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A Central Limit Theorem for Functions on Weighted Sparse Inhomogeneous Random Graphs

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Contents

Chapter 1

Introduction

A large number of real-world situations can be modelled in a fruitful way as a *graph* $G = (V, E)$ that is characterised by its vertex set V and its set of edges E, that form connections between the vertices.

Indeed, in what is commonly cited as the earliest work in graph theory [BM76, p. 51] the reduction of a complex problem to a simple structure made up of vertices and edges was key to its solution. In 1736 Euler showed that it is impossible to find a way to walk through the different parts of the city of Königsberg such that one ends up where one started and crosses each of the seven bridges over the river Prege exactly once. Euler realised that in order to analyse this problem of *the seven bridges of Königsberg* most of the characteristics of the walk through the city can be abstracted away: The only relevant piece of information is which bridges connect which parts of the city.

Often it is useful to assign additional attributes to the vertices and edges of a graph to model supplementary properties of the vertices and their connections. This results in a *weighted graph*.[1]

Many real world applications involve graphs that are so large or so complex that it becomes increasingly difficult to understand these objects in their entirety. It then becomes attractive to try and model such graphs as random objects so that one may analyse their typical behaviour.

The mathematical theory of *random graphs* is generally considered to have been founded by Erdős and Rényi in a series of papers published in 1959 and the early 1960s [ER59; ER60; ER64; ER66]. Indeed, one of the most well-known and well-studied random graph models, a graph with n vertices in which each pair of vertices is independently connected with a fixed probability p_n, is usually known as the Erdős–Rényi random graph, even though Gilbert introduced this model at around the same time [Gil59] and the formulation of the random graph investigated by Erdős and Rényi is slightly different.

Even sixty years after its inception the Erdős–Rényi random graph remains one of the standard models in random graph theory. The simple homogeneous structure of Erdős–Rényi random graphs lends itself particularly well to analysis, but real-

[1] Many of the well-known and well-studied problems in graph theory involve only weights on edges, so it is not uncommon to define weighted graphs as having weights only on edges [cf. BM76, § 1.8; Wil96, § 8]. We will be able to deal with weights on both edges and vertices.

world examples of graphs are not always this uniform. The class of *inhomogeneous random graphs*, in which the probability that an edge is present between two vertices depends on an attribute of the vertices, aims to generalise the approach of the Erdős–Rényi model to obtain more irregular graphs. Inhomogeneous random graphs were introduced by Söderberg [Söd02] and studied extensively by Bollobás, Janson and Riordan [BJR07]. For a general overview over basic properties of Erdős–Rényi graphs, other graph models and inhomogeneous random graphs we refer to two books by van der Hofstad [Hof18; Hof23].

In this thesis we will focus on a particular subclass of these inhomogeneous random graphs, namely those with sparse rank-one kernels. This class of inhomogeneous random graphs is related to the Chung–Lu model [CL02]. In the Chung–Lu model each vertex v has a weight $W_v \in (0, \infty)$ that determines its connectivity. Two vertices v and u are independently connected with probability $W_v W_u / (\sum_{u'} W_{u'})$ (assuming $\max_v W_v^2 < \sum_u W_u$ so that this always yields a probability). The average degree of a vertex v in the Chung–Lu model is approximately equal to W_v. This allows for a greater inhomogeneity in the graph when compared to the Erdős–Rényi model by choosing the W_v differently. Since every vertex has on average W_v neighbours, the total number of edges in the graph scales roughly like n (whereas the total possible number of edges in a graph with n vertices would scale like n^2). That is the reason why we call such a graph sparse.

The main aim of this work is to establish a central limit theorem for functions on weighted sparse rank-one inhomogeneous random graphs. This result extends recent work by Cao [Cao21] for the Erdős–Rényi model with weights only on edges.

While we formulate this theorem generally for functions on weighted graphs satisfying a certain *good local approximation property*, we will note that most interesting applications of this theorem will probably be related to combinatorial optimisation problems on weighted graphs. In a combinatorial optimisation problem we look for a particular substructure in the graph that is optimal according to a certain measure. This definition is sufficiently general (or vague) to beg for an example.

Consider a graph whose vertex set is a set of cities. Two cities u and v are connected via an edge if there is a road that takes one from u to v without visiting another city w on the way. The weight associated with that edge is the length of the road from u to v. In the *shortest path problem* we look for the shortest possible route to get from a city u to another city v, i.e. such that the sum of the length of the roads on which we reach our destination is minimal.

Another classical problem is the maximum matching problem, in which vertices are paired up with at most one partner along edges in such a way that the sum of weights of the edges is maximal. For this problem vertices may represent people on a party and there is an edge between two people if they share a common interest. The weight of an edge might model the length of the conversation between the two people. We are then interested in pairing up people so that the total conversation time is maximised.

In the Erdős–Rényi setting the first-order behaviour of a number of combinatorial optimisation problems is known [e.g. BGT81; GNS06; KS81]. There are also results for much more general sparse graph settings [BLS13]. A number of results are also known for the mean field model, where the underlying graph is a complete graph with i.i.d. edge weights, which reduces to an Erdős–Rényi setting for certain (relaxed) optimisation problems [e.g. Ald01; Wäs10; Wäs12]. As far as we are aware Cao's [Cao21] work provided the first general central limit theorem for these problems in the Erdős–Rényi setting. Our work aims to generalise this underlying setting.

This thesis is structured as follows. In Chapter 2 we introduce some basic notions, present the general setting in mathematical detail and state and discuss our central limit theorem (Theorem 2.3.5). Chapter 3 is dedicated to the analysis of the local structure of sparse rank-one inhomogeneous random graphs. These results are required to establish the central limit theorem in our setting, but they may also be of independent interest. We investigate the properties of the neighbourhoods of vertices in some detail and show that they are generally only weakly correlated. We also establish explicit coupling results between the local neighbourhood of a vertex in the graph and a limiting Galton–Watson tree. In Chapter 4 we will follow Cao's strategy and prove the central limit theorem via the (generalised) perturbative Stein's method introduced by Chatterjee [Cha08; Cha14]. We briefly present this method in Section 4.1 and then put it to use in the remainder of that chapter.

Chapter 2

Setting and Main Result

2.1 Basic notions

From now on the set of natural numbers includes 0 such that $\mathbb{N} = \{0, 1, \ldots\}$. We define $\mathbb{N}_+ = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}$.

In our calculations C will denote a numerical constant whose value may change (usually increase) from occurrence to occurrence.

We will briefly introduce some more definitions from graph theory. For a general introduction to the subject we refer the reader to works by Bondy and Murty [BM76], Bollobás [Bol98] and Wilson [Wil96].

From now on we only consider undirected graphs. In this setting a graph $G = (V, E)$ is an ordered pair of two disjoint sets such that E is a subset of $V^{(2)} = \{\{v, u\} : v, u \in V\}$, the set of subsets of V with two elements. In an undirected graph an edge $e \in E$ that connects the two vertices $u, v \in V$ does not have a direction and is written as $e = \{u, v\} = \{v, u\}$. We will usually assume that V is finite, but at times we may allow ourselves to consider graphs with a countable set of vertices V.

The following concepts arise by considering the relation that a single edge can have with vertices.

Definition 2.1.1 (Adjacency, incidence)**.** Let $G = (V, E)$ be a graph.

Two distinct vertices $v, u \in V$ are *adjacent* or *neighbours* if $\{v, u\} \in E$, i.e. if there exists an edge that joins v and u. Two distinct edges $e, e' \in E$ are *adjacent* if they have a vertex in common, i.e. if there exist vertices $v, u, u' \in V$ such that $e = \{v, u\}$ and $e' = \{v, u'\}$.

An edge $e \in E$ is called *incident* to a vertex $v \in V$ if there exists a vertex $u \in V$ such that $e = \{v, u\}$. In this case we will call v and u *endpoints* or *end vertices* of e.

We can use edges to move from one vertex to another.

Definition 2.1.2 (Walk, trail, path)**.** A sequence of edges $e_1, \ldots, e_n \in E$ is a *walk of length* n if there is a sequence of vertices $v_0, \ldots, v_n \in V$ such that $e_i = \{v_{i-1}, v_i\}$. In other words a walk of length n is a sequence of n edges that are adjacent to each other or identical. We will refer to the vertex v_0 as the start vertex of the walk and v_n as its end vertex. We say that this walk is a walk from v_0 to v_n.[1]

[1] It should be noted that we are working with undirected graphs. As such the choice to call v_0 start vertex and v_n is somewhat arbitrary, since the walk can be traversed in any direction.

A walk is called a *trail* if all its edges are distinct. A walk is called a *path* if all its vertices are distinct. (This immediately implies that all its edges are distinct.) A walk for which the sequence of vertices satisfies $v_0 = v_n$ is called *closed* and referred to as a *cycle*.

The graph G is called *connected* if for any two distinct vertices $v, u \in V$ there exists a walk from v to u.

An important subclass of graphs, in particular in the realms of sparse random graphs, is the class of trees.

Definition 2.1.3 (Tree)**.** A graph $G = (V, E)$ is a *tree* if it is connected and contains no cycles.

It is convenient to think of trees as 'starting' or 'growing' from somewhere.

Definition 2.1.4 (Rooted graphs, rooted trees)**.** A graph $G = (V, E)$ with a distinguished vertex $v^* \in V$, which we may call the *root*, is called a *rooted graph*.

Of particular interest are *rooted trees*. We usually denote the root of a tree by $\varnothing$.

Unless otherwise noted, all trees we consider will be rooted.

Every tree can be embedded into the following infinite object, which we may use to conveniently refer to vertices (individuals) in trees.

Definition 2.1.5 (Ulam–Harris tree)**.** Set $\mathbb{N}_+^0 = \{\varnothing\}$ and let the *Ulam–Harris tree* be

$$\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathbb{N}_+^n,$$

i.e. the set of all finite words over the positive natural numbers.

For an individual $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{U}$ we call $(\mathbf{i}, j) = (i_1, i_2, \ldots, i_k, j) \in \mathcal{U}$, $j \in \mathbb{N}_+$, a child of $\mathbf{i}$. Hence the parent of an individual $\mathbf{i} = (i_1, i_2, \ldots, i_{k-1}, i_k) \in \mathcal{U}$ is $(i_1, i_2, \ldots, i_{k-1}) \in \mathcal{U}$.

We obtain a tree structure on $\mathcal{U}$ if we draw an edge between every child and its parent.

Call $|\mathbf{i}| = |(i_1, i_2, \ldots, i_k)| = k$ the generation of $\mathbf{i} \in \mathcal{U}$ and set $|\varnothing| = 0$. Furthermore, we define an order on $\mathcal{U}$ by setting $\mathbf{i} \prec \mathbf{i}'$ if

(i) $|\mathbf{i}| < |\mathbf{i}'|$ or

(ii) $|\mathbf{i}| = |\mathbf{i}'|$, which implies $\mathbf{i} = (i_1, \ldots, i_k)$ and $\mathbf{i}' = (i'_1, \ldots, i'_k)$ for some $k \in \mathbb{N}_+$, and there is a $j \in \{1, \ldots, k\}$ such that $i_\ell = i'_\ell$ for all $\ell \in \{1, \ldots, j-1\}$ and $i_j < i'_j$.

It is easy to see that $\prec$ is transitive and that it is total, i.e., that for any two individuals $\mathbf{i}, \mathbf{i}' \in \mathcal{U}$ we have either $\mathbf{i} = \mathbf{i}'$, $\mathbf{i} \prec \mathbf{i}'$ or $\mathbf{i} \succ \mathbf{i}'$.

On the whole, these conventions imply a breadth-first way of thinking about the tree structure.

We may at times want to consider only a part of the graph or remove vertices from it. The following definition makes precise which objects we obtain in those cases.

Definition 2.1.6 (Subgraph, induced subgraph and 'reduced graph')**.** Let $G = (V, E)$ be a graph. A graph $G' = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

Let $V' \subseteq V$ be a subset of vertices of G. Then the subgraph of G whose vertex set is V' and whose edge set contains all edges in E whose endpoints both lie in V' is denoted by $G[V']$ and called the *subgraph of G induced by V'*.

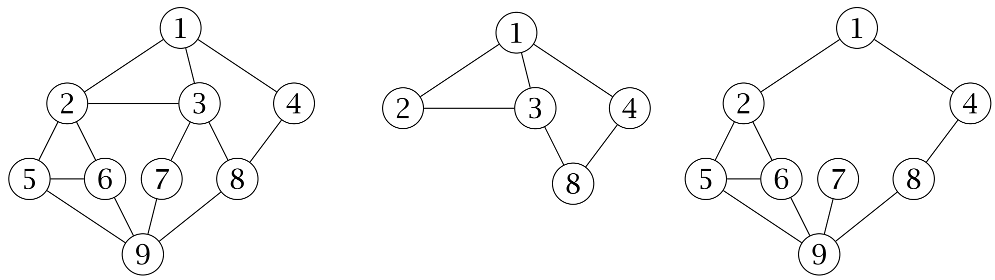
We write $G - V'$ for $G[V \setminus V']$, which we may call *G reduced by V'*. Then $G - V'$ is obtained from G by deleting all vertices in V' and all edges adjacent to those vertices.

Analogously, we can also induce subgraphs with edge sets. Let $E' \subseteq E$ be a subsets of edges of G (which could also be a set of paths or walks). Then the subgraph of G whose edge set is E' and whose vertex set contains exactly all the endpoints of edges in E' is denoted by $G[E']$ and called the *subgraph of G induced by E'*.

We write $G - E'$ for $G[E \setminus E']$, i.e. the subgraph of G for which we removed all edges of E'.

We will often write $G - v$ for $G - \{v\}$ and $G - e$ for $G - \{e\}$.

Examples of the definitions from Definition 2.1.6 are shown in Fig. 2.1.



(a) G (b) $G[\{1, 2, 3, 4, 8\}]$ (c) $G - 3$

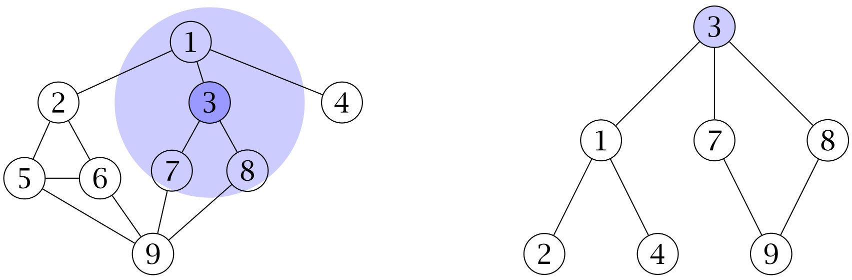
Figure 2.1: (a) A graph G with vertex set $\{1, \ldots, 9\}$, (b) the subgraph $G[\{1, 2, 3, 4, 8\}]$ induced by the vertices $\{1, 2, 3, 4, 8\}$ and (c) the graph $G - 3$.

One of the aims of Chapter 3 is to show a coupling between the local neighbourhood of a vertex and a limiting tree object. We first define what we mean by the local neighbourhood of a vertex.

Definition 2.1.7 (Local neighbourhood)**.** Let $G = (V, E)$ be a graph. For a vertex $v \in V$ and level $\ell \in \mathbb{N}$ denote by $B_\ell(v, G)$ the *(local) neighbourhood of the vertex v up to level ℓ* in the graph G. Formally, we define $B_\ell(v, G)$ as the subgraph of G induced by the union of all paths starting in v that are no longer than ℓ steps.

Unless otherwise noted, we usually regard $B_\ell(v, G)$ as a rooted graph with root v.

When the context is clear, we will sometimes drop the reference to the underlying subgraph G and will just write $B_\ell(v)$ instead of $B_\ell(v,G)$. If we define the graph distance $d_G(v,u)$ as the smallest number ℓ such that there is a path of length ℓ joining v and u, the ℓ-neighbourhood around v is the ℓ-ball around v with respect to the graph distance. It is tempting to draw $B_\ell(v)$ as a ball around the vertex v in a drawing of G (as is shown in Fig. 2.2a), but since the distance notion of the graph distance need not agree with the usual notion of distance in the space in which we draw G, this is rarely possible. We will therefore usually think of $B_\ell(v)$ just as a rooted graph (Fig. 2.2b).



(a) G and $B_1(3,G)$

(b) $B_2(3,G)$

Figure 2.2: A graph G with vertex set $\{1,\ldots,9\}$. The 1-neighbourhood of 3 is highlighted in blue on the left. The 2-neighbourhood of 3, $B_2(3,G)$, is shown on the right.

It is convenient to have a notation for the set of vertices in a local neighbourhood $B_\ell(v,G)$ of a vertex v and for the set of vertices at a certain level.

Definition 2.1.8. Let $G=(V,E)$ be a graph. For a level $\ell\in\mathbb{N}$ and a vertex $v\in V$ let $S_\ell(v,G)$ be the set of vertices in $B_\ell(v,G)$. Furthermore, set $S_{-1}(v,G)=\emptyset$.

Then let

$$D_\ell(v,G)=S_\ell(v,G)\setminus S_{\ell-1}(v,G)\quad\text{for } \ell\in\mathbb{N}$$

be the vertices of $B_\ell(v,G)$ at level ℓ.

The local neighbourhood can be generalised from the neighbourhood of a single vertex to the neighbourhood of a set of vertices.

Definition 2.1.9. Let $G=(V,E)$ be a graph and let $\mathcal{V}\subseteq V$ be a set of vertices. Then we let $B_\ell(\mathcal{V},G)$ be the subgraph of G that is induced by the union of all paths from $B_\ell(v,G)$ for $v\in\mathcal{V}$. The set of vertices in $B_\ell(\mathcal{V},G)$ is given by

$$S_\ell(\mathcal{V})=\bigcup_{v\in\mathcal{V}} S_\ell(v,G)$$

and we set

$$D_\ell(\mathcal{V},G)=S_\ell(\mathcal{V},G)\setminus S_{\ell-1}(\mathcal{V},G)$$

(with the convention $S_{-1}(\mathcal{V})=\emptyset$).

By this construction we only have $D_\ell(\mathcal{V}, G) \subseteq \bigcup_{v \in \mathcal{V}} D_\ell(v, G)$. Equality is only guaranteed if the $S_{\ell-1}(v, G)$ are disjoint.

For $\ell = 1$ the set $D_1(v, G)$ is the set of (direct) neighbours of v and gives rise to an important quantity of the vertex v: its degree.

Definition 2.1.10 (Degree)**.** Let $G = (V, E)$ be a graph and fix a vertex $v \in V$. Then the *degree* of the vertex v is the number of its neighbours

$$|\{u \in V : \{v, u\} \in E\}| = |D_1(v, G)|.$$

Formally, we have defined graphs as ordered pairs of two disjoint sets. The notion of equality that this induces on the set of graphs is not particularly enlightening, so we introduce the notion of a graph isomorphism.

Definition 2.1.11 (Graph isomorphism)**.** Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We write

$$G \cong G'$$

and say that G and G are *isomorphic* if there is a bijection φ from V to V' that preserves edges in the sense that $\{\varphi(v), \varphi(u)\} \in E'$ if and only if $\{v, u\} \in E$. Such a function G is called a *graph isomorphism*.

If G and G' are rooted graphs with root v and v', respectively, we additionally require that $\varphi(v) = v'$.

With those basic graph theoretic concepts in hand we finally turn to weighted graphs.

Definition 2.1.12 (Weighted graph)**.** A *weighted graph* $\mathbf{G}$ is an ordered tuple $\mathbf{G} = (V, E, \mathbf{w})$, where $G = (V, E)$ is a graph (the underlying graph) and $\mathbf{w} \colon V \cup E \to \mathbb{R}$ is a function that assigns a real-valued weight to each vertex $v \in V$ and each edge $e \in E$.

All the previous notions generalise naturally to weighted graphs. We will not restate them for the weighted case. For the avoidance of doubt and since it is a slight extension of a previous concept, we will explain what we mean by an isomorphism for graph *pairs* and make explicit reference to weights in that definition.

Definition 2.1.13 (Isomorphism for pairs of weighted rooted graphs)**.** Let $\mathbf{G} = (V, E, \mathbf{w})$ and $\mathbf{G}' = (V', E', \mathbf{w}')$ be two weighted rooted graphs with the same root $v^* \in V \cap V'$. Let similarly $\mathbf{H} = (U, F, \mathbf{x})$ and $\mathbf{H}' = (U', F', \mathbf{x}')$ be two further weighted rooted graphs with the same root $u^* \in U \cap U'$. We write

$$(\mathbf{G}, \mathbf{G}') \cong (\mathbf{H}, \mathbf{H}')$$

if there exists a bijection $\varphi \colon V \to U$ that satisfies $\varphi(v^*) = u^*$, preserves edges, i.e.

$$\{\varphi(v_1), \varphi(v_2)\} \in F \quad \text{if and only if} \quad \{v_1, v_2\} \in E,$$

and weights, i.e. $\mathbf{x}(\varphi(v)) = \mathbf{w}(v)$ for all $v \in V$ and $\mathbf{x}(\{\varphi(v_1), \varphi(v_2)\}) = \mathbf{w}(\{v_1, v_2\})$ for all $v_1, v_2 \in V$, and a bijection $\varphi' \colon V' \to U'$ that also maps v^* to u^* and preserves edges and weights in the way just described.

Finally we will introduce two stochastic concepts. First the mixed Poisson distribution and then the Galton–Watson tree.

Definition 2.1.14 (Mixed Poisson distribution)**.** Let μ be a measure on $(0, \infty)$, then a random variable Z has *mixed Poisson distribution* with mixing distribution μ if for all $k \in \mathbb{N}$

$$\mathbb{P}(Z = k) = \mathbb{E}\left[e^{-\Lambda}\frac{\Lambda^k}{k!}\right], \quad \text{where } \Lambda \sim \mu.$$

If Z has a mixed Poisson distribution with mixing distribution μ, we write $Z \sim \mathrm{MPoi}(\mu)$ or alternatively $Z \sim \mathrm{MPoi}(\Lambda)$ for $\Lambda \sim \mu$.

We write $\mathrm{Poi}(\lambda)$ for the regular Poisson distribution with parameter $\lambda > 0$. By construction $\mathrm{Poi}(\lambda) = \mathrm{MPoi}(\delta_\lambda)$, where δ_λ is the Dirac measure at λ.

For our intents and purposes the definition of a Galton–Watson tree based on the Ulam–Harris tree $\mathcal{U}$ (Definition 2.1.5) will be most convenient.

Definition 2.1.15 (Galton–Watson tree)**.** For any $k \in \mathbb{N}$ let $\nu^{(k)}$ be a probability measure on $\mathbb{N}$ and let $\{N_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}_+^k\}$ be a sequence of i.i.d. random variables with distribution $\nu^{(k)}$. Construct these sequences so that they are independent for every k, i.e. so that $\{N_{\mathbf{i}} : \mathbf{i} \in \mathcal{U}\}$ is a sequence of independent random variables.

Then the *Galton–Watson tree* with level-k offspring distributions $(\nu^{(k)})_{k\in\mathbb{N}}$ is defined as

$$\mathcal{T} = \bigcup_{k\in\mathbb{N}} \mathcal{T}_k \subseteq \mathcal{U},$$

where $\mathcal{T}_0 = \{\varnothing\}$ and

$$\mathcal{T}_{k+1} = \{(\mathbf{i}, j) \in \mathbb{N}_+^{k+1} : \mathbf{i} \in \mathcal{T}_k, 1 \le j \le N_{\mathbf{i}}\}.$$

As for the Ulam–Harris tree we draw an edge between each child $(\mathbf{i}, j) \in \mathcal{T}_{k+1}$ and its parent $\mathbf{i}$.

In what follows we will mainly need Galton–Watson trees in which the offspring distribution is the same for all levels except level 0, i.e. *at the root*.

2.2 Setting

We are now ready to describe our setting in more detail. First we define the rank-one inhomogeneous graph model (without additional vertex and edge weights) that we will focus on. We will work on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Graph structure For $n \in \mathbb{N}_+$ let $G_n = (V_n, E_n)$ be a graph with vertex set $V_n = [n] = \{1, \dots, n\}$.

Assign a possibly random *connectivity weight* $W_v^n \in (0, \infty)$ to each vertex $v \in V_n$. This weight will determine the connectivity of the vertex v in the graph G_n. Let $\mathbf{W}^n = (W_v^n)_{v \in V_n}$ be the collection of all connectivity weights for vertices in V_n and let $\mathcal{F}_n = \sigma(\mathbf{W}^n) = \sigma((W_v^n)_{v \in V_n})$ be the σ-algebra generated by all connectivity weights for vertices in V_n. From now on we will write

$$\mathbb{P}_n(\,\cdot\,) = \mathbb{P}(\,\cdot \mid \mathcal{F}_n) \quad \text{and} \quad \mathbb{E}_n(\,\cdot\,) = \mathbb{E}[\,\cdot \mid \mathcal{F}_n]$$

for the probability measure and expectation, respectively, conditioned on the connectivity weights $W_1^n, \dots, W_n^n$. We will also drop the superscript n from W_v^n to make formulas slightly easier on the eye.

Given these connectivity weights in $\mathcal{F}_n$ realise independent edges between all (unordered) pairs of vertices u and v with probability

$$p_{uv} = \frac{W_u W_v}{n\vartheta} \wedge 1, \tag{2.1}$$

where we define $\Lambda_n = \sum_{u \in V_n} W_u$ and assume that $\vartheta \in (0, \infty)$ satisfies

$$\frac{1}{n}\Lambda_n = \frac{1}{n} \sum_{u \in V_n} W_u \xrightarrow{\mathbb{P}} \vartheta \quad \text{as } n \to \infty,$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability. We will generally make stronger assumptions about the distribution of the connectivity weights (which will be detailed in Assumption 2.2.1 in just a moment), so we will not highlight this assumption here in more detail.

Formally, let $V_n^{(2)} = \{\{u, v\} : u, v \in V_n\}$ be the maximal set of edges that G_n could possibly have, i.e. the set of edges of the complete graph on V_n. Conditional on $\mathcal{F}_n$ let $X_{uv} \sim \mathrm{Bin}(1, p_{uv})$ for $1 \le u < v \le n$ be independent indicator functions (the *edge indicators*). We will write $X_{uv} = X_{vu}$ whenever $u, v \in V_n$, $u \neq v$, and set $X_{vv} = 0$ for all $v \in V_n$. The set of edges of G_n is then given by

$$E_n = \{\{u, v\} \in V_n^{(2)} : X_{uv} = 1\}.$$

Remark. This model is related – but in this formulation not exactly equal – to the Chung–Lu model [CL02], where vertices are connected independently with probability

$$\frac{W_u W_v}{\Lambda_n} \wedge 1,$$

and the Norros–Reittu model [NR06] where the edge probability is

$$1 - \exp(-W_u W_v / \Lambda_n).$$

By the assumed convergence of $n^{-1}\Lambda_n$ to ϑ, however, the edge probabilities will be very similar for large n.

The classical Erdős–Rényi model with $p_n = n^{-1}\lambda$ for some $\lambda \in (0,\infty)$ can easily be obtained by setting $W_v = \lambda$ for all $v \in V_n$. Then $n^{-1}\sum_{u' \in V_n} W_{u'} = \lambda$ is constant so that $\vartheta = \lambda$ and thus

$$p_{uv} = \frac{W_u W_v}{n\vartheta} = \frac{\lambda}{n} = p_n.$$

Our setting also includes Erdős–Rényi models in which $p_n = n^{-1}\lambda_n$ for a sequence of $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ with $\lambda_n \to \lambda \in (0,\infty)$. However, the intuitive idea of setting $W_v = \lambda_n$ yields $\vartheta = \lambda$ and edge probabilities of the form $(\lambda_n/n)(\lambda_n/\lambda) = p_n(\lambda_n/\lambda)$. The undesirable factor λ_n/λ tends to 1 and therefore does not matter in the limit. In fact all computations we make would still be valid if such a factor were present. Yet with a slightly different parametrisation we can obtain the desired edge probability p_n directly: Choosing $W_v = (\lambda_n\lambda)^{1/2}$ for any $v \in V_n$ we get again $\vartheta = \lambda$ so that the edge probability is equal to

$$p_{uv} = \frac{\lambda_n\lambda}{n\lambda} = \frac{\lambda_n}{n} = p_n$$

as desired.

In the framework of inhomogeneous sparse random graphs by Bollobás, Janson and Riordan [BJR07] our graph is a so-called rank-one model, since its *connection kernel* $\kappa_n(x,y) = xy/\vartheta$ has a simple product form.

In order to allow us to identify a limiting object for the graph G_n we will have to impose some conditions on the connectivity weight distribution.

Assumption 2.2.1. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of graphs as defined above.

Given $\mathcal{F}_n$ let

$$\nu_n(\,\cdot\,) = \frac{1}{n}\sum_{v\in V_n} \mathbb{1}_{\{W_v \in \,\cdot\,\}} \quad \text{and} \quad \hat{\nu}_n(\,\cdot\,) = \frac{1}{\Lambda_n}\sum_{v\in V_n} W_v \mathbb{1}_{\{W_v\in\,\cdot\,\}} \tag{2.2}$$

be the empirical measure of the connectivity weights and its size-biased version.

Assume that there is a measure ν on $(0,\infty)$ with mean in $(0,\infty)$ that satisfies the following properties.

(i) There exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ that converges to zero in probability such that

$$\mathcal{W}(\nu_n,\nu) \le \alpha_n,$$

where $\mathcal{W}(\mu,\nu)$ denotes the 1-Wasserstein distance between the measures μ and ν on $\mathbb{R}$

$$\mathcal{W}(\mu,\nu) = \inf\{\mathbb{E}_n[|X-Y|] : X\sim\mu, Y\sim\nu,\ X,Y \text{ defined on } (\Omega,\mathcal{F}_n,\mathbb{P}_n)\}.$$

(ii) Let $\hat{\nu}$ be the size-biased distribution of $W \sim \nu$ given by

$$\hat{\nu}(A) = \frac{1}{\mathbb{E}[W]}\mathbb{E}[W\mathbb{1}_{\{W\in A\}}].$$

Then with the same sequence $(\alpha_n)_{n\in\mathbb{N}}$ as in (i) we also have

$$\mathcal{W}(\hat{\nu}_n, \hat{\nu}) \leq \alpha_n.$$

(iii) Furthermore, we assume that the third moment of $W^{(n)} \sim \nu_n$

$$\mathbb{E}_n[(W^{(n)})^3] = \frac{1}{n}\sum_{u\in V_n} W_u^3$$

is bounded in probability.

The conditions we impose for the limiting object to exist are not minimal. Olvera-Cravioto [Olv22] constructed the couplings between the graph and the limiting object assuming only existence of the first moments. We decided to work under the stronger assumptions because they give us more explicit control over the rate of convergence and make the construction of the coupling slightly more natural.

Assumption (i) is in particular satisfied with an α_n of rate $n^{-1/2}$ if the weights W_v are drawn i.i.d. from the distribution ν assuming that ν has second moments [FG15, Thm. 1]. It is tempting to conjecture that the rate of convergence for $\mathcal{W}(\hat{\nu}_n, \hat{\nu})$ should be similar under the same moment conditions for $\hat{\nu}$, which would translate into the existence of third moments for ν. We do not attempt to address this question further in this thesis, we will just mention that Olvera-Cravioto [Olv22, in a slightly different setting in proof of Lem. 4.8] briefly argues that $\mathcal{W}(\nu_n, \nu) \xrightarrow{\mathbb{P}} 0$ implies $\mathcal{W}(\hat{\nu}_n, \hat{\nu}) \xrightarrow{\mathbb{P}} 0$ without claims on the rate of the latter convergence. Intuitively, this is true because size-biasing respects convergence in distribution [AGK19, Thm. 2.3] if the means converge as well and we can then use Skorohod's representation theorem to obtain coupled random variables with the desired distributions (possibly on a new probability space).

We will use the notation $W^{(n)} \sim \nu_n$ and $W \sim \nu$ to recall the definition of Λ_n

$$\Lambda_n = \sum_{u\in V_n} W_u = n\mathbb{E}_n[W^{(n)}]$$

If $W^{(n)}$ and W are constructed via the optimal coupling guaranteed by the Wasserstein distance [San15, Thm. 1.7], we have

$$\left|\frac{1}{n}\Lambda_n - \mathbb{E}_n[W]\right| \leq |\mathbb{E}_n[W^{(n)}] - \mathbb{E}_n[W]| \leq \mathbb{E}_n[|W^{(n)} - W|] \leq \alpha_n.$$

This implies that $n^{-1}\Lambda_n$ converges in probability to $\mathbb{E}_n[W]$, so that we can set $\vartheta = \mathbb{E}_n[W]$ for (2.1).

Fix $p \in (0, \infty)$. Now define

$$\Gamma_{p,n} = \frac{1}{n\vartheta} \sum_{u \in V_n} W_u^p = \frac{\mathbb{E}_n[(W^{(n)})^p]}{\vartheta} \tag{2.3}$$

for the average p-th power of the connectivity weights normalised with ϑ and

$$\kappa_{p,n} = \frac{1}{n\vartheta} \sum_{u \in V_n} W_u^p \mathbb{1}_{\{W_i > \sqrt{n\vartheta}\}} = \frac{1}{\vartheta} \mathbb{E}_n[(W^{(n)})^p \mathbb{1}_{\{W^{(n)} > \sqrt{n\vartheta}\}}] \tag{2.4}$$

for the average excess of p-th power of the connectivity weights above $\sqrt{n\vartheta}$ normalised with ϑ.

By (iii) we immediately have that $\Gamma_{p,n}$ is bounded in probability for all $p \in [0, 3]$.

For $\kappa_{p,n}$ observe that if $p \in [0, 3)$ by Hölder's and Markov's inequality

$$\begin{aligned}
\kappa_{p,n} &= \frac{1}{\vartheta} \mathbb{E}_n[(W^{(n)})^p \mathbb{1}_{\{W^{(n)} > \sqrt{n\vartheta}\}}] \\
&\leq \frac{1}{\vartheta} \mathbb{E}_n[(W^{(n)})^3]^{p/3} \mathbb{P}_n(W^{(n)} > \sqrt{n\vartheta})^{-p/3} \\
&\leq \frac{1}{\vartheta} \mathbb{E}_n[(W^{(n)})^3]^{p/3} \Big(\frac{\mathbb{E}_n[(W^{(n)})^3]}{\sqrt{n\vartheta}^3} \Big)^{1-p/3} \\
&= \frac{\Gamma_{3,n}}{\sqrt{n\vartheta}^{3-p}}.
\end{aligned}$$

This terms goes to zero in probability as $n \to \infty$ if $p \in [0, 3)$ since $\Gamma_{3,n}$ is bounded in probability.

Note that if $W_u \leq \sqrt{n\vartheta}$ and $W_v \leq \sqrt{n\vartheta}$ it follows that $W_u W_v \leq n\vartheta$. This implies that the minimum with 1 in the definition (2.1) of p_{uv} is not needed in this case. Hence, $\kappa_{p,n}$ measures the p-th moment of the connectivity weight of the vertices exceeding this 'safe' threshold.

Analogous to $\Gamma_{p,n}$ we also define Γ_p as the 'normalised' p-th moment of ν. Fix p and let $W \sim \nu$, then set

$$\Gamma_p = \frac{\mathbb{E}[W^p]}{\vartheta} = \frac{\mathbb{E}[W^p]}{\mathbb{E}[W]}. \tag{2.5}$$

Local limit We now describe the local limiting behaviour of the rank-one inhomogeneous graph model. This is done by showing that the local neighbourhood of a vertex v in G_n can be coupled with high probability to a 'delayed' Galton–Watson tree.

The limiting object can now be constructed as follows.

Definition 2.2.2. For a probability measure ν on $(0, \infty)$ and a connectivity weight $W \in (0, \infty)$ let $T(W, \nu)$ be a Galton–Watson tree in which the root has $\mathrm{Poi}(W)$ children and all other levels have offspring distribution $\mathrm{MPoi}(\hat{\nu})$.

For an integer $\ell \in \mathbb{N}$ let $T_\ell(W, \nu)$ be the subtree of $T(W, \nu)$ cut at height ℓ (or alternatively the ℓ-neighbourhood of the root $B_\ell(\varnothing, T(W, \nu))$).

Remark. Note that $T(W,\nu)$ can be constructed by joining $N \sim \mathrm{Poi}(W)$ independent Galton–Watson trees with offspring distribution $\mathrm{MPoi}(\hat{\nu})$ for all levels whose roots we call $\varnothing_1, \dots, \varnothing_N$ together at a root $\varnothing$ with edges $\{\varnothing, \varnothing_1\}, \dots, \{\varnothing, \varnothing_N\}$.

Olvera-Cravioto [Olv22] calls such a tree process 'delayed', because the root has a different offspring distribution than all other individuals [see also EHH08].

This limiting tree is closely connected to the local weak limit of the graph G_n [Hof18, Chap. 2]. The notion of local limits for graphs was introduced by Benjamini and Schramm [BS01] and later by Aldous and Steele [AS04], who used it extensively to develop the so-called objective method, in which the limiting properties of a sequence of finite problems are analysed in terms of local properties of a new infinite object. In our treatment we keep the vertex whose neighbourhood we explore fixed, whereas in the context of local weak limits this vertex is chosen uniformly at random. We can think of our setup as conditioning on the type of the root vertex, so that the usual local weak limit can then be recovered from our results by averaging over all vertices (and possibly adjusting the coupling of the root vertex). Specifically, the resulting tree would have a root with $\mathrm{MPoi}(\nu)$ children, while all other individuals have offspring distribution $\mathrm{MPoi}(\hat{\nu})$. Such a tree process is called *unimodular* [Hof23].

One of the main results of Chapter 3 is an explicit coupling construction that yields:

Proposition 2.2.3. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of rank-one inhomogeneous random graph that satisfies Assumption 2.2.1 for some measure ν on $(0,\infty)$.*

Let $\mathcal{V} \subseteq V_n$ be a set of vertices. Then for all $\ell \in \mathbb{N}$ the neighbourhoods around $v \in \mathcal{V}$ can be coupled to independent limiting trees $\mathcal{T}(v) \sim T(W_v, \nu)$ such that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}} \{B_\ell(v) \not\cong \mathcal{T}_\ell(v)\}\Big) \\
&\quad\le \frac{\Gamma_{2,n}}{n\vartheta}\sum_{v\in\mathcal{V}} W_v^2 + \Gamma_{1,n}\sum_{v\in\mathcal{V}} W_v \mathbb{1}_{\{W_v>\sqrt{n\vartheta}\}} \\
&\qquad + (\Gamma_{2,n}+1)^{\ell}\Big(\frac{\Gamma_{3,n}}{n\vartheta} + \kappa_{1,n} + \kappa_{2,n} + \frac{2+\Gamma_{1,n}}{k_n} + \frac{k_n}{n\vartheta}\Big)\sum_{v\in\mathcal{V}} W_v \\
&\qquad + |\mathcal{V}|\frac{1}{k_n} + \frac{k_n^2}{n\vartheta\Gamma_{1,n}} + \sum_{v\in\mathcal{V}} W_v\alpha_n\Big(\frac{1}{\vartheta} + (\Gamma_2+1)^{\ell-1}\Big(\frac{\Gamma_{2,n}}{\vartheta\Gamma_{1,n}}+1\Big)\Big),
\end{aligned}$$

where $(k_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ is an arbitrary sequence of positive real numbers.

By Assumption 2.2.1 $\Gamma_{1,n}$, $\Gamma_{2,n}$ and $\Gamma_{3,n}$ are bounded in probability and α_n, $\kappa_{1,n}$ and $\kappa_{2,n}$ converge to zero in probability. Additionally, $\sum_{v\in\mathcal{V}} W_v \mathbb{1}_{\{W_v>\sqrt{n\vartheta}\}}$ is zero if n is large enough for any finite set $\mathcal{V}$.

Then the probability that the coupling does not hold goes to zero in probability if $(k_n)_{n\in\mathbb{N}}$ is chosen appropriately. In particular the sequence needs to satisfy $k_n \to$

∞ as well as $k_n^2/n \to 0$. The choice $k_n \approx n^{1/3}$ balances the rate of $1/k_n$ and k_n^2/n so that both are of order $n^{-1/3}$.

This proposition can be used to obtain a coupling for neighbourhoods in the Erdős–Rényi model to a Galton–Watson tree with simple Poisson offspring distribution.

Example 2.2.4. Consider the Erdős–Rényi model with edge probability $p_n = \lambda_n/n$ (and $\lambda_n \to \lambda$). Recall that we had to set $W_v = (\lambda_n\lambda)^{1/2}$ to obtain the desired edge probabilities.

Then $\nu_n = \delta_{(\lambda_n\lambda)^{1/2}}$ and $\nu = \delta_\lambda$ and by basic properties of the Wasserstein distance

$$\mathcal{W}(\nu_n, \nu) = |\lambda_n^{1/2}\lambda^{1/2} - \lambda| = \lambda^{1/2}|\lambda_n^{1/2} - \lambda^{1/2}|.$$

Furthermore, the size-biased measures coincide with the original measures so that also

$$\mathcal{W}(\hat{\nu}_n, \hat{\nu}) = \mathcal{W}(\nu_n, \nu) = \lambda^{1/2}|\lambda_n^{1/2} - \lambda^{1/2}|.$$

Recall that $\lambda > 0$ and that $\lambda_n \to \lambda$ so that we may assume that $\lambda_n \geq \lambda/4$ for n large enough. Since the function $x \mapsto x^{1/2}$ is Lipschitz continuous with Lipschitz constant $1/2a^{-1/2}$ on the interval $[a, \infty)$, we have

$$\lambda^{1/2}|\lambda_n^{1/2} - \lambda^{1/2}| \leq |\lambda_n - \lambda|$$

as long as n is large enough. We thus set $\alpha_n = |\lambda_n - \lambda|$.

Furthermore, we may assume that $\kappa_{p,n} = 0$ and $\sum_{v\in\mathcal{V}} W_v \mathbb{1}_{\{W_v > \sqrt{n\vartheta}\}} = 0$, because $W_v > \sqrt{n\vartheta}$ if and only if $\lambda_n > n$ which is not the case for n large enough as $\lambda_n \leq 2\lambda < n$ for all n large enough.

Finally $\Gamma_{p,n} = \lambda_n^{1/2p}\lambda^{1/2p-1}$ and $\Gamma_p = \lambda^{p-1}$ in particular $\Gamma_{2,n} = \lambda_n$ and $\Gamma_2 = \lambda$.

The trees $\mathcal{T}(v)$ are just independent Galton–Watson trees with $\mathrm{Poi}(\lambda_n^{1/2}\lambda^{1/2})$ children at the root and offspring distribution $\mathrm{Poi}(\lambda)$ for all other individuals.

For $\mathcal{V} = \{v, u\}$ Proposition 2.2.3 then reduces to

$$\begin{aligned}
&\mathbb{P}_n(\{B_\ell(v) \neq \mathcal{T}_\ell(v)\} \cup \{B_\ell(u) \neq \mathcal{T}_\ell(u)\}\\
&\quad\leq 2\frac{\lambda_n^2}{n} + 2\lambda_n^{1/2}\lambda^{1/2}(\lambda_n+1)^\ell\left(\frac{\lambda_n^{3/2}}{n\lambda^{1/2}} + \frac{2+\lambda_n^{1/2}/\lambda^{1/2}}{k_n} + \frac{k_n}{n\lambda}\right)\\
&\qquad + 2\frac{1}{k_n} + 2\frac{k_n^2}{n\lambda_n^{1/2}\lambda^{1/2}} + 2\lambda_n^{1/2}\lambda^{1/2}|\lambda_n - \lambda|\left(\frac{1}{\lambda} + (\lambda+1)^{\ell-1}\left(\frac{\lambda_n^{1/2}}{\lambda^{1/2}} + 1\right)\right)
\end{aligned}$$

for n large enough. Choose $k_n = n^{1/3}$, then this bound can be estimated by

$$\mathbb{P}_n(\{B_\ell(v) \neq \mathcal{T}_\ell(v)\} \cup \{B_\ell(u) \neq \mathcal{T}_\ell(u)\}) \leq C\frac{(\lambda_n+1)^{\ell+2}}{\min\{1,\lambda\}n^{1/3}} + C\frac{(\lambda+1)^\ell}{\min\{1,\lambda\}}|\lambda_n - \lambda|,$$

which is of the same order as the coupling probability that Cao established for Erdős–Rényi random graphs [Cao21, Lem. 6.1].

Note that we coupled the neighbourhoods to Galton–Watson trees whose offspring distribution $\mathrm{Poi}((\lambda_n\lambda)^{1/2})$ at the root differs from the offspring distribution $\mathrm{Poi}(\lambda)$ of all other individuals. The classical coupling for Erdős–Rényi random graphs that Cao established couples the neighbourhood to Galton–Watson trees with offspring distribution $\mathrm{Poi}(\lambda)$ for all individuals. If we wanted to obtain this classical coupling, we would have to modify (or re-couple) the offspring distribution of the root at a cost of $\mathcal{W}(\delta_{\lambda_n\lambda}, \delta_\lambda) \le |\lambda_n - \lambda|$. This additional cost does not change the rate estimate.

Weighted graph We now add vertex and edge weights to our graph model to obtain a weighted graph. These weights are added independently of the underlying graph structure in an i.i.d. manner. An extension of the previous coupling result to the weighted setup is therefore straightforward.

Fix a graph G_n as defined above and fix two distributions on the non-negative real numbers $\mu_{E,n}$ and $\mu_{V,n}$. Assume that $\mu_{V,n}$ converges to some μ_V and $\mu_{E,n}$ to some μ_E in total variation distance, i.e. that

$$d_{\mathrm{TV}}(\mu_{V,n}, \mu_V) \to 0 \quad \text{and} \quad d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) \to 0,$$

where the total variation distance for two probability measures μ and λ on a measurable space $(\Omega', \mathcal{F}')$ is given by

$$d_{\mathrm{TV}}(\mu, \nu) = \inf\{\mathbb{P}(X \neq Y) : X \sim \mu, Y \sim \nu\}.$$

To obtain a weighted graph $\mathbf{G}_n$ from G_n, assign i.i.d. vertex weights $w_v^{(n)} \sim \mu_{V,n}$ to each $v \in V_n$ and i.i.d. edge weights $w_e^{(n)} \sim \mu_{E,n}$ to each $e \in V_n^{(2)}$.[2]

Given $\mathcal{F}_n$ the entire structure of $\mathbf{G}_n$ can be encoded in the following sequences of independent random variables

$$(\mathbf{X}^{(n)}, \mathbf{w}^{(n)}) = ((X_e^{(n)})_{e \in V_n^{(2)}}, (w_x^{(n)})_{x \in V_n \cup V_n^{(2)}}),$$

where $X_{\{u,v\}}^{(n)} \sim \mathrm{Bin}(1, p_{uv})$ for $\{u, v\} \in V_n^{(2)}$, $w_v^{(n)} \sim \mu_{V,n}$ for $v \in V_n$ and $w_e^{(n)} \sim \mu_{E,n}$ for $e \in V_n^{(2)}$ are all independent random variables.

We will usually drop the superscript (n) for all these objects. Additionally, we will use the notational convention that $X_{uv} = X_{vu} = X_{\{u,v\}}$ and $w_{uv} = w_{vu} = w_{\{u,v\}}$ for all $u \neq v$.

Local limit for the weighted graph In order to describe the local limit of the weighted graph we just need to add vertex and edge weight to the limiting object we identified for Proposition 2.2.3.

The limiting object will be the same as in Definition 2.2.2 just with added weights.

[2]Technically, we would only need to assign weights to edges $e \in E_n$ that are actually present in G_n, but it is more convenient to assign a weight to all 'possible' edges $e \in V_n^{(2)}$ 'just in case'.

Definition 2.2.5. Given two weight distributions μ_E and μ_V let $\mathbf{T}(W, \nu, \mu_E, \mu_V)$ be the Galton–Watson tree $T(W, \nu)$ endowed with i.i.d. edge weights drawn from μ_E and i.i.d vertex weights drawn from μ_V. For $\ell \in \mathbb{N}$ let $\mathbf{T}_\ell(W, \nu, \mu_E, \mu_V)$ denote the ℓ-level subtree of $\mathbf{T}(W, \nu, \mu_E, \mu_V)$.

The following object arises from the limiting object by conditioning on the presence of a certain edge.

Definition 2.2.6. Let ν be a probability measure on $(0, \infty)$ and $W, W' \in (0, \infty)$ two connectivity weights. Fix an edge weight distribution μ_E and a vertex weight distribution μ_V.

Let $\mathbf{T} \sim \mathbf{T}(W, \nu, \mu_E, \mu_V)$ with root $\varnothing$ and $\mathbf{T}' \sim \mathbf{T}(W', \nu, \mu_E, \mu_V)$ with root $\varnothing'$ be independent. Construct $\tilde{\mathbf{T}}(W, W', \nu, \mu_E, \mu_V)$ by grafting $\mathbf{T}'$ onto $\mathbf{T}$ via an edge of weight $w \sim \mu_E$ (independent of everything else) between $\varnothing$ and $\varnothing'$. (In particular $\varnothing$ is the root of $\tilde{\mathbf{T}}(W, W', \nu, \mu_E, \mu_V)$.) Let $\tilde{\mathbf{T}}_\ell(W, W', \nu, \mu_E, \mu_V)$ be the depth-ℓ subtree of $\tilde{\mathbf{T}}(W, W', \nu, \mu_E, \mu_V)$.

Alternatively, $\tilde{\mathbf{T}}_\ell \sim \tilde{\mathbf{T}}_\ell(W, W', \nu, \mu_E, \mu_V)$ can directly be constructed from independent trees $\mathbf{T}_\ell \sim \mathbf{T}_\ell(W, \nu, \mu_E, \mu_V)$ and $\mathbf{T}'_{\ell-1} \sim \mathbf{T}_{\ell-1}(W', \nu, \mu_E, \mu_V)$ with roots $\varnothing$ and $\varnothing'$, respectively, by grafting $\mathbf{T}'_{\ell-1}$ onto $\mathbf{T}_\ell$ via an edge between $\varnothing'$ and $\varnothing$ of weight $w \sim \mu_E$ (independent of everything else). Whenever $\tilde{\mathbf{T}}_\ell$ is defined via this procedure, we say it is constructed from $(\mathbf{T}_\ell, \mathbf{T}'_{\ell-1}, \varnothing, \varnothing', w)$.

As alluded to above, this object can be thought of as the limiting object of the neighbourhood of v if we condition on the presence of an edge between v and u.

We will not state and discuss the coupling results in the weighted setting here, because they are structurally similar to Proposition 2.2.3. The intuition should be that in a first step the underlying graph structure is coupled as in the unweighted case and then edge and vertex weights are added. Since the weight distributions converge in total variation distance, the weights can be coupled so that they are equal with high probability and because the number of vertices and edges in the neighbourhood can be estimated, the probability that the coupled weights are different can be controlled. We refer the reader to Section 3.5 for more details.

Perturbation Recall the representation of $\mathbf{G}_n$ as a sequence of Bernoulli random variables and weights $(\mathbf{X}, \mathbf{w})$. Let $\mathbf{X}'$ be an independent copy of $\mathbf{X}$ and likewise $\mathbf{w}'$ be an independent copy of $\mathbf{w}$.

Let F be a subset of $V_n \cup V_n^{(2)}$, i.e. sets of vertices and edges alike.

Let $\mathbf{G}_n^F$ be the weighted graph obtained from $\mathbf{G}_n$ by replacing

- X_e with X'_e whenever $e \in F$ and
- w_z with w'_z whenever $z \in F$.

For singleton sets F we often omit the curly brackets and simply write $\mathbf{G}_n^e$ for $\mathbf{G}_n^{\{e\}}$ and $\mathbf{G}_n^v$ for $\mathbf{G}_n^{\{v\}}$. We abuse notation even further to write $\mathbf{G}_n^{F\cup e}$ for $\mathbf{G}_n^{F\cup\{e\}}$ and $\mathbf{G}_n^{F\cup v}$ for $\mathbf{G}_n^{F\cup\{v\}}$.

For brevity we write

$$X^F_{\{u,v\}} = \begin{cases} X'_{\{u,v\}} & \text{if } \{u,v\} \in F, \\ X_{\{u,v\}} & \text{if } \{u,v\} \notin F, \end{cases} \quad \text{and} \quad w_z^A = \begin{cases} w'_z & \text{if } z \in F, \\ w_z & \text{if } z \notin F, \end{cases}$$

whenever $u, v \in V_n$ and $z \in V_n \cup V_n^{(2)}$. With this notation $\mathbf{G}_n^F$ is the weighted graph based on the sequences $(X_e^F)_{e\in V_n^{(2)}}$ and $(w_z^F)_{z\in V_n\cup V_n^{(2)}}$ instead of $(X_e)_{e\in V_n^{(2)}}$ and $(w_z)_{z\in V_n\cup V_n^{(2)}}$. Note that by construction $X_e^{\varnothing} = X_e$ for $e \in V_n^{(2)}$ and $w_z^{\varnothing} = w_z$ for $z \in V_n \cup V_n^{(2)}$. It follows that $\mathbf{G}_n^{\varnothing} = \mathbf{G}_n$.

2.3 Statement of the Central Limit Theorem

The proof of our central limit theorem relies on the analysis of the effect of a small perturbation to the weighted graph $\mathbf{G}_n$ on a function f. We introduce some notation to refer to the effect of this perturbation.

Definition 2.3.1. Let $\mathbf{G}_n$ be a weighted graph and let f be a function on weighted graphs. Recall the definition of the perturbed graph $\mathbf{G}_n^e$ and $\mathbf{G}_n^v$ for an edge $e \in V_n^{(2)}$ and a vertex $v \in V_n$, respectively. Then define

$$\Delta_e f = f(\mathbf{G}_n) - f(\mathbf{G}_n^e)$$

and

$$\Delta_v f = f(\mathbf{G}_n) - f(\mathbf{G}_n^v).$$

The main assumption of the theorem is that it is possible to approximate the effect of resampling perturbations on the function f by considering local neighbourhoods around the perturbed site, i.e. that we can find a *good local approximation* for the effects of the perturbation on f.

Assumption 2.3.2 (Property GLA)**.** Let f be a function on weighted graphs and let $(\mathbf{G}_n)_{n\in\mathbb{N}}$ be a sequence of weighted inhomogeneous random graphs. Then the pair $(f, (\mathbf{G}_n)_{n\in\mathbb{N}})$ has *property GLA* for ν, μ_E and μ_V if

(i) the underlying unweighted graph sequence $(G_n)_{n\in\mathbb{N}}$ satisfies Assumption 2.2.1,

(ii) the weight distributions satisfy $d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) \to 0$ and $d_{\mathrm{TV}}(\mu_{V,n}, \mu_V) \to 0$ as $n \to \infty$ and

(iii) the effects of perturbations of $\mathbf{G}_n$ on the function f can be approximated locally in the following sense.

For all $k \in \mathbb{N}$ there exist functions $\mathrm{LA}_k^{E,L}$, $\mathrm{LA}_k^{E,U}$, $\mathrm{LA}_k^{V,L}$ and $\mathrm{LA}_k^{V,U}$ from pairs of rooted weighted trees to the real numbers and furthermore there exist two sequences $(m_n^E)_{n\in\mathbb{N}}$ and $(m_n^V)_{n\in\mathbb{N}}$ of functions $m_n^E\colon V_n^2 \to \mathbb{R}$ and $m_n^V\colon V_n \to \mathbb{R}$ such that

$$(M_n^E)_{n\in\mathbb{N}} = \Big(n^{-2} \sum_{v,u\in V_n} m_n^E(v,u)\Big)_{n\in\mathbb{N}}$$

and

$$(M_n^V)_{n\in\mathbb{N}} = \Big(n^{-1} \sum_{v\in V_n} m_n^V(v)\Big)_{n\in\mathbb{N}}$$

are bounded in probability and two sequences $(\delta_k^E)_{k\in\mathbb{N}}$, $(\delta_k^V)_{k\in\mathbb{N}}$ with $\delta_k^E \to 0$ and $\delta_k^V \to 0$ as $k \to \infty$ such that the following conditions hold for any $k \in \mathbb{N}$.

(GLA 1) For any edge $e = \{u,v\} \in V_n^{(2)}$, if $B_k = B_k(v,\mathbf{G}_n)$ and $B_k^e = B_k(v,\mathbf{G}_n^e)$ are trees, then

$$\mathrm{LA}_k^{E,L}(B_k,B_k^e) \le \Delta_e f \le \mathrm{LA}_k^{E,U}(B_k,B_k^e).$$

(GLA 2) For any edge $e = \{u,v\} \in V_n^{(2)}$, if a pair of trees $(\mathbf{T},\mathbf{T}')$ satisfies $(B_k,B_k^e) = (B_k(v,\mathbf{G}_n),B_k(v,\mathbf{G}_n^e)) \cong (\mathbf{T},\mathbf{T}')$, then

$$\begin{aligned}\mathrm{LA}_k^{E,L}(\mathbf{T},\mathbf{T}') &= \mathrm{LA}_k^{E,L}(B_k,B_k^e),\\ \mathrm{LA}_k^{E,U}(\mathbf{T},\mathbf{T}') &= \mathrm{LA}_k^{E,U}(B_k,B_k^e).\end{aligned}$$

(GLA 3) For any two vertices $v,u \in V_n$ let $\tilde{\mathbf{T}}_k(v,u) \sim \tilde{\mathbf{T}}_k(W_v,W_u,\nu,\mu_E,\mu_V)$ be constructed from $(\mathbf{T}_k(v),\mathbf{T}_{k-1}(u),\varnothing,\varnothing',w)$ (cf. Definition 2.2.6). Then

$$\begin{aligned}\max\{&\mathbb{E}_n[(\mathrm{LA}_k^{E,U}(\tilde{\mathbf{T}}_k(v,u),\mathbf{T}_k(v)) - \mathrm{LA}_k^{E,L}(\tilde{\mathbf{T}}_k(v,u),\mathbf{T}_k(v)))^2],\\ &\quad\mathbb{E}_n[(\mathrm{LA}_k^{E,U}(\mathbf{T}_k(v),\tilde{\mathbf{T}}_k(v,u)) - \mathrm{LA}_k^{E,L}(\mathbf{T}_k(v),\tilde{\mathbf{T}}_k(v,u)))^2]\}\\ &\le m_n^E(v,u)\delta_k^E.\end{aligned}$$

(GLA 4) For any vertex $v \in V_n$, if $B_k = B_k(v,\mathbf{G}_n)$ and $B_k^v = B_k(v,\mathbf{G}_n^v)$ are trees, then

$$\mathrm{LA}_k^{V,L}(B_k,B_k^v) \le \Delta_v f \le \mathrm{LA}_k^{V,U}(B_k,B_k^v).$$

(GLA 5) For any vertex $v \in V_n$, if $(\mathbf{T},\mathbf{T}')$ are a pair of trees that satisfy $(B_k,B_k^v) = (B_k(v,\mathbf{G}_n),B_k(v,\mathbf{G}_n^v)) \cong (\mathbf{T},\mathbf{T}')$, then

$$\begin{aligned}\mathrm{LA}_k^{V,L}(\mathbf{T},\mathbf{T}') &= \mathrm{LA}_k^{V,L}(B_k,B_k^v),\\ \mathrm{LA}_k^{V,U}(\mathbf{T},\mathbf{T}') &= \mathrm{LA}_k^{V,U}(B_k,B_k^v).\end{aligned}$$

(GLA 6) For any vertex $v \in V_n$ let $\bar{\mathbf{T}}_k(v)$ be the weighted tree obtained by resampling the weight of the root of $\mathbf{T}_k(v) \sim \mathbf{T}_k(W_v, \nu, \mu_E, \mu_V)$. Then

$$\mathbb{E}_n[(\mathrm{LA}_k^{V,U}(\mathbf{T}_k(v), \bar{\mathbf{T}}_k(v)) - \mathrm{LA}_k^{V,L}(\mathbf{T}_k(v), \bar{\mathbf{T}}_k(v)))^2] \le m_n^V(v)\delta_k^V.$$

As discussed in the previous sections Assumption 2.2.1 and convergence of the weight distributions guarantee that the local neighbourhoods in $\mathbf{G}_n$ can be coupled to limiting Galton–Watson trees.

The three assumptions (GLA 4), (GLA 5) (GLA 6) for resampling at a vertex (which involves only resampling the weight at the vertex) are structurally analogous to (GLA 1), (GLA 2) and (GLA 3) for resampling at an edge (which involves resampling the edge indicator and its weight). It would have been possible to collect the conditions for edge and vertex resampling in a combined condition (even though a combination of (GLA 6) and (GLA 3) would be even more complex), but since it is more intuitive to think about the effect of resampling separately, we decided to present the conditions in this way. In the discussion of the interpretation of the conditions we will focus mainly on the first three properties, since the interpretation of the other three is analogous.

(GLA 1) implies that the effect of the perturbation of $\mathbf{G}_n$ on f can be approximated by the local quantity $\mathrm{LA}_k^{E,L}$ that only takes into account a k-neighbourhood of the perturbed site. The error of this approximation is bounded by $\mathrm{LA}_k^{E,U} - \mathrm{LA}_k^{V,L}$. (GLA 2) implies that the values of $\mathrm{LA}_k^{E,L}$ and $\mathrm{LA}_k^{E,U}$ only depend on properties that are preserved under graph isomorphisms, which means that we can substitute the limiting Galton–Watson trees for the local neighbourhoods in order to analyse the approximation error. (GLA 3) ensures that the approximation error goes to zero as the level k of the considered neighbourhood increases.

Note that property GLA does not need the function f to be local in the sense that $\Delta_e f$ and $\Delta_v f$ only depend on a fixed neighbourhood $B_k(v, \mathbf{G}_n)$. All that is required is that there be local approximations and that these approximations improve as k gets large. We will see the difference in Section 2.4, where we present a local function to get started and then a function for which $\Delta_e f$ can only be approximated locally.

(GLA 3) and (GLA 6) might look a bit daunting at first. Our proof relies on coupling the neighbourhood of fixed vertices, which as we remarked when we discussed the limiting object and the coupling Definition 2.2.2 and Proposition 2.2.3 makes for a slightly more complex situation at the root. The conditions state that the effect of the root can be separated from the approximation error that is due to the remaining tree structure. Essentially we can think of the averaging we apply as choosing the root uniformly, which transfers our setup to the *unimodular* setting. The boundedness assumptions then guarantee that even in this setting the approximation error goes to zero. We will show that the effect of the root can be separated out in a concrete example in Section 2.4.

In applications the function f will often be related to a combinatorial optimisation problem that has certain recursive properties so that the local approximation functions $\mathrm{LA}_k^{*,*}$ can be defined via a recursion on the graph by essentially cutting off everything that is not in the local neighbourhood of level k, imposing an arbitrary starting value for those vertices and then passing it down recursively towards the root vertex of the neighbourhood. Depending on the properties of the recursion in question natural lower and upper bounds may be found by selecting certain extremal values for the vertices that are cut or by exploiting that the recursion values oscillate for even and odd levels.

If applied to a Galton–Watson tree the recursive nature of f gives rise to a *recursive tree process* [AB05]. Briefly, a recursive tree process (RTP) is a Galton–Watson tree in which each individual $\mathbf{i}$ has an associated value $X_{\mathbf{i}}$ that is calculated by applying a function g to all the values $X_{\mathbf{i}1}, \dots, X_{\mathbf{i}N_{\mathbf{i}}}$ associated with the $N_{\mathbf{i}}$ children $\mathbf{i}1, \dots, \mathbf{i}N_{\mathbf{i}}$ of $\mathbf{i}$ and an independent noise $\xi_{\mathbf{i}}$ at $\mathbf{i}$. An RTP is called *endogenous* if the value at the root $X_\varnothing$ is measurable with respect to the σ-algebra generated by the noise $\xi_{\mathbf{i}}$ and number of children $N_{\mathbf{i}}$ for each individual in the tree. If the local approximations are defined via the recursion associated with f, (GLA 3) is closely related to the question of endogeny of the recursive tree process.

Properties related to (GLA 3) have also been called *long-range independence* by Gamarnik, Nowicki and Swirszcz [GNS06] and *replica symmetry* by Wästlund [Wäs12] and have been used to calculate limiting constants for the behaviour of some combinatorial optimisation problems.

In our statement of the main theorem we will encounter the following two sequences.

Definition 2.3.3. Let $(k_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ be any sequence.

Recall the definitions of α_n from Assumption 2.2.1 and the definition of $\Gamma_{p,n}$ and $\kappa_{p,n}$ from (2.3) and (2.4). For $n, \ell \in \mathbb{N}$ let

$$\begin{aligned}
\varepsilon_{n,\ell} = {}& \frac{\Gamma_{2,n}^2}{n} + \vartheta\kappa_{1,n}\Gamma_{1,n} \\
&+ \Gamma_{1,n}\vartheta(\Gamma_{2,n}+1)^{\ell}\left(\frac{\Gamma_{3,n}}{n\vartheta} + \kappa_{1,n} + \kappa_{2,n} + \frac{2+\Gamma_{1,n}}{k_n} + \frac{k_n}{n\vartheta}\right) \\
&+ \frac{1}{k_n} + \frac{k_n^2}{n\vartheta\Gamma_{1,n}} + \alpha_n(\Gamma_{1,n} + (\Gamma_2+1)^{\ell-1}(\Gamma_{2,n} + \vartheta\Gamma_{1,n})) \\
&+ (1 + \Gamma_{1,n}\vartheta(\Gamma_2+1)^{\ell})(d_{\mathrm{TV}}(\mu_{E,n},\mu_E) + d_{\mathrm{TV}}(\mu_{V,n},\mu_V))
\end{aligned}$$

and

$$\rho_{n,\ell} = \min\left\{\frac{\vartheta\Gamma_{2,n} + \vartheta\Gamma_{1,n} + 1}{n\vartheta}(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+C)^{2\ell+1}(\Gamma_{3,n}+1)^2, 1\right\}.$$

The sequence $\varepsilon_{n,\ell}$ arises from the coupling probability (cf. Proposition 2.2.3). The sequence $\rho_{n,\ell}$ absorbs the correlation between neighbourhoods of a fixed collection of vertices and bounds the probability of certain other undesirable events in our proofs.

As in the discussion of the convergence rate of Proposition 2.2.3, the terms $\Gamma_{1,n}$, $\Gamma_{2,n}$ and $\Gamma_{3,n}$ are bounded in probability and α_n, $\kappa_{1,n}$ and $\kappa_{2,n}$ converge to zero in probability. Hence both $\varepsilon_{n,\ell}$ and $\rho_{n,\ell}$ converge to 0 in probability as $n \to \infty$ for all $\ell \in \mathbb{N}$ if k_n is chosen appropriately, e.g. $k_n = n^{1/3}$.

In addition to the local approximation in property GLA we will assume a much simpler bound for the effect of the local perturbation.

Assumption 2.3.4. Assume that there are real-valued functions $H_E\colon [0,\infty)^4 \to \mathbb{R}$ and $H_V\colon [0,\infty)^2 \to \mathbb{R}$ that satisfy

$$J_E = \max\Big\{1, \sup_{n\in\mathbb{N}} \mathbb{E}[H_E(w_e, w_e', w_v, w_u)^6]\Big\} < \infty, \tag{2.6}$$

and

$$J_V = \max\Big\{1, \sup_{n\in\mathbb{N}} \mathbb{E}[H_V(w_v, w_v')^6]\Big\} < \infty. \tag{2.7}$$

Let $J = J_E + J_V + J_E J_V$.

Assume further that there exists a non-decreasing function $h\colon [0,\infty) \to \mathbb{R}$ such that for

$$\chi_n = \frac{1}{n}\sum_{v\in V_n} \zeta_n(v) \quad \text{with} \quad \zeta_n(v) = \mathbb{E}_n[h(|D_1(v)| + 4)^4] < \infty \tag{2.8}$$

we have that χ_n is bounded in probability.

Finally, assume that

$$|\Delta_e f| \le \mathbb{1}_{\{\max\{X_e, X_e'\}=1\}} H_E(w_e, w_e', w_v, w_u) \tag{2.9}$$

and

$$|\Delta_v f| \le h(|D_1(v)|) H_V(w_v, w_v'). \tag{2.10}$$

These bounds will allow us to make generous use of the Cauchy–Schwarz inequality in proofs especially when we are not on the event where the coupling holds.

Note that the dependence on n is only implicit in the terms inside the expectation in (2.6) and (2.7) because we have – as usual – dropped the superscript (n) for the weights w_e and w_v.

Property GLA and the simpler integrability bounds of Assumption 2.3.4 now finally yield explicit bounds for the Kolmogorov distance of $f(\mathbf{G}_n)$ to a normal distribution.

Theorem 2.3.5. *Suppose $(f, (\mathbf{G}_n)_{n\in\mathbb{N}})$ satisfies property GLA (Assumption 2.3.2) for ν, μ_E and μ_V. Assume that Assumption 2.3.4 holds with J and χ_n as defined there. Let $\sigma_n^2 = \mathrm{Var}_n(f(\mathbf{G}_n))$, set*

$$Z_n = \frac{f(\mathbf{G}_n) - \mathbb{E}_n[f(\mathbf{G}_n)]}{\sigma_n}$$

and let Φ *be the cumulative distribution function of the standard normal distribution. Then we have for all* $k, n \in \mathbb{N}$

$$\begin{aligned}
\sup_{t\in\mathbb{R}} &|\mathbb{P}_n(Z_n \le t) - \Phi(t)| \\
&\le C_0 J^{1/4}\Big[\Big(\frac{n}{\sigma_n^2}\Big)^{1/2}(\vartheta^{1/2} + \Gamma_{2,n} + \chi_n^{1/2})^2((M_n^E\delta_k)^{1/8} + (M_n^V\delta_k^V)^{1/8} + \varepsilon_{n,k}^{1/16} + \rho_{n,k}^{1/16}) \\
&\qquad + \Big(\frac{n}{\sigma_n^2}\Big)^{3/4}\frac{\vartheta\Gamma_{1,n} + \chi_n^{1/2}}{n^{1/4}}\Big].
\end{aligned} \tag{2.11}$$

Under the assumptions of the theorem $\Gamma_{1,n}$, $\Gamma_{2,n}$, χ_n, M_n^E and M_n^V are bounded in probability. Furthermore $\varepsilon_{n,k}$ and $\rho_{n,k}$ converge to zero in probability as $n \to \infty$ for all $k \in \mathbb{N}$.

If $n\sigma_n^{-2}$ is bounded in probability we can make all terms on the right hand-side of (2.11) arbitrarily small as follows. First choose k large enough so that the terms involving the bounded terms (whose bound is independent of n) and δ_k^E and δ_k^V are as small as desired. Then choose n large enough that for this k the terms $\varepsilon_{n,k}$ and $\rho_{n,k}$ are as small as desired.

This behaviour of the variance is in general not a given and will need to be verified separately in applications.

The following corollary replaces property GLA with a slightly simpler condition that is particularly suitable if f has a recursive structure.

Corollary 2.3.6. *Let* $(\mathbf{G}_n)_{n\in\mathbb{N}}$ *be a sequence of weighted inhomogeneous graphs and let* f *be a function defined on weighted graphs. Suppose that*

(i) *the underlying graph sequence* $(G_n)_{n\in\mathbb{N}}$ *satisfies Assumption 2.2.1,*

(ii) *there are two probability measures* μ_E *and* μ_V *on* $(0,\infty)$ *with* $d_{\mathrm{TV}}(\mu_{E,n},\mu_E) \to 0$ *and* $d_{\mathrm{TV}}(\mu_{V,n},\mu_V) \to 0$ *as* $n \to \infty$*,*

(iii) *Assumption 2.3.4 holds with* J *and* χ_n *as defined there and*

(iv) *the effects of perturbations of* $\mathbf{G}_n$ *on* f *can be approximated locally in the following sense.*

There exist functions g_k^L *and* g_k^U *defined on weighted rooted graphs for any* $k \in \mathbb{N}$ *and there exist two sequences* $(m_n)_{n\in\mathbb{N}}$ *and* $(\tilde{m}_n)_{n\in\mathbb{N}}$ *of functions* $m_n\colon V_n \to \mathbb{R}$ *and* $\tilde{m}_n\colon V_n^2 \to \mathbb{R}$ *such that*

$$(M_n)_{n\in\mathbb{N}} = \Big(n^{-1}\sum_{v\in V_n} m_n(v)\Big)_{n\in\mathbb{N}}$$

and

$$(\tilde{M}_n)_{n\in\mathbb{N}} = \Big(n^{-2}\sum_{v,u\in V_n} \tilde{m}_n(v,u)\Big)_{n\in\mathbb{N}}$$

are bounded in probability and furthermore two sequences $(\delta_k^E)_{k\in\mathbb{N}}$, $(\delta_k^V)_{k\in\mathbb{N}}$ *with* $\delta_k^E \to 0$ *and* $\delta_k^V \to 0$ *as* $k \to \infty$ *such that for all* $k \in \mathbb{N}$ *the following conditions are satisfied.*

(GLA′ 1) *For any* $v \in V_n$, *whenever* $B_k(v, \mathbf{G}_n)$ *is a tree, then*

$$g_k^L(B_k(v,\mathbf{G}_n)) \le f(\mathbf{G}_n) - f(\mathbf{G}_n - v) \le g_k^U(B_k(v,\mathbf{G}_n)).$$

(GLA′ 2) *For any* $v \in V_n$ *if* $B_k(v,\mathbf{G}_n) \cong \mathbf{T}$ *for some rooted weighted tree* $\mathbf{T}$, *then*

$$g_k^L(\mathbf{T}) = g_k^L(B_k(v,\mathbf{G}_n)) \quad \text{and} \quad g_k^U(\mathbf{T}) = g_k^U(B_k(v,\mathbf{G}_n)).$$

(GLA′ 3) *For any* $v, u \in V_n$ *if we have* $\mathbf{T}_k(v) \sim \mathbf{T}_k(W_v, \nu, \lambda, \mu_E, \mu_V)$ *and* $\tilde{\mathbf{T}}_k(v,u) \sim \tilde{\mathbf{T}}(W_v, W_u, \nu, \mu_E, \mu_V)$, *then*

$$\mathbb{E}_n[(g_k^U(\mathbf{T}_k(v)) - g_k^L(\mathbf{T}_k(v)))^2] \le m(v)\delta_k$$

and

$$\mathbb{E}_n[(g_k^U(\tilde{\mathbf{T}}_k(v,u)) - g_k^L(\tilde{\mathbf{T}}_k(v,u)))^2] \le \tilde{m}(v,u)\tilde{\delta}_k.$$

Let $\sigma_n^2 = \mathrm{Var}_n(f(\mathbf{G}_n))$, *set*

$$Z_n = \frac{f(\mathbf{G}_n) - \mathbb{E}_n[f(\mathbf{G}_n)]}{\sigma_n}$$

and let Φ *be the cumulative distribution function of the standard normal distribution. Then we have for all* $k, n \in \mathbb{N}$

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} |\mathbb{P}_n(Z_n \le t) - \Phi(t)| \\
&\quad \le C_0 J^{1/4} \Bigg[\left(\frac{n}{\sigma_n^2}\right)^{1/2} (\vartheta^{1/2} + \Gamma_{2,n} + \chi_n^{1/2})^2 ((M_n\delta_k)^{1/8} + (\tilde{M}_n\tilde{\delta}_k)^{1/8} + \varepsilon_{n,k}^{1/16} + \rho_{n,k}^{1/16}) \\
&\qquad + \left(\frac{n}{\sigma_n^2}\right)^{3/4} \frac{\vartheta\Gamma_{1,n} + \chi_n^{1/2}}{n^{1/4}} \Bigg].
\end{aligned}$$

The three conditions (GLA′ 1) to (GLA′ 3) together imply property GLA, so we may informally refer to them as *property GLA′*. Again, the intuition is that (GLA′ 1) can be used to approximate the effect of the perturbation on f locally with g_k^L with an approximation error at most $g_k^U - g_k^L$. Then (GLA′ 2) allows us to estimate this approximation error on the limiting Galton–Watson tree, where (GLA′ 3) ensures that the approximation error goes to 0 as $k \to \infty$.

We will prove Theorem 2.3.5 and Corollary 2.3.6 in Chapter 4.

These results extend the central limit theorem shown by Cao [Cao21]. We were able to include weights on the vertices and could prove the result in the more general setting of rank-one inhomogeneous random graphs. As far as we are aware

Cao's result is the only general central limit theorem for combinatorial optimisation problems in a sparse Erdős–Rényi random graph setting. In this setting, central limit theorems are known for certain graph statistics like subgraph counts [Ruc88] or the size of the giant component [BR12]. For the maximal matching (without weights) a central limit theorem can be shown in certain regimes [Kre17; Pit90]. Barbour and Röllin [BR19] recently proved a general central limit theorem for *local* graph statistics in the configuration model. The configuration model generates a random graph with a given degree sequence. In fact, conditional on its degrees the inhomogeneous graph model we considered here (and more general inhomogeneous graph models) have the same distribution as a configuration model conditioned on producing no loops or multiple edges [Hof18, Thm. 7.18].

The first-order behaviour of a number of combinatorial optimisation problems, on the contrary, has been studied extensively in the sparse Erdős–Rényi graph setting [e.g. BGT81; GNS06; KS81]. Some of the methods that were used to obtain limiting constants in this setting can in fact be used to verify property GLA, so that a first-order result together with our central limit theorem immediately also proves the second-order behaviour. Results for more general sparse graphs do not appear to be as abundant [BLS13].

2.4 Applications

In this section we will briefly present two applications of the central limit theorem. The first is a slightly contrived example in which we assign artificial edge weights to edges based on the weight at their end vertices. Contrary to the setup in which we assign edge weights as usual in our weighted graph model, this results in edge weights that are not independent, so that a standard central limit is not immediately applicable. The second example will be that of maximum weight matching.

2.4.1 A simple example

In order to whet our appetite here is a simple application of Theorem 2.3.5. Let $\mathbf{G}_n$ be a sequence of weighted inhomogeneous random graphs satisfying the assumptions of Theorem 2.3.5. We will assume that the connectivity weights are such that $\mathbb{E}_n[(W^{(n)})^4]$ converges in probability to a constant and that ν has fourth moments. In this example we do not place independent weights on the edges with distribution μ_E, which we ignore from now on. Instead we will use the weights we put on the vertices with $\mu_{V,n} = \mu_V$ to induce artificial weights on the edges by adding up the weights of their endpoints. We will assume that μ_V has at least sixth moments.

With the usual notation of $\mathbf{X}$ and $\mathbf{w}$ we are interested in the quantity

$$N(\mathbf{G}_n) = \sum_{u,v \in V_n} (w_v + w_u) X_{vu} = \sum_{e=\{u,v\} \in V_n^{(2)}} (w_v + w_u) X_e,$$

i.e. in twice the total sum of these artificial edge weights. Since the same w_v will appear for different edges $e \in V_n^{(2)}$, this is not a sum of independent random variables.

Observe that $N(\mathbf{G}_n)$ can be rewritten as

$$N(\mathbf{G}_n) = \sum_{v \in V_n} |D_1(v)| w_v,$$

but again that this is *not* a sum of independent random variables, since $|D_1(v)|$ is not independent for different v (take the simple example in which we consider a graph with just two vertices, if the degree of one of the vertices is 1, we know that the degree of the other must also be 1).

Since we could not easily rewrite $N(\mathbf{G}_n)$ as a sum of independent random variables, we cannot easily apply one of the standard central limit theorems. Hence, we will appeal to our central limit theorem Theorem 2.3.5.

We will first identify suitable local approximations for property GLA. Since the problem is 'truly local' in the sense that the effect of a local change can be fully estimated with local information, this is straightforward.

We observe that for any edge $e = \{u, v\} \in V_n^{(2)}$

$$\Delta_e N = N(\mathbf{G}_n) - N(\mathbf{G}_n^e) = (w_v + w_u)(X_e - X_e') \tag{2.12}$$

and any vertex $v \in V_n$

$$\Delta_v N = N(\mathbf{G}_n) - N(\mathbf{G}_n^v) = |D_1(v)|(w_v - w_v'). \tag{2.13}$$

To shorten notation from now on write $B_k = B_k(v, \mathbf{G}_n)$, $B_k^e = B_k(v, \mathbf{G}_n^e)$ and $B_k^v = B_k(v, \mathbf{G}_n^v)$. For $k \geq 1$ we can let

$$\mathrm{LA}^{E,L}(B_k, B_k^e) = \mathrm{LA}^{E,U}(B_k, B_k^e) = (w_v + w_u)(X_e - X_e')$$

and

$$\mathrm{LA}^{V,L}(B_k, B_k^v) = \mathrm{LA}^{V,U}(B_k, B_k^v) = |D_1(v)|(w_v - w_v').$$

Then

$$\mathrm{LA}^{E,L}(B_k, B_k^e) = \Delta_e N = \mathrm{LA}^{E,U}(B_k, B_k^e)$$

and

$$\mathrm{LA}^{V,L}(B_k, B_k^v) = \Delta_v N = \mathrm{LA}^{V,U}(B_k, B_k^v),$$

which immediately verifies (GLA 1) and (GLA 4). The construction of these functions relies only on properties that are preserved under isomorphisms for weighted graphs, so (GLA 2) and (GLA 5) are also satisfied. Since the upper and lower bound coincide, (GLA 3) and (GLA 6) are trivially satisfied. Hence, property GLA (Assumption 2.3.2) holds in our problem.

We turn to the simpler integrability bounds in Assumption 2.3.4. From (2.12) and (2.13) we obtain

$$|\Delta_e N| \leq \mathbb{1}_{\{\max\{X_e, X_e'\}=1\}}(w_v + w_u)$$

and

$$|\Delta_v N| \leq |D_1(v)|(w_v + w_v').$$

The sixth moments of the weights are bounded by assumption. Hence we can choose

$$H_E(w_v, w_u) = w_v + w_u \quad \text{and} \quad H_V(w_v, w_v') = w_v + w_v'$$

to satisfy (2.6) and (2.7) of Assumption 2.3.4. For (2.8) set $h(x) = x$ so that we need a bound on the fourth moment of $|D_1(v)|$, which can be found in Lemma 3.1.11. We then have that

$$\zeta_n(v) = \mathbb{E}_n[(|D_1(v)| + 4)^4] \leq C(W_v + 1)^4(\Gamma_{2,n} + 1)^4 < \infty$$

and that

$$\chi_n = \frac{1}{n}\sum_{v\in V_n}\zeta_n(v) \leq \frac{1}{n}\sum_{v\in V} C(W_v + 1)^4(\Gamma_{2,n} + 1)^4 = C(\Gamma_{2,n} + 1)^4\mathbb{E}_n[(W^{(n)} + 1)^4]$$

is bounded in probability because we assumed the existence of fourth moments for $W^{(n)}$.

With all assumptions verified we can now apply Theorem 2.3.5. As discussed in the remarks after Theorem 2.3.5 the bound for the Kolmogorov distance of the distribution of $\sigma_n^{-1}(N(\mathbf{G}_n) - \mathbb{E}_n[N(\mathbf{G}_n)])$ to a standard normal distribution goes to zero in probability if $n\sigma_n^{-2}$ is bounded in probability. Hence, in order to conclude convergence to a standard normal, we have to verify that the variance of $N(\mathbf{G}_n)$ is of sufficiently high order. A straightforward but tedious calculation, which we will not show here, verifies that indeed $\mathrm{Var}_n(N(\mathbf{G}_n))$ is of order n so that $n\sigma_n^{-2}$ is bounded in probability. This then allows us to conclude the desired convergence. The convergence rate depends on the rate of convergence of ν_n to ν and other properties of ν_n and ν.

2.4.2 Maximum weight matching

We will now apply Corollary 2.3.6 to the maximum weight matching problem on an inhomogeneous random graph satisfying Assumption 2.2.1 for some measure ν and with $\mathrm{Exp}(1)$ edge weights and no vertex weights. We will assume that ν has third moments, so that $\hat{\nu}$ has second moments.

Furthermore, we will assume that the following technical condition holds. Define an operator T on the space of probability distributions on $\mathbb{R}$ by letting $T(\mu)$ be the distribution of

$$\max_{i\in[N]}\{0, \xi_i - X_i\},$$

where $N \sim \hat{\nu}$, $\xi_1, \ldots, \xi_N \overset{\text{i.i.d.}}{\sim} \operatorname{Exp}(1)$ and $X_i, \ldots, X_N \overset{\text{i.i.d.}}{\sim} \mu$ are independent. We assume that the iterated operator T^2 (defined as $T^2(\mu) = T(T(\mu))$) has a unique fixed point. This condition is in essence what Gamarnik, Nowicki and Swirszcz [GNS06] call 'long-range independence' and verify for the case $\nu = \delta_c$ for some $c > 0$ [GNS06, Thm. 3]. We believe that this condition can be verified for a more general class of measures ν.

Definition 2.4.1 (Maximum weight matching)**.** Let $\mathbf{G}$ be an edge-weighted graph with vertex set V and edge set E.

A matching on $\mathbf{G}$ is a subset of edges $M \subseteq E$ in which no two edges have a vertex in common. In other words, for all $v \in V$ there is at most one $u \in V$ such that $\{v, u\} \in E$ (we call u the vertex matched to v).

The maximum weight matching is a matching M that maximises the sum of edge weights $\sum_{e \in M} w_e$.

Note that a maximum weight matching need not match every vertex to another vertex.

We follow the strategy used by Cao [Cao21] to tackle this problem. We want to apply Corollary 2.3.6 and so need to identify suitable functions g_k^L and g_k^U.

As alluded to before, recursive properties are usually a very good starting point to verify property GLA or its slightly simplified cousin in Corollary 2.3.6. Let $M(\mathbf{G})$ be the weight of the maximum weight matching on a graph $\mathbf{G}$ with vertex set V and edge set E. Then for any vertex $v \in V$ the weight of the maximum weight matching $M(\mathbf{G})$ satisfies the recursion

$$M(\mathbf{G}) = \max\Big\{M(\mathbf{G} - v), \max_{u:\{v,u\} \in E} \{w_{\{v,u\}} + M(\mathbf{G} - \{v, u\})\}\Big\}.$$

Essentially this formula says that we need to decide between not matching v to any partner vertex, so that the matching is in effect a matching on $\mathbf{G} - v$, or matching v to any of its neighbours u, upon which the weight of the matching increases by the weight of the edge between v and u and the remainder of the matching happens on the graph $\mathbf{G} - \{v, u\}$.

Now define

$$h(\mathbf{G}, v) = M(\mathbf{G}) - M(\mathbf{G} - v)$$

and note that

$$h(\mathbf{G}, v) = \max\Big\{0, \max_{u:\{v,u\} \in E} \{w_{\{v,u\}} - h(\mathbf{G} - v, u)\}\Big\}.$$

Intuitively, this expression quantifies how much better it is to match v to one of its neighbours rather than to leave it unmatched. If $h(\mathbf{G}, v) = 0$, then the weight of the maximum weight matching on $\mathbf{G}$ and $\mathbf{G} - v$ are the same, which means that v can remain unmatched in $\mathbf{G}$ and we still attain the maximum possible weight. If $h(\mathbf{G}, v) > 0$, then matching v to one of its neighbours means that the matching performs better than the matching that does not match v. Hence, v should be matched in the maximum weight matching on v.

Let $\mathbf{T}$ be a weighted tree of height at most k, let $\varnothing$ be the root of $\mathbf{T}$ and let $C(u)$ be the set of children of the vertex u in $\mathbf{T}$. Denote the edge weights of $\mathbf{T}$ by w_e.

Define $h_k(\,\cdot\,,\mathbf{T})\colon \mathbf{T} \to \mathbb{R}$ by setting $h_k(u,\mathbf{T}) = 0$ for all leaves u of $\mathbf{T}$ and by the recursion

$$h_k(u,\mathbf{T}) = \max\Big\{0, \max_{u' \in C(u)} \{w_{\{u,u'\}} - h_k(u',\mathbf{T})\}\Big\}$$

for all non-leaf vertices u of $\mathbf{T}$. Note that by this recursion the value $h_k(u,\mathbf{T})$ only ever depends on the subtree in $\mathbf{T}$ that is induced by the descendants of u.

A short induction argument shows that under the assumption that $B_k(v,\mathbf{G}_n)$ is a tree

$$h_k(v, B_k(v,\mathbf{G}_n)) \le h(\mathbf{G}_n, v) \quad \text{if } k \text{ is even}$$

and

$$h(\mathbf{G}_n, v) \le h_k(v, B_k(v,\mathbf{G}_n)) \quad \text{if } k \text{ is odd.}$$

Thus, if for $k \in \mathbb{N}$ we have that $B_{2k+1}(v,\mathbf{G}_n)$ is a tree (which naturally implies that $B_{2k}(v,\mathbf{G}_n)$ is a tree), we also have

$$h_{2k}(v, B_{2k}(v,\mathbf{G}_n)) \le h(\mathbf{G}_n, v) \le h_{2k+1}(v, B_{2k+1}(v,\mathbf{G}_n)).$$

This suggests the following definition for g_k^L and g_k^U: Let k_U be the largest odd number less than or equal to k and $k_L = k_U - 1$. Then $k_L \le k$ and $k_U \le k$ so that if $B_k(v,\mathbf{G}_n)$ is a tree, we can set

$$g_k^L(B_k(v,\mathbf{G}_n)) = h_{k_L}(v, B_{k_L}(v,\mathbf{G}_n))$$

and

$$g_k^U(B_k(v,\mathbf{G}_n)) = h_{k_U}(v, B_{k_U}(v,\mathbf{G}_n)).$$

Property GLA′ Immediately this construction ensures

$$g_k^L(B_k(v,\mathbf{G}_n)) \le h(\mathbf{G}_n, v) \le g_k^U(B_k(v,\mathbf{G}_n)),$$

which verifies (GLA′ 1) in Corollary 2.3.6.

The construction of g_k^L and g_k^U also ensures (GLA′ 2) of Corollary 2.3.6 because the definition relies only on structure that is preserved under isomorphisms on weighted rooted graphs, namely edges and weights.

The next step is to verify (GLA′ 3). We start with the first part Let $\mathbf{T}(v) \sim \mathbf{T}(W_v, \nu, \mathrm{Exp}(1), \,\cdot\,)$ and $\mathbf{T}_k(v)$ be its level-k subtree. Recall that $\mathbf{T}(v)$ is a delayed weighted Galton–Watson tree that can be constructed by joining together $N \sim \mathrm{Poi}(W_v)$ independent weighted Galton–Watson trees $\mathbf{T}^{(i)}$ for $i \in \{1,\dots,N\}$ with offspring distribution $\mathrm{MPoi}(\hat{\nu})$ via edges $\{\varnothing, \varnothing_i\}$ with independent edge weights according to Exp(1). The depth-k subtree $\mathbf{T}_k$ of $\mathbf{T}$ is then made up of the N subtrees $\mathbf{T}^{(i)}_{k-1}$ of $\mathbf{T}^{(i)}$ joined at the root $\varnothing$. By the construction of g^L_k and g^U_k we need to analyse

$$\mathbb{E}_n[(h_{2k+1}(\varnothing, \mathbf{T}_{2k+1}(v)) - h_{2k}(\varnothing, \mathbf{T}_{2k}(v)))^2]$$

in order to verify condition (GLA′ 3) of Corollary 2.3.6.

We evaluate one recursion step to find

$$\begin{aligned}
&|h_{2k+1}(\varnothing, \mathbf{T}_{2k+1}(v)) - h_{2k}(\varnothing, \mathbf{T}_{2k}(v))|^2 \\
&= \Big|\max\Big\{0, \max_{1\le i\le N}\{w_{\{\varnothing,\varnothing_i\}} - h_{2k+1}(\varnothing_i, \mathbf{T}_{2k+1}(v))\}\Big\} \\
&\qquad - \max\Big\{0, \max_{1\le i\le N}\{w_{\{\varnothing,\varnothing_i\}} - h_{2k}(\varnothing_i, \mathbf{T}_{2k}(v))\}\Big\}\Big|^2.
\end{aligned}$$

Recall that $h_k(u, \mathbf{T})$ only depends on the subtree of $\mathbf{T}$ induced by the descendants of u. If $h_k(\,\cdot\,, \mathbf{T}_k(v))$ assigns value 0 to the leaves of $\mathbf{T}_k(v)$, then $h_{k-1}(\,\cdot\,, \mathbf{T}^{(i)}_{k-1})$ does the same for the subtree induced by the descendants of $\varnothing_i$. Hence, the previous expression is equal to

$$\begin{aligned}
&= \Big|\max\Big\{0, \max_{1\le i\le N}\{w_{\{\varnothing,\varnothing_i\}} - h_{2k}(\varnothing_i, \mathbf{T}^{(i)}_{2k})\}\Big\} \\
&\qquad - \max\Big\{0, \max_{1\le i\le N}\{w_{\{\varnothing,\varnothing_i\}} - h_{2k-1}(\varnothing_i, \mathbf{T}^{(i)}_{2k-1})\}\Big\}\Big|^2.
\end{aligned}$$

Use that $|\max x_i - \max y_i| \le \max|x_i - y_i|$ to find that this difference is bounded by

$$\begin{aligned}
&\le \max_{1\le i\le N}|h_{2k}(\varnothing_i, \mathbf{T}^{(i)}_{2k}) - h_{2k-1}(\varnothing_i, \mathbf{T}^{(i)}_{2k-1})|^2 \\
&\le \sum_{i=1}^{N}|h_{2k}(\varnothing_i, \mathbf{T}^{(i)}_{2k}) - h_{2k-1}(\varnothing_i, \mathbf{T}^{(i)}_{2k-1})|^2.
\end{aligned}$$

Since all $\mathbf{T}^{(i)}$ are independent Galton–Watson trees with the same offspring distribution that are independent of $N \sim \mathrm{Poi}(W_v)$, we have

$$\begin{aligned}
&\mathbb{E}_n[(h_{2k+1}(\varnothing, \mathbf{T}_{2k+1}) - h_{2k}(\varnothing, \mathbf{T}_{2k}))^2] \\
&\quad\le \mathbb{E}_n\Big[\sum_{i=1}^{N}|h_{2k}(\varnothing_i, \mathbf{T}^{(i)}_{2k}) - h_{2k-1}(\varnothing_i, \mathbf{T}^{(i)}_{2k-1})|^2\Big] \\
&\quad\le \mathbb{E}_n[N]\mathbb{E}_n[|h_{2k}(\varnothing_1, \mathbf{T}^{(1)}_{2k}) - h_{2k-1}(\varnothing_1, \mathbf{T}^{(1)}_{2k-1})|^2] \\
&\quad\le W_v\,\mathbb{E}[|h_{2k}(\varnothing_1, \mathbf{T}^{(1)}_{2k}) - h_{2k-1}(\varnothing_1, \mathbf{T}^{(1)}_{2k-1})|^2].
\end{aligned} \tag{2.14}$$

We dropped the conditioning on $\mathcal{F}_n$ in the last expectation, because the random variables inside the expectation do not depend on $\mathcal{F}_n$ in any way.

Shorten $h_k(\varnothing_1, \mathbf{T}^{(1)}_k)$ to $h_k(\varnothing_1)$. With this notation it is enough to verify that

$$\delta_k = \mathbb{E}[(h_{2k}(\varnothing_1) - h_{2k-1}(\varnothing_1))^2] \to 0 \quad \text{as } k \to \infty, \tag{2.15}$$

because then together with $m_n(v) = W_v$, for which we have that

$$M_n = n^{-1} \sum_{v \in V_n} W_v = \vartheta \Gamma_1$$

is bounded in probability, and (2.14) we would have

$$\mathbb{E}_n[(h_{2k+1}(\varnothing, \mathbf{T}_{2k+1}(v)) - h_{2k}(\varnothing, \mathbf{T}_{2k}(v)))^2] \le m_n(v)\delta_k \tag{2.16}$$

as required for the first part of (GLA′ 3).

In order to verify (2.15), note first that as before a short induction argument shows that we have $h_{2k}(\varnothing_1) \le h_{2k-1}(\varnothing_1)$ for all $k \in \mathbb{N}_+$. Furthermore, we also have that $h_{2k-1}(\varnothing_1)$ is non-increasing in k and that $h_{2k}(\varnothing_1)$ is non-decreasing in k. Set

$$h^L = \lim_{k\to\infty} h_{2k}(\varnothing_1) \quad \text{and} \quad h^U = \lim_{k\to\infty} h_{2k-1}(\varnothing_1).$$

Then by the monotonicity of the sequences

$$h_{2k-1}(\varnothing_1) - h_{2k}(\varnothing_1) \searrow h^U - h^L$$

and

$$0 \le h_{2k-1}(\varnothing_1) - h_{2k}(\varnothing_1) \le h_{2k-1}(\varnothing_1) \le h_1(\varnothing_1) \le \max_{u \in C(\varnothing_1)} w_{\{\varnothing_1, u\}}.$$

Since the right-hand side has finite second moment (because we assumed that $\hat{\nu}$ has finite second moment and the w_e are exponentially distributed), we can apply Lebesgue's dominated convergence theorem and obtain the desired convergence for (2.15) if $h^U - h^L = 0$ almost surely. By definition we have $h^L \le h^U$, so the almost sure equality can be concluded from equality of the expectations.

Hence, (2.15) and with it the first part of (GLA′ 3) follow from the the following claim.

Claim 2.4.2. *We have* $\mathbb{E}[h^L] = \mathbb{E}[h^U]$.

Proof. Under the technical condition that the distributional operator T^2 has a unique fixed point, the arguments used by Gamarnik, Nowicki and Swirszcz [GNS06] to prove their Proposition 1 and Theorem 3 also apply in our setting, which implies that $h_k(\varnothing)$ converges in distribution to some limit H_∞. But this implies that h^L and h^U have the same distribution, namely H_∞. Then $\mathbb{E}[h^L] = \mathbb{E}[h^U]$ as claimed. □

This shows the first part of condition (GLA′ 3) in Corollary 2.3.6. For the second part we need to consider

$$\mathbb{E}_n[(h_{2k+1}(\varnothing, \tilde{\mathbf{T}}_{2k+1}(v,u)) - h_{2k}(\varnothing, \tilde{\mathbf{T}}_{2k}(v,u)))^2].$$

As explained in Definition 2.2.6 we may assume that the weighted tree $\tilde{\mathbf{T}}_k(v,u) \sim \tilde{\mathbf{T}}(W_v, W_u, \nu, \mu_E, \mu_V)$ is constructed from $(\mathbf{T}_k(v), \mathbf{T}_{k-1}(u), \varnothing, \varnothing', w)$. Again the idea is to unwrap what the recursion implies for the different subtrees. We only need to focus on the 'artifical' edge $\{\varnothing, \varnothing'\}$ of weight w between $\mathbf{T}_k(v)$ and $\mathbf{T}_{k-1}(u)$. We have

$$h_{2k+1}(\varnothing, \tilde{\mathbf{T}}_{2k+1}(v,u)) = \max\{h_{2k+1}(\varnothing, \mathbf{T}_{2k+1}(v)), w - h_{2k}(\varnothing', \mathbf{T}_{2k}(u))\}$$

and

$$h_{2k}(\varnothing, \tilde{\mathbf{T}}_{2k}(v,u)) = \max\{h_{2k}(\varnothing, \mathbf{T}_{2k+1}(v)), w - h_{2k-1}(\varnothing', \mathbf{T}_{2k-1}(u))\}.$$

Let

$$Y_k = h_{2k+1}(\varnothing, \mathbf{T}_{2k+1}(v)) - h_{2k}(\varnothing, \mathbf{T}_{2k+1}(v))$$

and

$$Y_k' = h_{2k-1}(\varnothing', \mathbf{T}_{2k-1}(u)) - h_{2k}(\varnothing', \mathbf{T}_{2k}(u)).$$

Then

$$|h_{2k+1}(\varnothing, \tilde{\mathbf{T}}_{2k+1}(v,u)) - h_{2k}(\varnothing, \tilde{\mathbf{T}}_{2k}(v,u))|^2 \le \max\{|Y_k|^2, |Y_k'|^2\}.$$

When we verified the first part of (GLA′ 3), we already showed that $\mathbb{E}_n[|Y_k|^2]$ satisfies (2.16), i.e.

$$\mathbb{E}_n[|Y_k|] \le W_v \delta_k.$$

The exact same reasoning can be used to show that

$$\mathbb{E}_n[|Y_k'|] \le W_u \delta_k.$$

Together this shows

$$\mathbb{E}_n[|h_{2k+1}(\varnothing, \tilde{\mathbf{T}}_{2k+1}(v,u)) - h_{2k}(\varnothing, \tilde{\mathbf{T}}_{2k}(v,u))|^2] \le (W_v + W_u)\delta_k,$$

so that the second part of (GLA′ 3) is satisfied with $\tilde{m}_n(v,u) = W_v + W_u$ for which $\tilde{M}_n = n^{-2} \sum_{v,u \in V_n} W_u + W_v = 2\vartheta \Gamma_{1,n}$ is bounded in probability.

This verifies the simplified version of property GLA from Corollary 2.3.6. Hence, we can apply Corollary 2.3.6 once we have verified Assumption 2.3.4.

Bounds for Assumption 2.3.4 Because there are no vertex weights, we only need to consider (2.6) and (2.9). Indeed, for (2.9) we only need to find a bound of the form

$$|M(\mathbf{G}_n) - M(\mathbf{G}_n^e)| \le \mathbb{1}_{\{\max\{X_e, X_e'\}=1\}} H_E(w_e, w_{e'}).$$

We identify a suitable bound by considering the cases separately. First we consider the case that perturbing the edge removes it from the graph, i.e. $X_e = 1$, but $X_e' = 0$. If e with weight w_e is part of the maximum weight matching, removing it from the graph (and therefore from the matching) can cost the maximum weight matching at most w_e, because the removal of e allows other previously blocked edges to participate again. If e is not part of the matching, removing it does not change the weight of the maximum weight matching at all. In case the perturbation adds a previously nonexistent edge, i.e. $X_e = 0$, $X'_e = 1$, the weight of the maximum weight matching can increase at most by the weight w'_e of this edge. If the edge is present in both the unperturbed and perturbed graph and only changes its weight, the weight of the maximum weight matching changes at most by the difference of the old and new weight. In any case the difference is bounded by the maximum of the old weight w_e and the new weight w'_e. In other words we can choose $H_E(w_e, w'_e) = \max\{w_e, w'_e\}$.

Since $w_e, w'_e \sim \mathrm{Exp}(1)$, we immediately have $\mathbb{E}[H_E(w_e, w'_e)^6]$, which implies (2.6), since our weight distribution is the same for all n.

Variance bound As in the previous example, an application of Corollary 2.3.6 now provides an estimate for the Kolmogorov distance of the distribution of $\sigma_n^{-1}(N(\mathbf{G}_n) - \mathbb{E}_n[N(\mathbf{G}_n)])$ to a standard normal distribution. In order to obtain a convergence we have to verify that the variance is at least of order n. This again a technical calculations. For the case $\nu = \delta_c$ we refer to the calculations done by Cao [Cao21, Lem. 3.2], whose approach carries over to $\nu \neq \delta_c$ under our technical condition.

2.5 Outlook

The main contribution of this thesis was to show that the framework used by Cao [Cao21] to establish the central limit theorem for the (homogeneous) edge-weighted Erdős–Rényi model can be extended to more inhomogeneous graph models and to models with weights on edges and vertices.

The setting we investigate in this thesis still exhibits a fair amount of uniformity in the limit. Yet still, the methods used in the proof for the Erdős–Rényi model had to be adapted not inconsiderably to apply to this case as well. It would be interesting to investigate which level of inhomogeneity – either in the graphs or the limiting objects – these methods can still support and at which point other methods need to be considered.

The inhomogeneous random graph models we investigate here do not exhibit a spatial structure, but many interesting real-world networks have inherent spatial and geometric properties that influence the graph structure. The local limiting behaviour of spatial inhomogeneous random graphs is known [HHM22] and may differ significantly from the local behaviour of sparse inhomogeneous random graphs. It is therefore doubtful that the sparsity/tree-based approach pursued here is directly suitable to these graphs. Nevertheless, the methods used here may be applicable to a subclass of spatial random graphs whose local properties are sufficiently similar to the sparse inhomogeneous graphs we considered. Moreover, the general approach of the perturbative Stein's method and local approximation has been used successfully in a spatial setting [CS17].

A final direction for future research would be to consider dynamic versions of the underlying random graph [see e.g. Man+19; ZMN17]. If it is possible to show a central limit theorem for each time-point of the evolving graph, one might hope for a functional central limit theorem for the entire process.

Chapter 3

Local Structure of Sparse Inhomogeneous Random Graphs

One of the fundamental results in the analysis of the behaviour of Erdős–Rényi random graphs is that their connected components can be described with branching processes [see, e.g. Hof18, Chap. 4]. Indeed, the entire local neighbourhood structure of a sparse Erdős–Rényi random graph can be related to a Galton–Watson tree with Poisson offspring distribution. In the realm of inhomogeneous random graphs similar branching process results have been known since the introduction of the model by Söderberg [Söd02] and extensive study by Bollobás, Janson and Riordan [BJR07]. Specifically, the local structure of a wide class of sparse inhomogeneous random graphs can be related to a class of multi-type branching processes [BJR07; Hof23, Chap. 3]. In our setting the limiting object will essentially turn out to be a single-type branching process with Poisson offspring distribution.

In this chapter we will analyse the local structure of sparse inhomogeneous random graphs of the form introduced in Section 2.2. We will start by showing relatively simple results about the sizes of the neighbourhoods of a fixed vertex and about the probability that a fixed vertex or edge is part of the neighbourhood of another vertex in Section 3.1. In a second step we will find explicit bounds for the correlation between neighbourhoods of different vertices in Section 3.2.

We will then use the graph exploration procedure introduced in Section 3.3 to explicitly couple neighbourhoods in the inhomogeneous graph to Galton–Watson trees in Section 3.4. Finally, in Section 3.5 we present several coupling results that are at first glance more complex, but follow directly from the coupling established in Section 3.4.

3.1 Neighbourhood size and path probabilities

In this section we will briefly establish a few results about the size of the neighbourhood of a vertex and for path probabilities. Since the strategy of coupling the cluster (i.e the neighbourhood of a vertex) to a branching process is well established [BJR07], these results are by no means surprising and are in principle known in a much more general setting. We were not able to locate all of the precise results we need in the literature, though, so we state all of them here in a consistent notation.

The estimates in this section only depend on the underlying graph structure and not on the additional vertex and edge weights in the weighted graph $\mathbf{G}_n$. Hence, all results in this section will be shown for $G_n = (V_n, E_n)$. Still, all results presented here will still hold if G_n is replaced with $\mathbf{G}_n$. Again, $V_n^{(2)} = \{\{u,v\} : u, v \in V_n\}$ is the set of possible edges.

Recall the definition of the local neighbourhood $B_\ell(v, G_n)$ of a vertex v in G_n up to level $\ell \in \mathbb{N}$ (cf. Definition 2.1.7). For the rest of this section we will drop the reference to G_n and will just write $B_\ell(v)$ for $B_\ell(v, G_n)$.

In a slight abuse of notation we will write both $u \in B_\ell(v)$ for a vertex $u \in V_n$ to mean that u is contained in the vertex set of $B_\ell(v)$, which means that there must be a path from v to u of length no more than ℓ, and $e \in B_\ell(v)$ to mean that the possible edge $e \in V_n^{(2)}$ is contained in the edge set of $B_\ell(v)$, which means that e is part of a path of length at most ℓ from v to an arbitrary vertex.

It will be useful to have an estimate of the expected number of vertices in a neighbourhood as well as of their 'total connectivity weight'. Amongst other things these quantities can be used to estimate the correlation between neighbourhoods of different vertices.

Definition 3.1.1 (Total p-connectivity weight)**.** Fix $p \geq 0$. For any set of vertices $\mathcal{U} \subseteq V_n$ let

$$\|\mathcal{U}\|_p = \sum_{u \in \mathcal{U}} W_u^p$$

denote the total sum of the p-th power of the connectivity weights of the vertices in $\mathcal{U}$. We also say that $\|\mathcal{U}\|_p$ is the *total p-connectivity weight of $\mathcal{U}$*.

We write $\|\mathcal{U}\| = \|\mathcal{U}\|_1$. Note also that the cardinality of a set can be written as its total 0-weight, that is to say $|\mathcal{U}| = \|\mathcal{U}\|_0$.

In a first step we estimate the expected number of vertices in $B_\ell(v)$.

Lemma 3.1.2. *For any level $\ell \in \mathbb{N}$ and vertex $v \in V_n$ we have*

$$\mathbb{E}_n[|S_\ell(v)|] \leq 1 + W_v \Gamma_{1,n}(\Gamma_{2,n} + 1)^{\ell-1}.$$

Proof. By construction $S_\ell(v) = \bigcup_{r=0}^{\ell} D_r(v)$ is a disjoint union so that

$$|S_\ell(v)| = \sum_{r=0}^{\ell} |D_r(v)|.$$

The number of vertices at level exactly $r \geq 1$ can be estimated by the number of vertices to which there exists a path of length r from v. In particular for $r \geq 1$ we have

$$\mathbb{E}_n[|D_r(v)|] \leq \sum_{\substack{u_1,\ldots,u_r \in V_n \setminus \{v\} \\ \text{pairwise different}}} \mathbb{E}_n[X_{vu_1} X_{u_1u_2} \cdots X_{u_{r-1}u_r}].$$

All edges are different and therefore independent, so that the edge probabilities factor

$$
\begin{aligned}
&\le \sum_{u_1 \in V_n} \frac{W_v W_{u_1}}{n\vartheta} \sum_{u_2 \in V_n} \frac{W_{u_1} W_{u_2}}{n\vartheta} \cdots \sum_{u_r \in V_n} \frac{W_{u_{r-1}} W_{u_r}}{n\vartheta} \\
&\le W_v \frac{1}{n\vartheta} \sum_{u_1 \in V_n} W_{u_1}^2 \frac{1}{n\vartheta} \sum_{u_2 \in V_n} W_{u_2}^2 \cdots \frac{1}{n\vartheta} \sum_{u_{r-1} \in V_n} W_{u_{r-1}}^2 \frac{1}{n\vartheta} \sum_{u_r \in V_n} W_{u_r} \\
&\le W_v \Big(\frac{1}{n\vartheta} \sum_{u \in V_n} W_u^2 \Big)^{r-1} \frac{1}{n\vartheta} \sum_{u \in V_n} W_u \\
&= W_v \Big(\frac{\mathbb{E}_n[(W^{(n)})^2]}{\vartheta} \Big)^{r-1} \frac{\mathbb{E}_n[W^{(n)}]}{\vartheta} \\
&= W_v \Gamma_{1,n} \Gamma_{2,n}^{r-1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}_n[|S_\ell(v)|] &= 1 + \sum_{r=1}^{\ell} |D_r(v)| \\
&\le 1 + W_v \Gamma_{1,n} \sum_{r=1}^{\ell} \Gamma_{2,n}^{r-1} \\
&\le 1 + W_v \Gamma_{1,n} (\Gamma_{2,n} + 1)^{\ell - 1}
\end{aligned}
$$

as claimed. □

Corollary 3.1.3. *For any $\mathcal{V} \subseteq V_n$ and $\ell \in \mathbb{N}$ we have*

$$\mathbb{E}_n[|S_\ell(\mathcal{V})|] \le |\mathcal{V}| + \|\mathcal{V}\| \Gamma_{1,n} (\Gamma_{2,n} + 1)^{\ell - 1}.$$

Proof. This follows directly from Lemma 3.1.2, because $|S_\ell(\mathcal{V})| \le \sum_{v \in \mathcal{V}} |S_\ell(v)|$. □

It is also instructive to calculate the expectation of the ‘total p-connectivity weight’ of the explored graph up to level ℓ. The proof replicates the ideas of Lemma 3.1.2, which is not surprising given that the following lemma actually implies Lemma 3.1.2 (by taking $p = 0$).

Lemma 3.1.4. *Let $p \ge 0$. For any level $\ell \in \mathbb{N}$ and vertex $v \in V_n$ we have*

$$\mathbb{E}_n[\|S_\ell(v)\|_p] \le W_v^p + W_v (\Gamma_{2,n} + 1)^{\ell - 1} \Gamma_{p+1,n}.$$

In case $p = 1$ this bound can be tweaked slightly to become

$$\mathbb{E}_n[\|S_\ell(v)\|] \le W_v (\Gamma_{2,n} + 1)^{\ell}.$$

Proof. By construction and the fact that the $D_r(v)$ are disjoint

$$\|S_\ell(v)\|_p = \sum_{r=0}^{\ell} \|D_r(v)\|_p.$$

Recall that $D_0(v) = \{v\}$ so that $\|D_0(v)\|_p = W_v^p$.
Analogous to the estimate for $|D_r(v)|$ we have for $r \geq 1$ that

$$\begin{aligned}
&\mathbb{E}_n[\|D_r(v)\|_p] && (3.1)\\
&\quad\leq \sum_{\substack{u_1,\dots,u_r \in V_n \setminus \{v\} \\ \text{pairwise different}}} \mathbb{E}_n[X_{vu_1} X_{u_1u_2} \cdots X_{u_{r-1}u_r} W_{u_r}^p] \\
&\quad\leq \sum_{u_1 \in V_n} \frac{W_v W_{u_1}}{n\vartheta} \sum_{u_2 \in V_n} \frac{W_{u_1} W_{u_2}}{n\vartheta} \cdots \sum_{u_r \in V_n} \frac{W_{u_{r-1}} W_{u_r}}{n\vartheta} W_{u_r}^p \\
&\quad\leq W_v \frac{1}{n\vartheta} \sum_{u_1 \in V_n} W_{u_1}^2 \frac{1}{n\vartheta} \sum_{u_2 \in V_n} W_{u_2}^2 \cdots \frac{1}{n\vartheta} \sum_{u_{r-1} \in V_n} W_{u_{r-1}}^2 \frac{1}{n\vartheta} \sum_{u_r \in V_n} W_{u_r}^{p+1} \\
&\quad\leq W_v \Big(\frac{1}{n\vartheta} \sum_{u \in V_n} W_u^2\Big)^{r-1} \Big(\frac{1}{n\vartheta} \sum_{u \in V_n} W_u^{p+1}\Big) \\
&\quad= W_v \Gamma_{2,n}^{r-1} \Gamma_{p+1,n}. && (3.2)
\end{aligned}$$

Now sum over r to obtain

$$\begin{aligned}
\mathbb{E}_n[\|S_\ell(v)\|_p] &= \mathbb{E}_n[\|D_0(v)\|_p] + \sum_{r=1}^{\ell} \mathbb{E}_n[\|D_r(v)\|_p] \\
&\leq W_v^p + W_v \Gamma_{p+1,n} \sum_{r=1}^{\ell} \Gamma_{2,n}^{r-1} \\
&\leq W_v^p + W_v \Gamma_{p+1,n} (\Gamma_{2,n} + 1)^{\ell-1}. \qquad (3.3)
\end{aligned}$$

This proves the first part of the claim.
For the second claim note that in case $p = 1$ the bound in (3.2) becomes

$$\mathbb{E}_n[\|D_r(v)\|] \leq W_v \Gamma_{2,n}^r.$$

But this bound also holds for $r = 0$, so that the summation in (3.3) becomes

$$\mathbb{E}_n[\|S_\ell(v)\|] \leq W_v \sum_{r=0}^{\ell} \Gamma_{2,n}^r \leq W_v (\Gamma_{2,n} + 1)^\ell,$$

which proves the second part of the claim. □

Again, summing the result gives:

Corollary 3.1.5. *Let $p \geq 0$. For any level $\ell \in \mathbb{N}$ and vertex set $\mathcal{V} \subseteq V_n$ we have*

$$\mathbb{E}_n[\|S_\ell(\mathcal{V})\|_p] \leq \|\mathcal{V}\|_p + \|\mathcal{V}\|(\Gamma_{2,n}+1)^{\ell-1}\Gamma_{p+1,n}$$

and thus also

$$\mathbb{E}_n[\|S_\ell(\mathcal{V})\|] \leq \|\mathcal{V}\|(\Gamma_{2,n}+1)^{\ell}.$$

We will need a similar result for the excess connectivity weight of vertices

Definition 3.1.6 (Excess connectivity weight)**.** Fix $n \in \mathbb{N}$. Let $\mathcal{U} \subseteq V_n$ a set of vertices in G_n. Then we say that

$$\|\mathcal{U}\|_+ = \sum_{u\in\mathcal{U}} W_u \mathbb{1}_{\{W_u > \sqrt{n\vartheta}\}}$$

is the *total sum of excess connectivity weights of $\mathcal{U}$*.

Replicating the exact same arguments as in Lemma 3.1.2 we can show a bound for $\|S_\ell(v)\|_+$.

Lemma 3.1.7. *For any level $\ell \in \mathbb{N}$ and vertex $v \in V_n$ we have*

$$\mathbb{E}_n[\|S_\ell(v)\|_+] \leq W_v \mathbb{1}_{\{W_v > \sqrt{n\vartheta}\}} + W_v(\Gamma_{2,n}+1)^{\ell-1}\kappa_{2,n}.$$

Proof. We follow the steps in the proof of Lemma 3.1.2.
By construction and the fact that the $D_r(v)$ are disjoint we have

$$\|S_\ell(v)\|_+ = \sum_{r=0}^{\ell} \|D_r(v)\|_+.$$

Recall that $D_0(v) = \{v\}$ so that $\|D_0(v)\|_+ = W_v \mathbb{1}_{\{W_v > \sqrt{n\vartheta}\}}$.
For $r \geq 1$ we have

$$\begin{aligned}
\mathbb{E}_n[\|D_r(v)\|_+] &\leq \sum_{\substack{u_1,\dots,u_r \in V_n\setminus\{v\} \\ \text{pairwise different}}} \mathbb{E}_n[X_{vu_1}X_{u_1u_2}\cdots X_{u_{r-1}u_r} W_{u_r}\mathbb{1}_{\{W_{u_r} > \sqrt{n\vartheta}\}}] \\
&\leq W_v \Big(\frac{1}{n\vartheta}\sum_{u\in V_n} W_u^2\Big)^{r-1}\Big(\frac{1}{n\vartheta}\sum_{u\in V_n} W_u^2 \mathbb{1}_{\{W_{u_r} > \sqrt{n\vartheta}\}}\Big) \\
&= W_v \Gamma_{2,n}^{r-1}\kappa_{2,n}.
\end{aligned}$$

Now sum over r as in the proof of Lemma 3.1.2 to prove the claim. □

For estimates involving the Cauchy–Schwarz inequality it will also be useful to have a bound on the second moment of $\|S_\ell(v)\|$.

Lemma 3.1.8. *For any level $\ell \in \mathbb{N}$ and vertex $v \in V_n$ we have*

$$\begin{aligned}&\mathbb{E}_n[\|S_\ell(v)\|_p^2]\\&\quad\le W_v^{2p} + 2W_v^{p+1}\Gamma_{p+1,n}(\Gamma_{2,n}+1)^{\ell-1}\\&\qquad + C(W_v+1)^2(\Gamma_{2,n}+2)^{2\ell-2}(\Gamma_{3,n}+1)(\Gamma_{p+1,n}+1)^2(\Gamma_{p+2,n}+1)(\Gamma_{2p+1,n}+1).\end{aligned}$$

For $p = 0$ this estimate can be slightly simplified further to

$$\mathbb{E}_n[|S_\ell(v)|^2] = \mathbb{E}_n[\|S_\ell(v)\|_0^2] \le C(W_v+1)^2(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+2)^{2\ell}(\Gamma_{3,n}+1)$$

and for $p = 1$ to

$$\mathbb{E}_n[\|S_\ell(v)\|^2] \le C(W_v+1)^2(\Gamma_{2,n}+2)^{2\ell}(\Gamma_{3,n}+1).$$

Proof. Recall that $D_0(v) = \{v\}$ so that $\|D_0(v)\|_p = W_v^p$. Then

$$\begin{aligned}&\|S_\ell(v)\|_p^2\\&\quad=\Big(\sum_{r=0}^{\ell}\|D_r(v)\|_p\Big)^2\\&\quad=\sum_{r=0}^{\ell}\|D_r(v)\|_p^2+\sum_{\substack{r,s=0\\r\ne s}}^{\ell}\|D_r(v)\|_p\|D_s(v)\|_p\\&\quad=W_v^{2p}+\sum_{r=1}^{\ell}\|D_r(v)\|_p^2+2W_v^p\sum_{r=1}^{\ell}\|D_r(v)\|_p+\sum_{\substack{r,s=1\\r\ne s}}^{\ell}\|D_r(v)\|_p\|D_s(v)\|_p\end{aligned}\tag{3.4}$$

The bound

$$\sum_{r=1}^{\ell}\mathbb{E}_n[\|D_r(v)\|_p] \le W_v\Gamma_{p+1,n}(\Gamma_{2,n}+1)^{\ell-1}\tag{3.5}$$

was already established in (3.3) for the proof of Lemma 3.1.4. It thus remains to bound $\sum_{r=1}^{\ell}\mathbb{E}_n[\|D_r(v)\|_p^2]$ and $\sum_{\substack{r,s=1\\r\ne s}}^{\ell}\mathbb{E}_n[\|D_r(v)\|_p\|D_s(v)\|_p]$.

For $r \in \mathbb{N}_+$ we can write

$$\|D_r(v)\|_p = \sum_u \mathbb{1}_{\{v\to_r u\}}W_u^p,$$

where $v \to_r u$ means that there is a path from v to u of length exactly r (that does not visit the same vertex twice) and that there exists no such path of length $s < r$. Then

$$\|D_r(v)\|_p^2 = \sum_u \mathbb{1}_{\{v\to_r u\}}W_u^{2p} + \sum_{u\ne u'}\mathbb{1}_{\{v\to_r u\}}\mathbb{1}_{\{v\to_r u'\}}W_u^pW_{u'}^p\tag{3.6}$$

and

$$\|D_r(v)\|_p\|D_s(v)\|_p = \sum_u \mathbb{1}_{\{v\to_r u\}}\mathbb{1}_{\{v\to_s u\}}W_u^{2p} + \sum_{u\ne u'}\mathbb{1}_{\{v\to_r u\}}\mathbb{1}_{\{v\to_s u'\}}W_u^pW_{u'}^p.\tag{3.7}$$

The first sum in (3.6) involves only paths to a single vertex and its expectation is easily estimated as follows

$$\begin{aligned}\mathbb{E}_n\Big[\sum_u \mathbb{1}_{\{v\to_r u\}} W_u^{2p}\Big] &\leq \sum_{\substack{u_1,u_2,\dots,u_r\\ \text{pairw. diff.}}} \frac{W_v W_{u_1}}{n\vartheta}\frac{W_{u_1}W_{u_2}}{n\vartheta}\cdots\frac{W_{u_{r-1}}W_{u_r}}{n\vartheta}W_{u_r}^{2p}\\ &\leq W_v\Big(\frac{\mathbb{E}_n[(W^{(n)})^2]}{\vartheta}\Big)^{r-1}\frac{\mathbb{E}_n[(W^{(n)})^{2p+1}]}{\vartheta}\\ &\leq W_v\Gamma_{2,n}^{r-1}\Gamma_{2p+1,n}. \end{aligned} \tag{3.8}$$

The first sum in (3.7) is zero for $r \neq s$, since the shortest path from v to u cannot have length both r and s. Hence,

$$\mathbb{E}_n\Big[\sum_u \mathbb{1}_{\{v\to_r u\}}\mathbb{1}_{\{v\to_s u\}} W_u^{2p}\Big] = 0 \tag{3.9}$$

The paths to u and u' in the second sums in (3.6) and (3.7) are not necessarily independent, so a priori their expectation does not factorise. A closer look at the indicators allows for a 'restricted factorisation'.

In particular we claim that if there is a path π of length r from v to u and a path π' of length s from v to u' (and no shorter path for either end-vertex), this implies there is a vertex z in the same position in π and π' such that π_{zu} and $\pi'_{zu'}$, the segments of π from z to u and π' and from z to u', respectively, do not share a vertex apart from z. That is to say, the paths π and π' agree up to a certain vertex and bifurcate afterwards.

If π and π' do not share any vertex apart from v, the claim is trivially true with $z = v$.

So suppose there is a vertex $z \neq v$ that is shared between π and π'. (If there are several such vertices, pick the one that appears last in π and π'.) Then z must appear in the same position in π and π'. If this were not the case, the path with the longer segment from v to z could be shortened by using the shorter segment from v to z from the other path. But this would be a contradiction to the minimality of the paths. By choice of z the path segments π_{zu} and $\pi'_{zu'}$ only share the vertex z.

It is easy to see that the path segments π_{zu} and $\pi'_{zu'}$ only have the vertex z in common with π_{vz}, the path segment of π connecting v to z. If the paths shared a vertex apart from z we could find a shorter path between v and u or u' by leaving out all vertices between the shared vertex contradicting the minimality assumption.

Hence, the event that u and u' (with $u \neq u'$) can be reached from v in r and s steps, respectively, can be estimated by counting these 'eventually bifurcating paths' of the form just described. We distinguish the two cases $r \neq s$ and $r = s$ (without loss of generality we may assume $r < s$ in the former case), because they differ in one detail.

In case $r < s$, the paths either bifurcate immediately at v or they split at a later step $t \in \{1, \dots, r\}$. Note that if the paths split at r, the path from v to u' includes the complete path from v to u. If we write $u_r = u$ and $u'_s = u'$ we thus have

$$\begin{aligned}\mathbb{1}_{\{v \to_r u\}} \mathbb{1}_{\{v \to_r u'\}} \leq & \sum_{\substack{u_1,\dots,u_{r-1},\\ u'_1,\dots,u'_{s-1}\\ \text{pairw. diff.}}} X_{vu_1} \cdots X_{u_{r-1}u_r} X_{vu'_1} \cdots X_{u'_{s-1}u'_s} \\ & + \sum_{t=1}^{r} \sum_{\substack{u_1,u_2,\dots,u_t,\\ u_{t+1},\dots,u_{r-1},\\ u'_{t+1},\dots,u'_{s-1}\\ \text{pairw. diff.}}} X_{vu_1} X_{u_1u_2} \cdots X_{u_{t-1}u_t} \\ & \qquad X_{u_tu_{t+1}} X_{u_{t+1}u_{t+2}} \cdots X_{u_{r-2}u_{r-1}} X_{u_{r-1}u_r} \\ & \qquad X_{u_tu'_{t+1}} X_{u'_{t+1}u'_{t+2}} \cdots X_{u'_{s-2}u'_{s-1}} X_{u'_{s-1}u'_s}.\end{aligned} \tag{3.10}$$

In case $r = s$, the paths similarly either bifurcate immediately at v or they split at a later step $t \in \{1, \dots, r-1\}$. Unlike in the previous case $r < s$, the bifurcation must happen before step r in the path, because otherwise the paths would be the same and would thus have the same endpoints, but we assumed that $u \neq u'$. Again we write $u_r = u$ and $u'_r = u'$ to obtain

$$\begin{aligned}\mathbb{1}_{\{v \to_r u\}} \mathbb{1}_{\{v \to_r u'\}} \leq & \sum_{\substack{u_1,\dots,u_{r-1},\\ u'_1,\dots,u'_{r-1}\\ \text{pairw. diff.}}} X_{vu_1} \cdots X_{u_{r-1}u_r} X_{vu'_1} \cdots X_{u'_{r-1}u'_r} \\ & + \sum_{t=1}^{r-1} \sum_{\substack{u_1,u_2,\dots,u_t,\\ u_{t+1},\dots,u_{r-1},\\ u'_{t+1},\dots,u'_{r-1}\\ \text{pairw. diff.}}} X_{vu_1} X_{u_1u_2} \cdots X_{u_{t-1}u_t} \\ & \qquad X_{u_tu_{t+1}} X_{u_{t+1}u_{t+2}} \cdots X_{u_{r-2}u_{r-1}} X_{u_{r-1}u_r} \\ & \qquad X_{u_tu'_{t+1}} X_{u'_{t+1}u'_{t+2}} \cdots X_{u'_{r-2}u'_{r-1}} X_{u'_{r-1}u'_r}.\end{aligned} \tag{3.11}$$

For each summand in (3.10) and (3.11) all involved edges are independent given $\mathcal{F}_n$ since apart from v or u_t no vertex appears multiple times.

For (3.10) this immediately implies

$$
\begin{aligned}
&\mathbb{E}_n\Big[\sum_{u\neq u'} \mathbb{1}_{\{v\to_r u\}}\mathbb{1}_{\{v\to_s u'\}} W_u^p W_{u'}^p\Big]\\
&\quad\leq \sum_{\substack{u_1,\dots,u_r,\\ u'_1,\dots,u'_s\\ \text{pairw. diff.}}} \frac{W_v W_{u_1}}{n\vartheta}\cdots\frac{W_{u_{r-1}}W_{u_r}}{n\vartheta} W_{u_r}^p \frac{W_v W_{u'_1}}{n\vartheta}\cdots\frac{W_{u'_{s-1}}W_{u'_s}}{n\vartheta}W_{u'_s}^p\\
&\qquad+\sum_{t=1}^{r}\sum_{\substack{u_1,u_2,\dots,u_t,\\ u_{t+1},\dots,u_r,\\ u'_{t+1},\dots,u'_s\\ \text{pairw. diff.}}} \frac{W_v W_{u_1}}{n\vartheta}\frac{W_{u_1}W_{u_2}}{n\vartheta}\cdots\frac{W_{u_{t-1}}W_{u_t}}{n\vartheta}\\
&\qquad\frac{W_{u_t}W_{u_{t+1}}}{n\vartheta}\frac{W_{u_{t+1}}W_{u_{t+2}}}{n\vartheta}\cdots\frac{W_{u_{r-2}}W_{u_{r-1}}}{n\vartheta}\frac{W_{u_{r-1}}W_{u_r}}{n\vartheta}W_{u_r}^p\\
&\qquad\frac{W_{u_t}W_{u'_{t+1}}}{n\vartheta}\frac{W_{u'_{t+1}}W_{u'_{t+2}}}{n\vartheta}\cdots\frac{W_{u'_{s-2}}W_{u'_{s-1}}}{n\vartheta}\frac{W_{u'_{s-1}}W_{u'_s}}{n\vartheta}W_{u'_s}^p.
\end{aligned}
$$

In the first sum the W_v as well as the $r-1$ terms $W_{u_1},\dots,W_{u_{r-1}}$ and the $s-1$ terms $W_{u'_1},\dots,W_{u'_{s'-1}}$ appear exactly twice, whereas W_{u_r} and $W_{u'_{s'}}$ appear to the power $p+1$.

In the second sum we distinguish the case $t<r$ and $t=r$. In any case W_v appears once. If $t\in\{1,\dots,r-1\}$, then all $t-1$ terms $W_{u_1},\dots,W_{u_{t-1}}$ appear twice, W_t appears three times, the $r-t-1$ terms $W_{u_{t+1}}\dots,W_{u_{r-1}}$ and the $s-t-1$ terms $W_{u'_{t+1}}\dots,W_{u'_{s-1}}$ appear twice and W_{u_r} and $W_{u'_s}$ appear to the power $p+1$. If $t=r$, then the r terms $W_{u_1},\dots,W_{u_{r-1}}$ appear twice, as do the $s-r-1$ terms $W_{u'_{r+1}},\dots,W_{u'_{s-1}}$, whereas W_{u_r} appears to the power $p+2$ and $W_{u'_s}$ appears to the power $p+1$.

Separate the sums over the us and u's and count the multiplicity of the respective W_us to obtain the bound

$$
\begin{aligned}
&W_v^2\Big(\frac{1}{n\vartheta}\sum_u W_u^2\Big)^{r+s-2}\Big(\frac{1}{n\vartheta}\sum_u W_u^{p+1}\Big)^2\\
&\quad+\sum_{t=1}^{r-1} W_v\Big(\frac{1}{n\vartheta}\sum_u W_u^2\Big)^{r+s-t-3}\Big(\frac{1}{n\vartheta}\sum_u W_u^3\Big)\Big(\frac{1}{n\vartheta}\sum_u W_u^{p+1}\Big)^2\\
&\quad+W_v\Big(\frac{1}{n\vartheta}\sum_u W_u^2\Big)^{s-2}\Big(\frac{1}{n\vartheta}\sum_u W_u^{p+1}\Big)\Big(\frac{1}{n\vartheta}\sum_u W_u^{p+2}\Big).
\end{aligned}
$$

The summation of $r + s - t - 3$ from $t = 1$ to $t = r - 1$ can be rewritten as a summation of t from $t = s - 2$ to $t = r + s - 4$, which in turn is a summation of $s - 2 + t$ from $t = 0$ to $r - 2$. If we additionally rewrite the sums with $\Gamma_{q,n}$ we obtain

$$\begin{aligned}
&\leq W_v^2 \Gamma_{2,n}^{r+s-2} \Gamma_{p+1,n}^2 + W_v \Gamma_{3,n} \Gamma_{p+1,n}^2 \Gamma_{2,n}^{s-2} \sum_{t=0}^{r-2} \Gamma_{2,n}^t + W_v \Gamma_{2,n}^{s-2} \Gamma_{p+1,n} \Gamma_{p+2,n} \\
&\leq W_v^2 \Gamma_{2,n}^{r+s-2} \Gamma_{p+1,n}^2 + W_v \Gamma_{3,n} \Gamma_{p+1,n}^2 \Gamma_{2,n}^{s-2} (\Gamma_{2,n} + 1)^{r-2} + W_v \Gamma_{2,n}^{s-2} \Gamma_{p+1,n} \Gamma_{p+2,n}.
\end{aligned}$$

In order to simplify this expression, we estimate very generously to factor out common terms

$$\leq W_v^2 \Gamma_{2,n}^{r+s-2} \Gamma_{p+1,n}^2 + W_v (\Gamma_{2,n} + 1)^{r+s-2} (\Gamma_{3,n} + 1)(\Gamma_{p+1,n} + 1)^2 (\Gamma_{p+2,n} + 1). \tag{3.12}$$

Note that the generous estimation can be slightly simplified in case $p = 1$, because then $\Gamma_{3,n}$ and $\Gamma_{p+2,n}$ coincide so that after factoring only one of the two terms needs to be part of the product.

The case $r = s$ in (3.11) can be treated similarly, but the sum does not include a term for $t = r$, so that

$$\begin{aligned}
&\mathbb{E}_n\Big[\sum_{u \neq u'} \mathbb{1}_{\{v \to_r u\}} \mathbb{1}_{\{v \to_r u'\}} W_u^p W_{u'}^p \Big] \\
&\quad \leq \sum_{\substack{u_1,\dots,u_r,\\ u'_1,\dots,u'_r \\ \text{pairw. diff.}}} \frac{W_v W_{u_1}}{n\vartheta} \cdots \frac{W_{u_{r-1}} W_{u_r}}{n\vartheta} W_{u_r}^p \frac{W_v W_{u'_1}}{n\vartheta} \cdots \frac{W_{u'_{r-1}} W_{u'_r}}{n\vartheta} W_{u'_r}^p \\
&\quad\quad + \sum_{t=1}^{r-1} \sum_{\substack{u_1,u_2,\dots,u_t,\\ u_{t+1},\dots,u_r,\\ u'_{t+1},\dots,u'_r \\ \text{pairw. diff.}}} \frac{W_v W_{u_1}}{n\vartheta} \frac{W_{u_1} W_{u_2}}{n\vartheta} \cdots \frac{W_{u_{t-1}} W_{u_t}}{n\vartheta} \\
&\quad\quad\quad \frac{W_{u_t} W_{u_{t+1}}}{n\vartheta} \frac{W_{u_{t+1}} W_{u_{t+2}}}{n\vartheta} \cdots \frac{W_{u_{r-2}} W_{u_{r-1}}}{n\vartheta} \frac{W_{u_{r-1}} W_{u_r}}{n\vartheta} W_{u_r}^p \\
&\quad\quad\quad \frac{W_{u_t} W_{u'_{t+1}}}{n\vartheta} \frac{W_{u'_{t+1}} W_{u'_{t+2}}}{n\vartheta} \cdots \frac{W_{u'_{r-2}} W_{u'_{r-1}}}{n\vartheta} \frac{W_{u'_{r-1}} W_{u'_r}}{n\vartheta} W_{u'_r}^p \\
&\quad \leq W_v^2 \Gamma_{2,n}^{2r-2} \Gamma_{p+1,n}^2 + W_v (\Gamma_{2,n} + 1)^{2r-2} (\Gamma_{3,n} + 1)(\Gamma_{p+1,n} + 1)^2.
\end{aligned} \tag{3.13}$$

Now (3.6) together with (3.8) and (3.13) and generous estimates of the involved terms imply

$$\begin{aligned}
&\mathbb{E}_n[\|D_r(v)\|_p^2] \\
&\quad \leq W_v^2 \Gamma_{2,n}^{2r-2} \Gamma_{p+1,n}^2 + W_v (\Gamma_{2,n}^{r-1} \Gamma_{2p+1,n} + (\Gamma_{2,n} + 1)^{2r-2} (\Gamma_{3,n} + 1)(\Gamma_{p+1,n} + 1)^2) \\
&\quad \leq W_v^2 \Gamma_{2,n}^{2r-2} \Gamma_{p+1,n}^2 + W_v (\Gamma_{2,n} + 1)^{2r-2} (\Gamma_{3,n} + 1)(\Gamma_{p+1,n} + 1)^2 (\Gamma_{2p+1,n} + 1).
\end{aligned} \tag{3.14}$$

This estimate can be simplified if $2p+1$ should happen to be equal to 3 or $p+1$. In those cases the $(\Gamma_{2p+1,n}+1)$ can be replaced by a constant.

Analogously (3.7) in conjuction with (3.9) and (3.12) and similarly generous estimates imply

$$\begin{aligned}&\mathbb{E}_n[\|D_r(v)\|_p\|D_s(v)\|_p]\\&\quad\le W_v^2\Gamma_{2,n}^{r+s-2}\Gamma_{p+1,n}^2+W_v(\Gamma_{2,n}+1)^{r+s-2}(\Gamma_{3,n}+1)(\Gamma_{p+1,n}+1)^2(\Gamma_{p+2,n}+1).\end{aligned} \tag{3.15}$$

As remarked after (3.12) the term $(\Gamma_{p+2,n}+1)$ may be dropped for $p=1$.

Briefly write

$$x_1=(\Gamma_{3,n}+1)(\Gamma_{p+1,n}+1)^2(\Gamma_{2p+1,n}+1)$$

and

$$x_2=(\Gamma_{3,n}+1)(\Gamma_{p+1,n}+1)^2(\Gamma_{p+2,n}+1)$$

to simplify (3.14) and (3.15), respectively. Then

$$x_1+x_2\le C(\Gamma_{3,n}+1)(\Gamma_{p+1,n}+1)^2(\Gamma_{p+2,n}+1)(\Gamma_{2p+1,n}+1).$$

By the comments after (3.12) and (3.14), the $(\Gamma_{p+2,n}+1)$ may be dropped if $p=1$ and the $(\Gamma_{2p+1}+1)$ may be replaced by a constant if $2p+1\in\{3,p+1\}$.

Now (3.4), (3.5), (3.14) and (3.15) and similarly rough estimates of the terms involved imply

$$\begin{aligned}&\mathbb{E}_n[\|S_\ell(v)\|_p^2]\\
&\quad\le W_v^{2p}+2W_v^pW_v\Gamma_{p+1,n}(\Gamma_{2,n}+1)^{\ell-1}+W_v^2\Gamma_{p+1,n}^2\sum_{r=0}^{\ell}\Gamma_{2,n}^{2r-2}\\
&\qquad+W_vx_1\sum_{r=0}^{\ell}(\Gamma_{2,n}+1)^{2r-2}+W_v^2\Gamma_{p+1,n}^2\sum_{r,s=0}^{\ell}\Gamma_{2,n}^{r+s-2}+W_vx_2\sum_{r,s=0}^{\ell}(\Gamma_{2,n}+1)^{r+s-2}\\
&\le W_v^{2p}+2W_v^{p+1}\Gamma_{p+1,n}(\Gamma_{2,n}+1)^{\ell-1}+W_v^2\Gamma_{p+1,n}^2\Big(\sum_{r=0}^{\ell}\Gamma_{2,n}^{2r-2}+\sum_{r,s=0}^{\ell}\Gamma_{2,n}^{r+s-2}\Big)\\
&\qquad+W_v\Big(x_1\sum_{r=0}^{\ell}(\Gamma_{2,n}+1)^{2r-2}+x_2\sum_{r,s=0}^{\ell}(\Gamma_{2,n}+1)^{r+s-2}\Big)\\
&\quad\le W_v^{2p}+2W_v^{p+1}\Gamma_{p+1,n}(\Gamma_{2,n}+1)^{\ell-1}+W_v^2\Gamma_{p+1,n}^2((\Gamma_{2,n}^2+1)^{\ell-1}+(\Gamma_{2,n}+1)^{2\ell-2})\\
&\qquad+W_v(x_1+x_2)((\Gamma_{2,n}+2)^{2\ell-2}+(\Gamma_{2,n}+2)^{2\ell-2})\\
&\quad\le W_v^{2p}+2W_v^{p+1}\Gamma_{p+1,n}(\Gamma_{2,n}+1)^{\ell-1}+CW_v^2\Gamma_{p+1,n}^2(\Gamma_{2,n}+2)^{2\ell-2}\\
&\qquad+CW_v(\Gamma_{3,n}+1)(\Gamma_{p+1,n}+1)^2(\Gamma_{p+2,n}+1)(\Gamma_{2p+1,n}+1)(\Gamma_{2,n}+2)^{2\ell-2}\\
&\quad\le W_v^{2p}+2W_v^{p+1}\Gamma_{p+1,n}(\Gamma_{2,n}+1)^{\ell-1}\end{aligned}$$

$$+ C(W_v + 1)^2(\Gamma_{2,n} + 2)^{2\ell-2}(\Gamma_{3,n} + 1)(\Gamma_{p+1,n} + 1)^2(\Gamma_{p+2,n} + 1)(\Gamma_{2p+1,n} + 1).$$

This shows the claim for general p.

Again, for $p = 1$ the term $(\Gamma_{p+2,n} + 1)$ may be dropped and the $(\Gamma_{2p+1} + 1)$ may be replaced by a constant if $2p + 1 \in \{3, p + 1\}$. Furthermore, for $p = 0$ and $p = 1$ the first terms can be absorbed into the last term by increasing C appropriately. This then proves the slightly simplified results for $p = 0$ and $p = 1$. □

We can sum the previous bound to obtain results for $S_\ell(\mathcal{V})$, where $\mathcal{V}$ is a set of vertices. In the following we will only need the results for $p = 0$ and $p = 1$, so we only work with the simplified results from Lemma 3.1.8

Corollary 3.1.9. *For any vertex set $\mathcal{V} \subseteq V_n$ and level $\ell \in \mathbb{N}$ we have*

$$\begin{aligned}\mathbb{E}_n[|S_\ell(\mathcal{V})|^2] &= \mathbb{E}_n[\|S_\ell(\mathcal{V})\|_0^2] \\ &\le C(\|\mathcal{V}\| + |\mathcal{V}|)^2(\Gamma_{1,n} + 1)^2(\Gamma_{2,n} + 2)^{2\ell}(\Gamma_{3,n} + 1)\end{aligned}$$

and

$$\mathbb{E}_n[\|S_\ell(\mathcal{V})\|^2] \le C(\|\mathcal{V}\| + |\mathcal{V}|)^2(\Gamma_{2,n} + 2)^{2\ell}(\Gamma_{3,n} + 1).$$

Proof. To simplify notation we will assume that for $p = 0$ and $p = 1$ Lemma 3.1.8 gives a bound of the form

$$\mathbb{E}_n[\|S_\ell(v)\|_p^2] \le C(W_v + 1)^2 x_{n,p},$$

where $x_{n,p}$ consists of terms involving $\Gamma_{2,n}$, $\Gamma_{3,n}$, $\Gamma_{p+1,n}$, $\Gamma_{p+2,n}$ and $\Gamma_{2p+1,n}$.

Note that

$$\begin{aligned}\mathbb{E}_n[\|S_\ell(\mathcal{V})\|_p^2] &\le \mathbb{E}_n\Big[\Big(\sum_{v\in\mathcal{V}} \|S_\ell(v)\|_p\Big)^2\Big] \\ &\le \sum_{v\in\mathcal{V}} \mathbb{E}_n[\|S_\ell(v)\|_p^2] + \sum_{\substack{v,v'\in\mathcal{V}\\ v\neq v'}} \mathbb{E}_n[\|S_\ell(v)\|_p\|S_\ell(v')\|_p].\end{aligned} \tag{3.16}$$

For the first sum we can apply Lemma 3.1.8 and obtain

$$\begin{aligned}\sum_{v\in\mathcal{V}} \mathbb{E}_n[\|S_\ell(v)\|_p^2] &\le \sum_{v\in\mathcal{V}} \mathbb{E}_n[\|S_\ell(v)\|_p^2] \\ &\le Cx_{p,n} \sum_{v\in\mathcal{V}} (W_v + 1)^2 \\ &\le Cx_{p,n}(\|\mathcal{V}\| + |\mathcal{V}|)^2.\end{aligned}$$

The terms in the second sum can be estimated using Cauchy–Schwarz and Lemma 3.1.8

$$\begin{aligned}\mathbb{E}_n[\|S_\ell(v)\|\|S_\ell(v')\|] &\le \mathbb{E}_n[\|S_\ell(v)\|^2]^{1/2}\mathbb{E}_n[\|S_\ell(v')\|^2]^{1/2} \\ &\le Cx_{p,n}(W_v + 1)(W_{v'} + 1).\end{aligned}$$

Summing over these terms we obtain

$$\sum_{\substack{v,v'\in\mathcal{V}\\ v\neq v'}} \mathbb{E}_n[\|S_\ell(v)\|\|S_\ell(v')\|] \leq Cx_{p,n} \sum_{\substack{v,v'\in\mathcal{V}\\ v\neq v'}} (W_v+1)(W_{v'}+1)$$
$$\leq Cx_{p,n}(\|\mathcal{V}\|+|\mathcal{V}|)^2.$$

Hence, both terms on the right-hand side of (3.16) can be estimated with the same bound. This finishes the proof. □

In fact the approach from (3.6) can be generalised to bound (higher) moments of the degree distribution of a given vertex v in G_n. Recall that with the notation from Definition 2.1.8 the degree of the vertex v in G_n can be written as $|D_1(v)|$.

In the calculations for higher moments of the degree we will encounter the Stirling numbers of the second kind [see e.g. AS72, § 24.1.4].

Definition 3.1.10 (Stirling numbers of the second kind)**.** Let $k\in\mathbb{N}$ and $j\in\{1,\ldots,k\}$. Then the number of ways of partitioning a set of k elements into j non-empty subsets is denoted by $S_k^{(j)}$ and called the Stirling number of the second kind for k, j.

Lemma 3.1.11. *Fix $v\in V_n$ and $k\in\mathbb{N}$. Then*

$$\mathbb{E}_n[|D_1(v)|^k] \leq \sum_{j=1}^k S_k^{(j)} W_v^j \Gamma_{1,n}^j \leq (W_v+1)^k(\Gamma_{1,n}+k)^k.$$

Proof. Note that

$$|D_1(v)|^k = \Big(\sum_{u\in V_n} X_{vu}\Big)^k$$
$$= \sum_{u_1,\ldots,u_k} X_{vu_1}\cdots X_{vu_k}.$$

Since the X_{vu_ℓ} are indicator functions, any product of X_{vu_ℓ} with identical u_ℓ collapses to one such indicator. We can therefore rewrite the sum over all possible combinations of k vertices into a sum over j pairwise different vertices to obtain

$$|D_1(v)|^k = \sum_{j=1}^k S_k^{(j)} \sum_{\substack{u_1,\ldots,u_j\\ \text{pairw. diff.}}} X_{vu_1}\cdots X_{vu_j},$$

because each product $X_{vu_1}\cdots X_{vu_j}$ with pairwise different u_ℓ is realised by exactly $S_k^{(j)}$ products of the form $X_{vu_1}\cdots X_{vu_k}$, where some of the u_ℓ may be equal. To see this in more detail, partition the k vertices $u_1,\ldots,u_k$ into subsets such that u_r and u_s are in the same subset of they are the same. The number of nonempty subsets of vertices that this procedure produces is exactly equal to the number vertices we need to write $X_{vu_1} \dots X_{vu_k}$ as a product over pairwise different vertices. In particular there are exactly $S_k^{(j)}$ to obtain a product of j pairwise different vertices.

Taking expectations and using that the X_{vu_ℓ} in the sums are independent since the u_ℓ are pairwise different, we obtain

$$\begin{aligned}
\mathbb{E}_n[|D_1(v)|^k] &= \sum_{j=1}^{k} S_k^{(j)} \sum_{\substack{u_1,\dots,u_j \\ \text{pairw. diff.}}} \mathbb{E}_n[X_{vu_1}] \cdots \mathbb{E}_n[X_{vu_j}] \\
&\leq \sum_{j=1}^{k} S_k^{(j)} W_v^j \left(\frac{1}{n\vartheta} \sum_u W_u \right)^j \\
&= \sum_{j=1}^{k} S_k^{(j)} W_v^j \left(\frac{\mathbb{E}_n[W^{(n)}]}{\vartheta} \right)^j \\
&= \sum_{j=1}^{k} S_k^{(j)} W_v^j \Gamma_{1,n}^j,
\end{aligned}$$

which proves the first part of the claim.

For the second part note that for $k \geq 2$ the Stirling numbers of the second kind can be bounded above as follows [RD69, Thm. 3]

$$S_k^{(j)} \leq \frac{1}{2} \binom{k}{j} j^{k-j} \leq \frac{1}{2} \binom{k}{j} k^{k-j},$$

so that

$$\begin{aligned}
\mathbb{E}_n[|D_1(v)|^k] &\leq \sum_{j=1}^{k} S_k^{(j)} W_v^j \Gamma_{1,n}^j \\
&\leq \frac{1}{2} \sum_{j=1}^{k} \binom{k}{j} (W_v \Gamma_{1,n})^j k^{k-j} \\
&\leq \frac{1}{2} (W_v \Gamma_{1,n} + k)^k \\
&\leq (W_v \Gamma_{1,n} + k)^k.
\end{aligned}$$

For $k = 1$ we have

$$\mathbb{E}_n[|D_1(v)|] \leq W_v \Gamma_{1,n} \leq W_v \Gamma_{1,n} + 1.$$

Thus we get for all $k \in \mathbb{N}$ that

$$\mathbb{E}_n[|D_1(v)|^k] \leq (W_v \Gamma_{1,n} + k)^k \leq (W_v + 1)^k (\Gamma_{1,n} + k)^k$$

as claimed. □

We will also need to bound the probability that there exists a path of given length between two subsets of vertices. To simplify notation we introduce a shorthand for this event.

Definition 3.1.12. Let $\mathcal{U}$ and $\mathcal{V}$ be two sets of vertices. Let $\mathcal{R}$ be a third set of vertices that is disjoint from $\mathcal{U}$ and $\mathcal{V}$

- Write $\mathcal{U} \leftrightsquigarrow_{\ell} \mathcal{V}$ if there exists a path of length ℓ between a vertex $u \in \mathcal{U}$ and a vertex $v \in \mathcal{V}$.

  In some cases it might be useful to restrict this event to paths that avoid a set of vertices $\mathcal{R}$, so we write $\mathcal{U} \leftrightsquigarrow_{(\mathcal{R})\ell} \mathcal{V}$ if there is a path of length ℓ between a vertex $u \in \mathcal{U}$ and a vertex $v \in \mathcal{V}$ that does not use any vertices from $\mathcal{R}$.

- Write $\mathcal{U} \leftrightsquigarrow_{\leq \ell} \mathcal{V}$ if there exists a path of length at most ℓ between a vertex $u \in \mathcal{U}$ and a vertex $v \in \mathcal{V}$.

  As above we write $\mathcal{U} \leftrightsquigarrow_{(\mathcal{R})\leq \ell} \mathcal{V}$ if there is a path between $\mathcal{U}$ and $\mathcal{V}$ that does not use any vertices from $\mathcal{R}$.

As we do so often, we drop the curly brackets if $\mathcal{V}$ or $\mathcal{U}$ is a set with a single element and write for example $v \leftrightsquigarrow_{\ell} u$ for $\{v\} \leftrightsquigarrow_{\ell} \{u\}$.

Lemma 3.1.13. *Fix $\ell \in \mathbb{N}$, $\ell \geq 1$. Let $\mathcal{U}, \mathcal{V} \subseteq V_n$ be two disjoint sets of vertices then*

$$\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{\ell} \mathcal{V}) \leq \frac{\|\mathcal{U}\| \|\mathcal{V}\|}{n\vartheta} \Gamma_{2,n}^{\ell-1}.$$

Proof. Recall that in a path no vertex can appear multiple times. This ensures that the edges of a path are all distinct and hence independent, so that

$$\begin{aligned}
\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{\ell} \mathcal{V}) &\leq \sum_{\substack{u_0 \in \mathcal{U} \\ u_1, \ldots, u_{\ell-1} \\ u_\ell \in \mathcal{V}}} \mathbb{E}_n[X_{u_0 u_1} X_{u_1 u_2} \cdots X_{u_{\ell-1} u_\ell}] \\
&\leq \sum_{\substack{u_0 \in \mathcal{U} \\ u_1, \ldots, u_{\ell-1} \\ u_\ell \in \mathcal{V}}} \frac{W_{u_0} W_{u_1}}{n\vartheta} \frac{W_{u_1} W_{u_2}}{n\vartheta} \cdots \frac{W_{u_{\ell-1}} W_{u_\ell}}{n\vartheta} \\
&\leq \frac{1}{n\vartheta} \sum_{u_0 \in \mathcal{U}} W_{u_0} \Big(\frac{1}{n\vartheta} \sum_{v} W_v^2 \Big)^{\ell-1} \sum_{u_\ell \in \mathcal{V}} W_{u_\ell} \\
&\leq \frac{\|\mathcal{U}\| \|\mathcal{V}\|}{n\vartheta} \Gamma_{2,n}^{\ell-1}.
\end{aligned}$$

This finishes the proof. □

In particular the probability that there is an edge between two sets of vertices $\mathcal{U}$ and $\mathcal{V}$ is bounded above by $(n\vartheta)^{-1} \|\mathcal{U}\| \|\mathcal{V}\|$.

Corollary 3.1.14. *Fix $\ell \in \mathbb{N}$, $\ell \geq 1$. Let $\mathcal{U}, \mathcal{V} \subseteq V_n$ be two disjoint sets of vertices, then*

$$\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{\leq \ell} \mathcal{V}) \leq \frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(1+\Gamma_{2,n})^{\ell-1}.$$

Proof. By Lemma 3.1.13 we have

$$\begin{aligned}\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{\leq \ell} \mathcal{V}) &\leq \sum_{r=1}^{\ell} \mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{r} \mathcal{V}) \\ &\leq \sum_{r=1}^{\ell} \frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}\Gamma_{2,n}^{r-1} \\ &\leq \frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(1+\Gamma_{2,n})^{\ell-1}\end{aligned}$$

as claimed. □

The following special case of Corollary 3.1.14 is of particular interest to us.

Corollary 3.1.15. *For any level $\ell \in \mathbb{N}$ and vertices $v, u \in V_n$ with $u \neq v$ we have*

$$\mathbb{P}_n(u \in B_\ell(v)) \leq \frac{W_u W_v}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}.$$

Proof. The vertex u is contained in the ℓ-neighbourhood $B_\ell(v)$ of v if and only if there is a path of length at most ℓ from v to u. Hence, we apply Corollary 3.1.14 with $\mathcal{V} = \{v\}$ and $\mathcal{U} = \{u\}$, so that

$$\mathbb{P}_n(u \in B_\ell(v)) = \mathbb{P}_n(\{v\} \leftrightsquigarrow_{\ell} \{u\}) \leq \frac{W_u W_v}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}$$

as desired. □

This result can be used to bound the probability that an edge e is contained in $B_\ell(v)$.

Corollary 3.1.16. *For any level $\ell \in \mathbb{N}$, vertex $v \in V_n$ and edge $e = \{u, u'\}$ whose endpoints are not equal to v we have*

$$\mathbb{P}_n(e \in B_\ell(v)) \leq \frac{W_v(W_u + W_{u'})}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}.$$

Proof. The edge $e = \{u, u'\}$ can only be contained in the ℓ-neighbourhood of v if one of its endpoints u or u' can be reached in at most $\ell - 1$ steps from v. Hence by Corollary 3.1.15

$$\begin{aligned}\mathbb{P}_n(e \in B_\ell(v)) &\leq \mathbb{P}_n(u \in B_{\ell-1}(v)) + \mathbb{P}_n(u' \in B_{\ell-1}(v)) \\ &\leq \frac{W_v(W_u + W_{u'})}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}\end{aligned}$$

as claimed. □

Corollary 3.1.14 still holds conditional on knowing the edges emanating from a fixed set of vertices. In order to formulate this in more detail, we need some more notation.

Definition 3.1.17. For any vertex $v \in V_n$ let

$$\tilde{S}_v = (X_{\{v,u\}})_{u \in V_n}$$

be the collection of all indicator functions of possible edges emanating from v.

For an edge $e = \{u, v\}$ define

$$\tilde{S}_e = (S_v, S_u)$$

so that $\tilde{S}_e$ contains information about all edges incident to e.

For the proof of the result it is convenient to estimate the set of neighbours of a vertex set in a way that makes this set independent of the neighbours of another vertex set.

Definition 3.1.18. Let $\mathcal{V}, \mathcal{U} \subseteq V_n$ be two disjoint subsets of vertices.

Set $S_1^{(\mathcal{U})}(\mathcal{V}) = S_1(\mathcal{V}) \setminus \mathcal{U}$ and $D_1^{(\mathcal{U})}(\mathcal{V}) = D_1(\mathcal{V}) \setminus \mathcal{U}$ so that

$$\begin{aligned} S_1^{(\mathcal{U})}(\mathcal{V}) &= \mathcal{V} \cup \{x \in V_n \setminus \mathcal{U} : X_{\{v,x\}} = 1 \text{ for some } v \in \mathcal{V}\} \\ D_1^{(\mathcal{U})}(\mathcal{V}) &= \{x \in V_n \setminus \mathcal{U} : X_{\{v,x\}} = 1 \text{ for some } v \in \mathcal{V}\} \end{aligned}$$

are independent of X_{vu} for all $v \in \mathcal{V}$ and $u \in \mathcal{U}$.

In particular then $D_1^{(\mathcal{U})}(\mathcal{V})$ and $D_1^{(\mathcal{V})}(\mathcal{U})$ are independent. The same holds for $S_1^{(\mathcal{U})}(\mathcal{V})$ and $S_1^{(\mathcal{V})}(\mathcal{U})$.

Clearly we also have

$$|S_1(v)| \leq |S_1^{(\mathcal{U})}(v)| + |\mathcal{U}| \quad \text{and} \quad \|S_1(v)\| \leq \|S_1^{(\mathcal{U})}(v)\| + \|\mathcal{U}\|.$$

as well as

$$|D_1(v)| \leq |D_1^{(\mathcal{U})}(v)| + |\mathcal{U}| \quad \text{and} \quad \|D_1(v)\| \leq \|D_1^{(\mathcal{U})}(v)\| + \|\mathcal{U}\|.$$

We write $D_1^{(u_1,\dots,u_m)}(v)$ instead of $D_1^{(\{u_1,\dots,u_m\})}(v)$ and $S_1^{(u_1,\dots,u_m)}(v)$ instead of $S_1^{(\{u_1,\dots,u_m\})}(v)$ for any fixed number of vertices $u_1, \dots, u_m \in V_n$.

Lemma 3.1.19. *Fix three disjoint sets of vertices $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{R}$ (the sets may be empty). Set $S = (\tilde{S}_w)_{w \in \mathcal{U} \cup \mathcal{V} \cup \mathcal{R}}$ and $\tilde{S} = (\tilde{S}_w)_{w \in \mathcal{U} \cup \mathcal{V}}$ (note the absence of vertices from $\mathcal{R}$ in $\tilde{S}$).*

Then there exists a function $\xi_\ell(\mathcal{U}, \mathcal{V}, \mathcal{R})$ that is $\sigma(\tilde{S}, \mathcal{F}_n)$-measurable and independent of all edges of the form $\{u, u'\}$ for $u, u' \in \mathcal{U}$, $\{v, v'\}$ for $v, v' \in \mathcal{V}$. This function satisfies

$$\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{(\mathcal{R}) \leq \ell} \mathcal{V} \mid \tilde{S}) \leq \xi_\ell(\mathcal{U}, \mathcal{V}, \mathcal{R}),$$

and

$$\mathbb{E}_n[\xi_\ell(\mathcal{U},\mathcal{V},\mathcal{R})] \le \min\Big\{\frac{\|\mathcal{U}\||\mathcal{V}|}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}, 1\Big\}$$

as well as

$$\begin{aligned}&\mathbb{E}_n[\xi_\ell(\mathcal{U},\mathcal{V},\mathcal{R})f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)]\\ &\quad\le C\beta_n(\mathcal{U},\mathcal{V})\min\Big\{\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}, 1\Big\}\end{aligned}$$

for all functions f that are non-decreasing in both arguments such that

$$\mathbb{E}_n[f(|D_1(\mathcal{U})|+|\mathcal{V}|+1, |D_1(\mathcal{V})|+|\mathcal{U}|+1)] \le \beta_n(\mathcal{U},\mathcal{V}).$$

Proof. Let $\tilde{G}_n$ be the graph that can be obtained from G_n by removing all vertices in $\mathcal{U}$ and $\mathcal{V}$ and introducing two new vertices $\tilde{u}$ and $\tilde{v}$ such that $\tilde{u}$ has an edge to a vertex z in $\tilde{G}$ if and only if z has an edge to some $u \in \mathcal{U}$ in G_n and analogous for $\tilde{v}$ and $\mathcal{V}$.

If there is a path from a $u \in \mathcal{U}$ to a $v \in \mathcal{V}$ in G_n, then this induces a path from $\tilde{u}$ to $\tilde{v}$ in $\tilde{G}_n$.

We write $x \sim y$ to indicate that there is an edge between x and y in G_n. In a slight abuse of notation we use the same expression to mean that two vertices are connected by an edge in $\tilde{G}_n$. We also write $\mathcal{U} \sim \mathcal{V}$ for subsets of vertices and mean that at least one vertex in $\mathcal{U}$ has an edge to at least one vertex in $\mathcal{V}$.

By definition of $\mathcal{U} \leftrightsquigarrow_{(\mathcal{R})\le\ell} \mathcal{V}$ and construction of $\tilde{G}_n$ we have

$$\begin{aligned}&\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{(\mathcal{R})\le\ell} \mathcal{V} \mid S)\\ &\quad\le \mathbb{P}_n(\tilde{u} \in B_\ell(\tilde{v}, \tilde{G}_n - \mathcal{R}) \mid S).\end{aligned}$$

Since path of length at most ℓ from $\tilde{v}$ to $\tilde{u}$ that avoids $\mathcal{R}$ consists either of a single edge between the two vertices or can be decomposed into a single edge from u to one of its neighbours $x \notin \mathcal{R}$, which is S-measurable, and a path from x to $\tilde{v}$ of length at most $\ell-1$ that avoids $\mathcal{R}$ and additionally also $\tilde{u}$, this can be bounded by

$$\le \sum_{x\notin\mathcal{R}\cup\{\tilde{u},\tilde{v}\}} \mathbb{1}_{\{\tilde{u}\sim x\}}\mathbb{P}_n(x \in B_{\ell-1}(\tilde{v}, \tilde{G}_n - \mathcal{R} - \{\tilde{u}\}) \mid S) + \mathbb{1}_{\{\tilde{u}\sim\tilde{v}\}}.$$

Similarly, a path of length at most $\ell-1$ from $\tilde{v}$ to x that avoids $\tilde{u}$ and $\mathcal{R}$ either consists of the single edge between the two vertices or can be composed of a single edge from $\tilde{v}$ to one of its neighbours $y \notin \mathcal{R}\cup\{\tilde{u},\tilde{v},x\}$ and a path of length at most $\ell-2$ from y to x that avoids $\mathcal{R}$, $\tilde{u}$ and additionally also $\tilde{v}$. We therefore have the bound

$$\begin{aligned}&\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{(\mathcal{R})\le\ell} \mathcal{V} \mid S)\\ &\quad\le \sum_{x\notin\mathcal{R}\cup\{\tilde{u},\tilde{v}\}} \mathbb{1}_{\{\tilde{u}\sim x\}} \sum_{y\notin\mathcal{R}\cup\{\tilde{u},\tilde{v},x\}} \mathbb{1}_{\{\tilde{v}\sim y\}}\mathbb{P}_n(x \in B_{\ell-2}(y, \tilde{G}_n - \mathcal{R} - \{\tilde{u},\tilde{v}\}) \mid S)\\ &\qquad+ \sum_{x\notin\mathcal{R}\cup\{\tilde{u},\tilde{v}\}} \mathbb{1}_{\{\tilde{u}\sim x\}}\mathbb{1}_{\{x\sim\tilde{v}\}} + \mathbb{1}_{\{\tilde{u}\sim\tilde{v}\}}\end{aligned} \tag{3.17}$$

Focus on the probability for a moment. Outside of $\mathcal{U} \cup \mathcal{V}$ and $\{\tilde{u}, \tilde{v}\}$ the graphs G_n and $\tilde{G}_n$ agree. Hence,

$$\begin{aligned}&\mathbb{P}_n(x \in B_{\ell-2}(y, \tilde{G}_n - \mathcal{R} - \{\tilde{u}, \tilde{v}\}) \mid S)\\&\quad\le \mathbb{P}_n(x \in B_{\ell-2}(y, G_n - \mathcal{R} - \mathcal{U} - \mathcal{V}) \mid S).\end{aligned}$$

The conditioning can be removed, since the graph $G_n - \mathcal{R} - \mathcal{U} - \mathcal{V}$ no longer depends on S.

$$\le \mathbb{P}_n(x \in B_{\ell-2}(y, G_n - \mathcal{R} - \mathcal{U} - \mathcal{V})).$$

The probability that there is a path from x to y in the graph $G_n - \mathcal{R} - \mathcal{U} - \mathcal{V}$ can be estimated by the probability that there is a path in G_n, so that together with Corollary 3.1.15 the probability is bounded by

$$\begin{aligned}&\le \mathbb{P}_n(x \in B_{\ell-2}(y, G_n))\\&\le \frac{W_x W_y}{n\vartheta}(\Gamma_{2,n} + 1)^{\ell-3}.\end{aligned}$$

Plugging this bound for the probability into (3.17) and going back from $\tilde{u}$ to $\mathcal{U}$ and from $\tilde{v}$ to $\mathcal{V}$, we arrive at

$$\begin{aligned}&\mathbb{P}_n(\mathcal{U} \leftrightsquigarrow_{(\mathcal{R}) \le \ell} \mathcal{V} \mid S)\\&\le \sum_{\substack{x,y \notin \mathcal{R} \cup \mathcal{U} \cup \mathcal{V}\\ x \ne y}} \frac{W_x W_y}{n\vartheta}(\Gamma_{2,n} + 1)^{\ell-3} + \sum_{x \notin \mathcal{R} \cup \mathcal{U} \cup \mathcal{V}} \mathbb{1}_{\{\mathcal{U} \sim x\}} \mathbb{1}_{\{\mathcal{V} \sim x\}} + \mathbb{1}_{\{\mathcal{U} \sim \mathcal{V}\}}\end{aligned} \tag{3.18}$$

Call the left-hand side $\xi_\ell(\mathcal{U}, \mathcal{V}, \mathcal{R})$. Then $\xi_\ell(\mathcal{U}, \mathcal{V}, \mathcal{R})$ is $\sigma(S, \mathcal{F}_n)$-measurable and conditionally on $\mathcal{F}_n$ its sole source of randomness is the collection X. Note additionally that the expression does not contain indicator functions for events of the form $\{X_{\{u,u'\}} = 1\}$ with $u, u' \in \mathcal{U}$, $\{X_{\{v,v'\}} = 1\}$ with $v, v' \in \mathcal{V}$ or $\{X_{\{x,y\}} = 1\}$ with $x \in \mathcal{R}$ and $y \in V_n$. Hence, $\xi_\ell(\mathcal{U}, \mathcal{V}, \mathcal{R})$ is $\sigma(\tilde{S}, \mathcal{F}_n)$-measurable.

Taking expectations we obtain

$$\begin{aligned}&\mathbb{E}_n[\xi_\ell(\mathcal{U}, \mathcal{V}, \mathcal{R})]\\&= \sum_{\substack{x,y \notin \mathcal{R} \cup \mathcal{U} \cup \mathcal{V}\\ x \ne y}} \mathbb{E}_n\Big[\mathbb{1}_{\{\mathcal{U} \sim x\}} \mathbb{1}_{\{\mathcal{V} \sim y\}} \frac{W_x W_y}{n\vartheta}(\Gamma_{2,n} + 1)^{\ell-3}\Big]\\&\quad + \sum_{x \notin \mathcal{R} \cup \mathcal{U} \cup \mathcal{V}} \mathbb{E}_n[\mathbb{1}_{\{\mathcal{U} \sim x\}} \mathbb{1}_{\{\mathcal{V} \sim x\}}] + \mathbb{E}_n[\mathbb{1}_{\{\mathcal{U} \sim \mathcal{V}\}}].\end{aligned}$$

The events in the indicators are independent, because they involve disjoint sets of edges, so that we can apply Lemma 3.1.13 to find the bound

$$\begin{aligned}
&\leq \sum_{\substack{x,y\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}\\ x\neq y}} \frac{\|\mathcal{U}\|W_x}{n\vartheta}\frac{\|\mathcal{V}\|W_y}{n\vartheta}\frac{W_xW_y}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-3}\\
&\quad+\sum_{x\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}} \frac{\|\mathcal{U}\|W_x}{n\vartheta}\frac{\|\mathcal{V}\|W_x}{n\vartheta}+\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}\\
&\leq \frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}\frac{1}{n\vartheta}\sum_x W_x^2\frac{1}{n\vartheta}\sum_y W_y^2(\Gamma_{2,n}+1)^{\ell-3}\\
&\quad+\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}\frac{1}{n\vartheta}\sum_x W_x^2+\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}\\
&\leq \frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(\Gamma_{2,n}^2(\Gamma_{2,n}+1)^{\ell-3}+\Gamma_{2,n}+1)\\
&\leq \frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}.
\end{aligned}$$

Since we take the expectation of a bound for a probability, we may assume that this expectation is also bounded above by 1.

Additionally,

$$\begin{aligned}
&\mathbb{E}_n[\xi_\ell(\mathcal{U},\mathcal{V},\mathcal{R})f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)]\\
&\quad=\sum_{\substack{x,y\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}\\ x\neq y}} \mathbb{E}_n\Big[f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)\mathbb{1}_{\{\mathcal{U}\sim x\}}\mathbb{1}_{\{\mathcal{V}\sim y\}}\frac{W_xW_y}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-3}\Big]\\
&\qquad+\sum_{x\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}} \mathbb{E}_n[f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)\mathbb{1}_{\{\mathcal{U}\sim x\}}\mathbb{1}_{\{\mathcal{V}\sim x\}}]\\
&\qquad+\mathbb{E}_n[f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)\mathbb{1}_{\{\mathcal{U}\sim\mathcal{V}\}}].
\end{aligned} \tag{3.19}$$

For the first of the three terms on the right-hand side of (3.19) note that $D_1(\mathcal{U}) \subseteq D_1^{(\mathcal{V}\cup\{x\})}(\mathcal{U})\cup\mathcal{V}\cup\{x\}$ and that by construction $D_1^{(\mathcal{V}\cup\{x\})}(\mathcal{U})$, $\mathbb{1}_{\{\mathcal{U}\sim x\}}$ and $\mathbb{1}_{\{\mathcal{V}\sim y\}}$ are independent as long as $x\neq y$ and $x,y\notin\mathcal{U}\cup\mathcal{V}$. Similarly, we have $D_1(\mathcal{V})\subseteq D_1^{(\mathcal{U}\cup\{y\})}(\mathcal{V})\cup\mathcal{U}\cup\{y\}$ and that $D_1^{(\mathcal{U}\cup\{y\})}(\mathcal{V})$, $\mathbb{1}_{\{\mathcal{U}\sim x\}}$ and $\mathbb{1}_{\{\mathcal{V}\sim y\}}$ are independent as long as $x\neq y$ and $x,y\notin\mathcal{U}\cup\mathcal{V}$.

Hence, using also that f is non-decreasing in both arguments, we can bound

$$\begin{aligned}
&\frac{(\Gamma_{2,n}+1)^{\ell-3}}{n\vartheta} \sum_{\substack{x,y\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}\\ x\neq y}} \mathbb{E}_n[f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)\mathbb{1}_{\{\mathcal{U}\sim x\}}W_x\mathbb{1}_{\{\mathcal{V}\sim y\}}W_y] \\
&\le \frac{(\Gamma_{2,n}+1)^{\ell-3}}{n\vartheta} \sum_{\substack{x,y\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}\\ x\neq y}} \mathbb{E}_n[\mathbb{1}_{\{\mathcal{U}\sim x\}}W_x]\mathbb{E}_n[\mathbb{1}_{\{\mathcal{V}\sim y\}}W_y] \\
&\qquad \mathbb{E}_n[f(|D_1^{(\mathcal{V}\cup\{x\})}(\mathcal{U})|+|\mathcal{V}|+1,|D_1^{(\mathcal{U}\cup\{y\})}(\mathcal{V})|+|\mathcal{U}|+1)].
\end{aligned}$$

Recall the definition of $\|D_1(\mathcal{U})\|$ and $\|D_1(\mathcal{V})\|$ and use that $D_1^{(\mathcal{V}\cup\{x\})}(\mathcal{U})\subseteq D_1(\mathcal{U})$ to bound this further by

$$\begin{aligned}
&\le \frac{(\Gamma_{2,n}+1)^{\ell-3}}{n\vartheta}\mathbb{E}_n[\|D_1(\mathcal{U})\|]\mathbb{E}_n[\|D_1(\mathcal{V})\|] \\
&\qquad \mathbb{E}_n[f(|D_1(\mathcal{U})|+|\mathcal{V}|+1,|D_1(\mathcal{U})|+|\mathcal{V}|+1)].
\end{aligned}$$

Now Corollary 3.1.5 and $D_\ell(\mathcal{V})\subseteq S_\ell(\mathcal{V})$ imply the bound

$$\le \beta_n(\mathcal{U},\mathcal{V})\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-2}.$$

For the second term in (3.19) we use a similar approach and Lemma 3.1.13

$$\begin{aligned}
&\sum_{x\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}} \mathbb{E}_n[f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)\mathbb{1}_{\{\mathcal{U}\sim x\}}\mathbb{1}_{\{\mathcal{V}\sim x\}}] \\
&\le \mathbb{E}_n\Big[\sum_{x\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}} \mathbb{1}_{\{\mathcal{U}\sim x\}}\mathbb{1}_{\{\mathcal{V}\sim x\}} \\
&\qquad f(|D_1^{(\mathcal{V}\cup\{x\})}(\mathcal{U})|+|V|+1,|D_1^{(\mathcal{U}\cup\{y\})}(\mathcal{V})|+|U|+1)\Big] \\
&\le \sum_{x\notin \mathcal{R}\cup\mathcal{U}\cup\mathcal{V}} \frac{W_x\|\mathcal{U}\|}{n\vartheta}\frac{W_x\|\mathcal{V}\|}{n\vartheta}\beta_n(\mathcal{U},\mathcal{V}) \\
&\le \beta_n(\mathcal{U},\mathcal{V})\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}\Gamma_{2,n}.
\end{aligned}$$

Similarly, the third term on the right-hand side of (3.19) can be estimated with Lemma 3.1.13

$$\begin{aligned}
&\mathbb{E}_n[\mathbb{1}_{\{\mathcal{U}\sim\mathcal{V}\}}f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)] \\
&\quad \le \mathbb{E}_n[\mathbb{1}_{\{\mathcal{U}\sim\mathcal{V}\}}f(|D_1^{(\mathcal{V})}(\mathcal{U})|+|\mathcal{V}|,|D_1^{(\mathcal{U})}(\mathcal{V})|+|\mathcal{U}|)] \\
&\quad \le \beta_n(\mathcal{U},\mathcal{V})\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}.
\end{aligned}$$

Together this proves

$$\mathbb{E}_n[\xi_\ell(\mathcal{U},\mathcal{V},\mathcal{R}) f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)] \le C\beta_n(\mathcal{U},\mathcal{V})\frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(\Gamma_{2,n}+1)^{\ell-1}.$$

Since ξ_ℓ is a probability, the left-hand side is trivially bounded by $\beta_n(\mathcal{U},\mathcal{V})$, the bound for the expectation of $f(|D_1(\mathcal{U})|,|D_1(\mathcal{V})|)$, which finishes the proof. □

These path and inclusion probabilities can now be used to bound the probability that the neighbourhood of a vertex is not tree shaped. Like the following section this proof is an extension of an idea by Cao [Cao21, Lem. 6.7].

Lemma 3.1.20. *For every vertex $v \in V_n$ and level $\ell \in \mathbb{N}$*

$$\mathbb{P}_n(B_\ell(v) \text{ is not a tree}) \le C(1+\Gamma_{2,n})^{2\ell+1}(\Gamma_{3,n}+1)\frac{(W_v+1)^2}{n\vartheta}.$$

Proof. Let A_ℓ be the event that $B_\ell(v)$ is a tree. If $B_{\ell-1}(v)$ is a tree, $B_\ell(v)$ can only fail to be a tree if there is an edge between two vertices in $D_{\ell-1}(v)$ or if there is a vertex not in $S_{\ell-1}(v)$ that is connected to two vertices in $D_{\ell-1}(v)$.

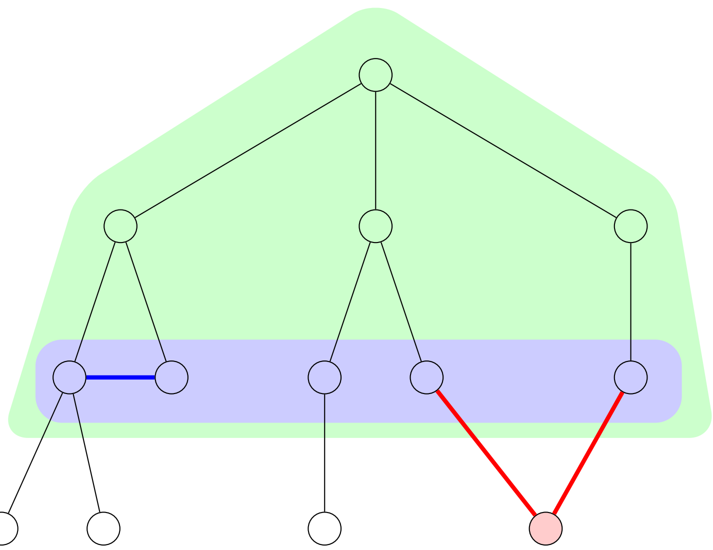


Figure 3.1: There are two ways $B_3(v)$ can fail to be a tree if $B_2(v)$ (shown in green) is a tree. Either two vertices in $D_2(v)$ are connected via an edge (shown in blue) or two vertices from $D_2(v)$ have an edge each (shown in red) to a vertex not in $S_2(v)$.

Thus

$$\begin{aligned}
&\mathbb{1}_{A_{\ell-1}} \mathbb{P}_n(A_\ell^c \mid B_{\ell-1}(v)) \\
&\quad\le \mathbb{1}_{A_{\ell-1}} \sum_{u,u' \in D_{\ell-1}(v)} \Big(\mathbb{P}_n(X_{uu'} = 1) + \sum_{x \notin S_{\ell-1}(v)} \mathbb{P}_n(X_{ux} = 1)\mathbb{P}_n(X_{xu'} = 1) \Big) \\
&\quad\le \mathbb{1}_{A_{\ell-1}} \sum_{u,u' \in D_{\ell-1}(v)} \Big(\frac{W_u W_{u'}}{n\vartheta} + \sum_{x \notin S_{\ell-1}(v)} \frac{W_u W_x}{n\vartheta} \frac{W_x W_{u'}}{n\vartheta} \Big) \\
&\quad\le \mathbb{1}_{A_{\ell-1}} (1 + \Gamma_{2,n}) \frac{\|D_{\ell-1}(v)\|^2}{n\vartheta}.
\end{aligned}$$

Take expectations and conclude that

$$\mathbb{P}_n(A_{\ell-1} \cap A_\ell^c) \le (1 + \Gamma_{2,n}) \frac{\mathbb{E}_n[\|D_{\ell-1}(v)\|^2]}{n\vartheta}.$$

Furthermore,

$$\begin{aligned}
\mathbb{P}_n(A_\ell^c) &= \mathbb{P}_n(A_\ell^c \cap A_{\ell-1}) + \mathbb{P}_n(A_\ell^c \cap A_{\ell-1}^c) \\
&\le \mathbb{P}_n(A_\ell^c \cap A_{\ell-1}) + \mathbb{P}_n(A_{\ell-1}^c).
\end{aligned}$$

Iterate, plug in the bound for $\mathbb{P}_n(A_\ell^c \cap A_{\ell-1})$ and use $\mathbb{P}_n(A_1^c) = 0$ to show

$$\mathbb{P}_n(A_\ell^c) \le (1 + \Gamma_{2,n}) \sum_{r=1}^{\ell-1} \frac{\mathbb{E}[\|D_{r-1}(v)\|^2]}{n\vartheta}.$$

Since the $D_{r-1}(v)$ are disjoint, and their union is contained in $S_{\ell-1}(v)$ the square of the sum of the weights of the $D_{r-1}(v)$ can be bounded by the square of the sum of weights in $S_\ell(v)$, which in turn can be bounded by Lemma 3.1.8

$$\begin{aligned}
&\le (1 + \Gamma_{2,n}) \frac{\mathbb{E}_n[\|S_{\ell-1}(v)\|^2]}{n\vartheta} \\
&\le C(1 + \Gamma_{2,n})^{2\ell+1}(\Gamma_{3,n} + 1) \frac{(W_v + 1)^2}{n\vartheta}.
\end{aligned}$$

This concludes the proof. □

3.2 Correlation between neighbourhoods

In this section we investigate the correlation between different neighbourhoods in the graph $\mathbf{G}_n$ more closely. Before we get into the formal argument, we will briefly recall Corollary 3.1.14, which bounds the probability that there is a path of length at most ℓ between two disjoint sets of vertices $\mathcal{U}$ and $\mathcal{V}$ by $(n\vartheta)^{-1}\|\mathcal{U}\|\|\mathcal{V}\|(1+\Gamma_{2,n})^{\ell-1}$. This implies that the probability that the ℓ-neighbourhoods $B_\ell(\mathcal{V}, G_n)$ and $B_\ell(\mathcal{U}, G_n)$

(here in the unweighted graph, but the argument is the same for the weighted graph) share a vertex is bounded by

$$\mathbb{P}_n(B_\ell(\mathcal{V}, G_n) \text{ and } B_\ell(\mathcal{U}, G_n) \text{ share a vertex}) \leq \frac{\|\mathcal{U}\|\|\mathcal{V}\|}{n\vartheta}(1+\Gamma_{2,n})^{2\ell-2}.$$

Intuitively, if the neighbourhoods do not share a vertex, their structure is determined by independent random variables, which would mean that they are independent. That argument is made more rigorous in the remainder of this section. As in the previous section, the results we show here are by no means surprising, but results for the exact setup we needed were not readily available in the literature.

We show that the ℓ-neighbourhoods of two disjoint sets of root vertices are relatively weakly correlated by constructing slightly altered independent versions of the $\ell+1$-neighbourhoods conditionally on the ℓ-neighbourhoods. An iterative argument that relates the correlation of the $\ell+1$-neighbourhoods to the correlation of the ℓ-neighbourhoods then finishes the argument.

The discussion here extends Cao's [Cao21, § 6] approach for edge-weighted Erdős–Rényi graphs to inhomogeneous random graphs with additional weights on edges and vertices. Our construction of the altered neighbourhoods needs to take into account both edge and vertex weights. We will do that in two separate steps. In a first step we ignore the vertex weights at level $\ell+1$ (formally, we do this by applying a function $\tau_{\ell+1}$ that removes these weights).

Given the ℓ-neighbourhoods, the $\ell+1$-neighbourhoods without vertex weights at level $\ell+1$ can be constructed by adding edges emanating from level ℓ vertices. If the edges that are added to the neighbourhoods are distinct, the added randomness is independent. If edges have to be used for neighbourhoods emanating from both root vertex sets, they can be replaced by independent copies for one of the two sets to make the added randomness independent. Provided that not too many edges have to be rerandomised in this way, the resulting objects are close enough to the original $\ell+1$-neighbourhoods.

Lemma 3.2.1. *Fix $m, m' \in \mathbb{N}$ and $m+m'$ distinct vertices $v_1, \ldots, v_m$ and $v'_1, \ldots, v'_{m'}$. Let $E_n = \{(i,j) : 1 \leq i < j \leq n\}$. Let $F \subseteq V_n \cup E_n$ and $F' \subseteq V_n \cup E_n$. For $r \in \mathbb{N}$ let*

$$\mathbf{B}_r = (B_r(v_1, \mathbf{G}_n), B_r(v_1, \mathbf{G}_n^F), \ldots, B_r(v_m, \mathbf{G}_n), B_r(v_m, \mathbf{G}_n^F)),$$
$$\mathbf{B}'_r = (B_r(v'_1, \mathbf{G}_n), B_r(v'_1, \mathbf{G}_n^{F'}), \ldots, B_r(v'_{m'}, \mathbf{G}_n), B_r(v'_{m'}, \mathbf{G}_n^{F'})).$$

Let $\mathbf{S}_r$ be the set of vertices in $\mathbf{B}_r$ and similarly let $\mathbf{S}'_r$ be the set of vertices in $\mathbf{B}'_r$. Then $\mathbf{D}_r = \mathbf{S}_r \setminus \mathbf{S}_{r-1}$ and $\mathbf{D}'_r = \mathbf{S}'_r \setminus \mathbf{S}'_{r-1}$ are the level r-vertices of $\mathbf{B}_r$ and $\mathbf{B}'_r$, respectively.

Let $\mathbf{I}_r$ be the event that the $\mathbf{S}_r$ and $\mathbf{S}'_r$ do not intersect. Let τ_r be the function that takes $m+m'$ weighted rooted graphs as input and removes the weight of the vertices at level r.

Fix any level $\ell \in \mathbb{N}$, then there is a coupling of $\tau_{\ell+1}(\mathbf{B}_{\ell+1})$ to $\tilde{\mathbf{B}}_{\ell+1}$ and of $\tau_{\ell+1}(\mathbf{B}'_{\ell+1})$ to $\tilde{\mathbf{B}}'_{\ell+1}$ such that $\tilde{\mathbf{B}}_{\ell+1}$ and $\tilde{\mathbf{B}}'_{\ell+1}$ are conditionally independent on $\mathbf{I}_\ell$ given $\mathbf{B}_\ell, \mathbf{B}'_\ell$.

Furthermore, on $\mathbf{I}_\ell$ *the law of* $\tilde{\mathbf{B}}_{\ell+1}$ *given* $\mathbf{B}_\ell, \mathbf{B}'_\ell$ *is equal to the law of* $\tau_{\ell+1}(\mathbf{B}_{\ell+1})$ *given* $\mathbf{B}_\ell$ *and the law of* $\tilde{\mathbf{B}}'_{\ell+1}$*, given* $\mathbf{B}_\ell, \mathbf{B}'_\ell$ *is equal to the law of* $\tau_{\ell+1}(\mathbf{B}'_{\ell+1})$ *given* $\mathbf{B}'_\ell$*.*

In formulas, for all bounded measurable functions g *and* g' *we have (almost surely)*

$$\begin{aligned}
\mathbb{1}_{\mathbf{I}_\ell} \operatorname{Cov}_n(g(\tilde{\mathbf{B}}_{\ell+1}), g'(\tilde{\mathbf{B}}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) &= 0,\\
\mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g(\tilde{\mathbf{B}}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] &= \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g(\tau_{\ell+1}(\mathbf{B}_{\ell+1})) \mid \mathbf{B}_\ell],\\
\mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g'(\tilde{\mathbf{B}}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] &= \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g'(\tau_{\ell+1}(\mathbf{B}'_{\ell+1})) \mid \mathbf{B}'_\ell].
\end{aligned}$$

Moreover,

$$\mathbb{1}_{\mathbf{I}_\ell} \mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \leq \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|}{n\vartheta}, \tag{3.20}$$

$$\mathbb{1}_{\mathbf{I}_\ell} \mathbb{P}_n(\tilde{\mathbf{B}}'_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \leq \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|}{n\vartheta}. \tag{3.21}$$

Proof. For $\mathbf{T}_1, \mathbf{T}_2 \subseteq V_n$ denote with

$$X(\mathbf{T}_1, \mathbf{T}_2) = \{(e, X_e, X'_e, w_e, w'_e) : e = \{v, u\}, v \in \mathbf{T}_1, u \in \mathbf{T}_2\}$$

the set of edges connecting $\mathbf{T}_1$ and $\mathbf{T}_2$ and all the information potentially associated to those edges in $\mathbf{G}_n$, $\mathbf{G}_n^F$ and $\mathbf{G}_n^{F'}$.

An edge that is new in $\mathbf{B}_{\ell+1}$, i.e. an edge that is present in $\mathbf{B}_{\ell+1}$, but not in $\mathbf{B}_\ell$, must connect a vertex in $\mathbf{D}_\ell$ to a vertex in $V_n \setminus \mathbf{S}_{\ell-1}$. (Let $e' = \{v', u'\}$ be a new edge in $\mathbf{B}_{\ell+1}$. W.l.o.g. we can assume that $v' \in \mathbf{S}_\ell$, because e' must be part of a path of length $\ell + 1$ in $\mathbf{G}_n$ or $\mathbf{G}_n^F$ from some v_i, which implies that one of its endpoints lies on a path of length ℓ from v_i. If $v' \in \mathbf{S}_{\ell-1}$ or $u' \in \mathbf{S}_{\ell-1}$, then it would already follow that $e \in \mathbf{B}_\ell$. In particular $v' \in \mathbf{S}_\ell \setminus \mathbf{S}_{\ell-1} = \mathbf{D}_\ell$ and $u' \in V_n \setminus \mathbf{S}_{\ell-1}$.)

Hence, there exists a function Ψ depending on F such that

$$\tau_{\ell+1}(\mathbf{B}_{\ell+1}) = \Psi(\mathbf{B}_\ell, X(\mathbf{D}_\ell, V_n \setminus \mathbf{S}_{\ell-1})).$$

Intuitively, this function just identifies the edges that are part of $\tau_{\ell+1}(\mathbf{B}_{\ell+1})$, but not of $\mathbf{B}_\ell$ and adds them to $\mathbf{B}_\ell$. Note that for an edge e between $\mathbf{D}_\ell$ and $V_n \setminus \mathbf{S}_{\ell-1}$ with $X_e = X'_e = 0$ the values of w_e and w'_e do not matter for Ψ, because such an edge cannot be part of $\mathbf{B}_{\ell+1}$. In other words, the values w_e and w'_e influence Ψ only if $X_e + X'_e \geq 1$. We call such an edge with $X_e + X'_e \geq 1$ *relevant* for Ψ.

There is a similar function Ψ' depending on F' such that

$$\tau_{\ell+1}(\mathbf{B}'_{\ell+1}) = \Psi'(\mathbf{B}'_\ell, X(\mathbf{D}'_\ell, V_n \setminus \mathbf{S}'_{\ell-1})).$$

Define

$$\begin{aligned}
X_1 &= X(\mathbf{D}_\ell, V_n \setminus (\mathbf{S}_{\ell-1} \cup \mathbf{S}'_\ell)),\\
X_2 &= X(\mathbf{D}'_\ell, V_n \setminus (\mathbf{S}'_{\ell-1} \cup \mathbf{S}_\ell)),\\
X_3 &= X(\mathbf{D}_\ell, \mathbf{D}'_\ell).
\end{aligned}$$

Then X_1 contains the information on all potential new edges of $\mathbf{B}_{\ell+1}$ that do not have an endpoint in $\mathbf{B}'_\ell$. Analogously, X_2 contains the information on all potential new edges of $\mathbf{B}'_{\ell+1}$ that do not have an endpoint in $\mathbf{B}_\ell$. Finally, X_3 contains the information on all potential new edges of $\mathbf{B}_{\ell+1}$ and $\mathbf{B}'_{\ell+1}$ that connect $\mathbf{B}_\ell$ and $\mathbf{B}'_\ell$. By construction

$$X_1 \cup X_3 = X(\mathbf{D}_\ell, V_n \setminus (\mathbf{S}_{\ell-1} \cup \mathbf{S}'_{\ell-1}))$$

and

$$X_2 \cup X_3 = X(\mathbf{D}'_\ell, V_n \setminus (\mathbf{S}'_{\ell-1} \cup \mathbf{S}_{\ell-1})).$$

On the event $\mathbf{I}_\ell$ the set of edges $X(\mathbf{D}_\ell, V_n \setminus \mathbf{S}_{\ell-1})$ coincides with $X_1 \cup X_3$, since there can be no edges between $\mathbf{D}_\ell$ and $\mathbf{S}'_{\ell-1}$, because that would imply that $\mathbf{D}_\ell \subseteq \mathbf{S}_\ell$ and $\mathbf{S}'_\ell$ have a nonempty intersection. An analogous consideration holds for $X_2 \cup X_3$. Hence, on $\mathbf{I}_\ell$ we have

$$\tau_{\ell+1}(\mathbf{B}_{\ell+1}) = \Psi(\mathbf{B}_\ell, X_1 \cup X_3) \quad \text{and} \quad \tau_{\ell+1}(\mathbf{B}'_{\ell+1}) = \Psi'(\mathbf{B}'_\ell, X_2 \cup X_3).$$

On $\mathbf{I}_\ell$ the edge collections X_1 and X_3 are conditionally independent given $\mathbf{B}_\ell, \mathbf{B}'_\ell$. Let $\tilde{X}_1$ be an independent copy of $X(\mathbf{D}_\ell, \mathbf{S}'_{\ell-1})$ and let $\tilde{X}_2$ be an independent copy of $X(\mathbf{D}'_\ell, \mathbf{S}_{\ell-1})$. Finally, let $\tilde{X}_3$ be an independent copy of X_3. Then define

$$\tilde{\mathbf{B}}_{\ell+1} = \Psi(\mathbf{B}_\ell, X_1 \cup \tilde{X}_1 \cup X_3) \quad \text{and} \quad \tilde{\mathbf{B}}'_{\ell+1} = \Psi'(\mathbf{B}'_\ell, X_2 \cup \tilde{X}_2 \cup \tilde{X}_3).$$

Given $\mathbf{B}_\ell, \mathbf{B}'_\ell$ the thus constructed $\tilde{\mathbf{B}}_{\ell+1}$ and $\tilde{\mathbf{B}}'_{\ell+1}$ are conditionally independent on $\mathbf{I}_\ell$. Furthermore, on $\mathbf{I}_\ell$ and conditionally on $\mathbf{B}_\ell, \mathbf{B}'_\ell$ the law of $X_1 \cup \tilde{X}_1 \cup X_3$ is equal to the law of $X(\mathbf{D}_\ell, V_n \setminus \mathbf{S}_{\ell-1})$ given only $\mathbf{B}_\ell$, because $\tilde{X}_1$ provides the 'missing source of randomness' for $X_1 \cup X_3$ when conditioning on both $\mathbf{B}_\ell$ and $\mathbf{B}'_\ell$, where the edges between $\mathbf{D}_\ell$ and $\mathbf{S}'_{\ell-1}$ are fixed, compared to conditioning $\mathbf{B}_{\ell+1}$ only on $\mathbf{B}_\ell$, where these edges are random. Then on $\mathbf{I}_\ell$ the law of $\tilde{\mathbf{B}}_{\ell+1}$ conditionally on $\mathbf{B}_\ell, \mathbf{B}'_\ell$ is the law of $\tau_{\ell+1}(\mathbf{B}_{\ell+1})$ given $\mathbf{B}_\ell$. An analogous result holds for $\tilde{\mathbf{B}}'_{\ell+1}$.

It remains to verify that $\tilde{\mathbf{B}}_{\ell+1}$ differs from $\tau_{\ell+1}(\mathbf{B}_{\ell+1})$ with small probability on $\mathbf{I}_\ell$ conditionally on $\mathbf{B}_\ell$ and $\mathbf{B}'_\ell$. Write

$$\tilde{X}_1 = \{(e, \tilde{X}_e, \tilde{X}'_e, \tilde{w}_e, \tilde{w}'_e) : e = \{u, v\}, v \in \mathbf{D}_\ell, u \in \mathbf{S}'_{\ell-1}\}.$$

By construction $\tilde{\mathbf{B}}_{\ell+1}$ and $\tau_{\ell+1}(\mathbf{B}_{\ell+1})$ differ only if there is an edge e in $\tilde{X}_1$ that is relevant for Ψ, which can only be the case if $\tilde{X}_e = 1$ or $\tilde{X}'_e = 1$. That is the same as saying that there is a path of length 1 between the (fixed) sets of vertices $\mathbf{D}_\ell$ and $\mathbf{S}'_\ell$ in a graph $\tilde{\mathbf{G}}_n$ or $\tilde{\mathbf{G}}'_n$, which are based on $\tilde{X}_e$ and $\tilde{X}'_e$, respectively. Hence, by Lemma 3.1.13

$$\begin{aligned}\mathbb{1}_{\mathbf{I}_\ell} \mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) &\leq \mathbb{1}_{\mathbf{I}_\ell}(\mathbb{P}_n(\mathbf{D}_\ell \overset{\tilde{\mathbf{G}}_n}{\leftrightsquigarrow}_1 \mathbf{S}'_\ell) + \mathbb{P}_n(\mathbf{D}_\ell \overset{\tilde{\mathbf{G}}'_n}{\leftrightsquigarrow}_1 \mathbf{S}'_\ell)) \\ &\leq \mathbb{1}_{\mathbf{I}_\ell} 2 \frac{\|\mathbf{D}_\ell\| \|\mathbf{S}'_{\ell-1}\|}{n\vartheta} \\ &\leq \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|}{n\vartheta}. \end{aligned} \tag{3.22}$$

Similarly we want to show that $\tilde{\mathbf{B}}'_{\ell+1}$ differs from $\tau_{\ell+1}(\mathbf{B}'_{\ell+1})$ with small probability on $\mathbf{I}_\ell$ conditionally on $\mathbf{B}_\ell$ and $\mathbf{B}'_\ell$. Write

$$\tilde{X}_2 = \{(e, \tilde{X}_e, \tilde{X}'_e, \tilde{w}_e, \tilde{w}'_e) : e = \{u, v\}, v \in \mathbf{D}'_\ell, u \in \mathbf{S}_{\ell-1}\}$$

and

$$\tilde{X}_3 = \{(e, \tilde{X}_e, \tilde{X}'_e, \tilde{w}_e, \tilde{w}'_e) : e = \{u, v\}, v \in \mathbf{D}_\ell, u \in \mathbf{D}'_\ell\}.$$

By construction $\tilde{\mathbf{B}}'_{\ell+1}$ and $\tau_{\ell+1}(\mathbf{B}'_{\ell+1})$ differ only if

- there is an edge e in $\tilde{X}_2$ that is relevant for Ψ' or
- there is an edge e in X_3 that differs in a relevant way between X_3 and $\tilde{X}_3$.

  An edge e differs in a relevant way between X_3 and $\tilde{X}_3$ if (X_e, X'_e, w_e, w'_e) differs from $(\tilde{X}_e, \tilde{X}'_e, \tilde{w}_e, \tilde{w}'_e)$, unless all of X_e, X'_e, $\tilde{X}_e$ and $\tilde{X}'_e$ are equal to zero, because that would mean that the edge is not relevant for Ψ'. In particular an edge can only differ in a relevant way if at least one of X_e, X'_e, $\tilde{X}_e$ or $\tilde{X}'_e$ is equal to one.

Hence, as in (3.22) Lemma 3.1.13 shows

$$\begin{aligned} \mathbb{1}_{\mathbf{I}_\ell} \mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) &\leq \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{D}_\ell\| \|\mathbf{S}'_{\ell-1}\|}{n\vartheta} + \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{D}_\ell\| \|\mathbf{D}'_\ell\|}{n\vartheta} \\ &\leq \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|}{n\vartheta}. \end{aligned}$$

This finishes the proof. □

We now add the missing weights to the vertices at level $\ell + 1$. Again the new weights are independent if no vertex appears for both sets of root vertices. If a vertex is needed for both root vertex sets, its weight can be rerandomised to still obtain independent random variables for both sets. As long as the number of rerandomised vertex weights is not too large, the independent versions of the neighbourhoods differ from the original $\ell + 1$-neighbourhoods with small enough probability.

Lemma 3.2.2. *Let $\mathbf{B}_r$, $\mathbf{B}'_r$, $\mathbf{S}_r$, $\mathbf{S}'_r$, $\mathbf{D}_r$, $\mathbf{D}'_r$, $\mathbf{I}_r$ and τ_r be as in Lemma 3.2.1.*

Fix any level $\ell \in \mathbb{N}$, then there is a coupling of $\bar{\mathbf{B}}_{\ell+1}$ with $\mathbf{B}_{\ell+1}$ and of $\mathbf{B}'_{\ell+1}$ with $\bar{\mathbf{B}}'_{\ell+1}$ such that $\bar{\mathbf{B}}_{\ell+1}$ and $\bar{\mathbf{B}}'_{\ell+1}$ are conditionally independent given $\mathbf{B}_\ell, \mathbf{B}'_\ell$ on $\mathbf{I}_\ell$. Furthermore, the law of $\bar{\mathbf{B}}_{\ell+1}$ given $\mathbf{B}_\ell, \mathbf{B}'_\ell$ is equal to the law of $\mathbf{B}_{\ell+1}$ given $\mathbf{B}_\ell$ and analogously for $\bar{\mathbf{B}}'_{\ell+1}$.

In formulas, for all functions g and g' we have almost surely

$$\begin{aligned} \mathbb{1}_{\mathbf{I}_\ell} \operatorname{Cov}_n(g(\bar{\mathbf{B}}_{\ell+1}), g'(\bar{\mathbf{B}}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) &= 0, \\ \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g(\bar{\mathbf{B}}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] &= \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell], \\ \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g'(\bar{\mathbf{B}}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] &= \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]. \end{aligned}$$

Moreover,

$$\mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\bar{\mathbf{B}}_{\ell+1} \neq \mathbf{B}_{\ell+1} \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \leq \mathbb{1}_{\mathbf{I}_\ell} C(1+\Gamma_{2,n})\frac{\|\mathbf{S}_\ell\|\|\mathbf{S}'_\ell\|}{n\vartheta},$$
$$\mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\bar{\mathbf{B}}'_{\ell+1} \neq \mathbf{B}'_{\ell+1} \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \leq \mathbb{1}_{\mathbf{I}_\ell} C(1+\Gamma_{2,n})\frac{\|\mathbf{S}_\ell\|\|\mathbf{S}'_\ell\|}{n\vartheta}.$$

Proof. Let $\mathbf{B}_{\ell+1}$, $\mathbf{B}'_{\ell+1}$, $\tilde{\mathbf{B}}_{\ell+1}$ and $\tilde{\mathbf{B}}'_{\ell+1}$) be as in Lemma 3.2.1. Let $\tilde{\mathbf{S}}_\ell$ be the union of the vertex sets of the constituent graphs of $\tilde{\mathbf{B}}_\ell$ and similarly $\tilde{\mathbf{S}}'_\ell$ be the corresponding vertex set of $\tilde{\mathbf{B}}'_\ell$. Define $\tilde{\mathbf{D}}_\ell = \tilde{\mathbf{S}}_\ell \setminus \tilde{\mathbf{S}}_{\ell-1}$ and $\tilde{\mathbf{D}}'_\ell = \tilde{\mathbf{S}}'_\ell \setminus \tilde{\mathbf{S}}'_{\ell-1}$.

Construct $\bar{\mathbf{B}}_{\ell+1}$ from $\tilde{\mathbf{B}}_{\ell+1}$ by adding the remaining vertex weights at level $\ell+1$ as follows

$$(\bar{w}_u, \bar{w}'_u) = \begin{cases} (\tilde{w}_u, \tilde{w}'_u) & u \in \tilde{\mathbf{S}}'_\ell \\ (w_u, w'_u) & u \notin \tilde{\mathbf{S}}'_\ell \end{cases} \quad u \in \tilde{\mathbf{D}}_{\ell+1}$$

and

$$(\bar{w}_u, \bar{w}'_u) = \begin{cases} (\tilde{w}_u, \tilde{w}'_u) & u \in \tilde{\mathbf{S}}_\ell \cup \tilde{\mathbf{D}}_{\ell+1} \\ (w_u, w'_u) & u \notin \tilde{\mathbf{S}}_\ell \cup \tilde{\mathbf{D}}_{\ell+1} \end{cases} \quad u \in \tilde{\mathbf{D}}'_{\ell+1},$$

where $(\tilde{W}, \tilde{W}')$ is an i.i.d. copy of (W, W').

Let $\bar{\Psi}$ be the function such that

$$\bar{\mathbf{B}}_{\ell+1} = \bar{\Psi}(\tilde{\mathbf{B}}_{\ell+1}, \tilde{\mathbf{B}}'_{\ell+1}, \tilde{\mathbf{W}}), \tag{3.23}$$

where $\tilde{\mathbf{W}}$ is the collection of random variables $(w_u, w'_u, \tilde{w}_u, \tilde{w}'_u)_{u \in V_n \setminus (\tilde{\mathbf{S}}_\ell \cup \tilde{\mathbf{S}}'_\ell)}$. The function $\bar{\Psi}$ endows $\tilde{\mathbf{B}}_{\ell+1}$ with weights on the $\ell+1$-level vertices from $\tilde{\mathbf{W}}$ and chooses (w_u, w'_u) or $(\tilde{w}_u, \tilde{w}'_u)$ according to the status of u in $\tilde{\mathbf{B}}'_{\ell+1}$. Because the alternatives (w_u, w'_u) and $(\tilde{w}_u, \tilde{w}'_u)$ have the same distribution and are both independent of $\tilde{\mathbf{B}}_{\ell+1}$, the realisation of $\tilde{\mathbf{B}}'_{\ell+1}$ does not matter for the *distribution* of the resulting object. This implies that for all realisations $\mathbf{b}'$ of $\tilde{\mathbf{B}}'_\ell$ we have

$$\bar{\mathbf{B}}_{\ell+1} \overset{\mathcal{D}}{=} \bar{\Psi}(\tilde{\mathbf{B}}_{\ell+1}, \mathbf{b}', \mathbf{W}), \tag{3.24}$$

where $\mathbf{W}$ is a collection of independent random variables with the same distribution as $(w_u, w'_u, \tilde{w}_u, \tilde{w}'_u)$. In particular this also holds if $\mathbf{b}'$ is empty. This equality in distribution also holds conditional on $\tilde{\mathbf{B}}_{\ell+1}$ and $\tilde{\mathbf{B}}'_{\ell+1}$. For the same reasons, the function also satisfies the following distributional equality

$$\mathbf{B}_{\ell+1} \overset{\mathcal{D}}{=} \bar{\Psi}(\tau_{\ell+1}(\mathbf{B}_{\ell+1}), \emptyset, \mathbf{W}). \tag{3.25}$$

The construction of $\bar{\mathbf{B}}_{\ell+1}$ and $\bar{\mathbf{B}}_{\ell+1}$ ensures that on $\mathbf{I}_\ell$ each vertex weight occurs only in one of $\bar{\mathbf{B}}_{\ell+1}$ or $\bar{\mathbf{B}}'_{\ell+1}$ and the decision where it occurs is deterministic given $\tilde{\mathbf{B}}_{\ell+1}$ and $\tilde{\mathbf{B}}'_{\ell+1}$. Thus $\bar{\mathbf{B}}_{\ell+1}$ and $\bar{\mathbf{B}}'_{\ell+1}$ are conditionally independent on $\mathbf{I}_\ell$ given $\tilde{\mathbf{B}}_{\ell+1}$ and $\tilde{\mathbf{B}}'_{\ell+1}$. In particular

$$\mathbb{1}_{\mathbf{I}_\ell} \operatorname{Cov}_n(g(\bar{\mathbf{B}}_{\ell+1}), g'(\bar{\mathbf{B}}'_{\ell+1}) \mid \tilde{\mathbf{B}}_{\ell+1}, \tilde{\mathbf{B}}'_{\ell+1}) = 0. \tag{3.26}$$

Furthermore, the observations (3.23) to (3.25) about $\bar{\Psi}$ show that we have for all functions g almost surely that

$$\begin{aligned}
\mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[g(\bar{\mathbf{B}}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] &= \mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[\mathbb{E}_n[g(\bar{\Psi}(\tilde{\mathbf{B}}_{\ell+1}, \tilde{\mathbf{B}}'_{\ell+1}, \tilde{\mathbf{W}})) \mid \tilde{\mathbf{B}}_{\ell+1}, \tilde{\mathbf{B}}'_{\ell+1}] \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] \\
&= \mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[\mathbb{E}_n[g(\bar{\Psi}(\tilde{\mathbf{B}}_{\ell+1}, \varnothing, \mathbf{W})) \mid \tilde{\mathbf{B}}_{\ell+1}, \tilde{\mathbf{B}}'_{\ell+1}] \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] \\
&= \mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[\mathbb{E}_n[g(\bar{\Psi}(\tilde{\mathbf{B}}_{\ell+1}, \varnothing, \mathbf{W})) \mid \tilde{\mathbf{B}}_{\ell+1}] \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] \\
&= \mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[g(\bar{\Psi}(\tilde{\mathbf{B}}_{\ell+1}, \varnothing, \mathbf{W})) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell] \\
&= \mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[g(\bar{\Psi}(\tau_{\ell+1}(\mathbf{B}_{\ell+1}), \varnothing, \mathbf{W})) \mid \mathbf{B}_\ell] \\
&= \mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell].
\end{aligned} \tag{3.27}$$

Analogous we also have

$$\mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[g'(\bar{\mathbf{B}}'_{\ell+1}) \mid \tilde{\mathbf{B}}_{\ell+1}, \tilde{\mathbf{B}}'_{\ell+1}] = \mathbb{1}_{\mathbf{I}_\ell}\mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]. \tag{3.28}$$

It remains to show that the probability that $\mathbf{B}_{\ell+1}$ and $\bar{\mathbf{B}}_{\ell+1}$ differ can be controlled as claimed. By construction $\mathbf{B}_{\ell+1}$ and $\bar{\mathbf{B}}_{\ell+1}$ differ only if the underlying edge structures $\tau(\mathbf{B}_{\ell+1})$ and $\tilde{\mathbf{B}}_{\ell+1}$ differ or if the underlying edge structures are the same, but the vertex weights differ in a relevant way due to rerandomisation. Vertex weights have to be rerandomised if $\tilde{\mathbf{D}}_{\ell+1}$ has a non-empty intersection with $\tilde{\mathbf{S}}'_\ell$. Hence,

$$\begin{aligned}
&\mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\mathbf{B}_{\ell+1} \neq \bar{\mathbf{B}}_{\ell+1} \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\quad \le \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\qquad + \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} = \tau_{\ell+1}(\mathbf{B}_{\ell+1}), \tilde{\mathbf{D}}_{\ell+1} \cap \tilde{\mathbf{S}}'_\ell \neq \varnothing \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\quad \le \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) + \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}'_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\qquad + \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} = \tau_{\ell+1}(\mathbf{B}_{\ell+1}), \tilde{\mathbf{B}}'_{\ell+1} = \tau_{\ell+1}(\mathbf{B}'_{\ell+1}), \tilde{\mathbf{D}}_{\ell+1} \cap \tilde{\mathbf{S}}'_\ell \neq \varnothing \mid \mathbf{B}_\ell, \mathbf{B}'_\ell).
\end{aligned}$$

The first and second term can be estimated by (3.20) and (3.21) from Lemma 3.2.1. In the third term we can replace $\tilde{\mathbf{D}}_{\ell+1}$ with $\mathbf{D}_{\ell+1}$ and $\tilde{\mathbf{S}}'_\ell$ with $\mathbf{S}'_\ell$ because the edge structures of $\tilde{\mathbf{B}}_{\ell+1}$ and $\mathbf{B}_{\ell+1}$ are the same, then the probability that $\mathbf{D}_{\ell+1}$ and $\mathbf{S}'_\ell$ intersect is given by the probability that there is an edge between $\mathbf{D}_\ell$ and $\mathbf{S}'_\ell$ so that by Lemma 3.1.13

$$\begin{aligned}
&\le \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|}{n\vartheta} + \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{D}_\ell\| \|\mathbf{S}'_\ell\|}{n\vartheta} \\
&\le \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|}{n\vartheta}.
\end{aligned}$$

The probability that $\bar{\mathbf{B}}'_{\ell+1}$ and $\mathbf{B}'_{\ell+1}$ differ can be estimated similarly, taking into account that rerandomisation of vertex weights happens additionally if $\tilde{\mathbf{D}}_{\ell+1}$ and $\tilde{\mathbf{D}}'_{\ell+1}$

have non-empty intersection, which is the case if there is a path consisting of two edges connecting $\mathbf{D}_\ell$ with $\mathbf{D}'_\ell$. Then Lemma 3.2.1 and Lemma 3.1.13 imply

$$\begin{aligned}
&\mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\mathbf{B}'_{\ell+1} \neq \bar{\mathbf{B}}'_{\ell+1} \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\quad \leq \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}'_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) + \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} \neq \tau_{\ell+1}(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\qquad + \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} = \tau_{\ell+1}(\mathbf{B}_{\ell+1}), \tilde{\mathbf{B}}'_{\ell+1} = \tau_{\ell+1}(\mathbf{B}'_{\ell+1}), \tilde{\mathbf{D}}'_{\ell+1} \cap \tilde{\mathbf{S}}_\ell \neq \emptyset \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\qquad + \mathbb{1}_{\mathbf{I}_\ell}\mathbb{P}_n(\tilde{\mathbf{B}}_{\ell+1} = \tau_{\ell+1}(\mathbf{B}_{\ell+1}), \tilde{\mathbf{B}}'_{\ell+1} = \tau_{\ell+1}(\mathbf{B}'_{\ell+1}), \tilde{\mathbf{D}}_{\ell+1} \cap \tilde{\mathbf{D}}'_{\ell+1} \neq \emptyset \mid \mathbf{B}_\ell, \mathbf{B}'_\ell) \\
&\quad \leq \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{S}_\ell\|\,\|\mathbf{S}'_\ell\|}{n\vartheta} + \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{D}'_\ell\|\,\|\mathbf{S}_\ell\|}{n\vartheta} + \mathbb{1}_{\mathbf{I}_\ell} C \frac{\|\mathbf{D}'_\ell\|\,\|\mathbf{D}_\ell\|}{n\vartheta}\Gamma_{2,n} \\
&\quad \leq \mathbb{1}_{\mathbf{I}_\ell} C(1+\Gamma_{2,n}) \frac{\|\mathbf{S}_\ell\|\,\|\mathbf{S}'_\ell\|}{n\vartheta}.
\end{aligned}$$

This completes the proof. □

Thanks to the previous constructions the covariance between $\mathbf{B}_{\ell+1}$ and $\mathbf{B}'_{\ell+1}$ can be bounded by a term involving the covariance between $\mathbf{B}_\ell$ and $\mathbf{B}'_\ell$ and an error term.

Lemma 3.2.3. *Let g and g' be measurable functions that are bounded by 1 in absolute value. Then*

$$\begin{aligned}
&\mathrm{Cov}_n(g(\mathbf{B}_{\ell+1}), g'(\mathbf{B}'_{\ell+1})) \\
&\quad \leq C\Big(\mathbb{P}_n(\mathbf{I}^c_\ell) + (1+\Gamma_{2,n})\frac{\mathbb{E}_n[\|\mathbf{S}_\ell\|\,\|\mathbf{S}'_\ell\|]}{n\vartheta}\Big) \\
&\qquad + \mathrm{Cov}_n(\mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell], \mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]).
\end{aligned}$$

Write $\bar{g}(\mathbf{B}_\ell) = \mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell]$ and $\bar{g}'(\mathbf{B}'_\ell) = \mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]$, then

$$\begin{aligned}
&\mathrm{Cov}_n(g(\mathbf{B}_{\ell+1}), g'(\mathbf{B}'_{\ell+1})) \\
&\quad \leq C\Big(\mathbb{P}_n(\mathbf{I}^c_\ell) + (1+\Gamma_{2,n})\frac{\mathbb{E}_n[\|\mathbf{S}_\ell\|\,\|\mathbf{S}'_\ell\|]}{n\vartheta}\Big) + \mathrm{Cov}_n(\bar{g}(\mathbf{B}_\ell), \bar{g}'(\mathbf{B}'_\ell)).
\end{aligned}$$

Proof. First split the covariance over $\mathbf{I}_\ell$

$$\begin{aligned}
&\mathrm{Cov}_n(g(\mathbf{B}_{\ell+1}), g'(\mathbf{B}'_{\ell+1})) \\
&\quad = \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\mathbf{B}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell} g'(\mathbf{B}'_{\ell+1})) + \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}^c_\ell} g(\mathbf{B}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell} g'(\mathbf{B}'_{\ell+1})) \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\mathbf{B}_{\ell+1}), \mathbb{1}_{\mathbf{I}^c_\ell} g'(\mathbf{B}'_{\ell+1})) + \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}^c_\ell} g(\mathbf{B}_{\ell+1}), \mathbb{1}_{\mathbf{I}^c_\ell} g'(\mathbf{B}'_{\ell+1})).
\end{aligned}$$

The covariances containing at least one factor $\mathbb{1}_{\mathbf{I}^c_\ell}$ can be bounded by bounding all other terms in the expectation making up the covariance by 1 and retaining only $\mathbb{P}_n(\mathbf{I}^c_\ell)$. Thus

$$\mathrm{Cov}_n(g(\mathbf{B}_{\ell+1}), g'(\mathbf{B}'_{\ell+1})) \leq \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\mathbf{B}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell} g'(\mathbf{B}_{\ell+1})) + C\mathbb{P}_n(\mathbf{I}^c_\ell). \tag{3.29}$$

Approximate $\mathbf{B}_{\ell+1}$ with $\bar{\mathbf{B}}_{\ell+1}$ and $\mathbf{B}'_{\ell+1}$ with $\bar{\mathbf{B}}'_{\ell+1}$ on $\mathbf{I}_\ell$

$$\begin{aligned}
&\mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\mathbf{B}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell} g'(\mathbf{B}'_{\ell+1})) \\
&\quad = \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\bar{\mathbf{B}}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell} g'(\bar{\mathbf{B}}'_{\ell+1})) \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell}(g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})), \mathbb{1}_{\mathbf{I}_\ell} g'(\bar{\mathbf{B}}'_{\ell+1})) \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\bar{\mathbf{B}}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell}(g'(\mathbf{B}'_{\ell+1}) - g'(\bar{\mathbf{B}}'_{\ell+1}))) \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell}(g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})), \mathbb{1}_{\mathbf{I}_\ell}(g'(\mathbf{B}'_{\ell+1}) - g'(\bar{\mathbf{B}}'_{\ell+1}))).
\end{aligned} \tag{3.30}$$

By the law of total covariance and using the fact that $\mathbb{1}_{\mathbf{I}_\ell}$ is $(\mathbf{B}_\ell, \mathbf{B}'_\ell)$-measurable the first term in the right-hand side of (3.30) is equal to

$$\begin{aligned}
&\mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\bar{\mathbf{B}}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell} g'(\bar{\mathbf{B}}'_{\ell+1})) \\
&\quad = \mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell} \mathrm{Cov}_n(g(\bar{\mathbf{B}}_{\ell+1}), g'(\bar{\mathbf{B}}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell)] \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g(\bar{\mathbf{B}}_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell], \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g'(\bar{\mathbf{B}}'_{\ell+1}) \mid \mathbf{B}_\ell, \mathbf{B}'_\ell]).
\end{aligned}$$

By Lemma 3.2.2 the first expectation vanishes and the conditional expectations in the covariance can be rewritten based on $\mathbf{B}_{\ell+1}$ and $\mathbf{B}'_{\ell+1}$ so that we obtain

$$= \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell], \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]). \tag{3.31}$$

The indicator in the covariance can be dropped at the cost of adding $C\mathbb{P}_n(\mathbf{I}_\ell^c)$ by the reverse of the argument we used above to introduce it. Hence, (3.31) implies

$$\begin{aligned}
&\mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} g(\bar{\mathbf{B}}_{\ell+1}), \mathbb{1}_{\mathbf{I}_\ell} g'(\bar{\mathbf{B}}_{\ell+1})) \\
&\quad = \mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell], \mathbb{1}_{\mathbf{I}_\ell} \mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]) \\
&\quad \le \mathrm{Cov}_n(\mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell], \mathbb{E}_n[g'(\mathbf{B}_{\ell+1}) \mid \mathbf{B}'_\ell]) + C\mathbb{P}_n(\mathbf{I}_\ell^c).
\end{aligned} \tag{3.32}$$

For the second term on the right-hand side of (3.30) the triangle inequality implies

$$\begin{aligned}
&|\mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell}(g(\mathbf{B}_{\ell+1}) - g_v(\bar{\mathbf{B}}_{\ell+1})), \mathbb{1}_{\mathbf{I}_\ell} g'(\bar{\mathbf{B}}'_{\ell+1}))| \\
&\quad \le |\mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell}(g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})) g'(\bar{\mathbf{B}}'_{\ell+1})]| \\
&\qquad + |\mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell}(g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1}))] \mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell} g'(\bar{\mathbf{B}}'_{\ell+1})]|.
\end{aligned} \tag{3.33}$$

Consider the first term and use that g' is bounded by 1 to bound the whole expectation

$$\begin{aligned}
|\mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell}(g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})) g'(\bar{\mathbf{B}}'_{\ell+1})]| &\le \mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell} |g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})| |g'(\bar{\mathbf{B}}'_{\ell+1})|] \\
&\le \mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell} |g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})|].
\end{aligned}$$

Use the tower property to condition on $\mathbf{B}_\ell, \mathbf{B}'_\ell$ and that $|g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})|$ is bounded above by 2, then apply Lemma 3.2.2 to bound the expectation further by

$$\begin{aligned}
&\le 2\mathbb{E}_n[\mathbb{1}_{\mathbf{I}_\ell} \mathbb{P}_n(\mathbf{B}_{\ell+1} \neq \bar{\mathbf{B}}_{\ell+1} \mid \mathbf{B}_\ell, \mathbf{B}'_\ell)] \\
&\le C(1 + \Gamma_{2,n}) \frac{\mathbb{E}_n[\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|]}{n\vartheta}.
\end{aligned}$$

The second term in (3.33) can be bounded similarly. Thus

$$|\mathrm{Cov}_n(\mathbb{1}_{\mathbf{I}_\ell}(g(\mathbf{B}_{\ell+1}) - g(\bar{\mathbf{B}}_{\ell+1})), \mathbb{1}_{\mathbf{I}_\ell} g'(\bar{\mathbf{B}}'_{\ell+1}))| \le C(1+\Gamma_{2,n}) \frac{\mathbb{E}_n[\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|]}{n\vartheta}. \tag{3.34}$$

The remaining terms in (3.30) can be bounded analogously.

Combine (3.29), (3.30) and (3.32) and (3.34) and the analogous results for the remaining terms to obtain the claimed bound.

□

The following lemma establishes bounds for the error terms from Lemma 3.2.3.

Lemma 3.2.4. *For $\ell \in \mathbb{N}$ we have*

$$\mathbb{P}_n(\mathbf{I}_\ell^c) \le \frac{\sum_{i,i'} W_{v_i} W_{v'_{i'}}}{n\vartheta} 2^{2\ell}(1+\Gamma_{2,n})^{2\ell} \tag{3.35}$$

and

$$\mathbb{E}_n[\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|] \le C \sum_{i,i'} (W_{v_i}+1)(W_{v'_{i'}}+1)(\Gamma_{3,n}+1)(\Gamma_{2,n}+2)^{2\ell}. \tag{3.36}$$

Proof. Recall that $\mathbf{I}_\ell$ is the event that $\mathbf{S}_\ell$ and $\mathbf{S}'_\ell$ intersect, where $\mathbf{S}_\ell$ is the set of vertices of neighbourhoods $B_\ell(v_i, \mathbf{G}_n)$ and $B_\ell(v_i, \mathbf{G}_n^F)$ for $i \in [m]$ and $\mathbf{S}'_\ell$ the set of vertices of of neighbourhoods $B_\ell(v'_{i'}, \mathbf{G}_n)$ and $B_\ell(v'_{i'}, \mathbf{G}_n^{F'})$ for $i' \in [m']$.

Fix $i \in [m], i' \in [m']$. If one of $B_\ell(v_i, \mathbf{G}_n)$ or $B_\ell(v_i, \mathbf{G}_n^F)$ intersects $B_\ell(v'_{i'}, \mathbf{G}_n)$ of $B_\ell(v'_{i'}, \mathbf{G}_n^{F'})$, then there exists a path of length 2ℓ from v_i to $v'_{i'}$ in a graph $\bar{\mathbf{G}}_n$ where an edge e is present if X_e or an independent copy X'_e is equal to 1. In particular the edge probability for $e = \{u, v\}$ in $\bar{\mathbf{G}}_n$ can be bounded by $2W_u W_v/(n\vartheta)$. Hence, the calculations for Corollary 3.1.15 imply that the probability of intersection is bounded by

$$\frac{W_{v_i} W_{v'_{i'}}}{n\vartheta} 2^{2\ell}(\Gamma_{2,n}+1)^{2\ell}.$$

Now sum over $i \in [m]$ and $i' \in [m']$ to obtain the first claim.

For the second inequality let $S_\ell(v_i)$ be the vertex set of of $B_\ell(v_i, \mathbf{G}_n)$ and let $S_\ell^F(v_i)$ the vertex set of $B_\ell(v_i, \mathbf{G}_n^F)$, similarly let $S_\ell(v'_i)$ be the vertex set of of $B_\ell(v'_i, \mathbf{G}_n)$ and $S_\ell^{F'}(v'_i)$ the vertex set of $B_\ell(v'_i, \mathbf{G}_n^F)$. Then Cauchy–Schwarz and Lemma 3.1.8 imply

$$\begin{aligned}\mathbb{E}_n[\|S_\ell(v_i)\| \|S_\ell(v'_{i'})\|] &\le \mathbb{E}_n[\|S_\ell(v)\|^2]^{1/2} \mathbb{E}_n[\|S_\ell(v'_{i'})\|^2]^{1/2} \\ &\le C(W_{v_i}+1)(W_{v'_{i'}}+1)(\Gamma_{3,n}+1)(\Gamma_{2,n}+2)^{2\ell}\end{aligned}$$

and the same bound for $S_\ell^F(v_i)$ instead of $S_\ell(v_i)$ or $S_\ell^{F'}(v'_{i'})$ instead of $S_\ell(v'_{i'})$. Now use that

$$\|\mathbf{S}_\ell\| \le \sum_{i=1}^m \|S_\ell(v_i)\| + \sum_{i=1}^m \|S_\ell^F(v_i)\|$$

and

$$\|\mathbf{S}'_\ell\| \leq \sum_{i'=1}^{m'} \|S_\ell(v'_{i'})\| + \sum_{i'=1}^{m'} \|S_\ell^{F'}(v'_{i'})\|$$

to obtain the second claim. □

Together the previous results establish a bound of order n^{-1} for the covariance between $\mathbf{B}_\ell$ and $\mathbf{B}'_\ell$.

Lemma 3.2.5. *For any $\ell \in \mathbb{N}$*

$$\sup_{g,g'} \operatorname{Cov}_n(g(\mathbf{B}_\ell), g'(\mathbf{B}'_\ell)) \leq \min\left\{ \frac{\sum_{i,i'} (W_{v_i}+1)(W_{v'_{i'}}+1)}{n\vartheta} (\Gamma_{3,n}+1)(\Gamma_{2,n}+C)^{2\ell+1}, 1 \right\},$$

where the supremum is taken over all functions g and g' that are bounded by 1. If we set $\mathcal{V} = \{v_1, \dots, v_m\}$ and $\mathcal{V}' = \{v'_1, \dots, v'_{m'}\}$, the result can be rewritten as

$$\begin{aligned}
&\sup_{g,g'} \operatorname{Cov}_n(g(\mathbf{B}_\ell), g'(\mathbf{B}'_\ell)) \\
&\quad \leq \min\left\{ \frac{(\|\mathcal{V}\| + |\mathcal{V}|)(\|\mathcal{V}'\| + |\mathcal{V}'|)}{n\vartheta} (\Gamma_{3,n}+1)(\Gamma_{2,n}+C)^{2\ell+1}, 1 \right\}.
\end{aligned}$$

Proof. Apply first Lemma 3.2.3 then use Lemma 3.2.4 to estimate the non-covariance 'error terms'. This gives the bound

$$\begin{aligned}
&\operatorname{Cov}_n(g(\mathbf{B}_{\ell+1}), g'(\mathbf{B}'_{\ell+1})) \\
&\leq C\left(\mathbb{P}_n(\mathbf{I}_\ell^c) + (1+\Gamma_{2,n}) \frac{\mathbb{E}_n[\|\mathbf{S}_\ell\| \|\mathbf{S}'_\ell\|]}{n\vartheta} \right) \\
&\quad + \operatorname{Cov}_n(\mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell], \mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]) \\
&\leq C\left(\frac{\sum_{i,i'} W_{v_i} W_{v'_{i'}}}{n\vartheta} 2^{2\ell}(1+\Gamma_{2,n})^{2\ell} + \frac{\sum_{i,i'} (W_{v_i}+1)(W_{v'_{i'}}+1)}{n\vartheta} (\Gamma_{2,n}+1)^{2\ell}(\Gamma_{3,n}+1) \right) \\
&\quad + \operatorname{Cov}_n(\mathbb{E}_n[g(\mathbf{B}_{\ell+1}) \mid \mathbf{B}_\ell], \mathbb{E}_n[g'(\mathbf{B}'_{\ell+1}) \mid \mathbf{B}'_\ell]).
\end{aligned}$$

Since $\mathbb{E}[g(\mathbf{B}_{\ell+1}) \,|\, \mathbf{B}_\ell]$ can be written as $\bar{g}(\mathbf{B}_\ell)$, where $\bar{g}_v$ is a measurable function that is bounded by 1, and similarly for g' with a function $\bar{g}'$ the term can be rewritten as a covariance of functions applied to $\mathbf{B}_\ell$ and $\mathbf{B}'_\ell$.

$$\begin{aligned}
&\leq C\left(\frac{\sum_{i,i'} W_{v_i} W_{v'_{i'}}}{n\vartheta} 2^{2\ell}(1+\Gamma_{2,n})^{2\ell} + \frac{\sum_{i,i'} (W_{v_i}+1)(W_{v'_{i'}}+1)}{n\vartheta} (\Gamma_{2,n}+1)^{2\ell}(\Gamma_{3,n}+1) \right) \\
&\quad + \operatorname{Cov}_n(\bar{g}(\mathbf{B}_\ell), \bar{g}'(\mathbf{B}'_\ell)) \\
&\leq \frac{\sum_{i,i'} (W_{v_i}+1)(W_{v'_{i'}}+1)}{n\vartheta} (\Gamma_{3,n}+1)(\Gamma_{2,n}+C)^{2\ell+1} + \operatorname{Cov}_n(\bar{g}(\mathbf{B}_\ell), \bar{g}'(\mathbf{B}_\ell)).
\end{aligned}$$

The claim follows by taking the supremum over all measurable bounded functions (first on the right-hand side and then on the left-hand side) and iteration. □

3.3 Graph exploration

We now define a procedure that allows us to explore the neighbourhood of a vertex in a graph. This procedure can be applied to arbitrary graphs, so for the remainder of this section, we shall not restrict ourselves to the sparse inhomogeneous graph setting and will work on a general graph $G = (V, E)$. The presentation of the graph exploration in this section is based on the formulation in lecture notes by Bordenave [Bor16, § 3.5.1]. The approach is also discussed by van der Hofstad [Hof18, § 4.1] who draws on work by Alon and Spencer [AS00, § 10.5]. In those discussions, however, the focus is on the cardinality of the connected component of a vertex v and not on the complete neighbourhood structure.

Fix a graph $G = (V, E)$ with vertex set V and edge set $E \subseteq \{\{u, v\} : u, v \in V\}$. The idea of the exploration on G is to discover which of the connections that could possibly be present in a graph with vertex set V are actually present in G. To this end let $V^{(2)} = \{\{u, v\} : u, v \in V\}$ be the set of edges in the complete graph on V and call elements of $V^{(2)}$ *possible edges of* G. The graph G is then completely determined by the edge indicators $(\mathbb{1}_E(e))_{e\in V^{(2)}}$ that tell us whether a possible edge $e \in V^{(2)}$ is present in the edge set E of G. Because we explore G by visiting vertices, it is slightly more convenient to think of these edge indicators as being indexed by pairs of vertices

$$X_{uv} = \mathbb{1}_E(\{u, v\}) \quad \text{for } u, v \in V.$$

To make notation a bit easier we will also define $X_{vv} = 0$ for all $v \in V$.

Formally, the graph exploration of G started in a vertex $v_0 \in V$ is given by a sequence of vertices $v_0, v_1, v_2, v_3, \dots \in V$ along with sets C_j, A_j and U_j as well as a function $\varphi\colon S \to G$, where $S \subseteq \mathcal{U}$ is a subtree of the Ulam–Harris tree (cf. Definition 2.1.5).

Algorithm 3.3.1. Start with a fixed vertex v_0 in G and set $C_{-1} = \emptyset$, $A_{-1} = \{v_0\}$, $U_{-1} = V \setminus \{v_0\}$ and on the Ulam–Harris side with $\mathbf{i}_0 = \varnothing$. Set $\varphi(\mathbf{i}_0) = \varphi(\varnothing) = v_0$.

For $j \in \{0, 1, \dots\}$ given C_{j-1}, A_{j-1} and U_{j-1} let $v_j = \varphi(\mathbf{i}_j)$ be the smallest element in A_{j-1} ('smallest' in the sense that its preimage $\mathbf{i}_j$ under φ is minimal in the order $\prec$ on the Ulam–Harris tree). Define $I_j = \{u \in U_{j-1} : X_{v_j u} = 1\}$ and let

$$\begin{aligned} C_j &= C_{j-1} \cup \{v_j\}, \\ A_j &= A_{j-1} \setminus \{v_j\} \cup I_j \quad \text{and} \\ U_j &= U_{j-1} \setminus I_j. \end{aligned}$$

Then set $N_{\mathbf{i}_j} = |I_j|$, enumerate the elements of I_j as $\{u_1, \dots, u_{N_{\mathbf{i}_j}}\}$ (if we want the exploration to always yield the same results, we need to impose an order on the vertices in the set, in our applications we can always assume that $V = V_n = [n]$ and use the natural order on $\mathbb{N}$) and extend φ by setting $\varphi((\mathbf{i}_j, 1)) = u_1, \dots, \varphi((\mathbf{i}_j, N_{\mathbf{i}_j})) = u_{N_{\mathbf{i}_j}}$ so that the image of φ now also covers all of I_j.

The exploration stops if $A_j = \emptyset$.

The set C_j can be seen as the set of explored vertices for which all neighbours have been seen, A_j is the set of active vertices (i.e. vertices that have been seen by the exploration, but whose neighbours may not all have been seen yet) and U_j the set of unexplored vertices.

The function φ catalogues all edges between v_j and the unexplored vertices U_{j-1}. It therefore encodes a spanning tree of the subgraph induced by C_j in G that uses only edges from v_j to vertices in $I_j \subseteq U_{j-1}$. The construction ensures that φ is an injective function. In particular we can invert φ on $C_j \cup A_j$ and obtain a subset of the Ulam–Harris tree $\mathcal{U}$.

Example 3.3.2. Consider the graph with nine vertices $\{1, \dots, 9\}$ shown in Fig. 3.2.

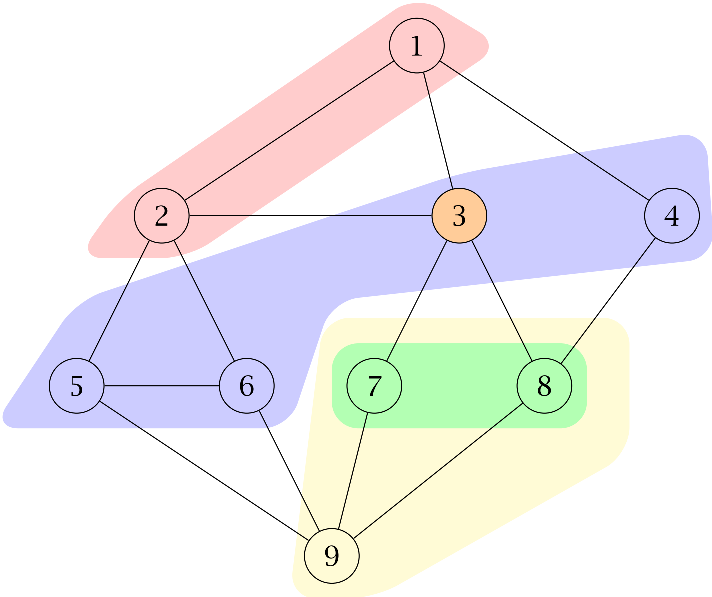


Figure 3.2: A graph with nine vertices. The sets C_1, A_1, U_1 of the exploration started in vertex $v_0 = 1$ are highlighted in red, blue and yellow, respectively. The vertex $v_2 = 3$ is highlighted in orange. Its unexplored neighbours are collected in the green set $I_2 \subseteq U_1$.

Starting in $v_0 = 1$ we explore the graph as described in Algorithm 3.3.1. If we have to order vertices in I_j, we choose to order them by their labels.

After step 1, the explored, active and unexplored sets are given by

$$C_1 = \{1,2\}, \quad A_1 = \{3,4,5,6\} \quad \text{and} \quad U_1 = \{7,8,9\}.$$

The function φ after this step is given by

$$\varphi(\varnothing) = 1 \quad \varphi((1)) = 2 \quad \varphi((2)) = 3$$
$$\varphi((1,1)) = 4 \quad \varphi((1,2)) = 5.$$

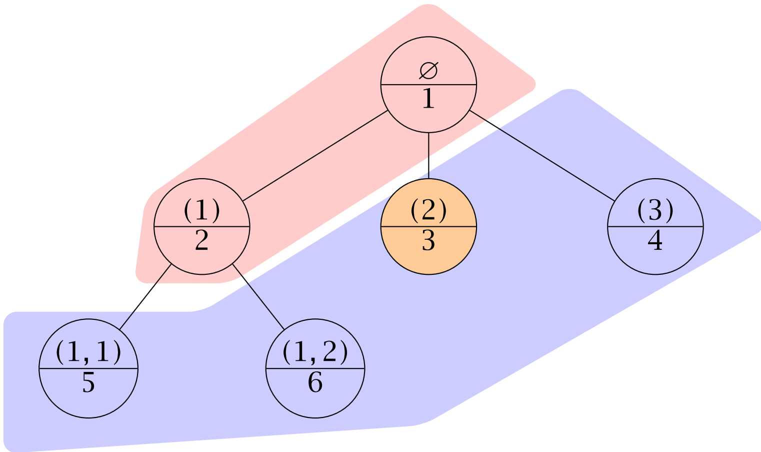


Figure 3.3: Representation of φ after exploration step $j = 1$ in Fig. 3.2. The upper part of each node is an individual $\mathbf{i}$ in the Ulam–Harris tree $\mathcal{U}$, the lower part corresponds to the vertex $\varphi(\mathbf{i})$ in G. The set C_1 is highlighted in red, the set A_1 in blue. The smallest element of $\varphi^{-1}(A_1)$ in the Ulam–Harris order is (2), which corresponds to the vertex 3 in G.

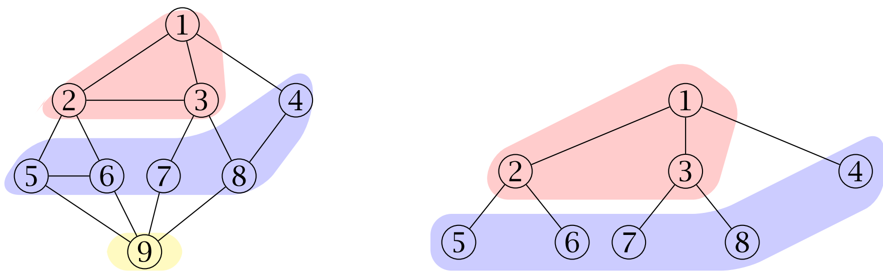
The spanning tree induced by φ is shown in Fig. 3.3.

In order to determine v_2 we find the smallest element in

$$\varphi^{-1}(A_j) = \{(2), (3), (1,1), (1,2)\} \subseteq \mathcal{U}$$

in the Ulam–Harris order. Here we have $\min \varphi^{-1}(A_j) = (2)$ and thus

$$v_2 = \varphi((2)) = 3.$$



(a) The graph G (b) Subtree induced by φ

Figure 3.4: The sets C_2, A_2 and U_2 after step 2 in G and the subtree induced by φ highlighted in blue, red and yellow, respectively.

We then collect all vertices in $U_1 = \{7, 8, 9\}$ that are connected to 3 via an edge in I_2. Hence,

$$I_2 = \{7, 8\}.$$

Order the elements in I_2 by their label and set

$$\varphi((2,1)) = 7 \quad \text{and} \quad \varphi((2,2)) = 8.$$

Then

$$\begin{aligned} C_2 &= \{1,2\} \cup \{3\} = \{1,2,3\}, \\ A_2 &= \{3,4,5,6\} \setminus \{3\} \cup \{7,8\} = \{4,5,6,7,8\} \end{aligned}$$

and

$$U_2 = \{7,8,9\} \setminus \{7,8\} = \{9\}.$$

The sets C_2, A_2 and U_2 as well as φ after exploration step 2 are shown in Fig. 3.4.

Knowledge of the exploration process up to step j in the form of C_ℓ, A_ℓ, U_ℓ for all $\ell \in \{-1,\dots,j\}$ and φ^{-1} on the set $C_j \cup A_j$ does not give us a complete picture of the explored graph, because the exploration does not explicitly keep track of edges between v_j and A_{j-1}.

This is easily seen in the simple example of the triangle graph.

Example 3.3.3. Consider the complete graph with three vertices 1, 2 and 3, the so-called *triangle graph* (see Fig. 3.5).

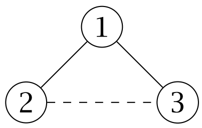


Figure 3.5: Triangle graph. The highlighted (dashed) edge $\{2,3\}$ is not seen by the exploration started in 1.

Start the exploration in $v_0 = 1$ (if required, we order vertices by their label) and obtain

Step 0 $v_0 = \varphi(\varnothing) = 1$, $I = \{2,3\}$, so that $C_0 = \{1\}$ $A_0 = \{2,3\}$,

Step 1 $v_1 = \varphi(1) = 2$, $I_1 = \varnothing$, so that $C_1 = \{1,2\}$, $A_1 = \{3\}$,

Step 2 $v_2 = \varphi(2) = 3$, $I_2 = \varnothing$, so that $C_2 = \{1,2,3\}$, $A_2 = \varnothing$.

The exploration stops after step 2 since $A_2 = \varnothing$.

We can recover the solid edges $\{1,2\}$ and $\{1,3\}$ from the spanning tree induced by φ. But the dashed edge between 2 and 3 is not catalogued by φ and cannot be recovered by looking at the sets C_ℓ, A_ℓ and U_ℓ for $\ell \in \{0,1,2\}$. Indeed, this edge is never considered by the exploration.

If we rely not only on the sets C_j, A_j, U_j and the function φ, but instead keep track of all edges between v_j and $A_{j-1} \cup U_{j-1}$, we can recover the entire subgraph structure and see each edge in the subgraph only once.

Lemma 3.3.4. *For all $j \in \mathbb{N}$ the following holds.*

(i) *The complete graph structure of the subgraph induced by C_j in G, i.e. the edge indicators for*

$$\mathcal{E}'_j = \{\{v_\ell, u\} \in V^{(2)} : \ell \in \{0, \dots, j\}, u \in V \setminus \{v_\ell\}\},$$

can be recovered from the edge indicators for

$$\mathcal{E}_j = \{\{v_\ell, u\} \in V^{(2)} : \ell \in \{0, \dots, j\}, u \in A_{\ell-1} \cup U_{\ell-1}\}.$$

In other words $\mathcal{E}_j = \mathcal{E}'_j$.

(ii) *Each edge in $\mathcal{E}'_j$ is contained in exactly one of the sets*

$$\mathcal{D}_\ell = \{\{v_\ell, u\} : u \in A_{\ell-1} \cup U_{\ell-1}\} \quad \text{for } \ell \in \{0, \dots, j\},$$

which means that it is seen exactly once by the exploration. In other words the $\mathcal{D}_\ell$ are disjoint and $\mathcal{E}'_j = \bigcup_{\ell=0}^{j} \mathcal{D}_\ell$.

Proof. **(i)** We use induction to show $\mathcal{E}_j = \mathcal{E}'_j$.

For $j = 0$ we have $A_{-1} \cup U_{-1} = V \setminus \{v_0\}$, which implies that $\mathcal{E}_0$ collects all $\{v_0, u\}$ for $u \in V \setminus \{v_0\}$, so that $\mathcal{E}_0 = \mathcal{E}'_0$.

Assume that the claim holds for some j. Then we have $\mathcal{E}_j = \mathcal{E}'_j$. In order to show that $\mathcal{E}_{j+1} = \mathcal{E}'_{j+1}$ it is enough to show that $\{v_{j+1}, u\} \in \mathcal{E}_{j+1}$ for all $u \in V \setminus \{v_{j+1}\}$. Note that v_{j+1} satisfies $v_{j+1} \in A_{\ell-1} \cup U_{\ell-1} = V \setminus C_{\ell-1}$ for all $\ell \in \{0, \dots, j\}$. Hence, $\{v_{j+1}, v_\ell\} = \{v_\ell, v_{j+1}\} \in \mathcal{E}_\ell \subseteq \mathcal{E}_j \subseteq \mathcal{E}_{j+1}$. In other words $\mathcal{E}_{j+1}$ contains $\{v_{j+1}, u\}$ for all $u \in C_j$. By construction $\mathcal{E}_{j+1}$ also contains $\{v_{j+1}, u\}$ for all $u \in A_j \cup U_j$. Together we thus have $\{v_{j+1}, u\} \in \mathcal{E}_{j+1}$ for $u \in V \setminus \{v_{j+1}\}$.

(ii) Since $\mathcal{E}_j = \bigcup_{\ell=0}^{j} \mathcal{D}_\ell$ the first part, which claims $\mathcal{E}_j = \mathcal{E}'_j$, implies that every edge in $\mathcal{E}'_j$ is contained in at least one of the $\mathcal{D}_\ell$.

It thus remains to show that each edge can be contained in only at most one $\mathcal{D}_\ell$. Assume for contradiction that $e \in \mathcal{D}_\ell \cap \mathcal{D}_{\ell'}$ for some $0 \le \ell < \ell' \le j$. Then $e = \{v_\ell, u\}$ for some $u \in A_{\ell-1} \cup U_{\ell-1}$, but at the same time $e = \{v_{\ell'}, u'\}$ for some $u' \in A_{\ell'-1} \cup U_{\ell'-1}$. Because $v_\ell \neq v_{\ell'}$ it follows that $e = \{v_{\ell'}, v_\ell\}$. But this is a direct contradiction to $e \in \mathcal{D}_{\ell'}$, because $v_\ell \in C_\ell \subseteq C_{\ell'-1}$ and thus $v_\ell \notin A_{\ell'-1} \cup U_{\ell'-1}$. It follows that the $\mathcal{D}_\ell$ are pairwise disjoint. This proves the claim. □

This observation motivates the following definition.

Definition 3.3.5. Consider the exploration of the neighbourhood of a vertex v_0 in a graph G as defined in Algorithm 3.3.1. For any j so that v_j is well-defined we call the neighbours of v_j that are in $A_{j-1} \cup U_{j-1}$ *exploration-relevant neighbours.*

Lemma 3.3.4 ensures that we can recover all neighbours of v_j even if we only ever keep track of exploration-relevant neighbours.

By construction there can be no edge between C_j and U_j. This is easily seen by induction, because the vertices of U_{j-1} that connect to the newly added vertex $v_j \in C_j$ are removed from U_j via I_j.

It follows that the subgraph of G induced by C_j spans a tree if there is no edge between a vertex in $A_{j'-1}$ and $v_{j'}$ for any $j' \leq j$.

Lemma 3.3.6. *If there is no edge between a vertex in $A_{j'-1}$ and $v_{j'}$ for any $j' \leq j$, that is to say if $J_{j'} = \{u \in A_{j'-1} : \{v_{j'}, u\} \in E\}$ is empty for all $j' \leq j$, then the subgraph of G induced by C_j is a tree.*

Proof. The construction of the exploration already embeds a tree into the subgraph induced by C_j via φ. This tree uses exactly the edges from v_k to U_{k-1} for $k \in \{0, \ldots, j\}$. We now show that under the assumption of this lemma all edges in the subgraph induced by C_j in G are of this form. This then shows that the complete subgraph is a tree.

Pick two vertices from C_j. By construction of C_j there exist indices $1 \leq k < k' \leq j$ such that the two vertices can be written as v_k and $v_{k'}$. We then also have that $v_k \in C_k$ and $v_{k'} \in A_k$ or $v_{k'} \in U_k$.

Now assume that v_k and $v_{k'}$ are connected via an edge. There are no edges between C_k and U_k, so it follows that $v_{k'} \in A_k = A_{k-1} \setminus \{v_k\} \cup I_k$. Since there are no edges between v_k and A_{k-1} by assumption, we can conclude that $v_{k'} \in I_k \subseteq U_{k-1}$. It follows that $v_{k'}$ is a child of v_k in the sense that $\varphi^{-1}(v_{k'}) = (\varphi^{-1}(v_k), \ell)$ for some $\ell \in \mathbb{N}$. Hence, edges only exist between direct descendant vertices, which implies the claimed tree structure because it makes cycles impossible. □

In case the subgraph induced by C_j is a tree, the complete information about this subgraph is contained in φ.

Let $\mathcal{G}_j$ be the σ-algebra generated by the edge indicators along the exploration sequence $X_{v_\ell, u}$ for $\ell \in \{0, \ldots, j\}$ and $u \in V$, i.e.

$$\mathcal{G}_j = \sigma(X_e)_{e \in \mathcal{E}'_j}.$$

Clearly the exploration process as recorded by C_ℓ, A_ℓ, U_ℓ for $\ell \in \{0, \ldots, j\}$ and φ^{-1} on the set $C_j \cup A_j$ is measurable with respect to $\mathcal{G}_j$.

Observe that given $\mathcal{G}_j$ the selection of v_{j+1} from A_j is deterministic, because we only need to know the preimages of the vertices in A_j under φ in order to pick v_{j+1}. This implies that v_{j+1} is $\mathcal{G}_j$-measurable.

On the other hand, $X_{v_{j+1}, u}$ for $u \in A_j \cup U_j$ are independent of $\mathcal{G}_j$, since the relevant possible edges are not included in $\mathcal{E}'_j$. To see this, note that $\mathcal{E}'_j$ only contains edges with at least one endpoint in C_j. Since $v_{j+1} \notin C_j$, it follows that u would have to be in C_j for $\{v_{j+1}, u\}$ to be contained in $\mathcal{E}'_j$. But $u \in A_j \cup U_j$ by definition, which is disjoint with C_j by construction.

Note further that the preimages of all vertices from I_j, i.e., the vertices that are added to A_j in step j, are larger in the order $\prec$ on the Ulam–Harris tree than the preimages of vertices in $C_{j-1} \cup A_{j-1}$. This implies that those vertices are only up for selection once all vertices from A_{j-1} have been fully explored. In particular the sequence $v_0, v_1, \ldots$ contains the vertices in exactly the order in which they were added to the active set (i.e., removed from the unexplored set). This means that the vertex sequences $(v_{\varphi(\mathbf{i})})_{\mathbf{i}\in S}$ (traversed in the order given by $\prec$) and $(v_j)_{j\in[|S|]}$ are exactly the same.

3.4 Neighbourhood coupling

We now construct a coupling between the neighbourhood of a vertex v in the unweighted inhomogeneous random graph G_n satisfying Assumption 2.2.1 and a Galton–Watson tree. We slightly modify the approach by Olvera-Cravioto [Olv22] by combining it with the exploration as described by Bordenave [Bor16]. As mentioned before we do not work under the minimal moment assumptions by Olvera-Cravioto [Olv22], instead we will assume that second and third moments of the connectivity weight distributions exist. This allows us to simplify some arguments and prove much more explicit bounds for the coupling probabilities. Couplings like this with explicit error bounds are interesting in the context of the objective method [AS04]. A related coupling was used by Fraiman, Lin and Olvera-Cravioto [FLO22] to define and analyse stochastic recursions (similar in principle to the RDE and RTP we briefly mentioned before) on directed random graphs.

The coupling is found in two steps. In a first step the neighbourhood is coupled to an intermediate tree in which the connectivity weights are still dependent on $\mathcal{F}_n$. The intermediate tree is then coupled to the desired limiting object in a second step.

Definition 3.4.1. Fix a vertex $v \in V_n$ and conditionally on $\mathcal{F}_n$ define the intermediate tree $\tilde{T}(v)$ via a sequence of random variables $\{(\tilde{W}_{\mathbf{i}}, \tilde{N}_{\mathbf{i}}) : \mathbf{i} \in \mathcal{U}\}$, where $\tilde{W}_{\mathbf{i}}$ is the type of individual $\mathbf{i}$ and $\tilde{N}_{\mathbf{i}}$ is its number of children. The distribution of $\{(\tilde{W}_{\mathbf{i}}, \tilde{N}_{\mathbf{i}}) : \mathbf{i} \in \mathcal{U}\}$ satisfies

- $\tilde{W}_\varnothing = W_v$ and $\tilde{N}_\varnothing \sim \mathrm{Poi}(\frac{W_v \Lambda_n}{n\vartheta})$,
- all other (non-root) individuals $\mathbf{i} \neq \varnothing$ have independent types and numbers of children $(\tilde{W}_{\mathbf{i}}, \tilde{N}_{\mathbf{i}})$ with distribution

  $$\mathbb{P}_n((\tilde{W}_{\mathbf{i}}, \tilde{N}_{\mathbf{i}}) \in \cdot\,) = \sum_{i=1}^{n} \frac{W_i}{\Lambda_n} \mathbb{P}((W_i, D_i) \in \cdot \mid W_i),$$

  where D_i is Poisson distributed with mean $\Lambda_n W_i/(n\vartheta)$ given W_i.

The tree structure on $\tilde{T}(v)$ is then obtained recursively from $\tilde{A}_0 = \{\varnothing\}$ and

$$\tilde{A}_k = \{(\mathbf{i}, j) : \mathbf{i} \in \tilde{A}_{k-1}, 1 \le j \le \tilde{N}_{\mathbf{i}}\} \quad \text{for } k \in \mathbb{N}, k \ge 1.$$

Intuitively, this defines $\tilde{T}(v)$ as a multi-type Galton–Watson process with n types corresponding to the vertices of G_n. Technically, we have defined the tree so that the type of a vertex is its weight $\tilde{W}_{\mathbf{i}}$. If the weights are all different, we can immediately infer which vertex v gave rise to this weight and call v the type. If some of the W_v are the same, we may assume that we sample $\tilde{W}_{\mathbf{i}}$ by partitioning the unit interval into n subintervals of length W_i/Λ_n, drawing a uniform random variable from $(0,1)$ and choosing $\tilde{W}_{\mathbf{i}}$ equal to the W_i into whose interval the uniform random variable falls.

We now use the exploration introduced in Section 3.3 to explore the neighbourhood of v in G_n and build the intermediate tree $\tilde{T}(v)$ at the same time coupling the two in the process. Broadly speaking the graph exploration is driven by Bernoulli random variables, which we couple to Poisson random variables in order to arrive at the Poisson structure of $\tilde{T}(v)$.

To simplify notation, set

$$p'_{vu} = \frac{W_v W_u}{n\vartheta},$$

such that $p_{vu} = p'_{vu} \wedge 1$.

Let $X_{vu} \sim \mathrm{Bin}(1, p_{vu})$ be the edge indicators in G_n. Couple $Z_{vu} \sim \mathrm{Poi}(p'_{vu})$ to X_{vu} (this coupling is constructed more explicitly in Lemma 3.4.3). Let Z^*_{vu} be i.i.d. copies of Z_{vu} that are independent of both Z_{vu} and X_{vu}.

Start the exploration in G_n in the vertex $v_0 = v$ (set $C_{-1} = \emptyset$, $A_{-1} = \{v\}$, $U_{-1} = V_n \setminus \{v\}$). In $\tilde{T}(v)$ give $\varnothing$ the type v and let $\mathbf{i}_0 = \varnothing$ be the first individual we visit (we keep track of the types of individuals we see via $\tilde{C}_{-1} = \emptyset$, $\tilde{A}_{-1} = \{v\}$, $\tilde{U}_{-1} = V_n \setminus \{v\}$).

In G_n the exploration process is governed by the random variables $X_{v_j u}$ for $u \in A_{j-1} \cup U_{j-1}$. In particular given C_{j-1}, A_{j-1}, U_{j-1}, we can select v_j and obtain I_j by collecting those vertices $u \in U_{j-1}$ for which $X_{v_j u} = 1$. We also record those vertices in A_{j-1} that satisfy $X_{v_j u} = 1$, which then allows us to recover all exploration-relevant neighbours and thus the complete graph structure (cf. Lemma 3.3.4). (Assume that we order elements in I_j by their vertex label.)

In $\tilde{T}(v)$ we have the analogous sets of types $\tilde{C}_{j-1}$, $\tilde{A}_{j-1}$, $\tilde{U}_{j-1}$ and obtain the children of the type-v_j vertex $\mathbf{i}_j$ by collecting $Z_{v_j u}$ children of type u for $u \in \tilde{U}_{j-1} \cup \tilde{A}_{j-1}$ and $Z^*_{v_j u}$ children of type u for $u \in \tilde{C}_j$. (As written, this procedure might suggest a certain ordering for the children that depends on the types in $\tilde{U}_{j-1}$, $\tilde{A}_{j-1}$ and $\tilde{C}_{j-1}$. The label of the children should not carry any information about its type, so we relabel the children of $\mathbf{i}_j$ randomly and add them to $\tilde{T}(v_0)$ as $(\mathbf{i}_j, t)$.) Let $\tilde{I}_j$ be the set of types of children of v_j.

Assuming that $C_{j-1} = \tilde{C}_{j-1}$, $A_{j-1} = \tilde{A}_{j-1}$ and $U_{j-1} = \tilde{U}_{j-1}$, the exploration-relevant neighbours of v_j in G can be identified with the children of the type-v_j vertex in $\tilde{T}(v_0)$ if

(i) $X_{v_j u} = Z_{v_j u}$ for all $u \in U_{j-1} \cup A_{j-1}$,

(ii) $Z_{v_j u} = 0$ for all $u \in A_{j-1}$ and

(iii) $Z^*_{v_j u} = 0$ for all $u \in C_{j-1}$.

Conditions (ii) and (iii) imply that $\tilde{I}_j$ only contains types from U_{j-1}. Together with condition (i) this implies $I_j = \tilde{I}_j$, so that we can conclude that also $C_j = \tilde{C}_j$, $A_j = \tilde{A}_j$ and $U_j = \tilde{U}_j$. Then condition (i) ensures that u is an exploration-relevant neighbour of v_j if and only if the type-v_j vertex in $\tilde{T}(v_0)$ has a unique child of type u. But this implies that the subgraph of G induced by C_j has the same graph structure as the $\tilde{T}(v_0)$ constructed so far.

In order to continue exploring G_n and building $\tilde{T}(v)$ at the same time, we now reorder the children of $\mathbf{i}_j$ so that their order matches the order in I_j. (This does not change the graph structure modulo graph isomorphism, which is all we are concerned with. It just ensures that the next exploration step continues with an individual of the correct type.)

We now verify that the procedure to generate children of $\mathbf{i}_j$ actually yields $\tilde{T}(v_0)$ as defined in Definition 3.4.1.

Lemma 3.4.2. *The tree generated via this coupling procedure has the distribution of the intermediate tree as defined in Definition 3.4.1.*

Proof. The distribution of the tree in Definition 3.4.1 is fully characterised by the offspring distribution for each individual.

From the definition it follows that the number of children of an individual of type u is Poisson distributed with mean $\Lambda_n W_u (n\vartheta)^{-1}$. The types of these children can be obtained by a thinning with the type distribution for non-root individuals: Each child has type W_i independently with probability $W_i \Lambda_n^{-1}$ (i.e. the type is chosen according to $\hat{v}_n$). This implies that the numbers of children of type i of a type-u individual are independently Poisson distributed with parameter

$$\frac{W_i}{\Lambda_n}\frac{\Lambda_n W_u}{n\vartheta} = \frac{W_u W_i}{n\vartheta} = p'_{ui}.$$

This coincides exactly with the offspring distribution induced by the coupling procedure, where a type-u individual has $\mathrm{Poi}(p'_{ui})$ many children of type i (depending on the status of the type i in the exploration so far this is either the random variable Z_{ui} or Z^*_{ui}). □

In order to estimate how long the coupling procedure can continue to produce isomorphic structures, we we first estimate the probability that for a fixed v the coupled random variables X_{vu} and Z_{vu} differ for any u.

Lemma 3.4.3. *Let $v \in V_n$, then we can couple $X_{vu} \sim \mathrm{Bin}(1, p_{vu})$ and $Z_{vu} \sim \mathrm{Poi}(p'_{vu})$ for $u \in V_n \setminus \{v\}$ such that for any $J \subseteq V_n \setminus \{v\}$*

$$\mathbb{P}_n(\max_{u \in J} |X_{vu} - Z_{vu}| \geq 1) \leq \sum_{u \in V_n \setminus \{v\}} ((p'_{vu})^2 - p'_{vu} \mathbb{1}_{\{p'_{vu} \geq 1\}}).$$

Proof. In order to couple $X_{vu} \sim \mathrm{Bin}(1, p_{vu})$ and $Z_{vu} \sim \mathrm{Poi}(p'_{vu})$ consider auxiliary random variables $Y_{vu} \sim \mathrm{Poi}(p'_{vu})$.

First we couple the Bernoulli random variable X_{vu} to a Poisson random variable Y_{vu} with the same parameter p_{uv} (see Lemma A.1.2). such that

$$\mathbb{P}_n(Y_{vu} \neq R_{vu}) \leq p_{vu}^2 \leq (p'_{uv})^2.$$

In a second step we couple the two Poisson random variables Y_{vu} and Z_{vu} with parameters p_{vu} and p'_{vu} (see Lemma A.1.3) such that

$$\begin{aligned}\mathbb{P}_n(Y_{vu} \neq Z_{vu}) &\leq p'_{vu} - p_{vu}\\ &= p'_{vu} - p'_{vu} \wedge 1\\ &\leq (p'_{vu} - 1)\mathbb{1}_{\{p'_{vu} \geq 1\}}\\ &\leq p'_{vu}\mathbb{1}_{\{p'_{vu} \geq 1\}}.\end{aligned}$$

With this chain of couplings $X_{vu} \neq Z_{vu}$ implies that $X_{vu} \neq Y_{vu}$ or $Y_{vu} \neq Z_{vu}$. Summing over the vertices in $J \subseteq V_n \setminus \{v\}$ we thus obtain

$$\begin{aligned}\mathbb{P}_n(\max_{u \in J}|X_{vu} - Z_{vu}| \geq 1) &\leq \sum_{u \in J} \mathbb{P}_n(|X_{vu} - Z_{vu}| \geq 1)\\ &\leq \sum_{u \in J} (\mathbb{P}_n(X_{vu} \neq Y_{vu}) + \mathbb{P}_n(Y_{vu} \neq Z_{vu}))\\ &\leq \sum_{u \in J} ((p'_{vu})^2 - p'_{vu}\mathbb{1}_{\{p'_{vu} \geq 1\}})\\ &\leq \sum_{u \in V_n \setminus \{v\}} ((p'_{vu})^2 - p'_{vu}\mathbb{1}_{\{p'_{vu} \geq 1\}})\end{aligned}$$

as claimed. □

The main result of this section is

Proposition 3.4.4. *For any vertex $v \in V_n$ of $G_n = (V_n, E_n)$ and any level $\ell \in \mathbb{N}$ it is possible to couple the neighbourhood of a vertex v to an intermediate tree $\tilde{T}(v)$ such that for any sequence $(k_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$*

$$\begin{aligned}&\mathbb{P}_n(B_\ell(v) \not\cong \tilde{T}_\ell(v))\\ &\quad \leq \mathbb{E}_n[\|S_\ell(v)\|_2]\frac{\Gamma_{2,n}}{n\vartheta} + \mathbb{E}_n[\|S_\ell(v)\|_+]\Gamma_{1,n} + \mathbb{E}_n[\|S_\ell(v)\|]\Big(\kappa_{1,n} + \frac{1}{k_n} + \frac{k_n}{n\vartheta}\Big),\end{aligned}$$

where $\tilde{T}_\ell(v)$ is $\tilde{T}(v)$ truncated at level ℓ.

Proof. Recall that $S_\ell(v)$ is the set of vertices in the ℓ-neighbourhood $B_\ell(v)$ of v in G_n and that we set $D_\ell(v) = S_\ell(v) \setminus S_{\ell-1}(v)$. Explore $S_\ell(v)$ with the exploration defined in Algorithm 3.3.1.

Let $\mathcal{G}_j$ be the σ-algebra generated by the edge indicators X_e in the exploration process on G_n up to step j along with the coupled Poisson random variables Z_e (cf. Lemma 3.4.3) and an independent copy Z_e^* of Z_e

$$\mathcal{G}_j = \sigma(X_e, Z_e, Z_e^*)_{e \in \mathcal{E}_j'}.$$

The Poisson random variables Z_e and Z_e^* will be used to construct $\tilde{T}(v)$.

For any $u \in S_\ell(v)$ let $C(u)$, $A(u)$ and $U(u)$ be the sets C_{j-1}, A_{j-1} and U_{j-1}, respectively, when u's neighbours are being explored, i.e. when $u = v_j$ for some $j \in \mathbb{N}$. Additionally we also define $\mathcal{G}(u) = \mathcal{G}_{j-1}$ and $\mathcal{G}^+(u) = \mathcal{G}_j$.

With this setup the step in which u's unexplored neighbours are explored is measurable with respect to $\mathcal{G}^+(u)$, but conditionally independent of $\mathcal{G}(u)$ given the sets $C(u)$, $A(u)$ and $U(u)$.

We say that the coupling between the neighbourhoods of v in G_n and $\tilde{T}(v)$ breaks in level ℓ if there is a $u \in D_{\ell-1}(v)$, i.e. a vertex at level $\ell - 1$, satisfying

(i) $X_{uu'} \neq Z_{uu'}$ for some $u' \in U(u) \cup A(u)$,

(ii) $Z_{uu'} \neq 0$ for some $u' \in A(u)$ or

(iii) $Z^*_{uu'} \neq 0$ for some $u' \in C(u)$.

For a fixed vertex $u \in D_{\ell-1}(v)$ let $N(u)$ event that at least one of these three conditions is true for u. Let M_ℓ be the event that the coupling *holds* up to level ℓ, i.e. that the coupling has not yet broken up to level ℓ. Note that if the coupling holds up to level ℓ, the graphs are isomorphic as rooted graphs up to level ℓ.

Let $\mathcal{G}_\ell'$ be the σ-algebra generated by the exploration process for all vertices up to level $\ell - 1$

$$\mathcal{G}_\ell' = \sigma\Big(\bigcup_{u \in S_{\ell-1}(v)} \mathcal{G}^+(u)\Big). \tag{3.37}$$

This σ-algebra already contains information about the vertices at level ℓ in $D_\ell(v)$ since

$$D_\ell(v) = V_n \setminus \Big(\bigcup_{u \in S_{\ell-1}(v)} C(u) \cup \bigcap_{u \in S_{\ell-1}(v)} U(u)\Big),$$

such that all of $B_\ell(v)$ is $\mathcal{G}_\ell'$-measurable. But the edges going from $D_\ell(v)$ to the as of yet not fully explored vertices $V_n \setminus S_{\ell-1}(v)$ are independent of $\mathcal{G}_\ell'$. Clearly, $M_{\ell-1}$ is $\mathcal{G}_{\ell-1}'$-measurable.

Additionally we have $\mathcal{G}(u) \supseteq \mathcal{G}_{\ell-1}'$ for all $u \in D_{\ell-1}(v)$.

Ultimately we want to estimate the probability $\mathbb{P}_n(M_\ell^c)$. We will do this by noting that $M_\ell^c = M_{\ell-1}^c \cup (M_{\ell-1} \cap M_\ell^c)$ so that

$$\begin{aligned}\mathbb{P}_n(M_\ell^c) &\leq \mathbb{P}_n(\|S_\ell(v)\| > k_n) + \mathbb{P}_n(M_\ell^c, \|S_\ell(v)\| \leq k_n)\\ &\leq \mathbb{P}_n(\|S_\ell(v)\| > k_n) + \sum_{j=1}^{\ell} \mathbb{P}_n(M_{j-1} \cap M_j^c, \|S_j(v)\| \leq k_n).\end{aligned} \tag{3.38}$$

Conditionally on $\mathcal{G}'_{\ell-1}$ the summands of the second sum can be split further by noting that $M_{\ell-1} \cap M_\ell^c$ can be written as a union over $N(u)$ for $u \in D_{\ell-1}(v)$. Additionally, we have that $\|C(u) \cup A(u)\| \le \|S_\ell(v)\|$, because all vertices in $C(u) \cup A(u)$ must be elements of $S_\ell(v)$ since they are neighbours of vertices in $S_{\ell-1}(v)$. Hence,

$$\begin{aligned}
&\mathbb{P}_n(\|S_\ell(v)\| \le k_n, M_{\ell-1} \cap M_\ell^c \mid \mathcal{G}'_{\ell-1}) \\
&\quad \le \mathbb{1}_{M_{\ell-1}} \sum_{u \in D_{\ell-1}(v)} \mathbb{P}_n(\|C(u) \cup A(u)\| \le k_n, N(u) \mid \mathcal{G}'_{\ell-1}).
\end{aligned} \tag{3.39}$$

For the individual summands for $u \in D_{\ell-1}(v)$ recall the definition of $N(u)$ and additionally condition on $\mathcal{G}(u) \supseteq \mathcal{G}'_{\ell-1}$ such that

$$\begin{aligned}
&\mathbb{P}_n(\|C(u) \cup A(u)\| \le k_n, N(u) \mid \mathcal{G}'_{\ell-1}) \\
&\quad \le \mathbb{E}_n\Big[\mathbb{P}_n\Big(\max_{u' \in U(u) \cup A(u) \setminus \{u\}} |X_{uu'} - Z_{uu'}| \ge 1 \;\Big|\; \mathcal{G}(u)\Big) \;\Big|\; \mathcal{G}'_{\ell-1}\Big] \\
&\qquad + \mathbb{E}_n\Big[\mathbb{1}_{\{\|C(u) \cup A(u)\| \le k_n\}} \mathbb{P}_n\Big(\sum_{u' \in A(u)} Z_{uu'} + \sum_{u' \in C(u)} Z^*_{uu'} \ge 1 \;\Big|\; \mathcal{G}(u)\Big) \;\Big|\; \mathcal{G}'_{\ell-1}\Big]
\end{aligned}$$

note that Lemma 3.4.3 still holds conditionally on $\mathcal{G}(u)$ for the $\mathcal{G}(u)$-measurable set $J = U(u) \cup A(u) \setminus \{u\}$, because $X_{uu'}$ and $Z_{uu'}$ are independent of $\mathcal{G}(u)$, furthermore by independence of the Poisson random variables $Z_{uu'}$ and $Z^*_{uu'}$ from each other (even conditional on $\mathcal{G}(u)$) and the previous steps in the exploration the sum of $\sum_{u' \in A(u)} Z_{uu'}$ and $\sum_{u' \in C(u)} Z^*_{uu'}$ has distribution $\mathrm{Poi}(\sum_{u' \in A(u) \cup C(u)} p'_{uu'})$

$$\begin{aligned}
&\le \mathbb{E}_n\Big[\sum_{u' \in V_n \setminus \{u\}} ((p'_{uu'})^2 + p'_{uu'} \mathbb{1}_{\{p'_{uu'} \ge 1\}}) \\
&\qquad + \mathbb{1}_{\{\|C(u) \cup A(u)\| \le k_n\}} (1 - e^{-\sum_{u' \in A(u) \cup C(u)} p'_{uu'}}) \;\Big|\; \mathcal{G}'_{\ell-1}\Big] \\
&\le \mathbb{E}_n\Big[\sum_{u' \in V_n \setminus \{u\}} ((p'_{uu'})^2 + p'_{uu'} \mathbb{1}_{\{p'_{uu'} \ge 1\}}) \\
&\qquad + \mathbb{1}_{\{\|C(u) \cup A(u)\| \le k_n\}} \sum_{u' \in A(u) \cup C(u)} p'_{uu'} \;\Big|\; \mathcal{G}'_{\ell-1}\Big],
\end{aligned} \tag{3.40}$$

where the last inequality follows from $1 - e^{-x} \le x$ for $x > -1$.

For the first inner sum in (3.40) we get

$$\begin{aligned}
&\mathbb{E}_n\Big[\sum_{u' \in V_n \setminus \{u\}} ((p'_{uu'})^2 + p'_{uu'} \mathbb{1}_{\{p'_{uu'} \ge 1\}}) \;\Big|\; \mathcal{G}'_{j-1}\Big] \\
&\quad \le \mathbb{E}_n\Big[\sum_{u' \in V_n \setminus \{u\}} \frac{W_u^2 W_{u'}^2}{\vartheta^2 n^2} + \sum_{u' \in V_n \setminus \{u\}} \frac{W_u W_{u'}}{n\vartheta} \mathbb{1}_{\{W_u W_{u'} \ge n\vartheta\}} \;\Big|\; \mathcal{G}'_{\ell-1}\Big].
\end{aligned}$$

The indicator function can be split by by noting that if $W_u W_{u'} \geq n\vartheta$, then $W_u \geq \sqrt{n\vartheta}$ or $W_{u'} \geq \sqrt{n\vartheta}$, in particular $\mathbb{1}_{\{W_u W_{u'} \geq n\vartheta\}} \leq \mathbb{1}_{\{W_u \geq \sqrt{n\vartheta}\}} + \mathbb{1}_{\{W_{u'} \geq \sqrt{n\vartheta}\}}$ so that the second sum splits into two sums with different indicators each

$$\begin{aligned}
&\leq \mathbb{E}_n\Big[\frac{W_u^2}{n\vartheta}\frac{1}{n\vartheta}\sum_{u\in V_n} W_{u'}^2 + W_u \mathbb{1}_{\{W_u \geq \sqrt{n\vartheta}\}}\frac{1}{n\vartheta}\sum_{u'\in V_n} W_{u'} \\
&\qquad + W_u \frac{1}{n\vartheta}\sum_{u'\in V_n} W_{u'}\mathbb{1}_{\{W_{u'} \geq \sqrt{n\vartheta}\}} \;\Big|\; \mathcal{G}'_{\ell-1}\Big].
\end{aligned}$$

Recall the definitions of $\Gamma_{p,n}$ and $\kappa_{p,n}$ to rewrite this as

$$\begin{aligned}
&\leq \mathbb{E}_n\Big[\frac{W_u^2}{n\vartheta}\Gamma_{2,n} + W_u \mathbb{1}_{\{W_u \geq \sqrt{n\vartheta}\}}\Gamma_{1,n} + W_u \kappa_{1,n} \;\Big|\; \mathcal{G}'_{\ell-1}\Big] \\
&\leq \frac{\Gamma_{2,n}}{n\vartheta}\mathbb{E}_n[W_u^2 \mid \mathcal{G}'_{\ell-1}] + \Gamma_{1,n}\mathbb{E}_n[W_u \mathbb{1}_{\{W_u > \sqrt{n\vartheta}\}} \mid \mathcal{G}'_{\ell-1}] + \kappa_{1,n}\mathbb{E}_n[W_u \mid \mathcal{G}'_{\ell-1}].
\end{aligned} \tag{3.41}$$

For the second inner sum in (3.40) we find

$$\begin{aligned}
&\mathbb{E}_n\Big[\mathbb{1}_{\{\|C(u)\cup A(u)\| \leq k_n\}} \sum_{u'\in A(u)\cup C(u)} p'_{uu'} \;\Big|\; \mathcal{G}'_{\ell-1}\Big] \\
&\leq \mathbb{E}_n\Big[\mathbb{1}_{\{\|C(u)\cup A(u)\| \leq k_n\}} \sum_{u'\in A(u)\cup C(u)} \frac{W_u W_{u'}}{n\vartheta} \;\Big|\; \mathcal{G}'_{\ell-1}\Big] \\
&\leq \mathbb{E}_n\Big[\mathbb{1}_{\{\|C(u)\cup A(u)\| \leq k_n\}} \frac{W_u}{n\vartheta}\|A(u)\cup C(u)\| \;\Big|\; \mathcal{G}'_{\ell-1}\Big] \\
&\leq \frac{\mathbb{E}_n[W_u \mid \mathcal{G}'_{\ell-1}]}{n\vartheta}k_n.
\end{aligned} \tag{3.42}$$

Then (3.40) together with (3.41) and (3.42) implies

$$\begin{aligned}
&\mathbb{P}_n(\|C(u)\cup A(u)\| \leq k_n, N(u) \mid \mathcal{G}'_{\ell-1}) \\
&\leq \mathbb{E}_n[W_u^2 \mid \mathcal{G}'_{\ell-1}]\frac{\Gamma_{2,n}}{n\vartheta} + \Gamma_{1,n}\mathbb{E}_n[W_u \mathbb{1}_{\{W_u > \sqrt{n\vartheta}\}} \mid \mathcal{G}'_{\ell-1}] \\
&\quad + \mathbb{E}_n[W_u \mid \mathcal{G}'_{\ell-1}]\Big(\kappa_{1,n} + \frac{k_n}{n\vartheta}\Big).
\end{aligned} \tag{3.43}$$

Hence, by (3.39), (3.40) and (3.43)

$$\begin{aligned}
&\sum_{j=1}^{\ell} \mathbb{P}_n(M_{j-1}\cap M_j^c, \|S_j(v)\| \leq k_n) \\
&\quad \leq \sum_{j=1}^{\ell} \mathbb{E}_n[\mathbb{P}_n(\|C(u)\cup A(u)\| \leq k_n, N(u) \mid \mathcal{G}'_{j-1})] \\
&\quad \leq \sum_{j=1}^{\ell} \mathbb{E}_n\Big[\sum_{u\in D_{j-1}(v)} \mathbb{E}_n[W_u^2 \mid \mathcal{G}'_{\ell-1}]\frac{\Gamma_{2,n}}{n\vartheta} + \Gamma_{1,n}\mathbb{E}_n[W_u \mathbb{1}_{\{W_u > \sqrt{n\vartheta}\}} \mid \mathcal{G}'_{\ell-1}] \\
&\qquad\qquad + \mathbb{E}_n[W_u \mid \mathcal{G}'_{\ell-1}]\Big(\kappa_{1,n} + \frac{k_n}{n\vartheta}\Big)\Big].
\end{aligned}$$

Recall the definitions of $\|\cdot\|_p$ and $\|\cdot\|_+$. Then use that the disjoint union of the $D_{j-1}(v)$ from $j = 1$ to ℓ is equal to $S_{\ell-1}(v)$ to estimate the sums over W_u^2 and W_u with $\|S_\ell(v)\|_2$ and $\|S_\ell(v)\|$, respectively. We obtain the bound

$$\leq \mathbb{E}_n[\|S_\ell(v)\|_2]\frac{\Gamma_{2,n}}{n\vartheta} + \mathbb{E}_n[\|S_\ell(v)\|_+]\Gamma_{1,n} + \mathbb{E}_n[\|S_\ell(v)\|]\Big(\kappa_{1,n} + \frac{k_n}{n\vartheta}\Big). \tag{3.44}$$

Hence, by (3.38) and (3.44) and Markov's inequality the probability that the coupling breaks can be bounded as follows

$$\begin{aligned}
&\mathbb{P}_n(B_\ell(v) \ncong \tilde{T}_\ell(v))\\
&\quad = \mathbb{P}_n(\text{coupling breaks up to level } \ell)\\
&\quad = \mathbb{P}_n(M_\ell^c)\\
&\quad \leq \mathbb{P}_n(\|S_\ell(v)\| > k_n) + \sum_{j=1}^{\ell} \mathbb{P}_n(M_{j-1} \cap M_j^c, \|S_j(v)\| \leq k_n)\\
&\quad \leq \mathbb{E}_n[\|S_\ell(v)\|]\frac{1}{k_n} + \mathbb{E}_n[\|S_\ell(v)\|_2]\frac{\Gamma_{2,n}}{n\vartheta} + \mathbb{E}_n[\|S_\ell(v)\|_+]\Gamma_{1,n}\\
&\qquad + \mathbb{E}_n[\|S_\ell(v)\|]\Big(\kappa_{1,n} + \frac{k_n}{n\vartheta}\Big)\\
&\quad \leq \mathbb{E}_n\|S_\ell(v)\|_2]\frac{\Gamma_{2,n}}{n\vartheta} + \mathbb{E}_n[\|S_\ell(v)\|_+]\Gamma_{1,n} + \mathbb{E}_n[\|S_\ell(v)\|]\Big(\kappa_{1,n} + \frac{1}{k_n} + \frac{k_n}{n\vartheta}\Big).
\end{aligned}$$

This concludes the proof. □

The coupling in Proposition 3.4.4 holds for a single ℓ-neighbourhood, but we would like to be able to couple the neighbourhoods of several distinct vertices to independent intermediate trees.

The neighbourhoods $B_\ell(v)$ of $m = |\mathcal{V}|$ distinct vertices $v \in \mathcal{V}$ to trees $T_\ell(v)$ are not guaranteed to be independent, because the same vertex and with it the same edges may appear in several neighbourhoods. By construction, the same holds for the coupled intermediate trees, since the edges in those trees are coupled to the original edges in G_n.

In Section 3.2 we have, however, already shown that the ℓ-neighbourhoods are asymptotically only weakly correlated. In much the same vein we can show that the intermediate trees can be altered (with sufficiently small probability) to make them independent. This then implies that we may assume that the coupled intermediate trees are independent at only a small additional penalty to the coupling probability.

Proposition 3.4.5. *Let $\mathcal{V} \subseteq V_n$ be a set of vertices from $G_n = (V_n, E_n)$. Then for all $\ell \in \mathbb{N}$ the neighbourhoods $B_\ell(v)$ around $v \in \mathcal{V}$ can be coupled to independent* *intermediate trees $\tilde{T}(v)$ such that*

$$\begin{aligned}
&\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}}\{B_\ell(v)\not\cong\tilde{T}_\ell(v)\}\Big)\\
&\quad\le \|\mathcal{V}\|_2\frac{\Gamma_{2,n}}{n\vartheta}+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell\Big(\frac{\Gamma_{3,n}}{n\vartheta}+\kappa_{1,n}+\kappa_{2,n}+\frac{2+\Gamma_{1,n}}{k_n}+\frac{k_n}{n\vartheta}\Big)\\
&\qquad+\|V\|_+\Gamma_{1,n}+|\mathcal{V}|\frac{1}{k_n}+\frac{k_n^2}{n\vartheta\Gamma_{1,n}}
\end{aligned}$$

for all sequences $(k_n)_{n\in\mathbb{N}}\subseteq(0,\infty)$.

Proof. Let $\tilde{S}_\ell(v)$ the set of individuals in the intermediate tree $\tilde{T}(v)$ up to level ℓ and $\tilde{D}_r(v)=\tilde{S}_r(v)\setminus\tilde{S}_{r-1}(v)$. As in Section 3.1 we calculate the expected number of individuals in $\tilde{S}_\ell(v)$ and their total weight.

Note that in the tree

$$\mathbb{E}_n[\tilde{N}_\varnothing]=\frac{W_v\Lambda_n}{n\vartheta}\le W_v\Gamma_{1,n}$$

and for $\mathbf{i}\neq\varnothing$

$$\mathbb{E}_n[\tilde{N}_{\mathbf{i}}]=\sum_{i=1}^n\frac{W_i}{\Lambda_n}\mathbb{E}_n[D_i]=\sum_{i=1}^n\frac{W_i}{\Lambda_n}\frac{\Lambda_nW_i}{n\vartheta}\le\frac{\mathbb{E}_n[(W^{(n)})^2]}{\vartheta}=\Gamma_{2,n},$$

where $D_i\sim\mathrm{Poi}(\Lambda_nW_i/(\vartheta n))$ given W_i. Then by construction

$$|\tilde{D}_0(v)|=1\quad\text{and}\quad\mathbb{E}_n[|\tilde{D}_1(v)|]=\mathbb{E}_n[\tilde{N}_\varnothing]\le W_v\Gamma_{1,n}$$

and furthermore by standard arguments for Galton–Watson trees

$$\mathbb{E}_n[|\tilde{D}_r(v)|]\le W_v\Gamma_{1,n}\Gamma_{2,n}^{r-1}.$$

Hence,

$$\mathbb{E}_n[|\tilde{S}_\ell(v)|]\le 1+W_v\Gamma_{1,n}(\Gamma_{2,n}+1)^{\ell-1},\tag{3.45}$$

which coincides with the bound Lemma 3.1.2 for the analogous quantity in G_n.

Similarly, by a standard argument for multi-type Galton–Watson processes

$$\mathbb{E}_n[\|\tilde{S}_\ell(v)\|]\le W_v(\Gamma_{2,n}+1)^\ell,\tag{3.46}$$

which coincides with the bound from Lemma 3.1.4 for G_n.

For a fixed set of vertices $\mathcal{V}$ we can use Proposition 3.4.4 to couple the neighbourhood of v in G_n to an intermediate tree $\tilde{T}'(v)$ up to level ℓ of each $v\in\mathcal{V}$. Together these couplings satisfy

$$\begin{aligned}
&\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}}\{B_\ell(v)\not\cong\tilde{T}'_\ell(v)\}\Big)\\
&\quad\le\sum_{v\in\mathcal{V}}\Big(\mathbb{E}_n[\|S_\ell(v)\|_2]\frac{\Gamma_{2,n}}{n\vartheta}+\mathbb{E}_n[\|S_\ell(v)\|_+]\Gamma_{1,n}+\mathbb{E}_n[\|S_\ell(v)\|]\Big(\kappa_{1,n}+\frac{1}{k_n}+\frac{k_n}{n\vartheta}\Big)\Big).
\end{aligned}$$

Bound the expectations of $\|S_\ell(v)\|$ and $\|S_\ell(v)\|_2$ with Lemma 3.1.4 and the expectation of $\|S_\ell(v)\|_+$ with Lemma 3.1.7 to further estimate the term by

$$\begin{aligned}
&\le \sum_{v\in\mathcal{V}} (W_v^2 + W_v(\Gamma_{2,n}+1)^{\ell-1}\Gamma_{3,n})\frac{\Gamma_{2,n}}{n\vartheta} + \sum_{v\in\mathcal{V}} W_v \mathbb{1}_{\{W_v>\sqrt{n\vartheta}\}} + W_v(\Gamma_{2,n}+1)^{\ell-1}\kappa_{2,n} \\
&\quad + \sum_{v\in\mathcal{V}} W_v(\Gamma_{2,n}+1)^{\ell}\Big(\kappa_{1,n} + \frac{1}{k_n} + \frac{k_n}{n\vartheta}\Big) \\
&\le \|\mathcal{V}\|_2\frac{\Gamma_{2,n}}{n\vartheta} + \|\mathcal{V}\|_+\Gamma_{1,n} + \|\mathcal{V}\|(\Gamma_{2,n}+1)^{\ell}\Big(\frac{\Gamma_{3,n}}{n\vartheta} + \kappa_{1,n} + \kappa_{2,n} + \frac{1}{k_n} + \frac{k_n}{n\vartheta}\Big).
\end{aligned} \tag{3.47}$$

The definition of these coupled trees does not ensure that the trees are independent, because the same (coupled) individual may appear in several trees. But given a family of intermediate trees the following procedure can generate independent trees $(\tilde{T}_\ell(v))_{v\in\mathcal{V}}$.

We construct level $r \in \{0,\dots,\ell\}$ of all trees $(\tilde{T}_\ell(v))_{v\in\mathcal{V}}$ in the same step. During our procedure we need to keep track of the individuals that we have seen so far.

Start by setting level 0 of each tree $\tilde{T}_\ell(v)$ to just v.

Assuming that we have already explored all vertices at level $r-1$ in all trees, we now use breadth-first search to completely explore level r of each $\tilde{T}'_\ell(v)$. Whenever we encounter an individual in $\tilde{T}'_\ell(v)$ that has not been seen before, it is copied over to the appropriate $\tilde{T}_\ell(v)$ and added to the set of individuals that have been seen. If the individual has been seen before, an independent Galton–Watson tree of appropriate height with offspring distribution $\hat{\nu}$ is added in its position to $\tilde{T}_\ell(v)$.

This procedure terminates with independent intermediate trees and the probability that $\tilde{T}'_\ell(v) \neq \tilde{T}_\ell(v)$ for any $v \in \mathcal{V}$ can be estimated by the probability that any individual added during the process was seen before. Let $\tilde{S}(v)$ be the set of individuals in the intermediate tree $\tilde{T}'_\ell(v)$ for $v \in \mathcal{V}$ and set $\tilde{S}_\ell(\mathcal{V}) = \bigcup_{v\in\mathcal{V}}\tilde{S}(v)$. Then $|\tilde{S}_\ell(\mathcal{V})|$ is the total number of individuals in all trees and $\|\tilde{S}_\ell(\mathcal{V})\|$ their total connectivity weight. The type of a non-root individual has distribution $\hat{\nu}$. Hence, the probability that during the breadth-first search a particular non-root individual has a type that has been seen before is bounded above by (cf. (2.2))

$$\hat{\nu}_n(\{W_i : i \in \tilde{S}_\ell(\mathcal{V})\}) \le \frac{\|\tilde{S}_\ell(\mathcal{V})\|}{\Lambda_n}.$$

If we have that $|\tilde{S}_\ell(\mathcal{V})| \le k_n$ and $\|\tilde{S}_\ell(\mathcal{V})\| \le k_n$ then the number of vertices whose type was already seen is dominated by a binomial distribution with parameters k_n and $k_n\Lambda_n^{-1}$. Let $Z \sim \mathrm{Bin}(k_n, k_n\Lambda_n^{-1})$ such that by Markov's inequality

$$\mathbb{P}_n(Z \ge 1) \le \frac{k_n^2}{\Lambda_n}.$$

It follows that

$$\begin{aligned}
&\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}}\{\tilde{T}'_\ell(v)\not\cong\tilde{T}_\ell(v)\}\Big)\\
&\quad=\mathbb{P}_n(\text{at least one individual appears in more than one tree})\\
&\quad\le\mathbb{P}_n(Z\ge 1,|\tilde{S}_\ell(\mathcal{V})|\le k_n,\|\tilde{S}_\ell(\mathcal{V})\|\le k_n)\\
&\qquad+\mathbb{P}_n(|\tilde{S}_\ell(\mathcal{V})|>k_n)+\mathbb{P}_n(\|\tilde{S}_\ell(\mathcal{V})\|>k_n)\\
&\quad\le\frac{k_n^2}{\Lambda_n}+\frac{\mathbb{E}_n[|\tilde{S}_\ell(\mathcal{V})|]}{k_n}+\frac{\mathbb{E}_n[\|\tilde{S}_\ell(\mathcal{V})\|]}{k_n}\\
&\quad\le\frac{k_n^2}{\Lambda_n}+\frac{\sum_{v\in\mathcal{V}}\mathbb{E}_n[|\tilde{S}_\ell(v)|]}{k_n}+\frac{\sum_{v\in\mathcal{V}}\mathbb{E}_n[\|\tilde{S}_\ell(v)\|]}{k_n}.
\end{aligned}$$

By (3.45) and (3.46) and $\sum_{v\in\mathcal{V}}W_v=\|\mathcal{V}\|$ this can be bounded by

$$\le\frac{k_n^2}{\Lambda_n}+\frac{|\mathcal{V}|+\|\mathcal{V}\|\Gamma_{1,n}(\Gamma_{2,n}+1)^{\ell-1}+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell}{k_n}\tag{3.48}$$

This shows that we can turn a family of intermediate trees $(\tilde{T}'(v))_{v\in\mathcal{V}}$ into independent intermediate trees $(\tilde{T}(v))_{v\in\mathcal{V}}$ at small cost by replacing a repeated individual and all its descendants by independent draws from $\hat{v}_n$.

In particular we can switch the trees $\tilde{T}'(v)$ coupled to the neighbourhoods of v to independent trees $\tilde{T}(v)$. By (3.47) and (3.48) the couplings between $B_\ell(v)$ and the independent $\tilde{T}(v)$ then satisfy

$$\begin{aligned}
&\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}}\{B_\ell(v)\not\cong\tilde{T}_\ell(v)\}\Big)\\
&\le\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}}\{B_\ell(v)\ne\tilde{T}'_\ell(v)\}\Big)+\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}}\{\tilde{T}'_\ell(v)\not\cong\tilde{T}_\ell(v)\}\Big)\\
&\le\|\mathcal{V}\|_2\frac{\Gamma_{2,n}}{n\vartheta}\|\mathcal{V}\|_+\Gamma_{1,n}\\
&\quad+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell\Big(\frac{\Gamma_{3,n}}{n\vartheta}+\kappa_{1,n}+\kappa_{2,n}+\frac{1}{k_n}+\frac{k_n}{n\vartheta}\Big)\\
&\quad+\frac{k_n^2}{\Lambda_n}+\frac{|\mathcal{V}|+\|\mathcal{V}\|\Gamma_{1,n}(\Gamma_{2,n}+1)^{\ell-1}+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell}{k_n}.
\end{aligned}$$

Recall that $\Lambda_n=n\vartheta\Gamma_{1,n}$. Then collect the terms for $\|\mathcal{V}\|_2$, $\|\mathcal{V}\|$ and $|\mathcal{V}|$ and estimate terms very generously to obtain the bound

$$\begin{aligned}
&\le\|\mathcal{V}\|_2\frac{\Gamma_{2,n}}{n\vartheta}+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell\Big(\frac{\Gamma_{3,n}}{n\vartheta}+\kappa_{1,n}+\kappa_{2,n}+\frac{2+\Gamma_{1,n}}{k_n}+\frac{k_n}{n\vartheta}\Big)\\
&\quad+\|\mathcal{V}\|_+\Gamma_{1,n}+|\mathcal{V}|\frac{1}{k_n}+\frac{k_n^2}{n\vartheta\Gamma_{1,n}}.
\end{aligned}$$

This finishes the proof. □

The construction of the intermediate trees relies heavily on the connectivity weights of vertices in G_n. Since the empirical distribution of those connectivity weights converges to a limiting distribution by assumption we now define (limiting) trees that draw from this limiting distribution and show that those trees can be coupled to the intermediate trees.

Definition 3.4.6. Fix a vertex $v \in V_n$ and define a tree $\mathcal{T}(v)$ via a sequence of random variables $\{(W_{\mathbf{i}}, N_{\mathbf{i}}) : \mathbf{i} \in \mathcal{U}\}$, where $W_{\mathbf{i}}$ is the type of individual $\mathbf{i}$ and $N_{\mathbf{i}}$ is its number of children. The distribution of $\{(\tilde{W}_{\mathbf{i}}, \tilde{N}_{\mathbf{i}}) : \mathbf{i} \in \mathcal{U}\}$ satisfies

- $W_\varnothing = W_v$ and $N_\varnothing \sim \mathrm{Poi}(W_v)$,
- all other (non-root) individuals $\mathbf{i} \neq \varnothing$ have independent types and numbers of children $(W_{\mathbf{i}}, N_{\mathbf{i}})$ with distribution

$$\mathbb{P}((W_{\mathbf{i}}, N_{\mathbf{i}}) \in \cdot\,) = \mathbb{P}((\widehat{W}, N) \in \cdot\,),$$

  where $\widehat{W} \sim \hat{\nu}$ and $N \sim \mathrm{MPoi}(\widehat{W})$.

The tree structure on $\mathcal{T}(v)$ is then obtained recursively from $\mathcal{A}_0 = \{\varnothing\}$ and

$$\mathcal{A}_k = \{(\mathbf{i}, j) : \mathbf{i} \in \mathcal{A}_{k-1}, 1 \leq j \leq N_{\mathbf{i}}\} \quad \text{for } k \in \mathbb{N}, k \geq 1.$$

Ignoring the root, which differs from all other individuals, the structure of this tree is given by a single-type branching process with a mixed Poisson offspring distribution.

Note also that a underlying structure of the tree $\mathcal{T}(v)$ constructed as described in Definition 3.4.6 has exactly the distribution $T(W_v, \nu)$ defined in Definition 2.2.2.

We now show that $\tilde{T}(v)$ can be coupled to $\mathcal{T}(v)$. This coupling relies on the coupling between ν_n and ν that can be obtained because the Wasserstein distance between the two measures is bounded by α_n. Furthermore, the Poisson random variables can easily be coupled once the vertex attributes are known.

Lemma 3.4.7. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of rank-one inhomogeneous random graphs that satisfies Assumption 2.2.1. Fix any $n \in naturals$. Let $\mathcal{V} \subseteq V_n$ be a set of vertices from $G_n = (V_n, E_n)$. We can couple the intermediate trees $(\tilde{T}(v))_{v\in\mathcal{V}}$ to the Poisson trees $(\mathcal{T}(v))_{v\in\mathcal{V}}$ defined as in Definition 3.4.6 such that*

$$\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}} \{\tilde{T}_\ell(v) \ncong \mathcal{T}_\ell(v)\}\Big) \leq \|\mathcal{V}\|\alpha_n\Big(\frac{1}{\vartheta} + (\Gamma_2 + 1)^{\ell-1}\Big(\frac{\Gamma_{2,n}}{\vartheta\Gamma_{1,n}} + 1\Big)\Big).$$

Proof. We couple the types and number of children for each individual separately. Since we can bound the expected number of individuals in the relevant trees, we can then give a bound for the probability that the trees have a different structure.

Clearly $\tilde{W}_\varnothing = W_\varnothing = W_v$ by construction, so the type at the root is the same in both trees. By Lemma A.1.3 we can couple the numbers of children of the root such that

$$\begin{aligned}\mathbb{P}_n(\tilde{N}_\varnothing \neq N_\varnothing) &\le \mathbb{E}_n\Big[\Big|\frac{\Lambda_n}{n\vartheta}W_v - W_v\Big|\Big]\\ &\le \mathbb{E}_n\Big[\Big|\frac{\Lambda_n}{n\vartheta} - 1\Big|W_v\Big]\\ &= W_v\mathbb{E}_n\Big[\Big|\frac{\Lambda_n}{n\vartheta} - 1\Big|\Big]\\ &\le W_v|\Gamma_{1,n} - 1|,\end{aligned}$$

where we used $\Lambda_n = n\vartheta\Gamma_{1,n}$ and $\mathcal{F}_n$-measurability of all involved terms in the last line.

For $\mathbf{i} \neq \varnothing$ first couple $\tilde{W}_\mathbf{i} \sim \hat{\nu}_n$ to $W_\mathbf{i} \sim \hat{\nu}$ optimally according to the Wasserstein distance, i.e.

$$\mathbb{E}_n[|\tilde{W}_\mathbf{i} - W_\mathbf{i}|] \le \alpha_n,$$

where α_n is $\mathcal{F}_n$-measurable and $\alpha_n \xrightarrow{\mathbb{P}} 0$. Then Lemma A.1.3 allows us to couple $\tilde{N}_\mathbf{i}$ and $N_\mathbf{i}$ such that

$$\begin{aligned}\mathbb{P}_n(\tilde{N}_\mathbf{i} \neq N_\mathbf{i}) &\le \mathbb{E}_n\Big[\Big|\frac{\Lambda_n}{n\vartheta}\tilde{W}_\mathbf{i} - W_\mathbf{i}\Big|\Big]\\ &\le \mathbb{E}_n\Big[\Big|\frac{\Lambda_n}{n\vartheta} - 1\Big|\tilde{W}_\mathbf{i}\Big] + \mathbb{E}_n[|\tilde{W}_\mathbf{i} - W_\mathbf{i}|]\\ &\le |\Gamma_{1,n} - 1|\mathbb{E}_n[\tilde{W}_\mathbf{i}] + \alpha_n.\end{aligned}$$

Since

$$\mathbb{E}_n[\tilde{W}_\mathbf{i}] = \sum_{i=1}^n W_i\frac{W_i}{\Lambda_n} = \sum_{i=1}^n \frac{W_i^2}{n\vartheta\Gamma_{1,n}} = \frac{1}{\Gamma_{1,n}}\Gamma_{2,n}$$

this implies

$$\mathbb{P}_n(\tilde{N}_\mathbf{i} \neq N_\mathbf{i}) \le |\Gamma_{1,n} - 1|\frac{\Gamma_{2,n}}{\Gamma_{1,n}} + \alpha_n.$$

Recall that $\Gamma_{1,n} = \vartheta^{-1}\mathbb{E}_n[W^{(n)}]$ and assume that $W^{(n)} \sim \nu_n$ is coupled optimally to $W \sim \nu$ according to the Wasserstein distance. Then we have $\mathbb{E}_n[W] = \vartheta$ and

$$|\Gamma_{1,n} - 1| = \vartheta^{-1}|\mathbb{E}_n[W^{(n)}] - \mathbb{E}_n[W]| \le \vartheta^{-1}\mathbb{E}_n[|W^{(n)} - W|] \le \vartheta^{-1}\alpha_n.$$

The tree structure of $\tilde{T}(v)$ and $\mathcal{T}(v)$ is determined only by $\tilde{N}_\mathbf{i}$ and $N_\mathbf{i}$, respectively. In particular $\tilde{T}_\ell(v)$ and $\mathcal{T}_\ell(v)$ can only disagree if there is an individual $\mathbf{i}$ at level $\ell-1$ in $\mathcal{T}_\ell(v)$ whose number of children $N_\mathbf{i}$ is different from $\tilde{N}_\mathbf{i}$. If we can control the number of vertices in $\mathcal{T}_\ell(v)$, a simple union bound and the fact that the distribution of $\tilde{N}_\mathbf{i}$ and $N_\mathbf{i}$ is the same for all $\mathbf{i} \neq \varnothing$ can be used to bound the probability that the coupling generates different tree structures. As in the proof of Proposition 3.4.4 we will work conditionally on the previous level.

Let $\mathcal{S}_r(v)$ be the set of individuals in $\mathcal{T}_r(v)$ and $\mathcal{D}_r(v) = \mathcal{S}_r(v) \setminus \mathcal{S}_{r-1}(v)$ the individuals at level $r \in \mathbb{N}$. Let $\mathcal{G}_\ell$ be the σ-algebra generated by $\tilde{W}_{\mathbf{i}}, W_{\mathbf{i}}, \tilde{N}_{\mathbf{i}}, N_{\mathbf{i}}$ up to level ℓ.

For $\ell = 1$ we have

$$\mathbb{P}_n(\tilde{T}_1(v) \ncong \mathcal{T}_1(v)) \leq \mathbb{P}_n(\tilde{N}_\varnothing \neq N_\varnothing) \leq W_v|\Gamma_{1,n} - 1| \leq W_v \vartheta^{-1}\alpha_n.$$

For $\ell \geq 2$ the coupling breaks if any of the individuals at level $\ell - 1$ creates a different number of children. Since all individuals apart from $\varnothing$ have the same distribution that is furthermore independent of other individuals (in particular independent from those in a lower level), that probability of generating different tree structures via this coupling is bounded by

$$\begin{aligned}
&\mathbb{1}_{\{\tilde{T}_{\ell-1}(v) \cong \mathcal{T}_{\ell-1}(v)\}} \mathbb{P}_n(\tilde{T}_\ell(v) \ncong \mathcal{T}_\ell(v) \mid \mathcal{G}_{\ell-1}) \\
&\quad \leq \mathbb{1}_{\{\tilde{T}_{\ell-1}(v) \cong \mathcal{T}_{\ell-1}(v)\}} \mathbb{P}_n\Big(\bigcup_{\mathbf{i} \in \mathcal{D}_{\ell-1}(v)} \{\tilde{N}_{\mathbf{i}} \neq N_{\mathbf{i}}\} \;\Big|\; \mathcal{G}_{\ell-1}\Big) \\
&\quad \leq \sum_{\mathbf{i} \in \mathcal{D}_\ell(v)} \mathbb{P}_n(\tilde{N}_{\mathbf{i}} \neq N_{\mathbf{i}}) \\
&\quad \leq |\mathcal{D}_{\ell-1}(v)| \mathbb{P}_n(\tilde{N}_{(1)} \neq N_{(1)}) \\
&\quad \leq |\mathcal{D}_{\ell-1}(v)| \alpha_n \Big(\frac{\Gamma_{2,n}}{\vartheta \Gamma_{1,n}} + 1 \Big).
\end{aligned}$$

Now sum over the probabilities that the tree structure is different for the first time at a specific level to find

$$\begin{aligned}
&\mathbb{P}_n(\tilde{T}_\ell(v) \ncong \mathcal{T}_\ell(v)) \\
&\quad \leq \mathbb{P}_n(\tilde{T}_1(v) \ncong \mathcal{T}_1(v)) + \sum_{r=2}^{\ell} \mathbb{E}_n[\mathbb{1}_{\{\tilde{T}_{r-1}(v) \cong \mathcal{T}_{r-1}(v)\}} \mathbb{P}_n(\tilde{T}_r(v) \ncong \mathcal{T}_r(v) \mid \mathcal{G}_{r-1})] \\
&\quad \leq W_v \vartheta^{-1} \alpha_n + \mathbb{E}_n[|\mathcal{S}_{\ell-1}(v)| - 1] \alpha_n \Big(\frac{\Gamma_{2,n}}{\vartheta \Gamma_{1,n}} + 1 \Big) \\
&\quad \leq W_v \vartheta^{-1} \alpha_n + W_v (\Gamma_2 + 1)^{\ell-1} \alpha_n \Big(\frac{\Gamma_{2,n}}{\vartheta \Gamma_{1,n}} + 1 \Big).
\end{aligned}$$

In the second to last step we used that $\mathbb{E}[N_\varnothing] = W_v$ and that $\mathbb{E}[N_{\mathbf{i}}] = \mathbb{E}[\widehat{W}] = \mathbb{E}[W^2]/\mathbb{E}[W] = \Gamma_2$ for $\mathbf{i} \neq \varnothing$ to conclude

$$\mathbb{E}[|\mathcal{S}_\ell(v)|] \leq 1 + W_v(\Gamma_2 + 1)^\ell.$$

Sum these bounds over $v \in \mathcal{V}$ to finish the proof. □

With this lemma it is now possible to couple the neighbourhoods to the limiting trees as claimed in Proposition 2.2.3. We restate the proposition in the notation of this section.

Proposition 2.2.3. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of rank-one inhomogeneous random graphs satisfying Assumption 2.2.1. Fix any $n \in \mathbb{N}$. Let $\mathcal{V} \subseteq V_n$ be a subset of vertices of $G_n = (V_n, E_n)$. Then for all $\ell \in \mathbb{N}$ the neighbourhoods around $B_\ell(v)$ can be coupled to independent limiting trees $\mathcal{T}(v)$ as defined in Definition 3.4.6 such that*

$$\begin{aligned}
&\mathbb{P}_n\Big(\bigcup_{v\in\mathcal{V}}\{B_\ell(v)\ncong\mathcal{T}_\ell(v)\}\Big)\\
&\le \|\mathcal{V}\|_2\frac{\Gamma_{2,n}}{n\vartheta}+\|\mathcal{V}\|_+\Gamma_{1,n}\\
&\quad+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell\Big(\frac{\Gamma_{3,n}}{n\vartheta}+\kappa_{1,n}+\kappa_{2,n}+\frac{2+\Gamma_{1,n}}{k_n}+\frac{k_n}{n\vartheta}\Big)\\
&\quad+|\mathcal{V}|\frac{1}{k_n}+\frac{k_n^2}{n\vartheta\Gamma_{1,n}}+\|\mathcal{V}\|\alpha_n\Big(\frac{1}{\vartheta}+(\Gamma_2+1)^{\ell-1}\Big(\frac{\Gamma_{2,n}}{\vartheta\Gamma_{1,n}}+1\Big)\Big)
\end{aligned}$$

for all sequences $(k_n)_{n\in\mathbb{N}}\subseteq(0,\infty)$.

Proof. This is a direct consequence of Proposition 3.4.5 and Lemma 3.4.7. □

3.5 More complex couplings

This section collects a few additional more specialised coupling results for the weighted inhomogeneous random graph $\mathbf{G}_n$ that follow directly from Proposition 2.2.3 for the unweighted model G_n. These results are extensions of lemmas shown by Cao [Cao21, § 6] for Erdős–Rényi graphs.

The following is just a reformulation of Proposition 2.2.3.

Lemma 3.5.1. *Fix $\ell \in \mathbb{N}$ and let $\mathcal{V} \subseteq V_n$ be a set of vertices vertices. Then there is a coupling $((B_\ell(v, G_n), \mathcal{T}_\ell(v))_{v\in\mathcal{V}})$ such that the $\mathcal{T}_\ell(v) \sim T_\ell(W_v, \nu)$ are independent limiting trees with*

$$\mathbb{P}_n(B_\ell(v,G_n)\cong\mathcal{T}_\ell(v)\text{ for all }v\in\mathcal{V})\ge 1-\eta_{n,\ell}(\mathcal{V}),$$

where

$$\begin{aligned}
\eta_{n,\ell}(\mathcal{V}) &= \|\mathcal{V}\|_2\frac{\Gamma_{2,n}}{n\vartheta}+\|\mathcal{V}\|_+\Gamma_{1,n}\\
&\quad+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell\Big(\frac{\Gamma_{3,n}}{n\vartheta}+\kappa_{1,n}+\kappa_{2,n}+\frac{2+\Gamma_{1,n}}{k_n}+\frac{k_n}{n\vartheta}\Big)\\
&\quad+|\mathcal{V}|\frac{1}{k_n}+\frac{k_n^2}{n\vartheta\Gamma_{1,n}}+\|\mathcal{V}\|\alpha_n\Big(\frac{1}{\vartheta}+(\Gamma_2+1)^{\ell-1}\Big(\frac{\Gamma_{2,n}}{\vartheta\Gamma_{1,n}}+1\Big)\Big).
\end{aligned}$$

Lemma 3.5.2. *Fix $\ell \in \mathbb{N}$ and let $\mathcal{V} \subseteq V_n$ be a set of vertices vertices. Then there is a coupling $((B_\ell(v, \mathbf{G}_n), \mathbf{T}_\ell(v))_{v\in\mathcal{V}})$ such that the $\mathbf{T}_\ell(v)$ are independent with distribution $\mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$ and*

$$\mathbb{P}_n(B_\ell(v,\mathbf{G}_n)\cong\mathbf{T}_\ell(v)\text{ for all }v\in\mathcal{V})\ge 1-\varepsilon_{n,\ell}(\mathcal{V}),$$

where

$$\varepsilon_{n,\ell}(\mathcal{V}) = \eta_{n,\ell}(\mathcal{V}) + (|\mathcal{V}| + \|\mathcal{V}\|(\Gamma_2+1)^{\ell})(d_{\mathrm{TV}}(\mu_{E,n},\mu_E) + d_{\mathrm{TV}}(\mu_{V,n},\mu_V))$$

with $\eta_{n,\ell}(\mathcal{V})$ *as in Lemma 3.5.1.*

Proof. Apply Lemma 3.5.1 to couple $((B_\ell(v,G_n),\mathcal{T}_\ell(v))_{v\in\mathcal{V}})$ such that the $\mathcal{T}_\ell(v)$ are independent with distribution $T_\ell(W_v,\nu)$ and

$$\mathbb{P}_n(A) = \mathbb{P}_n(B_\ell(v,G_n) \cong \mathcal{T}_\ell(v) \text{ for all } v \in \mathcal{V}) \geq 1 - \eta_{n,\ell}(\mathcal{V}),$$

where A is the event that $B_\ell(v,G_n) \cong \mathcal{T}_\ell(v)$ for all $v \in \mathcal{V}$. This provides the coupling of the underlying graph structure. It remains to also couple the edge and vertex weights.

For each edge $e \in V_n^{(2)}$ couple w_e, the weight in $\mathbf{G}_n$, to $\tilde{w}_e$, such that $\mathbb{P}_n(w_e \neq \tilde{w}_e) = d_{\mathrm{TV}}(\mu_{E,n},\mu_E)$. Similarly, for each each vertex $v \in V_n$ introduce a coupling $(w_v,\tilde{w}_v)$ such that $\mathbb{P}_n(w_v \neq \tilde{w}_v) = d_{\mathrm{TV}}(\mu_{V,n},\mu_V)$.

Let E^E be the event that there is an edge e in any of the $B_\ell(v,\mathbf{G}_n)$ such that $w_e \neq \tilde{w}_e$. Similarly let E^V be the event that there is a vertex u in any of the $B_\ell(v,\mathbf{G}_n)$ such that $w_v \neq \tilde{w}_v$. On the event A we have that $B_\ell(v,G_n)$ and $\mathcal{T}_\ell(v)$ are isomorphic, so that we can reformulate E^E and E^V in terms of edges or vertices in $\mathcal{T}_\ell(v)$. Let $S_\ell(v)$ be the set of vertices in $\mathcal{T}_\ell(v)$. Since a tree with n vertices has $n-1$ edges and the $\mathcal{T}_\ell(v)$ are trees, the number of edges and vertices that are relevant for E^E and E^V can be bounded by $\sum_{v\in\mathcal{V}}|S_\ell(v)|$.

In particular the same calculation as in Lemma 3.4.7 implies

$$\begin{aligned}\mathbb{P}_n(A \cap E^E) &\leq \sum_{v\in\mathcal{V}} \mathbb{E}_n[|S_\ell(v)|]d_{\mathrm{TV}}(\mu_{E,n},\mu_E)\\ &\leq (|\mathcal{V}| + \|\mathcal{V}\|(\Gamma_2+1)^{\ell})d_{\mathrm{TV}}(\mu_{E,n},\mu_E)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_n(A \cap E^V) &\leq \sum_{v\in\mathcal{V}} \mathbb{E}_n[|S_\ell(v)|]d_{\mathrm{TV}}(\mu_{V,n},\mu_V)\\ &\leq (|\mathcal{V}| + \|\mathcal{V}\|(\Gamma_2+1)^{\ell})d_{\mathrm{TV}}(\mu_{V,n},\mu_V).\end{aligned}$$

On the set A couple $B_\ell(v,\mathbf{G}_n)$ to $\mathbf{T}_\ell(v)$ by assigning edge weight $\tilde{w}_e$ to the edge isomorphic to e and vertex weight $\tilde{w}_v$ to the vertex isomorphic to v in $\mathcal{T}_\ell(v)$ resulting in $\mathbf{T}_\ell(v) \sim \mathbf{T}_\ell(W_v,\nu,\mu_E,\mu_V)$. This coupling satisfies

$$\begin{aligned}&\mathbb{P}_n(B_\ell(v,\mathbf{G}_n) \cong \mathbf{T}_\ell(v) \text{ for all } v \in \mathcal{V})\\ &\quad\geq \mathbb{P}_n(A) - \mathbb{P}_n(A \cap E_0)\\ &\quad\geq 1 - \eta_{n,\ell}(\mathcal{V}) - (|\mathcal{V}| + \|\mathcal{V}\|(\Gamma_2+1)^{\ell})(d_{\mathrm{TV}}(\mu_{E,n},\mu_E) + d_{\mathrm{TV}}(\mu_{V,n},\mu_V)).\end{aligned}$$

This proves the claim. □

Indeed the coupled tree structure of the neighbourhood of a vertex v can be manipulated slightly to be independent of a vertex $e' = \{u', v'\}$ which does not emanate from v. In essence we rerandomise the relevant edge and bound the probability that it actually occurs in the neighbourhood.

Lemma 3.5.3. *Let $v \in V_n$ be a vertex in $\mathbf{G}_n$ and let $e' = \{u', v'\} \in V_n^{(2)}$ be an edge with endpoints distinct from v. Given a coupling*

$$(B_\ell(v, \mathbf{G}_n), \mathbf{T}_\ell(v))$$

of $B_\ell(v, \mathbf{G}_n)$ with $\mathbf{T}_\ell(v) \sim \mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$ that satisfies

$$\varepsilon_{n,\ell}(\{v\}) \geq 1 - \mathbb{P}_n(B_\ell(v, \mathbf{G}_n) \cong \mathbf{T}_\ell(v))$$

it is is possible to couple $(B_\ell(v, \mathbf{G}_n), \tilde{\mathbf{T}}_\ell(v))$ such that $\tilde{\mathbf{T}}_\ell(v)$ is independent of $Y_{e'}$, where $Y_{e'} = (X_{e'}, X'_{e'})$, and

$$\mathbb{P}_n(B_\ell(v, \mathbf{G}_n) \cong \tilde{\mathbf{T}}_\ell(v)) \mid Y_{e'}) \geq 1 - \Big(\varepsilon_{n,\ell}(\{v\}) + C\frac{W_v(W_{v'} + W_{u'})}{n\vartheta}(\Gamma_{2,n} + 1)^{\ell+1}\Big).$$

Proof. Let X'' be a copy of X that is independent of everything else, in particular independent of (X, W) and (X', W'). Let $\mathbf{G}''_n$ be the weighted graph obtained from $\mathbf{G}_n$ by replacing $X_{e'}$ with $X''_{e'}$. Based on the initial coupling, couple $(B_\ell(v, \mathbf{G}''_n), \tilde{\mathbf{T}}_\ell(v))$, where $\tilde{\mathbf{T}}_\ell(v) \sim \mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$. By construction $B_\ell(v, \mathbf{G}''_n)$ is independent of $Y_{e'}$, so we may pick this coupling in a way such that $\tilde{\mathbf{T}}_\ell(v)$ is independent of $Y_{e'}$ as well.

Moreover,

$$\begin{aligned}
&\mathbb{P}_n(B_\ell(v, \mathbf{G}_n) \cong \tilde{\mathbf{T}}_\ell(v) \mid Y_{e'}) \\
&\quad \geq 1 - (\mathbb{P}_n(B_\ell(v, \mathbf{G}_n) \ncong B_\ell(v, \mathbf{G}''_n) \mid Y_{e'}) + \mathbb{P}_n(B_\ell(v, \mathbf{G}''_n) \ncong \tilde{\mathbf{T}}_\ell(v) \mid Y_{e'}))
\end{aligned}$$

By construction $B_\ell(v, \mathbf{G}''_n)$ and $\tilde{\mathbf{T}}_\ell(v)$ are independent of $Y_{e'}$, so by assumption

$$\begin{aligned}
\mathbb{P}_n(B_\ell(v, \mathbf{G}''_n) \ncong \tilde{\mathbf{T}}_\ell(v) \mid Y_{e'}) &= \mathbb{P}_n(B_\ell(v, \mathbf{G}''_n) \ncong \tilde{\mathbf{T}}_\ell(v)) \\
&\leq \varepsilon_{n,\ell}(\{v\}).
\end{aligned}$$

Finally, $\mathbf{G}''_n$ is independent of $Y_{e'}$ and differs from $\mathbf{G}_n$ only in e'. Hence, $B_\ell(v, \mathbf{G}''_n)$ can differ from $B_\ell(v, \mathbf{G}_n)$ only if e' is present in one and not the other. Thus a rough estimate yields

$$\mathbb{P}_n(B_\ell(v, \mathbf{G}''_n) \ncong B_\ell(v, \mathbf{G}_n) \mid Y_{e'}) \leq \mathbb{P}_n(e' \in B_\ell(v, \mathbf{G}''_n)) + \mathbb{P}_n(e' \in B_\ell(v, \mathbf{G}_n) \mid X_{e'}).$$

By Corollary 3.1.16 the first term can be bounded as follows:

$$\mathbb{P}_n(e' \in B_\ell(v, \mathbf{G}''_n)) \leq \frac{W_v(W_{u'} + W_{v'})}{n\vartheta}(\Gamma_{2,n} + 1)^\ell.$$

For the second probability note that since e' can only be present in the neighbourhood if one of its vertices u' or v' is present in the neighbourhood in its own right, i.e. when e' itself is ignored, we have by Corollary 3.1.15 that

$$\begin{aligned}
&\mathbb{P}_n(e' \in B_\ell(v, \mathbf{G}_n) \mid X_{e'}) \\
&\quad\le \mathbb{P}_n(v' \in B_\ell(v, \mathbf{G}_n - e')) + \mathbb{P}_n(u' \in B_\ell(v, \mathbf{G}_n - e')) \\
&\quad\le \mathbb{P}_n(v' \in B_\ell(v, \mathbf{G}_n)) + \mathbb{P}_n(u' \in B_\ell(v, \mathbf{G}_n)) \\
&\quad\le \frac{W_v(W_{v'} + W_{u'})}{n\vartheta}(\Gamma_{2,n} + 1)^\ell.
\end{aligned}$$

Together the last inequalities show the claim. □

Similarly we can find a coupling for the effect of flipping the edge $e = \{v, u\}$ between v and u and another edge $e' = \{v', u'\}$ that does not have any vertex in common with e on the neighbourhood of v.

Lemma 3.5.4. *Fix two vertices v and u and set $e = \{v, u\}$. Let $e' = \{v', u'\}$ be another edge with vertices distinct from u and v. Given a coupling*

$$(B_\ell(v, \mathbf{G}_n), B_\ell(u, \mathbf{G}_n), \mathbf{T}_\ell(v), \mathbf{T}_\ell(u))$$

with independent $\mathbf{T}_\ell(v) \sim \mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$ and $\mathbf{T}_\ell(u) \sim \mathbf{T}_\ell(W_u, \nu, \mu_E, \mu_V)$ that satisfies

$$\varepsilon_{n,\ell}(\{u, v\}) \ge 1 - \mathbb{P}_n(B_\ell(v, \mathbf{G}_n) \cong \mathbf{T}_\ell(v), B_\ell(u, \mathbf{G}_n) \cong \mathbf{T}_\ell(u))$$

it is possible to couple $(B_\ell(v, \mathbf{G}_n), B_\ell(v, \mathbf{G}_n^e), \tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v))$ such that

$$\begin{aligned}
(\tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v)) \mid (Y_e = (1, 0), Y_{e'}) &\overset{\mathcal{D}}{=} (\tilde{\mathbf{T}}_\ell, \mathbf{T}_\ell), \\
(\tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v)) \mid (Y_e = (0, 1), Y_{e'}) &\overset{\mathcal{D}}{=} (\mathbf{T}_\ell, \tilde{\mathbf{T}}_\ell),
\end{aligned}$$

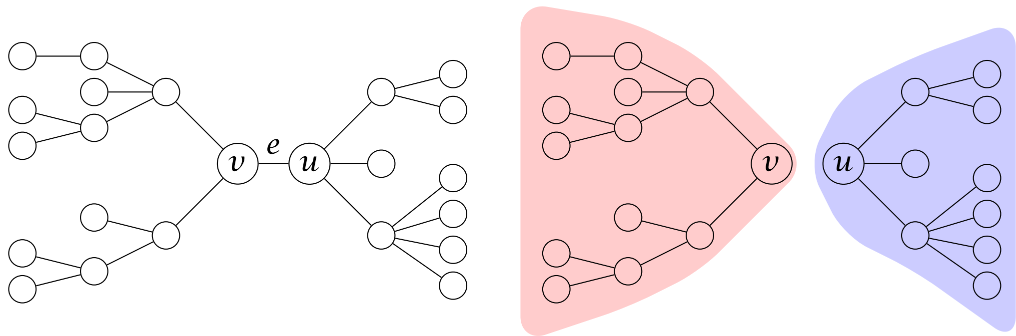
where $Y_e = (X_e, X_e')$ and $Y_{e'} = (X_{e'}, X_{e'}')$ as well as $\mathbf{T}_\ell \sim \mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$ and $\tilde{\mathbf{T}}_\ell \sim \tilde{\mathbf{T}}_\ell(W_v, W_u, \nu, \mu_E, \mu_V)$. Furthermore,

$$\begin{aligned}
&\mathbb{P}_n((B_\ell(v, \mathbf{G}_n), B_\ell(v, \mathbf{G}_n^e) \cong (\tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v))) \mid Y_e, Y_{e'}) \\
&\quad\ge 1 - \Big(\varepsilon_{n,\ell}(\{u, v\}) + 2d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) \\
&\qquad\qquad + C\frac{W_u W_v + (W_u + W_v)(W_{u'} + W_{v'})}{n\vartheta}(\Gamma_{2,n} + 1)^{2\ell}\Big).
\end{aligned}$$

Proof. Let (W'', X'') be a copy of (W, X) that is independent of everything else. Let $\mathbf{G}_n''$ be the weighted graph obtained from $\mathbf{G}_n$ by replacing X_e with X_e'' and $X_{e'}$ with $X_{e'}''$. Based on the initial coupling, couple $B_\ell(v, \mathbf{G}_n'')$ with $\mathbf{T}_\ell''(v)$ and $B_{k-1}(u, \mathbf{G}_n'')$ with $\mathbf{T}_{\ell-1}''(u)$, where $\mathbf{T}_\ell''(v) \sim \mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$ and $\mathbf{T}_{\ell-1}''(u) \sim \mathbf{T}_\ell(W_u, \nu, \mu_E, \mu_V)$ are independent. By construction $B_\ell(v, \mathbf{G}_n'')$ and $B_{\ell-1}(u, \mathbf{G}_n'')$ are independent of $Y_e, Y_{e'}$, so we may pick this coupling in a way that $\mathbf{T}''_\ell(v)$ and $\mathbf{T}''_{\ell-1}(v)$ are independent of $Y_e, Y_{e'}$ as well.

For brevity write $B_\ell = B_\ell(v, \mathbf{G}_n)$ and $B'_\ell = B_\ell(v, \mathbf{G}^e_n)$. These neighbourhoods can be constructed from the smaller neighbourhoods of v and u on $\mathbf{G}_n - e$ if we take into account X_e and w_e or X'_e and w'_e as required. That is to say there is a function Ψ that describes the procedure of possibly 'gluing together' the smaller neighbourhoods (see Fig. 3.6) such that

$$B_\ell = \Psi(B_\ell(v, \mathbf{G}_n - e), B_{\ell-1}(u, \mathbf{G}_n - e), X_e, w_e)$$
$$B'_\ell = \Psi(B_\ell(v, \mathbf{G}_n - e), B_{\ell-1}(u, \mathbf{G}_n - e), X'_e, w'_e).$$



(a) $B_3(v, \mathbf{G})$ (b) $B_3(v, \mathbf{G} - e)$ and $B_2(u, \mathbf{G} - e)$

Figure 3.6: Illustration of the gluing procedure. The neighbourhood $B_3(v, \mathbf{G})$ can be obtained by combining the two neighbourhoods $B_3(v, \mathbf{G} - e)$ and $B_2(u, \mathbf{G} - e)$ that cannot use e and information about e.

Let $(w_e, w'_e, \tilde{w}_e, \tilde{w}'_e)$ be a coupling independent of everything else that satisfies $w_e, w'_e \sim \mu_{E,n}$, $\tilde{w}_e, \tilde{w}'_e \sim \mu_E$ as well as

$$\mathbb{P}_n(w_e \neq \tilde{w}_e) \leq d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) \quad \text{and} \quad \mathbb{P}_n(w'_e \neq \tilde{w}'_e) \leq d_{\mathrm{TV}}(\mu_{E,n}, \mu_E).$$

Then define

$$\tilde{\mathbf{T}}_\ell(v) = \Psi(\mathbf{T}''_\ell(v), \mathbf{T}''_{\ell-1}(u), X_e, \tilde{w}_e),$$
$$\tilde{\mathbf{T}}^e_\ell(v) = \Psi(\mathbf{T}''_\ell(v), \mathbf{T}''_{\ell-1}(u), X'_e, \tilde{w}'_e).$$

Conditionally on $Y_e, Y_{e'}$ the pair $(\tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}^e_\ell(v))$ has the desired distribution.

Let E_0 be the event that $B_\ell(v, \mathbf{G}''_n)$ and $B_{\ell-1}(u, \mathbf{G}''_n)$ share a vertex. On the complement of E_0 it is possible to join $B_\ell(v, \mathbf{G}''_n)$ and $B_{\ell-1}(u, \mathbf{G}''_n)$ at the edge e to obtain a tree provided both $B_\ell(v,\mathbf{G}''_n)$ and $B_{\ell-1}(u,\mathbf{G}''_n)$ are trees. Then

$$\begin{aligned}\mathbb{P}_n((B_\ell,B'_\ell) &\cong (\tilde{\mathbf{T}}_\ell(v),\tilde{\mathbf{T}}^e_\ell(v)) \mid Y_e,Y_{e'}) \\ &\geq \mathbb{P}_n(B_\ell(v,\mathbf{G}''_n) \cong \mathbf{T}''_\ell(v), B_{\ell-1}(u,\mathbf{G}''_n) \cong \mathbf{T}''_{\ell-1}(u)) \\ &\quad - \mathbb{P}_n(E_0) \\ &\quad - \mathbb{P}_n(B_\ell(v,\mathbf{G}''_n) \neq B_\ell(v,\mathbf{G}_n - e) \mid Y_e, Y_{e'}) \\ &\quad - \mathbb{P}_n(B_{\ell-1}(u,\mathbf{G}''_n) \neq B_{\ell-1}(u,\mathbf{G}_n - e) \mid Y_e, Y_{e'}) \\ &\quad - \mathbb{P}_n(\tilde{w}_e \neq w_e) - \mathbb{P}_n(\tilde{w}'_e \neq w'_e).\end{aligned} \tag{3.49}$$

By assumption

$$\mathbb{P}_n(B_\ell(v,\mathbf{G}''_n) \cong \mathbf{T}_\ell(v), B_{\ell-1}(u,\mathbf{G}''_n) \cong \mathbf{T}_{\ell-1}(u)) \geq 1 - \varepsilon_{n,\ell}(\{u,v\}).$$

Furthermore, $B_\ell(v,\mathbf{G}''_n)$ and $B_{\ell-1}(u,\mathbf{G}''_n)$ share a vertex only if there is a path from v to u of length at most $2\ell - 1$. Hence, Corollary 3.1.14 implies

$$\mathbb{P}_n(E_0) \leq \mathbb{P}_n(v \rightsquigarrow_{2\ell-1} v) \leq \frac{W_v W_u}{n\vartheta}(\Gamma_{2,n}+1)^{2\ell-1}.$$

Finally, $\mathbf{G}''_n$ is independent of $Y_e, Y_{e'}$ and differs from $\mathbf{G}_n$ only in e and e'. Hence, the neighbourhood $B_\ell(v,\mathbf{G}''_n)$ can differ from $B_\ell(v,\mathbf{G}_n - e)$ only if e is present in the former or if e' is present in one and not the other. Thus

$$\begin{aligned}&\mathbb{P}_n(B_\ell(v,\mathbf{G}''_n) \neq B_\ell(v,\mathbf{G}_n - e) \mid Y_e, Y_{e'}) \\ &\quad \leq \mathbb{P}_n(e \in B_\ell(v,\mathbf{G}''_n)) + \mathbb{P}_n(e' \in B_\ell(v,\mathbf{G}''_n)) + \mathbb{P}_n(e' \in B_\ell(v,\mathbf{G}_n - e) \mid X_{e'}).\end{aligned}$$

For the first term note that e is present in $B_\ell(v,\mathbf{G}''_n)$ if and only if v is connected to u via e, i.e.

$$\mathbb{P}_n(e \in B_\ell(v,\mathbf{G}''_n)) = \mathbb{E}_n[X''_e] = \frac{W_u W_v}{n\vartheta}.$$

For the second term we we can apply Corollary 3.1.16

$$\mathbb{P}_n(e' \in B_\ell(v,\mathbf{G}''_n)) \leq \frac{W_v(W_{u'}+W_{v'})}{n\vartheta}(\Gamma_{2,n}+1)^\ell.$$

The third term contains a conditioning which can be removed as follows. Since e' can only be present in the neighbourhood if one of its vertices u' or v' is present in the neighbourhood in its own right, i.e. when e' itself is ignored, we can drop the conditioning on $X_{e'}$, so that by Corollary 3.1.15

$$\begin{aligned}&\mathbb{P}_n(e' \in B_\ell(v,\mathbf{G}_n - e) \mid X_{e'}) \\ &\quad \leq \mathbb{P}_n(v' \in B_\ell(v,\mathbf{G}_n - \{e,e'\})) + \mathbb{P}_n(u' \in B_\ell(v,\mathbf{G}_n - \{e,e'\})) \\ &\quad \leq \mathbb{P}_n(v' \in B_k(v,\mathbf{G}_n)) + \mathbb{P}_n(u' \in B_\ell(v,\mathbf{G}_n)) \\ &\quad \leq \frac{W_v(W_{u'}+W_{v'})}{n\vartheta}(\Gamma_{2,n}+1)^\ell.\end{aligned}$$

The remaining terms can be treated similarly. Putting everything together, this shows the claim. □

The result from Lemma 3.5.4 can be slightly simplified by dropping the conditioning on e'.

Lemma 3.5.5. *Fix two vertices v and u and set $e = \{v, u\}$. Given a coupling*

$$(B_\ell(v, \mathbf{G}_n), B_\ell(u, \mathbf{G}_n), \mathbf{T}_\ell(v), \mathbf{T}_\ell(u))$$

with independent $\mathbf{T}_\ell(v) \sim \mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$ and $\mathbf{T}_\ell(u) \sim \mathbf{T}_\ell(W_u, \nu, \mu_E, \mu_V)$ that satisfies

$$\varepsilon_{n,\ell}(\{u, v\}) \geq 1 - \mathbb{P}_n(B_\ell(v, \mathbf{G}_n) \cong \mathbf{T}_\ell(v), B_\ell(u, \mathbf{G}_n) \cong \mathbf{T}_\ell(u))$$

it is possible to couple $(B_\ell(v, \mathbf{G}_n), B_\ell(v, \mathbf{G}_n^e), \tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v))$ such that

$$\begin{aligned}
(\tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v)) \mid Y_e = (1, 0) &\overset{\mathcal{D}}{=} (\tilde{\mathbf{T}}_\ell, \mathbf{T}_\ell), \\
(\tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v)) \mid Y_e = (0, 1) &\overset{\mathcal{D}}{=} (\mathbf{T}_\ell, \tilde{\mathbf{T}}_\ell),
\end{aligned}$$

where $Y_e = (X_e, X_e')$ and $\mathbf{T}_\ell \sim \mathbf{T}_\ell(W_v, \nu, \mu_E, \mu_V)$ and $\tilde{\mathbf{T}}_\ell \sim \tilde{\mathbf{T}}_\ell(W_v, W_u, \nu, \mu_E, \mu_V)$. Furthermore,

$$\begin{aligned}
&\mathbb{P}_n((B_\ell(v, \mathbf{G}_n), B_\ell(v, \mathbf{G}_n^e) \cong (\tilde{\mathbf{T}}_\ell(v), \tilde{\mathbf{T}}_\ell^e(v))) \mid Y_e) \\
&\quad \geq 1 - \Big(\varepsilon_{n,\ell}(\{u, v\}) + 2d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) + C\frac{W_u W_v}{n\vartheta}(\Gamma_{2,n} + 1)^{2\ell}\Big).
\end{aligned}$$

Proof. Replicate the steps of the proof of Lemma 3.5.5, but do not modify the graph at e' and do not condition on $Y_{e'}$.

In particular the probabilities in (3.49) are only conditioned on Y_e and not on $Y_{e'}$. The relevant conditional probabilities on the right-hand side can then be estimated as

$$\begin{aligned}
\mathbb{P}_n(B_\ell(v, \mathbf{G}_n'') \neq B_\ell(v, \mathbf{G}_n - e) \mid Y_e) &\leq \mathbb{P}_n(e \in B_\ell(v, \mathbf{G}_n'')) \\
&\leq \mathbb{E}_n[X_e''] \\
&\leq \frac{W_u W_v}{n\vartheta}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}_n(B_{\ell-1}(u, \mathbf{G}_n'') \neq B_{\ell-1}(u, \mathbf{G}_n - e) \mid Y_e) &\leq \mathbb{P}_n(e \in B_{\ell-1}(u, \mathbf{G}_n'')) \\
&\leq \mathbb{E}_n[X_e''] \\
&\leq \frac{W_u W_v}{n\vartheta}.
\end{aligned}$$

Putting these results together we obtain the claimed bound. □

Chapter 4

Proof of the Main Result

The proof of Theorem 2.3.5 is based on the perturbative Stein's method. We will follow the approach used by Cao [Cao21, § 5] for the case of the (homogeneous) Erdős–Rényi graph. The same general strategy of combining the perturbative Stein's method with local approximations of the problem was also used by Chatterjee and Sen [CS17; see also Cha14, § 4] to show a central limit theorem for the minimal spanning tree in Euclidean space $\mathbb{R}^d$ and the lattice $\mathbb{Z}^d$ for dimensions $d \geq 2$.

Before we start with the proof, we will give a brief introduction to the perturbative approach to Stein's method.

4.1 Perturbative approach to Stein's method

This short introduction to Stein's method follows Chatterjee [Cha14], who formulated a (generalised) *perturbative* approach to Stein's method [first presented in Cha08] that is particularly suitable for applications to random graphs. We will not prove any of the results in this section, the proofs can be found in Chatterjee's survey paper [Cha14].

Stein's method [Ste72] was developed with the aim of finding explicit bounds for the error in the normal approximation to the distribution of a random variable. The underlying inspiration for the method is the observation that the only distribution satisfying the equation

$$\mathbb{E}[Zf(Z)] = \mathbb{E}[f'(Z)]$$

for all absolutely continuous f with derivative f' (existing almost everywhere) such that $\mathbb{E}[|f'(Z)|] < \infty$ is the standard normal distribution.

Since the standard normal distribution is the unique solution to this distributional equation, one can hope that a random variable W that satisfies this equation approximately is itself approximately normal. Stein makes this intuition rigorous as follows.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a measurable function and consider the differential equation

$$f'(x) - xf(x) = g(x) - \mathbb{E}[g(Z)] \tag{4.1}$$

for functions $f\colon \mathbb{R} \to \mathbb{R}$ where Z is a standard normal random variable. Stein showed that if g is bounded, then there always exists a bounded solution f_g of (4.1). Hence, taking expectations we have for any random variable W and bounded measurable function $g\colon \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[g(W)] - \mathbb{E}[g(Z)] = \mathbb{E}[f_g'(W) - Wf_g(W)].$$

This means that the expectation of $g(W)$ is close to the expectation of $g(Z)$ if W is close to satisfying the distributional equation $\mathbb{E}[Wf(W)] = \mathbb{E}[f'(W)]$ for $f = f_g$. To translate this observation into bounds for the distance of W to a standard normal distribution note that the supremum over a certain class of functions $\mathcal{G}$ on the left-hand side can be estimated by the supremum over the solutions f_g for $g \in \mathcal{G}$. In particular if we can somehow characterise to which class of functions $\mathcal{F}$ the solutions f_g belong, we can simply bound the supremum over $\mathcal{G}$ on the left-hand side by the supremum over $\mathcal{F}$ on the right-hand side.

We can use this idea to bound the Kolmogorov distance between W and a standard normal distribution.

Proposition 4.1.1 [Cha14, Prop. 1.1]**.** *Let $\mathcal{D}$ be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that are twice continuously differentiable with $|f(x)| \leq 1$, $|f'(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$. Let Z be a standard normal random variable and W any random variable, then*

$$\sup_{t\in\mathbb{R}} \bigl| \mathbb{P}(W \leq t) - \mathbb{P}(Z \leq t) \bigr| \leq 2\Bigl(\sup_{f\in\mathcal{D}} \bigl| \mathbb{E}[f'(W) - Wf(W)] \bigr| \Bigr)^{1/2}.$$

One way to show that the right-hand side is indeed small for a given W is the *method of exchangeable pairs* [Cha14, § 2; see also Ros11, § 3.3]. In this method we couple W to another random variable W' with the same distribution so that the vector (W, W') is exchangeable. If this pair additionally satisfies

(i) $\mathbb{E}[W' - W \mid W] = -\lambda W$,

(ii) $\mathbb{E}[(W' - W)^2 \mid W] = 2\lambda + o(\lambda)$ and

(iii) $\mathbb{E}[|W' - W|^3] = o(\lambda)$,

a small calculation shows that $\mathbb{E}[f'(W) - Wf(W)] = o(1)$, so that the right-hand side becomes small.

We mention briefly how this would work in the 'standard' central limit theorem setting.

Example 4.1.2. Let $X_1, \ldots, X_n$ be independent random variables with $\mathbb{E}[W_i] = 0$, $\mathbb{E}[W_i^2] = 1$ and $\mathbb{E}[|X_i|^4] < \infty$. We set $W = n^{-1/2} \sum_{i=1}^n X_i$.

Then we can obtain an exchangeable pair via resampling. Let $X_1', \ldots, X_n'$ be an independent copy of $X_1, \ldots, X_n$ and I be an independent uniform choice from $\{1, \ldots, n\}$. Then $(W, W') = (W, W - n^{-1/2}(X_I - X_I'))$ is an exchangeable pair that satisfies the additional assumptions (i) to (iii) for $\lambda = 1/n$.

The idea of the generalised perturbative approach to Stein's method generalises the resampling approach to obtain exchangeable pairs. It can be summarised as saying that if small perturbations or resamplings of a random variable are approximately independent, then the random variable is approximately normal. We make these notions of perturbation and approximately independent more rigorous in the following.

Let $\mathcal{X}$ be a measure space and suppose $X = (X_1, \dots, X_n)$ is a vector of independent $\mathcal{X}$-valued (not necessarily identically distributed) random variables. Let $f\colon \mathcal{X}^n \to \mathbb{R}$ be a measurable function and set $W = f(X)$. Suppose that $\mathbb{E}[W] = 0$ and $\mathbb{E}[W^2] = 1$.

Let $X' = (X_1', \dots, X_n')$ be an independent copy of X. Let $[n] = \{1, \dots, n\}$ and for each $A \subseteq [n]$ define the perturbed or resampled random vector X^A as

$$X_i^A = \begin{cases} X_i' & \text{if } i \in A, \\ X_i & \text{if } i \notin A. \end{cases}$$

We will write X^i instead of $X^{\{i\}}$ and $X^{A \cup i}$ instead of $X^{A \cup \{i\}}$.

Define a 'randomised derivative' of f along the i-th coordinate as

$$\Delta_i f = f(X) - f(X^i)$$

and for each $A \subseteq [n]$ and $i \notin A$

$$\Delta_i f^A = f(X^A) - f(X^{A \cup i}).$$

For any $i \in [n]$ and $A \subseteq [n] \setminus \{i\}$ define

$$\nu(A) = \frac{1}{n \binom{n-1}{|A|}}.$$

For a fixed $i \in [n]$ this defines a probability measure on the subsets of $[n] \setminus \{i\}$. Set

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{A \subseteq [n] \setminus \{i\}} \nu(A) \Delta_i f \Delta_i f^A.$$

With this notation, the Kolmogorov distance between W and a standard normal random variable can be estimated by controlling the (co)variance and third moment of the randomised derivative.

Theorem 4.1.3 [Cha14, Thm. 3.1]**.** *Let W be as before and Z be a standard normal random variable. Then*

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(W \le t) - \mathbb{P}(Z \le t)| \le 2\left(\sqrt{\operatorname{Var}(\mathbb{E}[T \mid W])} + \frac{1}{4} \sum_{i=1}^{n} \mathbb{E}[|\Delta_i f|^3]\right)^{1/2}.$$

On first glance the variance term on the right-hand side of the previous theorem might look a bit daunting. The following corollary simplifies the right-hand side to a simple sum of bounds for covariances.

Corollary 4.1.4 [Cha14, Cor. 3.2]**.** *In the setting of Theorem 4.1.3 suppose for each i, j there is a constant c_{ij} such that for all $A \subseteq [n] \setminus \{i\}$ and $B \subseteq [n] \setminus \{j\}$*

$$\operatorname{Cov}(\Delta_i f \Delta_i f^A, \Delta_j f \Delta_j f^B) \leq c_{ij}.$$

Then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(W \leq t) - \mathbb{P}(Z \leq t)| \leq \sqrt{2} \Big(\sum_{i,j=1}^{n} c_{ij} \Big)^{1/4} + \Big(\sum_{i=1}^{n} \mathbb{E}[|\Delta_i f|^3] \Big)^{1/2}.$$

Example 4.1.5. As a simple application of this result consider the classical central limit theorem (with the assumption of existing *fourth* moments).

Let $X_1, \ldots, X_n$ be i.i.d. random variables with $\mathbb{E}[|X_1|^4] < \infty$, $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$. Let $X_1', \ldots, X_n'$ be an independent copy of $X_1, \ldots, X_n$.

Define $W = f(X_1, \ldots, X_n) = n^{-1/2} \sum_{i=1}^{n} X_i$, then for any $i \in [n]$ and $A \subseteq [n] \setminus \{i\}$

$$\Delta_i f^A = f(X^A) - f(X^{A \cup i}) = \frac{1}{\sqrt{n}} (X_i - X_i').$$

In particular for $i, j \in [n]$ and $A \subseteq [n] \setminus \{i\}$, $B \subseteq [n] \setminus \{j\}$

$$\operatorname{Cov}(\Delta_i f \Delta_i f^A, \Delta_j f \Delta_j f^B) = \frac{1}{n^2} \operatorname{Cov}((X_i - X_i')^2, (X_j - X_j')^2).$$

Since we assumed that the X_i are i.i.d., the covariance on the right-hand side is zero for $i \neq j$ and can be bounded by a term involving fourth moments of X_i for $i = j$. In particular we can find c_{ij} so that $\sum_{ij} c_{ij}$ is of order $O(n^{-1})$.

Similarly, in

$$\mathbb{E}[|\Delta_i f|^3] = \frac{1}{n^{3/2}} \mathbb{E}[|X_i - X_i'|^3]$$

the expectation on the right can be bounded by a term involving third moments of X_i. This implies that the sum $\sum_i \mathbb{E}[|\Delta_i f|^3]$ is of order $O(n^{-1/2})$.

This shows that both terms on the right in Corollary 4.1.4 are of order $O(n^{-1/4})$ and thus the convergence of W to a standard normal. This rate of convergence of order $n^{-1/4}$ is suboptimal, since the Berry–Esseen theorem guarantees a rate of $n^{-1/2}$ in this case [Ber41].

We now adapt Corollary 4.1.4 to our setting. Recall that $\sigma_n^2 = \operatorname{Var}_n(f(\mathbf{G}_n))$.

Lemma 4.1.6. *If there exists a function $f \colon V_n^{(2)} \cup V_n \to \mathbb{R}$ such that for all $x, x' \in V_n^{(2)} \cup V_n$ we have*

$$\sigma_n^{-4} \operatorname{Cov}(\Delta_x f \Delta_x f^F, \Delta_{x'} f \Delta_{x'} f^{F'}) \leq c(x, x')$$

for all $F \subseteq (V_n^{(2)} \cup V_n) \setminus \{x\}$, $F' \subseteq (V_n^{(2)} \cup V_n) \setminus \{x'\}$. Then we have

$$\begin{aligned}\sup_{t\in\mathbb{R}} &|\mathbb{P}_n(Z_n \leq t) - \Phi(t)| \\ &\leq \sqrt{2}\Big(\sum_{x,x'\in V_n^{(2)}\cup V_n} c(x,x')\Big)^{1/4} + \sigma_n^{-3/2}\Big(\sum_{x\in V_n^{(2)}\cup V_n} \mathbb{E}[|\Delta_x f|^3]\Big)^{1/2}. \end{aligned} \tag{4.2}$$

Proof. Apply Corollary 4.1.4 to $\mathbf{G}_n$ interpreted as a sequence of independent random variables $((X_e, w_e)_{e\in V_n^{(2)}}, (w_v)_{v\in V_n})$ and the function $\sigma_n^{-1}(f(x) - \mathbb{E}[f(\mathbf{G}_n)])$. □

Theorem 2.3.5 can now be proved by finding a suitable function c and bounding the two sums in (4.2). In particular we need to identify bounds for $c(x,x')$ for all possible combinations of edges and vertices. This is what we set out to do in the remainder of this chapter, which can broadly be split into two parts. The first part is dedicated to identifying straightforward bounds for c and $\Delta_x f$. The second part of the chapter examines bounds for c, where we make use of the fact that edges and vertices that are not incident to each other are weakly correlated.

4.2 Straightforward moment and covariance bounds

The second sum on the right-hand side of (4.2) does not involve c and is easily bounded after separation into vertex and edge sums

$$\sum_{x\in V_n^{(2)}\cup V_n} \mathbb{E}_n[|\Delta_x f|^3] = \sum_{e\in V_n^{(2)}} \mathbb{E}_n[|\Delta_e f|^3] + \sum_{v\in V_n} \mathbb{E}_n[|\Delta_v f|^3].$$

Lemma 4.2.1. *Under the conditions of Theorem 2.3.5*

$$\sum_{e\in V_n^{(2)}} \mathbb{E}_n[|\Delta_e f|^3] \leq 2n\vartheta\Gamma_{1,n}^2 J_E^{1/2}.$$

Proof. Use the estimate (2.9) for $|\Delta_e f|$ to find

$$\mathbb{E}_n[|\Delta_e f|^3] \leq \mathbb{E}_n[\mathbb{1}_{\{\max\{X_e,X_e'\}=1\}} |H_E(w_e, w_e', w_v, w_u)|^3].$$

Note that the first term depends only on the collection of edge indicators $\mathbf{X}$ and $\mathbf{X}'$ and the second term on the weights $\mathbf{w}$ and $\mathbf{w}'$. Since these collections are independent, the expectations factor to

$$\begin{aligned} &= \mathbb{P}_n(\max\{X_e, X_e'\} = 1)\mathbb{E}_n[|H_E(w_e, w_e', w_v, w_u)|^3] \\ &\leq (\mathbb{P}_n(X_e = 1) + \mathbb{P}_n(X_e' = 1))\mathbb{E}_n[|H_E(w_e, w_e', w_v, w_u)|^6]^{1/2}. \end{aligned}$$

Now use the definition of X_e and X_e' as well as the moment bound (2.6) for H_E to bound these terms by

$$\leq 2\frac{W_u W_v}{n\vartheta} J_E^{1/2}.$$

Hence, summing over all edges or combinations of two vertices we obtain

$$\begin{aligned}\sum_{e\in E_n}\mathbb{E}_n[|\Delta_e f|^3] &\le \sum_{u,v\in V_n} 2\frac{W_uW_v}{n\vartheta}J_E^{1/2}\\ &\le 2n\vartheta\Big(\frac{1}{n\vartheta}\sum_{u\in V_n}W_u\Big)^2 J_E^{1/2}\\ &\le 2n\vartheta\Gamma_{1,n}^2 J_E^{1/2}.\end{aligned}$$

The claim follows. □

The vertex sum can be treated analogously.

Lemma 4.2.2. *Under the conditions of Theorem 2.3.5*

$$\sum_{v\in V_n}\mathbb{E}_n[|\Delta_v f|^3] \le J_V^{1/2}\sum_{v\in V_n}\zeta(v)^{3/4}.$$

Proof. By the bound (2.10) on $|\Delta_v f|$,

$$\mathbb{E}_n[|\Delta_v f|^3] \le \mathbb{E}_n[(h(|D_1(v)|)H_V(w_v,w_v'))^3].$$

The first term depends only on $\mathbf{X}$, the second term only on $\mathbf{w}$ and $\mathbf{w}'$. By independence these terms factor, so that with the bounds (2.7) and (2.8) for moments of H_V and $h(|D_1(v)|)$ we obtain

$$\begin{aligned}&\le \mathbb{E}_n[h(|D_1(v)|)^3]\mathbb{E}_n[H_V(w_v,w_v')^3]\\ &\le \zeta_n(v)^{3/4}J_V^{1/2}.\end{aligned}$$

Summing over v we obtain

$$\sum_{v\in V_n}\mathbb{E}_n[|\Delta_v f|^3] \le J_V^{1/2}\sum_{v\in V_n}\zeta_n(v)^{3/4}.$$

The claim follows. □

It remains to identify a suitable function c bounding the covariances of the randomised derivatives such that the sum over all vertex-edge combinations has a rate that still allows for the desired convergence.

The $O(n^2)$ 'diagonal terms' $c(e,e)$ are relatively easy to bound; and a bound of order $O(\sigma_n^{-4}n^{-1})$ is sufficient to ensure the desired convergence.

Lemma 4.2.3. *We may take*

$$c(e,e) = \sigma_n^{-4}\frac{W_uW_v}{n\vartheta}\Big(\min\Big\{\frac{W_uW_v}{n\vartheta},1\Big\}+1\Big)J_E^{2/3} \le \sigma_n^{-4}CJ_E^{2/3}\frac{W_uW_v}{n\vartheta}$$

for the edge $e=\{u,v\}$.

Proof. We will show

$$\mathrm{Cov}_n(\Delta_e f \Delta_e f^F, \Delta_e f \Delta_e f^{F'}) \le \frac{W_u W_v}{n\vartheta}\Big(\min\Big\{\frac{W_u W_v}{n\vartheta}, 1\Big\} + 1\Big) J_E^{2/3},$$

which immediately implies the claim.

In a first step we apply the Cauchy–Schwarz inequality to obtain separate expectations, we then note that $\Delta_e f$, $\Delta_e f^F$ and $\Delta_e f^{F'}$ have the same distribution. Hence,

$$\begin{aligned}
&\mathbb{E}_n[|(\Delta_e f)^2 \Delta_e f^F \Delta_e f^{F'}|] \\
&\quad \le \mathbb{E}_n[|\Delta_e f|^4]^{1/2} \mathbb{E}_n[|\Delta_e f^F|^4]^{1/4} \mathbb{E}_n[|\Delta_e f^{F'}|^4]^{1/4} \\
&\quad = \mathbb{E}_n[|\Delta_e f|^4].
\end{aligned}$$

We now proceed as in Lemma 4.2.1 with the bound (2.9), independence and finally (2.6) to estimate this term by

$$\begin{aligned}
&\le \mathbb{E}_n[\mathbb{1}_{\{\max\{X_e, X_e'\}=1\}} H_E(w_e, w_e', w_v, w_u)^4] \\
&\le \mathbb{P}_n(\max\{X_e, X_e'\} = 1)\mathbb{E}_n[H_E(w_e, w_e', w_v, w_u)^4] \\
&\le 2\min\Big\{\frac{W_u W_v}{n\vartheta}, 1\Big\} J_E^{2/3}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}_n[|\Delta_e f \Delta_e f^F|] &\le \mathbb{E}_n|\Delta_e f|^2]^{1/2} \mathbb{E}_n[|\Delta_e f^F|^2]^{1/2} \\
&= \mathbb{E}_n|\Delta_e f|^2] \\
&\le \mathbb{P}_n(\max\{X_e, X_e'\} = 1)\mathbb{E}_n H_E(w_e, w_e', w_v, w_u)^2] \\
&\le 2\min\Big\{\frac{W_u W_v}{n\vartheta}, 1\Big\} J_E^{1/3}.
\end{aligned}$$

The same bound holds for $\Delta_e f \Delta_e f^{F'}$

$$\mathbb{E}_n[|\Delta_e f \Delta_e f^{F'}|] \le 2\min\Big\{\frac{W_u W_v}{n\vartheta}, 1\Big\} J_E^{1/3}.$$

Putting all this together we get

$$\begin{aligned}
&\mathrm{Cov}_n(\Delta_e f \Delta_e f^F, \Delta_e f \Delta_e f^{F'}) \\
&\quad \le |\mathbb{E}_n[\Delta_e f \Delta_e f^F \Delta_e f \Delta_e f^{F'}]| + |\mathbb{E}_n[\Delta_e f \Delta_e f^F]\mathbb{E}_n[\Delta_e f \Delta_e f^{F'}]| \\
&\quad \le \mathbb{E}_n[|(\Delta_e f)^2 \Delta_e f^F \Delta_e f^{F'}|] + \mathbb{E}_n[|\Delta_e f \Delta_e f^F|]\mathbb{E}_n[|\Delta_e f \Delta_e f^{F'}|] \\
&\quad \le C\frac{W_u W_v}{n\vartheta}\Big(\min\Big\{\frac{W_u W_v}{n\vartheta}, 1\Big\} + 1\Big) J_E^{2/3}
\end{aligned}$$

as claimed. □

An analogous result can be shown for the pure vertex combination. Here c is of rate $O(\sigma_n^{-4})$ and there are $O(n)$ many terms.

Lemma 4.2.4. *We may take*

$$c(v,v) = \sigma_n^{-4} C J_V^{2/3} \zeta_n(v)$$

for each vertex v.

Proof. The proof proceeds as the proof of Lemma 4.2.3. We will show

$$\mathrm{Cov}_n(\Delta_v f \Delta_v f^F, \Delta_v f \Delta_v f^{F'}) \le C J_V^{2/3} \zeta_n(v)$$

to prove the claim.

As in Lemma 4.2.3 we first apply Cauchy–Schwarz and use that $\Delta_v f$, $\Delta_v f^F$ and $\Delta_v f^{F'}$ have identical distribution so that

$$\begin{aligned}
&\mathbb{E}_n[|(\Delta_v f)^2 \Delta_v f^F \Delta_v f^{F'}|] \\
&\quad \le \mathbb{E}_n[|\Delta_v f|^4]^{1/2} \mathbb{E}_n[|\Delta_v f^F|^4]^{1/4} \mathbb{E}_n[|\Delta_v f^{F'}|^4]^{1/4} \\
&\quad \le \mathbb{E}_n[|\Delta_v f|^4].
\end{aligned}$$

By the reasoning from Lemma 4.2.2 with the bound (2.10) for $|\Delta_v f|$, independence and the bounds (2.7) for H_V and (2.8) for $h(|D_1(v)|)$ this term can be estimated by

$$\begin{aligned}
&\le \mathbb{E}_n[h(|D_1(v)|)^4 H_V(w_v, w_v')^4] \\
&\le J_V^{2/3} \zeta_n(v).
\end{aligned}$$

In exactly the same way we show

$$\mathbb{E}_n[|\Delta_v f \Delta_v f^F|] \le \mathbb{E}_n[|\Delta_v f|^2]^{1/2} \mathbb{E}_n[|\Delta_v f^F|^2]^{1/2} \le J_V^{1/3} \zeta_n(v).$$

and

$$\mathbb{E}_n[|\Delta_v f \Delta_v f^{F'}|] \le J_V^{1/3} \zeta_n(v)^{1/4}.$$

Together this bounds the relevant covariance by

$$\begin{aligned}
&\mathrm{Cov}_n(\Delta_v f \Delta_v f^F, \Delta_v f \Delta_v f^{F'}) \\
&\quad \le |\mathbb{E}_n[\Delta_v f \Delta_v f^F \Delta_v f \Delta_v f^{F'}]| + |\mathbb{E}_n[\Delta_v f \Delta_v f^F] \mathbb{E}_n[\Delta_v f \Delta_v f^{F'}]| \\
&\quad \le \mathbb{E}[|(\Delta_v f)^2 \Delta_v f^F \Delta_v f^{F'}|] + \mathbb{E}_n[|\Delta_v f \Delta_v f^F|] \mathbb{E}_n[|\Delta_v f \Delta_v f^{F'}|] \\
&\quad \le J_V^{2/3} \zeta_n(v)^{1/2} + J_V^{1/3} \zeta_n(v)^{1/4} \cdot J_V^{1/3} \zeta_n(v)^{1/4} \\
&\quad \le C J_V^{2/3} \zeta_n(v)^{1/2}.
\end{aligned}$$

This proves the claim. □

For the $O(n^2)$ mixed terms involving a vertex and an edge emanating from that vertex we obtain an order of $O(\sigma_n^{-4}n^{-1})$.

Lemma 4.2.5. *We may take*

$$c(e,v) = \sigma_n^{-4}C\frac{W_uW_v}{n\vartheta}J_E^{1/3}J_V^{1/3}\zeta_n(v)^{1/4}$$

for any vertex v and all edges $e = \{v,u\}$ emanating from v.

Proof. We will show

$$\mathrm{Cov}_n(\Delta_e f\Delta_e f^F, \Delta_v f\Delta_v f^{F'}) \leq C\frac{W_uW_v}{n\vartheta}J_E^{1/3}J_V^{1/3}\zeta_n(v)^{1/4}.$$

The claim then follows.

Again, the proof follows along the lines of the proofs of Lemmas 4.2.3 and 4.2.4. The first term, however, needs a little more attention, since we do not want apply Cauchy–Schwarz immediately, because that would leave us with an exponent $1/2$ that we cannot remove easily.

By the bounds (2.9) and (2.10) for $\Delta_e f$ and $\Delta_v f$ we have

$$\begin{aligned}
&\mathbb{E}_n[|\Delta_e f\Delta_e f^F\Delta_v f\Delta_v f^{F'}|]\\
&\quad\leq \mathbb{E}_n[|\mathbb{1}_{\{\max\{X_e,X_e'\}=1\}}H_E(w_e,w_e',w_v,w_u)H_E(w_e,w_e',w_v^F,w_u^F)\\
&\qquad h(|D_1(v)|)H_V(w_v,w_v')h(|D_1^{F'}(v)|)H_V(w_v,w_v')|].
\end{aligned}$$

Collect terms that depend on the edges $\mathbf{X}$, $\mathbf{X}'$ and the weights $\mathbf{w}$, $\mathbf{w}'$ and use their independence to bound this term by

$$\begin{aligned}
&\leq \mathbb{E}_n[\mathbb{1}_{\{\max\{X_e,X_e'\}=1\}}h(|D_1(v)|)h(|D_1^{F'}(v)|)]\\
&\quad\mathbb{E}_n[H_E(w_e,w_e',w_v,w_u)H_E(w_e,w_e',w_v^F,w_u^F)H_V(w_v,w_v')^2].
\end{aligned}$$

Recall Definition 3.1.18 and estimate $h(|D_1(v)|)$ against $h(|D_1^{(u)}(v)+1|)$, which ignores the edge $e = \{u,v\}$, to create independence from X_e and X_e'

$$\begin{aligned}
&\leq \mathbb{E}_n[\mathbb{1}_{\{\max\{X_e,X_e'\}=1\}}(h(|D_1^{(u)}(v)|)+1)(h(|D_1^{(u),F'}(v)|)+1)]\\
&\quad\mathbb{E}_n[H_E(w_e,w_e',w_v,w_u)H_E(w_e,w_e',w_v^F,w_u^F)H_V(w_v,w_v')^2].
\end{aligned}$$

Use the independence of X_e, X_e' from $D_1^{(u)}$ and $D_1^{(u),F'}$, judiciously apply Cauchy–Schwarz and then use that $\mathbf{G}_n$ and $\mathbf{G}_n^F$ have the same distribution to find the bound

$$\begin{aligned}
&\leq \mathbb{P}_n(\max\{X_e,X_e'\}=1)\mathbb{E}_n[(h(|D_1^{(u)}(v)|)+1)^2]\\
&\quad\mathbb{E}_n[H_E(w_e,w_e',w_v,w_u)^4]^{1/4}\mathbb{E}_n[H_E(w_e,w_e',w_v^F,w_u^F)^4]^{1/4}\\
&\quad\mathbb{E}_n[H_V(w_v,w_v')^4]^{1/2}.
\end{aligned}$$

With (2.6) to (2.8) this term is further bounded by

$$\leq C\frac{W_v W_u}{n\vartheta}\zeta_n(v)^{1/4}J_E^{1/3}J_V^{1/3}.$$

Putting this together with the bounds shown in the proof of Lemmas 4.2.3 and 4.2.4 we obtain

$$\begin{aligned}
&\mathrm{Cov}_n(\Delta_e f\Delta_e f^F, \Delta_{v'} f\Delta_{v'} f^{F'}) \\
&\quad\leq |\mathbb{E}_n[\Delta_e f\Delta_e f^F\Delta_{v'} f\Delta_{v'} f^{F'}]| + |\mathbb{E}_n[\Delta_e f\Delta_e f^F]\mathbb{E}_n[\Delta_{v'} f\Delta_{v'} f^{F'}]| \\
&\quad\leq \mathbb{E}_n[|\Delta_e f\Delta_e f^F\Delta_{v'} f\Delta_{v'} f^{F'}|] + \mathbb{E}_n[|\Delta_e f\Delta_e f^F|]\mathbb{E}_n[|\Delta_{v'} f\Delta_{v'} f^{F'}|] \\
&\quad\leq C\frac{W_v W_u}{n\vartheta}\zeta_n(v)^{1/4}J_E^{1/3}J_V^{1/3} + 2J_E^{1/3}\frac{W_v W_u}{n\vartheta}J_V^{1/3}\zeta_n(v)^{1/4} \\
&\quad\leq C\frac{W_u W_v}{n\vartheta}J_E^{1/3}J_V^{1/3}\zeta_n(v)^{1/4}.
\end{aligned}$$

This proves the claim. □

The same general strategy can be used for the $O(n^3)$ edge pairs sharing one vertex. The resulting bound is of order $\sigma_n^{-4}n^{-2}$ and is thus sufficient for the desired convergence.

Lemma 4.2.6. *Let $e = \{u, v\}$ and $e' = \{u, v'\}$ be two edges in $V_n^{(2)}$ that share exactly one vertex. Then we may take*

$$c(e, e') = \sigma_n^{-4}CW_u^2\frac{W_v}{n\vartheta}\frac{W_{v'}}{n\vartheta}J_E^{2/3}.$$

Proof. As previously we bound

$$\mathrm{Cov}_n(\Delta_e f\Delta_e f^F, \Delta_{e'} f\Delta_{e'} f^{F'}).$$

As in Lemma 4.2.5 we first apply the bound (2.9) for $\Delta_e f$ so that

$$\begin{aligned}
&\mathbb{E}_n[|\Delta_e f\Delta_{e'} f\Delta_e f^F\Delta_{e'} f^{F'}|] \\
&\leq \mathbb{E}_n[\mathbb{1}_{\{\max\{X_e,X_e'\}=1\}}\mathbb{1}_{\{\max\{X_{e'},X_{e'}'\}=1\}} \\
&\qquad H_E(w_e, w_e', w_v, w_u)H_E(w_e, w_e', w_v^F, w_u^F) \\
&\qquad H_E(w_{e'}, w_{e'}', w_{v'}, w_{u'})H_E(w_{e'}, w_{e'}', w_{v'}^{F'}, w_{u'}^{F'})].
\end{aligned}$$

Since X_e, X_e', $X_{e'}$ and $X_{e'}'$ are independent from each other and from the weights $\mathbf{w}$ and $\mathbf{w}'$, this expectation factors and we can apply Cauchy–Schwarz to find the bound

$$\begin{aligned}
&\leq \mathbb{P}_n(\max\{X_e, X_e'\} = 1)\mathbb{P}_n(\max\{X_{e'}, X_{e'}'\} = 1) \\
&\qquad \mathbb{E}_n[H_E(w_e, w_e', w_v, w_u)^4]^{1/4}\mathbb{E}_n[H_E(w_e, w_e', w_v^F, w_u^F)^4]^{1/4} \\
&\qquad \mathbb{E}_n[H_E(w_{e'}, w_{e'}', w_{v'}, w_{u'})^4]^{1/4}\mathbb{E}_n[H_E(w_{e'}, w_{e'}', w_{v'}^{F'}, w_{u'}^{F'})^4]^{1/4} \\
&\leq 4\frac{W_u W_v}{n\vartheta}\frac{W_u W_{v'}}{n\vartheta}J_E^{2/3}.
\end{aligned}$$

From the proof of Lemma 4.2.3 we have directly that

$$\mathbb{E}_n[|\Delta_e f \Delta_e f^F|] \le 2\frac{W_u W_v}{n\vartheta} J_E^{1/3} \quad \text{and} \quad \mathbb{E}_n[|\Delta_{e'} f \Delta_{e'} f^{F'}|] \le 2\frac{W_u W_{v'}}{n\vartheta} J_E^{1/3}.$$

These estimates together show

$$\begin{aligned}
&\operatorname{Cov}_n(\Delta_e f \Delta_e f^F, \Delta_{e'} f \Delta_{e'} f^{F'}) \\
&\quad \le |\mathbb{E}_n[\Delta_e f \Delta_e f^F \Delta_{e'} f \Delta_{e'} f^{F'}]| + |\mathbb{E}_n[\Delta_e f \Delta_e f^F]\mathbb{E}_n[\Delta_{e'} f \Delta_{e'} f^{F'}]| \\
&\quad \le \mathbb{E}_n[|\Delta_e f \Delta_{e'} f \Delta_e f^F \Delta_{e'} f^{F'}|] + \mathbb{E}_n[|\Delta_e f \Delta_e f^F|]\mathbb{E}_n[|\Delta_{e'} f \Delta_{e'} f^{F'}|] \\
&\quad \le 4W_u^2 \frac{W_v}{n\vartheta}\frac{W_{v'}}{n\vartheta} J_E^{2/3} + 2\frac{W_u W_v}{n\vartheta} J_E^{1/3} \cdot 2\frac{W_u W_{v'}}{n\vartheta} J_E^{1/3} \\
&\quad \le C W_u^2 \frac{W_v}{n\vartheta}\frac{W_{v'}}{n\vartheta} J_E^{2/3}.
\end{aligned}$$

This proves the claim. □

4.3 Sparsity-based covariance bounds

In this section we will find bounds for c for the remaining cases in which the vertices and edges that are involved are not incident to each other. The mainly Cauchy–Schwarz-based approach of the previous section would not give the desired convergence rates here. We rely on property GLA (Assumption 2.3.2) and the coupling to the limiting Galton–Watson tree Definition 2.2.5 to bound the effect of a local change on the function by a local quantity. Property GLA and the coupling to the limiting tree structure ensure that the approximation error of using the local quantity goes to zero. The sparsity of the underlying graph ensures that local quantities are only very weakly correlated (cf. Lemma 3.2.5).

In our calculations we will use the following multi-purpose error term that absorbs the probability of various additional coupling events and the covariance bound for local neighbourhoods.

Definition 4.3.1. Let $n, k \in \mathbb{N}$. For any set of vertices $\mathcal{V} \subseteq V_n$ set

$$\rho_{n,k}(\mathcal{V}) = \min\Big\{\frac{(\|\mathcal{V}\| + |V|)^2}{n\vartheta}(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+C)^{2k+1}(\Gamma_{3,n}+1)^2, 1\Big\}.$$

Recall the definition of $\varepsilon_{n,k}(\mathcal{V})$ from Lemma 3.5.1 and Lemma 3.5.2

$$\begin{aligned}
\varepsilon_{n,k}(\mathcal{V}) = {}& \|\mathcal{V}\|_2 \frac{\Gamma_{2,n}}{n\vartheta} + \|\mathcal{V}\|_+ \Gamma_{1,n} \\
&+ \|\mathcal{V}\|(\Gamma_{2,n}+1)^k\Big(\frac{\Gamma_{3,n}}{n\vartheta} + \kappa_{1,n} + \kappa_{2,n} + \frac{2+\Gamma_{1,n}}{k_n} + \frac{k_n}{n\vartheta}\Big) \\
&+ |\mathcal{V}|\frac{1}{k_n} + \frac{k_n^2}{n\vartheta\Gamma_{1,n}} + \|\mathcal{V}\|\alpha_n\Big(\frac{1}{\vartheta} + (\Gamma_2+1)^{k-1}\Big(\frac{\Gamma_{2,n}}{\vartheta\Gamma_{1,n}} + 1\Big)\Big) \\
&+ (|\mathcal{V}| + \|\mathcal{V}\|(\Gamma_2+1)^k)(d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) + d_{\mathrm{TV}}(\mu_{V,n}, \mu_V)).
\end{aligned}$$

4.3.1 Edge-edge case

We will now bound $c(e,e')$ for edges $e = \{u,v\}$ and $e' = \{u',v'\}$ with all distinct vertices. There are $O(n^4)$ of these edge pairs and a $c(e,e')$ of rate $O(\sigma_n^{-4}n^{-2})$ (as in the previous case) would result in a convergence rate that is slightly worse than the rate that can be obtained by exploiting that with high probability the two edges do not influence each other. The remainder of this subsection is therefore dedicated to showing the following proposition.

Proposition 4.3.2. *Let $e = \{u,v\}$ and $e' = \{u',v'\}$ be two edges in $V_n^{(2)}$ with all distinct vertices. Then we may take*

$$c(e,e') = \sigma_n^{-4} C J_E \frac{W_u W_v}{n\vartheta} \frac{W_{u'} W_{v'}}{n\vartheta} ((m_n^E(v,u)\delta_k^E)^{1/2} + (m_n^E(v',u')\delta_k^E)^{1/2} + \varepsilon_{n,k}(\{u,v,u',v'\})^{1/4} + \rho_{n,k}(\{u,v,u',v'\})^{1/4}).$$

We will prove this result via the coupling to the limiting tree. In order to do this properly, we need to introduce some notation.

Let E_0 be the event that both $B_k(v,\mathbf{G}_n)$ and $B_k(v,\mathbf{G}_n^e)$ are trees and let

$$A_e = \{(X_e, X_e') = (1,0) \text{ or } (X_e, X_e') = (0,1)\}$$

be the event that e is switched by going from X_e to X_e'. Define

$$\tilde{L}_k^E(e) = \mathbb{1}_{E_0}\mathbb{1}_{A_e}\mathrm{LA}_k^{E,L}(B_k(v,\mathbf{G}_n), B_k(v,\mathbf{G}_n^e)).$$

Let $Q(\,\cdot\,,y)\colon \mathbb{R} \to [-y,y]$ be the function truncating x to level $y \ge 0$, i.e.

$$Q(x,y) = \max\{\min\{x,y\},-y\} = \begin{cases} -y & x < -y, \\ x & -y \le x \le y, \\ y & x > y. \end{cases}$$

For brevity let

$$\tilde{H}_E(e) = H_E(w_e, w_e', w_v, w_u), \quad \tilde{H}_E(F \cup e) = H_E(w_e, w_e', w_v^F, w_u^F)$$

and similarly for e' and $F' \cup e'$.

Then define a truncated version of $\tilde{L}_k^E(e)$

$$L_k^E(e) = Q(\tilde{L}_k^E(e), \tilde{H}_E(e)).$$

This will be our local approximation of $\Delta_e f$ (hence 'L'). The construction ensures

$$|L_k^E(e)| \le \mathbb{1}_{A_e}\tilde{H}_E(e). \tag{4.3}$$

Let $R_k^E(e)$ be the difference of $L_k^E(e)$ to $\Delta_e f$ (it is the remainder to $\Delta_e f$, hence 'R'), i.e. set

$$R_k^E(e) = \Delta_e f - L_k^E(e).$$

Let $\tilde{A}_e = \{\max\{X_e, X'_e\} = 1\}$, so that $\tilde{A}_e = A_e \cup \{X_e = X'_e = 1\}$. Then by (2.9) and (4.3)

$$\begin{aligned} |R^E_k(e)| &\le |\Delta_e f| + |L^E_k(e)| \\ &\le \mathbb{1}_{\tilde{A}_e}\tilde{H}_E(e) + \mathbb{1}_{A_e}\tilde{H}_E(e) \\ &\le 2\mathbb{1}_{\tilde{A}_e}\tilde{H}_E(e). \end{aligned} \tag{4.4}$$

For $F \subseteq (V^{(2)}_n \cup V_n) \setminus \{e\}$ define $L^E_k(F \cup e)$ with $B_k(v, \mathbf{G}^F_n)$ and $B_k(v, \mathbf{G}^{F\cup e}_n)$ instead of $B_k(v, \mathbf{G}_n)$ and $B_k(v, \mathbf{G}^e_n)$. Analogous to $R^E_k(e)$ define the remainder $R^E_k(F \cup e)$

$$R^E_k(F \cup e) = \Delta_e f^F - L^E_k(F \cup e).$$

With these definitions we want to bound

$$\begin{aligned} &\mathrm{Cov}_n(\Delta_e f \Delta_e f^F, \Delta_{e'} f \Delta_{e'} f^{F'}) \\ &\quad = \mathrm{Cov}_n((R^E_k(e) + L^E_k(e))(R^E_k(F \cup e) + L^E_k(F \cup e)), \\ &\qquad\qquad (R^E_k(e') + L^E_k(e'))(R^E_k(F' \cup e') + L^E_k(F' \cup e'))). \end{aligned} \tag{4.5}$$

Expand this expressions into sixteen terms of the form

$$\mathrm{Cov}_n(U^E_1(e)U^E_2(F \cup e), U^E_3(e')U^E_4(F' \cup e')),$$

where U^E_i for $i \in \{1, 2, 3, 4\}$ may be either L^E_k or R^E_k.

Recall that R^E_k corresponds to the difference of $\Delta_e f$ and L^E_k. In other words, R^E_k is the error of approximating $\Delta_e f$ locally. Hence, the fifteen covariances involving at least one R^E_k term can be bounded by coupling the neighbourhood to the limiting Galton–Watson tree and appealing to (GLA 3).

Lemma 4.3.3. *We have*

$$\begin{aligned} &\mathrm{Cov}(U^E_1(e)U^E_2(F \cup e), U^E_3(e')U^E_4(F' \cup e')) \\ &\quad \le CJ_E \frac{W_u W_v}{n\vartheta} \frac{W_{u'} W_{v'}}{n\vartheta} ((m^E_n(v, u)\delta^E_k)^{1/2} + (m^E_n(v', u')\delta^E_k)^{1/2} \\ &\qquad + \varepsilon_{n,k}(\{u, v, u', v\})^{1/4} + \rho_{n,k}(\{u, v, u', v'\})^{1/4}) \end{aligned}$$

if at least one of the U^E_i is an R^E_k-term.

Proof. We show the claim the case where U^E_1 is an R^E_k term, i.e. for

$$\mathrm{Cov}(R^E_k(e)U^E_2(F \cup e), U^E_3(e')U^E_4(F' \cup e')).$$

The proof for the other terms is analogous (the bound we show is symmetric in e and e').

By construction $J_E \ge 1$, so $J^r_E \le J_E$ for all $r \le 1$, so we will drop the exponents of J_E that are smaller than one.

Set

$$\tilde{H}_E = \max\{\tilde{H}_E(e), \tilde{H}_E(e'), \tilde{H}_E(F \cup e), \tilde{H}_E(F' \cup e')\}.$$

Then by (4.3) and (4.4) and noting that $\tilde{A}_e = A_e \cup \{X_e = X'_e = 1\}$

$$\begin{aligned}|R_k^E(e)U_2^E(F\cup e)U_3^E(e')U_4^E(F'\cup e')| &\le C\mathbb{1}_{\tilde{A}_e}\mathbb{1}_{\tilde{A}_{e'}}\tilde{H}_E^3|R_k^E(e)| \\ &\le C\mathbb{1}_{A_e}\mathbb{1}_{\tilde{A}_{e'}}\tilde{H}_E^3|R_k^E(e)| + C\mathbb{1}_{\{X_e=X'_e=1\}}\mathbb{1}_{\tilde{A}_{e'}}\tilde{H}_E^4.\end{aligned} \tag{4.6}$$

Since $\mathbf{w}$, $\mathbf{X}$ and $\mathbf{X}'$ are independent, the expectation of the second term in (4.6) can be bounded easily by

$$\begin{aligned}\mathbb{E}_n[\mathbb{1}_{\{X_e=X'_e=1\}}\mathbb{1}_{\tilde{A}_{e'}}\tilde{H}_E^4] &\le C\min\Big\{\frac{W_uW_v}{n\vartheta},1\Big\}^2\min\Big\{\frac{W_{u'}W_{v'}}{n\vartheta},1\Big\}J_E^{2/3} \\ &\le C\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}\min\Big\{\frac{W_uW_v}{n\vartheta},1\Big\}J_E^{2/3} \\ &\le C\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}J_E^{2/3}\rho_{n,k}(\{u,v,u',v'\}).\end{aligned}$$

Since $\rho_{n,k}(\{u,v,u',v'\}) \le 1$ we can make the bound worse by

$$\le C\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}J_E^{2/3}\rho_{n,k}(\{u,v,u',v'\})^{1/4}. \tag{4.7}$$

We move on to the first term in (4.6). Let $Y_e = (X_e, X'_e)$. Then the indicator functions of the first term in (4.6) are $Y_e, Y_{e'}$-measurable. The conditional expectation of the remainder of the term can be bounded with the Cauchy–Schwarz inequality

$$\mathbb{E}_n[\tilde{H}_E^3|R_k^E(e)| \mid Y_e, Y_{e'}] \le \mathbb{E}_n[\tilde{H}_E^6 \mid Y_e, Y_{e'}]^{1/2}\mathbb{E}_n[|R_k^E(e)|^2 \mid Y_e, Y_{e'}]^{1/2}$$

since $\mathbf{w}$, $\mathbf{w}'$ are independent of $\mathbf{X}$, $\mathbf{X}'$, the conditioning in the first term can be dropped and (2.6) implies

$$\le CJ_E^{1/2}\mathbb{E}_n[(R_k^E(e))^2 \mid Y_e, Y_{e'}]^{1/2}. \tag{4.8}$$

It remains to bound the second moment of $R_k^E(e)$ conditional on $Y_e, Y_{e'}$. Recall that $R_k^E(e) = \Delta_e f - L_k^E(e)$. Write $B_k = B_k(v, \mathbf{G}_n)$ and $B'_k = B_k(v, \mathbf{G}_n^e)$. Then on E_0

$$0 \le \Delta_e f - L_k^E(e) \le \mathrm{LA}_k^{E,U}(B_k, B'_k) - \mathrm{LA}_k^{E,L}(B_k, B'_k).$$

Verify this by considering the possible cases for the truncation of $\mathrm{LA}_k^{E,L}(B_k, B'_k)$ to level $\tilde{H}_E(e)$ separately.

Case (i) $\mathrm{LA}_k^{E,L}(B_k, B'_k) \in [-\tilde{H}_E(e), \tilde{H}_E(e)]$. Then $L_k^E(e) = \mathrm{LA}_k^{E,L}(B_k, B'_k)$ and (GLA 1) immediately implies both inequalities.

Case (ii) $\mathrm{LA}_k^{E,L}(B_k, B_k') < -\tilde{H}_E(e)$. Then $L_k^E(e) = -H_E(w_v, w_u)$. For the first inequality use that $\Delta_e f \geq -\tilde{H}_E(e)$ by (2.9). For the second inequality recall that $\Delta_e f \leq \mathrm{LA}_k^{E,U}(B_k, B_k')$ by (GLA 1) and use $\mathrm{LA}_k^{E,L}(B_k, B_k') < L_k^E(e)$, which follows directly from $L_k^E(e) = -\tilde{H}_E(e)$ in this case.

Case (iii) $\mathrm{LA}_k^{E,L}(B_k, B_k') > \tilde{H}_E(e)$. Then $L_k^E(e) = \tilde{H}_E(e)$. Combining (GLA 1) and (2.9) we obtain $\mathrm{LA}_k^{E,L}(B_k, B_k') \leq \Delta_e f \leq \tilde{H}_E(e)$, which forces $\Delta_e f = \tilde{H}_E(e)$. This shows that the first inequality holds (even with equality). But that immediately also proves the second inequality since $\mathrm{LA}_k^{E,L}(B_k, B_k') \leq \mathrm{LA}_k^{E,U}(B_k, B_k')$ by (GLA 1).

In particular $|R_k^E(e)| \leq \mathrm{LA}_k^{E,U}(B_k, B_k') - \mathrm{LA}_k^{E,L}(B_k, B_k')$ on E_0 and so together with (4.4) we have

$$\begin{aligned}\mathbb{1}_{A_e}|R_k^E(e)| &\leq \mathbb{1}_{A_e}\mathbb{1}_{E_0}\mathbb{1}_{E_1}|R_k^E(e)| + \mathbb{1}_{A_e}\mathbb{1}_{E_0^c \cup E_1^c}|R_k^E(e)| \\ &\leq \mathbb{1}_{A_e}\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{E,U}(B_k, B_k') - \mathrm{LA}_k^{E,L}(B_k, B_k')) + \mathbb{1}_{A_e} C\tilde{H}_E \mathbb{1}_{E_0^c \cup E_1^c}\end{aligned}$$

for an arbitrary set E_1 that will be chosen later. Taking conditional expectations, applying Cauchy–Schwarz on the second term and using the moment bound (2.6) for H_E and then in the next step $(x + y)^{1/2} \leq x^{1/2} + y^{1/2}$ we see

$$\begin{aligned}&\mathbb{1}_{A_e}\mathbb{E}_n[(R_k^E(e))^2 \mid Y_e, Y_{e'}] \\ &\quad\leq \mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{E,U}(B_k, B_k') - \mathrm{LA}_k^{E,L}(B_k, B_k'))^2 \mid Y_e, Y_{e'}] \\ &\qquad + \mathbb{1}_{A_e} C J_E^{1/3}(\mathbb{P}_n(E_0^c \mid Y_e, Y_{e'}) + \mathbb{P}_n(E_1^c \mid Y_e, Y_{e'}))^{1/2} \\ &\quad\leq \mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{E,U}(B_k, B_k') - \mathrm{LA}_k^{E,L}(B_k, B_k'))^2 \mid Y_e, Y_{e'}] \\ &\qquad + \mathbb{1}_{A_e} C J_E^{1/3}(\mathbb{P}_n(E_0^c \mid Y_e, Y_{e'})^{1/2} + \mathbb{P}_n(E_1^c \mid Y_e, Y_{e'})^{1/2}).\end{aligned} \tag{4.9}$$

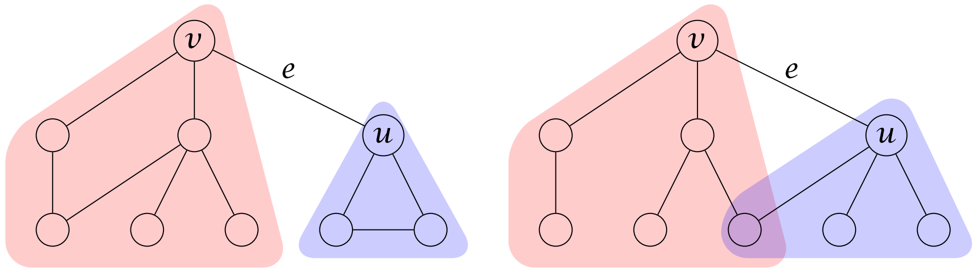
We turn first to the probability of E_0^c in (4.9). The probability that B_k or B_k' is a not tree can be bounded by the sum of the probabilities of those events. Thus

$$\mathbb{P}_n(E_0^c \mid Y_e, Y_{e'}) \leq \mathbb{P}_n(B_k \text{ is not a tree} \mid Y_e, Y_{e'}) + \mathbb{P}_n(B_k' \text{ is not a tree} \mid Y_e, Y_{e'}).$$

We consider the event that B_k is not a tree, the case for B_k' is analogous. If $B_k = B_k(v, \mathbf{G}_n)$ is not a tree, then $B_k(v, \mathbf{G}_n - e)$ is not a tree or $B_{k-1}(u, \mathbf{G}_n - e)$ is not a tree or the two (trees) $B_k(v, \mathbf{G}_n - e)$ and $B_{k-1}(u, \mathbf{G}_n - e)$ intersect (see Fig. 4.1). Therefore we have

$$\begin{aligned}&\mathbb{P}_n(B_k \text{ is not a tree} \mid Y_e, Y_{e'}) \\ &\quad\leq \mathbb{P}_n(B_k(v, \mathbf{G}_n - \{e\}) \text{ is not a tree} \mid Y_e, Y_{e'}) \\ &\qquad + \mathbb{P}_n(B_{k-1}(u, \mathbf{G}_n - \{e\}) \text{ is not a tree} \mid Y_e, Y_{e'}) \\ &\qquad + \mathbb{P}_n(B_k(v, \mathbf{G}_n - \{e\}) \text{ and } B_{k-1}(u, \mathbf{G}_n - \{e\}) \text{ intersect} \mid Y_e, Y_{e'}).\end{aligned}$$

The conditioning on $Y_e, Y_{e'}$ can be removed since $B_k(v, \mathbf{G}_n - e) \cong B_k(v, \mathbf{G}_n - \{e, e'\})$ and $B_{k-1}(u, \mathbf{G}_n - e) \cong B_{k-1}(u, \mathbf{G}_n - \{e, e'\})$ with high probability even conditionally



(a) $B_2(v, \mathbf{G}-e)$ and $B_1(u, \mathbf{G}-e)$ contain cycles.

(b) $B_2(v, \mathbf{G}-e)$ and $B_1(u, \mathbf{G}-e)$ intersect.

Figure 4.1: Examples of the three ways $B_2(v, \mathbf{G})$ can fail to be a tree. Either there is a cycle in $B_2(v, \mathbf{G}-e)$, there is a cycle in $B_1(u, \mathbf{G}-e)$ (a) or $B_2(v, \mathbf{G}-e)$ and $B_1(u, \mathbf{G}-e)$ overlap (b).

on Y_e, Y_e'. The events involving only $\mathbf{G}_n - \{e, e'\}$ are then independent of $Y_e, Y_{e'}$ so that

$$\begin{aligned}
&\mathbb{P}_n(B_k \text{ is not a tree} \mid Y_e, Y_{e'}) \\
&\quad \leq \mathbb{P}_n(B_k(v, \mathbf{G} - \{e, e'\}) \text{ is not a tree}) \\
&\qquad + \mathbb{P}_n(B_{k-1}(u, \mathbf{G} - \{e, e'\}) \text{ is not a tree}) \\
&\qquad + \mathbb{P}_n(B_k(v, \mathbf{G} - \{e, e'\}) \text{ and } B_{k-1}(u, \mathbf{G} - \{e, e'\}) \text{ intersect}) \\
&\qquad + 2\mathbb{P}_n(B_k(v, \mathbf{G}_n - e) \not\cong B_k(v, \mathbf{G}_n - \{e, e'\}) \mid Y_e, Y_{e'}) \\
&\qquad + 2\mathbb{P}_n(B_{k-1}(u, \mathbf{G}_n - e) \not\cong B_{k-1}(u, \mathbf{G}_n - \{e, e'\}) \mid Y_e, Y_{e'})
\end{aligned}$$

For the last two terms note that $B_k(v, \mathbf{G}_n - e) \not\cong B_k(v, \mathbf{G}_n - \{e, e'\})$, implies $e' \in B_k(v, \mathbf{G}_n - e)$, since otherwise all paths in $B_k(v, \mathbf{G}_n - e)$ would avoid e' and would thus already be in $B_k(v, \mathbf{G}_n - \{e, e'\})$. Hence,

$$\begin{aligned}
&\mathbb{P}_n(B_k(v, \mathbf{G}_n - e) \neq B_k(v, \mathbf{G}_n - \{e, e'\}) \mid Y_e, Y_{e'}) \\
&\quad \leq \mathbb{P}_n(e' \in B_k(v, \mathbf{G}_n - e) \mid Y_e, Y_{e'}).
\end{aligned}$$

The edge e' can only be present in $B_k(v, \mathbf{G}_n - e)$ if at least one of its endpoints v' or u' can be reached in $B_k(v, \mathbf{G}_n - \{e, e'\})$, which is independent of Y_e and $Y_{e'}$, so that Corollary 3.1.15 yields the bound

$$\begin{aligned}
&\leq \mathbb{P}_n(u' \in B_k(v, \mathbf{G}_n - \{e, e'\}) \mid Y_e, Y_{e'}) \\
&\quad + \mathbb{P}_n(v' \in B_k(v, \mathbf{G}_n - \{e, e'\}) \mid Y_e, Y_{e'}) \\
&\leq \mathbb{P}(u' \in B_k(v, \mathbf{G}_n - \{e, e'\})) + \mathbb{P}_n(v' \in B_k(v, \mathbf{G}_n - \{e, e'\})) \\
&\leq \mathbb{P}(u' \in B_k(v, \mathbf{G}_n)) + \mathbb{P}_n(v' \in B_k(v, \mathbf{G}_n)) \\
&\leq \frac{W_v W_{u'}}{n\vartheta}(\Gamma_{2,n} + 1)^k + \frac{W_v W_{v'}}{n\vartheta}(\Gamma_{2,n} + 1)^k
\end{aligned}$$

$$\leq \frac{W_v(W_{u'} + W_{v'})}{n\vartheta}(\Gamma_{2,n}+1)^k.$$

The analogous result applies to $B_{k-1}(u, \mathbf{G}_n - e)$ and $B_{k-1}(u, \mathbf{G}_n - \{e, e'\})$.

Now apply Lemma 3.1.20 to bound the (unconditional) probability that $B_k(v, \mathbf{G}_n - \{e, e'\})$ and $B_{k-1}(u, \mathbf{G}_n - \{e, e'\})$ are not trees and Corollary 3.1.15 to bound the probability that the neighbourhoods intersect. Then

$$\begin{aligned}
&\mathbb{P}_n(B_k \text{ is not a tree} \mid Y_e, Y_{e'}) \\
&\quad \leq C\frac{(W_v+1)^2 + (W_u+1)^2}{n\vartheta}(\Gamma_{3,n}+1)(\Gamma_{2,n}+1)^{2k+1} \\
&\qquad + C\frac{W_u W_v}{n\vartheta}(\Gamma_{2,n}+1)^{2k} + C\frac{(W_u+W_v)(W_{u'}+W_{v'})}{n\vartheta}(\Gamma_{2,n}+1)^k.
\end{aligned}$$

The same holds for the probability that B'_k is a tree, so that together with the definition of $\rho_{n,k}$

$$\mathbb{P}_n(E_0^c \mid Y_e, Y_{e'}) \leq \rho_{n,k}(\{u, v, u', v'\}). \tag{4.10}$$

For the second probability in the second term of (4.9) we use Lemmas 3.5.2 and 3.5.4 to couple $(B_k, B'_k, \mathbf{T}, \mathbf{T}')$ and set $E_1 = \{(B_k, B'_k) \cong (\mathbf{T}, \mathbf{T}')\}$ such that

$$\begin{aligned}
\mathbb{P}_n(E_1) &= \mathbb{P}_n((B_k, B'_k) \cong (\mathbf{T}, \mathbf{T}') \mid Y_e, Y_{e'}) \\
&\geq 1 - \Big(\varepsilon_{n,k}(\{u, v\}) + 2d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) \\
&\qquad + C\frac{W_u W_v + (W_u+W_v)(W_{u'}+W_{v'})}{n\vartheta}(\Gamma_{2,n}+1)^{2k}\Big)
\end{aligned}$$

Absorb $2d_{\mathrm{TV}}(\mu_{E,n}, \mu_E)$ into $\varepsilon_{n,k}(\{u, v, u', v\}) \geq \varepsilon_{n,k}(\{u, v\})$, and recall the definition of $\rho_{n,k}(\{u, v, u', v'\})$ to conclude

$$\mathbb{P}_n(E_1^c \mid Y_e, Y_{e'}) \leq C(\varepsilon_{n,k}(\{u, v, u', v\}) + \rho_{n,k}(\{u, v, u', v'\})). \tag{4.11}$$

Now consider the first term in (4.9). By the definition of E_1 and (GLA 2) we can replace B_k with $\mathbf{T}$ and B'_k with $\mathbf{T}'$. Then use Lemma 3.5.4 to show that $(\mathbf{T}, \mathbf{T}')$ is distributed like $(\tilde{\mathbf{T}}_k, \mathbf{T}_k)$ or $(\mathbf{T}_k, \tilde{\mathbf{T}}_k)$ on A_e given $(Y_e, Y_{e'})$, so that (GLA 3) gives us the following bound

$$\begin{aligned}
&\mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{E,U}(B_k, B'_k) - \mathrm{LA}_k^{E,L}(B_k, B'_k))^2 \mid Y_e, Y_{e'}] \\
&\quad = \mathbb{1}_{A_e}\mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{E,U}(\mathbf{T}, \mathbf{T}') - \mathrm{LA}_k^{E,L}(\mathbf{T}, \mathbf{T}'))^2 \mid Y_e, Y_{e'}] \\
&\quad \leq m_n^E(v, u)\delta_k^E.
\end{aligned} \tag{4.12}$$

Together (4.9) to (4.12) imply

$$\begin{aligned}
&\mathbb{1}_{A_e}\mathbb{E}_n[(R_k^E(e))^2 \mid Y_e, Y_{e'}] \\
&\quad \leq C(m_n^E(v, u)\delta_k^E + \varepsilon_{n,k}(\{u, v, u', v'\})^{1/2} + \rho_{n,k}(\{u, v, u', v'\})^{1/2}),
\end{aligned}$$

so that together with by (4.8) the expectation of the first term in (4.6) becomes

$$\begin{aligned}
&\mathbb{E}_n[C\mathbb{1}_{A_e}\mathbb{1}_{\tilde{A}_{e'}}\tilde{H}_E^3|R_k^E(e)|] \\
&\quad\le \mathbb{E}_n[C\mathbb{1}_{A_e}\mathbb{1}_{\tilde{A}_{e'}}J_E^{1/2}(\mathbb{E}_n[(R_k^E(e))^2 \mid Y_e, Y_{e'}])^{1/2}] \\
&\quad\le C\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}J_E^{1/2} \\
&\qquad\quad((m_n^E(v,u)\delta_k^E)^{1/2} + \varepsilon_{n,k}(\{u,v,u',v'\})^{1/4} + \rho_{n,k}(\{u,v,u',v'\})^{1/4}).
\end{aligned} \tag{4.13}$$

Putting (4.7) and (4.13) together we obtain

$$\begin{aligned}
&\mathbb{E}_n[|R_k^E(e)U_2^E(F\cup e)U_3^E(e')U_4^E(F'\cup e')|] \\
&\quad\le CJ_E\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta} \\
&\qquad((m_n^E(v,u)\delta_k^E)^{1/2} + \varepsilon_{n,k}(\{u,v,u',v'\})^{1/4} + \rho_{n,k}(\{u,v,u',v'\})^{1/4}) \\
&\quad\le CJ_E\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}((m_n^E(v,u)\delta_k^E)^{1/2} + (m_n^E(v',u')\delta_k^E)^{1/2} \\
&\qquad + \varepsilon_{n,k}(\{u,v,u',v'\})^{1/4} + \rho_{n,k}(\{u,v,u',v'\})^{1/4}).
\end{aligned}$$

Here we introduced the additional term $m_n^E(v',u')\delta_k^E$ in order to make the bound symmetric in e and e'.

The other two expectations in the covariance can be bounded similarly.

This proves the claim. □

The last of the sixteen covariance terms does not involve an R_k^E term and thus has to be bounded differently. The key idea here is that $L_k^E(e)$ is a function of the neighbourhood around e and $L_k^E(e)$ a function of the neighbourhood around e', by sparsity the neighbourhood around e should only be very weakly correlated to the neighbourhood around e' (cf. Section 3.2), so that $L_k^E(e)$ and $L_k^E(e')$ are also only weakly correlated. The formal proof proceeds by constructing independent approximations to $L_k^E(e)$ and $L_k^E(e')$.

Lemma 4.3.4. *We have*

$$\mathrm{Cov}_n(L_k^E(e)L_k^E(F\cup e), L_k^E(e')L_k^E(F'\cup e')) \le CJ_E^{2/3}\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}\rho_{n,k}(\{u,v,u',v'\}).$$

Proof. For the proof of this lemma we will need some more notation.

For any vertex $v \in V_n$ let

$$S_v = ((X_{\{v,x\}}, X'_{\{v,x\}}, w_{\{v,x\}}, w'_{\{v,x\}})_{x\in V_n}, w_v, w'_v).$$

be the collection of random variables of $\mathbf{X}$, $\mathbf{X}'$, $\mathbf{w}$ and $\mathbf{w}'$ at v and edges emanating from v. For an edge $e = \{u,v\}$ define

$$S_e = (S_v, S_u)$$

the collection of random variables associated with the two end vertices u and v of the edge e. Let $U_v = (X_{\{v,x\}}, X'_{\{v,x\}})_{x \in V_n}$ be the collection of random variables that describe the presence or absence of edges emanating from v.

Let

$$A_e = \{(X_e, X'_e) = (1,0) \text{ or } (X_e, X'_e) = (0,1)\},$$

then set

$$X^E(e) = L_k^E(e) L_k^E(F \cup e) \quad \text{and} \quad X^E(e') = L_k^E(e') L_k^E(F' \cup e').$$

Then $X^E(e) = \mathbb{1}_{A_e} X^E(e)$ and $X^E(e') = \mathbb{1}_{A_{e'}} X^E(e')$. By the law of total covariance

$$\begin{aligned}
&\mathrm{Cov}_n(X^E(e), X^E(e')) \\
&\quad = \mathrm{Cov}_n(\mathbb{1}_{A_e} \mathbb{E}_n[X^E(e) \mid S_e, S_{e'}], \mathbb{1}_{A_{e'}} \mathbb{E}_n[X^E(e') \mid S_e, S_{e'}]) \\
&\qquad + \mathbb{E}_n[\mathbb{1}_{A_e} \mathbb{1}_{A_{e'}} \mathrm{Cov}_n(X^E(e), X^E(e') \mid S_e, S_{e'})].
\end{aligned}$$

The claim then follows from the following two lemmas 4.3.5 and 4.3.6. □

Lemma 4.3.5. *We have*

$$\begin{aligned}
&\mathrm{Cov}_n(\mathbb{1}_{A_e} \mathbb{E}_n[X^E(e) \mid S_e, S_{e'}], \mathbb{1}_{A_{e'}} \mathbb{E}_n[X^E(e') \mid S_e, S_{e'}]) \\
&\quad \le C J_E^{2/3} \frac{W_u W_v}{n\vartheta} \frac{W_{u'} W_{v'}}{n\vartheta} \rho_{n,k}(\{u, v, u', v'\}).
\end{aligned}$$

Proof. The random variable $X^E(e)$ can be written as a function of

$$(B_k(v, \mathbf{G}_n), B_k(v, \mathbf{G}_n^e), B_k(v, \mathbf{G}_n^F), B_k(v, \mathbf{G}_n^{F \cup e})).$$

Define an approximation $\tilde{X}^E(e)$ of $X^E(e)$ as the same function, but applied to

$$\begin{aligned}
&(B_k(v, \mathbf{G}_n - \{u', v'\}), B_k(v, \mathbf{G}_n^e - \{u', v'\}), \\
&\ B_k(v, \mathbf{G}_n^F - \{u', v'\}), B_k(v, \mathbf{G}_n^{F \cup e} - \{u', v'\})).
\end{aligned}$$

By construction $\tilde{X}^E(e)$ is independent of $S_{e'}$, since the vertices u' and v' are completely removed from the underlying graph objects. Define $\tilde{X}^E(e')$ analogously as a function applied to the neighbourhoods of v' in graphs ignoring u and v'. Set

$$Z^E(e) = \mathbb{E}_n[X^E(e) \mid S_e, S_{e'}]$$

and

$$\tilde{Z}^E(e) = \mathbb{E}_n[\tilde{X}^E(e) \mid S_e, S_{e'}] = \mathbb{E}_n[\tilde{X}^E(e) \mid S_e],$$

where the last equality is justified by the fact that $\tilde{X}^E(e)$ is based on the original graph with the edge e' and its endpoints u' and v' completely removed, so that $\tilde{X}^E(e)$ is independent of $S_{e'}$. Note that the fact that $\tilde{X}^E(e)$ is independent of $S_{e'}$ is slightly stronger than the equality above, which only claims that $\tilde{Z}^E(e)$ is a function of S_e, since S_e and $S_{e'}$ overlap.

Define $Z^E(e')$ and $\tilde{Z}^E(e')$ similarly for e'.
By construction and (4.3)

$$|X^E(e)|, |\tilde{X}^E(e)| \le \mathbb{1}_{A_e}\tilde{H}_E(e)\tilde{H}_E(F \cup e) \tag{4.14}$$

and therefore also

$$\begin{aligned} |Z^E(e)| &\le \mathbb{E}_n[|X^E(e)| \mid S_e, S_{e'}] \\ &= \mathbb{E}_n[\mathbb{1}_{A_e}\tilde{H}_E(e)\tilde{H}_E(F \cup e) \mid S_e, S_{e'}] \\ &\le \mathbb{1}_{A_e}\tilde{H}_E(e)\tilde{H}_E(F \cup e), \end{aligned}$$

since $\tilde{H}^E(e)$ and $\tilde{H}^E(F \cup e)$ are measurable with respect to S_e. Similarly

$$|\tilde{Z}^E(e)| \le \mathbb{1}_{A_e}\tilde{H}_E(e)\tilde{H}_E(F \cup e).$$

Analogous results hold for $X^E(e')$, $\tilde{X}^E(e')$, $Z^E(e')$ and $\tilde{Z}^E(e')$.
With this notation the covariance of interest is

$$\begin{aligned} &\mathrm{Cov}_n(\mathbb{1}_{A_e}\mathbb{E}_n[X^E(e) \mid S_e, S_{e'}], \mathbb{1}_{A_{e'}}\mathbb{E}_n[X^E(e') \mid S_e, S_{e'}]) \\ &\quad = \mathrm{Cov}_n(\mathbb{1}_{A_e}Z^E(e), \mathbb{1}_{A_{e'}}Z^E(e')). \end{aligned}$$

Approximate $Z^E(e)$ with $\tilde{Z}^E(e)$ and $Z^E(e')$ with $\tilde{Z}^E(e')$ so that this covariance becomes

$$\begin{aligned} &= \mathrm{Cov}_n(\mathbb{1}_{A_e}\tilde{Z}^E(e), \mathbb{1}_{A_{e'}}\tilde{Z}^E(e')) \\ &\quad + \mathrm{Cov}_n(\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)), \mathbb{1}_{A_{e'}}\tilde{Z}^E(e')) \\ &\quad + \mathrm{Cov}_n(\mathbb{1}_{A_e}\tilde{Z}^E(e), \mathbb{1}_{A_{e'}}(Z^E(e') - \tilde{Z}^E(e'))) \\ &\quad + \mathrm{Cov}_n(\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)), \mathbb{1}_{A_{e'}}(Z^E(e') - \tilde{Z}^E(e'))). \end{aligned} \tag{4.15}$$

By construction $\mathbb{1}_{A_e}\tilde{Z}_e$ is a function of S_e that is independent of $S_{e'}$ and $\mathbb{1}_{A_{e'}}\tilde{Z}_{e'}$ is a function of $S_{e'}$ that is independent of S_e. Hence, $\mathbb{1}_{A_e}\tilde{Z}_e$ and $\mathbb{1}_{A_{e'}}\tilde{Z}_{e'}$ are independent and the first term vanishes.

We will now bound the remaining three covariances. The argument is similar for all of three terms, so we will only spell out the argument for the first of the remaining three covariances. By the bound on $\tilde{Z}^E(e')$

$$|\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}(Z^E(e)-\tilde{Z}^E(e))\tilde{Z}^E(e')| \le \mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}\tilde{H}_E(e')\tilde{H}_E(F' \cup e')|(Z^E(e)-\tilde{Z}^E(e))|. \tag{4.16}$$

Furthermore, by the bounds (4.14) for $X^E(e)$ and $\tilde{X}^E(e)$ and measurability of $\tilde{H}_E(e')$ and $\tilde{H}_E(F' \cup e')$ we have

$$\begin{aligned} |Z^E(e) - \tilde{Z}^E(e)| &= |\mathbb{E}[X^E(e) - \tilde{X}^E(e) \mid S_e, S_{e'}]| \\ &\le 2\tilde{H}_E(e)\tilde{H}_E(F \cup e)\mathbb{P}_n(X^E(e) \ne \tilde{X}^E(e) \mid S_e, S_{e'}). \end{aligned} \tag{4.17}$$

We now claim that the probability that $X^E(e)$ and $\tilde{X}^E(e)$ differ can be bounded by

$$\mathbb{P}_n(X^E(e) \neq \tilde{X}^E(e) \mid S_e, S_{e'}) \leq C\Xi \tag{4.18}$$

for a random variable Ξ that is independent of X_e, X'_e, $X_{e'}$ and $X'_{e'}$ and satisfies $\mathbb{E}_n[\Xi] \leq C\rho_{n,k}(\{u,v,u',v'\})$. To verify this, note that $X^E(e) \neq \tilde{X}^E(e)$ implies that $B_k(v, \mathbf{G} - \{u',v'\}) \neq B_k(v,\mathbf{G})$ for at least one of $\mathbf{G} = \mathbf{G}_n, \mathbf{G}_n^e, \mathbf{G}_n^F, \mathbf{G}_n^{F\cup e}$. For $\mathbf{G} = \mathbf{G}_n$ we can use Lemma 3.1.19 with $\mathcal{V} = \{u,v\}$, $\mathcal{U} = \{u',v'\}$ and $\mathcal{R} = \emptyset$ to find a random variable ξ_k that is independent of X_e, X'_e, $X_{e'}$ and $X'_{e'}$ such that

$$\begin{aligned}
&\mathbb{P}_n(B_k(v, \mathbf{G}_n - \{u',v'\}) \neq B_k(v,\mathbf{G}_n) \mid S_e, S_{e'}) \\
&\quad \leq \mathbb{P}_n(u' \in B_k(v,\mathbf{G}_n) \mid S_e, S_{e'}) + \mathbb{P}_n(v' \in B_k(v,\mathbf{G}_n) \mid S_e, S_{e'}) \\
&\quad \leq \xi_k(\{u,v\},\{u',v'\},\emptyset)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_n[\xi_k(\{u,v\},\{u',v'\},\emptyset)] &\leq \min\left\{\frac{(W_u+W_v)(W_{u'}+W_{v'})}{n\vartheta}(\Gamma_{2,n}+1)^k, 1\right\} \\
&\leq \rho_{n,k}(\{u,v,u',v'\}).
\end{aligned}$$

Similar bounds hold for the other cases, which are not based on $\mathbf{G}_n$, but on $\mathbf{G}_n^e$, $\mathbf{G}_n^F$ or $\mathbf{G}_n^{F\cup e}$. Sum these bounds to obtain Ξ, which is independent of $\mathbf{X}$ and $\mathbf{X}'$ at e and e' and has expectation bounded by $C\rho_{n,k}(\{u,v,u',v'\})$

Put (4.16) to (4.18) together to find

$$\begin{aligned}
&|\mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}(Z^E(e) - \tilde{Z}^E(e))\tilde{Z}^E(e')]| \\
&\quad \leq C\mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}\tilde{H}_E(e)\tilde{H}_E(F\cup e)\tilde{H}_E(e')\tilde{H}_E(F'\cup e')\Xi].
\end{aligned}$$

Now $\tilde{H}_E$ depends only on $\mathbf{w}$ and $\mathbf{w}'$, that Ξ depends on $\mathbf{X}$ and $\mathbf{X}'$ except at e and e' and A_e and $A_{e'}$ depends only on $\mathbf{X}$ and $\mathbf{X}'$ at e and e'. This means that the expectation factors, so that the definitions of A_e, $A_{e'}$, the bound for $\tilde{H}_E$ Lemma 3.1.19 and for the expectation of Ξ yield

$$\leq CJ_E^{2/3}\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}\rho_{n,k}(\{u,v,u',v'\}).$$

The same holds for $|\mathbb{E}_n[\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e))]\mathbb{E}_n[\mathbb{1}_{A_{e'}}\tilde{Z}^E(e')]|$ by analogous arguments. This bound is symmetric in e and e' so that it also holds for the other two remaining covariance terms in (4.15) involving a difference of Z^E and $\tilde{Z}^E$.

Putting these results together proves the claim. □

Lemma 4.3.6. *We have*

$$\mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}\operatorname{Cov}_n(X^E(e), X^E(e') \mid S_e, S_{e'})] \leq CJ_E^{2/3}\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}\rho_{n,k}(\{u,v,u',v'\}).$$

Proof. Similar to the previous proof define the following objects

$$\begin{aligned}
\mathbf{B}_k(-\{u,v\}) &= (B_{k-1}(w,\mathbf{G}_n - \{u,v\}), B_{k-1}(w,\mathbf{G}_n^F - \{u,v\}))_{w\in V_n},\\
\mathbf{B}'_k(-\{u',v'\}) &= (B_{k-1}(w,\mathbf{G}_n - \{u',v'\}), B_{k-1}(w,\mathbf{G}_n^{F'} - \{u',v'\}))_{w\in V_n},\\
\mathbf{B}_k(-) &= (B_{k-1}(w,\mathbf{G}_n - \{u,v,u',v'\}), B_{k-1}(w,\mathbf{G}_n^F - \{u,v,u',v'\}))_{w\in V_n},\\
\mathbf{B}'_k(-) &= (B_{k-1}(w,\mathbf{G}_n - \{u,v,u',v'\}), B_{k-1}(w,\mathbf{G}_n^{F'} - \{u,v,u',v'\}))_{w\in V_n},
\end{aligned}$$

With this notation $X^E(e)$ can be written as a function of $(\mathbf{B}_k(-\{u,v\}), S_e)$, because the entire k-neighbourhood of u and v can easily be obtained from S_e and $\mathbf{B}_k(-\{u,v\})$.

We can define an approximation $\tilde{X}^E(e)$ of $X^E(e)$ as the same function applied to $(\mathbf{B}_k(-), S_e)$. Formally the function is defined taking all of $\mathbf{B}_k(-\{u,v\})$ or $\mathbf{B}_k(-)$ as argument, but given S_e it can be recast into a form that only uses neighbourhoods of vertices w that are connected to u or v by an edge in the original graph $\mathbf{G}_n$ or in $\mathbf{G}_n^F$. We will call such vertices ws – and by extension their neighbourhoods – *relevant*. See Fig. 4.2 for an illustration. Define $\tilde{X}^E(e')$ similarly.

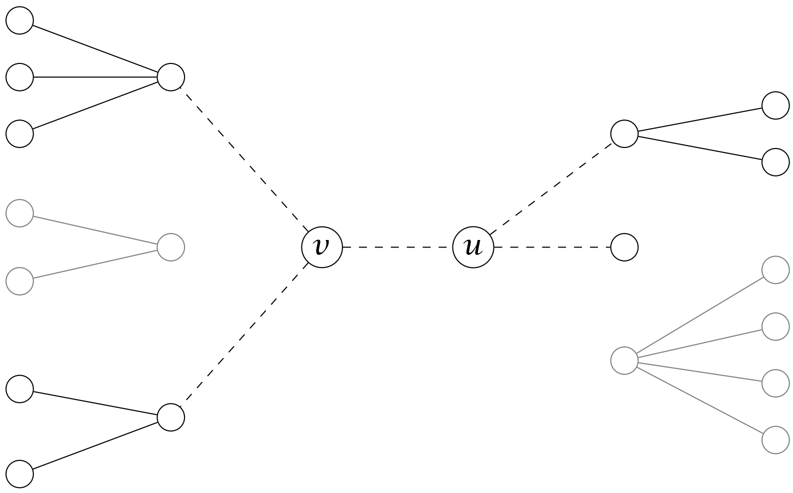


Figure 4.2: The neighbourhood around v in a graph G can be obtained from the information on edges incident to u and v (dashed) and the neighbourhoods (in $G - \{v,u\}$) of vertices that connect to v or u via dashed edges (solid black). Neighbourhoods (in $G - \{v,u\}$) of vertices that do not connect to v or u are irrelevant (shown in grey).

First estimate $\mathbb{P}_n(\tilde{X}^E(e) \neq X^E(e) \mid S_e, S_{e'})$. Since $X^E(e)$ and $\tilde{X}^E(e)$ are obtained by applying the same function to slightly modified neighbourhoods, the two can only differ if at least one of the relevant neighbourhoods differs between $\mathbf{B}_k(-\{u,v\})$ and $\mathbf{B}_k(-)$. Such a neighbourhood of a vertex w, say, can only differ if u' or v' is contained in the $k-1$-neighbourhood of w in $\mathbf{G}_n - \{u,v\}$ or $\mathbf{G}_n^F - \{u,v\}$. That is to say, there is a path of length at most $k-1$ from w to $\{u',v'\}$ in $\mathbf{G}_n - \{u,v\}$

or $\mathbf{G}_n^F - \{u, v\}$. Since only neighbourhoods of vertices w that neighbour u or v are relevant, this immediately implies that there is a path of length no more than k steps from $\{u, v\}$ to $\{u', v'\}$ in $\mathbf{G}_n$ or $\mathbf{G}_n^F$. Hence by Lemma 3.1.19

$$\begin{aligned}
&\mathbb{P}_n(\tilde{X}^E(e) \neq X^E(e) \mid S_e, S_{e'}) \\
&\quad \leq \mathbb{P}_n(\{u, v\} \leftrightsquigarrow_{\leq k} \{u', v'\} \mid S_e, S_{e'}) + \mathbb{P}_n(\{u, v\} \leftrightsquigarrow^F_{\leq k} \{u', v'\} \mid S_e, S_{e'}) \\
&\quad \leq \xi_k(\{u', v'\}, \{u, v\}, \varnothing) + \xi_k^F(\{u', v'\}, \{u, v\}, \varnothing).
\end{aligned}$$

Call the right-hand side of the last equation Ξ. Note that Ξ is independent of X_e, X'_e, $X_{e'}$ and $X'_{e'}$. Furthermore, its expectation can be bounded as follows

$$\mathbb{E}_n[\Xi] \leq C \min\left\{ \frac{(W_u + W_v)(W_{u'} + W_{v'})}{n\vartheta} (\Gamma_{2,n} + 1)^k, 1 \right\} \leq C\rho_{n,k}(\{u, v, u', v'\}).$$

An analogous bound Ξ' with the same expectation and independence properties can be found for $\mathbb{P}_n(\tilde{X}^E(e') \neq X^E(e') \mid S_e, S_{e'})$

Let

$$\tilde{H}_E = \max\{\tilde{H}_E(e), \tilde{H}_E(F \cup e), \tilde{H}_E(e'), \tilde{H}_E(F' \cup e')\}.$$

Write $X^E(e) = X^E(e) - \tilde{X}^E(e) + \tilde{X}^E(e)$ and $X^E(e) = X^E(e') - \tilde{X}^E(e') + \tilde{X}^E(e')$, then expand $\mathrm{Cov}(X^E(e), X^E(e') \mid S_e, S_{e'})$ to obtain

$$\begin{aligned}
&\mathrm{Cov}_n(X^E(e), X^E(e') \mid S_e, S_{e'}) \\
&\quad = \mathrm{Cov}_n(\tilde{X}^E(e), \tilde{X}^E(e') \mid S_e, S_{e'}) \\
&\qquad + \mathrm{Cov}_n(X^E(e) - \tilde{X}^E(e), \tilde{X}^E(e') \mid S_e, S_{e'}) \\
&\qquad + \mathrm{Cov}_n(\tilde{X}^E(e), X^E(e') - \tilde{X}^E(e') \mid S_e, S_{e'}) \\
&\qquad + \mathrm{Cov}_n(X^E(e) - \tilde{X}^E(e), X^E(e') - \tilde{X}^E(e') \mid S_e, S_{e'}).
\end{aligned} \tag{4.19}$$

First focus on the three covariances containing $X^E(e) - \tilde{X}^E(e)$ or $X^E(e') - \tilde{X}^E(e')$. To this end note that by measurability of terms depending only on information at e or e', i.e. on $S_e, S_{e'}$, we have

$$\begin{aligned}
&|\mathbb{E}_n[(X^E(e) - \tilde{X}^E(e))X^E(e') \mid S_e, S_{e'}]| \\
&\quad \leq \mathbb{E}_n[|X^E(e) - \tilde{X}^E(e)||X^E(e')| \mid S_e, S_{e'}] \\
&\quad \leq \mathbb{E}_n[(|X^E(e)| + |\tilde{X}^E(e)|)\mathbb{1}_{\{X^E(e) \neq \tilde{X}^E(e)\}}|X^E(e')| \mid S_e, S_{e'}] \\
&\quad \leq \mathbb{E}_n[C\mathbb{1}_{A_e}\tilde{H}_E(e)\tilde{H}_E(F \cup e)\mathbb{1}_{\{X^E(e) \neq \tilde{X}^E(e)\}}\mathbb{1}_{A_{e'}}\tilde{H}_E(e')\tilde{H}_E(F' \cup e') \mid S_e, S_{e'}] \\
&\quad \leq C\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}\tilde{H}_E(e)\tilde{H}_E(F \cup e)\tilde{H}_E(e')\tilde{H}_E(F' \cup e')\mathbb{P}_n(X^E(e) \neq \tilde{X}^E(e) \mid S_e, S_{e'}) \\
&\quad \leq C\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}\tilde{H}_E^4\Xi.
\end{aligned}$$

The same holds for the other terms in the definition of the covariance. With (4.19) this shows

$$|\mathrm{Cov}_n(X^E(e), X^E(e') \mid S_e, S_{e'}) - \mathrm{Cov}_n(\tilde{X}^E(e), \tilde{X}^E(e') \mid S_e, S_{e'})| \leq C\tilde{H}_E^4(\Xi + \Xi')$$

and thus by independence of $\Xi + \Xi'$ from A_e and A'_e

$$\begin{aligned}&\mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}|\mathrm{Cov}_n(X^E(e),X^E(e') \mid S_e,S_{e'}) - \mathrm{Cov}_n(\tilde{X}^E(e),\tilde{X}^E(e') \mid S_e,S_{e'})|]\\ &\quad \le CJ_E^{2/3}\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}\rho_{n,k}(\{u,v,u',v'\}).\end{aligned} \tag{4.20}$$

It thus remains to investigate $\mathrm{Cov}_n(\tilde{X}^E(e),\tilde{X}^E(e') \mid S_e,S_{e'})$.

By construction $\mathbf{B}_k(-)$ and $\mathbf{B}'_k(-)$ are independent of S_e and $S_{e'}$. Hiding the dependence of $\tilde{X}^E(e)$ and $\tilde{X}^E(e')$ on S_e and $S_{e'}$ in functions Ψ and Ψ' we may use Lemma A.2.3 to write

$$\mathrm{Cov}_n(\tilde{X}^E(e),\tilde{X}^E(e') \mid S_e,S_{e'}) = \mathrm{Cov}_n(\Psi(\mathbf{B}_k(-)),\Psi'(\mathbf{B}'_k(-))).$$

By definition $|\Psi| \le \tilde{H}_E(e)\tilde{H}_E(F \cup e)$, which is constant conditioned on S_e and $S_{e'}$. A similar bound holds for $|\Psi'|$.

Recall that Ψ and Ψ' depend not on all of $B_k(w,\mathbf{G}_n)$ for $w \in V_n$, but only on the neighbourhoods of vertices w for which there is a connection to u or v and u' or v', respectively. This data is known conditioned on S_e and $S_{e'}$. More formally, the relevant vertices for Ψ are contained in

$$D = D_1(v) \cup D_1(u) \cup D_1^F(v) \cup D_1^F(u),$$

and the vertices relevant for Ψ' are contained in

$$D' = D_1(v') \cup D_1(u') \cup D_1^{F'}(v') \cup D_1^{F'}(u').$$

While D and D' are not independent of A_e and $A_{e'}$,

$$\bar{D} = S_1^{(u)}(v) \cup S_1^{(v)}(u) \cup S_1^{(u),F}(v) \cup S_1^{(v),F}(u)$$

and

$$\bar{D}' = S_1^{(u')}(v') \cup S_1^{(v')}(u') \cup S_1^{(u'),F'}(v') \cup S_1^{(v'),F'}(u')$$

are independent of A_e and $A_{e'}$ and satisfy $D \subseteq \bar{D}$ and $D' \subseteq \bar{D}'$. Furthermore by Corollary 3.1.9

$$\begin{aligned}\mathbb{E}_n[(\|\bar{D}\| + |\bar{D}|)^2] &\le \mathbb{E}_n[\|\bar{D}\|^2] + 2\mathbb{E}_n[\|\bar{D}\||\bar{D}|] + \mathbb{E}_2[|\bar{D}|^2]\\ &\le \mathbb{E}_n[\|\bar{D}\|^2] + 2\mathbb{E}_n[\|\bar{D}\|^2]^{1/2}\mathbb{E}_n[|\bar{D}|^2]^{1/2} + \mathbb{E}_2[|\bar{D}|^2]\\ &\le C(\|\{u,v\}\| + 2)^2(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+1)^2(\Gamma_{3,n}+1),\end{aligned}$$

so that

$$\begin{aligned}&\mathbb{E}_n[(\|\bar{D}\| + |\bar{D}|)(\|\bar{D}'\| + |\bar{D}'|)]\\ &\quad \le \mathbb{E}_n[(\|\bar{D}\| + |\bar{D}|)^2]^{1/2}\mathbb{E}_n[(\|\bar{D}'\| + |\bar{D}'|)^2]^{1/2}\\ &\quad \le C(\|\{u,v\}\| + 2)(\|\{u',v'\}\| + 2)(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+1)^2(\Gamma_{3,n}+1).\end{aligned}$$

Now Lemma 3.2.5 implies

$$\begin{aligned}
&\mathrm{Cov}_n(\Psi(\mathbf{B}_k(-)), \Psi'(\mathbf{B}'_k(-))) \\
&\quad\le C\tilde{H}_E^4 \min\Big\{\frac{(\|D\| + |D|)(\|D'\| + |D'|)}{n\vartheta}(\Gamma_{3,n}+1)(\Gamma_{2,n}+C)^{2k+1}, 1\Big\} \\
&\quad\le C\tilde{H}_E^4 \min\Big\{\frac{(\|\bar{D}\| + |\bar{D}|)(\|\bar{D}'\| + |\bar{D}'|)}{n\vartheta}(\Gamma_{3,n}+1)(\Gamma_{2,n}+C)^{2k+1}, 1\Big\}.
\end{aligned}$$

Taking the expectation and using independence we obtain

$$\begin{aligned}
&\mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}\mathrm{Cov}_n(\tilde{X}^E(e), \tilde{X}^E(e') \mid S_e, S_{e'})] \\
&\quad= \mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{A_{e'}}\mathrm{Cov}(\Psi(\mathbf{B}_k(-)), \Psi'(\mathbf{B}'_k(-)))] \\
&\quad\le C\mathbb{P}_n(A_e)\mathbb{P}_n(A_{e'})\mathbb{E}_n[\tilde{H}_E^4] \\
&\qquad\min\Big\{\frac{\mathbb{E}_n[(\|\bar{D}\| + |\bar{D}|)(\|\bar{D}'\| + |\bar{D}'|)]}{n\vartheta}(\Gamma_{3,n}+1)(\Gamma_{2,n}+C)^{2k+1}, 1\Big\} \\
&\quad\le CJ_E^{2/3}\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}\min\Big\{\frac{(W_u+W_v+2)(W_{u'}+W_{v'}+2)}{n\vartheta} \\
&\qquad\quad(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+C)^{2k+1}(\Gamma_{3,n}+1)^2, 1\Big\} \\
&\quad\le CJ_E^{2/3}\frac{W_uW_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}\rho_{n,k}(\{u,v,u',v'\}). \qquad (4.21)
\end{aligned}$$

Now (4.20) and (4.21) imply the claim. □

With all these results in hand we can verify Proposition 4.3.2.

Proof of Proposition 4.3.2. Recall (4.5)

$$\begin{aligned}
&\mathrm{Cov}_n(\Delta_e f \Delta_e f^F, \Delta_{e'} f \Delta_{e'} f^{F'}) \\
&\quad= \mathrm{Cov}_n((R_k^E(e) + L_k^E(e))(R_k^E(F\cup e) + L_k^E(F\cup e)), \\
&\qquad\qquad (R_k^E(e') + L_k^E(e'))(R_k^E(F'\cup e') + L_k^E(F'\cup e'))).
\end{aligned}$$

Expand this into sixteen terms then apply Lemmas 4.3.3 and 4.3.4 to these terms as appropriate. □

4.3.2 Vertex-vertex case

The combination of two different vertices can be handled like the case of two edges which do not share any vertices.

Proposition 4.3.7. *Let v and v' be two distinct vertices in V_n, then we can choose*

$$\begin{aligned}
c(v,v') = \sigma_n^{-4}CJ_V\zeta_n(v)\zeta_n(v')((m_n^V(v)\delta_k^V)^{1/2} + (m_n^V(v')\delta_k^V)^{1/2} + \\
\varepsilon_{n,k}(\{v,v'\})^{1/4} + \rho_{n,k}(\{v,v'\})^{1/4})
\end{aligned}$$

As previously, the proof is split over several auxiliary lemmas and needs some notation first. The ideas are analogous to what we did in Section 4.3.1.

Let E_0 be the event that $B_k(v, \mathbf{G}_n)$ and thus also $B_k(v, \mathbf{G}_n^v)$ is a tree. Set

$$\tilde{L}_k^V(v) = \mathbb{1}_{E_0} \mathrm{LA}_k^{V,L}(B_k(v, \mathbf{G}_n), B_k(v, \mathbf{G}_n^v))$$

and

$$L_k^V(v) = Q(\tilde{L}_k^V, h(|D_1(v)|) H_V(w_v, w_v')).$$

Furthermore, define

$$R_k^V(v) = \Delta_v f - L_k^V(v).$$

With these definitions and (2.10) we have

$$\begin{aligned} |L_k^V(v)| &\leq h(|D_1(v)|) H_V(w_v, w_v'), \\ |R_k^V(v)| &\leq 2h(|D_1(v)|) H_V(w_v, w_v'). \end{aligned} \tag{4.22}$$

As before also define $L_k^V(F \cup v)$ and $R_k^V(F \cup v)$ based on $B_k(v, \mathbf{G}^F)$ and $B_k(v, \mathbf{G}_n^{F \cup v})$ instead of $B_k(v, \mathbf{G})$ and $B_k(v, \mathbf{G}_n^v)$. With these definitions we can bound

$$\begin{aligned} &\mathrm{Cov}_n(\Delta_v f \Delta_v f^F, \Delta_{v'} f \Delta_{v'} f^{F'}) \\ &\quad = \mathrm{Cov}_n((R_k^V(v) + L_k^V(v))(R_k^V(F \cup v) + L_k^V(F \cup v)), \\ &\qquad\qquad (R_k^V(v') + L_k^V(v'))(R_k^V(F' \cup v') + L_k^V(F' \cup v'))) \end{aligned} \tag{4.23}$$

in order to find $c(v, v')$.

As in the previous section the fifteen term involving at least one R_k^V term can be bounded by appealing to (GLA 6).

Lemma 4.3.8.

$$\begin{aligned} &\mathrm{Cov}(U_1^V(v) U_2^V(F \cup v), U_3^V(v'), U_4^V(F' \cup v')) \\ &\quad \leq C J_V \zeta_n(v)^{1/4} \zeta_n(v')^{1/4} ((m_n^V(v) \delta_k^V)^{1/2} + (m_n^V(v') \delta_k^V)^{1/2} + \\ &\qquad \varepsilon_{n,k}(\{v, v'\})^{1/4} + \rho_{n,k}(\{v, v'\})^{1/4}) \end{aligned}$$

if at least one of the U_i^V terms is R_k^V.

Proof. We show the claim for a term of the form

$$\mathrm{Cov}_n(R_k^V(v) U_2^V(F \cup v), U_3^V(v'), U_4^V(F' \cup v')).$$

The proof for the other terms is analogous and yields the same estimate, since we obtain a bound that is symmetric in v and v'.

Set

$$\tilde{H}_V = \max\{H_V(w_v, w_v'), H_V(w_{v'}, w_{v'}')\}$$

as well as

$$\tilde{D}(v) = h(|D_1^{(v')}(v)| + 1) + h(|D_1^{(v'),F}(v)| + 1)$$

and

$$\tilde{D}(v') = h(|D_1^{(v)}(v')| + 1) + h(|D_1^{(v),F'}(v')| + 1).$$

This construction ensures that $\tilde{D}(v)$ and $\tilde{D}(v')$ are independent and still satisfy

$$h(|D_1(v)|) \le \tilde{D}(v) \quad \text{and} \quad h(|D_1(v')|) \le \tilde{D}(v')$$

as well as a condition like (2.8)

$$\mathbb{E}_n[\tilde{D}(v)^4] \le C\zeta_n(v) \quad \text{and} \quad \mathbb{E}_n[\tilde{D}(v')^4] \le C\zeta_n(v').$$

By the bounds (4.22) for L_k^V and R_k^V, Cauchy–Schwarz and independence of $\tilde{D}$, which depends only on $\mathbf{X}$ and $\mathbf{X}'$, from $\tilde{H}_V$, which depends only on $\mathbf{w}$ and $\mathbf{w}'$, as well as the bounds (2.7) and (2.8) for $h(|D_1(v)|)$ and H_V we have

$$\begin{aligned}
&\mathbb{E}_n[|R_k^V(v)U_2^V(F \cup v)U_3^V(v')U_4^V(F' \cup v')|] \\
&\quad \le C\mathbb{E}_n[\tilde{D}(v)\tilde{D}(v')^2\tilde{H}_V^3|R_k^V(v)|] \\
&\quad \le C\mathbb{E}_n[\tilde{D}(v)^2\tilde{D}(v')^4\tilde{H}_V^6]^{1/2}\mathbb{E}_n[|R_k^V(v)|^2]^{1/2} \\
&\quad \le C(\mathbb{E}_n[\tilde{D}(v)^2\tilde{D}(v')^4]\mathbb{E}_n[\tilde{H}_V^6])^{1/2}\mathbb{E}_n[|R_k^V(v)|^2]^{1/2} \\
&\quad \le CJ_V^{1/2}\zeta_n(v)^{1/4}\zeta_n(v')^{1/2}\mathbb{E}_n[|R_k^V(v)|^2]^{1/2}.
\end{aligned} \tag{4.24}$$

Let $B_k = B_k(v, \mathbf{G}_n)$ and $B_k' = B_k(v, \mathbf{G}_n^v)$. Use Lemma 3.5.2 to couple the neighbourhood $B_k(v, \mathbf{G}_n)$ to $\mathbf{T} \sim \mathbf{T}_k(W_v, \nu, \mu_E, \mu_V)$. Exchange the weight of the root of $\mathbf{T}$ for a random variable $\tilde{w}_v$ with distribution μ_V coupled to the weight of v in $\mathbf{G}_n^v$ such that $\tilde{w}_v \neq w_v$ with probability at most $d_{\mathrm{TV}}(\mu_{V,n}, \mu_V)$ and call the resulting weighted tree $\tilde{\mathbf{T}}$.

Let E_1 be the event that $B_k \cong \mathbf{T}$ and $B_k' \cong \tilde{\mathbf{T}}$ and let E_0 be the event that $B_k(v, \mathbf{G}_n)$ is a tree. Then

$$\begin{aligned}
|R_k^V(v)| &\le \mathrm{LA}_k^{V,U}(B_k, B_k') - \mathrm{LA}_k^{V,L}(B_k, B_k') \\
&\le \mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B_k') - \mathrm{LA}_k^{V,L}(B_k, B_k')) + \mathbb{1}_{E_0^c \cup E_1^c} C\tilde{D}(v)\tilde{H}_V.
\end{aligned}$$

Square and take expectations, then apply Cauchy–Schwarz and use independence of $\tilde{D}$ from $\tilde{H}_V$ as well as (2.7) and (2.8) to show

$$\begin{aligned}
\mathbb{E}[(R_k^V(v))^2] &\le \mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B_k') - \mathrm{LA}_k^{V,L}(B_k, B_k'))^2] \\
&\quad + C\mathbb{E}_n[\tilde{D}(v)^2\tilde{H}_V^2\mathbb{1}_{E_0^c \cup E_1^c}] \\
&\le \mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B_k') - \mathrm{LA}_k^{V,L}(B_k, B_k'))^2] \\
&\quad + CJ_V^{1/3}\zeta_n(v)^{1/2}(\mathbb{P}_n(E_0^c) + \mathbb{P}_n(E_1^c))^{1/2}.
\end{aligned} \tag{4.25}$$

Since the rerandomisation of the vertex weight does not change the underlying tree structure, B_k is a tree if and only if B'_k is a tree. Hence, Lemma 3.1.20 implies

$$\mathbb{P}_n(B_k(v, \mathbf{G}_n) \text{ is not a tree}) \leq C(\Gamma_{2,n}+1)^{2k+1}(\Gamma_{3,n}+1)\frac{(W_v+1)^2}{n\vartheta},$$

so that recalling the definition of $\rho_{n,k}$ we have

$$\mathbb{P}_n(E_0^c) = \mathbb{P}_n(B_k(v, \mathbf{G}_n) \text{ is not a tree}) \leq C\rho_{n,k}(\{v, v'\}). \tag{4.26}$$

By construction of the coupling $B_k \cong \mathbf{T}$ implies $B'_k \cong \bar{\mathbf{T}}$ unless $\tilde{w}_v \neq w_v$. Hence,

$$\begin{aligned}\mathbb{P}_n(E_1^c) &\leq \mathbb{P}_n(B_k \ncong \mathbf{T}) + \mathbb{P}_n(\tilde{w}_v \neq w_v)\\ &\leq \varepsilon_{n,k}(\{v\}) + d_{\mathrm{TV}}(\mu_{V,n}, \mu_V).\end{aligned}$$

Absorb $d_{\mathrm{TV}}(\mu_{V,n}, \mu_V)$ into $\varepsilon_{k,n}(v) \leq \varepsilon_{n,k}(\{v, v'\})$ to find the shorter bound

$$\leq C\varepsilon_{n,k}(\{v, v'\}). \tag{4.27}$$

On E_1 the neighbourhoods B_k and B'_k can be replaced with $\mathbf{T}$ and $\bar{\mathbf{T}}$. Then by (GLA 6)

$$\begin{aligned}&\mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B'_k) - \mathrm{LA}_k^{V,L}(B_k, B'_k))^2]\\ &\quad\leq \mathbb{E}_n[(\mathrm{LA}_k^{V,U}(\mathbf{T}, \bar{\mathbf{T}}) - \mathrm{LA}_k^{V,L}(\mathbf{T}, \bar{\mathbf{T}}))^2]\\ &\quad\leq m_n^V(v)\delta_k^V. \end{aligned}\tag{4.28}$$

Put together (4.24) with (4.25) and (4.26) and (4.27) as well as (4.28) to obtain

$$\begin{aligned}&\mathbb{E}[|R_k^V(v)U_2^V(F\cup v)U_3^V(v')U_4^V(F'\cup v')|]\\ &\quad\leq CJ_V\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}((m_n^V(v)\delta_k^V)^{1/2} + \varepsilon_{n,k}(\{v, v'\})^{1/4} + \rho_{n,k}(\{v, v'\})^{1/4})\\ &\quad\leq CJ_V\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\\ &\qquad((m_n^V(v)\delta_k^V)^{1/2} + (m_n^V(v')\delta_k^V)^{1/2} + \varepsilon_{n,k}(\{v, v'\})^{1/4} + \rho_{n,k}(\{v, v'\})^{1/4}).\end{aligned}$$

Here, we added an additional $m_n^V(v')\delta_k^V$ to make the bound symmetric in v and v'.

The remaining components of the covariance can be bounded similarly. This concludes the proof. □

It remains to bound the covariance involving only L_k^V terms

Lemma 4.3.9. *We have*

$$\mathrm{Cov}_n(L_k^V(v)L_k^V(F\cup v), L_k^V(v')L_k^V(F'\cup v')) \leq CJ_V^{2/3}\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\rho_{n,k}(\{v, v'\}).$$

Proof. Set $X^V(v) = L_k^V(v)L_k^V(F \cup v)$ and $X^V(v') = L_k^V(v')L_k^V(F' \cup v')$. Recall the definition of S_v and $S_{v'}$ in the proof of Lemma 4.3.4 and use the law of total covariance to show

$$\begin{aligned}\mathrm{Cov}_n(X^V(v), X^V(v')) &= \mathrm{Cov}_n(\mathbb{E}_n[X^V(v) \mid S_v, S_{v'}], \mathbb{E}_n[X^V(v') \mid S_v, S_{v'}]) \\ &\quad + \mathbb{E}_n[\mathrm{Cov}_n(X^V(v), X^V(v') \mid S_v, S_{v'})].\end{aligned}$$

The claim will follow from Lemmas 4.3.10 and 4.3.11. □

Lemma 4.3.10. *We have*

$$\begin{aligned}&\mathrm{Cov}_n(\mathbb{E}_n[X^V(v) \mid S_v, S_{v'}], \mathbb{E}_n[X^E(v') \mid S_v, S_{v'}]) \\ &\leq C J_V^{2/3} \zeta_n(v)^{1/2} \zeta_n(v')^{1/2} \rho_{n,k}(\{v, v'\}).\end{aligned}$$

Proof. The random variable $X^V(v)$ can be written as a function of

$$(B_k(v, \mathbf{G}_n), B_k(v, \mathbf{G}_n^v), B_k(v, \mathbf{G}_n^F), B_k(v, \mathbf{G}_n^{F\cup v})).$$

Define an approximation $\tilde{X}^V(v)$ of $X^V(v)$ as the same function, but applied to

$$(B_k(v, \mathbf{G}_n - v'), B_k(v, \mathbf{G}_n^v - v'), B_k(v, \mathbf{G}_n^F - v'), B_k(v, \mathbf{G}_n^{F\cup v} - v')).$$

By construction $\tilde{X}^V(v)$ is independent of $S_{v'}$, since the vertex v' and with it the edges emanating from v' were completely removed from the underlying graph objects. Define $\tilde{X}^V(v')$ similarly. Set $Z^V(v) = \mathbb{E}_n[X^V(v) \mid S_v, S_{v'}]$ and $\tilde{Z}^V(v) = \mathbb{E}_n[\tilde{X}^V(v) \mid S_v, S_{v'}] = \mathbb{E}_n[\tilde{X}^V(v) \mid S_v]$, similarly for $Z^V(v')$ and $\tilde{Z}^V(v')$.

Let

$$\tilde{H}_V = \max\{H_V(w_v, w_v'), H_V(w_{v'}, w_{v'}')\}$$

as well as

$$\tilde{D}(v) = h(|D_1(v)|) + h(|D_1^F(v)|) \quad \text{and} \quad \tilde{D}(v') = h(|D_1(v')|) + h(|D_1^{F'}(v')|).$$

By construction and the bounds on L_k^V from (4.22)

$$|X^V(v)|, |\tilde{X}^V(v)| \leq \tilde{D}(v)^2 \tilde{H}_V^2$$

and therefore also

$$\begin{aligned}|Z^V(v)| &= |\mathbb{E}[X^V(v) \mid S_v, S_{v'}]| \\ &\leq \mathbb{E}[\tilde{D}(v)^2 \tilde{H}_V^2 \mid S_v, S_{v'}] \\ &= \tilde{D}(v)^2 \tilde{H}_V^2,\end{aligned}$$

because $D(v)$ is S_v-measurable. Similarly

$$|\tilde{Z}^V(v)| \leq \tilde{D}(v)^2 \tilde{H}_V^2.$$

The same bounds with $\tilde{D}(v)$ replaced by $\tilde{D}(v')$ also hold for $X^V(v')$, $\tilde{X}^V(v')$, $Z^V(v')$ and $\tilde{Z}^V(v')$.

Split the relevant covariance

$$\begin{aligned}\mathrm{Cov}_n(Z^V(v), Z^V(v')) = \mathrm{Cov}_n(\tilde{Z}^V(v), \tilde{Z}^V(v'))\\ &+ \mathrm{Cov}_n((Z^V(v) - \tilde{Z}^V(v)), \tilde{Z}^V(v'))\\ &+ \mathrm{Cov}_n(\tilde{Z}^V(v), (Z^V(v') - \tilde{Z}^V(v')))\\ &+ \mathrm{Cov}_n((Z^V(v) - \tilde{Z}^V(v)), (Z^V(v') - \tilde{Z}^V(v'))).\end{aligned}$$

By construction $\tilde{Z}^V(v)$ is a function of S_v, but since $\tilde{X}^V(v)$ and thus also $\tilde{Z}^V(v)$ ignores v', it is actually a function of

$$((X_{\{v,u\}}, X'_{\{v,u\}}, w_{\{v,u\}}, w'_{\{v,u\}})_{u \in V_n \setminus \{v'\}}, w_v, w'_v).$$

In the same vein $\tilde{Z}^V(v')$ is a function of

$$((X_{\{v',u\}}, X'_{\{v',u\}}, w_{\{v',u\}}, w'_{\{v',u\}})_{u \in V_n \setminus \{v\}}, w_{v'}, w'_{v'}).$$

These two collections of random variables are disjoint, so that $\tilde{Z}^V(v)$ and $\tilde{Z}^V(v')$ are independent. Hence, the first term vanishes.

We will investigate the bound for the first of the remaining three covariances. The other terms can be bounded similarly (the bound we obtain is symmetric in v and v'). By the bound on $\tilde{Z}^V(v')$

$$|(Z^V(v) - \tilde{Z}^V(v))\tilde{Z}^V(v')| \leq \tilde{D}(v')^2 \tilde{H}_V^2 |(Z^V(v) - \tilde{Z}^V(v))|. \tag{4.29}$$

Furthermore, by the bounds on $X^V(v)$ and $\tilde{X}^V(v)$ and measurability

$$\begin{aligned}|Z^V(v) - \tilde{Z}^V(v)| &= |\mathbb{E}_n[X^V(v) - \tilde{X}^V(v) \mid S_v, S_{v'}]|\\ &\leq \mathbb{E}_n[2\tilde{D}^2(v)\tilde{H}_V^2 \mathbb{1}_{\{X^V(v) \neq \tilde{X}^V(v)\}} \mid S_v, S_{v'}]\\ &\leq 2\tilde{D}^2(v)\tilde{H}_V^2 \mathbb{P}_n(X^V(v) \neq \tilde{X}^V(v) \mid S_v, S_{v'}).\end{aligned} \tag{4.30}$$

We now claim that probability that $X^V(v)$ and $\tilde{X}^V(v)$ differ can be bounded by

$$\mathbb{P}_n(X^V(v) \neq \tilde{X}^V(v) \mid S_v, S_{v'}) \leq \Xi, \tag{4.31}$$

where Ξ depends only on X and X'.

To see (4.31), note that $X^V(v) \neq \tilde{X}^V(v)$ implies $B_k(v, \mathbf{G} - v') \neq B_k(v, \mathbf{G})$ for at least one of $\mathbf{G} = \mathbf{G}_n, \mathbf{G}_n^v, \mathbf{G}_n^F, \mathbf{G}_n^{F \cup v}$. For $\mathbf{G} = \mathbf{G}_n$ we apply Lemma 3.1.19 we obtain

$$\begin{aligned}&\mathbb{P}_n(B_k(v, \mathbf{G}_n - v') \neq B_k(v, \mathbf{G}_n) \mid S_v, S_{v'})\\ &\quad = \mathbb{P}_n(v' \notin B_k(v, \mathbf{G}_n) \mid S_v, S_{v'})\\ &\quad \leq \xi_k(\{v\}, \{v'\}, \varnothing)\end{aligned}$$

The other cases are analogous. Let Ξ be the sum of the $\xi_k(v, v')$s. Together with (2.8) Lemma 3.1.19 additionally guarantees that

$$\begin{aligned}
&\mathbb{E}_n[\tilde{D}(v)^2\tilde{D}(v')^2\xi_k(\{v\}, \{v'\}, \varnothing)] \\
&\quad \le C\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\min\left\{\frac{W_v W_{v'}}{n\vartheta}(\Gamma_{2,n}+1)^k, 1\right\},
\end{aligned}$$

so that we have

$$\begin{aligned}
&\mathbb{E}_n[\tilde{D}(v)^2\tilde{D}(v')^2\Xi] \\
&\quad \le C\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\min\left\{\frac{W_v W_{v'}}{n\vartheta}(\Gamma_{2,n}+1)^k, 1\right\} \\
&\quad \le C\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\rho_{n,k}(\{v, v'\}). \qquad (4.32)
\end{aligned}$$

Putting (4.29), (4.30) and then (4.31) together we obtain

$$\begin{aligned}
&|\mathbb{E}_n[(Z^V(v) - \tilde{Z}^V(v))\tilde{Z}^V(v')]| \\
&\quad \le C\mathbb{E}_n[\tilde{D}(v)^2\tilde{H}_V^2\tilde{D}(v')^2\tilde{H}_V^2\mathbb{P}_n(X^V(v) \neq \tilde{X}^V(v) \mid S_v, S_{v'})] \\
&\quad \le C\mathbb{E}_n[\tilde{H}_V^4\tilde{D}(v)^2\tilde{D}(v')^2\Xi].
\end{aligned}$$

Use independence of $\tilde{H}_V$ from the other terms and then (4.32) to bound

$$\begin{aligned}
&\le C\mathbb{E}_n[\tilde{H}_V^4]\mathbb{E}_n[\tilde{D}(v)^2\tilde{D}(v')^2\Xi] \\
&\le CJ_V^{2/3}\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\rho_{n,k}(\{v, v'\}).
\end{aligned}$$

The same holds for $|\mathbb{E}[\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e))]\mathbb{E}[\mathbb{1}_{A_{e'}}\tilde{Z}^E(e')]|$ by analogous arguments.
Putting these results together proves the claim. □

Lemma 4.3.11. *We have*

$$\mathbb{E}_n[\mathrm{Cov}_n(X^V(v), X^V(v') \mid S_v, S_{v'})] \le CJ_V^{2/3}\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\rho_{n,k}(\{v, v'\}).$$

Proof. Let

$$\tilde{H}_V = \max\{H_V(w_v, w'_v), H_V(w_{v'}, w'_{v'})\}$$

as well as

$$\tilde{D}(v) = h(|D_1(v)|) + h(|D_1^F(v)|) \quad \text{and} \quad \tilde{D}(v') = h(|D_1(v')|) + h(|D_1^{F'}(v')|).$$

Similar to the previous we proof define the following objects

$$\begin{aligned}
\mathbf{B}_k(-v) &= ((B_{k-1}(w, \mathbf{G}_n - v), B_{k-1}(w, \mathbf{G}_n^F - v))_{w \in V_n}), \\
\mathbf{B}_k(-v') &= ((B_{k-1}(w, \mathbf{G}_n - v'), B_{k-1}(w, \mathbf{G}_n^F - v'))_{w \in V_n}), \\
\mathbf{B}_k(-) &= ((B_{k-1}(w, \mathbf{G}_n - \{v, v'\}), B_{k-1}(w, \mathbf{G}_n^F - \{v, v'\}))_{w \in V_n}), \\
\mathbf{B}'_k(-) &= ((B_{k-1}(w, \mathbf{G}_n - \{v, v'\}), B_{k-1}(w, \mathbf{G}_n^{F'} - \{v, v'\}))_{w \in V_n}).
\end{aligned}$$

With this notation $X^V(v)$ can be written as a function of $(\mathbf{B}_k(-v), S_v)$. Note that $X^V(v)$ is completely determined by the neighbourhoods $B_{k-1}(w, \mathbf{G}_n)$ of vertices w which are connected to v in $\mathbf{G}_n$ or $\mathbf{G}_n^F$ via an edge. Define an approximation $\tilde{X}^V(v)$ of $X^V(v)$ as the same function applied to $(\mathbf{B}_k(-), S_v)$. Define $\tilde{X}^V(v')$ analogously.

By construction $X^V(v) \neq \tilde{X}^V(v)$ implies that there is a vertex w connected to v via an edge such that its $k-1$-neighbourhood differs between $\mathbf{B}_k(-v)$ and $\mathbf{B}_k(-)$. These neighbourhoods can differ only if v' is present in $B_{k-1}(w, \mathbf{G}_n)$ or $B_{k-1}(w, \mathbf{G}_n^F)$. In other words, there is a path of length at most $k-1$ from w to v' in $\mathbf{G}_n$ or $\mathbf{G}_n^F$. Since only those w which are connected to v via an edge are relevant for $X^V(v)$, it follows that there is a path of length at most k from v to v'. Thus Lemma 3.1.19 implies

$$\begin{aligned}
&\mathbb{P}_n(X^V(v) \neq \tilde{X}^V(v) \mid S_v, S_{v'}) \\
&\quad \leq \mathbb{P}_n(v' \in B_k(v, \mathbf{G}_n) \mid S_v, S_{v'}) + \mathbb{P}_n(v' \in B_k(v, \mathbf{G}_n^F) \mid S_v, S_{v'}) \\
&\quad \leq \xi_k(v, v', \varnothing) + \xi_k^F(v, v', \varnothing).
\end{aligned}$$

Call the right-hand side of the last equation Ξ. Clearly Ξ is independent of $\mathbf{w}$ and $\mathbf{w}'$. Additionally Lemma 3.1.19 together with (2.8) ensures

$$\begin{aligned}
\mathbb{E}_n[\tilde{D}(v)^2\tilde{D}(v')^2\Xi] &\leq C\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\min\left\{\frac{W_v W_{v'}}{n\vartheta}(\Gamma_{2,n}+1)^k, 1\right\} \\
&\leq C\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\rho_{n,k}(\{v, v'\}). \qquad (4.33)
\end{aligned}$$

A similar bound holds for $X^V(v')$ and $\tilde{X}^V(v')$ with Ξ' instead of Ξ.

Write $X^V(v) = X^V(v) - \tilde{X}^V(v) + \tilde{X}^V(v)$ and $X^V(v) = X^V(v') - \tilde{X}^V(v') + \tilde{X}^V(v')$, then expand $\mathrm{Cov}_n(X^V(v), X^V(v') \mid S_v, S_{v'})$ to

$$\begin{aligned}
&\mathrm{Cov}_n(X^V(v), X^V(v') \mid S_v, S_{v'}) \\
&\quad = \mathrm{Cov}_n(\tilde{X}^V(v), \tilde{X}^V(v') \mid S_v, S_{v'}) \\
&\qquad + \mathrm{Cov}_n(X^V(v) - \tilde{X}^V(v), \tilde{X}^V(v') \mid S_v, S_{v'}) \\
&\qquad + \mathrm{Cov}_n(\tilde{X}^V(v), X^V(v') - \tilde{X}^V(v') \mid S_v, S_{v'}) \\
&\qquad + \mathrm{Cov}_n(X^V(v) - \tilde{X}^V(v), X^E(v') - \tilde{X}^V(v') \mid S_v, S_{v'}).
\end{aligned}$$

We claim that

$$|\mathrm{Cov}_n(X^V(v) - \tilde{X}^V(v), X^V(v') \mid S_v, S_{v'})| \leq C\tilde{D}(v)^2\tilde{D}(v')^2\tilde{H}_V^4\Xi$$

and similarly for the other covariances involving $X^V(v) - \tilde{X}^V(v)$. To this end note that by measurability of $\tilde{H}_V$ and $\tilde{D}$ with respect to $S_v, S_{v'}$ we have

$$
\begin{aligned}
&|\mathbb{E}_n[(X^V(v) - \tilde{X}^V(v))X^V(v') \mid S_v, S_{v'}]| \\
&\quad \le \mathbb{E}_n[|X^V(v) - \tilde{X}^V(v)||X^V(v')| \mid S_v, S_{v'}] \\
&\quad \le \mathbb{E}_n[(|X^V(v)| + |\tilde{X}^V(v)|)\mathbb{1}_{\{X^V(v) \ne \tilde{X}^V(v)\}}|X^V(v)| \mid S_v, S_{v'}] \\
&\quad \le C\mathbb{E}[\tilde{D}(v)^2\tilde{D}(v')^2\tilde{H}_V^4\mathbb{1}_{\{X^V(v) \ne \tilde{X}^V(v)\}} \mid S_v, S_{v'}] \\
&\quad \le C\tilde{D}(v)^2\tilde{D}(v')^2\tilde{H}_V^4\mathbb{P}_n(X^V(v) \ne \tilde{X}^V(v) \mid S_v, S_{v'}) \\
&\quad \le C\tilde{H}_V^4\tilde{D}(v)^2\tilde{D}(v')^2\Xi.
\end{aligned}
$$

The same holds for the other terms in the definition of the covariance. This shows

$$
\begin{aligned}
&|\mathrm{Cov}_n(X^V(v), X^V(v') \mid S_v, S_{v'}) - \mathrm{Cov}_n(\tilde{X}^V(v), \tilde{X}^V(v') \mid S_v, S_{v'})| \\
&\quad \le C\tilde{D}(v)^2\tilde{D}(v')^2\tilde{H}_V^4(\Xi + \Xi').
\end{aligned}
$$

We can then use (4.33) to estimate the expectation

$$
\begin{aligned}
&\mathbb{E}_n[|\mathrm{Cov}_n(X^V(v), X^V(v') \mid S_v, S_{v'}) - \mathrm{Cov}_n(\tilde{X}^V(v), \tilde{X}^V(v') \mid S_v, S_{v'})|] \\
&\quad \le CJ_V^{2/3}\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\min\left\{\frac{W_v W_{v'}}{n\vartheta}(\Gamma_{2,n} + 1)^k, 1\right\} \\
&\quad \le CJ_V^{2/3}\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\rho_{n,k}(\{v, v'\}).
\end{aligned} \tag{4.34}
$$

It remains to bound the expectation of $\mathrm{Cov}_n(\tilde{X}^V(v), \tilde{X}^V(v') \mid S_v, S_{v'})$.

By construction $\mathbf{B}_{k-1}(-)$ and $\mathbf{B}'_{k-1}(-)$ are independent of S_v and $S_{v'}$. Hiding the dependence of $\tilde{X}^V(v)$ and $\tilde{X}^V(v')$ on S_v and $S_{v'}$ in functions Ψ and Ψ' we may appeal to Lemma A.2.3 to write

$$
\mathrm{Cov}_n(\tilde{X}^V(v), \tilde{X}^V(v') \mid S_v, S_{v'}) = \mathrm{Cov}_n(\Psi(\mathbf{B}_{k-1}(-)), \Psi'(\mathbf{B}'_{k-1}(-))).
$$

Moreover, the values of Ψ and Ψ' do not depend on the collection of neighbourhoods of all vertices in $\mathbf{G}_n$ and $\mathbf{G}_n^F$. Instead the value of Ψ is completely determined by the neighbourhoods of vertices that are connected to v via an edge. Similarly, the value of Ψ' is determined by the neighbourhoods of vertices w that have an edge to v'. Since we fixed S_v and $S_{v'}$ for Ψ and Ψ', the selection of those neighbourhoods is deterministic. More precisely let

$$
D = D_1(v) \cup D_1^F(v) \quad \text{and} \quad D' = D_1(v') \cup D_1^{F'}(v').
$$

Then Ψ and Ψ' only depend on the neighbourhoods of vertices in D and D', respectively, and D and D' are $(S_v, S_{v'})$-measurable. Note that D and D' are not independent, but

$$
\bar{D} = D_1^{(v')}(v) \cup D_1^{(v'),F}(v) \cup \{v'\} \quad \text{and} \quad \bar{D}' = D_1^{(v)}(v') \cup D_1^{(v),D'}(v') \cup \{v\}
$$

are independent and satisfy $D \subseteq \bar{D}$ and $D' \subseteq \bar{D}'$. Additionally, Lemma 3.1.11 implies

$$\begin{aligned}
\mathbb{E}_n[|\bar{D}|^r] &\leq \mathbb{E}_n[(|D_1(v)| + |D_1^F(v)| + 1)^r] \\
&\leq \sum_{j=0}^{r} \binom{r}{j} \mathbb{E}_n[(|D_1(v)| + |D_1^F(v)|)^j] \\
&\leq \sum_{j=0}^{r} \binom{r}{j} \sum_{\ell=0}^{j} \binom{j}{\ell} \mathbb{E}_n[|D_1(v)|^{\ell} |D_1^F(v)|^{j-\ell}] \\
&\leq \sum_{j=0}^{r} \binom{r}{j} \sum_{\ell=0}^{j} \binom{j}{\ell} \mathbb{E}_n[|D_1(v)|^{2\ell}]^{1/2} \mathbb{E}_n[|D_1^F(v)|^{2(j-\ell)}]^{1/2} \\
&\leq \sum_{j=0}^{r} \binom{r}{j} \sum_{\ell=0}^{j} \binom{j}{\ell} (W_v + 1)^{\ell} (\Gamma_{1,n} + r)^{\ell} (W_v + 1)^{j-\ell} (\Gamma_{1,n} + r)^{j-\ell} \\
&\leq \sum_{j=0}^{r} \binom{r}{j} 2^j (W_v + 1)^j (\Gamma_{1,n} + r)^j \\
&\leq (2(W_v + 1)(\Gamma_{1,n} + r) + 1)^r \\
&\leq 4^r (W_v + 1)^k (\Gamma_{1,n} + r)^r.
\end{aligned}$$

If r is bounded above, we can absorb it into the constant to obtain

$$\mathbb{E}_n[|\bar{D}|^r] \leq C(W_v + 1)^r (\Gamma_{1,n} + 1)^r.$$

Similarly by Lemmas 3.1.4 and 3.1.8 we have

$$\begin{aligned}
\mathbb{E}_n[\|\bar{D}\|^2] &\leq \mathbb{E}_n[(\|D_1(v)\| + \|D_1^F(v)\| + W_{v'})^2] \\
&\leq \mathbb{E}_n[\|D_1(v)\|^2] + 2\mathbb{E}_n[\|D_1(v)\| \|D_1^F(v)\|] + \mathbb{E}_n[\|D_1^F(v)\|^2] \\
&\qquad + 2W_{v'} \mathbb{E}_n[\|D_1(v)\| + \|D_1^F(v)\|] + W_{v'}^2 \\
&\leq \mathbb{E}_n[\|D_1(v)\|^2] + 2(\mathbb{E}_n[\|D_1(v)\|^2])^{1/2} (\mathbb{E}_n[\|D_1^F(v)\|^2])^{1/2} \\
&\qquad + \mathbb{E}_n[\|D_1^F(v)\|^2] + 2W_{v'} \mathbb{E}_n[\|D_1(v)\| + \|D_1^F(v)\|] + W_{v'}^2 \\
&\leq C(W_v + 1)^2 (\Gamma_{3,n} + 1)(\Gamma_{2,n} + 1)^2 + CW_v(\Gamma_{2,n} + 1) + W_{v'}^2,
\end{aligned}$$

which we estimate crudely by

$$\leq C(W_v + W_{v'} + 1)^2 (\Gamma_{3,n} + 1)(\Gamma_{2,n} + 1)^2.$$

This implies

$$\begin{aligned}\mathbb{E}_n[(\|\bar{D}\| + |\bar{D}|)^2] &= \mathbb{E}_n[\|\bar{D}\|^2] + 2\mathbb{E}_n[\|\bar{D}\||\bar{D}|] + \mathbb{E}_n[|\bar{D}|^2] \\ &\le \mathbb{E}_n[\|\bar{D}\|^2] + 2\mathbb{E}_n[\|\bar{D}\|^2]^{1/2}\mathbb{E}_n[|\bar{D}|^2]^{1/2} + \mathbb{E}_n[|\bar{D}|^2] \\ &\le C(W_v + W_{v'} + 1)^2(\Gamma_{3,n} + 1)(\Gamma_{2,n} + 1)^2 \\ &\quad + C(W_v + W_{v'} + 1)(W_v + 1)(\Gamma_{1,n} + 1)(\Gamma_{3,n} + 1)(\Gamma_{2,n} + 1) \\ &\quad + C(W_v + 1)^2 w(\Gamma_{1,n} + 1)^2 \\ &\le C(W_v + W_{v'} + 1)^2(\Gamma_{1,n} + 1)^2(\Gamma_{3,n} + 1)(\Gamma_{2,n} + 1)^2\end{aligned}$$

and similarly for $\bar{D}'$. Since $\bar{D}$ and $\bar{D}'$ are independent, this immediately implies

$$\begin{aligned}&\mathbb{E}_n[(\|\bar{D}\| + |\bar{D}|)^2(\|\bar{D}'\| + |\bar{D}'|)^2] \\ &\quad \le C(W_v + W_{v'} + 1)^4(\Gamma_{1,n} + 1)^4(\Gamma_{3,n} + 1)^2(\Gamma_{2,n} + 1)^4.\end{aligned} \tag{4.35}$$

By definition $|\Psi| \le \tilde{D}(v)^2\tilde{H}_V^2$ and $|\Psi'| \le \tilde{D}(v')^2\tilde{H}_V^2$. Hence Lemma 3.2.5 shows

$$\begin{aligned}&\mathrm{Cov}_n(\Psi(\mathbf{B}_{k-1}(-)), \Psi'(\mathbf{B}'_{k-1}(-))) \\ &\le \tilde{D}(v)^2\tilde{D}(v')^2\tilde{H}_V^4 \min\left\{\frac{(\|D\| + |D|)(\|D'\| + |D'|)}{n\vartheta}(\Gamma_{3,n} + 1)(\Gamma_{2,n} + 1)^{2k+1}, 1\right\}.\end{aligned}$$

Take the expectation and use independence of $\tilde{H}_V$ from the other terms as well as Cauchy–Schwarz to separate the expectations of the terms involving $\tilde{D}$ from D and D' and (4.35) to obtain

$$\begin{aligned}&\mathbb{E}_n[\mathrm{Cov}_n(\tilde{X}^V(v), \tilde{X}^V(v') \mid S_v, S_{v'})] \\ &\quad = \mathbb{E}_n[\mathrm{Cov}_n(\Psi(\mathbf{B}_{k-1}(-)), \Psi'(\mathbf{B}'_{k-1}(-)))] \\ &\quad \le C\mathbb{E}_n[\tilde{H}_V^4]\mathbb{E}_n[\tilde{D}(v)^4\tilde{D}(v')^4]^{1/2} \\ &\qquad \min\left\{\frac{\mathbb{E}_n[(\|D\| + |D|)^2(\|D'\| + |D'|)^2]^{1/2}}{n\vartheta}(\Gamma_{3,n} + 1)(\Gamma_{2,n} + 1)^{2k+1}, 1\right\} \\ &\quad \le CJ_V^{2/3}\zeta_n(v)^{1/2}\zeta_n(v')^{1/2}\rho_{n,k}(\{v, v'\}),\end{aligned} \tag{4.36}$$

where we used (2.7) and (2.8) and the definition of $\rho_{n,k}$ in the last step.
Now (4.34) and (4.36) imply the claim. □

Proof of Proposition 4.3.7. Recall (4.23)

$$\begin{aligned}&\mathrm{Cov}_n(\Delta_v f\Delta_v f^F, \Delta_{v'} f\Delta_{v'} f^{F'}) \\ &\quad = \mathrm{Cov}_n((R_k^V(v) + L_k^V(v))(R_k^V(F \cup v) + L_k^V(F \cup v)), \\ &\qquad\qquad (R_k^V(v') + L_k^V(v'))(R_k^V(F' \cup v') + L_k^V(F' \cup v'))).\end{aligned}$$

Expand this covariance into sixteen terms, then apply Lemmas 4.3.8 and 4.3.9 to these terms as appropriate. □

4.3.3 Vertex-edge case

A fusion of the ideas for the edge-edge and vertex-vertex case can be used to treat the vertex-edge case, in which we estimate $c(e, v')$ for an edge $e = \{u, v\}$ that is not incident to the vertex v'.

Proposition 4.3.12. *Let $e = \{v, u\}$ be an edge in $V_n^{(2)}$ and $v' \in V_n$ be a vertex not incident to e. Then we can choose*

$$c(e, v') = \sigma_n^{-4} C J_E J_V \frac{W_u W_v}{n\vartheta} \zeta_n(v')^{1/2} ((m_n^E(v, u)\delta_k^E)^{1/2} + (m_n^V(v')\delta_k^V)^{1/2} + \varepsilon_{n,k}(\{u, v, v'\})^{1/4} + \rho_{n,k}(\{u, v, v'\})^{1/4}).$$

The steps are as before. We want to bound

$$\begin{aligned} \mathrm{Cov}_n(&(R_k^E(e) + L_k^E(e))(R_k^E(F \cup e) + L_k^E(F \cup e)), \\ &(R_k^V(v') + L_k^V(v'))(R_k^V(F' \cup v') + L_k^V(F' \cup v'))). \end{aligned} \tag{4.37}$$

Again we expand the covariance and need to bound sixteen terms of the form

$$\mathrm{Cov}(U_1^E(e) U_2^E(F \cup e), U_3^V(v') U_4^V(F' \cup v')).$$

For brevity let

$$\tilde{H}_E = \max\{H_E(w_e, w_e', w_v, w_u), H_E(w_e, w_e', w_v^F, w_u^F)\}$$

and

$$\tilde{H}_V = H_V(w_{v'}, w_{v'}') \quad \text{and} \quad \tilde{D}(v') = h(|D_1(v')|) + h(|D_1^{F'}(v')|).$$

Note that these random variables are independent. Note further that $\tilde{D}(v')$ is independent of A_e, since the former involves edges emanating from v' and A_e is only concerned with the edge e, which by assumption does not emanate from v'.

Again there are fifteen terms involving R_k^E or R_k^V. These can be bounded by appealing to (GLA 3) and (GLA 6), respectively. We first deal with terms involving an R_k^E.

Lemma 4.3.13.

$$\begin{aligned} &\mathrm{Cov}_n(U_1^E(e) U_2^E(F \cup e), U_3^V(v') U_4^V(F' \cup v')) \\ &\quad \le C J_E J_V \frac{W_u W_v}{n\vartheta} \zeta_n(v')^{1/2} ((m_n^E(v, u)\delta_k^E)^{1/2} \\ &\qquad + \varepsilon_{n,k}(\{u, v, v'\})^{1/4} + \rho_{n,k}(\{u, v, v'\})^{1/4}) \end{aligned}$$

if U_1^E or U_2^E is an R_k^E-term.

Proof. The proof is essentially a simplification of the calculations done in the proof of Lemma 4.3.3. Assume that the covariance is of the form

$$\operatorname{Cov}_n(R_k^E(e)U_2^E(F \cup e), U_3^V(v')U_4^V(F' \cup v'))$$

the remaining cases are analogous.

By (4.3), (4.4) and (4.22)

$$\begin{aligned}
&|R_k^E(e)U_2^E(F \cup e)U_3^V(v')U_4^V(F' \cup v')| \\
&\quad \leq C\mathbb{1}_{\tilde{A}_e}\tilde{H}_E\tilde{D}(v')^2\tilde{H}_V^2|R_k^E(e)| \\
&\quad \leq C\mathbb{1}_{A_e}\tilde{H}_E\tilde{D}(v')^2\tilde{H}_V^2|R_k^E(e)| + C\mathbb{1}_{\{X_e=X_e'=1\}}\tilde{H}_E^2\tilde{D}(v')^2\tilde{H}_V^2.
\end{aligned} \tag{4.38}$$

The expectation of the second term in (4.38) can be bounded using independence and then the Cauchy–Schwarz inequality

$$\begin{aligned}
C\mathbb{E}_n[\mathbb{1}_{\{X_e=X_e'=1\}}\tilde{H}_E^2\tilde{D}(v')^2\tilde{H}_V^2] &\leq C\mathbb{P}_n(X_e = X_e' = 1)\mathbb{E}_n[\tilde{H}_E^2]\mathbb{E}_n[\tilde{D}(v')^2\tilde{H}_V^2] \\
&\leq C\min\left\{\frac{W_vW_u}{n\vartheta}, 1\right\}^2 J_E^{1/3}J_V^{1/3}\zeta_n(v')^{1/2} \\
&\leq J_E^{1/3}J_V^{1/3}\zeta_n(v')^{1/2}\frac{W_uW_v}{n\vartheta}\rho_{n,k}(\{u,v,v'\}).
\end{aligned}$$

The remainder of the first term can be bounded with Cauchy–Schwarz as in (4.8)

$$\begin{aligned}
\mathbb{E}_n[\mathbb{1}_{A_e}\tilde{H}_E\tilde{D}(v')^2\tilde{H}_V^2|R_k^E(e)| \mid Y_e] &\leq \mathbb{1}_{A_e}\mathbb{E}_n[\tilde{H}_E^2\tilde{D}(v')^4\tilde{H}_V^4]^{1/2}\mathbb{E}_n[R_k^E(e)^2 \mid Y_e]^{1/2} \\
&\leq \mathbb{1}_{A_e}J_E^{1/6}J_V^{1/3}\zeta_n(v')^{1/2}\mathbb{E}_n[R_k^E(e)^2 \mid Y_e]^{1/2}.
\end{aligned}$$

Write $B_k = B_k(v, \mathbf{G}_n)$ and $B_k' = B_k(v, \mathbf{G}_n^e)$. Let E_0 be the event that B_k and B_k' are trees. Then use Lemmas 3.5.2 and 3.5.5 to couple $(B_k, B_k', \mathbf{T}, \mathbf{T}')$ and set $E_1 = \{(B_k, B_k') \cong (\mathbf{T}, \mathbf{T}')\}$.

Analogous to (4.9) we have

$$\begin{aligned}
&\mathbb{1}_{A_e}\mathbb{E}_n[(R_k^E(e))^2 \mid Y_e] \\
&\quad \leq \mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{E,U}(B_k, B_k') - \mathrm{LA}_k^{E,L}(B_k, B_k'))^2 \mid Y_e] \\
&\qquad + \mathbb{1}_{A_e}CJ_E^{1/3}(\mathbb{P}_n(E_0^c \mid Y_e)^{1/2} + \mathbb{P}_n(E_1^c \mid Y_e)^{1/2}).
\end{aligned} \tag{4.39}$$

Simplify the argument for (4.10) by conditioning only on Y_e and not also on $Y_{e'}$ to obtain

$$\mathbb{P}_n(E_0^c \mid Y_e) \leq C\rho_{n,k}(\{u,v\}) \leq C\rho_{n,k}(\{u,v,v'\}).$$

Similarly, the calculations for (4.11) can be based on Lemma 3.5.5 instead of Lemma 3.5.4 so that

$$\mathbb{P}_n(E_1^c) \leq C(\varepsilon_{n,k}(\{u,v,v'\}) + \rho_{n,k}(\{u,v,v'\})).$$

Lemma 3.5.5 and (GLA 3) then give that

$$\mathbb{E}_n[\mathbb{1}_{A_e}\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{E,U}(B_k,B_k') - \mathrm{LA}_k^{E,L}(B_k,B_k'))^2 \mid Y_e] \le m_n^E(v,u)\delta_k^E.$$

Hence, the expectation of the first part of (4.38) can be bounded by

$$\begin{aligned}
&\mathbb{E}_n[C\mathbb{1}_{A_e}\tilde{H}_E\tilde{H}_V^2|R_k^E(e)|] \\
&\quad\le \frac{W_uW_v}{n\vartheta}CJ_EJ_V\zeta_n(v')^{1/2}((m_n^E(v,u)\delta_k^E)^{1/2} \\
&\qquad + \varepsilon_{n,k}(\{u,v,v'\})^{1/4} + \rho_{n,k}(\{u,v,v'\})^{1/4}).
\end{aligned}$$

Putting this together we obtain

$$\begin{aligned}
&\mathbb{E}_n[|R_k^E(e)U_2^E(F\cup e)U_3^V(v')U_4^V(F'\cup v')|] \\
&\quad\le CJ_EJ_V\frac{W_uW_v}{n\vartheta}\zeta_n(v')^{1/2}((m_n^E(v,u)\delta_k^E)^{1/2} \\
&\qquad + \varepsilon_{n,k}(\{u,v,v'\})^{1/4} + \rho_{n,k}(\{u,v,v'\})^{1/4}).
\end{aligned}$$

The other two terms in the covariance can be bounded similarly, which proves the claim. □

Now we treat those terms involving an R_k^V.

Lemma 4.3.14.

$$\begin{aligned}
&\mathrm{Cov}_n(U_1^E(e)U_2^E(F\cup e), U_3^V(v')U_4^V(F'\cup v')) \\
&\quad\le CJ_VJ_E\frac{W_uW_v}{n\vartheta}\zeta_n(v')^{1/2}((m_n^V(v)\delta_k^V)^{1/2} \\
&\qquad + \varepsilon_{n,k}(\{u,v,v'\})^{1/4} + \rho_{n,k}(\{u,v,v'\})^{1/4})
\end{aligned}$$

if U_3^V or U_4^V is an R_k^V-term.

Proof. The main line of the argument follows Lemma 4.3.8. Here, however, we have to condition on Y_e, which makes the proof slightly more complicated at first glance.

Assume that the covariance is of the form

$$\mathrm{Cov}_n(U_1^E(e)U_2^E(F\cup e), R_k^V(v')U_4^V(F'\cup v'))$$

the remaining cases are analogous.

By (4.3), (4.4) and (4.22), measurability, independence and Cauchy–Schwarz we have

$$\begin{aligned}
&\mathbb{E}_n[|U_1^E(e)U_2^E(F\cup e)R_k^V(v')U_4^V(F'\cup v')| \mid Y_e] \\
&\quad\le C\mathbb{1}_{\tilde{A}_e}\mathbb{E}_n[\tilde{H}_E^2\tilde{D}(v')\tilde{H}_V|R_k^V(v')| \mid Y_e] \\
&\quad\le C\mathbb{1}_{\tilde{A}_e}\mathbb{E}_n[\tilde{D}(v')^2\tilde{H}_E^4\tilde{H}_V^2]^{1/2}\mathbb{E}_n[|R_k^V(v')|^2 \mid Y_e]^{1/2} \\
&\quad\le C\mathbb{1}_{\tilde{A}_e}J_E^{1/3}J_V^{1/6}\zeta_n(v')^{1/4}\mathbb{E}_n[|R_k^V(v')|^2 \mid Y_{e'}]^{1/2}.
\end{aligned} \tag{4.40}$$

Let $B_k = B_k(v', \mathbf{G}_n)$ and $B'_k = B_k(v', \mathbf{G}_n^v)$. Use Lemma 3.5.2 to couple $B_k(v', \mathbf{G}_n)$ and $\mathbf{T} \sim \mathbf{T}_k(W_{v'}, v, \mu_E, \mu_V)$. Exchange the weight of the root of $\mathbf{T}$ for a random variable $\tilde{w}_{v'}$ with distribution μ_V coupled to the weight of v' in $\mathbf{G}_n^{v'}$ such that $\tilde{w}_{v'} \neq w_{v'}$ with probability at most $d_{\mathrm{TV}}(\mu_{V,n}, \mu_V)$ and call the resulting weighted tree $\tilde{\mathbf{T}}$.

Let E_1 be the event that $B_k \cong \mathbf{T}$ and $B'_k \cong \tilde{\mathbf{T}}$ and let E_0 be the event that $B_k(v, \mathbf{G}_n)$ is a tree. Then

$$\begin{aligned}|R_k^V(v')| &\leq \mathrm{LA}_k^{V,U}(B_k, B'_k) - \mathrm{LA}_k^{V,L}(B_k, B'_k) \\ &\leq \mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B'_k) - \mathrm{LA}_k^{V,L}(B_k, B'_k)) + \mathbb{1}_{E_0^c \cup E_1^c} C\tilde{D}(v')\tilde{H}_V.\end{aligned}$$

Square this inequality, take conditional expectations, use independence and the Cauchy–Schwarz inequality to find

$$\begin{aligned}\mathbb{E}[(R_k^V(v'))^2 \mid Y_e] &\leq \mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B'_k) - \mathrm{LA}_k^{V,L}(B_k, B'_k))^2 \mid Y_e] \\ &\quad + C\mathbb{E}_n[\tilde{D}(v')^2\tilde{H}_V^2\mathbb{1}_{E_0^c \cup E_1^c} \mid Y_{e'}] \\ &\leq \mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B'_k) - \mathrm{LA}_k^{V,L}(B_k, B'_k))^2 \mid Y_e] \\ &\quad + C(\mathbb{E}_n[\tilde{D}(v')^4\tilde{H}_V^4])^{1/2}(\mathbb{P}_n(E_0^c \mid Y_e) + \mathbb{P}_n(E_1^c \mid Y_e))^{1/2} \\ &\leq \mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B'_k) - \mathrm{LA}_k^{V,L}(B_k, B'_k))^2 \mid Y_e] \\ &\quad + C\zeta_n(v')^{1/2}J_V^{1/3}(\mathbb{P}_n(E_0^c \mid Y_e) + \mathbb{P}_n(E_1^c \mid Y_e))^{1/2}.\end{aligned} \tag{4.41}$$

First we bound the probability of E_0^c in (4.41). Since the rerandomisation of the vertex weight does not change the underlying tree structure, B_k is a tree if and only if B'_k is a tree. Now $B_k(v', \mathbf{G}_n)$ is equal to $B_k(v', \mathbf{G}_n - e)$ with high probability. More precisely by Corollary 3.1.15

$$\begin{aligned}&\mathbb{P}_n(B_k(v', \mathbf{G}_n) \neq B_k(v', \mathbf{G}_n - e) \mid Y_e) \\ &\quad = \mathbb{P}_n(e \in B_k(v', \mathbf{G}_n) \mid Y_e) \\ &\quad \leq \mathbb{P}_n(u \in B_k(v', \mathbf{G}_n - e) \mid Y_e) + \mathbb{P}_n(v \in B_k(v', \mathbf{G}_n - e) \mid Y_e) \\ &\quad \leq \mathbb{P}_n(u \in B_k(v', \mathbf{G}_n - e)) + \mathbb{P}_n(v \in B_k(v', \mathbf{G}_n - e)) \\ &\quad \leq \mathbb{P}_n(u \in B_k(v', \mathbf{G}_n)) + \mathbb{P}_n(v \in B_k(v', \mathbf{G}_n)) \\ &\quad \leq \frac{(W_u + W_v)W_{v'}}{n\vartheta}(\Gamma_{2,n} + 1)^k.\end{aligned}$$

Hence, we can look at $B_k(v', \mathbf{G}_n - e)$ instead of $B_k(v', \mathbf{G}_n)$ to calculate the conditional probability of E_0^c given Y_e so that Lemma 3.1.20 together with the last inequality yields

$$\begin{aligned}&\mathbb{P}_n(E_0^c \mid Y_e) \\ &\quad = \mathbb{P}_n(B_k(v', \mathbf{G}_n) \text{ is not a tree} \mid Y_e) \\ &\quad \leq \mathbb{P}_n(B_k(v', \mathbf{G}_n) \neq B_k(v', \mathbf{G}_n - e) \mid Y_e) + \mathbb{P}_n(B_k(v', \mathbf{G}_n - e) \text{ is not a tree}) \\ &\quad \leq \frac{(W_u + W_v)W_{v'}}{n\vartheta}(\Gamma_{2,n} + 1)^k + C(\Gamma_{2,n} + 1)^{2k+1}(\Gamma_{3,n} + 1)\frac{(W_{v'} + 1)^2}{n\vartheta}.\end{aligned}$$

Hence,

$$\mathbb{P}_n(E_0^c \mid Y_e) \le \rho_{n,k}(\{u, v, v'\}). \tag{4.42}$$

By construction of the coupling $B_k \cong \mathbf{T}$ implies $B'_k \cong \bar{\mathbf{T}}$ unless $\tilde{w}_v \ne w_v$. Then by Lemma 3.5.3

$$\begin{aligned}\mathbb{P}_n(E_1^c \mid Y_e) &\le \mathbb{P}_n(B_k \ncong \mathbf{T} \mid Y_e) + \mathbb{P}_n(\tilde{w}_v \ne w_v) \\ &\le \varepsilon_{n,k}(\{v'\}) + C\frac{W_{v'}(W_v + W_u)}{n\vartheta}(\Gamma_{2,n} + 1)^{k+1} + d_{\mathrm{TV}}(\mu_{V,n}, \mu_V).\end{aligned}$$

Absorb $d_{\mathrm{TV}}(\mu_{V,n}, \mu_V)$ into $\varepsilon_{n,k}(\{v'\}) \le \varepsilon_{n,k}(\{u, v, v'\})$ to obtain

$$\mathbb{P}_n(E_1^c \mid Y_e) \le C(\varepsilon_{n,k}(\{u, v, v'\}) + \rho_{n,k}(\{u, v, v'\})). \tag{4.43}$$

In fact the construction of Lemma 3.5.3 allows us to assume that $\mathbf{T}$ does not depend on Y_e at all.

On E_1 the neighbourhoods B_k and B'_k can be replaced with $\mathbf{T}$ and $\bar{\mathbf{T}}$, which are independent of Y_e. Then by (GLA 6)

$$\begin{aligned}&\mathbb{E}_n[\mathbb{1}_{E_0}\mathbb{1}_{E_1}(\mathrm{LA}_k^{V,U}(B_k, B'_k) - \mathrm{LA}_k^{V,L}(B_k, B'_k))^2 \mid Y_e] \\ &\quad\le \mathbb{E}_n[(\mathrm{LA}_k^{V,U}(\mathbf{T}, \bar{\mathbf{T}}) - \mathrm{LA}_k^{V,L}(\mathbf{T}, \bar{\mathbf{T}}))^2] \\ &\quad\le m_n^V(v')\delta_k^V.\end{aligned} \tag{4.44}$$

Putting (4.40) to (4.44) together we have

$$\begin{aligned}&\mathbb{E}_n[|U_1^E(e)U_2^E(F \cup e)R_k^V(v')U_4^V(F' \cup v')|] \\ &\quad\le CJ_EJ_V\frac{W_uW_v}{n\vartheta}\zeta_n(v')^{1/2}((m_n^V(v')\delta_k^V)^{1/2} \\ &\qquad\quad + \varepsilon_{n,k}(\{u, v, v'\})^{1/4} + \rho_{n,k}(\{u, v, v'\})^{1/4}).\end{aligned}$$

The other two terms in the covariance can be bounded similarly, which proves the claim. □

It remains to bound the covariance involving only L_k terms.

Lemma 4.3.15. *We have*

$$\begin{aligned}&\mathrm{Cov}_n(L_k^E(e)L_k^E(F \cup e), L_k^V(v')L_k^V(F' \cup v')) \\ &\quad\le CJ_V^{1/3}J_E^{1/3}\frac{W_uW_v}{n\vartheta}\zeta_n(v')^{1/2}\rho_{n,k}(\{u, v, v'\}).\end{aligned}$$

Proof. Use the notation from the proofs of Lemmas 4.3.4 and 4.3.9 and use the law of total variance to obtain.

$$
\begin{aligned}
&\mathrm{Cov}_n(L_k^E(e)L_k^E(F\cup e), L_k^V(v')L_k^V(F'\cup v'))\\
&\quad = \mathrm{Cov}_n(X^E(e), X^V(v'))\\
&\quad = \mathrm{Cov}_n(\mathbb{E}_n[X^E(e) \mid S_e, S_{v'}], \mathbb{E}_n[X^V(v') \mid S_e, S_{v'}])\\
&\qquad + \mathbb{E}_n[\mathrm{Cov}_n(X^E(e), X^V(v') \mid S_e, S_{v'})]\\
&\quad = \mathrm{Cov}_n(\mathbb{1}_{A_e}\mathbb{E}_n[X^E(e) \mid S_e, S_{v'}], \mathbb{E}_n[X^V(v') \mid S_e, S_{v'}])\\
&\qquad + \mathbb{E}_n[\mathbb{1}_{A_e}\,\mathrm{Cov}_n(X^E(e), X^V(v') \mid S_e, S_{v'})].
\end{aligned}
$$

The claim will follow from Lemmas 4.3.16 and 4.3.17. □

Lemma 4.3.16. *We have*

$$
\begin{aligned}
&\mathrm{Cov}_n(\mathbb{1}_{A_e}\mathbb{E}_n[X^E(e) \mid S_e, S_{v'}], \mathbb{E}_n[X^V(v') \mid S_e, S_{v'}])\\
&\quad \le C J_V^{1/3} J_E^{1/3} \frac{W_u W_v}{n\vartheta} \zeta_n(v')^{1/2} \rho_{n,k}(\{u,v,v'\}).
\end{aligned}
$$

Proof. Write $X^E(e)$ as a function of

$$(B_k(v,\mathbf{G}_n), B_k(v,\mathbf{G}_n^e), B_k(v,\mathbf{G}_n^F), B_k(v,\mathbf{G}_n^{F\cup e}))$$

and define an approximation $\tilde{X}^E(e)$ as the same function applied to

$$(B_k(v,\mathbf{G}_n - v'), B_k(v,\mathbf{G}_n^e - v'), B_k(v,\mathbf{G}_n^F - v'), B_k(v,\mathbf{G}_n^{F\cup e} - v')).$$

Similarly, $X^V(v')$ can be written as a function of

$$(B_k(v',\mathbf{G}_n), B_k(v',\mathbf{G}_n^{v'}), B_k(v',\mathbf{G}_n^{F'}), B_k(v',\mathbf{G}_n^{F'\cup v'}))$$

and can be approximated by $\tilde{X}^V(v')$ that is defined as the same function, but applied to

$$
\begin{aligned}
&(B_k(v',\mathbf{G}_n - \{u,v\}), B_k(v',\mathbf{G}_n^{v'} - \{u,v\}),\\
&\quad B_k(v',\mathbf{G}_n^{F'} - \{u,v\}), B_k(v',\mathbf{G}_n^{F'\cup v'} - \{u,v\})).
\end{aligned}
$$

By construction $\tilde{X}^E(e)$ is independent of $S_{v'}$ and $\tilde{X}^V(v')$ is independent of S_e. Set

$$Z^E(e) = \mathbb{E}_n[X^E(e) \mid S_e, S_{v'}] \text{ and } \tilde{Z}^E(e) = \mathbb{E}_n[\tilde{X}^E(e) \mid S_e, S_{v'}] = \mathbb{E}_n[\tilde{X}^E(e) \mid S_e]$$

and analogously

$$Z^V(v') = \mathbb{E}_n[X^V(v') | S_e, S_{v'}] \text{ and } \tilde{Z}^V(v') = \mathbb{E}_n[\tilde{X}^V(v') | S_e, S_{v'}] = \mathbb{E}_n[\tilde{X}^V(v') | S_{v'}].$$

The bounds from (4.3), (4.4) and (4.22) imply

$$|X^E(e)|, |\tilde{X}^E(e)| \le \mathbb{1}_{A_e}\tilde{H}_E^2$$

and

$$|X^V(v')|, |\tilde{X}^V(v')| \le \tilde{D}(v')^2 \tilde{H}_V^2.$$

The conditional versions then satisfy

$$|Z^E(e)| = |\mathbb{E}_n[X^E(e) \mid S_e, S_{v'}]| \le \mathbb{E}_n[\mathbb{1}_{A_e} \tilde{H}_E^2 \mid S_e, S_{v'}] \le \mathbb{1}_{A_e} \tilde{H}_E^2$$

by measurability of $\tilde{H}_E$ and A_e and in exactly the same way

$$|\tilde{Z}^E(e)| \le \mathbb{1}_{A_e} \tilde{H}_E^2$$

as well as

$$|Z^V(v')| = |\mathbb{E}_n[X^V(v') \mid S_e, S_{v'}]| \le \mathbb{E}_n[\tilde{D}(v')^2 \tilde{H}_V^2 \mid S_e, S_{v'}] \le \tilde{D}(v')^2 \tilde{H}_V^2 \tag{4.45}$$

by measurability of $\tilde{H}_V$ and $\tilde{D}(v')$ and

$$|\tilde{Z}^V(v')| \le \tilde{D}(v')^2 \tilde{H}_V^2.$$

Split the relevant covariance

$$\begin{aligned}
&\mathrm{Cov}_n(\mathbb{1}_{A_e} Z^E(e), Z^V(v')) \\
&\quad = \mathrm{Cov}_n(\mathbb{1}_{A_e} \tilde{Z}^E(e), \tilde{Z}^V(v')) \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)), \tilde{Z}^V(v')) \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{A_e} \tilde{Z}^E(e), Z^V(v') - \tilde{Z}^V(v')) \\
&\qquad + \mathrm{Cov}_n(\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)), (Z^V(v') - \tilde{Z}^V(v'))).
\end{aligned}$$

Since $\mathbb{1}_{A_e} \tilde{Z}^E(e)$ is a function of S_e that is completely independent of $S_{v'}$ and $\tilde{Z}^V(v')$ is a function of $S_{v'}$ that is completely independent of S_e, the two approximations are independent and thus the first term vanishes.

Now bound the three remaining covariances. We will only show the argument for

$$\mathrm{Cov}_n(\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)), \tilde{Z}^V(v')),$$

the argument for the other two covariances is similar.

By (4.45)

$$|\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)) Z^V(v')| \le \mathbb{1}_{A_e} \tilde{D}(v')^2 \tilde{H}_V^2 |Z^E(e) - \tilde{Z}^E(e)|.$$

By construction

$$\begin{aligned}
|Z^E(e) - \tilde{Z}^E(e)| &\le \mathbb{E}_n[|X^E(e) - \tilde{X}^E(e)| \mid S_e, S_{v'}] \\
&\le 2\tilde{H}_E^2 \mathbb{P}_n(X^E(e) \ne \tilde{X}^E(e) \mid S_e, S_{v'}).
\end{aligned}$$

Similar to previous proofs $X^E(e) \neq \tilde{X}^E(e)$ implies that $B_k(v, \mathbf{G} - v') \neq B_k(v, \mathbf{G})$ for at least one $\mathbf{G}$ of $\mathbf{G} = \mathbf{G}_n, \mathbf{G}_n^e, \mathbf{G}_n^F, \mathbf{G}_n^{F \cup e}$. Then apply Lemma 3.1.19

$$\begin{aligned}
&\mathbb{P}_n(B_k(v, \mathbf{G}_n - v') \neq B_k(v, \mathbf{G}_n) \mid S_e, S_{v'}) \\
&\quad \leq \mathbb{P}_n(v' \in B_k(v, \mathbf{G}_n) \mid S_e, S_{v'}) \\
&\quad \leq \xi_k(\{v'\}, \{u, v\}, \varnothing).
\end{aligned}$$

Sum these probabilities for all four graphs to Ξ with

$$\mathbb{P}_n(X^E(e) \neq \tilde{X}^E(e) \mid S_e, S_{v'}) \leq \Xi$$

and

$$\mathbb{E}_n[\tilde{D}(v')^2 \Xi] \leq C \zeta_n(v')^{1/2} \min\left\{ \frac{W_{v'}(W_u + W_v)}{n\vartheta} (\Gamma_{2,n} + 1)^k, 1 \right\}.$$

Take the expectation and use independence to find

$$\begin{aligned}
\mathbb{E}_n[|\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)) Z^V(v')|] &\leq C \mathbb{E}_n[\mathbb{1}_{A_e} \tilde{D}(v')^2 \tilde{H}_V^2 \tilde{H}_E^2 \Xi] \\
&\leq C \frac{W_u W_v}{n\vartheta} J_V^{1/3} J_E^{1/3} \zeta_n(v')^{1/2} \rho_{n,k}(\{u, v, v'\}).
\end{aligned}$$

The other terms in the covariance can be bounded similarly. Hence,

$$\mathrm{Cov}_n(\mathbb{1}_{A_e}(Z^E(e) - \tilde{Z}^E(e)), \tilde{Z}^V(v')) \leq C \frac{W_u W_v}{n\vartheta} J_V^{1/3} J_E^{1/3} \zeta_n(v')^{1/2} \rho_{n,k}(\{u, v, v'\}).$$

The same bound holds for the other covariances. This finishes the proof. □

Lemma 4.3.17.

$$\mathbb{E}_n[\mathbb{1}_{A_e} \mathrm{Cov}_n(X^E(e), X^V(v') \mid S_e, S_{v'})] \leq C J_E^{1/3} J_V^{1/3} \frac{W_u W_v}{n\vartheta} \zeta_n(v')^{1/2} \rho_{n,k}(\{u, v, v'\}).$$

Proof. Define

$$\begin{aligned}
\mathbf{B}_k(-\{u, v\}) &= (B_{k-1}(w, \mathbf{G}_n - \{u, v\}), B_{k-1}(w, \mathbf{G}_n^F - \{u, v\}))_{w \in V_n}, \\
\mathbf{B}_k(-v') &= (B_{k-1}(w, \mathbf{G}_n - \{v'\}), B_{k-1}(w, \mathbf{G}_n^{F'} - \{v'\}))_{w \in V_n}, \\
\mathbf{B}_k(-) &= (B_{k-1}(w, \mathbf{G}_n - \{u, v, v'\}), B_{k-1}(w, \mathbf{G}_n^F - \{u, v, v'\}))_{w \in V_n}, \\
\mathbf{B}'_k(-) &= (B_{k-1}(w, \mathbf{G}_n - \{u, v, v'\}), B_{k-1}(w, \mathbf{G}_n^{F'} - \{u, v, v'\}))_{w \in V_n}.
\end{aligned}$$

With those definitions $X^E(e)$ can be written as a function of $(\mathbf{B}_k(-\{u, v\}), S_e)$, we define an approximation $\tilde{X}^E(e)$ as the same function but applied to $(\mathbf{B}_k(-), S_e)$. As in the proof of Lemma 4.3.6 $\tilde{X}^E(e)$ actually only depends on the neighbourhoods of vertices w that are connected to v or u via an edge. Analogously, $X^V(v')$ is a function of $(\mathbf{B}'_k(-v'), S_{v'})$, so define its approximation $\tilde{X}^V(v')$ as that function applied to $(\mathbf{B}'_k(-), S_{v'})$. As in the proof of Lemma 4.3.11 $X^V(v')$ depends only on the neighbourhoods of vertices w that are connected to v' via an edge.

On the event $X^E(e) \neq \tilde{X}^E(e)$ there is at least one relevant vertex w connected to u or v such that $B_{k-1}(w, \mathbf{G}_n - \{u,v\})$ differs from $B_{k-1}(w, \mathbf{G}_n - \{u,v,v'\})$ or $B_{k-1}(w, \mathbf{G}_n^F - \{u,v\})$ differs from $B_{k-1}(w, \mathbf{G}_n^F - \{u,v,v'\})$. That is to say there is a path of length at most $k-1$ from w to v' that avoids u and v. Since w is directly connected to u or v by an edge, it follows that there is a path from u or v to v' of no more than k steps. Thus by Lemma 3.1.19

$$\begin{aligned} \mathbb{P}_n(\tilde{X}^E(e) \neq X^E(e) \mid S_e, S_{v'}) &\leq \mathbb{P}_n(\{u,v\} \leftrightsquigarrow_{\leq k} v') + \mathbb{P}_n(\{u,v\} \leftrightsquigarrow^F_{\leq k} v') \\ &\leq \xi_k(\{u,v\},\{v'\},\varnothing) + \xi_k^F(\{u,v\},\{v'\},\varnothing) \end{aligned}$$

Call the right-hand side of the last equation Ξ and note that

$$\mathbb{E}_n[\tilde{D}(v')^2 \Xi] \leq C\zeta_n(v')^{1/2} \frac{(W_u + W_v)W_{v'}}{n\vartheta}(\Gamma_{2,n}+1)^k.$$

Similarly $X^V(v')$ differs from $\tilde{X}^V(v')$ only if there is a vertex w connected to v' by an edge such that $B_{k-1}(w, \mathbf{G}_n - \{v'\})$ and $B_{k-1}(w, \mathbf{G}_n - \{u,v,v'\})$ or $B_{k-1}(w, \mathbf{G}_n^{F'} - \{v'\})$ and $B_{k-1}(w, \mathbf{G}_n^{F'} - \{u,v,v'\})$ differ. This implies that there is a path in $\mathbf{G}_n$ or $\mathbf{G}_n^{F'}$ of length at most $k-1$ from w to u or v that avoids v'. Hence, there is a path of length at most k in $\mathbf{G}_n$ or $\mathbf{G}_n^{F'}$ from v' to u or v. Thus by Lemma 3.1.19

$$\begin{aligned} \mathbb{P}_n(\tilde{X}^V(v') \neq X^V(v') \mid S_e, S_{v'}) &\leq \mathbb{P}_n(v' \leftrightsquigarrow_{\leq k} \{u,v\}) + \mathbb{P}_n(v' \leftrightsquigarrow^F_{\leq k} \{u,v\}) \\ &\leq \xi_k(\{v'\},\{u,v\},\varnothing) + \xi_k^F(\{v'\},\{u,v\},\varnothing) \end{aligned}$$

Call the right-hand side of the last equation Ξ' and note that

$$\mathbb{E}_n[\tilde{D}(v')^2 \Xi'] \leq C\zeta_n(v')^{1/2} \min\Big\{\frac{(W_u + W_v)W_{v'}}{n\vartheta}(\Gamma_{2,n}+1)^k, 1\Big\}.$$

Expand the covariance of interest

$$\begin{aligned} &\mathrm{Cov}_n(X^E(e), X^V(v') \mid S_e, S_{v'}) \\ &\quad = \mathrm{Cov}_n(\tilde{X}^E(e), \tilde{X}^V(v') \mid S_e, S_{v'}) \\ &\qquad + \mathrm{Cov}_n(X^E(e) - \tilde{X}^E(e), \tilde{X}^V(v') \mid S_e, S_{v'}) \\ &\qquad + \mathrm{Cov}_n(\tilde{X}^E(e), X^V(v') - \tilde{X}^V(v') \mid S_e, S_{v'}) \\ &\qquad + \mathrm{Cov}_n(X^E(e) - \tilde{X}^E(e), X^V(v') - \tilde{X}^V(v') \mid S_e, S_{v'}). \end{aligned}$$

As previously, bound the covariances involving an approximation error by estimating the approximation error by the bounds on X^E and X^V times the probability that the approximation is different from the original function. We obtain

$$|\mathrm{Cov}_n(X^E(e) - \tilde{X}^E(e), X^V(v') \mid S_e, S_{v'})| \leq C\tilde{H}_E^2 \tilde{D}(v')^2 \tilde{H}_V^2 \Xi$$

and similar results for the other terms. Thus

$$\begin{aligned} &|\mathrm{Cov}_n(X^E(e), X^V(v') \mid S_e, S_{v'}) - \mathrm{Cov}_n(\tilde{X}^E(e), \tilde{X}^V(v') \mid S_e, S_{v'})| \\ &\quad \leq C\tilde{H}_E^2 \tilde{D}(v')^2 \tilde{H}_V^2 (\Xi + \Xi') \end{aligned}$$

and therefore also

$$\begin{aligned}&\mathbb{E}_n[\mathbb{1}_{A_e}|\mathrm{Cov}_n(X^E(e),X^V(v')\mid S_e,S_{v'})-\mathrm{Cov}_n(\tilde{X}^E(e),\tilde{X}^V(v')\mid S_e,S_{v'})|]\\&\quad\le CJ_E^{1/3}J_V^{1/3}\zeta_n(v')^{1/2}\rho_{n,k}(\{u,v,v'\}).\end{aligned}\tag{4.46}$$

It remains to bound $\mathrm{Cov}_n(\tilde{X}^E(e),\tilde{X}^V(v')\mid S_e,S_{v'})$. Recall that the construction ensured that $\mathbf{B}_k(-)$ and $\mathbf{B}'_k(-)$ are independent of S_e and $S_{v'}$. Hide the dependence of $\tilde{X}^E(e)$ on S_e and the dependence of $\tilde{X}^V(v')$ on $S_{v'}$ in functions Ψ and Ψ', respectively, so that by Lemma A.2.3

$$\mathrm{Cov}_n(\tilde{X}^E(e),\tilde{X}^V(v')\mid S_e,S_{v'})=\mathrm{Cov}_n(\Psi(\mathbf{B}_k(-)),\Psi'(\mathbf{B}_k(-))).$$

By definition $|\Psi|\le\tilde{H}_E^2$ and $|\Psi'|\le\tilde{D}(v')^2\tilde{H}_V^2$.

As in Lemmas 4.3.6 and 4.3.11 Ψ and Ψ' depend not on all w-neighbourhoods for $w\in V_n$, but only on the neighbourhoods of ws for which there is an edge to u or v and v', respectively. This data is known conditioned on S_e and $S_{v'}$. More formally, the relevant vertices for Ψ are contained in

$$D=D_1(v)\cup D_1(u)\cup D_1^F(v)\cup D_1^F(u),$$

and the vertices relevant for Ψ' are contained in

$$D'=D_1(v')\cup D_1^{F'}(v').$$

While D is not independent of A_e and D',

$$\bar{D}=S_1^{(u,v')}(v)\cup S_1^{(v,v')}(u)\cup S_1^{(u,v'),F}(v)\cup S_1^{(v,v'),F}(u)\cup\{v'\}$$

is independent of A_e and of

$$\bar{D}'=S_1^{(u,v)}(v')\cup S_1^{(u,v),F'}(v')\cup\{u,v\}.$$

Then as before

$$\begin{aligned}&\mathbb{E}_n[(\|\bar{D}\|+|\bar{D}|)^2(\|\bar{D}'\|+|\bar{D}'|)^2]\\&\quad\le C(W_u+W_v+W_{v'}+2)^4(\Gamma_{1,n}+1)^4(\Gamma_{2,n}+1)^4(\Gamma_{3,n}+1)^2\end{aligned}$$

and by Lemma 3.2.5

$$\begin{aligned}&\mathrm{Cov}_n(\Psi(\mathbf{B}_k(-)),\Psi'(\mathbf{B}_k(-)))\\&\quad\le C\tilde{H}_E^2\tilde{D}(v')^2\tilde{H}_V\min\left\{\frac{(\|\bar{D}\|+|\bar{D}|)(\|\bar{D}'\|+|\bar{D}'|)}{n\vartheta}(\Gamma_{3,n}+1)(\Gamma_{2,n}+1)^{2k+1},1\right\}.\end{aligned}$$

Take expectations as in Lemma 4.3.11 to obtain

$$\mathbb{E}_n[\mathbb{1}_{A_e}\mathrm{Cov}_n(\Psi(\mathbf{B}_k(-)),\Psi'(\mathbf{B}_k(-)))]\le CJ_V^{1/3}\frac{W_uW_v}{n\vartheta}\zeta_n(v')^{1/2}\rho_{n,k}(\{u,v,v'\}).$$

Put these covariance estimates together to conclude the claim. □

Proof of Proposition 4.3.12. Recall (4.37)

$$\begin{aligned}&\mathrm{Cov}_n((R_k^E(e)+L_k^E(e))(R_k^E(F\cup e)+L_k^E(F\cup e)),\\ &\qquad (R_k^V(v')+L_k^V(v'))(R_k^V(F'\cup v')+L_k^V(F'\cup v'))).\end{aligned}$$

Expand this covariance into sixteen terms, then apply Lemmas 4.3.13 to 4.3.15 to these terms as appropriate. □

4.4 Summing the bounds

We are almost ready to prove our main result.

Lemma 4.4.1. *Recall $\eta_{n,\ell}(\mathcal{V})$ and $\varepsilon_{n,\ell}(\mathcal{V})$ from Lemma 3.5.1 and Lemma 3.5.2, respectively. Then*

$$\sum_{\substack{u_1,\ldots,u_m\in V_n\\ \text{pairw. diff.}}} \varepsilon_{n,\ell}(\{u_1,\ldots,u_m\}) \le mn^m\varepsilon_{n,\ell},$$

where $\varepsilon_{n,\ell}$ is as defined in Definition 2.3.3.

For any $r>1$ we also have

$$\frac{1}{n^m}\sum_{\substack{u_1,\ldots,u_m\in V_n\\ \text{pairw. diff.}}} \varepsilon_{n,\ell}(\{u_1,\ldots,u_m\})^{1/r} \le m^{1/r}\varepsilon_{n,\ell}^{1/r},$$

Proof. We have

$$\begin{aligned}\varepsilon_{n,\ell}(\mathcal{V}) &= \|\mathcal{V}\|_2\frac{\Gamma_{2,n}}{n\vartheta}+\|\mathcal{V}\|_+\Gamma_{1,n}\\ &\quad+\|\mathcal{V}\|(\Gamma_{2,n}+1)^\ell\left(\frac{\Gamma_{3,n}}{n\vartheta}+\kappa_{1,n}+\kappa_{2,n}+\frac{2+\Gamma_{1,n}}{k_n}+\frac{k_n}{n\vartheta}\right)\\ &\quad+|\mathcal{V}|\frac{1}{k_n}+\frac{k_n^2}{n\vartheta\Gamma_{1,n}}+\|\mathcal{V}\|\alpha_n\left(\frac{1}{\vartheta}+(\Gamma_2+1)^{\ell-1}\left(\frac{\Gamma_{2,n}}{\vartheta\Gamma_{1,n}}+1\right)\right)\\ &\quad+(|\mathcal{V}|+\|\mathcal{V}\|(\Gamma_2+1)^\ell)(d_{\mathrm{TV}}(\mu_{E,n},\mu_E)+d_{\mathrm{TV}}(\mu_{V,n},\mu_V)).\end{aligned}$$

In order to understand the sum over $\varepsilon_{n,\ell}(\mathcal{V})$ it is therefore enough to understand the sum over $|\mathcal{V}|$, $\|\mathcal{V}\|$, $\|\mathcal{V}\|_+$ and $\|\mathcal{V}\|_2$. Recall that $|\mathcal{V}|=\|\mathcal{V}\|_0$.

For $p \geq 0$ we have

$$\begin{aligned}
\sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \|\{u_1,\ldots,u_m\}\|_p &= \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \sum_{i=1}^{m} W_{u_i}^p \\
&\leq \sum_{u_1,\ldots,u_m \in V_n} \sum_{i=1}^{m} W_{u_i}^p \\
&\leq m \sum_{u_1,\ldots,u_m \in V_n} W_{u_1}^p \\
&= mn^m \vartheta \left(\frac{1}{n\vartheta} \sum_{u_1 \in V_n} W_{u_1}^p \right) \left(\frac{1}{n} \sum_{u \in V_n} 1 \right)^{m-1} \\
&= mn^m \vartheta \Gamma_{p,n}.
\end{aligned}$$

In exactly the same way we also obtain

$$\begin{aligned}
\sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \|\{u_1,\ldots,u_m\}\|_+ &= \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \sum_{i=1}^{m} W_{u_i} \mathbb{1}_{\{W_i > \sqrt{n\vartheta}\}} \\
&= mn^m \vartheta \left(\frac{1}{n\vartheta} \sum_{u_1 \in V_n} W_{u_1} \mathbb{1}_{\{W_1 > \sqrt{n\vartheta}\}} \right) \left(\frac{1}{n} \sum_{u \in V_n} 1 \right)^{m-1} \\
&= mn^m \vartheta \kappa_{1,n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \varepsilon_{n,\ell}(\{u_1,\ldots,u_m\}) \\
&= mn^m \frac{\Gamma_{2,n}^2}{n} + mn^m \vartheta \kappa_{1,n} \Gamma_{1,n} \\
&\quad + mn^m \Gamma_{1,n} \vartheta (\Gamma_{2,n} + 1)^{\ell} \left(\frac{\Gamma_{3,n}}{n\vartheta} + \kappa_{1,n} + \kappa_{2,n} + \frac{2 + \Gamma_{1,n}}{k_n} + \frac{k_n}{n\vartheta} \right) \\
&\quad + mn^m \frac{1}{k_n} + n^m \frac{k_n^2}{n\vartheta \Gamma_{1,n}} \\
&\quad + mn^m \Gamma_{1,n} \vartheta \alpha_n \left(\frac{1}{\vartheta} + (\Gamma_2 + 1)^{\ell-1} \left(\frac{\Gamma_{2,n}}{\vartheta \Gamma_{1,n}} + 1 \right) \right) \\
&\quad + mn^m (1 + \Gamma_{1,n} \vartheta (\Gamma_2 + 1)^{\ell}) (d_{\mathrm{TV}}(\mu_{E,n}, \mu_E) + d_{\mathrm{TV}}(\mu_{V,n}, \mu_V)) \\
&\leq mn^m \varepsilon_{n,\ell}.
\end{aligned}$$

This finishes the proof of the first part of the claim.

For the second part of the claim let $s > 1$ such that $1/r + 1/s = 1$ and apply Hölder's inequality so that the first part of the claim implies

$$\begin{aligned}
&\frac{1}{n^m} \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \varepsilon_{n,\ell}(\{u_1,\ldots,u_m\})^r \\
&\quad \le \Bigg(\frac{1}{n^m} \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} (\varepsilon_{n,\ell}(\{u_1,\ldots,u_m\})^{1/r})^r \Bigg)^{1/r} \Bigg(\frac{1}{n^m} \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} 1 \Bigg)^{1/s} \\
&\le \Bigg(\frac{1}{n^m} \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \varepsilon_{n,\ell}(\{u_1,\ldots,u_m\}) \Bigg)^{1/r} \\
&\le m^{1/r} \varepsilon_{n,k}^{1/r}.
\end{aligned}$$

This shows the second claim. □

Lemma 4.4.2. *Recall $\rho_{n,k}(\mathcal{V})$ from Definition 4.3.1. Then*

$$\sum_{\substack{u_1,\ldots,u_m \in V_n \\ \textit{pairw. diff.}}} \rho_{n,k}(\{u_1,\ldots,u_m\}) \le m n^m \rho_{n,k},$$

where $\rho_{n,\ell}$ is as defined in Definition 2.3.3.

Furthermore, for any $r > 1$ we have

$$\frac{1}{n^m} \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \textit{pairw. diff.}}} \rho_{n,k}(\{u_1,\ldots,u_m\})^{1/r} \le m^{1/r} \rho_{n,k}^{1/r}.$$

Proof. As in the proof of Lemma 4.4.1 we just need to understand the relevant sum over $(\|\mathcal{V}\| + |\mathcal{V}|)^2 = \|\mathcal{V}\|^2 + 2|\mathcal{V}|\|\mathcal{V}\| + |\mathcal{V}|^2$.

We use the Cauchy–Schwarz inequality to estimate the square of the sum by the number of summands times the sum of squares and then proceed as in the proof of Lemma 4.4.1 to obtain

$$\begin{aligned}
\sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \|\{u_1,\ldots,u_m\}\|^2 &\le \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} \Bigg(\sum_{i=1}^m W_{u_i} \Bigg)^2 \\
&\le \sum_{\substack{u_1,\ldots,u_m \in V_n \\ \text{pairw. diff.}}} m \sum_{i=1}^m W_{u_i}^2 \\
&\le m^2 n^m \vartheta \Gamma_{2,n}.
\end{aligned}$$

The sum of the other terms is known from the proof of Lemma 4.4.1.

Ignore the minimum with 1 for now to find

$$\begin{aligned}
&\sum_{\substack{u_1,\dots,u_m\in V_n\\ \text{pairw. diff.}}} \rho_{n,k}(\{u_1,\dots,u_m\})\\
&\le \sum_{\substack{u_1,\dots,u_m\in V_n\\ \text{pairw. diff.}}} \frac{\|\{u_1,\dots,u_m\}\|^2+2m\|\{u_1,\dots,u_m\}\|+m^2}{n\vartheta}\\
&\qquad(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+C)^{2k+1}(\Gamma_{3,n}+1)^2\\
&\le m^2n^m\frac{\vartheta\Gamma_{2,n}+\vartheta\Gamma_{1,n}+1}{n\vartheta}(\Gamma_{1,n}+1)^2(\Gamma_{2,n}+C)^{2k+1}(\Gamma_{3,n}+1)^2.
\end{aligned}$$

But clearly it also holds that

$$\sum_{\substack{u_1,\dots,u_m\in V_n\\ \text{pairw. diff.}}} \rho_{n,k}(\{u_1,\dots,u_m\}) \le n^m \le mn^m.$$

The first claim now follows immediately from the last two inequalities.
The second claim follows as in the proof of Lemma 4.4.1. □

Finally, we can proceed to prove the main result.

Proof of Theorem 2.3.5. In the previous sections we identified a function c defined on both vertices and edges with

$$\sigma_n^{-4}\operatorname{Cov}(\Delta_x f\Delta_x f^F, \Delta_{x'} f\Delta_{x'} f^{F'}) \le c(x,x')$$

for all $x, x' \in V_n \cup V_n^{(2)}$ and $F \subseteq (V_n \cup V_n^{(2)}) \setminus \{x\}$, $F' \subseteq (V_n \cup V_n^{(2)}) \setminus \{x'\}$. Apply Lemma 4.1.6 to obtain that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}|\mathbb{P}_n(Z_n\le t)-\Phi(t)|\\
&\le \sqrt{2}\Big(\sum_{x,x'\in V_n\cup V_n^{(2)}} c(x,x')\Big)^{1/4} + \Big(\sigma_n^{-3}\sum_{x\in V_n\cup E_n}\mathbb{E}[|\Delta_x f|^3]\Big)^{1/2}.
\end{aligned}$$

All that is left to do is to calculate these sums. We start with the first sum over $c(x,x)$ and split the sum over all the different cases we handled.
Lemma 4.2.3 and the definition of $\Gamma_{p,n}$ imply

$$\begin{aligned}
\sum_{e\in E_n} c(e,e) &\le \sigma_n^{-4}CJ_E^{2/3}\sum_{e\in E_n}\frac{W_uW_v}{n\vartheta}\\
&\le \sigma_n^{-4}CJ_E^{2/3}n\vartheta\Big(\frac{1}{n\vartheta}\sum_{v\in V_n}W_v\Big)^2\\
&\le \frac{n}{\sigma_n^4}CJ_E^{2/3}\vartheta\Gamma_{1,n}^2. \qquad (4.47)
\end{aligned}$$

With Lemma 4.2.4 and (2.8) we can show

$$\begin{aligned}\sum_{v\in V_n} c(v,v) &\le \sigma_n^{-4} C J_V^{2/3} \sum_{v\in V_n} \zeta_n(v)\\ &\le \frac{n}{\sigma_n^4} C J_V^{2/3} \chi_n. \end{aligned}\tag{4.48}$$

From Lemma 4.2.5 we get

$$\begin{aligned}&\sum_{v,u\in V_n} c(\{v,u\},v)\\ &\le \sigma_n^{-4} C J_E^{1/3} J_V^{1/3} \sum_{v,u\in V_n} \frac{W_u W_v}{n\vartheta} \zeta_n(v)^{1/2}\\ &\le \frac{n}{\sigma_n^4} C J_E^{1/3} J_V^{1/3} \Big(\frac{1}{n\vartheta}\sum_{u\in V_n} W_u\Big)\Big(\frac{1}{n}\sum_{v\in V_n} W_v \zeta_n(v)^{1/2}\Big)\\ &\le \frac{n}{\sigma_n^4} C J_E^{1/3} J_V^{1/3} \Big(\frac{1}{n\vartheta}\sum_{u\in V_n} W_u\Big)\Big(\frac{1}{n}\sum_{v\in V_n} W_v^2\Big)^{1/2}\Big(\frac{1}{n}\sum_{v\in V_n} \zeta_n(v)\Big)^{1/2}\\ &\le \frac{n}{\sigma_n^4} C J_E^{1/3} J_V^{1/3} \Gamma_{1,n} \vartheta^{1/2} \Gamma_{2,n}^{1/2} \chi_n^{1/2}. \end{aligned}\tag{4.49}$$

By Lemma 4.2.6

$$\begin{aligned}\sum_{u,v,v'\in V_n} c(\{u,v\},\{u,v'\}) &\le \sigma_n^{-4} C J_E^{2/3} \sum_{u,v,v'\in V_n} W_u^2 \frac{W_v}{n\vartheta}\frac{W_{v'}}{n\vartheta}\\ &\le \frac{n}{\sigma_n^4} C J_E^{2/3} \vartheta \Big(\frac{1}{n\vartheta}\sum_u W_u^2\Big)\Big(\frac{1}{n\vartheta}\sum_v W_v\Big)\Big(\frac{1}{n\vartheta}\sum_{v'} W_{v'}\Big)\\ &\le \frac{n}{\sigma_n^4} C J_E^{2/3} \vartheta \Gamma_{2,n} \Gamma_{1,n}^2. \end{aligned}\tag{4.50}$$

The bounds (4.47) to (4.50) can in turn be bounded above by

$$\frac{n}{\sigma_n^4} C (J_E + J_V + J_E J_V)(\vartheta^{1/2} + \chi_n^{1/2})^2 (\Gamma_{1,n}+1)^2(\Gamma_{2,n}+1).$$

We note that the last two terms can be bounded by $n\rho_{n,k}$ if n is large enough, so that with $J = J_E + J_V + J_E J_V$ this term is bounded by

$$\begin{aligned}&\le \frac{n^2}{\sigma_n^4} C J (\vartheta^{1/2} + \chi_n^{1/2})^2 \rho_{n,k}\\ &\le \frac{n^2}{\sigma_n^4} C J (\vartheta^{1/2} + \Gamma_{2,n} + \chi_n^{1/2})^2 \rho_{n,k}. \end{aligned}\tag{4.51}$$

For the more complex bound from Proposition 4.3.2 we apply Cauchy-Schwarz to separate the δ, ρ and ε terms in the sum. Then apply Lemmas 4.4.1 and 4.4.2 and the definition of m_n^E and M_n^E to obtain

$$
\begin{aligned}
&\sum_{\substack{e,e'\in E_n\\ e\cap e'=\emptyset}} c(e,e')\\
&\le \sigma_n^{-4} C J_E \sum_{\substack{u,v,u',v'\in E_n\\ \text{pairwise different}}} \frac{W_u W_v}{n\vartheta}\frac{W_{u'}W_{v'}}{n\vartheta}((\delta_k^E(v,u)+\delta_k^E(v',u'))^{1/2}\\
&\quad + \varepsilon_{n,k}(\{u,v,u',v'\})^{1/4} + \rho_{n,k}(\{u,v,u',v'\})^{1/4})\\
&\le \frac{n^2}{\sigma_n^4} C J_E \frac{1}{n^4\vartheta^2}\sum_{u,v,u',v'} W_u W_v W_{u'} W_{v'}((m_n^E(v,u)\delta_k^E)^{1/2} + (m_n^E(v',u')\delta_k^E)^{1/2}\\
&\quad + \varepsilon_{n,k}(\{u,v,u',v'\})^{1/4} + \rho_{n,k}(\{u,v,u',v'\})^{1/4})\\
&\le \frac{n^2}{\sigma_n^4} C J_E \Big(\frac{1}{n^4\vartheta^4}\sum_{u,v,u',v'} W_u^2 W_v^2 W_{u'}^2 W_{v'}^2\Big)^{1/2}\Big(\frac{1}{n^4}\sum_{u,v,u',v'} m_n^E(v,u)\delta_k^E\\
&\quad + m_n^E(v',u')\delta_k^E + \varepsilon_{n,k}(\{u,v,u',v'\})^{1/2} + \rho_{n,k}(\{u,v,u',v'\})^{1/2}\Big)^{1/2}\\
&\le \frac{n^2}{\sigma_n^4} C J_E \Gamma_{2,n}^2\Big(\delta_k^E \frac{2}{n^2}\sum_{u,v} m_n^E(v,u) + \frac{1}{n^4}\sum_{u,v,u',v'} \varepsilon_{n,k}(\{u,v,u',v'\})^{1/2}\\
&\quad + \frac{1}{n^4}\sum_{u,v,u',v'} \rho_{n,k}(\{u,v,u',v'\})^{1/2}\Big)^{1/2}\\
&\le \frac{n^2}{\sigma_n^4} C J_E \Gamma_{2,n}^2((M_n^E\delta_k^E)^{1/2} + \varepsilon_{n,k}^{1/4} + \rho_{k,n}^{1/4}).
\end{aligned} \tag{4.52}
$$

Similarly by Proposition 4.3.7 we have

$$
\begin{aligned}
&\sum_{\substack{v,v'\in V_n\\ v\neq v'}} c(v,v')\\
&\le \sigma_n^{-4} C J_V \sum_{\substack{v,v'\in V_n\\ v\neq v'}} \zeta_n(v)^{1/2}\zeta_n(v')^{1/2}((m_n^V(v)\delta_k^V)^{1/2} + (m_n^V(v')\delta_k^V)^{1/2}+\\
&\quad \varepsilon_{n,k}(\{v,v'\})^{1/4} + \rho_{n,k}(\{v,v'\})^{1/4})\\
&\le \frac{n^2}{\sigma_n^4} C J_V \Big(\frac{1}{n^2}\sum_{v,v'\in V_n}\zeta_n(v)\zeta_n(v')\Big)^{1/2}\Big(\frac{1}{n^2}\sum_{v,v'\in V_n}(m_n^V(v)\delta_k^V)^{1/2}\\
&\quad + (m_n^V(v')\delta_k^V)^{1/2} + \varepsilon_{n,k}(\{v,v'\})^{1/2} + \rho_{n,k}(\{v,v'\})^{1/2}\Big)^{1/2}\\
&\le \frac{n^2}{\sigma_n^4} C J_V \chi_n((M_n^V\delta_k^V)^{1/2} + \varepsilon_{n,k}^{1/4} + \rho_{k,n}^{1/4}).
\end{aligned} \tag{4.53}
$$

Finally Proposition 4.3.12 yields

$$
\sum_{\substack{u,v,v'\in V_n\\ \text{pairw. diff.}}} c(\{u,v\},v')
$$

$$\leq \sigma_n^{-4} C J_E J_V \sum_{u,v,v'} \frac{W_u W_v}{n\vartheta} \zeta_n(v')^{1/2} ((m_n^E(v,u)\delta_k^E)^{1/2} + (m_n^V(v)\delta_k^V)^{1/2}$$
$$+ \varepsilon_{n,k}(\{u,v,v'\})^{1/4} + \rho_{n,k}(\{u,v,v'\})^{1/4})$$
$$\leq \frac{n^2}{\sigma_n^4} C J_E J_V \Big(\frac{1}{n^3\vartheta^2} \sum_{u,v,v' \in V_n} W_u^2 W_v^2 \zeta_n(v')\Big)^{1/2} \Big(\frac{1}{n^3} \sum_{u,v,v' \in V_n} (m_n^E(v,u)\delta_k^E)^{1/2}$$
$$+ (m_n^V(v')\delta_k^V)^{1/2} + \varepsilon_{n,k}(\{u,v,v'\})^{1/2} + \rho_{n,k}(\{u,v,v'\})^{1/2}\Big)^{1/2}$$
$$\leq \frac{n^2}{\sigma_n^4} C J_E J_V \Gamma_{2,n} \chi_n^{1/2} ((M_n^E\delta_k^E)^{1/2} + (M_n^V\delta_k^V)^{1/2} + \varepsilon_{n,k}^{1/4} + \rho_{k,n}^{1/4}). \tag{4.54}$$

The bounds (4.52) to (4.54) can all be estimated by

$$\frac{n^2}{\sigma_n^4} C(J_E + J_V + J_E J_V)(\Gamma_{2,n} + \chi_n^{1/2})^2 ((M_n^E\delta_k^E)^{1/2} + (M_n^V\delta_k^V)^{1/2} + \varepsilon_{n,k}^{1/4} + \rho_{k,n}^{1/4}) \tag{4.55}$$
$$\leq \frac{n^2}{\sigma_n^4} C J(\vartheta^{1/2} + \Gamma_{2,n} + \chi_n^{1/2})^2 ((M_n^E\delta_k^E)^{1/2} + (M_n^V\delta_k^V)^{1/2} + \varepsilon_{n,k}^{1/4} + \rho_{k,n}^{1/4}). \tag{4.56}$$

Putting all these terms together and using the observations from (4.51) and (4.56) we obtain

$$\Big(\sum_{x,x'} c(x,x')\Big)^{1/4}$$
$$\leq C J^{1/4} \Big(\frac{n}{\sigma_n^2}\Big)^{1/2} (\vartheta^{1/2} + \Gamma_{2,n} + \chi_n^{1/2})^2 ((M_n^E\delta_k^E)^{1/8} + (M_n^V\delta_k^V)^{1/8} + \varepsilon_{n,k}^{1/16} + \rho_{n,k}^{1/16}).$$

Furthermore, by Lemmas 4.2.1 and 4.2.2 and from $\chi_n = n^{-1}\sum_{v \in V_n} \zeta_n(v)$

$$\Big(\sigma_n^{-3}\sum_x \mathbb{E}[|\Delta_x f|^3]\Big)^{1/2} = \Big(\sigma_n^{-3}\sum_e \mathbb{E}[|\Delta_x f|^3] + \sigma_n^{-3}\sum_v \mathbb{E}[|\Delta_x f|^3]\Big)^{1/2}$$
$$\leq (\sigma_n^{-3} J_E^{1/2} 2n\vartheta\Gamma_{1,n}^2 + \sigma_n^{-3} J_V^{1/2} n\chi_n)^{1/2}$$
$$\leq (J_E + J_V)\Big(\frac{n}{\sigma_n^2}\Big)^{3/4} \frac{\vartheta\Gamma_{1,n} + \chi_n^{1/2}}{n^{1/4}}.$$

Together these two terms give the required bound. □

With Theorem 2.3.5 shown, we can prove Corollary 2.3.6.

Proof of Corollary 2.3.6. Let $e = \{u,v\}$. When $B_k(v,\mathbf{G}_n)$ and $B_k(v,\mathbf{G}_n^e)$ are trees, define

$$\mathrm{LA}_k^{E,L}(B_k(v,\mathbf{G}_n), B_k(v,\mathbf{G}_n^e)) = g_k^L(B_k(v,\mathbf{G}_n)) - g_k^U(B_k(v,\mathbf{G}_n^e))$$

and

$$\mathrm{LA}_k^{E,U}(B_k(v,\mathbf{G}_n), B_k(v,\mathbf{G}_n^e)) = g_k^U(B_k(v,\mathbf{G}_n)) - g_k^L(B_k(v,\mathbf{G}_n^e)).$$

When $B_k(v,\mathbf{G}_n)$ and $B_k(v,\mathbf{G}_n^v)$ are trees, define

$$\mathrm{LA}_k^{V,L}(B_k(v,\mathbf{G}_n),B_k(v,\mathbf{G}_n^v)) = g_k^L(B_k(v,\mathbf{G}_n)) - g_k^U(B_k(v,\mathbf{G}_n^v))$$

and

$$\mathrm{LA}_k^{V,U}(B_k(v,\mathbf{G}_n),B_k(v,\mathbf{G}_n^v)) = g_k^U(B_k(v,\mathbf{G}_n)) - g_k^L(B_k(v,\mathbf{G}_n^v)).$$

We now verify that property GLA holds for this choice of functions $\mathrm{LA}_k^{E,L}$, $\mathrm{LA}_k^{E,U}$, $\mathrm{LA}_k^{V,L}$, $\mathrm{LA}_k^{V,U}$ and then apply Theorem 2.3.5. We will only verify (GLA 1), (GLA 2) and (GLA 3) for the edge perturbation. The proof for the vertex perturbation (GLA 4), (GLA 5) and (GLA 6) is analogous.

(GLA 1) follows from (GLA′ 1). We use that $\mathbf{G}_n^e - v = \mathbf{G}_n - v$, since the vertex v and all edges incident to v are not present in those graphs, so that rerandomisation at e, which is incident to v do not have any effect. Hence,

$$\begin{aligned}
&\mathrm{LA}_k^{E,L}(B_k(v,\mathbf{G}_n),B_k(v,\mathbf{G}_n^e))\\
&\quad= g_k^L(B_k(v,\mathbf{G}_n)) - g_k^U(B_k(v,\mathbf{G}_n^e))\\
&\quad\le (f(\mathbf{G}_n) - f(\mathbf{G}_n - v)) - (f(\mathbf{G}_n^e) - f(\mathbf{G}_n^e - v))\\
&\quad= f(\mathbf{G}_n) - f(\mathbf{G}_n^e)\\
&\quad= \Delta_e f
\end{aligned}$$

and similarly

$$\mathrm{LA}_k^{E,U}(B_k(v,\mathbf{G}_n),B_k(v,\mathbf{G}_n^e)) \ge \Delta_e f.$$

(GLA 2) follows directly from (GLA′ 2).

For (GLA 3) observe that

$$\begin{aligned}
\left|\mathrm{LA}^{E,U}(\mathbf{T},\mathbf{T}') - \mathrm{LA}^{E,L}(\mathbf{T},\mathbf{T}')\right| &\le \left|(g_k^U(\mathbf{T}) - g_k^L(\mathbf{T}')) - (g_k^L(\mathbf{T}) - g_k^U(\mathbf{T}))\right|\\
&\le |g_k^U(\mathbf{T}) - g_k^L(\mathbf{T})| + |g_k^U(\mathbf{T}') - g_k^L(\mathbf{T}')|.
\end{aligned}$$

Thus $(x+y)^2 \le 2x^2 + 2y^2$ and (GLA′ 3) yield

$$\begin{aligned}
&\mathbb{E}_n[|\mathrm{LA}^{E,U}(\mathbf{T},\mathbf{T}') - \mathrm{LA}^{E,L}(\mathbf{T},\mathbf{T}')|^2]\\
&\quad\le 2\mathbb{E}_n[|g_k^U(\mathbf{T}) - g_k^L(\mathbf{T})|^2] + 2\mathbb{E}_n[|g_k^U(\mathbf{T}') - g_k^L(\mathbf{T}')|^2]\\
&\quad\le 2m_n(v)\delta_k + 2\tilde{m}_n(v,u)\tilde{\delta}_k.
\end{aligned}$$

Set $\delta_k^E = \delta_k + \tilde{\delta}_k$ and $m_n^E(v,u) = 2(m_n(v) + \tilde{m}(v,u))$. Then the previous term can be bounded by $m_n^E(v,u)\delta_k^E$. Now

$$M_n^E = \frac{1}{n^2}\sum_{v,u\in V_n} m_n^E(v,u) = \frac{2}{n^2}\sum_{v,u\in V_n}(m_n(v) + \tilde{m}(v,u))$$

is bounded in probability by (GLA′ 3). Also by (GLA′ 3) we have $\delta_k^E \to 0$ as $k\to\infty$, which finally verifies (GLA 3).

Now the claim follows with an application of Theorem 2.3.5. □

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A Auxiliary Results

A.1 Couplings

In this section we will briefly list a few standard results that allow us to couple Binomial and Poisson random variables. For the simple results we will just present the well-known proofs ourselves.

Lemma A.1.1. *Fix $p, p' \in (0,1)$ with $p \leq p'$. Then we can couple $X \sim \mathrm{Bin}(1,p)$ and $X' \sim \mathrm{Bin}(1,p')$ such that $X \leq X'$ a.s. and*

$$\mathbb{P}(X' \neq X) = p' - p.$$

Proof. We construct X' from X and an independent random variable.

Let $\xi \sim \mathrm{Bin}(1,(p'-p)/(1-p))$ be independent of $X \sim \mathrm{Bin}(1,p)$. Since $p \leq p' \in (0,1)$, we have that $(p'-p)/(1-p) \in [0,1)$, so ξ is a well-defined random variable.

Now set

$$X' = \max\{X, \xi\}.$$

Clearly, this definition ensures that $X \leq X'$ a.s.

By construction of X' and independence of X and ξ we have

$$\mathbb{P}(X' = 0) = \mathbb{P}(X = 0, \xi = 0) = (1-p)\Big(1 - \frac{p'-p}{1-p}\Big) = 1 - p'.$$

Since X' can only take the values 0 and 1, this shows $X' \sim \mathrm{Bin}(1,p')$ as desired.

Finally, we have that

$$\begin{aligned}\mathbb{P}(X' \neq X) &= \mathbb{P}(X = 0, X' = 1)\\ &= \mathbb{P}(X = 0, \xi = 1)\\ &= (1-p)\frac{p'-p}{1-p}\\ &= p' - p\end{aligned}$$

as claimed. □

The following lemma is a well-known building block of the coupling between Binomial and Poisson random variables [Tho00, § 1.5.1].

Lemma A.1.2. *Fix $p \in (0,1)$. Then we can couple $X \sim \mathrm{Bin}(1,p)$ and $Z \sim \mathrm{Poi}(p)$ such that*

$$\mathbb{P}(Z \neq X) \leq p^2.$$

Lemma A.1.3. *Fix $0 < \lambda < \mu < \infty$. Then we can couple $X \sim \mathrm{Poi}(\lambda)$ and $Y \sim \mathrm{Poi}(\mu)$ such that $X \leq Y$ a.s. and*

$$\mathbb{P}(X \neq Y) \leq \mu - \lambda.$$

Proof. Let $\Delta \sim \mathrm{Poi}(\mu - \lambda)$ be independent of $X \sim \mathrm{Poi}(\lambda)$. Then Δ is well-defined, since $\mu - \lambda > 0$. Set

$$Y = X + \Delta.$$

Since the sum of independent Poisson random variable is again a Poisson random variable whose parameter is given by the sum of parameters, it follows that $Y \sim \mathrm{Poi}(\mu)$. Clearly, the definition ensures $X \leq Y$ a.s.

Additionally,

$$\mathbb{P}(X \neq Y) = \mathbb{P}(\Delta \geq 1) = 1 - \exp(-(\mu - \lambda)) \leq \mu - \lambda,$$

where the last inequality follows from $1 - \exp(-x) \leq x$ for all $x > -1$. □

Formally, all these couplings are defined on new probability spaces that need not have any connection to the probability spaces that supported the (or one of the) involved random variables. If we want to have several of these couplings at once, it is a priori not clear that there should be a probability space on which all the relevant random variables can be defined. A more careful analysis of the construction of these couplings shows that it is possible to construct the coupled random variables on straightforward extensions of the original probability space of one of the involved random variables. This allows us to define chains of couplings on the same probability space. We will not pursue this here any further; for the purposes of the constructions to follow it is enough to employ a *gluing lemma* for couplings.

Lemma A.1.4 [Aco82, Cor. A.2]**.** *Let V_1, V_2, V_3 be Polish spaces. Let (X_1, X_2) be a random vector with values in $V_1 \times V_2$ and (Y_2, Y_3) a random vector in $V_2 \times V_3$ such that X_2 and Y_2 have the same distribution. Then there exists a random vector (Z_1, Z_2, Z_3) with values in $V_1 \times V_2 \times V_3$ such that (Z_1, Z_2) has the same law as (X_1, X_2) and (Z_2, Z_3) has the same law as (Y_2, Y_3).*

This result for two couplings can easily be extended to n couplings.

Lemma A.1.5. *Fix $n \in \mathbb{N}$, $n \geq 3$, and let $V_1, \dots, V_n$ be Polish spaces. For $i \in \{1, \dots, n-1\}$ let $(X_i^{(i)}, X_{i+1}^{(i)})$ be a random vector with values in $V_i \times V_{i+1}$ such that $X_{i+1}^{(i)}$ and $X_{i+1}^{(i+1)}$ have the same distribution. Then there exists a random vector $(Z_1, \dots, Z_n)$ with values in $V_1 \times \cdots \times V_n$ such that (Z_i, Z_{i+1}) has the same distribution as $(X_i^{(i)}, X_{i+1}^{(i)})$ for all $i \in \{1, \dots, n-1\}$*

Proof. The proof proceeds by induction.

The base case $n = 3$ is Lemma A.1.4.

Assume that the claim holds for some n. We now want to show that the claim also holds for $n+1$. Hence, we seek to define a random vector $(Z_1, \dots, Z_{n+1})$ with values in $V_1 \times \dots \times V_{n+1}$ such that (Z_i, Z_{i+1}) and $(X_i^{(i)}, X_{i+1}^{(i)})$ have the same distribution for all $i \in \{1, \dots, n\}$ under the assumption that $X_i^{(i)}$ and $X_{i+1}^{(i)}$ have the same distribution for all $i \in \{1, \dots, n\}$.

Because the claim holds for n, there is a random vector $(Z'_1, \dots, Z'_n)$ with values in $V_1 \times \dots \times V_n$ such that (Z'_i, Z'_{i+1}) has the same distribution as $(X_i^{(i)}, X_{i+1}^{(i)})$ for all $i \in \{1, \dots, n-1\}$. In particular Z'_n has the same distribution as $X_n^{(n-1)}$, which by assumption coincides with the distribution of $X_n^{(n)}$. Set $\tilde{Z} = (Z'_1, \dots, Z'_{n-1})$ so that $(Z'_1, \dots, Z'_n) = (\tilde{Z}', Z'_n)$. Now apply Lemma A.1.4 to $(\tilde{Z}', Z'_n)$ and $(X_n^{(n)}, X_{n+1}^{(n)})$ to obtain a random vector $(\tilde{Z}, Z_n, Z_{n+1})$ such that $(\tilde{Z}, Z_n)$ has the same distribution as $(\tilde{Z}', Z'_n)$ and (Z_n, Z_{n+1}) has the same distribution as $(X_n^{(n)}, X_{n+1}^{(n)})$. Now write $(Z_1, \dots, Z_{n-1})$ for $\tilde{Z}$ and note that $(\tilde{Z}, Z_n) = (Z_1, \dots, Z_n)$ has the same distribution as $(\tilde{Z}', Z'_n) = (Z'_1, \dots, Z'_n)$, so that the distributions of (Z_i, Z_{i+1}) and $(X_i^{(i)}, X_{i+1}^{(i)})$ coincide for all $i \in \{1, \dots, n-1\}$. Since we already have that (Z_n, Z_{n+1}) has the same distribution as $(X_n^{(n)}, X_{n+1}^{(n)})$ we have shown that $(Z_1, \dots, Z_{n+1})$ satisfies that (Z_i, Z_{i+1}) has the same distribution as $(X_i^{(i)}, X_{i+1}^{(i)})$ for all $i \in \{1, \dots, n\}$.

This concludes the proof. □

With the help of this lemma we can now construct chains of couplings that retain the properties of each involved coupling pair.

A.2 Measurability

The aim of this section is to establish that the conditional covariance in Lemmas 4.3.6, 4.3.11 and 4.3.17 can be rewritten as a measurable function

Lemma A.2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra and X and Y be random variables with values in $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that X is independent of $\mathcal{G}$ and Y is $\mathcal{G}$-measurable.*

Then

$$\mathbb{E}[\phi(X, Y) \mid \mathcal{G}] = g_\phi(Y),$$

where

$$g_\phi(y) = \mathbb{E}[\phi(X, y)]$$

whenever $\mathbb{E}[|\phi(X, Y)|] < \infty$.

Proof. Let

$$\mathcal{D} = \{A \in \mathcal{X} \otimes \mathcal{Y} : \mathbb{E}[\mathbb{1}_A(X, Y) \mid \mathcal{G}] = g_A(Y),\ g_A(y) = \mathbb{E}[\mathbb{1}_A(X, y)]\}.$$

First we show that $\mathcal{D}$ is a Dynkin system.

(i) $\Omega \in \mathcal{D}$ since $\mathbb{1}_\Omega = 1$;

(ii) if $A \in \mathcal{D}$, then also $A^c \in \mathcal{D}$, since $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$;

(iii) let $A_i \in \mathcal{D}$, then $\mathbb{1}_{\bigcup_i A_i} = \sum_i \mathbb{1}_{A_i}$ and by monotone convergence

$$\mathbb{E}[\mathbb{1}_{\bigcup_i A_i}(X,Y) \mid \mathcal{G}] = \sum_i \mathbb{E}[\mathbb{1}_{A_i}(X,Y) \mid \mathcal{G}] = \sum_i g_{A_i}(Y)$$

where again by monotone convergence

$$\sum_i g_{A_i}(y) = \sum_i \mathbb{E}[\mathbb{1}_{A_i}(X,y)] = \mathbb{E}\Big[\sum_i \mathbb{1}_{A_i}(X,y)\Big] = \mathbb{E}[\mathbb{1}_{\cup_i A_i}(X,y)].$$

Now show that $\{A \times B : A \in \mathcal{X}, B \in \mathcal{Y}\} \subseteq \mathcal{D}$. To this end note that since Y is $\mathcal{G}$-measurable and X is independent of $\mathcal{G}$

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{A\times B}(X,Y) \mid \mathcal{G}] &= \mathbb{E}[\mathbb{1}_A(X)\mathbb{1}_B(Y) \mid \mathcal{G}] \\ &= \mathbb{1}_B(Y)\mathbb{E}[\mathbb{1}_A(X) \mid \mathcal{G}] \\ &= \mathbb{1}_B(Y)\mathbb{E}[\mathbb{1}_A(X)] \\ &= g_{A\times B}(Y)\end{aligned}$$

where

$$g_{A\times B}(y) = \mathbb{1}_B(y)\mathbb{E}[\mathbb{1}_A(X)] = \mathbb{E}[\mathbb{1}_A(X)\mathbb{1}_B(y)] = \mathbb{E}[\mathbb{1}_{A\times B}(X,y)].$$

as desired.

Now apply Dynkin's π-λ-theorem to conclude that

$$\mathcal{X} \otimes \mathcal{Y} = \sigma(A \times B) \subseteq \mathcal{D}.$$

Hence, the claim holds for all indicator functions.

Use linearity and monotone convergence to show that the claim also holds for elementary and nonnegative measurable function. For general integrable functions split into positive and negative part. □

Lemma A.2.2. *Let X_1, X_2, Y_1 and Y_2 be independent random variables. Then*

$$\mathbb{E}[f(X_1,X_2)g(Y_1,Y_2) \mid X_1,Y_1] = \mathbb{E}[f(X_1,X_2) \mid X_1]\mathbb{E}[g(Y_1,Y_2) \mid Y_1].$$

Proof. By Lemma A.2.1 and the fact that X_2, Y_2 is independent of (X_1, Y_1)

$$\mathbb{E}[f(X_1,X_2)g(Y_1,Y_2) \mid X_1,Y_1] = h(X_1,Y_1),$$

where

$$h(x,y) = \mathbb{E}[f(x,X_2)g(y,Y_2)]$$

since X_2 and Y_2 are independent

$$\begin{aligned} &= \mathbb{E}[f(x, X_2)]\mathbb{E}[g(y, Y_2)] \\ &= h_f(x)h_g(y). \end{aligned}$$

Lemma A.2.1 implies

$$\mathbb{E}[f(X_1, X_2) \mid X_1] = h_f(X_1) \quad \text{and} \quad \mathbb{E}[g(Y_1, Y_2) \mid Y_1] = h_g(Y_1)$$

and so

$$\begin{aligned} \mathbb{E}[f(X_1, X_2)g(Y_1, Y_2) \mid X_1, Y_1] &= h(X_1, Y_1) \\ &= h_f(X_1)h_g(Y_1) \\ &= \mathbb{E}[f(X_1, X_2) \mid X_1]\mathbb{E}[g(Y_1, Y_2) \mid Y_1] \end{aligned}$$

as claimed. □

Lemma A.2.3. *Let X_1 and X_2 be two random variables that are (jointly) independent of Y. Then*

$$\mathrm{Cov}(f_1(X_1, Y), f_2(X_2, Y) \mid Y) = h(Y)$$

where

$$h(y) = \mathrm{Cov}(f_1(X_1, y), f_2(X_2, y)).$$

Proof. By definition of the conditional covariance

$$\begin{aligned} &\mathrm{Cov}(f_1(X_1, Y), f_2(X_2, Y) \mid Y) \\ &\quad = \mathbb{E}[f_1(X_1, Y)f_2(X_2, Y) \mid Y] - \mathbb{E}[f_1(X_1, Y) \mid Y]\mathbb{E}[f_2(X_2, Y) \mid Y] \end{aligned}$$

by Lemma A.2.1

$$= h_{12}(Y) - h_1(Y)h_2(Y),$$

where

$$h_{12}(y) = \mathbb{E}[f_1(X_1, y)f_2(X_2, y)]$$

and

$$h_1(y) = \mathbb{E}[f_1(X_1, y)] \quad \text{and} \quad h_2(y) = \mathbb{E}[f_2(X_2, y)],$$

so that

$$\begin{aligned} &h_{12}(y) - h_1(y)h_2(y) \\ &\quad = \mathbb{E}[f_1(X_1, y)f_2(X_2, y)] - \mathbb{E}[f_1(X_1, y)]\mathbb{E}[f_2(X_2, y)] \\ &\quad = \mathrm{Cov}(f_1(X_1, y), f_2(X_2, y)). \end{aligned}$$

Set $h(y) = h_{12}(y) - h_1(y)h_2(y)$ and the claim follows. □