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WEAKLY REGULAR HYPERBOLIC BOUNDARY VALUE PROBLEMS OF REAL TYPE

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To Mercedes and Margarita

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Abstract

In this thesis, we derive energy estimates for weakly regular hyperbolic boundary value problems of real type, for which the Lopatinskiĭ condition degenerates in a specific way in the so-called hyperbolic region. Such boundary problems, commonly known in the literature as $\mathcal{WR}$, are easily seen to be stable under small perturbations of the coefficients and the initial data. Moreover, their applications include many relevant physical situations like the formation of shock waves in isentropic gas dynamics and the subsonic phase transitions in a van der Waals fluid. In these and other $\mathcal{WR}$ problems, the failure of the uniform Lopatinskiĭ condition plays a major role since it is associated to a loss of regularity in the scale of Sobolev spaces, eventually leading to energy estimates that might be ill-suited for dealing with nonlinear problems when solved by iteration. To circumvent this problem, one option is to apply a Nash-Moser-Hörmander iterative scheme consisting of a two-step procedure that includes a smoothing operator to compensate for the loss of regularity at each iteration. Another alternative is to modify the underlying function spaces so that the a priori estimates do not experience a loss of regularity. In the course of this dissertation, we adopt the latter technique as a starting point and derive linear estimates for the model case that are comparable to existing results, but using a more robust approach that we later generalise to some extent to variable coefficients. The result represents a significant progress towards the ultimate goal of having a one-step technique applicable to nonlinear problems of this kind.

Zusammenfassung

In dieser Arbeit leiten wir Energieabschätzungen für schwach reguläre hyperbolische Randwertprobleme vom reellen Typ her, für die die Lopatinskiĭ -Bedingung auf eine bestimmte Art und Weise in der sogenannten hyperbolischen Region entartet. Diese Randwertprobleme, in der Literatur allgemein als WR bekannt, sind stabil unter kleinen Störungen der Koeffizienten und der Anfangsdaten. Darüber hinaus finden sie in vielen relevanten physikalischen Situationen Anwendung, wie beispielsweise bei der Formierung von Stoßwellen in der isentropen Gasdynamik und bei Unterschall-Phasenübergängen in Van-der-Waals-Gasen. Bei diesen und anderen WR-Problemen spielt das Verletztsein der gleichmäßigen Lopatinskiĭ -Bedingung eine wichtige Rolle, da dies mit einem Regularitätsverlust in der Skale der Sobolevräume einhergeht. Dies führt schließlich zu Energieabschätzungen, die für Behandlung von nichtlinearen Problemen wenig geeignet sind, wenn die Lösung dieser Probleme durch Iteration erfolgt. Eine Möglichkeit, die genannte Schwierigkeit zu umgehen, ist die Verwendung eines Nash-Moser-Hörmander-Iterationsschemas. Dieses besteht aus einem zweistufigen Verfahren, wobei ein Glättungsoperator eingeführt wird, um den Regularitätsverlust in jedem Iterationsschritt zu kompensieren. Eine andere Möglichkeit besteht darin, die zugrundeliegenden Funktionenräume so zu modifizieren, dass die A-priori-Abschätzungen ohne einen Regularitätsverlust auskommen. In dieser Dissertation nehmen wir letztgenannten Zugang als Ausgangspunkt und leiten entsprechende lineare Abschätzungen für einen Modellfall her. Die Abschätzungen sind mit bestehenden Resultaten vergleichbar, jedoch robuster, sodass wir diese später in gewissem Umfang auf den Fall variabler Koeffizienten übertragen können. Unser Ergebnis stellt einen bedeutenden Fortschritt auf dem Weg zu einem einstufigen Iterationsverfahren dar, das schießlich auf nichtlineare Probleme der beschriebenen Art anwendbar sein wird.

Contents

Symbols

Symbol	Description	Page
D_j	$D_j := -i\partial_j$	1
$\mathbb{R}^d_+$	The set of elements $(y, x) \in \mathbb{R}^{d-1} \times \mathbb{R}$ with $x_d > 0$	11
$\mathcal{M}_{n\times m}$	Set of matrices with dimensions $n \times m$ and entries from $\mathbb{K} \in \{\mathbb{R}, \mathbb{R}\}$ $\mathcal{M}_{n\times m}(\mathbb{K})$	12
$L(t, x, D_t, D_x)$	Matrix form first-order differential operator/Scalar differential operator of order m	12
π_L	Characteristic polynomial of L	12
B	A mapping encoding the boundary conditions	13
$\cdot\,\|_{x_d=0}$	Trace operator. Also represented by evaluating at $x_d = 0$	13
ζ	Frequency covariable $\zeta = (\tau - i\gamma, \eta)$	14
λ^s	Japanese bracket $\lambda^s = (1 + \tau^2 + \eta^2)^{s/2}$ of order s	14
λ^s_γ	Parameter-dependent Japanese bracket $\lambda^s_\gamma = (\gamma^2 + \tau^2 + \eta^2)^{s/2}$ of order s	14
H^s	Standard Sobolev space	14
$\underline{H}^s_\gamma$	Set of elements u such that $\\|u\\|_{\underline{H}^s_\gamma} = \\|\lambda^s_\gamma \hat{u}\\|_{L^2} < \infty$	14
S^m_γ	Class of parameter-dependent tangential pseudo-differential of order m	15
$\mathrm{Op}_\gamma(a)$	Family of parameter-dependent pseudodifferential operators with symbol a	15
Λ^m_γ	Pseudodifferential operator with symbol γ^m_γ	15
L^2_γ	Weighted-in-time L^2 space	16
H^s_γ	Weighted-in-time Sobolev space	17
$\\|u\\|^2_{s,\gamma}$	$\\|u\\|^2_{s,\gamma} = \int_\Omega e^{-2\gamma t} \|\Lambda^s_\gamma u\|^2 \, d\mu$.	17
$\mathcal{A}_\gamma$	Conjugated pseudodifferential operator $\mathcal{A}_\gamma(t, x, D_t, D_y) = e^{\gamma t} \mathcal{A}(t, x, D_t, D_y, \gamma) e^{-\gamma t}$	17
$\mathcal{D}$	Set of vector-value test functions	17
$\mathcal{F}$	Fourier transform with respect to the tangential variables (t, y). Also represented by $\hat{\cdot}$	17

Symbol	Description	Page
OPS^m_γ	Class of $\gamma-$dependent pseudodifferential operators	18
$\mathcal{A}_\gamma$	Classical pseudodifferential operator of order 1	20
Ξ	The set of elements $(\eta,\tau,\gamma) \in \mathbb{R}^{d+1} \setminus \{0\}$ such that $\gamma \geq 0$	20
Ξ_0	$\Xi \cap \{\gamma = 0\}$	20
$\mathbb{X}$	The set of elements $X = (t, y, x_d, \eta, \tau, \gamma)$ such that $(t, y, x_d) \in \mathbb{R} \times \mathbb{R}^d_+$ and $(\eta, \tau, \gamma) \in \Xi$	20
$\mathbb{X}_0$	The set of elements $X \in \mathbb{X}_S$ such that $X = (t, y, x_d, \eta, \tau, 0)$	20
S^d	The set of elements $(\tau, \eta, \gamma) \in \Xi$ such that $\gamma^2 + \tau^2 + \|\eta\|^2 = 1$	20
$\mathbb{X}_S$	The set of elements $X \in \mathbb{X}$ such that $(\eta, \tau, \gamma) \in S^d$	21
$\mathbb{Y}$	The set of elements $X \in \mathbb{X}$ such that $X = (t, y, 0, \eta, \tau, \gamma)$	21
$\mathbb{Y}_S$	The set of elements $X \in \mathbb{Y}$ such that $(\eta, \tau, \gamma) \in S^d$	21
X	Generic point $X = (t, y, x_d, \eta, \tau, \gamma) \in \mathbb{X}$	21
π_P	Characteristic polynomial of $a(X)$	21
$\mathbb{E}^-$	Stable subspace	22
$\mathbb{E}^+$	Unstable subspace	22
Π^-	Projector onto $\mathbb{E}^-(X)$	23
$\mathcal{E}$	Elliptic region	24
$\mathcal{H}$	Hyperbolic region	24
$\mathcal{EH}$	Mixed region	24
$\mathcal{G}$	Glancing region	24
$\underline{\Delta}$	Lopatinskiĭ determinant	27
$\mathbb{Y}_0$	The set of elements $X \in \mathbb{Y}_S$ such that $X = (t, y, x_d, \eta, \tau, 0)$	33
$\Phi_{i,\gamma}$	Microlocal partition of unity	34
L^*	Adjoint of L	38
$\mathcal{T}$	The set of pairs (L, B) such that L is constantly hyperbolic and (L, B) is normal	44
$\mathcal{WR}$	The class of weakly regular initial boundary value problems of real type	46
Γ	Critical set	47
$\underline{\underline{\Delta}}$	Second Lopatinskiĭ determinant	48
Δ_γ	Lopatinskiĭ family of operators	51
L^2_Δ	Lopatinskiĭ L^2-space	51
$\mathcal{S}'$	The space of tempered distributions	51
ℓ	Critical direction	52
K_ℓ	Smallest invariant subspace of a containing ℓ	53

Symbol	Description	Page
Σ_γ	$\mathcal{WR}$ functional symmetriser	54
$\tilde{e}_0(X)$	Change of variables predicted by the block structure condition	59
δ	Lopatinskiĭ symbol	59
$\mathcal{E}_{0,\gamma}$	Pseudodifferential operator with symbol $\tilde{e}_0(X)$	64
$\mathcal{E}_{-1,\gamma}$	Correction to $\mathcal{E}_{0,\gamma}$	64
H^s_Δ	Lopatinskiĭ H^s-space	66
C_c^∞	Set of smooth, compactly supported functions	66
$\phi_{x_d,j}$	Hamiltonian flow map associated to $a_{1,j}$	71
H_p	Hamiltonian vector field associated with $p(x,\xi)$	76
WF	C^∞ wavefront set	77
WF_s	Sobolev wavefront set	77
$e_+^{isQ}g$	Approximate solution to $(D_s - Q)w = 0$ with Cauchy data $w(0) = g$	78

CHAPTER ONE
Introduction

Partial differential equations are powerful tools that can be used to describe a wide range of physical phenomena, including wave propagation, fluid flow, and heat transfer. Their study has a long and rich tradition and is still a very active area of research in modern analysis. Within this field, the treatment of initial boundary value problems occupies a prominent place, largely justified by the immense variety of applications in science and engineering. In particular, recent years have seen a growing interest in a class of boundary problems associated with the formation of shock waves in systems of hyperbolic conservation laws. In this thesis, we focus on investigating the properties of this class, and seek to derive energy estimates with potential applications to nonlinear problems.

1.1 Background

Let $\mathbb{R}^d_+$ be the half-space $\{x = (y, x_d) \in \mathbb{R}^d : x_d > 0\}$ and suppose that $L(t, x, D_t, D_x)$ is a first-order linear differential operator of the form

$$L(t, x, D_t, D_x) = D_t + \sum_{j=1}^{d} A_j(t, x) D_j.$$

Here, $D_j := -i\partial_{x_j}$ and $A_1(t, x), \cdots, A_d(t, x)$ are $n \times n$ matrix-valued functions with real entries depending on $(t, x) \in \mathbb{R} \times \overline{\mathbb{R}}^d_+ \simeq \mathbb{R}^{1+d}_+$. In addition, denote by $\mathcal{M}_{m\times n}(\mathbb{R})$ the set of matrices with dimensions $m \times n$ and real entries.

Consider the initial boundary value problem

$$\begin{cases} Lu(t,x) = f(t,x) & (t,x) \in (0,T) \times \mathbb{R}^d_+, \\ Bu|_{x_d=0}(t,y) = g(t,y) & (t,y) \in (0,T) \times \mathbb{R}^{d-1}, \\ u(0,x) = u_0 & x \in \mathbb{R}^d_+, \end{cases} \tag{1.1}$$

where L is a hyperbolic operator, $B \in C^\infty(\mathbb{R} \times \mathbb{R}^{d-1}, \mathcal{M}_{p\times n}(\mathbb{R}))$, $\cdot|_{x_d=0} : C^\infty(\overline{\mathbb{R}}^d_+) \to C^\infty(\mathbb{R}^{d-1})$ means restriction to the boundary (properly extended to larger function spaces), and the source terms f, g, u_0 are chosen from suitable function spaces to be specified later. We assume the boundary to be non-characteristic for L, i.e., $\det A_d(t,x) \neq 0$ for every $(t,x) \in \mathbb{R} \times \overline{\mathbb{R}}^d_+$ and that the number p of boundary conditions is the number of positive eigenvalues of A_d.

It has been known since the seminal works of Lopatinskiĭ, Kreiss, and Sakamoto that the weak Lopatinskiĭ condition is both necessary and sufficient for the C^∞ well-posedness of (1.1), while the uniform Lopatinskiĭ condition is necessary and sufficient for the H^s well-posedness of (1.1) without loss of derivatives (see [BGS07], [Kre70], [Ser99], [Ben14], and [Sak70] for details). But this lossless scenario is the exception rather than the rule, and it is observed in many interesting examples that the Lopatinskiĭ condition is met weakly but not uniformly [1], thus raising the question of whether it is possible to systematically classify such examples. The answer to this issue proves to be positive, and fits indeed in a more general perspective: besides the two stable classes of hyperbolic initial boundary value problems (L, B) where either the weak Lopatinskiĭ condition fails (strongly unstable) or the uniform Lopatinskiĭ holds (strongly stable), Benzoni–Gavage, Rousset, Serre, and Zumbrun have identified in [BGRSZ02] a third stable class that they named *weakly regular of real type*, or $\mathcal{WR}$ for short, for which the Lopatinskiĭ condition degenerates to the first order as one approaches the so-called hyperbolic region.

When it comes to energy estimates, strongly unstable and strongly stable initial boundary value problems are well understood. In the former, there is no hope for any satisfactory theory; in the latter, the uniform Lopatinskiĭ condition has been shown to be equivalent to an energy estimate of the type

$$\begin{aligned} &e^{-2\gamma T}\int_{\mathbb{R}^d_+} |u(T,x)|^2\,dx\,dt + \int_0^T\int_{\mathbb{R}^{d-1}} e^{-2\gamma t}|u|_{x_d=0}(t,y)|^2\,dy\,dt + \int_0^T\int_{\mathbb{R}^d_+} e^{-2\gamma t}|u(t,x)|^2\,dx\,dt \\ &\lesssim \int_{\mathbb{R}^d_+} |u(0,x)|^2\,dx\,dt + \frac{1}{\gamma}\int_0^T\int_{\mathbb{R}^d_+} e^{-2\gamma t}|Lu(t,x)|^2\,dx\,dt + \int_0^T\int_{\mathbb{R}^{d-1}} e^{-2\gamma t}|Bu|_{x_d=0}(t,y)|^2\,dy\,dt, \end{aligned} \tag{1.2}$$

where we have the remarkable feature that both the input (f, g, u_0) and the solution are estimated in the same norms (L^2 in this example, or any other Sobolev norm). In an attempt to establish energy inequalities for the $\mathcal{WR}$ class, we examine the pure boundary value problem

$$\begin{cases} Lu(t,x) = f(t,x) & (t,x) \in \mathbb{R} \times \mathbb{R}^d_+, \\ Bu|_{x_d=0}(t,y) = g(t,y) & (t,y) \in \mathbb{R} \times \mathbb{R}^{d-1}, \end{cases} \tag{1.3}$$

where t runs along the whole real line rather than in the interval $[0, T)$. Alternatively, if we multiply both sides of $Lu = f$ in (1.3) by A_d^{-1} and solve for D_d, we get the equivalent

[1] this failure being characterised in a distinctive way.

boundary value problem

$$\begin{cases} Pu(t,x) := D_{x_d} + \mathcal{A}(t,y,D_t,D_y) = A_d^{-1} f(t,x) & (t,x) \in \mathbb{R} \times \mathbb{R}^d_+, \\ Bu|_{x_d=0}(t,y) = g(t,y) & (t,y) \in \mathbb{R} \times \mathbb{R}^{d-1}. \end{cases} \tag{1.4}$$

It turns out that energy estimates for (1.1) are mainly based on those for (1.4) in weighted spaces $L^2_\gamma \equiv e^{\gamma t} L^2$ (see Section 2.2.2), so we shall merely analyse Problem (1.4) hereafter.

1.2 Problem

We shall deal in the sequel with *constantly hyperbolic operators*. Roughly speaking, an operator L is said to be symmetrisable of constant multiplicities if every root in τ of $L(t,x,\tau,\xi)$ is real and its multiplicity is locally constant and equal to the dimension of $\ker L(t,x,\tau,\xi)$. These operators meet the *block structure condition*, a central notion in the construction of microlocal symmetrisers for $D_d + \mathcal{A}$ that allow us to distinguish four different situations according to the spectrum of $\mathcal{A}$, more precisely,

- ▷ the set of elliptic points $\mathcal{E}$ for which $\mathcal{A}$ is diagonalisable with complex conjugate eigenvalues,
- ▷ the set of hyperbolic points $\mathcal{H}$ for which $\mathcal{A}$ is diagonalisable only with purely real eigenvalues,
- ▷ the set of mixed points $\mathcal{EH}$ for which $\mathcal{A}$ is diagonalisable with a combination of complex conjugate and purely real eigenvalues, and
- ▷ the set of glancing points $\mathcal{G}$ for which $\mathcal{A}$ is not diagonalisable, but exhibits at least one Jordan block.

The definition of the $\mathcal{WR}$ class is typically given in terms of the Lopatinskiĭ determinant $\underline{\Delta}$, a function that vanishes exactly at points where the Lopatinskiĭ condition fails. Equipped with $\underline{\Delta}$, we say that a pair (L,B) is $\mathcal{WR}$ if the following conditions are fulfilled:

- ▷ The weak Lopatinskiĭ condition holds,
- ▷ the uniform Lopatinskiĭ condition is violated to the first order in the hyperbolic region $\mathcal{H}$, that is to say, $\partial_\tau \underline{\Delta} \neq 0$ whenever $\underline{\Delta} = 0$ in $\mathcal{H}$.

Though formula (1.2) cannot apply to the $\mathcal{WR}$ class by its very definition, it is certainly possible to deduce energy inequalities with loss of derivatives. For instance[2],

- ▷ in [Cou02] and [Cou04], Coulombel studies the linear stability of multidimensional shock waves for hyperbolic systems of conservation laws assuming that

[2]We warn the reader that the symbols defined here may have a completely different meaning throughout the text, as we follow in this chapter the authors' notation in each case to facilitate comparison with the original references.

Majda's uniform condition is violated (see [Maj83b]). Two concrete examples motivate his research: planar Lax shocks in isentropic gas dynamics and phase transitions in isothermal fluids. In both cases, Coulombel modifies Kreiss original construction of a microlocal symmetriser and derives energy estimates for the linearised boundary value problem with constant and variable coefficients. Although not explicitly stated in any of these papers, but in a later work from 2010 (see [CG10]), these problems belong to the $\mathcal{WR}$ class and satisfy energy estimates of the type

$$\gamma|||u|||^2_{0,\gamma} + \|u|_{x_d=0}\|^2_{0,\gamma} \lesssim \frac{1}{\gamma^3}|||f|||^2_{1,\gamma} + \frac{1}{\gamma^2}\|g\|^2_{1,\gamma}, \tag{1.5}$$

where

$$|||u|||^2_{s,\gamma} := \int_0^\infty \|u(\cdot, x_d)\|^2_{s,\gamma}\, dx_d,$$

and

$$\|u\|^2_{s.\gamma} := \frac{1}{(2\pi)^d}\int_0^\infty (\gamma^2 + |\xi|^2)^s |\hat{u}(\xi)|^2\, d\xi.$$

Formula (1.5) remains true when the coefficients have limited regularity (since Coulombel uses paradifferential calculus) and they are optimal on the scale of Sobolev spaces, as shown in [CG10] using geometric optics expansions. In the long run, the loss of one derivative on the interior and one derivative on the boundary causes some difficulties when applied to nonlinear problems where iterations are expected.

▷ Initially in [Ser05] and then in collaboration with Benzoni-Gavage in [BGRSZ02], Serre proposes another approach to deriving energy estimates for the $\mathcal{WR}$ class, for which he applies a specific operator to the solution and the data alike. His method covers two different scenarios, namely, boundary value problems for second-order scalar hyperbolic operators with constant and variable coefficients, and boundary value problems for first-order systems with constant coefficients. In the scalar case, Serre investigates the wave operator $L := -\partial_d - c^2\Delta_x$ in a half-space supplemented by a boundary condition $B := -\partial_d - \beta - \partial_t - v\nabla_y$. Then, through an intricate factorisation, (L, B) is decomposed as follows:

$$\begin{cases} Lz = Pf & (t,x) \in \mathbb{R}^{1+d}_+, \\ Pu = z & (t,x) \in \mathbb{R}^{1+d}_+, \\ Ez = \dfrac{(\epsilon^2 - 1)}{c^2} - 4Rg & (t,y) \in \mathbb{R}^d, \end{cases} \tag{1.6}$$

where R is a tangential operator (i.e., acting on the variables (t, y)) and P is a "filter" operator that vanishes at points on the boundary where the Lopatinskiĭ is violated. The idea is to choose ϵ in (1.6) so that (L, E) satisfies the uniform Lopatinskiĭ condition and $Pu = z$ can be solved, leading eventually to energy estimates like

$$\gamma\|\nabla_{t,x}Pu\|_{L^2_\gamma(Q)} + \|\nabla_{t,x}Pu\|^2_{L^2_\gamma(\partial Q)} \leq C\left(\frac{1}{\gamma}\|Pf\|^2_{L^2_\gamma(Q)} + \|Rg\|^2_{L^2_\gamma(\partial Q)}\right),$$

or

$$\gamma\|\nabla_{t,x}u\|_{L^2_\gamma(Q)} + \|\nabla_{t,x}u\|^2_{L^2_\gamma(\partial Q)} \leq \frac{C}{\gamma^2}\left(\frac{1}{\gamma}\|Pf\|^2_{L^2_\gamma(Q)} + \|Rg\|^2_{L^2_\gamma(\partial Q)}\right).$$

The variable coefficient case for a general hyperbolic operator of second order L adheres to the same principle, except that one needs to account for lower order terms at each step, as the composition formulae are no longer exact.

As for the system case, the treatment is done by taking the Fourier transform to (1.4) and then writing the unknown $\hat{u}$ as $\hat{u} = \pi_+\hat{u} + \pi_-\hat{u}$, where $\pi_-\hat{u}$ (resp. $\pi_+\hat{u}$) is the projection of $\hat{u}$ onto the stable (resp. unstable) subspace (see Definition 2.4.1). From here, the divergent term $\hat{u}_s(0)$ (see Section 8.4.1 in [BGS07]) is controlled by multiplying $\hat{u}_s(0)$ by the Lopatinskiĭ determinant $\Delta(\tau,\eta)$, thus paving the way to obtain energy estimates for $\hat{u}$ after some manipulations. It should be noted that one of the inequalities used for this part is only valid in certain region Γ of the frequency space (see Lemma 8.3 in [BGRSZ02]), a fact that necessarily limits the scope of the statement. In the end, using Plancherel's theorem we have

$$\begin{aligned}
&\gamma\int_{\Omega\times\mathbb{R}} e^{-2\gamma t}\|P_\gamma u\|^2\,dx\,dt + \int_{\partial\Omega\times\mathbb{R}} e^{-2\gamma t}\|P_\gamma\gamma_0 u\|^2\,dx\,dt \\
&\leq C\left(\frac{1}{\gamma}\int_{\Omega\times\mathbb{R}} e^{-2\gamma t}\|P_\gamma Lu\|^2\,dx\,dt + \int_{\partial\Omega\times\mathbb{R}} e^{-2\gamma t}|P_\gamma\gamma_0(A^d)^{-1}M^TBu|^2\,dx\,dt\right),
\end{aligned} \tag{1.7}$$

where P_γ is a pseudo-differential operator with symbol $p(\tau,\eta) = \pi_+ + \Delta(\tau,\eta)\pi_-$, γ_0 is the trace operator and M is some matrix such that $(A^d)^{-1}M^TB$ is a projector. Finally, the author claims that P_γ is an operator of real principal type, meaning in practice that the pseudodifferential problem $P_\gamma u = z$ is solvable for u.

In summary, while Serre's philosophy serves as the starting point for this work, his techniques rely on ad-hoc steps that barely admit any generalisation beyond the model problem presented in each case.

1.3 Main results

We now outline the results of this thesis and explain how they address the issues identified in the references mentioned above. These findings provide a significant improvement in the overall understanding of various aspects of the $\mathcal{WR}$ class. To start with, let $P := D_d + A(D_t, D_y)$ be an $n\times n$ first-order differential operator with constant coefficients and B a $p\times n$ full rank matrix, with p identical to the dimension of the stable space of $A(\tau,\eta)$ (see Definition 2.4.1). In Section 3.2, we begin by exploring the model problem

$$\begin{cases} Pu(t,x) := \big(D_d + A(D_t, D_y)\big)\, u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ Bu|_{x_d=0}(t,y) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{1.8}$$

subject to a set of auxiliary hypothesis that we make precise in Assumption 2.3.1. Although we deduce energy estimates for (1.8) that are basically the same as those in [Ser05] and [BGS07], our approach based on the construction of a symmetriser is more robust because it extends the validity of (1.7) beyond Γ, and because it reveals a necessary condition that must be satisfied by any further generalisation of this symmetriser. Essentially, we prove the existence of two families of pseudo-differential operators $\Delta_\gamma = \mathrm{Op}_\gamma(\delta)$ and $\Sigma_\gamma = \mathrm{Op}_\gamma(\sigma)$ so that:

- ▹ the set of points where the Lopatinskiĭ condition fails is included in the characteristic set of Δ_γ,
- ▹ Σ_γ is self-adjoint,
- ▹ if
  $$L_\Delta := \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Delta_\gamma v \in L^2_\gamma(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\},$$
  then for every $v_1, v_2 \in L^2_\Delta$ there exists a positive constant C satisfying
  $$\langle \Sigma_\gamma v_1, v_2 \rangle \leq C|\Delta_\gamma v_1||\Delta_\gamma v|,$$
- ▹ there is a positive constant c such that
  $$\mathrm{Im}\, \langle \Sigma_\gamma A_\gamma v, v \rangle \geq c\gamma |\Delta_\gamma v|^2$$
  for each $v \in L^2_\Delta$,
- ▹ there exist positive constants α and β for which
  $$\langle \Sigma_\gamma v(0), v(0) \rangle \geq \alpha |\Delta_\gamma v(0)|^2 - \beta |Bv(0)|^2$$
  holds true for every $v \in L^2_\Delta$.

As we shall see in Chapter 3, when the Lopatinskiĭ condition fails at one point, the intersection of the stable subspace and the kernel of B is a one-dimensional subspace of $\mathbb{C}^n$. Being a special one, we call it the *critical direction* and denote it by ℓ. Interestingly, we are in a position to explain in Proposition 3.2.1 how Σ_γ degenerates around ℓ: if a is the principal symbol of $A(D_t, D_y)$ and $\underline{\Delta}(\zeta_0) = 0$, then $v \mapsto \langle \sigma(\zeta_0) v, v \rangle$ vanishes on the Krylov space of $\ell(\zeta_0)$ with respect to $a(\zeta_0)$, i.e., the smallest invariant subspace of $a(\zeta_0)$ containing $\ell(\zeta_0)$. We take this observation into account in subsequent generalisations of Σ_γ.

Having prepared the ground with the previous case study, we tackle the main problem.

In order to include higher-order scalar systems as well as first-order matrix systems, we propose the more general problem

$$\begin{cases} P_\gamma u_\gamma(t,x) := \big(D_d + A_\gamma(t,y,x_d,D_t,D_y)\big)\, u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{1.9}$$

where f_γ, g_γ are chosen at least in L^2_γ, and $A_\gamma \in \mathrm{OPS}^1_\gamma$ is a classical pseudo-differential operator with parameter whose symbol a admits an asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_{1-j},$$

each a_{1-j} being an $n \times n$ homogeneous matrix of degree $1-j$. In the same spirit, $B_\gamma \in \mathrm{OPS}^0_\gamma$ is a classical $\gamma-$dependent pseudo-differential operator with a $p \times n$ matrix b as principal part. If we suppose that (P_γ, B_γ) satisfies the definition of the $\mathcal{WR}$ class (besides some other structural assumptions), we show that there exists symbols $\tilde{e}_0$ and δ with the following features:

- $\tilde{e}_0$ and δ are homogeneous of degree 0,
- $\tilde{e}_0 \in GL_n(\mathbb{C})$ and
  $$\dot{a}_1 := \tilde{e}_0^{-1} a_1 \tilde{e}_0$$
  is diagonal with entries $a_{1,1}, \cdots, a_{1,n}$,
- there exits $s \leq p$ so that δ is diagonal with respect to the basis $\tilde{e}_0$ and given by
  $$\delta = \mathrm{diag}(\delta^-, I_{n-p}), \tag{1.10}$$
  where
  $$\delta^- := \begin{pmatrix} \begin{pmatrix} \delta_1^- & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_s^- \end{pmatrix} & \\ & I_{p-s} \end{pmatrix},$$
  with each δ_j^- being the solution of the transport equation
  $$\begin{cases} \partial_d \delta_j^- + \{\delta_j^-, a_{1,j}\} = 0, \\ \delta_j^-|_{x_d=0} = \underline{\Delta}, \end{cases} \tag{1.11}$$
- If $x_d = 0$, there exist smooth matrix-valued functions q and m with dimensions $p \times p$ and $p \times n$, respectively, so that if $\dot{b} := b\tilde{e}_0$, there holds the identity
  $$q\dot{b} = m\delta, \tag{1.12}$$

- $\triangleright$ when nontrivial, $\ker \delta$ is an $\mathfrak{s}-$dimensional invariant subspace of a_1 containing the critical direction ℓ.

Next, we generalise the $\mathcal{WR}$ symmetriser Σ_γ that was postulated earlier for the model problem. Certainly, if $\Delta_\gamma = \mathrm{Op}_\gamma(\delta)$ and $Q_\gamma = \mathrm{Op}_\gamma(q)$, we claim that it is possible to find a family Σ_γ of C^1 operator-valued mappings depending on x_d so that for γ sufficiently large,

- $\triangleright$ $\Sigma_\gamma(x_d)$ is self-adjoint.
- $\triangleright$ for every $v_1, v_2 \in L^2_\Delta$, there is a positive constant C satisfying

$$\langle \Sigma_\gamma(x_d)v_1, v_2\rangle \leq C|\Delta_\gamma(x_d)v_1||\Delta_\gamma(x_d)v|.$$

- $\triangleright$ There is a positive constant c, independent of x_d, such that

$$\langle \partial_d \Sigma_\gamma(x_d)v, v\rangle + 2\,\mathrm{Im}\,\langle \Sigma_\gamma(x_d)A_\gamma(x_d)v, v\rangle \geq c\gamma|\Delta_\gamma(x_d)v|^2$$

for each $v \in L^2_\Delta$.

- $\triangleright$ There exist positive constants α and β for which

$$\langle \Sigma_\gamma(0)v(0), v(0)\rangle \geq \alpha|\Delta_\gamma(0)v(0)|^2 - \beta|Q_\gamma B_\gamma v(0)|^2$$

holds true for every $v \in L^2_\Delta$.

In contrast to the standard case where the uniform Lopatinskiĭ condition is satisfied, lower order terms in a $\mathcal{WR}$ problem are problematic and require some attention. To deal with them, we have to resort to finer tools, including a technical lemma (see Lemma 3.3.2) and the following statement.

Proposition (Chapter 3 - Proposition 3.3.1). *The norms $\|\Delta_\gamma \cdot\|$ lie between L^2_γ and H^{-1}_λ for γ larger than $\gamma_0 \geq 1$, that is to say, there exist positive constants C_1 and C_2 such that*

$$C_1\|\cdot\|_{\gamma,-1} \leq \frac{1}{\gamma_0}\|\Delta_\gamma \cdot\| \leq C_2\|\cdot\|_\gamma, \tag{1.13}$$

for every $\gamma \in (\gamma_0, +\infty]$.

Due to Proposition 3.3.1, we are able to recover Coulombel's estimates with loss of one derivative. Overall, by combining Σ_γ with a partition of unity, we can prove the main theorem of this work.

Theorem (Chapter 3 - Theorem 3.3.2). *Let*

$$\begin{cases} P_\gamma u_\gamma(t,x) := \big(D_d + A_\gamma(t,y,x_d,D_t,D_y)\big)\, u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{1.14}$$

where $A_\gamma \in \mathrm{OPS}^1_\gamma(\mathbb{R}^{1+d}_+ \times [1,+\infty))$ and $B_\gamma \in \mathrm{OPS}^0_\gamma(\mathbb{R}^d \times [1,+\infty))$ are classical pseudo-

differential operators with matrix-valued symbols $a(X)$ and $b(X)$ of dimensions $n \times n$ and $p \times n$, respectively. Suppose that P_γ is hyperbolic in the sense of Definition 2.3.1, P_γ and B_γ satisfy Property **(C)** *(see Assumption 2.3.1), and that $p = \dim \mathbb{E}^-(X)$. Then there exist*

(i) $\gamma_0 \geq 1$,

(ii) a family of pseudodifferential operators

$$\Delta_\gamma(t, x, D_t, D_y) \in \mathrm{OPS}^0_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0, +\infty)),$$

(iii) function spaces

$$L^2_\Delta := \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Delta_\gamma v \in L^2_\gamma(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\},$$
$$H^s_\Delta := \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Lambda^s_\gamma v \in L^2_\Delta(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\},$$

(iv) and a positive constant C such that,

if $f \in L^2_\gamma(\mathbb{R}^{1+d}_+)$ and $g \in L^2_\gamma(\mathbb{R}^d)$, then for all $\gamma > \gamma_0$ and every $u \in \mathcal{D}(\mathbb{R}^{1+d}_+)$ the following estimate holds

$$\gamma \|\Delta_\gamma u\|^2_{0,\gamma} + |\Delta_\gamma u(0)|^2_{0,\gamma} \leq C \left(\frac{1}{\gamma} \|f\|^2_{0,\gamma} + |g|^2_{0,\gamma} \right). \tag{1.15}$$

More generally, if $f \in H^s_\gamma(\mathbb{R}^{1+d}_+)$ and $g \in H^s_\gamma(\mathbb{R}^d)$,

$$\gamma \|\Delta_\gamma u\|^2_{s,\gamma} + |\Delta_\gamma u(0)|^2_{s,\gamma} \leq C \left(\frac{1}{\gamma} \|f\|^2_{s,\gamma} + |g|^2_{s,\gamma} \right). \tag{1.16}$$

The last result of this thesis concerns the existence, uniqueness and regularity of the solution of the $\mathcal{WR}$ problem. To answer these and other relevant questions about u, we show that $\Delta_\gamma u$ is indeed the solution to a problem satisfying the uniform Lopatinskiĭ condition. This insight, which is encoded in Theorem 3.3.2, allows us to use all the machinery developed in Chapter 2 and, ultimately, to draw conclusions for u by studying the operator Δ_γ. As the proposition below indicates, the key element is that Δ_γ is an operator of principal type.

Proposition (Chapter 4 - Proposition 4.2.1). *Let $\delta(X) = \mathrm{diag}(\delta^-(X), I_{n-p})$,*

$$\delta^-(X) := \begin{pmatrix} \begin{pmatrix} \delta^-_1(X) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta^-_\jmath(X) \end{pmatrix} & \\ & I_{p-\jmath} \end{pmatrix}, \tag{1.17}$$

where $\delta_1(X), \cdots, \delta_\jmath(X)$ are solutions of (1.11). Then the operator $\mathrm{Op}_\gamma(\delta_j)$ is of principal type for every $j \in \{1, \cdots, \jmath\}$.

We conclude with a concise description of the propagation of singularities for a $\mathcal{WR}$ problem.

THEOREM (CHAPTER 4 - THEOREM 4.2.1). *Consider*

$$\begin{cases} \mathrm{Op}_\gamma\,(\delta_j)\tilde{u}_j = \tilde{w}_j, \\ \qquad \tilde{u}_j|_{t=0} = 0, \end{cases} \tag{1.18}$$

where $\tilde{w} \in H^s_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ *and* $\mathrm{Op}_\gamma\,(\delta_j)$ *is as defined in Proposition 4.2.1. Then there exists a unique solution of* (4.10) *modulo* C^∞ *for which*

$$\mathrm{WF}_{s+m-1}(\tilde{u}_j) \setminus \mathrm{WF}_s(\mathrm{Op}_\gamma\,(\delta_j)\tilde{u}_j) \subset \delta_j^{-1}(0). \tag{1.19}$$

1.4 Thesis structure

This thesis is organised as follows: in Chapter 2, we set much of the notation that will be used throughout the document and recapitulate some well-known facts on boundary value problems. This includes, but it is not limited to, a precise notion of hyperbolicity, parameter-dependent pseudo-differential operators, weighted-in-time Sobolev spaces, the block structure condition, the Lopatinskiĭ condition, L^2-energy estimates and its consequences. In Chapter 3, we examine the $\mathcal{WR}$ problem in depth. Firstly, we establish the concept of a stable class and give some examples. Secondly, we exhibit an alternative characterisation of the $\mathcal{WR}$ class that fits better our purposes when constructing a symmetriser for the model problem. After that, we implement the subsequent generalisations to variable coefficients and construct families Δ_γ and Σ_γ having all the necessary properties to get a priori estimates. In Chapter 4, we define operators of (real) principal type and introduce the main theorems in this direction. We then verify that Δ_γ is of this kind and elaborate on how to solve $\Delta_\gamma u = w$. Lastly, we finish with an appendix that covers miscellaneous results that are of some use over the course of this discussion.

CHAPTER TWO

Regular boundary problems in a half-space

In this chapter we synthesise the main aspects of the theory of boundary value problems in a half-space. In the interest of balancing precision and fluency, we refrain from including all the proofs and provide instead only those that are of some use to the discussion, feeling free to cite the rest. As far as possible, we shall introduce the most relevant concepts from scratch, without assuming any specific knowledge beyond the classical ideas of mathematical analysis and the standard theory of pseudodifferential operators.

Let us briefly describe the content. In Section 2.1, we specify some basic notation and state the target problem of the chapter. In Section 2.2, we explore function spaces and certain classes of pseudodifferential operators that constitute the basis for subsequent arguments. In Section 2.4, we detail further results on boundary value problems, including two cornerstones of this thesis: the block structure condition and the Lopatinskiĭ condition. In Section 2.5, we derive energy estimates using previously established facts and some structural assumptions. Finally, we investigate in Section 2.6 the existence, uniqueness, and regularity of the solution by combining the energy estimates and other tools from functional analysis.

2.1 Problem set-up

2.1.1 Background and notation

Let $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $t \in \mathbb{R}$. If we denote by y the variables $(x_1, \cdots, x_{d-1})$, the half-space corresponds to

$$\mathbb{R}^d_+ := \{x = (y, x_d) \in \mathbb{R}^d : x_d > 0\}.$$

As may be verified by inspection, the boundary is given by

$$\partial\mathbb{R}^d_+ = \{x = (y, x_d) \in \mathbb{R}^d : x_d = 0\} \simeq \mathbb{R}^{d-1}.$$

Other domains (as long as they are good enough) may be reduced to this situation by using local charts and a partition of unity, so there is no loss of generality in working with flat boundaries right from the beginning. Besides the spatial coordinate x, one of the variables in the type of problems we are interested in may be identified with time, even though it is defined all over $\mathbb{R}$. Mathematically, we take this into account by writing the *space-time* as $\mathbb{R}^{1+d}_+ = \mathbb{R} \times \mathbb{R}^d_+$.

Except for the scalar case, which is discussed in Section 2.3, the operators involved are matrix form and hence it will be useful to fix some basic notation for this purpose. For example, we denote by $\mathcal{M}_{n\times m}(\mathbb{R})$ (resp. $\mathcal{M}_{n\times m}(\mathbb{C})$) the set of matrices with dimension $n \times m$ and real entries (resp. complex entries), by I_m the identity matrix of order $m \in \mathbb{N}$, and by $M = \operatorname{diag}(m_1, \cdots, m_n)$ a generic diagonal matrix with elements $m_1, \cdots, m_n$. As for the differential operators, we often write D_j instead of D_{x_j}.

Let us establish the concrete problem we wish to tackle. Consider the first order differential operator

$$L(t, x, D_t, D_x) = D_t + \sum_{j=1}^{d} A_j(t, x) D_j, \tag{2.1}$$

whose coefficients $A_j(t, x)$ are smooth matrix-valued functions with real entries. For future reference, we take $(\tau, \xi) = (\tau, \eta, \xi_d)$ as the set of covariables of $(t, x) = (t, y, x_d)$ and define

$$A(t, x, \eta, \xi_d) = \sum_{j=1}^{d-1} \eta_j A_j(t, x) + \xi_d A_d.$$

The symbol of $L(t, x, D_t, D_x)$ is then

$$L(t, x, \tau, \eta, \xi_d) = \tau + A(t, x, \eta, \xi_d),$$

and the characteristic polynomial $\pi_L(t, x, \tau, \xi)$ of L is given by

$$\pi_L(t, x, \tau, \xi) = \det L(t, x, \tau, \eta, \xi_d) = \det\left(\tau + A(t, x, \eta, \xi_d)\right). \tag{2.2}$$

In the sequel, we shall deal with constantly *symmetrisable hyperbolic operators* in the t-direction, the precise meaning of which is addressed below.

Definition 2.1.1. *The operator L is symmetrisable hyperbolic of constant multiplicities or constantly hyperbolic for short, if there are positive integers $\alpha_1, \cdots, \alpha_q$ and real analytic, pairwise different functions $\lambda_1(t, x, \xi), \cdots, \lambda_q(t, x, \xi)$ defined on $\mathbb{R}^{1+d}_+ \times \mathbb{R}^d$ such that*

$$\pi_L(t, x, \tau, \xi) = \prod_{k=1}^{q} (\tau + \lambda_k(t, x, \xi))^{\alpha_k}, \tag{2.3}$$

and the mappings $\lambda_1(t,x,\xi), \cdots, \lambda_q(t,x,\xi)$*, when understood as the eigenvalues of* $A(t,x,\eta,\xi_d)$*, are semi-simple. In particular, when* $\alpha_k = 1$ *for every* $k \in \{1, \cdots, q\}$*, the eigenvalues are simple and* L *is said to be strictly hyperbolic.*

A first compatibility condition follows from the above definition: when L is constantly hyperbolic, $A_d(t,x)$ must be diagonalisable over the reals since $A(t,x,0,1) = A_d(t,x)$ for all $(t,x) \in \mathbb{R}^{1+d}_+$. We shall soon impose the stronger condition that the eigenvectors of A_d are either positive or negative, but in no case equal to 0.

2.1.2 A boundary value problem in a half-space

After this preparation, we shall study a boundary value problem

$$\begin{cases} L(t,x,D_t,D_x)u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ Bu|_{x_d=0}(t,y) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.4}$$

where $B \in C^\infty(\mathbb{R}^d, \mathcal{M}_{p\times n}(\mathbb{R}))$, f and g are functions chosen from suitable functions spaces to be defined, and $\cdot|_{x_d=0}$ stands for the trace operator on $\mathbb{R}^d$ (see Theorem 2.1.1). Also, we supplement Problem 2.4 with extra hypotheses.

Assumption 2.1.1.

(i) *L is symmetrisable of constant multiplicities.*

(ii) *The boundary* $\partial\mathbb{R}^{1+d}_+ \simeq \mathbb{R}^d$ *is non-characteristic for* L*, meaning that* A_d *is nonsingular for all* $(t,y,0) \in \mathbb{R}^d$*.*

(iii) *B is assumed to be everywhere of maximal rank* p*, with* p *being the number of positive eigenvalues of* A_d *(the number of incoming characteristics).*

(iv) (L,B) *satisfies for all* $(t,y) \in \mathbb{R}^d$ *the normality condition*

$$\mathbb{R}^n = \ker B(t,y) \oplus E^s(A_d(t,y,0)),$$

where $E^s(A_d(t,y,0))$ *is the stable subspace of* $A_d(t,y,0)$*, i.e., the subspace spanned by eigenvectors associated with negative eigenvalues.*

While the first three items are typical assumptions for (2.4), the last one may seem less natural. The reason for Condition (iv) is twofold: in the present context, it facilitates the classification of boundary value problems in Chapter 3; in the broader sense of initial boundary value problems, *normality* is a necessary condition to ensure well-posedness in spaces of considerable importance like L^2 (see Section 4.1.2 in [BGS07] for an ample discussion on this matter).

To bring this section to an end, we formalise the intuition behind restricting u to the boundary $\{x_d = 0\}$.

Theorem 2.1.1 (Theorem 9.8, [BGS07]). *Let* (L,B) *be a boundary value problem subject to*

Assumption 2.1.1. If the subspace

$$E := \{w \in L^2(\mathbb{R}^{1+d}_+) : Pw \in L^2(\mathbb{R}^{1+d}_+)\}$$

is endowed with the graph norm, then $\mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$ *is dense in* E *and the trace operator*

$$\begin{aligned} \cdot|_{x_d=0} : \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+) &\longrightarrow \mathcal{D}(\mathbb{R}^d) \\ w &\longmapsto w|_{\mathbb{R}^d} \end{aligned}$$

extends in a unique way to a continuous map from E *to* $H^{-1/2}(\mathbb{R}^d)$.

Provided there is no room for confusion, we shall refer to the trace operator acting on u as $u|_{x_d=0}$ or $u(\cdot, 0)$ interchangeably.

2.2 Function spaces and parameter-dependent operators

2.2.1 Pseudodifferential calculus with parameter

In this section, we shall introduce the basic ideas behind the pseudodifferential calculus with parameter. To make the exposition easier, we stick to the following convention:

- ▷ we set $\zeta := (\tau - i\gamma, \eta) \simeq (\tau, \eta, \gamma)$,
- ▷ we use $|\cdot|$ or $\|\cdot\|$ depending on whether we are dealing with matrix norms (single bars) or norms in function spaces (double bars),
- ▷ by an abuse of notation, we represent both the inner product of $L^2(\mathbb{R}^{1+d}_+)$ and $L^2(\mathbb{R}^d)$ by $\langle \cdot, \cdot \rangle$.

Just as the Japanese bracket $\lambda^s(\tau, \eta) = (1 + \tau^2 + \eta^2)^{s/2}$ is the object to compare with in the conventional pseudodifferential calculus, the parameter-dependent Japanese bracket

$$\lambda^s_\gamma(\tau, \eta) \equiv \lambda^s(\zeta) := (\gamma^2 + \tau^2 + \eta^2)^{s/2} \tag{2.5}$$

serves as the model symbol in the new setting. In practice, substituting $\lambda^s(\tau, \eta)$ by $\lambda^s(\zeta)$ does not change the results significantly and virtually any concept from the theory of pseudodifferential operators *without* parameter may be reformulated into a parameter-dependent version with only a few changes, usually. Notably, a $\gamma-$variant of the familiar Sobolev spaces H^s can be obtained by replacing $\lambda^s(\tau, \eta)$ by $\lambda^s_\gamma(\tau, \eta)$ to get

$$\|u\|_{\underline{H}^s_\gamma} = \|\lambda^s_\gamma \hat{u}\|_{L^2}. \tag{2.6}$$

As might be expected, the hat on top of u in (2.6) refers to the Fourier transform of u with respect to the variables (t, y). It turns out that Formula (2.6) defines a family of γ-indexed norms that is equivalent to the standard $\|\cdot\|_{H^s}$ norm. Actually, the interactions between $\|\cdot\|_{H^s}$ and $\|\cdot\|_{\underline{H}^s_\lambda}$ are generally simple and merely require some γ factor to be taken into account. For instance,

$$\begin{aligned} \|u\|_{H^s} \le \|u\|_{\underline{H}^s_\gamma} &\le \gamma^s \|u\|_{H^s} && s > 0, \\ \gamma^s \|u\|_{H^s} \le \|u\|_{\underline{H}^s_\gamma} &\le \|u\|_{H^s} && s < 0, \\ \|u\|_{H^s} \le \|u\|_{\underline{H}^s_\gamma} &\le \gamma^{s-m} \|u\|_{H^m_\gamma} && s \le m, \end{aligned}$$

are common inequalities. We continue with the definition of a class $S^m_\gamma(\mathbb{R}^d \times \mathbb{R}^d \times [\gamma_0, +\infty))$ of tangential pseudodifferential operators depending on the parameter γ.

Definition 2.2.1. *Let $m \in \mathbb{R}$ and $\gamma_0 \ge 1$. Let us denote by $S^m_\gamma(\mathbb{R}^d \times \mathbb{R}^d \times [\gamma_0, +\infty))$ the set of functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \times [\gamma_0, +\infty), \mathcal{M}_{n\times m}(\mathbb{R}))$ so that for all multi-indices $\alpha, \beta \in \mathbb{N}^d$ there exists a positive constant $C_{\alpha,\beta}$ such that*

$$|\partial^\alpha_{(t,y)} \partial^\beta_\zeta a(t, y, \zeta)| \le C_{\alpha,\beta} \lambda^{m-|\beta|}(\zeta) \tag{2.7}$$

for all $\gamma \ge \gamma_0$.

To any symbol $a(t, y, \tau, \eta, \gamma) \in S^m_\gamma(\mathbb{R}^d \times \mathbb{R}^d \times [\gamma_0, +\infty))$, we may associate a family $\mathrm{Op}_\gamma(a)$ of pseudodifferential operators depending on γ through the expression

$$(\mathrm{Op}_\gamma(a)u)(t, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t\tau + y\eta)} a(t, y, \tau, \eta, \gamma) \hat{u}(\tau, \eta)\, d\tau\, d\eta. \tag{2.8}$$

As in the traditional calculus, the set $\mathrm{OPS}^m_\gamma(\mathbb{R}^d \times [\gamma_0, +\infty])$ consists of families of pseudodifferential operators $\{\mathrm{Op}_\gamma(a)\}_{\gamma \ge \gamma_0}$ for which the mapping property

$$\|\mathrm{Op}_\gamma(a)u\|_{\underline{H}^{s-m}_\gamma} \le C\|u\|_{\underline{H}^s_\gamma} \tag{2.9}$$

holds for some positive constant C, every $\gamma \ge \gamma_0 \ge 1$, and all $s \in \mathbb{R}$. A prototypical example of a family of order m is $\Lambda^m_\gamma := \mathrm{Op}_\gamma(\lambda^m_\gamma)$, as may be seen from the definition and the elementary computation

$$\|\Lambda^m_\lambda u\|_{\underline{H}^{s-m}_\gamma} = \|\Lambda^{s-m}_\gamma \Lambda^m_\gamma u\|_{L^2} = \|\Lambda^s_\gamma u\|_{L^2} = \|u\|_{\underline{H}^s_\gamma}.$$

As the next theorem shows, compositions and adjoints obey similar rules as in the ordinary case.

Theorem 2.2.1 (Theorem C.6, [BGS07]). *Let $a \in S^m_\gamma$, $b \in S^n_\gamma$. For some $\gamma_0 \ge 1$, it is true that*

(i) $\{\mathrm{Op}_\gamma(a)\}_{\gamma\ge\gamma_0} \in \mathrm{OPS}^m_\gamma$,

(ii) $\{\mathrm{Op}_\gamma(a)^* - \mathrm{Op}_\gamma(a^*)\}_{\gamma\ge\gamma_0} \in \mathrm{OPS}^{m-1}_\gamma$,

(iii) $\{\mathrm{Op}_\gamma(a) \circ \mathrm{Op}_\gamma(b) - \mathrm{Op}^\gamma(ab)\}_{\gamma\ge\gamma_0} \in \mathrm{OPS}^{m+n-1}_\gamma$,

(iv) $\{[\mathrm{Op}_\gamma(a),\mathrm{Op}_\gamma(b)]-\mathrm{Op}_\gamma([a,b])\}_{\gamma\geq\gamma_0}\in \mathrm{OPS}^{m+n-1}_\gamma$.

We shall also make extensive use of the relation

$$[\mathrm{Op}_\gamma(a),\mathrm{Op}_\gamma(b)]\in \mathrm{OPS}^{m+n-1}_\gamma,$$

if the symbols a and b happen to be commutative. Looking ahead to Section 2.5 in this chapter, we state a parameter-dependent Gårding's inequality in its normal and sharp version.

THEOREM 2.2.2 (GÅRDING'S INEQUALITY - THEOREM C.7, [BGS07]). *Let $a\in S^m_\gamma$. Suppose that for some positive constant α*

$$\mathrm{Im}\,\langle a(t,y,\zeta)v,v\rangle\geq \alpha\lambda^m(\zeta)|v|^2,$$

for all $v\in\mathbb{C}^n$ and every $(t,y,\tau,\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}^d\times[1,+\infty]$. Then there there exists $\gamma_0\geq 1$ such that for every $\gamma\geq\gamma_0$ and each $u\in \underline{H}^{m/2}_\gamma$,

$$\mathrm{Im}\,\langle \mathrm{Op}_\gamma(a)u,u\rangle\geq\frac{\alpha}{4}\|u\|^2_{\underline{H}^{m/2}_\gamma}.$$

THEOREM 2.2.3 (SHARP GÅRDING'S INEQUALITY - THEOREM C.8, [BGS07] - CHAPTER VII, [TAY06]). *Let $a\in S^m_\gamma$. Suppose that*

$$\mathrm{Im}\,\langle a(t,y,\zeta,\gamma)v,v\rangle\geq 0,$$

for all $v\in\mathbb{C}^n$ and every $(t,y,\tau,\eta,\gamma)\in\mathbb{R}^d\times\mathbb{R}^d\times[1,+\infty]$. Then there there exist $\gamma_0\geq 1$ and $C>0$ such that for every $\gamma\geq\gamma_0$ and each $u\in \underline{H}^{m/2}_\gamma$,

$$\mathrm{Im}\,\langle \mathrm{Op}_\gamma(a)u,u\rangle\geq -C\|u\|^2_{\underline{H}^{(m-1)/2}_\gamma}.$$

2.2.2 Weighted spaces and conjugated operators

As we shall see in due course, energy estimates for (2.4) are given in weighted-in-time spaces $e^{\gamma t}L^2(\mathbb{R}^{1+d}_+)$, with $\gamma>0$. Intuitively, the presence of a large parameter γ allows us to absorb errors due to lower order terms and focus exclusively on the leading parts of the operators, just as in the constant coefficients case, where only the principal symbols are of interest (see [Maj83a] and [Maj83b]). Specifically, we have:

DEFINITION 2.2.2. *Let Ω be either $\mathbb{R}^{1+d}_+$ or $\partial\mathbb{R}^{1+d}_+\simeq\mathbb{R}^d$ with $d\mu$ being the appropriate measure in each case. If $\gamma\in\mathbb{R}$, the space $L^2_\gamma(\Omega)\equiv e^{\gamma t}L^2(\Omega)$ consists of functions u in Ω such that $e^{-\gamma t}u\in L^2(\Omega)$. Furthermore, $L^2_\gamma(\Omega)$ is a Hilbert space endowed with the inner product*

$$\langle u,v\rangle_\gamma:=\langle e^{-\gamma t}u,e^{-\gamma t}v\rangle=\int_\Omega e^{-2\gamma t}u\bar{v}\,d\mu,$$

and corresponding norm

$$\|u\|_{0,\gamma}^2 = \int_\Omega e^{-2\gamma t}|u|^2\, d\mu.$$

We can define weighted Sobolev spaces $H^s_\gamma(\Omega) \equiv e^{\gamma t}\underline{H}^s_\lambda(\Omega)$ *in the same manner as the set of elements* $e^{-\gamma t}u \in H^s_\lambda(\Omega)$*, endowed with the norm*

$$\|u\|_{s,\gamma}^2 = \int_\Omega e^{-2\gamma t}|\Lambda^s_\gamma u|^2\, d\mu.$$

When $\gamma > 0$, the elements of $L^2_\gamma(\Omega)$ must vanish as $t \to -\infty$ to compensate the exponential factor $e^{-\gamma t}$, so the choice of a positive parameter γ is consistent with an orientation of time (see [Méto4]).

We now describe the operators that we will be using hereafter. Given a family of pseudodifferential operators with parameter $\mathcal{A}(t,x,D_t,D_y,\gamma) \in \mathrm{OPS}^m_\gamma(\mathbb{R}^d \times [\gamma_0,+\infty])$, we define the conjugated operator $\mathcal{A}_\gamma$ by the formula

$$\mathcal{A}_\gamma(t,x,D_t,D_y) = e^{\gamma t}\mathcal{A}(t,x,D_t,D_y,\gamma)e^{-\gamma t}. \tag{2.10}$$

When expressed as an oscillatory integral, $\mathcal{A}_\gamma$ applied to $u \in \mathcal{D}(\mathbb{R}^{1+d}_+)$ reads

$$\begin{aligned}\mathcal{A}_\gamma u(t,x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(t(\tau-i\gamma)+y\eta)}\mathcal{A}(t,y,\tau,\eta,\gamma)\mathcal{F}[e^{-\gamma t}u(t,y)]\,d\tau\,d\eta \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(t\rho+y\eta)}\mathcal{A}(t,y,\rho+i\gamma,\eta,\gamma)(\mathcal{F}u)(\rho,\eta)\,d\rho\,d\eta, \end{aligned} \tag{2.11}$$

where $\rho := \tau - i\gamma$ and $\mathcal{F} \equiv \mathcal{F}_{(t,y)}$ is the Fourier transform with respect to (t,y). In fact, we may deduce from (2.11) that

$$\mathcal{A}_\gamma = \mathcal{A}(t,y,D_t+i\gamma,D_y,\gamma),$$

or simply

$$\mathcal{A}_\gamma = \underline{\mathcal{A}}(t,y,D_t,D_y),$$

if $\underline{\mathcal{A}}(t,y,\rho,\eta)$ is a polynomial in ρ and η. Properties such as (2.9) and the like remain valid for $\mathcal{A}_\gamma$, as can be shown in an elementary fashion for operators $\mathcal{P}_\gamma$ and $\mathcal{Q}_\gamma$ of order 0 and -1, for which it is established that (see [CP82, pp. 413])

$$\|\mathcal{P}_\gamma u\|_{0,\gamma} \le C\|u\|_{0,\gamma} \qquad \text{and} \qquad \|\mathcal{Q}_\gamma u\|_{0,\gamma} \le \frac{C}{\gamma}\|u\|_{0,\gamma},$$

for all $u \in \mathcal{D}(\mathbb{R}^{1+d}_+)$ and γ sufficiently large. To close this section, let us present a class of pseudodifferential operators that captures the idea of the upper half-space as a foliation of horizontal lines. Denoted by $\mathrm{OPS}^m_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0,+\infty))$, it consists of γ-dependent tangential pseudodifferential operators parametrised by x_d, of the form

$$\mathcal{A}_\gamma(t,y,x_d) = \mathcal{A}(t,y,x_d,D_t+i\gamma,D_y,\gamma).$$

For each $x_d \geq 0$, $\mathcal{A}(t,y,x_d,\tau,\eta,\gamma) \in S^m_\gamma(\mathbb{R}^d \times \mathbb{R}^d \times [\gamma_0,+\infty))$ and the estimate (2.7) is satisfied uniformly in x_d. Interestingly, the properties of $\mathrm{OPS}^m_\gamma(\mathbb{R}^d \times \mathbb{R}^d \times [\gamma_0,+\infty))$ transfer to $\mathrm{OPS}^m_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0,+\infty))$ via standard arguments and the next proposition.

Proposition 2.2.1 (Proposition 2.4 - Chapter 7, [CP82]).
Let $\mathcal{A}_\gamma(t,y,x_d) \in \mathrm{OPS}^1_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0,+\infty))$. *If* $u \in L^2(\mathbb{R}_+, H^s_\gamma(\mathbb{R}^d))$ *is such that* $(D_{x_d} + \mathcal{A}_\gamma(t,y,x_d))u \in H^{s-1}(\overline{\mathbb{R}}^{1+d}_+)$, *then* $u \in H^s(\overline{\mathbb{R}}^{1+d}_+)$.

2.3 The scalar case

Let L be an m-th order differential operator

$$L(t,x,D_t,D_x) = D_t^m + \sum_{i=0}^{m-1} A_{m-i}(t,x,D_x)D_t^i, \tag{2.12}$$

whose coefficients are scalar classical pseudodifferential operators $A_{m-j} \in \mathrm{OPS}^{m-j}$ with principal symbols $a_{m-j} \in S^{m-j}$, each a_{m-j} being homogeneous of degree $m-j$ and independent of (t,x) outside some compact set of $\overline{\mathbb{R}}^{1+d}_+$. As in the system case, we are interested in the boundary value problem

$$\begin{cases} L(t,x,D_t,D_x)u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.13}$$

where Bu is defined by p scalar equations $(B_j u)_{j=1,\cdots,p}$ so that

$$B_j u := \sum_{k=1}^{m} B_{j,k}(t,y,D_t,D_y)\gamma_{k-1}u.$$

Here above, $B_{j,k}$ is a differential operator of order $r_j - k + 1$ with smooth coefficients and

$$\gamma_k u := (-i\nu)^k\, u|_{\mathbb{R}^d},$$

where ν is as a fixed vector field transversal to the boundary. To reduce (2.13) to a matrix system, we assume that the coefficient of D_d in (2.12) is non-vanishing (i.e., L is non-characteristic with respect to B) and write L as

$$P = D^m_{x_d} + \sum_{i=0}^{m-1} P_{m-i}(t,x,D_t,D_y)D^i_{x_d},$$

with P_{m-j} being homogeneous of degree $m-j$. If $u_j := \Lambda_\lambda^{m-j} D_{x_d}^{j-1} u$ and $g_j := \Lambda_\gamma^{m-1-r_j} g_j$, let us define $u_\gamma := (u_1, \cdots, u_m)$ and $G_\gamma := (g_1, \cdots, g_d)$. In the same spirit, set $F_\gamma = (0, \cdots, 0, f)$. In the end, we can formulate problem (2.13) as

$$\begin{cases} (D_d + \mathcal{A}_\gamma(t, y, x_d, D_t, D_y))\, u(t,x) = F(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma(t,y))u(t,y,0) = G(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.14}$$

where

$$\mathcal{A}_\gamma := \begin{pmatrix} 0 & \Lambda_\gamma & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_\gamma & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & \Lambda_\gamma \\ A_{1\gamma} & A_{2\gamma} & \cdots & \cdots & A_{m\gamma} \end{pmatrix} \quad \text{and} \quad B_\gamma := (\mathcal{B}_{j,k,\gamma})_{\substack{j=1,\cdots,p \\ k=1,\cdots,m}}. \tag{2.15}$$

The entries in (2.15) are given by $A_{j,\gamma} := P_{m-j+1}\Lambda_\gamma^{-m+j}$ and $\mathcal{B}_{j,k,\gamma} := \Lambda^{m-1-r_j,\gamma} B_{j,k} \Lambda_\gamma^{-m+k}$. If $a(X)$ and $b(X)$ are the principal symbols of A_γ and B_γ, respectively, it is clear that $a(X)$ is homogeneous of degree 1 and that $b(X)$ is homogeneous of degree 0. The notion of hyperbolicity that fits this situation is therefore the one in Definition 2.3.1.

2.3.1 A more general problem

In order to cover both the scalar and the system case, we shall examine a more general problem in the rest of the chapter. To put things in perspective, let us first analyse (2.4) under the Assumption 2.1.1. If $u_\gamma := e^{-\gamma t} u$, a quick calculation yields

$$L(e^{\gamma t} u) = e^{\gamma t}(L - i\gamma)u,$$

and we see that (2.4) is equivalent to

$$\begin{cases} L(t,x,D_t,D_y,\gamma)u_\gamma(t,x) = f_\gamma(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B(t,x,\gamma)u_\gamma(t,y,0) = g_\gamma(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.16}$$

where $f_\gamma = e^{-\gamma t} f$, $g_\gamma = e^{-\gamma t} g$,

$$L(t,x,D_t,D_y,\gamma) := \left(D_t - i\gamma + \sum_{j=1}^{d} A(t,x) D_j \right) \quad \text{and} \quad B(t,x,\gamma) := B(t,x).$$

Evidently, (2.16) can be written in terms of conjugated operators L_γ and B_γ as well, like

$$\begin{cases} L_\gamma(t,x,D_t,D_y)u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma(t,x)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d. \end{cases} \tag{2.17}$$

At this level, we take into account the special role of x_d and recast the differential equation in (2.17) as

$$D_d u + e^{\gamma t} A_d^{-1} \left(D_t - i\gamma + \sum_{j=1}^{d-1} A_j(t,x) D_j \right) e^{-\gamma t} u = A_d^{-1} f(t,x),$$

so we get in the end

$$\begin{cases} P_\gamma u(t,x) := \left(D_d + \mathcal{A}_\gamma(t,y,x_d,D_t,D_y)\right) u(t,x) = A_d^{-1} f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma(t,y) u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.18}$$

with

$$\mathcal{A}_\gamma(t,y,x_d,D_t,D_y) := e^{\gamma t} A_d^{-1} \left(D_t - i\gamma I_n + \sum_{j=1}^{d-1} A_j(t,x) D_j \right) e^{-\gamma t} \tag{2.19}$$

and

$$B_\gamma(t,y,D_t,D_y) := B(t,y).$$

A reasonable generalisation of (2.18) is therefore the boundary value problem

$$\begin{cases} P_\gamma u(t,x) := \left(D_d + \mathcal{A}_\gamma(t,y,x_d,D_t,D_y)\right) u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma(t,y) u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.20}$$

where $\mathcal{A}_\gamma \in \mathrm{OPS}^1_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0, +\infty))$ is a classical pseudodifferential operator whose principal part $a(t,y,x_d,\tau,\eta,\gamma)$ is an $n \times n$ matrix-valued function that is homogeneous of degree 1, $B_\gamma \in \mathrm{OPS}^0_\gamma(\mathbb{R}^d \times [\gamma_0, +\infty))$ is a classical pseudodifferential operator whose principal symbol $b(t,y,\tau,\eta,\gamma)$ is a $p \times n$ matrix-valued function that is homogeneous of degree 0, and the initial data f and g are chosen at least in L^2_γ. It should be noted that by writing D_d alone in (2.20), we are assuming implicitly that the new problem is non-characteristic.

Let us suggest some notation before proceeding further. The frequency set and its projection onto $\{\gamma = 0\}$ are characterised by

$$\Xi := \{\zeta = (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus \{0,0\} : \gamma \geq 0\}, \quad \Xi_0 := \Xi \cap \{\gamma = 0\},$$

whereas the space-time-frequency set and its projection onto $\{\gamma = 0\}$ are given by

$$\mathbb{X} := \{(t,y,x_d,\tau,\eta,\gamma) : (t,y,x_d) \in \mathbb{R}^{1+d}_+, (\tau - i\gamma, \eta) \in \Xi\}, \quad \mathbb{X}_0 := \mathbb{X} \cap \{\gamma = 0\}.$$

As we have seen already, the symbols in this work are classical and thus reducible to homogeneous pieces in $(\tau - i\gamma, \eta)$, so it may be advantageous to concentrate on the sphere

$$S^d := \{(\tau - i\gamma, \eta) \in \Xi : \gamma^2 + \tau^2 + |\eta|^2 = 1\},$$

or when required, on the set

$$\mathbb{X}_S := \{X \in \mathbb{X} : (\tau, \eta, \gamma) \in S^d\}.$$

In the same vein, we define

$$\mathbb{Y} := \{X \in \mathbb{X} : X = (t, y, 0, \eta, \tau, \gamma)\} \qquad \text{and} \qquad \mathbb{Y}_S := \{X \in \mathbb{Y} : (\tau, \eta, \gamma) \in S^d\}.$$

Let $X = (t, y, x_d, \eta, \tau, \gamma)$ be here and everywhere a generic point in $\mathbb{X}$. We shall adopt the following notion of hyperbolicity for Problem 2.20 (see [Méo0] for more details).

Definition 2.3.1. *Let* $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}, \underline{\eta}, \underline{\gamma}) \in \mathbb{X}_S$ *and set* $\pi_P(X, \xi_d) = \det(\xi_d I_n + a(X))$.

(i) *When* $\underline{\gamma} \neq 0$, $\pi_P(\underline{X}, \xi_d) \neq 0$ *for every* $\xi_d \in \mathbb{R}$. *This may be rephrased by saying that* $a(X)$ *has no real eigenvalues when* $\underline{\gamma} \neq 0$.

(ii) *If* $\underline{\xi_d} \in \mathbb{R}$ *is such that* $\pi_P(\underline{X}, \underline{\xi_d}) = 0$, *there exist* $\alpha \in \mathbb{N}$ *together with smooth functions* $\lambda(t, x, \eta, \xi_d)$ *and* $e(X, \xi_d)$ *defined locally around* $(\underline{t}, \underline{x}, \underline{\eta}, \underline{\xi_d})$ *and* $(\underline{X}, \underline{\xi}_d)$, *respectively, such that they are holomorphic in* ξ_d,

$$\pi_P(X, \xi_d) = e(X, \xi_d)(\tau - i\gamma + \lambda(t, x, \eta, \xi_d))^\alpha, \tag{2.21}$$

and $e(X, \xi_d)$ *is nonvanishing at* $(\underline{X}, \underline{\xi_d})$. *Moreover,* λ *is real when* ξ *is real and there is a smooth matrix-valued function* $\Pi(X)$ *on a neighbourhood of* $\underline{X}$, *holomorphic with respect to* ξ_d, *such that* Π *is a projector of rank* α *for which* $\ker(\xi_d I_n + a(X)) = \Pi(X)\mathbb{C}^n$ *when* $\tau - i\gamma + \lambda(t, x, \eta, \xi_d) = 0$.

We shall refer to the concrete situation when $\alpha = 1$ in (2.21) as the *strictly hyperbolic case*.

When $\mathcal{A}_\gamma$ is the differential operator (2.19), Definitions 2.1.1 and 2.3.1 are compatible, meaning that Conditions (i) and (ii) above hold if and only if the operator L is hyperbolic. Indeed, if we assume that $\xi_d = -\mu$ is a purely real eigenvalue of

$$a(X) \equiv a(t, x, \tau, \eta, \gamma) = A_d^{-1}\left((\tau - i\gamma)I_n + \sum_{j=1}^{d-1} \eta_j A_j(t, x)\right),$$

then

$$\begin{aligned} 0 = \pi_P(X, \xi_d) = \det(a(X) - \mu I_n) &= \det\left(A_d^{-1}\right)\det\left(\xi_d A_d + (\tau - i\gamma)I_n + \sum_{j=1}^{d-1} \eta_j A_j(t, x)\right) \\ &= \det\left(\tau - i\gamma + A(t, x, \eta, \xi_d)\right), \end{aligned}$$

which contradicts the hyperbolicity of L unless $\gamma = 0$. This clever observation is known as Hersh's lemma. In addition, if $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}, \underline{\eta}, 0) \in \mathbb{X}_0$ and $\underline{\xi}_d$ verify that $\pi_P(\underline{X}, \underline{\xi}_d) = 0$, then $(\underline{\eta}, \underline{\xi}_d) \neq 0$ because $(\underline{\tau}, \underline{\eta}_d) \neq 0$, and there exists a unique eigenvalue λ_j

so that $\underline{\tau} + \lambda_k(\underline{t}, \underline{x}, \underline{\eta}, \underline{\xi}_d) = 0$ in view of (2.3). Under these circumstances, both λ_j and the eigenprojector Π_j extend to functions of $\xi_d \in \mathbb{C}$ for which Condition (ii) is applicable.

Back to Problem 2.20, we impose the following set of hypotheses.

Assumption 2.3.1.

(i) P_γ *is hyperbolic as in Definition 2.3.1.*

(ii) The symbols $a(X)$ *and* $b(X)$ *are independent of* (t, x) *outside certain compact set* K*. We shall designate this as property* **(C)***.*

(iii) $b(X)$ *is everywhere of maximal rank* p*.*

2.4 Additional results on boundary value problems

2.4.1 Stable and unstable subspaces

As a first step, let us note that Part (i) in Definition 2.3.1 implies that if $\gamma > 0$, $a(X)$ is a hyperbolic matrix in the sense of dynamical systems, i.e., all its eigenvalues have non-zero imaginary part [3] when $\gamma > 0$. In this way, $\mathbb{C}^n$ may be decomposed at each $X \in \mathbb{X}$ as the direct sum of two invariant subspaces of $a(X)$, which we now explain.

Definition 2.4.1. *Let* $X \in \mathbb{X}$ *with* $\gamma > 0$*. The stable (resp. unstable) subspace* $\mathbb{E}^-(X)$ *(resp.* $\mathbb{E}^+(X)$*) of* $a(X)$ *is the subspace generated by the generalised eigenspaces of* $a(X)$ *associated with eigenvalues with negative (resp. positive) imaginary part.*

Observe that $\pi_P(X, \mu) = \det(a(X) - \mu I_n)$ is a polynomial of degree n in μ varying smoothly on X, for which the number of roots with negative imaginary part (counted with their multiplicities) remains locally constant. What is more, the connectedness of $\mathbb{X}$ ensures that this number persists globally and that $\dim \mathbb{E}^-(X)$ is independent of X. In particular, when

$$a(X) = A_d^{-1}\left((\tau - i\gamma)I_n + \sum_{i=1}^{d-1} \eta_i A_i(t, x)\right) \tag{2.22}$$

as in Problem 2.18, one can choose $X_0 = (t, x, 1, 0, \xi_d)$ and conclude from

$$0 = \pi_P(X_0, \xi_d) = \pi_P(t, x, 1, 0, \xi_d) = \det\left(-a(t, x, 1, 0) - \xi_d I_n\right) = \det\left(-A_d^{-1} - \xi_d I_n\right) \tag{2.23}$$

that $\dim \mathbb{E}^-(X)$ equals p, the number of incoming characteristics. For the abstract problem (P_γ, B_γ), which does not necessarily have an analogue (L, B), we add an extra assumption.

Assumption 2.4.1. *The number of required boundary conditions* p *agrees with* $\dim \mathbb{E}^-(X)$*.*

[3] Depending on the convention used, one could say that $a(X)$ is hyperbolic if all its eigenvalues have non-zero real part.

The stable and unstable subspaces may be alternatively written using spectral projectors. For example, if C^- is a Jordan curve (positively oriented) that encloses the eigenvalues of $a(X)$ with negative imaginary part, the expression

$$\Pi^-(X) = \frac{1}{2i\pi}\int_{C^-}(a(X) - \mu I_n)\,d\mu \tag{2.24}$$

defines a projector onto $\mathbb{E}^-(X)$, along $\mathbb{E}^+(X)$, for which

$$\mathbb{E}^-(X) = \{\Pi^-(X) : X \in \mathbb{X} \cap \{\gamma > 0\}\} \quad \text{and} \quad \mathbb{E}^+(X) = \ker \Pi^-(X).$$

Notice that the map $\Pi^-(X)$ is automatically smooth in (t,x), and since we may slightly perturb the argument $X \in \mathbb{X} \cap \{\gamma > 0\}$ in (2.24) while keeping the same contour C^-, it is also holomorphic in $\tau - i\gamma$ and real analytic in η. Last but not least, it is possible to find a basis for $\mathbb{E}^-(X)$ whose elements are homogeneous of degree 0 in (τ, η, γ), as illustrated by Kato in [Kat13]. Needless to say, none of these conclusions has to be true when $\gamma = 0$. These questions, along with the actual behaviour of $\mathbb{E}^-(X)$ when $\gamma \to 0$ are yet to be explored.

2.4.2 The block structure condition

In this section, we study a fundamental idea in the construction of symmetrisers: the *block structure condition*. This notion was originally introduced by Kreiss in [Kre70] for the strictly hyperbolic case, and later adapted by Métivier in [Mé00] to the wider class of constantly hyperbolic operators.

Definition 2.4.2 (Block structure condition). *Let $\underline{X} \in \mathbb{X}$. The matrix $a(X)$ verifies the block structure condition if there exists a neighbourhood $\mathcal{V}$ of $\underline{X}$ in $\mathbb{X}$, an integer $q \geq 1$, a partition $n = \nu_1 + \cdots + \nu_q$ with $\nu_i \geq 1$, and a smooth nonsingular map $\mathfrak{e}_0(X)$ defined on $\mathcal{V}$ such that for every $X \in \mathcal{V}$*

$$\mathfrak{e}_0^{-1}(X)a(X)\mathfrak{e}_0(X) = \operatorname{diag}(a_1(X), \cdots, a_q(X)),$$

with blocks $a_k(X)$ of size $\nu_k \times \nu_k$ that fall exactly into one of the following categories:

(i) The spectrum of $a_k(X)$ is contained in $\mathbb{C} \setminus \mathbb{R}$.

(ii) $\nu_j = 1$, $a_k(X) \in \mathbb{R}$ when $\gamma = 0$, and $i\partial_\gamma a_k(X) \in \mathbb{R} \setminus \{0\}$.

(iii) $\nu_j > 1$, $a_k(X)$ has real coefficients when $\gamma = 0$, and there is $\mu_j \in \mathbb{R}$ such that

$$a_k(\underline{X}) = \begin{pmatrix} \mu_j & 1 & 0 & \cdots & 0 \\ 0 & \mu_j & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \mu_j \end{pmatrix}.$$

Additionally, the entry at the lower left corner of $i\partial_\gamma a_k(X)$ *is nonvanishing and real.*

PROPOSITION 2.4.1. *Let* $\underline{X} \in \mathbb{X}$*. If* $a(X)$ *is hyperbolic as in Definition 2.3.1, then* $a(X)$ *satisfies the block structure condition.*

The proof of this fact is lengthy and highly technical, so we feel free to skip it. If interested, the reader may find useful the discussions in Chapter 7 in [CP82] (for strictly hyperbolic operators), Chapter 5 in [BGS07], and the remarkable paper [Mé00] by Métivier from which we emphasise Lemmas 2.5 and 2.6, as they support the upcoming assertion.

PROPOSITION 2.4.2. *The stable subspace* $\mathbb{E}^-(X)$ *defines a smooth vector bundle over* $\mathbb{X} \cap \{\gamma > 0\}$ *that extends into a continuous vector bundle (again denoted by* $\mathbb{E}^-(X)$*) over* $\mathbb{X}$ *with the same rank.*

We stress two things concerning Proposition 2.4.2:

▷ Although the notation could be misleading, the extension of the stable space to a point $\underline{X} \in \mathbb{X}_0$ should be read as $\mathbb{E}^-(\underline{X}) = \lim_{X \to \underline{X}} \mathbb{E}^-(X)$, for $X \in \mathbb{X} \cap \{\gamma > 0\}$, and in no case as the stable subspace of $a(\underline{X})$, which is usually smaller or even empty.

▷ For non-constantly hyperbolic operators, the continuous extension of $\mathbb{E}^-(X)$ to $\{\gamma = 0\}$ may not exist, as shown in [BGS07], Theorem 8.2.

Based on Definition 2.4.2 and Proposition 2.4.1, $\mathbb{X}_0$ may be divided into four regions as indicated below.

DEFINITION 2.4.3.

▷ *The set of elliptic points* $\mathcal{E}$ *consists of those* $X \in \mathbb{X}_0$ *for which Definition 2.4.2 is satisfied with blocks of type (i) exclusively (complex conjugate pairs).*

▷ *The set of hyperbolic points* $\mathcal{H}$ *consists of those* $X \in \mathbb{X}_0$ *for which Definition 2.4.2 is satisfied with blocks of type (ii) exclusively.*

▷ *The set of mixed points* $\mathcal{EH}$ *consists of those* $X \in \mathbb{X}_0$ *for which Definition 2.4.2 is satisfied with blocks of type (i) and (ii), but no blocks of type (iii).*

▷ *The set of glancing points* $\mathcal{G}$ *consists of those* $X \in \mathbb{X}_0$ *for which Definition 2.4.2 is satisfied with at least one block of type (iii).*

The hyperbolic region will be discussed in more detail in Chapter 3, as it is there where the $\mathcal{WR}$ problem occurs.

2.4.3 The weak Lopatinskiĭ condition

To derive a necessary condition for well-posedness, we focus on a concrete, elementary example. Let us assume for simplicity that P_γ and B_γ have constant coefficients, P_γ is strictly hyperbolic and $f = 0$, so that Problem 2.20 becomes

$$\begin{cases} \left(D_d + \mathcal{A}_\gamma(D_t, D_y)\right) u = 0 & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma u|_{x_d=0} = g & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.25}$$

for some $g \in L^2_\gamma(\mathbb{R}^d)$. Putting (2.25) in the form (2.16) and applying Fourier transform to the tangential variables (t,y) produces

$$\begin{cases} (D_d + a(\zeta))\hat{u}(x_d,\zeta) = 0, \\ B\hat{u}(0,\zeta) = \hat{g}(\zeta). \end{cases} \tag{2.26}$$

We know from the standard theory of differential equations that

$$\hat{u}(x_d,\zeta) = e^{-ix_d a(\zeta)}\hat{u}(0,\zeta) \tag{2.27}$$

is the unique solution of (2.26) provided that $B\hat{u}(0,\zeta) = \hat{g}(\zeta)$. The spectrum of $a(\zeta)$ brings valuable information about $\hat{u}(x_d,\zeta)$ because the exponential function is well-behaved with respect to matrix conjugation. That said, if $\mu_1, \cdots, \mu_q$ are pairwise different eigenvalues of $a(\zeta)$ with corresponding multiplicities $\beta_1, \cdots, \beta_q$, the whole space $\mathbb{C}^n$ may be decomposed into a sum of generalised eigenspaces as

$$\mathbb{C}^n = \bigoplus_{k=1}^{q} \ker\left(a(\zeta) - \mu_k\right)^{\beta_k}. \tag{2.28}$$

Moreover, if w_k represents the components of an arbitrary element $w \in \mathbb{C}^n$ in the basis prescribed by (2.28), it may be deduced that

$$\hat{u}(x_d,\zeta) = \sum_{k=1}^{q} e^{-i\mu_k x_d} \sum_{j=0}^{\beta_k-1} \frac{x_d^j}{j!}(a(\zeta) - \mu_k I_n)^j w_k \tag{2.29}$$

after a simple computation (see [CP82, pp. 422]). Recall that the hyperbolicity of P_γ guarantees that $\mu_k \in \mathbb{C} \setminus \mathbb{R}$ whenever $\gamma > 0$ and therefore that $u(x_d)$ is bounded for x_d large if and only if $w_j = 0$ for every $\operatorname{Im}\mu_j > 0$. This leads us to the next reasoning. We claim that (2.26) has a unique bounded solution for every $\hat{g}(\zeta)$ if and only if $\mathbb{E}^-(\zeta) \cap \ker B = \{0\}$. As a matter of fact, suppose that there is a unique bounded solution $\hat{u}(x_d,\zeta)$ of (2.26) for every $\hat{g}(\zeta)$. According to (2.29), $\hat{u}(0,\zeta)$ must be an element of $\mathbb{E}^-(\zeta)$ (otherwise it would be unbounded) and $B\hat{u}(0,\zeta) = \hat{g}(\zeta)$ must be solvable at an algebraic level. Together, this means that B restricted to $\mathbb{E}^-(\zeta)$ is an isomorphism. To prove the converse, let us suppose that B restricted to $\mathbb{E}^-(\zeta)$ is an isomorphism. Then, for any $\hat{g}(\zeta)$,

$$\hat{u}(0,\zeta) = (B|_{\mathbb{E}^-(\zeta)})^{-1} g(\zeta) \in \mathbb{E}^-(\zeta),$$

and consequently $\hat{u}(x_d)$ is bounded in light of (2.29). The preceding analysis motivates the following definition.

DEFINITION 2.4.4. *A boundary value problem (P_γ, B_γ) satisfies the weak Lopatinskiĭ condition if for every $X \in \mathbb{Y}_S \cap \{\gamma > 0\}$,*

$$\mathbb{E}^-(X) \cap \ker b(X) = \{0\}.$$

Definition 2.4.4, when satisfied, implies *normality* in Assumption 2.1.1, as can be readily confirmed by a similar calculation to the one outlined in equation (2.23). The validity of the Lopatinskiĭ condition comprises more general spaces, as the following proposition reveals.

PROPOSITION 2.4.3 (PROPOSITION 4.2, [BGS07] - CHAPTER 14 [SER99]). *The weak Lopatinskiĭ condition is necessary for the boundary value problem (P_γ, B_γ) to be well-posed in Hölder and Sobolev spaces. When it fails, there is no chance of having estimates in these norms, even at the price of a loss of regularity.*

2.4.4 The uniform Lopatinskiĭ condition

Let $\mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$ be the set of vector-valued test functions on $\overline{\mathbb{R}}^{1+d}_+$. For reasons that will become apparent later, we shall say that a boundary value problem (P, B) is *strongly well-posed* in L^2 if, for some $\gamma_0 > 1$, there exists a positive constant $C > 0$ such that

$$\gamma \int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|u(t,x)|^2 \, dt \, dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|u(t,y,0)|^2 \, dt \, dy \le C \left(\frac{1}{\gamma} \int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|Pu(t,x)|^2 \, dt \, dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|Bu(t,y,0)|^2 \, dt \, dy \right), \tag{2.30}$$

for each $\gamma \ge \gamma_0$ and every $u \in \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$.

The word "strong" (or "strongly") refers to u and the source data (f, g) being estimated in the same norms. To make sure this happens, a stricter version of the Lopatinskiĭ condition is needed.

DEFINITION 2.4.5. *A boundary value problem (P_γ, B_γ) satisfies the uniform Lopatinskiĭ condition if for every $X \in \mathbb{Y}_S$,*

$$\mathbb{E}^-(X) \cap \ker b(X) = \{0\}, \tag{2.31}$$

that is to say, equation (2.31) *holds up to the frequency boundary* [4] $\{\gamma = 0\}$.

In practice, we do not rely on Definition 2.4.4 (resp. Definition 2.4.5) to check whether a given (P_γ, B_γ) verifies the weak Lopatinskiĭ condition (resp. uniform Lopatinskiĭ condition). Instead, we utilise the so-called Lopatinskiĭ determinant $\underline{\Delta}$, a special map possessing the following features:

[4] This extension relies on Proposition 2.4.2.

- for $\gamma > 0$, $X \mapsto \underline{\Delta}(X)$ is smooth with respect to (t, x), holomorphic in $\tau - i\gamma$ and real analytic in η,
- $\underline{\Delta}$ is homogeneous of degree 0,
- $\underline{\Delta}(X)$ vanishes only at points $X \in \mathbb{X}$ where the Lopatinskiĭ condition is violated.

To construct such a function, we take the homogeneous basis $\{\mathbf{e}_1(X), \cdots, \mathbf{e}_p(X)\}$ briefly described after Assumption 2.4.1 and put

$$\underline{\Delta}(X) := \det\left(b(X)\mathbf{e}_1(X), \cdots b(X)\mathbf{e}_p(X)\right). \tag{2.32}$$

Clearly, it makes sense to evaluate the Lopatinskiĭ determinant wherever the subspace $\mathbb{E}^-(X)$ is defined, and in such case $\underline{\Delta}(X)$ inherits all the properties of $\mathbb{E}^-(X)$.

2.5 A priori estimates

Energy estimates assuming the uniform Lopatinskiĭ condition were first derived by Kreiss in the system case (see [Kre70]) and by Sakamoto in the (higher-order) scalar case (see [Sak70]). As anticipated at the beginning of Section 2.4.2, the strategy involves constructing a microlocal symmetriser in the first place, and then using the theory of pseudodifferential operators to find the desired energy inequalities. Let us start with the most intricate task.

2.5.1 Construction of a microlocal symmetriser

This section is entirely devoted to the construction of a microlocal symmetriser, assuming that the uniform Lopatinskiĭ condition is fulfilled. In developing the proofs, we closely follow Chapter 7 in [CP82].

Theorem 2.5.1. *Let (P_γ, B_γ) be as in Problem 2.20, subject to Assumptions 2.3.1 and 2.4.1. Suppose that the uniform Lopatinskiĭ condition is satisfied. If $\underline{X} \in \mathbb{X}$, there exist two smooth matrix-valued symbols r and $\mathfrak{e}_0$ of dimensions $n \times n$, both defined in a conic neighbourhood $\mathcal{V}$ of $\underline{X}$ and homogeneous of order 0, such that*

(i) $\mathfrak{e}_0(X)$ is nonsingular,

(ii) there is $c > 0$, independent of $X \in \mathcal{V}$, so that if $\dot{a}(X) := \mathfrak{e}_0^{-1}(X)a(X)\mathfrak{e}_0(X)$,

$$\operatorname{Im}\left(r(X)\dot{a}(X)\right) \geq \gamma c I_n,$$

and

(iii) when $\underline{X} \in \mathbb{Y}$, there exist positive constants α and β, so that

$$r(X) + \beta \dot{b}(X)^* \dot{b}(X) \geq \alpha I_n,$$

where $\dot{b}(X) := b(X)\mathfrak{e}_0(X)$.

Proof. Although we shall only use in the future the ideas behind the construction of a symmetriser when $\underline{X} \in \mathcal{H}$, we shall include the other regions for the sake of completeness. In any case, we stick to the following convention:

- C and c are positive constants that may vary from line to line throughout the proof.
- Thanks to homogeneity, we just need to pay attention to points on $\mathbb{X}_s$.
- Given any $X = (t, y, x_d, \tau, \eta, \gamma) \in \mathbb{X}_s$, the projection of X onto the frequency boundary $\{\gamma = 0\}$ is $\tilde{X} := (t, y, x_d, \tau, \eta, 0) \in \mathbb{X}_0$.
- To avoid breaking the thread of the proof, we shall postpone the proofs of some of the lemmas used to Section A.1 in the appendix.

Elliptic region. Let $\underline{X} \in \mathcal{E}$. According to Proposition 2.4.1 and the classification proposed in Definition 2.4.3, there exists a basis $\mathfrak{e}_0(X)$ depending smoothly on X in which the matrix $a(X)$ has locally the diagonal form

$$\dot{a}(X) = \mathfrak{e}_0^{-1}(X)a(X)\mathfrak{e}_0(X) = \begin{pmatrix} a^-(X) & 0 \\ 0 & a^+(X) \end{pmatrix}. \tag{2.33}$$

Here above, $a^-(X)$ (resp. $a^+(X)$) is a diagonal matrix of size p (resp. $n-p$) containing the eigenvalues with negative (resp. positive) imaginary part. Bearing in mind Condition (ii), we introduce a lemma that allow us to find a lower bound for the imaginary part of $a^{\mp}(X)$.

Lemma 2.5.1. *Let M be a square matrix of dimensions $n \times n$. If M has eigenvalues with strictly positive imaginary part, then one may find a positive definite matrix H such that* $\operatorname{Im}(HM) > 0$ *in the sense of matrices.*

Without further delay, let us apply Lemma 2.5.1 to $a^{\mp}(X)$ to get

$$\mp \operatorname{Im}(a^{\mp}(X)) \geq cI,$$

for some positive constant c. Let $\rho > 1$ to be fixed large enough later on. If $r(X)$ is taken as

$$r(X) = \begin{pmatrix} -I_p & 0 \\ 0 & \rho I_{n-p} \end{pmatrix}, \tag{2.34}$$

then

$$\operatorname{Im}(r(X)a(X)) = \begin{pmatrix} -\operatorname{Im}(a^-(X)) & 0 \\ 0 & \rho \operatorname{Im}(a^+(X)) \end{pmatrix} \geq \begin{pmatrix} cI_p & 0 \\ 0 & \rho c I_{n-p} \end{pmatrix} \geq c\gamma I,$$

considering that $0 \leq \gamma \leq 1$. This completes Condition (ii). In order to prove Condition (iii), let $\underline{X} \in \mathbb{Y}_S$ and denote by $b^-(X)$ (resp. $b^+(X)$) the restriction of $b(X)$ to $\mathbb{E}^-(X)$ (resp. $\mathbb{E}^-(X)$). Likewise, let us represent by v^- (resp. v^+) the projection onto $\mathbb{E}^-(X)$ (resp. $\mathbb{E}^+(X)$) of $v \in \mathbb{C}^n$ relative to the basis $\epsilon_0(X)$, so that

$$v = \begin{pmatrix} v^- \\ v^+ \end{pmatrix}.$$

For the rest of the argument, it suffices to focus on $\underline{X}$ by continuity. Note that if we solve $\dot{b}(\underline{X}) = \dot{b}^-(\underline{X})v^- + \dot{b}^+(\underline{X})v^+$ for $\dot{b}^-(\underline{X})v^-$ initially, and then use that $\dot{b}^-(\underline{X})$ restricted to $\mathbb{E}^-(\underline{X})$ is an isomorphism, there must be a constant C such that, for all $v \in \mathbb{C}$,

$$|v^-|^2 \leq C(|v^+|^2 + |\dot{b}(\underline{X})v|^2).$$

Thus,

$$\begin{aligned} \langle r(X)v, v\rangle &= |v^-|^2 + \rho|v^+|^2 - 2|v^-|^2 && (2.35) \\ &\geq |v^-|^2 + \rho|v^+|^2 - 2C|v^+|^2 - 2C|\dot{b}(\underline{X})v|^2 && (2.36) \\ &= |v^-|^2 + (\rho - 2C)|v^+|^2 - 2C|\dot{b}(\underline{X})v|^2. \end{aligned}$$

Finally, choosing ρ so that $\rho - 2C > 0$ gives the required inequality.

Hyperbolic region. Having placed ourselves at $\underline{X} \in \mathcal{H}$, Proposition 2.4.2 indicates that $a(X)$ is smoothly diagonalisable in a neighbourhood $\mathcal{V}$ of $\underline{X}$ with eigenvalues $a_1(X), \cdots, a_n(X)$ (counted according to their multiplicities) and eigenvectors $e_1(X), \cdots, e_n(X)$ ordered as the columns of a nonsingular matrix $\epsilon_0(X)$, so that

$$\dot{a}_1(X) = \epsilon_0^{-1}(X)a_1(X)\epsilon_0(X) = \begin{pmatrix} a_{1,1}(X) & 0 & \cdots & 0 \\ 0 & a_{1,2}(X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{1,n}(X) \end{pmatrix}. \tag{2.37}$$

Let

$$r(X) = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix} \tag{2.38}$$

with every $r_j \in \mathbb{R}$, $1 \leq j \leq n$, to be chosen later. If $\kappa_j(\widetilde{X}) := -i\partial a_j(\widetilde{X})/\partial\gamma$, Taylor's theorem with respect to γ yields

$$a_j(X) = a_j(\widetilde{X}) + i\gamma\kappa_j(\widetilde{X}) + \gamma^2 w_j(X), \tag{2.39}$$

where $w_j(X)$ is a smooth function. Now, given that $a(X)$ and $r(X)$ are diagonal, proving

Part (iii) amounts to analyzing

$$\operatorname{Im} r_j a_j(X) = \operatorname{Im}(r_j a_j(\widetilde{X}) + i\gamma r_j \kappa_j(\widetilde{X}) + r_j \gamma^2 w_j(X)). \tag{2.40}$$

The block structure condition implies that $a_j(\tilde{X}) \in \mathbb{R}$, so we may discard $r_j a_j(\tilde{X})$ in (2.40) and write

$$\operatorname{Im} r_i a_j(X) = \gamma r_i \operatorname{Re} \kappa_j(\widetilde{X}) + \gamma^2 r_i \operatorname{Im} w_j(X). \tag{2.41}$$

Notice that (2.41) is bounded away from zero because of the continuity of $\kappa_j(\widetilde{X})$ and the fact that $\underline{\kappa}_j := \kappa(\underline{X}) \in \mathbb{R} \setminus \{0\}$ (also ensured by Proposition 2.4.2). We can say a little bit more about $\underline{\kappa}_j$, actually. By definition,

$$\underline{\kappa}_j = \operatorname{Re}\left(-i\frac{\partial a_j}{\partial \gamma}(\underline{X})\right) = \operatorname{Im}\left(\frac{\partial a_j}{\partial \gamma}(\underline{X})\right) = \operatorname{Im}\left(\lim_{\gamma \to 0^+} \frac{a_j(X) - a_j(\underline{X})}{\gamma}\right) = \lim_{\gamma \to 0^+} \frac{\operatorname{Im} a_j(X)}{\gamma},$$

showing that $\underline{\kappa}_j$ and $\operatorname{Im} a(X)$ have the same sign for $X = (t, y, x_d, \tau, \eta, \gamma)$ sufficiently close to $\underline{X}$. Hence,

$$\mathbb{E}^-(X) = \bigoplus_{\operatorname{Im} a_j(X) < 0} \ker\left(a(X) - a_j(X)\right) = \{v \in \mathbb{C}^n : v_j = 0 \quad \text{if} \quad \underline{\kappa}_j > 0\},$$

and it is straightforward to check that Condition (ii) is satisfied when

$$r_j = \begin{cases} -1 & \text{for} \quad \underline{\kappa}_j > 0, \\ \rho & \text{for} \quad \underline{\kappa}_j < 0, \end{cases} \tag{2.42}$$

with ρ being a positive constant to be specified when meeting Condition (iii). The remaining portion of the proof follows exactly the same philosophy as its elliptic counterpart, so we feel free to omit it.

Mixed region. It is a combination of the two previous cases.

Glancing region. If $\underline{X} \in \mathcal{G}$, Proposition 2.4.2 and Definition 2.4.3 imply that $a(X)$ can be written locally around $\underline{X}$ in the block-diagonal form

$$a(X) = \begin{pmatrix} a_1(X) & & & & \\ & \ddots & & & \\ & & a_m(X) & & \\ & & & a^-(X) & \\ & & & & a^+(X) \end{pmatrix}, \tag{2.43}$$

where at least one block, say $a_j(X)$, is such that when evaluated at $\underline{X}$, it becomes the

Jordan block

$$a_j(\underline{X}) = \begin{pmatrix} \mu_j & 1 & 0 & \cdots & 0 \\ 0 & \mu_j & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \mu_j \end{pmatrix}.$$

We seek to decompose $a_j(X)$ into simpler pieces that are easier to analyse. For this purpose, we recast Taylor's expansion with respect to γ conveniently as

$$a_j(X) = a_j(\tilde{X}) + i\gamma\kappa_j(\tilde{X}) + \gamma^2 w_j(X) = (\mu_j I + \mathfrak{a}_j(\tilde{X}) + \mathfrak{b}_j(\tilde{X})) + i\gamma\kappa_j(\tilde{X}) + \gamma^2 w_j(X), \tag{2.44}$$

with

$$\mathfrak{a}_j(\tilde{X}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}, \tag{2.45}$$

$\mathfrak{b}_j(\tilde{X}) := a_j(\tilde{X}) - a_j(\underline{X})$, and $\kappa_j(\tilde{X}) := -i\partial_\gamma a_j(\tilde{X})$. A quick inspection confirms that $\mathfrak{b}_j, \kappa_j, w_j$ are smooth, $\mathfrak{b}_j(\underline{X}) = 0$ and the bottom left element α_j of $\kappa_j(\underline{X})$ is real and nonvanishing. Yet, nothing suggests that $\mathfrak{b}_j(\tilde{X})$ has real entries. That this is a mild assumption is the content of the following lemma.

Lemma 2.5.2. *The basis $\mathfrak{e}_0(X)$ in Definition 2.4.2 may be chosen so that*

$$\mathfrak{b}_j(\tilde{X}) = \begin{pmatrix} b_1(\tilde{X}) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ b_{\nu_j}(\tilde{X}) & 0 & \cdots & 0 \end{pmatrix}, \tag{2.46}$$

with real entries $b_1(\tilde{X}), \cdots, b_{\nu_j}(\tilde{X})$.

With this simplification at hand, we postulate the block-diagonal matrix

$$r(X) = \begin{pmatrix} r_1(X) & & & & \\ & \ddots & & & \\ & & r_m(X) & & \\ & & & -I & \\ & & & & \rho I \end{pmatrix} \tag{2.47}$$

as a candidate for a symmetriser, where each $r_j(X)$ is hermitian and $\mathrm{diag}(-I, \rho I)$

behaves like (2.34). It is clear that only r_j is of interest at the moment, as the others are covered by the derivations made in previous situations. We opt for

$$r_j(X) = E_j(X) + F_j(\tilde{X}) + i\gamma G_j(X),$$

where both E_j and F_j are real symmetric matrices, F_j is such that $F_j(\underline{X}) = 0$, and G_j is a real, skew-symmetric matrix. When combined with a_j, the resulting $r_j(X)$ produces

$$\mathrm{Im}(r_j a_j) = \mathrm{Im}((E_j + F_j)(\alpha_j + \beta_j)) + \gamma\,\mathrm{Re}(E_j\kappa_j + G_j\alpha_j) + \gamma W(X),$$

for some smooth $W(X)$ such that $O(\gamma) + O(|X - \underline{X}|)$. At this stage, some restrictions must be imposed on $(E_j + F_j)(\alpha_j + \beta_j)$ and $E_j\kappa_j + G_j\alpha_j$ to make sure Condition (ii) is met. To do so, let us select E_j and F_j so that

$$\mathcal{P}_j := (E_j + F_j)(\alpha_j + \beta_j) = E_j\alpha_j + E_j\beta_j + F_J\alpha_j + F_j\beta_j$$

is symmetric on one hand (thus implying that $\mathrm{Im}(\mathcal{P}_j) = 0$) , and on the other that

$$\mathcal{Q}_j := \mathrm{Re}(E_j\kappa_j + G_j\alpha_j)$$

is positive definite. In an attempt to understand the precise structure of the matrices involved, let us evaluate the whole expression at $\underline{X}$ where we may exploit the identities $\beta(\underline{X}) = 0$ and $F(\underline{X}) = 0$. The conclusion is summarised in the statement below.

Lemma 2.5.3. *Let α_j be as in equation* (2.45) *and suppose that E_j is symmetric. If $E_j\alpha_j$ is symmetric, then E_j must be triangular of the form*

$$E_j(X) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & e_1 \\ \vdots & & & \iddots & e_2 \\ \vdots & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ e_1 & e_2 & \cdots & \cdots & e_{m_j} \end{pmatrix}. \tag{2.48}$$

Back to $\mathcal{P}_j$, we claim that it is possible to find a $(\nu_j - 1) \times (\nu_j - 1)$ symmetric sub-matrix $\Phi_j(X)$ such that

$$F_j(\tilde{X}) = \begin{pmatrix} \Phi_j(\tilde{X}) & 0 \\ 0 & 0 \end{pmatrix},$$

$F(\underline{X}) = 0$, and $E_j\beta_j + F_J\alpha_j + F_j\beta_j$ is symmetric. Indeed, by plugging the ansatz and computing the products, we arrive at a linear system of equations of dimension $(\nu_j - 1) \times (\nu_j - 1)$ whose coefficients are smooth in $\tilde{X}$ and such that $F(\underline{X}) = 0$ when $\tilde{X}$ is close to $\underline{X}$, as desired.

Having settled the first problem completely, we address the second question. To control $\mathcal{Q}_j = \mathrm{Re}(E_j\kappa_j + G_j a_j)$ from below, let us note that due to the special shape of the matrix $E_j \equiv E_j(X)$, there exists a constant C depending exclusively on the coefficients of E_j so that for every $w \in C^{\nu_j}$,

$$\mathrm{Re}\,(E_j\kappa_j(\underline{X})w, w) \geq e_1\alpha_j|w_1|^2 - (|w_1|^2 + C|w'|^2),$$

where $w' := (w_2, \cdots, w_{\nu_j})^t$. Moreover,

$$\mathrm{Re}\,(E_j\kappa_j(X)w, w) \geq e_1\alpha_j|w_1|^2 - (|w_1|^2 + C|w'|^2) \tag{2.49}$$

by continuity. As such, we are left with the single term $\mathrm{Re}(G_jA_j)$ for which the next lemma applies.

Lemma 2.5.4. *Let C be an arbitrary constant. IF a_j is defined as in equation* (2.45)*, there exists a real, skew-symmetric matrix G_j such that*

$$\mathrm{Re}(G_j a_j w, w) \geq -|w_1|^2 + C|w'|^2, \tag{2.50}$$

for every $w \in \mathbb{C}^{\nu_j}$. As before, $w' := (w_2, \cdots, w_{\nu_j})^t$.

In essence, if we merge (2.49) and (2.50) and pick $e_1\alpha_j$ sufficiently large (for instance, $e_1\alpha_j \geq 3$), we see that Condition (ii) is fulfilled. We still have to realise Condition (ii), which will be enough to check at $\underline{X} \in \mathbb{Y}_0 := \mathbb{Y} \cap \mathbb{X}_0$ thanks to continuity. That being so, using that $F_j(\underline{X}) = 0$, we get

$$r_j(\underline{X}) = E_j(\underline{X}),$$

and consequently,

$$r(\underline{X}) = \begin{pmatrix} E_1(\underline{X}) & & & & \\ & \ddots & & & \\ & & E_m(\underline{X}) & & \\ & & & I & \\ & & & & -\rho I \end{pmatrix}. \tag{2.51}$$

Finally, if we take into account the structure of $\mathbb{E}^-(X)$ at $\{\gamma = 0\}$ (see Proposition 2.4.2) and represent by v^- (resp. v^+) the projection onto $\mathbb{E}^-(X)$ (resp. $\mathbb{E}^+(X)$), the remaining part of the proof proceeds almost identically as in the elliptic case, provided we admit the following lemma.

Lemma 2.5.5 (p. 171, [Méto4]). *Let $\rho > 0$. There exist matrices $E_1, \cdots, E_m$ such that*

(i) each E_j is defined as in equation (2.5.3)*,*

(ii) the constraint $e_1\alpha_j \geq 3$ holds true,

(iii) $r(\underline{X})$ *as in* (2.51) *verifies the estimate*

$$-\langle r(\underline{X})w, w\rangle \geq C(-|w^+|^2 + \rho|w^-|^2),$$

for some positive constant C *and for all* $w \in \mathbb{C}^n$.

□

2.5.2 Energy inequalities

In this section, we shall explore the role of the microlocal symmetriser $r(X)$ in connection with energy estimates. To achieve this, we shall distinguish two Sobolev norms, namely, $|\cdot|_{0,\gamma}$ for $L^2_\gamma(\mathbb{R}^d)$ and $\|\cdot\|_{0,\gamma}$ for $L^2_\gamma(\mathbb{R}^{1+d}_+)$.

Theorem 2.5.2. *Let* $r(X)$ *be a microlocal symmetriser as indicated in Theorem 2.5.1. Then there exist positive constants* C *and* $\gamma_0 \geq 1$ *such that for every* $\gamma \geq \gamma_0$ *and all* $u \in \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$,

$$\gamma\|u\|^2_{0,\gamma} + |u(0)|^2_{0,\gamma} \leq C\left(\frac{1}{\gamma}\|P_\gamma u\|^2_{0,\gamma} + |B_\gamma u(0)|^2_{0,\gamma}\right), \tag{2.52}$$

or more generally,

$$\gamma\|u\|^2_{s,\gamma} + |u(0)|^2_{s,\gamma} \leq C\left(\frac{1}{\gamma}\|P_\gamma u\|^2_{s,\gamma} + |B_\gamma u(0)|^2_{s,\gamma}\right).$$

Proof. Due to Property **(C)** and homogeneity of the symbols under consideration, we may work in principle on a compact set $K \times S^d \subset \mathbb{X}$, and then extend everything beyond $K \times S^d$ by means of a standard argument (see [BGS07, pp.231]). Having said that, let us cover $K \times S^d$ with finitely many neighbourhoods $\{\mathcal{V}_i\}_{i\in I}$ as stated in Theorem 2.5.1. Subordinated to $\{\mathcal{V}_i\}_{i\in I}$, there is a partition of unity $\{\varphi_i\}_{i\in I}$, together with functions $\{\theta_i\}_{i\in I}$ such that $\theta_i \equiv 1$ on supp φ_i. If we denote by the same letters the extensions of φ_i and θ_i to functions that are homogeneous of degree 0 in (τ, η, γ), we may associate pseudodifferential operators $\Phi_{i,\gamma} := \mathrm{Op}_\gamma(\varphi_i)$ and $\Theta_{i,\gamma} := \mathrm{Op}_\gamma(\theta_i)$ that we can use to localise u and the operators involved. In fact, if we set on one hand[5]

$$\dot{\mathcal{A}}_\gamma := \mathrm{Op}_\gamma(\theta e_0^{-1})\mathcal{A}_\gamma \,\mathrm{Op}_\gamma(\theta e_0), \quad \dot{B}_\gamma := B_\gamma \,\mathrm{Op}_\gamma(\theta e_0), \quad \dot{u} := \mathrm{Op}_\gamma(\theta e_0^{-1})u,$$

and on the other $f_i := \Phi_{i,\gamma} f$, $g_i := \Phi_{i,\gamma} g$, we shall first prove the target inequality (2.52) for $u_i := \Phi_i \dot{u}$ such that

$$\begin{cases} \dot{P}u_i := \left(D_d + \dot{\mathcal{A}}_\gamma\right) u_i(t,x) = f_i(t,x), \\ \qquad\qquad\qquad \dot{B}_\gamma u_i(t,y,0) = g_i(t,y), \end{cases} \tag{2.53}$$

and then see that the general result follows from controlling the commutators $[P_\gamma, \Phi_\gamma]$

[5] e_0 is understood as in Theorem 2.5.1.

and $[B_\gamma, \Phi_\gamma]$. In order to check that u_i satisfies (2.52), we introduce the notion of a functional symmetriser R_γ.

Definition 2.5.1. *A (local) functional symmetriser for* (2.53) *is a family R_γ of C^1 operator-valued maps parameterised by x_d so that, for $\gamma \geq \gamma_0 \geq 1$,*

i) $R_\gamma(x_d)$ and $\partial_d R_\gamma(x_d)$ are L^2-bounded operators with uniform bounds in x_d and γ.

ii) $R_\gamma(x_d)$ is self-adjoint.

iii) There is a positive constant c, independent of x_d and γ, such that

$$\operatorname{Im} \langle R_\gamma(x_d)\dot{\mathcal{A}}_\gamma(x_d)v, v\rangle \geq c\gamma |v|^2_{0,\gamma}$$

for every $v \in L^2(\mathbb{R}^{1+d}_+)$.

iv) There exist positive constants α and β so that

$$\langle R_\gamma(0)v, v\rangle \geq \alpha |v|^2_{0,\gamma} - \beta |\dot{B}_\gamma v|^2_{0,\gamma}$$

is valid for each $v \in L^2(\mathbb{R}^d)$.

We claim that $R_\gamma = \mathrm{Op}_\gamma(r)$ fulfils Definition 2.5.1. The first two properties follow directly from the definition of the symbol $r(X)$, whereas Condition (iii) is obtained by applying the sharp Gårding inequality to the first-order symbol $r(X)\dot{a}(X) - c\gamma I_n$. Analogously, inequality

$$r(X) + \beta \dot{b}(X)^* \dot{b}(X) \geq \alpha I_n$$

along with Gårding's inequality implies Condition (iv). It then remains to confirm that the existence of R_γ gives the promised energy estimates. In doing so, we shall drop the parameter γ and the variable x_d to keep the calculations as legible as possible. As is customary, we start with

$$\begin{aligned}\frac{d}{dx_d}\langle Ru_i, u_i\rangle &= \langle \partial_d R u_i, u_i\rangle + \langle R\partial_d u_i, u_i\rangle + \langle Ru_i, \partial_d u_i\rangle \\ &= \langle \partial_d R u_i, u_i\rangle + 2\operatorname{Re}\langle R\partial_d u_i, u_i\rangle \\ &= \langle \partial_d R u_i, u_i\rangle + 2\operatorname{Re}\langle R(f_i - i\dot{\mathcal{A}}u), u_i\rangle \\ &= \langle \partial_d R u_i, u_i\rangle + 2\operatorname{Im}\langle R\dot{\mathcal{A}}_\gamma u_i, u_i\rangle + 2\operatorname{Re}\langle Rf_i, u_i\rangle,\end{aligned}$$

where in the third line we have used the differential equation in (2.53). Integrating with respect to x_d over $[0, +\infty)$ and multiplying by -1 leads to

$$\begin{aligned}\langle Ru_i, u_i\rangle|_{x_d=0} = -\int_0^\infty \langle \partial_d R u_i, u_i\rangle\, dx - \int_0^\infty \left(2\operatorname{Im}\langle R\dot{\mathcal{A}}_\gamma u_i, u_i\rangle\right) dx_d \\ - 2\int_0^\infty \operatorname{Re}\langle Rf_i, u_i\rangle\, dx_d,\end{aligned} \tag{2.54}$$

an expression that we can bound from above via Condition (i) in Definition 2.5.1 to get

$$\langle Ru_i, u_i\rangle|_{x_d=0} \leq -c\gamma \int_0^\infty |u_i|^2\, dx_d + C_1 \int_0^\infty |f_i||u_i|\, dx_d.$$

From Condition (iv) and Young's inequality we have

$$\alpha|u_i(0)|^2 - \beta|\dot{B}u_i(0)|^2 \leq (-c\gamma + \varepsilon\gamma) \int_0^\infty |u_i|\, dx_d + \frac{C_1}{4\varepsilon\gamma} \int_0^\infty |f_i|^2\, dx_d,$$

or by shrinking $\varepsilon > 0$,

$$\alpha|u_i(0)|^2 - \beta|\dot{B}u_i(0)|^2 \leq -C_2\gamma \int_0^\infty |u_i|^2\, dx_d + \frac{C_3}{\gamma} \int_0^\infty |f_i|^2\, dx_d.$$

Ultimately, after rearranging terms, we find that

$$\begin{aligned} \gamma \int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|u_i|^2\, dt\, dx &+ \int_{\mathbb{R}^d} e^{-2\gamma t}|u_i(0)|^2\, dt\, dy \\ &\leq C\left(\frac{1}{\gamma}\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|f_i|^2\, dt\, dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|\dot{B}u_i(0)|^2\, dt\, dy\right). \end{aligned} \tag{2.55}$$

It is time to generalise (2.55) by showing that it is possible to absorb the contribution of $[\dot{P}_\gamma, \Phi_\gamma]$ and $[\dot{B}_\gamma, \Phi_\gamma]$. More precisely,

$$\begin{aligned} \gamma\|u_i\|^2 + |u_i(0)|^2 &\lesssim \left(\frac{1}{\gamma}\|\dot{P}u_i\|^2 + |\dot{B}u_i(0)|^2\right) = C\left(\frac{1}{\gamma}\|\dot{P}\Phi_i\dot{u}\|^2 + |\dot{B}\Phi_i\dot{u}(0)|^2\right) \\ &\lesssim \left(\frac{1}{\gamma}\|\Phi_i\dot{P}\dot{u}\|^2 + \frac{1}{\gamma}\|[\dot{P},\Phi_i]\dot{u}\|^2 + |\Phi_i\dot{B}\dot{u}(0)|^2 + |[\dot{B},\Phi_i]\dot{u}(0)|^2\right). \end{aligned}$$

Since the symbols commute, $[\dot{P}, \Phi_i]$ and $[\dot{B}, \Phi_i]$ are pseudodifferential operators of order 0 and -1, respectively, so we may write

$$\gamma\|u_i\|^2 + |u_i(0)|^2 \lesssim \left(\frac{1}{\gamma}\|\dot{P}u_i\|^2 + |\dot{B}u_i(0)|^2\right) \lesssim \frac{1}{\gamma}\|\dot{P}\dot{u}\|^2 + \frac{1}{\gamma}\|\dot{u}\|^2 + |\dot{B}\dot{u}|^2 + \frac{1}{\gamma^2}|\dot{u}(0)|^2. \tag{2.56}$$

As for the left-hand side of (2.56), the triangle inequality and the convexity of the power function $x \mapsto x^2$ show that

$$\begin{aligned} \|u\|^2 &\lesssim \sum_i \|u_i\|^2, \\ |u(0)| &\lesssim \sum_i |u_i(0)|^2. \end{aligned}$$

In the end, putting all the pieces together yields

$$\gamma\|u\|^2 + |u(0)|^2 \lesssim \frac{1}{\gamma}\|Pu\|^2 + \frac{1}{\gamma}\|u\|^2 + |Bu|^2 + \frac{1}{\gamma^2}|u(0)|^2,$$

from which we may deduce that

$$\gamma\|u\|^2 + |u(0)|^2 \leq C\left(\frac{1}{\gamma}\|Pu\|^2 + |Bu(0)|^2\right) \tag{2.57}$$

by taking γ sufficiently large. Let us conclude by extending the above inequality to Sobolev spaces. As might be expected, the key observation is that $v := \Lambda^s_\gamma u \in L^2$ whenever $u \in H^s_\gamma$. After all, if we substitute v in (2.57), the energy estimate becomes

$$\gamma\|\Lambda^s_\gamma u\|^2 + |\Lambda^s_\gamma u(0)|^2 \leq C\left(\frac{1}{\gamma}\|P\Lambda^s_\gamma u\|^2 + |B\Lambda^s_\gamma u(0)|^2\right). \tag{2.58}$$

The new commutators are not significantly different from the ones already investigated. Actually, we know that

$$\begin{aligned}\|P\Lambda^s_\gamma u\|^2 &\lesssim \|\Lambda^s_\gamma Pu\|^2 + \|[P,\Lambda^s_\gamma]u\|^2\\ &\lesssim \|\Lambda^s_\gamma Pu\|^2 + \|u\|^2,\end{aligned}$$

and

$$\begin{aligned}|B\Lambda^s_\gamma u(0)|^2 &\lesssim |\Lambda^s_\gamma Bu(0)|^2 + |[B,\Lambda^s_\gamma]u(0)|^2\\ &\lesssim |\Lambda^s_\gamma Bu|^2 + |u(0)|^2_{-1}\\ &\lesssim |\Lambda^s_\gamma Bu|^2 + \frac{1}{\gamma^2}|u(0)|^2.\end{aligned}$$

Once again, the result follows from choosing γ sufficiently large. □

2.6 Existence, uniqueness, and regularity

We shall utilise a duality argument together with classical results from functional analysis to establish existence, uniqueness, and regularity for the solution of (L, B). To simplify the exposition, we shall deal with differential operators only, so that Problem (2.4) reads

$$\begin{cases} Lu(t,x) = \left(D_t + \sum_{j=1}^{d} A_j(t,x)D_j\right)u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+,\\ B(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d.\end{cases} \tag{2.59}$$

As usual, we furnish (2.59) with Assumption 2.1.1 and seek to define and adjoint problem $(L^*, \tilde{B})$.

2.6.1 An adjoint boundary value problem

As we did in Section 2.2.1, we abuse the notation and represent both the inner product of $L^2(\mathbb{R}^{1+d}_+)$ and $L^2(\mathbb{R}^d)$ by $\langle \cdot\,,\cdot\rangle$. Recall that the adjoint operator L^* obeys the Green formula

$$\langle Lw, v\rangle - \langle w, L^* v\rangle = i\langle A_d w, v\rangle \tag{2.60}$$

for every $v, w \in \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+, \mathbb{R}^n)$, and it is easily seen to be

$$L^* \cdot = D_t \cdot + \sum_{j=1}^{d} D_j(A_j^T(t,x)\cdot). \tag{2.61}$$

For reasons that will become apparent soon, we need a decomposition of the right-hand side of (2.60) taking account of the boundary matrix $B(t,y)$. In general, this decomposition relies upon the existence of a smooth basis for $\ker B(t,x)$, a fact that may not follow from the assumptions we have made when dealing with more general domains than the upper half-space (or any other contractile manifold). In such cases, it must be considered part of the hypotheses (see [Hir12] and [BGS07]). After this remark, let us continue with a lemma that we state without proof.

Lemma 2.6.1 (Proposition 6.3, [CP82]). *There exists a map $N \in C^\infty(\mathbb{R}^d, \mathcal{M}_{(n-p)\times n}(\mathbb{C}))$ such that for every $(t,y) \in \mathbb{R}^d$,*

$$\mathbb{C}^n = \ker B \oplus \ker N.$$

In addition, for $N(t,y)$ fixed, there exist unique functions $\tilde{B} \in C^\infty(\mathbb{R}^d, \mathcal{M}_{(n-p)\times n}(\mathbb{C}))$ and $M \in C^\infty(\mathbb{R}^d, \mathcal{M}_{p\times n}(\mathbb{C}))$ so that for every $(t,y) \in \mathbb{R}^d$,

$$(A_d w, v) = (Bw, Mv) + (Nw, \tilde{B}v), \tag{2.62}$$

and

$$\ker \tilde{B} = (A_d \ker B)^\perp.$$

The matrix-valued function N in lemma 2.6.1 is not unique, and by extension there is also considerable freedom in choosing $\tilde{B}$ and M too. Given equations (2.60), (2.61) and Lemma 2.6.1, let us now make precise what we understand by an *adjoint problem*.

Definition 2.6.1. *Let (L, B) be a boundary value problem as in* (2.59)*. A pair $(L^*, \tilde{B})$ such that*

$$\langle Lw, v\rangle - \langle w, L^* v\rangle - i(Bw, Mv) - i(Nw, \tilde{B}v) = 0 \tag{2.63}$$

for every $w, v \in \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$ is called an adjoint problem of (2.59).

We proceed with a short but important statement.

Proposition 2.6.1. *Under the notation of Definition 2.6.1, it is true that*

$$\{Mv : v \in \mathcal{D}(\mathbb{R}^d, \mathbb{C}^n), \tilde{B}v = 0\} = \mathcal{D}(\mathbb{R}^d, \mathbb{C}^p).$$

Proof. It suffices to check that the matrix M is onto. Certainly, if $v \in \mathcal{D}(\mathbb{R}^d, \mathbb{C}^n)$ belongs to $\ker M \cap \ker \tilde{B}$, then $\langle A_d w, v\rangle = 0$ for every w. But A_d is nonsingular, so the latter means that $w = 0$, and consequently that $\ker M \cap \ker \tilde{B} = \{0\}$. Since the dimension of $\ker \tilde{B}$ is p, the intersection above may be rephrased as saying that

$$M : \ker \tilde{B} \longmapsto \mathbb{C}^p$$

is an isomorphism, which finishes the proof. $\square$

So far, we have studied the forward Lopatinskiĭ condition. Now, we shall introduce its complementary notion, i.e., the backward Lopatinskiĭ condition.

Definition 2.6.2. *The boundary value problem $(L^*, \tilde{B})$ satisfies the backward weak (resp. uniform) Lopatinskiĭ condition if for every $X \in \mathbb{Y}_S \cap \{\gamma < 0\}$ (resp. $X \in \mathbb{Y}_S \cap \{\gamma \leq 0\}$),*

$$\mathbb{E}^-(X) \cap \ker \tilde{B} = \{0\}.$$

Proposition 2.6.2 (Proposition 6.6 - Chapter 7, [CP82]). *Suppose that (L, B) is a boundary value problem satisfying Assumption 2.1.1 and that $(L^*, \tilde{B})$ is an adjoint problem of (L, B). Then (L, B) satisfies the weak Lopatinskiĭ condition (resp. the uniform Lopatinskiĭ condition) if and only if $(L^*, \tilde{B})$ satisfies the backward weak Lopatinskiĭ condition (resp. the backward uniform Lopatinskiĭ condition).*

2.6.2 Well-posedness of the boundary problem (L, B)

Let us formulate the central theorem of this section.

Theorem 2.6.1. *Consider the boundary value problem* (2.59) *under Assumption 2.1.1. If the uniform Lopatinskiĭ condition is satisfied, it is possible to find a constant $\gamma_0 \geq 1$ such that the following assertion holds for every $\gamma \geq \gamma_0$: if $f \in L^2_\gamma(\mathbb{R}^{1+d}_+)$ and $g(t,x) \in L^2_\gamma(\mathbb{R}^d)$, there is a unique $u(t,x) \in L^2_\gamma(\mathbb{R}^{1+d}_+)$ with the properties listed below:*

(i) u is a solution of

$$\begin{cases} L(t,x,D_t,D_y)u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.64}$$

(ii) the trace of $u|_{x_d=0}$ is an element of $L^2_\gamma(\mathbb{R}^d)$, and

(iii) u satisfies the energy estimate

$$\gamma \int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|u(t,x)|^2\,dt\,dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|u(t,y,0)|^2\,dt\,dy \\ \leq C\left(\frac{1}{\gamma}\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|Lu(t,x)|^2\,dt\,dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|Bu(t,y,0)|^2\,dt\,dy\right) \tag{2.65}$$

for some C which only depends on γ_0.

Moreover, there exists $\gamma_k \geq \gamma_0$ so that if $\gamma \geq \gamma_k$, it is true that for $f \in H^k_\gamma(\mathbb{R}^{1+d}_+)$ and $g \in H^k_\gamma(\mathbb{R}^d)$, there is a unique solution $u \in H^k_\gamma(\mathbb{R}^{1+d}_+)$ of (L,B) whose trace on $\mathbb{R}^d$ belongs to $H^k_\gamma(\mathbb{R}^d)$, and so that u satisfies

$$\gamma\|u\|^2_{s,\gamma} + |u(0)|^2_{s,\gamma} \leq c\left(\frac{1}{\gamma}\|f\|^2_{s,\gamma} + |Bu(0)|^2_{s,\gamma}\right).$$

Proof. We shall address the fundamental questions of existence, uniqueness, and regularity for Problem (2.59) following [CP82] and [BGS07] closely. Since the ideas arise most naturally by first proving existence, then regularity and finally uniqueness, we shall adopt this specific order throughout.

Existence. Let us assume that $f_\gamma \in L^2_\gamma(\mathbb{R}^{1+d}_+)$ and $g_\gamma \in L^2_\gamma(\mathbb{R}^d)$. We show the existence of a solution $u \in L^2_\gamma(\mathbb{R}^{1+d}_+)$ having the prescribed properties. To begin with, suppose that $(L^*, \tilde{B})$ is an adjoint problem of (L,B) and define the spaces

$$\tilde{E} = \{v \in \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+) : \tilde{B}v = 0\} \quad \text{and} \quad L^*\tilde{E} = \{L^*v : v \in \tilde{E}\},$$

both endowed with the topology induced by the norm $\|\cdot\|_\gamma$. From Proposition 2.6.2, $(L^*, \tilde{B})$ meets the backward uniform Lopatinskiĭ condition and therefore enjoys the energy estimate

$$\gamma\|v\|_{-\gamma} + |v|^2_{-\gamma} \leq \frac{C}{\gamma}\|L^*v\|_{-\gamma} + |\tilde{B}v|_{-\gamma}, \tag{2.66}$$

for some positive constant C and every $\gamma \geq \gamma_0 > 1$. In particular, inequality (2.66) reduces to

$$\gamma\|v\|_{-\gamma} + |v|^2_{-\gamma} \leq \frac{C}{\gamma}\|L^*v\|_{-\gamma} \tag{2.67}$$

when $v \in \tilde{E}$, revealing that L^* restricted to $\tilde{E}$ is injective. Thus, the map $\ell : L^*\tilde{E} \to \mathbb{C}$ given by

$$\ell(L^*v) = \langle f, v\rangle - i\langle g, Mv\rangle \tag{2.68}$$

is a well-defined, linear form on L^*E such that

$$|\ell(L^*v)| \leq \|f\|_\gamma\|v\|_{-\gamma} + |g|_\gamma\|v\|_{-\gamma}. \tag{2.69}$$

To obtain an upper bound for (2.69) in terms of L^*v, we combine (2.67) and (2.69) appropriately to see that

$$|\ell(L^*v)| \leq \|f\|_\gamma \|v\|_{-\gamma} + |g|_\gamma \|v\|_{-\gamma} \leq C\left(\frac{1}{\gamma}\|f\|_\gamma + \frac{1}{\sqrt{\gamma}}|g|_\gamma\right)\|L^*v\|_{-\gamma}.$$

That being the case, we infer that ℓ is a continuous linear functional on $L^*\tilde{E}$ that we can extend to $L^2_{-\gamma}(\mathbb{R}^{1+d}_+)$ owing to the Hahn-Banach theorem. Next, we invoke the Riesz representation theorem to predict the existence of a unique element $u \in L^2_\gamma(\mathbb{R}^{1+d}_+)$ such that

$$\langle u, L^*v\rangle = \ell(L^*v) = \langle f, v\rangle - i\langle g, Mv\rangle \tag{2.70}$$

for all $v \in \tilde{E}$. Notice that if $v \in \mathcal{D}(\mathbb{R}^{1+d}_+)$, we may discard the boundary term in (2.70) and get $Lu = f$ in the sense of distributions. On the other hand, as $\mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$ is dense in $L^2_\gamma(\mathbb{R}^{1+d}_+)$, we may approximate u by a sequence of elements in $\mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$ and conclude from the continuity of (2.63) that

$$\langle Lu, v\rangle - \langle u, L^*v\rangle - i(Bu, Mv) - i(Nu, \tilde{B}v) = 0 \tag{2.71}$$

for every $v \in \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$. To finalise the argument, let us insert (2.70) into (2.71) and take advantage of the identity $Lu = f$, so

$$(Bu, Mv) + (g, Mv) = 0 \tag{2.72}$$

for all $v \in \mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$. The surjectivity of M (see Proposition 2.6.1) then indicates that $Bu = g$ in the sense of distributions.

Regularity. At the heart of this segment there is a technical result that we assume without demonstration.

Proposition 2.6.3 (Proposition 6.8 - Chapter 7, [CP82]). *Under the assumptions of Theorem 2.6.1, for every integer $k \geq -1$, there exists a positive constant γ_k such that for every $\gamma \geq \gamma_k$ the following implication holds true: if $u \in H^k_\gamma(\overline{\mathbb{R}}^{1+d}_+) \cap L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$, $u|_{x_d=0} \in H^k_\gamma(\mathbb{R}^d)$, $L_\gamma u \in H^{k+1}_\gamma(\overline{\mathbb{R}}^{1+d}_+)$, and $B_\gamma u \in H^{k+1}_\gamma(\mathbb{R}^d)$, then $u \in H^{k+1}(\overline{\mathbb{R}}^{1+d}_+)$ and $u|_{x_d=0} \in H^{k+1}_\gamma(\mathbb{R}^d)$.*

Suppose that $u \in H^{-1}_\gamma(\overline{\mathbb{R}}^{1+d}_+) \cap L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ satisfies

$$\begin{cases} L_\gamma u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma u|_{x_d=0} = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{2.73}$$

with $f \in L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ and $g \in L^2_\gamma(\mathbb{R}^d)$. In the present context, Proposition 2.6.3 guarantees that $u|_{x_d=0} \in L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$, as claimed. Repeating this process inductively for $k \geq 1$, we arrive at the conclusion that whenever $f \in H^k_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ and $g \in H^k_\gamma(\mathbb{R}^d)$, it occurs that $u \in H^k_\gamma$ and $u|_{x_d=0} \in H^k_\gamma(\overline{\mathbb{R}}^{1+d}_+)$.

Uniqueness. In view of the linearity of (L, B), let us assume that $u \in L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ is a solution of

$$\begin{cases} L_\gamma u(t,x) = 0 & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma u|_{x_d=0} = 0 & (t,y) \in \mathbb{R}^d. \end{cases} \tag{2.74}$$

At first glance, the validity of (2.65) for test functions merely extends to elements $u \in H^1_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ rather than to $u \in L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$, which forces us to check the regularity of u before using

$$\gamma \int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t} |u|^2 \, dt \, dx + \int_{\mathbb{R}^d} e^{-2\gamma t} |u(0)|^2 \, dt \, dy \\ \leq C \left(\frac{1}{\gamma} \int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t} |f|^2 \, dt \, dx + \int_{\mathbb{R}^d} e^{-2\gamma t} |Bu(0)|^2 \, dt \, dy \right). \tag{2.75}$$

That said, as the initial data $f \equiv 0$ and $g \equiv 0$ trivially belong to H^1_γ, Proposition 2.6.3 ensures that $u \in H^1_\gamma$, meaning that we may use the energy inequality (2.75). It is clear then that u vanishes almost everywhere, hence establishing the uniqueness for (L, B).

Energy estimates in L^2_γ. To complete the proof, we only need to verify that (2.75) applies to $u \in L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$. As a matter of fact, from the density of $\mathcal{D}(\overline{\mathbb{R}}^{1+d}_+)$ (resp. $\mathcal{D}(\mathbb{R}^d)$) in $L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ (resp. $L^2_\gamma(\mathbb{R}^d)$), we may find sequences $\{f_j\}$ and $\{g_j\}$ converging to $f \in L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ and $g \in L^2_\gamma(\mathbb{R}^d)$ with respect to the norms of $L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ and $L^2_\gamma(\mathbb{R}^d)$. If we examine the boundary value problem (L, B) with data (f_j, g_j), it is known from the ideas discussed in previous paragraphs that there is a unique solution $u_j \in H^1_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ for γ large enough. The resulting sequences $\{u_j\}$ and $\{u_j(0)\}$ thus formed are Cauchy sequences in $L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ and $L^2_\gamma(\mathbb{R}^d)$, respectively, as we may deduce from the a priori estimates (2.75) applied to $u_j - u_k$. Let $\underline{u} \in L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ and $\underline{u}_0 \in L^2_\gamma(\mathbb{R}^d)$ be the corresponding limits of $\{u_j\}$ and $\{u_j(0)\}$. Note that $u_j(0) \to \underline{u}(0)$ in $H^1_\gamma(\mathbb{R}^d)$ as $j \to \infty$ due to Theorem 2.1.1, so it must be the case that $\underline{u}(0) = u_0$ and, eventually, that $B\underline{u}(0) = g$. On the other side, since $Lu_j = f_j$ and $Bu_j = g_j$ converge to $Lu = f$ and $Bu = g$ in $L^2_\gamma(\overline{\mathbb{R}}^{1+d}_+)$, from the uniqueness of the limit in $L^2_\gamma(\mathbb{R}^d)$, it necessarily follows that $\underline{u} = u$. In the end, taking the limit when $u_j \to u$ in (2.75) gives the result. □

CHAPTER THREE

Weakly regular boundary problems of real type in a half-space

It was Kreiss' belief that initial boundary value problems that fulfilled the weak Lopatinskiĭ condition and were stable under perturbations were necessarily strong, in the sense that they satisfied an estimate of the type (2.30) (see Theorem 2 in [Kre70]). It was eventually shown, first through examples and then through a complete characterisation in [BGRSZ02], that certain problems for which the Lopatinskiĭ condition holds weakly but not uniformly preserve their defining properties under small perturbations, thus conforming a new stable class. The main goal of this chapter is to study this class in depth and ultimately to derive energy estimates applicable to such case.

We provide a brief description of the content. In Section 3.1, we introduce the concept of a stable class and give some illustrative examples before moving on to a detailed study of the $\mathcal{WR}$ class. In Section 3.2, as a prelude to more general constructions, we first examine a $\mathcal{WR}$ problem with constant coefficients, for which we derive energy estimates equivalent to those in [BGS07] using a different and more robust approach. Finally, in Section 3.3, we broaden the discussion and derive energy estimates applicable to a $\mathcal{WR}$ problem with variable coefficients.

3.1 Classification of linear boundary value problems

To illustrate the central ideas of this section, we shall focus initially on

$$\begin{cases} Lu(t,x) := \left(D_t + \sum_{j=1}^{d} A_j(t,x) D_j \right) u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_{+}, \\ \qquad\qquad\qquad\qquad\qquad B(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^{d}, \end{cases} \tag{3.1}$$

equipped with Assumption 2.1.1. As in Chapter 2, the source terms f, g are taken, in principle, from weighted Sobolev spaces H^k_γ, $k \geq 0$.

We know from Section 2.3.1 that (3.1) can be recast as

$$\begin{cases} P_\gamma u(t,x) := \left(D_d + \mathcal{A}_\gamma(t,y,x_d,D_t,D_y)\right) u(t,x) = A_d^{-1} f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ \qquad\qquad\qquad\qquad\qquad B_\gamma(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{3.2}$$

where $\mathcal{A}_\gamma(t,y,x_d,D_t,D_y)$ and $B_\gamma(t,y)$ are differential operators whose symbols are

$$a(X) \equiv a(t,x,\tau,\eta,\gamma) = A_d^{-1}\left((\tau - i\gamma)I_n + \sum_{j=1}^{d-1} \eta_j A_j(t,x)\right) \quad \text{and} \quad b(X) = B_\gamma(t,y).$$

Recall that the Lopatinskiĭ determinant is defined in terms a smooth, homogeneous of degree 0 basis $\{\mathbf{e}_1(X), \cdots, \mathbf{e}_p(X)\}$ of $\mathbb{E}^-(X)$ by the expression

$$\underline{\Delta}(X) = \det\left(b(X)\mathbf{e}_1(X), \cdots b(X)\mathbf{e}_p(X)\right). \tag{3.3}$$

3.1.1 Stable classes

We shall use the Lopatinskiĭ condition to classify boundary value problems (L,B). To this end, let $\mathcal{T}$ be the space of pairs (L,B) such that (L,B) is normal, as explained in Assumption 2.1.1. The relevant notions for our purposes are those that remain stable under small perturbations of the coefficients of B for a fixed constantly hyperbolic operator L. For instance, the *unstable* and *strongly stable* classes listed below are robust.

- ▷ The set of boundary problems (L,B) for which the weak Lopatinskiĭ condition fails. Certainly, when (L,B) is ill-posed, there exists $X_0 = (t_0,y_0,0,\tau_0,\eta_0,\gamma_0) \in \mathbb{X}$, $\gamma_0 > 0$, such that $\underline{\Delta}(X_0) = 0$. If in addition B depends continuously on a parameter ε, the Lopatinskiĭ determinant does so, and it follows from Rouché's theorem [6] that the roots of $\underline{\Delta}(\cdot,\varepsilon)$ persist for ε small. Given that (L,B) exhibits a Hadamard instability, there is no hope of any satisfactory theory for general data (f,g).
- ▷ The set of boundary problems (L,B) satisfying the uniform Lopatinskiĭ condition. In such a situation, $\underline{\Delta}(\cdot,0)$ is nonvanishing, homogeneous of degree 0 in (τ,η,γ), and since $b(X)$ does not depend on (t,x) for $|t|$ and $|x|$ large, $|\underline{\Delta}(\cdot,0)|$ may be regarded as a smooth function on a compact set $K \times S^d$. Now, considering that $\underline{\Delta}(\cdot,\varepsilon)$ is continuous in ε too, we conclude that it cannot vanish for ε small.

As briefly mentioned at the beginning, in [BGRSZ02] Benzoni-Gavage, Rousset, Serre and Zumbrun identified a third stable class of boundary value problems (L,B) for which the Lopatinskiĭ condition holds weakly but not uniformly, so completing the preceding description. This has been called *weakly regular of real type*, or $\mathcal{WR}$ for short,

[6] Striclty speaking, we need to verify that $f(\tau - i\gamma) := \underline{\Delta}(\cdot,\tau,\eta_0,\gamma)$ is a nontrivial holomorphic function of $\tau - i\gamma$ before applying Rouché's theorem. Fortunately, this is a straighforward consequence of homogeneity, continuity, and normality. See Lemma 8.1 in [BGS07].

and will be defined soon in Section 3.1.3. It is worth saying that, besides defining the $\mathcal{WR}$ class, the authors also simplify the analysis of transitions between stable classes, which proves particularly valuable in some physical and mathematical problems (see Section 4 in [BGRSZ02] for a fully worked example).

3.1.2 Hyperbolic frequency boundary points

According to Definition 2.4.3, the hyperbolic region $\mathcal{H}$ comprises the elements $X \in \mathbb{X}$ such that $a(X)$ is diagonalizable with purely real eigenvalues, meaning that $\mathcal{H}$ is necessarily confined to the frequency boundary $\{\gamma = 0\}$ since $a(X)$ is known to have no real eigenvalues when $\gamma > 0$ (see Hersh's lemma). Our next goal is to explore in detail the stable subspace $\mathbb{E}^-(X)$ when restricted to $\mathcal{H}$.

Definition 3.1.1. *A complex vector space is of real type if it possesses a basis consisting entirely of real vectors.*

Proposition 3.1.1. *Let $X \in \mathbb{X}_0$. If $X \in \mathcal{H}$, then $\mathbb{E}^-(X)$ is of real type.*

Proof. Our strategy heavily relies on that discussed in Lemma 3.1 in [CG10], which is in turn inspired by the more general arguments in [Mé00]. To begin with, let $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}, \underline{\eta}, 0) \in \mathcal{H}$ and $\underline{\xi}_d$ be such that $\pi_P(\underline{X}, \underline{\xi}_d) = \det(a(\underline{X}) + \underline{\xi}_d I_n) = 0$. Since L is constantly hyperbolic, its characteristic polynomial $\pi_L(X, \xi_d)$ factors as

$$\pi_L(X, \xi_d) = \prod_{k=1}^{q} (\tau + \lambda_k(t, x, \eta, \xi_d))^{\alpha_k}, \tag{3.4}$$

where $\alpha_1, \cdots, \alpha_q$ are positive integers and $\lambda_1, \cdots, \lambda_q$ are pairwise distinct, real analytic functions on $\mathbb{R}^{1+d}_+ \times \mathbb{R}^d$ admitting holomorphic extensions in a complex neighbourhood of $\underline{\xi}_d$. The computations performed right after Assumption 2.3.1 validate the dispersion formula

$$\det(a(X) + \xi_d I_n) = \det\left(\tau - i\gamma + A(t, x, \eta, \xi_d)\right),$$

whose roots in ξ_d when $X = \underline{X}$ are real and equal to $\underline{\xi}_d$. Meanwhile, there is a unique λ_k so that $\underline{\tau} + \lambda_k(\underline{t}, \underline{x}, \underline{\eta}, \underline{\xi}_d) = 0$ and $\partial_{\xi_d} \lambda_k(\underline{t}, \underline{x}, \underline{\eta}, \underline{\xi}_d) \neq 0$, for which the eigenspace of $a(\underline{X})$ associated with $\underline{\xi}_d = -\underline{\mu}$ agrees with the α_k−dimensional subspace

$$\ker\left(\underline{\tau} + A(\underline{t}, \underline{x}, \underline{\eta}, -\underline{\mu})\right).$$

We may deduce from the Weierstrass preparation theorem (see Theorem A.2.1 in Appendix) that there are functions $e(X, \xi_d)$ and $\xi(X)$ so that:

- $e(X, \xi_d)$ and $\xi(X)$ are smooth in (t, x) and real analytic in η,
- $e(X, \xi_d)$ is holomorphic with respect to $(\tau - i\gamma, \xi_d)$, whereas $\xi(X)$ is holomorphic with respect to $(\tau - i\gamma)$,

▷ $\tau - i\gamma + \lambda_k(t,x,\eta,\xi)$ factors as

$$\tau - i\gamma + \lambda_k(t,x,\eta,\xi) = e(X,\xi_d)(\xi_d - \xi_k(X)),$$

with $\xi_k(\underline{X}) = \underline{\xi}_d$ and $e(X,\xi_d)$ such that it does not vanish on a neighbourhood of $(\underline{X},\underline{\xi}_d)$.

It is then clear that for every X in the vicinity of $\underline{X}$, $\xi_1(X), \cdots, \xi_q(X)$ are pairwise real eigenvalues of $a(X)$ with algebraic multiplicity α_k. That $\xi_k(X) = -\mu_k(X)$ are semisimple and their eigenspaces depend holomorphically on $\tau - i\gamma$ and analytically on η is a delicate point at the heart of Metivier's work in [Mé00], so we feel free to skip the details and draw the reader's attention to Lemmas 2.5 and 2.6 therein. In the end, we can show that for $X \in \mathbb{X} \cap \{\gamma > 0\}$ close to $\underline{X}$,

$$\mathbb{E}^-(X) = \bigoplus_{\operatorname{Im}\mu_k<0} \ker\left(a(X) - \mu_k(X)I_n\right) = \bigoplus_{\operatorname{Im}\mu_k<0} \ker\left(\tau - i\gamma + A(t,x,\eta,-\mu_k(X))\right). \tag{3.5}$$

Taking the limit of (3.5) as γ goes to zero[7], we get a matrix $\tau + A(t,x,\eta,-\mu_k(X))$ with real entries that is diagonalisable in the reals. This ensures that $\mathbb{E}^-(X)$ is of real principal type. □

Corollary 3.1.1. *When restricted to $\mathcal{H}$, the Lopatinskiĭ determinant $\underline{\Delta}$ is real-valued.*

Proof. The result follows immediately from Proposition 3.1.1, the definition of the Lopatinskiĭ determinant, and the fact that $b(X)$ is real-valued. □

We complement the last Proposition and its corollary with an assertion that we state without proof.

Proposition 3.1.2 ([Mé00], [CG10]). *The hyperbolic region $\mathcal{H}$ is open and contains no glancing points. What is more, $\mathbb{E}^-(X)$ depends smoothly on (t,x), holomorphically on $(\tau - i\gamma, \xi_d)$ and analytically on η if $X \in \mathcal{H}$.*

3.1.3 The $\mathcal{WR}$ class

We now proceed to define the $\mathcal{WR}$ class and examine its main properties.

Definition 3.1.2. *Let (L,B) be as in Problem (3.1), subject to Assumption 2.1.1. The boundary value problem (L,B) is of class $\mathcal{WR}$ if the following conditions are met:*

(i) The weak Lopatinskiĭ condition holds,

(ii) The level set $\underline{\Delta}^{-1}(0)$ is non-void and contained in the hyperbolic region $\mathcal{H}$. Moreover, $\partial_\tau\underline{\Delta}(X) \neq 0$ whenever $\underline{\Delta}(X) = 0$.

Theorem 3.1.1 (Theorem 2.10, [BGRSZ02]). *Definition 3.1.2 describes an open class.*

[7] via Theorem 2.4.2.

Proof. Suppose that $B(t,y)$ can be parameterised by ε in a smooth way. Then the Lopatinskiĭ determinant depends smoothly on (t,x,ε), holomorphically on $\tau - i\gamma$, and real analytically on η. If we assume that (L,B) is $\mathcal{WR}$ for $\varepsilon = 0$, by the same reasoning as the one sketched for the strongly stable class, the domain of $\underline{\Delta}$ is compact and therefore the zero set of $\Delta(\cdot,\varepsilon)$ tends to the zero set of $\underline{\Delta}(\cdot,0)$ as the parameter ε goes to 0. But $\Delta^{-1}(\cdot,0)$ is by definition a smooth, real variety, so $\Delta^{-1}(\cdot,\varepsilon)$ has to be a smooth, real variety too for ε sufficiently small. Consequently, no new components of $\Delta^{-1}(\cdot,\varepsilon)$ appear in the hyperbolic region $\mathcal{H}$ and $\Delta^{-1}(\cdot,\varepsilon)$ does not move into the interior $\gamma > 0$. □

Instead of using Definition 3.1.2, we shall utilise an equivalent characterisation of the the $\mathcal{WR}$ class which is more appropriate for our future discussion.

Proposition 3.1.3 (Proposition B.1, [CG10] - Lemma 5.3, [OS75]). *Let (L,B) as in Problem (3.1). The pair (L,B) defines a $\mathcal{WR}$ boundary value problem if and only if:*

(i) for every $X \in \mathbb{Y}_S \cap \{\gamma > 0\}$, it is true that $\mathbb{E}^-(X) \cap \ker b(X) = \{0\}$. In other words, the weak Lopatinskiĭ condition is fulfilled.

(ii) The critical set $\Gamma := \{X \in \mathbb{Y}_S : \mathbb{E}^-(X) \cap \ker b(X) \neq \{0\}\}$ is nonempty and included in the hyperbolic region $\mathcal{H}$. Furthermore, for every $\underline{X} \in \Gamma$, there exist a conic neighbourhood $\mathcal{V}$ of $\underline{X}$, and mappings

- ▷ $\underline{e}_1, \cdots, \underline{e}_p \in C^\infty(\mathcal{V}, \mathbb{C}^n)$,
- ▷ $\mathfrak{p}(X) \in C^\infty(\mathcal{V}, GL_p(\mathbb{C}))$,
- ▷ $\omega \in C^\infty(\mathcal{V}, \mathbb{R})$

such that, for every $X \in \mathcal{V}$, $\underline{e}^-(X) = \{\underline{e}_1(X), \cdots, \underline{e}_p(X)\}$ is a basis for $\mathbb{E}^-(X)$,

$$b^-(X) := b(X)(\underline{e}_1(X), \cdots, \underline{e}_p(X)) = \mathfrak{p}(X) \begin{pmatrix} \lambda^{-1}(\zeta)(\gamma + i\omega(X)) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

and $\partial_\tau \omega(X) \neq 0$ when $\omega(X) = 0$.

Proof. Let us first assume that the boundary value problem (L,B) satisfies Conditions (i) and (ii) in the statement above. As is customary, we shall focus on a compact set $K \times S^d \subset \mathbb{X}_S$ in view of the Property **(C)** an the homogeneity of $\underline{\Delta}(X)$. Since by definition the Lopatinskiĭ determinant vanishes at points where the Lopatinskiĭ condition fails,

$$\Gamma = \{X \in \mathbb{Y}_S : \mathbb{E}^-(X) \cap \ker b(X) \neq \{0\}\} = \{X \in \mathbb{Y}_S : \underline{\Delta}(X) = 0\},$$

and hence $\{X \in \mathbb{Y}_S : \underline{\Delta}(X) = 0\} \subset \mathcal{H}$. On top of that, as $\mathcal{H}$ is known to be open and disjoint from the set of glancing points $\mathcal{G}$ (see Proposition 3.1.2), the Lopatinskiĭ determinant extends analytically to the frequency boundary $\{\gamma = 0\}$ where it becomes real-valued, and then similar arguments to the ones offered in the proof of Proposition 3.1.1 reveal that Γ is a real analytic submanifold of $K \times S^d$. There is still the question of whether the roots of the Lopatinskiĭ determinant are simple or not. To give an answer, let us invoke Condition (ii) to define a second Lopatinskiĭ determinant by

$$\Delta'(X) := \det(b^-(X)) = \det\Big(b(X)\underline{e}_1(X), \cdots, b(X)\underline{e}_p(X)\Big) = (\gamma + i\omega(X)) \det \mathfrak{p}(X),$$

or just by

$$\underline{\underline{\Delta}}(X) = \gamma + i\omega(X), \tag{3.6}$$

if we divide by $\det \mathfrak{p}(X) \neq 0$. That said, as both $\mathfrak{e}(X)$ and $\underline{\mathfrak{e}}(X)$ span the stable subspace $\mathbb{E}^-(X)$, there must be a nonvanishing complex-valued function $v(X)$ defined locally around $\underline{X}$ such that

$$\underline{\Delta}(X) = v(X)\underline{\underline{\Delta}}(X).$$

It is then easy to check that $\partial_\tau \omega(X) \neq 0$ implies that $\partial_\tau \underline{\Delta}(X) \neq 0$ when $\omega(X) = 0$, thereby completing the first part of the proof. For the converse, let us suppose that (L, B) belongs to the $\mathcal{WR}$ class. It is a direct consequence of the definition that $\mathbb{E}^-(X) \cap \ker b(X) = \{0\}$ for every $X \in \mathbb{Y} \cap \{\gamma > 0\}$, and that Γ is non-void and contained in $\mathcal{H}$. It is yet to be proved that the vanishing of the Lopatinskiĭ determinant at first order yields the existence a of a basis $\underline{\mathfrak{e}}(X)$ with the properties indicated in Proposition 3.1.3, Part (ii). This is addressed in the following proposition and its corollary. □

Proposition 3.1.4. *If (L, B) belongs to the $\mathcal{WR}$ class, there exist a neighbourhood $\mathcal{V}$ and two mappings $\mathfrak{p}, c \in C^\infty(\mathcal{V}, GL_p(\mathbb{C}))$ homogeneous of degree 0 in $\zeta = (\tau - i\gamma, \eta) \simeq (\tau, \eta, \gamma)$ such that the factorisation*

$$b^-(X) = \mathfrak{p}(X) \begin{pmatrix} \lambda^{-1}(\zeta)\,(\gamma + i\omega(X)) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} c^{-1}(X) \tag{3.7}$$

holds true.

Proof. Let $\underline{X} \in \Gamma$ be such that $\underline{\Delta}(\underline{X}) = 0$ and $\partial_\tau \underline{\Delta}(\underline{X}) \neq 0$. Since $\underline{\Delta}(X)$ is holomorphic in $\rho := \tau - i\gamma$ and homogeneous of degree 0 with respect to ζ, the implicit function theorem (see Theorem A.2.3) characterises the zeros of $\underline{\Delta}(X)$ in a conic neighbourhood $\mathcal{V}$ of $\underline{X}$ through an equation $\rho = \nu(t, y, \eta)$, where $\nu(t, y, \eta)$ is a smooth, homogeneous function of degree 1 in η. If $\mathfrak{e}(X) = \{e_1(X), \cdots, e_p(X)\}$ is any basis for $\mathbb{E}^-(X)$, set

$$b^-(X) := (b(X)e_1(X), \cdots, b(X)e_p(X)),$$

and denote each $b(X)e_i(X)$ by $b_i^-(X)$ in the sequel. We claim that $b^-(X)$ has a non-singular cofactor matrix $h(X)$ of order $p-1$ on $\mathcal{V}$. Indeed, shrinking $\mathcal{V}$ if necessary, $\partial_\tau \underline{\Delta}(\underline{X}) \neq 0$ guarantees that

$$\operatorname{rank} b^-(X) \geq p-1 \quad \text{on} \quad \mathcal{V}, \tag{3.8}$$

the equality being realised only when $X \in \mathcal{V} \cap \Gamma$. In practice, there is no loss of generality in assuming that $h(X)$ is the resulting block after deleting the first column and the first row of $b^-(X)$. Let $b_1'(X)$ be the vector $b_1^-(X)$ without its first entry. In such case, for every $X \in \mathcal{V}$, the linear system $h(X)d(X) = b_1'(X)$ possesses a unique solution

$$\begin{pmatrix} d_2(X) \\ \vdots \\ d_p(X) \end{pmatrix} = h^{-1}(X)b_1'(X),$$

whose entries $d_2(X), \cdots, d_p(X)$ are smooth in (t,x) and homogeneous of degree 0 in (ρ,η). Suppose now that

$$k(X) = b_1^-(X) - \sum_{i=2}^{p} d_i(X)b_i^-(X).$$

Except for the first component of $k(X)$, which only vanishes when $X \in \{\rho = \nu(t,y,\eta)\}$, all other entries are identically zero by construction. Thus, owing to the Weierstrass preparation theorem (see Theorem A.2.1 in Appendix), there exists a nonvanishing function $z_1(X) \equiv z(t,y,\rho,\eta)$ in $\mathcal{V}$, analytic with respect to ρ and homogeneous of degree -1 in (ρ,η), such that

$$k_1(X) = (\rho - \nu(t,y,\eta))z_1(X) \quad \text{for} \quad X \in \mathcal{V}.$$

As a result, if we put

$$z(X) := \begin{pmatrix} z_1(X) \\ \vdots \\ z_p(X) \end{pmatrix},$$

with $z_2(X) = \cdots = z_p(X) = 0$ along $\mathcal{V}$, we can write $k(X) = (\rho - \nu(t,y,\eta))z(X)$ and with it,

$$b_1^-(X) = (\rho - \nu(t,y,\eta))z(X) + \sum_{i=2}^{p} d_i(X)b_i^-(X).$$

Let

$$c(X) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -d_2(X) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_p(X) & 0 & \cdots & 1 \end{pmatrix}. \tag{3.9}$$

Observe that $c(X)$ is homogeneous of order 0 in (ρ, η) and that

$$\begin{aligned} b^-(X) &= \begin{pmatrix} b_1^-(X) & b_2^-(X) & \cdots & b_p^-(X) \end{pmatrix} \\ &= \begin{pmatrix} (\rho - \nu(t,y,\zeta))z(X) & b_2^-(X) & \cdots & b_p^-(X) \end{pmatrix} c^{-1}(X), \end{aligned}$$

after a straightforward computation. Notably, the latter amounts to writing in matrix notation

$$\begin{aligned} b^-(X) &= \mathfrak{p}(X) \begin{pmatrix} \lambda^{-1}(\zeta) i(\rho - \nu(t,y,\eta)) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} c^{-1}(X) \\ &= \mathfrak{p}(X) \begin{pmatrix} \lambda^{-1}(\zeta)(\gamma + i(\tau - \nu(t,y,\eta))) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} c^{-1}(X), \end{aligned}$$

where

$$\mathfrak{p}(X) := \begin{pmatrix} -iz(X)\lambda(\zeta) & b_2^-(X) & \cdots & b_p^-(X) \end{pmatrix}$$

is homogeneous of degree 0 in ζ and nonsingular. Lastly, setting $\omega(t,y,\tau,\eta) := \tau - \nu(t,y,\eta)$, it is easily seen that $\partial_\tau \omega(t,y,\tau,\eta) \neq 0$ and that

$$b^-(X) = \mathfrak{p}(X) \begin{pmatrix} \lambda^{-1}(\zeta)(\gamma + i\omega(t,y,\tau,\eta)) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} c^{-1}(X). \tag{3.10}$$

□

Corollary 3.1.2. *Let $\omega(X) \equiv \omega(t,y,\tau,\eta)$ and $\underline{\underline{\Delta}}(X) = (\gamma + i\omega(X))/\lambda(\zeta)$ be as in (3.6) (the $\lambda^{-1}(\zeta)$ factor has to do with $\underline{\underline{\Delta}}$ being originally defined in $\mathbb{X}_S$). Under the assumptions of Proposition 3.1.4, there is a basis*

$$\underline{e}^-(X) = \{\underline{e}_1(X), \cdots, \underline{e}_p(X)\},$$

for which

$$b^-(X) = \mathfrak{p}(X) \begin{pmatrix} \underline{\underline{\Delta}}(X) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{3.11}$$

Proof. Notice that (3.7) attains its simplest form when choosing

$$\begin{aligned} \underline{e}_1(X) &:= e_1(X) - d_2(X)e_2(X) - \cdots - d_p(X)e_p(X), \\ \underline{e}_2(X) &:= e_2(X), \\ &\vdots \\ \underline{e}_p(X) &:= e_p(X), \end{aligned} \tag{3.12}$$

as a basis for $\mathbb{E}^-(X)$. □

The one-dimensional subspace spanned by $\underline{e}_1(X)$ is special, as it points in the direction in which the Lopatinskiĭ condition degenerates when $X \in \Gamma$. In what follows, we shall write $\ell(X)$ to denote this subspace and refer to it as the *critical direction*.

3.2 The constant coefficients case

Even though we shall derive deeper results in future sections, exploring the model case (i.e. when P and B have constant coefficients and (P, B) is posed in the half-space) still has its value, especially as it leads to an observation that is relevant for more general constructions to come.

3.2.1 A $\mathcal{WR}$ symmetriser for the model problem

Let (P, B) be a $\mathcal{WR}$ boundary value problem of the form

$$\begin{cases} Pu(t,x) := \big(D_d + \mathcal{A}(D_t, D_y)\big)\, u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ \qquad\qquad\qquad\qquad\qquad Bu(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{3.13}$$

subject to Assumptions 2.3.1 and 2.4.1. We continue with the definitions of a Lopatinskiĭ family of operators and a $\mathcal{WR}$ symmetriser.

Definition 3.2.1. *Let $\delta(\zeta) \in C^\infty(\Xi, \mathcal{M}_{n\times n}(\mathbb{C}))$ such that*

(i) $\delta(\zeta)$ is homogeneous of degree 0 in ζ,

(ii) If $\zeta \in S^d$, $\ker \delta(\zeta)$ is trivial provided that $\gamma > 0$,

(iii) when $\zeta \in \Gamma$, $\ker \delta(\zeta)$ is nontrivial and $\ell(\zeta) \subseteq \ker \delta(\zeta)$.

We shall call $\Delta_\gamma := \mathrm{Op}_\gamma(\delta) \in \mathrm{OPS}^0(\mathbb{R}^{1+d}_+ \times [1, +\infty))$ a Lopatinskiĭ family of operators.

Having the Lopatinskiĭ operator as a reference, we define the function space

$$L^2_\Delta = \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Delta_\gamma v \in L^2_\gamma(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\}.$$

Definition 3.2.2. *A $\mathcal{WR}$ symmetriser for Problem 3.13 is a family of pseudodifferential operators $\Sigma_\gamma \in \mathrm{OPS}^0(\mathbb{R}^{1+d}_+ \times [1, +\infty))$ so that* *(i)* Σ_γ *is Hermitian,*

(ii) *for every* $v_1, v_2 \in L^2_\Delta$, *there is a positive constant* C *satisfying*

$$\langle \Sigma_\gamma v_1, v_2 \rangle \leq C|\Delta_\gamma v_1||\Delta_\gamma v_2|,$$

(iii) *there exists a positive constant* c *such that*

$$\operatorname{Im}(\Sigma_\gamma \mathcal{A} v, v) \geq c\gamma |\Delta_\gamma v|^2,$$

for each $v \in L^2_\Delta$,

(iv) *there are positive constants* α *and* β, *together with a family of pseudodifferential operators* $Q_\gamma \in \mathrm{OPS}^0(\mathbb{R}^d_+ \times [1, +\infty))$ *such that*

$$\langle \Sigma_\gamma v(0), v(0) \rangle \geq \alpha |\Delta_\gamma v(0)|^2 - \beta |Q_\gamma B v(0)|^2.$$

If a $\mathcal{WR}$ symmetriser exists, we expect the symbol $\sigma(\zeta)$ of Σ_γ to be somewhat degenerate on the critical set Γ. More precisely, we have

Proposition 3.2.1. *If* $\underline{\zeta} \in \Gamma$, $v \mapsto \langle \sigma(\underline{\zeta}) v, v \rangle$ *vanishes on the Krylov space of*

$$\ell(\underline{\zeta}) = \mathbb{E}^-(\underline{\zeta}) \cap \ker b(\underline{\zeta})$$

with respect to $a(\underline{\zeta})$, *that is to say, on the smallest invariant subspace of* $a(\underline{\zeta})$ *containing* $\ell(\underline{\zeta})$.

Proof. Recall from Section 2.3.1 that the symbol of $A(D_t, D_y)$ is

$$a(\zeta) \equiv a(\tau - i\gamma, \eta) := (A_d)^{-1} \left((\tau - i\gamma) I_n + \sum_{i=1}^{d-1} \eta_i A_i \right). \tag{3.14}$$

We prove initially that $v \mapsto \langle \sigma(\zeta) v, v \rangle$ restricted to $\mathbb{E}^-(\zeta)$ is positive definite for $\gamma > 0$. To do so, let $u \in \mathbb{E}^-(\zeta)$ be such that $u \neq 0$ and consider the initial value problem

$$\begin{cases} D_s v + a(\zeta) v = 0, \\ \qquad v(0) = u, \end{cases}$$

whose solution is well known and equal to

$$v(s) = e^{-ia(\zeta)s} u \tag{3.15}$$

in the sense of matrices. Then,

$$\begin{aligned}\frac{d}{ds}\langle\sigma(\zeta)v,v\rangle = \langle\sigma(\zeta)\partial_s v,v\rangle + \langle\sigma(\zeta)v,\partial_s v\rangle &= i\langle\sigma(\zeta)D_s v,v\rangle - i\langle\sigma(\zeta)v,D_s v\rangle\\ &= -2\,\mathrm{Im}\,\langle D_s a(\zeta)v,v\rangle\\ &= 2\,\mathrm{Im}\,\langle\sigma(\zeta)a(\zeta)v,v\rangle\\ &\geq 2c\gamma|\delta(\zeta)v|^2 > 0,\end{aligned} \tag{3.16}$$

The first inequality being due to (iii) in Definition 3.2.2 and Plancherel's theorem (see Theorem A.2.9 in Appendix), while the second one is being due to Condition (ii) in Definition 3.2.1. Since $u \in \mathbb{E}^-(\zeta)$, $v \to 0$ decreases exponentially fast as $s \to \infty$ and thus, when integrating over $\mathbb{R}^+$,

$$-\langle\sigma(\zeta)v(0),v(0)\rangle = \int_0^\infty \frac{d}{ds}\langle\sigma(\zeta)v(s),v(s)\rangle\,ds \geq 2C\gamma\int_0^\infty |\delta(\zeta)v(s)|^2\,ds > 0,$$

or $\langle\sigma(\zeta)u,u\rangle < 0$, as claimed. Back to the original assertion, let us fix $\underline{\zeta} \in \Gamma$. As $\mathbb{E}^-(\underline{\zeta}) \cap \ker b(\zeta)$ is nontrivial, for every $v \in \mathbb{E}^-(\underline{\zeta}) \cap \ker b(\zeta)$ such that $v \neq 0$, it must happen on one hand that $\langle\sigma(\underline{\zeta})v,v\rangle \leq 0$ by the opening argument of this proof and by continuity in γ, and on the other that $\langle\sigma(\underline{\zeta})v,v\rangle \geq 0$ because of a combination of Plancherel's theorem and Part (iv) in Definition 3.2.2, which ensures that the restriction of $\sigma(\underline{\zeta})$ to $\ker b(\zeta)$ is non-negative. Together, both facts indicate that $\sigma(X)|_{\ell(X)} = 0$.

In order to prove that $\sigma(\zeta)$ certainly vanishes in a larger subspace, we argue in a similar fashion as above and integrate (3.16) from 0 to a positive real number t,

$$\langle\sigma(\underline{\zeta})v(t),v(t)\rangle - \langle\sigma(\underline{\zeta})v(0),v(0)\rangle = \int_0^t \frac{d}{ds}\langle\sigma(\underline{\zeta})v(s),v(s)\rangle\,ds \geq 0,$$

or equivalently,

$$\langle\sigma(\underline{\zeta})v(t),v(t)\rangle \geq \langle\sigma(\underline{\zeta})v(0),v(0)\rangle = \langle\sigma(\underline{\zeta})u,u\rangle. \tag{3.17}$$

If we pick $u \in \ell(\underline{\zeta})$, the right-hand side of (3.17) is automatically zero and it is safe to say that

$$\langle\sigma(\underline{\zeta})v(t),v(t)\rangle \geq 0. \tag{3.18}$$

Let $K_\ell(\underline{\zeta})$ be the smallest invariant subspace of $a(\underline{\zeta})$ containing $\ell(\underline{\zeta})$. As $\ell(\underline{\zeta})$ is included in $\mathbb{E}^-(\underline{\zeta})$ and $\mathbb{E}^-(\underline{\zeta})$ is invariant under $a(\underline{\zeta})$, we necessarily have that $K_\ell(\underline{\zeta}) \subseteq \mathbb{E}^-(\underline{\zeta})$. Furthermore, as the solution of a first-order autonomous differential equation whose initial value belongs to an invariant space remains within the invariant space (see Theorem A.2.5), it is true that

$$\langle\sigma(\underline{\zeta})v(t),v(t)\rangle \geq 0. \tag{3.19}$$

The reverse inequality may be inferred from two facts, namely, that $\langle \sigma(\underline{\zeta}) \cdot, \cdot \rangle \leq 0$ when restricted to $\mathbb{E}^-(\underline{\zeta})$ (already verified!), and that $v(t) \in \mathbb{E}^-(\underline{\zeta})$ for every t. $\square$

Remark 3.2.1. The latter means, roughly speaking, that in general there is no hope that $\sigma(\zeta)$ kills only the critical direction $\ell(X)$, but a larger subspace containing $\ell(X)$.

3.2.2 Construction of a $\mathcal{WR}$ symmetriser and energy estimates for the model problem

As the present analysis is merely intended to motivate future results, we shall keep things simple and assume at this early stage that $K_\ell(\zeta) = \mathbb{E}^-(\zeta)$, leaving the more general case $K_\ell(\zeta) \subset \mathbb{E}(\zeta)$ for the next section where problems with variable coefficients are explored. Also, considering that the construction proposed in this section is global in nature and not a set of pieces assembled with a partition of unity, we shall assume that there are no glancing points for simplicity. Before looking at the construction of Σ_γ, let us see how Conditions (i) to (iv) in Definition 3.2.2 imply energy estimates for the $\mathcal{WR}$ class in the current situation. To shorten the notation, we shall often omit the independent variables and the parameter γ in the calculations ahead. We pursue the same strategy as in Chapter 2 and expand the term $d\langle \Sigma u, u\rangle / dx_d$ as shown:

$$\begin{aligned}
\frac{d}{dx_d}\langle \Sigma u, u\rangle &= \langle \Sigma \partial_d u, u\rangle + \langle \Sigma u, \partial_d u\rangle \\
&= 2\,\mathrm{Re}\,\langle \Sigma i D_d u, u\rangle \\
&= 2\,\mathrm{Re}\,\langle \Sigma i (f - \mathcal{A}u), u\rangle \\
&= 2\,\mathrm{Im}\,\langle \Sigma a u, u\rangle - 2\,\mathrm{Im}\,\langle \Sigma f, u\rangle.
\end{aligned}$$

Keeping in mind that $u \in \mathcal{D}(\mathbb{R}^{1+d})$ vanishes at infinity, an integration over $[0,\infty)$ with respect to x_d produces

$$\langle \Sigma u(0), u(0)\rangle = -2\int_0^\infty \mathrm{Im}\,\langle \Sigma \mathcal{A}u, u\rangle\, dx_d + 2\int_0^\infty \mathrm{Im}\,\langle \Sigma f, u\rangle\, dx_d.$$

To bound both integrals, we exploit Definition 3.2.2 directly. For example, from Condition (ii),

$$2\,\mathrm{Im}\,\langle \Sigma f, u\rangle \leq 2|\langle \Sigma f, u\rangle| \leq C_1|\Delta f||\Delta u|$$

for some positive constant C_1, whereas from (iii) it is clear that

$$\langle \Sigma u(0), u(0)\rangle \leq -c\gamma \int_0^\infty |\Delta u|^2\, dx_d + C_1\int_0^\infty |\Delta f||\Delta u|\, dx_d. \tag{3.20}$$

We can control (3.20) from below by means of (iv) in Definition 3.2.2, and from above via Young's inequality (see Theorem A.2.6), so that

$$\alpha|\Delta u(0)|^2 - \beta|QBu(0)|^2 \leq (-c\gamma + \varepsilon\gamma)\int_0^\infty |\Delta u|\, dx_d + \frac{C_1}{4\varepsilon\gamma}\int_0^\infty |\Delta f|^2\, dx_d.$$

Taking a sufficiently small value for ε implies that

$$\alpha|\Delta u(0)|^2 - \beta|QBu(0)|^2 \leq -C_2\gamma\int_0^\infty |\Delta u|^2\, dx_d + \frac{C_3}{\gamma}\int_0^\infty |\Delta f|^2\, dx_d,$$

for some constant C_2, or what is the same,

$$\gamma\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|\Delta u|^2\, dt\, dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|\Delta u(0)|^2\, dt\, dy \tag{3.21}$$
$$\leq C\left(\frac{1}{\gamma}\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|\Delta f|^2\, dt\, dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|QBu(0)|^2\, dt\, dy\right),$$

for some $C > 0$.

Having established energy estimates, let us exhibit a pseudodifferential operator Σ_γ with the properties listed in Definition 3.2.2. As we shall confirm soon, this can be achieved by slightly modifying the first part of the proof of Theorem 2.5.1, which we now quickly summarise for the benefit of the reader.

- ▷ We pick a hyperbolic frequency $\underline{\zeta} \in S^d$, and realise that $a(\zeta)$ is diagonalisable around $\underline{\zeta}$ with eigenvalues $a_1(\zeta), \cdots, a_n(\zeta)$ (counted according to their multiplicities) and eigenvectors

$$\mathbb{e}_0(\zeta) = \begin{pmatrix} e_1(\zeta) & \cdots & e_n(\zeta)\end{pmatrix}$$

  that depend smoothly on ζ.

- ▷ We can expand each $a_j(\zeta)$ as

$$a_j(\zeta) = a_j(\tilde{\zeta}) + i\gamma\kappa_j(\tilde{\zeta}) + \gamma^2 w_j(\zeta) \tag{3.22}$$

  using Taylor's theorem with respect to γ. Here, $\tilde{\zeta} \simeq (\tau, \eta)$.

- ▷ If $\kappa_j' := \kappa_j(\underline{\zeta})$, then $r := \operatorname{diag}(r_1, \cdots, r_n)$ with

$$r_j = \begin{cases} -1 & \text{for} \quad \kappa_j' > 0, \\ \rho & \text{for} \quad \kappa_j' < 0, \end{cases} \tag{3.23}$$

  where $\rho > 0$ is to be determined later.

Returning to the main question, let us set

$$\delta(\zeta) \equiv \operatorname{diag}(\delta_1(\zeta), \cdots, \delta_n(\zeta)) := \operatorname{diag}\left(\underline{\underline{\Delta}}(\zeta) I_p, I_{n-p}\right). \tag{3.24}$$

We seek σ in the form $\sigma = \delta^* r \delta$ and affirm that $\Sigma_\gamma = \mathrm{Op}(\sigma)$ meets Definition 3.2.2. For instance, Condition (i) is immediate from

$$\langle \Sigma v_1, v_2 \rangle = \langle \mathrm{Op}\,(r)\Delta v_1, \Delta v_2 \rangle \leq C|\Delta v_1||\Delta v_2|,$$

which is valid for arbitrary test functions v_1, v_2. In contrast, equations (3.22) and (3.23) yield component-wise

$$\mathrm{Im}\,\overline{\delta}_j r_j a_j \delta_j(\zeta) = \mathrm{Im}\left(r_j a_j(\widetilde{\zeta})|\delta_j(\zeta)|^2 + i\gamma r_j \kappa_j(\widetilde{\zeta})|\delta(\zeta)|^2 + r_j\gamma^2 w_j(\zeta)|\delta_j(\zeta)|^2 \right) \geq C\gamma|\delta_j(\zeta)|^2,$$

for the same reasons as the ones already outlined in Theorem 2.5.1. Plancherel's formula then completes Part (iii). To check the last item in Definition 3.2.2, let us suppose that v^- (resp. v^+) is the projection component of $v \in \mathbb{C}^n$ onto $\mathbb{E}^-(\zeta)$ (resp. $\mathbb{E}^+(\zeta)$) so that

$$v = \begin{pmatrix} v^- \\ v^+ \end{pmatrix}.$$

A direct calculation gives

$$\begin{aligned} \langle \sigma(\zeta)v, v \rangle = \langle \delta^*(\zeta) r \delta(\zeta) v, v \rangle &= -|\underline{\underline{\Delta}}(\zeta)v^-|^2 + \rho|v^+|^2 \\ &= \quad |\underline{\underline{\Delta}}(\zeta)v^-|^2 + \rho|v^+|^2 - 2|\underline{\underline{\Delta}}(\zeta)v^-|^2, \end{aligned} \tag{3.25}$$

which suggests using equation (3.7) to link (3.25) with the boundary matrix b. Indeed, borrowing $\mathfrak{p}(\zeta)$ and $c(\zeta)$ from Proposition 3.1.4, one has

$$\begin{aligned} \underline{\underline{\Delta}} I_p &= c(\zeta)(\underline{\underline{\Delta}} I_p)c^{-1}(\zeta) \\ &= c(\zeta) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \underline{\underline{\Delta}}(\zeta) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \underline{\underline{\Delta}}(\zeta) \end{pmatrix} \begin{pmatrix} \underline{\underline{\Delta}}(\zeta) & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} c^{-1}(\zeta) \qquad (3.26) \\ &= c(\zeta) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \underline{\underline{\Delta}}(\zeta) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \underline{\underline{\Delta}}(\zeta) \end{pmatrix} \mathfrak{p}^{-1}(\zeta)\mathfrak{p}(\zeta) \begin{pmatrix} \underline{\underline{\Delta}}(\zeta) & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} c^{-1}(\zeta) \\ &= q(\zeta) b^-(\zeta), \qquad (3.27) \end{aligned}$$

where $q(\zeta) := c(\zeta)\mathrm{diag}(1, \underline{\underline{\Delta}}(\zeta), \cdots, \underline{\underline{\Delta}}(\zeta))\mathfrak{p}^{-1}(\zeta)$. Armed with (3.26) and

$$b^+(\zeta) := (b(\zeta)e_{p+1}(\zeta), \cdots, b(\zeta)e_n(\zeta)),$$

we see that the first term in (3.25) unfolds as

$$
\begin{aligned}
|\underline{\underline{\Delta}}(\zeta)v^-|^2 &= \left| \begin{pmatrix} \underline{\underline{\Delta}}(\zeta)I_p & 0 \\ 0 & I_{n-p} \end{pmatrix} \begin{pmatrix} v^- \\ 0 \end{pmatrix} \right|^2 \\
&= \left| \begin{pmatrix} \underline{\underline{\Delta}}(\zeta)I_p & q(\zeta)b^+(\zeta) \\ 0 & I_{n-p} \end{pmatrix} \begin{pmatrix} v^- \\ 0 \end{pmatrix} \right|^2 \\
&= \left| \begin{pmatrix} \underline{\underline{\Delta}}(\zeta)I_p & q(\zeta)b^+(\zeta) \\ 0 & I_{n-p} \end{pmatrix} \begin{pmatrix} v^- \\ v^+ \end{pmatrix} - \begin{pmatrix} \underline{\underline{\Delta}}(\zeta)I_p & q(\zeta)b^+(\zeta) \\ 0 & I_{n-p} \end{pmatrix} \begin{pmatrix} 0 \\ v^+ \end{pmatrix} \right|^2 \\
&= \left| \begin{pmatrix} q(\zeta)b^-(\zeta) & q(\zeta)b^+(\zeta) \\ 0 & I_{n-p} \end{pmatrix} \begin{pmatrix} v^- \\ v^+ \end{pmatrix} - \begin{pmatrix} \underline{\underline{\Delta}}(\zeta)I_p & q(\zeta)b^+(\zeta) \\ 0 & I_{n-p} \end{pmatrix} \begin{pmatrix} 0 \\ v^+ \end{pmatrix} \right|^2 \\
&\le \left(\left| \begin{pmatrix} q(\zeta)b(\zeta)v \\ v^+ \end{pmatrix} \right| + C|v^+| \right)^2 \le C\left(|q(\zeta)b(\zeta)v|^2 + |v^+|^2 \right), \qquad (3.28)
\end{aligned}
$$

where the final line accounts for the convexity of the power function $x \mapsto x^2$. In the end, combining (3.25) and (3.28), we get

$$
\langle \sigma(\zeta)v, v \rangle \ge |\underline{\underline{\Delta}}(\zeta)v^-|^2 + (\rho - 2C)|v^+|^2 - 2C|q(\zeta)b(\zeta)v|^2,
$$

from which the result follows from choosing ρ such that $\alpha := \rho - 2C = 1$, $\beta := 2C$, and from Plancherel's theorem.

To close this section, we show how to recover estimates analogous to those in [BGS07], Chapter 8. In there, each term in the energy inequality is "filtered" by Δ, even the one containing the boundary condition Bu. For this purpose, let

$$
m(\zeta) := \begin{pmatrix} I_p & q(\zeta)b^+(\zeta) \end{pmatrix},
$$

and observe that

$$
\begin{aligned}
q(\zeta)b(\zeta) = \begin{pmatrix} q(\zeta)b^-(\zeta) & q(\zeta)b^+(\zeta) \end{pmatrix} &= \begin{pmatrix} I_p & q(\zeta)b^+(\zeta) \end{pmatrix} \begin{pmatrix} \delta^-(\zeta) & \\ & I_{n-p} \end{pmatrix} \\
&= m(\zeta)\delta(\zeta).
\end{aligned}
$$

If we take any $(n-p) \times n$ matrix $x(\zeta)$ such that $x(\zeta)$ is surjective and

$$
\begin{pmatrix} b(\zeta) \\ x(\zeta) \end{pmatrix}
$$

is nonsingular, there exist matrices $y(\zeta)$ and $\mathscr{d}(\zeta)$ with respective dimensions $n \times p$ and

$n \times (n-p)$, so that

$$I_n = \begin{pmatrix} b(\zeta) \\ x(\zeta) \end{pmatrix} \begin{pmatrix} y(\zeta) & d(\zeta) \end{pmatrix} = \begin{pmatrix} b(\zeta)y(\zeta) & b(\zeta)d(\zeta) \\ x(\zeta)y(\zeta) & x(\zeta)d(\zeta) \end{pmatrix}. \tag{3.29}$$

From (3.29), we may deduce that $b(\zeta)y(\zeta) = I_p$, and then that

$$q(\zeta)b(\zeta) = q(\zeta)b(\zeta)y(\zeta)b(\zeta) = m(\zeta)\delta(\zeta)y(\zeta)b(\zeta).$$

Finally, if $Y := \mathrm{Op}(y)$, $Q := \mathrm{Op}(q)$, and $M := \mathrm{Op}(m)$, by invoking Plancherel's theorem we conclude that $QB = M\Delta YB$ and

$$\begin{aligned} \gamma \int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|\Delta u|^2\,dt\,dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|\Delta u(0)|^2\,dt\,dy \\ \lesssim \left(\frac{1}{\gamma}\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|\Delta f|^2\,dt\,dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|\Delta QBu(0)|^2\,dt\,dy\right) \\ \lesssim \left(\frac{1}{\gamma}\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|\Delta f|^2\,dt\,dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|\Delta YBu(0)|^2\,dt\,dy\right), \end{aligned}$$

as desired.

3.3 The variable coefficients case

Our starting point will be once again the general boundary value problem

$$\begin{cases} P_\gamma u_\gamma(t,x) := \left(D_d + \mathcal{A}_\gamma(t,y,x_d,D_t,D_y)\right)u(t,x) = f(t,x) & (t,x) \in \mathbb{R}^{1+d}_+, \\ B_\gamma(t,y)u(t,y,0) = g(t,y) & (t,y) \in \mathbb{R}^d, \end{cases} \tag{3.30}$$

where $\mathcal{A}_\gamma \in \mathrm{OPS}^1(\mathbb{R}^{1+d}_+ \times [\gamma_0, +\infty))$ is a classical pseudodifferential operator whose symbol $a \in S^1_\gamma(\mathbb{R}^{1+d}_+ \times \mathbb{R}^d \times [\gamma_0, +\infty))$ is a matrix-valued function of dimension $n \times n$ that admits an asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_{1-j},$$

each a_{1-j} being homogeneous of degree $1-j$. Likewise, $B_\gamma \in \mathrm{OPS}^0(\mathbb{R}^d \times [\gamma_0, +\infty))$ is a classical pseudodifferential operator with a $p \times n$ principal part $b(X) \in S^0(\mathbb{R}^d \times \mathbb{R}^d \times [\gamma_0, +\infty))$. In addition, the source data f and g are chosen at least in L^2_γ.

We supplement Problem 3.30 with the following hypothesis.

Assumption 3.3.1.

(i) *P_γ is hyperbolic as in Definition 2.3.1.*

(ii) *P_γ and B_γ satisfy Property* **(C)**, *that is, a_1 and b do not depend on (t,x) outside certain compact set K.*

(iii) $b(X)$ *is everywhere of maximal rank* $p = \dim \mathbb{E}^-(X)$.

(iv) (P_γ, B_γ) *is* $\mathcal{WR}$.

3.3.1 Construction of a $\mathcal{WR}$ symmetriser and energy estimates for the general problem

Taking into account Proposition 3.2.1, we shall generalise the notion of a Lopatinskiĭ multiplier to fit Problem 3.3.1

Theorem 3.3.1. *Suppose that* (P_γ, B_γ) *is a* $\mathcal{WR}$ *boundary value problem furnished with Assumption 3.3.1. Let* $\underline{X} \in \mathcal{H}$. *Then there exist symbols* $\tilde{\mathfrak{e}}_0(X)$ *and* $\delta(X)$ *defined in some conic neighbourhood* $\mathcal{V}$ *of* $\underline{X}$ *such that for every* $X \in \mathcal{V}$,

(i) $\tilde{\mathfrak{e}}_0(X)$ *and* $\delta(X)$ *are homogeneous of degree* 0,

(ii) $\tilde{\mathfrak{e}}_0(X) \in GL_n(\mathbb{C})$ *and*

$$\dot{a}_1(X) := \tilde{\mathfrak{e}}_0^{-1}(X) a_1(X) \tilde{\mathfrak{e}}_0(X)$$

is diagonal with entries $a_{1,1}(X), \cdots, a_{1,n}(X)$,

(iii) there is $\mathfrak{s} \leq p$ *so that* $\delta(X)$ *is diagonal with respect to the basis* $\tilde{\mathfrak{e}}_0(X)$ *and given by*

$$\delta(X) = \operatorname{diag}(\delta^-(X), I_{n-p}),$$

where

$$\delta^-(X) := \begin{pmatrix} \begin{pmatrix} \delta_1^-(X) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_{\mathfrak{s}}^-(X) \end{pmatrix} & \\ & I_{p-\mathfrak{s}} \end{pmatrix},$$

with each δ_j^- *being the solution of the transport equation*

$$\begin{cases} \partial_d \delta_j^- + \{\delta_j^-, a_{1,j}\} = 0, \\ \delta_j^-|_{x_d=0} = \underline{\underline{\Delta}}, \end{cases} \tag{3.31}$$

(iv) when $X \in \mathbb{Y}_S \cap \mathcal{H}$, *there exist matrices* $q(X)$ *and* $m(X)$ *depending smoothly on* $X \in \mathcal{V} \cap \mathbb{Y}$ *with dimensions* $p \times p$ *and* $p \times n$, *respectively, so that if* $\dot{b}(X) := b(X)\tilde{\mathfrak{e}}_0(X)$, *there holds*

$$q(X)\dot{b}(X) = m(X)\delta(X), \tag{3.32}$$

(v) $\ker \delta(X) \neq \{0\}$ *if and only if* $X \in \Gamma$. *When nontrivial,* $\ker \delta(X)$ *is an* $\mathfrak{s}-$*dimensional invariant subspace of* $a_1(X)$ *containing the critical direction* $\ell(X)$.

Before entering into the proof of Theorem 3.3.1, we shall state an auxiliary result whose proof can be found in the appendix.

Lemma 3.3.1. *Let* V *be a finite dimensional vector space. Suppose* $T \in \operatorname{End}(V)$ *is diagonalis-*

able with distinct eigenvalues $\lambda_1, \cdots, \lambda_q$, and corresponding eigenspaces $V_{\lambda_1}, \cdots, V_{\lambda_q}$. Then every $T-$invariant subspace W can be decomposed as

$$W = (W \cap V_{\lambda_1}) \oplus \cdots \oplus (W \cap V_{\lambda_q}).$$

Proof of Theorem 3.3.1. For simplicity's sake, we shall split the argument into several steps.

Step 1. We classify points in $\mathcal{H}$. Our strategy focuses on defining $\delta(X)$ initially for $X = (t, y, x_d, \tau, \eta, \gamma)$ with x_d small, and then extend $\delta(X)$ in a constant way to larger values of x_d. That being said, let us assume that $\underline{X} \in \mathcal{H} \cap \mathbb{Y}$. We distinguish two cases, namely when $\underline{X}$ belongs to Γ and when it does not. In the latter, we choose $\tilde{e}_0(X)$ as predicted by the block structure condition (see Definition 2.4.2 and Proposition 2.4.1) and notice that the uniform Lopatinskiĭ condition is fulfilled. As a result, each $\delta_j(X)$, —which is guaranteed to exist locally by Picard-Lindelöf's theorem—, never vanishes in a small neighbourhood $\mathcal{V}$ of $\underline{X}$, for $\underline{\underline{\Delta}}(X)$ never vanishes in $\mathcal{V} \cap \mathbb{Y}$ either. Hence, $\delta(X)$ is nonsingular and it follows that equation (3.32) holds by choosing $q(X) = I_p$ and $m(X) = \dot{b}(X)\delta^{-1}(X)$. The remaining and most interesting case occurs therefore around points $\underline{X} \in \mathbb{Y} \cap \mathcal{H}$ where the Lopatinskiĭ determinant vanishes to the first order. We devote the rest of the proof to examine this situation.

Step 2. We find a suitable basis $\tilde{e}(X)$. Recall that $a_1(X)$ is smoothly diagonalisable around $\underline{X}$ in view of Proposition 2.4.2, meaning that for every X in a neighbourhood $\mathcal{V}$ of $\underline{X}$ there exist eigenvalues

$$a_{1,1}(X), \cdots, a_{1,n}(X)$$

(counted according to their multiplicities) and eigenvectors $e_1(X), \cdots, e_n(X)$ organised as the columns of a nonsingular matrix $\mathfrak{e}_0(X)$, so that

$$\mathfrak{e}_0^{-1}(X) a_1(X) \mathfrak{e}_0(X) = \begin{pmatrix} a_{1,1}(X) & 0 & \cdots & 0 \\ 0 & a_{1,2}(X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{1,n}(X) \end{pmatrix}.$$

For $X \in \mathcal{V}$, let $K_\ell(X)$ be the Krylov space of $\ell(X)$ with respect to $a_1(X)$. Admitting that the first p columns of $\mathfrak{e}_0(X)$ span the stable subspace $\mathbb{E}^-(X)$ and that repeated eigenvalues are adjacent, we explain how to pick a different basis for $\mathbb{E}^-(X)$ that interacts nicely with $K_\ell(X)$. To this end, let $\mu_1(X), \cdots, \mu_q(X)$ be pairwise different eigenvalues of $a_1(X)$ with multiplicities $\alpha_1, \cdots, \alpha_q$. For every $k \in \{1, \cdots, q\}$, we can find a positive integer $i_k \leq n$ such that

$$\mu_k(X) = a_{1,i_k}(X) = a_{1,i_k+1}(X) = \cdots = a_{1,i_k+\alpha_k-1}(X),$$

with associated eigenspace $V_k(X) = \operatorname{span}\{e_{i_k}(X), \cdots, e_{i_k+\alpha_k-1}(X)\}$. With this at hand, Lemma 3.3.1 suggests that $K_\ell(X)$ can be decomposed as

$$K_\ell(X) = K_\ell(X) \cap V_1(X) \oplus \cdots \oplus K_\ell(X) \cap V_q(X),$$

where $K_\ell(X) \cap V_k(X)$ is trivial for every $V_k(X) \subset \mathbb{E}^+(X)$, since $K_\ell(X) \subseteq \mathbb{E}^-(X)$ for every $X \in \mathcal{V}$. If $K_\ell(X) \cap V_k(X) \neq \{0\}$, we can choose $\tilde{e}_k(X) \in K_\ell(X) \cap V_k(X)$ and use it to replace an existing element in $e(X)$ in such a way that the resulting set $\tilde{e}_0(X)$ is still a basis of $\mathbb{C}^n$. Thus, if $\mathscr{s} = \mathscr{s}(X)$ is the number of non-zero coefficients from $d_1(X), \cdots, d_p(X)$ in (3.12), after rearranging components if necessary, the new basis $\tilde{e}_0(X)$ consists of eigenvectors of $a_1(X)$ whose first s elements span $K_\ell(X)$. To put it differently,

$$\begin{aligned}\mathbb{C}^n &= \operatorname{span}\{\tilde{e}_1(X), \cdots, \tilde{e}_{\mathscr{s}}(X)\} \oplus \operatorname{span}\{\tilde{e}_{\mathscr{s}+1}(X), \cdots, \tilde{e}_n(X)\} \\ &= K_\ell(X) \oplus \operatorname{span}\{\tilde{e}_{\mathscr{s}+1}(X), \cdots, \tilde{e}_n(X)\}.\end{aligned}$$

Looking ahead to future stages of this proof, it is of primary interest to us that only one value of $\mathscr{s}$ is chosen for the whole neighbourhood $\mathcal{V}$. This is generally the case, save possibly when a coefficient d_i vanishes point-wise at X while not being identically zero in $\mathcal{V}$. In these circumstances, $K_\ell(X) := \operatorname{span}\{\tilde{e}_1(X), \cdots, \tilde{e}_{\mathscr{s}}(X)\}$ with $\mathscr{s}$ being the largest value we encounter while X ranges on $\mathcal{V}$ (although $K_\ell(X)$ is no longer the smallest subspace containing $\ell(X)$ for every $X \in \mathcal{V}$, this will suffice for our plans).

Step 3. We define $\delta(X)$. Having picked an appropriate basis in the previous passage, we can define

$$\delta(X) = \operatorname{diag}(\delta^-(X), I_{n-p}),$$

with

$$\delta^-(X) := \begin{pmatrix} \begin{pmatrix} \delta_1^-(X) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_{\mathscr{s}}^-(X) \end{pmatrix} & \\ & I_{p-\mathscr{s}} \end{pmatrix}. \tag{3.33}$$

Each $\delta_i^-(X)$ in (3.33) solves locally the transport equation

$$\begin{cases} \partial_d \delta_j^- + \{\delta_j^-, a_{1,j}\} = 0, \\ \qquad \delta_j^-|_{x_d=0} = \underline{\underline{\Delta}}, \end{cases} \tag{3.34}$$

whose characteristic curves coincide with the integral curves of the Hamiltonian system

$$\begin{cases} \dfrac{dy_k}{dx_d} = \dfrac{\partial a_{1,j}}{\partial \eta_k}, \\ \dfrac{d\eta_k}{dx_d} = -\dfrac{\partial a_{1,j}}{\partial y_k}, \\ (y_0, \cdots, y_{d-1}, 0, \eta_0, \cdots, \eta_{d-1}, \gamma) \in \mathcal{V} \cap \mathbb{Y}, \end{cases} \tag{3.35}$$

provided that we interpret x_0 as t and η_0 as τ. The preceding set of equations does not impose any restriction on γ, so we are free to complement (3.35) with the natural assumption that $d\gamma/dx_d = 0$ along the bicharacteristic curves. As before, the existence of such $\delta_j^-(X)$ in (perhaps a smaller) $\mathcal{V}$ is justified by Picard-Lindelöf's theorem (see Theorem A.2.4).

Step 4. We link $\delta(X)$ and $b^-(X)$. To investigate the behaviour of $\delta(X)$ on the boundary $\{x_d = 0\}$, we place ourselves at any point $X \in \mathcal{V} \cap \mathbb{Y}$ and take

$$c(X) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -d_2(X) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_p(X) & 0 & \cdots & 1 \end{pmatrix} \tag{3.36}$$

as in the proof of Proposition 3.1.4 (with $\tilde{e}(X)$ as the underlying basis). In Step 2, we arranged the columns of $\tilde{e}(X)$ in a way that all the nonvanishing elements in $d_1, \cdots, d_p$ are written in the upper left part of $c(X)$. In other words, $c(X)$ can be seen as a block diagonal matrix

$$c(X) = \operatorname{diag}(c_{\mathscr{s}}(X), I_{p-\mathscr{s}})$$

with

$$c_{\mathscr{s}}(X) := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -d_2(X) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_{\mathscr{s}}(X) & 0 & \cdots & 1 \end{pmatrix}. \tag{3.37}$$

Meanwhile, since $\delta_j^-(X) = \Delta(X)$ when $X \in \mathcal{V} \cap \mathbb{Y}$ for every $j \in \{1, \cdots, \mathscr{s}\}$,

$$\delta(X) = \operatorname{diag}\left(\underline{\underline{\Delta}}(X) I_{\mathscr{s}}, I_{p-\mathscr{s}}, I_{n-p}\right).$$

We make use of $c(X)$ and its properties to define

$$\mathfrak{s}(X) = \operatorname{diag}\left(c(X), I_{n-p}\right),$$

which is nonsingular and hence could be regarded as a legitimate change of variables. What is more, $\mathfrak{s}(X)\delta(X)\mathfrak{s}^{-1}(X)$ is the product of commuting diagonal blocks, from which it easily follows that $\delta(X)$ is invariant under conjugation by $\mathfrak{s}(X)$. Let $v_1(X)$ and $v_2(X)$ be square matrices of dimension $\mathscr{s} \times \mathscr{s}$ such that

$$v_1(X) := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \underline{\underline{\Delta}}(X) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \underline{\underline{\Delta}}(X) \end{pmatrix} \quad \text{and} \quad v_2(X) := \begin{pmatrix} \underline{\underline{\Delta}}(X) & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Note that $\delta(X)$ can be factored as

$$\delta(X) = \begin{pmatrix} \underline{\underline{\Delta}}(X) I_{\mathscr{s}} & & \\ & I_{p-\mathscr{s}} & \\ & & I_{n-p} \end{pmatrix} = \begin{pmatrix} v_1(X) v_2(X) & & \\ & I_{p-\mathscr{s}} & \\ & & I_{n-p} \end{pmatrix}, \tag{3.38}$$

so

$$\begin{aligned} \delta(X) &= \mathfrak{s}(X) \begin{pmatrix} v_1(X) & & \\ & I_{p-\mathscr{s}} & \\ & & I_{n-p} \end{pmatrix} \begin{pmatrix} v_2(X) & & \\ & I_{p-\mathscr{s}} & \\ & & I_{n-p} \end{pmatrix} \mathfrak{s}^{-1}(X) \qquad (3.39) \\ &= \begin{pmatrix} \begin{pmatrix} c_s(X) v_1(X) & \\ & I_{p-\mathscr{s}} \end{pmatrix} & \\ & I_{n-p} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} v_2(X) & \\ & I_{p-\mathscr{s}} \end{pmatrix} c^{-1}(X) & \\ & I_{n-p} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} c_{\mathscr{s}}(X) v_1(X) & \\ & I_{p-\mathscr{s}} \end{pmatrix} \mathfrak{p}^{-1}(X) & \\ & I_{n-p} \end{pmatrix} \begin{pmatrix} \mathfrak{p}(X) \begin{pmatrix} \underline{\underline{\Delta}}(X) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} c^{-1}(X) & \\ & I_{n-p} \end{pmatrix}. \end{aligned}$$

In this manner, if

$$q(X) := \begin{pmatrix} c_{\mathscr{s}}(X) v_1(X) & \\ & I_{p-\mathscr{s}} \end{pmatrix} \mathfrak{p}^{-1}(X),$$

Proposition 3.1.4 and equation (3.39) enable us to conclude that

$$\delta(X) = \operatorname{diag}\left(q(X) b^-(X), I_{n-p}\right).$$

To finalise, let

$$b^+(X) := \begin{pmatrix} b(X)\tilde{e}_{p+1}(X) & \cdots & b(X)\tilde{e}_n(X) \end{pmatrix}.$$

If $m(X) := \begin{pmatrix} I_p & q(X) b^+(X) \end{pmatrix}$, then

$$\begin{aligned} q(X) b(X) = \begin{pmatrix} q(X) b^-(X) & q(X) b^+(X) \end{pmatrix} &= \begin{pmatrix} I_p & q(X) b^+(X) \end{pmatrix} \begin{pmatrix} \delta^-(X) & \\ & I_{n-p} \end{pmatrix} \\ &= m(X) \delta(X). \end{aligned}$$

Step 6. $\delta(X)$ degenerates at critical points. By construction, $\ker \delta(X)$ is nontrivial if and only if $\underline{\underline{\Delta}}(X) = 0$, i.e., if and only if $X \in \Gamma$. That $\ker \delta(X)$ is $\mathfrak{s}-$dimensional is evident from Step 2. $\square$

Definition 3.3.1. *A collection of pseudodifferential operators $\Delta_\gamma(t, x, D_t, D_y) \in \mathrm{OPS}^0_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0, +\infty))$ is a Lopatinskiĭ family of operators if $\Delta_\gamma := \mathrm{Op}_\gamma(\delta)$ and $\delta(X) \equiv \delta_\gamma(t, x, \tau, \eta)$ satisfies Theorem 3.3.1.*

Before continuing further, we present and prove a crucial lemma that simplifies the zeroth-order terms in Problem 3.30. Morally, if $\mathcal{E}_{0,\gamma}$ is a pseudodifferential operator whose symbol is the nonsingular matrix $\tilde{e}_0(X)$ found in Theorem 3.3.1, we look for a correction of $\mathcal{E}_{0,\gamma}$ by an operator of order -1, say $\mathcal{E}_{-1,\gamma}$, for which $D_d + A_\gamma$ is block diagonal up to an error of order -1. Specifically, we have:

Lemma 3.3.2 (Lemma 1, [Cou04]). *Consider (P_γ, B_γ) as in Problem (3.30) with a_1 and a_0 being the first two elements of the asymptotic expansion of a. Under the notation of Theorem 3.3.1, we can define a symbol $e_{-1}(X)$ on $\mathcal{V}$ such that $e_{-1}(X)$ is homogeneous of order -1 and*

$$(\tilde{e}_0 + e_{-1})(a_1 + a_0) - (\dot{a}_1 + \ddot{a}_0)(\tilde{e}_0 + e_{-1}) + D_d\tilde{e}_0 + \frac{1}{i}\sum_{k=0}^{d-1}\left(\partial_{\eta_k}\tilde{e}_0\partial_{x_k}\dot{a}_1 - \partial_{\eta_k}\dot{a}_1\partial_{x_k}\tilde{e}_0\right)$$

is a symbol of order -1, where $\ddot{a}_0$ is a block diagonal symbol of order 0 with blocks having dimensions $\alpha_1, \cdots, \alpha_q$ as those of $\dot{a}_1$.

Proof. Let e_{-1} be a symbol of order -1 to be determined. A first-order approximation of $(\tilde{e}_0 + e_{-1})^{-1}$ shows that

$$(\tilde{e}_0 + e_{-1})(\tilde{e}_0^{-1} - \tilde{e}_0^{-1}e_{-1}\tilde{e}_0^{-1}) = I_n \mod S^{-1}_\gamma, \tag{3.40}$$

so $(\tilde{e}_0 + e_{-1})(\xi_d I_n + a_1 + a_0)(\tilde{e}_0 + e_{-1})^{-1}$ can be estimated up to an error of order -1 by

$$\begin{aligned}&(\tilde{e}_0 + e_{-1})(\xi_d I_n + a_1 + a_0)(\tilde{e}_0^{-1} - \tilde{e}_0^{-1}e_{-1}\tilde{e}_0^{-1})\\ &\quad = (\xi_d I_n + \tilde{e}_0 a_1\tilde{e}_0^{-1} - \tilde{e}_0 a_1\tilde{e}_0^{-1}e_{-1}\tilde{e}_0^{-1} + e_{-1}a_1\tilde{e}_0^{-1} + \tilde{e}_0 a_0\tilde{e}_0^{-1}) \mod S^{-1}_\gamma.\end{aligned}$$

Since $\tilde{e}_0 a_1 \tilde{e}_0^{-1} = \dot{a}_1$, it is true that

$$\tilde{e}_0 a_1\tilde{e}_0^{-1}e_{-1}\tilde{e}_0^{-1} - e_{-1}a_1\tilde{e}_0^{-1} = \dot{a}_1 e_{-1}\tilde{e}_0^{-1} - e_{-1}\tilde{e}_0^{-1}\dot{a}_1 = [\dot{a}_1, e_{-1}\tilde{e}_0^{-1}],$$

and consequently

$$(\tilde{e}_0 + e_{-1})(\xi_d I_n + a_1 + a_0)(\tilde{e}_0^{-1} - \tilde{e}_0^{-1}e_{-1}\tilde{e}_0^{-1}) = \xi_d I_n + \dot{a}_1 - [\dot{a}_1, e_{-1}\tilde{e}_0^{-1}] + \tilde{e}_0 a_0\tilde{e}_0^{-1}$$

modulo S^{-1}_γ. As we always do with involved computations, we shall omit the parameter γ to facilitate the exposition. That being so, let $\dot{A}_1 := \mathrm{Op}(\dot{a}_1)$ and $\ddot{A}_0 := \mathrm{Op}(\ddot{a}_0)$. We now utilise the usual symbolic calculus on the operator equation

$$(\mathcal{E}_0+\mathcal{E}_{-1})(D_d+\mathcal{A})-(D_d+(\dot{A}_1+\dot{A}_0))(\mathcal{E}_0+\mathcal{E}_{-1})=0 \mod \Psi_{-1} \tag{3.41}$$

to derive precise conditions on $\ddot{a}_0$. As a matter of fact, a first-order expansion of the symbol of (3.41) yields

$$\begin{aligned}(\tilde{e}_0+e_{-1})(a_1+a_0)-(\dot{a}_1+\ddot{a}_0)(\tilde{e}_0+e_{-1})\\ +D_d\tilde{e}_0+\frac{1}{i}\sum_{k=0}^{d-1}\left(\partial_{\eta_k}\tilde{e}_0\partial_{x_k}a_1-\partial_{\eta_k}\dot{a}_1\partial_{x_k}\tilde{e}_0\right)=0,\end{aligned} \tag{3.42}$$

or more concisely,

$$-[\dot{a}_1,e_{-1}\tilde{e}_0^{-1}]\tilde{e}_0+\tilde{e}_0a_0-\ddot{a}_0\tilde{e}_0+D_d\tilde{e}_0+\frac{1}{i}\sum_{k=0}^{d-1}\left(\partial_{\eta_k}\tilde{e}_0\partial_{x_k}a_1-\partial_{\eta_k}\dot{a}_1\partial_{x_k}\tilde{e}_0\right)=0 \mod \Psi_{-1}. \tag{3.43}$$

Alternatively, if we multiply (3.43) from the right by $\tilde{e}_0^{-1}$ and put $\dot{a}_0=e_0a_0e_0^{-1}$, then

$$\ddot{a}_0=-[\dot{a}_1,e_{-1}\tilde{e}_0^{-1}]+\dot{a}_0+(D_d\tilde{e}_0)\tilde{e}_0^{-1}+\frac{1}{i}\sum_{k=0}^{d-1}\left(\partial_{\eta_k}\tilde{e}_0\partial_{x_k}a_1-\partial_{\eta_k}\dot{a}_1\partial_{x_k}\tilde{e}_0\right)\tilde{e}_0^{-1} \qquad \mod \Psi_{-1}.$$

In general, there is no reason to expect that $\ddot{a}_0$ above is block diagonal. Yet, we can choose the off-diagonal entries of $[\dot{a}_1,e_{-1}\tilde{e}_0^{-1}]$ (it is worth remembering that $[\dot{a}_1,e_{-1}\tilde{e}_0^{-1}]$ has zero diagonal) to compensate those of

$$\dot{a}_0+(D_d\tilde{e}_0)\tilde{e}_0^{-1}+\frac{1}{i}\sum_{k=0}^{d-1}\left(\partial_{\eta_k}\tilde{e}_0\partial_{x_k}a_1-\partial_{\eta_k}\dot{a}_1\partial_{x_k}\tilde{e}_0\right)\tilde{e}_0^{-1}.$$

As all the terms we have neglected so far are of lower order, the operator $D_d+\dot{A}_1+\ddot{A}_0$ is a block diagonalisation of $D_d+\mathcal{A}_\gamma$ modulo an error of order -1. □

We continue with the main result of this chapter, namely, the derivation of energy inequalities for the $\mathcal{WR}$ class. The details are summarised in the statement below.

Theorem 3.3.2. *Let*

$$\begin{cases}P_\gamma u_\gamma(t,x):=\big(D_d+\mathcal{A}_\gamma(t,y,x_d,D_t,D_y)\big)\,u(t,x)=f(t,x) & (t,x)\in\mathbb{R}^{1+d}_+,\\ B_\gamma(t,y)u(t,y,0)=g(t,y) & (t,y)\in\mathbb{R}^d,\end{cases} \tag{3.44}$$

*where $\mathcal{A}_\gamma\in\mathrm{OPS}^1_\gamma(\mathbb{R}^{1+d}_+\times[1,+\infty))$ and $B_\gamma\in\mathrm{OPS}^0_\gamma(\mathbb{R}^d\times[1,+\infty))$ are classical pseudo-differential operators with matrix-valued symbols $a(X)$ and $b(X)$ of dimensions $n\times n$ and $p\times n$, respectively. Suppose that P_γ is hyperbolic in the sense of Definition 2.3.1, P_γ and B_γ satisfy Property **(C)**, and that $p=\dim\mathbb{E}^-(X)$. Then there exist* *(i)* $\gamma_0 \geq 1$,

(ii) a family of pseudodifferential operators

$$\Delta_\gamma(t, x, D_t, D_y) \in \mathrm{OPS}^0_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0, +\infty)),$$

(iii) function spaces

$$L^2_\Delta := \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Delta_\gamma v \in L^2_\gamma(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\},$$
$$H^s_\Delta := \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Lambda^s_\gamma v \in L^2_\Delta(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\},$$

(iv) and a positive constant C such that,

if $f \in L^2_\gamma(\mathbb{R}^{1+d}_+)$ and $g \in L^2_\gamma(\mathbb{R}^d)$, then for all $\gamma > \gamma_0$ and every $u \in \mathcal{D}(\mathbb{R}^{1+d}_+)$ the following estimate holds

$$\gamma\|\Delta_\gamma u\|^2_{0,\gamma} + |\Delta_\gamma u(0)|^2_{0,\gamma} \leq C\left(\frac{1}{\gamma}\|f\|^2_{0,\gamma} + |g|^2_{0,\gamma}\right). \tag{3.45}$$

More generally, if $f \in H^s_\gamma(\mathbb{R}^{1+d}_+)$ and $g \in H^s_\gamma(\mathbb{R}^d)$,

$$\gamma\|\Delta_\gamma u\|^2_{s,\gamma} + |\Delta_\gamma u(0)|^2_{s,\gamma} \leq C\left(\frac{1}{\gamma}\|f\|^2_{s,\gamma} + |g|^2_{s,\gamma}\right). \tag{3.46}$$

Proof. In the interest of not overloading the notation, we adhere to the following conventions:

- ▷ We shall suppress the parameter γ all through the calculations, except when its presence is relevant to the point being made (e.g. when we wish to emphasise the existence of a parameter-dependent family of pseudodifferential operators).
- ▷ Ψ_m represents an error of order m that may be different from line to line.
- ▷ When it comes to norms, we shall write $\|\cdot\|_{s,\gamma} = \|\cdot\|_s$ or $\|\cdot\|_{0,\gamma} = \|\cdot\|$ when $s = 0$ (resp. $|\cdot|_{s,\gamma} = |\cdot|_s$ or $|\cdot|_{0,\gamma} = |\cdot|$ when $s = 0$).
- ▷ We shall adopt $\mathcal{A}_\gamma(x_d)$ and $\Delta_\gamma(x_d)$ as a substitute for

$$\mathcal{A}_\gamma(t, y, x_d, D_t, D_y) \qquad \text{and} \qquad \Delta_\gamma(t, y, x_d, D_t, D_y).$$

Once again, we shall divide the analysis into steps for ease of explanation.

Step 1. We pick a pseudodifferential partition of unity. Thanks to homogeneity and Property **(C)**, we can restrict our attention to a compact region $K \times S^d \subset \mathbb{X}_S$, which we may cover with finitely many neighbourhoods $\{\mathcal{V}_i\}_{i\in I}$ as shown in Theorem 3.3.1. Let $\{\varphi_i\}_{i\in I}$ be a partition of unity subordinate to $\{\underline{\mathcal{V}}_i\}_{i\in I}$ and $\{\theta_i\}_{i\in I}$ be a system of functions such that $\theta_i \in C^\infty_c(\mathcal{V}_i)$, and $\theta_i \equiv 1$ in a vicinity of supp φ_i. In addition, let us assume that Θ_i and Φ_i are pseudodifferential operators whose symbols are the extensions of θ_i and φ_i

to homogeneous functions of degree 0 in $\zeta = (\tau, \eta, \gamma)$.

Step 2. We perform a change of variables in Problem 3.44. To begin with, let us fix $\mathcal{V} \equiv \mathcal{V}_i$, $\theta \equiv \theta_i$, and take $\tilde{e}_0 \equiv \tilde{e}_{i,0}$ as in Theorem 3.3.1. Putting $\mathcal{E}_0 := \mathrm{Op}\,(\theta \tilde{e}_0)$ and $\mathcal{E}_0^{-1} := \mathrm{Op}\,(\theta \tilde{e}_0^{-1})$, it is readily verified that $\mathcal{E}_0 \mathcal{E}_0^{-1} = I_n \mod \Psi_{-1}$, which justifies the abuse of notation in writing $\mathcal{E}_0^{-1}$ to refer to $\mathrm{Op}\,(\theta \tilde{e}_0^{-1})$ (in rigour, only the inverse of $\mathcal{E}_0$ mod Ψ_{-1}). Applying $\mathcal{E}_0^{-1}$ on both sides of $(D_d + \mathcal{A})u = f$ bring us to

$$\mathcal{E}_0^{-1} D_d u + \mathcal{E}_0^{-1} \mathcal{A} u = \mathcal{E}_0^{-1} f, \tag{3.47}$$

or alternatively to

$$D_d \mathcal{E}_0^{-1} u + \mathcal{E}_0^{-1} \mathcal{A} u + [\mathcal{E}_0^{-1}, D_d] u = \mathcal{E}_0^{-1} f. \tag{3.48}$$

The equivalence

$$\mathcal{E}_0^{-1}\mathcal{A}u = \mathcal{E}_0^{-1}\mathcal{A}\mathcal{E}_0\mathcal{E}_0^{-1}u + \mathcal{E}_0^{-1}\mathcal{A}\Psi_{-1}u = \mathcal{E}_0^{-1}\mathcal{A}\mathcal{E}_0\mathcal{E}_0^{-1}u + \mathcal{E}_0^{-1}\mathcal{A}\Psi_{-1}\mathcal{E}_0\mathcal{E}_0^{-1}u + \mathcal{E}_0^{-1}\mathcal{A}\Psi_{-2}u$$

modulo an error of order -1 let us recast (3.48) succinctly as

$$(D_d + \dot{\mathcal{A}} + E_0)\dot{u} = \dot{f} \qquad \mod \Psi_{-1},$$

with $\dot{u} := \mathcal{E}_0^{-1} u$, $\dot{\mathcal{A}} := \mathcal{E}_0^{-1} \mathcal{A} \mathcal{E}_0$, $E_0 := [\mathcal{E}_0^{-1}, D_d]\mathcal{E}_0 + \mathcal{E}_0^{-1}\mathcal{A}\Psi_{-1}\mathcal{E}_0$, and $\dot{f} := \mathcal{E}_0^{-1} f$. If we think of E_0 as part of $\dot{\mathcal{A}}$, Lemma 3.3.2 implies the existence of a refined basis $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_{-1}$ with respect to which $D_d + \underline{\mathcal{A}}$ is a block diagonalisation of $D_d + \dot{\mathcal{A}} + E_0$ modulo Ψ_{-1}. In the same vein, if $\dot{B} := B\mathcal{E}$, we notice that $\dot{B}\mathcal{E}^{-1} := (B\mathcal{E})\mathcal{E}^{-1}$ differs from B by an error of order -1. The resulting operator is

$$\begin{cases} \underline{P}\dot{u}(t,x) := (D_d + \underline{\mathcal{A}})\,\dot{u}(t,x) = \dot{f}(t,x), \\ \qquad\qquad\qquad\qquad \dot{B}\dot{u}(t,y,0) = \dot{g}(t,y), \end{cases} \tag{3.49}$$

where $\dot{g} \equiv g$, and $\underline{\mathcal{A}}$ is a classical pseudodifferential operator with symbol

$$\underline{a} \sim \dot{a}_1 + \ddot{a}_0 + \cdots,$$

with $\dot{a}_1$ and $\ddot{a}_0$ being block diagonal.

Step 3. We localise u by means of Φ. To do so, we fix i such that $\Phi \equiv \Phi_i$ and observe that the commutator relations

$$\begin{aligned} (D_d + \underline{\mathcal{A}})\Phi &= \Phi(D_d + \underline{\mathcal{A}}) + [(D_d + \underline{\mathcal{A}}), \Phi], \\ \dot{B}\Phi &= \Phi\dot{B} + [\dot{B}, \Phi], \end{aligned}$$

enable us to formulate (3.49) in terms of $\tilde{u} = \Phi\dot{u}$ at the expense of a zeroth-order term $[\underline{P}, \Phi]$ and a harmless error $[\dot{B}, \Phi]$ of order -1 to be analysed shortly. Thus, we are left with

$$\begin{cases} \underline{P}\tilde{u} = (D_d + \underline{\mathcal{A}})\,\tilde{u}(t,x) = \tilde{f}(t,x) \\ \qquad\qquad \dot{B}\tilde{u}(t,y,0) = \tilde{g}(t,y), \end{cases} \tag{3.50}$$

where the $\sim$ everywhere refers to the application of the operator Φ. In this setting, we can define pseudodifferential operators $Q := \mathrm{Op}(\theta q)$, $\Delta := \mathrm{Op}(\theta\delta)$, $M := \mathrm{Op}(\theta m)$ and use them to set the auxiliary system

$$\begin{cases} (D_d + \underline{\mathcal{A}})\,\tilde{w}(t,x) = \Delta\tilde{f}(t,x), \\ \qquad M\tilde{w}(t,y,0) = Q\tilde{g}(t,y). \end{cases} \tag{3.51}$$

Interestingly, (3.51) satisfies the uniform Lopatinskiĭ condition, for $m(X)$ restricted to the stable subspace of $\dot{a}_1$ is the identity I_p (see Step 5 in Theorem 3.3.1). This observation paves the way for the rest of the proof, as we shall soon confirm.

Step 4. We establish the existence of a local $\mathcal{WR}$ symmetriser. To leave no room for ambiguity, let us momentarily reintegrate the parameter γ to describe the spaces

$$\begin{aligned} L^2_\Delta(\mathbb{R}^{1+d}_+) &:= \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Delta_\gamma v \in L^2_\gamma(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\}, \\ H^s_\Delta(\mathbb{R}^{1+d}_+) &:= \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Lambda^s_\gamma v \in L^2_\Delta(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\}, \end{aligned}$$

and introduce the next definition.

Definition 3.3.2. *A $\mathcal{WR}$ symmetriser for Problem* (3.50) *is a family of pseudodifferential operators $\Sigma_\gamma \in \mathrm{OPS}^0_\gamma(\mathbb{R}^{1+d}_+ \times [\gamma_0, +\infty))$ for some $\gamma_0 \geq 1$, such that for all $\gamma \geq \gamma_0 \geq 1$,*

i) $\Sigma_\gamma(x_d)$ is self-adjoint,

ii) for every $v_1, v_2 \in L^2_\Delta$, there is a positive constant C satisfying

$$\langle \Sigma_\gamma(x_d)v_1, v_2\rangle \leq C|\Delta_\gamma(x_d)v_1||\Delta_\gamma(x_d)v_2|,$$

iii) there is a positive constant c, independent of x_d, so that

$$\langle \partial_d\Sigma_\gamma(x_d)v, v\rangle + 2\,\mathrm{Im}\,\langle \Sigma_\gamma(x_d)\underline{\mathcal{A}}_\gamma(x_d)v, v\rangle \geq c\gamma|\Delta_\gamma(x_d)v|^2$$

for each $v \in L^2_\Delta(\mathbb{R}^{1+d}_+)$,

iv) there exist positive constants α, β and a family of pseudodifferential operators Q_γ for which

$$\langle \Sigma_\gamma(0)v, v\rangle \geq \alpha|\Delta_\gamma(0)v|^2 - \beta|Q_\gamma\dot{B}_\gamma v|^2$$

holds true for every $v \in L^2_\Delta(\mathbb{R}^d)$.

Let us drop x_d and γ once more to make the idea smoother. A quick glance at the proof of Theorem 2.5.1 indicates that is possible to find a functional symmetriser for the auxiliary problem (3.51). If $R := \mathrm{Op}(\theta r)$ is such a symmetrizer, we claim that $\Sigma = \Delta^* R\Delta$ meets (i) to (iv) in Definition 3.3.2. That Σ is self-adjoint follows because R is self-adjoint. Condition (ii), on the other hand, stems from the elementary computation

$$\langle \Sigma v_1, v_2\rangle = \langle \Delta^* R\Delta v_1, v_2\rangle = \langle R\Delta v_1, \Delta v_2\rangle \leq C|\Delta v_1||\Delta v_2|,$$

valid for test functions v_1, v_2 supported on $\mathcal{V}$ and some constant $C > 0$. The remaining properties can be obtained from those of R and Δ as explained hereafter. Firstly,

$$\begin{aligned}\langle \partial_d \Sigma v, v\rangle + 2\,\mathrm{Im}\,\langle \Sigma \underline{\mathcal{A}} v, v\rangle &= \langle (\partial_d R)\Delta v, \Delta v\rangle + 2\,\mathrm{Re}\,\langle R(\partial_d \Delta) v, \Delta v\rangle + 2\,\mathrm{Im}\,\langle R\Delta\underline{\mathcal{A}} v, \Delta v\rangle \\ &= \langle (\partial_d R)\Delta v, \Delta v\rangle + 2\,\mathrm{Re}\,\langle R(\mathrm{Op}\,(\partial_d \delta)) v, \Delta v\rangle + 2\,\mathrm{Im}\,\langle R\Delta\underline{\mathcal{A}} v, \Delta v\rangle.\end{aligned}$$

Secondly,

$$\begin{aligned}\Delta\underline{\mathcal{A}} &= \underline{\mathcal{A}}\Delta + [\Delta, \underline{\mathcal{A}}] \\ &= \underline{\mathcal{A}}\Delta + \mathrm{Op}([\delta, \dot{a}_1] + [\delta, \ddot{a}_0] - i\{\dot{a}_1, \delta\}) + \Psi_{-1} = \underline{\mathcal{A}}\Delta - i\,\mathrm{Op}(\{\dot{a}_1, \delta\}) + \Psi_{-1},\end{aligned} \tag{3.52}$$

given that $[\delta, \dot{a}_1]$ and $[\delta, \ddot{a}_0]$ vanish identically in light of Lemma 3.3.2. Inserting (3.52) into $2\,\mathrm{Im}\,\langle R\Delta\underline{\mathcal{A}} v, \Delta v\rangle$ gives

$$\begin{aligned}2\,\mathrm{Im}\,\langle R\Delta\underline{\mathcal{A}} v, \Delta v\rangle &= 2\,\mathrm{Im}\,\langle R\underline{\mathcal{A}}\Delta v, \Delta v\rangle + 2\,\mathrm{Im}\,\langle R[\Delta, \underline{\mathcal{A}}] v, \Delta v\rangle \\ &= 2\,\mathrm{Im}\,\langle R\underline{\mathcal{A}}\Delta v, \Delta v\rangle - 2\,\mathrm{Re}\,\langle R\,\mathrm{Op}(\{\dot{a}_1, \delta\}) v, \Delta v\rangle + \langle \Psi_{-1} v, \Delta v\rangle, \\ &= 2\,\mathrm{Im}\,\langle R\underline{\mathcal{A}}\Delta v, \Delta v\rangle + 2\,\mathrm{Re}\,\langle R\,\mathrm{Op}(\{\delta, \dot{a}_1\}) v, \Delta v\rangle + \langle \Psi_{-1} v, \Delta v\rangle,\end{aligned}$$

and eventually,

$$\begin{aligned}\langle \partial_d \Sigma v, v\rangle + 2\,\mathrm{Im}\,\langle \Sigma\underline{\mathcal{A}} v, v\rangle = \langle (\partial_d R)\Delta v, \Delta v\rangle &+ 2\,\mathrm{Im}\,\langle R\underline{\mathcal{A}}\Delta v, \Delta v\rangle \\ &+ 2\,\mathrm{Re}\,\langle R\,\mathrm{Op}(\partial_d \delta + \{\delta, \dot{a}_1\}) v, \Delta v\rangle,\end{aligned} \tag{3.53}$$

modulo a negligible error $\langle \Psi_{-1} v, \Delta v\rangle$ (to be seen!). Now, considering that R is a strong functional symmetriser, the second term in (3.53) obeys the inequality

$$2\,\mathrm{Im}\,(R\underline{\mathcal{A}}\Delta v, \Delta v) \geq C\gamma|\Delta v|^2,$$

whereas the last bracket is null because $\partial_d \delta + \{\delta, \dot{a}_1\} = 0$ by construction. Finally, the identity $M\Delta = Q\dot{B} \mod \Psi_{-1}$ from Theorem 3.3.1 and Condition (iv) in Definition 2.5.1 result in

$$\langle \Sigma(0)v(0), v(0)\rangle \geq \alpha|\Delta(0)v(0)|^2 - \beta|M\Delta v(0)|^2 = \alpha|\Delta(0)v(0)|^2 - \beta|Q\dot{B}v(0)|^2 - |\Psi_{-1}v(0)|^2,$$

for some positive constants α, β, and a perturbation $|\Psi_{-1}v(0)|^2$ to be absorbed.

Step 5. We deduce energy estimates for $\tilde{u}$ via Σ. We proceed analogously as in the model case:

$$\begin{aligned}\frac{d}{dx_d}\langle \Sigma\tilde{u},\tilde{u}\rangle &= \langle \partial_d\Sigma\tilde{u},\tilde{u}\rangle + \langle \Sigma\partial_d\tilde{u},\tilde{u}\rangle + \langle \Sigma\tilde{u},\partial_d\tilde{u}\rangle \\ &= \langle \partial_d\Sigma\tilde{u},\tilde{u}\rangle + 2\,\mathrm{Re}\,\langle \Sigma i D_d\tilde{u},\tilde{u}\rangle \\ &= \langle \partial_d\Sigma\tilde{u},\tilde{u}\rangle + 2\,\mathrm{Re}\,\langle \Sigma i(\tilde{f}-\underline{\mathcal{A}}\tilde{u}),\tilde{u}\rangle \\ &= \langle \partial_d\Sigma\tilde{u},\tilde{u}\rangle + 2\,\mathrm{Im}\,\langle \Sigma\underline{\mathcal{A}}\tilde{u},\tilde{u}\rangle - 2\,\mathrm{Im}\,\langle \Sigma\tilde{f},\tilde{u}\rangle.\end{aligned}$$

Integrating in x_d over $[0,\infty)$ and bearing in mind that $\tilde{u}$ vanishes at infinity leads to

$$\langle \Sigma\tilde{u},\tilde{u}\rangle|_{x_d=0} = -\int_0^\infty \left(\langle \partial_d\Sigma\tilde{u},\tilde{u}\rangle + 2\,\mathrm{Im}\,\langle \Sigma\underline{\mathcal{A}}\tilde{u},\tilde{u}\rangle\right) dx_d + 2\int_0^\infty \mathrm{Im}\,\langle \Sigma\tilde{f},\tilde{u}\rangle\, dx_d.$$

From Condition (ii) in Definition 3.3.2, it is clear that

$$2\,\mathrm{Im}\,\langle \Sigma\tilde{f},\tilde{u}\rangle \le 2|\langle \Sigma\tilde{f},\tilde{u}\rangle| \le C_1|\Delta\tilde{f}||\Delta\tilde{u}|,$$

while from Part (iii) in Definition 3.3.2,

$$\langle \Sigma\tilde{u},\tilde{u}\rangle|_{x_d=0} \le -c\gamma\int_0^\infty |\Delta\tilde{u}|^2\, dx_d + C_1\int_0^\infty |\Delta\tilde{f}||\Delta\tilde{u}|\, dx_d.$$

Meanwhile, if we use (iv) in Definition 3.3.2 plus Young's inequality, we arrive at the conclusion that

$$\alpha|\Delta\tilde{u}(0)|^2 - \beta|Q\dot{B}u(0)|^2 \le (-c\gamma+\varepsilon\gamma)\int_0^\infty |\Delta\tilde{u}|^2\, dx_d + \frac{C_1}{4\varepsilon\gamma}\int_0^\infty |\Delta\tilde{f}|^2\, dx_d$$

for all $\varepsilon > 0$. By making the parameter ε small enough, we initially infer that

$$\alpha|\Delta\tilde{u}(0)|^2 - \beta|Q\dot{B}\tilde{u}(0)|^2 \le -C_2\gamma\int_0^\infty |\Delta\tilde{u}|^2\, dx_d + \frac{C_3}{\gamma}\int_0^\infty |\Delta\tilde{f}|^2\, dx_d,$$

and then by rescaling constants if necessary,

$$\begin{aligned}\gamma\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|\Delta\tilde{u}|^2\, dt\, dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|\Delta\tilde{u}(0)|^2\, dt\, dy \qquad\qquad (3.54)\\ \le C\left(\frac{1}{\gamma}\int_{\mathbb{R}^{1+d}_+} e^{-2\gamma t}|\Delta\tilde{f}|^2\, dt\, dx + \int_{\mathbb{R}^d} e^{-2\gamma t}|Q\dot{B}\tilde{u}(0)|^2\, dt\, dy\right)\end{aligned}$$

for some $C > 0$.

Step 6. We embed L^2_Δ into H^s_γ. More precisely:

Proposition 3.3.1. *The norms $\|\Delta_\gamma\cdot\|_{0,\gamma}$ lie between L^2_γ and H^{-1}_γ for a sufficiently large γ,* *that is to say, for $\gamma_0 \geq 1$ there exist positive constants C_1, C_2 such that*

$$C_1\|\cdot\|_{-1,\gamma} \leq \frac{1}{\gamma_0}\|\Delta_\gamma \cdot\|_{0,\gamma} \leq C_2\|\cdot\|_{0,\gamma}, \tag{3.55}$$

for every $\gamma \in [\gamma_0, +\infty)$.

Proof. The upper inequality in (3.55) is straightforward as soon as one realises that Δ_γ is a family of pseudodifferential operators of order 0, so only the lower inequality needs to be checked. Let us fix $\gamma_0 \geq 1$. Back to Section 3.1.3, we know that

$$\lambda_\gamma(\zeta) \equiv \lambda(\zeta) = (\gamma^2 + \tau^2 + \eta^2)^{1/2},$$

and

$$\underline{\underline{\Delta}}(X) \equiv \underline{\underline{\Delta}}(t, y, \zeta) = \frac{\gamma + i\omega(t, y, \zeta)}{\lambda(\zeta)},$$

for $\gamma \geq \gamma_1 > \gamma_0$. Let us study the cases $x_d = 0$ and $x_d > 0$ independently for greater clarity. If $x_d = 0$, then

$$\frac{1}{\lambda(\zeta)} = \frac{\gamma_0}{\gamma_0\lambda(\zeta)} < \frac{\sqrt{\gamma^2 + \omega^2(t, y, \zeta)}}{\gamma_0\lambda(\tau, \eta, \gamma)} = \frac{|\underline{\underline{\Delta}}(X)|}{\gamma_0}. \tag{3.56}$$

In the same spirit,

$$\frac{1}{\lambda} \leq \frac{1}{\gamma} < \frac{1}{\gamma_0}, \tag{3.57}$$

and a direct comparison reveals that

$$\begin{aligned}\frac{1}{\lambda^2}|\hat{u}|^2 &= \frac{1}{\lambda^2}\hat{u}_1^2 + \cdots + \frac{1}{\lambda^2}\hat{u}_p^2 + \frac{1}{\lambda^2}\hat{u}_{p+1}^2 + \cdots + \frac{1}{\lambda^2}\hat{u}_n^2 \\ &< \frac{1}{\gamma_0^2}|\underline{\underline{\Delta}}(X)\hat{u}_1|^2 + \cdots + \frac{1}{\gamma_0^2}|\underline{\underline{\Delta}}(X)\hat{u}_p|^2 + \frac{1}{\gamma_0^2}\hat{u}_{p+1}^2 + \cdots + \frac{1}{\gamma_0^2}\hat{u}_n^2 = \frac{1}{\gamma_0^2}|\delta(X)\hat{u}|^2.\end{aligned}$$

Moreover, if

$$C := \frac{1}{\gamma_0^2}\delta^*(X)\delta(X) - \frac{1}{\lambda^2} > 0,$$

a simple calculation shows that $\sqrt{C}$ is bounded as well as all its derivatives, so $\sqrt{C} \in S^0 \in \mathrm{OPS}^0_\gamma(\mathbb{R}^d \times [1, +\infty))$. Lastly, Hörmander's square root trick allow us to conclude that

$$\|\cdot\|_{-1,\gamma} \leq \frac{1}{\gamma_0}\|\Delta_\gamma \cdot\|_{0,\gamma}.$$

When $x_d > 0$, the situation is more intricate and require some effort. Certainly, as $\delta_j^-(X)$ solves the transport equation (3.31), $\delta_j^-(X)$ is the composition of $\underline{\underline{\Delta}}(X)$ with the inverse of the Hamiltonian flow map $\phi_{x_d,j}$ associated with the eigenvalue $a_{1,j}(X)$ (see the proof of Theorem 3.3.1). This means that if $X_\flat = (t_\flat, y_\flat, 0, \tau_\flat, \eta_\flat, \gamma_\flat) \in \mathbb{Y}$ is such that

$$\delta_j^-(X) = (\phi^*_{-x_d,j}\underline{\underline{\Delta}})(X) = \underline{\underline{\Delta}}(X_\flat),$$

then

$$\frac{1}{\lambda(\zeta)} = \frac{\gamma_0}{\gamma_0\lambda(\zeta)} \leq \frac{\sqrt{\gamma_\flat^2 + \omega^2(t_\flat, y_\flat, \zeta_\flat)}}{\gamma_0\lambda(\zeta)} = \frac{\sqrt{\gamma_\flat^2 + \omega^2(t_\flat, y_\flat, \zeta_\flat)}}{\gamma_0\lambda(\zeta_\flat)}\frac{\lambda(\zeta_\flat)}{\lambda(\zeta)}$$
$$= \frac{|\Delta(X_\flat)|}{\gamma_0}\frac{\lambda(\zeta_\flat)}{\lambda(\zeta)} = \frac{|\delta_j^-(X)|}{\gamma_0}\frac{\lambda(\zeta_\flat)}{\lambda(\zeta)}. \tag{3.58}$$

In an attempt to control (3.58) appropriately, we must find an upper bound for $\lambda(\zeta_\flat)/\lambda(\zeta)$. This is addressed in the lemma below.

LEMMA 3.3.3. *Let $\phi^{-1} \equiv \phi_{-x_d,j}$ be the inverse of the Hamiltonian map ϕ_{x_d} encoded in equation (3.35). Under the assumptions of Theorem 3.3.1, there exists a constant $C > 0$ such that for every $X = (t, y, x_d, \zeta) \in \mathcal{V}$,*

$$\frac{\lambda(\phi^{-1}(X))}{\lambda(\zeta)} \leq C.$$

Proof. Let us argue by contradiction and suppose that for every $n \in \mathbb{N}$, there is a $X_n = (t_n, y_n, x_{d,n}, \zeta_n) \in \mathcal{V}$ such that

$$\frac{\lambda(\phi^{-1}(t_n, y_n, x_{d,n}, \zeta_n))}{\lambda(\zeta_n)} > n. \tag{3.59}$$

Since λ and ϕ^{-1} are homogeneous of degree 1 in ζ, we can equally write (3.59) as

$$\lambda(\phi^{-1}(t_n, y_n, x_{d,n}, \zeta'_n)) > n, \tag{3.60}$$

where $\zeta'_n := \zeta_n/|\zeta_n| = \zeta_n/\lambda(\zeta_n)$. If we look at the covariables as elements on the sphere S^d, the new neighbourhood $\mathcal{V}' \subset \mathbb{X}_S$ is compact (as $\mathcal{V}$ can be taken compact in (t, y, x_d) in view of Property **(C)**), so the sequence $\{X_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, say, $\{X_k\}_{k\in\mathbb{N}}$, so that $X_k \to \underline{X} = (\underline{t}, \underline{y}, \underline{x}_d, \underline{\zeta}')$ as k goes to infinity. Then

$$\begin{aligned} k &< |\lambda(\phi^{-1}(t_k, y_k, x_{d,k}, \zeta'_k))| \\ &= |\lambda(\phi^{-1}(t_k, y_k, x_{d,k}, \zeta'_k)) - \lambda(\phi^{-1}(\underline{t}, \underline{y}, \underline{x}_d, \underline{\zeta}'))| + \lambda(\phi^{-1}(\underline{t}, \underline{y}, \underline{x}_d, \underline{\zeta}')). \end{aligned} \tag{3.61}$$

For k sufficiently large, the difference on the right-hand side of (3.61) can be made arbitrarily small because of the continuity of ϕ^{-1} and λ, meaning that the whole expression can be bounded by some constant C' for every k large enough, which is a contradiction. □

Returning to equations (3.57) and (3.58), we see that

$$\frac{1}{\lambda(\zeta)} \leq \frac{C}{\gamma_0}|\delta_j^-(X)|$$

and

$$\frac{1}{\lambda} \leq \frac{C}{\gamma_0}$$

for some positive constant C. The rest of the proof is identical to that for $x_d = 0$. $\square$

Proposition 3.3.1 allows us to handle $\langle \Psi_{-1}v, \Delta v\rangle$ and $|\Psi_{-1}v(0)|^2$ effectively using the philosophy illustrated in the next step.

Step 7. We glue the pieces. To start with, we take note of the inequalities

$$\gamma\|\Delta\tilde{u}\|^2 + |\Delta\tilde{u}(0)|^2 \leq C\left(\frac{1}{\gamma}\|\Delta\underline{P}\tilde{u}\|^2 + |Q\dot{B}\tilde{u}(0)|^2\right) = C\left(\frac{1}{\gamma}\|\Delta\underline{P}\Phi\dot{u}\|^2 + |Q\dot{B}\Phi\dot{u}(0)|^2\right),$$

and

$$\begin{aligned}
\|\Delta\underline{P}\Phi\dot{u}\|^2 &\lesssim \|\Phi\Delta\underline{P}\dot{u}\|^2 + \|[\Delta,\Phi]\underline{P}\dot{u}\|^2 + \|\Delta[\underline{P},\Phi]\dot{u}\|^2 \\
&\lesssim \|\Phi\Delta\underline{P}\dot{u}\|^2 + \|\underline{P}\dot{u}\|_{-1}^2 + \|\Delta[\underline{P},\Phi]\dot{u}\|^2 \\
&\lesssim \|\Delta\dot{f}\|^2 + \frac{1}{\gamma_0^2}\|\Delta\dot{f}\|^2 + \|\Delta[\underline{P},\Phi]\dot{u}\|^2,
\end{aligned}$$

where we have appealed to Proposition 3.3.1 in the third inequality. It remains to check that $\Delta[\underline{P},\Phi]$ can be controlled satisfactorily. In essence, taking into account that $\underline{P}$ is block diagonal modulo Ψ_{-1}, $[\underline{P},\Phi]$ is an operator of order 0 with principal symbol $i\{\Phi, \xi_d + \dot{a}_1\}$, also diagonal, so

$$\Delta[\underline{P},\Phi] = [\underline{P},\Phi]\Delta + \Psi_{-1},$$

and therefore,

$$\begin{aligned}
\|\Delta\underline{P}\Phi\dot{u}\|^2 &\lesssim \|\Delta\dot{f}\|^2 + \frac{1}{\gamma_0^2}\|\Delta\dot{f}\|^2 + \|[\underline{P},\Phi]\Delta\dot{u}\|^2 + \|\dot{u}\|_{-1}^2 \\
&\lesssim \|\Delta\dot{f}\|^2 + \frac{1}{\gamma_0^2}\|\Delta\dot{f}\|^2 + \|\Delta\dot{u}\|^2 + \frac{1}{\gamma_0^2}\|\Delta\dot{u}\|^2 \\
&\lesssim \|\Delta\dot{f}\|^2 + \|\Delta\dot{u}\|^2.
\end{aligned} \tag{3.62}$$

Let us now examine $Q\dot{B}\Phi\dot{u}$. In this case,

$$\begin{aligned}
|Q\dot{B}\Phi\dot{u}(0)|^2 &\lesssim |\Phi Q\dot{B}\dot{u}(0)|^2 + |[Q,\Phi]\dot{B}\dot{u}(0)|^2 + |Q[\dot{B},\Phi]\dot{u}(0)|^2 \\
&\lesssim |\Phi Q g|^2 + |\dot{u}(0)|_{-1}^2 \\
&\lesssim |Qg|^2 + \frac{1}{\gamma_0^2}|\Delta\dot{u}(0)|^2,
\end{aligned} \tag{3.63}$$

the second line being a consequence of the fact that $[Q,\Phi]\dot{B}$ and $Q[\dot{B},\Phi]$ are operators of order -1. In summary, we have

$$\gamma\|\Delta\tilde{u}\|^2 + |\Delta\tilde{u}(0)|^2 \leq C\left(\frac{1}{\gamma}\|\Delta\dot{f}\|^2 + \frac{1}{\gamma}\|\Delta\dot{u}\|^2 + |Q\dot{g}|^2 + \frac{1}{\gamma_0^2}|\Delta\dot{u}(0)|^2\right),$$

provided that we pick γ_0 (and thereby γ) sufficiently large,

$$\gamma\|\Delta\tilde{u}\|^2 + |\Delta\tilde{u}(0)|^2 \leq C\left(\frac{1}{\gamma}\|\Delta\dot{f}\|^2 + |Q\dot{g}|^2\right). \tag{3.64}$$

Recall that Δ, Q, $\tilde{u}$, $\tilde{f}$, and $\tilde{g}$ in (3.64) are indexed by i, so we can add the pieces to get

$$\gamma\sum_i\|\Delta_i\tilde{u}_i\|^2 + \sum_i|\Delta_i\tilde{u}_i(0)|^2 \leq C\left(\frac{1}{\gamma}\sum_i\|\Delta_i\dot{f}_i\|^2 + \sum_i|Q_i\dot{g}_i|^2\right). \tag{3.65}$$

Let

$$\Delta := \sum_i \Delta_i\mathcal{E}_i^{-1}\Phi_i = \sum_i \Delta_i\Phi_i\mathcal{E}_i^{-1} \mod \Psi^{-1}.$$

From Proposition 3.3.1, the triangle inequality, and the convexity of the power function $x \to x^2$, it follows that

$$\|\Delta u\|^2 \lesssim \sum_i\|\Delta_i\tilde{u}_i\|^2 \qquad \text{and} \qquad |\Delta u(0)|^2 \lesssim \sum_i|\Delta_i\tilde{u}_i(0)|^2, \tag{3.66}$$

and consequently,

$$\gamma\|\Delta u\|^2 + |\Delta u(0)|^2 \leq C\left(\frac{1}{\gamma}\sum_i\|\Delta_i\tilde{f}_i\|^2 + \sum_i|Q_i\dot{g}_i|^2\right). \tag{3.67}$$

The obvious relations $\|\Delta_i\dot{f}_i\| \lesssim \|\dot{f}_i\| \lesssim \|f\|$ and $|Q_i\dot{g}| \lesssim |\dot{g}| \lesssim |g|$ then yield

$$\gamma\|\Delta u\|^2 + |\Delta u(0)|^2 \leq C\left(\frac{1}{\gamma}\|f\|^2 + |g|^2\right). \tag{3.68}$$

To close the argument, we extend (3.68) to Sobolev spaces. Indeed, $\Lambda_\gamma^s\tilde{u} \in L_\Delta^2$ when $\tilde{u} \in H_\Delta^s$, and as such it fulfils (3.54):

$$\gamma\|\Delta_i\Lambda^s\tilde{u}_i\|^2 + |\Delta_i\Lambda^s\tilde{u}_i(0)|^2 \leq C\left(\frac{1}{\gamma}\|\Delta_i\underline{P}_i\Lambda^s\tilde{u}_i\|^2 + |Q_i\dot{B}_i\Lambda^s\tilde{u}_i|^2\right). \tag{3.69}$$

A similar reasoning to that in (3.62) and (3.63) combined with Proposition (3.3.1) gives upper bounds

$$\|\Delta_i\underline{P}_i\Lambda^s\tilde{u}_i\|^2 \lesssim \|\Delta_i\tilde{f}_i\|_s^2 + \|\Delta_i\tilde{u}_i\|_s^2, \tag{3.70}$$

$$|Q_i\dot{B}_i\Lambda^s\tilde{u}_i(0)|^2 \lesssim |Q_i\tilde{g}_i|_s^2 + \frac{1}{\gamma_0^2}|\Delta_i\tilde{u}_i(0)|_s^2. \tag{3.71}$$

For the left-hand side of (3.69), we exploit $\Lambda^s\Delta = \Delta\Lambda^s + [\Lambda^s, \Delta]$ to derive

$$\|\Lambda^s \Delta \tilde{u}\|^2 \lesssim \|\Delta\Lambda^s \tilde{u}\|^2 + \|\Psi_{s-1}\tilde{u}\|^2.$$
$$\|\Delta\tilde{u}\|_s^2 - \frac{1}{\gamma_0^2}\|\Delta\tilde{u}\|_s^2 \lesssim \|\Delta\Lambda^s\tilde{u}\|^2.$$

In the end, collecting terms and choosing γ_0 large enough produce

$$\gamma\|\Delta\tilde{u}\|_s^2 + |\Delta\tilde{u}(0)|_s^2 \leq C\left(\frac{1}{\gamma}\|\Delta\tilde{f}\|_s^2 + |Q\tilde{g}|_s^2\right). \tag{3.72}$$

From this point on, we continue in the same way as we did for $\|\cdot\|$. □

Remark 3.3.1. Formula (3.68) is not a strict generalisation of (3.21), since neither Δ nor Q are present on the right hand side of the inequality. This is due to the intrinsic difficulty in controlling $\|\Delta_i\|$ (resp. $\|Q_i\|$) in terms of $\|\Delta\|$ (resp. $\|Q\|$), as may be easily seen by considering two adjacent charts $\mathcal{V}_1, \mathcal{V}_2$, and the difference

$$\|\Delta_1\mathcal{E}_1^{-1}f + \Delta_2\mathcal{E}_2^{-1}f\|^2 - \left(\|\Delta_1\mathcal{E}_1^{-1}f\|^2 + \|\Delta_2\mathcal{E}_2^{-1}f\|^2\right) = 2\,\mathrm{Re}\,\langle\Delta_1\mathcal{E}_1^{-1}f, \Delta_2\mathcal{E}_2^{-1}f)\rangle. \tag{3.73}$$

For instance, when dealing with constant coefficients, local and global constructions coincide, so $\Delta = \Delta_1 = \Delta_2$ and $\mathcal{E}^{-1} = \mathcal{E}_1^{-1} = \mathcal{E}_2^{-1}$. Hence, (3.73) is positive and we can recover the original estimate (3.21). Whether (3.73) can always be absorbed in the variable coefficient setting is still an open question.

CHAPTER FOUR

The well-posedness of the $\mathcal{WR}$ problem

In this chapter, we address the classical questions of existence, uniqueness, and regularity for solutions of $\mathcal{WR}$ problems through the lens of Chapter 3 and the results therein. We anticipate, however, that we do not use the energy inequalities in the same way as in Chapter 1, but follow an alternative strategy suggested by the very nature of the estimates. The idea is to verify that $w = \Delta_\gamma \tilde{u}$ is locally the unique solution of a strong boundary value problem (i.e., of a problem satisfying the uniform Lopatinskiĭ condition), after which the properties of $\tilde{u}$ can be inferred from the analysis of the pseudodifferential equation $w = \Delta_\gamma \tilde{u}$.

4.1 Operators of real principal type

For a better understanding, we first have a look at the theory of operators of real principal type in $\Omega \subset \mathbb{R}^n$ and only then we deal with our particular case study.

4.1.1 Definitions and main properties

Definition 4.1.1. *Let $P(x, D)$ be a scalar, classical pseudodifferential operator of order m with principal symbol $p(x,\xi)$. P is said to be of real principal type if $p(x,\xi)$ is real-valued and the Hamiltonian vector field*

$$H_p := \sum_{i=1}^{n} (\partial_{\xi_i} p(x,\xi)\partial_{x_i} - \partial_{x_i} p(x,\xi)\partial_{\xi_i})$$

is both non-vanishing and not proportional to the radial vector field $\xi\partial\xi$.

Some remarks on principal type operators:

- Since $p(x,\xi)$ is homogeneous of degree m, Euler's theorem (see Theorem A.2.8 or [SR18]) yields

$$mp(x,\xi) = \partial_\xi p(x,\xi) \cdot \xi.$$

For that reason, we just need to check that $\partial_\xi p(x,\xi) \neq 0$ when $p(x,\xi) = 0$ to meet Definition 4.1.1.

- ▷ The previous condition can be equivalently formulated by saying that no bicharacteristic curve stays over a compact subset K indefinitely ([DH94]).
- ▷ Owing to Euler's theorem, elliptic operators trivially satisfy Definition 4.1.1 and thereby are of principal type.
- ▷ The canonical example of a non-elliptic operator of real principal type is $D_{x_1} \equiv D_1$. Interestingly, any operator of real principal type can be reduced to the composition of an elliptic operator $Q(x, D)$ and D_1 with the help of a canonical transformation (see [Ivr13], [CP82], and [Ivr13] for an ample discussion).

We close this section with the remarkable theorem on the propagation of singularities by Hörmander and Duistermaat (see [DH94] and [HH87] for an comprehensive discussion).

Theorem 4.1.1 (Theorem 3.1, [Ivr13]). *Suppose that P is an m-th order pseudodifferential operator of real principal type. If $u \in \mathcal{D}'(\Omega)$ and J is a connected piece of a null bicharacteristic curve such that $J \cap \mathrm{WF}(Pu) = \varnothing$, then either $J \subset \mathrm{WF}(u)$ or $J \cap \mathrm{WF}(u) = \varnothing$.*

Theorem 4.1.2 (Theorem 3.2, [Ivr13]). *Let $u \in \mathcal{D}'(\Omega)$ and assume that P is an m-th order pseudodifferential operator of real principal type. Suppose that WF_s represents the Sobolev wavefront set. If J is a connected piece of a bicharacteristic curve so that $J \cap \mathrm{WF}_s(Pu) = \varnothing$, then either $J \subset \mathrm{WF}_{s+m-1}(Pu)$ or $J \cap \mathrm{WF}_{s+m-1}(u) = \varnothing$.*

4.1.2 Construction of a parametrix

Theorem 4.1.3 (Lemma 3.2, [Tay79]). *Let Ω be a compact manifold and suppose that $\mathbb{R} \times \Omega$ has coordinates (t, x). Assume that P is a scalar zeroth-order pseudodifferential operator with real principal symbol $p(t, x, \tau, \xi)$. If $f \in \mathcal{E}'(\mathbb{R} \times \Omega)$ is supported in $\{t > 0\}$ and $\partial_\tau p$ is nonvanishing whenever $p = 0$, then the pseudodifferential equation*

$$Pu = f \tag{4.1}$$

has a unique solution modulo C^∞ that vanishes for $\{t < 0\}$. Moreover, $\mathrm{WF}(u)$ is contained in the union of $\mathrm{WF}(Pu)$ and the set of positively time-oriented null bicharacteristics of p passing over $\mathrm{WF}(u)$.

Proof. Let Ψ_1 be a first-order pseudodifferential operator whose principal symbol ψ is a scalar positive function. It suffices to examine the equivalent problem

$$Qu = g, \tag{4.2}$$

where $Q := \Psi_1 P \in \mathrm{OPS}^1(\mathbb{R} \times \Omega)$ and $g := \Psi_1 f$. We shall construct a solution of (4.2) modulo C^∞. A first attempt could be to investigate the formal expressions

$$\begin{cases} u_+ = Q^{-1}g = (iQ)^{-1}(ig) = -i\int_0^\infty e^{isQ}g\,ds, \\ u_- = (Q)^{-1}g = (iQ)^{-1}(ig) = i\int_{-\infty}^0 e^{isQ}g\,ds, \end{cases} \tag{4.3}$$

which might be ill-defined. The integrand $w(s) := e^{isQ}g$, however, verifies the hyperbolic equation

$$\begin{cases} D_s w - Qw = 0, \\ \qquad w(0) = g, \end{cases} \tag{4.4}$$

which can be solved up to a smoothing remainder using geometrical optics techniques. Denote by $e_+^{isQ}g$ this approximate solution. Now, observe that $\partial_\tau p \neq 0$ implies that $\dot{t}$ is bounded away from zero along the null bicharacteristics of P (and hence along the null bicharacteristics of Q), so we can write

$$\mathrm{Char}(Q) = \mathrm{Char}(P) = S_+ \cup S_-,$$

where S_+ and S_- correspond to the sets of integral curves for which $\dot{t} > 0$ and $\dot{t} < 0$, respectively. Additionally, assume that S_0 is the complement in $T^*(\mathbb{R}\times\Omega)$ of a $\varepsilon-$neighbourhood of $\mathrm{Char}(Q)$. After this preparation, let us decompose the identity $I \in \mathcal{D}'(\mathbb{R}\times\Omega)$ as the sum of three zeroth-order pseudodifferential operators $\{P_k\}_{k\in\{+,-,0\}}$, each being supported on a small conic neighbourhood $\mathcal{V}_k \subset S_k$ and such that the principal symbol $\sigma(P_k)$ satisfies $\sigma(P_k) \equiv 1$ on a smaller conic neighbourhood $\tilde{\mathcal{V}}_k \subset \mathcal{V}_k$. Thus,

$$g = P_+g + P_-g + P_0g,$$

and since Q is elliptic on $\mathrm{supp}(P_0)$, we can solve

$$Pu_0 = P_0g$$

modulo C^∞. As for the contributions of $P_\pm g$, the situation is more complicated. To this end, let $T_1 < \infty$, and suppose that $T_0 < \infty$ is such that for $|s| \leq T_0$, the image of all $\underline{\zeta} = (\underline{t}, \underline{x}, \underline{\tau}, \underline{\xi}) \in \mathrm{WF}(g)$ obtained by following the Hamiltonian flow for s units of time has a $t-$coordinate that is larger than T_1. Consequently, if $\theta \in C_0^\infty(\mathbb{R})$ is a cutoff function so that $\theta \equiv 1$ when $|s| \leq T_0$, then

$$u = u_0 - i\int_0^\infty \theta(s)e_+^{isQ}P_+g\,ds + i\int_{-\infty}^0 \theta(s)e_+^{isQ}P_-g\,ds.$$

Finally, a direct computation shows that u satisfies (4.2), while the standard propagation of singularities theorem enable us to conclude that u is smooth for $t < 0$, as desired. □

4.2 The $\mathcal{WR}$ problem

4.2.1 The operator Δ_γ revisited

As the proof of Theorem 3.3.2 and the author's comments in [BGS07] on the model case tentatively suggest, problem (3.50) (one for each $\tilde{u}_i$ piece) can be recast into a strong system by somehow inserting $\Delta_{i,\gamma}$ into the norms. Eventually, if we are able to use the properties of $\Delta_{i,\gamma}$ to solve the pseudodifferential problem $\tilde{w}_i := \Delta_{i,\gamma}\tilde{u}_i$ for $\tilde{u}_i$, we can reconstruct u as the sum $\sum_i \tilde{u}_i$. Having outlined the strategy, let us revisit Step 3 in Theorem 3.3.2 to confirm that, under the conditions and notation specified there, $\tilde{w} = \Delta_\gamma \tilde{u}$ is a solution of the auxiliary problem

$$\begin{cases} (D_d + \underline{A}_\gamma)\, \tilde{w}(t,x) = \Delta_\gamma \tilde{f}(t,x), \\ \qquad M_\gamma \tilde{w}(t,y,0) = Q_\gamma \tilde{g}(t,y). \end{cases} \tag{4.5}$$

This follows from applying Δ_γ to both sides of (3.50) and from the next identities modulo Ψ_{-1}:

$$\Delta_\gamma \underline{A}_\gamma = \underline{A}_\gamma \Delta_\gamma - i\operatorname{Op}_\gamma(\{\dot{a}_1, \delta\}),$$

$$\Delta_\gamma D_d = D_d \Delta_\gamma - i \operatorname{Op} \delta,$$

and

$$M_\gamma \Delta_\gamma = Q_\gamma B_\gamma.$$

At this point, let us suppose that $\Delta_\gamma \tilde{f} \in H^s_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ and $\Delta_\gamma \tilde{g} \in H^s_\gamma(\overline{\mathbb{R}}^d)$. Theorem 2.6.1 asserts that there exists a unique $\tilde{w} \in H^s_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ with $\tilde{w}|_{x_d=0} \in H^s_\gamma(\mathbb{R}^d)$ fulfilling (4.5), so there is some hope of establishing existence, uniqueness, and regularity for $\tilde{u}$ by studying $\Delta\tilde{u} = \tilde{w}$ equipped with $u|_{t=0} = 0$. The major step in this direction is given by the statement below.

Proposition 4.2.1. *Suppose that* $\delta(X) = \operatorname{diag}(\delta^-(X), I_{n-p})$ *with*

$$\delta^-(X) := \begin{pmatrix} \begin{pmatrix} \delta_1^-(X) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_s^-(X) \end{pmatrix} & \\ & I_{p-s} \end{pmatrix}, \tag{4.6}$$

and $\delta_1(X), \cdots, \delta_s(X)$ *as described in Theorem 3.3.1. Then the pseudodifferential operators* $\operatorname{Op}_\gamma(\delta_1), \cdots, \operatorname{Op}_\gamma(\delta_s)$ *are of real principal type.*

Proof. Recall that the unique solution of the transport equation

$$\begin{cases} \partial_d \delta_i^- + \{\delta_j^-, a_{1,j}\} = 0, \\ \delta_j^-|_{x_d=0} = \underline{\underline{\Delta}}, \end{cases} \tag{4.7}$$

is obtained by composing $\underline{\underline{\Delta}}$ with the inverse of the Hamiltonian flow map ϕ_{x_d}, i.e., $\delta_j(X) = (\phi^*_{-x_d}\underline{\underline{\Delta}})(X)$. In the sequel, we distinguish two scenarios, namely, when $\gamma > 0$ and when $\gamma = 0$. In the former, $\underline{\underline{\Delta}}$ and $\phi^*_{-x_d}$ are never vanishing and so $\delta_j(X)$ is elliptic. In the latter, we verify the definition for $-i\delta_j(X)$ rather than for $\delta(X)$. Certainly, note that

$$-i\delta_j(\tilde{X}) = -i(\phi^*_{-x_d}\underline{\underline{\Delta}})(\tilde{X}) = \frac{\omega(t_\flat, y_\flat, \zeta_\flat)}{\lambda}$$

is real-valued and its zeroes match those of $\omega(t_\flat, y_\flat, \zeta_\flat)$ (see Theorem 3.3.2). To validate that these roots are simple is more involved, as we see now. For $x_d > 0$, $\mathrm{Char}(\mathrm{Op}_\gamma(\delta_j))$ is the orbit of Γ under ϕ_{x_d}, that is to say,

$$\mathrm{Char}(\delta_j) = \Gamma_j := \{\Gamma \text{ transported along the flow } \phi_{x_d}\}.$$

Therefore, we have to show that $\partial_\tau \delta_j(X)$ is non-vanishing on Γ_j, which amounts to proving that

$$\partial_\tau \delta_j = (\partial_\tau \phi^*_{-x_d}\underline{\underline{\Delta}}) = (\partial_\tau \underline{\underline{\Delta}})(\phi_{-x_d})\partial_\tau \phi_{-x_d} \neq 0 \tag{4.8}$$

on Γ_i. The first factor in (4.8) is different from zero by the very definition of the $\mathcal{WR}$ class, whereas $\partial_\tau \phi_{-x_d}$ can be computed via (see [Arn92])

$$\partial_\tau \phi_{-x_d} = \exp\left(\int_0^t \partial_\tau H_\phi(s)(\phi_{-s})\, ds\right) \neq 0,$$

where $H_\phi(x_d)(\cdot)$ stands for the vector field associated with the flow ϕ_{x_d}. □

4.2.2 Existence, uniqueness, and regularity

Let us focus on the pseudodifferential system

$$\Delta_\gamma \tilde{u} = \begin{pmatrix} \begin{pmatrix} \mathrm{Op}_\gamma(\delta_1) & & \\ & \ddots & \\ & & \mathrm{Op}_\gamma(\delta_s) \end{pmatrix} & \\ & I_{n-s} \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \vdots \\ \tilde{u}_n \end{pmatrix} = \begin{pmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = \tilde{w} \tag{4.9}$$

equipped with homogeneous Cauchy data. Reasoning component-wise, we get s (nontrivial) initial value problems that can be solved according to Theorem 4.1.3. More precisely, there holds

THEOREM 4.2.1. *Consider*

$$\begin{cases} \mathrm{Op}_\gamma(\delta_j)\tilde{u}_j = \tilde{w}_j, \\ \tilde{u}_j|_{t=0} = 0, \end{cases} \tag{4.10}$$

where $\tilde{w} \in H^s_\gamma(\overline{\mathbb{R}}^{1+d}_+)$ *and* $\mathrm{Op}_\gamma(\delta_j)$ *is defined as in Proposition 4.2.1. Then there exists a unique solution of* (4.10) *modulo* C^∞ *for which*

$$\mathrm{WF}_{s-1}(\tilde{u}_j) \setminus \mathrm{WF}_s(\mathrm{Op}_\gamma(\delta_j)\tilde{u}_j) \subset \delta_j^{-1}(0). \tag{4.11}$$

Proof. A direct application of Theorem 4.1.3 and Proposition 4.2.1. □

Formula (4.11) and Proposition 3.3.1 allow us to characterise H^s_Δ alternatively.

PROPOSITION 4.2.2. *Let*

$$H^s_\Delta := \{v \in \mathcal{S}'(\mathbb{R}^{1+d}_+, \mathbb{R}^n) : \Lambda^s_\gamma v \in L^2_\Delta(\mathbb{R}^{1+d}_+, \mathbb{C}^n)\}$$

be the function space introduced in Theorem 3.3.2. Then

$$H^s_\Delta = \{u \in H^{s-1}_\gamma : \Delta_\gamma u \in H^s_\gamma\}.$$

Proof. Let $u \in \underline{H}^s_\Delta := \{u \in H^{s-1}_\gamma : \Delta_\gamma u \in H^s_\gamma\}$. By definition, $u \in H^{s-1}_\gamma$ and $\Lambda^s \Delta_\gamma u \in L^2_\gamma$. We need to prove that $\Delta_\gamma \Lambda^s_\gamma u \in L^2_\gamma$. Indeed, from

$$\Delta_\gamma \Lambda^s_\gamma u = \Lambda^s_\gamma \Delta_\gamma u + [\Delta_\gamma, \Lambda_\gamma]u, \tag{4.12}$$

it will be enough to check that $[\Delta_\gamma, \Lambda_\gamma]u \in L^2_\gamma$. But this is a straightforward consequence of the fact that $[\Delta_\gamma, \Lambda^s_\gamma]$ is an operator of order $s-1$ and $u \in H^{s-1}_\gamma$. Suppose now that $u \in H^s_\Delta$, meaning that $\Delta_\gamma \Lambda^s_\gamma u \in L^2_\gamma$. From Proposition 3.3.1,

$$\|u\|_{s-1} \le \|\Lambda^s_\gamma u\|_{-1} \le \|\Delta_\gamma \Lambda^s_\gamma u\| < \infty,$$

and thus it is clear that $\Lambda^s_\gamma \Delta_\gamma u \in L^2_\gamma$ from (4.12) and the same argument as before. □

We can interpret Theorem 4.2.1 and Proposition 4.2.2 as follows: given that $\tilde{w} \in H^s_\gamma$, the first $\mathfrak{z}$ components experience a loss of regularity of one derivative, while the others remain unchanged. This supports Serre's observation in [Ser05] that the solution $\tilde{u}$ exhibits a polarisation effect around the critical set Γ.

Appendix

This appendix contains two sections with relevant material that complements the ideas presented in this thesis. Let us briefly outline what each of them consists of: in Section A.1, we include several auxiliary results whose proofs have been skipped in the main body of the document in order to facilitate the discussion. In Section A.2, on the other hand, we list for the reader's reference some classical theorems from real, complex, and Fourier analysis that have been used throughout the text.

A.1 Auxiliary results

For Lemmas 3 to 5, we shall adopt the hypothesis and notation of Theorem 2.5.1.

Lemma A.1.3 (Lemma 5.4 - Chapter 7, [CP82]). *There exists a basis $\mathfrak{e}_0(X)$ with the properties described in Definition 2.4.2 (the block structure condition) for which*

$$\mathfrak{b}_j(\tilde{X}) = \begin{pmatrix} b_1(\tilde{X}) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ b_{\nu_j}(\tilde{X}) & 0 & \cdots & 0 \end{pmatrix}, \tag{13}$$

and $b_1(\tilde{X}), \cdots, b_{\nu_j}(\tilde{X}) \in \mathbb{R}$.

Proof. The key ingredient in the proof is to find a non-singular matrix $U(\tilde{X})$ of dimensions $\nu_j \times \nu_j$ such that it behaves smoothly around $\underline{X}$ and the identity

$$U^{-1}(\tilde{X})(\mathfrak{a}_j + \mathfrak{b}_j(\tilde{X}))U(\tilde{X}) = \mathfrak{a}_j + \mathfrak{b}'_j(\tilde{X})$$

holds with $\mathfrak{b}'_j$ as in (13). With this goal in mind, let us suppose that $e_1, \cdots, e_{\nu_k}$ is the canonical basis of $\mathbb{C}^{\nu_j}$ and notice that

$$\mathfrak{a}_j^i e_m = e_{m-i}$$

for every $1 \leq i \leq \nu_j - 1$. Thus, $\mathcal{T} := \{e_m, \mathfrak{a} e_{m-1}, \cdots, \mathfrak{a}_j^{m-1} e_1\}$ is also a basis for $\mathbb{C}^{\nu_j}$.

What is more, since $\mathcal{b}(\underline{X}) = 0$, we can slightly perturb $\mathcal{T}$ with $\mathcal{b}$ and still get a basis for $\mathbb{C}^{\nu_j}$ provided that $\tilde{X}$ remains sufficiently close to $\underline{X}$. In other words, the mapping

$$U(\tilde{X}) : \mathbb{C}^{\nu_j} \to \mathbb{C}^{\nu_j}$$

defined by the rule $U(\tilde{X})a_j^i e_m = (a_j^i)e_m$ is non-singular. It is easy to check that $U(X)$ has all the required characteristics. □

Lemma A.1.4 (Lemma 5.5 - Chapter 7, [CP82]). *Let a_j as in equation* (2.45) *and suppose that E_j is symmetric. If $E_j a_j$ is symmetric, then E_j must be of the form*

$$E_j(X) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & e_1 \\ \vdots & & & \iddots & e_2 \\ \vdots & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ e_1 & e_2 & \cdots & \cdots & e_{m_j} \end{pmatrix}. \tag{14}$$

Proof. The effect of multiplying a symmetric matrix E_j by a_j from the right is simple: the first column is zero and the others are obtained by shifting the columns of E_j to the right. The result follows from comparing the off-diagonal entries. □

Lemma A.1.5 (Lemma 5.7 - Chapter 7, [CP82]). *Let C be an arbitrary constant. We can find a real, skew-symmetric matrix G_j such that*

$$\mathrm{Re}(G_j a_j w, w) \geq -|w_1|^2 + C|w'|^2, \tag{15}$$

for every $w \in \mathbb{C}^{\nu_j}$.

Proof. Let G_j be a real, skew-symmetric, block tridiagonal matrix as depicted below:

$$G_j = \begin{pmatrix} 0 & & & 0 \\ & \ddots & \ddots & \\ & \ddots & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

The product $a_j G_j$ looks similar to G_j with its components shifted to the right, except for the first column which has zero entries. Hence, a direct calculation reveals that

$$\mathrm{Re}\langle G_j a_j w, w\rangle = \sum_{i=2}^{\nu_j} g_{i,i-1}|w_i|^2 - \mathrm{Re}\left(\sum_{i=1}^{\nu_j-2} g_{i+1,i} w_{i+2}\overline{w}_i\right).$$

As for the second term, notice that

$$\mathrm{Re}\left(\sum_{i=1}^{\nu_j-2} g_{i+1,i}w_{i+2}\overline{w}_i\right) \le \left|\sum_{i=1}^{\nu_j-2} g_{i+1,i}w_{i+2}\overline{w}_i\right| \le \sum_{i=1}^{\nu_j-2} |g_{i+1,i}||w_{i+2}||w_i|,$$

so it is enough to invoke Young's inequality and choose sufficiently large elements $g_{i,i-1}$ to close the argument. □

LEMMA A.1.8. *Let V be a finite dimensional vector space and $T : V \to V$ a diagonalisable linear map. Suppose that $\lambda_1, \cdots, \lambda_q$ are pairwise different eigenvalues with corresponding eigenspaces $V_{\lambda_1}, \cdots, V_{\lambda_q}$. Then, every $T-$invariant subspace W can be decomposed as*

$$W = (W \cap V_{\lambda_1}) \oplus \cdots \oplus (W \cap V_{\lambda_q}). \tag{16}$$

Proof. To start with, note that $W_i := W \cap V_{\lambda_i}$ is the intersection of two subspaces of V, so formula (16) is meaningful. Now, to prove the assertion, we need to show that $W_i \cap W_j = \{0\}$ for $i \neq j$ and that $W = W_1 \oplus W_2 \oplus \cdots \oplus W_k$. Let $w \in W$. To verify the first part, let us assume that $w \in W_i \cap W_j = \{0\}$ for different indices i and j. The latter necessarily implies that $w \in E_i \cap E_j = \{0\}$, and eventually that $w = 0$ since $\lambda_i \neq \lambda_j$. For the remaining claim, let us write w_i for the projection onto V_{λ_i}. Clearly, $w_i \in W_i$, and consequently w can be expressed as $w = w_1 + \cdots + w_k$. Lastly, if $w \in W_i \cap \breve{W}$ with

$$\breve{W} := W_1 \oplus W_2 \oplus W_{i-1} \oplus W_{i+1} \oplus W_k,$$

then $w = 0$ for $W_i \cap W_j = \{0\}$ with $i \neq j$. In conclusion, $W = W_1 \oplus \cdots \oplus W_k$, as desired. □

A.2 Some results in Analysis

A.2.1 Complex Analysis

THEOREM A.2.1 (WEIERSTRASS PREPARATION THEOREM - THEOREM 7.5.1, [HÖR15] - CHAPTER 2, [KK11]). *Let $f(t, z)$ be an analytic function of the variables $(t, z) \in \mathbb{C} \times \mathbb{C}^n$ in a neighbourhood of $(0, 0)$. If*

$$f = \frac{\partial f}{\partial t} = \cdots = \frac{\partial^{k-1} f}{\partial t^{k-1}} = 0 \quad \text{and} \quad \frac{\partial^{k-1} f}{\partial t^{k-1}} \neq 0$$

at $(0, 0)$, then there exists a unique factorisation

$$f(t, z) = c(t, z)(t^k + a_{k-1}(z)t^{k-1} + \cdots + a_0(z)),$$

with the coefficients a_j and c being analytic functions in a neighbourhood of the origin, and such that $c(0, 0)$ and $a_j(0)$ are non-vanishing.

Theorem A.2.2 (Rouché's theorem, [GAM01] - [CKP05]). *Suppose that $D \subset \mathbb{C}$ is a bounded domain having a piece-wise smooth boundary ∂D. If $f(z)$ and $h(z)$ are analytic on $D \cup \partial D$ and $|h(z)| \leq |f(z)|$ on ∂D, then the functions $f(z)$ and $f(z) + h(z)$ have the same number of zeroes in D counting their multiplicities.*

A.2.2 Real Analysis

Theorem A.2.3 (Theorem 9.2, [MUN18]). *Let $\Omega_1 \subset \mathbb{R}^k$ and $\Omega_2 \subset \mathbb{R}^n$ be open sets. Suppose that*

$$\begin{aligned} f : \Omega_1 \times \Omega_2 &\longrightarrow \quad \mathbb{R}^n \\ (x, y) \quad &\longmapsto f(x, y) \end{aligned}$$

is of class C^r. If $(x_0, y_0) \in \Omega_1 \times \Omega_2$ is sucht that

$$f(x_0, y_0) = 0 \quad \text{and} \quad \det \frac{\partial f}{\partial y}(a, b) \neq 0,$$

there exist a neighbourhood $\mathcal{V} \subset \mathbb{R}^k$ of x_0 and a unique function $g : \mathcal{V} \to \mathbb{R}^n$ of class C^r for which $g(x_0) = y_0$ and

$$f(x, g(x)) = 0$$

for every $x \in \mathcal{V}$.

Theorem A.2.4 (Picard-Lindelöf - Theorem 3.1, [CLT56]). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\chi : (a, b) \times \Omega \to \mathbb{R}^n$ be a continuous function which is Lipschitz continuous with respect to the second variable. Then for every $t_0 \in (a, b)$ and $x_0 \in \Omega$, there exists a positive δ such that the Cauchy problem*

$$\begin{cases} \dfrac{du}{dt} = \chi(t, u), \\ u(t_0) = x_0, \end{cases} \tag{17}$$

has a unique solution in $[t_0 - \delta, t_o + \delta]$.

Theorem A.2.5. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Suppose that E is an invariant subspace under A, meaning that $A(E) \subseteq E$ and consider the Cauchy problem*

$$\begin{cases} \dfrac{dx}{dt} = Ax, \\ x(t_0) = x_0. \end{cases} \tag{18}$$

If $x_0 \in E$, then $x(t) \in E$ for every $t \in \mathbb{R}$.

Proof. From the standard theory of differential equations, the solution of (18) is

$$u(t) = e^{tA} x_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x_0.$$

If $x_0 \in E$, then $A^k x_0 \in E$ for every $k \in \mathbb{N} \cup \{0\}$. In addition, since E is finite dimensional, E must be closed and the result follows. $\square$

THEOREM A.2.6 (YOUNG). *For every $a, b \in \mathbb{R}$ and $\varepsilon > 0$, it is true that*

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2.$$

Proof. Observe that

$$0 \leq \left(\sqrt{2\varepsilon} a - \frac{b}{\sqrt{2\varepsilon}}\right)^2 = 2\varepsilon a^2 - 2ab + \frac{b^2}{2\varepsilon},$$

and therefore

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2.$$

$\square$

THEOREM A.2.7 (THEOREM 2.23, [LL12]). *Let M be a smooth manifold and $(\mathcal{V}_\alpha)_{\alpha \in \mathrm{A}}$ an indexed open cover of M. Then there exists a partition of unity subordinate to $(\mathcal{V}_\alpha)_{\alpha \in \mathrm{A}}$, i.e., a family $(\varphi_\alpha)_{\alpha \in \mathrm{A}}$ of real-valued functions on M with the following properties:*

(i) for each $\alpha \in \mathrm{A}$, $\varphi_\alpha : M \to \mathbb{R}$ is a continuous function such that $0 \leq \varphi_\alpha \leq 1$,

(ii) for all $\alpha \in \mathrm{A}$, φ_α is supported in $\mathcal{V}_\alpha$,

(iii) for every point $x \in M$, there exists a neighbourhood $\mathcal{V}_x$ such that $\mathcal{V}_x \cap \operatorname{supp} \varphi_\alpha$ is non-void only for finitely many indices α,

(iv) $\sum_{\alpha \in \mathrm{A}} \varphi_\alpha(x) = 1$ for every $x \in M$.

THEOREM A.2.8 (REMARK 6.7, (IV), [NIG18]). *A function $f \in C^1(\mathbb{R}^n)$ is homogeneous of degree m if and only if*

$$\sum_{j=1}^{n} x_j (\partial_j f)(x) = m f(x) \tag{19}$$

for all $x \in \mathbb{R}^n$.

A.2.3 Harmonic Analysis

THEOREM A.2.9 (PLANCHEREL'S THEOREM, LEMMA 1.5.1, [MÉT04]). *The following two statements are equivalent:*

(i) $u \in e^{\gamma t} L^2$,

(ii) $\mathcal{F}(e^{-\gamma t} u) \in L^2$. Moreover,

$$\|e^{-\gamma t} u\|_{L^2(\mathbb{R})} = \|\mathcal{F} u\|_{L^2(\{\operatorname{Re} \xi = \gamma\})}.$$

Bibliography

[Arn92] Vladimir I Arnold. *Ordinary differential equations*. Springer Science & Business Media, 1992.

[Ben14] Antoine Benoit. Geometric optics expansions for linear hyperbolic boundary value problems and optimality of energy estimates for surface waves. *Differential and Integral Equations*, 27(5/6), 2014.

[BGRSZ02] Sylvie Benzoni-Gavage, Frédéric Rousset, Denis Serre, and K. Zumbrun. Generic types and transitions in hyperbolic initial–boundary-value problems. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 132(5):1073–1104, 2002.

[BGS07] Sylvie Benzoni-Gavage and Denis Serre. *Multi-dimensional hyperbolic partial differential equations: first-order systems and applications*. Claerendon Press, 2007.

[CG10] Jean-François Coulombel and Olivier Guès. Geometric optics expansions with amplification for hyperbolic boundary value problems: Linear problems. In *Annales de l'institut Fourier*, volume 60, page 2183–2233, 2010.

[CKP05] George F Carrier, Max Krook, and Carl E Pearson. *Functions of a complex variable: theory and technique*. SIAM, 2005.

[CLT56] Earl A Coddington, Norman Levinson, and T Teichmann. Theory of ordinary differential equations. *Physics Today*, 9(2):18, 1956.

[Cou02] Jean-François Coulombel. Weak stability of nonuniformly stable multidimensional shocks. *SIAM journal on mathematical analysis*, 34(1):142–172, 2002.

[Cou04] Jean-François Coulombel. Weakly stable multidimensional shocks. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 21(4):401–443, 2004.

[CP82] J. Chazarain and A. Piriou. *Introduction to the Theory of Linear Partial Differential Equations*. Elsevier, Amsterdam, 1982.

[DH94] Johannes J Duistermaat and Lars Hörmander. *Fourier integral operators. II*. Springer, 1994.

[Gam01] Theodore W Gamelin. Complex analysis, springer-verlag new york, 2001.

[HH87] Lars Hörmander and Lars Hörmander. The analysis of linear partial differential operators. iii. *Bulletin of the American Mathematical Society*, 16:161–167, 1987.

[Hir12] Morris W Hirsch. *Differential topology*, volume 33. Springer Science & Business Media, 2012.

[Hör15] Lars Hörmander. *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis*. Springer, 2015.

[Ivr13] V Ya Ivrii. *Partial Differential Equations IV: Microlocal Analysis and Hyperbolic Equations*, volume 33. Springer Science & Business Media, 2013.

[Kat13] Tosio Kato. *Perturbation theory for linear operators*, volume 132. Springer Science & Business Media, 2013.

[KK11] Ludger Kaup and Burchard Kaup. *Holomorphic functions of several variables: an introduction to the fundamental theory*, volume 3. Walter de Gruyter, 2011.

[Kre70] Heinz-Otto Kreiss. Initial boundary value problems for hyperbolic systems. *Communications on Pure and Applied Mathematics*, 23(3):277–298, 1970.

[LL12] John M Lee and John M Lee. *Smooth manifolds*. Springer, 2012.

[Maj83a] Andrew Majda. *The existence of multi-dimensional shock fronts*, volume 281. American Mathematical Soc., 1983.

[Maj83b] Andrew Majda. *The stability of multi-dimensional shock fronts*, volume 275. American Mathematical Soc., 1983.

[Mét04] Guy Métivier. *Small viscosity and boundary layer methods: theory, stability analysis, and applications*. Springer Science & Business Media, 2004.

[Mun18] James R Munkres. *Analysis on manifolds*. CRC Press, 2018.

[Mé00] Guy Métivier. The block structure condition for symmetric hyperbolic systems. *Bulletin of the London Mathematical Society*, 32(6):689–702, 2000.

[Nig18] Eduard A Nigsch. Pseudodifferential operators. 2018.

[OS75] Toshio Ohkubo and Taira Shirota. On structures of certain l^2-well-posed mixed problems for hyperbolic systems of first order. *Hokkaido Mathematical Journal*, 4(1):82–158, 1975.

[Sak70] Reiko Sakamoto. Mixed problems for hyperbolic equations i energy inequalities. *Journal of Mathematics of Kyoto University*, 10(2):349–373, 1970.

[Ser99] Denis Serre. *Systems of Conservation Laws 2: Geometric Structures, Oscillations, and Initial-Boundary Value Problems*, volume 2. Cambridge University Press, 1999.

[Ser05] Denis Serre. Solvability of hyperbolic ibvps through filtering. *Methods and Applications of Analysis, 12(3)*, page 253–266, 2005.

[SR18] Xavier Saint Raymond. *Elementary introduction to the theory of pseudodifferential operators*. Routledge, 2018.

[Tay79] Michael E Taylor. Rayleigh waves in linear elasticity as a propagation of singularities phenomenon. In *Proc. Conf. on PDE and Geometry, Marcel Dekker, New York*, pages 273–291, 1979.

[Tay06] Michael Taylor. Pseudo differential operators. *Pseudo Differential Operators*, pages 19–44, 2006.