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Propagation of polarization sets for systems of generalized transverse type and for systems of MHD type

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Abstract

Polarization sets were introduced by Dencker (1982) as a refinement of wavefront sets to the vector-valued case. He also clarified the propagation of polarization sets when the characteristic variety of the pseudodifferential system under study consists of two hypersurfaces intersecting tangentially (1992), or transversally (1995). In this thesis, we consider the case of more than two intersecting characteristic hypersurfaces that are interesting transversally (and we give a note on the tangential case).

Mainly, we consider two types of systems which we name "systems of generalized transverse type" and "systems of MHD type", and we show that we can get a result for the propagation of polarization set similar to Dencker's result for systems of transverse type. Furthermore, we give an application to the MHD equations.

Zusammenfassung

Polarizationsmengen wurden von Dencker (1982) als eine Verfeinerung der Wellenfrontmengen im vektorwertigen Fall eingeführt. Er klärte auch die Ausbreitung von Polarizationsmengen im Fall, dass die charakteristische Varität des betrachteten pseudodifferenziellen Systems aus zwei Hyperflächen besteht, die sich tangential (1992) oder transversal (1995) schneiden.

Wir betrachten hauptsächlich zwei Typen von Systemen, die wir als "Systeme vom verallgemeinerten transversalen Typ" und "Systeme vom MHD-Typ" bezeichnen, und beweisen ein Ergebnis über die Ausbreitung von Polarizationsmengen, das ähnlich Denckers Resultat für Systeme vom transversalen Typ ist. Außerdem geben wir eine Anwendung auf die MHD-Gleichungen.

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Contents

1 Introduction

As is known, the singular support is the set of points at which a distribution fails to be a smooth function. In [Hö03], Hörmander defined the wavefront set of a distribution u, denoted by $\mathrm{WF}(u)$, which is a refinement of the singular support of a distribution. The wavefront sets does not only show the location of singularity, but also the direction in which the singularity occurs. For any smooth manifold X, the wavefront set is closed conical subset of the cotangent bundle $T^*(X)$. The projection of the wavefront set on X is equal to the singular support of the distribution. In [Hö03], Hörmander also gave the result for the propagation of the wavefront set for the solutions of partial differential equations, when considering the partial differential operator to be of real principal type, where he stated that the wavefront set is invariant under the bicharacteristic flow. Note that Hörmader's theorem is still satisfied if one considers pseudodifferential operators instead of partial differential operators. Similarly, one can define the H^s-wavefront set, which will be showing the location and direction where the distribution is not in the Sobolev space $H^s(X)$, and we have a similar propagation result regarding the H^s-wavefront sets for pseudodifferential operators of real principal type. In [Den89], Dencker studied the propagation of singularities for pseudodifferential operators $P \in \Psi^m(X)$ having characteristics of variable multiplicity. He considered the characteristic set to be the union of hypersurfaces S_j, $j = 1, ..., r_0$ intersecting tangentially at $\cap_{j=1}^{r_0} S_j$ of order $k_0 \geq 1$. Under some assumptions he proved that the wavefront set of u; the solution of the considered pseudodifferential operator, is invariant under the union of the Hamilton flows on S_j, $j = 1, ..., r_0$, given that Pu is smooth on X.

In [Hö03], Hörmander defined locally the wavefront set of distributional sections $u \in \mathcal{D}'(X; E)$, where $E \to X$ is a vector bundle over the manifold X. He defined the wavefront set of u locally as $\bigcup \mathrm{WF}(u_j)$ where $(u_1, ..., u_N)$ are the components of u with respect to a local trivialization of E. However, this definition does not specify in which component u is singular, that is why Dencker defined in [Den82a] the polarization set for vector-valued distribution u that we will be denoting it by Pol(u). The polarization set still shows the location and the direction of the singularity as the wavefront set, but it additionally shows the most singular components of a distribution. Hence, the polarization set of a distribution is a refinement of the wavefront set, and the projection of Pol(u) $\setminus 0$ on the cotangent bundle T^*X gives the wavefront set of u. Similarly, the H^s-polarization set is defined as a refinement of the H^s-wavefront set.

In [Den82a], Dencker defined systems of pseudodifferential operators of real principal type; note that the definition of systems of pseudodifferential operators of real principal type differs from the case of scalar pseudodifferential operators of real principal type. In this article, Dencker also defined Hamilton orbits for systems of real principal type which are certain line bundles, and then he proved that the polarization set of a solution u of systems of real principal type P will be union of Hamilton orbits, given that Pu is smooth. In [G\'86], Gérard pointed out that the above result also holds for H^s-polarization sets.

Moreover, in [Den92], Dencker considered pseudodifferential system having its characteristic set is union of two non-radial hypersurfaces intersecting tangentially at an involutive manifold of exactly order $k_0 \geq 1$. He also assumed that the principal symbol vanishes of first order on the two-dimensional kernel at the intersection, and he assumed a Levi type of condition. Then, he defined systems satisfying these conditions to be systems of uniaxial type. Outside the intersection of the hypersurfaces the system will be of real principal type, hence the propagation result of the polarization set is already known there. In this article, Dencker has also proved a propagation result of the polarization set at the intersection. In [Den95], Dencker considered pseudodifferential system having its characteristic set is union of two non-radial hypersurfaces intersecting transversally at an involutive manifold of codimension 2. He also assumed that the principal symbol vanishes of first order on the two-dimensional kernel at the intersection. Systems satisfying these conditions are systems of transverse type. In this article, Dencker has also proved a propagation result of the polarization set at the intersection.

We worked on extending Dencker's result stated above to pseudodifferential systems having their characteristic sets are union of several non-radial hypersurfaces intersecting transversally at an involutive manifold; not necessary just two hypersurfaces as in the case of systems of transverse type and systems of uniaxial type. Note that even if we assumed that the hypersurfaces are intersecting tangentially of exactly order $k_0 \geq 1$ instead of intersecting transversally, we get a similar result, and for the proof we use the same weight and metric introduced by Dencker in [Den92] for the symbol classes $S(\vartheta, g)$ of the Weyl calculus. We have considered two cases for that: the first case is the case where we have r_0 hypersufaces, and we assumed that r_0th-differential of the determinant of the principal symbol is different than zero at the intersection, and the ith-differential of the determinat of the principal symbol vanishes at the intersection for $i < r_0$. Moreover, we assumed the dimension of the kernel of the principal symbol to be r_0 at the intersection, and we also assumed a condition similar to the Levi type condition given for systems of uniaxial type. We called systems satisfying the above conditions systems of generalized transverse type, and we proved that we have a similar propagation result of the polarization set as that for systems of uniaxial type. The second case, is the case where we also have r_0 hypersurfaces, and condition similar to the Levi type condition given to systems of uniaxial type, but here we assumed that the $(r_0 + 1)$th-differential of the determinant of the principal symbol is different than zero at the intersection, and the ith-differential of the determinate of the principal symbol vanishes at the intersection for $i < r_0 + 1$. Moreover, we assumed the dimension of the kernel of the principal symbol to be $r_0 + 1$ at the intersection. We also assumed some additional conditions that we did not assume in the case of systems of generalized transverse type. We defined systems satisfying these conditions to be systems of MHD type. We named them systems of MHD type because we have first noticed such systems when we considered the linearized ideal MHD equations. Thus we will have chapter in which we study the propagation of polarization sets for the linearized ideal MHD equations.

In our work, we will assume that we have P an $N \times N$ system of pseudodifferential operators. Let $p = \sigma(P)$ be the principal symbol, $\det p$ the determinant of p, and $\Sigma = (\det p)^{-1}(0)$ the characteristics of P. We consider Σ to be union of several non-radial hypersurfaces intersecting transversally at an involutive manifold Σ_2. Now, we state our main theorem in this thesis regarding the propagation of polarization sets for systems of generalized transverse type, and systems of MHD type, but its proof will be postponed to sections 3.4, and 3.5 to prove it for systems of generalized transverse type, and systems of MHD type respectively. Let

$$r_u^*(\nu) = \sup\{r \in \mathbb{R} : u \in H^r \text{ at } \nu\} \qquad \nu \in T^*X \setminus 0 \tag{1.1}$$

be the regularity function.

Theorem 1.1. *Let $P \in \Psi^m_{phg}$ be an $N \times N$ system of generalized transverse type (or of MHD type) at $\nu_0 \in \Sigma_2$, and let $A \in \Psi^0_{phg}$ be an $N \times N$ system such that the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at ν_0, and let $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ be a hypersurface near ν_0, where $\pi_1 : T^*X \times \mathbb{C}^N \to T^*X$ is the projection along the fibers. Assume that $u \in \mathcal{D}'(X, \mathbb{C}^N)$ satisfies $\min(r^*_{Pu} + m - 1, r^*_{Au}) > r$ at ν_0. Then,* $\mathrm{Pol}^r(u)$ *is a union of $\mathcal{C}^\infty$ line bundles in $\mathcal{N}_A \cap \mathcal{N}_P$ over* *bicharacteristics of* $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ *near* ν_0.

The plan of this thesis is as follows. In the first chapter, the first two sections 2.1, and 2.2 will be about the wavefront sets, and about the H^s-wavefront set respectively. In these two sections we give the definition of the wavefront set (and of the H^s-wavefront set), and some properties of the wavefront set (and of the H^s-wavefront sets). Moreover, we will state Hörmander's propagation result of the wavefront set for differnetial operators of real principal type (and the propagation result for the H^s-wavefront set). In sections 2.3, and 2.4, we will give the definition of the polarization sets and H^s-polarization sets respectively, and state some of their proprieties. In chapter 3, we will discuss the propagation of polarization sets for different types of systems. In section 3.1, we will state Dencker's result for the propagation of polarization sets for systems of real principal type; see [Den82a]. Moreover, we will state some results proven in [HR04], where they used the calculus of Fourier integral operators to construct Lagrangian solutions and parametrices for systems of real principal type. In sections 3.2, and 3.3, we will state Dencker's result regarding the propagation of polarization sets for systems of uniaxial type; see [Den92], and for systems of transverse type; [Den95] respectively. Note that in [Den92] and [Den95], Dencker proved several results for the propagation of polarization sets under different conditions. Here we just mention the result which is similar to the result in our main theorem. In sections 3.4, and 3.5, we will define systems of generalized transverse type, and systems of MHD type respectively, and we prove Theorem 1.1 for both types of systems.

As we have mentioned, we first observed systems of MHD type when considering linearized ideal MHD equations. Chapter 4 will be an application for the results in [HR04], and for the propagation of polarization sets for systems of MHD type. First, we give the set of equations describing the ideal MHD, and we linearize it. In section 4.1, we write the linearized ideal MHD equations in the form of a wave equation, and we calculate the characteristic variety of this wave equation under some assumptions as done in [Sch09]. Then, we calculate the transport equation under these assumptions as an application to Hansen's and Röhrig's results mentioned in subsection 3.1.2. In section 4.2, we return to the linearized ideal MHD equations, and we calculate the eigenvalues and their multiplicities which are not constant. Then, we study the propagation of polarization sets, where we observe different cases, some in which our system is of real principal type, some in which our system is of uniaxial type, and one where our system is of MHD type.

In addition, we will have four appendices. Because in our work we will be considering involutive manifolds, the first appendix, Appendix A will be about symplectic geometry. Section A.1 will be about symplectic linear algebra. In section A.2 we give the definition of symplectic manifolds, and state Darboux's theorem. In section A.3 we will give a special case of the symplectic manifold which is the conic symplectic manifold. Section A.4 will be about the characteristic foliation, where we state Frobenius theorem. In chapter 3, we will be using the symbol classes $S(\vartheta, g)$ of the Weyl calculus, that is why we included appendix B which is about Hörmander Weyl calculus and estimates of pseudodifferential operators. In the first section, section B.1 we give the definition of the symbol class $S(\vartheta, g)$, and in section B.2 we give Hörmander Weyl calculus, and in the last section, section B.3 we give the estimates of pseudodifferential operators. Moreover, as in chapter 3 we will be using the Hörmander spaces $H^{r,s}$ we added appendix C which will be about the spaces $B_{p,k}$ which are generalization of the spaces $H^{r,s}$. In section C.1, we define these spaces, and in section C.2 we will discuss the localization of these spaces. In appendix D we include some tools that we will be using to prove our result. In section D.1, we give the generalization of the Malgrange preparation theorem, which was proven by Dencker in [Den92], and in section D.2 we give some calculus lemma similar to that given in [Den89] by Dencker, but here using different weight and metric.

2 Polarization sets

The wavefront sets describe the singularities of a distribution with respect to location and direction. In [Hö03], Hörmander gave the definition of the wavefront set, and stated many properties of it. Also, in [Hö03], Hörmander proved a propagation result of the wavefront set for solutions of partial differential equations that are of real principal type. When we consider vector-valued distribution, the wavefront set of $u = (u_j) \in \mathcal{D}'(X, \mathbb{C}^N)$ is defined as union of $\mathrm{WF}(u_j)$, but it does not specify in which components u is singular. In [Den82a], Dencker gave the definition of polarization set, which refines the notion of wavefront set for vector-valued distributions, and it indicates the component that have the strongest singularity. Moreover, he gave some properties of the polarization sets. In [G\'86], Gérard have considered H^s-polarization set, and gave also some properties of the H^s-polarization set.

In this chapter, we will give the definition of the wavefront set (and the H^s-wavefront set), and state some of their properties. Moreover, we will state Hörmader's propagation result for the wavefront sets (and for the H^s-wavefront sets). In addition, we will state the definition of the polarization set (and H^s-polarization set), and we will state some of their properties.

2.1 Wavefront sets

While the singular support of a distribution shows at which points the distribution is singular, Hörmander defined in [Hö03] the wavefront set which does not only tell where the distribution is singular, but also it shows in which direction the singularity occurs. In this section, we will introduce the definition of the wavefront set, and some properties of the wavefront sets.

First, we will remind the reader of the definition of the singular support of a distribution.

Definition 2.1. (Singular Support) Let $u \in \mathcal{D}'(X)$, and let X to be an open set in $\mathbb{R}^n$, then the singular support of u is defined as

$$\operatorname{singsupp}(u) := X \setminus \{x_0 \in X; \exists \text{ an open neighborhood } U \text{ of } x_0 \text{ such that } u|_U \in \mathcal{C}^\infty(U)\}. \tag{2.1}$$

However, we have that if $u \in \mathcal{E}'(\mathbb{R}^n)$, then u is smooth if and only if the Fourier transform of u is rapidly decreasing, that is, for every $n \in \mathbb{N}$, there exists a constant $C_N > 0$ such that

$$|\hat{u}(\xi)| \leq C_N(1 + |\xi|)^{-N}, \tag{2.2}$$

for all $\xi \in \mathbb{R}^n$. Hence, we have the following proposition which can be used as another definition of the singular support.

Proposition 2.2. *Let $u \in \mathcal{D}'(X)$, and let X to be an open set in $\mathbb{R}^n$, and consider $x_0 \in X$. Then we say x_0 is not in the singular support of u if and only if there exists $\varphi \in \mathcal{D}(X)$, with $\varphi(x_0) \neq 0$ and $\widehat{\varphi u}$ is rapidly decreasing.*

Now, we come to the definition of the wavefront set. The wavefront set consists of all the directions such that the local Fourier transform of the distribution is not rapidly decaying in this direction.

Definition 2.3. (Wavefront set in Euclidean Space) Assuming that X is an open set in $\mathbb{R}^n$, we say that a point $(x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus 0)$ does not belong to the wavefront set of u; denoted by $\mathrm{WF}(u)$, if there exists a function $\varphi \in \mathcal{C}_c^\infty(X)$ with $\varphi(x_0) \neq 0$ and a conic neighborhood $\Gamma \subseteq \mathbb{R}^n \setminus 0$ of ξ_0 such that, for each $N \in \mathbb{N}_0$, we have

$$\sup_{\xi \in \Gamma}(1 + |\xi|)^N |\widehat{\varphi u}(\xi)| < \infty. \tag{2.3}$$

The wavefront set is a refinement of the notion of singular support of a distribution in the following sense: We have $\mathrm{WF}(u) \subset X \times (\mathbb{R}^n \setminus 0)$ for $u \in \mathcal{D}'(X)$ and

$$\pi \, \mathrm{WF}(u) = \operatorname{singsupp} u, \tag{2.4}$$

where $\pi : X \times (\mathbb{R}^n \setminus 0) \to X$ is the projection onto the first component.

We can also define the wavefront set of a distribution on a smooth manifold by localization on coordinate patches.

Definition 2.4. (Wavefront set in a smooth manifold) Assume that X is a smooth manifold, $\kappa : U \to V$ a diffeomorphism between open coordinate patch U of the manifold X, and V an open set in $\mathbb{R}^n$, $u \in \mathcal{D}'(V)$, and $\tilde{u} \in \mathcal{D}'(U)$ the distributions with $\tilde{u}(f) := u(f \circ \kappa^{-1})$ for $f \in \mathcal{D}(U)$. Then

$$\mathrm{WF}(\tilde{u}) = \kappa_* \mathrm{WF}(u) := \{(\kappa^{-1}(x), \ {}^t\kappa'(x)\xi); (x, \xi) \in \mathrm{WF}(u)\}. \tag{2.5}$$

To state some properties of the wavefront set, we need to give the definition of properly supported pseduodifferential operators, but first we have to state the kernel theorem.

Assume that X_j is an open set in $\mathbb{R}^{n_j}$ for $j = 1, 2$. Every function $K \in \mathcal{C}(X_1 \times X_2)$ defines an integral operator $\mathcal{K}$ from $\mathcal{C}_0(X_2)$ to $\mathcal{C}(X_1)$ by the formula

$$\int (\mathcal{K}\phi)(x_1) = \int K(x_1, x_2)\phi(x_2)dx_2, \quad \phi \in \mathcal{C}_0(X_2),\ x_1 \in X_1. \tag{2.6}$$

This can be extended to arbitrary distribution K if ϕ is restricted to $\mathcal{C}_0^\infty$ and $\mathcal{K}\phi$ is a distribution. We observe that we have

$$\langle \mathcal{K}\phi, \psi \rangle = K(\psi \otimes \phi); \quad \psi \in \mathcal{C}_0^\infty(X_1),\ \phi \in \mathcal{C}_0^\infty(X_2), \tag{2.7}$$

when $K \in \mathcal{C}(X_1 \times X_2)$.

Theorem 2.5. *The Schwartz kernel theorem. Every distribution $K \in \mathcal{D}'(X_1 \times X_2)$ defines accroding to* (2.7) *a linear map $\mathcal{K}$ from $\mathcal{C}_0^\infty(X_2)$ to $\mathcal{D}'(X_1)$ which is continuous in the sense $\mathcal{K}\phi_j \to 0$ in $\mathcal{D}'(X_1)$ if $\phi_j \to 0$ in $\mathcal{C}_0^\infty(X_2)$.*

Conversely, for every such linear map $\mathcal{K}$ there is one and only one distribution K such that (2.7) *is valid. K is called the kernel of $\mathcal{K}$.*

From now on, assume that X is a smooth manifold.

Definition 2.6. A pseudodifferential operator A in X is said to be properly supported if both projections from the support of the kernel in $X \times X$ to X are proper maps, that is for every compact set $K \subset X$ there is a compact set $K' \subset X$ such that

$$\operatorname{supp} u \subset K \Rightarrow \operatorname{supp} Au \subset K'; u = 0 \text{ at } K' \Rightarrow Au = 0 \text{ at } K. \tag{2.8}$$

For P a pseudodifferential operator, we give the definition of the characteristic set of P by

$$\operatorname{Char} P = \{(x, \xi) \in T^*(X) \setminus 0;\ p(x, \xi) = 0\}, \tag{2.9}$$

where p is the principal symbol of P.

Theorem 2.7. *If $A \in \Psi^m$ is properly supported, and $(x_0, \xi_0) \notin \operatorname{Char} A$ then there exists $B \in \Psi^{-m}$ properly supported such that $(x_0, \xi_0) \notin \operatorname{WF}(BA - \operatorname{Id})$ and $(x_0, \xi_0) \notin \operatorname{WF}(AB - \operatorname{Id})$; these conditions are equivalent.*

Because of the above theorem, Hörmander gave another description of the wavefront set in [Hö07]:

Theorem 2.8. *If $u \in \mathcal{D}'(X)$, we have*

$$\mathrm{WF}(u) = \bigcap \operatorname{Char} A, \tag{2.10}$$

where the intersection is taken over all properly supported $A \in \Psi^0(X)$ such that $Au \in \mathcal{C}^\infty(X)$.

Remark 2.9. For $u \in \mathcal{D}'(X)$, the wavefront set of u is a closed subset in $T^*X \setminus \{0\}$, and conic in the sense that it is invariant under multiplication of the second variable by positive scalars. That is, $(x, \xi) \in \mathrm{WF}(u)$ implies $(x, t\xi) \in \mathrm{WF}(u)$ for all $t > 0$.

Theorem 2.10. *If $P \in \Psi^m(X)$ is properly supported, and $u \in \mathcal{D}'(X)$ then*

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u) \subseteq \mathrm{WF}(Pu) \cup \operatorname{Char} P. \tag{2.11}$$

A Special case is when P is elliptic. In this case we get $\mathrm{WF}(Pu) = \mathrm{WF}(u)$, as $\operatorname{Char} P = \emptyset$.

We will give two simple examples of the wavefront set of a distribution.

Example 2.11. (Wavefront set of the Dirac distribution) For $\delta(x) \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$\mathrm{WF}(\delta) = \{(0, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0);\ \text{such that } \xi \in \mathbb{R}^n \setminus 0\}. \tag{2.12}$$

Example 2.12. (Wavefront set of the Heaviside function) For $H(x)$ being the Heaviside function on $\mathbb{R}$ we have

$$\mathrm{WF}(H) = \{(0, \xi) \in \mathbb{R} \times (\mathbb{R} \setminus 0);\ \text{such that } \xi \in \mathbb{R} \setminus 0\}. \tag{2.13}$$

Here we used Theorem 2.10 as we have $H' = \delta$.

To state Hörmander's theorem about the propagation of wavefront set for a pseudodifferential operator of real principal type, we have to introduce some definitions.

Definition 2.13. (Hamiltonian Vector field) Let $P \in \Psi^m(X)$, and suppose that the principal symbol p of P is real valued. The Hamiltonian vector field of p of $T^*X \setminus 0$ is defined locally by

$$H_p = \sum_j \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j} - \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j}. \tag{2.14}$$

Definition 2.14. The integral curves of H_p along which $p = 0$ are called null bicharacteristics.

Remark 2.15. Clearly, $H_p p = 0$, thus p is constant along each bicharacteristic. The null bicharacteristics run in the characteristic set of P.

Definition 2.16. We say that H_p is nowhere radial if $\partial_\xi p \neq 0$ or $\partial_x p \nparallel \xi$, when $p = 0$.

Definition 2.17. We say that $P \in \Psi^m(X)$ is of real principal type if the principal symbol p is real and the Hamilton field H_p is nowhere radial.

Notice that when $P \in \Psi^m(X)$ is of real principal type then the characteristic set of P is foliated by null bicharacteristics.

Now, after defining operators of real principal type we can state Hörmander's theorem about the propagation of the wavefront sets for pseudodifferential operators of real principal type.

Theorem 2.18. *(Hörmander's theorem) Let $P \in \Psi^m(X)$ be of real principal type, and let γ be a null bicharacteristic such that $\gamma \cap \mathrm{WF}(Pu) \neq \emptyset$. Then $\gamma \subseteq \mathrm{WF}(u)$ or $\gamma \cap \mathrm{WF}(u) = \emptyset$.*

2.2 Sobolev wavefront sets

We can consider another type of wavefront set relative to the sobolev space which is called H^s-wavefront set. For H^s-wavefront sets we have similar results as for the wavefront sets. In this section we are also assuming X to be a smooth manifold.

Definition 2.19. If $u \in \mathcal{D}'(X)$ we have

$$\mathrm{WF}^s(u) = \bigcap \mathrm{Char}\, A, \tag{2.15}$$

where the intersection is taken over all properly supported $A \in \Psi^0(X)$ such that $Au \in H^s_{\mathrm{loc}}(X)$.

For $u \in H^s_{\mathrm{loc}}(X)$ we get $\mathrm{WF}^s(u) = \emptyset$.

Proposition 2.20. *For $u \in \mathcal{D}'(X)$, we have*

$$\pi\, \mathrm{WF}^s(u) = \{x \in X |\ u \notin H^s \text{ at } x\}, \tag{2.16}$$

where $\pi : T^*X \setminus 0 \to X$ is the canonical projection.

Notice that we have $\mathrm{WF}(u) = \bigcap_s \mathrm{WF}^s(u)$.

Theorem 2.21. *For $A \in \Psi^m(x)$ we have*

$$\mathrm{WF}^s(Au) \subseteq \mathrm{WF}^{s+m}(u) \subseteq \mathrm{WF}^s(A) \cup \mathrm{Char}\, A. \tag{2.17}$$

A special case of this is when A is elliptic. In this case we have $\mathrm{WF}^{s+m}(u) = \mathrm{WF}^s(Au)$.

We also have a result for the propagation of H^s wavefront set for pseudodifferential operators of real principal type.

Theorem 2.22. *(Hörmander's theorem) If $P \in \Psi^m$ is of real principal type, γ is a null bicharacteristic, then either $\gamma \subseteq \mathrm{WF}^{s+m-1}(u) \setminus \mathrm{WF}^s(Pu)$ or $\gamma \cap (\mathrm{WF}^{s+m-1}(u) \setminus \mathrm{WF}^s(Pu)) = \emptyset$.*

2.3 Polarization sets

In this section, we are going to consider vector-valued distribution instead of scalar distribution. In [Hö03], Hörmander stated that if E is a $\mathcal{C}^\infty$ vector bundle over X, and $u \in \mathcal{D}'(X, E)$, then the wavefront set of u is defined locally as union of $\mathrm{WF}(u_j)$ where $(u_1, ..., u_N)$ are the components of u with respect to a local trivialization of E. Passage to another local trivialization only means that $(u_1, ..., u_N)$ is multiplied by an invertible $\mathcal{C}^\infty$ matrix, so the definition is independent of the choice of local trivialization. However, this definition does not specify the component of u which have the strongest singularity. That is why Dencker defined in [Den82a] the polarization sets, which are refinement of the wavefront set for vector-valued distributions.

In this section, we will give the definition of the polarization set and state some of its properties. The propagation of polarization sets, will be postponed to the next chapter.

Definition 2.23 (Polarization set)**.** Let $E \to X$ be a vector bundle over a smooth manifold X, and $\pi : T^*X \setminus 0 \to X$ is the canonical projection. For $u \in \mathcal{D}'(X, E)$, we define the polarization set of u as

$$\mathrm{Pol}(u) = \bigcap_{Au \in \mathcal{C}^\infty(X)} \mathcal{N}_A, \tag{2.18}$$

where

$$\mathcal{N}_A = \{(x, \xi; \nu) \in \pi^*E;\ \nu \in \ker a(x, \xi)\}. \tag{2.19}$$

Here $A \in \Psi^0(X; E, \mathbb{C})$ with principal symbol a, and $\pi^*E \to T^*X \setminus 0$ is the induced (pullback) bundle.

Note that $u \in \mathcal{D}'(X; E)$ (distributional sections) is locally just a vector valued distribution. Hence, the definition is locally the following:

Definition 2.24. For $u \in \mathcal{D}'(X, \mathbb{C}^N)$, we define the polarization set of u as

$$\mathrm{Pol}(u) = \bigcap_{Au \in \mathcal{C}^\infty(X)} \mathcal{N}_A, \tag{2.20}$$

where

$$\mathcal{N}_A = \{(x, \xi; \nu) \in (T^*X \setminus 0) \times \mathbb{C}^N;\ \nu \in \ker a(x, \xi)\}. \tag{2.21}$$

Here, A is $1 \times N$ system of pseudodifferential operators of order zero with principal symbol a.

The polarization set is closed, conical in the ξ variables and linear in the fiber.

Example 2.25. Let $u = (u_1, u_2) \in \mathcal{D}'(X, \mathbb{C}^2)$ and assume that $(y, \eta) \notin \mathrm{WF}(u_1)$. Then

$$\mathrm{Pol}(u) \subseteq \{(x, \xi; (0, z));\ z \in \mathbb{C}\}$$

over a conical neighborhood of (y, η).

Example 2.26. Let $u = (v, \Delta v) \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^2)$ where $v \in \mathcal{D}'(\mathbb{R}^n)$ and Δ is the Laplacian operator. Then

$$\mathrm{Pol}(u) \subseteq \{(x, \xi; (0, z));\ z \in \mathbb{C}\},$$

since if we choose $A = (\mathrm{Id}, -\Delta^{-1})$ we get $Au = 0 \in \mathcal{C}^\infty(\mathbb{R}^n)$. For $(z_1, z_2) \in \ker a = \sigma(A)$ we get $z_1 = 0$.

Hence, the polarization set indicates the component of the distribution having the strongest singularity. The following proposition shows how the polarization set is a refinement of the notion of the wavefront set.

Proposition 2.27. *If $u \in \mathcal{D}'(X, \mathbb{C}^N)$, then*

$$\pi(\mathrm{Pol}(u) \setminus 0) = \mathrm{WF}(u), \tag{2.22}$$

where π is the projection on T^*X; that is $\pi(x, \xi, \nu) = (x, \xi)$.

Proposition 2.28. *Let A be an $M \times N$ system of pseudodifferential operators on X with principal symbol $a(x, \xi)$, and let $u \in \mathcal{D}'(X, \mathbb{C}^N)$. Then*

$$a(\mathrm{Pol}(u)) \subseteq \mathrm{Pol}(Au), \tag{2.23}$$

where a operates on the fiber; $a(x, \xi; \nu) = (x, \xi; a(x, \xi)\nu)$.

Corollary 2.29. *Let E be an $N \times N$ system of pseudodifferential operators on X with principal symbol $e(x, \xi)$, and let $u \in \mathcal{D}'(X, \mathbb{C}^N)$. If E is elliptic at $(y, \eta) \in T^*X$; that is $e(y, \eta) \neq 0$, then*

$$\mathrm{Pol}(Eu) = e(\mathrm{Pol}(u)) \tag{2.24}$$

over a conical neighborhood of (y, η).

2.4 Sobolev polarization sets

In [Ǵ86], Gérard considered the H^s-polarization set, and stated that it has similar properties as that of the polarization set. The propagation of the H^s-polarization set will be postponed to the next chapter also.

Definition 2.30. For $u \in \mathcal{D}'(X, \mathbb{C}^N)$, we define the H^s-polarization set of u as

$$\mathrm{Pol}^s(u) = \bigcap_{Au \in H^s_{\mathrm{loc}}(X)} \mathcal{N}_A, \tag{2.25}$$

where

$$\mathcal{N}_A = \{(x, \xi; \nu) \in (T^*X \setminus 0) \times \mathbb{C}^N;\ \nu \in \ker a(x, \xi)\}. \tag{2.26}$$

Here, A is $1 \times N$ systems of pseudodifferential operators with $\sigma(A) = a$ being its principal symbol.

The H^s Polarization set is a refinement for the H^s wavefront set in the following sense:

Proposition 2.31. *If $u \in \mathcal{D}'(X, \mathbb{C}^N)$, then*

$$\pi(\mathrm{Pol}^s(u) \setminus 0) = \mathrm{WF}^s(u), \tag{2.27}$$

where π is the projection on T^*X; that is $\pi(x, \xi; \nu) = (x, \xi)$.

Proposition 2.32. *If A is $M \times N$ system of pseudodifferential operators on X of order m with principal symbol $a(x, \xi)$ and $u \in \mathcal{D}'(X, \mathbb{C}^N)$, then*

$$a(\mathrm{Pol}^s(u)) \subseteq \mathrm{Pol}^{s-m}(Au), \tag{2.28}$$

where a operates on the fiber; $a(x, \xi; \nu) = (x, \xi; a(x, \xi)\nu)$.

Corollary 2.33. *Let E be an $N \times N$ system of pseudodifferential operators on X of order m with principal symbol $e(x, \xi)$, and let $u \in \mathcal{D}'(X, \mathbb{C}^N)$. If E is elliptic at $(y, \eta) \in T^*X$, then*

$$\mathrm{Pol}^{s-m}(Eu) = e(\mathrm{Pol}^s(u)) \tag{2.29}$$

over conical neighborhood of (y, η).

3 Propagation of polarization sets

In this chapter, we will state the results of the propagation of polarization sets for different types of systems. In [Den82a], Dencker have defined systems of real principal type and he gave the propagation result of the polarization sets of such systems. In [Den92], and [Den95] he gave the propagation result for systems of uniaxial type, and for systems of transverse type. We will state Dencker's results in the first three sections. Moreover, we will give the definition of two new kinds of systems, one which we named system of generalized transverse type, and the other one we named system of MHD type, and we will prove our main theorem; Theorem 1.1 for these two types of systems.

3.1 Systems of real principal type

We diveded this section into two parts. In the first part, we state the definition of real principal type, and state Dencker's result regarding the propagation of polarization sets for systems of real principal type; see [Den82a]. In the second part, we state some of the results mentioned in [HR04] by Hansen and Röhrig, who merged the theory of real principal type system with the calculus of Fourier integral operators. They showed that the principal symbol of a Lagrangian distribution solving a real principal type system satisfies a transport equation, and they constructed a Lagrangian solution and parametrices.

3.1.1 Propagation of polarization sets for systems of real principal type

For the definition of real principal type, we will differentiate between two cases, the scalar case, and the case of system of pseudodifferential operators. We aleady defined pseudodifferential operators of real principal type for the scalar case; see Definition 2.17. For the case of system of pseudofifferential operators we have the following definition:

Definition 3.1 (Case of system of Pseudodifferential operators)**.** An $N \times N$ system P of pseudodifferential operators on X with principal symbol $p(x,\xi)$ is of real principal type at $(y,\eta) \in T^*X \setminus 0$ if there exists an $N \times N$ symbol $\tilde{p}(x,\xi)$ such that

$$\tilde{p}(x,\xi)p(x,\xi) = q(x,\xi) \cdot \mathrm{Id}_N$$

in a neighborhood of (y,η) where $q(x,\xi)$ is a scalar symbol of real principal type and Id_N is the identity in $\mathbb{C}^N$.

Assuming $P(x,D)$ to be an $N \times N$ system of pseudodifferential operators on an n-dimensional smooth manifold X of order m, the symbol of P is an asymptotic sum of homogeneous terms: $p(x,\xi)+p_{m-1}(x,\xi)+p_{m-2}(x,\xi)+...$ where p is the principal symbol of P and p_j is homogeneous of degree j. The characteristic set of P is

$$\Sigma = \{(x,\xi); \det p(x,\xi) = 0\}, \tag{3.1}$$

and the subprincipal symbol of P is by definition

$$p^s := p_{m-1} - \frac{1}{2i}\sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}. \tag{3.2}$$

To state the result of the propagation of polarization set given by Dencker in [Den82a], we have to introduce first the connection he defined, and give the definition of the Hamilton orbit. In [Den82a], Dencker defined the following connection

$$D_p w = H_q w + \frac{1}{2}\{\tilde{p}, p\}w + i\tilde{p}p^s_{m-1}w, \tag{3.3}$$

for systems of real principal type, where w is $\mathcal{C}^\infty$ function on $T^*X \setminus 0$ with values in $\mathbb{C}^N$. $\{,\}$ is the Poisson bracket, where $\{\tilde{p}, p\} = H_{\tilde{p}}p$. D_P is a connection on $\mathcal{N}_P$ over Σ, i.e. if $w \in \ker p$ at one point of a bicharacteristic of Σ, then $D_P w \in \ker p$ along the bicharacteristic if and only if $w \in \ker p$ there. Hence, each parallel section (that is w such that $D_P w = 0$) is uniquely determined by one point. D_P depends on the choice of $\tilde{p}$ and q, however Dencker showed that different choices of $\tilde{p}$ and q only change the solution of $D_P w = 0$ in $\mathcal{N}_P$ by a scalar factor.

Definition 3.2 (Hamilton Orbit)**.** A Hamilton orbit of a system P of real principal type is a line bundle $L \subseteq \mathcal{N}_P|_\gamma$, where γ is an integral curve of the Hamilton field of Σ, and L is spanned by $\mathcal{C}^\infty$ section w satisfying $D_p w = 0$.

Theorem 3.3 (Dencker's propagation result)**.** *Let P be an $N \times N$ system of pseudodifferential operators on a manifold X and let $u \in \mathcal{D}'(X, \mathbb{C}^N)$. Assume that P is of real principal type at $(y,\eta) \in \Sigma$, and that $(y,\eta) \notin \mathrm{WF}(Pu)$. Then, over a neighborhood of (y,η) in Σ,* $\mathrm{Pol}(u)$ *is a* *union of Hamilton orbits of P.*

In [Ǵ86] Gérard stated that we have similar propagation result for the H^s-polarization sets for systems of real principal type.

Theorem 3.4. *Let P be an $N \times N$ system of pseudodifferential operators on a manifold X of order m, and let $u \in \mathcal{D}'(X, \mathbb{C}^N)$. Assume that P is of real principal type at $(y, \eta) \in \Sigma$, and that $(y, \eta) \notin \mathrm{WF}^s(Pu)$. Then over a neighborhood of $(y, \eta) \in \Sigma$, $\mathrm{Pol}^{s+m-1}(u)$ is a union of Hamilton orbits of P.*

3.1.2 Lagrangian solutions to systems of real principal type

Before stating some results proven by Hansen and Röhrig in [HR04], we remind the reader by the definition of Lagrnagian distribution, the definition of conormal distribution, and by the definition of Fourier integral operator. Then we state some of the results mentioned in [HR04].

Definition 3.5 (Lagrangian distribution)**.** Let X be an n-dimensional $\mathcal{C}^\infty$ manifold, and $\Lambda \subset T^*X \setminus 0$ be a closed conic Lagrangian submanifold. The class of Lagrangian distribution denoted by $I^\mu(X, \Lambda)$ for $\mu \in \mathbb{C}$ consists of all $u \in \mathcal{D}'(X)$ with $WF(u) \subseteq \Lambda$ such that, near any $\lambda^0 \in \Lambda$, u is microlocally of the form

$$u(x) = (2\pi)^{-\frac{n+2N}{4}} \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta) d\theta, \quad x \in U,$$

where $a \in S^{\mu + \frac{n-2N}{4}}(U \times \mathbb{R}^N)$, and $\phi \in \mathcal{C}^\infty(U \times (\mathbb{R}^N \setminus 0))$ is a phase function. Here, $\Lambda = \Lambda_\phi$ microlocally near λ^0, where

$$\Lambda_\phi = \{(x, \phi'_x(x,\theta)) |\ \phi'_\theta(x,\theta) = 0\} \subset U \times (\mathbb{R}^n \setminus 0) \cong T^*U \setminus 0.$$

Definition 3.6 (Conormal distribution)**.** Let X be an n-dimensional $\mathcal{C}^\infty$ manifold, $Y \subset X$, and $\Lambda = N^*Y \setminus 0$, where

$$N^*Y = \{(y, \xi) \in T^*_y X |\ \xi_{T_y Y} = 0\}.$$

We write $u \in I^\mu_{cl}(X, Y) = I^\mu_{cl}(X, N^*Y \setminus 0)$ to denote classical conormal distributions of order μ.

Definition 3.7 (Fourier integral operator)**.** Let X, Y be $\mathcal{C}^\infty$ manifolds, and $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ a homogeneous canonical relation, that is

$$\Lambda = C' = \{(x, y, \xi, \eta) \in T^*(X \times Y) \setminus 0 |\ (x, \xi, y, -\eta) \in C\},$$

is closed conic Lagrangian submanifold of $T^*(X \times Y) \setminus 0$. A linear operator $A : \mathcal{C}^\infty_c(Y) \to \mathcal{D}'(X)$ is a (classical) Fourier integral operator with underlying canonical relation C, of order $\mu \in \mathbb{C}$, if its kernel belongs to $I^\mu_{cl}(X \times Y, \Lambda)$. We write $A \in I^\mu_{cl}(X, Y, C)$.

The following settings will be assumed from now on till the end of this section. Let F and G be complex $\mathcal{C}^\infty$ vector bundles with the same fiber dimension N over n-dimensional smooth manifold X. In [HR04], they modified the definition of real principal type slightly in a way that they chose explicitly a Hamilton field of the characteristic variety of the operator. Let $P \in \Psi^m(X; F, G)$ be a properly supported and polyhomogeneous pseudodifferential operator mapping sections of F to sections of G. We say P is elliptic at the subset of $T^*X \setminus 0$ where the principal symbol $p \in S^m(T^*X, \mathrm{Hom}(F, G))$ is an isomorphism. The complement of the elliptic set is the characteristic variety of P, $\mathrm{Char}\, P \subset T^*X \setminus 0$. Assume that for every $\gamma \in T^*X \setminus 0$, there exists symbols $\tilde{p} \in S^{1-m}(T^*X \setminus 0, \mathrm{Hom}(G, F))$ and $q \in S^1(T^*X \setminus 0)$, homogeneous of degree $1 - m$, and 1, respectively, such that in a conic neighborhood of γ, q is of scalar real principal type with $\mathrm{Char}\, P = q^{-1}(0)$, Hamilton field $H = H_q$, and $p\tilde{p} = q\,\mathrm{Id}$. We then say P is of real principal type with Hamilton field H. See Definition 3.1.

Using local coordinates, and local frames, P becomes an $N \times N$ matrix of scalar pseudodifferential operators in an open subset of $\mathbb{R}^n$. Then the full symbol of P is an asymptotic sum of homogeneous terms: $p + p_{m-1} + p_{m-2} + \ldots$. The subprincipal symbols of the components of P is given by (3.2). Let $\Lambda \subset T^*X \setminus 0$ be a closed, conic Lagrangian submanifold of the characteristic variety of P, $\Lambda \subset \mathrm{Char}\, P$. Let M_Λ and $\Omega_\Lambda^{1/2}$ be the Maslov, and the half-density bundle of Λ, respectively. $S^{\mu+n/4}(\Lambda; M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F)$ is the space of symbols of the space of Lagrangian distribution sections, $I^\mu(X, \Lambda; \Omega_X^{1/2} \otimes F)$. See [Hö09, Section 25.1] to read more about Lagrangian distributions.

Given local coordinates $\lambda_1, \ldots, \lambda_n$ on Λ, the Lie derivative of half-densities with respect to H ($H = H_q$) is defined as follows:

$$\mathcal{L}_H(u|d\lambda|^{1/2}) = \big(Hu + \frac{1}{2}\,\mathrm{div}(H)u\big)|d\lambda|^{1/2}. \tag{3.4}$$

Here $\mathrm{div}(H) = \sum_j \partial H_j/\partial\lambda_j$ when $H = \sum_j H_j \partial/\partial\lambda_j$. Over a (conic) neighborhood of a point in Λ we trivialize bundles by choosing local frames of F, G, and M_Λ:

$$M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F \cong (\Omega_\Lambda^{1/2})^N \cong M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes G. \tag{3.5}$$

Thus, we represent sections by N-vectors of half-densities.

Proposition 3.8. *There is a first order differential operator $\mathcal{T}_{P,H}$ on Λ, uniquely determined by P and H, which maps $\mathcal{C}^\infty$ sections of $M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes \ker p$ to $\mathcal{C}^\infty$ sections of $M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F$. That is*

$$\mathcal{T}_{P,H} : \mathcal{C}^\infty(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes \ker p) \to \mathcal{C}^\infty(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F).$$

$\mathcal{T}_{P,H}$ is given as follows

$$\mathcal{T}_{P,H}a = \mathcal{L}_H a + \frac{1}{2}\{\tilde{p}, p\}a + i\tilde{p}p^s a. \tag{3.6}$$

Trivializing bundles as in (3.5)*, a is an N vector of half-densities with $pa = 0$.*

Let $E_r \in \Psi^k(X; F_r, F)$ and $E_l \in \Psi^k(X; G, G_l)$ be elliptic with principal symbols e_r and e_l, respectively. Let $f \in \mathcal{C}^\infty(T^*X \setminus 0)$ be a real-valued and non-vanishing in Char P*. Then,*

$$\mathcal{T}_{PE_r,H}(e_r^{-1}a) = e_r^{-1}\mathcal{T}_{P,H}(a), \quad \mathcal{T}_{E_lP,H}(a) = \mathcal{T}_{P,H}(a), \quad \mathcal{T}_{P,fH}(a) = f\mathcal{T}_{P,H}(a), \tag{3.7}$$

holds for every $a \in \mathcal{C}^\infty(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F)$ with $pa = 0$.

Remark 3.9. The transport equation $\mathcal{T}_{P,H}$ acts on symbol spaces

$$\mathcal{T}_{P,H} : S^{\mu+n/4}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes \ker p) \to S^{\mu+n/4}(\Lambda, M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F).$$

This follows from (3.6), since H is homogeneous of degree 0, and $\tilde{p}$ is homogeneous of degree $1 - m$.

An equation $\mathcal{T}_{P,H} = 0$ for a section a of $\ker p$ is called homogeneous transport equation.

Lemma 3.10. *Let P_1, P_2 be $N \times N$ systems of pseudodifferential operators with principal symbols p_1, p_2 and subprincipal symbols p_1^s, p_2^s. Then P_1P_2 has principal symbol p_1p_2 and subprincipal symbol*

$$p_1^s p_2 + p_1 p_2^s + \frac{1}{2i}\{p_1, p_2\}.$$

Proof. The proof follows from the composition formula of the Weyl calculus [Hö07, Section 18.5]. □

Proof of Proposition 3.8. Assume trivialization of the bundles as in (3.5. Using $\tilde{p}p = q\,\mathrm{Id}_N$, we can get that

$$\{f\tilde{p}, p\} = f\{\tilde{p}, p\} - (Hf)\,\mathrm{Id}_N - (H_f\tilde{p})p.$$

Using this, and using that $\mathcal{L}_{fH}a = f\mathcal{L}_H a + \frac{1}{2}(Hf)a$, we get the last formula in (3.7).

Let $E \in \Psi^k$ be $N \times N$ elliptic with principal symbol e. Set $P_1 = PE$. P_1 is of real principal type with Hamilton field H, and principal symbol $p_1 = pe$. Observe that $\tilde{p}_1 p_1 = q\,\mathrm{Id}_N$ holds with $\tilde{p}_1 = e^{-1}\tilde{p}$. Using

$$(\partial q)\,\mathrm{Id}_N = (\partial\tilde{p})p + \tilde{p}(\partial p), \tag{3.8}$$

and $He^{-1} = e^{-1}(He)e^{-1}$, we get

$$\{\tilde{p}_1, p_1\} = e^{-1}\{\tilde{p}, p\}e + 2e^{-1}He - e^{-1}\tilde{p}\{p, e\} + r_1.$$

Here r_1 vanishes on $\ker p_1$. Using Lemma 3.10, we have

$$\tilde{p}_1 p_1^s = e^{-1}\big(\tilde{p}p^s e + \frac{1}{2i}\tilde{p}\{p,e\}\big) + r_2.$$

Here r_2 vanishes on $\{q=0\}$. Let b be a section of $(\Omega_\Lambda^{1/2})^N$ with $pa = p_1 b = 0$, $a = eb$. Then

$$\mathcal{L}_H b = e^{-1}\mathcal{L}_H a - e^{-1}(He)b.$$

Summing up we have

$$\mathcal{L}_H b + \frac{1}{2}\{\tilde{p}_1, p_1\}b + i\tilde{p}_1 p_1^s b = e^{-1}\big(\mathcal{L}_H a + \frac{1}{2}\{\tilde{p}, p\}a + i\tilde{p}p^s a\big).$$

So, we get the first formula in (3.7). Similarly, we can prove the second formula by multiplying P from the right by an elliptic operator. □

Remark 3.11. A transport equation $\mathcal{T}_{P,H} a = 0$ for a section a over a Lagrangian manifold $\Lambda \subset \operatorname{Char} P$ can be viewed as a family of ordinary differential equations associated with the bicharacteristic curves which foliate Λ.

If A is a Lagrangian distribution of order μ with principal symbol a, then PA has order (at most) $m+\mu$ and principal symbol pa. If $pa = 0$ then PA is of order $m+\mu-1$. In the following theorem, we will state the relation between its principal symbol and the transport connection $\mathcal{T}_{P,H}$.

Theorem 3.12. *Assume Λ a homogeneous Lagrangian submanifold of $T^*(X)\setminus 0$ such that $\Lambda \subset \operatorname{Char} P$. If $A \in I^\mu(X,\Lambda;\Omega_X^{1/2}\otimes F)$, and $a \in S^{\mu+n/4}(\Lambda; M_\Lambda \otimes \Omega_\Lambda^{1/2}\otimes F)$ is a principal symbol of A such that $pa = 0$, it follows that $B = PA \in I^{m+\mu-1}(X,\Lambda;\Omega_X^{1/2}\otimes G)$ has a principal symbol $b \in S^{m+\mu+n/4-1}(\Lambda; M_\Lambda\otimes\Omega_\Lambda^{1/2}\otimes G)$ satisfying*

$$i^{-1}\mathcal{T}_{P,H} a = \tilde{p}b. \tag{3.9}$$

Proof. We want to show that (3.9) holds microlocally near $\gamma \in \Lambda$. We assume that X is an open subset of $\mathbb{R}^n$, and that F and G are trivial N-bundles. Hence, P is a $N\times N$ matrix of scalar pseudodifferential operators, and a, and b are N-vectors of half-densities. We have to prove that, in a conic neighbourhood of γ,

$$\frac{1}{i}\mathcal{L}_H a + \frac{1}{2i}\{\tilde{p}, p\}a + \tilde{p}p^s a = \tilde{p}b.$$

First, consider P is diagonal, $p = q\operatorname{Id}_N$ ($\tilde{p} = \operatorname{Id}_N$), and $p^s = 0$. Then (3.9) follows from the scalar case [Hö07, Theorem 25.2.4].

Next, consider when the principal symbol of P is diagonal $p = q\operatorname{Id}_N$, (we assumed here $\tilde{p} = \operatorname{Id}_N$). Choose $P_\Delta \in \Psi^m$ diagonal $N\times N$ with principal symbol p, and vanishing subprincipal symbol. Choose $E \in \Psi^0$ elliptic $N \times N$ with principal symbol e which solves

$$\frac{1}{i} H_q e + p^s e = 0 \tag{3.10}$$

in a conic neighbourhood of γ. PE and EP_Δ have the same principal symbol qe. Using Lemma 3.10, the subprincipal symbols of PE and EP_Δ are

$$qe^s + p^s e + \frac{1}{2i} H_q e \quad \text{and} \quad qe^s - \frac{1}{2i} H_q e,$$

respectively. It follows from (3.10) that the subprincipal symbols are equal. Hence $PE \equiv EP_\Delta$ modulo Ψ^{m-2}. Let E^{-1} be a parametrix for E. Microlocally near γ, $B = PA \equiv EP_\Delta E^{-1} A$ holds modulo $I^{m+\mu-2}(X, \Lambda; \Omega_X^{1/2} \otimes G)$. From the previous case we have that the principal symbol of $P_\Delta E^{-1} A$ equals $i^{-1} \mathcal{L}_H (e^{-1} a)$. Now, using (3.10), we get:

$$b = i^{-1} e \mathcal{L}_H (e^{-1} a) = i^{-1} \mathcal{L}_H a + i^{-1} e (H e^{-1}) a = i^{-1} \mathcal{L}_H a + p^s a.$$

Now, consider the general case. $P_1 = \tilde{P} P$ is of real principal type with diagonal principal symbol $p_1 = q \operatorname{Id}_N$. By Lemma 3.10 the subprincipal symbol is $p_1^s = \tilde{p}^s p + \tilde{p} p^s + (2i)^{-1} \{\tilde{p}, p\}$. We apply the previous case to $\tilde{P} B = P_1 A$ with $\tilde{p}_1 = \operatorname{Id}_N$ and obtain

$$\tilde{p} b = i^{-1} \mathcal{L}_H a + p_1^s a = i^{-1} \mathcal{L}_H a + \tilde{p} p^s a + (2i)^{-1} \{\tilde{p}, p\} a.$$

□

Apply Theorem 3.12 with P the lift to $X \times Y$ under the canonical projection onto the factor X and $\Lambda = C'$ the twisted canonical relation.

Corollary 3.13. *Assume that C is a homogeneous canonical relation from $T^*(Y)\backslash 0$ to $T^*(X)\backslash 0$ such that the projection of C in $T^*(X) \backslash 0$ is contained in* $\operatorname{Char} P$. *If $A \in I^\mu(X \times Y, C'; \Omega_{X \times Y}^{1/2} \otimes \operatorname{Hom}(E, F))$, and $a \in S^{\mu + (n_X + n_Y)/4}(C; M_C \otimes \Omega_C^{1/2} \otimes \operatorname{Hom}(E, F))$ is a principal symbol for A such that $pa = 0$, it follows that $B = PA \in I^{m+\mu-1}(X \times Y, C'; \Omega_{X \times Y}^{1/2} \otimes \operatorname{Hom}(E, G))$ has principal symbol $b \in S^{m+\mu+(n_X+n_Y)/4-1}(C; M_C \otimes \Omega_C^{1/2} \otimes \operatorname{Hom}(E, G))$ which satisfies*

$$i^{-1} \mathcal{T}_{P,H} a = \tilde{p} b.$$

Let $\Lambda \subset \operatorname{Char} P$ be a closed, conic Lagrangian submaniold. We want to show how $PA \equiv 0$ modulo $\mathcal{C}^\infty$ with non-trivial Lagrangian distribution A associated with Λ is solved.

Lemma 3.14. *Let $\Lambda_0 \subset \Lambda$ be a conic isotropic submanifold of codimension one such that evey bicharacteristic curve in Λ intersects Λ_0 in exactly one point and transversally. Let $a_0 \in S^{\mu+n/4}(\Lambda; M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F)$ and $b \in S^{m+\mu+n/4-1}(\Lambda; M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes G)$. Then there is a unique symbol $a \in S^{\mu+n/4}(\Lambda; M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F)$ which satisfies $a = a_0$ on Λ_0 and the transport equation*

$i^{-1}\mathcal{T}_{P,H}a = \tilde{p}b$ *on* Λ*. If* $pa_0 = 0$ *on* Λ_0 *then* $pa = 0$ *on* Λ*.*

Now, we show how $B \equiv PA$ is solved within the framework of Lagrangian distributions when the product of principal symbols vanishes and the transport equation holds.

Proposition 3.15. *Let* $B \in I^{m+\mu-1}(X, \Lambda; \Omega_X^{1/2} \otimes G)$ *have principal symbol* b*. Let* $a \in S^{\mu+n/4}(\Lambda; M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F)$ *with* $pa = 0$ *be such that the transport equation* $i^{-1}\mathcal{T}_{P,H}a = \tilde{p}b$ *holds on* Λ*. Then there exists* $A \in I^\mu(X, \Lambda; \Omega_X^{1/2} \otimes F)$ *with principal symbol* a *such that* $B - PA \in I^{m+\mu-2}(X, \Lambda; \Omega_X^{1/2} \otimes G)$*.*

Proof. Choose $A' \in I^\mu(X, \Lambda)$ with principal symbol a. By Theorem 3.12, $B' := B - PA' \in I^{m+\mu-1}(X, \Lambda)$ with principal symbol b' which satisfies $\tilde{p}b' = 0$. From [Den82a], we have $\ker \tilde{p} = \operatorname{im} p$. Therefore we find $a'' \in S^{\mu-1+n/4}(\Lambda)$ with $b' = pa''$. $A = A' + A''$ has the requested properties when $A'' \in I^{\mu-1}(X, \Lambda)$ is chosen with principal symbol a''. □

Theorem 3.16. *Let* $\Lambda_0 \subset \Lambda$ *be a conic isotopic submanifold of codimension one such that every* H *bicharacteristic curve in* Λ *intersects* Λ_0 *in exactly one point and transversally. Let* $a_0 \in S^{\mu+n/4}(\Lambda; M_\Lambda \otimes \Omega_\Lambda^{1/2} \otimes F)$ *be such that* $pa_0 = 0$ *on* Λ_0*. Then there exists* $A \in I^\mu(X, \Lambda; \Omega_X^{1/2} \otimes F)$ *which satisfies* $PA \equiv 0$ *modulo* $\mathcal{C}^\infty(X; F)$ *and has principal symbol* a *such that* $a = a_0$ *on* Λ_0 *and* $pa = 0$*,* $\mathcal{T}_{P,H}a = 0$ *on* Λ*.*

Proof. Set $B_1 = 0$. From Lemma 3.14, and Proposition 3.15, we know that there exists $A_1 \in I^\mu(X, \Lambda; \Omega_\Lambda^{1/2} \otimes F)$ such that $B_2 := B_1 - PA_1 \in I^{m+\mu-2}(X, \Lambda; \Omega_X^{1/2} \otimes G)$ and the principal symbol of A_1, a_1, satisfies $a_1 = a_0$ on Λ_0 and $pa_1 = 0$, $i^{-1}\mathcal{T}_{P,H}a_1 = \tilde{p}b_1 = 0$ on Λ. Recursively for $j \in \mathbb{N}$, we obtain $A_j \in I^{\mu+1-j}(X, \Lambda; \Omega_X^{1/2} \otimes F)$ and $B_{j+1} := B_j - PA_j \in I^{m+\mu-1-j}(X, \Lambda; \Omega_X^{1/2} \otimes G)$ by solving inhomogeneous transport equations $i^{-1}\mathcal{T}_{P,H}a_j = \tilde{p}b_j$ on Λ. A is constructed as an asymptotic sum $A \sim A_1 + A_2 + ...$ □

3.2 Propagation of polarization sets for systems of uniaxial type

In this section, we will state the definition of systems of uniaxial type, and give the result of propagation of polarization sets for systems of uniaxial type given by Dencker in [Den92] . We are going to assume that the characteristic set is a union of two non-radial hypersurfaces, which are tangent of exactly order $k_0 \geq 1$ at an involutive manifold. We will also assume that the principal symbol of our system vanishes of first order on the two-dimensional kernel at the intersection, and we will assume a Levi-type of condition.

Let $P \in \Psi^m_{phg}(X)$ be an $N \times N$ system of classical pseudodifferential operators on a smooth manifold X. Let p be the principal symbol of P. Let $\Sigma = (\det p)^{-1}(0)$, and let

$$\Sigma_2 = \{(x,\xi) \in \Sigma : d(\det p) = 0 \text{ at } (x,\xi)\}, \tag{3.11}$$

and $\Sigma_1 = \Sigma \setminus \Sigma_2$. Assume that we have

$$\begin{aligned} &\Sigma = S_1 \cup S_2, \text{ where } S_1 \text{ and } S_2 \text{ are non-radial hypersurfaces tangent at} \\ &\Sigma_2 = S_1 \cap S_2 \text{ of exactly order } k_0 \geq 1, \end{aligned} \tag{3.12}$$

microlocally near $\nu_0 \in \Sigma_2$. This means that the Hamilton field of S_j does not have the radial direction $\langle \xi, \partial_\xi \rangle$, and it means also that the k_0th jets of S_1 and S_2 coincide on Σ_2, but no (k_0+1)th jet does. Note that we have P is of real principal type at Σ_1, since $d(\det p) \neq 0$ there; see Definition 3.1. Moreover, (3.12) gives us that Σ_2 have to be a manifold of codimension ≥ 2. We assume that

$$\Sigma_2 \text{ is an involutive manifold of codimension } d_0 \geq 2, \tag{3.13}$$

let $\mathcal{N}_p = \ker p \subseteq (T^*X \setminus 0) \times \mathbb{C}^N$. We will assume that

$$\text{the (complex) dimension of the fiber of } \mathcal{N}_p \text{ is equal to 2 at } \Sigma_2, \tag{3.14}$$

and

$$d^2(\det p) \neq 0 \text{ at } \Sigma_2, \tag{3.15}$$

that is p vanishes of first order on the kernel. We want to consider the limits of $\mathcal{N}_P|_{\Sigma_1}$ when we approach Σ_2, so let

$$\mathcal{N}^j_p = \mathcal{N}_p|_{S_j \setminus \Sigma_2}, \tag{3.16}$$

$T_{\Sigma_2}\Sigma := T_{\Sigma_2}S_1 = T_{\Sigma_2}S_2$ (note here Σ is not a manifold), and $\partial\Sigma_1 := T_{\Sigma_2}\Sigma / T\Sigma_2$. Here $\partial\Sigma_1$ is the normal bundle of Σ_2 in S_1 which is equal to the normal bundle of Σ_2 in S_2. Let $i_0 : \Sigma_2 \to \partial\Sigma_1$ denotes the zero section of $\partial\Sigma_1$. By the tubular neighborhood theorem we know that there exists a diffeomorphism Φ from some neighborhood $\mathcal{U} \subset S_j$ of Σ_2 to a neighborhood $\mathcal{U}_0 \in \partial\Sigma_1$ of the zero section of $\partial\Sigma_1$, and Φ identifies Σ_2 itself with the zero section.

Before giving the definition of systems of uniaxial type, we need to give the definition of the limit polarizations.

Definition 3.17. For $j = 1, 2$, we define the limit polarizations

$$\partial\mathcal{N}^j_p = \{(\nu, \rho, z) \in \partial\Sigma_1 \times \mathbb{C}^N : \rho \neq 0 \text{ and } z = \lim_{k\to\infty} z_k\}, \tag{3.17}$$

where $z_k \in \ker p(\nu_k)$ and $\nu_k \in S_j \setminus \Sigma_2$ satisfy $(\nu - \nu_k)/|\nu - \nu_k| \to \rho/|\rho|$ when $k \to \infty$.

$\partial\mathcal{N}_P^j$ is conical in ξ and ρ, and homogeneous in the fiber, but it may have (complex) dimension > 1 at (ν, ρ). We assume that the fiber of

$$\partial\mathcal{N}_p^1 \cap \partial\mathcal{N}_p^2 = \{0\} \text{ over } \partial\Sigma_1 \setminus (\Sigma_2 \times 0). \tag{3.18}$$

This condition means that no element in $\mathcal{N}_P|_{\Sigma_2}$ can be the limit of polarization vectors on both characteristic surfaces, along the same direction. Dencker showed that (3.18) implies that $\partial\mathcal{N}_P^j$ is a complex line bundle over $\partial\Sigma_1 \setminus (\Sigma_2 \times 0)$, if we assume (3.12)-(3.15). We will show this in the proof of proposition 3.20. Now, we give the definition of systems of uniaxial type:

Definition 3.18. The system P is of uniaxial type at $\nu_0 \in \Sigma_2$, if (3.12)-(3.15) and (3.18) hold microlocally near ν_0.

If $P \in \Psi^m_{phg}$ is of uniaxial type and $Pu \in H^r$ near $\nu \in \Sigma_1$, then we already know the result as Dencker showed in [Den82a] that $\mathrm{Pol}^{r+m-1}(u)$ is a union of Hamilton orbits in $\mathcal{N}_P$ near ν because P is of real principal type at Σ_1, since $d(\det p) \neq 0$. Now, we want to give Dencker's result for the propagation of the polarization set when we approach Σ_2; see [Den92]. Note that in this section we will not be writing precisely what Dencker wrote to prove the result, but it is almost the same process. Dencker applied a change of variable while writing the normal form, which we do not apply here.

Proposition 3.19. *Let $P \in \Psi^1_{phg}$ be an $N \times N$ system of uniaxial type at $\nu_0 \in \Sigma_2$. Then by choosing suitable symplectic coordinates, we may assume that $X = \mathbb{R} \times \mathbb{R}^{n-1}$, $\nu_0 = (0; (0, ..., 1))$,*

$$S_1 = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau = 0\}, \tag{3.19}$$

and

$$S_2 = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta(t, x, \xi) = 0\}, \tag{3.20}$$

microlocally near ν_0. Here β is real and homogeneous of degree 1 in ξ, and satisfies in a conical neighborhood of ν_0

$$c|\xi'|^{k_0+1}/|\xi|^{k_0} \leq |\beta| \leq C|\xi'|^{k_0+1}/|\xi|^{k_0}, \quad 0 < c < C, \tag{3.21}$$

where $(\tau, \xi', \xi'') \in \mathbb{R} \times \mathbb{R}^{d_0-1} \times \mathbb{R}^{n-d_0}$, which gives $\Sigma_2 = \{\tau = 0, \xi' = 0\}$. By conjugating P with elliptic, scalar Fourier integral operators, and multiplying with elliptic $N \times N$ systems of order 0, we may assume that

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \mod \mathcal{C}^\infty, \tag{3.22}$$

microlocally near ν_0*, where* $E \in \Psi^1_{phg}$ *is an elliptic* $(N-2)\times(N-2)$ *system and*

$$F \cong \mathrm{Id}_2\, D_t + K(t,x,D_x) \mod \mathcal{C}^\infty. \tag{3.23}$$

Here $K(t,x,D_x) \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{phg})$ *is a* 2×2 *system,* $\det k \equiv 0$*, and trace* $\operatorname{tr} k = \beta$ *where* k *is the principal symbol of* K*.*

Proof. Since the result is local, we may assume $X = \mathbb{R}^n$. Because Σ_2 is involutive, we may choose symplectic, homogeneous coordinates $(x,\xi) \in T^*\mathbb{R}^n$ near $\nu_0 \in \Sigma_2$, so that $\nu_0 = (0;(0,...,1))$ and

$$\Sigma_2 = \{(x,\xi) \in T^*\mathbb{R}^n : \xi' = 0\}, \tag{3.24}$$

where $\xi = (\xi', \xi'') \in \mathbb{R}^{d_0} \times \mathbb{R}^{n-d_0}$. We may also assume that

$$S_1 = \{(x,\xi) \in T^*\mathbb{R}^n : \xi_1 = 0\}, \tag{3.25}$$

near ν_0. Now, we rename $x_1 = t$, $(x_2,..,x_{d_0}) = x'$, and $(x_{d_0+1},...,x_n) = x''$. Since S_2 is tangent to S_1 at Σ_2, we obtain

$$S_2 = \{(t,x;\tau,\xi) \in T^*(\mathbb{R}\times\mathbb{R}^{n-1}) : \tau + \beta(t,x,\xi) = 0\}, \tag{3.26}$$

with β real and homogeneous of degree 1 in ξ, in a conical neighborhood of ν_0, and β satisfies (3.21).

By choosing suitable homogeneous bases for $\ker p$ and orthogonal complement of $\operatorname{Im} p$ in $\mathbb{C}^N$ on Σ_2, and extending to a neighborhood, we obtain P on the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in S^1. \tag{3.27}$$

Here P_{22} is an elliptic $(N-2)\times(N-2)$ system and the principal symbols of P_{11}, P_{12}, and P_{21} vanish on Σ_2. By constructing a parametrix for P_{22} and multiplying P from left and right with suitable elliptic systems of order 0, we obtain P on the form

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \mod \mathcal{C}^\infty \tag{3.28}$$

microlocally near ν_0, where $E \in \Psi^1_{phg}$ is an elliptic $(N-2)\times(N-2)$ system. To know more about how this is done, you can check either [Den88, Proposition 2.5], or you can check the proof of Proposition 3.31 in the next section, as we will explain this more when writing the normal form for systems of generalized transverse type.

If f is the principal symbol for $F = P_{11}$, the conditions (3.15), (3.19), and (3.20) imply

$$\det f = c\tau(\tau + \beta), \quad 0 \neq c \in S^{-1}, \tag{3.29}$$

thus $\partial_\tau^2(\det f) = \det(\partial_\tau f) \neq 0$ at Σ_2. By Theorem D.1, and by homogeneity, we may find homogeneous system $C_0 \in S^0$ such that

$$f = C_0(\tau \operatorname{Id}_2 + k(t, x, \xi)), \tag{3.30}$$

where $\det C_0 \neq 0$ at Σ_2. By multiplication with an elliptic system, we may assume $C_0 \equiv \operatorname{Id}_2$. Thus, $\det f = \tau(\tau + \beta)$, which implies $\det k \equiv 0$, and $\operatorname{tr} k = \beta$. If $f_0 \in S^0$ is the term homogeneous of degree 0 in the expansion of F, then Theorem D.1 and homogeneity give

$$f_0 = B_{-1} f + B_0, \tag{3.31}$$

where $B_0 \in \mathcal{C}^\infty(\mathbb{R}, S^0)$ is independent of τ, and $B_{-1} \in S^{-1}$. By multiplying f with an operator with symbol $\operatorname{Id}_2 - B_{-1}$, we may assume $B_{-1} \equiv 0$. By induction over lower order terms we obtain (3.23). □

We want to introduce symbol classes adapted to the function β defined in (3.21). Let

$$\vartheta^2(\xi) = 1 + |\xi'|^{2k_0+2} \langle \xi \rangle^{-2k_0}, \tag{3.32}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, thus $\vartheta \approx 1 + |\beta|$. Consider the metric

$$g(dx, d\xi) = |dx|^2 + |d\xi'|^2/(\langle \xi \rangle^\mu + |\xi'|)^2 + |d\xi''|^2/\langle \xi \rangle^2 \quad \text{at} \quad (x, \xi), \tag{3.33}$$

where $\mu = k_0/(k_0 + 1)$, which gives $h^2 = \sup g/g^\sigma = (\langle \xi \rangle^\mu + |\xi'|)^{-2} \leq 1$. We get that g is σ temperate, ϑ is a weight for g, and $\beta \in S(\vartheta, g)$; see Appendix B. In fact, Taylor's formula gives

$$\beta = \sum_{|\alpha| = k_0 + 1} a^\alpha \xi'^\alpha \tag{3.34}$$

where $a^\alpha \in S^{-k_0}$ are homogeneous in ξ. Hence, we get

$$\langle \xi \rangle^{|\gamma''|} |\partial_x^\alpha \partial_{\xi'}^{\gamma'} \partial_{\xi''}^{\gamma''} \beta| \leq C_{\alpha\gamma} \langle \xi \rangle^{-k_0 - (|\gamma'| - k_0 - 1)_+} |\xi'|^{(k_0 + 1 - |\gamma'|)_+} \leq C'_{\alpha\gamma} \vartheta h^{|\gamma'|}, \tag{3.35}$$

by considering the cases $|\xi'| \gtrless \langle \xi \rangle^\mu$. Similarly, we get that if we have $a(t, x, \xi)$ is homogeneous of degree j in ξ and $|a| \leq c\vartheta^k$, then $a \in S(\vartheta^j, g)$. Moreover, if $k < j$, then $a \equiv 0$, otherwise a vanishes of order $\geq j(k_0 + 1)$ at Σ_2.

Proposition 3.20. *Let*

$$P = \operatorname{Id}_2 D_t + K(t, x, D_x) \tag{3.36}$$

be a 2×2 *system with* $K \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{phg})$, *and assume that* $k = \sigma(K)$ *satisfies* $\det k \equiv 0$, *and*

$\operatorname{tr} k = \beta$. *Then P is of uniaxial type if and only if $k \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g))$.*

Proof. We will write k in the following form

$$k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \tag{3.37}$$

and let $\alpha = (k_{11}, k_{12}, k_{21}, k_{22}) \in \mathcal{C}^\infty(\mathbb{R}, S^1)$, homogeneous of degree 1 in ξ. By homogeneity; see the discussion above,

$$k \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g)) \Leftrightarrow \alpha = O(\beta) \tag{3.38}$$

Assume that $\alpha = O(\beta)$ and $(\nu, \rho) \in \partial\Sigma_1$, $\rho \neq 0$. Choose $\Sigma_1 \in \nu_j \to \nu$ such that $(\nu - \nu_j)|\nu - \nu_j|^{-1} \to \rho/|\rho|$, $j \to \infty$. Let us define

$$\gamma^{ij}(\nu, \rho) := \lim_{\nu_j \to \nu} \frac{k_{ij}}{\beta}(\nu_j). \tag{3.39}$$

Since $\alpha = O(\beta)$ does not depend on τ, so the above definition (3.39) is independent of the choice of ν_j. We get

$$\partial\mathcal{N}^1_P(\nu, \rho) = \ker((\gamma^{ij}(\nu, \rho))_{i,j=1,2}), \tag{3.40}$$

where $(\gamma^{ij}(\nu, \rho))_{i,j=1,2}$ denotes the matrix with entries $\gamma^{ij}(\nu, \rho)$ for $i, j = 1, 2$. We also have

$$\partial\mathcal{N}^2_P(\nu, \rho) = \ker(-\operatorname{Id}_2 + (\gamma^{ij}(\nu, \rho))_{i,j=1,2}). \tag{3.41}$$

It is easy to see that the condition $\partial\mathcal{N}^1_P(\nu, \rho) \cap \partial\mathcal{N}^2_P(\nu, \rho) = \{0\}$ is satisfied. Moreover, using that $\det k = 0$, and trace $k = \beta$, we have that $\det(\gamma^{ij}(\nu, \rho))_{i,j=1,2}) = \det(-\operatorname{Id}_2 + (\gamma^{ij}(\nu, \rho))_{i,j=1,2}) = 0$. Also, we get that $(\gamma^{ij}(\nu, \rho))_{i,j=1,2}) \neq \{0\}$ and $(-\operatorname{Id}_2 + (\gamma^{ij}(\nu, \rho))_{i,j=1,2}) \neq \{0\}$. Hence, by the rank-nullity theorem we get $\dim \partial\mathcal{N}^1_P(\nu, \rho)$ and $\dim \partial\mathcal{N}^2_P(\nu, \rho)$ is equal to 1 for all $(\nu, \rho) \in \partial\Sigma_1$, $\rho \neq 0$.

On the other hand, assume that $\alpha \neq O(\beta)$ at $\nu \in \Sigma_2$. Then there exists a sequence $\nu_l = (t_l, x_l; 0, \xi_l) \to \nu$, such that

$$|\alpha(\nu_l)| > l|\beta(\nu_l)|, \quad \forall l \in \mathbb{N}. \tag{3.42}$$

It is no restriction to assume that $\{(\nu - \nu_l)|\nu - \nu_l|^{-1}\}$ has a limit $0 \neq \rho \in \partial\Sigma_1|_\nu$ as $l \to \infty$, and that

$$\varepsilon^{i,j} = \lim_{l \to \infty} k^{ij}(\nu_l)/|\alpha(\nu_l)| \quad \text{exists}. \tag{3.43}$$

Since $\beta(\nu_l)/|\alpha(\nu_l)| \to 0$, we get that

$$\partial\mathcal{N}^s_P(\nu, \rho) \supseteq \ker((\varepsilon^{i,j})_{i,j=1,2}) \quad \text{for } s = 1, 2. \tag{3.44}$$

Now, we want to show that $\ker((\varepsilon^{i,j})_{i,j=1,2}) \neq \{0\}$. As we have $\det k = 0$, we get

$$\lim_{\nu_l \to \nu} \big((k_{11}k_{22} - k_{12}k_{21})/(|\alpha(\nu_l)|^2\big) = 0. \tag{3.45}$$

So, we get the rank of the matrix $\big(\lim_{\nu_l \to \nu} k_{ij}/|\alpha(\nu_l)|\big)$ is strictly less than 2. By the rank-nullity theorem we get that the dimension of the kernel of $\big(\lim_{\nu_l \to \nu} k_{ij}/|\alpha(\nu_l)|\big)$ is greater or equal to 1. By that we get $\ker((\varepsilon^{i,j})_{i,j=1,2}) \neq \{0\}$. □

We will introduce some spaces that we are going to use. Let $H^{r,s}$ be the space of $u \in \mathcal{S}'$ satisfying

$$\|u\|^2_{(r,s)} = (2\pi)^{-n} \int |\hat{u}(\tau,\xi)|^2 \langle(\tau,\xi)\rangle^{2r} \langle(\tau,\vartheta)\rangle^{2s} d\tau d\xi < \infty, \tag{3.46}$$

where ϑ is given by (3.32). We say that $u \in H^{r,s}$ at $\nu \in T^*\mathbb{R}^n \setminus 0$, that is, $\nu \notin \mathrm{WF}^{r,s}(u)$, if $u = u_1 + u_2$, where $u_1 \in H^{r,s}$ and $\nu \notin \mathrm{WF}(u_2)$. Note that $H^{r,s-t} \subseteq H^{r,s}$ when $t \geq 0$.

Proposition 3.21. *Assume that P is a 2×2 system of pseudodifferential operators of order 1 on $\mathbb{R}^n$, on the form (3.23) with $K \in \mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}(S(\vartheta, g)))$ near $\nu_0 \in \Sigma_2$. Let $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^2)$ and assume $Pu \in H^{r,s}$ at ν_0. Then, for every $\delta > 0$ we can find $c_\delta > 0$ and $v_\delta \in H^{r,s+1}$ at ν_0, such that $u_\delta = u - v_\delta$ satisfies*

$$|\hat{u}_\delta(\tau,\xi)| \leq C_{\delta,N} \langle(\tau,\xi)\rangle^{-N}, \quad \forall N, \tag{3.47}$$

when $|\tau| > c_\delta(\langle\xi\rangle^\delta + \langle\vartheta\rangle)$.

Proof. Without loss of generality, we will assume that $\delta \leq \mu$ is fixed, where $\mu = k_0/(k_0+1)$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfy $\chi(r) = 1$ when $|r| \leq 1$. Then we have

$$\phi_{\epsilon,\delta}(\tau,\xi) = \chi(\epsilon|\tau|(\langle\xi\rangle^\delta + \langle\vartheta\rangle)^{-1}) \in S^0_{\delta,0}, \quad \forall \epsilon\delta > 0, \tag{3.48}$$

since $d\phi_{\epsilon,\delta}$ is supported, where $|\tau| \approx \langle\xi\rangle^\delta + \langle\vartheta\rangle$. Put $v_\delta = (1 - \phi_{\epsilon,\delta})(D)u$, then $u_\delta = \phi_{\epsilon,\delta}(D)u$ satisfies (3.47), $\forall \epsilon\delta > 0$.

In the support of $1 - \phi_{\epsilon,\delta}$ we find $|\det p| > c\vartheta_\delta^2$ for small ϵ, where

$$\vartheta_\delta = \langle(\tau,\xi)\rangle^\delta + \langle(\tau,\vartheta)\rangle \tag{3.49}$$

is a weight for the metric $g_\delta = |dt|^2 + |dx|^2 + (|d\tau|^2 + |d\xi|^2)/\langle(\tau,\xi)\rangle^{2\delta}$. Since $\delta \leq \mu$, we find $P \in \mathrm{Op}\, S(\vartheta_\delta, g_\delta)$ when $|\tau| \leq C|\xi|$. Thus, we may construct $E \in \mathrm{Op}\, S(\vartheta_\delta^{-1}, g_\delta) \subseteq \Psi^{-\delta}_{\delta,0}$ such that $EP \cong (1 - \phi_{\epsilon,\delta}(D))\boldsymbol{I} \bmod \mathcal{C}^\infty$, microlocally near ν_0. Since E preserves wavefront sets, and $\langle(\tau,\vartheta)\rangle\sigma(E) \in S^0_{0,0}$, we find $v_\delta \cong EPu \in H^{r,s+1}$ at ν_0. □

Let $H_*^{r,s}$ be the Banach space of $u \in \mathcal{S}'$, satisfying

$$(\|u\|_*^{r,s})^2 = (2\pi)^{-n} \int |\hat{u}(\tau,\xi)|^2 \langle\xi\rangle^{2r} \langle\vartheta\rangle^{2s} d\tau d\xi < \infty. \tag{3.50}$$

If $u \in H_*^{r,s}$, then we get $u|_{t=\rho} \in H^{r,s}$ for almost all ρ, by Fubini's theorem. If $u \in \mathcal{S}'$ satisfies (3.47), then

$$\|u\|_*^{r-\delta s_-,s} \le C_{r,s}(\|u\|_{r,s}+1) \le C'_{r,s}(\|u\|_*^{r+\delta s_+,s}+1),\ \forall r,s \in \mathbb{R}, \tag{3.51}$$

where $s_\pm = \max(\pm s, 0)$. Hence, we lose only $O(\delta)$ derivatives when taking restriction to $\{t = r\}$, for almost all r.

Definition 3.22. Let $u \in \mathcal{S}'(\mathbb{R}^n)$, and assume $\xi \neq 0$ in $\mathrm{WF}(u)$. We say that $u \in H_*^{r,s}$ at (t_0, x_0, ξ_0), that is, $(t_0, x_{,0}, \xi_0) \notin \mathrm{WF}_*^{r,s}(u)$, if there exists $\phi(t,x,\xi) \in \mathcal{C}^\infty(\mathbb{R}, S^0_{1,0})$ such that $\phi(t,x,D_x)u \in H_*^{r,s}$ and $\underline{\lim}_{\lambda\to\infty}|\phi(t_0,x_0,\lambda\xi_0)| \neq 0$.

We have

$$(t_0,x_0,\xi_0) \notin \mathrm{WF}_*^{r,s}(u) \Rightarrow (x_0,\xi_0) \notin \mathrm{WF}^{r,s}(u_\rho), \tag{3.52}$$

for almost all ρ close to t_0, where $u_\rho = u|_{t=\rho}$. If $\xi \neq 0$ in $\mathrm{WF}(u)$, then from [Den82b, Lemma 2.3], we get that

$$\pi_0(\mathrm{WF}^{r,0})(u) = \mathrm{WF}_*^{r,0}(u), \tag{3.53}$$

where $\pi_0(t,x;\tau,\xi) = (t,x,\xi)$. The following lemma gives the result for the more general wave-front sets:

Lemma 3.23. *Assume that $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies* (3.47)*. Then Au satisfies* (3.47)*, for any $A \in \mathcal{C}^\infty(\mathbb{R}, \Psi^\nu_{\delta,0})$. We also obtain*

$$\mathrm{WF}_*^{r-\delta s_-,s}(u) \subseteq \pi_0(\mathrm{WF}^{r,s}(u)) \subseteq \mathrm{WF}_*^{r+\delta s_+,s}(u), \tag{3.54}$$

where $s_\pm = \max(\pm s, 0)$ and $\pi_0(t,x;\tau,\xi) = (t,x,\xi)$. Since $u \in \mathcal{C}^\infty$ in $\pi_0^{-1}(\Sigma_2)\setminus\Sigma_2$ by (3.47)*, we find*

$$\iota_0(\mathrm{WF}_*^{r-\delta s_-,s}(u)) \subseteq \mathrm{WF}^{r,s}(u) \subseteq \iota_0(\mathrm{WF}_*^{r+\delta s_+,s}(u)) \quad \text{on } \Sigma_2, \tag{3.55}$$

where $\iota_0(t,x,\xi) = (t,x;0,\xi)$.

Note that $H^r = H^{r,0}$ is the usual Sobolev space.

Changing the notation, let $x_1 = t$, $x' = (x_2, ..., x_{d_0})$, then $x = (x_1, x', x'') \in \mathbb{R}\times\mathbb{R}^{d_0-1}\times\mathbb{R}^{n-d_0}$. Introduce the symbol classes $S^{r,s} = S(\langle\xi\rangle^r h^{-s}, g)$ where $h^{-2} = 1 + |\xi_1|^2 + |\xi'|^{2k_0+2}/\langle\xi\rangle^{2k_0}$ and

$\langle\xi\rangle$ are weights for the metric g defined by

$$g_{x,\xi}(dx, d\xi) = |dx|^2 + |d\xi|^2 h^2. \tag{3.56}$$

Let $\Psi^{r,s} = \operatorname{Op} S^{r,s}$ be the corresponding pseudodifferential operators, which maps $H^{r,s}$ into L^2. Returning to the old notation where using t instead of x_1, and assume that P be of the form in proposition 3.19, we get $P \in \Psi^{0,1}$ when $K \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g))$ because we can notice that $h^{-2} = \vartheta^2$ and by that we have $S^{0,1} = S(\vartheta, g)$.

We are going to consider the following $N \times N$ system

$$Q = q \operatorname{Id}_N + Q_1 + Q_0. \tag{3.57}$$

Here q is a scalar operator with symbol

$$q(t, x; \tau, \xi) = \tau(\tau + \beta(t, x, \xi)), \tag{3.58}$$

where $\beta \in S(\vartheta, g)$ is homogeneous and satisfies (3.21). We will assume

$$Q_1 = A_0 D_t + A_1, \tag{3.59}$$

with $A_j \in S(\vartheta^j, g)$, and $Q_0 \in S(1, g)$. We are going to study the following Cauchy Problem:

$$\begin{aligned} Qu &= f \\ (u, D_t u)|_{t=0} &= (u_0, u_1). \end{aligned} \tag{3.60}$$

Since we are going to assume that $\xi \neq 0$ in $\mathrm{WF}(u)$, the restrictions are well defined.

Proposition 3.24. *Assume that* $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$ *satisfies* (3.60)*, and* $\xi \neq 0$ *in* $\mathrm{WF}(u)$*. If* $u_0 \in H^{r,s}$*,* $u_1 \in H^{r,s-1}$ *at* (x_0, ξ_0)*,* $f \in H_*^{r,s-1}$ *at* (t, x_0, ξ_0) *for* $0 \leq t \leq t_0$*, and* $\xi_0' = 0$*, then* $u \in H_*^{r,s}$ *at* (t_0, x_0, ξ_0)*.*

Proof. By conjugating with an elliptic, scalar operator with symbol in $S(\langle\xi\rangle^r \vartheta^s, g)$, we may assume that $r = s = 0$. We will reduce to a first order symmetric system. Let $v_1 = u$, and $v_2 = \lambda D_t u$, where $\lambda \in \operatorname{Op} S(\vartheta^{-1}, g)$ has symbol ϑ^{-1}, so $v_2 = \lambda D_t v_1$. Then $V = {}^t(v_1, v_2) \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^{2N})$, $\xi \neq 0$ in $\mathrm{WF}(V)$, and V satisfies

$$\begin{aligned} PV &= F \\ V|_{t=0} &= V_0. \end{aligned} \tag{3.61}$$

Here $P = \operatorname{Id}_{2N} D_t + K$, $F = {}^t(0, \lambda f)$, $V_0 = {}^t(u_0, \lambda u_1)$, and $K \in \operatorname{Op} S(\vartheta, g)$ has symbol equal to

$$k \cong \begin{pmatrix} 0 & -\vartheta \operatorname{Id}_N \\ 0 & \beta \operatorname{Id}_N \end{pmatrix} \mod S(1, g). \tag{3.62}$$

We find $V_0 \in H^{0,0} = L^2$ at (x_0, ξ_0) and $F \in H^{0,0}_*$ at (t, x_0, ξ_0) for $0 \le t \le t_0$. Thus the result follows from the following proposition. ☐

Proposition 3.25. *Let $P = D_t \operatorname{Id}_N + K$ where $K \in \operatorname{Op} S(\vartheta, g)$ has symbol which is diagonalizable in $S(1, g)$ with eigenvalues 0 and β mod $S(1, g)$, and $\beta \in \mathcal{C}^\infty$ is homogeneous, satisfying* (3.21). *Assume that $V \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$, $\xi \neq 0$ in $\mathrm{WF}(V)$ and V satisfies* (3.61). *If $V_0 \in L^2$ at (x_0, ξ_0), $F \in H^{0,0}_*$ at (t, x_0, ξ_0) for $0 \le t \le t_0$, and $\xi_0' = 0$, then $V \in H^{0,0}_*$ at (t_0, x_0, ξ_0).*

Proof. Using that $\xi \neq 0$ in $\mathrm{WF}(V)$ and $\xi_0' = 0$, we get that $V \in H^{1,0}$ at $(t, x_0; \tau, \xi_0)$ when $\tau \neq 0$ and $0 \le t \le t_0$. By cut-off we may assume $u \in \mathcal{E}'$, then by [Den82b, Lemma 2.3] we only have to prove that $V \in L^2$ at $(t_0, x_0; 0, \xi_0)$. By the assumptions and Definition 3.22, we may find $\Psi(t, x, \xi) \in S(1, g)$ such that $\Psi \cong 1$ in a conical neighborhood of $\{(t, x_0, \xi_0) : 0 \le t \le t_0\}$, $\Psi F \in L^2$ when $0 \le t \le t_0$, and $\Psi V|_{t=0} \in L^2$. Let $\Psi_0 = \Psi|_{t=0}$ and consider the Cauchy problem

$$\begin{aligned} PU &= \Psi F \quad \text{for } 0 \le t \le t_0 \\ U|_{t=0} &= \Psi_0 V_0. \end{aligned} \tag{3.63}$$

By [Den88, Lemma 5.4], this has a unique solution $U \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}, \mathbb{C}^N)) \cap L^2$. Put $W = V - U$, then from they proof of Lemma 3.23; check [Den92], we know that $(t, x_0; \tau, \xi_0) \notin \mathrm{WF}(PW) = \mathrm{WF}((1 - \Psi)F)$, $\forall \tau$, $0 \le t \le t_0$, and $(x_0, \xi_0) \notin \mathrm{WF}(W|_{t=0}) = \mathrm{WF}((1 - \Psi_0)V_0)$. By using the parametrix in [Den89, Proposition 3.4], and the microlocal uniqueness obtained in [Den89, Section 5], we get that $(t_0, x_0; 0, \xi_0) \notin \mathrm{WF}(W)$. ☐

Let

$$r_u^*(\nu) = \sup\{r \in \mathbb{R} : u \in H^r \text{ at } \nu\}, \qquad \nu \in T^*X \setminus 0, \tag{3.64}$$

be the regularity function. Note that as S_1 and S_2 are tangent at Σ_2, so their Hamilton fields are parallel on Σ_2, and since Σ_2 is involutive, the Hamilton fields are tangent to Σ_2. Therefore Σ and Σ_2 are foliated by the bicharacteristics of Σ. Also, Dencker proved that $\partial\Sigma_1 \setminus (\Sigma_2 \times 0)$ is foliated by limit bicharactersitcs, which are liftings of bicharactersitics in Σ_2, and that $\partial\mathcal{N}_P^1 \cup \partial\mathcal{N}_P^2$ is foliated by limit Hamilton orbits, which are liftings of limits of Hamilton orbits, and are unique line bundles over limit bicharacteristics.

Theorem 3.26. *Let $P \in \Psi^m_{phg}$ be an $N \times N$ system of uniaxial type at $\nu_0 \in \Sigma_2$, and let $A \in \Psi^0_{phg}$ be a $1 \times N$ system such that the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at ν_0. Assume that $u \in \mathcal{D}'(X, \mathbb{C}^N)$ satisfies $\min(r^*_{Pu} + m - 1, r^*_{Au}) > r$ at ν_0. Then, $\mathrm{Pol}^r(u)$ is a union of $\mathcal{C}^\infty$ line bundles in $\mathcal{N}_A \cap \mathcal{N}_P$ over bicharacteristics of Σ in $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ near ν_0, where*

$\pi_1 : T^*X \setminus 0 \times \mathbb{C}^N \to T^*X$ *is the projection along the fibers.*

Proof. We may assume $s = 0$, $N = 2$, and P is of the form in proposition 3.19. By using Theorem D.2 for all terms in the expansion of A, we obtain that $A \in \mathcal{C}^\infty(\mathbb{R}, \Psi^0_{phg})$. Since $\sigma(A) \neq 0$ at $\nu_0 \in \Sigma_2$ (because if $\sigma(A) = 0$ we do not get that the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1), we can conjugate by suitable elliptic systems in $\mathcal{C}^\infty(\mathbb{R}, \Psi^0_{\mathrm{phg}})$ to get that $Au \cong u_1 \in H^\epsilon$ in a conical neighborhood U of ν_0, for some $\epsilon > 0$. Then, we find $\pi_1(\mathrm{Pol}^0(u)) = \mathrm{WF}^0(u_2)$ in U. By shrinking U and decreasing ϵ, we may assume $Pu \in H^\epsilon$ in U, (note that $P \in \Psi^{0,1}$) which gives $Qu = {}^tP^{\mathrm{co}}Pu \in H^{\epsilon,-1}$ there. Let $P = (P_{ij})_{i,j=1,2}$, and $Q = (q_{ij})_{i,j=1,2}$. Since $q_{21} = [P_{11}, P_{21}] \in \mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}\, S(\vartheta, g))$, we find $q_{22}u_2 \in H^{\epsilon,-1}$ in U. Similarly, we find $P_{22}u_2 \in H^{\epsilon,-1} \in U$. Since the result holds on Σ_1, we only have to prove that $u_2 \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ for $t < t_0$, implies $u_2 \in H^0$ at $(t_0, x_0; 0, \xi_0) = \nu_0$.

Thus assume that $u_2 \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ when $t < t_0$. Since $S(\vartheta, g) \in S^1_{\mu,0}$, and we may assume that $\delta \leq \mu$, Lemma 3.23 gives that u_2, $P_{22}u_2$, and $q_{22}u_2$ satisfies (3.47). Then $\xi \neq 0$ in $\mathrm{WF}(u_2)$, if we assume also that $\delta \leq \epsilon$ in (3.47) we find that $P_{22}u_2$, and $q_{22}u_2$ are in $H^{0,-1}_*$ in $\pi_0(U \cap \Sigma_2)$, and $u_2 \in H^0_*$ at $(t, x_0, \xi_0) \in \pi_0(U \cap \Sigma_2)$ for $t < t_0$ by (3.55). We also get that $D_t u_2 \in H^{0,-1}_*$ at $(t, x_0, \xi_0) \in \pi_0(U \cap \Sigma_2)$ when $t < t_0$, since $P_{22} \cong D_t \mod \mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}\, S(\vartheta, g))$. This gives

$$(u_2, D_t u_2)|_{\{t=r\}} \in H^0_* \times H^{0,-1}_* \quad \text{at} \quad (x_0, \xi_0), \tag{3.65}$$

for almost all $r < t_0$, close to t_0. Proposition 3.24 (with $N = 1$ and $Q = q_{22}$) gives $u_2 \in H^0_*$ at (t_0, x_0, ξ_0), and *Lemma* 3.23 gives $u_2 \in H^0$ at $(t_0, x_0; 0, \xi_0)$. □

Note that in [Den92], Dencker showed additionally under what assumptions we get $\mathrm{Pol}^r(u)$ is union of limits of Hamilton orbits in $\mathcal{N}_A \cap \mathcal{N}_P$ near $\nu_0 \in \Sigma_2$.

3.3 Propagation of polarization sets for systems of transverse type

We want to state Dencker's result regarding the propagation of polarization sets for systems of transverse type; see [Den95]. Let $P \in \Psi^m_{phg}$ be an $N \times N$ system of classical pseudodifferential operators on a smooth manifold X, $p = \sigma(P)$ be the principal symbol, and $\Sigma = (\det p)^{-1}(0)$ be the characteristics of P. Let

$$\Sigma_2 = \{(x,\xi) \in \Sigma : d(\det p) = 0 \text{ at } (x,\xi)\}, \tag{3.66}$$

and $\Sigma_1 = \Sigma \setminus \Sigma_2$. For systems of transverse type we have Σ is a union of two non radial hypersurfaces intersecting transversally at Σ_2. More precisely, the systems of transverse type is defined as the following

Definition 3.27. The system P is of transverse type at $\nu_0 \in \Sigma_2$ if

$$\Sigma_2 \text{ is a non-radial involutive manifold of codimension 2,} \tag{3.67}$$

$$\det p = e \cdot q, \text{ where } e \neq 0 \text{ and } q \text{ is real valued with Hessian having rank 2 and positivity 1,} \tag{3.68}$$

$$\dim \ker p = 2 \text{ on } \Sigma_2, \tag{3.69}$$

microlocally near ν_0.

Similar to the case of systems of uniaxial type, if $P \in \Psi^m_{phg}$ is of transverse type and $Pu \in H^r$ near $\nu \in \Sigma_1$, then P is of real principal type at ν. Let $\mathcal{N}^j_P$ be as in (3.16). In [Den95], Dencker modified slightly the definition of limit polarizations.

Definition 3.28. For $j = 1, 2$, the limit polarizations is defined by

$$\partial\mathcal{N}^j_P = \{(\nu, z) \in \Sigma_2 \times \mathbb{C}^N : z = \lim_{k\to\infty} z_k\}, \tag{3.70}$$

where $z_k \in \ker p(\nu_k)$ and $S_j \setminus \Sigma_2 \ni \nu_k \to \nu$.

$\partial\mathcal{N}^j_P$ is conical in ξ and linear in the fibers. Dencker showed that $\partial\mathcal{N}^j_P$ is a $\mathcal{C}^\infty$ line bundle over Σ_2, $j = 1, 2$, and that

$$\partial\mathcal{N}^1_P \cap \partial\mathcal{N}^2_P = \{0\} \quad \text{over } \Sigma_2. \tag{3.71}$$

Here, S_1 and S_2 are transverse at Σ_2, so their Hamilton fields are non-parallel on Σ_2. Since Σ_2 is involutive of codimension 2, the Hamilton fields of S_j are tangent to Σ_2 and generate the two-dimensional foliation of Σ_2. Moreover, $\partial\mathcal{N}^j_P$ is foliated by limit Hamilton orbits which are limits of Hamilton orbits in $\mathcal{N}^j_P$, and are unique line bundles over bicharacteristics in S_j at Σ_2 for $j = 1, 2$.

Theorem 3.29. *Let $P \in \Psi^m_{phg}$ be an $N \times N$ system of transverse type at $\nu_0 \in \Sigma_2$, and let $A \in \Psi^0_{phg}$ be a $1 \times N$ system such that the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at ν_0, and $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ is a hypersurface near ν_0. Assume that $u \in \mathcal{D}'(X, \mathbb{C}^N)$ such that $Pu \in H^{r-m+1}$ and $Au \in H^r$ at ν_0. Then $\mathrm{Pol}^r(u)$ is a union of (limit) Hamilton orbits in $\mathcal{N}_A \cap \mathcal{N}_P$. Here $\pi_1 : T^*X \times \mathbb{C}^N \to T^*X$ is the projection along the fibers.*

Note that in this case $M_A = S_j$ for some j, and $\mathcal{N}_A \cap \mathcal{N}_P$ is a union of (limit) Hamilton orbits.

3.4 Propagation of polarization sets for systems of generalized transverse type

In this section, we generalize Dencker's result stated in the previous section by considering the system to have its characteristic set is union of r_0 hypersurfaces intersecting transversally at an involutive manifold of codimension $d_0 \geq 2$, with $r_0 \geq 2$. Let $P \in \Psi^m_{phg}(X)$ be an $N \times N$ system of classical pseudodifferential operators on a smooth manifold X. Let $P \in \Psi^m_{phg}(X)$ be a pseudodifferential operator on a smooth manifold X. Let $p = \sigma(P)$ be the principal symbol and $\Sigma = p^{-1}(0)$ the characteristic set. Assume microlocally near $(x_0, \xi_0) \in \Sigma$,

$$\begin{aligned} &\Sigma = \cup_{j=1}^{r_0} S_j,\ r_0 \geq 2,\ \text{where } S_j \text{ are non-radial hypersurfaces intersecting transversally at} \\ &\quad \Sigma_2 = \cap_{j=1}^{r_0} S_j. \end{aligned} \tag{3.72}$$

Assume microlocally near (x_0, ξ_0),

$$\Sigma_2 \text{ is an involutive manifold of codimension } d_0 \geq 2. \tag{3.73}$$

Assume that

$$\text{the dimension of the fiber of } \mathcal{N}_P \text{ is equal to } r_0 \text{ at } \Sigma_2, \tag{3.74}$$

and

$$d^i(\det p) = 0 \text{ for } i < r_0 \text{ and } d^{r_0}(\det p) \neq 0 \quad \text{at } \Sigma_2. \tag{3.75}$$

Let

$$\mathcal{N}_P^j = \mathcal{N}_P|_{S_j \setminus \Sigma_2}, \tag{3.76}$$

$T_{\Sigma_2}\Sigma := \cup_{j=1}^{r_0} T_{\Sigma_2} S_j$, and let $\partial\Sigma_1 = T_{\Sigma_2}\Sigma / T\Sigma_2$. We let $\partial\mathcal{N}_P^j$ as in Definition 3.17, for $j = 1, ..., r_0$. We will assume that the fiber of

$$\partial N_P^1 \cap ... \cap \partial N_P^{r_0} = \{0\} \quad \text{over } \ \partial\Sigma_1 \setminus (\Sigma_2 \times 0). \tag{3.77}$$

Definition 3.30. The system P is of generalized transverse type at $\nu_0 \in \Sigma_2$, if (3.72)-(3.75) and (3.77) hold microlocally near ν_0.

Proposition 3.31. *Let $P \in \Psi^1_{phg}$ be an $N \times N$ system of generalized transverse type at $\nu_0 \in \Sigma_2$. Then by choosing suitable symplectic coordinates, we may assume that $X = \mathbb{R} \times \mathbb{R}^{n-1}$, $\nu_0 = (0; (0, ..., 1))$, and*

$$S_j = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta_j(t, x, \xi) = 0\}, \quad j = 1, ..., r_0, \tag{3.78}$$

microlocally near ν_0. Here β_j are real and homogeneous of degree 1 in ξ; with $\beta_1 \equiv 0$, satisfies in a conical neighborhood of ν_0

$$c|\xi'| \le |\beta_j| \le C|\xi'|, \quad j = 2, ..., r_0, \quad 0 < c < C, \tag{3.79}$$

where $(\tau, \xi', \xi'') \in \mathbb{R} \times \mathbb{R}^{d_0 - 1} \times \mathbb{R}^{n-d_0}$, which gives $\Sigma_2 = \{\tau = 0, \xi' = 0\}$. By conjugating P with an elliptic, scalar Fourier integral operators, and multiplying with elliptic $N \times N$ systems of order 0, we may assume that

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \mod \mathcal{C}^\infty, \tag{3.80}$$

microlocally near ν_0, where $E \in \Psi^1_{phg}$ is an elliptic $(N - r_0) \times (N - r_0)$ system and

$$F \cong \mathrm{Id}_{r_0} D_t + K(t, x, D_x) \mod \mathcal{C}^\infty. \tag{3.81}$$

Here $K(t, x, D_x) \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{phg})$ is a $r_0 \times r_0$ system, such that $k = \sigma(K)$ has 0, β_2,...,β_{r_0} as eigenvalues.

Proof. We will prove it in the same way Dencker proved the normal form for systems of uniaxial type. Since the result is local, we may assume $X = \mathbb{R}^n$. Because Σ_2 is involutive, we may choose symplectic, homogeneous coordinates $(x, \xi) \in T^*\mathbb{R}^n$ near $\nu_0 \in \Sigma_2$, so that $\nu_0 = (0; (0, ..., 1))$ and

$$\Sigma_2 = \{(x, \xi) \in T^*\mathbb{R}^n : \xi' = 0\}, \tag{3.82}$$

where $\xi = (\xi', \xi'') \in \mathbb{R}^{d_0} \times \mathbb{R}^{n-d_0}$. We may also assume

$$S_1 = \{(x, \xi) \in T^*\mathbb{R}^n : \xi_1 = 0\}, \tag{3.83}$$

near ν_0. Now, we rename $x_1 = t$, $(x_2, .., x_{d_0}) = x'$, and $(x_{d_0+1}, ..., x_n) = x''$. Since S_j is intersecting transversally with S_1 at Σ_2, we obtain

$$S_j = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta_j(t, x, \xi) = 0\}, \tag{3.84}$$

with β_j real and homogeneous of degree 1 in ξ, $\beta_1 \equiv 0$, and

$$c|\xi'| \le |\beta_j - \beta_k| \le C|\xi'|, \quad j \ne k, \tag{3.85}$$

in a conical neighborhood of ν_0. By taking $k = 1$, we obtain

$$c|\xi'| \leq |\beta_j| \leq C|\xi'|, \quad j = 2, ..., r_0. \tag{3.86}$$

Using that $\dim \mathcal{N}_P = r_0$ at Σ_2, we can find an $N \times N$ elliptic matrix b homogeneous of degree 0 in the ξ variables which maps $\operatorname{Im} p$ to $\{z \in \mathbb{C}^N; z_j = 0,\ j \leq r_0\}$ over Σ_2 near ν_0, and we can choose an $N \times N$ matrix a homogeneous of degree 0 in the ξ variables such that a^{-1} maps $\ker p$ onto $\{z \in \mathbb{C}^N; z_j = 0,\ j > r_0\}$ over Σ_2 near ν_0. Then we have

$$bpa = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & e \end{pmatrix} \tag{3.87}$$

such that e is an $(N - r_0) \times (N - r_0)$ matrix which is elliptic at ν_0, and s_{11}, s_{12}, s_{21}, vanish on Σ_2 near ν_0.

Now, we choose $N \times N$ systems of pseudodifferential operators A and B with principal symbols a, and b respectively. Then

$$BPA = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & E \end{pmatrix} \tag{3.88}$$

where its principal symbol is given by (3.87). As E is a system of order 1 which is elliptic at Σ_2, choose J to be its microlocal parametrix of order -1. Multiply BPA from the left with

$$B_1 = \begin{pmatrix} \mathrm{Id}_{r_0} & -S_{12}J \\ 0 & \mathrm{Id}_{N-r_0} \end{pmatrix}. \tag{3.89}$$

Multiply also $B_1 BPA$ from the right by

$$A_1 = \begin{pmatrix} \mathrm{Id}_{r_0} & 0 \\ -JS_{21} & \mathrm{Id}_{N-r_0} \end{pmatrix}. \tag{3.90}$$

By that we get

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \quad \mathrm{mod}\ \mathcal{C}^\infty, \tag{3.91}$$

microlocally near ν_0, where $E \in \Psi^1_{\mathrm{phg}}$ is an elliptic $(N - r_0) \times (N - r_0)$ system. If f is the principal symbol for F, then conditions (3.75) and (3.78) imply

$$\det f = c\tau \prod_{i=2}^{r_0} (\tau + \beta_i),\ 0 \neq c \in S^{-1}, \tag{3.92}$$

thus $\partial_\tau^{r_0}(\det f) = \det(\partial_\tau f) \neq 0$ at Σ_2. By Theorem D.1, and homogeneity, we may find homogeneous system $C_0 \in S^0$ such that

$$f = C_0(\tau \,\mathrm{Id}_{r_0} + k(t, x, \xi)), \tag{3.93}$$

where $\det C_0 \neq 0$ at Σ_2. By multiplication with an elliptic system, we may assume $C_0 \equiv \mathrm{Id}_{r_0}$. Thus, $\det f = \tau \prod_{i=2}^{r_0}(\tau + \beta_j)$, which implies that $k(t, x, \xi)$ has the eigenvalues 0, β_2,...,β_{r_0}. If $f_0 \in S^0$ is the term homogeneous of degree 0 in the expansion of F, then Theorem D.2, and homogeneity give

$$f_0 = B_{-1} f + B_0, \tag{3.94}$$

where $B_0 \in \mathcal{C}^\infty(\mathbb{R}, S^0)$ is independent of τ, and $B_{-1} \in S^{-1}$. By multiplying f with an operator with symbol $\mathrm{Id}_{r_0} - B_{-1}$, we may assume $B_{-1} \equiv 0$. By induction over lower order terms we obtain (3.81).

□

We want to introduce symbol classes adapted to the functions β_j defined in (3.79) for $j = 2, ..., r_0$. Let

$$\vartheta(\xi) = \langle \xi' \rangle, \tag{3.95}$$

where $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$, thus $\vartheta \approx 1 + |\beta_j|$. Consider the metric

$$g(dx, d\xi) = |dx|^2 + |d\xi'|^2 \langle \xi' \rangle^{-2} + |d\xi''|^2 \langle \xi \rangle^{-2}, \tag{3.96}$$

and $h^2 = \sup g/g^\sigma = \langle \xi' \rangle^{-2}$. We get that g is σ temperate, and $\beta_j \in S(\vartheta, g)$. See Appendix B. Moreover, using Taylor's formula we can write

$$\beta_j = a_j \xi', \tag{3.97}$$

with $a_j \in S^0$ is homogeneous of degree 0 in ξ.

Proposition 3.32. *Let*

$$P = \mathrm{Id}_{r_0} D_t + K(t, x, D_x) \tag{3.98}$$

be an $r_0 \times r_0$ system with $K \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{\mathrm{phg}})$, such that the eigenvalues of $k = \sigma(K)$ are 0, β_2,...,β_{r_0}. Then P is of generalized transverse type if and only if $k \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g))$.

Proof. First, let $k = (k_{ij})_{1 \le i,j \le r_0}$. Let $\alpha = (\alpha_1, ..., \alpha_{1r_0}) \in \mathcal{C}^\infty(\mathbb{R}, S^1)$, with $\alpha_i = (k_{i1}, ..., k_{ir_0})$ for $i = 1, ..., r_0$ homogeneous of degree 1 in ξ. By homogeneity,

$$k \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g)) \quad \Leftrightarrow \alpha = O(\beta_j), \tag{3.99}$$

for every $j = 2, ..., r_0$.

Assume that $\alpha = O(\beta_j)$ and $(\nu, \rho) \in \partial\Sigma_1$, $\rho \neq 0$. Choose $\Sigma \setminus \Sigma_2 \in \nu_l \to \nu$ such that

$(\nu-\nu_l)|\nu-\nu_l|^{-1} \to \rho/|\rho|$, $l \to \infty$. Let us define

$$\gamma_s^{ij}(\nu,\rho) := \lim_{\nu_l \to \nu} \frac{k_{ij}}{\beta_s}(\nu_l) \text{ for } s \in \{2,...,r_0\} \text{ and } 1 \leq i,j \leq r_0. \tag{3.100}$$

Since $\alpha = O(\beta_j)$ does not depend on τ, so the above definition; (3.100), is independent of the choice of ν_l. We get

$$\partial \mathcal{N}_P^1(\nu,\rho) = \ker((\gamma_s^{ij}(\nu,\rho))_{1\leq i,j\leq r_0}),\ \forall s \in \{2,...,r_0\}, \tag{3.101}$$

where $(\gamma_s^{ij}(\nu,\rho))_{1\leq i,j\leq r_0}$ denote the matrix with entries $\gamma_s^{ij}(\nu,\rho)$ for $1 \leq i,j \leq r_0$.

$$\partial \mathcal{N}_P^s(\nu,\rho) = \ker(-\operatorname{Id}_{r_0} + (\gamma_s^{ij}(\nu,\rho))_{1\leq i,j\leq r_0}) \text{ for } s \in \{2,...,r_0\}. \tag{3.102}$$

It is easy to see that the condition (3.77) is satisfied.

On the other hand, assume that $\alpha \neq O(\beta_j)$ at $\nu \in \Sigma_2$. Then, there exists a sequence $\nu_l = (t_l, x_l; 0, \xi_l) \to \nu$, such that

$$|\alpha(\nu_l)| > l|\beta_j(\nu_l)|, \quad \forall l \in \mathbb{N}. \tag{3.103}$$

It is no restriction to assume that $\{(\nu-\nu_l)|\nu-\nu_l|^{-1}\}$ has a limit $0 \neq \rho \in \partial\Sigma_1|_\nu$ as $l \to \infty$, and that

$$\varepsilon^{i,j} = \lim_{l\to\infty} k_{ij}(\nu_l)/|\alpha(\nu_l)| \text{ exists.} \tag{3.104}$$

Since $\beta_j(\nu_l)/|\alpha(\nu_l)| \to 0$, we get that

$$\partial \mathcal{N}_P^s(\nu,\rho) \supseteq \ker((\varepsilon^{i,j})_{1\leq i,j\leq r_0}) \quad \text{for } s = 1,...,r_0. \tag{3.105}$$

Now we want to show that $\ker((\varepsilon^{i,j})_{1\leq i,j\leq r_0}) \neq \{0\}$. Since, we have $\det k = 0$, we get $\det\left((\varepsilon^{i,j})_{1\leq i,j\leq r_0}\right) = 0$. Hence, we get that the rank of $(\varepsilon^{i,j})_{1\leq i,j\leq r_0}$ is less than or equal to $r_0 - 1$, which gives in turn that $\dim\ker((\varepsilon^{i,j})_{1\leq i,j\leq r_0})$ is greater than or equal to 1. □

Proposition 3.33. *Assume that P is an $r_0 \times r_0$ system of pseudodifferential operators of order 1 on $\mathbb{R}^n$, on the form (3.98), with $K \in \mathcal{C}^\infty(\mathbb{R}, \operatorname{Op}(S(\vartheta,g)))$ near $\nu_0 \in \Sigma_2$. Let $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^{r_0})$ and assume $Pu \in H^{r,s}$ at ν_0. Then, for every $\delta > 0$ we can find $c_\delta > 0$ and $v_\delta \in H^{r,s+1}$ at ν_0, such that $u_\delta = u - v_\delta$ satisfies*

$$|\hat{u}_\delta(\tau,\xi)| \leq C_{\delta,N}\langle(\tau,\xi)\rangle^{-N}, \quad \forall N, \tag{3.106}$$

when $|\tau| > c_\delta(\langle\xi\rangle^\delta + \langle\xi'\rangle)$.

Proof. Same proof as in Proposition 2.15 in [Den88]. We can assume that $\delta \leq 1$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$

satisfy $\chi(r) = 1$ when $|r| \leq 1$. Then we have

$$\phi_{\varepsilon,\delta}(\tau, \xi) = \chi(\varepsilon|\tau|(\langle\xi\rangle^{\delta} + \langle\xi'\rangle)^{-1}) \in S^0_{\delta,0}, \quad \forall\varepsilon\delta > 0, \tag{3.107}$$

since $\langle\xi\rangle \cong \langle(\tau, \xi)\rangle$ in $\operatorname{supp} d\phi_{\varepsilon,\delta}$. Put $v_\delta = (1 - \phi_{\varepsilon,\delta})(D)u$, then $u_\delta = \phi_{\varepsilon,\delta}(D)u$ satisfies (3.106), $\forall\varepsilon\delta > 0$.

In the support of $1 - \phi_{\varepsilon,\delta}$, we have $|\det p| > c\vartheta_\delta^{r_0}$ for small ε, where

$$\vartheta_\delta(\tau, \xi) = \langle(\tau, \xi)\rangle^{\delta} + \langle(\tau, \xi')\rangle, \tag{3.108}$$

is a weight for the metric $g_\delta = |dt|^2 + |dx|^2 + (|d\tau|^2 + |d\xi|^2)/\langle(\tau, \xi)\rangle^{2\delta}$. Hence, for small ε we get $P \in \operatorname{Op} S(\vartheta_\delta, g_\delta)$ when $|\tau| \leq C|\xi|$. Therefore, we can construct $E \in \operatorname{Op} S(\vartheta_\delta^{-1}, g_\delta) \subseteq \Psi^{-\delta}_{\delta,0}$ such that $EP \cong (1 - \phi_{\varepsilon,\delta}(D)) \operatorname{Id} \bmod \mathcal{C}^\infty$, microlocally near ν_0. As E preserves wavefront sets, and we have $\langle(\tau, \xi')\rangle\sigma(E) \in S^0_{0,0}$, we get that $v_\delta \cong EPu \in H^{r,s+1}$ at ν_0. □

We change notation, and put $x_1 = t$, $x' = (x_2, ..., x_{d_0})$, which gives $x = (x_1, x', x'') \in \mathbb{R} \times \mathbb{R}^{d_0-1} \times \mathbb{R}^{n-d_0}$. Introduce the symbol classes $S^{r,s} = S(\langle\xi\rangle^r h^{-s}, g)$ where $h^{-2} = 1 + |\xi_1|^2 + |\xi'|^2$ and $\langle\xi\rangle$ are weights for the metric g defined by

$$g_{x,\xi}(dx, d\xi) = |dx|^2 + |d\xi|^2 h^2. \tag{3.109}$$

Let $\Psi^{r,s} = \operatorname{Op} S^{r,s}$ be the corresponding pseudodifferential operators, which maps $H^{r,s}$ into L^2. Returning to the old notation where using t instead of x_1, and assume that $P \in \mathcal{C}^\infty(\mathbb{R}, \operatorname{Op}(S(\vartheta, g)))$ be as in Proposition 3.32, we get $P \in \Psi^{0,1}$.

We are going to consider the following $N \times N$ system

$$Q = q \operatorname{Id}_N + \sum_{i=0}^{r_0-1} Q_i. \tag{3.110}$$

Here q is a scalar operator with symbol

$$q(t, x; \tau, \xi) = \tau \prod_{j=2}^{r_0} (\tau + \beta_j), \tag{3.111}$$

where $\beta_j \in S(\vartheta, g)$ is homogeneous and satisfies (3.79). We will assume

$$Q_i = \sum_{k=0}^{i} A^i_{i-k} D^k_t, \tag{3.112}$$

with $A^i_{k'} \in \operatorname{Op} S(\vartheta^{k'}, g)$. We are going to study the following Cauchy Problem:

$$\begin{aligned} Qu &= f \\ D^k_t u|_{t=0} &= u_k, \text{ for } k = 0, ..., r_0 - 1. \end{aligned} \tag{3.113}$$

We are going to assume that the $\xi \neq 0$ in $\operatorname{WF}(u)$. Hence, the restrictions are well defined.

Proposition 3.34. *Assume that $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$ satisfies* (3.113), *and $\xi \neq 0$ in* WF(u). *If $u_k \in H^{r,s-k}$ at (x_0, ξ_0) for $k = 0, ..., r_0 - 1$, $f \in H_*^{r,s-r_0+1}$ at (t, x_0, ξ_0) for $0 \leq t \leq t_0$, and $\xi_0' = 0$, then $u \in H_*^{r,s}$ at (t_0, x_0, ξ_0).*

Proof. By conjugating with an elliptic, scalar operator with symbol in $S(\langle\xi\rangle^r \vartheta^s, g)$, we may assume that $r = s = 0$. We will reduce to a first order symmetric system. Let $v_k = \lambda^{k-1} D_t^{k-1} u$ for $k = 1, ..., r_0$, where $\lambda \in \operatorname{Op} S(\vartheta^{-1}, g)$ has symbol ϑ^{-1}. Hence, $v_k = \lambda D_t v_{k-1}$ for $k = 2, ..., r_0$. Then $V = {}^t(v_1, ..., v_{r_0}) \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^{r_0 N})$, $\xi \neq 0$ in WF(V), and V satisfies

$$\begin{aligned} PV &= F \\ V|_{t=0} &= V_0. \end{aligned} \tag{3.114}$$

Here $P = \operatorname{Id}_{r_0 N} D_t + K$, $F = (0, ..., 0, \lambda^{r_0-1} f)$, $V_0 = (u_0, \lambda u_1, ..., \lambda^{r_0-1} u_{r_0-1})$, and $K \in \operatorname{Op} S(\vartheta, g)$ has principal symbol $k = (k_{ij})_{1 \leq i,j \leq r_0}$ such that

$$k_{ii+1} = -\vartheta \operatorname{Id}_N \text{ for } 1 \leq i \leq r_0 - 1, \tag{3.115}$$

$$k_{r_0 j} = \sum \beta_{i_1} ... \beta_{i_{r_0-j+1}} \vartheta^{j-r_0} \text{ for } 2 \leq j \leq r_0, \tag{3.116}$$

$$k_{ij} = 0 \text{ elsewhere} \tag{3.117}$$

where the sum in (3.116) is such that $2 \leq i_1 \leq j$ and $i_{k-1}+1 \leq i_k \leq j+k-1$ for $2 \leq k \leq r_0-j+1$.

We find $V_0 \in H^{0,0} = L^2$ at (x_0, ξ_0) and $F \in H_*^{0,0}$ at (t, x_0, ξ_0) for $0 \leq t \leq t_0$. Thus the result follows from the next proposition, which we will state after the following Lemma. □

Lemma 3.35. *When the above assumptions are satisfied we get*

$$\int_0^\varepsilon \|V\|(t) dt \leq C\varepsilon^{1/2} \left(\int_0^\varepsilon \|D_t V + KV\|(t) dt + \|V\|(0) \right) \tag{3.118}$$

for $V \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^{r_0 N})$, if $\varepsilon > 0$ is sufficiently small. Here $\|.\|(t)$ denotes the L^2 norm in the x variables, depending on t.

Proof. We are going to prove it in a similar way as the proof of Lemma 5.2 in [Den88]. As k is diagonalizable in $S(1, g)$, we get that $k = \sum_{j=1}^{r_0} \beta_j \pi_j$, with $\beta_1 = 0$, where $\pi_j(t, x, \xi) \in S(1, g)$ is the projection on the eigenvectors corresponding to the eigenvalues β_j along the others when $\xi' \neq 0$. k is symmetrizable with symmetrizer $M = \sum \pi_j^* \pi_j$, that is $M > 0$ amd Mk is symmetric. Note that k is symmetrizable means there exists symmetric $N \times N$ system $M(t, x, \xi) \in S(1, g)$ such that $0 < c \leq M$ and $Mk - (Mk)^* \in S(1, g)$. If we put $\|V\|(t)$ to be the L^2 norm in the x-variables, depending on t, and we put

$$\|V\|_M^2(t) = \langle MV, V \rangle = \int \langle MV(t,x), \overline{V(t,x)} \rangle dx \tag{3.119}$$

then

$$c \leq \frac{\|V\|_M^2(t)}{\|V\|^2(t)} \leq C \tag{3.120}$$

If $D_tV + KV = F$, we obtain

$$\begin{aligned}\partial_t\|V\|_M^2(t) &= \langle(\partial_t M - i(MK - K^*M))V, V\rangle(t) + \langle MF, V\rangle(t) + \langle MV, F\rangle(t)\\ &\leq C(\|V\|_M^2(t) + \|F\|_M^2(t)).\end{aligned} \tag{3.121}$$

By Grönwall's inequality we get, for bounded t,

$$\|V\|_M^2(t) \leq C\Big(\|V\|_M^2(0) + \int_0^t \|F\|_M^2(s)ds\Big), \tag{3.122}$$

so (3.120) and integration gives the result. □

Proposition 3.36. *Let $P = D_t \operatorname{Id}_N + K$ where $K \in \operatorname{Op} S(\vartheta, g)$ has symbol which is diagonalizable in $S(1,g)$ with eigenvalues $0, \beta_2, ..., \beta_{r_0}$ mod $S(1,g)$, and $\beta_j \in \mathcal{C}^\infty$ is homogeneous, satisfying (3.79) for $j = 2, ..., r_0$. Assume that $V \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$, $\xi \neq 0$ in $\mathrm{WF}(V)$ and V satisfies (3.114). If $V_0 \in L^2$ at (x_0, ξ_0), $F \in H_*^{0,0}$ at (t, x_0, ξ_0) for $0 \leq t \leq t_0$, and $\xi_0' = 0$, then $V \in H_*^{0,0}$ at (t_0, x_0, ξ_0).*

The condition on k means that there exists a basis of eigenvectors $\{v_j\} \in S(1,g)$, with eigenvalues 0, β_2,...,β_{r_0} mod $S(1,g)$.

The above proposition is similar to Proposition 3.25 in case systems of uniaxial type. To prove Proposition 3.25, Dencker used the parametrix constructed in [Den89] for $P = D_t \operatorname{Id}_N + K$, where $K \in \operatorname{Op} S(\vartheta, g)$ has principal symbol k satisfying the conditions in Proposition 3.25, and ϑ, and g are as in (3.32), and (3.33) respectively, and he used the microlocal uniqueness; see [Den89]. In our case; case of generalized transverse type, we are using different weight and metric, but we can still construct a parametrix for the $N_0 \times N_0$ matrix, where $K \in \operatorname{Op} S(\vartheta, g)$ has principal symbol k satisfying the conditions in Proposition 3.36, and we can prove microlocal uniqueness as in [Den89]. We will show this before giving the proof of Proposition 3.36. The steps are similar to that in [Den89], except some details are changed.

Consider $P = D_t \operatorname{Id}_{N_0} + K(t, x, D_x)$ as in Proposition 3.36; that is, $K \in \operatorname{Op} S(\vartheta, g)$, has principal symbol $k(t, x, \xi)$ satisfying

$$\begin{aligned}&k \text{ is diagonalizable in } S(1,g) \text{ with real eigenvalues } \{\beta_j\}_{j=1,\dots,r_0} \text{ homogenous of degree 1}\\ &\text{in } \xi,\ \text{satisfying (3.79) and } \beta_1 = 0.\end{aligned} \tag{3.123}$$

Hence, there exists a base of left (right) eigenvectors in $S(1,g)$. The dimension of the eigenspace corresponding to β_j is constant outside Σ_2. Let $\pi_j(t,x,\xi) \in S(1,g)$ be the projection on the eigenvectors corresponding to the eigenvalue β_j along the others when $\xi' \neq 0$, and extended by continuity. Then we have

$$k = \sum_{j=1}^{r_0} \beta_j \pi_j, \tag{3.124}$$

and k is symmetrizable with symmetrizer $M = \sum \pi_j^* \pi_j$, that is $M > 0$, and MK is symmetric.

We want to solve the microlocal Cauchy problem

$$\begin{cases} D_t E + KE \cong 0 \\ E|_{t=0} \cong \mathrm{Id}_{N_0} \end{cases} \tag{3.125}$$

microlocally near $(0,(0,\xi_0),(0,\xi_0))$, $\xi_0' = 0$, with $E : \mathcal{E}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^n)$. Hence, we should solve the eiconal equations

$$\begin{cases} \partial_t \phi_j + \beta_j(t,x,d_x\phi_j) = 0 \\ \phi_j(0,x,\eta) = \langle x,\eta\rangle \end{cases} \quad \text{for } j = 1,...,r_0. \tag{3.126}$$

By Hamilton-Jacobi, this has a unique local solution, homogeneous of degree 1 in η.

Lemma 3.37. *Let ϕ_j solve* (3.126) *with β_j satisfying* (3.79)*. Then we get that $\varphi_j(t,x,\eta) = \phi_j(t,x,\eta) - \langle x,\eta\rangle$ satisfies*

$$\varphi_j \equiv 0 \quad \textit{when } \eta' = 0 \quad \forall j. \tag{3.127}$$

Proof. Using (3.97), the eiconal equation gives

$$\partial_t \varphi_j + a_j(t,x,\eta + d_x\varphi_j)(\eta' + d_{x'}\varphi_j) = 0, \tag{3.128}$$

and $\varphi_j(0,x,\eta) \equiv 0$. When $\eta' = 0$ we get $\partial_t\varphi_j + a_j(t,x,\eta+d_x\varphi_j)(d_{x'}\varphi_j) = 0$, so uniqueness gives $\varphi_j \equiv 0$ when $\eta' = 0$. □

Let $E_j : \mathcal{E}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^n)$, $j = 1,...,r_0$ be oscillatory integrals defined by

$$E_j u(t,x) = (2\pi)^{1-n} \int\int e^{i(\phi_j(t,x,\eta) - \langle y,\eta\rangle)} a_j(t,x,\eta) u(y)\, dy d\eta, \tag{3.129}$$

with $a_j \in S(1,g)$. Assume that a_j is supported in canonical neighborhood of $\{\eta' = 0\}$.

By Lemma D.3, we get

$$PE_j u(t,x) = (2\pi)^{1-n} \int\int e^{i(\phi_j(t,x,\eta) - \langle y,\eta\rangle)} b_j(t,x,\eta) u(y)\, dy d\eta, \tag{3.130}$$

where

$$b_j(t,x,\eta) = (\partial_t\phi_j \,\mathrm{Id}_{N_0} + k(t,x,d_x\phi_j))a_j + L_j a_j + R_j a_j, \tag{3.131}$$

R_j is continuous $S(\vartheta^i h^l, g) \to S(\vartheta^i h^{l+1}, g)$, $\forall i, j, l$, with $h^2 = \langle \xi' \rangle^{-2}$, and

$$L_j a_j = D_t a_j + \sum_i (\partial_{\xi_i} k)(t, x, d_x \phi_j) D_{x_i} a_j + M_j a_j, \tag{3.132}$$

with $M_j \in S(1, g)$.

Lemma 3.38. *Assuming* (3.123), *we can find $a_j \in S(1, g)$ such that $b_j \in S(\vartheta^{-N}, g)$, $\forall N$, in* (3.131), *$j = 1, ..., r_0$ and*

$$\sum_j a_j|_{t=0} \equiv \operatorname{Id}_{N_0}. \tag{3.133}$$

Proof. Let $a_j \sim a_j^0 + a_j^{-1} + ...$, where $a_j^{-k} \in S(\vartheta^{-k}, g)$. The dominant term in (3.131) is

$$(\partial_t \phi_j \operatorname{Id}_{N_0} + k(t, x, d_x \phi_j)) a_j^0 = \sum_i \phi_j^* ((\beta_i - \beta_j) \pi_i) a_j^0,$$

where $\phi_j^* f = f(t, x, d_x \phi_j)$, so we get $a_j^0 \in \operatorname{Im} \phi_j^* \pi_j = \cap_{i \neq j} \ker \phi_j^* \pi_i$. If we take $a_j^0 = \pi_j(0, x, \eta)$ at $t = 0$, we obtain $\sum a_j^0|_{t=0} = \operatorname{Id}_{N_0}$. The term in $S(\vartheta^{-r}, g)$, $r \geq 0$ in the expansion (3.131) is given by

$$\sum_i \phi_j^* ((\beta_i - \beta_j) \pi_i) a_j^{-r-1} + L_j a_j^{-r} + R_j a_j^{1-r}, \tag{3.134}$$

since $h \leq \vartheta^{-1}$ ($a_j^1 \equiv 0$). Now, $\phi_j^*(\beta_i - \beta_j) \in S(\vartheta, g)$ is invertible modulo $S(\vartheta^{-\infty}, g)$ according to (3.79) when $j \neq i$, since $d_{x'} \phi_j = O(|\eta'|)$ by (3.127). Hence, it suffices to solve successively, with suitable initial data,

$$(\phi_j^* \pi_j)(L_j a_j^{-r} + \tilde{R}_j a_j^{1-r}) = 0, \quad r \geq 0, \tag{3.135}$$

where $a_j^1 \equiv 0$, and $(\operatorname{Id}_{N_0} - \phi_j^* \pi_j) a_j^{-r}$ has been determined in the previous step. Here $\tilde{R}_j$ is continuous $S(\vartheta^i, g) \to S(\vartheta^{i-1}, g)$, $\forall i$.

Now, let $\{v_j^i\}_i \in S(1, g)$ be a base for $\operatorname{Im} \phi_j^* \pi_j$, and consider $\sum_i \alpha_i v_j^i$, $\alpha_i \in S(\vartheta^{-r}, g)$. (Such a base exists since it follows from the proof of Lemma D.3 that ϕ_j^* preserves the metric g). Using $\pi_j \pi_l = \delta_{jl} \pi_l$, we get

$$(\partial \pi_j) \pi_l + \pi_j \partial \pi_l = \delta_{jl} \partial \pi_l \quad \forall i, j, l,$$

which gives

$$(\phi_j^* \pi_j) \phi_j^* (\partial \pi_l) v_j^i = \delta_{jl} \phi_j^* (\partial \pi_l) v_j^i - \phi_j^* (\partial \pi_j) \phi_j^* (\pi_l) v_j^i = 0 \quad \forall i, j, l.$$

Since $\partial k = \sum_i ((\partial \beta_i) \pi_i + \beta_i \partial \pi_i)$, we get

$$(\phi_j^* \pi_j) L_j \sum_i \alpha_i v_j^i = \sum_i \gamma_i v_j^i,$$

where

$$\gamma_i = D_t \alpha_i + \sum_l \phi_j^*(\partial_{\xi_l}\beta_j) D_{x_l}\alpha_i + \sum_l \mu_i^l \alpha_l \in S(1,g), \tag{3.136}$$

with $\mu_i^l \in S(1,g)$. If we introduce local g orthogonal coordinates, then $\sum_l \phi_j^*(\partial_{\xi_l}\beta_j)D_{x_l}$ transforms into a uniformly bounded smooth vector field. Therefore, by adding a suitable linear combination of v_j^i to each column of a_j^{-r} we may solve (3.135) for all $1 \le j \le r_0$, with initial data making

$$\pi_l \sum_j a_j^{-r} = \begin{cases} \pi_l & r = 0 \\ 0 & r > 0 \end{cases} \quad \forall l, \text{ at } t = 0. \tag{3.137}$$

If we do this recursively for $r > 0$, we obtain the lemma. □

Lemma 3.39. *If $a(t,x,\eta) \in \bigcap_N S(\vartheta^{-N}, g)$ has support where $|\eta'| \le c|\eta''|$, and $\varphi(t,x,\eta)$ is homogeneous of degree 1 satisfying (3.127), then*

$$\tilde{a}(t,x,y',\eta'') = \int e^{i(\varphi(t,x,\eta)+\langle x'-y',\eta'\rangle)} a(t,x,\eta)d\eta' \in S^0_{1,0,0}. \tag{3.138}$$

Here $S^v_{1,0,0}$ is defined by

$$|D_t^k D_x^\alpha D_{y'}^{\beta'} D_{\eta''}^{\gamma''} b(t,x,y',\eta'')| \le C_{\alpha\beta\gamma k}\langle\eta''\rangle^{v-|\gamma''|}. \tag{3.139}$$

Proof. If $N \ge d_0 + |\alpha|$, we obtain

$$\int_{|\eta'|\le c|\eta''|} \eta'^\alpha (1+|\eta'|)^{-N} d\eta' \le C_\alpha, \tag{3.140}$$

so we get $|\tilde{a}| \le C$. When differentiating (3.138) the derivatives falling on a give the right factors (here we use that $|\eta'| \le c|\eta''|$). The derivatives falling on the exponent give either η' factors, or factors

$$|\partial_t^k \partial_x^\alpha \partial_{\eta''}^{\gamma''} \varphi(t,x,\eta)| \le C_{k\alpha\gamma''}\langle\eta\rangle^{-|\gamma''|}\vartheta \tag{3.141}$$

since we have by (3.127) and homogeneity that $|\varphi(t,x,\eta)| \le C\vartheta$. The η' factors can be seen in (3.140), and the ϑ factors are harmless since $a \in S(\vartheta^{-N}, g)$, $\forall N$. □

The lemma gives

$$\begin{cases} PE_j u = (2\pi)^{d_0-n} \int\int e^{i\langle x''-y'',\eta''\rangle} r_j(t,x,y',\eta'')u(y)dyd\eta'' \\ \sum_j E_j u|_{t=0} \equiv u, \end{cases} \tag{3.142}$$

where $r_j \in S^0_{1,0,0}$, $j = 1, ..., r_0$. We add to these terms $E_0 : \mathcal{E}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^n)$ defined by

$$E_0 u(t,x) = (2\pi)^{d_0-n} \int\int e^{i\langle x''-y'',\eta''\rangle} a_0(t,x,y',\eta'')u(y)dyd\eta'', \tag{3.143}$$

with $a_0 \in S^0_{1,0,0}$. By Lemma D.4, we have

$$PE_0u(t,x) = (2\pi)^{d_0-n} \int\int e^{i\langle x''-y'',\eta''\rangle} b_0(t,x,y',\eta'')u(y)dyd\eta'', \tag{3.144}$$

where $b_0 \in S^0_{1,0,0}$ is given by

$$b_0 = D_t a_0 + e^{i\langle D_y, D_\xi\rangle}\tilde{k}(t,x,\xi)a_0(t,y,z',\eta'')|_{\substack{y=x\\ \xi=(0,\eta'')}} \tag{3.145}$$

if $\tilde{k}$ is the full symbol of K. By using Proposition 3.41, we may solve

$$\begin{cases} b_0 + \sum r_j \cong 0, \quad 0 < t < c, \\ a_0|_{t=0} \cong 0, \end{cases} \tag{3.146}$$

modulo $S^{-\infty}$. Therefore, we obtain the solution to the Cauchy problem (3.125). Naturally, this can be done with t replaced by $t-s$, for small s, which gives the following

Proposition 3.40. *Let $K(t,x,D_x) \in \operatorname{Op} S(\vartheta, g)$ be an $N_0 \times N_0$ system with principal symbol $k(t,x,\xi)$ satisfying* (3.123). *Then then Cauchy problem for $|s| \le \varepsilon$*

$$\begin{cases} D_t E^{(s)} + K(t,x,D_x)E^{(s)} \cong 0, \; t > s, \\ E^{(s)}|_{t=s} \cong \operatorname{Id}_{N_0}, \end{cases} \tag{3.147}$$

microlocally near $(0,(0,\xi_0),(0,\xi_0))$, $\xi_0' = 0$, has a solution $E^{(s)} : \mathcal{E}'(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^n)$ in the form

$$E^{(s)} = \sum_{j=0}^{r_0} E^{(s)}_j.$$

Here

$$E^{(s)}_j u(t,x) = (2\pi)^{1-n} \int\int e^{i\phi_j(t,x,\eta)-\langle y,\eta\rangle} a_j(t,x,\eta)u(y)dyd\eta, \; j \ge 1,$$

ϕ_j solves (3.126)*, $a_j \in S(1,g)$, and*

$$E^{(s)}_0 u(t,x) = (2\pi)^{d_0-n} \int\int e^{i\langle x''-y'',\eta''\rangle} a_0(t,x,y',\eta'')u(y)dyd\eta'',$$

where $a_0 \in S^0_{1,0,0}$, $d_0 = \operatorname{codim}\Sigma_2$.

We are going to study the system

$$\begin{cases} D_t f + e^{i\langle D_y, D_\xi\rangle}k(t,x,\xi)f(t,y,z',\eta'')|_{\substack{y=x\\ \xi=(0,\eta'')}} \cong r(t,x,z',\eta''), \; t > 0, \\ f(0,x,z',\eta'') \cong f_0(x,z',\eta''), \end{cases} \tag{3.148}$$

modulo $S^{-\infty}$, where f_0, $r \in S^v_{1,0,0}$ have values in C^{N_0}, and $k \in S(\vartheta, g)$ is $N_0 \times N_0$ system as in Proposition 3.36. By Lemma D.4, we have $r \in S^v_{1,0,0}$ if $f \in S^v_{1,0,0}$.

Proposition 3.41. *Assume that $k(t,x,\xi) \in S(\vartheta, g)$ is a symmetrizable $N_0 \times N_0$ system. Then,* *for every* f_0, $r \in S^v_{1,0,0}$, *the equation* (3.148) *has a solution* $f \in S^v_{1,0,0}$ *in a canonical neighborhood of* $(0,0,(0,\eta_0'')) \in \mathbb{R} \times \mathbb{R}^{2d_0-2} \times T^*\mathbb{R}^{n-d_0}$.

Proof. We will solve (3.148) by iteration modulo $S^{v-1}_{1,0,0}$. By Lemma D.4, we have

$$\begin{aligned} &e^{i\langle D_y, D_\xi\rangle} k(t,x,\xi) f(t,y,z',\eta'')|_{\substack{y=x\\ \xi=(0,\eta'')}} \\ &\cong e^{i\langle D_y', D_{\xi'}\rangle} k(t,x,\xi',\eta'') f(t,y',x'',z',\eta'')_{\substack{y'=x'\\ \xi'=0}} = k(t,x,D_{x'},\eta'') f(t,x,z',\eta''), \end{aligned} \tag{3.149}$$

modulo terms in $S^{v-1}_{1,0,0}$. Also, we have k supported where $|\xi - (0,\eta'')| < \varepsilon\langle\eta''\rangle$ and $|t| < c$. By cutting off, we may assume k, f supported where $\langle\eta''\rangle \approx \langle\eta_0''\rangle$, and f_0, r having compact support. Let $\lambda = \langle\eta_0''\rangle^{-1}$, and let

$$w = (x'', z', \lambda\eta''). \tag{3.150}$$

Then (3.148) becomes, by (3.149)

$$\begin{cases} D_t f(t,x',w) + k(t,x',w,D_{x'}) f(t,x',,w) \cong r(t,x',w), \ t > 0, \\ f(0,x',w) \cong f_0(x',w), \end{cases} \tag{3.151}$$

mod $S(\lambda^{1-v}, |dx'|^2 + |dw|^2)$, where f_0, $r \in S(\lambda^{-v}, |dx'|^2 + |dw|^2)$, and

$$k \in S(\langle\xi'\rangle, |dx'|^2 + |d\xi'|^2/\langle\xi'\rangle^2 + |dw|^2).$$

We may assume that $v = 0$. Let

$$\begin{cases} y = x' \\ \eta = \xi', \end{cases} \tag{3.152}$$

then it suffices to solve the system

$$\begin{cases} D_t f(t,y,w) + k(t,y,w,D_y) f(t,y,w) \cong r(t,y,w), \ t > 0, \\ f(0,y,w) \cong f_0(y,w), \end{cases} \tag{3.153}$$

modulo $S(\lambda, |dw|^2 + |dy|^2)$, where $k(t,y,w,\eta) \in S(\langle\eta\rangle, |dy|^2 + |d\eta|^2/\langle\eta\rangle^2 + |dw|^2)$, and f_0, $r \in S(1, |dw|^2 + |dy|^2)$. By assumption, there exists a symmetric $N_0 \times N_0$ system $0 < c \le M(t,y,w,\eta) \in S(1, |dy|^2 + |d\eta|^2/\langle\eta\rangle^2 + |dw|^2)$, such that Mk is symmetric modulo $S(1, |dy|^2 + |d\eta|^2/\langle\eta\rangle^2 + |dw|^2)$. To complete the proof we need to solve (3.153) with $f \in S(1, |dw|^2 + |dy|^2)$. Going back, we obtain a solution in $S^v_{1,0,0}$ to (3.148) modulo $S^{v-1}_{1,0,0}$.

Choose a partition of unity $\{\chi_j(y,w)\} \in S(1, |dw|^2 + |dy|^2)$, such that there is a fixed bound of the diameter of the supports of χ_j, and on the number of overlapping supports. Replacing f_0, r with $\chi_j f_0$, $\chi_j r$, and translating in y, it suffices to solve (3.153) with $f \in \mathcal{S}$, when f_0,

$r \in \mathcal{C}_0^\infty$ with fixed support. Since

$$(k(t,y,w,\eta) - k(t,0,w,\eta)) \in S(\langle y\rangle\langle\eta\rangle, |dy|^2 + |d\eta|^2/\langle\eta\rangle^2 + |dw|^2),$$

we can replace $k(t,y,w,D_y)$ by $k(t,w,D_y) = k(t,0,w,D_y)$ in the system (3.153). By taking $M(t,w,\eta) = M(t,0,w,\eta)$ we obtain that Mk is symmetric, mod $S(1,|dy|^2+|d\eta|^2/\langle\eta\rangle^2+|dw|^2)$.

Now, taking the Fourier transform with respect to y in (3.153), we want to solve

$$\begin{cases} D_t\tilde{f}(t,\eta,w) + k(t,w,\eta)\tilde{f}(t,\eta,w) = \tilde{r}(t,\eta,w), \ t > 0, \\ \tilde{f}(0,\eta,w) = \tilde{f}_0(\eta,w). \end{cases} \tag{3.154}$$

The unique temperate solution to (3.154) is given by

$$\tilde{f}(t,\eta,w) = F(t,\eta,w)(\tilde{f}_0(\eta,w) + i\int_0^t F^{-1}(s,\eta,w)\tilde{r}(s,\eta,w)ds), \tag{3.155}$$

if $F(t,\eta,w)$ is temperate solution to

$$\begin{cases} D_tF(t,\eta,w) + k(t,w,\eta)F(t,\eta,w) = 0, \ t > 0, \\ F(0,\eta,w) = \mathrm{Id}_{N_0}. \end{cases} \tag{3.156}$$

Since Fourier transformation and integration are continuous in $\mathcal{S}$, and $\mathcal{S}$ is closed under multiplication we get that $f \in \mathcal{S}$. □

Now, we shall construct a microlocal parametrix for the $N_0\times N_0$ system $P = D_t \mathrm{Id}_{N_0} + K(t,x,D_x)$, where $K \in \mathrm{Op}\, S(\vartheta, g)$ has principal symbol k satisfying (3.123), and study the propagation of singularities. This will be done by using Duhamel's principle and the parametrix for the Cauchy problem constructed by Proposition 3.40.

It suffices to consider $w = (0,0,\eta_0'') \in \Sigma_2$. Let ϱ_s be the restriction to $\{t = s\}$ and $\phi \in S^0_{1,0}$ have support in a conical neighborhood of w, such that $w \notin \mathrm{WF}(\phi-1)$ and $N^*\{t = s\}\cap \mathrm{WF}\,\phi = \emptyset$, $\forall s$, where N^* is the conormal bundle. Then the composition $\varrho_s \circ \phi$ is well defined, and we put

$$Ef = \int_{-\varepsilon}^t E^{(s)} \circ \varrho_s \circ \phi f ds, \ f \in \mathcal{D}'(\mathbb{R}^n), \tag{3.157}$$

$t \in]-\varepsilon, \varepsilon[$, where $E^{(s)}$ is the solution to (3.147) for sufficiently small $\varepsilon > 0$. Then E is a microlocal parametrix near w, since

$$PEf = E^{(t)} \circ \varrho_t \circ \phi f + \int_{-\varepsilon}^t (PE^{(s)}) \circ \varrho_s \circ \phi f ds \cong \phi f \mod \mathcal{C}^\infty.$$

We want to study the singularities of this parametrix. Recall that $\Sigma = \cup_{j=1}^{r_0} S_j$, where S_j is non-radial hypersurfaces. Let $C_j \subset S_j \times S_j$ be the forward (in t) Hamilton flow on S_j, $j = 1,...,r_0$, and Δ^* the diagonal in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$.

Proposition 3.42. *Let $P = D_t + K(t,x,D_x)$ be an $N_0 \times N_0$ system with $K \in \operatorname{Op} S(\vartheta, g)$ having principal symbol k satisfying* (3.123). *If E is the parametrix for P defined by* (3.157), *then $\mathrm{WF}' E \subset (\cup_{j=1}^{r_0} C_j) \cup \Delta^*$, microlocally near $(w,w) \in \Sigma_2 \times \Sigma_2$.*

Proof. We have $\mathrm{WF}(\varrho_s \phi f) = \pi(\mathrm{WF}(\phi f))|_{t=s}$, where $\pi : (t,x;\tau,\xi) \to (t,x,\xi)$ is the projection. Hence, it is enough to show

$$\mathrm{WF}(E^{(s)} f_0)|_{t>s} \subset \bigcup_{j=1}^{r_0} C_j \circ \iota_s^{*-1}(\mathrm{WF}\, f_0), \quad f_0 \in \mathcal{D}'(\mathbb{R}^{n-1}), \tag{3.158}$$

where $\iota_s^* : T^*_{t=s}\mathbb{R}^n \to T^*\mathbb{R}^{n-1}$ is the dual to the inclusion of $\mathbb{R}^{n-1}$ as the surface $\{t = s\}$ in $\mathbb{R}^n$. Now, (3.158) holds for $E_j^{(s)} f_0$, $j > 0$, since ϕ_j solves (3.126). Also, we have

$$\mathrm{WF}(E_0^{(s)} f_0)|_{t>s} \subset C_0 \circ \iota_s^{*-1}(\mathrm{WF}\, f_0), \quad f_0 \in \mathcal{D}'(\mathbb{R}^{n-1}),$$

where $C_0 \in \Sigma_2 \times \Sigma_2$ is the set of (w_1, w_2) such that w_1 and w_2 are in the same leaf of Σ_2 and $t(w_1) > t(w_2)$. Thus it suffices to prove that $E_0^{(s)} \in \mathcal{C}^\infty$ microlocally near $(t, x, (0, \eta_0''), z, (0, \eta_0''))$ when $x' \neq z'$. By translation we may assume $s = 0$.

Now, applying P to $E_0^{(0)}$ we obtain by (3.143)-(3.146) and Lemma D.4

$$\begin{cases} D_t a_0 + e^{i\langle D_{y'}, D_{\xi'}\rangle} \tilde{k}(t,x,\xi',\eta'') a_0(t,y',x'',z',\eta'')|_{\substack{\xi'=0 \\ y'=x'}} \cong R_0 a_0, \ t > 0, \\ a_0(0,x,z',\eta'') \cong 0, \end{cases} \tag{3.159}$$

mod $S^{-\infty}$, microlocally when $|x' - z'| \geq \varepsilon > 0$, $\forall \varepsilon > 0$. Here $R_0 : S^v_{1,0,0} \to S^{v-1}_{1,0,0}$, $\forall v$, and $\tilde{k}$ is the full symbol of K. (This follows since (3.158) holds for $E_j^{(0)}$, $j > 0$.) Also, (3.159) is determined modulo $S^{-\infty}$ by the restriction of a_0 to $\{|y' - z'| > \varepsilon/2\}$, and $\tilde{k}$ to $\{|\xi'| \leq C\langle \eta'' \rangle\}$. We shall prove $a_0 \in S^{-\infty}$ in $\{x' \neq z'\}$, by showing that $a_0 \in S^v_{1,0,0} \Rightarrow S^{v-1/2}_{1,0,0}$ $\forall v$, there.

Assume $a_0 \in S^v_{1,0,0}$ near $(t_0, x_0, z_0', \eta_0'')$, $|x_0' - z_0'| \geq \varrho > 0$. By translation and localization, we can assume that $z_0' = 0$, $a_0 \in S^v_{1,0,0}$ supported where $\langle \eta'' \rangle \cong \langle \eta_0'' \rangle$, and $\tilde{k}$ supported where $|\xi'| \leq C\langle \eta'' \rangle \cong C\langle \eta_0'' \rangle$. Let $\lambda = \langle \eta_0'' \rangle^{-1}$, and we consider the change of variables (3.150) and (3.152). Then $a_0(t,y,w) \in S(\lambda^{-v}, e)$, $\tilde{k}(t,y,w,\eta) \in S(\langle \eta \rangle, |dy|^2 + |d\eta|^2/\langle \eta \rangle^2 + |dw|^2)$, where e is equal to the euclidean metric and we can assume $v = 0$. Clearly, $|w| > \varrho\lambda^{-1}$. (3.159) holds mod $S(\lambda^N, e)$, $\forall N$, when $|y| = |x'| < \varrho/2$. Choose $\Phi(s) \in \mathcal{C}_0^\infty(\mathbb{R})$, such that $\Phi(s) = 1$ when $|s| \leq 1/2$, $\Phi(s) = 0$ when $|s| > 1$, and let

$$\chi(y,w) = \lambda \Phi(4|y|^2/\varrho^2 + C\lambda^2|w|^2) \in S(\lambda, |dy|^2 + |dw|^2).$$

Then $b_0 = \lambda^{-1/2}\chi a_0$ satisfies

$$\begin{cases} D_t b_0 + \tilde{k}_0(t, w, D_y) b_0 = r_1, \ 0 < t < \varepsilon, \\ b_0|_{t=0} = r_0, \end{cases} \tag{3.160}$$

where $\tilde{k}_0(t, w, \eta) = \tilde{k}(t, 0, w, \eta)$ and $r_j \in \mathcal{C}_0^\infty$. In fact, $\chi a_0 \in S(\lambda^N, e)$, $\forall N$, at $t = 0$. Also, the calculus gives

$$\lambda^{-1/2}[\tilde{k}_0(t, w, D_y), \chi] \in \operatorname{Op} S(1, \tilde{g}_\lambda),$$

and

$$\lambda^{-1/2}\chi(\tilde{k}(t, y, w.D_y) - \tilde{k}_0(t, w, D_y)) \in \operatorname{Op} S(\langle\eta\rangle, \tilde{g}_\lambda),$$

where $\tilde{g}_\lambda = \lambda|dy|^2 + |dw|^2 + |d\eta|^2/\langle\eta\rangle^2$. Then we get that b_0 is in $\mathcal{S}$, $0 \le t < \varepsilon$. Thus $\chi a_0 \in S(\lambda^{1/2}, e)$, and since this is uniformly in λ when $|x' - z'| \ge \varrho > 0$, we obtain the proposition. □

Proof of Proposition 3.36. The argument is all the same as in the proof of Proposition 3.25, except that to get $U \in \mathcal{C}^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}, \mathbb{C}^N)) \cap L^2$, we need to prove (5.6) in [Den88] for our case, which we proved in Lemma 3.35, and we need to prove the parametrix and the microlocal uniquness proven in [Den89] for our used weight and metric, which we also proved. □

Now, we will prove Theorem 1.1 for the case of systems of generalized transverse type. Note that for systems of uniaxial type (tangential case), we have S_j are tangent at Σ_2, so their Hamilton fields are parallel on Σ_2, and since Σ_2 is involutive, the Hamilton fields are tangent to Σ_2. Therefore Σ and Σ_2 are foliated by the bicharacteristics of Σ. However, in case of systems of transverse type, S_j are transverse so their Hamilton fields are non parallel at Σ_2, that is why we assumed that M_A is a hypersurface near ν_0 in the main theorem, so we could consider $M_A = S_j$ for some j.

Proof. By multiplication and conjugation with elliptic, scalar pseudodifferential operators we may assume that $m = 1$ and $r = 0$, and using the normal form we can assume that $N = r_0$, and P is of the form in Proposition 3.32. By using Theorem D.2 for all terms in the expansion of A, we obtain that $A \in \mathcal{C}^\infty(\mathbb{R}, \Psi^0_{\text{phg}})$. As the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is 1 at $\nu_0 \in \Sigma_2$, and the dimension of the fiber of $\mathcal{N}_P$ is r_0 at Σ_2, we get $\operatorname{rank}\sigma(A) = r_0 - 1$ at Σ_2. Hence, we can conjugate by suitable elliptic systems in $\mathcal{C}^\infty(\mathbb{R}, \Psi^0_{\text{phg}})$ to get that $Au \cong (u_1, ...u_{r_0-1}, 0) \in H^\epsilon$ in a conical neighborhood U of ν_0, for some $\epsilon > 0$. Then, we find $\pi_1(\mathrm{Pol}^0(u)) = \mathrm{WF}^0(u_{r_0})$ in U. By shrinking U and decreasing ϵ, we may assume $Pu \in H^\epsilon$ in U. Remember that we have

$P \in \Psi^{0,1}$, hence we get $Qu = {}^tP^{\mathrm{co}}Pu \in H^{\epsilon,-r_0+1}$ there. Let $Q = (q_{ij})_{i,j=1}^{r_0}$. Since $q_{r_0 i}$ are in $\mathcal{C}^\infty(\mathbb{R}, \operatorname{Op} S(\vartheta^{r_0-1}, g))$ for $i = 1, ...r_0 - 1$, we find that $q_{r_0r_0}u_{r_0+1} \in H^{\epsilon,-r_0+1}$ in U. Similarly, we find that $P_{r_0r_0}u_{r_0} \in H^{\epsilon,-1}$, which in turns gives $D_t^{k-1}P_{r_0r_0}u_{r_0} \in H^{\epsilon,-k}$ for $k = 1, ..., r_0 - 1$. We want to prove that $u_{r_0} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ for $t < t_0$, implies $u_{r_0} \in H^0$ at $(t_0, x_0; 0, \xi_0) = \nu_0$.

Thus assume that $u_{r_0} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ when $t < t_0$. We may assume that $\delta \leq 1$, then Lemma 3.23 gives that u_{r_0}, $P_{r_0r_0}u_{r_0}$, $D_t^{k-1}P_{r_0r_0}u_{r_0}$ for $k = 2, ..., r_0 - 1$ and $q_{r_0r_0}u_{r_0}$ satisfies (3.106). Then $\xi \neq 0$ in $\mathrm{WF}(u_{r_0})$, and assuming that $\delta \leq \epsilon$ in (3.106) we find that $P_{r_0r_0}u_{r_0} \in H_*^{0,-1}$, $D_t^{k-1}P_{r_0r_0}u_{r_0} \in H_*^{0,-k}$ for $k = 1, ..., r_0-1$ and $q_{r_0r_0}u_{r_0} \in H_*^{0,-r_0+1}$ in $\pi_0(U\cap\Sigma_2)$, and $u_{r_0} \in H_*^0$ at $(t, x_0, \xi_0) \in \pi_0(U \cap \Sigma_2)$ for $t < t_0$. Since $P_{r_0r_0} \cong D_t \bmod \mathcal{C}^\infty(\mathbb{R}, \operatorname{Op} S(\vartheta, g))$, we get $P_{r_0r_0}u_{r_0} \cong D_tu_{r_0} \in H_*^{0,-1}$ which implies, using $D_t^{k-1}P_{r_0r_0}u_{r_0} \in H_*^{0,-k}$, that $D_t^k u_{r_0} \in H_*^{0,-k}$ for $k = 1, ..., r_0 - 1$. This gives

$$(D_t^k u_{r_0})|_{t=r} \in H_*^{0,-k} \quad \text{at} \quad (x_0, \xi_0), \text{ for } k = 0, ..., r_0 - 1, \tag{3.161}$$

for almost all $r < t_0$, close to t_0. Proposition 3.34 (with $N = 1$ and $Q = q_{r_0r_0}$) gives $u_{r_0} \in H_*^0$ at (t_0, x_0, ξ_0), and Lemma 3.23 gives $u_{r_0} \in H^0$ at $(t_0, x_0; 0, \xi_0)$. □

3.5 Propagation of polarization sets for systems of MHD type

In this section, we define systems of MHD type which are also systems having their characteristic sets are union of r_0 hypersufaces intersecting transversally at an involutive manifold of codimension $d_0 \geq 2$, with $r_0 \geq 2$. However, they satisfy some assumptions different than that in case of systems of generalized transverse type. We named them systems of MHD type because we first noticed these types of systems when we considered studying the propagation of polarization sets of the linearized ideal MHD equations. Let $P \in \Psi_{phg}^m(X)$ be an $N \times N$ system of pseudodifferential operators of order m on a smooth manifold X. Let $p = \sigma(P)$ be the principal symbol of P, $\det p$ the determinant of p and $\Sigma = (\det p)^{-1}(0)$ the characteristic set of P. Assume microlocally near $\nu_0 = (x_0, \xi_0) \in \Sigma$, that

$$\Sigma = \cup_{j=1}^{r_0} S_j, \ r_0 \geq 2, \text{ where } S_j \text{ are non-radial hypersurfaces intersecting transversally at } \Sigma_2 = \cap_{j=1}^{r_0} S_j. \tag{3.162}$$

We are interested in finding the propagation of polarization set at Σ_2, as in our application; see chapter 4, we know the result on $\Sigma \setminus \Sigma_2$. We assume that

$$\Sigma_2 \text{ is an involutive manifold of codimension } d_0 \geq 2, \tag{3.163}$$

$$d^j(\det p) = 0 \text{ for } j \leq r_0 \text{ and } d^{r_0+1}(\det p) \neq 0 \quad \text{at } \Sigma_2. \tag{3.164}$$

Moreover, let $\mathcal{N}_p = \ker p \subseteq (T^*X \setminus 0) \times \mathbb{C}^N$ and assume that

$$\text{the dimension of the fiber of } \mathcal{N}_P \text{ is equal to } r_0 + 1 \text{ at } \Sigma_2, \tag{3.165}$$

$$d(\det p) = 0 \text{ and } d^2(\det p) \neq 0 \text{ at } S_{i_0} \setminus \Sigma_2, \text{ for only one } i_0 \in \{1, ...r_0\}, \tag{3.166}$$

and

$$d(\det p) \neq 0 \text{ at } S_j \setminus \Sigma_2 \text{ for each } j \in \{1, ..., r_0\}, \text{ such that } j \neq i_0. \tag{3.167}$$

Moreover, assume that ${}^tP^{\text{co}}$ the adjugate matrix of P can be written as

$${}^tP^{\text{co}} = RL_1 + L_2, \tag{3.168}$$

with R being a scalar pseudodifferential operator of order m, with $\sigma(R)$ vanishing on S_{i_0} to the first order. L_1, and L_2 are $N \times N$ matrices of order $m(N-2)$, and $m(N-3)$ respectively. Assume also that

$$\sigma(L_1)p = f\,\text{Id}_N, \tag{3.169}$$

with $\Sigma = \{f = 0\}$. We are using same notation as the previous section.

Definition 3.43. The system P is of MHD type at $\nu_0 \in \Sigma_2$, if (3.162)-(3.169) and (3.77) hold microlocally near ν_0.

We want to write systems of MHD in a normal form.

Proposition 3.44. *Let $P \in \Psi^1_{phg}$ be an $N \times N$ system of MHD type at $\nu_0 \in \Sigma_2$. Then by choosing suitable symplectic coordinates, we may assume that $X = \mathbb{R} \times \mathbb{R}^{n-1}$, $\nu_0 = (0; (0, ..., 1))$, and*

$$S_j = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta_j(t, x, \xi) = 0\}, \quad j = 1, ..., r_0, \tag{3.170}$$

microlocally near ν_0. Here β_j are real and homogeneous of degree 1 in ξ; with $\beta_1 \equiv 0$, satisfies *in a conical neighborhood of* ν_0

$$c|\xi'| \le |\beta_j| \le C|\xi'| \quad j = 2, ..., r_0, \quad 0 < c < C, \tag{3.171}$$

where $(\tau, \xi', \xi'') \in \mathbb{R} \times \mathbb{R}^{d_0-1} \times \mathbb{R}^{n-d_0}$, *which gives* $\Sigma_2 = \{\tau = 0, \xi' = 0\}$. *By conjugating* P *with elliptic, scalar Fourier integral operators, and multiplying with elliptic* $N \times N$ *systems of order* 0, *we may assume that*

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \quad \mathrm{mod}\ \mathcal{C}^\infty, \tag{3.172}$$

microlocally near ν_0, *where* $E \in \Psi^1_{\mathrm{phg}}$ *is an elliptic* $(N - r_0 - 1) \times (N - r_0 - 1)$ *system, and*

$$F \cong \mathrm{Id}_{r_0+1} D_t + K(t, x, D_x) \quad \mathrm{mod}\ \mathcal{C}^\infty. \tag{3.173}$$

Here $K(t, x, D_x) \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{\mathrm{phg}})$ *is an* $(r_0+1) \times (r_0+1)$ *system, and the eigenvalues of* $k(t, x, \xi)$; *the principal symbol of* $K(t, x, D_x)$, *are* 0 *(double),* $\beta_2, ... \beta_{r_0}$.

Proof. The proof is similar to the proof of Proposition 3.31, with some changes. Since the result is local, we may assume $X = \mathbb{R}^n$. Because Σ_2 is involutive, we may choose symplectic, homogeneous coordinates $(x, \xi) \in T^*\mathbb{R}^n$ near $\nu_0 \in \Sigma_2$, so that $\nu_0 = (0; (0, ..., 1))$ and

$$\Sigma_2 = \{(x, \xi) \in T^*\mathbb{R}^n : \xi' = 0\}, \tag{3.174}$$

where $\xi = (\xi', \xi'') \in \mathbb{R}^{d_0} \times \mathbb{R}^{n-d_0}$. We may also assume

$$S_1 = \{(x, \xi) \in T^*\mathbb{R}^n : \xi_1 = 0\}, \tag{3.175}$$

near ν_0. Now, we rename $x_1 = t$, $(x_2, .., x_{d_0}) = x'$, and $(x_{d_0+1}, ..., x_n) = x''$. Since S_j is intersecting transversally with S_1 at Σ_2, we obtain

$$S_j = \{(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1}) : \tau + \beta_j(t, x, \xi) = 0\}, \tag{3.176}$$

with β_j real and homogeneous of degree 1 in ξ, $\beta_1 \equiv 0$, and

$$c|\xi'| \le |\beta_j - \beta_k| \le C|\xi'|, \quad j \ne k, \tag{3.177}$$

in a conical neighborhood of ν_0. By taking $k = 1$, we obtain

$$c|\xi'| \le |\beta_j| \le C|\xi'|, \quad j = 2, ..., r_0. \tag{3.178}$$

Moreover, we can assume $i_0 = 1$ in (3.166).

Using that $\dim \mathcal{N}_P = r_0 + 1$ at Σ_2, we can find an $N \times N$ elliptic matrix b homogeneous of degree 0 in the ξ variables which maps $\mathrm{Im}\, p$ onto $\{z \in \mathbb{C}^N ; z_j = 0,\ j \le r_0 + 1\}$ over Σ_2 near ν_0, and we can choose an $N \times N$ matrix a homogeneous of degree 0 in the ξ variables such that

a^{-1} maps $\ker p$ onto $\{z \in \mathbb{C}^N; z_j = 0,\ j > r_0 + 1\}$ over Σ_2 near ν_0. Then we have

$$bpa = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & e \end{pmatrix}, \tag{3.179}$$

such that e is an $(N - r_0 - 1) \times (N - r_0 - 1)$ matrix which is elliptic at Σ_2, and s_{11}, s_{12}, s_{21}, vanish on Σ_2 near ν_0.

Now, we choose $N \times N$ systems of pseudodifferential operators A and B with principal symbols a and b. Then

$$BPA = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & E \end{pmatrix}, \tag{3.180}$$

where its principal symbol is given by (3.179). As E is a system of order 1 which is elliptic at ν_0, choose J to be its microlocal parametrix of order -1. Multiply BPA from the left with

$$B_1 = \begin{pmatrix} \mathrm{Id}_{r_0+1} & -S_{12}J \\ 0 & \mathrm{Id}_{N-r_0-1} \end{pmatrix}. \tag{3.181}$$

Multiply also B_1BPA from the right by

$$A_1 = \begin{pmatrix} \mathrm{Id}_{r_0+1} & 0 \\ -JS_{21} & \mathrm{Id}_{N-r_0-1} \end{pmatrix}. \tag{3.182}$$

By that we get

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix} \quad \mathrm{mod}\ \mathcal{C}^\infty, \tag{3.183}$$

microlocally near ν_0, where $E \in \Psi^1_{\mathrm{phg}}$ is an elliptic $(N - r_0 - 1) \times (N - r_0 - 1)$ system. If f is the principal symbol for F, then conditions (3.166), (3.167), and (3.176) imply

$$\det f = c\tau^2 \prod_{i=2}^{r_0} (\tau + \beta_i),\ 0 \neq c \in S^{-1}, \tag{3.184}$$

thus $\partial_\tau^{r_0+1}(\det f) = \det(\partial_\tau f) \neq 0$ at Σ_2. By Theorem D.1, and homogeneity, we may find homogeneous system $C_0 \in S^0$ such that

$$f = C_0(\tau \,\mathrm{Id}_{r_0+1} + k(t, x, \xi)), \tag{3.185}$$

where $\det C_0 \neq 0$ at Σ_2. By multiplication with an elliptic system, we may assume $C_0 \equiv \mathrm{Id}_{r_0+1}$. Thus, $\det f = \tau^2 \prod_{i=2}^{r_0}(\tau + \beta_j)$, which implies that $k(t, x, \xi)$ has the eigenvalues 0 (double), β_2, ...β_{r_0}. If $f_0 \in S^0$ is the term homogeneous of degree 0 in the expansion of F, then Theorem D.2, and homogeneity give

$$f_0 = B_{-1}f + B_0, \tag{3.186}$$

where $B_0 \in \mathcal{C}^\infty(\mathbb{R}, S^0)$ is independent of τ, and $B_{-1} \in S^{-1}$. By multiplying f with an operator with symbol $\mathrm{Id}_{r_0+1} - B_{-1}$, we may assume $B_{-1} \equiv 0$. By induction over lower order terms we obtain (3.173). □

We use the same weight and metric introduced in the previous section; see (3.95), and (3.96). Thus, we have $\beta_j \in S(\vartheta, g)$.

Proposition 3.45. *Let*

$$P = \mathrm{Id}_{r_0+1} D_t + K(t, x, D_x) \tag{3.187}$$

be an $(r_0 + 1) \times (r_0 + 1)$ *system with* $K \in \mathcal{C}^\infty(\mathbb{R}, \Psi^1_{\mathrm{phg}})$*, such that the eigenvalues of* $k = \sigma(K)$ *are* 0 *(double),* β_2*,...,*β_{r_0}*. Then* P *satisfies* (3.77) *if and only if* $k \in \mathcal{C}^\infty(\mathbb{R}, S(\vartheta, g))$*.*

Proof. Same proof as in Proposition 3.32, just we replace r_0 by $r_0 + 1$ when needed. □

Proposition 3.46. *Assume that* P *is an* $(r_0+1)\times(r_0+1)$ *system of pseudodifferential operators of order* 1 *on* $\mathbb{R}^n$*, on the form* (3.173)*, with* K *is in* $\mathcal{C}^\infty(\mathbb{R}, \mathrm{Op}(S(\vartheta, g)))$ *near* $\nu_0 \in \Sigma_2$*, and* $k(t, x, \xi)$ *has the eigenvalues* 0 *(double),* β_1*,...,* β_{r_0}*. Let* $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^{r_0+1})$ *and assume* $Pu \in H^{r,s}$ *at* ν_0*. Then, for every* $\delta > 0$ *we can find* $c_\delta > 0$ *and* $v_\delta \in H^{r,s+1}$ *at* ν_0*, such that* $u_\delta = u - v_\delta$ *satisfies*

$$|\hat{u}_\delta(\tau, \xi)| \leq C_{\delta,N} \langle(\tau, \xi)\rangle^{-N}, \quad \forall N, \tag{3.188}$$

when $|\tau| > c_\delta(\langle\xi\rangle^\delta + \langle\xi'\rangle)$*.*

Proof. Same proof as for Proposition 3.33, with replacing r_0 by $r_0 + 1$ when needed. □

We will be using the same symbol classes used in the previous section. That is, changing notation, we put $x_1 = t$, $x' = (x_2, ..., x_{d_0})$, which gives $x = (x_1, x', x'') \in \mathbb{R} \times \mathbb{R}^{d_0-1} \times \mathbb{R}^{n-d_0}$. $S^{r,s} = S(\langle\xi\rangle^r h^{-s}, g)$ where $h^{-2} = 1 + |\xi_1|^2 + |\xi'|^2$ and $\langle\xi\rangle$ are weight for the metric g defined by

$$g_{x,\xi}(dx, d\xi) = |dx|^2 + |d\xi|^2 h^2. \tag{3.189}$$

Let $\Psi^{r,s} = \mathrm{Op}\, S^{r,s}$ be the corresponding pseudodifferential operators, which maps $H^{r,s}$ into L^2. Returning to the old notation where using t instead of x_1, and assume that P be of the form in proposition 3.45, we get $P \in \Psi^{0,1}$.

Now, we are ready to prove Theorem 1.1 for systems of MHD type:

Proof of Theorem 1.1. By multiplication and conjugation with elliptic, scalar pseudodifferential operators we may assume that $m = 1$ and $r = 0$, and using the normal form we can assume that $N = r_0 + 1$, and P is of the form (3.187). By using Theorem D.2 for all terms in the expansion of A, we obtain that $A \in \mathcal{C}^\infty(\mathbb{R}, \Psi^0_{\text{phg}})$. As the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is 1 at $\nu_0 \in \Sigma_2$, and the dimension of the fiber of $\mathcal{N}_P$ is $r_0 + 1$ at Σ_2, we get $\operatorname{rank} \sigma(A) = r_0$ at Σ_2. Hence, we can conjugate by suitable elliptic systems in $\mathcal{C}^\infty(\mathbb{R}, \Psi^0_{\text{phg}})$ to get that $Au \cong {}^t(u_1, ...u_{r_0}, 0) \in H^\epsilon$ in a conical neighborhood U of ν_0, for some $\epsilon > 0$. Then, we find $\pi_1(\text{Pol}^0(u)) = \text{WF}^0(u_{r_0+1})$ in U. By shrinking U and decreasing ϵ, we may assume $Pu \in H^\epsilon$ in U. Remember that we have $P \in \Psi^{0,1}$. Let $L = L_1 + L_2$ and $Q = LP$. We have $Qu = LPu \in H^{\epsilon, -r_0+1}$ there. Let $Q = (q_{ij})_{i,j=1}^{r_0+1}$. By (3.168) and (3.169) we have q_{ij} are in $\mathcal{C}^\infty(\mathbb{R}, \operatorname{Op} S(\vartheta^{r_0-1}, g))$ for $i \neq j$, we find that $q_{r_0+1 r_0+1} u_{r_0+1} \in H^{\epsilon, -r_0+1}$ in U. Similarly, we find that $P_{r_0+1 r_0+1} u_{r_0+1} \in H^{\epsilon, -1}$, which in turns gives $D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1} \in H^{\epsilon, -k}$ for $k = 1, ..., r_0 - 1$. We want to prove that $u_{r_0+1} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ for $t < t_0$, implies $u_{r_0+1} \in H^0$ at $(t_0, x_0; 0, \xi_0) = \nu_0$.

Thus assume that $u_{r_0+1} \in H^0$ at $(t, x_0; 0, \xi_0) \in U \cap \Sigma_2$ when $t < t_0$. We may assume that $\delta \leq 1$, then Lemma 3.23 gives that u_{r_0+1}, $P_{r_0+1 r_0+1} u_{r_0+1}$, $D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1}$ for $k = 2, ..., r_0 - 1$ and $q_{r_0+1 r_0+1} u_{r_0+1}$ satisfies (3.188). Then $\xi \neq 0$ in $\text{WF}(u_{r_0+1})$, and assuming that $\delta \leq \epsilon$ in (3.106) we find that $P_{r_0+1 r_0+1} u_{r_0+1} \in H_*^{0,-1}$, $D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1} \in H_*^{0,-k}$ for $k = 1, ..., r_0 - 1$ and $q_{r_0+1 r_0+1} u_{r_0+1} \in H_*^{0, -r_0+1}$ in $\pi_0(U \cap \Sigma_2)$, and $u_{r_0+1} \in H_*^0$ at $(t, x_0, \xi_0) \in \pi_0(U \cap \Sigma_2)$ for $t < t_0$. Since $P_{r_0+1 r_0+1} \cong D_t \mod \mathcal{C}^\infty(\mathbb{R}, \operatorname{Op} S(\vartheta, g))$, we get $P_{r_0+1 r_0+1} u_{r_0+1} \cong D_t u_{r_0+1} \in H_*^{0,-1}$ which implies using $D_t^{k-1} P_{r_0+1 r_0+1} u_{r_0+1} \in H_*^{0,-k}$ that $D_t^k u_{r_0+1} \in H_*^{0,-k}$ for $k = 1, ..., r_0 - 1$. This gives

$$(D_t^k u_{r_0+1})|_{t=r} \in H_*^{0,-k} \quad \text{at} \quad (x_0, \xi_0), \text{ for } k = 0, ..., r_0 - 1 \tag{3.190}$$

for almost all $r < t_0$, close to t_0. Proposition 3.34 (with $N = 1$ and $Q = q_{r_0+1 r_0+1}$) gives $u_{r_0+1} \in H_*^0$ at (t_0, x_0, ξ_0), and Lemma 3.23 gives $u_{r_0+1} \in H^0$ at $(t_0, x_0; 0, \xi_0)$. □

4 Application

Magnetohydrodynamics, or MHD couples Maxwell's equations with hydrodynamics to describe the behavior of electrically conducting fluids under the influence of electromagnetic fields. In this chapter, we want to consider the simplest form of MHD, which is the Ideal MHD to study the propagation of polarization set of the solution of these equations. Ideal MHD, assumes that the fluid has so little resistivity that it can be treated as a perfect conductor. See [Sch09] to know more about MHD equations.

We will show, under some assumptions, that the linearized ideal MHD equation is of real principal type. As we mentioned before, systems of real principal type were defined by Dencker; see [Den82a], who studied the propagation of their solutions, and showed that the propagation of polarization sets is governed by a certain connection on sections of the kernel subbundle, $\ker p_2$ where p_2 is the principal symbol of the system. In [HR04], Hansen and Röhrig merged the theory of real principal type systems with the calculus of Fourier integral operators and constructed a Fourier integral solution for system of real principal type, and derived a transport equation for the principal symbol of this solution (note that disregarding half densities this transport equation is the connection introduced by Dencker).

The plan of this chapter is as follows: first, we introduce the ideal MHD equations and its linearization. In section 4.1, we write the linearized ideal MHD equations in the form of a wave equation $P\beta = 0$ where β is the displacement vector and P is a second order 3×3 system; see [Sch09, Lecture 20], and we show that under some assumptions, the characteristic variety of P is disjoint union of the Shear Alfvén wave, the slow magnetosonic wave and the fast magnetosonic wave; see [Sch09, Lecture 24], and [MZ05, Appendix A]. Moreover, we show that, under the considered assumptions, P is of real principal type and we calculate the transport equation on $\operatorname{Char} P$. In section 4.2, we return to the linearized ideal MHD equations, and we study the propagation of polarization sets in general. It turns out that we can consider different cases, some in which we have our system is of real principal type, some in which our system is of uniaxial type, and we have a case where our system is of MHD type.

The set of equations describing the ideal MHD are

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \operatorname{div} u = 0, \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p + H \times \operatorname{curl} H = 0, \\ \partial_t H - \nabla \times (u \times H) = 0, \\ \partial_t p + u \cdot \nabla p + \gamma p \operatorname{div} u = 0, \end{cases} \tag{4.1}$$

where ρ, $p \in \mathbb{R}$ denotes the density and the pressure respectively. $u \in \mathbb{R}^3$ is the fluid velocity, $H \in \mathbb{R}^3$ is the magnetic field, and γ is the adiabatic index, see [Sch09, Lecture 20]. Assuming a stationary equilibrium the linearized equations of (4.1) about (ρ, H, p) is:

$$\partial_t \dot{\rho} = -\rho \operatorname{div} \dot{u} - \dot{u} \cdot \nabla \rho, \tag{4.2a}$$

$$\rho \partial_t \dot{u} = -\nabla \dot{p} - H \times \operatorname{curl} \dot{H} - \dot{H} \times \operatorname{curl} H, \tag{4.2b}$$

$$\partial_t \dot{H} = \nabla \times (\dot{u} \times H), \tag{4.2c}$$

$$\partial_t \dot{p} = -\gamma p \operatorname{div} \dot{u} - \dot{u} \cdot \nabla p, \tag{4.2d}$$

where (ρ, H, p) are the values in the equilibrium state (that is the solutions of the Ideal MHD equations when $\partial/\partial t = 0$, and as we assumed stationary equilibrium we have $u = 0$). Note that we used that

$$\nabla p + H \times \operatorname{curl} H = 0, \tag{4.3}$$

which we get from the stationary equilibrium assumption.

4.1 The ideal MHD wave equation and the transport equation

We can write the linearized ideal MHD equations (4.2) in the form of a wave equation; see [Sch09, Lecture 20]. Consider the displacement β to be defined by

$$\dot{u} = \frac{\partial \beta}{\partial t}. \tag{4.4}$$

Substituting (4.4) in (4.2a), (4.2c), and (4.2d) respectively, and integrating with respect to t we get

$$\dot{\rho} = -\rho \operatorname{div} \beta - \beta \cdot \nabla \rho, \tag{4.5a}$$

$$\dot{H} = \nabla \times (\beta \times H). \tag{4.5b}$$

$$\dot{p} = -\gamma p \operatorname{div} \beta - \beta \cdot \nabla p, \tag{4.5c}$$

Replace now (4.5b), (4.5c), and (4.4) in (4.2b) to get

$$\rho \frac{\partial^2 \beta}{\partial t^2} = \gamma \nabla (p \operatorname{div} \beta) + \nabla(\beta \cdot \nabla p) + (\nabla \times (\nabla \times (\beta \times H))) \times H + (\nabla \times H) \times (\nabla \times (\beta \times H)). \tag{4.6}$$

Equation (4.6) is the ideal MHD wave equation. Consider from now on P where

$$\begin{aligned} P\beta = & -\rho \frac{\partial^2 \beta}{\partial t^2} + \gamma \nabla (p \operatorname{div} \beta) + \nabla(\beta \cdot \nabla p) + (\nabla \times (\nabla \times (\beta \times H))) \times H \\ & + (\nabla \times H) \times (\nabla \times (\beta \times H)) \\ = & \ 0. \end{aligned} \tag{4.7}$$

Now, we want to calculate the Characteristic variety of P under some assumption; for this part we refer to [Sch09, Lectures 23 and 24], and [MZ05, Appendix A].

Lemma 4.1. *Assume $c^2 = \gamma p/\rho > 0$, $0 < |H|^2 \neq \rho c^2$, $\xi \cdot H \neq 0$, and $\xi \times H \neq 0$. The characteristic variety of P is disjoint union of the Shear Alfvén wave, the slow magnetosonic wave, and the fast magnetosonic wave characteristic varieties $\{q_1 = 0\}$, $\{q_2 = 0\}$, and $\{q_3 = 0\}$ respectively, where*

$$\begin{cases} q_1 = \rho\tau^2 - (H.\xi)^2, \\ q_2 = \rho(\tau^2 - c_s^2(x,\xi)) \\ q_3 = \rho(\tau^2 - c_f^2(x,\xi)), \end{cases} \tag{4.8}$$

with

$$c_f^2(x,\xi) := \frac{1}{2}\big((c^2+h^2)\xi^2 + \sqrt{(c^2-h^2)^2\xi^4 + 4b^2c^2\xi^2}\big),$$

$$c_s^2(x,\xi) := \frac{1}{2}\big((c^2+h^2)\xi^2 - \sqrt{(c^2-h^2)^2\xi^4 + 4b^2c^2\xi^2}\big),$$

where $c^2 = \gamma p/\rho > 0$ is the square of the sound speed, $h^2 = |H|^2/\rho$ is the square of the Alfvén speed, $b^2 = |\xi \times H|^2/\rho$.

Proof. We have

$$p_2\beta = \rho\tau^2\beta - \gamma p\xi(\xi\cdot\beta) - (\xi\times(\xi\times(\beta\times H)))\times H, \tag{4.9}$$

with p_2 is the principal symbol of P. Considering $v = H/\sqrt{\rho}$, equation (4.9) can be written as

$$\rho(\tau^2 - (\xi\cdot v)^2)\beta = \rho((c^2+h^2)(\xi\cdot\beta) - (v\cdot\beta)(\xi\cdot v))\xi - \rho v(\xi\cdot\beta)(\xi\cdot v). \tag{4.10}$$

Without loss of generality, let $v = |H|/\sqrt{\rho}\ \hat{e}_z$, $\xi = \xi_\perp\hat{e}_x + \xi_\parallel\hat{e}_z$ with $\xi^2 = \xi_\perp^2 + \xi_\parallel^2$, and $\beta = \beta_x\hat{e}_x + \beta_y\hat{e}_y + \beta_z\hat{e}_z$ with $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ are unit vectors that points in the direction of the x-axis, y-axis, and z-axis respectively. Substituting this in equation (4.10), we find

$$\text{x-component:}\quad \rho\tau^2\beta_x = \rho c^2\xi_\parallel\xi_\perp\beta_z + \rho(h^2\xi^2 + c^2\xi_\perp^2)\beta_x, \tag{4.11a}$$

$$\text{y-component:}\quad \rho\tau^2\beta_y = \rho h^2\xi_\parallel^2\beta_y, \tag{4.11b}$$

$$\text{z-component:}\quad \rho\tau^2\beta_z = \rho(c^2\xi_\parallel\xi_\perp)\beta_x + \rho c^2\xi_\parallel^2\beta_z. \tag{4.11c}$$

Notice that the y-component decouples from the x- and z-components. This immediately gives

$$\rho\tau^2 = \rho h^2\xi_\parallel^2, \tag{4.12}$$

This is the shear Alfvén wave. The characteristic equation for the coupled x-and z-component is

$$\rho^2\tau^4 - \rho^2\tau^2\xi^2(c^2+h^2) + \rho^2c^2h^2\xi_\parallel^2\xi^2 = 0. \tag{4.13}$$

Hence, we get

$$\rho\tau^2 = \frac{\rho}{2}\left((c^2+h^2)\xi^2 \pm \sqrt{(c^2-h^2)\xi^4 + 4c^2h^2\xi_\perp^2\xi^2}\right). \tag{4.14}$$

Still we want to prove that $\{q_1 = 0\}$, $\{q_2 = 0\}$, and $\{q_3 = 0\}$ are disjoint. Dividing (4.13) by ρ^2, it can be written as

$$(\tau^2 - h^2\xi_\parallel^2)(\tau^2 - c^2\xi^2) - \tau^2h^2\xi_\perp^2. \tag{4.15}$$

Consider $R(X) = (X^2 - h^2\xi_\parallel^2)(X^2 - c^2\xi^2) - X^2h^2\xi_\perp^2$, $\{R \le 0\} = [c_s^2, c_f^2]$ and $R(X) \le 0$ for $X \in [\min(h^2\xi_\parallel^2, c^2\xi^2), \max(h^2\xi_\parallel^2, c^2\xi^2)]$. Thus,

$$c_f^2 \ge \max(h^2\xi_\parallel^2, c^2\xi^2) \ge h^2\xi_\parallel^2, \tag{4.16a}$$

$$c_s^2 \le \min(h^2\xi_\parallel^2, c^2\xi^2) \le h^2\xi_\parallel^2. \tag{4.16b}$$

As $h^2\xi_\parallel^2 \neq 0$, we have $R(h^2\xi_\parallel^2) = -h^2\xi_\parallel^2c^2\xi^2 \le 0$. Hence, $c_s^2 < h^2\xi_\parallel^2 < c_f^2$. □

Suppose that the conditions of *Lemma* 4.1 are satisfied. Now, we are interested in calculating the transport equation as in [HR04], which we stated in the subsection 3.1.2. The full symbol of P is $p_2 + p_1 + p_0$, with p_2 is the principal symbol of P homogeneous of degree 2, and p_1 and p_0 are homogeneous terms of degree 1 and 0 respectively. One can check that the principal symbol of P is

$$p_2 = (\rho\tau^2 - (H.\xi)^2)\,\mathrm{Id}_3 - (\gamma p + |H|^2)\xi\otimes\xi + (H.\xi)(\xi\otimes H + H\otimes\xi), \tag{4.17}$$

and

$$\begin{aligned}p_1 =& i\gamma(\nabla p)\otimes\xi + i\xi\otimes(\nabla p) + i(\nabla(H\cdot\xi)\cdot H + (H\cdot\xi)\operatorname{div}H)\,\mathrm{Id}_3 + \frac{i}{2}(\xi\otimes\nabla|H|^2) + i(\nabla|H|^2\otimes\xi)\\ &- 2i(H\cdot\nabla)(H\otimes\xi) - i(\nabla(H\cdot\xi))\otimes H - i(H\cdot\xi)(\nabla\otimes H) - i(\nabla\cdot H)(\xi\otimes H).\end{aligned}$$

Using (4.3), we get

$$\begin{aligned}p_1 =& i\gamma(\nabla p)\otimes\xi + i\xi\otimes(\nabla p) - i(\nabla p)\otimes\xi + i(\nabla(H\cdot\xi)\cdot H + (H\cdot\xi)\operatorname{div}H)\,\mathrm{Id}_3 + \frac{i}{2}(\xi\otimes\nabla|H|^2)\\ &+ \frac{i}{2}(\nabla|H|^2\otimes\xi) - i(H\cdot\nabla)(H\otimes\xi) - i(\nabla(H\cdot\xi))\otimes H - i(H\cdot\xi)(\nabla\otimes H) - i(\nabla\cdot H)(\xi\otimes H).\end{aligned} \tag{4.18}$$

One can check (4.17), (4.18, and the calculations given below by using "mathematica" for example.

Let Γ_1, Γ_2, and Γ_3 be disjoint conic neighborhoods of $\{q_1 = 0\}$, $\{q_2 = 0\}$, and $\{q_3 = 0\}$ respectively. Set $q = q_1$ in Γ_1, $q = q_2$ in Γ_2, and $q = q_3$ in Γ_3.

Proposition 4.2. *P is of real principal type with respect to the Hamilton field H_q of* $\operatorname{Char} P = \{q = 0\}$.

Proof. We have q_1, q_2, and q_3 are scalar real principal type. Let

$$w^2 := |H|^2\xi\otimes\xi + |\xi|^2 H\otimes H - (H\cdot\xi)(H\otimes\xi + \xi\otimes H) \tag{4.19}$$

In Γ_1 we take

$$\tilde{p}_2 = \mathrm{Id}_3 + \frac{q_1}{q_2q_3}(\gamma p + |H|^2)\xi\otimes\xi - \frac{q_1}{q_2q_3}((H\cdot\xi)(\xi\otimes H + H\otimes\xi)) + \frac{(H\cdot\xi)^2w^2}{q_2q_3} \tag{4.20}$$

to get $\tilde{p}_2p_2 = q_1\,\mathrm{Id}_3$. In Γ_2 we take

$$\tilde{p}_2 = \frac{q_2}{q_1}\mathrm{Id}_3 + \frac{1}{q_3}(\gamma p + |H|^2)\xi\otimes\xi - \frac{1}{q_3}((H\cdot\xi)(\xi\otimes H + H\otimes\xi)) + \frac{(H\cdot\xi)^2w^2}{q_1q_3} \tag{4.21}$$

to get $\tilde{p}_2p_2 = q_2\,\mathrm{Id}_3$. In Γ_3 we take

$$\tilde{p}_2 = \frac{q_3}{q_1}\mathrm{Id}_3 + \frac{1}{q_2}(\gamma p + |H|^2)\xi\otimes\xi - \frac{1}{q_2}((H\cdot\xi)(\xi\otimes H + H\otimes\xi)) + \frac{(H\cdot\xi)^2w^2}{q_1q_2} \tag{4.22}$$

to get $\tilde{p}_2p_2 = q_3\,\mathrm{Id}_3$. □

Remark 4.3. • The principal symbol of P calculated before can be written as

$$p_2 = q_1\pi_1 + q_2\pi_2 + q_3\pi_3, \tag{4.23}$$

where

$$\pi_1 = \mathrm{Id}_3 + \frac{w^2}{(H\cdot\xi)^2 - |H|^2|\xi|^2}, \tag{4.24}$$

$$\pi_2 = \frac{1}{\rho c_s^2(x,\xi) - \rho c_f^2(x,\xi)}\left((\gamma p + |H|^2)\xi\otimes\xi - (H\cdot\xi)(\xi\otimes H + H\otimes\xi) + \frac{(H\cdot\xi)^2 w^2}{\rho c_s^2(x,\xi) - (H\cdot\xi)^2}\right), \tag{4.25}$$

and

$$\pi_3 = \frac{1}{\rho c_f^2(x,\xi) - \rho c_s^2(x,\xi)}\left((\gamma p + |H|^2)\xi\otimes\xi - (H\cdot\xi)(\xi\otimes H + H\otimes\xi) + \frac{(H\cdot\xi)^2 w^2}{\rho c_f^2(x,\xi) - (H\cdot\xi)^2}\right), \tag{4.26}$$

with π_1, π_2 and π_3 are orthogonal projectors and $\pi_1 + \pi_2 + \pi_3 = \mathrm{Id}_3$

• In Γ_1,

$$\tilde{p}_2 = \pi_1 + \frac{q_1}{q_2}\pi_2 + \frac{q_1}{q_3}\pi_3, \tag{4.27}$$

and set $\pi = \pi_1$.

In Γ_2,

$$\tilde{p}_2 = \pi_2 + \frac{q_2}{q_1}\pi_1 + \frac{q_2}{q_3}\pi_3, \tag{4.28}$$

and set $\pi = \pi_2$.

In Γ_3,

$$\tilde{p}_2 = \pi_3 + \frac{q_3}{q_1}\pi_1 + \frac{q_3}{q_2}\pi_2, \tag{4.29}$$

and set $\pi = \pi_3$.

On $\mathrm{Char}\, P$, $\tilde{p}_2 = \pi$, $p_2\pi = 0 = \pi p_2$, and $p_2 a = 0$ if and only if $a = \pi a$.

In what follows, let $X = \mathbb{R}\times\mathbb{R}^3$, $\Lambda \subset T^*X \setminus 0$ be a closed Lagrangian submanifold of the characteristic set of P, and let $\Omega_\Lambda^{1/2}$ denote the half-density bundle of Λ. $S^{\mu+1}(\Lambda, (\Omega_\Lambda^{1/2})^3)$ is the space of symbols of the space of Lagrangian distributions $I^\mu(X, \Lambda; (\Omega_X^{1/2})^3)$; see [Hö07].

From Subsection 3.1.2, we know that there is a first order differential operator $\mathcal{T}_{P,H_q}$ on Λ, uniquely determined by P and H_q which maps a a 3-vector of half densities with $p_2 a = 0$ to 3-vector of half densities where

$$\mathcal{T}_{P,H_q} a = \mathcal{L}_{H_q} a + \frac{1}{2}\{\tilde{p}_2, p_2\}a + i\tilde{p}_2 p^s a. \tag{4.30}$$

Lemma 4.4. *$\tilde{p}_2 p^s \pi = 0$ on* $\operatorname{Char} P$.

Proof. Differentiating p_2 we get

$$\sum_{j=1}^{3} \frac{\partial^2 p_2}{\partial x_j \partial \xi_j} = -2(\nabla(H\cdot\xi)\cdot H + (H\cdot\xi)\operatorname{div} H)\operatorname{Id}_3 - (\gamma\nabla p + \nabla(|H|^2))\otimes\xi - \xi\otimes(\gamma\nabla p + \nabla|H|^2)$$
$$+ (\operatorname{div} H + H\cdot\nabla)(\xi\otimes H + H\otimes\xi) + (\nabla(H\cdot\xi))\otimes H + (H\cdot\xi)(\nabla\otimes H + (\nabla\otimes H)^\intercal)$$
$$+ H\otimes(\nabla(H\cdot\xi))$$

Therefore, the subprincipal $p^s = p_1 - \frac{1}{2i}\sum_j \frac{\partial^2 p_2}{\partial x_j \partial \xi_j}$ is given as follows:

$$2ip^s = \gamma(\xi\otimes\nabla p - \nabla p\otimes\xi) + \operatorname{div} H(\xi\otimes H - H\otimes\xi) + (H\cdot\xi)(-(\nabla\otimes H)^\intercal + (\nabla\otimes H))$$
$$+ (H\cdot\nabla)(H\otimes\xi - \xi\otimes H) + ((\nabla(H\cdot\xi))\otimes H - H\otimes\nabla(H\cdot\xi)) + 2(\nabla p\otimes\xi - \xi\otimes\nabla p).$$

We have

$$2i\tilde{p}_2 p^s \pi = 2i\pi p^s \pi \quad \text{on } \operatorname{Char} P. \tag{4.31}$$

Using that $2ip^s$ is a 3×3 skew-symmetric matrix with zero entries on the diagonal, and π is a symmetric matrix we get that $2i\pi p^s \pi$ is a 3×3 skew-symmetric matrix. Therefore to prove that it vanishes, it suffices to show that its rank is < 2. Since π is projection we have rank π =trace π. Calculating the trace of π we get that rank π=1 and hence we proved the lemma. ☐

Lemma 4.5. *Let $\Lambda \subset \operatorname{Char} P$ be a conic Lagrangian submanifold. Let $a \in S^{\mu+1}(\Lambda, (\Omega_\Lambda^{1/2})^3)$. Then, $\{\tilde{p}_2, p_2\}\pi a = -2(H\pi)a$ on* $\operatorname{Char} P$.

Proof. We will prove the result for $\pi = \pi_1$ and the same argument applies for π_2 and π_3. We have in a conic neighborhood of $\{q_1 = 0\}$, $p_2 = q_1\pi_1 + q_2\pi_2 + q_3\pi_3$, and $\tilde{p}_2 = \pi_1 + \frac{q_1}{q_2}\pi_2 + \frac{q_1}{q_3}\pi_3$. Using that $\pi^2 = \pi$, and $\{q_1, q_1\} = 0$ we get

$$\{\tilde{p}_2, p_2\}\pi_1 a = \{\pi_1, q_1\}\pi_1 a + q_2\{\pi_1, \pi_2\}\pi_1 a + q_3\{\pi_1, \pi_3\}\pi_1 a + \pi_2\{q_1, \pi_2\}\pi_1 a + \frac{q_3}{q_2}\pi_2\{q_1, \pi_3\}\pi_1 a$$
$$+ \frac{q_2}{q_3}\pi_3\{q_1, \pi_2\}\pi_1 a + \pi_3\{q_1, \pi_3\}\pi_1 a.$$

Using that $H\pi_j = H\pi_j^2 = (H\pi_j)\pi_j + \pi_j(H\pi_j)$ for $j = 2, 3$, we get $\pi_j\{q_1, \pi_j\}\pi_1 a = \{q_1, \pi_j\}\pi_1 a$ for $j = 2, 3$, and using $0 = H(\pi_2\pi_3) = \pi_2(H\pi_3) + (H\pi_2)\pi_3$ and $0 = H(\pi_3\pi_2) = \pi_3(H\pi_2) + (H\pi_3)\pi_2$ we get $\pi_2\{q_1, \pi_3\}\pi_1 a = \pi_3\{q_1, \pi_2\}\pi_1 a = 0$.
Using $\pi_2 = \operatorname{Id}_3 - \pi_1 - \pi_3$, $\{\pi_1, \operatorname{Id}_3\} = \{q_1, \operatorname{Id}_3\} = 0$ and $\{\pi_1, \pi_1\} = 0$ we get

$$\{\tilde{p}_2, p_2\}\pi_1 a = -2(H\pi_1)\pi_1 a + (q_3 - q_2)\{\pi_1, \pi_3\}\pi_1 a.$$

Now, we want to prove that $\{\pi_1, \pi_3\}\pi_1 a = 0$. We have $\partial_{\xi_i}\pi_1 = \partial_{\xi_i}\pi_1^2 = \pi_1\partial_{\xi_i}\pi_1 + \partial_{\xi_i}\pi_1\pi_1$, and similarly $\partial_{x_i}\pi_1 = \partial_{x_i}\pi_1^2 = \pi_1\partial_{x_i}\pi_1 + \partial_{x_i}\pi_1\pi_1$. Also, $0 = \partial_{\xi_i}(\pi_1\pi_3) = \pi_1\partial_{\xi_i}\pi_3 + \partial_{\xi_i}\pi_1\pi_3$. Hence, $\pi_1\partial_{\xi_i}\pi_3 = -\partial_{\xi_i}\pi_1\pi_3$. Similarly, we have $\pi_1\partial_{x_i}\pi_3 = -\partial_{x_i}\pi_1\pi_3$. Combining these together we get

$$\{\pi_1, \pi_3\}\pi_1 a = \pi_1\{\pi_1, \pi_3\}\pi_1 a.$$

We have

$$\pi_1\{\pi_1, \pi_3\}\pi_1 a = \sum_{i=1}^{3} \pi_1(\partial_{\xi_i}\pi_1\partial_{x_i}\pi_3 - \partial_{x_i}\pi_1\partial_{\xi_i}\pi_3)\pi_1 a.$$

Moreover,

$$\begin{aligned}\pi_1\partial_{\xi_i}\pi_1\partial_{x_i}\pi_3\pi_1 a &= \pi_1\partial_{\xi_i}\pi_1(\pi_3\partial_x\pi_3 + \partial_{x_i}\pi_3\pi_3)\pi_1 a = \pi_1(\partial_{\xi_i}\pi_1\pi_3)\partial_x\pi_3\pi_1 a = -\pi_1\partial_{\xi_i}\pi_3(\partial_{x_i}\pi_3\pi_1)a \\ &= \pi_1(\partial_{\xi_i}\pi_3\pi_3)\partial_{x_i}\pi_1 a = \pi_1(\partial_{\xi_i}\pi_3 - \pi_3\partial_{\xi_i}\pi_3)\partial_{x_i}\pi_1 a = \pi_1\partial_{\xi_i}\pi_3\partial_{x_i}\pi_1 a.\end{aligned}$$

Using that $\pi_1 a = a$ on $\operatorname{Char} P$, we get $\pi_1\partial_{\xi_i}\pi_1\partial_{x_i}\pi_3\pi_1 a = \pi_1\partial_{\xi_i}\pi_3\partial_{x_i}\pi_1\pi_1 a$. Therefore,

$$\pi_1\{\pi_1, \pi_3\}\pi_1 a = \sum_{i=1}^{3} \pi_1(\partial_{\xi_i}\pi_3\partial_{x_i}\pi_1 - \partial_{x_i}\pi_1\partial_{\xi_i}\pi_3)\pi_1 a.$$

We have $\partial_{\xi_i}\pi_3\partial_{x_i}\pi_1 - \partial_{x_i}\pi_1\partial_{\xi_i}\pi_3$ is a 3×3 skew-symmetric matrix with the entries in the diagonal equal to zero. So same as before we get $\pi_1\{\pi_1, \pi_3\}\pi_1 a = 0$ as the rank of π_1 equal to 1. Hence, the lemma is proved. □

Proposition 4.6. *Let $\Lambda \subset \operatorname{Char} P$ be a conic Lagrangian submanifold. Let $a \in S^{\mu+1}(\Lambda, (\Omega_\Lambda^{1/2})^3)$ with $p_2 a = 0$. Then*

$$\mathcal{T}_{P,H} a = \mathcal{L}_H a - (H\pi)a \quad \text{on} \quad \Lambda. \tag{4.32}$$

4.2 Propagation of polarization sets for the linearized ideal MHD equations

Note that (4.2) is hyperbolic symmetric with symmetrizer

$$S = \begin{pmatrix} \gamma p & 0 & 0 & 0 & 0 & 0 & 0 & -\rho \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\rho & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\rho^2}{\gamma p} \end{pmatrix}. \tag{4.33}$$

The principal symbol of (4.2) is

$$\begin{cases} \tau\dot{\rho} + \rho(\xi \cdot \dot{u}) = 0, \\ \tau\dot{u} + \rho^{-1}\dot{p}\xi + \rho^{-1}H \times (\xi \times \dot{H}) = 0, \\ \tau\dot{H} + (\xi \cdot \dot{u})H - (H \cdot \xi)\dot{u} = 0, \\ \tau\dot{p} + \gamma p(\xi \cdot \dot{u}) = 0. \end{cases} \tag{4.34}$$

We use here the notation $\xi = (\xi_1, \xi_2, \xi_3)$ for the spatial frequencies and $\xi = |\xi|\hat{\xi}$, $\dot{u}_\parallel = \hat{\xi} \cdot \dot{u}$, $\dot{u}_\perp = \dot{u} - \dot{u}_\parallel\hat{\xi} = -\hat{\xi} \times (\hat{\xi} \times \dot{u})$.

We write (4.34) in the general form $\tau\dot{U} + A(U,\xi)\dot{U} = 0$ with parameters $U = (\rho, H, p)$, and $\dot{U} = (\dot{\rho}, \dot{u}, \dot{H}, \dot{p})$.

We have the following result

Lemma 4.7. *Assume that $c^2 = \gamma p/\rho > 0$. The eigenvalues of $A(U,\xi)$ are*

$$\begin{cases} \lambda_0 = \lambda_4 = 0, \\ \lambda_{\pm 1} = \pm c_s(\hat{\xi})|\xi|, \\ \lambda_{\pm 2} = \pm(\xi \cdot H)/\sqrt{\rho}, \\ \lambda_{\pm 3} = \pm c_f(\hat{\xi})|\xi|, \end{cases} \tag{4.35}$$

with $\hat{\xi} = \xi/|\xi|$ and

$$c_f^2(\hat{\xi}) := \frac{1}{2}\big((c^2 + h^2) + \sqrt{(c^2 - h^2)^2 + 4b^2c^2}\big), \tag{4.36}$$

$$c_s^2(\hat{\xi}) := \frac{1}{2}\Big((c^2+h^2) - \sqrt{(c^2-h^2)^2+4b^2c^2}\Big), \tag{4.37}$$

where $h^2 = |H|^2/\rho$, $b^2 = |\hat{\xi}\times H|^2/\rho$.

Moreover, if we assume that $0 < |H|^2 \neq \rho c^2$, *then we have*

(i) When $\xi \cdot H \neq 0$ *and* $\xi \times H \neq 0$*:*

$\lambda_0 = \lambda_4$ *is double eigenvalue of* $A(U,\xi)$, *and the eigenvalues* $\lambda_{\pm 1}, \lambda_{\pm 2}$ *and* $\lambda_{\pm 3}$ *are simple eigenvalues of* $A(U,\xi)$.

(ii) When $\xi \cdot H = 0$, $\xi \neq 0$*:*

$\lambda_{\pm 3}$ *are simple eigenvalues, while* $\lambda_0 = \lambda_{\pm 1} = \lambda_{\pm 2} = \lambda_4$ *is a multiple eigenvalue.*

(iii) When $\xi \times H = 0$, $\xi \neq 0$*:*

when $|H|^2 < \rho c^2$ *(resp.* $|H|^2 > \rho c^2$*),* $\lambda_{\pm 3}$ *(resp.* $\lambda_{\pm 1}$*) are simple;* $\lambda_{+2} \neq \lambda_{-2}$ *are double, equal to* $\lambda_{\pm 1}$ *(resp.* $\lambda_{\pm 3}$*) depending on* $\xi \cdot H$, λ_0 *is double equal to* λ_4.

The proof of this lemma is very similar to the explanation given in [MZ05, Appendix A] except here we have the additional eigenvalue $\lambda_4 = 0$. Also, here we will not state all the eigenspaces as in [MZ05].

Proof. Let $\dot{U} = (\dot{\rho}, \dot{u}, \dot{H}, \dot{p})$. The eigenvalue equation $A(U,\xi)\dot{U} = \lambda\dot{U}$ reads

$$\begin{cases} \lambda\dot{\rho} = \rho\dot{u}_{\parallel}, \\ \rho\lambda\dot{u}_{\parallel} = \dot{p} + H_{\perp}\cdot\dot{H}_{\perp}, \\ \lambda\rho\dot{u}_{\perp} = -H_{\parallel}\dot{H}_{\perp}, \\ \lambda\dot{H}_{\perp} = \dot{u}_{\parallel}H_{\perp} - H_{\parallel}\dot{u}_{\perp}, \\ \lambda\dot{H}_{\parallel} = 0, \\ \lambda\dot{p} = \gamma p\dot{u}_{\parallel}. \end{cases} \tag{4.38}$$

On $\{\dot{\rho} = 0, \dot{u} = 0, \dot{H}_{\perp} = 0, \dot{p} = 0\} = \mathbb{E}_0(\xi)$, A is equal to $\lambda = 0$. From now on we work on $\mathbb{E}_0^{\perp} = \{\dot{H}_{\parallel} = 0\}$ which is invariant by $A(p,\xi)$.

Consider $v = H/\sqrt{\rho}$, $\dot{v} = \dot{H}/\sqrt{\rho}$, $\dot{\alpha} = \dot{p}/\rho$, $\alpha = p/\rho$, and $\dot{\sigma} = \dot{\rho}/\rho$. The characteristic system reads:

$$\begin{cases} \lambda\dot{\sigma} = \dot{u}_{\parallel}, \\ \lambda\dot{u}_{\parallel} = \dot{\alpha} + v_{\perp}\cdot\dot{v}_{\perp}, \\ \lambda\dot{u}_{\perp} = -v_{\parallel}\dot{v}_{\perp}, \\ \lambda\dot{v}_{\perp} = \dot{u}_{\parallel}v_{\perp} - v_{\parallel}\dot{u}_{\perp}, \\ \lambda\dot{\alpha} = \gamma\alpha\dot{u}_{\parallel}. \end{cases} \tag{4.39}$$

Take a basis of $\xi^{\perp}$ such that $v_{\perp} = (b, 0)$ and let $a = v_{\parallel}$. In such a basis, the matrix of the system is

$$\lambda - \tilde{A} := \begin{pmatrix} \lambda & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & -b & 0 & -1 \\ 0 & 0 & \lambda & 0 & a & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & a & 0 \\ 0 & -b & a & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & a & 0 & \lambda & 0 \\ 0 & -\gamma\alpha & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}. \tag{4.40}$$

The characteristic roots satisfy

$$\lambda(\lambda^2 - a^2)((\lambda^2 - a^2)(\lambda^2 - c^2) - \lambda^2 b^2) = 0 \tag{4.41}$$

Thus either

$$\lambda = 0 \text{ or} \tag{4.42}$$

$$\lambda^2 = a^2 \text{ or} \tag{4.43}$$

$$\lambda^2 = c_f^2(\hat{\xi}) = \frac{1}{2}(c^2 + h^2 + \sqrt{(c^2 - h^2)^2 + 4b^2c^2}) \text{ or} \tag{4.44}$$

$$\lambda^2 = c_s^2(\hat{\xi}) = \frac{1}{2}(c^2 + h^2 - \sqrt{(c^2 - h^2)^2 + 4b^2c^2}), \tag{4.45}$$

with $h^2 = a^2 + b^2 = |H|^2/\rho$.

As in Lemma 4.1, if we consider $R(X) = (X - a^2)(X - c^2) - b^2X$, $\{R \le 0\} = [c_s^2(\hat{\xi}), c_f^2(\hat{\xi})]$, and $R(X) \le 0$ for $X \in [\min(a^2, c^2), \max(a^2, c^2)]$. Thus,

$$c_f^2(\hat{\xi}) \ge \max(a^2, c^2) \ge a^2, \tag{4.46}$$

$$c_s^2(\hat{\xi}) \le \min(a^2, c^2) \le a^2. \tag{4.47}$$

At the case $v_{\perp} \neq 0$ that is $w = \hat{\xi} \times v \neq 0$: we have the basis such that (4.40) holds is smooth in ξ. In this basis, $w = (0, b)$, $b = |v_\perp| > 0$. Since $R(c^2) = -b^2c^2 < 0$ there holds $c_s^2(\hat{\xi}) < c^2 < c_f^2(\hat{\xi})$. Suppose that $a \neq 0$. Then $R(a^2) = -a^2c^2 < 0$ and $c_s^2(\hat{\xi}) < a^2 < c_f^2(\hat{\xi})$. Moreover, $c_s^2(\hat{\xi})c_f^2(\hat{\xi}) = a^2c^2$ and $c_s^2(\hat{\xi}) > 0$. However, when $a = 0$, we get $c_s^2(\hat{\xi}) = 0$, but $c_f^2(\hat{\xi}) > c^2 > 0$.

When $a \neq 0$, and $b = 0$, the eigenvalues of $\tilde{A}$ are $\pm c$ (simple), 0 (simple), and $\pm h$ (double). Assume that $c^2 \neq h^2$. Note that when $b = 0$, then $|a| = h$ and

$$\text{when } c^2 > h^2 : c_f(\hat{\xi}) = c,\ c_s(\hat{\xi}) = h,$$

$$\text{when } c^2 < h^2 : c_f(\hat{\xi}) = h,\ c_s(\hat{\xi}) = c.$$

□

Let Q be a pseudodifferential operator of order 1, such that $Q\dot{U} = 0$ be the system of the linearized ideal MHD equations, and $q = \sigma(Q)$ be its principal symbol. We have $\det q = \tau^2(\tau^2 - c_s^2(\hat{\xi})|\xi|^2)(\tau^2 - c_f^2(\hat{\xi})|\xi|^2)(\tau^2 - (\xi \cdot H)^2/\rho)$.

Proposition 4.8. *When we have Σ is disjoint union of the hypersurfaces $S_1 = \{q_1 = \tau = 0\}$, $S_2 = \{q_2 = \tau - c_s(\hat{\xi})|\xi| = 0\}$, $S_3 = \{q_3 = \tau + c_s(\hat{\xi})|\xi| = 0\}$, $S_4 = \{q_4 = \tau - (\xi \cdot H)/\sqrt{\rho} = 0\}$, $S_5 = \{q_5 = \tau + (\xi \cdot H)/\sqrt{\rho} = 0\}$, $S_6 = \{q_6 = \tau - c_f(\hat{\xi})|\xi| = 0\}$, and $S_7 = \{q_7 = \tau + c_f(\hat{\xi})|\xi| = 0\}$; that is when we are outside the intersection of any of these hypersurfaces then Q is of real principal type. Note that we have this case when $\xi \cdot H \neq 0$, and $\xi \times H \neq 0$.*

Proof. Let Γ_1,....Γ_7 be the disjoint conic neighborhoods of S_1,...,S_7 respectively.

Let ${}^tq^{\mathrm{co}}$ be the adjugate matrix (transpose of the cofactor matrix) of q. We can check by using "Mathematica" for example that ${}^tq^{\mathrm{co}}$ can be written as

$$ {}^tq^{\mathrm{co}} = \tau M, \tag{4.48}$$

with M being an 8×8 matrix.

In Γ_1, we choose

$$\tilde{q} = \left(1/\prod_{i=2}^{7} q_j\right) M, \tag{4.49}$$

so we get

$$\tilde{q}q = \tau \,\mathrm{Id}_8 . \tag{4.50}$$

In Γ_j, for $j = 2, ..., 7$ we choose

$$\tilde{q} = \Big(1/\Big(q_1^2 \prod_{\substack{i=2\\ i\neq j}}^{7} q_i\Big)\Big)\ {}^t q^{\mathrm{co}}, \tag{4.51}$$

so we get

$$\tilde{q}q = q_j \,\mathrm{Id}_8\,. \tag{4.52}$$

As q_j for $j = 1, ...7$ are of real principal type we get the result.

□

Remember that from lemma 4.7, we know that $c_f^2(\hat{\xi}) \neq c_s^2(\hat{\xi})$ and that $c_f^2(\hat{\xi}) \neq 0$.

Proposition 4.9. *If $\tau \neq 0$, and $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$, or if $\tau \neq 0$, and $\tau^2 \neq c_s^2(\hat{\xi})|\xi|^2$, then our system is of uniaxial type at Σ_2.*

Proof. First case: If $\tau \neq 0$, and $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$, we have Σ is union of two hypersurfaces $S_1 = \{\tau - c_s(\hat{\xi})|\xi| = 0\} \sqcup \{\tau + c_s(\hat{\xi})|\xi| = 0\}$, and $S_2 = \{\tau - (\xi\cdot H)/\sqrt{\rho} = 0\} \sqcup \{\tau + (\xi\cdot H)/\sqrt{\rho} = 0\}$ intersecting at $\Sigma_2 = \{\tau = |\xi||H|/\sqrt{\rho},\ \xi\times H = 0, |H|^2 < \rho c^2, \xi \neq 0\} \sqcup \{\tau = -|\xi||H|/\sqrt{\rho},\ \xi\times H = 0, |H|^2 < \rho c^2, \xi \neq 0\}$.
Second case: If $\tau^2 \neq c_s^2(\hat{\xi})|\xi|^2$, and $\tau \neq 0$, we have Σ is union of two hypersurfaces $S_1 = \{\tau - c_f(\hat{\xi})|\xi| = 0\} \sqcup \{\tau + c_f(\hat{\xi})|\xi| = 0\}$ and $S_2 = \{\tau - (\xi \cdot H)/\sqrt{\rho} = 0\} \sqcup \{\tau + (\xi \cdot H)/\sqrt{\rho} = 0\}$ intersecting at $\Sigma_2 = \{\tau = |\xi||H|/\sqrt{\rho}, \xi \times H = 0, |H|^2 > \rho c^2, \xi \neq 0\} \sqcup \{\tau = -|\xi||H|/\sqrt{\rho}, \xi \times H = 0, |H|^2 > \rho c^2, \xi \neq 0\}$.

In the first and in the second case we have: S_1 and S_2 are tangent of order 1 at Σ_2, the codimension of Σ_2 is three, the (complex) dimension of $\mathcal{N}_Q$ is equal to 2 at Σ_2, $d^2(\det q) \neq 0$ at Σ_2, and $d^i(\det q) = 0$ at Σ_2 for $i < 2$. Hence, the conditions (3.12)-(3.15) are satisfied. It remains only to prove (3.18). In [Den92], Dencker mentioned that by proposition 3.2 in [Den92], we only have to verify

$$\partial_\rho q : \ker q \mapsto \mathrm{Im}\, q \text{ at } \Sigma_2 \tag{4.53}$$

when $\rho \in T_{\Sigma_2}\Sigma$, since the order of tangency of S_1 and S_2 is 1. $T_{\Sigma_2}\Sigma$ is characterized as those $\rho \in T_{\Sigma_2}X$ such that $\partial_\rho^2(\det q) = 0$. Thus $T_{\Sigma_2}\Sigma$ is spanned by $D_1 = \xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2}$, $D_2 = \xi_3\partial_{\xi_1} - \xi_1\partial_{\xi_3}$, $D_3 = \xi_2\partial_{\xi_3} - \xi_3\partial_{\xi_2}$, $D_4 = H_2\partial_{\xi_1} - H_1\partial_{\xi_2}$, $D_5 = H_1\partial_{\xi_3} - H_3\partial_{\xi_1}$, $D_6 = H_2\partial_{\xi_3} - H_3\partial_{\xi_2}$, $D_7 = \partial_t$, $D_8 = \xi_1\tau\partial_\tau + |\xi|^2\partial_{\xi_1}$, $D_9 = \xi_2\tau\partial_\tau + |\xi|^2\partial_{\xi_2}$, $D_{10} = \xi_3\tau\partial_\tau + |\xi|^2\partial_{\xi_3}$, $D_{11} = \tau H_1\partial_\tau + (\xi\cdot H)\partial_{\xi_1}$, $D_{12} = \tau H_2\partial_\tau + (\xi\cdot H)\partial_{\xi_2}$, $D_{13} = \tau H_3\partial_\tau + (\xi\cdot H)\partial_{\xi_3}$. We can check that if ${}^t(\nu_1, ..., \nu_8) \in \ker q$

at Σ_2, then we find

$$D_j q^t(\nu_1, ..., \nu_8) = \ {}^t(0, ..., 0), \qquad j = 1, ..., 13, \tag{4.54}$$

so, (3.18) is satisfied. For D_7 we clearly have $D_7 q^t \nu = 0$. We will show how one can get (4.54) for the other D_j's, in particular we show the proof of D_1, and we one can apply similar way to the others. Let $\nu \in \ker q$ at Σ_2 so we have

$$\begin{cases} \pm\frac{|H||\xi|}{\sqrt{\rho}}\nu_1 + \rho\xi_1\nu_2 + \rho\xi_2\nu_3 + \rho\xi_3\nu_4 = 0 \\ \pm\frac{|H||\xi|}{\sqrt{\rho}}\nu_2 - \rho^{-1}(H_3\xi_3 + H_2\xi_2)\nu_5 + \rho^{-1}\xi_1 H_2\nu_6 + \rho^{-1}\xi_1 H_3\nu_7 + \rho^{-1}\xi_1\nu_8 = 0 \\ \pm\frac{|H||\xi|}{\sqrt{\rho}}\nu_3 + \rho^{-1}\xi_2 H_1\nu_5 - \rho^{-1}(H_3\xi_3 + H_1\xi_1)\nu_6 + \rho^{-1}\xi_2 H_3\nu_7 + \rho^{-1}\xi_2\nu_8 = 0 \\ \pm\frac{|H||\xi|}{\sqrt{\rho}}\nu_4 + \rho^{-1}\xi_3 H_1\nu_5 + \rho^{-1}\xi_3 H_2\nu_6 - \rho^{-1}(H_2\xi_2 + H_1\xi_1)\nu_7 + \rho^{-1}\xi_3\nu_8 = 0 \\ -(H_2\xi_2 + H_3\xi_3)\nu_2 + \xi_2 H_1\nu_3 + \xi_3 H_1\nu_4 \pm \frac{|H||\xi|}{\sqrt{\rho}}\nu_5 = 0 \\ \xi_1 H_2\nu_2 - (H_1\xi_1 + H_3\xi_3)\nu_3 + \xi_3 H_2\nu_4 \pm \frac{|H||\xi|}{\sqrt{\rho}}\nu_6 = 0 \\ \xi_1 H_3\nu_2 + \xi_2 H_3\nu_3 - (H_1\xi_1 + H_2\xi_2)\nu_4 \pm \frac{|H||\xi|}{\sqrt{\rho}}\nu_7 = 0 \\ \gamma p\xi_1\nu_2 + \gamma p\xi_2\nu_3 + \gamma p\xi_3\nu_4 \pm \frac{|H||\xi|}{\sqrt{\rho}}\nu_8 = 0. \end{cases} \tag{4.55}$$

We have

$$D_1 q\nu = D_1 \begin{pmatrix} \tau\nu_1 + \rho\xi_1\nu_2 + \rho\xi_2\nu_3 + \rho\xi_3\nu_4 \\ \tau\nu_2 - \rho^{-1}(H_3\xi_3 + H_2\xi_2)\nu_5 + \rho^{-1}\xi_1 H_2\nu_6 + \rho^{-1}\xi_1 H_3\nu_7 + \rho^{-1}\xi_1\nu_8 \\ \tau\nu_3 + \rho^{-1}\xi_2 H_1\nu_5 - \rho^{-1}(H_3\xi_3 + H_1\xi_1)\nu_6 + \rho^{-1}\xi_2 H_3\nu_7 + \rho^{-1}\xi_2\nu_8 \\ \tau\nu_4 + \rho^{-1}\xi_3 H_1\nu_5 + \rho^{-1}\xi_3 H_2\nu_6 - \rho^{-1}(H_2\xi_2 + H_1\xi_1)\nu_7 + \rho^{-1}\xi_3\nu_8 \\ -(H_2\xi_2 + H_3\xi_3)\nu_2 + \xi_2 H_1\nu_3 + \xi_3 H_1\nu_4 + \tau\nu_5 \\ \xi_1 H_2\nu_2 - (H_1\xi_1 + H_3\xi_3)\nu_3 + \xi_3 H_2\nu_4 + \tau\nu_6 \\ \xi_1 H_3\nu_2 + \xi_2 H_3\nu_3 - (H_1\xi_1 + H_2\xi_2)\nu_4 + \tau\nu_7 \\ \gamma p\xi_1\nu_2 + \gamma p\xi_2\nu_3 + \gamma p\xi_3\nu_4 + \tau\nu_8 \end{pmatrix} \tag{4.56}$$

$$= D_1 \begin{pmatrix} \tau\nu_1 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_1 \\ \tau\nu_2 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_2 \\ \tau\nu_3 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_3 \\ \tau\nu_4 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_4 \\ \tau\nu_5 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_5 \\ \tau\nu_6 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_6 \\ \tau\nu_7 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_7 \\ \tau\nu_8 \mp \frac{|H||\xi|}{\sqrt{\rho}}\nu_8 \end{pmatrix} = {}^t(0, ..., 0),$$

for $\nu \in \ker q$ at Σ_2. □

Proposition 4.10. *If $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$ then our system is of MHD type at Σ_2.*

Proof. When $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$, then Σ is union of the five hypersurfaces $S_1 = \{\tau = 0\}$, $S_2 = \{\tau - c_s(\hat{\xi})|\xi| = 0\}$, $S_3 = \{\tau + c_s(\hat{\xi})|\xi| = 0\}$, $S_4 = \{\tau - (\xi \cdot H)/\sqrt{\rho} = 0\}$, and $S_5 = \{\tau + (\xi \cdot H)/\sqrt{\rho} = 0\}$, intersecting at $\Sigma_2 = \cap_{j=1}^5 S_j = \{\tau = 0, \xi \cdot H = 0, \xi \neq 0\}$. We want to prove that our system is of MHD type at Σ_2. We have S_j intersect transversally at Σ_2, the codimension of Σ_2 is equal to two, $d^6(\det q) \neq 0$, and $d^i(\det q) = 0$ for $i < 6$ at Σ_2, and dim of the fiber of $\mathcal{N}_Q$ is equal to 6 at Σ_2. (3.166)is satisfied for $i_0 = 1$. Hence, still we want to check (3.77). Again, we will prove this by proving the following

$$\partial_\rho q : \ker q \mapsto \operatorname{Im} q \text{ at } \Sigma_2, \tag{4.57}$$

when $\rho \in T_{\Sigma_2}\Sigma$. $T_{\Sigma_2}\Sigma$ is spanned by $D_1 = \xi_1\partial_{\xi_1} + \xi_2\partial_{\xi_2} + \xi_3\partial_{\xi_3}$, $D_2 = \tau\partial_\tau$, $D_3 = \partial_t$, $D_4 = \tau\partial_{\xi_1}$, $D_5 = \tau\partial_{\xi_2}$, $D_6 = \tau\partial_{\xi_3}$, $D_7 = \tau\partial_{x_1}$, $D_8 = \tau\partial_{x_2}$, $D_9 = \tau\partial_{x_3}$, $D_{10} = (\xi \cdot H)\partial_{x_1}$, $D_{11} = (\xi \cdot H)\partial_{x_2}$, $D_{12} = (\xi \cdot H)\partial_{x_3}$, $D_{13} = (\xi \cdot H)\partial_{\xi_1}$, and $D_{14} = (\xi \cdot H)\partial_{\xi_3}$ (note that we have not mentioned $(\xi \cdot H)\partial_{\xi_2}$ as it can be written in terms of D_{13}, D_{14}, and D_5). We have for $(\nu_1, ..., \nu_8) \in \ker q$ at

Σ_2

$$D_i q\ {}^t(\nu_1, ..., \nu_8) = {}^t(0, ..., 0) \text{ at } \Sigma_2 \text{ for } i = 1, ..., 14. \tag{4.58}$$

From (4.48) we know that (3.168), and (3.169) are satisfied with $R = D_t$, $\sigma(L_1) = M$, and $f = \tau(\tau^2 - c_s^2(\hat{\xi})|\xi|^2)(\tau^2 - c_f^2(\hat{\xi})|\xi|^2)(\tau^2 - (\xi \cdot H)^2/\rho)$. □

Remark 4.11. When $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$, on $\Sigma \setminus \Sigma_2$ we have either $\tau \neq 0$ and for this case we have Proposition 4.9, or $\tau = 0$ but $\xi \cdot H \neq 0$ so our system is of real principal type in this case.

If $\tau^2 \neq c_s^2(\hat{\xi})|\xi|^2$, but $\tau = 0$ then we know that $\tau^2 \neq c_f^2(\hat{\xi})|\xi|^2$ and again our system is of real principal type in this case.

Note: As an application for systems of generalized transverse type, one can consider the linearized isentropic MHD equations, which is 7×7 matrix; check [MZ05, Appendic A] where the first order term of the linearized isentropic MHD equations and its eigenvalues are given, and then we can easily check the type of the system as we did in this section for linearized ideal MHD equations.

A Symplectic geometry

In this appendix, we will study about symplectic geometry. It will be divided into four sections. The first section is about symplectic linear algebra. In the second section, we give the definition of symplectic manifolds, and we state Darboux theorem. In the third section, we give the definition of conic symplectic manifolds. And, in the last one, we discuss characteristic foliation, where Frobenius theorem is also stated. To study more about symplectic geometry, one can check [Hö07].

A.1 Symplectic linear algebra

Let V be a real vector space of dimension $2n$, σ a bilinear form on V.

Definition A.1. (V, σ) is said to be a symplectic vector space if σ is non-degenerate and skew-symmetric. That is, we have

- $\sigma(u, v) = 0$ for all $v \in V$ implies that $u = 0$,
- $\sigma(u, v) + \sigma(v, u) = 0$ for all u, $v \in V$.

Note that the second condition is equivalent to $\sigma(u, u) = 0$ for all $u \in V$ because

$$0 = \sigma(u, v) + \sigma(v, u) = \sigma(u + v, u + v) - \sigma(u, u) - \sigma(v, v), \tag{A.1}$$

so by choosing $v = u$ we get $4\sigma(u, u) = 2\sigma(u, u)$ which gives $\sigma(u, u) = 0$.

Example A.2. $V = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates (x, ξ), where the two factors are understood to be dual to each other under the Euclidean inner product, and

$$\sigma((x, \xi), (y, \eta)) = x \cdot \eta - y \cdot \xi. \tag{A.2}$$

Using the language of differential forms, we have

$$\sigma = dx \wedge d\xi = \sum\nolimits_{i=1}^{n} dx_i \wedge d\xi_i. \tag{A.3}$$

Let $W \subseteq V$ be a linear subspace, we define the set

$$W^\sigma = \{v \in V \mid \forall w \in W : \sigma(v, w) = 0\}. \tag{A.4}$$

Note that $\dim W + \dim W^\sigma = 2n$, and $(W^\sigma)^\sigma = W$.

Definition A.3. Let $W \subseteq V$ be a linear subspace. Then W is said to be

- isotropic if $W \subseteq W^\sigma$, which implies $\dim W \leq n$,
- Lagrangian if $W = W^\sigma$, which implies $\dim W = n$,
- involutive (or co-isotropic) if $W \supseteq W^\sigma$, which implies $\dim W \geq n$.

Proposition A.4. *All symplectic vector spaces of the same dimension $2n$ are symplectomorphic. That is, there exists a linear map $T : (V, \sigma) \to (V', \sigma')$ such that*

$$\sigma(u, v) = \sigma'(Tu, Tv), \quad u, v \in V. \tag{A.5}$$

Definition A.5. A linear basis e_1,..., e_n, f_1,..., f_n of V is said to be symplectic if $\sigma(e_j, e_k) = \sigma(f_j, f_k) = 0$, and $\sigma(e_j, f_k) = \delta_{jk}$ for $1 \leq j, k \leq n$.

Example A.6. Let ε_1,...ε_n denote the standard basis of $\mathbb{R}^n$. Then $e_1 = (\varepsilon_1, 0)$,...,$e_n = (\varepsilon_n, 0)$, $f_1 = (0, \varepsilon_1)$,..., $f_n = (0, \varepsilon_n)$ is a symplectic basis of $(\mathbb{R}^{2n}, dx \wedge d\xi)$.

It is easy to find a symplectic basis to a given symplectic vector space (V, σ): We choose two complementary Lagrangian subspaces W, W' of (V, σ); that is $V = W + W'$ or, equivalently $W \cap W' = \{0\}$. W' can be seen as the dual to W under the identification

$$W' \xrightarrow{\cong} L(W, \mathbb{R}),$$
$$w' \mapsto (w \mapsto \sigma(w, w')).$$

Then, we find the symplectic basis $e_1, ..., e_n, f_1, ..., f_n$ by choosing $e_1, ..., e_n$ to be a linear basis of W and $f_1, ..., f_n$ to be a linear basis of W'.

On the other hand, if we have $e_1, ..., e_n, f_1, ..., f_n$ is a symplectic basis of (V, σ) then $W = \mathrm{span}\{e_1, ..., e_n\}$ and $W' = \mathrm{span}\{f_1, ..., f_n\}$ are complementary Lagrangian subspaces of (V, σ).

Lemma A.7. *Let $W \subset (V, \sigma)$ be isotropic, with $\dim W = m \leq n$. Then any linear basis $e_1, ..., e_m$ of W can be completed to a symplectic basis e_1,...e_n, f_1,...,f_n of (V, σ).*

A.2 Symplectic manifolds

Let σ be a differential 2-form on a manifold M, that is, for each $p \in M$, the map $\sigma_p : T_pM \times T_pM \to \mathbb{R}$ is skew symmetric bilinear on tangent space to M at p, and σ_p varies smoothly in p. If $d\sigma = 0$, then we say that σ is closed, where here d denotes the exterior derivative.

Definition A.8. The 2-form σ is called symplectic if σ is closed and σ_p is symplectic for all $p \in M$.

Note that if σ is symplectic, then $\dim T_pM = \dim M$ have to be even.

Definition A.9. A symplectic manifold is a pair (M, σ) where M is a manifold and σ is a symplectic form.

If (M, σ) is a symplectic manifold, then (T_pM, σ_p) is a symplectic vector space for all $p \in M$.

Example A.10. The simplest example of a symplectic manifold is $\mathbb{R}^{2n}$ equipped with the form $\sum_{i=1}^n dx_i \wedge d\xi_i$.

Theorem A.11. *(Darboux) All symplectic manifolds of the same dimension are locally symplectomorphic. More precisely, around each point of a symplectic manifold of dimension $2n$ there are local coordinates $x_1, ..., x_n, \xi_1, ..., \xi_n$ such that $\sigma = \sum_{i=1}^n dx_i \wedge d\xi_i$.*

Definition A.12. Let (M, σ) be a symplectic manifold, and $f : M \to \mathbb{R}$ smooth. Then we define H_f a vector field on M, by $df = \sigma(H_f, .)$.

In conical coordinates,

$$H_f = \sum_i \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial \xi_i}. \tag{A.6}$$

Definition A.13. Let (M, σ) be a symplectic manifold, and $f, g \in \mathcal{C}^\infty(M; \mathbb{R})$. The Poisson bracket of the two functions f and g is defined as

$$\{f, g\} = H_f g = \sigma(H_g, H_f). \tag{A.7}$$

In canonical coordinates,

$$\{f, g\} = \sum_i \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i}. \tag{A.8}$$

Lemma A.14. *We have $[H_f, H_g] = H_{\{f,g\}}$.*

Proposition A.15. *$(\mathcal{C}^\infty(M; \mathbb{R}); \{\ ,\ \})$ forms a Lie algebra, for (M, σ) is a symplectic manifold.*

In particular, the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$$

holds.

Definition A.16. Let (M,σ) be a symplectic manifold. Then a $\mathcal{C}^\infty$ submanifold $N\subset M$ is isotropic, coisotropic, symplectic, Lagrangian, if $T_pN\subset(T_pM,\sigma_p)$ has the corresponding property for all $p\in\mathbb{N}$.

Example A.17. Let $Y\subseteq X$ be a smooth submanifold. Then the conormal bundle $N^*Y=\{(y,z)\in T_y^*X|\ \xi_{T_yY}=0\}$ is Lagrangian.

A.3 Conic symplectic manifold

Definition A.18. A symplectic manifold (M,σ) is said to be conic if it admits a $\mathcal{C}^\infty$ free proper $\mathbb{R}_+$-action $\{\chi_t\}_{t>0}$ such that $\chi_t^*\sigma=t\sigma$ for any $t>0$.

Example A.19. $T^*X\setminus 0$ with its natural $\mathbb{R}_+$-action in the fibers.

A version of Darboux's theorem holds: (M,σ) is locally conically symplectomorphic to $T^*\mathbb{R}^n\setminus 0\cong\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$ with coordinates (x,ξ) and $\mathbb{R}_+$ action $\chi_t(x,\xi)=(x,t\xi)$.

Definition A.20. The radial vector field R is the generator of the $\mathbb{R}_+$ action, that is,

$$Rf=\frac{d}{dt}\chi_t^*f|_{t=1}. \tag{A.9}$$

The canonical 1-form $\alpha\in\Omega^1(M)$ is then defined as $\alpha=\sigma(.,R)$.

Example A.21. For $M=\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$, we have the radial vector field is $R=\xi\partial/\partial\xi$, and the canonical 1-form is $\alpha=\xi dx$.

Proposition A.22. *We have $\sigma=-d\alpha$.*

Proposition A.23. *Let $N\subset(M,\sigma)$ be an n-dimensional closed submanifold. Then N is conic Lagrangian if and only if α vanishes on N.*

A.4 The characteristic foliation

Let X be a manifold, $\dim X=n$, and $\mathcal{F}\subseteq TX$ be a subbundle of rank k. $\mathcal{F}$ is also called a distribution in differential topology. A k-dimensional submanifold $Y\subseteq X$ is said to be an integral manifold of $\mathcal{F}$ if $T_yY=\mathcal{F}_y$ for all $y\in Y$ as subspaces of T_yX.

For $k=1$, we have $\mathcal{F}=\operatorname{span}H$ locally for some non-degenerate vector field H on X, and integral manifolds of $\mathcal{F}$ are the integral curves of H.

For $2 \le k < n$, integral manifolds need not always exists even locally.

Definition A.24. The distribution $\mathcal{F}$ is said to be integrable (or involutive) if, locally, $[H, K](x) \in \mathcal{F}_x$ whenever H, K are vector fields on X such that $H(x)$, $K(x) \in \mathcal{F}_x$ for any $x \in X$.

Theorem A.25. ($Frobenius$) *The distribution $\mathcal{F}$ is integrable if and only if there is an integral manifold of $\mathcal{F}$ through every point of X. In this case, X is regularly foliated by maximal integral manifolds of $\mathcal{F}$.*

Let (M, σ) be a symplectic manifold of dimension $2n$, and let V be a submanifold of M.

Lemma A.26. *Let V be a submanifold (M, σ) of codimension r. Then V is involutive, that is, $(T_vV)^\sigma \subseteq (T_vV)$ for all $v \in V$, if and only if*

$$\{f_i, f_j\} = 0 \text{ on } V \text{ for } 1 \le i < j \le r, \tag{A.10}$$

where $V = \{f_1 = ... = f_r = 0\}$ locally with $df_1, ..., df_r$ is linearly independent on V.

If V is of dimension $n + p$ where $0 \le p \le n$. Then $(TV)^\sigma$ is a distribution of rank $n - p$.

Proposition A.27. *If V is involutive, then $(TV)^\sigma \subseteq TV$ is integrable.*

Example A.28. Let (X, σ) be a symplectic manifold, let $p \in S^{(m)}(T^*X \setminus 0)$ be a real-valued such that $dp \neq 0$ everywhere in $T^*X \setminus 0$, and let p be the principal symbol of a pseudodifferential operator P. Then $\operatorname{Char} P = p^{-1}(0) \subset T^*X \setminus 0$ is a hypersurface, so it is involutive, since every hypersurface of a symplectic manifold is involutive. Moreover, $(T \operatorname{Char} P)^\sigma$ is spanned by H_p, and the integral curves of this integrable distribution are the integral curves of H_p.

B Hörmander Weyl calculus and estimates of pseudodifferential operators

This appendix will be divided into three sections. In the first section, we give the definition of the general symbol classes of the Weyl calulus. In the second section, we discuss the Weyl calculus. And in the last section, we give some estimates of the Weyl pseudodifferential operators. For the proofs of the results mentioned in this appendix check [Hö07].

B.1 The symbol class $S(\vartheta, g)$

Let V be a finite dimensional vector space, and g a metric on V. In [Hö07], it was mentioned that it is not a restriction to assume that the metric is Riemannian that is for every $x \in V$ we have a positive definite quadratic form $g_x(y)$ in $y \in V$.

Definition B.1. g is said to be slowly varying if there are positive constants c and C such that

$$g_x(y) \leq c \Rightarrow g_{x+y}(t) \leq C g_x(t). \tag{B.1}$$

Decreasing c we can replace (B.1) by a symmetric form:

$$g_x(y) \leq c \Rightarrow g_x(t)/C \leq g_{x+y}(t) \leq C g_x(t). \tag{B.2}$$

An example is the metric

$$|dx|^2 + |d\xi|^2/(1 + |\xi|^2), \tag{B.3}$$

or more generally

$$|dx|^2(1 + |\xi|^2)^{\delta} + |d\xi|^2(1 + |\xi|^2)^{-\rho}, \tag{B.4}$$

if $\rho \leq 1$.

If G is a fixed quadratic form, and $u \in \mathcal{C}^k$ in a neighborhood of $x \in V$ we define the norm

$$|u|_k^G(x) = \sup_{t_j \in V} |u^{(k)}(x; t_1, ..., t_k)| / \prod_1^k G(t_j)^{1/2}, \tag{B.5}$$

where $u^{(k)}$ denotes the k^{th} differential of u. For fixed k an equivalent norm is the maximum of the derivatives of order k with respect to a G orthonormal coordinate system. When g is Riemannian metric We will write $|u|_k^g(x)$ instead of $|u|_k^G(x)$ when $G = g_x$.

Now, we give the definition of the general symbol classes of the Weyl calculus.

Definition B.2. If g is slowly varying, then a positive real-valued function ϑ in V is said to be g continuous if there are positive constants c and C such that

$$g_x(y) < c \Rightarrow \vartheta(x)/C \le \vartheta(x+y) \le C\vartheta(x). \tag{B.6}$$

We define $S(\vartheta, g)$ to be the set of all $u \in \mathcal{C}^\infty(V)$ such that, for every integer $k \ge 0$

$$\sup |u|_k^g(x)/\vartheta(x) < \infty. \tag{B.7}$$

$S(\vartheta, g)$ is a Fréchet space with the topology defined in (B.7).

Now, we remind the reader of the definition of the symbol class $S^\mu_{\rho,\delta}$, and then we show in which case we have $S^\mu_{\rho,\delta}$ is same as $S(\vartheta, g)$.

Definition B.3. Let $X \subset \mathbb{R}^n$ be open, let μ, ρ, δ be real numbers with $0 < \rho \le 1$, and $0 \le \delta < 1$. Then $S^\mu_{\rho,\delta}(X \times \mathbb{R}^N)$ denotes the set of all $a \in \mathcal{C}^\infty(X \times \mathbb{R}^N)$ such that for every compact set $K \subset X$, and all α, β we have

$$|D_x^\beta D_\theta^\alpha a(x, \theta)| \le C_{\alpha,\beta,K}(1 + |\theta|)^{\mu - \rho|\alpha| + \delta|\beta|}, \quad x \in K, \ \theta \in \mathbb{R}^N, \tag{B.8}$$

is valid for some constant $C_{\alpha,\beta,K}$.

If g is the metric (B.4), and we let $\vartheta = (1 + |\xi|^2)^{\mu/2}$ for any real number μ. Then, $S(\vartheta, g)$ becomes the symbol $S^\mu_{\rho,\delta}$.

Lemma B.4. *If $u \in S(\vartheta, g)$ and $v \in S(\vartheta', g)$ then $uv \in S(\vartheta\vartheta', g)$. If $1/|u| < C/\vartheta$ for some C, then $1/u \in S(1/\vartheta, g)$.*

Note that we have $\mathcal{C}_0^\infty(V) \subset S(\vartheta, g)$, for g_x and $\vartheta(x)$ are bounded from above and from below when x is in a compact set.

Theorem B.5. *We have $S(\vartheta, g) \subset S(\vartheta', g)$ if and only if ϑ/ϑ' is bounded.*

Let A be a real-valued quadratic form in the dual space V' of V. Then $A(D)$ is a differential operator in V characterized by

$$A(D) \exp\langle ix, \xi\rangle = A(\xi) \exp\langle ix, \xi\rangle, \quad x \in V,$$

for every fixed $\xi \in V'$. When u is in $\mathcal{S}$ or in $\mathcal{S}'$ we can define $\exp(iA(D))u$ as the inverse Fourier transform of $\exp(iA(\xi))\hat{u}(\xi)$ where $\hat{u}$ is the Fourier transform of u. Let g be a positive definite quadratic form in V, and let

$$K = \{x;\ g(x) < 1\}$$

be the corresponding unit ball. Let

$$g^A(x) = \sup_{g(A\xi)<1} \langle x, \xi \rangle^2 \tag{B.9}$$

be the dual form of $\xi \to g(A\xi)$.

Proposition B.6. *Let g be a positive definite quadratic form in V, and A a real quadratic form in V'. Denote by K the unit ball with respect to g, and define g^A by* (B.9)*. Then for $u \in \mathcal{C}_0^\infty(K)$ we have*

$$|\exp(iA(D))u - \sum_{j<k} (iA(D))^j u/j!| \leq C \sup_{j\leq s} \sup_{y\in K} |A(D)^k u|_j^g(y)/k!, \tag{B.10}$$

and

$$|\exp(iA(D))u(x)| \leq C_{k,R}(1 + \inf_{y\in RK} g^A(x-y))^{-k/2} \sup_{j\leq s+k} \sup |u|_j^g \tag{B.11}$$

for all $k \geq 0$ and $R > 1$ if $2s > \dim V$.

Definition B.7. The Riemannain metric g (and the positive function ϑ) in V is said to be A temperate (resp. A, g temperate) with respect to $x \in V$ if g is slowly varying (and ϑ is g continuous) and there exist constants C and N such that for all $y, t \in V$

$$g_y(t) \leq C g_x(t)(1 + g_y^A(x-y))^N, \tag{B.12}$$

$$\vartheta(y) \leq C\vartheta(x)(1 + g_y^A(x-y))^N. \tag{B.13}$$

Now, we want to extend the definition of $\exp(iA(D))$ from $\mathcal{C}_0^\infty$ to $S(\vartheta, g)$. For this we introduce first the following definition.

Definition B.8. A continuous linear form on $S(\vartheta, g)$ will be called weakly continuous if the restriction to a bounded subset is continuous in the $\mathcal{C}^\infty$ topology.

Theorem B.9. *The map $\mathcal{C}_0^\infty \ni u \mapsto \exp(iA(D))u(x) \in \mathbb{C}$ has a unique extension to a weakly continuous linear form on $S(\vartheta, g)$ for every x such that g is A temperate, $g_x \leq g_x^A$, and ϑ is A, g temperate with respect to x. We have*

$$|\exp(iA(D))u(x)| \leq \vartheta(x)\|u\|, \tag{B.14}$$

where the seminorm $\|u\|$ in $S(\vartheta, g)$ only depends on the constants in (B.2)*,* (B.6)*,* (B.12)*, and*

(B.13).

Theorem B.10. *Assume that the hypothesis of Theorem* B.9 *are fulfilled uniformly for all x in a linear subspace V_0 of V. Then the map*

$$S(\vartheta, g) \ni u \mapsto \exp(iA(D))u|_{V_0}$$

is weakly continuous with values in the space $S(\vartheta, g)|_{V_0}$ of symbols in V_0 corresponding to the restrictions of ϑ and of g.

The preceding results can be improved when

$$h(x)^2 = \sup_t g_x(t)/g_x^A(t) \tag{B.15}$$

is not only less than or equal to 1, but is small. Let

$$R_N = \exp(i(A(D))u - \sum_{j<N}(i(A(D))^j u/j!. \tag{B.16}$$

In [Hö07], it was shown that R_N has the bound

$$|R_N^{(k)}(x; t_1, ..., t_k)| \leq h(x)^N \vartheta(x) \prod_1^k g_x(t_j)^{\frac{1}{2}} \|u\|, \tag{B.17}$$

where $\|u\|$ is a fixed seminorm in $S(\vartheta, g)$.

Theorem B.11. *Assume the hyothesis of Theorem* B.9 *are fulfilled uniformly for all x in a linear subspace V_0 of V. Then we have*

$$S(\vartheta, g) \ni u \mapsto R_N \in S(\vartheta h^N, g)|_{V_0}$$

is weakly continuous. The seminorm in (B.17) *depends only on N, k, and the constants in* (B.2)*,* (B.6)*,* (B.12)*, and* (B.13)*.*

B.2 Hörmander Weyl calculus

Let V be an n dimensional vector space over $\mathbb{R}$ and V' its dual. Given $a \in \mathscr{S}(W)$, where $W = V \oplus V'$. First, we remind the reader about the operator $a(x, D)$ for $a \in \mathscr{S}(W)$:

$$a(x, D)u(x) = (2\pi)^{-n} \int\int a(x, \xi) e^{i\langle x-y, \xi\rangle} u(y)\, dy\, d\xi, \quad u \in \mathscr{S}. \tag{B.18}$$

Here dy is a Lebesgue (Haar) measure in V and $d\xi$ is the dual one in V'. The weak version of (B.18) is

$$\begin{aligned}\langle a(x,D)u,v\rangle &= (2\pi)^{-n}\int\int\int a(x,\xi)e^{i\langle x-y,\xi\rangle}u(y)v(x)dy\ dx\ d\xi \\ &= (2\pi)^{-n}\int\int\int a(x,\xi)e^{i\langle t,\xi\rangle}u(x-t)v(x)\ dx\ dt\ d\xi,\end{aligned} \tag{B.19}$$

makes sense for any $a\in\mathscr{S}'(W)$ and defines a continuous operator from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$. The adjoint operator of $\bar{a}(x,D)$ is given by

$$\tilde{a}(x,D)u(x) = (2\pi)^{-n}\int\int a(y,\xi)e^{i\langle x-y,\xi\rangle}u(y)\ dy\ d\xi, \tag{B.20}$$

interpreted in the weak sense too.

If $a\in S^m$ then $a(x,D)$ maps $\mathcal{S}$ to $\mathcal{S}$. Also, when $a\in S^m$, we have the class of operator (B.18) is the same as the class of operators (B.20), so they can be extended to continuous operators from $\mathcal{S}'$ to $\mathcal{S}'$.

The operator $a^w(x,D)$ is defined, via the Weyl calculus as

$$a^w(x,D)u = (2\pi)^{-n}\int\int a((x+y)/2,\xi)e^{i\langle x-y,\xi\rangle}u(y)\ dy\ d\xi. \tag{B.21}$$

The Schwartz kernel is

$$K(x,y) = (2\pi)^{-n}\int a((x+y)/2,\xi)e^{i\langle x-y,\xi\rangle}\ d\xi, \tag{B.22}$$

so

$$K(x+t/2,x-t/2) = (2\pi)^{-n}\int a(x,\xi)e^{i\langle t,\xi\rangle}\ d\xi \tag{B.23}$$

is the inverse Fourier Transform of a with respect to ξ. Hence, we have

$$a(x,\xi) = \int K(x+t/2,x-t/2)e^{-i\langle t,\xi\rangle}\ dt. \tag{B.24}$$

The adjoint of a^w is equal to $\bar{a}^w$. Hence, we have a^w is its own adjoint if a is real valued. Now, we want to give a formula for the composition of $a_1^w(x,D)$ and $a_2^w(x,D)$ when a_1 and a_2 are in $\mathscr{S}(W)$. Using (B.22) we get that the kernel of $a_1^w(x,D)a_2^w(x,D)$ is equal to

$$(2\pi)^{-n}\int\int\int a_1((x+z)/2,\zeta)a_2((z+y)/2,\tau)e^{i\langle x-z,\zeta\rangle+i\langle z-y,\tau\rangle}\ dz\ d\zeta\ d\tau, \tag{B.25}$$

so it follows from (B.24) that $a_1^w a_2^w = a^w$ where

$$a(x,\xi) = (2\pi)^{-2n}\int\int\int\int a_1((x+z+t/2)/2,\zeta)a_2((x+z-t/2)/2,\tau)e^{iE}\ dz\ d\zeta\ dt\ d\tau, \tag{B.26}$$

with

$$\begin{aligned} E &= \langle x - z + t/2, \zeta\rangle + \langle z - x + t/2, \tau\rangle - \langle t, \xi\rangle \\ &= \langle x - z + t/2, \zeta - \xi\rangle + \langle z - x + t/2, \tau - \xi\rangle. \end{aligned} \tag{B.27}$$

Introducing $\zeta - \xi$, $\tau - \xi$, $(z - x + t/2)/2$ and $(z - x - t/2)/2$ as new variables instead of ζ, τ, z, and t, we get

$$a(x,\xi) = \pi^{-2n} \int\int\int\int a_1(x+z,\xi+\zeta)a_2(x+t,\xi+\tau)e^{2i\sigma(t,\tau;z,\zeta)}\, dz\, d\zeta\, dt\, d\tau, \tag{B.28}$$

where

$$\sigma(t,\tau;z,\zeta) = \langle\tau, z\rangle - \langle t,\zeta\rangle. \tag{B.29}$$

Here σ is the symplectic form regarded as a quadratic form on $W \oplus W$, and notice that the Jacobian after the change of variables is 2^{2n}.

For $f \in \mathscr{S}(\mathbb{R}^2)$, it follows from the Fourier inversion formula if $f(x,y) = g(x)h(y)$ that

$$\int\int f(x,y)e^{2ixy}\, dx\, dy = (4\pi)^{-1} \int\int \hat{f}(\xi,\eta)e^{-i\xi\eta/2}\, d\xi\, d\eta. \tag{B.30}$$

Using that, we can write $a(x,\xi)$ given above in the following form

$$a(x,\xi) = \exp(i\sigma(D_x, D_\xi; D_y, D_\eta)/2)a_1(x,\xi)a_2(y,\eta)|_{(x,\xi)=(y,\eta)}. \tag{B.31}$$

We want to study a when a_1 and a_2 belong to suitable symbol classes. Hence, we will encounter quadratic forms in $W \oplus W$ of the form

$$G(t_1,t_2) = g_1(t_1) + g_2(t_2),$$

where g_1 and g_2 are quadratic forms in W. If $(x,\xi,y,\eta) \in W \oplus W$ and $(\hat{x},\hat{\xi},\hat{y},\hat{\eta})$ are the dual variables, let

$$A = 2\sigma(\hat{x},\hat{\xi},\hat{y},\hat{\eta}) = 2\langle\hat{\xi},\hat{y}\rangle - 2\langle\hat{x},\hat{\eta}\rangle$$

If we write $(x,\xi) = w$ and $(\hat{\xi}, -\hat{x}) = w' \in W$ then

$$\langle x,\hat{x}\rangle + \langle\xi,\hat{\xi}\rangle = \sigma(w,w').$$

Let

$$g_j^\sigma(w) := \sup|\sigma(w,w')|^2/g_j(w'). \tag{B.32}$$

Then, we get that

$$G^A(w_1,w_2) = g_1^\sigma(w_2) + g_2^\sigma(w_1).$$

Definition B.12. The metric g in $W = V \oplus V'$ is called σ temperate if it is slowly varying and there exist constants C and N such that for all $w, w' \in W$

$$g_{w_1}(t) \leq C g_w(t)(1 + g^{\sigma}_{w_1}(w_1 - w))^N \tag{B.33}$$

is valid. A positive function ϑ in W is called σ, g temperate if ϑ is g continuous and there exist constants C and N such that for all $w, w' \in W$ we have

$$\vartheta(w_1) \leq C\vartheta(w)(1 + g^{\sigma}_{w_1}(w - w_1))^N, \quad w, w_1 \in W. \tag{B.34}$$

Remark that (B.33) is equivalent to

$$g^{\sigma}_w(t) \leq C g^{\sigma}_{w_1}(t)(1 + g^{\sigma}_{w_1}(w_1 - w))^N. \tag{B.35}$$

Using (B.35) we get $1/\vartheta$ is σ, g temperate if ϑ is σ, g temperate.

Proposition B.13. *If g is a σ temperate and ϑ_1, ϑ_2 are σ, g temperate in $W = V \oplus V'$ then the metric $G = g_1 \oplus g_2$ in $W \oplus W$, where $g_1 = g_2 = g$, and the weight function $\vartheta = \vartheta_1 \otimes \vartheta_2$ are uniformly A temperate and A, G temperate with respect to the diagonal. If $h(w)^2 = \sup\ g_w/g^{\sigma}_w$ then $\sup\ G_{w,w}/G^A_{w,w} = h(w)^2$ also.*

Now, we consider the general case where g_1 and g_2 are different.

Proposition B.14. *Let g_1 and g_2 be σ temperate in W. Then $G = g_1 \oplus g_2$ is uniformly A temperate with respect to the diagonal in $W \oplus W$ if and only if*

$$\begin{aligned} g^{\sigma}_{1w} &\leq C g^{\sigma}_{1w_1}(t)(1 + g^{\sigma}_{2w}(w_1 - w))^N; \quad t, w, w_1 \in W, \\ g^{\sigma}_{2w} &\leq C g^{\sigma}_{2w_2}(t)(1 + g^{\sigma}_{1w}(w_2 - w))^N; \quad t, w, w_2 \in W. \end{aligned} \tag{B.36}$$

The metric $g = (g_1 + g_2)/2$ is then σ temperate in W. If we set

$$h_j(w)^2 = \sup\ g_{jw}/g^{\sigma}_{jw}; \quad H(w)^2 = \sup\ g_{1w}/g^{\sigma}_{2w} = \sup\ g_{2w}/g^{\sigma}_{1w},$$

then

$$\max(h_1(w)^2, h_2(w)^2, H(w)^2) \leq 4 \sup\ g_w/g^{\sigma}_w \leq h_1(w)^2 + h_2(w)^2 + 2H(w)^2. \tag{B.37}$$

If ϑ_j is σ, g_j temperate then $\vartheta = \vartheta_1 \otimes \vartheta_2$ is uniformly A, G temperate with respect to the diagonal in $W \oplus W$ if and only if

$$\begin{aligned} \vartheta_1(w_1) &\leq C\vartheta_1(w)(1 + g^{\sigma}_{2w}(w - w_1))^N; \quad w, w_1 \in W; \\ \vartheta_2(w_2) &\leq C\vartheta_2(w)(1 + g^{\sigma}_{1w}(w - w_2))^N; \quad w, w_2 \in W. \end{aligned} \tag{B.38}$$

These conditions are equivalent to ϑ_j being σ, g temperate.

The following theorem is tha main theorem of the Weyl calculus.

Theorem B.15. *Let g be a σ temperate Riemannian metric in $W = V \oplus V'$ with $g \leq g^{\sigma}$, and let ϑ_1, ϑ_2 be σ, g temperate weight functions in W. Then the composition formula* (B.31) *can* *be extended to a weakly continuous bilinear map* $(a_1, a_2) \mapsto a = a_1 \# a_2$ *from* $S(\vartheta_1, g) \times S(\vartheta_2, g)$ *to* $S(\vartheta_1 \vartheta_2, g)$*. If*

$$h^2(x,\xi) := \sup g_{x,\xi}/g^{\sigma}_{x,\xi}, \tag{B.39}$$

then the map from $a_1 \in S(\vartheta_1, g)$ *and* $a_2 \in S(\vartheta_2, g)$ *to the remainder term*

$$a_1 \# a_2(x,\xi) - \sum_{j<N} (i\sigma(D_x, D_\xi; D_y, D_\eta)/2)^j a_1(x,\xi) a_2(x,\xi)/j! \tag{B.40}$$

evaluated for $(x,\xi) = (y,\eta)$ *is continuous with values in* $S(h^N \vartheta_1 \vartheta_2, g)$ *for every integer* N*. It is zero if* a_1 *or* a_2 *is a polynomial of degree less than* N*.*

The terms with j even (resp. odd) are symmetric (resp. skew symmetric) in a_1, a_2. This implies

$$a_1 \# a_2 - a_2 \# a_1 - \{a_1, a_2\}/i \in S(h^3 \vartheta_1 \vartheta_2, g),$$

$$a_1 \# a_2 + a_2 \# a_1 - 2a_1 a_2 \in S(h^2 \vartheta_1 \vartheta_2, g).$$

We have the following more general result:

Theorem B.16. *Let* g_1 *and* g_2 *be* σ *temperate metrices in* $W = V \oplus V'$ *satisfying*

$$\begin{aligned} g^{\sigma}_{1w}(t) &\le C g^{\sigma}_{1w_1}(1 + g_{2w}(w_1 - w))^N, \quad t, w, w_1 \in W \\ g^{\sigma}_{2w}(t) &\le C g^{\sigma}_{2w_2}(1 + g_{1w}(w_1 - w))^N, \quad t, w, w_1 \in W, \end{aligned} \tag{B.41}$$

and assume that

$$H(x,\xi)^2 = \sup g_{1x,\xi}/g^{\sigma}_{2x,\xi} = \sup g_{2x,\xi}/g^{\sigma}_{1x,\xi} \le 1. \tag{B.42}$$

Let $g = (g_1 + g_2)/2$*, and let* ϑ_j *be* g_j *continuous* σ, g *temperate weight functions for* $j = \{1, 2\}$*. Then the composition formula* (B.31) *can be extended to a weakly continuous bilinear map* $(a_1, a_2) \mapsto a = a_1 \# a_2$ *from* $S(\vartheta_1, g_1) \times S(\vartheta_2, g_2)$ *to* $S(\vartheta_1 \vartheta_2, g)$*. The map to the* N^{th} *remainder term is continuous with values in* $S(H^N \vartheta_1 \vartheta_2, g)$ *for every integer* N*.*

Note that the error terms in the calculus improve by powers of H which may be smaller than h defined (B.39).

Proposition B.17. *Assume that* g *is* σ *temperate, that* $G = \vartheta g$*, where the function* $\vartheta \ge 1$*, is slowly varying, and that* $G \le G^{\sigma}$*. Then we get that* G *is* σ *temperate.*

The following proposition simplifies the condition in Theorem B.16 for conformal metrics:

Proposition B.18. *Assume that* g_1 *and* g_2 *are conformal (that is, we have* $g_2 = \vartheta g_1$ *for some function* ϑ*)* σ *temperate metrics with* $h_j(w)^2 = \sup g_{jw}/g^{\sigma}_{jw} \le 1$ *for* $j = \{1, 2\}$*. Then* (B.41) *is* *valid and the function H in* (B.42) *is* $(h_1h_2)^{1/2}$.

Now, we want to show the invariance of the Weyl calculus under affine symplectic transformations χ, that is, affine maps χ in W with $\chi^*\sigma = \sigma$. Having this invariance we may assume $V = \mathbb{R}^n$.

Lemma B.19. *Every affine symplectic map is a composition of maps of the following types:*

(*a*) *The translation $x \mapsto x + x_0$ in V.*

(*b*) *The translation $\xi \mapsto \xi + \xi_0$ in V'.*

(*c*) *The map $\chi(x,\xi)$ replacing x_j, ξ_j by ξ_j, $-x_j$, leaving the other coordinates unchanged.*

(*d*) *The map $\chi(x,\xi) = (Tx,\ {}^tT^{-1}\xi)$ where T is a linear bijection in $\mathbb{R}^n$.*

(*e*) *The map $\chi(x,\xi) = (x, \xi - Ax)$ where A is a symmetric matrix.*

Theorem B.20. *for every affine symplectic transformation χ in $W = V \oplus V'$ there is a unitary transformation U in $L^2(V)$, uniquely determined apart from a constant factor of modulus* 1*, such that for all linear forms L in W we have*

$$U^{-1}L(x,D)U = (L \circ \chi)(x,D). \tag{B.43}$$

U is also an automorphism of $\mathscr{S}$ and of $\mathscr{S}'$, and

$$U^{-1}a^w(x,D)U = (a \circ \chi)^w(x,D) \tag{B.44}$$

for every $a \in \mathscr{S}'(W)$.

Notice that we have $L(x,D) = L^w(x,D) = \tilde{L}(x,D)$ when L is linear.

If $a \in \mathscr{S}(W)$, we have its kernel $K \in \mathscr{S}$ is

$$K(x,y) = (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} a(x,\xi)\, d\xi, \tag{B.45}$$

then we can write $a(x,D) = b^w(x,D)$ where $b \in \mathscr{S}(W)$ is given by (B.24), and $b(x,\xi) = e^{\langle D_x, D_\xi\rangle/2i} a(x,\xi)$. Moreover, if $\tilde{c}(x,D)$ is defined by (B.20) then we have $a(x,D) = \tilde{c}(x,D)$ if $a(x,\xi) = e^{i\langle D_x, D_\xi\rangle} c(x,\xi)$. Because of continuity these observations remain true if $a, b, c \in \mathscr{S}'$.

The following theorem shows under which conditions we have $a(x,D) = b^w(x,D) = \tilde{c}(x,D)$ if $a, b, c \in S(\vartheta, g)$.

Theorem B.21. *Let g be σ temperate, $g \le g^\sigma$, and let ϑ be σ, g temperate. If $g_{x,\xi}(t,\tau) = g_{x,\xi}(t,-\tau)$ then $\exp\langle i\kappa D_x, D_\xi\rangle$ is a weakly continuous isomorphism of $S(\vartheta, g)$ for every $\kappa \in \mathbb{R}$,*

$$e^{i\kappa\langle D_x, D_\xi\rangle} a(x,\xi) - \sum_{j<N} \langle i\kappa D_x, D_\xi\rangle^j a(x,\xi)/j! \in S(h^N\vartheta, g) \tag{B.46}$$

for every integer N if h is defined by (B.39)*. If $a, b, c \in S(\vartheta, g)$ then $a(x, D) = b^w(x, D) = \tilde{c}(x, D)$ if and only if*

$$
\begin{aligned}
b(x,\xi) &= e^{-i\langle D_x, D_\xi\rangle/2} a(x,\xi) = e^{i\langle D_x, D_\xi\rangle/2} c(x,\xi), \\
a(x,\xi) &= e^{i\langle D_x, D_\xi\rangle/2} b(x,\xi) = e^{i\langle D_x, D_\xi\rangle} c(x,\xi), \\
c(x,\xi) &= e^{-i\langle D_x, D_\xi\rangle} a(x,\xi) = e^{-i\langle D_x, D_\xi\rangle/2} b(x,\xi).
\end{aligned}
\tag{B.47}
$$

If $g_{x,\xi}(0,\tau) \leq |\tau|^2$, then the bilinear maps $(a, u) \mapsto a(x, D)u$, $(b, u) \mapsto b^w(x, D)u$, and $(c, u) \mapsto \tilde{c}(x, D)u$ are continuous from $S(\vartheta, g) \times \mathscr{S}$ to $\mathscr{S}$ and from $S(\vartheta, g) \times \mathscr{S}'$ to $\mathscr{S}'$.

Under the assumptions given in the above theorem, we have if $a_j \in S(\vartheta_j, g)$ for $j = \{1, 2\}$ and $u \in \mathscr{S}$ that

$$a_1^w(x, D)a_2^w(x, D)u = (a_1 \# a_2)^w(x, D)u. \tag{B.48}$$

Also, if $a(x, D)$ is a pseudodifferential operator with polyhomogeneous symbol

$$a(x, \xi) \sim a_m(x, \xi) + a_{m-1}(x, \xi) + ..., \tag{B.49}$$

where a_j is homogeneous in ξ of degree j. Then by using Theorem B.21, we can write $a(x, D) = b^w(x, D)$ where $b(x, \xi) \sim \sum b_{m-j}(x, \xi)$. Then,

$$b_m(x, \xi) = a_m(x, \xi), \quad b_{m-1}(x, \xi) = a_{m-1}(x, \xi) + i \sum \partial^2 a_m(x, \xi)/\partial x_j \partial \xi_j/2, \tag{B.50}$$

where b_{m-1} is the subprincipal symbol.

B.3 Estimates of pseudodifferential operators

Theorem B.22. *If g is σ temperate, and ϑ is σ, g temperate then $a^w(x, D)$ is continuous map from $\mathscr{S}$ to $\mathscr{S}$ and from $\mathscr{S}'$ to $\mathscr{S}'$, and it is weakly continuous as a function of a.*

The following theorem shows the L^2 continuity:

Theorem B.23. *If $g \leq g^\sigma$, g is σ temperate, and ϑ is σ, g temperate, then the operator $a^w(x, D)$ is L^2 continuous for every $a \in S(\vartheta, g)$ if and only if ϑ is bounded. The L^2 norm of $a^w(x, D)$ is then a continuous seminorm in $S(\vartheta, g)$.*

Theorem B.24. *If $g \leq g^\sigma$, g is σ temperate, and ϑ is σ, g temperate, then the operators $a^w(x, D)$ with $a \in S(\vartheta, g)$ are all compact in L^2 if and only if $\vartheta \to 0$ at ∞.*

Above we have considered scalar pesudodifferential operators, but the calculus developed in Section B.2 is not changed if the functions u take their values in a Banach space B_1 and the symbol a takes its values in $\mathscr{L}(B_1, B_2)$, so that $a^w(x, D)u$ takes its values in B_2. However, for the L^2 estimates we need Hilbert spaces. Hence, for theorems B.23 and B.24 we need B_1 and B_2 to be Hilbert spaces, and in the second case we need it also to be finite dimensional.

Theorem B.25. *If g is σ temperate,*

$$h^2(x,\xi) = \sup g_{x,\xi}/g^\sigma_{x,\xi} \leq 1, \tag{B.51}$$

and $0 \leq a \in S(1/h, g)$, then

$$(a^w(x,D)u, u) \geq -C\|u\|^2, \quad u \in \mathcal{S}, \tag{B.52}$$

with scalar product and norm in $L^2(\mathbb{R}^n)$.

We have the following stronger Fefferman-Phong inequality :

Theorem B.26. *If g is σ temperate, and* (B.51) *is satisfied, then* (B.52) *is valid for every $a \in S(1/h^2, g)$ with $a \geq 0$.*

Considering the metric (B.4) we get the following result:

Corollary B.27. *If $0 \leq a \in S^{2(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 \leq \delta < \rho \leq 1$ then $a^w(x,D)$ is bounded from below, and so $a(x.D) + a(x,D)^*$.*

Note that Theorem B.25 remains valid in the vector valued case.

Theorem B.28. *Let g be a σ temperate metric and assume that* (B.51) *holds. If $a \in S(1/h, g)$ takes non-negative values in $\mathscr{L}(H,H)$ where H is a Hilbert space, then we have*

$$(a^w(x,D)u, u) \geq -C\|u\|^2, \quad u \in \mathscr{S}(\mathbb{R}^n, H). \tag{B.53}$$

C Generalization of the spaces $H^{r,s}$

The spaces $H^{r,s}$ introduced in Chapter 3, see (3.46) is a particular case of the spaces $B_{p,k}$ introduced by Hörmander; check [Hö05], where $p = 2$ and $k(\tau, \xi) = \langle \tau, \xi \rangle^r \langle \tau, \vartheta \rangle^s$ with ϑ given by (3.32). Although in our work we just deal with the case $p = 2$, in this appendix we will give the definition of these spaces with general p, and state some of their properties. Also, we will give some properties for the localization of these spaces.

C.1 The spaces $B_{p,k}$

Definition C.1. A positive function k defined in $\mathbb{R}^n$ is said to be a temperate weight function if there exist positive constants C and N such that

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^n. \tag{C.1}$$

The set of all such functions k will be denoted by $\mathscr{K}$.

For $k \in \mathscr{K}$ we write

$$M_k(\xi) = \sup_{\eta} k(\xi + \eta)/k(\eta). \tag{C.2}$$

Hence, M_k is the smallest function such that

$$k(\xi + \eta) \leq M_k(\xi) k(\eta). \tag{C.3}$$

M_k is submultiplicative, that is

$$M_k(\xi + \eta) \leq M_k(\xi) M_k(\eta), \tag{C.4}$$

and since $M_k(\xi) \leq (1 + C|\xi|)^N$ we get that $M_k \in \mathscr{K}$.

Example C.2. Let P be a polynomial of degree m, and define the function $\tilde{P}$ as

$$\tilde{P}(\xi)^2 = \sum_{|\alpha| \geq 0} |\partial^\alpha P(\xi)|^2. \tag{C.5}$$

This function is in $\mathscr{K}$, since using Taylor's formula we have

$$\tilde{P}(\xi+\eta) \leq (1+C|\xi|)^m \tilde{P}(\eta), \tag{C.6}$$

where C is a constant depending only on m and the dimension n.

Theorem C.3. *If k_1 and k_2 belong to $\mathscr{K}$, it follows that k_1+k_2, k_1k_2, $\sup(k_1,k_2)$, and $\inf(k_1,k_2)$ are also in $\mathscr{K}$. If $k \in \mathscr{K}$ then $k^s \in \mathscr{K}$ for every real s. Moreover, if μ is a positive measure we have either $\mu * k \equiv \infty$ or $\mu * k \in \mathscr{K}$.*

Theorem C.4. *If $k \in \mathscr{K}$ we can for every $\delta > 0$ find a function $K_\delta \in \mathscr{K}$ and a constant C_δ such that*

$$1 \leq k_\delta(\xi)/k(\xi) \leq C_\delta, \quad \xi \in \mathbb{R}^n, \tag{C.7}$$

$$M_{k_\delta}(\xi) \leq (1+C|\xi|)^N, \quad \xi \in \mathbb{R}^n, \tag{C.8}$$

where C and N are independent of δ, and M_{k_δ} tends to 1 uniformly on compact subsets of $\mathbb{R}^n$ when δ tends to 0.

Now, we give the definition of the spaces $B_{p,k}$.

Definition C.5. If $k \in \mathscr{K}$ and $1 \leq p \leq \infty$, we denote by $B_{p,k}$ the set of all distributions $u \in \mathscr{S}'$ such that $\hat{u}$ is a function and

$$\|u\|_{p,k} = ((2\pi)^{-n} \int |k(\xi)\hat{u}(\xi)|^p \, d\xi)^{1/p} < \infty. \tag{C.9}$$

When $p = \infty$, we interpret $\|u\|_{p,k}$ as $\operatorname{ess.sup} |k(\xi)\hat{u}(\xi)|$.

Theorem C.6. *$B_{p,k}$ is a Banach space with the norm* (C.9). *We have*

$$\mathscr{S} \subset B_{p,k} \subset \mathscr{S}'. \tag{C.10}$$

in the topological sense, that is, the topology in $\mathscr{S}$ is stronger than that induced there by $B_{p,k}$, and the topology in $B_{p,k}$ is stronger than the one induced by $\mathscr{S}'$. Also, we have $\mathcal{C}_0^\infty$ is dense in $B_{p,k}$.

Theorem C.7. *If k_1, $k_2 \in \mathscr{K}$ and*

$$k_2(\xi) \leq Ck_1(\xi), \quad \xi \in \mathbb{R}^n, \tag{C.11}$$

then we have $B_{p,k_1} \subset B_{p,k_2}$. Conversely, if there exists non-empty open set X such that $B_{p,k_1} \cap \mathcal{E}'(X) \subset B_{p,k_2}$, then (C.11) *is valid.*

Corollary C.8. *If k_1, $k_2 \in \mathscr{K}$, then we have*

$$B_{p,k_1} \cap B_{p,k_2} = B_{p,k_1+k_2}, \tag{C.12}$$

and for $u \in B_{p,k_1} \cap B_{p,k_2}$ *we have*

$$\max_{j=1,2} \|u\|_{p,k_j} \leq \|u\|_{p,k_1+k_2} \leq \|u\|_{p,k_1} + \|u\|_{p,k_2}. \tag{C.13}$$

Now, we state when the inclusion mapping in Theorem C.7 is compact.

Theorem C.9. *If* K *is a compact set in* $\mathbb{R}^n$*, the inclusion mapping of* $B_{p,k_1} \cap \mathcal{E}'(K)$ *into* B_{p,k_2} *is compact if*

$$k_2(\xi)/k_1(\xi) \to 0, \text{ as } \xi \to \infty. \tag{C.14}$$

Conversely, if the mapping is compact for one set K *with interior points, then* (C.14) *is valid.*

Now, we will show how differential operators with constant coefficients act in the spaces $B_{p,k}$.

Theorem C.10. *If* $u \in B_{p,k}$*, then* $P(D)u \in B_{p,k/\tilde{P}}$*, with* $\tilde{P}$ *is given by* (C.5)*.*

Theorem C.11. *If* $u_1 \in B_{p,k_1} \cap \mathcal{E}'$ *and* $u_2 \in B_{\infty,k_2}$*, then* $u_1 * u_2 \in B_{p,k_1k_2}$*, and we have that*

$$\|u_1 * u_2\|_{p,k_1k_2} \leq \|u_1\|_{p,k_1}\|u_2\|_{\infty,k_2}. \tag{C.15}$$

Theorem C.12. *If* $k \in \mathscr{K}$*, and* j *is a non-negative integer such that*

$$(1 + |\xi|)^j/k(\xi) \in L^{p'}, \quad 1/p + 1/p' = 1, \tag{C.16}$$

then $B_{p,k} \subset \mathcal{C}^j$*. Conversely, if* $B_{p,k} \cap \mathcal{E}'(X) \subset \mathcal{C}^j$ *for some non-empty open set* X*, then* (C.16) *is valid.*

In the follwoing theorem, we determine the dual space of $B_{p,k}$ when $p < \infty$. Since $\mathcal{S}$ is dense in $B_{p,k}$ if $p < \infty$ (Theorem C.6), a continuous linear form on $B_{p,k}$ is uniquely determined in that case by its restriction to $\mathcal{S}$.

Theorem C.13. *If* L *is a continuous linear map on* $B_{p,k}$*,* $p < \infty$*, we have for some* $v \in B_{p',1/k}$*,* $1/p' + 1/p = 1$*,*

$$L(u) = \check{v}(u), \quad u \in \mathcal{S}.$$

The norm of this linear form is $\|v\|_{p',1/k}$*. Thus,* $B_{p',1/k}$ *is the dual space of* $B_{p,k}$ *and the canonical bilinear form in* $B_{p,k} \times B_{p',1/k}$ *is the continuous extension of the bilinear form* $\check{v}(u)$*;* $v \in B_{p',1/k}$*,* $u \in \mathcal{S}$*. Here* $\check{v}$ *is the composition of* v *with* $x \to -x$*.*

Theorem C.14. *If* $u \in B_{p,k}$ *and* $\phi \in \mathscr{S}$*, then we get that* $\phi u \in B_{p,k}$ *and*

$$\|\phi u\|_{p,k} \leq \|\phi\|_{1,M_k}\|u\|_{p,k}. \tag{C.17}$$

The following theorem shows how to approxiamte by elements with compact support.

Theorem C.15. *Let* $\psi \in \mathcal{C}_0^\infty$ *and assume that* $\psi(0) = 1$*. Set* $\psi^\varepsilon(x) = \psi(\varepsilon x)$*. If* $u \in B_{p,k}$ *and*

$p < \infty$, then we have that $\psi^\varepsilon u \to u$ in $B_{p,k}$ when $\varepsilon \to 0$.

We can also approximate by the usual regularization:

Theorem C.16. *Let $\phi \in \mathcal{C}_0^\infty$ such that $\int \phi \, dx = 1$. Set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. If $u \in B_{p,k}$ and $p < \infty$, then the regularization $u * \phi_\epsilon$ converge to u in $B_{p,k}$ when $\varepsilon \to 0$.*

C.2 Localization of the spaces $B_{p,k}$

Definition C.17. A linear subspace $\mathcal{F}$ of $\mathcal{D}'(X)$ is called semi-local if $\phi u \in \mathcal{F}$ when $u \in \mathcal{F}$ and $\phi \in \mathcal{C}_0^\infty(X)$. It is called local if, in addition, $\mathcal{F}$ contains every distribution u such that $\phi u \in \mathcal{F}$ for every $\phi \in \mathcal{C}_0^\infty$.

From Theorem C.14, we get that the set of restrictions to X of distributions in $B_{p,k}$ is semi-local.

Theorem C.18. *If $\mathcal{F}$ is semi-local, the smallest local space containing $\mathcal{F}$ is the space*

$$\mathcal{F}^{\text{loc}} = \{u;\ u \in \mathcal{D}'(X),\ \phi u \in \mathcal{F} \text{ for every } \phi \in C_0^\infty(X)\}.$$

Theorem C.19. *Let $\mathcal{F}$ be a local subspace of $\mathcal{D}'(X)$. If $u \in \mathcal{D}'(X)$ and to every point $x_0 \in X$ there exists a function $\phi \in \mathcal{C}_0^\infty(X)$ such that $\phi u \in \mathcal{F}$ and $\phi(x_0) \neq 0$, it follows that $u \in \mathcal{F}$.*

We denote by $\mathcal{F}^c$ the set of distributions in $\mathcal{F}$ with compact support in X. If $\mathcal{F}$ is semi-local, we have

$$\mathcal{F}^c = \mathcal{F}^{\text{loc}} \cap \mathcal{E}'(X), \quad \mathcal{F}^{\text{loc}} = (\mathcal{F}^c)^{\text{loc}}. \tag{C.18}$$

Most of the results for the spaces $B_{p,k}$ stated in the previous section carry over immediately to the local spaces $B_{p,k}^{\text{loc}}(X)$ corresponding to the set of restrictions to X of distributions in $B_{p,k}$ (or, equivalently, corresponding to $B_{p,k} \cap \mathcal{E}'(X)$).

Theorem C.20. *We have $B_{p,k_1}^{\text{loc}}(X) \subset B_{p,k_2}^{\text{loc}}(X)$ if and only if* (C.11) *is valid.*

Theorem C.21. *If $u \in B_{p,k}^{\text{loc}}(X)$, then $P(D)u \in B_{p,k/\tilde{P}}^{\text{loc}}(X)$.*

Theorem C.22. *If $u \in B_{p,k}^{\text{loc}}(X)$ and $\phi \in \mathcal{C}^\infty(X)$ then $\phi u \in B_{p,k}^{\text{loc}}(X)$.*

Theorem C.23. *If $u_1 \in B_{p,k_1}(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$ and $u_2 \in B_{\infty,k_2}^{\text{loc}}(\mathbb{R}^n)$, then $u_1 * u_2 \in B_{p,k_1 k_2}^{\text{loc}}(\mathbb{R}^n)$.*

From Theorem C.12, we obtain:

Theorem C.24. *We have $B_{p,k}^{\text{loc}}(X) \subset \mathcal{C}^j(X)$ for j is a non-negative integer if and only if* (C.16) *is valid.*

Theorem C.6 leads to the following:

Theorem C.25. *$B^{\text{loc}}_{p,k}(X)$ is a Fréchet space with the topology defined by the semi-norms $u \to \|\phi u\|_{p,k}$, $\phi \in \mathcal{C}^\infty_0(X)$, and we have*

$$\mathcal{C}^\infty(X) \subset B^{\text{loc}}_{p,k}(X) \subset \mathcal{D}'^j(X) \tag{C.19}$$

for some j in the topological sense. $\mathcal{D}'^j(X)$ is the set of distributions of order $\leq j$.

If $k_\mu \in \mathcal{K}$ and $1 \leq p_\mu \leq \infty$, $\mu = 1, 2, ...$ Consider the space $\bigcap_1^\infty B^{\text{loc}}_{p_\mu,k_\mu}(X)$ with the topology which is the least upper bound of the topologies in the spaces $B^{\text{loc}}_{p_\mu,k_\mu}(X)$, that is, defined by the semi-norms

$$u \to \|\phi u\|_{p_\mu,k_\mu}, \quad \phi \in \mathcal{C}^\infty_0(X), \quad \mu = 1, 2, ...$$

This is a Fréchet space. Note that Theorem C.24 gives that

$$\mathcal{C}^\infty(X) = \bigcap_1^\infty B^{\text{loc}}_{p_\mu,k_\mu}(X) \;\; \text{if } k_\mu(\xi) = (1 + |\xi|)^\mu. \tag{C.20}$$

Now, we extend Theorem C.9.

Theorem C.26. *Every bounded set in $B^{\text{loc}}_{p,k_1}(X)$ is precompact in $B^{\text{loc}}_{p,k_2}(X)$ if and only if* (C.14) *holds.*

D Background material

In the appendix, we will state some theorems and lemmas needed in proving results in chapter 3. In the first section, we will state a generalization of the Malgrange preparation theorem, and in the second section, we state some calculus lemmas.

D.1 Generalization of the Malgrange preparation theorem

For the proofs of the lemmas introduced in the section check the appendix in [Den92].

Theorem D.1. *Let $F(t,x)$ be a $\mathcal{C}^\infty$ function of (t,x) in a neighborhood of the origin in $\mathbb{R}\times\mathbb{R}^n$ with values in $\mathscr{L}(\mathbb{C}^n,\mathbb{C}^n)$, satisfying*

$$F(0,0)=0, \quad \det(\partial_t F(0,0)) \neq 0. \tag{D.1}$$

Then we may factorize

$$F(t,x) = C(t,x)(t\operatorname{Id}_N + B(x)) \tag{D.2}$$

near the origin, where C and B are $\mathcal{C}^\infty$ functions with values in $\mathscr{L}(\mathbb{C}^n,\mathbb{C}^n)$. We also get $\det(C(0,0)) \neq 0$ *and $B(0)=0$. If F is real (matrix) valued, we may choose C and B real (matrix) valued.*

Theorem D.2. *Let $F(t,x)$ satisfy the hypothesis in Theorem* D.1*. If $G(t,x)$ is $\mathcal{C}^\infty$ function in a neighborhood of the origin in $\mathbb{R}\times\mathbb{R}^n$ with values in $\mathscr{L}(\mathbb{C}^n,\mathbb{C}^n)$, then we can write*

$$G(t,x) = Q(t,x)F(t,x) + R(x) \tag{D.3}$$

near the origin, with $\mathcal{C}^\infty$ functions Q and R.

D.2 Some calculus lemmas

In the appendix in [Den89], Dencker proved some calculus lemmas while considering the weight and the metric given by (3.32), and (3.33) respectively. We will follow his steps to prove similar calculus lemmas while using the weight and metric given by (3.95), and (3.96) respectively.

We are going to study the composition of conormal distributions having nonstandard symbols. Let $a_\varphi(x,D) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ be given by

$$a_\varphi(x,D)u(x) = (2\pi)^{-n} \int e^{i(\langle x-y,\eta\rangle + \varphi(x,\eta))} a(x,\eta)u(y)dyd\eta, \tag{D.4}$$

$u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, where $a \in S(\vartheta^k, g)$, $\varphi(x,\eta) \in \mathcal{C}^\infty(T^*\mathbb{R}^n \setminus 0)$ is homogeneous of degree 1 in the η variables and satisfies (3.127). Here ϑ, and g are defined by (3.95), and (3.96) respectively. The composition with $p(x,D)$ given by

$$\begin{aligned} p(x,D)a_\varphi(x,D)u(x) &= (2\pi)^{-2n} \int\int e^{i(\langle x-y,\xi\rangle + \langle y-z,\eta\rangle + \varphi(y,\eta))} p(x,\xi)a(y,\eta)u(z)dzd\eta dyd\xi \\ &= b_\varphi(x,D)u(x), \end{aligned} \tag{D.5}$$

if p, $a \in \mathcal{S}$, where

$$b(x,\eta) = (2\pi)^{-n} \int e^{-iE} p(x,\xi)a(y,\eta)dyd\xi \tag{D.6}$$

and $E = \langle y-x, \xi-\eta\rangle - \varphi(y,\eta) + \varphi(x,\eta) = \langle y-x, \theta-\eta\rangle$, if we put

$$\theta = \xi - \int_0^1 \partial_x\varphi(x+s(y-x),\eta)ds. \tag{D.7}$$

Now, $\chi : (x,\xi;y,\eta) \to (x,\theta;y,\eta)$ is a diffeomorphism. Thus, if we let

$$f(x,\theta;y,\eta) = p(x,\xi)a(y,\eta),$$

we obtain

$$b(x,\eta) = e^{i\langle D_y, D_\theta\rangle} f(x,\theta;y,\eta))|_{\substack{\theta=\eta \\ y=x}} \tag{D.8}$$

since

$$\left|\frac{d(y,\xi)}{d(y,\theta)}\right| \equiv 1.$$

This can be extended to general symbols by the following

Lemma D.3. *Assume $\varphi(x,\eta) \in \mathcal{C}^\infty(T^*\mathbb{R}^n \backslash 0)$ is homogeneous of degree 1 in the η variables and satisfies* (3.127). *If $a \in S(\vartheta^k, g)$, $k \in \mathbb{Z}$, has support in a sufficiently small conical neighborhood of $\{\eta' = 0\}$ and $p \in S(\vartheta, g)$, then the composition is given by* (D.5) *where $b \in S(\vartheta^{k+1}, g)$ satisfies*

(D.8)*, and has expansion*

$$b(x,\eta) \cong \sum_{j=0}^{N-1} (i\langle D_\xi, D_y - (\partial\theta/\partial y)D_\xi\rangle)^j p(x,\xi)a(y,\eta)/j!|_{\substack{y=x\\ \xi=\eta+d_x\varphi(x,\eta)}} \tag{D.9}$$

modulo $S(\vartheta^{k+1}h^N, g)$*, with* θ *given by* (D.7)*, and* $h^2 = \langle\xi'\rangle^{-2}$.

Proof. If $\varphi \equiv 0$ then (D.8)-(D.9) follows from the Weyl calculus, since $g(t,-\tau) = g(t,\tau)$ (see Theorems B.15 and B.21. We have $p(x,\xi)a(y,\eta) \in S(M,G)$ where $M(\xi,\eta) = \vartheta(\xi)\vartheta^k(\eta)$ is a weight for $G = g_{x,\xi}(dx,d\xi) + g_{y,\eta}(dy,d\eta)$. Hence, if we prove that $\chi^*S(M,G) = S(M,G)$, we get (D.9), since $\partial_\xi\chi = (0,\mathrm{Id};0,0)$ and $\partial_y\chi = (0,\partial\theta/\partial y;\mathrm{Id},0)$. We only have to consider the case when

$$|\theta - \eta| \le \varepsilon, \tag{D.10}$$

since otherwise we may integrate by parts with respect to y in (D.6) to obtain $b \in S^{-\infty}$. (3.127) gives $|\theta - \xi| \le \rho|\eta'|$, when η is in a small conical neighborhood of $\{\eta' = 0\}$, with ρ is some constant. We have

$$\langle\theta'\rangle \le 1 + |\theta' - \xi'| + |\xi'| \le \langle\xi'\rangle + \rho c\langle\theta'\rangle, \tag{D.11}$$

so, for ρ small we get $\langle\theta'\rangle \le \langle\xi'\rangle$. Also, we have

$$\langle\xi'\rangle \le 1 + |\xi' - \theta'| + |\theta'| \le 1 + \rho|\eta'| + |\theta'| \le \langle\theta'\rangle + c\rho\langle\theta'\rangle, \tag{D.12}$$

so, for ρ small we have $\langle\xi'\rangle \le c\langle\theta'\rangle$. Hence, we get $\chi^*M \approx M$ for η in a small conical neighborhood of $\{\eta' = 0\}$. As we have $\partial\theta/\partial\eta'' = O(|\eta'|)$, $\partial\theta/\partial\eta' = O(1)$, and $\partial\theta/\partial x = O(|\eta'|)$, we get $\chi^*G \approx G$ in a small conical neighborhood of $\{\eta' = 0\}$. Thus by Lemma 8.2 in [Hö79], we obtain $\chi^*S(M,G) = S(M,G)$ if

$$G_{\chi(w)}(\chi^{(k)}(w;t_1,...,t_k)) \le C_k \prod_{i=1}^{k} G_{\chi(w)}(\chi'(w,t_i)) \tag{D.13}$$

for $k > 1$, where χ^k is the kth differential. This means that

$$\begin{cases} |\partial_y^\alpha\partial_\eta^\beta\partial_x^\gamma\theta'(x,y,\eta)| \le C_{\alpha\beta\gamma}\langle\eta\rangle^{-|\beta''|}\langle\eta'\rangle^{1-|\beta'|} \\ |\partial_y^\alpha\partial_\eta^\beta\partial_x^\gamma\theta''(x,y,\eta)| \le C_{\alpha\beta\gamma}\langle\eta\rangle^{1-|\beta''|}\langle\eta'\rangle^{-|\beta'|}, \end{cases} \tag{D.14}$$

for $|\alpha| + |\beta| + |\gamma| > 1$, with θ given by (D.7). Since θ is homogeneous of degree 1, the second inequality follows easily by using $\langle\eta\rangle^{-1} \le \langle\eta'\rangle^{-1}$. Similarly, we get the first when $|\beta'| > 0$, and otherwise

$$|\partial_y^\alpha\partial_{\eta''}^{\beta''}\partial_x^\gamma\theta'(x,y,\eta)| \le C_{\alpha\beta''\gamma}|\eta'|\langle\eta\rangle^{-|\beta''|},$$

according to (3.127), which proves (D.14) and the lemma. □

Now, let $S^v_{1,0,0}$ be the symbol defined by (3.139), and $d_0 = \operatorname{codim} \Sigma_2$. For $a \in S^v_{1,0,0}$ we define $a(x, D'') \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$a(x, D'')u(x) = (2\pi)^{d_0-n} \int\int e^{i\langle x''-y'',\eta''\rangle} a(x, y', \eta'')u(y)dyd\eta'', \tag{D.15}$$

$u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. If p, $a \in \mathcal{S}$, then the composition is given by

$$\begin{aligned} p(x, D)a(x, D'')u(x) &= (2\pi)^{d_0-2n} \int\int e^{i(\langle x-y,\xi\rangle+\langle y''-z'',\eta''\rangle)} p(x, \xi)a(y, z', \eta'')u(z)dzd\eta''dyd\xi \\ &= b(x, D'')u(x) \end{aligned} \tag{D.16}$$

where

$$\begin{aligned} b(x, z', \eta'') &= (2\pi)^{-n} \int\int e^{i\langle x-y,\xi-(0,\eta'')\rangle} p(x, \xi)a(y, z', \eta'')dyd\xi \\ &= e^{i\langle D_y,D_\xi\rangle} p(x, \xi)a(y, z', \eta'')|_{\substack{y=x \\ \xi=(0,\eta'')}}. \end{aligned} \tag{D.17}$$

For more general symbols we obtain the following lemma.

Lemma D.4. *If $p \in S(\vartheta, g)$ and $a \in S^v_{1,0,0}$, then the composition is given by* (D.16) *where $b \in S^v_{1,0,0}$ satisfies* (D.17) *and*

$$b(x, z', \eta'') = e^{i\langle D_{y'},D_{\xi'}\rangle} p(x, \xi', \eta'')a(y', x'', z', \eta'')|_{\substack{\xi'=0 \\ y'=x'}} + Ra, \tag{D.18}$$

where $R : S^v_{1,0,0} \to S^{v-1}_{1,0,0}$ is continuous. Also, b and Ra are determined modulo $S^{-\infty}$ by the restriction of a to $\{|y - x| < \varepsilon\}$, and p to $\{|\xi - (0, \eta'')| < \varepsilon\langle\eta''\rangle\}$, $\forall\varepsilon > 0$.

Proof. Let

$$G_{(x,\xi,y,z',\eta'')} = |dx|^2 + |d\xi'|/\langle\xi'\rangle^2 + |d\xi''|^2/\langle\xi\rangle^2 + |dy|^2 + |dz'|^2 + |d\eta''|^2/\langle\eta''\rangle^2,$$

and $A(x, \xi, y, z', \eta'') = \langle y, \xi\rangle$. Then the dual metric is given by

$$G^A_{(x,\xi,y,z',\eta'')}(0, d\xi, dy, 0) = |d\xi|^2 + |dy'|^2\langle\xi\rangle^2 + |dy''|^2\langle\xi\rangle^2, \tag{D.19}$$

and equal to $+\infty$ otherwise. We have $p(x, \xi)a(y, z', \eta'') \in S(M, G)$ where $M(\xi, \eta'') = \vartheta(\xi)\langle\eta''\rangle^v$. In the following we will suppress the z' variables, which are not important.

G is slowly varying, $G \le G^A$ at $\Delta = \{\xi = (0, \eta''), y = x\}$ and G is A temperate with respect to Δ, that is

$$G_{(x,\xi,y,\eta'')} \le CG_{(x,(0,\eta''),x,\eta'')}(1 + G^A_{(x,\xi,y,\eta'')}(0, \xi - (0, \eta''), y - x, 0))^N.$$

This follows since

$$1/\langle\xi'\rangle + \langle\eta''\rangle/\langle\xi\rangle \le C(1+|\xi''-\eta''|),$$

and similarly M is A, G temperate with respect to Δ, since

$$M(\xi,\eta'')/M((0,\eta''),\eta'') = \langle\xi'\rangle$$

and we can easily check that

$$M(\xi,\eta'')/M((0,\eta''),\eta'') \le C(1+G^A_{(x,\xi,y,\eta'')}(0,\xi-(0,\eta''),y-x,0)). \tag{D.20}$$

By Theorem B.10, we obtain that $b\in S^v_{1,0,0}$ satisfies (D.17).

In order to prove (D.18), we observe that

$$e^{i\langle D_y,D_\xi\rangle} = e^{i\langle D_{y'},D_{\xi'}\rangle}\circ e^{i\langle D_{y''},D_{\xi''}\rangle}.$$

If $A''(x,\xi,y,\eta'') = \langle y'',\xi''\rangle$ we obtain

$$G^{A''}(0,d\xi'',0,dy'',0) = G^A(0,d\xi'',0,dy'',0),$$

and equal to $+\infty$ otherwise. We have $G\le\langle\eta''\rangle^{-2}G^{A''}$ at $\Delta''=\{\xi''=\eta'',y''=x''\}$, and G is A'' temperate with respect Δ'', since

$$(1+|\xi'|+|\eta''|)/\langle\xi\rangle \le C(1+|\xi''-\eta''|).$$

Similarly, M is A'', G temperate with respect to Δ'', so Theorem B.11 gives

$$c = e^{i\langle D_{y''},D_{\xi''}\rangle}p(x,\xi)a(y,\eta'')|_{\Delta''}\in S(\tilde M,\tilde G),$$

where $\tilde G$, $\tilde M$ are the restrictions of G, M to Δ''. Here $c\cong p(x,\xi)a(y,\eta'')|_{\Delta''}$, modulo $S(\tilde M_1,\tilde G)$ where $\tilde M_1 = \tilde M\langle\eta''\rangle^{-1}$.

If $A'(x,\xi,y,\eta'') = \langle y',\xi'\rangle$, we get

$$\tilde G^{A'}(0,d\xi',dy',0) = G^A(0,d\xi',dy',0)|_{\Delta''},$$

and equal to $+\infty$ otherwise. Then $\tilde G\le\tilde G^{A'}$ at $\Delta'=\{\xi'=0,y'=x'\}$ in Δ'', and $\tilde G$ is clearly A' temperate with respect to Δ', since it is the restriction of an A temperate metric. Similarly, $\tilde M$ is A', $\tilde G$ temperate with respect to Δ', since it is restriction of an A, G temperate weight. As before, we obtain that

$$b(x,\eta'') = e^{i\langle D_{y'},D_{\xi'}\rangle}c(x,\xi',y',\eta'')|_{\Delta'}\in S^v_{1,0,0}$$

satisfies (D.18), since

$$\tilde G|_{\Delta'} = G|_\Delta \approx |dx'|^2+|dx''|^2+|dz'|^2+|d\eta''|^2/\langle\eta''\rangle^2.$$

Outside the support of the integrand in (D.17), the symbol decays as any power of the G^A distance to the support (see [Hö07, section 18.4]). Therefore, the last statement follows from the fact that

$$G^A(y-x, \xi-(0,\eta'')) \geq |y-x|^2 + |\xi-(0,\eta'')|^2,$$

at $(x,(0,\eta''),x,\eta'')$. □

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