Cartan automorphisms from twisted groupoid automorphisms and vice versa. We also show that these constructions are mutually inverse if the twist is over an effective groupoid, giving an isomorphism of the automorphism group of an essential Cartan pair $A \subseteq B$ to the automorphism group of the corresponding Weyl pair $(G(A, B), \Sigma(A, B))$. In particular we show that all automorphisms of Weyl twists come from automorphisms of the induced Cartan pair and vice versa.

Throughout this section we assume (G, Σ) is a twisted groupoid giving rise to an essential Cartan pair $C_0(G^{(0)}) \subseteq C^*_{\text{ess}}(G, \Sigma)$.

Definition 6.4.1. Let (A_1, B_1) and (A_2, B_2) be regular non-degenerate inclusions with local conditional expectations $E_i : B_i \to M_{\text{loc}}(A_i)$ for $i = 1, 2$. A *morphism of Cartan pairs* or *Cartan morphism* $(A_1, B_1) \to (A_2, B_2)$ is a *-homomorphism $\varphi : B_1 \to B_2$ such that

1. $\varphi(A_1) \subseteq A_2$;
2. $\varphi(N(A_1, B_1)) \subseteq N(A_2, B_2)$; and
3. $E_2 \circ \varphi(b) = \varphi \circ E_1(b)$ for all $b \in B_1$ with $E_1(b) \in A_1$.

An *isomorphism of Cartan pairs* or *Cartan isomorphism* is a Cartan morphism that is a *-isomorphism.

If $\varphi : (A_1, B_1) \to (A_2, B_2)$ is an isomorphism of Cartan pairs as above, then $\varphi(A_1) = A_2$ will hold since A_1 is masa in B_1. That is, φ restricts to an isomorphism $A_1 \cong A_2$.

Lemma 6.4.2. *Let (A_1, B_1) and (A_2, B_2) be regular non-degenerate inclusions with local conditional expectations $E_i : B_i \to M_{\mathrm{loc}}(A_i)$ for $i = 1, 2$ and let $\varphi : B_1 \to B_2$ be a *-homomorphism. If φ entwines conditional expectations and $\varphi(A_1) = A_2$, then φ is a Cartan morphism.*

Proof. We need only show that φ maps normalisers to normalisers. Let $n \in N(A_1, B_1)$ be a normaliser and fix $a \in A_2$. Since $\varphi(A_1) = A_2$, there exists $a' \in A_1$ with $\varphi(a') = a$. Then $\varphi(n)^* a \varphi(n) = \varphi(n^* a' n) \in \varphi(A_1) = A_2$. □

If the pairs (A_1, B_1) and (A_2, B_2) are aperiodic inclusions then the conditional expectations are unique by [17, Theorem 3.6]. Thus the expectations are entwined by any *-homomorphism that restricts to an isomorphism of the subalgebras A_1 and A_2.

Lemma 6.4.3. *The assignment (A, B) to $(G(A, B), \Sigma(A, B))$ taking an essential Cartan pair to its Weyl groupoid and twist is functorial for Cartan isomorphisms. That is, if φ and ψ are automorphisms of (A, B) then $\beta_{\varphi\circ\psi} = \beta_\varphi \circ \beta_\psi$, where β_φ, β_ψ, and $\beta_{\varphi\circ\psi}$ are as in Theorem* 6.3.1.

Proof. We observe that $\beta_\varphi \circ \beta_\psi[n, x] = [\varphi(\psi(n)), ((\varphi \circ \psi)|^*_{C_0(G^{(0)})})^{-1}(x)] = \beta_{\varphi\circ\psi}[n, x]$. □

Proposition 6.4.4. *Let $\beta : (G, \Sigma) \to (H, \Omega)$ be an automorphism of twists. Then there is an isomorphism of Cartan pairs $\Phi_\beta : (C_0(G^{(0)}), C^*_{\mathrm{ess}}(G, \Sigma)) \to (C_0(H^{(0)}), C^*_{\mathrm{ess}}(H, \Omega))$ mapping a section $f \in \mathcal{C}_c(G, \Sigma)$ to $f \circ \beta^{-1}$.*

Proof. Since β is an isomorphism of twists, so is β^{-1}, and in particular β^{-1} is a homeomorphism mapping bisections to bisections, so maps sums of compactly supported sections to sums of compactly supported sections. Let $\Phi^0_\beta : \mathcal{C}_c(G, \Sigma) \to \mathcal{C}_c(H, \Omega)$ be the map $\Phi^0_\beta(f) = f \circ \beta^{-1}$. That Φ^0_β is a *-isomorphism follows from the fact that β is an isomorphism of groupoids.

The isomorphism Φ^0_β then extends to an isomorphism Φ_β of the full twisted groupoid C^*-algebras. Let R_G and R_H be the restriction maps Proposition 6.2.9 for the twists (G, Σ) and (H, Ω), and note that R_G and R_H extend to the canonical conditional expectations $EL_G : C^*(G, \Sigma) \to M_{\mathrm{loc}}(C_0(G^{(0)}))$ and $EL_H : C^*(H, \Omega) \to M_{\mathrm{loc}}(C_0(H^{(0)}))$. Then clearly $R_H \circ \Phi^0_\beta = \Phi^0_\beta \circ R_G$, so Φ_β descends to an isomorphism of the quotients $C^*_{\mathrm{ess}}(G, \Sigma) \to C^*_{\mathrm{ess}}(H, \Omega)$.

To see that Φ_β is an isomorphism of Cartan pairs, note that β restricts to a homeomorphism of the unit spaces, and so for $f \in C_0(G^{(0)})$ we have $\Phi_\beta(f) = f \circ \beta^{-1}$ which belongs to $C_0(H^{(0)})$. That Φ_β maps normalisers to normalisers follows from Lemma 6.4.2. □

Corollary 6.4.5. *The assignment (G, Σ) to $(C_0(G^{(0)}), C^*_{\mathrm{ess}}(G, \Sigma))$ is functorial lifting isomorphisms of twists to Cartan isomorphisms. That is, the map $\beta \mapsto \Phi_\beta$ from twisted groupoid automorphisms of (G, Σ) to Cartan automorphisms of $(C_0(G^{(0)}), C^*_{\mathrm{ess}}(G, \Sigma))$ is a group homomorphism.*

Proof. Let $\beta, \beta' \in \mathrm{Aut}(G, \Sigma)$. For each slice $U \subseteq G$ and $f \in \mathcal{C}_c(U, \Sigma)$ we have $\Phi_\beta(\varphi_{\beta'}(f)) = f \circ (\beta'^{-1} \circ \beta^{-1}) = \Phi_{\beta\circ\beta'}(f)$. Such normalisers span a dense subalgebra of $C^*_{\mathrm{ess}}(G, \Sigma)$ and both $\Phi_\beta \circ \Phi_{\beta'}$ and $\Phi_{\beta\circ\beta'}$ are linear and isometric so the equality extends to all elements of $C^*_{\mathrm{ess}}(G, \Sigma)$. □

Proposition 6.4.6. *Let β be a twisted groupoid automorphism of (G, Σ), and let $\tilde{\beta} : G \to G$ be its descent to G. If the induced Cartan automorphism Φ_β is equal to the identity on $C^*_{\mathrm{ess}}(G, \Sigma)$, then there is a dense subset $D \subseteq G$ with $\tilde{\beta}|_D = \mathrm{id}_D$. Moreover, if $\tilde{\beta} \neq \mathrm{id}_G$ then there exists $\eta \in \overline{G^{(0)}}^\circ \backslash G^{(0)}$, and in particular $G^{(0)} \neq \overline{G^{(0)}}^\circ$.*

Proof. Suppose that $U \subseteq G$ is an open bisection with $\tilde{\beta}(\gamma) \neq \gamma$ for all $\gamma \in U$. We claim that there is a non-empty open subset $V \subseteq U$ with $\tilde{\beta}^{-1}(V) \cap V = \varnothing$. If $\beta^{-1}(U) \cap U = \varnothing$ then the choice of $V := U$ suffices. Else, if $\beta^{-1}(U) \cap U \neq \varnothing$ then there exists some $\gamma \in U$ with $\tilde{\beta}^{-1}(\gamma) \in U$. Since U is Hausdorff, we can pick open $V_1, V_2 \subseteq U$ with $\gamma \in V_1$, $\tilde{\beta}^{-1}(\gamma) \in V_2$, and $V_1 \cap V_2 = \varnothing$. Set $V := V_1 \cap \tilde{\beta}(V_2)$. Then

$$V \cap \tilde{\beta}^{-1}(V) = V_1 \cap \tilde{\beta}(V_2) \cap \tilde{\beta}^{-1}(V_1) \cap V_2 \subseteq V_1 \cap V_2 = \varnothing$$

as required. In particular, since V and $\tilde{\beta}(V)$ belong to the same bisection and are disjoint the product slices $V^{-1}\tilde{\beta}^{-1}(V)$ and $\tilde{\beta}^{-1}(V)^{-1}V$ are empty. Thus for any $f \in \mathcal{C}_c(V, \Sigma)$ the compositions $f^*\Phi_\beta(f)$ and $\Phi_\beta(f)^*f$ are zero as their supports are $V^{-1}\tilde{\beta}(V)$ and $\tilde{\beta}^{-1}(V)^{-1}V$, respectively. Since Φ_β is the identity on $C^*_{\mathrm{ess}}(G, \Sigma)$, we then have $0 = ||f - \Phi_\beta(f)||^2 = ||f^*f - f^*\Phi_\beta(f) - \Phi_\beta(f)^*f + \Phi_\beta(f^*f)|| = ||f^*f + \Phi_\beta(f^*f)||$, and so $f^*f = -\Phi_\beta(f^*f)$. But both f^*f and $\Phi_\beta(f^*f)$ are positive elements of $C^*_{\mathrm{ess}}(G, \Sigma)$, so $f^*f = 0$ must hold. Thus $f = 0$ for all $f \in \mathcal{C}_c(V, \Sigma)$, whereby $V = \varnothing$, which is a contradiction. Hence there are no open bisections $U \subseteq G$ with $\tilde{\beta}(\gamma) \neq \gamma$ for all $\gamma \in U$, and so the subset D of G on which $\tilde{\beta}$ acts trivially is dense in G.

Suppose $\tilde{\beta} \neq \mathrm{id}_G$. Then there exists $\gamma \in G$ with $\tilde{\beta}(\gamma) \neq \gamma$. Let $U \subseteq G$ be an open bisection with $\gamma \in U$. Then $U \cap D$ is dense in U, and so $(U \cap D)^{-1}\tilde{\beta}(U \cap D) = (U \cap D)^{-1}(U \cap D) = s(U \cap D)$ is dense in the bisection $U^{-1}\tilde{\beta}(U)$. In particular $U^{-1}\tilde{\beta}(U)$ is an open bisection contained in the closure of the unit space $G^{(0)}$, so in particular $U^{-1}\tilde{\beta}(U) \subseteq \overline{G^{(0)}}^\circ$. Since $\tilde{\beta}(\gamma) \neq \gamma$, we have that $\eta := \gamma^{-1}\tilde{\beta}(\gamma)$ is not a unit, and so $\eta \in \overline{G^{(0)}}^\circ \backslash G^{(0)}$. □

Corollary 6.4.7. *Let (G, Σ) be a twist over an étale effective groupoid with locally compact Hausdorff unit space. Suppose that $C_0(G^{(0)}) \subseteq C^*_{\mathrm{ess}}(G, \Sigma)$ is an essential commutative Cartan pair. Then an automorphism β of (G, Σ) gives rise to the identity automorphism Φ_β of $C^*_{\mathrm{ess}}(G, \Sigma)$ if and only if β is the identity map.*

Proof. Since G has Hausdorff unit space, the closure of the unit space is contained in the isotropy of G, and hence so is the interior. The interior of the isotropy is again the unit space of G since G is effective. If β is not the identity but $\Phi_\beta = \mathrm{id}_{C^*_{\mathrm{ess}}(G,\Sigma)}$ then Proposition 6.4.6 gives that $\overline{G^{(0)}}^\circ$ is strictly larger than $G^{(0)}$, leading to a contradiction. □

Corollary 6.4.8. *If G is effective then the homomorphism*

$$\Phi : \mathrm{Aut}(G, \Sigma) \to \mathrm{Aut}(C_0(G^{(0)}), C^*_{\mathrm{ess}}(G, \Sigma))$$

is injective.

Proposition 6.4.9. *For the Weyl twist (G, Σ) of an essential Cartan pair (A, B) the constructions in Theorem* 6.3.1 *and Proposition* 6.4.4 *are mutually inverse. That is, for a twisted groupoid automorphism $\beta : (G, \Sigma) \to (G, \Sigma)$ we have $\beta_{\Phi_\beta} = \beta$ and for a Cartan automorphism $\Phi : (C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma) \to (C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma))$ we have $\Phi_{\beta_\Phi} = \Phi$. Hence* $\text{Aut}(G, \Sigma)$ *and* $\text{Aut}(C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma))$ *are isomorphic as groups via these constructions.*

Proof. Let $\Phi : (C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma)) \to (C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma))$ be a Cartan automorphism. By Lemma 6.4.3 we have $\beta_\Phi[n, x] = [\Phi(n), \theta_\Phi(x)]$, and so $\beta_\Phi^{-1}[n, x] = [\Phi^{-1}(n), \theta_\Phi^{-1}(x)]$. For a normaliser $n \in N(C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma))$ let $U_n := \{[\alpha_n, x] : x \in \text{dom}(n)\}$ be the bisection of G specified by n, and fix $f \in \mathcal{C}_c(U_n, \Sigma)$. Then $f = \hat{n}h$ for some $h \in C_0(G^{(0)})$ by Lemma 6.2.7, and moreover for all $[n, x]$ in the open support of f we have $[n, x] = [znh, x]$ for some fixed $z \in \mathbb{T}$ (namely $z = \overline{h(x)}$). Thus

$$\begin{aligned}\Phi(f)[\Phi(f), \theta_\Phi(x)] &= \Phi(\hat{n}h)[\Phi(\hat{n}h), \theta_\Phi(x)] \\ &= \sqrt{\Phi((nh)^*(nh))(\theta_\Phi(x)} \\ &= \sqrt{(nh)^*(nh)(x)} \\ &= \sqrt{f^*f(x)} \\ &= f[f, x],\end{aligned}$$

and

$$\begin{aligned}\Phi_{\beta_\Phi}(f)[\Phi(f), \theta_\Phi(x)] &= f(\beta_\Phi^{-1}[\Phi(f), \theta_\Phi(x)]) \\ &= f[\Phi^{-1}\Phi(f), \theta_\Phi^{-1}\theta_\Phi(x)] \\ &= f[f, x].\end{aligned}$$

Hence $\Phi = \Phi_{\beta_\Phi}$, so the map $\text{Aut}(G, \Sigma) \to \text{Aut}(C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma))$ is surjective. The Weyl groupoid is effective by Lemma 5.1.6, and so Corollary 6.4.8 gives that this map $\text{Aut}(G, \Sigma) \to \text{Aut}(C_0(G^{(0)}), C^*_{\text{ess}}(G, \Sigma))$ is injective, hence an isomorphism. The above calculation shows that the construction in Theorem 6.3.1 is a one-sided inverse to this isomorphism, hence is the two-sided inverse and is itself an isomorphism. □

As previously mentioned, for any C^*-inclusion $A \subseteq B$ there is always a pseudo-expectation $B \to I(A)$ taking values in Hamana's injective hull $I(A)$ of A (cf. [12]). For commutative C^*-algebras the injective hull and the local multiplier algebra agree by [9, Theorem 1], so commutative inclusions $A \subseteq B$ always have a local multiplier algebra valued conditional expectation. This expectation is however not necessarily faithful a priori, but by the work of Kwaśniewski and Meyer [16] a certain quotient can be taken to make this conditional expectation faithful. If the image of the masa subalgebra in this quotient is again masa, then one would acquire an essential Cartan pair. Unfortunately it is not known when the image of of the subalgebra in such a quotient is still masa.

6.5 Outlook

We are able to show that aperiodic dynamical inclusions where the subalgebra is commutative are exactly essential commutative Cartan pairs, and the analysis of such pairs is accessible via geometric techniques associated to groupoids. Thus we are able to find stronger classification results in this case. We have shown that the twist over an effective groupoid defining such an inclusion is unique up to canonical isomorphism among twists over effective groupoids, and gained a universal property relating the Weyl twist to other possible groupoid representations that are not effective. We recover and generalise many of the results of Renault [22], as well as some of their noncommutative counterparts in [7] and [15].

We were however not able to show that a twist over an effective groupoid gives rise to a masa inclusion, and we only gain strong results for Weyl twists arising from essential commutative Cartan pairs. The proof for Hausdorff groupoids relies heavily on continuity of functions on the groupoid, and so these techniques are not available in the essential Cartan setting. We expect that effective groupoids do give rise to masa inclusions, but are not yet able to prove this.

Other possible extensions of this work could include generalising results of Li [19] about inductive limits of Cartan pairs, as well as exploring more generally what kinds of morphisms of twists over groupoids give rise to morphisms of (essential) Cartan pairs.

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