Surface waves as Fourier integral operators with complex phase

Dissertation
zur Erlangung des mathematisch-naturwissenschaftlichen Doktorgrades

"Doctor rerum naturalium" (Dr.rer.nat.)

der Georg-August-Universität Göttingen

im Promotionsprogramm

Mathematical Science (Ph.D)

der Georg-August University School of Science (GAUSS)

vorgelegt von

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aus Barranquilla, Kolumbien

Göttingen, 2023

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Tag der mündlichen Prüfung: 8. März 2023

Abstract

This thesis studies linear hyperbolic boundary value problems that admit surface waves as solutions. Surface waves are related to a specific type of weakly regular hyperbolic boundary value problems, where the precise meaning of weak is to be determined. With this thesis, we aim to provide a theoretical framework for rigorously analyzing such problems. To this end, we show that, under appropriate assumptions, a solution of a hyperbolic boundary value problem can be approximated by an oscillatory integral with complex-valued phase function. Then, we use the theory of Fourier integral operators with complex phase to study the properties of that particular solution. Following this approach, we are able to provide a refined description of the propagation of singularities as well as a preliminary result concerning the regularity of the solution in the context of Sobolev spaces H^s.

Furthermore, we present some original results that complement the existing theory of Fourier integral operators with complex phase. In particular, we propose an alternative construction of the principal symbol of the operators, and use it to compute the principal symbol after composition under the assumption of clean intersection.

Zusammenfassung

Diese Arbeit untersucht lineare hyperbolische Randwertprobleme, die Oberflächenwellen als Lösung zulassen. Oberflächenwellen sind mit einer bestimmten Art von schwach regulieren hyperbolischen Randwertproblemen verbunden, wo die genaue Bedeutung von schwach regulieren zu bestimmen ist. Mit dieser Doktorarbeit wollen wir einen theoretischen Rahmen für die rigorose Analyse dieser Art von Problemen entwickeln. Dazu zeigen wir, dass eine Lösung des hyperbolischen Randwertproblems unter geeigneten Annahmen durch ein Oszillationsintegral mit komplexwertiger Phasenfunktion approximiert werden kann. Dann verwenden wir die Theorie der Fourierintegraloperatoren mit komplexer Phasenfunktion, um die Eigenschaften dieser Lösung zu untersuchen. Mit diesem Zugang ent wir eine verfeinerte Beschreibung der Ausbreitung von Singularitäten stellen, sowie ein vorläufiges Ergebnis über die Regularität der Lösung in den Sobolev-Räumen H^s.

Darüber hinaus präsentieren wir einige Resultate, die die existierende Theorie der Fourierintegraloperatoren mit komplexer Phasenfuktion ergänzen. Insbesondere schlagen wir einen alternativen Zugang zur Definition des Hauptsymbols vor und benutzen diesen, um die Komposition von Fourierintegraloperatoren mit komplexer Phasenfuktion unter eine Clean-Intersection-Annahme zu untersuchen.

Acknowledgements

First and foremost, I would like to express my gratitude to my supervisor Prof. Dr. Ingo Witt. His guidance, patience and feedback were invaluable for the completion of this project. I also want to show my appreciation for the members of my advisory committee, Prof. Dr. Bahns and Dr. Jäh, who generously offered their time and expertise to support this endeavor. And I would like to thank the examination board for their time.

I am extremely grateful to the DAAD, who financed my research. This PhD project would not have been possible without their funding. Their continuous support helped me navigate what has been, without a doubt, the biggest challenge of my life so far.

I am also thankful for the friends I made along the way. To be away from home was difficult at times, but having met so many interesting people made it easier. To my fellow PhD students, many thanks for the time spend together, for the insightful discussions and the shared laughs. To all the people that made my time in Göttingen an unforgettable experience, thank you.

A special thank you goes to my family, who offered emotional support from afar. Thank you for believing in me when I doubted myself.

Contents

1 | Introduction

The present thesis is concerned with the study of hyperbolic surface waves. The term refers to a special type of solutions that leads to some loss of regularity. Even though the regularity theory of hyperbolic boundary value problems is quite developed, surface wave solutions represent a critical case for which the standard techniques do not fully apply. To overcome this issue, we first show how surface waves can be represented as an oscillatory integral with complex-valued phase function. Later, we use the theory of Fourier integral operators with complex phase to analyse these solutions.

This document covers both the theory of Fourier integral operators with complex phase and the theory of hyperbolic boundary value problems. It contains a comprehensive summary of the existent theories, as well as original results concerning the principal symbol map of Fourier integral operators with complex phase.

1.1 Background and problem setting

Let $\mathbb{R}^{1+d}_+$ denote the half-space $\{(x,y) \in \mathbb{R}^{1+d} \colon x \geq 0\}$, and let L be a first order symmetrizable hyperbolic operator with constant multiplicities. Consider an initial boundary value problem, or IBVP for short,

$$\begin{cases} Lu = f(t,x,y) & \text{in} \quad (0,T) \times \mathbb{R}^{1+d}_+, \\ Bu = g(t,y) & \text{on} \quad (0,T) \times \mathbb{R}^{d}, \\ u|_{t=0} = u_0(x,y) & \text{in} \quad \mathbb{R}^{d+1}_+, \end{cases} \tag{1.1.1}$$

The well-posedness of this problem is linked to the uniform Kreiss-Lopatinskii (UKL) condition, which will be explained in Chapter 3. To put it simply, this condition is the hyperbolic version of the Lopatinskii condition for elliptic operators. It was introduced by Kreiss in [15] for strictly hyperbolic systems. With the help of a symbolic symmetrizer, Kreiss proved maximal energy estimates for the solutions of the BVP. It was later shown by Mètivier in [18], that the symbolic construction can be carried out for a broader class of hyperbolic problems, which includes many physically relevant systems: the class of symmetrizable hyperbolic systems with constant multiplicities, also called *constantly hyperbolic*.

The UKL condition is a necessary and sufficient condition for the stability of the problem (1.1.1) in Sobolev spaces. In this case, the problem is said to be *strongly stable*, because the solution satisfies maximal energy estimates,

$$\|u\|_{L^2}^2 + \|u|_{x=0}\|_{L^2}^2 \leq C\left(\|u_0\|_{L^2}^2 + \|f\|_{L^2}^2 + \|g\|_{L^2}^2\right). \tag{1.1.2}$$

A weaker version of this condition, known as the Kreiss-Lopatinskii condition (KL) allows us to classify hyperbolic IBVPs into three classes, which are stable under perturbation of the coefficients. When the (KL) condition is satisfied, the problem is stable in $\mathscr{C}^\infty$, but a loss of regularity is to be expected in Sobolev spaces. Then, we have the following generic classification:

1. The *strongly stable class* (SS), when the UKL condition holds.
2. The *unstable class* (SU), when the KL condition fails.
3. The *weakly regular* problems, when the KL condition holds, but the UKL fails.

In [4], the authors introduce the class of *weakly stable IBVPs of real type* (WRR). This class consists of weakly regular problems for which the UKL fails at the so-called *hyperbolic region* (see Definition 3.4). The same paper also provides a generic description of the transition between classes. The transition between the classes SS and SU is described as the class of IBVP with *surface wave* solutions. That is, problems of type 3, for which the UKL condition fails at the *elliptic region*.

An important illustration of this classification is given by the symmetric hyperbolic problems with dissipative boundary condition. The following proposition is proved in [3].

Proposition 1.1. *Let L be a Friedrichs symmetric hyperbolic operator.*

1. *If B is dissipative, then the IBVP* (1.1.1) *satisfy the KL condition.*
2. *If B is strictly dissipative, then the IBVP* (1.1.1) *satisfy the UKL condition.*

Note that the statement does not imply that a symmetric hyperbolic system with dissipative boundary condition cannot be strongly stable, just that further analysis is required. Many physically relevant systems fit into this category. They are symmetrizable, but their boundary condition is maximally dissipative. Then, according to the proposition above, they may be *weakly regular* and a loss of regularity is to be expected. However, there is no general result that accounts for this loss of regularity, and it has to be computed on a case-by-case basis.

On the other hand, it was shown in [6, Theorem 4.4] that if B is maximally dissipative, the UKL condition can only fail at *glancing* points or at isolated points in the *elliptic* region. In the second case, the failure is due to the existence of surface waves. Since surface waves are important in applications, we chose to focus on them. Namely, we study hyperbolic IBVPs that violate the UKL condition at the elliptic region. Thus, one of our main goals is to account for the loss of regularity caused by surface waves.

Furthermore, it is shown in [4], Chapter 3, that when the boundary condition is homogeneous, i.e., $Bu = 0$, maximally dissipative IBVPs satisfy the energy estimate

$$\|u\|_{L^2}^2 \leq C\left(\|u_0\|_{L^2}^2 + \|f\|_{L^2}^2\right). \tag{1.1.3}$$

Then, if a maximally dissipative problem violates the UKL condition, the loss of regularity with respect to the strongly stable case can only occur at the boundary $x = 0$. For this reason, we only treat the homogeneous boundary value problem, i.e., we assume $f \equiv 0$.

Our approach is one that has been previously used for studying weakly regular problems: Geometric optics approximation. See for example [6] and [2], where rigorous geometric optics approximation has been conducted for hyperbolic boundary value problems violating the UKL. The main difference is that we treat the solution as a Fourier distribution with complex phase. This allow us to rigorously study the properties of the solution. To further emphasize the advantage of this approach, we present below, in Section 1.3, the example of Rayleigh waves in linear elasticity. The two-dimensional BVP serves as a model for the general problem, yet the resulting equations are simple enough to be solved explicitly.

The main tool in our analysis is the theory of Fourier distributions with complex phase. Because we aim for this thesis to be as self-contained as possible, we dedicate a part of this document to review the theory. Since the theory is highly involved, the review is rather long. We followed the approach developed in 1975 by Melin and Sjöstrand in their paper [16], and subsequently improved in [17]. The most challenging part of the theory is to provide meaningful global characterizations of the distributions. The authors managed to do so by using *almost analytic extensions* to obtain a complex-valued analogue of the Lagrangian manifolds use in the globalization of the real case. However, the transition to the complex domain comes with a price. In actuality, the new "Lagrangian manifolds" are an equivalence class of *almost analytic* manifolds that may not be Lagrangian when restricted to the real domain. Similarly, the analogue to the Maslov line bundle, denoted by $\mathscr{L}$, is only a "virtual" line bundle, which seems to be closer to a sheaf than the usual definition of a vector bundle.

As a result of this intricate construction, the authors showed in [16] the existence of a principal symbol map that behave somewhat like the real-valued principal symbol map. To be precise, they showed the existence of a, rather complicated, bijection

$$\mathcal{P} : \Gamma^{m+n/4}(\Lambda, \mathscr{L}) \longrightarrow \frac{I^m(X, \Lambda; \Omega^{\frac{1}{2}})}{I^{m-1}(X, \Lambda; \Omega^{\frac{1}{2}})}. \tag{1.1.4}$$

Here, $\Gamma^m(\Lambda, \mathscr{L})$ denotes the space of sections of the almost analytic line bundle $\mathscr{L}$, which are homogeneous of degree m. The construction is technical and abstract, due to the limitations of working with the virtual line bundle. In fact, [16] does not give an explicit characterization of the principal symbol of a given distribution, instead the authors define it as the pre-image under the map $\mathcal{P}$. In other words, the principal symbol of $A \in I^m_{cl}(X, \Lambda; \Omega^{\frac{1}{2}})$ is the homogeneous section

$$\sigma(A) := \mathcal{P}^{-1}([A]). \tag{1.1.5}$$

While this is an excellent first result, it would be useful to have a more natural definition of the principal symbol of a given distribution. Specially if one wants to able to compute it after performing operations on the distributions, in particular, composition.

Since the publication of [16] and [17], other authors have contribute to the theory. For instance, in [12] Hörmander proposes an alternative approach, using *complex Lagrangian ideals* instead of almost analytic extensions. He also provides necessary and sufficient conditions for an operator of order zero to be L^2 continuous (see [11]). In [20], Trèves show a general method for solving the complex eikonal equations, and thus finding the phase function of the solution operators associated to hyperbolic problems. But, to the best of our knowledge, there is no intrinsic formulation of the principal symbol of a Lagrangian distribution with complex phase in the literature. With this thesis, we attempted to fill this gap in the literature. Our construction, as well as some consequences, are presented in Section 2.2.

1.2 Main results

The construction of an approximated solution for a general homogeneous BVP, that is a problem like (1.1.1) with $f \equiv 0$, is presented in Chapter 3. We also present the analysis of this solution operator in light of the theory of Fourier distributions with complex phase. Specifically, Theorem 3.7 states that, assuming the existence of surface wave solutions, it is possible to represent the solution operator of the BVP as a Fourier integral operator with complex phase. Furthermore, we provide a complete description of the wave front set of such solution,

Theorem (Theorem 3.20). *The wave front sent of the solution u is contained in the set*

$$\left\{ \left(x, z, \lambda_j(\zeta), \zeta\right) \in T^*X \setminus 0\colon \ \left(z + \nabla\lambda_j(\zeta)x, \zeta\right) \in \mathrm{WF}\,(g) \right\}.$$

Regarding the theory of Fourier integral operators with complex phase, our main results deal with the principal symbol of such operators. To be precise, we extend to the complex-value case the approach by Duistermaat in [8]. That is, first we identify the principal symbol of a given Lagrangian distribution with the leading order term of the asymptotic expansion. Then, we showed that the resulting expression defines a section of $\mathscr{L}$, which is equivalent to $\mathcal{P}^{-1}([A])$. This allowed us to see that, similarly to the real-valued case, the principal symbol map fits into a short exact sequence

$$0 \to I^{m-1}_{cl}(X, \Lambda; \Omega^{\frac{1}{2}}) \to I^{m}_{cl}(X, \Lambda; \Omega^{\frac{1}{2}}) \xrightarrow{\sigma} S^{(m+d)}(\Lambda, \mathscr{L}) \to 0.$$

In addition, we study the composition of two Lagrangian distributions under the assumption of *clean composition*. In [16], the authors studied the composition under the assumption of *transverse intersection* and gave a rough description of the principal symbol of the resulting distribution. With our characterization of the principal symbol, we are able to generalize their result, and prove the following theorem.

Theorem (Theorem 2.51). *Let $A_1 \in I^{m_1}_{cl}(X \times Y, C_1; \Omega^{\frac{1}{2}})$, $A_2 \in I^{m_2}_{cl}(Y \times Z, C_2; \Omega^{\frac{1}{2}})$ be such that the clean composition $B = A_1 \circ A_2$ defines a distribution in the class $I^{m_1+m_2+e/2}_{cl}(X \times Z, C_1 \circ C_2; \Omega^{\frac{1}{2}})$. Suppose that, for $\gamma \in C_{\mathbb{R}}$, the set C_γ is compact. Then, the principal symbol of B is*

$$\sigma^{m+e/2}(B) \sim \int_{C_\gamma} (a_1)_0 (a_2)_0 (\theta^2 + \sigma^2)^{\frac{-n_Y}{2}} \sqrt{d\Phi}\, dy'' d\theta'' d\sigma'' \in S^{(m-e/2+n/4)}(\Lambda, \mathscr{L}), \tag{1.2.1}$$

with $n = n_X + n_Z$, $m = m_1 + m_2$ and $\sqrt{d\Phi}$ defined as in Lemma 2.44. Here $(a_1)_0$, $(a_2)_0$ are the principal parts of the amplitudes of A_1 and A_2, respectively.

Since we deal with two fairly different topics, the present document is divided into two parts. The first of them, Chapter 2, is dedicated to the theory of Fourier distribution with complex phases. This chapter is further subdivided into two sections. In Section 2.1, a summary of the theory is presented, following mainly [16]. In Section 2.2, we present our construction of the principal symbol map and the proof of Theorem 2.51.

Chapter 3 is devoted to the study hyperbolic surface waves. There, the theory of well posedness of hyperbolic boundary value problems is introduced. To make the presentation as clear as possible, we focus first on operators with constant coefficients. In Section 3.2, we construct an approximated solution to the BVP, which is a distribution with complex phase. In Section 3.3, we use the results of Chapter 2 to analyse this solution. Finally, in Section 3.4, we consider operators with variable coefficients.

1.3 A model case: Rayleigh waves in linear elasticity

The aim of this section is to illustrate the relevance of Fourier distributions with complex phases in the study of surface waves. To do so, we consider the equation of linear elasticity in an isotropic medium as a model case. Let λ and μ be positive constants. We study the BVP

$$\begin{aligned} Lu = \partial_t^2 u - (\lambda+\mu)\nabla(\text{div } u) - \mu\Delta u = 0 \quad &\text{in} \quad \mathbb{R}\times K, \\ Bu = \sum_i (n_i \sigma_{ij}) = f_j \quad &\text{on} \quad \mathbb{R}\times \partial K, \end{aligned} \tag{1.3.1}$$

where K is the half space $\{x \in \mathbb{R}^2 \colon x_1 \geq 0\}$, n is the normal vector to ∂K and σ is stress tensor

$$\sigma_{ij} = \lambda(\text{div } u)\delta_{ij} + \mu(\partial_{x_j} u_i + \partial_{x_i} u_j).$$

In the context of elasticity, surface waves are known as *Rayleigh wave*. They have been thoroughly studied due to their considerable importance in different fields. For instance, in seismology they are known to be responsible for most of the damage during an earthquake. Other examples of surface waves can be found in [13], where the author gives an informal overview of non linear hyperbolic surface waves.

We will follow the approach proposed by Taylor in [19]. Namely, assuming that $f_j \in \mathcal{E}'(\mathbb{R} \times \partial K)$ vanishes for $t < 0$, our goal is to construct an approximate solution to this problem. Hence, we consider the ansatz

$$u = \int e^{i\phi(t,x,\zeta)} a(t,x,\zeta) \widehat{F}(\zeta)\, d\zeta + \int e^{i\psi(t,x,\zeta)} b(t,x,\zeta) \widehat{G}(\zeta)\, d\zeta, \tag{1.3.2}$$

where $\zeta = (\tau, \xi_2) \in \mathbb{R}^2$ is the dual variable to $z = (t, x_2)$ and F and G are scalar valued distributions to be determined from the boundary condition. The phase functions, ϕ and ψ, and the vector-valued amplitudes, a and b, will be found using the method of geometric optics.

Construction of the approximated solution

The first step in the construction is to determine the phase functions ϕ and ψ. In order to use the argument of geometric optics approximation, one needs to assume that they satisfy the eikonal equations. In other words, we need to solve

$$\begin{aligned} \phi'_{x_1} &= \alpha(x_1, z, \nabla_z \phi), & \psi'_{x_1} &= \beta(x_1, z, \nabla_z \psi) \\ \phi|_{x_1=0} &= z\zeta, & \psi|_{x_1=0} &= z\zeta \end{aligned} \tag{1.3.3}$$

where α, β are roots in ξ_1 of $\det \sigma(L)(x_1, z, \xi_1, \zeta)$, here $\sigma(L)$ denotes the principal symbol of the operator L. A straightforward computation shows

$$\alpha(\zeta) = \alpha(\tau, \xi_2) = \sqrt{\frac{\tau^2}{\mu} - \xi_2^2} \quad \text{and} \quad \beta(\zeta) = \beta(\tau, \xi_2) = \sqrt{\frac{\tau^2}{\lambda + 2\mu} - \xi_2^2}.$$

We can now rewrite the first line in equation (1.3.3) as

$$\phi'_{x_1} = \alpha(\nabla_z \phi) \quad \text{and} \quad \psi'_{x_1} = \beta(\nabla_z \psi).$$

Then, it is easy to see that

$$\phi = c(z, \zeta) + \alpha(\nabla_z \phi) x_1 \Rightarrow \phi(x_1, z, \zeta) = z\zeta + \alpha(\zeta) x_1,$$

$$\psi = c(z, \zeta) + \beta(\nabla_z \psi) x_1 \Rightarrow \psi(x_1, z, \zeta) = z\zeta + \beta(\zeta) x_1.$$

Note that α and β, thus ϕ and ψ, are not real valued. In fact, $T^*(\mathbb{R} \times \partial K)$ is divided into three regions

I. The **hyperbolic region**: $|\tau| > (\lambda + 2\mu)^{1/2}|\xi_2|$. Here both α and β are real numbers.

II. The **mixed region**: $\mu^{1/2}|\xi_2| < |\tau| < (\lambda + 2\mu)^{1/2}|\xi_2|$. Here α is real but β is not.

III. The **elliptic region**: $|\tau| < \mu^{1/2}|\xi_2|$. Here both α and β are purely imaginary.

Since we want the phase functions to be smooth, we are excluding the regions

$$|\tau| = \mu^{1/2}|\xi_2| \quad \text{and} \quad |\tau| = (\lambda + 2\mu)^{1/2}|\xi_2|$$

from their domains. Finally, note that the integral (1.3.2) is defined, as long as $\Im\alpha \geq 0$ and $\Im\beta \geq 0$. Hence, we take the positive branch of the square root.

We can now focus on the next step of our construction, to determine the amplitudes a and b. This is usually done by solving the additional transport equations that arise when replacing the ansatz u into the equation, however our problem is particularly easy. In fact, it is enough to consider a and b such that

$$L(\alpha, \zeta)a(x_1, z, \zeta) = 0 \quad \text{and} \quad L(\beta, \zeta)b(x_1, z, \zeta) = 0. \tag{1.3.4}$$

Indeed, denoting by u_1 the first integral in (1.3.2) and replacing into the equation (1.3.1), we see

$$Lu_1 = \int \left[L\left(e^{i\phi(t,x,\zeta)}a(t,x,\zeta)\right)\right]\widehat{F}(\zeta)\,d\zeta = 0 \iff L\left(e^{i\phi(t,x,\zeta)}a(t,x,\zeta)\right) = 0.$$

It follows that

$$L\left(e^{i\phi(t,x,\zeta)}a(t,x,\zeta)\right) = e^{i\phi(t,x,\zeta)} \sum_{|\gamma|+|\delta|\leq 2} (\partial^\gamma L)(\phi'_{x_1}, \nabla_z\phi)\left(D^\delta a\right)(x_1, z, \zeta),$$

here the derivatives are taken with respect to (t, x). Note that $(\phi'_{x_1}, \nabla_z\phi) = (\alpha(\zeta), \zeta)$, so for $|\gamma| + |\delta| = 0$, we have

$$L(\alpha(\zeta), \zeta)a(x_1, z, \zeta) = 0 \iff a(x_1, z, \zeta) = a(\zeta) = (\xi_2, -\alpha(\zeta)).$$

With this choice of a, all the other terms in the sum vanish and we conclude that u_1 solves the equation of linear elasticity. The same argument works for the second integral in (1.3.2), in this case one obtains $b(\zeta) = (\beta(\zeta), \xi_2)$.

Remark 1.2. *Even though we have not computed their asymptotic expansion, the amplitudes a and b are classical symbols. To see the full asymptotic sum, we should require that only their principal parts satisfy* (1.3.4)*, as stated in* [19].

At this point we know that the the ansatz (1.3.2) satisfises the first equation in (1.3.1). The link to the boundary condition is given by the distributions F and G. Since $\phi(0,z,\zeta) = \psi(0,z,\zeta) = z\zeta$, we can write $u|_{x_1=0}$ as one oscillatory integral

$$u(0,z) = \int e^{iz\zeta} A(\zeta) \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix} d\zeta$$

where $A(\zeta)$ denotes the 2×2 matrix with columns $a(\zeta)$ and $b(\zeta)$. Then, the boundary condition $Bu(0,z) = (f_1, f_2)$ reads

$$T \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix} := \int e^{iz\zeta} \mathcal{M}(\zeta) \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix} d\zeta = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\zeta, \quad \text{with} \quad \mathcal{M}(\zeta) = BA(\zeta). \tag{1.3.5}$$

Thus, to guarantee that u satisfises the boundary condition, F and G need to solve the previous pseudodifferential equation. The following lemma, which is proven in [19], tells us that it is possible to find such distributions.

Lemma 1.3. *The operator T is elliptic in the hyperbolic and mixed regions. In the elliptic region, the real-valued symbol $p = \det \sigma(T)$ has a simple zero on a hypersurface in $T^*(\mathbb{R} \times \partial K)$. On this hypersurface, $\partial_\tau p \neq 0$.*

This lemma says that the operator with symbol $p = \det \sigma(T)$ is of real principal type at the elliptic region. It is well know that if the operator P is of real principal type, one can always find an approximated solution to the equation $Pu = f$. Assuming that f, which is defined for all t, has support contained in $\{t > 0\}$, the approximated solution is actually unique modulo smooth functions. Thus, Lemma 1.3 implies the existence of a forward fundamental solution for the operator T. Equivalently, one can find distributions F and G that solve equation (1.3.5) up to a smooth factor, assuming that $\operatorname{supp}(f_j) \subseteq \{(t,x,y) \colon t > 0, x > 0\}$. From there, we can conclude that the distribution u in (1.3.2) defines an approximated solution to the BVP (1.3.1), which is unique up to a smooth factor.

Remark 1.4. *The general parametrix construction, due to Duistermaat and Hörmander [9], uses heavier machinery than the one presented in [19], as the operator T corresponds to a special case. An outline of the construction from [19] is presented in Appendix A.*

Why should we use Fourier distributions?

Although the previous construction presents some technical difficulties, particularly when constructing the parametrix for T, writing the solution u as a Fourier integral with complex phase have considerable advantages. The main one being that it offers a way of rigorously studying the behaviour of the solution and the singularities travelling along the boundary ∂K, i.e. the wave front set of the Rayleigh wave. On this topic, Taylor proves in [19] the following two results

Lemma 1.5. *Let $S = \mathrm{WF}(f_1) \cup \mathrm{WF}(f_2)$. The sets $\mathrm{WF}(F)$ and $\mathrm{WF}(G)$ are contained in*

$$\Sigma = S \cup \{\textit{null-bicharacteristics of } p = \det \sigma(T) \textit{ passing over } S\}.$$

Moreover, the wave front of the Rayleigh wave is exactly this set Σ.

As an immediate consequence of the lemma, we have the propagation of singularities for the solution u.

Theorem 1.6. *Let u be the solution to (1.3.1). Assume that $\mathrm{WF}(f_j)$ avoids the characteristics variety. Then, in $\mathbb{R} \times \operatorname{int} K$, $\mathrm{WF}(u)$ is contained in the set of null-bicharacteristics of L passing over S, travelling in the positive t direction. Moreover, $\mathrm{WF}(u|_{\mathbb{R}\times\partial K}) \subseteq \Sigma$.*

This theorem has some important consequences that, as we will see later, also hold for more general BVP. First of all, if the null-bicharacteristics of L do not pass over $\mathbb{R} \times \partial K$ twice, there is no reflection of singularities into the interior. Furthermore, such rays cannot pass over the elliptic region (see [16] or Subsection 2.1.3 for details), which meas that no more Rayleigh waves are produced.

Another advantage of this approach is that we can compute the regularity of the solution in Sobolev spaces with respect to the boundary data f_1 and f_2. By construction we know that T is a pseudodifferential operator of order 0, then if $f_j \in H^s(\mathbb{R}^2)$, it follows that both F and G are also in $H^s(\mathbb{R}^2)$. Now, from the definition of the phase functions, it is clear that $u(0,z)$ is given by the action on F and G of a pseudodifferential operator of order 0. Then, $u(0,z) \in H^s$ with $z = (t, x_2)$. The same holds if we consider u a function of z with x_1 fixed. Finally, let us compute the H^t norm of u in x_1,

$$\begin{aligned}
\|u(\cdot,z)\|_{H^t}^2 &= \int \langle \xi_1 \rangle^{2t} |\hat{u}(\xi_1, z)|^2 \, d\xi_1 \\
&\leq \int \langle \xi_1 \rangle^{2t} |a(t,x,\zeta)|^2 |\widehat{F}(\zeta)|^2 \, d\zeta \, d\xi_1 + \int \langle \xi_1 \rangle^{2t} |b(t,x,\zeta)|^2 |\widehat{G}(\zeta)|^2 \, d\zeta \, d\xi_1 \\
&\leq \int \langle \xi_1 \rangle^{2t} |\widehat{F}(\zeta)|^2 \, d\zeta \, d\xi_1 + \int \langle \xi_1 \rangle^{2t} |\widehat{G}(\zeta)|^2 \, d\zeta \, d\xi_1 \\
&= \left(\|F\|_{L^2}^2 + \|G\|_{L^2}^2 \right) \int \langle \xi_1 \rangle^{2t} \, d\xi_1,
\end{aligned}$$

which is finite as long as $s \geq 0$ and $t > -1/2$. We have proven the following proposition,

Proposition 1.7. *Assume that $f_1 \in H^s(\mathbb{R}^2)$, $j = 1, 2$, for some $s \geq 0$. Then, there exist a solution u to the problem* (1.3.1)*, unique up to a smooth remainder, such that*

$$u(x_1, z) \in H^t \left(\mathbb{R}_+; H^s \left(\mathbb{R}^2 \right) \right), \quad t > -1/2.$$

In general, one may not be able to get such a clear description of the phase functions. But the theory of Fourier distributions with complex phase will allow us to obtain results comparable to the ones presented in this section.

Remark 1.8. *This proposition is compatible with the results obtained for general hyperbolic surface waves* (*Theorem* 3.25 *and Theorem* 3.29)*. Indeed, if we set $t = s = 1/2$, we obtain $u \in H^{1/2}(\mathbb{R}_+ \times \mathbb{R}^2) \subseteq L^2(\mathbb{R}_+ \times \mathbb{R}^2)$, which is the two-dimensional version of Theorem* 3.25*.*

2 | Fourier integral operators with complex phase

As stated above, we wish to use Fourier integral operators with complex phase to study certain type of hyperbolic boundary value problems. Thus, the present chapter is devoted to the study of this type of operators.

A theory for Fourier integral operators with complex phase was develop by Melin and Sjöstrand in their paper [16], and subsequently improved in [17]. The first section of this chapter summarizes their work. Most of the results are presented without proofs, as they are highly technical. Only the constructions that offer some insights into the theory are included. With the purpose of making the presentation as clear as possible, we also provide some examples and additional comments.

The second section consists of some original improvements on the theory. In their paper [16], the authors follow Hörmander's ideas in [10] closely. By using *almost analytic extensions*, they construct a complex-valued analog of the real-valued theory. Unfortunately, the almost analytic machinery introduces technical difficulties. As a result, the constructions are more abstract than one would want. The difficulty is particularly prominent when working with the principal symbol of the operators. We present an alternative construction of the principal symbol that allows us to overcome this difficulty. In addition, we study the composition of Fourier integral operators under the assumption of clean intersection.

2.1 Previously known results

As an introductory remark, note that the expression

$$I = \int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta = \int e^{i\Re\phi(x,\theta)} e^{-\Im\phi(x,\theta)} a(x,\theta)\, d\theta$$

is defined in the sense of oscillatory integrals if $\Re\phi$ is a *non-degenerate phase function* and $\Im\phi \geq 0$. Unfortunately, this fact alone is not enough to obtain a complete analog theory. The main idea in [16] is to use an almost analytic extension of ϕ to formulate a complex-valued version of the geometrical objects involved in the real-valued theory. This will allow us to develop a global theory for Fourier distributions with complex phase function.

It is worth mentioning that this is not the only possible approach. For instance, Hörmander developed an alternative theory using complex Lagrangian ideals instead of the almost analytic machinery, see [12] for details.

The content of this section is taken from [16], although some of the statements has been reformulated to facilitate the reading process.

2.1.1 Almost analytic functions and manifolds

Let $\overline{\partial}_z$ denote the Cauchy-Riemann operator for $z = x + iy \in \mathbb{C}$. If f is a smooth function in $\mathbb{C}^n$, we denote by ∂f and $\overline{\partial} f$ the operators

$$\partial f = \sum \partial_{z_j} f\, dz_j \quad \text{and} \quad \overline{\partial} f = \sum \overline{\partial}_{z_j} f\, d\overline{z_j}.$$

Definition 2.1. *Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $\Omega_{\mathbb{R}} = \Omega \cap \mathbb{R}^n$. We say that:*

1. *$f \in \mathscr{C}^\infty(\Omega)$ is* ***almost analytic****, if $\overline{\partial} f$ vanish to the infinite order near $\Omega_{\mathbb{R}}$. This means that, for all close set $K \subset \Omega_{\mathbb{R}}$, there exists a constant $C_{N,K} > 0$ such that for all $z \in \Omega$ with $\Re z \in K$, and all $N \in \mathbb{Z}_+$,*

$$|\overline{\partial} f(z)| \leq C_{N,K} |\Im z|^N.$$

2. *$f_1, f_2 \in \mathscr{C}^\infty(\Omega)$ are* ***equivalent****, if $f_1 - f_2$ is almost analytic. In this case, we write $f_1 \sim f_2$.*

By an ***almost analytic extension*** *of a function* $f \in \mathscr{C}^\infty(\Omega_\mathbb{R})$*, we mean an almost analytic function* $\widetilde{f} \in \mathscr{C}^\infty(\Omega)$ *such that* $\widetilde{f}|_{\Omega_\mathbb{R}} = f$*.*

A relatively simple construction shows that every $f \in \mathscr{S}(\mathbb{R})$ admits an almost analytic extension, see for example [21, Theorem 3.6]. An extension of this result was proven in [16]. Given a function $a \in S^m(\Gamma)$, defined in a conic set $\Gamma \subseteq \mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$, there exists an almost analytic extension $\tilde{a} \in S^m(\widetilde{\Gamma})$, which is unique up to equivalence . Here $\widetilde{\Gamma} \subseteq \mathbb{C}^n \times (\mathbb{C}^N \setminus 0)$ satisfies $\widetilde{\Gamma}_\mathbb{R} = \Gamma$ and it is a cone in the sense that, for all $t \in \mathbb{R}_+$, we have $(z, t\zeta) \in \widetilde{\Gamma}$ if $(z, \zeta) \in \widetilde{\Gamma}$.

Remark 2.2. *The class* $S^m(\widetilde{\Gamma})$ *is defined in* [16] *as the space of smooth functions* $a(z, \zeta)$*, with* $z = x + iy \in \mathbb{C}^n$ *and* $\zeta = \xi + i\eta \in \mathbb{C}^N$*, that vanish for large* $|y|, |\eta|$ *and satisfy an estimate of the form*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_y^\mu \partial_\eta^\nu a| \leq C \langle \xi \rangle^{m - |\beta + \nu|}, \quad \forall \alpha, \mu \in \mathbb{N}_0^n,\ \forall \beta, \nu \in \mathbb{N}_0^N$$

on every compact set $\Gamma' \Subset \widetilde{\Gamma}$*.*

The notion of *almost analytic manifold* can now be presented. They are of great importance as they will play the role that smooth manifolds play in the real-valued case.

Definition 2.3. *Let* $\Omega \subseteq \mathbb{C}^n$ *be an open set and* $M \subseteq \Omega$ *a closed submanifold (in the real sense) of real dimension* $2k$*. We say that* M *is an* ***almost analytic manifold*** *if for every real point* $z_0 \in M$*, one can find an open neighborhood* $\mathcal{O} \subseteq \Omega$ *of* z_0 *and almost analytic functions* $f_{k+1}, \dots, f_n$ *such that, in* $\mathcal{O}$*,*

- *M is defined by* $f_{k+1} = \cdots = f_n = 0$*,*
- *and the differentials* $\partial f_{k+1}(z), \dots, \partial f_n(z)$ *are linearly independent over* $\mathbb{C}$*.*

The neighborhood $\mathcal{O}$ can be interpreted as a complex analog of a coordinate neighborhood. However, the term *coordinate neighborhood* would always refer to a real coordinate patch. In contrast, we will refer to an appropriate set of complex coordinates as *admissible coordinates*. The exact definition is presented in Subsection 2.1.3 because a special construction is required.

The following theorem gives a useful description of almost analytic manifolds.

Theorem 2.4. *Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $M \subseteq \Omega$ an almost analytic manifold. Then, for every real point $z_0 \in M$, one can find a neighborhood $\mathcal{O} = \mathcal{O}' \times \mathcal{O}'' \subseteq \mathbb{C}^k \times \mathbb{C}^{n-k}$ of z_0 in M and an almost analytic function h on $\mathcal{O}'$ such that, for all $z = (z', z'') \in \mathcal{O}$, $z'' = h(z')$.*

Example 2.5. As an example of an almost analytic manifold, we can consider an *extension* of a real manifold. Let $\Lambda \subseteq \mathbb{R}^n$ be a manifold, locally described by

$$x = (x', x'') \in U \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}; \quad x'' = (f_{k+1}(x'), \dots, f_n(x')).$$

Then, for every $x \in \Lambda$, take a complex neighborhood $\mathcal{O} \subseteq \mathbb{C}^n$ such that $\mathcal{O}_{\mathbb{R}} \subseteq U$ and almost analytic extensions $\tilde{f}_j$ of f_j to $\mathcal{O}$, $j = k+1, \dots, n$. Then, thanks to the previous theorem, the set $\tilde{\Lambda} \subseteq \mathbb{C}^n$ given locally, around every real point, by

$$z'' = h(z'), \quad h(z') = (\tilde{f}_{k+1}(z'), \dots, \tilde{f}_n(z'')), \quad \text{for } z \in \mathcal{O},$$

defines an almost analytic manifold, with the property $\tilde{\Lambda}_{\mathbb{R}} = \Lambda$. $\square$

The manifold above is not unique because the almost analytic extension of functions are only unique up to equivalence. This motivates the notion of *equivalent almost analytic manifolds*.

Definition 2.6. *Let $M_1, M_2 \subseteq \Omega \subseteq \mathbb{C}^n$ be two closed manifolds of the same dimension. We say that M_1 and M_2 are equivalent, denoted $M_1 \sim M_2$, if they have the same intersection with $\mathbb{R}^n$ and, for every compact subset $K \Subset \Omega$ and $N \in \mathbb{Z}_+$, we have*

$$d(z, M_2) \leq C_{N,K} |\Im z|^N, \quad z \in K \cap M_1.$$

In the case of almost analytic manifolds, we also have a notion of *local* equivalence, considering only neighborhoods of real points.

Proposition 2.7. *Let $M_1, M_2 \subseteq \Omega \subseteq \mathbb{C}^n$ be almost analytic manifolds of the same dimension with $M_{1\mathbb{R}} = M_{2\mathbb{R}}$. Let h_1, h_2 be the defining functions in Theorem* 2.4*. Then, the following conditions are equivalent:*

i. $M_1 \sim M_2$

ii. For all $K \Subset \Omega$ and $N \in \mathbb{Z}_+$ there is a constant $C_{N,K} > 0$ such that,

$$|h_1(x') - h_2(x')| \leq C_{N,K} |\Im h_2(x')|^N, \qquad (x', h_j(x')) \in K, \ x' \in \mathbb{R}^k.$$

iii. For all $K \Subset \Omega$ and $N \in \mathbb{Z}_+$ there is a constant $C_{N,K} > 0$ such that,

$$|h_1(z') - h_2(z')| \leq C_{N,K} |(z', \Im h_2(z'))|^N, \qquad (z', h_j(z')) \in K.$$

The notion of equivalent almost analytic manifolds allows us to have a theory independent of the choice of almost analytic extensions. This means that, when working with almost analytic manifolds, one should keep in mind that they form equivalence classes. It may be useful to think about them as *germs* of almost analytic functions at real points.

To finish the section, let us consider a result that will be useful later: a complex-valued version of the so called *stationary phase formula*. Although the usual real-valued result can be applied to oscillatory integrals with complex-valued phase functions, we need a stronger result in order to understand the asymptotic behaviour of our distributions. That being said, the following construction is also interesting on its own, as it shows that working with almost analytic functions and manifolds requires, for the most part, only simple algebraic arguments.

Let $F(x, w) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^k)$ be defined in a neighborhood of $(0, 0)$. Assume that $\Im F \geq 0$ with equality only at $(0, 0)$ and that

$$\partial_x F(0, 0) = 0, \qquad \det(\partial_x^2 F(0, 0)) \neq 0.$$

Lemma 2.8. *Let F be as above and $\widetilde{F}(z, \omega)$, $z = x + iy$, $\omega \in \mathbb{C}^k$, be an almost analytic extension to a complex neighborhood of $(0, 0)$. Then, the equations*

$$\partial_z \widetilde{F}(z, \omega) = 0, \quad \nabla_{(x,y)} \Re \widetilde{F}(z, \omega) = 0, \quad \nabla_{(x,y)} \Im \widetilde{F}(z, \omega) = 0,$$

define three equivalent almost analytic manifolds which are of the form $z = Z(w)$. Moreover, let M be any of these equivalent manifolds, then there exists $C > 0$ such that

$$\Im \widetilde{F}(z, w) \geq C |\Im z|^2, \qquad (z, w) \in M, \ w \in \mathbb{R}^k.$$

Theorem 2.9 (The stationary phase formula)**.** *Let F be as above. There are neighborhoods of the origin $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^k$ and differential operators $C_{\nu,w}(D)$, which are $\mathscr{C}^\infty$ functions of $w \in V$, and have order at most 2ν, such that*

$$\int e^{itF(x,w)} u_t(x)\, dx \sim \sum_{\nu=0}^{\infty} t^{-\nu-n/2} e^{it\widetilde{F}(Z(w),w)} \left(C_{\nu,w}(D)\tilde{u}_t\right) Z(w), \quad t \to +\infty \quad (2.1.1)$$

in $S^{-n/2}_{\mathrm{cl}}(V \times \mathbb{R}_+)$. Here $u_t(x) \in S^0_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}_+)$ is supported in $U \times \mathbb{R}_+$ and the function $(2\pi)^{-n/2} C_{0,w}$ is the branch of the square root of $\left(\det \frac{1}{i}\partial_z^2 F(Z(w),w)\right)^{-1}$ that continuously deform into 1 under the homotopy

$$[0,1] \ni s \mapsto \frac{1}{i}(1-s)\partial_z^2 F + sI \in GL(n,\mathbb{C}).$$

Proof. ([16, Section 2]) As one would expect, the proof relies on well-known techniques to estimate the integral. The main difficulty is to find suitable coordinates $\tilde{z} \in \mathbb{C}^n$ that allow us to find such estimates. Let $z = Z(w)$ be the manifold given by Lemma 2.8 and consider

$$h(z,w) = F(z + Z(w), w) - F(Z(w), w), \quad z \in \mathbb{C}^n,\ w \in \mathbb{R}^k,$$

defined in a neighborhood of the origin. Using Taylor's formula, we can write

$$h(z,w) = \frac{1}{2}\left(z, R(z,w)\cdot z\right) + \rho(z + Z(w), w),$$

where $R(z,w) = 2\int_0^1 (1-t) h''_{zz}(tz,w)\, dt$ satisfies

$$|\overline{\partial_z} R(z,w)| + |\rho(z,w)| \leq C_N \left(|\Im z|^N + |\Im Z(w)|^N\right), \quad \text{for } N \in \mathbb{N}.$$

Our goal is to find coordinates $\tilde{z} \in \mathbb{C}^n$ for which $F(z,w) - F(Z(w),w)$ differs from a quadratic form by a smooth factor. Note that $R(0,0)$ is a non-degenerate quadratic form in $\mathbb{C}^n$, thus there exists a matrix A such that $A^T R(0,0) A = iI$. Now, suppose for a moment that there exists a matrix Q such that

$$iQ(z,w)^T Q(z,w) = R(z,w), \quad Q(0,0) = A^{-1},$$
$$|\overline{\partial_z} Q(z,w)| \leq C_N \left(|\Im z|^N + |\Im Z(w)|^N\right).$$

Then, the map $z \mapsto \tilde{z}(z) = Q(z - Z(w), w) \cdot (z - Z(w))$ would define the coordinates we need. Indeed, in these new coordinates, we can write

$$F(z, w) = F(Z(w), w) + \frac{i}{2}(\tilde{z}, \tilde{z}) + \rho(z, w) \tag{2.1.2}$$

and $\mathbb{R}^n$ is given by the equation $\tilde{y} = g(\tilde{x}, w)$, where $\tilde{z} = \tilde{x} + i\tilde{y}$ and g is a $\mathscr{C}^\infty$ function near $(0, 0)$. Finding such matrix Q is always possible, because the map

$$GL(n, \mathbb{C}) \ni Q \mapsto iQ^T Q \in Sym(n, \mathbb{R})$$

is analytic with surjective differential. Hence, our first goal is completed.

Now let $z(\tilde{z})$ be the inverse map to $z \mapsto \tilde{z}(z)$, the next step is to examine the behaviour of F along the chains $\Gamma_{w,s}$ given by

$$\tilde{x} \mapsto z(\tilde{z}_s), \quad \tilde{z}_s = \tilde{z}_s(\tilde{x}) = \tilde{x} + isg(\tilde{x}, w), \quad 0 \leq s \leq 1.$$

The idea is to first integrate along $\Gamma_{w,0}$ and then estimate the difference between this integral and the left hand side of (2.1.1). Without entering in much detail, the key steps of the process are presented below.

Let $\tilde{u}_t(z)$ be an almost analytic extension of u_t, supported in a fixed neighborhood of the origin. We can write

$$\int_{\mathbb{R}^n} e^{itF(x,w)} u_t(x)\, dx = \int_{\Gamma_{w,1}} e^{itF(z,w)} \tilde{u}_t(z)\, dz_1 \wedge \cdots \wedge dz_n.$$

It follows form the Stoke's formula, the fact that $\tilde{u}_t$ is almost analytic, and the properties of F that, for w is in some fixed neighborhood of the origin $W \subseteq \mathbb{R}^k$ independent of u_t,

$$\left| \int_{\mathbb{R}^n} e^{itF(x,w)} u_t(x)\, dx - \int_{\Gamma_{w,0}} e^{itF(z,w)} \tilde{u}_t(z)\, dz_1 \wedge \cdots \wedge dz_n \right| \leq C_N t^{-N}, \quad \forall N \in \mathbb{Z}_+. \tag{2.1.3}$$

Note that the change of variables that leads to (2.1.2) allows us to compute

$$\int_{\Gamma_{w,1}} e^{itF(z,w)} \tilde{u}_t(z)\, dz_1 \wedge \cdots \wedge dz_n = \int_U e^{iF(Z(w),w) + \frac{i}{2}|\tilde{x}|^2 + \rho(z(\tilde{x}),w)} u_t(z(\tilde{x})) J_w(\tilde{x})\, d\tilde{x}, \tag{2.1.4}$$

where $J_w(\tilde{x}) = \det\left(\frac{dz}{d\tilde{x}}\right)$ and $U \subseteq \mathbb{R}^n$ is a small enough neighborhood of the origin. On the other hand, for any function $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, we have

$$\left| \int e^{-t\frac{|x|^2}{2}} f(x)\, dx - \sum_{\nu=0}^{k-1} \left(\frac{2\pi}{t}\right)^{n/2} \left(\frac{\Delta}{2t}\right)^{\nu} \frac{f(0)}{\nu!} \right| \leq C_k' t^{-k-n/2} \sum_{|\alpha|\leq 2k+n+1} \int |D^\alpha f|\, dx. \tag{2.1.5}$$

One can verify the asymptotic formula (2.1.1) by combining equations (2.1.3),(2.1.4) and (2.1.5) with $f = u_t J_w$. Moreover, we see that $C_{0,w}(D) = (2\pi)^{n/2} J_w(0)$. Computing the Hessian of both sides of this equation and setting $\tilde{z} = 0$, we obtain

$$(2\pi)^{-n/2} C_{0,w}(D) = \pm \left(\det \frac{1}{i} \partial_z^2 F(Z(w), w)\right)^{-1/2}.$$

Finally, the branch of the square root is chosen in a way that $\left(\det \frac{1}{i}\partial_z^2 F\right)^{1/2} = 1$ when $\partial_z^2 F = iI$. $\square$

2.1.2 Lagrangian manifolds and complex-valued phase functions

Lagrangian manifolds are a major component of the theory of Lagrangian distributions. As they are associated with the phase functions, and we are allowing the phase functions to be complex-valued, we need to extend the idea of symplectic manifold to the complex domain. Before doing so, we need a more general notion of almost analytic manifolds. The following definitions are taken from [16].

Let M be a real paracompact $\mathscr{C}^\infty$ manifold of dimension n. We say that an almost analytic manifold N is associated to M, formally $N \subseteq \widetilde{M}$ if:

1. $N_{\mathbb{R}} \subseteq M$ is locally closed, i.e., every point of $N_{\mathbb{R}}$ has an open neighborhood $U \subseteq M$ such that $N_{\mathbb{R}} \cap U$ is closed in U.
2. One can find a covering of $N_{\mathbb{R}}$ by coordinate neighborhoods
$$M \supset X_\alpha \xrightarrow{\mathcal{H}_\alpha} \Omega_\alpha \subseteq \mathbb{R}^n,$$
and almost analytic manifolds $N_\alpha \subseteq \widetilde{\Omega}_\alpha \subseteq \mathbb{C}^n$, such that $N_{\alpha\mathbb{R}} = \mathcal{H}_\alpha(X_\alpha \cap N_{\mathbb{R}})$. Here $\widetilde{\Omega}_\alpha$ is some open set with $\widetilde{\Omega}_\alpha \cap \mathbb{R}^n = \Omega_\alpha$.

3. The local representatives N_α satisfy the following compatibility condition:

$$\widetilde{\mathcal{H}}_{\beta\alpha}(N_\alpha) \text{ and } N_\beta \text{ are equivalent near } \mathcal{H}_\beta(X_\alpha \cap X_\beta \cap N_{\mathbb{R}}).$$

Where $\mathcal{H}_{\beta\alpha} = \mathcal{H}_\beta \circ \mathcal{H}_\alpha^{-1}$ and $\widetilde{\mathcal{H}}_{\beta\alpha}$ is an almost analytic extension.

Let M be a real symplectic manifold of dimension $2n$, fix a point $\rho_0 \in M$ and consider a coordinate neighborhood $W \subseteq \mathbb{R}^{2n}$ of ρ_0. Assuming that $\Lambda \subseteq \widetilde{M}$ is an almost analytic manifold containing ρ_0, we want to extend the symplectic structure of M to $\widetilde{M}$. This is done locally, so we identify the manifold Λ with its local representative in $\widetilde{W}$, where $\widetilde{W} \subseteq \mathbb{C}^{2n}$ is an open set with $\widetilde{W}_{\mathbb{R}} = W$.

Note that, given symplectic coordinates (x, ξ) near ρ_0 in M, we can have coordinates in Λ by taking almost analytic extensions $\left(\widetilde{x}, \widetilde{\xi}\right)$ to $\widetilde{W}$.

Definition 2.10. *The manifold Λ is called **positive Lagrangian** if, near every real point (x_0, ξ_0), it is equivalent to a manifold of the form*

$$\widetilde{\xi} = \frac{\partial h}{\partial \widetilde{x}}(\widetilde{x}), \quad \widetilde{x} \in \mathbb{C}^n,$$

where h is an almost analytic function satisfying $\Im h \geq 0$ on $\mathbb{R}^n$, with equality at x_0.

This definition does not fully extend the symplectic structure. So far, we have no information on the symplectic form σ. In fact, the manifolds on which σ vanish represent a special case.

Definition 2.11. *An almost analytic manifold $\Lambda \subseteq \widetilde{M}$, of real dimension $2n$, is called **strictly positive Lagrangian** if*

i. $\Lambda_{\mathbb{R}}$ is a submanifold of M.

ii. $\sigma_\alpha|_{\Lambda_\alpha} \sim 0$ for all local representatives Λ_α and all local almost analytic extensions σ_α of σ.

iii. $i^{-1}\sigma(v, v) > 0$ for all $v \in T_\rho(\Lambda) \setminus \left(T_\rho(\Lambda_{\mathbb{R}})\right)^{\sim}$, $\rho \in \Lambda_{\mathbb{R}}$.

In practice, we will consider $M = T^*X \setminus 0$, for some smooth manifold X, and $\Lambda \subseteq (T^*X \setminus 0)^{\sim}$ positive Lagrangian.

Lemma 2.12. *Let $\Lambda \subseteq (T^*X \setminus 0)^{\sim}$ be a conic almost analytic manifold such that $\sigma_\alpha|_{\Lambda_\alpha}$ vanishes for all local representatives Λ_α and all local almost analytic extensions σ_α of σ. Then, for every $\rho_0 \in \Lambda_{\mathbb{R}}$, there are local coordinates $x \in X$ such that, near ρ_0, Λ has a local representative of the form*

$$\widetilde{x} = \frac{\partial g(\widetilde{\xi})}{\partial \widetilde{\xi}}.$$

The function g is almost analytic and homogeneous of degree 1. Furthermore, Λ is a positive Lagrangian manifold if $\Im g(\xi) \leq 0$ for real ξ.

Due to this lemma, we will sometimes claim without proof that a positive Lagrangian manifold $\Lambda \subseteq (T^*X \setminus 0)^{\sim}$ is defined near every real point by

$$\widetilde{x} = H(\widetilde{\xi}),$$

where H is positive homogeneous of degree 0 with $\Im H(\xi) \leq 0$ for ξ real.

The complex version of a non-degenerate phase function is straightforward.

Definition 2.13. *A complex-valued function $\phi(x, \theta)$, smooth in a conic subset V of $\mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$, is called* ***non-degenerate phase function of positive type*** *if $\Im\phi \geq 0$ and*

- *$d\phi \neq 0$,*
- *ϕ is homogeneous of degree 1 in θ,*
- *the differentials $\left\{ d(\frac{\partial \phi}{\partial \theta_j}) \right\}_{i=1}^{N}$ are linearly independent over $\mathbb{C}$ on*

$$C_{\phi\mathbb{R}} = \left\{ (x, \theta) \in V \colon \phi'_\theta = 0 \right\}.$$

Remark 2.14. *In [16], complex-valued non-degenerate phase functions are called regular phase functions.*

Let $\tilde{\phi}(\tilde{x}, \tilde{\theta})$ be an almost analytic extension of a non-degenerate phase function ϕ, defined in a conic neighborhood $W \subseteq \mathbb{C}^n \times (\mathbb{C}^N \setminus 0)$ of the point $(x_0, \theta_0) \in C_{\phi\mathbb{R}}$. Then the critical set

$$C_{\tilde{\phi}} = \left\{ (\tilde{x}, \tilde{\theta}) \in W \colon \partial_{\tilde{\theta}} \tilde{\phi}(\tilde{x}, \tilde{\theta}) = 0 \right\}$$

is a conic almost analytic manifold of dimension $2n$. The image $\Lambda_{\tilde{\phi}}$ of $C_{\tilde{\phi}}$ under the map

$$(\tilde{x}, \tilde{\theta}) \mapsto \left(\tilde{x}, \partial_{\tilde{x}}\tilde{\phi}(\tilde{x}, \tilde{\theta})\right)$$

is locally, near $\rho_0 = (x_0, \phi'_x(x_0, \theta_0))$, a manifold of dimension $2n$. Moreover, the image of $C_{\tilde{\phi}\mathbb{R}}$ is precisely $\Lambda_{\tilde{\phi}\mathbb{R}}$.

It was shown in [16, Theorem 3.6] that $\Lambda_{\tilde{\phi}}$ is locally a conic positive Lagrangian manifold, whose equivalence class does not depend on the choice of almost analytic extension $\widetilde{\phi}$. For this reason, in the following we will write Λ_ϕ instead of $\Lambda_{\tilde{\phi}}$.

Remark 2.15. *One can easily check that every local almost analytic extension σ_α vanishes over the local representatives $(\Lambda_\phi)_\alpha$, but Λ_ϕ do not satisfy Definition* 2.11 *in general. In some cases, the restriction to the real domain, $\Lambda_{\tilde{\phi}\mathbb{R}}$ is not a manifold. Thus, it is not true in general that $\Lambda_{\tilde{\phi}\mathbb{R}} \subseteq \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is a Lagrangian manifold in the usual sense.*

Definition 2.16. *Let ϕ be a non-degenerate phase function, and fix $(x_0, \theta_0) \in C_{\phi\mathbb{R}}$. We say that ϕ parametrizes a positive Lagrangian manifold Λ if, for any choice of almost analytic extension $\tilde{\phi}$,*

$$\Lambda \sim \Lambda_{\tilde{\phi}} \text{ near } (x_0, \phi'_x(x_0, \theta_0)).$$

Remark 2.17. *A positive Lagrangian manifold $\Lambda \subseteq (T^*X \setminus 0)^\sim$ that satisfies the assumptions of Lemma* 2.12, *can always be parametrized by a non-degenerate phase function of the form*

$$\phi(x, \xi) = x \cdot \xi - g(\xi).$$

Notice that, the equivalence of almost analytic manifolds naturally induces an equivalence relation on the phase functions that parameterize such manifolds. Namely, two non-degenerate phase functions are equivalent if they parameterize equivalent positive Lagrangian manifolds. This equivalence is make precise in the following definition.

Definition 2.18. *Let ϕ and ψ be non-degenerate phase functions defined in small conic neighborhoods of $(x_0, \theta_0) \in \mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$ and $(x_0, \omega_0) \in \mathbb{R}^n \times (\mathbb{R}^M \setminus 0)$, respectively.*

*We say that ϕ and ψ are **equivalent at** (x_0, ξ_0) if $\xi_0 = \phi'_x(x_0, \theta_0) = \psi'_x(x_0, \omega_0)$ and $\Lambda_\phi \sim \Lambda_\psi$ in a neighborhood of (x_0, ξ_0).*

2.1.3 Fourier distributions and their principal symbol

Let $V \subseteq \mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$ be a conic open set, $\phi = \phi(x, \theta) \in \mathscr{C}^\infty(V)$ be a non-degenerate phase function, and $a = a(x, \theta) \in S^m_{\mathrm{cl}}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ be supported in a closed conic subset of V. We define a Fourier distribution $A = I(\phi, a) \in \mathcal{D}'(\mathbb{R}^n)$ by

$$(I(\phi, a), u) = \int\int e^{i\phi(x,\theta)} a(x, \theta) u(x)\, dx d\theta, \quad u \in \mathscr{C}^\infty_0(\mathbb{R}^n). \tag{2.1.6}$$

Formally,

$$I(\phi, a) = \int e^{i\phi(x,\theta)} a(x, \theta)\, d\theta.$$

Note that the contribution of $\Im\phi$ to the integral is an exponentially decreasing factor, which cannot influence the singularities of the distribution. Thus, we get, directly from the real case, a result about the wave front set of A. Namely,

$$\mathrm{WF}\,(A) \subseteq \left\{ (x, \phi'_x(x, \theta)) \colon (x, \theta) \in \mathrm{supp}(a) \cap C_{\phi\mathbb{R}} \right\} \subseteq \Lambda_{\phi\mathbb{R}}.$$

The following proposition tells us that equivalent phase functions yield the same kind of distribution. A sketch of the proof is presented in order to illustrate the relevance of the complex-valued stationary phase formula (Theorem 2.9). For a detailed construction see [16, Theorem 4.2].

Proposition 2.19. *Let $\phi(x, \theta)$ and $\psi(x, \sigma)$ be non-degenerate phase functions equivalent at the real point $\rho_0 = (x_0, \xi_0)$. Then, there exist $a \in S^{m+(n-2N)/4}_{\mathrm{cl}}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ and $b \in S^{m+(n-2M)/4}_{\mathrm{cl}}(\mathbb{R}^n \times (\mathbb{R}^M \setminus 0))$, such that the distributions $A = I(\phi, a)$ and $B = I(\psi, b)$ are microlocally equivalent near ρ_0, this means*

$$\rho_0 \notin \mathrm{WF}\,(A - B)\,.$$

Proof. First of all, recall that by definition of equivalence of phase functions, ϕ and ψ parameterize equivalent positive Lagrangian manifolds Λ_ϕ and Λ_ψ. This means, among other things, that

$$\Lambda_{\phi\mathbb{R}} = \Lambda_{\psi\mathbb{R}} =: \Lambda_\mathbb{R}.$$

Then, we have $\mathrm{WF}(A) \subseteq \Lambda_{\mathbb{R}}$ and $\mathrm{WF}(B) \subseteq \Lambda_{\mathbb{R}}$. We need to show that the distribution $\widehat{A} - \widehat{B}$ is smooth at $\rho_0 \in \Lambda_{\mathbb{R}}$.

Assume that in a conic complex neighborhood of ρ_0, Λ_ϕ is given by the equation $x = x(\xi) \in \mathbb{C}^n$, where x is homogeneous of degree 0 for real ξ. Let $(x(\xi), \theta(\xi)) \in \mathbb{C}^n \times (\mathbb{C}^N \setminus 0)$ be a critical point of the function

$$(x, \theta) \mapsto \tilde{\phi}(x, \theta) - x\xi,$$

and suppose that the amplitude a is supported in a neighborhood of (x_0, θ_0). Then, we can write $\widehat{A}(t\xi) = I_1(t\xi) + I_2(t\xi)$, where

$$I_1(t\xi) = t^N \int e^{it(\phi(x,\theta)-x\xi)} a(x, t\theta)(1 - \chi(x, \theta))\, dx d\theta,$$
$$I_2(t\xi) = t^N \int e^{it(\phi(x,\theta)-x\xi)} a(x, t\theta)\chi(x, \theta)\, dx d\theta,$$

and $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ equals one near (x_0, θ_0). Note that, when ξ is in a small enough neighborhood W of ξ_0, the phase function of I_1 has no critical points in the support of a. It follows, by repeated partial integration, that $I_1 \in S^{-\infty}(W)$. Thus, it does not contribute to the wave front set of A.

On the other hand, the fact that ϕ is a non-degenerate phase function implies that the function $F(x, \theta, \xi) = \tilde{\phi}(x, \theta) - x\xi$ satisfies the assumptions of Theorem 2.9. Thus, we can apply the stationary phase formula to I_2 with respect to (x, θ). Setting $t = 1$, we get

$$\widehat{A}(\xi) \sim \sum_{\nu=0}^{\infty} e^{-ix(\xi)\xi} |\xi|^{(N-n)/2-\nu} \left(C_{\nu, \frac{\xi}{|\xi|}}(D_x, |\xi| D_\theta)\tilde{a} \right) (x(\xi), \theta(\xi)),$$

for ξ in a small conic neighborhood V of ξ_0, independent of a. Now, if we take $Q_{-l}(a) \in S_{\mathrm{cl}}^{m+(n-2N)/4}(V)$ such that

$$Q_{-l}(a) \sim \sum_{\nu=1}^{\infty} \left(C_{0, \frac{\xi}{|\xi|}} \right)^{-l} |\xi|^{-\nu} \left(C_{\nu, \frac{\xi}{|\xi|}}(D_x, |\xi| D_\theta)\tilde{a} \right) (x(\xi), \theta(\xi)),$$

we can write

$$\widehat{A}(\xi) \sim e^{-ix(\xi)\xi} C_{0, \frac{\xi}{|\xi|}} + Q_{-l}(a).$$

Since the factor $e^{-ix(\xi)\xi}$ is independent of the local representatives of $\Lambda_\phi \sim \Lambda_\psi$ (modulo $S^{-\infty}$), we can have a similar expansion for $\widehat{B}$, with the same exponential factor and some $b \in S_{\mathrm{cl}}^{m+(n-2M)/4}$. Finally, note that we can approximate each term of the asymptotic sum of b in a way that

$$e^{-ix(\xi)\xi}\tilde{a} + Q_{-l}(a) \sim e^{-ix(\xi)\xi}\tilde{b},$$

which concludes the proof. $\square$

We can now define Fourier distributions globally. Let X be a $\mathscr{C}^\infty$ paracompact manifold of dimension n and denote by $\mathcal{D}'(X;\Omega^{\frac{1}{2}})$ be the space of 1/2-densities in X. From this point forward, unless stated otherwise, $\Lambda \subseteq (T^*X \setminus 0)^\sim$ denotes a positive Lagrangian manifold.

Definition 2.20 (Fourier distributions)**.** *We say that a distribution $A \in \mathcal{D}'(X;\Omega^{\frac{1}{2}})$ belongs to the class $I_{\mathrm{cl}}^m(X,\Lambda;\Omega^{\frac{1}{2}})$, if $\mathrm{WF}\,(A) \subseteq \Lambda_{\mathbb{R}}$ and there exists a phase function $\phi \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ and an amplitude $a \in S_{\mathrm{cl}}^{m+(n-2N)/4}(\mathbb{R}^n \times \mathbb{R}^N)$ such that*

- *for every $\rho_0 = (x_0, \xi_0) \in \Lambda_{\mathbb{R}}$ and every choice of local coordinates, A is microlocally, near ρ_0, of the form $I(\phi, a)$,*
- *Λ is parametrized by ϕ, that is $\Lambda \sim \Lambda_\phi$ near ρ_0,*
- *$\mathrm{supp}(a)$ is contained in a small conic neighborhood of $(x_0, \theta_0) \in C_{\phi\mathbb{R}}$.*

In analogy with the real case, one would like to define the principal symbol of $A \in I^m(X,\Lambda;\Omega^{\frac{1}{2}})$ as a section of the tensor product of the bundle of 1/2-densities in Λ and the *Maslov* line bundle. But, it turns out that it is impossible to replicate this construction for complex manifolds in a way that is invariant under a change of local coordinates. To avoid this difficulty, Melin and Sjöstrand [16] introduce *admissible coordinates* and define a special line bundle over Λ, which is somewhat similar to the product bundle described above.

This is an intricate construction, so we need to consider the linear situation first. Let $\widetilde{M}$ be the complex extension of a real symplectic vector space M of dimension $2n$, and denote by $\mathcal{L}^-$ the set of negative definite Lagrangian planes in M. Let $F \subseteq M$

be a fixed real Lagrangian plane and $\widetilde{F}$ its complexification. We denote by $B(F)$ the set of all real bases of F.

Definition 2.21. *Let $N \subseteq \widetilde{M}$ be a positive semi-definite Lagrangian plane. A basis $e = \{e_1, \ldots, e_n\}$ of N is said to be* ***admissible*** *if there exists a basis $f = \{f_1, \ldots, f_n\}$ of F and a plane $L \in \mathcal{L}^-$ such that, for each j, e_j is the projection of f_j along L. We write*

$$e = E(f, L) = E_N(f, L), \quad (f, L) \in B(F) \times \mathcal{L}^-,$$

and denote by $\mathscr{B}(N)$ the set of all admissible bases for N.

Given a set $S \subseteq B(F) \times \mathcal{L}^-$, we write $E_N(S)$ to denote the set of all admissible bases $e = E(f, L)$ of N with $(f, L) \in S$.

Proposition 2.22. *The set $\mathscr{B}(N)$ is the union of two disjoint arcwise-connected subsets. Two admissible bases $e = E(f, L)$ and $e' = E(f', L')$ belong to the same set if and only if $f, f' \in B(F)$ have the same orientation. Moreover, there exists a unique function*

$$s = s_N : \mathscr{B}(N) \times \mathscr{B}(N) \to \mathbb{C} \setminus 0$$

with the following properties

i. For all compact set $K \Subset B(F) \times \mathcal{L}^-$, the function $s(e, e')$ restricted to $E_N(K) \times E_N(K)$ is a continuous function of e, e' and N.

ii. If $e, e', e'' \in \mathscr{B}(N)$, then $s(e, e')s(e', e'') = s(e, e'')$.

iii. If e, e' have the same $L \in \mathcal{L}^-$, then $s(e, e') > 0$.

iv. $s^2(e, e') = \pm e/e'$ with the plus sign precisely when f, f' have the same orientation. Here,

$$e/e' = e_1 \wedge \cdots \wedge e_n / e'_1 \wedge \cdots \wedge e'_n, \quad e = E(f, L),\ e' = E(f', L') \in \mathscr{B}(N).$$

Proof. ([16, Theorem 6.2]) In order to prove this result, we need a description of $\mathscr{B}(N)$ that is easier to handle. We start by choosing symplectic coordinates (x, ξ) on M, such that F is given by $x = 0$ and N by $\widetilde{x} = A\widetilde{\xi}$. Here A is a symmetric matrix with $\Im A \leq 0$ and $(\widetilde{x}, \widetilde{\xi})$ are the corresponding coordinates in $\widetilde{M}$. In the same coordinates, a plane $L \in \mathcal{L}^-$ is of the form $\widetilde{\xi} = B\widetilde{x}$, for some symmetric matrix B with $\Im B < 0$.

This allows us to establish one-to-one correspondence between the set of projections $\widetilde{F} \to N$ along planes in $\mathcal{L}^-$ and the matrices of the form $(I - BA)^{-1}$. Indeed, if $(0, \widetilde{\xi}_0) \in \widetilde{F}$, the projected coordinates $(\widetilde{x}, \widetilde{\xi}) \in N$ satisfy

$$\widetilde{\xi} = (I - BA)^{-1}\widetilde{\xi}_0.$$

Then, the set of admissible bases $\mathscr{B}(N)$ can be identified with the set of matrices $\mathscr{M}$, of the form

$$C = C_A(B, R) = (I - BA)^{-1}R, \tag{2.1.7}$$

where $R \in GL(n, \mathbb{R})$ relates to the basis f of F, and A, B are determined by the Lagrangian planes N, L as above. But, this representation is not unique, as the plane L and the basis f can be chosen freely. Then, there could be matrices B, B', R and R' such that

$$C_A(B, R) = C_A(B', R'). \tag{2.1.8}$$

It can be shown that these two representations are connected by an arc on the set $\mathscr{M}$. This arc defines a homotopy relation amongst the different representations (2.1.7). Explicitly, let $0 \leq t \leq 1$, and put

$$\begin{aligned} B_t &= (1 - t)B + tB', \\ R_t &= (I - B_t A)(I - B'A)^{-1}R' \in GL(n, \mathbb{R}). \end{aligned}$$

Then, for equation (2.1.8) to hold, we need $(\det R) \,/\, (\det R') > 0$. This shows that $\mathscr{B}(N)$ splits into two disjoint arc-connected components, which are determined by the sign of $\det R$ in the representation (2.1.7). Or, equivalently, by the orientation of f as a basis of F in the representation $e = E(f, L)$.

For the second, and most important, part of the theorem, define

$$p(A, B, R) = \left(\det(I - BA)^{-1}\right)^{1/2} |\det R|^{1/2},$$

with C, A, B and R as in equation (2.1.7). Here, the square root in the first factor is taken as the branch that deforms into a positive real number when both A and B go to iI. Note that, thanks to the homotopy above, the map p is continuous along the fibers of

$$(A, B, R) \mapsto (A, (I - BA)^{-1}R).$$

Then, we can define

$$s_A(C) = p(A, B, R) \quad C = (I - BA)^{-1}R.$$

Finally, for two admissible bases e and e' with representation C and C' as in (2.1.8), the quotient e/e' is

$$\frac{\det(I - BA)^{-1}R}{\det(I - B'A)^{-1}R'}.$$

So, it is natural to define

$$s_N(e, e') := \frac{s_A(C)}{s_A(C')}.$$

Properties $i. - iv.$ follow from the construction, while the uniqueness of s is a consequence of ii. □

Consider now a positive Lagrangian manifold $\Lambda \subseteq (T^*X \setminus 0)^{\sim}$ and $\rho \in \Lambda_{\mathbb{R}}$. Then, $M = T_\rho(T^*X)$ and $N = T_\rho(\Lambda)$ satisfy the conditions of Definition 2.21. Namely, $T_\rho(T^*X)$ is a symplectic vector space and $T_\rho(\Lambda) \subset T_\rho(T^*X)$ is a positive semi-definite Lagrangian plane. Taking $F \subseteq M$ as the tangent space to the fiber, we can define $\mathscr{B}(T_\rho(\Lambda))$ as above.

Keeping in mind the previous notation, we now define admissible coordinate systems on Λ. Recall that, $E_{T_\rho(\Lambda)}(S)$ is the set of all admissible bases $e = E(f, L)$ of $T_\rho(\Lambda)$ with $(f, L) \in S \subseteq B(F) \times \mathcal{L}^-$.

Definition 2.23. *Let $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ be almost analytic functions on Λ, defined in some complex neighborhood $\mathcal{U}^\lambda$ of a real point. We say that $\lambda_1, \ldots, \lambda_n$ are **admissible coordinates** on Λ if*

1. *The differentials $d\lambda_1, \ldots, d\lambda_n$ are linearly independent over $\mathbb{C}$ at real points.*

2. *$\delta\lambda = \{\delta\lambda_1, \dots, \delta\lambda_n\}$ belongs locally to $E_{T_\rho(\Lambda)}(K)$ with $\rho \in \mathcal{U}^\lambda \cap \Lambda_{\mathbb{R}}$, for some $K \Subset B(F) \times \mathcal{L}^-$. Here $\delta\lambda$ is the dual basis of $d\lambda$ in $T_\rho(\Lambda)^*$.*

*We refer to the neighborhoods $\mathcal{U}^\lambda$ as **admissible coordinate systems**.*

Note that each λ_j depends on some local coordinates (x, ξ) on Λ, defined near the point $\rho \in \Lambda_{\mathbb{R}}$, where x denotes a choice of local coordinates on the manifold X and ξ its dual coordinate. Another set of admissible coordinates, defined near some ρ' would depend on different local coordinates (x', ξ').

One can show that it is always possible to find admissible coordinates locally, however, this construction is not unique. By definition, the functions $\lambda = \{\lambda_1, \dots, \lambda_n\}$ are chosen in a way that $\delta\lambda$, the dual to $d\lambda$, is an admissible basis to the Lagrangian plane $T_\rho(\Lambda)$ in the sense of Definition 2.21. And, as stated in the proof of Proposition 2.22, one can have many admissible bases for the same plane. It is precisely this property what allows us to define the "line bundle" we want, as each basis would have its own admissible coordinate system.

All we need now is a way to define the transition functions between two coordinate systems $\mathcal{U}^\lambda$ and $\mathcal{U}^\mu$. Thanks to Proposition 2.22, we know that

- $s(\delta\lambda, \delta\mu)$ is continuous in $\mathcal{U}^\lambda \cap \mathcal{U}^\mu \cap \Lambda_{\mathbb{R}}$.
- $s^2 = \pm \dfrac{d\mu}{d\lambda} = \pm \det \left[\left(\dfrac{\partial \mu_j}{\partial \lambda_k} \right)_{j,k} \right]$, where $\partial\mu_j / \partial\lambda_k$ is defined by

$$d\mu_j = \sum_k \left(\frac{\partial \mu_j}{\partial \lambda_k} \right) d\lambda_k + \sum_k \left(\frac{\partial \mu_j}{\partial \overline{\lambda}_k} \right) d\overline{\lambda}_k.$$

  The positive sign occurs when $d\lambda$ and $d\mu$ have the same orientation as bases of $T_\rho(\Lambda)$.

Consider now an almost analytic extension $\mathbf{S}$ of $s(\delta\lambda, \delta\mu)$, defined in a small complex neighborhood of $\mathcal{U}^\lambda \cap \mathcal{U}^\mu \cap \Lambda_{\mathbb{R}}$ in Λ. Thanks to the previous properties, $\mathbf{S}$ can be chosen to satisfy

$$\left(\mathbf{S}_{\lambda,\mu}\right)^2 \sim \pm\frac{d\mu}{d\lambda}, \tag{2.1.9}$$

$$\mathbf{S}_{\lambda,\lambda} \sim 1, \quad \mathbf{S}_{\lambda,\mu}\mathbf{S}_{\mu,\omega} \sim \mathbf{S}_{\lambda,\omega}. \tag{2.1.10}$$

Additionally, the functions $\mathbf{S}_{\lambda,\mu}$ are continuous under small perturbations of $\lambda,\ \mu$ for which $\delta\lambda,\ \delta\mu$ stay in the same component of $E_{T_\rho(\Lambda)}(K)$. For all of these, the functions $\mathbf{S}_{\lambda,\mu}$ are the ideal choice of transition functions in the new almost analytic line bundle.

Definition 2.24. *The bundle $\mathscr{L} \to \Lambda$ is defined as the family of admissible coordinate systems $\mathcal{U}^\lambda$ on Λ with transition functions $\mathbf{S}_{\lambda,\mu}$. A section $f \in \Gamma(\Lambda, \mathscr{L})$ is an almost analytic function on Λ such that, the restriction to each $\mathcal{U}^\lambda$ satisfy*

$$f_\lambda \sim \mathbf{S}_{\lambda,\mu} f_\mu.$$

The space of homogeneous section of degree m is denote by $\Gamma^m(\Lambda, \mathscr{L})$.

At this point, it is necessary to clarify what we mean by homogeneous section. Given $t \in \mathbb{R}_+$, denote by $\mathbf{t} : \Lambda \to \Lambda$ the multiplication by t in the second coordinate. Then, if $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ are admissible coordinates near $\mathbf{t}\rho \in \Lambda_{\mathbb{R}}$, the pullback

$$\mathbf{t}^*\lambda = \{\lambda_1 \circ \mathbf{t}, \ldots, \lambda_n \circ \mathbf{t}\}$$

defines admissible coordinates μ near ρ. We say that $f \in \Gamma(\Lambda, \mathscr{L})$ is homogeneous of degree m if for all $\rho \in \Lambda_{\mathbb{R}}$, all $t \in \mathbb{R}_+$ and all coordinates λ near $\mathbf{t}\rho$

$$f_{\mathbf{t}^*\lambda} \sim t^m \mathbf{t}^*\left(f_\lambda\right), \text{ near } \rho. \tag{2.1.11}$$

Remark 2.25. *By definition of $\Gamma(\Lambda, \mathscr{L})$, in particular by the property (2.1.9) of the transition functions, it holds in general that $f_{\mathbf{t}^*\lambda} \sim t^{n/2} f_\lambda$.*

The following example show us that homogeneous functions on Λ define homogeneous section of $\Gamma(\Lambda, \mathscr{L})$. To avoid any confusion further on, the definition of homogeneity is shown explicitly.

Example 2.26. Let g be an almost analytic function in Λ, homogeneous of degree m in ξ. We know that for each $\rho \in \Lambda_{\mathbb{R}}$, it is possible to find and admissible coordinate system $\mathcal{U}^{\lambda}$, locally near ρ. Then, we can define a function f such that $f_\lambda = g|_{\mathcal{U}^\lambda}$. Clearly, this defines a section of $\mathscr{L}$.

Moreover, it can be shown that $f \in \Gamma^m(\Lambda, \mathscr{L})$. Indeed, fix $\rho = (x, \xi) \in \Lambda_{\mathbb{R}}$ and $t \in \mathbb{R}_+$. Then, near $\mathbf{t}\rho$, points in Λ are of the form $(x, t\xi)$. Let $\lambda = \{\lambda_1, \dots, \lambda_n\}$ and $\mu = \left(\mathbf{t}^{-1}\right)^* \lambda$ be admissible coordinates near ρ and $\mathbf{t}\rho$, respectively. Explicitly, for each $j = 1, \dots, n$,

$$\mu_j(x, t\xi) = \left(\lambda_j \circ \mathbf{t}^{-1}\right)(x, t\xi) = \lambda_j(x, \xi).$$

Then, near ρ, we have $\left(\mathbf{t}^{-1}\right)^* (f_\lambda)(x, \xi) \sim f\left(x, \frac{1}{t}\xi\right) \sim t^{-n/2} f_\lambda(x, \xi)$ and

$$\begin{aligned} f_{\left(\mathbf{t}^{-1}\right)^*\lambda} &\sim f(x, t\xi) = t^m f(x, \xi) = t^{n/2} t^{-n/2} t^m f(x, \xi) \\ &\sim t^{m-n/2} f_\lambda(x, \xi) \sim t^m \left(\mathbf{t}^{-1}\right)^* (f_\lambda) . \end{aligned}$$

In other words, f satisfies relation (2.1.11). We conclude that it defines a section of $\mathscr{L}$, homogeneous of degree m. $\square$

The last step before defining the principal symbol of an distribution in $I^m_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}})$ is to assign to each non-degenerate phase function ϕ a non-vanishing section in $\Gamma^{N/2}(\Lambda, \mathscr{L})$. This is accomplished by Lemma 2.28. As we will see below, one can easily define $d\phi$ uniquely as a n-form on C_ϕ, the difficulty of the construction lies in obtaining the correct homogeneity. The construction presented here is part of the proof of Theorem 6.4 in [16].

Remark 2.27. *A p-form on an almost analytic manifold is completely defined by its local representatives. In other words, all p-forms can be understood as*

$$\sum a_k \, df_{1,k} \wedge \dots \wedge df_{p,k},$$

where a_k and $df_{j,k}$ are almost analytic functions.

For the rest of the section, $\tilde{f}$ would always denote an almost analytic extension of some function f.

Lemma 2.28. *Given a non-degenerate phase function $\phi(x,\theta)$ that parameterizes Λ near $\rho_0 \in \Lambda_{\mathbb{R}}$, there is a section $\sqrt{d\phi} \in \Gamma^{N/2}(\Lambda, \mathscr{L})$, defined by*

$$(\sqrt{d\phi})_\tau \sim \left[\det \frac{1}{i}\begin{pmatrix} \tilde{\phi}''_{xx} - \tilde{\psi}''_{xx} & \tilde{\phi}''_{x\theta} \\ \tilde{\phi}''_{\theta x} & \tilde{\phi}''_{\theta\theta} \end{pmatrix}\right]^{-1/2}, \tag{2.1.12}$$

where τ is an admissible coordinate system on Λ and $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfy $\tilde{\psi}''_{xx} < 0$. The branch of the square root is chosen as in Theorem 2.9.

Proof. [16, Theorem 6.4] Denote by $d\widetilde{\phi}$ the almost analytic n-form on C_ϕ that satisfy

$$d\widetilde{\phi} \wedge d\left(\frac{d\widetilde{\phi}}{d\widetilde{\theta}_1}\right) \wedge \cdots \wedge d\left(\frac{d\widetilde{\phi}}{d\widetilde{\theta}_N}\right) \sim i^{n+N} d\widetilde{x}_1 \wedge \cdots \wedge d\widetilde{x}_n \wedge d\widetilde{\theta}_1 \wedge \cdots \wedge d\widetilde{\theta}_N. \tag{2.1.13}$$

The condition is satisfied if $d\widetilde{\phi} \sim a(\lambda) d\lambda_1 \wedge \cdots \wedge \lambda_n$, where $\lambda = (\lambda_1 \ldots, \lambda_n)$ are admissible coordinates and

$$a(\lambda) \sim \left[\det \frac{1}{i}\begin{pmatrix} \dfrac{\partial\lambda}{\partial\widetilde{x}} & \dfrac{\partial\lambda}{\partial\widetilde{x}} \\ \dfrac{\partial^2\widetilde{\phi}}{\partial\widetilde{x}\partial\widetilde{\theta}} & \dfrac{\partial^2\lambda}{\partial\widetilde{\theta}} \end{pmatrix}\right]^{-1}.$$

This shows that the differential form $d\widetilde{\phi}$ always exists and that it is unique up to equivalence. Because of this, we write only $d\phi$. Thanks to the local identification of $\Lambda \sim \Lambda_\phi$ with C_ϕ, we can consider $d\phi$ an n-form on Λ, defined in a conic neighborhood of some real point $\rho_0 \in \Lambda$.

Now let $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfy $\tilde{\psi}''_{xx} < 0$. The restriction to Λ of $\tau = \widetilde{\xi} - \tilde{\psi}'_{\widetilde{x}}$ defines admissible coordinates in Λ. In these coordinates, $d\phi$ takes the form

$$d\phi \sim \left[\det \frac{1}{i}\begin{pmatrix} \tilde{\phi}''_{xx} - \tilde{\psi}''_{xx} & \tilde{\phi}''_{x\theta} \\ \tilde{\phi}''_{\theta x} & \tilde{\phi}''_{\theta\theta} \end{pmatrix}\right]^{-1} d\tau_1 \wedge \cdots \wedge d\tau_n.$$

After taking the the square root in the previous n-form as in Theorem 2.9, we obtain exactly equation (2.1.12). Moreover, any other function ψ' results in admissible

coordinates τ' for which
$$\frac{(\sqrt{d\phi})_\tau}{(\sqrt{d\phi})_{\tau'}} \sim \mathbf{S}_{\tau,\tau'}.$$
Recall that $\mathbf{S}_{\tau,\tau'}$ is a transition function on $\mathscr{L}$. Thus, we have correctly defined $\sqrt{d\phi} \in \Gamma(\Lambda, \mathscr{L})$.

It is important to note that while $\sqrt{d\phi}$ is independent of the function ψ, it does depend on the choice of local coordinates x. In fact, if y is some new coordinate on X and $\phi_1(y, \theta)$ also parameterize Λ near ρ_0, we get
$$\sqrt{d\phi_1} \sim \left|\frac{d\widetilde{y}}{d\widetilde{x}}\right|^{1/2} \sqrt{d\phi}.$$

To complete the proof, we need to verify the homogeneity. As before, we denote by $\mathbf{t} : \Lambda \to \Lambda$ the multiplication by $t \in \mathbb{R}_+$ in the second coordinate. Now we consider $\tau = \widetilde{\xi} - \tilde{\psi}'_{\widetilde{x}}$ to be admissible coordinates near a point $\mathbf{t}\rho_0 = (x_0, t\xi_0)$. Then,
$$\mathbf{t}^*\tau = t\widetilde{\xi} - \tilde{\psi}'_{\widetilde{x}} = t\left(\widetilde{\xi} - \frac{1}{t}\tilde{\psi}'_{\widetilde{x}}\right)$$
defines admissible coordinates near ρ_0. Proceeding as in Example 2.26, one can see that $\sqrt{d\phi}$ has the correct homogeneity. Indeed,
$$\begin{aligned}
\mathbf{t}^*\left(\sqrt{d\phi}\right)_\tau &\sim \left[\det \frac{1}{i}\begin{pmatrix} \tilde{\phi}''_{xx}(\tilde{x}, t\tilde{\theta}) - \tilde{\psi}''_{xx}(\tilde{x}) & \tilde{\phi}''_{x\theta}(\tilde{x}, t\tilde{\theta}) \\ \tilde{\phi}''_{\theta x}(\tilde{x}, t\tilde{\theta}) & \tilde{\phi}''_{\theta\theta}(\tilde{x}, t\tilde{\theta}) \end{pmatrix}\right]^{-1/2} \\
&\sim t^{(n+N)/2}\left[\det \frac{1}{i}\begin{pmatrix} \tilde{\phi}''_{xx}(\tilde{x}, \tilde{\theta}) - \frac{1}{t}\tilde{\psi}''_{xx}(\tilde{x}) & \tilde{\phi}''_{x\theta}(\tilde{x}, \tilde{\theta}) \\ \tilde{\phi}''_{\theta x}(\tilde{x}, \tilde{\theta}) & \tilde{\phi}''_{\theta\theta}(\tilde{x}, \tilde{\theta}) \end{pmatrix}\right]^{-1/2} \\
&\sim t^{N/2}\left(\sqrt{d\phi}\right)_{\mathbf{t}^*\tau}.
\end{aligned}$$
We conclude that $\sqrt{d\phi} \in \Gamma^{N/2}(\Lambda, \mathscr{L})$. □

The next theorem presents what we refer to as the *principal symbol map*. The proof is long and technical and will be omitted. Instead, we offer a short explanation on how the map acts locally.

Theorem 2.29 (Principal symbol map)**.** *There is a linear bijection*

$$\mathcal{P} : \Gamma^{m+n/4}(\Lambda, \mathscr{L}) \longrightarrow \frac{I^m(X, \Lambda; \Omega^{\frac{1}{2}})}{I^{m-1}(X, \Lambda; \Omega^{\frac{1}{2}})}.$$

Proof. See Theorem 6.4 in [16]. □

Let $s \in \Gamma^{m+n/4}(\Lambda, \mathscr{L})$ be supported in a small conic neighborhood of some ρ_0. We can find a non-degenerate phase function ϕ that parameterizes Λ near ρ_0, and an almost analytic function b on Λ homogeneous of degree $m + n/4 - N/2$, such that $s \sim b\sqrt{d\phi}$. Then, define $\mathcal{P}(s)$ as the oscillatory integral

$$\mathcal{P}(s) = \int e^{i\phi(x,\theta)} B(x, \theta)\, d\theta,$$

where $B(x, \theta)$ is an almost analytic extension of b to $\mathbb{C}^n \times \mathbb{C}^N$.

As a direct consequence of the theorem, we have a formulation of the principal symbol of a given distribution, as a section of the line bundle $\mathscr{L}$.

Definition 2.30. *Let $A \in I^m_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}})$. Then, **the principal symbol** of A is the section*

$$\sigma^m(A) := \mathcal{P}^{-1}([A]) \in \Gamma^{m+n/4}(\Lambda, \mathscr{L}),$$

where $[A]$ denotes the equivalent class of A in I^m/I^{m-1}.

Since we do not have information on the action of $\mathcal{P}^{-1}$, it is rather difficult to work with this definition. In analogy with the real-valued theory, one may expect to be able to obtain the principal symbol of a distribution $A \in I^m_{\mathrm{cl}}(X, \Lambda)$ from any local representation $I(\phi, a)$. Unfortunately, this is not clear from Definition 2.30. To be precise, we know that

$$\sigma^m(A) = s \sim b\sqrt{d\phi} \in \Gamma^{m+n/4}(\Lambda, \mathscr{L}), \tag{2.1.14}$$

for some non-degenerate phase function ϕ that parameterizes Λ, and some almost analytic function b. Then, the question is whether one can find a local representation $I(\phi, a)$ for A, where the main homogeneous component of $a \in S^{m+(n-2N)/4}_{\mathrm{cl}}$ equals b. The answer to this question is presented in Subsection 2.2.1

2.1.4 Composition of Fourier integral operators

Let $X,\ Y$ be paracompact $\mathscr{C}^\infty$ manifolds of dimension n_X, n_Y respectively. A standard result tells us that $T^*(X\times Y)\cong T^*X\times T^*Y$ via

$$T^*(X\times Y)\ni(x,y,\xi,\eta)\longleftrightarrow(x,\xi,y,-\eta)\in T^*X\times T^*Y.$$

Then, similar to the real case, if $C\subseteq(T^*X\setminus 0)^{\sim}\times(T^*Y\setminus 0)^{\sim}$ is an arbitrary submanifold, we denote by C' the manifold

$$\{(x,y,\xi,-\eta)\colon(x,\xi,y,\eta)\in C\}\,.$$

Definition 2.31. *We say $C\subseteq((T^*X\setminus 0)\times(T^*Y\setminus 0))^{\sim}$ is a (strictly)* ***positive canonical relation*** *if $C'\subseteq(T^*(X\times Y)\setminus 0)^{\sim}$ is a closed conic (strictly) positive Lagrangian manifold.*

Here *closed* means that $C'_{\mathbb{R}}$ is a closed set of $T^*(X\times Y)\setminus 0$. But, the set $C_{\mathbb{R}}$ does not need to be close in $(T^*X\setminus 0)\times(T^*Y\setminus 0)$. Finally, we define a special class of operators.

Definition 2.32. *An operator $A:\mathscr{C}_0^\infty(Y;\Omega^{\frac12})\to\mathcal{D}'(X;\Omega^{\frac12})$ is called a* ***Fourier integral operator with complex phase*** *if its distributional kernel K_A belongs to $I^m_{\mathrm{cl}}(X\times Y,\Lambda;\Omega^{\frac12})$. We denote this class by $I^m_{\mathrm{cl}}(X\times Y,C;\Omega^{\frac12})$, where $C'=\Lambda\subseteq(T^*(X\times Y)\setminus 0)^{\sim}$ is a closed conic positive Lagrangian manifold.*

It follows from the real case that if C is a canonical relation, the operator A maps

$$\mathscr{C}_0^\infty(Y;\Omega^{\frac12})\to\mathscr{C}^\infty(X;\Omega^{\frac12}),$$

and it can be extended to a continuous operator from $\mathscr{E}'(Y;\Omega^{\frac12})$ to $\mathcal{D}'(X;\Omega^{\frac12})$.

Now let $X,\ Y,\ Z$ be paracompact $\mathscr{C}^\infty$ manifolds of dimension n_X, n_Y, n_Z respectively. Suppose that

$$A_1\in I^{m_1}_{\mathrm{cl}}(X\times Y,C_1;\Omega^{\frac12})\quad\text{and}\quad A_2\in I^{m_2}_{\mathrm{cl}}(Y\times Z,C_1;\Omega^{\frac12}),$$

are properly supported operators, and that $C_j=\Lambda'_j$ are positive canonical relations. The next step is to investigate under which conditions the composition $A_1\circ A_2$

defines a Fourier integral operator according to Definition 2.32. To this end, denote by Δ_Y the diagonal

$$\Delta_Y = \{(y,y) \in Y \times Y\}.$$

Set $D = T^*X \times \Delta_{T^*Y} \times T^*Z$ and let $\widetilde{D}$ be its almost analytic continuation. The following condition was considered in [16].

Assumption 2.33. *Suppose that:*

a. The intersection of $C_1 \times C_2$ and $\widetilde{D}$ is transverse at points in $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap D$.

b. The natural projection $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \to (T^*X \setminus 0) \times (T^*Z \setminus 0)$ is injective and proper.

Remark 2.34. *Two smooth submanifolds Y_1 and Y_2 of a smooth manifold X are said to intersect transversally if, at every point $x \in Y_1 \cap Y_2$, it holds that $T_xY_1 + T_xY_2 = T_xX$.*

Proposition 2.35. *Let $\Lambda_1 = C_1'$ and $\Lambda_2 = C_2'$ be parametrized by the non-degenerate phase functions $\phi_1(x,y,\theta)$ and $\phi_2(y,z,\sigma)$ respectively, and suppose that Assumption 2.33 holds. Then, there exists a positive canonical relation $C \subseteq (T^*X \setminus 0)^\sim \times (T^*Z \setminus 0)^\sim$ such that $C_\mathbb{R} = C_{1\mathbb{R}} \circ C_{2\mathbb{R}}$. Moreover, the manifold $\Lambda = C'$ is parametrized by the phase function $\Phi = \phi_1 + \phi_2$. We write $C = C_1 \circ C_2$.*

Proof. Let $N(\Delta_Y) \subseteq T^*(X \times Y)$ be the normal bundle of Δ_Y and denote by $(N^*)^\sim$ the almost analytic continuation of $N^* = T^*X \times N(\Delta_Y) \times T^*Z$. In terms of the manifolds Λ_1 and Λ_2, Assumption 2.33 reads:

a. $\Lambda_1 \times \Lambda_2$ and $(N^*)^\sim$ intersect transversely at real points.

b. The projection $N^* \cap (\Lambda_{1\mathbb{R}} \times \Lambda_{2\mathbb{R}}) \to (T^*X \setminus 0) \times (T^*Z \setminus 0)$ is injective and proper.

Let ϕ_1 and ϕ_2 be non-degenerate phase function paramterizing Λ_1 and Λ_2 near $\rho_1 = (x_0, \xi_0, y_0, -\eta_0)$ and $\rho_2 = (y_0, \eta_0, z_0, \zeta_0)$, respectively. Then, the first condition implies that the map

$$\Lambda_1 \times \Lambda_2 \ni (\widetilde{x}, \widetilde{\xi}, \widetilde{y'}, \widetilde{\eta'}, \widetilde{y''}, \widetilde{\eta''}, \widetilde{z}, \widetilde{\zeta}) \mapsto (\widetilde{y'} - \widetilde{y''}, \widetilde{\eta'} + \widetilde{\eta''}) \in \mathbb{C}^{2n_Y}$$

has surjective differential near (ρ_1, ρ_2). It follows, from this and the identification $C_i \longleftrightarrow \Lambda_i$, that the map

$$\mathbb{C}^{n_X+n_Y+N_1+N_2} \ni (\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\theta}, \widetilde{\sigma}) \mapsto \left(\frac{\partial \widetilde{\phi}_1}{\partial \widetilde{y}} + \frac{\partial \widetilde{\phi}_2}{\partial \widetilde{y}}, \frac{\partial \widetilde{\phi}_1}{\partial \widetilde{\theta}}, \frac{\partial \widetilde{\phi}_2}{\partial \widetilde{\sigma}} \right) \in \mathbb{C}^{n_Y+N_1+N_2}$$

also has surjective differential. Which means that the function

$$\Phi(x, z, (y, \theta, \sigma)) = \phi_1(x, y, \theta) + \phi_2(y, z, \sigma)$$

satisfy almost all the conditions to be a non-degenerate phase function near the real point $(x_0, z_0, y_0, \theta_0, \sigma_0)$, with (y, θ, σ) taken as the fiber variables. The only problem is the homogeneity. Following the usual trick, let us introduce a homogeneous function $\omega = \omega(y, \theta, \sigma)$ as the new fiber variable. In [16], the authors choose

$$\omega(y, \theta, \sigma) = \left(y(\theta^2 + \sigma^2)^{1/2}, \theta, \sigma \right).$$

We can now treat $\Phi(x, z, \omega)$ as a non-degenerate phase function, and identify C_Φ with

$$C_\Phi^0 = \left\{ \left(\left(\widetilde{x}, \widetilde{y}', \widetilde{\theta} \right), \left(\widetilde{y''}, \widetilde{z}, \widetilde{\sigma} \right) \right) \in C_{\phi_1} \times C_{\phi_2} : \widetilde{y}' = \widetilde{y''},\ \frac{\partial \widetilde{\phi}_1}{\partial \widetilde{y}} + \frac{\partial \widetilde{\phi}_2}{\partial \widetilde{y}} = 0 \right\}.$$

Finally, define $C_1 \circ C_2$ as the almost analytic manifold with $(C_1 \circ C_2)_\mathbb{R} = C_{1\mathbb{R}} \circ C_{2\mathbb{R}}$, whose local representation is given by the map

$$(C_1 \times C_2) \cap \widetilde{D} \ni (\widetilde{x}, \widetilde{\xi}, \widetilde{y}, \widetilde{\eta}, \widetilde{y}, \widetilde{\eta}, \widetilde{z}, \widetilde{\zeta}) \mapsto (\widetilde{x}, \widetilde{\xi}, \widetilde{z}, \widetilde{\zeta}) \in \mathbb{C}^{2n_X+2n_Y}.$$

In other words, there is a natural identification between $C_1 \circ C_2$ and $(C_1 \times C_2) \cap \widetilde{D}$. That $C_1 \circ C_2$ is a positive canonical relation, locally generated by Φ, follows from the fact that the projection in Assumption 2.33 can be factored as

$$(N^*)^\sim \cap (\Lambda_1 \times \Lambda_2) \to C_\Phi^0 \xrightarrow{\approx} C_\Phi \to \Lambda_\Phi \hookrightarrow \mathbb{C}^{2n_X+2n_Z}.$$

□

Theorem 2.36. *Let $C_1 \subseteq (T^*X \setminus 0)^\sim \times (T^*Y \setminus 0)^\sim$ and $C_2 \subseteq (T^*Y \setminus 0)^\sim \times (T^*Z \setminus 0)^\sim$ be positive canonical relations satisfying Assumption 2.33. Suppose that the operators $A_1 \in I^{m_1}_{\mathrm{cl}}(X \times Y, C_1; \Omega^{\frac{1}{2}})$ and $A_2 \in I^{m_2}_{\mathrm{cl}}(Y \times Z, C_2; \Omega^{\frac{1}{2}})$ are properly supported. Then, $A_1 \circ A_2 \in I^{m_1+m_2}_{\mathrm{cl}}(X \times Z, C_1 \circ C_2; \Omega^{\frac{1}{2}})$.*

Proof. Suppose that $A_1 = I(\phi_1, a_1)$ and $A_2 = I(\phi_2, a_2)$, near ρ_1 and ρ_2, where ϕ_1, ϕ_2 are the phase functions above and

$$a_1 \in S_{\mathrm{cl}}^{m_1+(n_X+n_Y-2N_1)/4}\left(\mathbb{R}^{n_X+n_Y} \times \mathbb{R}^{N_1}\right),$$

$$a_2 \in S_{\mathrm{cl}}^{m_2+(n_Y+n_Z-2N_2)/4}\left(\mathbb{R}^{n_Y+n_Z} \times \mathbb{R}^{N_2}\right).$$

One can prove, following standard methods (see for instance [12]), that the composition $A = A_1 \circ A_2$ is, modulo $\mathscr{C}^\infty$, locally given by

$$\int e^{i\Phi(x,y,z,\theta,\sigma)} a(x,y,z,\theta,\sigma)\, dy\, d\theta\, d\sigma,$$

where $\Phi = \phi_1(x,y,\theta) + \phi_2(y,z,\sigma)$, and

$$a = a_1(x,y,\theta)a_2(y,z,\sigma) \in S_{\mathrm{cl}}^{m'}(R^{n_X+n_Y+n_Z} \times \mathbb{R}^{N_1+N_2}),$$

with $m' = m_1 + m_2 + (n_X + n_Z + 2(n_Y - N_1 - N_2))/4$. We wish to consider y a parameter. So, as before, we take

$$\omega(y,\theta,\sigma) = \left(y(|\theta|^2 + |\sigma|^2)^{1/2}, \theta, \sigma\right).$$

Then,

$$\Phi(x,z,\omega) = \phi_1(x,y,\theta) + \phi_2(y,z,\sigma),$$

is the resulting phase function in Proposition 2.35 and

$$a(x,y,z,\theta,\sigma) \in \frac{D\omega}{D(y,\theta,\sigma)} S_{\mathrm{cl}}^{m'-n_Y}(R^{n_X+n_Z} \times \mathbb{R}^{n_Y+N_1+N_2}).$$

After computing $D\omega/D(y,\theta,\sigma) = (|\theta|^2 + |\sigma|^2)^{n_Y/2}$, we see that A is microlocally of the form $I(\Phi, b)$, where the amplitude

$$b(x,z,\omega) = a_1(x,y,\theta)a_2(y,z,\sigma)(|\theta|^2 + |\sigma|^2)^{-n_Y/2}$$

belongs to the space

$$S_{\mathrm{cl}}^{m_1+m_2+(n_X+n_Z-2(n_Y+N_1+N_2))/4}\left(\mathbb{R}^{n_Y+n_Z} \times \mathbb{R}^{N_2}\right).$$

The result follows from this and Definition 2.20. $\square$

It is clear that the principal symbol of A is given by some map

$$\gamma : \Gamma^{m_1'}(C_1'; \mathscr{L}) \times \Gamma^{m_2'}(C_2'; \mathscr{L}) \to \Gamma^{m'}((C_1 \circ C_2)'; \mathscr{L}),$$

where $m_1' = m_1 + (n_X + n_Y)/4$, $m_2' = m_2 + (n_Y + n_Z)/4$ and $m' = m_1 + m_2 + (n_X + n_Z)/4$. Unfortunately, once again [16] fails to provide an explicit description of this map. Instead, the authors describe the square of the principal symbol.

To understand this description, we first need to clarify some terminology and notation. Let $\mathcal{M}$ be a manifold, $\mathcal{N} \subseteq \mathcal{M}$ a submanifold, and ω a differential form on $\mathcal{M}$. We say that

- ω is a form **on** $\mathcal{N}$, when considering the pull-back $i^*\omega$. Here $i : \mathcal{N} \to \mathcal{M}$ is the inclusion map.
- ω is a form **along** $\mathcal{N}$, when considering the equivalence class $[\omega]$ with respect to the relation
$$\omega_1 \sim \omega_2 \text{ if } \omega_1 - \omega_2 \equiv 0 \text{ on } \mathcal{N}.$$

Lemma 2.37. *For any choice of local coordinates $y_1, \dots, y_{n_Y}$,*

$$\omega = d(y' - y'') \wedge d(\eta' + \eta'')$$

defines an invariant $2n_Y$-form along $N(\Delta_Y)$. Here

$$d(y' - y'') = d(y_1' - y_1'') \wedge \cdots \wedge d(y_{n_Y}' - y_{n_Y}''),$$
$$d(\eta' + \eta'') = d(\eta_1' + \eta_2'') \wedge \cdots \wedge d(\eta_{n_Y}' + \eta_{n_Y}'').$$

Let Ω be the differential form on $T^*X \times T^*Y \times T^*Y \times T^*Z$, and along N^*, defined by the pull back $\pi^*\omega$, where $\pi : T^*X \times T^*Y \times T^*Y \times T^*Z \to T^*Y \times T^*Y$ is the natural projection. Denote by $\widetilde{\Omega}$ an almost analytic extension of Ω. Locally, we can write

$$\widetilde{\Omega} = d(\widetilde{y'} - \widetilde{y''}) \wedge d(\widetilde{\eta'} + \widetilde{\eta''}).$$

Denoting by α_1, α_2 and α the principal symbols of A_1, A_2 and A respectively, the following equivalence is proven in [16].

Theorem 2.38. *There is an equivalence of forms on $C_1' \times C_2'$ along $(C_1 \times C_2) \cap (N^*)^{\sim}$,*

$$\alpha_1^2 \wedge \alpha_2^2 \sim \pm\alpha^2 \wedge \Omega.$$

2.2 Original results

In this section, we present an alternative construction of the principal symbol map, as well as, an explicit formulation of the principal symbol of $A \in I^m_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}})$. With this new formulation, we revise the theorems in Subsection 2.1.4. In addition, we generalize Assumption 2.33 to include the case of *clean intersection*. To do so, we introduce the notion of complex-valued *clean phase functions*. This is a simple, yet useful, generalization of the non-degenerate phase functions.

2.2.1 Alternative construction of the principal symbol

The following construction shows how the stationary phase formula can be used to give an explicit description of the principal symbol of a distribution in $I^m_{\mathrm{cl}}(X, \Lambda)$. In particular, given a distribution $A \in I^m_{\mathrm{cl}}(X, \Lambda)$, which is microlocally of the form $I(\phi, a)$ near some real point ρ, we are able to see the relation between the amplitude a and the function b in Definition 2.30.

We follow the ideas of Duistermaat for the real case (see [8]) and adapt them to the complex domain. Namely, we use the asymptotic expansion in Theorem 2.9 to provide a description of the principal symbol in coordinates. Later, we show that this description corresponds to the pre-image, under the map $\mathcal{P}$ (Theorem 2.29), of some equivalence class in $[A]$. Finally, we prove this is the same equivalence class as the one in Definition 2.30.

Lemma 2.39. *Let $\phi \in \mathscr{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ be a non-degenerate phase function and $(x_0, \theta_0) \in \mathrm{C}_{\phi\mathbb{R}}$ fixed. Suppose that $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ is real-valued and*

$$\psi(x_0) = 0, \quad \psi_x'(x_0) = \phi_x'(x_0, \theta_0), \quad \psi_{xx}'' < 0.$$

Then, the function $F(x, \theta) = \phi(x, \theta) - \psi(x)$ satisfies the assumptions of Theorem 2.9 *around (x_0, θ_0).*

Proof. It is clear that $F(x,\theta)$ is smooth with $\Im F \geq 0$. From the definition of $\mathbb{C}_{\phi\mathbb{R}}$, one sees that $\Im F(x_0,\theta_0) = 0$. Thus, we only need to verify that

$$\partial_x F(x_0,\theta_0) = 0, \qquad \det(\partial_x^2 F(x_0,\theta_0)) \neq 0.$$

A short computation shows

$$\partial_x F(x_0,\theta_0) = \phi'_x(x_0,\theta_0) - \psi'_x(x_0) = 0,$$
$$\partial_x^2 F = \begin{pmatrix} \phi''_{xx} - \psi''_{xx} & \phi''_{x\theta} \\ \phi''_{\theta x} & \phi''_{\theta\theta} \end{pmatrix}.$$

This matrix is the same as the one in the definition of $\sqrt{d\phi}$ (Lemma 2.28). The result is proven by the same arguments presented there. □

Now let X be a smooth manifold of dimension n and consider a distribution $A \in I^m_{\mathrm{cl}}(X,\Lambda)$, which is of the form $I(\phi,a)$ microlocally near $\rho_0 = (x_0,\xi_0)$, with ϕ a non-degenerate phase function, $\xi_0 = \phi'_x(x_0,\theta_0)$ and $a(x,\theta)$ an amplitude function in $S^{m+(n-2N)/4}_{\mathrm{cl}}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$. Let $u \in C_c^\infty$ be supported in a neighborhood of x_0 and ψ be as in the previous lemma (Lemma 2.39). Set $v(x) = e^{-it\psi(x)}u(x)$. We want to understand the asymptotic behaviour of $I := (I(\phi,a),v)_{L^2}$ as $t \to \infty$. By definition,

$$\begin{aligned}(I(\phi,a),v)_{L^2} &= \int e^{i\phi(x,\theta)}a(x,\theta)e^{-it\psi(x)}u(x)\,dxd\theta \\ &= \int e^{i[\phi(x,\theta)-t\psi(x)]}a(x,\theta)u(x)\,dxd\theta.\end{aligned}$$

After the change of variables $\theta = t\eta$, we obtain

$$I = \int e^{it(\phi(x,\eta)-\psi(x))}t^N a(x,t\eta)u(x)\,d\eta = t^N \int e^{itF(x,\eta)}u_t(x,\eta)\,d\eta\,dx,$$

with $F = \phi - \psi$ and $u_t = a(x,t\eta)u(x)$. Thanks to Lemma 2.39, we know that the *complex-valued* stationary formula (Theorem 2.9) applies here. We then get

$$e^{-it\widetilde{F}(Z(\widetilde{\eta}),\widetilde{\theta})}I \sim \sum_{\nu=0}^{\infty} t^{-\nu-(n+N)/2}\left(C_{\nu,\eta}(D)\widetilde{u_t}\right)Z(\widetilde{\eta}),$$

where $x = Z(\widetilde{\eta})$ is the almost analytic manifold described by $\partial_x\widetilde{F}(\widetilde{x},\widetilde{\eta}) = 0$.

We can now describe the principal symbol of A as the map that assigns to each ψ the top-order term of the asymptotic expansion of I. Since $C_{0,w}(D) = (2\pi)^{\frac{n+N}{2}}\sqrt{d\phi}$, we can write the *principal symbol map* explicitly

$$\mathcal{T}_A : \psi \mapsto (2\pi)^{\frac{n+N}{2}} \widetilde{a}_0(Z(\widetilde{\eta}), \widetilde{\eta})\widetilde{u}(Z(\widetilde{\eta}))\sqrt{d\phi}, \tag{2.2.1}$$

where $\sqrt{d\phi} \in \Gamma^{N/2}(\Lambda, \mathscr{L})$ and a_0, the highest order term in the asymptotic sum of a, is an homogeneous function of degree $m + (n - 2N)/4$ in η. The next step is to relate this expression with the formulation of the principal symbol in [16], which will give us a complete description of the principal symbol.

To do so, we need to introduce further notation. Denote by $S^{(d)}(\Lambda, \mathscr{L})$, $d = d_1 + d_2$, the space of almost analytic functions $f \sim bs$, where $s \in \Gamma^{d_1}(\Lambda, \mathscr{L})$ and b is an homogeneous function of degree d_2 in θ.

Theorem 2.40. *Let $A = I(\phi, a)$, with $\phi(x, \theta) \in \mathscr{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ a non-degenerate phase function and $a(x, \theta) \in S^{m+(n-2N)/4}_{\mathrm{cl}}$. Then, the principal symbol of a distribution $A \in I^m_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}})$ is the homogeneous section*

$$\sigma^m(A) \sim \widetilde{a}_0\sqrt{d\phi} \in S^{(m+n/4)}(\Lambda, \mathscr{L}), \tag{2.2.2}$$

where a_0 is the top order term of the asymptotic expansion of a.

Proof. Recall that, according to Definition 2.30, $\sigma^m(A) = \mathcal{P}^{-1}([A])$, where $\mathcal{P}$ is the bijective map in Theorem 2.29. Then, we need to show that

$$\mathcal{T}_A(\psi) \text{ and } \mathcal{P}^{-1}([A]) \text{ are equivalent as sections of } \mathscr{L}. \tag{2.2.3}$$

First of all, we need to verify that $\mathcal{T}_A(\psi) \in \Gamma^{m+n/4}(\Lambda, \mathscr{L})$. In the previous section, we showed that almost analytic homogeneous functions define homogeneous sections in $\mathscr{L}$. This, together with the local identification of C_ϕ and Λ_ϕ, allows us to interpret $\widetilde{a}_0$ and $\widetilde{u}$ as elements of $\Gamma^{m+(n-2N)/4}(\Lambda, \mathscr{L})$ and $\Gamma^0(\Lambda, \mathscr{L})$, respectively. Then, it follows that the right hand side of (2.2.1) defines a homogeneous section of degree $m + n/4$.

Recall that the sections of the line bundle form an equivalent class, and that the classes are independent of the choice of almost analytic extension of a_0 and u. Recall also that $\mathcal{T}_A(\psi) \sim \mathcal{P}^{-1}([A])$ means they belong to the same equivalence class. In other words, we need to show that $\mathcal{P} \circ \mathcal{T}_A$ is the identity.

Given a distribution $A = I(\phi, a)$, the equivalent class $[A]$ is determined by the oscillatory integral $I(\phi, a_0)$, where a_0 is the top order term of the asymptotic expansion of a. On the other hand, the action of $\mathcal{P}$, tells us that

$$\mathcal{P}(s) = I(\phi, \overline{b}), \qquad s \sim b\sqrt{d\phi} \in \Gamma^{m+n/4}(\Lambda, \mathscr{L}),$$

with $\overline{b}$ an extension of b to $\mathbb{C}^n \times \mathbb{C}^N$. Then, taking $s = \mathcal{T}_A(\psi)$, we have $\mathcal{P}(s) = I(\phi, \overline{b})$ with $b = \widetilde{a_0}\widetilde{u}$. Since this is valid for any $u \in C_c^\infty$ and any almost analytic extensions, we conclude that

$$\mathcal{P}(s) \sim I(\phi, \widetilde{a_0}) = [A].$$

Equivalently, $\mathcal{P}^{-1}([A])$ is a scalar multiple of $\mathcal{T}_A(\psi) = \widetilde{a_0}\sqrt{d\phi}$, which gives the result. Once again, there is some freedom in choosing the extension $\overline{b}$, but another extension will result in another representative of the same equivalent class. $\square$

This description of the principal symbol has some important consequences. First note that σ^m maps

$$I^m_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}}) \to S^{(m+n/4)}(\Lambda, \mathscr{L}), \tag{2.2.4}$$

Moreover, as a direct consequence of (2.2.3) and the bijectivity of $\mathcal{P}$, we see that $\sigma^{m+n/4}$ is surjective. Thus, similarly to the real-valued case, the map $\sigma^{m+n/4}$ fits into a short exact sequence

$$0 \to I^{m-1}_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}}) \to I^m_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}}) \xrightarrow{\sigma^m} S^{(m+n/4)}(\Lambda, \mathscr{L}) \to 0.$$

On the other hand, the equation (2.2.2) allows us to refine the formulation of Theorem 2.38, because now we see the multiplicative behaviour of the principal symbol.

Theorem 2.41. *Let* $A_1 \in I^{m_1}_{\mathrm{cl}}(X \times Y, C_1; \Omega^{\frac{1}{2}})$, $A_2 \in I^{m_2}_{\mathrm{cl}}(Y \times Z, C_2; \Omega^{\frac{1}{2}})$ *be as in* Theorem 2.36 *and denote by B the composition* $A_1 \circ A_2 \in I^{m_1+m_2}_{\mathrm{cl}}(X \times Z, C_1 \circ C_2; \Omega^{\frac{1}{2}})$. *Then,*

$$\sigma^m(B) \sim (a_1)_0 (a_2)_0 (\theta^2 + \sigma^2)^{\frac{-n_Y}{2}} \sqrt{d\Phi} \in S^{(m+n/4)}(\Lambda, \mathscr{L}),$$

with $m = m_1 + m_2$, $n = n_X + n_Z$ and $(C_1 \circ C_2)' = \Lambda \sim \Lambda_\Phi$. Here $(a_1)_0$, $(a_2)_0$ are the principal parts of the amplitudes of A_1 and A_2, respectively.

Proof. Theorem 2.40 applied to $B = A_1 \circ A_2$, tells us that

$$\sigma^m(B) \sim \widetilde{b_0}\sqrt{d\Phi}, \tag{2.2.5}$$

with $b = (a_1)(a_2)(\theta^2 + \sigma^2)^{\frac{-n_Y}{2}}$ the amplitude of B. Then, the proof consists in showing that $\sqrt{d\Phi}$ defines a section in $\Gamma(\Lambda, \mathscr{L})$, for $\Lambda = (C_1 \circ C_2)'$. Luckily, all the necessary steps were proved in [16]. In general, the square of a section $\sigma \in \Gamma(\Lambda, \mathscr{L})$ defines, up to a sign, an almost analytic form on Λ of maximal degree. That is

$$\sigma^2 \sim \pm\omega, \quad \text{for some } n\text{-form } \omega.$$

By construction (see Lemma 2.28), we know that given a phase function ϕ, there is a n-form $\omega = d\phi$. Moreover,

$$\sigma^2 \sim \pm d\phi, \quad \text{for } \sigma = \sqrt{d\phi}. \tag{2.2.6}$$

On the other hand, the proof of Theorem 2.38 states

$$d\phi_1 \wedge d\phi_2 \sim \pm(\theta^2 + \sigma^2)^{-n_Y} d\Phi \wedge \Omega$$

defines an almost analytic form on $\Lambda = (C_1 \circ C_2)'$, and that $\alpha_1^2 \wedge \alpha_2^2 \sim \pm\alpha^2 \wedge \Omega$, for some Ω. Here $\alpha_1, \alpha_2, \alpha$ are the principal symbols of A_1, A_2, B, respectively. Combing these facts with the expressions for α_1, α_2 according to Theorem 2.40, we get

$$\begin{aligned}\alpha_1^2 \wedge \alpha_2^2 &\sim ((a_1)_0^2\, d\phi_1) \wedge ((a_2)_0^2\, d\phi_2) \sim ((a_1)_0(a_2)_0)^2\, d\phi_1 \wedge d\phi_2 \\ &\sim \pm\left((a_1)_0(a_2)_0(\theta^2 + \sigma^2)^{-\frac{n_Y}{2}}\right)^2 d\Phi \wedge \Omega,\end{aligned}$$

Then, $\alpha^2 \sim \pm b_0^2 d\Phi$. From equation (2.2.6), we see that $\alpha \sim b_0\sqrt{d\Phi}$. The result follows from equation (2.2.5) and the definition of b_0. □

2.2.2 Clean phase functions and composition

In [16], the case of *transverse composition* was considered. We wish to relax their assumptions to include a slightly more general geometric situation. In this subsection, we consider the case of *clean composition*. To do so, we first need to consider complex-valued clean phase functions. They are a natural generalization of the concept of non-degenerate phase functions, but we were not able to find them in the existing literature.

Definition 2.42. *A complex-valued function $\phi(x,\theta)$, smooth in a conic set $V \subset \mathbb{R}^n \times \mathbb{R}^N \setminus 0$, is called **clean phase function of positive type** if $\Im\phi \geq 0$ and*

- *$d\phi \neq 0$,*
- *ϕ is homogeneous of degree 1 in θ,*
- *there exist $M \leq N$, such that M of the differentials $\left\{d(\frac{\partial\phi}{\partial\theta_j})\right\}_{j=1}^{N}$ are linearly independent over $\mathbb{C}$ on*
$$C_{\phi\mathbb{R}} = \left\{(x,\theta) \in V \colon \phi'_\theta = 0\right\}.$$
The number $e = N - M$ is called the excess *of ϕ.*

Note that, whenever the excess $e = 0$, the function ϕ is a non-degenerate phase function. Also note that, after reorganizing the variables, it is possible to split $\theta \in \mathbb{R}^N \setminus 0$ as $(\theta', \theta'') \in (\mathbb{R}^M \times \mathbb{R}^e) \setminus 0$, where the differentials $\left\{d(\partial\phi/\partial\theta'_j)\right\}$, $j = 1, \dots, M$ are the ones satisfying Definition 2.42.

As usual, we denote by Λ_ϕ the manifold
$$\left\{(\tilde{x}, \partial_{\tilde{x}}\tilde{\phi}(\tilde{x},\tilde{\theta})) \in \mathbb{C}^n \times \mathbb{C}^n \setminus 0 \colon (\tilde{x},\tilde{\theta}) \in C_{\tilde{\phi}}\right\},$$
for some almost analytic extension $\tilde{\phi}$ of ϕ to a complex extension of V. One can verify that
$$C_\phi = \left\{(\tilde{x},\tilde{\theta}) \in \mathbb{C}^n \times \left(\mathbb{C}^N \setminus 0\right) : \partial_{\tilde{x}}\tilde{\phi}(\tilde{x},\tilde{\theta}) = 0\right\}$$
is an almost analytic manifold of dimension $2(n+e)$. Moreover, the the map
$$C_\phi \ni (\tilde{x},\tilde{\theta}) \to \left(\tilde{x}, \partial_{\tilde{x}}\tilde{\phi}(\tilde{x},\tilde{\theta})\right) \in \Lambda_\phi \tag{2.2.7}$$

is almost analytic and bijective, which makes Λ_ϕ a positive Lagrangian manifold. Then, in view of Remark 2.17, it can be parameterized by a non-degenerate phase function,

$$\widetilde{\psi}(\tilde{x},\tilde{\xi}) = \tilde{x}\tilde{\xi} - g(\tilde{\xi}), \quad (\tilde{x},\tilde{\xi}) \in \mathbb{C}^n \times (\mathbb{C}^n \setminus 0), \tag{2.2.8}$$

for some almost analytic function g with $\Im g \leq 0$ at $\xi \in \mathbb{R}^n \setminus 0$. Denoting by ψ the restriction of $\widetilde{\psi}$ to the real domain, we see that $\phi \sim \psi$ in the sense of Definition 2.18. This equivalence of phase functions allows us to associate distributions in I^m_{cl} with a microlocal representation $I(\phi, a)$, where ϕ is a clean phase function instead of a non-degenerate one.

Proposition 2.43. *Let $\phi(x,\theta) \in \mathscr{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ be a clean phase function of excess e. Then, for $a \in S^{m+(n-2N-2e)/4}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$, the oscillatory integral $I(\phi, a)$ defines a Fourier distribution of order m.*

Proof. Fix a point $(x_0, \theta_0) \in C_{\phi\mathbb{R}}$ and set $\xi_0 = \phi'_x(x_0, \theta_0)$. We know that, near (x_0, ξ_0), the almost analytic manifold $\Lambda \sim \Lambda_\phi$ is equivalent to a manifold Λ_ψ, with ψ the non-degenerate phase function (2.2.8). We wish to use the construction in Proposition 2.19 to show that there exists an amplitude $b \in S^{m+(n-2N')/4}(\mathbb{R}^n \times (\mathbb{R}^{N'} \setminus 0))$, $N' = n$, such that the oscillatory integrals $I(\phi, a)$ and $I(\psi, b)$ are microlocally equivalent near (x_0, ξ_0). This will imply that $I(\phi, a)$ defines a Fourier distribution.

The proof of Proposition 2.19 is based on the stationary phase formula. There, we could apply Theorem 2.9 with respect to the variables (x, θ), because the phase functions were assumed to be non-degenerate. This is no longer true for a clean phase function ϕ. Instead, we need to consider $\theta = (\theta', \theta'') \in (\mathbb{R}^M \times \mathbb{R}^e) \setminus 0$, and

$$I(\phi, a) \sim \int \left(\int e^{i\phi(x,\theta',\theta'')} a(x, \theta', \theta'')\, d\theta' \right) d\theta''. \tag{2.2.9}$$

Since the differentials $\left\{ d(\partial\phi/\partial\theta'_j) \right\}$ are linearly independent over $\mathbb{C}$ at real points, we can apply the stationary phase formula to the inner integral in (2.2.9). The rest of the argument in Proposition 2.19 applies without further modification. The desired conclusion follows after integrating out the excess variables θ''.

Finally, note that applying the stationary formula in e less variables increases the order of the distribution by $e/2$. Now the asymptotic sum (2.1.1) is in $S^{-(n+N)/2+e/2}$ instead of $S^{-(n+N)/2}$, as it was the case in Proposition 2.19. □

Since the construction that leads to Theorem 2.40 is also based on the stationary phase formula, the description of the principal symbol given by equation (2.2.2) applies only to the inner integral in (2.2.9). So, a priori, the principal symbol of the distribution $A = I(\phi, a)$ above is given by the integral, with respect to θ'' , of the principal symbol of the inner distribution. But, for this to be correctly defined, we first need to modify the definition of $\sqrt{d\phi}$.

Lemma 2.44. *Let $\phi(x,\theta)$ be a clean phase function with excess e that parameterizes Λ. Then, there is a section $\sqrt{d\phi} \in \Gamma^{(N-e)/2}(\Lambda, \mathscr{L})$, defined by*

$$(\sqrt{d\phi})_\tau \sim \left[\det \frac{1}{i} \begin{pmatrix} \tilde{\phi}''_{xx} - \tilde{\psi}''_{xx} & \tilde{\phi}''_{x\theta'} \\ \tilde{\phi}''_{\theta' x} & \tilde{\phi}''_{\theta'\theta'} \end{pmatrix}\right]^{-1/2}, \tag{2.2.10}$$

where θ'' are the excess variables in the splitting $\theta = (\theta', \theta'')$. Here τ, ψ and the branch of the square root are chosen as in Lemma 2.28.

Proof. Note that, for θ'' fixed, ϕ defines a non-degenerate phase function with respect to the variables (x, θ'). Then, it follows from Lemma 2.28, that $\sqrt{d\phi}$ defines a section of $\mathscr{L}$. Since the matrix in (2.2.10) is now of dimension $(n+N-e) \times (n+N-e)$, we see that $\sqrt{d\phi}$ is homogeneous of degree $(N-e)/2$, which completes the proof. □

With this new meaning for $\sqrt{d\phi}$, we can apply Theorem 2.40 to the inner integral in equation (2.2.9). It follows that $\widetilde{a_0}\sqrt{d\phi}$ defines an element of $S^{(m-e+n/4)}(\Lambda, \mathscr{L})$, but, we still need to integrate out the excess variables θ''. In principle, this integral may not be defined. Thus, similar to the real case, we restrict the domain of integration.

Let $\pi : \Lambda_\phi \to \mathbb{C}^n$ be the projection $\pi(\tilde{x}, \tilde{\xi}) = \tilde{\xi}$. The composition of π with the map (2.2.7) defines a fiber bundle over Λ with fiber

$$C_{\tilde{\xi}} = \left\{(\tilde{x}, \tilde{\theta})\colon \partial_{\tilde{\theta}}\tilde{\phi}(\tilde{x}, \tilde{\theta}) = 0,\ \partial_{\tilde{x}}\tilde{\phi}(\tilde{x}, \tilde{\theta}) = \tilde{\xi}\right\}.$$

The fiber $C_{\tilde{\xi}}$ can be interpreted as an almost analytic manifold of dimension $2e$, if the differentials $\left\{d(\frac{\partial\phi}{\partial x_j})\right\}_{j=1}^{n}$ are linearly independent at real points. In any case, we can compute $\int_{C_{\tilde{\xi}}} \widetilde{a_0}\sqrt{d\phi}\, d\theta''$ if we assume that $C_{\xi\mathbb{R}}$, the restriction of $C_{\tilde{\xi}}$ to the real domain, is compact.

Definition 2.45. *Let $\phi(x,\theta) \in \mathscr{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ be a clean phase function with excess e, such that the set $C_{\xi\mathbb{R}}$ above is compact. Then, the principal symbol of $A \in I^m_{\mathrm{cl}}(X, \Lambda; \Omega^{\frac{1}{2}})$, locally $A = I(\phi, a)$, is*

$$\sigma^m(A) = \int_{C_{\tilde{\xi}}} \widetilde{a_0}\sqrt{d\phi}\, d\theta'' \in S^{(m-e+n/4)}(\Lambda, \mathscr{L}). \tag{2.2.11}$$

Here a_0 is the principal part of $a(x,\theta) \in S^{m+(n-2N-2e)/4}_{\mathrm{cl}}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$.

We consider now a generalization of Assumption 2.33. As before, let $X,\ Y,\ Z$ be manifolds of dimension n_X, n_Y, n_Z respectively, $\Delta = T^*X \times \mathrm{diag}(T^*Y) \times T^*Z$ and $\tilde{\Delta}$ its almost analytic continuation. Consider

$$A_1 \in I^{m_1}_{\mathrm{cl}}(X \times Y, C_1; \Omega^{\frac{1}{2}}), \quad A_2 \in I^{m_2}_{\mathrm{cl}}(Y \times Z, C_2; \Omega^{\frac{1}{2}}),$$

where the C_j, $j = 1, 2$, are positive canonical relations. Recall that $\Lambda_j = C_j'$ are positive Lagrangian manifolds.

Assumption 2.46. *Suppose that:*

a. At the real points of $(C_1 \times C_2) \cap \Delta$, the intersection is clean with excess e.

b. The natural projection $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \to (T^*X \setminus 0) \times (T^*Z \setminus 0)$ is injective and proper.

Remark 2.47. *The intersection of two smooth submanifolds Y_1 and Y_2 of a smooth manifold X is said to be* clean *if, $Y_1 \cap Y_2$ defines a manifold and, at every point $x \in Y_1 \cap Y_2$, it holds that $T_xY_1 \cap T_xY_2 = T_x(Y_1 \cap Y_2)$.*
The non-negative integer e satisfying $\operatorname{codim} Y_1 + \operatorname{codim} Y_2 = \operatorname{codim}(Y_1 \cap Y_2) + e$ *is called the excess of the intersection.*

As expected, the generalized assumption leads to clean phase functions. The following results follow form arguments similar to those in Subsection 2.1.4. Here we only present the parts where the proofs are different.

Proposition 2.48. *Let $C_1 \subseteq (T^*X \setminus 0)^\sim \times (T^*Y \setminus 0)^\sim$, $C_2 \subseteq (T^*Y \setminus 0)^\sim \times (T^*Z \setminus 0)^\sim$ be positive canonical relations satisfying Assumption* 2.46*. Then, there exists a manifold $C' := (C_1 \circ C_2)'$, parameterized by a clean phase function Φ, such that $C_{\mathbb{R}} = C_{1\mathbb{R}} \circ C_{2\mathbb{R}}$. The excess of Φ is equal to the excess of the intersection at real points.*

Proof. Since the Λ_j, $j = 1, 2$, are almost Lagrangian manifolds, there are coordinates such that, in a neighborhood of real points $(x_0, \xi_0, y_0, -\eta_0) \in \Lambda_{1\mathbb{R}} = C'_{1\mathbb{R}}$ and $(y'_0, \eta'_0, z_0, \zeta_0) \in \Lambda_{2\mathbb{R}} = C'_{2\mathbb{R}}$, the manifolds are given by the vanishing of

$$\tilde{x} - \frac{\partial H_1}{\partial \tilde{\xi}}(\tilde{\xi}, \tilde{\eta}), \quad \tilde{y} + \frac{\partial H_1}{\partial \tilde{\eta}}(\tilde{\xi}, \tilde{\eta}); \qquad \tilde{y}' - \frac{\partial H_2}{\partial \tilde{\eta}'}(\tilde{\eta}', \tilde{\zeta}), \quad \tilde{z} + \frac{\partial H_2}{\partial \tilde{\zeta}}(\tilde{\eta}', \tilde{\zeta}).$$

The intersection $(C_1 \times C_2) \cap \tilde{\Delta}$ is completely described by these functions together with

$$\tilde{y} = \tilde{y}', \qquad \tilde{\eta} = \tilde{\eta}',$$

and its tangent plane is given by the vanishing of these differentials. Clean intersection means that $T(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap T\Delta$ is described by the equations

$$\begin{aligned} &d\left(x - \frac{\partial H_1}{\partial \xi}(\xi, \eta)\right) = 0, \quad d\left(y + \frac{\partial H_1}{\partial \eta}(\xi, \eta)\right) = 0, \quad d(y - y') = 0, \\ &d\left(y' - \frac{\partial H_2}{\partial \eta'}(\eta', \zeta)\right) = 0, \; d\left(z + \frac{\partial H_2}{\partial \zeta}(\eta', \zeta)\right) = 0, \; d\left(\eta - \eta'\right) = 0, \end{aligned} \tag{2.2.12}$$

and has dimension $n_X + n_Z + e$, where e is the excess of the intersection. Like in to the transversal case (Proposition 2.35), we define $C_1 \circ C_2$ as the manifold that satisfies $(C_1 \circ C_2)_{\mathbb{R}} = C_{1\mathbb{R}} \circ C_{2\mathbb{R}}$, where

$$C_{1\mathbb{R}} \circ C_{2\mathbb{R}} = \left\{((x, y, \xi, \eta), (y', z, \eta', \zeta)) \in C_{1\mathbb{R}} \times C_{2\mathbb{R}} \colon y = y',\ \eta + \eta' = 0\right\}$$

can be identified with $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap \Delta$. The main difference is that now the Lagrangian manifold $\Lambda = (C_1 \circ C_2)'$ is of dimension $n_X + n_Z + e$. Suppose now that Λ_1 and Λ_2 are parameterized by the non-degenerate phase functions

$$\phi_1(x, y, \xi, \eta) = x \cdot \xi - y \cdot \eta + H_1(\xi, \eta), \quad \phi_2(y, z, \eta, \zeta) = y \cdot \eta - z \cdot \zeta + H_2(\eta, \zeta).$$

The previous analysis shows that the function

$$\Phi(x,z,\omega) = \phi_1(x,y,\xi,\eta) + \phi_2(y,z,\eta,\zeta), \qquad \omega = \omega(y,\xi,\eta,\zeta),$$

defines a clean phase function with excess e, because the differentials $d(\partial\Phi/\partial\omega_j)$ are exactly those in (2.2.12). Then, the excess of the phase function Φ is

$$\dim\Lambda - (n_X + n_Z) = e.$$

Recall that y needs to be considered a parameter, so we take Φ as a function of (x,z,ω), with $\omega = \omega(y,\xi,\eta,\zeta)$ some homogeneous function of degree 1. Finally, note that there is a one-to-one correspondence between $C_{1\mathbb{R}} \circ C_{2\mathbb{R}}$ and

$$C_{\Phi\mathbb{R}} = \left\{((x,y,\xi,\eta),(y,z,\eta,\zeta)) \in C_{\phi_1} \times C_{\phi_2} \colon \partial_y(\phi_1+\phi_2) = 0\right\},$$

so the manifold $\Lambda = (C_1 \circ C_2)'$ can be parameterized by the clean function Φ. $\square$

To make the notation consistent with the results presented in Section 2.1, we put $\theta = (\xi,\eta)$ and $\sigma = (\eta,\zeta)$. Then, we consider Φ as a clean phase function depending on (x,z,y,θ,σ), that is

$$\Phi(x,z,\omega) = \phi_1(x,y,\theta) + \phi_2(y,z,\sigma), \qquad \omega = \omega(y,\theta,\sigma),$$

where ω is some homogeneous function of degree 1.

The next theorem tells us that under Assumption 2.46, the composition of Fourier integral operators is well-defined. Note that when the excess $e = 0$, we land in the case of transverse composition. Thus, the following is a generalization of Theorem 2.36.

It follows from the remark before Definition 2.45, that to compute the principal symbol of the resulting distribution, we would need to integrate out the excess variables. To do so, we take advantage of the identification between $C_1 \circ C_2$ and $(C_1 \times C_2) \cap \tilde{\Delta}$.

Assumption 2.49. *The image C_γ of a point $\gamma \in (C_1 \circ C_2)_{\mathbb{R}}$ in $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap \Delta$, defines a compact fiber of dimension e over γ.*

Theorem 2.50. *Let $C_1 \subseteq (T^*X \setminus 0)^\sim \times (T^*Y \setminus 0)^\sim$ and $C_2 \subseteq (T^*Y \setminus 0)^\sim \times (T^*Z \setminus 0)^\sim$ be positive canonical relations satisfying Assumption 2.46 and Assumption 2.49. Suppose that $A_1 \in I^{m_1}_{\mathrm{cl}}(X \times Y, C_1; \Omega^{\frac{1}{2}})$ and $A_2 \in I^{m_2}_{\mathrm{cl}}(Y \times Z, C_2; \Omega^{\frac{1}{2}})$ are properly supported. Then, the composition $A_1 \circ A_2$ defines a distribution in $I^{m+e/2}_{\mathrm{cl}}(X \times Z, C_1 \circ C_2; \Omega^{\frac{1}{2}})$, $m = m_1 + m_2$, where e is the excess of the intersection.*

The proof is omitted because it is essentially the same as the proof of Theorem 2.36, but uses the canonical transformation defined in Proposition 2.48, instead of the one in Proposition 2.35. The order of the distribution is a direct consequence of Proposition 2.43 and the fact that $A_1 \circ A_2 \sim I(\Phi, b)$, for

$$b(x,z,\omega) = a_1(x,y,\theta)a_2(y,z,\sigma)(|\theta|^2 + |\sigma|^2)^{-n_Y/2} \in S^{m'}_{\mathrm{cl}}\left(\mathbb{R}^{n_X+n_Z} \times \mathbb{R}^N\right), \tag{2.2.13}$$

where $A_1 \sim I(\phi_1, a_1)$, $A_2 \sim I(\phi_2, a_2)$, $N = n_Y + N_1 + N_2$ and $m' = m_1 + m_2 + (n_X + n_Z - 2N)/4$. This last claim, can be proven by the same calculations presented in Theorem 2.36.

To compute the principal symbol of the distribution $A_1 \circ A_2$, we need further assumptions. We can now present an extension of Theorem 2.41.

Theorem 2.51. *Let $A_1 \in I^{m_1}_{\mathrm{cl}}(X \times Y, C_1; \Omega^{\frac{1}{2}})$ and $A_2 \in I^{m_2}_{\mathrm{cl}}(Y \times Z, C_2; \Omega^{\frac{1}{2}})$ satisfy the assumptions of Theorem 2.50. Let $m = m_1 + m_2$ and denote by B the composition $A_1 \circ A_2 \in I^{m+e/2}_{\mathrm{cl}}(X \times Z, C; \Omega^{\frac{1}{2}})$. Then,*

$$\sigma^{m+e/2}(B) \sim \int_{C_\gamma} (a_1)_0 (a_2)_0 (\theta^2 + \sigma^2)^{\frac{-n_Y}{2}} \sqrt{d\Phi}\, dy'' d\theta'' d\sigma'' \in S^{(m-e/2+n/4)}(\Lambda, \mathscr{L}), \tag{2.2.14}$$

with $n = n_X + n_Z$ and $\sqrt{d\Phi}$ defined as in Lemma 2.44.

Proof. A direct consequence of Theorem 2.41 and Definition 2.45. We only need to compute the order of homogeneity. Since $\sqrt{d\Phi}$ is defined according to Lemma 2.44, it is homogeneous of degree $(N-e)/2$, with $N = n_Y + N_1 + N_2$ and e the excess of the intersection. Then,

$$(a_1)_0 (a_2)_0 (\theta^2 + \sigma^2)^{\frac{-n_Y}{2}} \sqrt{d\Phi} \in S^{(m'')}(\Lambda, \mathscr{L})$$

for $m'' = m + (N - e)/2 = m_1 + m_2 - e/2 + (n_X + n_Z)/4$. Equation (2.2.14) follows after integration with respect to the excess variables $\omega'' = \omega''(y, \theta, \sigma)$. Note that it is possible to organize the variables in a way that $\omega'' = (y'', \theta'', \sigma'') \in \mathbb{R}^e$, for some splitting $y = (y', y'')$, $\theta = (\theta', \theta'')$ and $\sigma = (\sigma', \sigma'')$. $\square$

We conclude the chapter with an important observation: Proposition 2.48 implies that $\Lambda \sim (C_1 \circ C_2)'$ is a positive Lagrangian manifold. As such, it can be parametrized by a non-degenerate phase function. This re-parametrization could potentially save us the difficulty of working with the implicit variables $\omega = \omega(y, \theta, \sigma)$, but the amplitude b will no longer be as in (2.2.13). In Subsection 3.3.2, we take advantage of this fact to verify the H^s-continuty of our solution operator.

3 | Hyperbolic surface waves

At the beginning of this thesis, the equation of linear elasticity was presented as a motivation to study Fourier distributions with complex phase functions. It was shown, following ideas from [19], that it is possible to approximate the solution operator by a sum of two Fourier integral operators with complex phase functions. Approach that allowed us to rigorously study the properties of these type of solutions. The present chapter is devoted to extend that construction, and the subsequent analysis, to general hyperbolic surface waves.

Since surface waves are related to certain type of weakly regular hyperbolic boundary value problems (BVP), the first section of this chapter is an overview of the regularity theory of hyperbolic BVP. Specifically, we consider a constantly hyperbolic first order boundary value problem in the half-space

$$\mathbb{R}^{1+d}_{+} := \{(x,y) \in \mathbb{R}^{1+d} \colon x > 0,\ y \in \mathbb{R}^{d}\}.$$

Definition 3.1. *A first order differential operator L, of the form*

$$L = \partial_t - A(t,x,y,\partial_x,\partial_y),$$

is called symmetrizible hyperbolic with constant multiplicities, or constantly hyperbolic, if the matrix $A(t,x,y,\xi,\eta)$ have only real, semi-simple eigenvalues, and their multiplicity is independent of (t,x,y,ξ,η).

In order to make the presentation as clear as possible, we start by making a series of simplifying assumptions. Assume that L has constant coefficients and that the boundary is non-characteristic. In other words, we consider the general IBVP

$$\begin{cases} Lu = \partial_t u - A_0 \partial_x u - \sum_{j=1}^{d} A_j \partial_{y_j} u = f(t,x,y) & \text{in} \quad (0,T) \times \mathbb{R}^{1+d}_+, \\ Bu = g(t,y) & \text{on} \quad (0,T) \times \mathbb{R}^d, \\ u|_{t=0} = u_0(x,y) & \text{in} \quad \mathbb{R}^{d+1}_+, \end{cases} \tag{3.0.1}$$

where the matrices $A_j \in M_{N\times N}(\mathbb{C})$ and $B \in M_{p\times N}(\mathbb{C})$ have constant entries and A_0 is non-singular. Furthermore, we assume that p is the number of incoming characteristics, this is the number of negative eigenvalues of A_0.

The well-posedness of this problem is connected to the so called *uniform Kreiss-Lopatinskii condition*. When this condition is satisfied, the IBVP is strongly well-posed, with estimates comparable to those obtained for the Cauchy problems. Having said that, the existence of surface waves is related to the *failure* of the condition in a controlled manner. And it is precisely this way of not satisfying the condition, which naturally leads us to Fourier distributions whit complex phases.

This chapter consist of four sections. The first one is a summary of the theory of well-posedness of hyperbolic boundary value problem. There, the uniform Kreiss-Lopatinskii condition is introduced, and its relation to the existence of surface waves is explained. On the second section, we show how the construction in [19] can be used to obtain an approximated solution to a certain type of *weakly regular* BVP. The third section is devoted to analyse this approximated solution. With the help of the results from the previous chapter, we provide a refined description of the propagation of singularities and a preliminary result concerning the regularity of the solution. Finally, in the last section, we get rid of the assumption of constant coefficients and study a more general operator L.

3.1 The Kreiss-Lopatinskii condition

The uniform Kreiss-Lopatinskii condition, or UKL for short, is necessary and sufficient for the L^2 well posedness of the hyperbolic IBVP. A detailed presentation of this condition can be found in [3] and [5]. For the sake of completeness, this section contains a rough sketch of the ideas leading to UKL and its consequences.

We start by considering the IBVP (3.0.1), with $f \equiv 0$. After taking Fourier transform on (t, y), we search for a solution that decays away from the boundary. To be precise, we look for a solution $v(x) = \hat{u}(\tau, x, \eta)$ to

$$v' = i\mathcal{A}(\tau,\eta)v, \qquad \mathcal{A}(\tau,\eta) = (A_0)^{-1}\left(\tau I_n - \sum_{j=1}^{d} A_j \eta_j\right),$$

such that $v \to 0$ when $x \to \infty$. Here $(\tau, \eta) \in \mathbb{R}^n$, with $n = 1 + d$, and $x \in \mathbb{R}_+$. The solution to this ODE, with initial condition $v(0) = w$, is

$$v(x) = e^{i\mathcal{A}(\tau,\eta)x} w.$$

Suppose now that $\tau = \rho - i\gamma$. Then, the hyperbolicity assumption implies that, for $\gamma > 0$, $\mathcal{A}(\tau, \eta)$ has no real eigenvalues and we can decompose $\mathbb{C}^N$ as a direct sum of eigenspaces. Then, v can be written as $v(x) = v^+(x) + v^-(x)$, where

$$v^{\pm}(x) = \frac{1}{2i\pi} \int_{C^{\pm}} e^{ix\xi} [\xi I_n - \mathcal{A}(\tau,\eta)]^{-1} w \, d\xi$$

and $C^{\pm}$ are curves enclosing the eigenvalues of $\mathcal{A}(\tau, \eta)$ with positive/negative imaginary part. Since we need our solution to be bounded for $x > 0$, we consider only v^+. It is clear that $v^+(0) = Pw$, where P is the projector

$$P = \frac{1}{2i\pi} \int_{C^{+}} [\xi I_n - \mathcal{A}(\tau,\eta)]^{-1} w \, d\xi.$$

Let $S(\tau, \eta)$ be the image of P. Then, stability of the BVP (3.0.1) requires that the restriction of B to $S(\tau, \eta)$ is an isomorphism. When this holds, we say that the BVP satisfy the uniform Kreiss-Lopatinskii condition, UKL for short.

Remark 3.2. *For the weaker (non-uniform) version of the condition, the KL condition, one needs to consider the boundary case $\Im\tau = 0$. While the space $S(\tau, \eta)$ is no longer defined, it is possible to consider the limit space as $\Im\tau \to 0$. Further details can be found in [3].*

Let $\lambda_j(\tau, \eta) \in \mathbb{C}$ denote the eigenvalues of $\mathcal{A}(\tau, \eta)$ with $\Im\lambda_j \geq 0$ and $r_j(\tau, \eta) \in \mathbb{C}^N$ is the corresponding eigenvector. Since the operator L is constantly hyperbolic, we know that for $\Im\tau \leq 0$, there are exactly p of such eigenvectors counted with their multiplicities. Thus, we can write the stable function v^+ as

$$v^+(x) = \sum_{j=1}^{p} e^{i\lambda_j(\tau,\eta)x} r_j(\tau,\eta). \tag{3.1.1}$$

The solution u to the BVP in (3.0.1) is then

$$u(t,x,y) = \sum_{j=1}^{p} \int e^{i(\tau t+\eta y)} e^{i\lambda_j(\tau,\eta)x} r_j(\tau,\eta)\, d\tau, d\eta.$$

Roughly speaking, there are two possible behaviours for u:

1. There are no solutions that grow or oscillate in time, i.e. $\Im\tau < 0$. In this case, the BVP satisfy the UKL condition.
2. There is a solution that oscillates in time, i.e. $\Im\tau = 0$, and it is a limit of solutions that grow in time and decay away from the boundary. In this case the BVP satisfy the KL condition.

3.1.1 The Lopatinskii determinant

It is often difficult to verify the UKL condition in practice. But, there is a practical, and somewhat easier, tool to verify this condition: the **Lopatinskii determinant**. This is a function Δ, that vanishes precisely at the points (τ, η) where either KL or UKL are violated.

To construct the determinant Δ, we first need a basis of the stable subspace $S(\tau, \eta)$ which is jointly analytic in (τ, η) and, therefore, holomorphic in τ. By *stable subspace* we mean the sum of the generalized eigenspaces corresponding to eigenvalues of $\mathcal{A}(\tau, \eta)$ with non-negative imaginary part. Let $\{X_1(\tau, \eta), \ldots, X_p(\tau, \eta)\}$ be such basis and define

$$\Delta(\tau,\eta) = \det(BX_1(\tau,\eta), \ldots, BX_p(\tau,\eta)). \tag{3.1.2}$$

Thanks to hyperbolicity and the assumption of constant multiplicities, it is possible to take this basis as a basis of eigenvectors. This can be done following the procedure described in [14], Chapter II, Section 4.2. It follows that we can take the X_j to be the eigenvectors r_j in (3.1.1). At this point, there are two alternatives for the BVP,

1. UKL holds, meaning $\Delta(\tau, \eta) \neq 0$ for all (τ, η) with $\Im\tau \leq 0$. So the problem is strongly well-posed in any Sobolev space. Precise energy estimates can be found in [3], Chapter 4.

2. UKL fails but KL holds. This means $\Delta(\tau, \eta) \neq 0$ for all (τ, η) with $\Im\tau < 0$, but it vanishes for some $\tau \in \mathbb{R}$. Hence, the problem is generically *weakly* well-posed.

In the second case, the problem is C^∞ well-posed but there is a loss of regularity, which has to be worked out in a case-by-case study. We are interested in one of those cases, when the IBVP admits **surface waves** as solutions. Which, according to [4], represents the transition between the classes of stable and unstable problems.

3.1.2 The block structure

Before giving the exact definition of surfaces waves, we need to understand another important property of the symmetrizable and constantly hyperbolic BVPs, which was introduced by Kreiss and improved by Mètivier [18] . Set $\tau = \rho - i\gamma$, and denote by $\mathscr{X}$ and $\mathscr{X}_0$ the set of frequencies

$$\mathscr{X} = \{\zeta = (\rho - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^d \colon \gamma \geq 0\}, \quad \text{and} \quad \mathscr{X}_0 = \{\zeta \in \mathscr{X} : \gamma = 0\}.$$

Theorem 3.3 (Block structure). *If the IBVP* (3.0.1) *is symmetrizable hyperbolic with constant multiplicities, then for all $\underline{\zeta} \in \mathscr{X}$ there is a neighborhood $\mathcal{O}$ of $\underline{\zeta}$ in $\mathbb{C} \times \mathbb{R}^d$, an integer $L \geq 1$ and a invertible matrix $T(\zeta)$ defined in $\mathcal{O}$ such that*

$$T^{-1}(\zeta)\mathcal{A}(\zeta)T(\zeta) = \operatorname{diag}\left(\mathcal{A}_1(\zeta), \ldots, \mathcal{A}_L(\zeta)\right),$$

where the blocks $\mathcal{A}_i$ are $\nu_i \times \nu_i$ matrices that satisfy one of the following conditions:

i. *All elements of the spectrum of $\mathcal{A}_i(\zeta)$ have strictly positive imaginary part.*

ii. *All elements of the spectrum of $\mathcal{A}_i(\zeta)$ have strictly negative imaginary part.*

iii. $\nu_i = 1$, $\mathcal{A}_i(\zeta)$ *is real when* $\gamma = 0$ *and* $\partial_\gamma \mathcal{A}_i(\zeta) \neq 0$.

iv. $\nu_i > 1$, $\mathcal{A}_i(\zeta)$ *has real coefficients when* $\gamma = 0$, *there exists* $k_i \in \mathbb{R}$ *such that*

$$\mathcal{A}_i(\zeta) = \begin{pmatrix} k_i & 1 & 0 \\ \vdots & \ddots & 1 \\ 0 & \dots & k_i \end{pmatrix}$$

and the lower left hand corner of $\partial_\gamma \mathcal{A}_i(\zeta)$ *does not vanish.*

Thanks to this result, we can classify the frequencies in $\mathscr{X}_0$ into four groups.

Definition 3.4 (Boundary frequencies)**.** *We say* $\zeta \in \mathscr{X}_0$ *is*

- ***elliptic*** *if Theorem* 3.3 *is satisfied with exactly one block of type* i. *and, consequently, one block of type* ii. *The set of elliptic frequencies is denoted by* $\mathcal{E}$.
- ***hyperbolic*** *if Theorem* 3.3 *is satisfied exclusively with blocks of type* iii. *The set of hyperbolic frequencies is denoted by* $\mathcal{H}$.
- ***mixed*** *if Theorem* 3.3 *is satisfied with one block of type* i., *one block of type* ii. *and at least one block of type* iii., *but no blocks of type* iv. *The set of mixed frequencies is denoted by* $\mathcal{M}$.
- ***glancing*** *if Theorem* 3.3 *is satisfied with at least one block of type* iv. *The set of glancing frequencies is denoted by* $\mathcal{G}$.

It should be clear that we have a partition of the frequency space

$$\mathscr{X}_0 = \mathcal{E} \cup \mathcal{H} \cup \mathcal{M} \cup \mathcal{G},$$

however we will neglect the glancing frequencies, as we did in Section 1.3. Mainly because their presence means there is at least one pair of crossing eigenvalues. As explained in [2], if $\underline{\zeta} \in \mathscr{X}_0 \setminus \mathcal{G}$, we can write

$$\mathbb{C}^N = S(\underline{\zeta}) \bigoplus U(\underline{\zeta}),$$

where S and U refer to the *stable* and *unstable* subspaces of $\mathcal{A}$, respectively. Note that

$$S(\underline{\zeta}) = S^e(\underline{\zeta}) \bigoplus S^h(\underline{\zeta}),$$

where S^e is the generalized eigenspace associated to eigenvalues of positive imaginary part, while S^h is associated to some real eigenvalues. Moreover, near glancing points, it is no longer true that $\dim S = p$, which is essential for the construction.

3.1.3 Surface waves

Let $(\rho_0, \eta_0) \in \mathcal{E}$ be a point where the BVP (3.0.1) violates the UKL condition. A *surface wave* is a non-trivial solution of the form

$$e^{i(\rho_0 t + \eta_0 y)} V(x), \quad V \in L^2(\mathbb{R}_+).$$

This solution represents a wave travelling in the direction parallel to the boundary. More importantly, they have finite energy density and can be used to construct exact solutions of the BVP (see [3], Chapter 7).

Clearly, a solution like this one can only exists if the Lopatinkii determinant vanishes at some elliptic point (ρ_0, η_0), however a little more control is required. To be precise, an IBVP admits a surface wave solution, if the following assumption is satisfied.

Assumption 3.5. *The BVP satisfies KL, but the Lopatinskii determinant Δ vanishes, at the first order only, at some elliptic point. Explicitly, there exists some $\zeta_0 = (\rho_0, \eta_0) \in \mathcal{E}$ such that*

$$\Delta(\zeta_0) = 0, \quad \partial_\gamma \Delta(\zeta_0) \neq 0,$$

and $\Delta(\zeta) \neq 0$ everywhere else.

In general, because the UKL condition is not satisfied, surface waves are associated with some loss of regularity. However, their presence does not implies instability. For this reason, this type of problems are sometimes called *weakly regular* in the literature.

Remark 3.6. *Recall that we are considering the BVP in (3.0.1) with $f \equiv 0$ and $g \neq 0$. If we assume $f \neq 0$ and $g \equiv 0$, the problem can be strongly well-posed in L^2. See, for instance, [3], Section 7.2.*

As mention before, the authors in [4] showed that this condition generically describes the transition between the classes of strongly stable and strongly unstable problems. They also showed that the BVP satisfying Assumption 3.5 are not stable under perturbation of the coefficients, thus a small change in parameters can lead to instability. An example of this can be found in [1], where a small change in the coefficients forces the wave speed to become complex, which makes the solution exponentially unstable.

3.2 An approximated solution of the BVP

In this section, we extend the construction used for Rayleigh waves to the case of general hyperbolic surface waves. Specifically, we will show the following theorem

Theorem 3.7. *Assume that the boundary value problem* (3.0.1)*, with* $f \equiv 0$*, satisfies Assumption* 3.5*. Denote by* $\lambda_j(\zeta)$*,* $j = 1, \dots, p$*, the eigenvalues of* $\mathcal{A}$ *with positive imaginary part, counted with their multiplicities, and by* $r_j(\zeta)$ *the corresponding eigenvectors. Then, one can approximate the solution of the BVP by*

$$u = \sum_{j=1}^{p} S_j(F_j), \tag{3.2.1}$$

where the distributions F_j *satisfy equation* (3.2.7) *below, and each* S_j *denotes the Fourier integral operator with complex-valued phase function* $\phi_j(x,z,\zeta) = z\zeta + \lambda_j(\zeta)x$ *and amplitude*

$$a_j(x,z,\zeta) \sim r_j + \sum_{l \geq 1} \kappa(x,z) \frac{r_j(\zeta)}{|\zeta|^l}.$$

Under the assumptions of the theorem, the arguments presented in Section 1.3 can be applied, but we should expect the computations to be more involved. We now present a sketch of the proof, the detailed construction can be found in the next few subsections.

Sketch of the proof. Suppose that the BVP (3.0.1) satisfies Assumption 3.5 and take the ansatz

$$u = \sum_{j=1}^{p} u_j, \quad u_j = \int e^{i\phi_j(x,z,\zeta)} a_j(x,z,\zeta) \widehat{F}_j(\zeta)\, d\zeta =: S_j(F_j). \tag{3.2.2}$$

The phase functions ϕ_j and the amplitudes a_j will come from a geometric optics approximation. Namely, for each j, ϕ_j should satisfy the eikonal equation (see Subsection 3.2.1) and a_j solves certain transport equations (see Subsection 3.2.2). Finally, the distribution F_j solves the pseudodifferential equation $T(F) = g$, where the operator T is determined by the boundary condition. (see Subsection 3.2.3). □

3.2.1 The phase functions

Following the ideas of geometric optics, for each j, the phase function ϕ_j needs to satisfy the eikonal equation

$$\begin{aligned} \partial_x \phi_j &= \lambda_j(\nabla_z \phi_j), \\ \phi_j|_{x=0} &= z\zeta, \end{aligned} \tag{3.2.3}$$

where λ_j is a root in ξ of $\det \sigma^1(L)(\xi, \zeta)$ or, equivalently, an eigenvalue of $\mathcal{A}(\zeta)$. Note that, because λ_j can take imaginary values, ϕ_j must also be complex-valued, so it is possible that the composition $\lambda_j(\nabla_z \phi)$ is not defined. To avoid this, we would need to consider an almost analytic extension $\tilde{\lambda}_j$ of λ_j. In which case, the eikonal equation (3.2.3) can be solved as long as $\tilde{\lambda}_j$ is smooth and positive homogeneous of degree one. A method for solving this complex-valued eikonal equation can be found in [20]. The general solution to (3.2.3) is given by the formula

$$\phi_j(x, z, \zeta) = z\zeta + \int_0^x \widetilde{\lambda}_j(\theta)\, ds,$$

where $\theta = \theta(s, y, \zeta)$ solves the Hamilton equation,

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\partial \widetilde{\lambda}_j}{\partial \zeta}, & \frac{d\theta}{dx} &= \frac{\partial \widetilde{\lambda}_j}{\partial z}, \\ y|_{x=0} &= z, & \theta|_{x=0} &= \zeta. \end{aligned}$$

Thanks to our simplifying assumptions, this generic situation is not relevant for the model BVP (3.0.1), as the eigenvalues of $\mathcal{A}(\zeta)$ are known to be analytic and we only need to consider λ_j as a function on $\mathbb{C}^n$. Since we are assuming constant coefficients, the Hamilton equation is solved by the curve $(y, \theta) = (z + \nabla\lambda_j(\zeta)x, \zeta)$. Hence, for each j,

$$\phi_j(x, z, \zeta) = z\zeta + \lambda_j(\zeta)x, \tag{3.2.4}$$

with the variables (x, z, ζ) now taken in $\mathbb{C}^{1+n} \times \mathbb{C}^n$. For the oscillatory integrals in (3.2.2) to be defined, we need $\Im\phi_j \geq 0$. Thus, we consider only eigenvalues λ_j with non negative imaginary part. It is easy to check that this expression defines, for each j, a regular phase function in the sense of Definition 2.13.

3.2.2 The amplitudes

We now follow the argument of geometric optics to determine the amplitude functions a_j. Since the real-valued stationary phase formula remains valid for complex phase functions, we can directly replace each u_j into the first equation of (3.0.1), with $f \equiv 0$. Before doing so, recall that $z = (t, y)$ and $\zeta = (\tau, \eta)$. Next, note that we can write $\mathcal{A}(\tau, \eta) = Q\tau + P\eta$, for matrices Q and P. Then, $v = Lu_j$ is equivalent to $v = \left(\partial_x - Q\partial_t - P\partial_y\right) u_j$. Computing this, with ϕ_j as in (3.2.4), one sees that

$$v = \int e^{i\phi_j(x,z,\zeta)} c_j(x, z, \zeta) \widehat{F}_j(\zeta)\, d\zeta, \quad c_j = i\lambda_j(\tau, \eta)a_j + \partial_x a_j - b_j,$$

where b_j is given by the relation $e^{i\phi_j} b_j = \left(Q\partial_t + P\partial_y\right)\left(e^{i\phi_j} a_j\right)$. Assuming that each $a_j \in S^m_{cl}$, for some m, with $a_j \sim \sum_{l\geq 0} a_j^{(l)}$, it follows that

$$\begin{aligned} b_j =& i\left(Q(\partial_t\phi_j) + P(\partial_y\phi_j)\right) \sum_{l\geq 0} a_j^{(l)} + Q \sum_{l\geq 0} \partial_t a_j^{(l)} + P \sum_{l\geq 0} \partial_y a_j^{(l)} \\ =& i\left(Q\tau + P\eta\right) \sum_{l\geq 0} a_j^{(l)} + Q \sum_{l\geq 0} \partial_t a_j^{(l)} + P \sum_{l\geq 0} \partial_y a_j^{(l)} \\ =& i\mathcal{A}(\tau, \eta) \sum_{l\geq 0} a_j^{(l)} + Q \sum_{l\geq 0} \partial_t a_j^{(l)} + P \sum_{l\geq 0} \partial_y a_j^{(l)} \end{aligned}$$

After sorting by homogeneity, and momentarily omitting the index j, we get that b has an asymptotic sum $b \sim \sum_{l\geq 0} b^{(l)}$, with

$$b^{(l)} = i\mathcal{A}(\tau, \eta)a^{(l-1)} + Q\partial_t a^{(l)} + P\partial_y a^{(l)}.$$

Then,

$$\begin{aligned} c^{(l)} &= i\lambda_j(\tau,\eta)a^{(l-1)} + \partial_x a^{(l)} - b^{(l)} \\ &= i\left(\lambda_j(\tau,\eta) - \mathcal{A}(\tau,\eta)\right) a^{(l-1)} + \partial_x a^{(l)} - Q\partial_t a^{(l)} - P\partial_y a^{(l)}. \end{aligned}$$

Setting $a^{(-1)} = 0$ and $c^{(l)} = 0$ for all l, we have

$$\begin{aligned} &c^{(0)} = \partial_x a^{(0)} - Q\partial_t a^{(0)} - P\partial_y a^{(0)} = 0, \\ &c^{(1)} = i\left(\lambda_j(\tau,\eta) - \mathcal{A}(\tau,\eta)\right) a^{(0)} - \partial_x a^{(1)} - Q\partial_t a^{(1)} - P\partial_y a^{(1)} = 0. \end{aligned}$$

Both equations are satisfied if $a^{(0)}(\tau,\eta)$ is an λ_j-eigenvector of $\mathcal{A}$ and $a^{(1)}$ is any solution of

$$\partial_x w + Q\partial_t w + P\partial_y w = 0. \tag{3.2.5}$$

The next value of l gives

$$c^{(2)} = i\left(\lambda_j(\tau,\eta) - \mathcal{A}(\tau,\eta)\right) a^{(1)} - \partial_x a^{(2)} - Q\partial_t a^{(2)} - P\partial_y a^{(2)} = 0,$$

which shows that $a^{(1)}$ must also be in the λ_j-eigenspace, and $a^{(2)}$ should be solution of (3.2.5). Continuing this way, we see that each a_j is of the form

$$a_j(x,z,\zeta) \sim r_j(\zeta)\chi(\zeta) + \sum_{l\geq 1} \kappa(x,z)\frac{r_j(\zeta)}{|\zeta|^l}\chi(\zeta), \tag{3.2.6}$$

where κ solves equation (3.2.5), r_j is an eigenvector of $\mathcal{A}$ corresponding to λ_j and $\chi \in \mathscr{C}^\infty$ satisfy

$$\chi(\zeta) = 0 \text{ for } |\zeta| \leq 1/2, \quad \chi(\zeta) = 1 \text{ for } |\zeta| > 1.$$

Note that, for all j, r_j is homogeneous of degree one, then the role of the factor $|\zeta|^{-l}$ is to fix the homogeneity, so that $r_j(\zeta)/|\zeta|^l$ is homogeneous of degree $1-l$. One can verify that

$$a_j(x,z,\zeta) \in S^0_{\mathrm{cl}}(\mathbb{R}^{1+n} \times (\mathbb{R}^n \setminus 0)).$$

Remark 3.8. *We can pick κ to be a constant. In this case, the amplitudes a_j would be independent of (x,z), satisfying equation* (3.2.5) *trivially.*

3.2.3 The boundary condition

The last step is to show that one can find distributions F_j such that the ansatz (3.2.2) satisfies the boundary condition. As before, this will be done by solving, mod $\mathscr{C}^\infty$, the equation

$$T(F_1, \dots, F_p)^T = g,$$

where T is determined by the boundary matrix B.

We start by replacing $u|_{x=0}$ into the boundary condition, that is the second equation of (3.0.1).

Remark 3.9. *By $u|_{x=0}$ we mean the restriction to the boundary of $\mathbb{R}_+ \times \mathbb{R}^n$. That is the trace of the distribution u.*

A short calculation shows that

$$u|_{x=0} = \sum_{j=1}^{p} \int e^{iz\cdot\zeta} a_j(0, z, \zeta) \widehat{F}_j(\zeta)\, d\zeta,$$

$$g = Bu|_{x=0} = \int e^{iz\cdot\zeta} \mathcal{M}(z, \zeta) \widehat{F}(\zeta)\, d\zeta, \tag{3.2.7}$$

where $F = (F_1, \dots, F_p)^T$ and $\mathcal{M} \in M_{p\times p}(\mathbb{C})$ is the matrix with columns $Ba_j(0, z, \zeta)$, with a_j the amplitudes in (3.2.6). Since each $a_j \in S^0_{cl}$ and B can be seen as a differential operator of order 0, it follows that $\mathcal{M}$ can be seen as a classical amplitude of order 0. Then, the right hand side of equation (3.2.7) defines a pseudodifferential operator T, of order 0, acting on the vector value distribution F. Thus, the boundary condition $Bu|_{x=0} = g$, can be seen as the equation $T(F) = g$. Moreover, the principal symbol of T is

$$\sigma(T) = \left(Br_1(\zeta), \dots, Br_p(\zeta)\right),$$

and $q(\zeta) := \det \sigma(T)$ is exactly the Lopatinskii determinant Δ in (3.1.2). Thanks to Assumption 3.5, we know that $q(\zeta)$ has exactly one simple zero at some elliptic point ζ_0. Based on the construction of the parametrix for Rayleigh waves (see Section 1.3) and other examples, we have reason to believe that Δ is real-valued in the elliptic region. Thus, we make the following additional assumption.

Assumption 3.10. *The determinant of $\sigma(T)$, the principal symbol of T, is real-valued at the elliptic region.*

It then follows that the operator T is of real principal type near $\zeta_0 \in \mathcal{E}$, and elliptic away from $\mathcal{E}$. The precise meaning of real principal type is given by the following definition.

Definition 3.11. *1.* [9] *A pseudodifferential operator P with real-valued principal symbol $p(x,\xi)$ is of real principal type, if no complete null-bicharacteristic curve stays in a compact set. Equivalently, P is of real principal type if the Hamiltonian vector field H_p is nowhere radial on* $\mathrm{Char}(P)$.

2. [7] *An $N \times N$ system P of pseudodifferential operators with principal symbol $p(x,\xi)$ is of real principal type at (x_0,ξ_0), if there exists an $N \times N$ symbol $\widetilde{p}(x,\xi)$ such that*

$$\widetilde{p}(x,\xi)p(x,\xi) = q(x,\xi)I_N,$$

in a neighborhood of (x_0,ξ_0), where $q(x,\xi)$ is a scalar symbol of real principal type and I_N is the identity matrix. We say that P is of real principal type in $\Omega \subseteq \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, if it is at every $(x,\xi) \in \Omega$.

To see that T satisfies part 2 of this definition, it suffices to take $p = \sigma(T)$ and $\widetilde{p}$ as its co-factor matrix. Assumption 3.5 tells us that q has exactly one simple zero in $\mathcal{E}$, precisely at ζ_0. Which implies that no null- bicharacteristic curve passing through (z,ζ_0) can stay in a compact set. This fact, combined with Assumption 3.10, makes $q = \det \sigma(T)$ an operator of real principal type near (z,ζ_0).

Remark 3.12. *A pseudodifferential operator P with real-valued principal symbol $p(x,\xi)$ satisfying $d_\xi p \neq 0$ on* $\mathrm{Char}(P)$ *is of real principal type. However, the two conditions are not equivalent, as being of real principal type is more general.*

The following theorem, which is proven in [19], tells us that we can find $F \bmod \mathscr{C}^\infty$. Because we aim for this thesis to be as self-contained as possible, a short proof is presented in Appendix A.

Theorem 3.13. *Let T be a $k \times k$ system of pseudodifferential operators of order 0, with principal symbol $\sigma(T)(t,x,\tau,\xi)$. Assume that $q = \det \sigma(T)$ is real and $\partial_\tau q \neq 0$ whenever*

$q = 0$. Then, one can construct a forward fundamental solution to the equation T.

We conclude that the function u in (3.2.2) solves the BVP (3.0.1) up to a smooth remainder. Which completes the proof of Theorem 3.7.

We finish the section with an elementary consequence of the construction.

Proposition 3.14. *Let T be the operator in (3.2.7). Suppose that $g \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, has support contained in $\{t > 0\}$. Then, the equation $T(F) = g$, has a solution $F \in H^s(\mathbb{R}^n)$, which is unique* $\operatorname{mod} \mathscr{C}^\infty$ *and vanish for $t < 0$.*

3.3 Analysis of the solution

We have shown that (3.2.2) defines an approximation of the solution of the BVP (3.0.1), which differs from the real solution by a smooth remainder. Such solution is given by the action of an operator S on the distribution F determined by Proposition 3.14. The kernel of this operator is the distribution

$$K(x,z,y) = \sum_{j=1}^{p} K_j(x,z,y) = \sum_{j=1}^{p} \int e^{i\psi_j(x,z,y,\zeta)} a_j(x,z,\zeta)\, d\zeta, \tag{3.3.1}$$

where $\psi_j(x,z,y,\zeta) = (z-y)\zeta + \lambda_j(\zeta)x$ and $a_j(x,z,\zeta)$ is as in (3.2.6). In principle, $(x,z) \in \mathbb{R}^{1+n}_+$ and $\zeta, y \in \mathbb{R}^n$, but we need to consider $(x,z) \in \mathbb{C}^{1+n}$ and $\zeta, y \in \mathbb{C}^n$, as explained in Subsection 3.2.1. Which makes our solution operator S a sum of Fourier integral operators with complex phase of order $-1/4$. We denote by S_j the operator with kernel K_j.

In this section, we use the tools from Chapter 2 to analyse the solution u. The theory of Fourier operators with complex phase allows us to study the propagation of singularities, as Taylor did in [19]. Furthermore, we are able to provide an refined description of the wave front set of the solution u, as well as the wave front sent of the surface waves. And, thanks to our result about clean intersection of Fourier integral operators with complex phase, Theorem 2.50, we are able to proof the continuity of the solution operator on the Sobolev space H^s. We then formulate a theorem that could account for the lost of regularity under Assumption 3.5 with respect to the case where the UKL condition holds.

3.3.1 Propagation of singularities

We know that all the meaningful information about the singularities of the solution operator comes from the points where the phase functions

$$\psi_j(x,z,y,\zeta) = (z-y)\zeta + \lambda_j(\zeta)x$$

are real-valued. However, different eigenvalues of $\mathcal{A}(\zeta)$ can have different behaviours. At any given point, some eigenvalues can be real while some others have imaginary parts different from zero. Recall that, in view of Definition 3.4, the frequency space is divided into three zones

1. The hyperbolic region $\mathcal{H}$, where the matrix $\mathcal{A}(x,z,\zeta)$ is diagonalizable with real eigenvalues. Then, when $\zeta \in \mathcal{H}$, all the $\psi_j(x,z,\zeta)$ are real-valued.

2. The elliptic region $\mathcal{E}$, where p eigenvalues of $\mathcal{A}(x,z,\zeta)$ have strictly positive imaginary parts, while the rest have strictly negative imaginary part. Thanks to the choices we made in Subsection 3.2.1, $\zeta \in \mathcal{E}$ implies that $\Im\psi_j(x,z,\zeta) > 0$, for all j.

3. The mixed region $\mathcal{M}$, where both kinds of eigenvalues are possible. Therefore, all we can tell is that $\Im\psi_j \geq 0$.

Since we need to study each oscillatory integral separately, we need to further subdivide these regions. For each j, denote by $\mathcal{H}_j$ the set of frequencies $\zeta \in \mathbb{R}^n \setminus 0$ where the function ψ_j is real valued; and by $\mathcal{E}_j$ the set where $\Im\psi_j > 0$. It follows that

$$\mathcal{H} = \cap_{j=1}^{p}\mathcal{H}_j \quad \text{and} \quad \mathcal{E} = \cap_{j=1}^{p}\mathcal{E}_j.$$

With this in mind, we compute the critical set of each phase function and the positive Lagrangian manifold associated to each oscillatory integral. Denoting by C_j the critical set of ψ_j, one can easily see that, for each j,

$$C_j = \left\{(x, z, z + \nabla\lambda_j(\zeta)x, \zeta) \colon (x, z) \in \mathbb{C}^{1+n}, \zeta \in \mathbb{C}^n \setminus 0\right\},$$
$$C_{j\mathbb{R}} = \left\{(x, z, z + \nabla\lambda_j(\zeta)x, \zeta) \colon (x, z) \in \mathbb{R}^{1+n}_+, \zeta \in \mathcal{H}_j\right\}.$$

Then the underlying positive Lagrangian manifold is

$$\Lambda_j = \left\{(x, z, z + \nabla\lambda_j(\zeta)x, \lambda_j(\zeta), \zeta, -\zeta) \colon (x, z, \zeta) \in \mathbb{C}^{1+n} \times (\mathbb{C}^n \setminus 0)\right\},$$

whose intersection with the real domain is a real manifold,

$$\Lambda_{j\mathbb{R}} = \left\{(x, z, z + \nabla\lambda_j(\zeta)x, \lambda_j(\zeta), \zeta, -\zeta) \colon (x, z) \in \mathbb{R}^{1+n}_+,\ \zeta \in \mathcal{H}_j\right\}. \tag{3.3.2}$$

Unfortunately this manifold is not good enough. Not only it is open, but its closure $\overline{\Lambda_{j\mathbb{R}}}$ may not be smooth, as we have no way to guarantee that $\nabla\lambda_j(\zeta)$ is defined at the boundary. To illustrate this, lets go back momentarily to the equation of linear elasticity in Section 1.3. There, we had

$$\lambda_1(\zeta) = \sqrt{\frac{\tau^2}{\mu} - \eta^2}, \qquad \lambda_2(\zeta) = \sqrt{\frac{\tau^2}{\lambda + 2\mu} - \eta^2}$$

and

$$\mathcal{H}_1 = \left\{|\tau| > \mu^{1/2}|\eta|\right\} \Rightarrow \partial\mathcal{H}_1 = \left\{|\tau| = \mu^{1/2}|\eta|\right\},$$
$$\mathcal{H}_2 = \left\{|\tau| > (\lambda + 2\mu)^{1/2}|\eta|\right\} \Rightarrow \partial\mathcal{H}_2 = \left\{|\tau| = (\lambda + 2\mu)^{1/2}|\eta|\right\}.$$

Taking the closure $\overline{\Lambda_{j\mathbb{R}}}$ requires evaluating $\nabla\lambda_j$, at points where the $\lambda_j = 0$ and their gradients

$$\nabla\lambda_j(\zeta) = \frac{1}{\lambda_j(\zeta)}\,(\tau, -\eta)$$

are singular. In this case, the way around the obstacle is fairly easy, and we have inadvertently made the correct adjustments. Note that, for $j = 1, 2$,

$$\partial\mathcal{H}_j \subseteq \mathcal{G} = \{(\tau, \eta) \colon \lambda_1 = 0 \text{ or } \lambda_2 = 0\},$$

and that the support of the amplitude functions does not meet this region, because we have carry on the construction away from the glancing points. Furthermore, the regions $\mathcal{H}_j$ are separated from the corresponding $\mathcal{E}_j$ by $\mathcal{G}$, as it is precisely at the glancing region where the eigenvalues λ_j change from real to pure imaginary. In other words

$$\mathcal{G} = \bigcup_j \partial\mathcal{H}_j = \bigcup_j \partial\mathcal{E}_j.$$

Another nice aspect of the problem of linear elasticity is that the manifolds $\Lambda_{j\mathbb{R}}$ are Lagrangian, which, as explained in Chapter 2, is not always true. These useful features may not be true in general but, since Rayleigh waves serve as a model for hyperbolic surface waves, we believe it is reasonable to assume it. In summary, we make the following assumptions

Assumption 3.15. *The BVP* (3.0.1) *is such that the following hold:*

1. *The set of glancing frequencies satisfy* $\bigcup_{j=1}^{p} \partial\mathcal{H}_j \subseteq \mathcal{G}$ *and* $\bigcup_{j=1}^{p} \partial\mathcal{E}_j \subseteq \mathcal{G}$.
2. *For each* j*, the manifold* $\Lambda_{j\mathbb{R}}$ *in* (3.3.2) *is Lagrangian.*

Remark 3.16. *Under this condition, the elliptic region does not meet any of the* $\mathcal{H}_j$*.*

Remark 3.17. *The conditions of Assumption* 3.15 *would be satisfied whenever there are square root eigenvalues.*

On the other hand, the wave front set of K is the union $\bigcup_{j=1}^{p} \mathrm{WF}\left(K_j\right)$, with

$$\mathrm{WF}\left(K_j\right) \subseteq \left\{(x,z,z+\nabla\lambda_j(\zeta)x,\lambda_j(\zeta),\zeta,-\zeta)\colon (x,z,\zeta) \in \mathrm{supp}(a_j),\ \zeta \in \mathcal{H}_j\right\}.$$

This set is closed if we assume that, for each j, $\mathrm{supp}(a_j)$ can be decomposed into two disjoint sets

$$\mathrm{supp}(a_j)_{\mathcal{H}} = \left\{(x,z,\zeta) \in \mathrm{supp}(a_j)\colon \zeta \in \mathcal{H}_j\right\},$$
$$\mathrm{supp}(a_j)_{\mathcal{E}} = \left\{(x,z,\zeta) \in \mathrm{supp}(a_j)\colon \zeta \in \mathcal{E}_j\right\},$$

which can be done thanks to Assumption 3.15. We then have $\mathrm{WF}\,(K) \subseteq M$, with

$$M = \bigcup_{j=1}^{p} \left\{ (x, z, z + \nabla\lambda_j(\zeta)x, \lambda_j(\zeta), \zeta, -\zeta) \colon (x, z, \zeta) \in \operatorname{supp}(a_j)_{\mathcal{H}} \right\}. \tag{3.3.3}$$

We now turn our attention to understanding the wave front set of the surface waves. The analysis in [19] can be applied, with increased difficulty of having many eigenvalues, which are real on different open sets. From construction, we know that F solves the equation $T(F) = g$, where T is a pseudodifferential operator of order 0. Thus, the standard results apply and we have

$$\begin{aligned} &\mathrm{WF}\,(F) \subseteq \mathrm{Char}(T) \cup \mathrm{WF}\,(g), \\ &\mathrm{Char}(T) = \{(z, \zeta) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) : \det \sigma(T) = 0\} = \{(z, \zeta_0) \colon z \in \mathbb{R}^n\}. \end{aligned} \tag{3.3.4}$$

This description can be refined if we consider the bicharachteristics flow. This way, we obtain an analog to Lemma 1.5.

Lemma 3.18. *The wave front set of the surface waves is contained in the set*

$$\Sigma := \mathrm{WF}\,(g) \cup \{\textit{null-bicharacteristics of } \Delta = \det \sigma(T) \textit{ passing over } \mathrm{WF}\,(g)\}.$$

Proof. For a pseudodifferential operator, the propagation of singularities is a well-known phenomenon. Hence, we know that $\mathrm{WF}\,(F) \subseteq \Sigma$. On the other hand, surface waves solutions are of the form

$$u_{sw}(x, z) = e^{iz\zeta_0} V, \quad V = e^{-\Im\lambda_j(\zeta_0)x} \mathcal{F}^{-1} \left(a_j \widehat{F} \right)(x, z), \text{ for some } j$$

where $\zeta_0 \in \mathcal{E}$ is the point where the Lopatinskii determinant Δ vanishes, according to Assumption 3.5. Recall that, in $\mathcal{E}$, all $\lambda_j = \Im\lambda_j$ are strictly positive and the amplitudes a_j are smooth. Then, it follows that the singularities of $u_{sw}(x, z)$ are exactly the singularities of F. Thus, $\mathrm{WF}\,(u_{sw}) = \mathrm{WF}\,(F) \subseteq \Sigma$. □

Remark 3.19. *The null-bicharacteristic curves mention in the lemma are of the form*

$$(z(s), \zeta_0) = \left(\left(\frac{\partial \Delta}{\partial \zeta} \right) s + z(0), \zeta_0 \right),$$

with $\zeta_0 \in \mathcal{E}$ determined by Assumption 3.5, as before.

Finally, we give a description of the wave front set of the solution of the BVP (3.0.1). Since our approximation u differs from the real solution only by a smooth remainder, it is enough to consider $\mathrm{WF}(u)$. To simplify the notation, write

$$(x,z) \in X = \mathbb{R}_+ \times \mathbb{R}^n \quad \text{and} \quad y \in Y = \mathbb{R}^n.$$

Then, the solution u satisfies the equation $u = S(F)$, where S is the continuous operator from $\mathscr{C}_0^\infty(Y)$ to $\mathcal{D}'(X)$ with distributional kernel K in (3.3.1).

Theorem 3.20. *The wave front sent of the solution u is contained in the set*

$$\bigcup_{j=1}^{p} \left\{ \left(x,z,\lambda_j(\zeta),\zeta\right) \in T^*X \setminus 0\colon\ \left(z+\nabla\lambda_j(\zeta)x,\zeta\right) \in \mathrm{WF}(g) \right\}.$$

Proof. First recall that $u = \sum_{j=1}^{p} u_j$. Then, $\mathrm{WF}(u) \subseteq \bigcup_{j=1}^{p} \mathrm{WF}\left(u_j\right)$. The theorem follows form this and the so called *wave front relations* (see [8] or Appendix C.2), which state that, for each j,

$$\mathrm{WF}\left(u_j\right) \subseteq \mathrm{WF}'(S_j) \circ \mathrm{WF}\left(F_j\right) \cup \mathrm{WF}'_X(S_j).$$

In this case, thanks to equation (3.3.3), we have

$$\begin{aligned}\mathrm{WF}'(S_j) = \{ &\left(\left(x,z,\lambda_j(\zeta),\zeta\right), \left(z+\nabla\lambda_j(\zeta)x,\zeta\right)\right) \in (T^*X \times T^*Y) \setminus 0\colon j = 1,\dots,p \\ &\text{and } (x,z,z+\nabla\lambda_j(\zeta)x,\lambda_j(\zeta),\zeta,-\zeta) \in \mathrm{WF}(K)\},\end{aligned}$$

and

$$\begin{aligned}\mathrm{WF}'_X(S_j) = \{ &\left(x,z,\lambda_j(\zeta),\zeta\right) \in T^*X \setminus 0\colon \\ &\left(\left(x,z,\lambda_j(\zeta),\zeta\right), \left(z+\nabla\lambda_j(\zeta)x,0\right)\right) \in \mathrm{WF}'(S)\} = \emptyset.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{WF}\left(u_j\right) &\subseteq \mathrm{WF}'(S_j) \circ \mathrm{WF}\left(F_j\right) \\ &= \left\{ \left(x,z,\lambda_j(\zeta),\zeta\right) \in T^*X \setminus 0\colon\ \left(z+\nabla\lambda_j(\zeta)x,\zeta\right) \in \mathrm{WF}(F) \subseteq \Sigma \right\}.\end{aligned}$$

Finally, note that due to the definition of Σ, the points in $\mathrm{WF}\left(u_j\right)$ must satisfy one of two conditions: either $\left(z+\nabla\lambda_j(\zeta)x,\zeta\right) \in \mathrm{WF}\left(g\right)$ or it lies in some null-bicharacteristic curve. But, the second condition means that $\zeta=\zeta_0 \in \mathcal{E}$, which is forbidden by equation (3.3.3). □

3.3.2 H^s continuity

As already mentioned, surface wave solutions are responsible for some loss of regularity. The aim of this section is to give a first result in this direction. We beging by showing the continuity of the solution operator in the Sobolev spaces H^s.

To this end, we keep the notation of the previous section and study the composition of the solution operator with its adjoint. For the composition to be well defined, either Assumption 2.33 or Assumption 2.46 should hold. To verify these assumptions, we need to know the phase function of the adjoint operator. Consider an operator $A : \mathscr{C}_c^\infty(Y) \to \mathscr{C}^\infty(X)$, with distributional kernel

$$K(x,z,y,\zeta) = \int e^{i\psi(x,z,y,\zeta)} a(x,z,\zeta)\, d\zeta.$$

Then $A^* : \mathscr{C}_c^\infty(X) \to \mathscr{C}^\infty(Y)$ is the operator with kernel

$$K^*(y,x,z,\zeta) = \overline{K(x,z,y,\zeta)} = \int e^{-i\overline{\psi}(y,x,z,\zeta)} \overline{a(x,z,\zeta)}\, d\zeta,$$

where $\overline{a(x,z,\zeta)}$ denotes the complex conjugate and $-\overline{\psi} = -\Re\psi + i\Im\psi$. In our case, we have a sum of integrals K_j of this form, with $\psi_j(x,z,y,\zeta) = (z-y)\zeta + \lambda_j(\zeta)x$. Then, the phase function of the corresponding K_j^* is

$$\varphi_j(y,x,z,\zeta) := -\overline{\psi_j}(y,x,z,\zeta) = (y-z)\zeta - \overline{\lambda_j}(\zeta)x.$$

For each j, we want to understand the composition $A_j := S_j^* \circ S_j$, where S_j denotes the operator with kernel K_j. Note that, in principle, A_j maps $\mathscr{C}_c^\infty(Y) \to \mathscr{C}^\infty(Y)$.

In the following, $\Lambda_j^* \subseteq (T^*(Y\times X)\setminus 0)^\sim$ and $\Lambda_j \subseteq (T^*(X\times Y)\setminus 0)^\sim$ denote the positive Lagrangian manifolds associated to each K_j^* and K_j, respectively. Namely,

$$\begin{aligned} \Lambda_j^* &= \left\{ (y,x,y-\nabla\lambda_j(\zeta)x,\zeta,-\lambda_j(\zeta),-\zeta) \colon (x,y,\zeta) \in \mathbb{C}^{1+n} \times (\mathbb{C}^n\setminus 0) \right\}, \\ \Lambda_j &= \left\{ (x,y-\nabla\lambda_j(\zeta)x,y,\lambda_j(\zeta),\zeta,-\zeta) \colon (x,y,\zeta) \in \mathbb{C}^{1+n} \times (\mathbb{C}^n\setminus 0) \right\}. \end{aligned} \tag{3.3.5}$$

We also need to consider the canonical relations

$$C_j^* = \{((y,\zeta),(x,y-\nabla\lambda_j(\zeta)x,-\lambda_j(\zeta),-\zeta))\} \subseteq (T^*Y\setminus 0)^\sim \times (T^*X\setminus 0)^\sim,$$
$$C_j = \{((x,y-\nabla\lambda_j(\zeta)x,\lambda_j(\zeta),\zeta),(y,-\zeta))\} \subseteq (T^*X\setminus 0)^\sim \times (T^*Y\setminus 0)^\sim.$$

Recall that $C_j^* = (\Lambda_j^*)'$ and $C_j = (\Lambda_j)'$.

Lemma 3.21. *The operators $S_j^* \in I_{cl}^{-1/4}(Y\times X,\Lambda_j^*)$ and $S_j \in I_{cl}^{-1/4}(X\times Y,\Lambda_j)$ satisfy Assumption* 2.46. *In particular, $C_j^*\times C_j$ and $\Delta_X = (T^*Y\times \operatorname{diag}(T^*X)\times T^*Y)^\sim$ intersect cleanly at real points.*

Proof. For any j, denote by $C_{\mathbb{R}}$ and $\Delta_{X\mathbb{R}}$ the restriction of $C = C_j^*\times C_j$ and Δ_X to the real domain. Then, $p\in C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}}$ has the form

$$p = ((y,\zeta,x,y-\nabla\lambda_j(\zeta)x,-\lambda_j(\zeta),-\zeta),(x,y-\nabla\lambda_j(\zeta)x,-\lambda_j(\zeta),-\zeta,y,\zeta))$$

It is not hard to check that the intersection $C\cap\Delta_X$ is clean at real points. Indeed, given $p\in C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}}$ any vector $v\in T_p(C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}})$ is by construction tangent to both $C_{\mathbb{R}}$ and $\Delta_{X\mathbb{R}}$ at p. Then, $T_p(C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}})\subseteq T_p(C_{\mathbb{R}})\cap T_p(\Delta_{X\mathbb{R}})$. The opposite relation follows after observing that the only difference between a vector $v\in T_p(C_{\mathbb{R}})$ and a vector $w\in T_p(C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}})$ is that v depends on $(y,x,z,\zeta,y',x',z',\zeta')$ and w depends on (y,x,z,ζ) . But, the intersection of $T_p(C_{\mathbb{R}})$ with $T_p(\Delta_{X\mathbb{R}})$ reduces the dimension, and we lose the extra variables. Thus, $T_p(C_{\mathbb{R}})\cap T_p(\Delta_{X\mathbb{R}})$ consists of the subset of vectors $v'\in T_p(C_{\mathbb{R}})$ that depend only on (y,x,z,ζ). Which implies that $T_p(C_{\mathbb{R}})\cap T_p(\Delta_{X\mathbb{R}})\subseteq T_p(C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}})$.

Since it will be important later, we compute here the excess of the intersection. By definition,

$$e = \operatorname{codim}(C_{\mathbb{R}}) + \operatorname{codim}(\Delta_{X\mathbb{R}}) - \operatorname{codim}(C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}}).$$

The set $C_{\mathbb{R}}$ has dimension $2(1+2n)$, because both of the canonical relations have dimension $1+2n$. And the intersection $C_{\mathbb{R}}\cap\Delta_{X\mathbb{R}}$ has dimension $1+2n$. Then,

$$e = (2+4n)+(2+3n)-3(1+2n) = 1+n.$$

Finally, note that the dimension of X is precisely $1+n$. So, we expect (x,z) to be the excess variables. $\square$

A similar argument shows that the composition $S_j \circ S_j^*$ is also defined. In that case, Assumption 2.33 is satisfied with $C_1 \times C_2 = C_j \times C_j^*$ and $D = T^*X \times \operatorname{diag}(T^*Y) \times T^*X$. But, the resulting operator is not helpful for our goal.

Proposition 3.22. *The composition $A_j := S_j^* \circ S_j$ defines a pseudodifferential operator of order $n/2$.*

Proof. The previous lemma and Theorem 2.50 tell us that $A_j := S_j^* \circ S_j$ defines an operator in the class $I_{cl}^{m+e/2}(Y \times Y, \Lambda)$, with

$$m = 2\left(-\frac{1}{4}\right), \quad e = 1+n,$$

and Λ the positive Lagrangian manifold parameterized by the clean phase function

$$\Phi(y,y',\omega) = -\overline{\psi}_j(y,x,z,\zeta) + \psi_j(x,z,y',\eta), \quad \omega = \omega(x,z,\zeta,\eta).$$

More precisely, Λ is the image under a linear transformation of the set

$$C_\Phi \sim \{((y,x,z,\zeta),(x,z,y',\eta)) \colon -\partial_x\overline{\psi}_j + \partial_x\psi_j = 0,\ -\partial_z\overline{\psi}_j + \partial_z\psi_j = 0\}.$$

Restricting to real points, we see from the definition of ψ, that

$$C_{\Phi\mathbb{R}} \sim \{((y,x,z,\zeta),(x,z,y',\eta)) \colon \zeta = \eta,\ \lambda(\zeta) = \lambda(\eta) \in \mathbb{R}\}.$$

Under these conditions, $\Phi \sim (y-y')\eta$, which makes A_j a pseudo differential operator of order $m + e/2 = n/2$. $\square$

Remark 3.23. *For each j, the operator A_j is elliptic. We can see from equations* (2.2.13) *and* (3.2.6)*, that the principal part of the amplitude of A_j is $b_{j0} = |r_j(\zeta)|^2|\zeta|^{-(1+n)}$. Since the eigenvectors $r_j(\zeta)$ cannot be zero, b_{j0} never vanishes.*

We can now use this result to study the mapping properties of the solution operator S. Recall that S is equal to the sum $\sum_{j=1}^{p} S_j$.

Theorem 3.24. *The solution operator S is continuous from $H^{\frac{n}{4}}(Y)$ to $L^2(X)$.*

Proof. Let A_j denote the operator $S_j^* S_j$. From the previous proposition, we know that, for each j, A_j is a pseudodifferential operator of order n. As such, it maps $H^{\frac{n}{2}}(Y) \to L^2(Y)$ continuously. Then, if $u \in H^{\frac{n}{2}}(Y)$, it holds

$$\begin{aligned}\|S_j u\|^2_{L^2(X)} &= (S_j^* S_j u, u)_{L^2(Y)} = (A_j u, u)_{L^2(Y)} \\ &\leq \|A_j u\|_{H^{-\frac{n}{4}}(Y)} \|u\|_{H^{\frac{n}{4}}(Y)} \leq C\|u\|^2_{H^{\frac{n}{4}}(Y)}.\end{aligned}$$

It follows that $S : H^{\frac{n}{4}}(Y) \to L^2(X)$ continuously. □

Thanks to the ellipticity of the operators A_j, the previous result is sharp. We conclude the chapter with a result concerning the regularity of the solution u.

Theorem 3.25. *Assume that $f \equiv 0$ and $g \in H^{\frac{n}{4}}(\mathbb{R}^n)$. Then, there exists a solution u to the BVP* (3.0.1)*, unique mod $\mathscr{C}^\infty$, such that*

$$u(x,z) \in L^2\left(\mathbb{R}_+ \times \mathbb{R}^n\right) \quad \textit{and} \quad u|_{x=0} \in H^{\frac{n}{4}}(\mathbb{R}^n).$$

Proof. From our construction, we know that the solution u is given by the equation

$$u(x,z) = \sum_{j=1}^{p} S_j(F_j),$$

where $F = (F_1, \ldots, F_p)^T$ solves $T(F) = g$, for a pseudodifferential operator T of order 0. Then, if $g \in H^{\frac{n}{4}}(Y)$, each F_j also belongs to $H^{\frac{n}{4}}(Y)$. Then, we get from the action of S that $u \in L^2(\mathbb{R}_+ \times \mathbb{R}^n)$.

On the other hand, it is clear from the definition of the phase functions, that $u(0,z)$ is given by the action of a pseudodifferential operator of order 0 on $F \in H^{\frac{n}{4}}(Y)$. Then, $u|_{x=0} \in H^{\frac{n}{4}}(\mathbb{R}^n)$. □

This theorem does not imply that the problem is strongly stable. For that, we would need to establish an energy estimate of the form

$$\|u\|_{L^2} + \|u|_{x=0}\|_{L^2} \leq C\|g\|_{L^2},$$

which is not the case here. In fact, we know that the problem is *weakly* stable, but we are not able to provide a suitable energy estimate. The reason is that we cannot control the norm of the solution u by the norm of the boundary data g. From our construction, we only know that

$$\|u\|_{L^2} \leq \|F\|_{H^{\frac{n}{4}}} \quad \text{and} \quad \|g\|_{H^{\frac{n}{4}}} = \|T(F)\|_{H^{\frac{n}{4}}} \leq \|F\|_{H^{\frac{n}{4}}}.$$

These inequalities tell us that the lost of regularity must come from the action of the boundary operator T. We believe that the properties of the operator T will lead to a priori estimates for the equation $Tv = g$, which in turn will give a suitable definition of *weak stability* for hyperbolic problems with surface waves. A sketch of a possible argument is presented in Appendix B. The approach is based on a fact that T is of real principal type in the sense of Definition 3.11. However, this estimate is only valid for problems satisfying Assumption 3.10, which may not hold in general.

3.4 Boundary value problems with variable coefficients

At the beginning of this chapter, we restricted ourselves to the analysis of a first order constantly hyperbolic operator with constant coefficients. This was done in order to simplify the presentation, but the assumption is not essential for our construction. We now consider the boundary value problem of the form

$$\begin{aligned} &L(t,x,y,\partial_t,\partial_x,\partial_y)u = 0 \quad \text{in} \quad (0,T)\times\mathbb{R}^{1+d}_+, \\ &B(t,y)u = g(t,y) \qquad\quad\;\; \text{on} \quad (0,T)\times\mathbb{R}^d, \end{aligned} \tag{3.4.1}$$

where L is a symmetrizable hyperbolic differential operator with constant multiplicities and symbol

$$L(t,x,y,\tau,\xi,\eta) = \tau - A_0(t,x,y)\xi - \sum_{j=1}^{d} A_j(t,x,y)\eta_j.$$

The coefficients $A_j \in \mathscr{C}^\infty\left((0,T)\times\mathbb{R}^{1+d}_+\right)$, $j = 0,1,\ldots,d$, and $B \in \mathscr{C}^\infty\left((0,T)\times\mathbb{R}^d\right)$ are matrices of size $N \times N$ and $p \times N$, respectively. As before, we assume that the boundary is non-characteristic, and that p is the number of incoming characteristics. Moreover, we assume that the coefficients of A_j and B are constant outside a compact set. Then, the problem (3.4.1) can be reduced to

$$\begin{aligned} \left[\partial_x - \mathcal{A}(t,x,y,\partial_t,\partial_y)\right] u = 0 \quad &\text{in} \quad (0,T)\times\mathbb{R}^{1+d}_+, \\ B(t,y)u = g(t,y) \quad &\text{on} \quad (0,T)\times\mathbb{R}^d, \end{aligned} \tag{3.4.2}$$

with

$$\mathcal{A}(t,x,y,\tau,\eta) = A_0^{-1}(t,x,y)\left(\tau I_N - \sum_{j=1}^{d} A_j(t,x,y)\eta_j\right).$$

In this section, we show that the arguments presented in the case of constant coefficients, can be applied to the BVP (3.4.2). The construction extends in a straightforward manner, the main difference is that the new phase functions and amplitudes are more involved.

Additionally, the results from Section 3.3 are reviewed and reformulated to fit the boundary value problem with variable coefficient. We should clarify that the behaviour of the solution is essentially the same.

3.4.1 Assumptions

Since we are interested in the BVPs that admit surface waves, we assume that the problem (3.4.2) violates the (UKL) condition in a particular way. To avoid any confusion with the notation, all the necessary conditions are presented below.

For simplicity, we put $z = (t,y) \in \mathbb{R}^n$, $\tau = \sigma - i\gamma$ and $\zeta = (\tau,\eta)$. Also, thanks to the homogeneity of the symbols, we can work on the semi-spheres

$$\begin{aligned} \mathscr{X} &= \left\{\zeta = (\tau,\eta) \in \mathbb{R}^n \colon \sigma^2 + \gamma^2 + |\eta|^2 = 1,\ \gamma \geq 0\right\}, \\ \mathscr{X}_0 &= \{\zeta \in \mathscr{X} : \gamma = 0\}. \end{aligned}$$

The hyperbolicity now reads

Assumption 3.26 ([18]). *The eigenvalues of the matrix $A_0(x,z)\xi + \sum_{j=1}^{d} A_j(x,z)\eta_j$ are all real and semi-simple, and their multiplicity is independent of (x,z,ξ,η).*

Under this assumption, the boundary value problem (3.4.2) satisfies the block structure condition (Theorem 3.3). Thus, the classification of boundary points given by Definition 3.4 remains valid for the points $(x, z, \zeta) \in \mathbb{R}^{1+n}_{+} \times \mathscr{X}_0$.

Let $E_{+}(x, z, \zeta)$ denote the stable subspace of $\mathcal{A}(x, z, \zeta)$, that is the direct sum of the generalized eigenspaces of $\mathcal{A}$ associated to eigenvalues of positive imaginary part. We can locally choose a basis $\{r_1(x, z, \zeta), \ldots, r_q(x, z, \zeta)\}$ of $E_{+}(x, z, \zeta)$, which is smooth when $\gamma > 0$ and extends continuously to $\gamma = 0$. Then, for all (z, ζ) in $\mathbb{R}^n \times \mathscr{X}$ we can define the Lopatinskii determinant

$$\Delta(z, \zeta) := \det(B(z, \zeta) r_1(0, z, \zeta), \ldots, B(z, \zeta) r_p(0, z, \zeta)). \tag{3.4.3}$$

Now our *weakly regularity* condition, that is the assumption that guarantees the existence of surface waves, reads

Assumption 3.27. *Points of the form $(0, z, \zeta)$, satisfy the following conditions:*

(KL) ([5]) *For all $\zeta \in \mathscr{X}$ with $\gamma > 0$, and all $\zeta \in \mathscr{X}_0$ outside the elliptic region, the Lopatinskii determinant $\Delta(z, \zeta)$ does not vanish. Moreover,*

$$\dim E_{+}(0, z, \zeta) = p.$$

(SW) *If there exists an elliptic point (z_0, ζ_0) such that $\Delta(z_0, \zeta_0) = 0$, then $\partial_\sigma \Delta(z_0, \zeta_0) \neq 0$.*

3.4.2 Construction of the solution

As before, we want to find an approximated solution to the BVP (3.4.2) of the form

$$U = \sum_{j=1}^{p} U_j, \quad U_j = \int e^{i\phi_j(x,z,\zeta)} a_j(x, z, \zeta) \widehat{F}_j(z, \zeta)\, d\zeta, \tag{3.4.4}$$

where the functions ϕ_j, a_j and F_j satisfy the same conditions as in the constant coefficient case. Namely, the complex-valued phase functions ϕ_j should satisfy certain complex-valued eikonal equations, the amplitudes a_j solve the transport equations arising from the method of geometric optics, and the scalar valued distributions F_j are determined by the boundary condition.

Unlike the constant coefficient case, the equations involving ϕ_i and a_i cannot be solved explicitly. Instead, we follow the ideas of Trèves in [20, Chapter XI]. This allows us to determine the phase functions and amplitudes up to a smooth remainder.

First, we need to determine the phase functions. For each j, ϕ_j must satisfy the eikonal equation,

$$\begin{aligned} \partial_x \phi_j &= \lambda_j(x, z, \nabla_z \phi_j), \\ \phi_j|_{x=0} &= z\zeta, \end{aligned} \tag{3.4.5}$$

where λ_j is a root in ξ of $D(x, z, \xi, \zeta) = \det(\xi I_N - \mathcal{A}(t, z, \zeta))$, with positive imaginary part. For this equation to make sense, we need to consider an almost analytic extension $\widetilde{\lambda}_j$, which introduces an error. The following arguments are taken from [20], but adapted to our situation.

Since we are considering almost analytic extensions, let $(z, \zeta) \in \mathbb{C}^n \times (\mathbb{C}^n \setminus 0)$. Consider also the *almost* Hamilton-Jacobi equation

$$\begin{aligned} \frac{d\tilde{z}}{dx} &= -\frac{\partial \widetilde{\lambda}_j}{\partial \zeta}, \qquad & \frac{d\tilde{\zeta}}{dx} &= \frac{\partial \widetilde{\lambda}_j}{\partial z}, \\ \tilde{z}|_{x=0} &= z, \qquad & \tilde{\zeta}|_{x=0} &= \zeta. \end{aligned}$$

Whose unique solution is given by

$$\widetilde{z} = \widetilde{z}(x, z, \zeta) \quad \text{and} \quad \widetilde{\zeta} = \widetilde{\zeta}(x, z, \zeta). \tag{3.4.6}$$

These equations describe the flow of the complex vector field

$$H_{\lambda_j} = \sum_{k=1}^{n} \frac{\partial \widetilde{\lambda}_j}{\partial z_k} \frac{\partial}{\partial \zeta_k} - \frac{\partial \widetilde{\lambda}_j}{\partial \zeta_k} \frac{\partial}{\partial z_k}.$$

Note that, unless λ_j is analytic, this is not a Hamiltonian vector field, however, the solution of (3.4.5) can still be obtained by following the flow of H_{λ_j}. Then, we can write

$$\widetilde{\phi}_j(x, z, \zeta) = z\zeta + \int_0^x \widetilde{\lambda}_j(s, z, \widetilde{\zeta}(x, \widetilde{z}(x, z, \eta), \eta))\, ds. \tag{3.4.7}$$

One can easily check that $\widetilde{\phi}_j$ solves, modulo almost analytic functions, the equation

$$\partial_x w = \widetilde{\lambda}_j(x, z, \nabla_z w),$$
$$w|_{x=0} = z\zeta.$$

Denoting by ϕ_i the restriction of $\widetilde{\phi}_j$ to the real domain, a short calculation shows that

$$\partial_x \phi_j - \lambda_j(x, z, \nabla_z \phi_j) = -\lambda_j(0, z, \zeta) \in \mathscr{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)). \tag{3.4.8}$$

We now focus on constructing the amplitudes. Once again, we wish to use arguments from geometric optics. That is, we need to find $c_i \in S^m_{cl}$, for some m, such that

$$(\partial_x - \mathcal{A}(x, z, \partial_z))\, U_j = \int e^{i\phi_j(x,z,\zeta)} c_j(x, z, \zeta) \widehat{F}_j(z, \zeta)\, d\zeta = 0,$$

with

$$c_j = i\left((\partial_x \phi_j) a_j - \mathcal{A}(x, z, \partial_z \phi_j) a_j\right) + \left(\partial_x a_j - \mathcal{A}(x, z, \partial_z) a_j\right) = 0. \tag{3.4.9}$$

Thanks to equation (3.4.5), we can solve

$$(\partial_x \phi_j) a_j - \mathcal{A}(x, z, \partial_z \phi_j) a_j = 0$$

by taking a_j in the λ_j-eigenspace of $\mathcal{A}$. But, since ϕ_j is only an approximated solution, the equality holds only up to a smooth remainder. In view of (3.4.8), we take $a_j^{(0)}(z, \zeta)$, the principal part of a_j to be an eigenvector of $\lambda_j(0, z, \zeta)$. In this case, the previous equation is satisfied modulo a smooth function. Then, the rest of the terms in the asymptotic sum of a_j need to be eigenvectors of $\lambda_j(x, z, \zeta)$ satisfying

$$(\partial_x - \mathcal{A}(x, z, \partial_z))\, a_j^{(l)} = i\lambda_j(0, z, \zeta) a_j^{(l-1)}. \tag{3.4.10}$$

Thus, we can determine the amplitudes $a_j \sim \sum_{l \leq 0} a_j^{(l)}$ satisfying (3.4.9), by solving equation (3.4.10) for each l.

The last step in the construction is to find the distributions F_i, which are the link to the boundary condition. But, since $\phi_j|_{x=0} = z\zeta$ and $a_j^{(0)}$ takes values in the eigenspace of $\lambda_j(0,z,\zeta)$, the arguments in Subsection 3.2.3 remain valid. In particular Theorem 3.13 holds, so we can approximate F, mod $\mathscr{C}^\infty$.

Going forward, we assume that the function U in (3.4.4) is an approximated solution of the boundary value problem (3.4.2). As before, we denote by S the operator satisfying $S(F) = U$. Explicitly, S is the operator with distributional kernel

$$K = \sum_{j=1}^{p} K_j = \sum_{j=1}^{p} \int e^{i\psi_j(x,z,y,\zeta)} a_j(x,z,\zeta)\, d\zeta, \tag{3.4.11}$$

where $\psi_j(x,z,y,\zeta) = (z-y)\zeta + \int_0^x \lambda_j(s,z,\widetilde{\zeta})\, ds$, with $\widetilde{\zeta}$ given by (3.4.6), and $a_j(x,z,\zeta)$ as above.

3.4.3 Analysis of the solution operator

At the beginning of this section, we assumed that the coefficients of $\mathcal{A}$ and B are constant outside a compact set. Thus, we only need to consider (x,z) inside that compact set, lets call it V, because for $(x,z) \in \mathbb{R}^{1+n}_+ \setminus V$, the analysis on the previous section applies. We now consider the sets

$$\begin{aligned} \mathcal{H}_j &= \left\{(x,z,\zeta) \in V \times \mathscr{X}_0 \colon \Im\lambda_j(x,z,\zeta) = 0\right\}, \\ \mathcal{E}_j &= \left\{(x,z,\zeta) \in V \times \mathscr{X}_0 \colon \Im\lambda_j(x,z,\zeta) > 0\right\}. \end{aligned}$$

Then, the real part of the critical set C_j of each ψ_j, is

$$\begin{aligned} C_{j\mathbb{R}} &= \left\{\left(x,z,z+\partial_\zeta\left(\int_0^x \lambda_j(s,z,\widetilde{\zeta})\, ds\right),\zeta\right) : (x,z,\zeta) \in \mathcal{H}_j\right\}, \\ &= \left\{(x,z,2z-\widetilde{z}(x,z,\widetilde{\zeta}),\zeta) \colon (x,z,\zeta) \in \mathcal{H}_j \text{ and } \widetilde{z} \in \mathbb{R}^n\right\}. \end{aligned}$$

And, then the underlying positive Lagrangian manifold is

$$\Lambda_j = \left\{\left(x,z,z+\partial_\zeta\left(\int_0^x \lambda_j(s,z,\widetilde{\zeta})\, ds\right),\lambda_j(x,z,\widetilde{\zeta}) - \lambda_j(0,z,\zeta),\widetilde{\zeta},-\zeta\right)\right\},$$

whose intersection with the real domain is a real manifold,

$$\Lambda_{j\mathbb{R}} = \Big\{ (x, z, 2z - \widetilde{z}(x, z, \widetilde{\zeta}), \lambda_j(x, z, \widetilde{\zeta}) - \lambda_j(0, z, \zeta), \widetilde{\zeta}, -\zeta) \colon \\ (x, z, \zeta) \in \mathcal{H}_j,\ (\widetilde{z}, \widetilde{\zeta}) \in \mathbb{R}^n \times \mathbb{R}^n \Big\}.$$

Recall that $\widetilde{z} = \widetilde{z}(x, z, \zeta)$ and $\widetilde{\zeta} = \widetilde{\zeta}(x, z, \zeta)$ denote the solution to the almost Hamilton equation in (3.4.6). Let us denote by $\mathcal{H}_j^*$ the points in $\mathcal{H}_j$ for which the flow $(\widetilde{z}, \widetilde{\zeta})$ remains real, and by $\operatorname{supp}_{\mathcal{H}_j^*}(a_j)$ the intersection $\operatorname{supp}(a_j) \cap \mathcal{H}_j^*$. If we keep Assumption 3.15, we can see that $\mathrm{WF}\left(K_j\right) \subseteq L_j$, where

$$L_j = \Big\{ (x, z, 2z - \widetilde{z}(x, z, \widetilde{\zeta}), \lambda_j(x, z, \widetilde{\zeta}) - \lambda_j(0, z, \zeta), \widetilde{\zeta}, -\zeta) \colon (x, z, \zeta) \in \operatorname{supp}_{\mathcal{H}_j^*}(a_j) \Big\}.$$

Concerning the singularities of the surface waves, Lemma 3.18 remains valid, however, the statement of Theorem 3.20 needs to be modified to fit the new manifolds $\Lambda_{j\mathbb{R}}$. Recall that

$$\mathrm{WF}\,(U) \subseteq \bigcup_{j=1}^{p} \mathrm{WF}\left(U_j\right).$$

Theorem 3.28. *For each j, we have*

$$\mathrm{WF}\left(U_j\right) \subseteq \Big\{ \Big(x, z, \lambda_j(x, z, \widetilde{\zeta}) - \lambda_j(0, z, \zeta), \widetilde{\zeta}\Big) \in T^*X \setminus 0 \colon \Big(2z - \widetilde{z}, \widetilde{\zeta}\Big) \in \mathrm{WF}\,(g) \Big\}.$$

To finish the analysis of the solution operator, one needs to verify the statement of Theorem 3.25 when the operators are assumed to have variable coefficients. To do so, we first need to show that the composition $S_j^* \circ S_j$, with S_j the operator with kernel K_j in equation (3.4.11), still defines a pseudodifferential operator. Whether or not this is true, depends completely on the properties of the phase functions ψ_j and the positive Lagrangian manifolds Λ_j.

Retracing the arguments presented in the previous section, we obtain the following result.

Theorem 3.29. *Assume that $f \equiv 0$ and $g \in H^{\frac{n}{4}}(\mathbb{R}^n)$. Then, there exists a solution u to the BVP* (3.4.1), *unique mod $\mathscr{C}^\infty$, such that*

$$u(x, z) \in L^2\left(\mathbb{R}_+ \times \mathbb{R}^n\right) \quad \text{and} \quad u|_{x=0} \in H^{\frac{n}{4}}(\mathbb{R}^n).$$

The proof is omitted because it follows the same arguments presented in Section 3.3. In particular, one can see that Lemma 3.21 and Proposition 3.22 hold for the operator S_j with kernel K_j as in equation $(3.4.11)$. It also holds, that the composition $S_j^* \circ S_j$ defines an elliptic pseudodifferential operator.

Remark 3.30. *The arguments in Appendix B remain valid for BVPs with variable coefficients. Thus, the energy estimate proposed there covers the problems studied in this section.*

4 | Concluding remarks

In this thesis, the theory of Fourier integral operators with complex phase was used to study hyperbolic surface waves. By representing the solution operator of the corresponding hyperbolic boundary value problem as an oscillatory integral with complex phase, we are able to improve the analysis of this type of solutions. Specifically, we showed that if a constantly hyperbolic boundary value problem violates the uniform Kreiss-Lopatinskii condition at an isolated elliptic frequency, then the solution to the problem can be approximated by a distribution of the form

$$S(F) := \sum_{j=1}^{p} S_j(F_j) = \sum_{j=1}^{p} \int e^{i\phi_j(x,z,\zeta)} a_j(x,z,\zeta) \widehat{F}_j(\zeta)\, d\zeta,$$

where the vector value distribution $F = (F_1, \ldots, F_p)$ is determined by the boundary condition. After analyzing the operator S, we refined the previously known description of the propagation of singularities (Theorem 3.20 and Theorem 3.28). We also proved a preliminary result concerning the regularity of the solution (Theorem 3.25 and Theorem 3.29). However, the loss of regularity caused by surface waves remains an open question. In Appendix B, we propose a possible answer to this question. Moreover, we conjecture an estimate that is compatible with the results found in the literature for less general settings. Further investigation of the proposed estimate is reserved for future studies. It would also be interesting to consider a less controlled failure of the UKL condition.

An important part of this thesis was devoted to the study of Fourier integral operators with complex phase function. Particularly, we focus on the principal symbol map for this type of operators. After adapting a method from the real-valued theory, we provide in Theorem 2.40 an explicit description of the principal symbol. Furthermore, in Theorem 2.51, we compute the principal symbol after composition under the assumption of clean intersection.

While the study of the clean composition of Fourier integral operators with complex phase is interesting on its own, it also proved useful in applications. The proof of Theorem 3.29 is based on the fact that the clean composition of S_j^* and S_j defines a pseudodifferential operator. In future work, it would be interesting to study the composition under more general geometric assumptions. Geometrical situations that appear in the study of physically relevant problems would be particularly interesting.

Appendices

A Proof of Theorem 3.11

The purpose of this appendix is to show that we can construct a parametrix for the boundary equation $T(F) = g$ in Subsection 3.2.3. Or, equivalently, we show a sketch of the proof of Theorem 3.13, which is taken from [19].

Theorem. *Let T be a $k \times k$ system of pseudodifferential operators of order 0, with principal symbol $\sigma^0(T)(t, x, \tau, \xi)$. Assume that $q(t, x, \tau, \xi) = \det \sigma^0(T)$ is real and $\partial_\tau q \neq 0$ whenever $q = 0$. Suppose that $\operatorname{supp}(g)$ is contained in $t > 0$. Then, one can construct an approximated solution to the equation $Tv = g$, which is unique modulo smooth functions and vanish for $t < 0$.*

Proof. ([19, Section 4]) Let R be the operator with the co-factor matrix of $\sigma^0(T)$ as its symbol. Then, $RT = q + Q$, where Q is a pseudodifferential operator of order -1. With this in mind, we can rewrite the equation $Tv = g$ as

$$(q + Q)v = h, \qquad h = Rg.$$

Denote by $S_\pm$ the sets of characteristic curves of q with $\pm t > 0$, respectively. It is possible to write the identity operator as $I = P_+ + P_- + P_0$, where $P_\pm$ have support inside $S_\pm$, and their principal parts are equal to 1 in a smaller conic neighborhood. Then, the support of P_0 necessarily consist of the points where $q \neq 0$. Hence, $q + Q$ is elliptic on $\operatorname{supp}(P_0)$, and the equation $(q + Q)v_0 = h_0$ can be solved.

We now focus on solving the non-elliptic part. Formally, the solution to the equation $(q+Q)v_+ = h_+$ is

$$v_+ = (q+Q)^{-1}h_+ = i\int_0^\infty \exp\left(is(q+Q)\right)h_+ \, ds.$$

The expression is not defined, but the function $w(s) = \exp\left(is(q+Q)\right)h_+$ solves the equation

$$\partial_s w = i(q+Q)w, \quad w(0) = h_+.$$

Since this is a hyperbolic equation, we can find an approximated solution w_+ by means of geometric optics approximation. It follows that $(q+Q)w_+ = v_+$ modulo smooth functions. Repeating the argument over the support of P_-, we see that

$$v = v_0 + i\int_0^\infty \psi(s)w_+(s)\, ds - i\int_\infty^0 \psi(s)w_-(s)\, ds, \tag{A.1}$$

where $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ is 1 for $|s|$ sufficiently small.

Finally, we check that this function actually solves $Tv = g, \quad \mathrm{mod}\ \mathscr{C}^\infty$. Note that $(q+Q)v = h$ implies $R(Tv-g) = 0$. Multipliying by $\sigma^0(T)$, we get

$$0 = (\sigma^0(T)R)(Tv-g) = q(Tv-g).$$

Then, the result follows from the fact that q has isolated zeros. □

B An energy estimate

The problems studied in this thesis are *weakly regular* in the sense that they cannot satisfy maximal energy estimates in L^2, like the one in equation (1.1.2). Instead, there exists a loss of regularity with respect to the boundary data g. It remains an open problem to establish a suitable notion of week regularity for constantly hyperbolic BVPs that violate the UKL condition at the elliptic region.

A first step in this direction is to find appropriate energy estimates that account for the loss of regularity with respect to the case when the UKL condition holds. In this appendix we propose an energy estimate for BVPs satisfying Assumption 3.27 and Assumption 3.10. The theorem is an extension of Theorem 3.25.

Theorem B.1. *Assume that $f \equiv 0$ and $g \in H^s(\mathbb{R}^n)$. Then, there exists a solution $u \in H^{s-\frac{n}{4}}(\mathbb{R}_+ \times \mathbb{R}^n)$ of the BVP* (3.0.1) *which is unique $\mod \mathscr{C}^\infty$. Moreover, it satisfies*

$$\|u\|_{H^{s-\frac{n}{4}}(\mathbb{R}_+\times\mathbb{R}^n)} + \|u|_{x=0}\|_{H^{s-1}(\mathbb{R}^n)} \leq C\|g\|_{H^s(\mathbb{R}^n)}. \tag{B.1}$$

Proof. The proof relies on the the properties of the solution operator $S = \sum_{j=1}^p S_j$ and the boundary operator T constructed in Section 3.2. First recall that the approximated solution u has the form $\sum_{j=1}^p u_j$, with

$$u_j = \sum_{j=1}^p S_j(F_j), \quad \text{and} \quad T(F) = g,\ F = (F_1, \dots, F_p).$$

The pseudodifferential operator T is of real principal type at the elliptic region $\mathcal{E}$, and elliptic everywhere else. Then, assuming that $g \in H^s(\mathbb{R}^n)$, it follows that $F \in H^s(\mathbb{R}^n)$ and that

$$\|F\|_{s-1} \leq C\|g\|_s. \tag{B.2}$$

We also know that $A_j = S_j^* \circ S_j$ is an elliptic pseudodifferential operator of order $n/2$ (see Remark 3.23). Then, for each j, it holds $S_j^*(u_j) = A_j(F_j)$ and

$$\|F_j\|_s \leq C\left(\|S_j^*(u_j)\|_{s-\frac{n}{2}} + \|F_j\|_{s-1}\right). \tag{B.3}$$

Furthermore, it follows from Theorem 3.24 that

$$\|S_j^*(u_j)\|_{s-\frac{n}{2}} \leq C\|u_j\|_{s-\frac{n}{4}} \quad \text{and} \quad \|u_j\|_{s-\frac{n}{4}} = \|S_j(u_j)\|_{s-\frac{n}{4}} \leq C\|F_j\|_s. \tag{B.4}$$

Combining inequalities (B.2)-(B.4), we obtain $\|u_j\|_{s-\frac{n}{4}} \leq C\|g\|_s$. It follows that

$$\|u\|_{s-\frac{n}{4}} \leq C\|g\|_s. \tag{B.5}$$

On the other hand, the restriction of S to the boundary $x = 0$ is a pseudodifferential operator of order 0. Thus,

$$\|u|_{x=0}\|_{s-1} \leq C\|F\|_{s-1} \leq C\|g\|_s. \tag{B.6}$$

Estimate (B.2) is obtained by adding (B.5) and (B.6). □

C Further theoretical background

This Appendix contains additional results that are relevant for the context of this thesis. The theory presented here is essential in the study of pseudodifferential operators and Fourier integral operators. As such, it was used in the previous chapters without further explanation. In the spirit of making this thesis as self-contained as possible, we collect some of the most important aspects of the necessary theoretical background.

C.1 Overview of symplectic geometry

In this section, we present the basic definitions concerning *Lagrangian manifolds*, which are an important part of the theory of Fourier integral operators. In a way, this appendix represents the standard geometrical setting for the real-valued theory that was generalized in the first two sections of the second chapter of this thesis. The content of this section is completely taken from [20, Chapter VII.].

Definition C.1. *A vector space V, over $\mathbb{R}$ or $\mathbb{C}$, with a non-degenerate antisymmetric bilinear form ω is called a symplectic vector space. We write (V, ω) to denote this space.*

That ω is non-degerate means that, if $\omega(x, y) = 0$ for all $y \in V$, then $x = 0$. It can be shown that every finite dimensional symplectic vector space is of even dimension $2n$. Now, let W be any subspace of V, we denote by $W^\perp$ the orthogonal complement of W with respect to ω. Namely,

$$W^\perp = \{x \in V \colon \omega(x, y) = 0,\ \forall y \in W\}.$$

Definition C.2. *A subspace W of a vector space V is called Lagrangian (resp. isotropic, resp. coisotropic) if $W = W^\perp$ (resp. $W \subset W^\perp$, resp. $W^\perp \subset W$).*

By construction, Lagrangian subspaces are of dimension $n = (\dim V)/2$. We say that two Lagrangian subspaces L_1 and L_2 are *transverse* if $V = L_1 \oplus L_2$.

Definition C.3. *A symplectic basis in (V, ω) is a basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of V, such* *that*

$$\omega(e_i, e_j) = \omega(f_i, f_j) = \omega(e_i, f_j) - \delta_{ij}.$$

Proposition C.1. *Let L_1 and L_2 be transverse Lagrangian subspaces of V. Then, there are bases $e = (e_1, \ldots, e_n)$ of L_1 and $f = (f_1, \ldots, f_n)$ of L_2, such that (e, f) is a symplectic basis of V.*

Definition C.4. *A symplectic manifold is a smooth manifold X, equipped with a closed non-degenerate 2-form ω, such that for all $x \in X$, $(T_x X, \omega_x)$ define a symplectic vector space.*

It follows from the results above that a symplectic manifold X is of even dimension. The fundamental example of a symplectic manifold is given by the cotangent bundle T^*M of an arbitrary smooth manifold M of dimension n. Let $(x_1, \ldots, x_n)$ be local coordinates in M and $(\xi_1, \ldots, \xi_n)$ the associated coordinates in the tangent space. Then, the symplectic form in T^*M is

$$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx_j.$$

Definition C.5. *Let (X, ω) be a symplectic manifold. A submanifold Y is called Lagrangian (resp. isotropic, resp. coisotropic) if, for every $y \in Y$, this is true for $T_y Y$ as a subspace of the symplectic space $(T_y X, \omega_y)$.*

Proposition C.2. *If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are symplectic coordinates in an open subset U of X. Then, in U, the canonical volume form of X takes the form*

$$\Omega = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n.$$

C.2 The wave front set

Broadly speaking, the wave front set $\mathrm{WF}(u)$ characterizes the singularities of a distribution u. The concept is useful to study the propagation of singularities due to the action of an operator, as was done in Chapter 3. In this section, we collect the definitions necessary to understand the analysis presented there. The content of the section is taken from [8, Chapter 1].

Let X denote an open set in $\mathbb{R}^n$ and u a distribution in $\mathcal{D}'(X)$. We also denote by $\widehat{u}$ the Fourier transform of u.

Definition C.6. *The wave front set* WF (u) *is defined as the complement in* $X \times (\mathbb{R}^n \setminus 0)$ *of the collections of all* $(x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus 0)$ *such that, for some neighborhoods* U *of* x_0 *and* V *of* ξ_0*, we have*

$$\forall \varphi \in \mathscr{C}_0^\infty(U), \forall N > 0, \quad \widehat{\varphi u}(\tau\xi) = O(\tau^{-N}) \quad \textit{for } \tau \to \infty, \textit{ uniformly in } \xi \in V.$$

It can be shown that WF (u) is a closed conic subset of $X \times (\mathbb{R}^n \setminus 0)$. If X is a manifold, it follows from the appropriate coordinate invariant definition that WF $(u) \subseteq T^*X \setminus 0$.

For the rest of the section, we focus on continuous maps $A : \mathscr{C}_0^\infty(Y) \to \mathcal{D}'(X)$ with distributional kernel $K_A \in \mathcal{D}'(X \times Y)$. That is,

$$Au(x) = \int K_A(x,y)u(y)\,dy, \quad \forall u \in \mathscr{C}_0^\infty(Y).$$

Definition C.7. *The wave front relation* $\mathrm{WF}'(A)$ *of the operator* A *is the set*

$$\mathrm{WF}'(A) = \{((x,\xi),(y,\eta)) \in (T^*X \times T^*Y) \setminus 0\colon (x,y,\xi,-\eta) \in \mathrm{WF}\,(K_A)\}\,.$$

We need to introduce further notation. Denote the projection of $\mathrm{WF}'(A)$ to $T^*X \setminus 0$ and $T^*Y \setminus 0$, respectively, by

$$\mathrm{WF}'_X(A) = \{(x,\xi) \in T^*X \setminus 0\colon \exists y \in Y,\ (x,y,\xi,0) \in \mathrm{WF}\,(K_A)\}\,,$$

$$\mathrm{WF}'_Y(A) = \{(y,\eta) \in T^*Y \setminus 0\colon \exists x \in X,\ (x,y,0,\eta) \in \mathrm{WF}\,(K_A)\}\,.$$

Theorem C.1. *Let* $\Gamma \subseteq T^*Y \setminus 0$ *be a closed conic set that does not meet* $\mathrm{WF}'_Y(A)$*. Then,* A *can be extended to a continuous map from* $\mathcal{E}'(Y)$ *to* $\mathcal{D}'(X)$ *and*

$$\mathrm{WF}\,(Au) \subseteq \big(\mathrm{WF}'(A) \circ \mathrm{WF}\,(u)\big) \cup \mathrm{WF}'_X(A),$$

for all $u \in \mathcal{E}'(Y)$ *with* WF $(u) \subseteq \Gamma$.

Here $\mathrm{WF}'(A) \circ \mathrm{WF}\,(u)$ is the set

$$\big\{(x,\xi) \in T^*X \setminus 0\colon \exists (y,\eta) \in T^*Y,\ ((x,\xi),(y,\eta)) \in \mathrm{WF}'(A) \text{ and } (y,\eta) \in \mathrm{WF}\,(u)\big\}\,.$$

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