TWISTED EQUIVARIANT K-THEORY
AND
EQUIVARIANT T-DUALITY

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ABSTRACT

This thesis is about two things: twisted equivariant K-theory and equivariant topological T-duality. First, we prove a fixed point decomposition theorem for twisted equivariant K-theory, generalising a result of Atiyah and Segal. This is a description of joint work with Thomas Schick and Mario Velásquez. Next, we generalise the Atiyah-Segal completion theorem for families of subgroups to the twisted case. This is an extension of work by Lahtinen, who generalised the original Atiyah-Segal theorem to the twisted case. Thirdly, we explicitly define the pushforward map in twisted equivariant K-theory and apply it to the case of equivariant principal circle bundles. This is an application of techniques that are well-known to non-commutative geometers but have not gained widespread attention among topologists. In the second half of the thesis, we formulate equivariant topological T-duality and prove that the T-duality transformation in twisted equivariant K-theory is an isomorphism for all compact Lie groups.

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DECLARATION

The content of Chapter 2 was created together with Thomas Schick (TS) and Mario Velásquez (MV). As per the thesis guidelines, the following is a declaration of the contributions of each author. The candidate, Thomas Dove, will henceforth be referred to as TD.

The collaboration started in the summer of 2021 during a research visit by MV to Göttingen. Due to his interest in K-theory, MV was invited to join the regular meetings between TS and TD. The subject of the meetings was certain computations in twisted equivariant K-theory and a desire for a generalisation of Atiyah and Segal's fixed point formula became apparent. Through a series of meetings, the generalisation of the decomposition formula was devised and a proof was cooperatively developed. Each author contributed equally to the formulation and proof of the result. A first draft of the paper was subsequently written by TD and shared with TS and MV. After some minor revisions by each author, the paper was uploaded to ArXiv and submitted for publication.

CONTENTS

CHAPTER 1
INTRODUCTION

Mathematicians' study of twisted K-theory and topological T-duality is a result of the constant exchange of ideas between physics and mathematics. The former, a mathematical construction initially defined by Donovan-Karoubi [DK70] and later for general twists by Rosenberg [Ros89], first gained recognition in physics as the home of D-brane charges in the presence of a B-field [Wit98]. It is also conjectured to play a role in condensed matter theory, where it is involved in the classification of topological insulators [SS22]. The latter, a duality of space-time models in string theory, is of interest to mathematicians because of the T-duality transformation, which is a specific isomorphism relating the twisted K-theory groups of circle and torus bundles.

This thesis studies these concepts in the equivariant setting; that is, for spaces equipped with a group action. Twisted equivariant K-theory first gained major attention in the work of Freed, Hopkins and Teleman [FHT11], who prove a close relationship between the twisted equivariant K-theory of a compact Lie group with its conjugation action and the Verlinde algebra of its loop group. There are also links to elliptic cohomology, in particular twisted equivariant Tate K-theory, which is built on the twisted equivariant K-theory of loop spaces and has connections with the generalised Moonshine conjecture [Gan09, Dov19]. Twisted equivariant K-theory also arises naturally when considering symmetries in quantum field theory [FM13].

We contribute to the theory of twisted equivariant K-theory in three ways; firstly, we provide a fixed point decomposition theorem for twisted finite group equivariant K-theory in the style of Atiyah and Segal. This is a description of joint work with Thomas Schick and Mario Velásquez [DSV22]. Secondly, we extend the work of Adams, Haeberly, Jackowski, and May [AHJM88a] on the completion theorem in equivariant K-theory for families of subgroups to the twisted setting. For this, we draw on the methods of Lahtinen [Lah12], who already used the methods of Adams et al. to generalise Atiyah and Segal's completion theorem to twisted K-theory. Thirdly, we describe a precise construction of the pushforward in twisted equivariant K-theory. All the necessary results are already present in the non-commutative geometry literature. Indeed, the relevant Thom isomorphism exists in the more general setting of "real" groupoid-equivariant KK-theory [Mou13]; we simply apply this to our chosen setting. The main motivation is to use the pushforward to define the T-duality transformation in equivariant K-theory.

Topological T-duality over a base space X is a relation between two pairs (E, P) and $(\hat{E}, \hat{P})$ consisting of T^n-bundles $E \to X$ and $\hat{E} \to X$ with twists $P \to E$

and $\hat{P} \to \hat{E}$. A twist in this context is the same as in twisted equivariant K-theory; one could, for example, consider principal $PU(\mathcal{H})$-bundles. Such a setup models the underlying topology of spacetime models used in string theory and quantum field theory; the space E is the background spacetime in which strings propagate and the twist is the "H-flux" representing the B-field. Two pairs (E, P) and $(\hat{E}, \hat{P})$ are T-dual if there is an isomorphism between the pullbacks of P and $\hat{P}$ to $E \times_X \hat{E}$ that satisfies a certain "Poincaré bundle" condition.

$$
\begin{tikzcd}
 & p^*P \arrow[rr, "\cong"] \arrow[dl] \arrow[dr] & & \hat{p}^*\hat{P} \arrow[dl] \arrow[dr] & \\
P \arrow[dr] & & E \times_X \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & & \hat{P} \arrow[dl] \\
 & E \arrow[dr] & & \hat{E} \arrow[dl] & \\
 & & X & &
\end{tikzcd}
$$

Coming with the T-duality relation is an isomorphism (with degree shift) between the P-twisted K-theory of E and the $\hat{P}$-twisted K-theory of $\hat{E}$. Such an isomorphism is an essential component of any formulation of T-duality. The physical motivation is that two T-dual spacetime models are physically equivalent; the twisted K-theory groups classify certain physical properties of the model and so such an isomorphism must exist because the physical quantities measured by the twisted K-theory groups on each pair must be equivalent. More generally, there is the notion of "T-admissible" cohomology theories; these are those for which there is a T-duality isomorphism. Among these are twisted K-theory and twisted periodic de Rham cohomology.

Equivariant (topological) T-duality has until now not been formulated. We have done so; in Chapter 5 we extend the work of Bunke and Schick on topological T-duality [BS05, BRS06] to the equivariant setting. This includes a formulation of the T-duality relation between G-equivariant circle bundles equipped with a G-equivariant twist. We define the notion of G-T-admissibility and show that G-T-admissibility of a twisted G-equivariant cohomology theory implies that the T-duality transformation is an isomorphism. The main theorem of this section states that twisted equivariant K-theory is T-admissible, and hence the T-duality transformation is an isomorphism, for all compact Lie groups.

Building symmetries into an established theory to produce an equivariant formulation of said theory is a process that is very natural to mathematicians. With that said, equivariant topological T-duality is expected by physicists as well. The main point is that T-duality makes sense on orbifolds, which string theory has been formulated for. Thus we would expect equivariant T-duality to be included in the consideration of global quotient orbifolds. Thus, equivariant T-duality is a reasonable expectation, at least for finite groups.[1]

1.1 Twisted Equivariant K-Theory

Here we establish the definition and properties of twisted equivariant K-theory that we shall use throughout the thesis. Let X be a locally compact space acted on by a compact group G and let $P \to X$ be a stable G-equivariant principal $PU(\mathcal{H})$-bundle. For simplicity, we call these bundles G-equivariant twists. Stable equivariant projective unitary bundles are defined, for instance, in [BEJU14, Def 2.2]. We also give the definitions and main properties in Appendix A.

Let $\mathcal{K}$ denote the space of compact operators on $\mathcal{H}$. $PU(\mathcal{H})$ acts on $\mathcal{K}$ via conjugation, so for each G-equivariant $PU(\mathcal{H})$-bundle $P \to X$ there is an associated G-equivariant bundle of compact operators $P \times_{PU(\mathcal{H})} \mathcal{K}$. Let $\Gamma_0(P \times_{PU(\mathcal{H})} \mathcal{K})$ denote the C*-algebra of sections of $P \times_{PU(\mathcal{H})} \mathcal{K}$ that vanish at infinity. This is a G-equivariant C*-algebra with action

$$(g \cdot \sigma)(x) = g^{-1}\sigma(g \cdot x),$$

where $g \in G$, $\sigma \in \Gamma_0(P \times_{PU(\mathcal{H})} \mathcal{K})$, and $x \in X$.

The P-twisted G-equivariant K-theory of X is defined as the G-equivariant K-theory of $\Gamma_0(P \times_{PU(\mathcal{H})} \mathcal{K})$.

Definition 1.1. $K^*_G(X, P) := K^G_*\big(\Gamma_0(P \times_{PU(\mathcal{H})} \mathcal{K})\big)$

This definition is motivated by Rosenberg's definition of twisted K-theory [Ros89, §2]. The equivariant version appears in, for instance, [Kar08, §5.4] and [Mei09, §2.3]. There are of course other formulations of twisted equivariant K-theory; for example, via equivariant sections of the bundle of Fredholm operators associated with P [AS04, §7]. The formulation we use allows us to access techniques and results from non-commutative geometry. Indeed, by the Green-Julg theorem, twisted equivariant K-theory simply becomes the ordinary K-theory of some C^*-algebra, namely the crossed product of G with the above algebra of sections. For this reason, we will occasionally work in the more general context of K-theory for C*-algebras, and then restrict to the case important to us.

The basic properties of twisted equivariant K-theory are as follows:

Proposition 1.2. *Twisted equivariant K-theory satisfies the following:*

1. *Functoriality with respect to maps of spaces: if $f\colon X \to Y$ is a continuous map of G-spaces, then there is a natural map*

$$f^*\colon K^*_G(Y, f^*P) \to K^*_G(X, P).$$

[1] We thank Urs Schreiber for his comments on this in a brief email correspondence.

2. *Functoriality with respect to group morphisms: if $\alpha\colon H \to G$ is a group homomorphism then there is a natural map*

$$\alpha^*\colon K_H^*(X, P) \to K_G^*(X, P),$$

where H acts on X and P via α.

3. *Functoriality with respect to twist isomorphisms: if $u\colon P \to P'$ is a morphism of G-twists, then there is a natural homomorphism*

$$u^*\colon K_G(X, P) \to K_G(X, P').$$

4. *Mayer-Vietoris sequence: if U and V are G-invariant open sets of X such that $X = U \cup V$, then there is a long exact sequence*

$$\begin{tikzcd}[column sep=small] \cdots \to K_G^{n-1}\big(U \cap V, P|_{U\cap V}\big) \arrow[dr] & & K_G^n(U, P|_U) \oplus K_G^n(V, P|_U) \arrow[dr] & \\ & K_G^n(X, P) \arrow[ur] & & K_G^n\big(U \cap V, P|_{U\cap V}\big) \to \cdots \end{tikzcd}$$

5. *Bott Periodicity: there are natural isomorphisms*

$$K_G^*(X, P) \cong K_G^{*+2}(X, P).$$

We shall prove one additional property of twisted equivariant K-theory: the so-called induction isomorphism.

Proposition 1.3. *Let $H \subseteq G$ be a closed subgroup, X a H-space, and P a G-equivariant twist. Then there is a natural isomorphism*

$$K_H\big(X, P\big) \cong K_G\big(G \times_H X, G \times_H P\big).$$

Proof. There is a C*-algebraic analogue to the construction $X \mapsto G \times_H X$: given a H-C*-algebra A one constructs a G-C*-algebra $\mathrm{Ind}_H^G(A)$ such that $A \rtimes H$ and $\mathrm{Ind}_H^G(A) \rtimes G$ are Morita equivalent. This construction is introduced in [Gre78], but we refer to a textbook account [CELY17, §2.6]. The induced algebra $\mathrm{Ind}_H^G(A)$ consists of bounded functions $f\colon G \to A$ such that $f(gh) = h^{-1}f(g)$ for all $h \in H$ and $g \in G$ and such that the function $G/H \to \mathbb{C}$ defined as $gH \mapsto \|f(g)\|$ vanishes at infinity.

We consider $A = \Gamma_0(X, P)$ and show that, in this case, $\mathrm{Ind}_H^G(A)$ is isomorphic to $\Gamma_0(G \times_H X, G \times_H P)$. Then, we will have that $\Gamma_0(X, P) \rtimes H$ is Morita equivalent to $\Gamma_0(G \times_H X, G \times_H P)$, which proves the result. For $A = \Gamma_0(X, P)$, an element of

$\mathrm{Ind}_H^G(A)$ is a map $f\colon G \to \Gamma_0(X,P)$ such that

$$f(gh)(x) = \big(h^{-1}f(g)\big)(x) = h \cdot f(g)(h^{-1}x) \tag{1.1}$$

with $gH \to \|f(g)\|$ vanishing at infinity. We consider two maps,

$$\Phi\colon \mathrm{Ind}_H^G\big(\Gamma_0(X,P)\big) \leftrightarrows \Gamma_0\big(G\times_H X, G\times_H P\big)\colon \Psi,$$

where $\Phi(f)([g,x]) = [g, f(g)(x)]$ and $\Psi(\sigma)(g) = p$ where $[g,p] := \sigma([g,x])$. The first map is well-defined because

$$\begin{aligned}\Phi(f)\big([gh,hx]\big) &= \big[gh, f(gh)(hx)\big] = \big[gh,(h^{-1}f(g))(hx)\big]\\ &= \big[gh, hf(g)(x)\big] = \big[g, f(g)(x)\big] = \Phi(f)\big([g,x]\big).\end{aligned}$$

Ψ is also well-defined: $\Psi(\sigma)(gh)(x) = p$ and $h\cdot\Psi(\sigma)(g)(h^{-1}x) = h\cdot q$ where $[gh,p] := \sigma([gh,x])$ and $[g,q] := \sigma([g,h^{-1}x])$. Then

$$[g,q] = \sigma([g,h^{-1}x]) = \sigma([gh,x]) = [gh,p],$$

so $q = h^{-1}p$ and we can conclude that $\Psi(\sigma)(gh)(x) = h\cdot\Psi(\sigma)(g)(h^{-1}x)$, which is the condition (1.1). Both sides have a "vanishing at infinity" condition, and one can check that these correspond with each other. It is also straightforward to check that Φ and Ψ are inverse to each other. The two C*-algebras are thus isomorphic; this completes the proof. $\square$

Before moving on, we introduce one more definition, which is twisted equivariant K-theory that is further twisted by a bundle of Clifford algebras.

This is relevant when discussing the Thom isomorphism; a vector bundle that is not K-oriented still has a Thom isomorphism, except that we need to add a twist coming from its Clifford bundle.

Definition 1.4. Let $V \to X$ be a G-equivariant vector bundle, $\mathbb{C}l(V)$ the associated bundle of complex Clifford algebras, and $P \to X$ a G-equivariant principal $PU(\mathcal{H})$-bundle. We define

$$K_G^*(X, P + \mathbb{C}l(V)) := K_*^G\big(\Gamma_0(X, P_{\mathcal{K}} \otimes \mathbb{C}l(V))\big)$$

where $P_{\mathcal{K}} := P \times_{PU(\mathcal{H})} \mathcal{K}$. Here, the C*-algebra of sections is viewed as a $\mathbb{Z}_2$-graded C*-algebra and we use Kasparov's KK-theory for graded C*-algebras.

Another way of thinking about this is that the obstruction to V being K-oriented lies in $H_G^3(X)$, so by adding the twist classified by this obstruction, $\mathbb{C}l(V)$, we still have a Thom isomorphism.

1.2 Topological T-Duality

Here, we will provide a history and overview of topological T-duality. T-duality has its origins in string theory, where it is a duality of spacetime models. From a mathematical viewpoint, the two models may appear very different but are in fact physically equivalent. In topological T-duality, only the underlying topological information of the model is considered. In this case, the objects of study are pairs (E, P) consisting of S^1-bundles $\pi\colon E \to X$ over a fixed base space X together with a twist $P \to E$.

The first paper on topological T-duality was by Bouwknegt, Evslin, and Mathai [BEM04]. The main observation was that the presence of a non-trivial twist P (the so-called *H-flux*) changes the topology of spacetime when taking the T-dual. In particular, if (E, P) is T-dual to $(\hat{E}, \hat{P})$, there is an exchanging of Chern classes with the fiberwise integral of the twist:

$$\pi_!([P]) = c_1(\hat{E}), \ \ \hat{\pi}_!([\hat{P}]) = c_1(\hat{E}), \tag{1.2}$$

Here, $[P] \in H^3(E;\mathbb{Z})$ and $[\hat{P}] \in H^3(\hat{E},\mathbb{Z})$ are the characteristic classes classifying the twists, $c_1(-)$ denotes the first Chern class, and $\pi_!, \hat{\pi}_!$ are the pushforward maps for the respective S^1-bundles. It was already known from geometric considerations that T-duality changes the global topology of the background spacetime, but this was the first formal description of the change.

A formalised definition of topological T-duality was made by Bunke and Schick using a Thom class on an associated S^3-bundle, namely the sphere bundle of $L \oplus \hat{L}$, where L and $\hat{L}$ are the line bundles associated with E and $\hat{E}$ [BS05]. In this setting, the authors confirmed the relation (1.2) and proved that every pair (E, P) has a uniquely defined T-dual, up to isomorphism. The key part of the proof was the construction of a classifying space R together with a homeomorphism $T\colon R \to R$ inducing the T-duality relation.

The same authors, with the addition of Rumpf, further generalised to principal T^n-bundles in [BRS06], introducing the notion of T-duality triples. They consider pairs (E, P) consisting of a principal T^n-bundle $\pi\colon E \to X$ and a twist $P \to E$, with the additional assumption that P is trivialisable when restricted to the fibers of E. There was one further assumption on the characteristic class of P, but this turned out to be redundant [DS23]. Here, two pairs (E, P) and $(\hat{E}, \hat{P})$ are T-dual if they belong to a T-duality triple $\big((E, P), (\hat{E}, \hat{P}), u\big)$, where u is a twist morphism fitting into the following diagram, which was already shown in the first part of the introduction:

$$\begin{tikzcd}[column sep=small]
 & p^*P \arrow[rr, "u"] \arrow[dl] \arrow[dr] & & \hat{p}^*\hat{P} \arrow[dl] \arrow[dr] & \\
P \arrow[dr] & & E \times_X \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & & \hat{P} \arrow[dl] \\
 & E \arrow[dr] & & \hat{E} \arrow[dl] & \\
 & & X & &
\end{tikzcd} \tag{1.3}$$

The morphism u must satisfy the *Poincaré bundle condition*, which we describe now. First, recall that automorphisms of the trivial twist are in bijection with degree 2 cohomology of the base space. Now, if $x \in X$, then by choosing trivialisations of $P|_{E_x}$ and $\hat{P}|_{\hat{E}_x}$, u gives an automorphism of the trivial twist on $E_x \times \hat{E}_x \cong T^{2n}$:

$$P_{\text{trivial}} \cong P|_{E_x} \xrightarrow{u} \hat{P}|_{\hat{E}_x} \cong P_{\text{trivial}}$$

If this automorphism corresponds to the generator of $H^2(T^n;\mathbb{Z})$ up to the choice of trivialisations of $P|_{E_x}$ and $\hat{P}|_{\hat{E}_x}$, then u satisfies the Poincaré bundle condition. This condition is named after the Poincaré line bundle, which arises in algebraic geometry and is the canonical line bundle on the product of an abelian variety with its dual.

For $n > 1$, the situation is quite different to the circle case: the T-dual of a pair (E, P) need not exist and if it does exist it need not be unique. Bunke, Rumpf and Schick provide simple criteria for when a T-dual exists and what the T-duals are [BRS06, Theorem 2.24]. When the twist is not trivialisable on fibers, there is a non-classical interpretation of the T-dual as a bundle of noncommutative tori [MR05]. This fits into an interpretation of T-duality via noncommutative geometry, which we will not discuss.

Given a diagram (1.3) and a twisted cohomology theory $h^*(-)$, we can define the following composition:

$$h^*(E,P) \xrightarrow{p^*} h^*(E \times_X \hat{E}, p^*P) \xrightarrow{u^*} h^*(E \times_X \hat{E}, \hat{p}^*\hat{P}) \xrightarrow{\hat{p}_!} h^{*-1}(\hat{E}, \hat{P}).$$

In words, we pull back along p, apply the twist automorphism u, and then push forward along $\hat{p}$. This is called the T-duality transformation and is an essential aspect of T-duality. Since a pair and its dual should represent equivalent spacetime models, cohomological information about each model should be equivalent. Thus the T-duality transformation should be an isomorphism for cohomological groups that carry physical information. This is true of twisted K-theory and twisted de Rham cohomology; see the aforementioned T-duality papers. As a general framework, Bunke and Schick introduced the notion of T-admissibility; the T-transformation can be defined for any twisted cohomology theory (satisfying some prescribed axioms) and is an isomorphism if the theory is T-admissible [BS05].

In general, one would like to allow for singularities, that is, cases where the T^n-action is not free. The main approach for this is to consider T-duality in the context of stacks, groupoids, and/or orbispaces [BS06, BSS11, Pan18]. A full theory of T-duality for stacks, including a T-duality isomorphism for the K-theory of stacks, would of course include the equivariant case considered in this thesis. The aforementioned papers, however, do not consider this type of T-duality transformation. [BS06] considers Borel equivariant K-theory, defined by taking the non-equivariant K-theory of the Borel construction, whereas [BSS11] considers periodic twisted cohomology, which is a generalisation of de Rham cohomology. Non-free actions have also been dealt with by passing to the Borel construction [LM18].

CHAPTER 2

A FIXED POINT DECOMPOSITION THEOREM

In this chapter, we describe joint work with Thomas Schick and Mario Velásquez that resulted in the paper [DSV22]. We generalise Atiyah and Segal's decomposition formula for equivariant K-theory, which states that the equivariant K-theory of a compact space X acted on by a finite group G can, after tensoring with $\mathbb{C}$, be decomposed into the non-equivariant K-theory of its fixed point spaces [AS89, Theorem 2]. That is, there is a natural isomorphism

$$K_G(X) \otimes \mathbb{C} \xrightarrow{\cong} \Big[\bigoplus_{g\in G} K(X^g) \otimes \mathbb{C}\Big]^G.$$

We have generalised this to twisted equivariant K-theory. For twists coming only from the group, that is, those with characteristic class belonging to the image of $H^3(G;\mathbb{Z}) = H^3_G(*;\mathbb{Z}) \to H^3_G(X;\mathbb{Z})$, Atiyah and Segal's decomposition has already been generalised to twisted K-theory by Adem and Ruan [AR03, Theorem 7.4]. Our decomposition map is defined by restriction to the fixed point spaces X^g; we have

$$K_G(X,P) \otimes \mathbb{Q} \to \Big[\bigoplus_{g\in G} K_{\langle g\rangle}(X^g, P|_{X^g}) \otimes \mathbb{Q}\Big]^G,$$

where $K_G(X,P)$ denotes the G-equivariant K-theory of X twisted by an equivariant principal $PU(\mathcal{H})$-bundle P. Our main theorem is that this is an isomorphism onto a subspace defined by a simple relation between the summands required by the naturality of the restriction maps. This is described in detail in Section 2.1.

Our result has to be distinguished from the Atiyah-Segal *completion* theorem which describes equivariant K-theory completed at the augmentation ideal as the (representable) non-equivariant K-theory of the Borel construction. This has been generalised to the K-theory of C^*-algebras and in particular to twisted equivariant K-theory in [Phi89]. Some of our decomposition results also hold for the equivariant K-theory of an arbitrary G-C^*-algebra; these are explained in Section 2.2 and might be useful in this generality.

The initial motivation for this decomposition is the T-duality transformation in equivariant K-theory. In Chapter 5, we describe how this decomposition result helps prove that the T-duality transformation is rationally an isomorphism for finite group actions. This is a key step in proving the result for compact Lie groups.

2.1 The Decomposition Theorem

Let us formulate the decomposition theorem in detail. For each $g \in G$, the inclusion $X^g \to X$ of the fixed point set $X^g = \{x \in X \mid g \cdot x = x\}$ induces a map

$$K_G(X, P) \to K_{\langle g \rangle}(X^g, P_g)$$

by first restricting the group from G to $\langle g \rangle$ and then restricting to X^g. Here, P_g is the restriction of P to X^g. Applying this to each $g \in G$ gives

$$K_G(X, P) \to \bigoplus_{g \in G} K_{\langle g \rangle}(X^g, P_g). \tag{2.1}$$

The inclusion of cyclic subgroups $\langle h \rangle \subset \langle g \rangle$ induces the following commutative diagram:

$$\begin{tikzcd} K_G(X,P) \arrow[r] \arrow[d] & K_{\langle g \rangle}(X^g, P_g) \arrow[d] \\ K_{\langle h \rangle}(X^h, P_h) \arrow[r] & K_{\langle h \rangle}(X^g, P_g). \end{tikzcd}$$

Thus there is a relation in the image of (2.1): for each $g, h \in G$ with $\langle h \rangle \subseteq \langle g \rangle$, the factors in the g- and h-summands map to the same element in $K_{\langle h \rangle}(X^g, P_g)$.

A G-action on the right-hand side of (2.1) is defined as follows. The action of $k \in G$ induces a homeomorphism $k \colon X^g \to X^{kgk^{-1}}$. We then obtain the composition of isomorphisms

$$K_{\langle kgk^{-1} \rangle}(X^{kgk^{-1}}, P_{kgk^{-1}}) \xrightarrow{\cong} K_{\langle g \rangle}(X^g, k^* P_{kgk^{-1}}) \xrightarrow{\cong} K_{\langle g \rangle}(X^g, P_g). \tag{2.2}$$

The second isomorphism is obtained by the canonical identification $k^* P_{kgk^{-1}} \cong P_g$ given by the action of k on P. In this way, we get an action of G on the direct sum of all the $K_{\langle g \rangle}(X^g, P_g)$. Moreover, the image of (2.1) takes values in the G-invariants due to the following commutative diagram:

$$\begin{tikzcd} K_G(X,P) \arrow[r] \arrow[d, "k"'] & K_{\langle kgk^{-1} \rangle}(X^{kgk^{-1}}, P_{kgk^{-1}}) \arrow[d, "k"] \\ K_G(X,P) \arrow[r] & K_{\langle g \rangle}(X^g, P_g) \end{tikzcd}$$

Our main theorem states that, after tensoring with the rationals, these two conditions on the image of (2.1) are the right ones to produce an isomorphism. From now on we write $(-)_{\mathbb{Q}} = (-) \otimes \mathbb{Q}$, and similarly for $\mathbb{C}$.

Theorem 2.1. *Let G be a finite group, X a finite G-CW-complex, and P a G-equivariant* *twist on X. Then, there is an isomorphism*

$$K_G(X,P)_{\mathbb{Q}} \xrightarrow{\cong} CSC_G(X,P) \subseteq \Big[\bigoplus_{g\in G} K_{\langle g\rangle}(X^g, P_g)_{\mathbb{Q}}\Big]^G$$

onto the subspace $CSC_G(X,P)$ defined by the following relation:

If $g,h \in G$ and $\langle h\rangle \subseteq \langle g\rangle$, then the g-summand and the h-summand map to the same element in $K_{\langle h\rangle}(X^g, P_g)_{\mathbb{Q}}$.

We shall call this the cyclic subgroup compatibility condition.

Remark 2.2. The subspace $CSC_G(X,P)$ can be described as a limit over all the spaces $K_{\langle h\rangle}(X^g, P_g)_{\mathbb{Q}}$ with $\langle h\rangle \subseteq \langle g\rangle$ and arrows coming from the cyclic subgroup compatibility relation.

The theorem is proved in the standard way. First, we must show that the theorem holds for homogeneous spaces G/H; for this, we will compare our formulation to Adem and Ruan's. Then we want to use Mayer-Vietoris to do induction on the G-cells of X. The difficulty is showing that, after imposing the cyclic subgroup compatibility condition, the Mayer-Vietoris sequence is still exact. The full proof is in Section 2.4.

Example: Klein four group acting on a point

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ act trivially on a point. In this case, since $H^3(\mathbb{Z}_2\times\mathbb{Z}_2;\mathbb{Z}) \cong \mathbb{Z}_2$, there is one non-trivial twist up to isomorphism. Let τ be a non-trivial twist; the restriction τ_g is isomorphic to the trivial twist for all $g \in G$. We have one summand for each element of G and each of these summands is preserved by the G-action because the group is abelian. In the following, all K-theory groups are concentrated in degree 0. For each non-identity element of G, we have a copy of $\mathbb{Q}\oplus\mathbb{Q}$ in $\bigoplus_{g\in G} K_{\langle g\rangle}(*,\tau_g)_{\mathbb{Q}}$ because $K_{\langle g\rangle}(*,\tau_g)_{\mathbb{Q}} \cong \mathbb{Q}\oplus\mathbb{Q}$ when g has order 2.

g	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$K_{\langle g\rangle}(*,\tau_g)_{\mathbb{Q}}$	$\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}$

The action of $g \in G\setminus\{e\}$ on $K_{\langle g\rangle}(*,\tau_g)_{\mathbb{Q}}$ is non-trivial even though the induced map g^* on the space is the identity and τ_g is isomorphic to the trivial twist. The action is described in (2.2); we get

$$K_{\langle g\rangle}(*,\tau_g) \xrightarrow{g^*} K_{\langle g\rangle}(*,\tau_g) \xrightarrow{\cdot g^{-1}} K_{\langle g\rangle}(*,\tau_g).$$

The first map is the identity. The second map is induced by a twist automorphism coming from multiplication by $g^{-1} = g$. This automorphism is the one corresponding to the non-trivial element of $H^2(\mathbb{Z}_2;\mathbb{Z}) \cong \mathbb{Z}_2$. This exchanges the two factors in

$K_{\langle g\rangle}(*, \tau_g)_{\mathbb{Q}} \cong \mathbb{Q}\oplus\mathbb{Q}$. Therefore the G-invariant subspace of each $\mathbb{Q}\oplus\mathbb{Q}$ is precisely the diagonal $\Delta\mathbb{Q} \subset \mathbb{Q} \oplus \mathbb{Q}$.

For each $g \in G$, we have that $\langle e\rangle \subset g$, and so the cyclic subgroup relation implies that for each of the $(\mathbb{Q} \oplus \mathbb{Q})$-summands, one of the factors is determined by the e-summand $K(*)_{\mathbb{Q}} \cong \mathbb{Q}$. Together with the G-invariance, we conclude that all the summands are determined by the e-summand; hence, $K_G(*, \tau)_{\mathbb{Q}} \cong \mathbb{Q}$. This example also appears in [AR03, Example 7.8].

Example: D_8 acting trivially on S^1

We calculate the twisted equivariant K^0-groups of S^1 with trivial D_8 action, where $D_8 = \langle r, s \mid r^4 = s^2 = e, srs = r^3\rangle$ is the dihedral group of order 8. The twists are classified by

$$\begin{aligned} H^3_{D_8}(S^1;\mathbb{Z}) &\cong H^3(S^1 \times BD_8;\mathbb{Z}) \\ &\cong H^3(D_8;\mathbb{Z}) \oplus H^2(D_8;\mathbb{Z}) \\ &\cong \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2). \end{aligned}$$

We only consider the twists coming from $H^2(D_8;\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. One can use Adem and Ruan's formula for twists coming from $H^3(D_8;\mathbb{Z})$. Using the isomorphism $H^2(D_8;\mathbb{Z}) \cong H^1(D_8;S^1)$, the twists we consider are induced from 1-cocycles $D_8 \to S^1$. Such maps must factor through the abelianisation $D_8/\langle r^2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The following describes the four possibilities, denoted τ_1, τ_2, τ_3, and τ_4:

	$1, r^2$	r, r^3	s, r^2s	rs, r^3s
τ_1	1	1	1	1
τ_2	1	-1	1	-1
τ_3	1	1	-1	-1
τ_4	1	-1	-1	1

The restrictions to the cyclic subgroups of D_8 can be read from the table; a twist restricts to a non-trivial twist whenever there is a -1. Since $H^2(\mathbb{Z}_2;\mathbb{Z}) \cong \mathbb{Z}_2$ there is, up to isomorphism, only one non-trivial twist on the subgroups of order 2. The non-trivial twist on the r-summand corresponds to the order 2 element of $H^2(\mathbb{Z}_4;\mathbb{Z}) \cong \mathbb{Z}_4$. Using a Mayer-Vietoris argument, one can compute that

$$K^0_{\mathbb{Z}_2}(S^1, \xi_1)_{\mathbb{Q}} \cong \mathbb{Q} \quad \text{and} \quad K^0_{\mathbb{Z}_4}(S^1, \xi_2)_{\mathbb{Q}} \cong \mathbb{Q}^2,$$

where ξ_1 and ξ_2 are these non-trivial $\mathbb{Z}_2$- and $\mathbb{Z}_4$-equivariant twists, respectively. The Mayer-Vietoris argument is the same as the computation in [FHT11, Example 1.6]. Here, one can compute the integral twisted equivariant K-theory, not just the rationalisation. The untwisted K-theory is $K^0_{\mathbb{Z}_2}(S^1) \cong R(\mathbb{Z}_2)$ and $K^0_{\mathbb{Z}_4}(S^1) \cong R(\mathbb{Z}_4)$, the complex representation rings.

Let τ be one of the twists τ_1, τ_2, τ_3, or τ_4. We want to determine

$$CSC_{D_8}(S^1, \tau) \subseteq \Big[\bigoplus_{g \in D_8} K^0_{\langle g \rangle}(S^1, \tau_g)_{\mathbb{Q}}\Big]^{D_8}$$

We summarise the contribution of each of the group elements:

	e	r^2	r	s	rs
τ_1	$\mathbb{Q}$	$\mathbb{Q}^2$	$\mathbb{Q}^4$	$\mathbb{Q}^2$	$\mathbb{Q}^2$
τ_2	$\mathbb{Q}$	$\mathbb{Q}^2$	$\mathbb{Q}^2$	$\mathbb{Q}^2$	$\mathbb{Q}$
τ_3	$\mathbb{Q}$	$\mathbb{Q}^2$	$\mathbb{Q}^4$	$\mathbb{Q}$	$\mathbb{Q}$
τ_4	$\mathbb{Q}$	$\mathbb{Q}^2$	$\mathbb{Q}^2$	$\mathbb{Q}$	$\mathbb{Q}^2$

Note that we only need to write one element of each conjugacy class, as summands corresponding to conjugate elements are identified by the group action.

To see the conditions imposed by the cyclic subgroup condition, we investigate the relations between the cyclic subgroups. These are shown in the following diagram:

$$\begin{tikzcd}
\langle s \rangle & \langle rs \rangle & \langle r^2 s \rangle & \langle r^3 s \rangle & & \\
& & \langle e \rangle \arrow[ull, hook] \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] \arrow[rr, hook] & & \langle r^2 \rangle \arrow[r, hook] \arrow[dr, hook] & \langle r \rangle \arrow[d, equal] \\
& & & & & \langle r^3 \rangle
\end{tikzcd}$$

One can check that the maps induced by restriction to subgroups are all injective. The trivial subgroup is a subgroup of every cyclic group, so the e-summand determines one factor of each of the other summands. We also have $\langle r^2 \rangle \subset \langle r \rangle$, so two factors of the r-summand are determined by the r^2-summand. The final relation is that $\langle r \rangle = \langle r^3 \rangle$. This means that the $\langle r \rangle$- and $\langle r^3 \rangle$-summands are equal. The elements r and r^3 are also related by conjugation by s, and when the twist τ_r is trivial this action induces an automorphism on $K^0_{\langle r \rangle}(S^1, \tau_r) \cong \mathbb{Z}_4$ that swaps the two order 4 elements. The result is that the invariant subspace of $K^0_{\langle r \rangle}(S^1, \tau)_{\mathbb{Q}} \cong \mathbb{Q}^4$ is of rank 3. We can now compute the rational twisted equivariant K-theory by counting dimensions:

τ	$K^0_{D_8}(S^1, \tau)_{\mathbb{Q}}$
τ_1	$\mathbb{Q}^5$
τ_2	$\mathbb{Q}^3$
τ_3	$\mathbb{Q}^3$
τ_4	$\mathbb{Q}^3$

As a sanity check, we know that $K^0_{D_8}(S^1, \tau_1) \cong R(D_8)$, which is of rank 5 because

D_8 has 5 conjugacy classes. We remark that this example can be computed integrally using the Mayer-Vietoris technique in [FHT11, Example 1.6]; our discussion serves as a demonstration of the decomposition theorem.

Comparison with Atiyah-Segal

Before proceeding, we explain how our result is a generalisation of the decompositions of Atiyah-Segal and Adem-Ruan. The Atiyah-Segal map,

$$K_G(X)_{\mathbb{C}} \to \Big[\bigoplus_{g\in G} K(X^g)_{\mathbb{C}}\Big]^G,$$

is defined as a direct sum of maps

$$K_G(X)_{\mathbb{C}} \to K_{\langle g\rangle}(X^g)_{\mathbb{C}} \cong K(X^g) \otimes R(\langle g\rangle)_{\mathbb{C}} \to K(X^g)_{\mathbb{C}},$$

where the isomorphism is because $\langle g\rangle$ acts trivially on X^g and the final map is induced by sending a character ϕ to $\phi(g)$. The resulting map factors through the map in our decomposition theorem (using $\mathbb{C}$ instead of $\mathbb{Q}$),

$$K_G(X)_{\mathbb{C}} \to \Big[\bigoplus_{g\in G} K_{\langle g\rangle}(X^g)_{\mathbb{C}}\Big]^G \to \Big[\bigoplus_{g\in G} K(X^g)_{\mathbb{C}}\Big]^G. \tag{2.3}$$

Given an element in the right-most space, we recover the g-summand in the middle space as follows. For each $k \in \{0, \dots, |g| - 1\}$, the element in the g^k-summand of the Atiyah-Segal space is a sum $F_k = \sum_i E_i \xi^{ik}$, where E is a G-vector bundle on X, E_i is the g^i-isotopic component of $E|_{X^g}$, and ξ is the $|g|$th root of unity. By appropriately weighting each term by powers of ξ, we can recover each E_i by adding together the F_k. This is essentially an inverse discrete Fourier transform. The g-summand in the middle space is then just $\sum_i E_i \otimes \chi_i$, where χ_i is the representation $\chi_i(g) = \xi^i$. Repeating this for every summand gives us a split of the second map in (2.3). The image is precisely the cyclic subgroup compatible elements, giving us an isomorphism between our decomposition and that of Atiyah and Segal. If we use twisted characters, then we can similarly recover the decomposition theorem of Adem and Ruan, as explained in detail in the next subsection. Note that, to use the Fourier decomposition of Atiyah-Segal or Adem-Ruan, one is forced to work with $\mathbb{C}$ instead of $\mathbb{Q}$.

Comparison with Adem-Ruan

For a twist P classified by an element in the image of $H^3(G;\mathbb{Z}) \to H^3_G(X;\mathbb{Z})$, that is, represented by a $\mathbb{C}^*$-valued group 2-cocycle α, we are in the context of Adem and

Ruan's paper [AR03]. Their decomposition also factors through ours:

$$K_G(X,P)_{\mathbb{C}} \to \Big[\bigoplus_{g\in G} K_{\langle g\rangle}(X^g,P_g)_{\mathbb{C}}\Big]^G \to \bigoplus_{[g]}\big[K(X^g)\otimes L_g\big]^{C_g}. \tag{2.4}$$

Here L_g is a one-dimensional representation of the centraliser C_g of g defined by the map $h \mapsto \alpha(h,g)\alpha(g,h)^{-1}$. The final direct sum is defined over the conjugacy classes of G; a representative of each conjugacy class is chosen. The second map is given by the following composition:

$$K_{\langle g\rangle}(X^g,P_g)_{\mathbb{C}} \cong K(X^g)\otimes R_{\mathrm{res}(\alpha)}(\langle g\rangle)_{\mathbb{C}} \to K(X^g)\otimes L_g.$$

To explain:

- $R_{\mathrm{res}(\alpha)}(\langle g\rangle)$ is the ring of $\mathrm{res}(\alpha)$-twisted characters of $\langle g\rangle$, where $\mathrm{res}(\alpha)$ is the restriction of α to $\langle g\rangle$.
- The isomorphism exists because $\langle g\rangle$ acts trivially on X^g and the twist comes only from the group; see [AR03, Lemma 7.3].
- The second map is given by evaluating twisted characters at g, that is, $\chi \mapsto \chi(g)$.

A splitting of the second map in (2.4) can be constructed in the same way as in the Atiyah-Segal case, except that α-twisted characters are used. Since $H^3(\langle g\rangle;\mathbb{Z}) = 0$, we know that $\mathrm{res}(\alpha) = \delta\beta$ where $\delta\beta$ is the boundary map of cocycles applied to a 1-cochain $\beta\colon \langle g\rangle \to \mathbb{C}^*$. Every α-twisted character is then of the form $\beta\cdot\chi$ for χ an untwisted character of $\langle g\rangle$. One now performs the same calculation as in the previous section, inserting β in the relevant places. We apply the group action to get the summands corresponding to elements that are not one of the chosen conjugacy class representatives.

2.2 Restriction and Induction Maps

For this section, let G be a finite abelian group and A a G-C^*-algebra. We emphasise that this G is not the same G as in the decomposition theorem; rather, it will be one of the finite cyclic groups $\langle g\rangle$. In our discussion, there are two important maps. Let $H \subseteq G$ be a subgroup. The two maps, to be defined, are

$$i_H\colon K^H(A) \to K^G(A) \quad \text{and} \quad r_H\colon K^G(A) \to K^H(A).$$

The first is simple to define. By the Green-Julg theorem, we can identify equivariant K-theory with the K-theory of crossed products. The map $K(A\rtimes H) \to K(A\rtimes G)$

corresponding to i_H is induced by the inclusion of $A \rtimes H$ into $A \rtimes G$.

The restriction map r_H is defined as follows. First, suppose that A is unital. Then $A \rtimes G$ is also unital and there is a canonical isomorphism $A \rtimes G = \mathrm{End}_{A \rtimes G}(A \rtimes G)$, where $A \rtimes G$ is viewed as a left-$(A \rtimes G)$-module. We have the following maps:

$$A \rtimes G = \mathrm{End}_{A\rtimes G}(A \rtimes G) \hookrightarrow \mathrm{End}_{A\rtimes H}(A \rtimes G) \leftarrow A \rtimes H$$

The final map sends $f_0 \in A \rtimes H$ to the $(A \rtimes H)$-module map

$$\sum_{g\in G} a_g g \longmapsto \Big(\sum_{h \in H} a_h h\Big) \cdot f_0.$$

This function is isomorphic to the standard block inclusion $A \rtimes H \hookrightarrow M_{[G:H]}(A \rtimes H)$ after identifying the matrix algebra with $\mathrm{End}_{A\rtimes H}(A \rtimes G)$, and hence induces an isomorphism on K-theory. Its inverse allows us to define r_H as the composition

$$r_H \colon K\big(A \rtimes G\big) = K\big(\mathrm{End}_{A\rtimes G}(A \rtimes G)\big) \to K\big(\mathrm{End}_{A\rtimes H}(A \rtimes G)\big) \to K\big(A \rtimes H\big).$$

When A is non-unital, one defines the map for the unitalisation $A_+ = A \oplus \mathbb{C}$ and then restricts it to the K-theory of A.

Since H is normal in G, there is a G-action on both $A \rtimes G$ and $A \rtimes H$ given by conjugation by G considered as unitary elements in $A \rtimes G$. Indeed, on $A \rtimes G$ this is the inner action. Hence the induced action on $K^G(A) = K(A \rtimes G)$ is trivial, but the action on $K^H(A) = K(A \rtimes H)$ may not be. For $x \in A \rtimes H$ and $g \in G$ we write x^g for the element x acted on by g. The maps used to define r_H are all G-equivariant (after defining suitable actions on the endomorphism algebras) and $K^G(A)^G = K^G(A)$, so r_H takes values in $K^H(A)^G$, the G-invariant component of $K^H(A)$.

Lemma 2.3. *i_H and r_H satisfy the following properties:*

(a) If $H_1 \subseteq H_2 \subseteq G$, then the following commute:

$$\begin{tikzcd} K^{H_1}(A) \arrow[r, "i_{H_1}"] \arrow[dr, "i_{H_1}"'] & K^{H_2}(A) \arrow[d, "i_{H_2}"] \\ & K^G(A) \end{tikzcd} \qquad \begin{tikzcd} K^G(A) \arrow[r, "r_{H_2}"] \arrow[dr, "r_{H_1}"'] & K^{H_2}(A) \arrow[d, "r_{H_2}"] \\ & K^{H_1}(A) \end{tikzcd}$$

(b) If $H \subseteq G$ and $f \colon A \to B$ is a G-equivariant morphism, then the following

diagrams commute:

$$\begin{array}{ccc} K^H(A) & \xrightarrow{i_H} & K^G(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K^H(B) & \xrightarrow{i_H} & K^G(B) \end{array} \qquad \begin{array}{ccc} K^G(A) & \xrightarrow{r_H} & K^H(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K^G(B) & \xrightarrow{r_H} & K^H(B) \end{array}$$

(c) *If* $0 \to J \to A \to A/J \to 0$ *is an exact sequence of* G-C^**-algebras, then* i_H *and* r_H *commute with the index maps* $\partial\colon K^*(A/J) \to K^{*-1}(J)$*, that is, the following commute:*

$$\begin{array}{ccc} K^H_*(A/J) & \xrightarrow{i_H} & K^G_*(A/J) \\ \partial\downarrow & & \downarrow\partial \\ K^H_{*-1}(J) & \xrightarrow{i_H} & K^G_{*-1}(J) \end{array} \qquad \begin{array}{ccc} K^G_*(A/J) & \xrightarrow{r_H} & K^H_*(A/J) \\ \partial\downarrow & & \downarrow\partial \\ K^G_{*-1}(J) & \xrightarrow{r_H} & K^H_{*-1}(J) \end{array}$$

As a consequence, the K-theory long exact sequence and Mayer-Vietoris sequence are natural with respect to the maps i_H *and* r_H.

Proof. i_H and r_H are defined using the functoriality of K-theory for certain maps between naturally constructed C^*-algebras. Proving these properties comes down to writing the relevant diagrams of C^*-algebras and showing that the maps commute. We leave the details to the reader. $\square$

Lemma 2.4. *The composition* $r_H \circ i_H\colon K^H(A)_{\mathbb{Q}} \to K^H(A)_{\mathbb{Q}}$ *is given by*

$$r_H \circ i_H(x) = \frac{[G:H]}{|G|} \sum_{g \in G} g \cdot x.$$

In other words, $r_H \circ i_H$ *is* $[G:H]$ *times the averaging map. In particular,* $r_H \circ i_H$ *is multiplication by* $[G:H]$ *when restricted to* $K^H(A)^G$.

Proof. The map $i_H\colon K^H(A)_{\mathbb{Q}} \to K^G(A)_{\mathbb{Q}}$ takes values in the G-invariants because G acts trivially on $K^G(A)$. Therefore i_H factors through $K^H(A)^G_{\mathbb{Q}}$ via the averaging map. This is where taking the tensor product with $\mathbb{Q}$ is necessary. It now suffices to show that $r_H \circ i_H$ restricted to $K^H(A)^G_{\mathbb{Q}}$ is multiplication by $[G:H]$.

Assume that A is unital; it is sufficient to prove the theorem in this case. Choosing representatives g_i for elements of G/H, one has an isomorphism

$$\mathrm{End}_{A \rtimes H}(A \rtimes G) \cong M_{[G:H]}(A \rtimes H).$$

The induced isomorphism on K-theory does not depend on the choice of representatives. The composition

$$A \rtimes H \hookrightarrow A \rtimes G \hookrightarrow \mathrm{End}_{A \rtimes H}(A \rtimes G) \cong M_{[G:H]}(A \rtimes H)$$

is given by

$$x \mapsto \begin{bmatrix} x^{g_1} & 0 & \cdots & 0 \\ 0 & x^{g_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x^{g_{[G:H]}} \end{bmatrix}.$$

The induced map on K-theory is

$$K^H(A) \to K^H(A), \quad x \mapsto \sum_{i=1}^{[G:H]} x^{g_i}.$$

After restricting to G-invariants, this is multiplication by $[G : H]$, as required. □

Lemma 2.5. *Let H_1 and H_2 be subgroups of G. The following diagram commutes up to multiplication by $[G : H_1H_2]$:*

$$\begin{tikzcd} K^{H_1}(A)^G \arrow[r, "i_{H_1}"] \arrow[d, "r_{H_1 \cap H_2}"'] & K^G(A) \arrow[d, "r_{H_2}"] \\ K^{H_1 \cap H_2}(A)^G \arrow[r, "i_{H_1 \cap H_2}"] & K^{H_2}(A)^G \end{tikzcd} \tag{2.5}$$

Proof. First, we prove the result when $G = H_1H_2$. Consider the following diagram:

$$\begin{tikzcd} A \rtimes H_1 \arrow[r] \arrow[d] & A \rtimes H_1H_2 \arrow[d] \\ \mathrm{End}_{A \rtimes H_1 \cap H_2}(A \rtimes H_1) \arrow[r] & \mathrm{End}_{A \rtimes H_2}(A \rtimes H_1H_2) \\ A \rtimes H_1 \cap H_2 \arrow[r] \arrow[u] & A \rtimes H_2 \arrow[u] \end{tikzcd} \tag{2.6}$$

The map in the middle row is defined as follows. Let $f\colon A \rtimes H_1 \to A \rtimes H_1$ be an $(A \rtimes H_1 \cap H_2)$-module morphism and $x = \sum_{g \in H_1H_2} a_g g \in A \rtimes H_1H_2$. For each $h_2 \in H_2$ we set $x_{h_2} := \sum_{h_1 \in H_1} a_{h_1h_2} h_1 \in A \rtimes H_1$. Applying f to x_{h_2} gives another element, say $f(x_{h_2}) = \sum_{h_1 \in H_1} \tilde{a}_{h_1,h_2} h_1$. Thus, given $f\colon A \rtimes H_1 \to A \rtimes H_1$, we define an endomorphism of $A \rtimes H_1H_2$ by

$$x = \sum_{g \in H_1H_2} a_g g \longmapsto \sum_{h_1h_2 \in H_1H_2} \tilde{a}_{h_1,h_2} h_1h_2.$$

One must check that if $h_1h_2 = h'_1h'_2 \in H_1H_2$ then $\tilde{a}_{h_1,h_2} = \tilde{a}_{h'_1,h'_2}$. This follows from the fact that f is an $(A \rtimes H_1 \cap H_2)$-module map. One also checks that the resulting map is an $(A \rtimes H_2)$-module map.

Diagram (2.6) commutes, and the two outer paths from $A \rtimes H_1$ to $A \rtimes H_2$ define the two maps from $K^{H_1}(A)^G$ to $K^{H_2}(A)^G$ described in the theorem. Thus, the result holds for $G = H_1H_2$.

The general case is implied by the case $G = H_1H_2$. Consider the following diagram, which describes the composition $K^{H_1}(A)^G \to K^G(A) \to K^{H_2}(A)^G$:

$$\begin{tikzcd}
 & & K^G(A) \arrow[dr, "r_{H_1H_2}"] \arrow[drr, "r_{H_2}"] & & \\
K^{H_1}(A)^G \arrow[r, "i_{H_1}"'] \arrow[urr, "i_{H_1}"] & K^{H_1H_2}(A)^G \arrow[ur, "i_{H_1H_2}"] \arrow[rr, "\cdot[G:H_1H_2]"'] & & K^{H_1H_2}(A)^G \arrow[r, "r_{H_2}"'] & K^{H_2}(A)^G
\end{tikzcd}$$

The commutativity of the left and right triangles follows from Lemma 2.3 and the commutativity of the center triangle is a result of the Lemma 2.4. The composition $K^{H_1}(A)^G \to K^{H_1 \cap H_2}(A)^G \to K^{H_2}(A)^G$ remains unchanged, so we conclude that diagram (2.6) commutes up to multiplication by $[G : H_1H_2]$. $\square$

2.3 The Subgroup Independent Component of $K^G(A)$

Let G remain a finite abelian group. The collection of maps $r_H \colon K^G(A) \to K^H(A)^G$ give rise to a map $K^G(A) \to \bigoplus_{H \subsetneq G} K^H(A)^G$. Denote its kernel as follows:

Definition 2.6.

$$K^G_>(A) := \ker\Big(K^G(A) \to \bigoplus_{H \subsetneq G} K^H(A)^G\Big)$$

$K^G_>(A)$ is the component of $K^G(A)$ that does not depend on any subgroups of G. As an example, if $A = \mathbb{C}$, then $K^G(A) = R(G)$ and $K^G_>(A)$ is the subspace of characters $G \to \mathbb{C}$ that vanish on all proper subgroups of G.

Note that, by definition, the restriction of $r_H \colon K^G(A) \to K^H(A)$ to $K^G_>(A)$ is the zero map. Lemma 2.5 then implies that for $H_1, H_2 \subsetneq G$ and $x \in K^{H_1}_>(A)^G$,

$$r_{H_2} \circ i_{H_1}(x) = \begin{cases} [G : H_1] \cdot x & H_1 = H_2, \\ 0, & H_1 \neq H_2. \end{cases}$$

Lemma 2.7. *There is a canonical isomorphism*

$$K^G_>(A)_{\mathbb{Q}} \cong K^G(A)_{\mathbb{Q}} \ / \sum_{H \subsetneq G} i_H\big(K^H_>(A)^G_{\mathbb{Q}}\big).$$

Proof. The isomorphism is the quotient map restricted to $K^G_>(A)$. The inverse is

$$K^G(A)_{\mathbb{Q}} \;/ \sum_{H\subsetneq G} i_H\big(K^H_>(A)^G_{\mathbb{Q}}\big) \longrightarrow K^G_>(A)_{\mathbb{Q}}, \quad [x] \mapsto x - \sum_{H\subsetneq G} \frac{1}{[G:H]} i_H \circ r_H(x).$$

This is well defined, since if $x_H \in K^H_>(A)$, then

$$\begin{aligned}\sum_{H\subsetneq G} i_H(x_H) &\mapsto \sum_{H\subsetneq G} i_H(x_H) - \sum_{H_1\subsetneq G}\sum_{H_2\subsetneq G} \frac{1}{[G:H_1]} i_{H_1}\circ r_{H_1}\circ i_{H_2}(x_{H_2}) \\ &= \sum_{H\subsetneq G} i_H(x_H) - \sum_{H\subsetneq G} i_H(x_H) \\ &= 0,\end{aligned}$$

It is straightforward to check that this is indeed the inverse. $\square$

This lemma determines a preferred splitting projection onto $K^G_>(A)_{\mathbb{Q}}$ given by

$$K^G(A)_{\mathbb{Q}} \to K^G_>(A)_{\mathbb{Q}}, \qquad x \mapsto x - \sum_{H\subsetneq G} \frac{1}{[G:H]} i_H \circ r_H(x).$$

The following lemma shows that the $K_>$-groups inherit exactness properties from the K-groups:

Lemma 2.8. *Consider a map $K^G(A)_{\mathbb{Q}} \to K^G(B)_{\mathbb{Q}}$ and an element $x \in K^G_>(B)_{\mathbb{Q}}$. If there exists a lift of x to $K^G(A)_{\mathbb{Q}}$, then there exists a lift of x to $K^G_>(A)_{\mathbb{Q}}$.*

Proof. Let $\tilde{x}$ be a lift of x to $K^G(A)$. For each $H \subsetneq G$, let $\tilde{x}^>_H$ the projection of $r_H(\tilde{x})$ onto $K^>_H(A)$. Then $\tilde{x} - \sum_H \frac{1}{[G:H]} i_H(\tilde{x}^>_H)$ is a lift of x to $K^G_>(A)$. $\square$

Now for the main theorem of this section:

Theorem 2.9. *There is the following split short exact sequence:*

$$0 \longrightarrow \bigoplus_{H\subsetneq G} K^H_>(A)^G_{\mathbb{Q}} \longrightarrow K^G(A)_{\mathbb{Q}} \longrightarrow K^G_>(A)_{\mathbb{Q}} \to 0 \tag{2.7}$$

This is natural with respect to maps of C^*-algebras and boundary maps in the K-theory long exact sequence.

Proof. Lemma 2.7 gives the short exact sequence

$$0 \longrightarrow \sum_{H\subsetneq G} i_H(K^H_>(A)^G_{\mathbb{Q}}) \longrightarrow K^G(A)_{\mathbb{Q}} \longrightarrow K^G_>(A)_{\mathbb{Q}} \to 0.$$

We obtain (2.7) by noting that the i_H-maps give an isomorphism between the direct sum $\bigoplus_{H\subsetneq G} K^H_>(A)^G_{\mathbb{Q}}$ and $\sum_{H\subsetneq G} i_H(K^H_>(A)^G_{\mathbb{Q}})$. The splitting can be described in two ways. A right split is given by the inclusion $K^G_>(A) \to K^G(A)$. A left split is given by the restriction maps r_H, followed by the projection map onto each $K^H_>(A)$. Naturality follows from the naturality of i_H and r_H, proved in Lemma 2.3. □

Since the $K_>$-groups are no easier to compute than the K_G-groups, this is not exactly a useful result for calculating $K^G(A)$. However, it is useful in our proof of the decomposition theorem because it allows us to isolate the component of $K^G(A)$ that depends on the subgroups of G and the "free" component that does not depend on any of these subgroups.

2.4 Proof of the Decomposition Theorem

Before beginning the proof, we restate the theorem.

Theorem 2.10. *Let G be a finite group, X a finite G-CW-complex, and P a G-equivariant twist on X. Then, there is an isomorphism*

$$K_G(X,P)_{\mathbb{Q}} \xrightarrow{\cong} CSC_G(X,P) \subseteq \Big[\bigoplus_{g\in G} K_{\langle g\rangle}(X^g, P_g)_{\mathbb{Q}}\Big]^G \tag{2.8}$$

onto the subspace $CSC_G(X,P)$ defined by the following relation:

If $g,h \in G$ and $\langle h\rangle \subseteq \langle g\rangle$, then the g-summand and the h-summand map to the same element in $K_{\langle h\rangle}(X^g, P_g)_{\mathbb{Q}}$.

Proof. First, consider the homogeneous space $X = G/H$. This is discrete, so every G-equivariant twist comes from a group cocycle. The result is then true by comparison with the isomorphism of Adem and Ruan [AR03]; we compared our decomposition with Adem and Ruan's in §2.1.

Now, we wish to use induction on the G-cells of X via a Mayer-Vietoris argument. For this, it is required that the functor CSC_G satisfies the Mayer-Vietoris property. The direct sum in (2.8) satisfies this property and restricting to the cyclic subgroup compatible elements certainly preserves the property that the composition of two consequent morphisms is trivial. It is thus sufficient to show that if a cyclic subgroup compatible element lies in the kernel of a map in the Mayer-Vietoris sequence, then it can be lifted to an element that is also cyclic subgroup compatible.

Let $\big[\bigoplus_{g\in G} K_{\langle g\rangle}(X^g, P_g)_{\mathbb{Q}}\big]^G \to \big[\bigoplus_{g\in G} K_{\langle g\rangle}(Y^g, Q_g)_{\mathbb{Q}}\big]^G$ be a map in the Mayer-Vietoris sequence. It is the restriction of a direct sum of maps in the Mayer-Vietoris sequence for each $K_{\langle g\rangle}$. Let $y = (y_g)_{g\in G}$ be a cyclic subgroup compatible element in the kernel of the next map in the Mayer-Vietoris sequence. We construct a lift of this element to the cyclic subgroup compatible elements.

Consider the partial order $<$ on G where $h < g$ when $\langle h\rangle \subsetneq \langle g\rangle$. We construct a lift by induction on this poset. The smallest element in this poset is the identity, and

$e < g$ for every $g \in G$. Start by choosing a lift x_e of y_e under the map $K(X,P)^G_{\mathbb{Q}} \to K(Y,Q)^G_{\mathbb{Q}}$. Now consider $g \in G$. Assume that for every $h < g$ we have compatible lifts x_h of y_h - compatible meaning that for every $h' < h$ the lifts satisfy the correct compatibility relation between the h' and h-summands. Note that since the elements are G-invariant, they are also $\langle g\rangle$-invariant. By Theorem 2.9 we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{h<g} K^{>}_{\langle h\rangle}(X^g,P_g)^{\langle g\rangle}_{\mathbb{Q}} & \longrightarrow & K_{\langle g\rangle}(X^g,P_g)_{\mathbb{Q}} & \longrightarrow & K^{>}_{\langle g\rangle}(X^g,P_g)_{\mathbb{Q}} & \longrightarrow & 0 \\
 & & \downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \bigoplus_{h<g} K^{>}_{\langle h\rangle}(Y^g,Q_g)^{\langle g\rangle}_{\mathbb{Q}} & \longrightarrow & K_{\langle g\rangle}(Y^g,Q_g)_{\mathbb{Q}} & \longrightarrow & K^{>}_{\langle g\rangle}(Y^g,Q_g)_{\mathbb{Q}} & \longrightarrow & 0
\end{array}
$$

The $K^>$-groups were defined in the previous section (we now write the $>$ as a superscript because we are using the K-theory of spaces instead of algebras). We have been given elements in the bottom row and, by assumption, we have a lift $(x_h)_{h<g}$ of $(y_h)_{h<g}$ on the left-hand side of the diagram. Let y' be the projection of y onto $K^{>}_{\langle g\rangle}(Y^g,Q_g)_{\mathbb{Q}}$. By Lemma 2.8 there exists a lift $x' \in K^{>}_{\langle g\rangle}(X^g,P_g)_{\mathbb{Q}}$ of y'. Then $(x_h)_{h<g}$ and x' together form an element in $K_{\langle g\rangle}(X^g,P_g)_{\mathbb{Q}}$ that is a lift of y and satisfies cyclic subgroup compatibility. Thus, by induction, we can always construct the necessary lift and, by averaging over G, we can further ensure that the lift is G-invariant. This completes the proof. $\square$

As a corollary, we can also decompose the twisted equivariant K-theory of a G-equivariant fiber bundle $E \to X$. Note that this is precisely the situation encountered in T-duality, where the twisted K-theory of the total space of a $U(1)$-bundle or T^n-bundle has to be analysed. We decompose the rational equivariant K-theory of E into the twisted cyclic group equivariant K-theory of $E|_{X^g}$, the restriction of E to the fixed point spaces of X. The maps are induced by the inclusions $E|_{X^g} \to E$, and the resulting map

$$K_G(E,P)_{\mathbb{Q}} \to \bigoplus_{g\in G} K_{\langle g\rangle}(E|_{X^g}, P|_{E|_{X^g}})_{\mathbb{Q}}$$

has image the G-invariant, cyclic subgroup compatible elements.

Corollary 2.11. *Let $E \to X$ be a G-equivariant fiber bundle with fiber and base both G-CW-complexes. Let P be a G-equivariant twist on E. Then, there is an isomorphism*

$$K_G(E,P)_{\mathbb{Q}} \to \widetilde{CSC}_G(E,P) \subseteq \Big[\bigoplus_{g\in G} K_{\langle g\rangle}(E|_{X^g}, P|_{E|_{X^g}})_{\mathbb{Q}}\Big]^G$$

onto the subspace $\widetilde{CSC}_G(E,P)$ of cyclic subgroup compatible elements.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
K_G(E,P)_{\mathbb{Q}} \arrow[r] \arrow[d] & \bigoplus_{g\in G} K_{\langle g\rangle}(E|_{X^g}, P|_{E|_{X^g}})_{\mathbb{Q}} \arrow[d] \\
\bigoplus_{g\in G} K_{\langle g\rangle}(E^g, P|_{E^g})_{\mathbb{Q}} \arrow[r] & \bigoplus_{g\in G}\bigoplus_{i=0}^{|g|-1} K_{\langle g^i\rangle}(E^{g^i}, P|_{E^{g^i}})_{\mathbb{Q}}
\end{tikzcd} \tag{2.9}$$

The upper horizontal map is as described in the theorem. The vertical maps are obtained by applying the decomposition theorem to E and each $E|_{X^g}$ respectively. These are isomorphisms onto their images.

The bottom right group is a direct sum over tuples (g,i) with $g \in G$ and $0 \leq i < |g|$. Cyclic subgroup compatibility implies that the (h,i)- and (k,j)-summands must agree whenever $h^i = k^j$.

The lower horizontal map is obtained by sending the element in the g-summand of the left-hand side to each (h,i)-summand on the right-hand side with $g = h^i$. This induces an isomorphism between the cyclic subgroup compatible elements of each side. Diagram (2.9) therefore induces a diagram of isomorphisms:

$$\begin{tikzcd}
K_G(E,P)_{\mathbb{Q}} \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \widetilde{CSC}_G(E,P)_{\mathbb{Q}} \arrow[dl, "\cong"] \\
& CSC_G(E,P)_{\mathbb{Q}} &
\end{tikzcd}$$

This completes the proof. $\square$

CHAPTER 3
COMPLETION THEORY

This chapter is dedicated to completion theorems in twisted equivariant K-theory. The first and most famous theorem of this type is Atiyah and Segal's completion theorem, which states that there is an isomorphism

$$K_G(X)\widehat{_{I_G}} \xrightarrow{\cong} K(X \times_G EG)\widehat{_{I_G}}, \tag{3.1}$$

where $(-)\widehat{_{I_G}}$ denotes the completion with respect to the topology generated by the augmentation ideal I_G [AS69]. This was generalised to families of subgroups by Adams, Haeberly, Jackowski, and May in [AHJM88a], with (3.1) being the case of the trivial family. We are of course interested in its generalisations to twisted equivariant K-theory. Using the methods of [AHJM88a], Lahtinen generalised Atiyah and Segal's theorem to twisted equivariant K-theory. We shall use these same methods to prove the twisted version for families of subgroups.

There is a related theorem by Jackowski that a map $K_G(X) \to K_G(Y)$ is an isomorphism if it is an isomorphism when restricted to all the cyclic subgroups of G [Jac77]. This was later reproved as a corollary of a more general result in [AHJM88a]. We use their techniques to generalise the result to twisted equivariant K-theory, but only for twists coming from a central extension.

We start by establishing some definitions for families of subgroups.

3.1 Families of Subgroups

In this section, we give the minimum amount of background information needed to describe the statement and proof of the completion theorem in twisted equivariant K-theory. For a more thorough discussion of families of subgroups, their classifying spaces, and their applications, we recommend the survey article [Lüc05].

Definition 3.1. Let G be a group. A family of subgroups of a group G is a collection $\mathcal{F}$ of subgroups of G satisfying the following two properties:

1. $\mathcal{F}$ is closed under conjugation: If $H \in \mathcal{F}$ then $gHg^{-1} \in \mathcal{F}$ for all $g \in G$.
2. $\mathcal{F}$ is closed under subgroups: If $H \in \mathcal{F}$ and $K \subseteq H$ is a subgroup, then $K \in \mathcal{F}$.

In other words, $\mathcal{F}$ is closed under subconjugation.

Definition 3.2. A G-space X is $\mathcal{F}$-free if $G_x \in \mathcal{F}$ for all $x \in X$.

Note that if X is $\mathcal{F}$-free and H is a subgroup not in $\mathcal{F}$, then $X^H = \varnothing$.

Definition 3.3. A G map $f\colon X \to Y$ is an $\mathcal{F}$-homotopy equivalence if the induced map $f^H\colon X^H \to Y^H$ is a homotopy equivalence for all $H \in \mathcal{F}$.

Definition 3.4. The classifying space for a family $\mathcal{F}$ is the $\mathcal{F}$-free space $E\mathcal{F}$ such that for any $\mathcal{F}$-free space X there exists a G-map $X \to E\mathcal{F}$ that is unique up to G-homotopy.

Theorem 3.5. [Lüc05, Theorem 1.9] *Let $\mathcal{F}$ be a family of subgroups of a group G. $E\mathcal{F}$ is characterised up to G-homotopy by the following property: $(E\mathcal{F})^H$ is weakly contractible when $H \in \mathcal{F}$ and empty otherwise.*

In other words, $E\mathcal{F}$ is the unique $\mathcal{F}$-free space that is $\mathcal{F}$-homotopy equivalent to a point. The following are some examples:

- If $\mathcal{F}$ contains only the trivial subgroup, then an $\mathcal{F}$-free space is a just a space with free G-action and $E\mathcal{F} = EG$.
- If $\mathcal{F}$ is the family of all subgroups of G, then all G-spaces are $\mathcal{F}$-free and $E\mathcal{F}$ is a point.
- If $\mathcal{F}$ is the family of all finite subgroups, then $E\mathcal{F}$ is called the classifying space for proper group actions. This appears in the Baum-Connes conjecture.

Definition 3.6. Let $\mathcal{F}$ be a family of subgroups. The $\mathcal{F}$-topology on $R(G)$ is the topology whose neighbourhood basis around 0 is

$$I(\mathcal{F}) := \big\{ I(H_1) \cdots I(H_k) \mid H_1, \dots, H_k \in \mathcal{F} \big\},$$

where $I(H) = \ker(R(G) \to R(H))$ for $H \in \mathcal{F}$.

Equivariant K-theory groups, both twisted and untwisted, are $R(G)$-modules. This chapter is about their completion with respect to the $\mathcal{F}$-topology.

3.2 The Completion Theorem

We prove the following theorem regarding the completion of twisted equivariant K-theory with respect to a family of subgroups.

Theorem 3.7. *Let G be a compact Lie group, X a finite G-CW-complex, P a G-equivariant twist on X, and $\mathcal{F}$ a family of subgroups of G. Then, the projection map $\pi\colon E\mathcal{F} \times X \to X$ induces an isomorphism*

$$K^*_G(X, P)^{\widehat{}}_{\mathcal{F}} \xrightarrow{\cong} K^*_G(E\mathcal{F} \times X, \pi^* P),$$

where $(-)^{\widehat{}}_{\mathcal{F}}$ denotes completion with respect to the $\mathcal{F}$-topology.

The special case of untwisted K-theory with $\mathcal{F}$ containing only the trivial subgroup is Atiyah and Segal's completion theorem [AS69]. This was generalised to general families of subgroups by Adams, Haeberly, Jackowski, and May in [AHJM88a, Theorem 1.1]. Using the same techniques, the twisted version of the theorem was proven for the trivial family of subgroups by Lahtinen [Lah12], who also included twists coming from $H^1_G(X; \mathbb{Z}_2)$.

The theorem is proved via induction on the number of cells of X. For this, we must consider the case where X is of the form G/H for H a closed subgroup of G. By general induction arguments, this will reduce the theorem to the case where X is a point. In this case, the twist comes in the form of an S^1-central extension of G. We first prove the result for these kinds of twists.

Theorem 3.8. *Let G be a compact Lie group, X a finite G-CW complex, τ a twist on X coming from a central extension*

$$1 \to S^1 \to G^\tau \to G \to 1,$$

and $\mathcal{F}$ a family of subgroups of G. Then, the projection map $\pi\colon E\mathcal{F} \times X \to X$ induces an isomorphism

$$K^*_G(X, \tau)^\wedge_{\mathcal{F}} \xrightarrow{\cong} K^*_G(E\mathcal{F} \times X, \pi^*\tau),$$

where $(-)^\wedge_{\mathcal{F}}$ denotes completion with respect to the $\mathcal{F}$-topology.

For the main result, we technically only need this theorem in the case that X is a point. However, since the proof only uses that π is an $\mathcal{F}$-homotopy equivalence, which is true for all X, the more general case does not require any extra work.

When the twist comes from a central extension, the K-theory of a finite G-CW complex can be described by G^τ-equivariant vector bundles in which the central S^1 acts via scalar multiplication. The following make this precise:

Definition 3.9. Let π be an irreducible representation of S^1. For a finite G-CW-complex X, define $K^0_{G^\tau}(X)(\pi)$ to be the Grothendieck group of isomorphism classes of G^τ-vector bundles on X on which S^1 acts via the representation π. Define the other K-groups in the usual way via suspension.

Theorem 3.10. *Let X be a finite G-CW-complex and P a G-equivariant twist coming from a central extension. There is an isomorphism*

$$K^*_G(X, P) \cong K^*_{G^\tau}(X)(1),$$

where 1 denotes the representation given by the identity $S^1 \xrightarrow{\mathrm{id}} S^1 \subseteq \mathbb{C}^*$.

This theorem is a particular case of [FHT11, Proposition 3.5]; further details can be found there.

The completion results in [AHJM88a] and [Lah12] are often stated in terms of pro-group valued K-theory. This is a more general setting because one does not need to worry about spaces being finite. Where possible, we have restricted to finite G-CW-complexes; in this context, the pro-groups satisfy the Mittag-Lefler condition and so taking the limit gives an isomorphism between the true K-theory groups. In the proof of Theorem 3.16, however, a non-finite G-space is considered and so we cannot completely avoid speaking of pro-groups. We refer the reader to [AS69, §2] for an introduction to pro-group valued K-theory.

Definition 3.11. Let X be a G-CW-complex and π a representation of S^1. Define the pro-group valued K-theory

$$\mathbf{K}_G^*(X)(\pi) = \left\{K_G^*(X_\alpha)(\pi)\right\}_\alpha,$$

where X_α runs along the finite subcomplexes of X. If $\mathcal{F}$ is a family of subgroups of G then let

$$\mathbf{K}_G^*(X)(\pi)\widehat{_\mathcal{F}} = \left\{K_G^*(X_\alpha)/JK_G^*(X_\alpha)\right\}_{\alpha,J}$$

where X_α is as before and J is indexed by $I(\mathcal{F})$ (see Definition 3.6).

Theorem 3.8 will follow from the following result, which comes from [Lah12, Theorem 9].

Theorem 3.12. *Let $f\colon X_1 \to X_2$ be a G-map between G-CW-complexes such that the induced map*

$$f^*\colon \mathbf{K}_{H^\tau}^*(X_2)(\pi) \to \mathbf{K}_{H^\tau}^*(X_1)(\pi)$$

is an isomorphism of pro-$R(G)$-modules for all $H \in \mathcal{F}$. Then, the induced map

$$f^*\colon \mathbf{K}_{G^\tau}^*(X_2)(\pi)\widehat{_\mathcal{F}} \to \mathbf{K}_{G^\tau}^*(X_1)(\pi)\widehat{_\mathcal{F}}$$

is an isomorphism of pro-$R(G)$-modules.

Corollary 3.13. *Let $f\colon X_1 \to X_2$ be a G-map between finite G-CW-complexes such that the induced map*

$$f^*\colon K_{H^\tau}^*(X_2)(\pi) \to K_{H^\tau}^*(X_1)(\pi)$$

is an isomorphism for all $H \in \mathcal{F}$. Then, the induced map

$$f^*\colon K_{G^\tau}^*(X_2)(\pi)\widehat{_\mathcal{F}} \to K_{G^\tau}^*(X_1)(\pi)\widehat{_\mathcal{F}}$$

is an isomorphism.

Theorem 3.8 follows from Theorem 3.12 by considering the canonical projection map $\pi\colon E\mathcal{F}\times X \to X$. We refer the reader to Lahtinen's paper for the detailed proof. We comment on the necessary changes: At the beginning of the proof, he uses a result, Theorem 9, that in our case is replaced by Theorem 3.12. We also need to replace the inverse system representing EG with the inverse system representing $E\mathcal{F}$. The only other result needed is the following, which replaces Lemma 6 in Lahtinen's paper.

Lemma 3.14. *If the compact G-space X is $\mathcal{F}$-free then the $\mathcal{F}$-topology in $K^*_{G^\tau}(X)(\pi)$ is discrete.*

Proof. Since X is a compact, $\mathcal{F}$-free G^τ-space, Proposition 3.1 from [Jac77] implies that $K^*_{G^\tau}(X)$ is discrete in the $\mathcal{F}$-topology. Therefore, $K^*_{G^\tau}(X)(\pi)$ is also discrete with respect to the $\mathcal{F}$-topology. □

Proof of Theorem 3.12. The proof is once again the same as Lahtinen's. By considering the mapping cone of f, it suffices to show that if Z is a space such that Z^H is contractible for each $H \in \mathcal{F}$, then $K^*_{G^\tau}(X)(\pi)^\wedge_{\mathcal{F}} = 0$. Using induction on the subgroups of G, an argument first used in [AHJM88a], works in our case as well. The only change is that we complete with respect to the $\mathcal{F}$-topology instead of just the augmentation ideal. We note that when Lahtinen uses his Lemma 11, we instead use Lemma 3.2 in [AHJM88a], which is the same result but for families of subgroups. □

The next step is the case of homogeneous spaces G/H:

Theorem 3.15. *Let H be a subgroup of G, P a G-equivariant twist on G/H, and $\mathcal{F}$ a family of subgroups of G. Then, the projection map $\pi\colon E\mathcal{F}\times X \to X$ induces an isomorphism*

$$K^*_G(G/H,P)^\wedge_{\mathcal{F}} \xrightarrow{\cong} K^*_G(E\mathcal{F}\times G/H, \pi^*P).$$

where $(-)^\wedge_{\mathcal{F}}$ denotes completion with respect to the $\mathcal{F}$-topology.

Proof. Let $\mathcal{F}|_H = \{K \in \mathcal{F} : K \subseteq H\}$ be the restriction of $\mathcal{F}$ to H. As shown in Proposition 1.3, we have an induction isomorphism

$$K^*_G(G/H,P) \cong K^*_H(*,P_H),$$

where P_H is the restriction of P to the neutral element in G/H. By [AHJM88a, Lemma 3.4], the $\mathcal{F}$-adic topology on $R(H)$ coincides with the $\mathcal{F}|_H$-adic topology, so this isomorphism respects the completions:

$$K^*_G(G/H,P)^\wedge_{\mathcal{F}} \cong K^*_H(*,P_H)^\wedge_{\mathcal{F}|_H}.$$

Next, note that $E\mathcal{F}\times G/H \cong E\mathcal{F}\times_H G$, so using the induction isomorphism we get

$$K^*_G(E\mathcal{F}\times G/H,\pi^*P) \cong K^*_G(E\mathcal{F}\times_H G,\pi^*P) \cong K^*_H(E\mathcal{F}|_H,\pi^*P_H).$$

Now, we have the following commuting square:

$$\begin{array}{ccc} K^*_G(G/H,P)^{\widehat{}}_{\mathcal{F}} & \longrightarrow & K^*_G(E\mathcal{F}\times G/H,\pi^*P) \\ \cong\big\downarrow & & \big\downarrow\cong \\ K^*_H(*,P_H)^{\widehat{}}_{\mathcal{F}|_H} & \xrightarrow{\cong} & K^*_H(E\mathcal{F}|_H,\pi^*P_H). \end{array}$$

The lower isomorphism is a result of Theorem 3.8. We conclude that the upper map is an isomorphism, as required. □

Now we are ready for the proof of the full completion theorem.

Proof of Theorem 3.7. We argue by induction on the number of G-cells of X. The base case is covered by the previous theorem. Assume that X has k cells and that the theorem holds for all spaces containing less than k cells. Then, X is obtained from a G-CW-complex X' by gluing a single G-cell, that is,

$$X = X' \cup_\varphi (e^n \times G/H).$$

Let U be a small open neighbourhood around X' and V a small open neighbourhood of the newly added cell, so that

$$U \simeq X', \quad V \simeq G/H, \quad \text{and} \quad U\cap V \simeq S^{n-1}\times G/H.$$

Using $X = U \cup V$ and $E\mathcal{F} \times X = (E\mathcal{F} \times U) \cup (E\mathcal{F} \times V)$, we consider the corresponding Mayer-Vietoris sequences. By assumption, we have the completion isomorphism for X', G/H, and $S^{n-1}\times G/H$, so applying the five lemma gives the desired result. Note that the isomorphism for $S^{n-1}\times G/H$ does not come directly from the induction assumption, but we get it by decomposing the sphere into two cells and applying Mayer-Vietoris. □

3.3 Restriction to Cyclic Groups

We can generalise the second main result of [AHJM88a], Theorem 1.2, to twisted K-theory for twists coming from the group.

For a prime ideal $\mathfrak{p}$ of $R(G)$, define $\mathrm{Supp}(\mathfrak{p})$ to be the set of minimal subgroups H such that $\mathfrak{p}$ is the pre-image of a prime ideal under the restriction map $R(G) \to R(H)$. This was defined by Segal in [Seg68b], who also showed that $\mathrm{Supp}(\mathfrak{p})$ is a collection of conjugate cyclic subgroups of G. We note that Segal considered $\mathrm{Supp}(\mathfrak{p})$ to be a choice of one of these cyclic subgroups; we follow [AHJM88a] and view it as a set of conjugate subgroups.

Theorem 3.16. *Let $S \subseteq R(G)$ be a multiplicative set, $I \subseteq R(G)$ an ideal, and define*

$$\mathcal{H} = \bigcup \bigl\{ \mathrm{Supp}(\mathfrak{p}) : S \cap \mathfrak{p} = \varnothing \textit{ and } \mathfrak{p} \supseteq I \bigr\}.$$

Let $f\colon X \to Y$ be a G-map between finite G-CW-complexes such that the induced map $K_H^*(Y, Q) \to K_H^*(X, f^*Q)$ is an isomorphism for all $H \in \mathcal{H}$, where Q is a twist on Y coming from a central extension of G. Then, the induced map

$$S^{-1} K_G^*\bigl(Y, Q\bigr)^{\widehat{}}_I \to S^{-1} K_G^*\bigl(X, f^*Q\bigr)^{\widehat{}}_I$$

is an isomorphism, where $S^{-1}(-)$ denotes the localisation at S and $(-)^{\widehat{}}_I$ is the I-adic completion.

Remark 3.17. In [AHJM88a], the authors do not require the spaces to be finite. Instead, their theorem states that the isomorphism is an isomorphism of pro-groups. For finite complexes, the pro-groups satisfy the Mittag-Leffler condition, so taking the limit gives us an isomorphism of the true K-theory groups.

Our motivation for proving this theorem is the following corollary. Consider the case $S = \{1\}$ and $I = \{0\}$. Segal showed that, in this case, $\mathcal{H}$ is the family of all cyclic subgroups of G, so we get:

Corollary 3.18. *If $f\colon X \to Y$ is a map of finite G-CW-complexes, Q is a twist on Y coming from a central extension of G, and the induced map*

$$K_H^*(Y, Q) \to K_H^*(X, f^*Q)$$

is an isomorphism for all cyclic subgroups H of G, then the induced map

$$f^*\colon K_G^*(Y, Q) \to K_G^*(X, f^*Q)$$

is an isomorphism.

The untwisted version of this theorem was proven by Jackowski [Jac77]. Our proof of Theorem 3.16 directly follows the proof in [AHJM88a]. Ideally, we would like to generalise it to arbitrary twists. The difficulty is that the mapping cone argument does not work for such twists and without this one cannot use the usual inductive argument since there are two different spaces to consider.

Proof of Theorem 3.16. The twist Q comes from an S^1-central extension of G:

$$1 \to S^1 \to G^\tau \to G \to 1$$

By Theorem 3.10, there are isomorphisms

$$K_G^*(Y,Q) \cong \widetilde{K}_{G^\tau}^*(Y)(1) \;\text{ and }\; K_G^*(X,f^*Q) \cong \widetilde{K}_{G^\tau}^*(X)(1).$$

By considering the mapping cone of f, it is again sufficient to show that

$$\widetilde{K}_{H^\tau}^*(X)(\pi) = 0 \text{ for all } H \in \mathcal{H} \implies S^{-1}\widetilde{K}_{G^\tau}^*(X)(\pi)_I^\wedge = 0.$$

By general algebraic results, for instance [AHJM88b, Lemma 2.3], we have that

$$S^{-1}\widetilde{K}_{G^\tau}^*(X)(\pi)_I^\wedge = 0$$
$$\text{if and only if}$$
$$S_\mathfrak{p}^{-1}\widetilde{K}_{G^\tau}^*(X)(\pi)_\mathfrak{p}^\wedge = 0 \text{ for all primes } \mathfrak{p} \text{ with } S \cap \mathfrak{p} = \emptyset \text{ and } \mathfrak{p} \supseteq I.$$

For such a $\mathfrak{p}$, let $H \in \mathrm{Supp}(\mathfrak{p})$ and let $\mathcal{F}$ be the family of subgroups of G subconjugate to H. By Corollary 3.13, for any finite G-space X we have

$$\widetilde{K}_{K^\tau}^*(X)(\pi) = 0 \text{ for all } K \in \mathcal{F} \implies \widetilde{K}_{G^\tau}(X)(\pi)_\mathcal{F}^\wedge = 0. \tag{3.2}$$

This follows from the theorem by considering the constant map $X \to *$. The prime $\mathfrak{p}$ contains $\ker(R(G) \to R(H))$, which implies that the $\mathfrak{p}$-adic topology is coarser than the $\mathcal{F}$-topology. Then, $\widetilde{K}_{G^\tau}(Y)(\pi)_\mathcal{F}^\wedge = 0$ implies that $\widetilde{K}_{G^\tau}(Y)(\pi)_\mathfrak{p}^\wedge = 0$, and hence $S_\mathfrak{p}^{-1}\widetilde{K}_{G^\tau}^*(Y)(\pi)_\mathfrak{p}^\wedge = 0$.

In our case, $\widetilde{K}_{H^\tau}^*(X)(\pi) = 0$, but the same is not necessarily true for subgroups of H. This is remedied by embedding X into a space Y such that $X^K \simeq Y^K$ for all K containing a conjugate of H and Y^K is contractible otherwise. Such a Y satisfies the assumption in (3.2), hence $S_\mathfrak{p}^{-1}\widetilde{K}_{G^\tau}^*(Y)(\pi)_\mathfrak{p}^\wedge = 0$ via the previous discussion. One can, for instance, take $Y = X \wedge S(E\mathcal{G})$, where $S(-)$ denotes the unreduced suspension and $\mathcal{G}$ is the family of subgroups not containing a conjugate of H as a subgroup. One observes that this Y is not finite; we instead use the pro-group valued K-theory version of (3.2) given in Theorem 3.12.

Now, the proof will be complete after constructing an isomorphism

$$S_\mathfrak{p}^{-1}\widetilde{\mathbf{K}}_{G^\tau}^*(Y)(\pi) \to S_\mathfrak{p}^{-1}\widetilde{\mathbf{K}}_{G^\tau}(X)(\pi). \tag{3.3}$$

Then, after completing with the $\mathfrak{p}$-adic topology and using that $S_\mathfrak{p}^{-1}\widetilde{\mathbf{K}}_{G^\tau}^*(Y)(\pi)_\mathfrak{p}^\wedge = 0$, we have the result.

Let $\{Y_\alpha\}$ denote the inductive system of finite sub-complexes of Y and let $X_\alpha = X \cap Y_\alpha$. For each α, an isomorphism

$$S_\mathfrak{p}^{-1}\widetilde{K}_{G^\tau}^*(Y_\alpha)(\pi) \to S_\mathfrak{p}^{-1}\widetilde{K}_{G^\tau}(X_\alpha)(\pi).$$

can be constructed via induction along the finitely many cells of Y_α that are not in X_α. Such cells are of orbit type G/K with $K \in \mathcal{G}$. We just need to check that $S_{\mathfrak{p}}^{-1}\widetilde{K}_{G^\tau}(G/H)(\pi) = 0$. This follows from the result [Seg68b, 3.7 (iv)], noting that this is a twisted representation ring; the proof works the same with the twist. $\square$

CHAPTER 4
THOM ISOMORPHISM AND PUSHFORWARDS

In this chapter, we describe the Thom isomorphism in twisted equivariant K-theory and use it to define the pushforward map along equivariant principal S^1-bundles. The main result we use comes from a paper by Moutuou [Mou13], who proves that there is a Thom isomorphism in the "real" twisted K-theory of groupoids. Although the "real" case is of independent interest, for example in the study of topological insulators (see for instance [MJ22]), we do not require this level of generality and so we restrict the result to twisted equivariant K-theory. If $\pi\colon V \to X$ is a G-equivariant vector bundle (not necessarily K-oriented) and P is a G-equivariant twist on X, then there is a Thom isomorphism

$$K^*_G(X, P + \mathbb{C}l(V)) \cong K^*_G(E, \pi^*P).$$

This first group is the twisted K-theory that has been further twisted by the complex Clifford bundle associated with V, see Definition 1.4. Since Moutuou's result is in the context of groupoid equivariant KK-theory, we first show that, at least for compact spaces, the twisted K-theory groups we consider are examples of $(G \ltimes X)$-equivariant KK-groups. The groupoid equivariant setting is useful to us because the C*-algebras we use – mainly sections of equivariant bundles – have a natural $(G \ltimes X)$-module structure that we can take advantage of. In particular, we can take tensor products over $C_0(X)$ instead of just $\mathbb{C}$.

One should notice that this Thom isomorphism exists for all vector bundles, not just K-oriented vector bundles like in the untwisted setting. This phenomenon occurs in ordinary cohomology as well; to define the Thom isomorphism for a non-oriented vector bundle one must use cohomology with local coefficients. Thus, one could consider this as a motivation for defining twisted K-theory.

In the non-equivariant setting, the Thom isomorphism and pushforward in twisted K-theory has already been constructed in [CW08], incorporating a variety of approaches to twisted K-theory; bundle gerbe modules, bundles of Fredholm operators, and a small amount of KK-theory. Some other work on the Thom isomorphism includes the thesis of Garvey [Gar22], who proves that the Thom class in groupoid equivariant KK-theory defined in two ways, by pulling back the Bott element or by using spin representations and Clifford multiplications are the same.

As a final remark, we comment on the fact that the Thom isomorphism in twisted equivariant K-theory in the above form does not, to our knowledge, explicitly exist yet in the algebraic topology literature. That it is just a special case of a broader KK-theoretic result is a commendation of the richness of Kasparov's KK-theory and its groupoid equivariant generalisations. This should serve as an advertisement to algebraic topologists about the usefulness of KK-theory when working with K-theory.

To make this chapter as self-contained as possible, we start with some background information on groupoid equivariant KK-theory. We then prove that our twisted equivariant K-groups can be written as $(G \ltimes X)$-equivariant K-groups before outlining the proof of the Thom isomorphism theorem. Finally, motivated by T-duality, we apply the Thom isomorphism to pushforwards along principal S^1-bundles.

4.1 Groupoid Equivariant KK-Theory

Groupoid equivariant KK-theory was introduced by Le Gall [LG99]. For completeness, we use this section to introduce the main definitions and properties. We do not provide proofs that can be found elsewhere.

Definition 4.1. Let X be a locally compact space. A $C_0(X)$-algebra is a C*-algebra A together with a nondegenerate homomorphism $\theta\colon C_0(X) \to M(A)$, meaning that

$$\theta\big(C_0(X)\big)A = \big\{\theta(f)(a) : f \in C_0(X), a \in A\big\}$$

is dense in A, where $M(A)$ is the multiplier algebra of A.

Using the characterisation of $M(A)$ as the adjointable maps $A \to A$, one can view a $C_0(X)$-algebra as a C*-algebra with an action of $C_0(X)$ via adjointable maps. Requiring θ to be nondegenerate allows us to extend a $*$-homomorphism $f\colon A \to B$ to a map $M(f)\colon M(A) \to M(B)$. A $C_0(X)$-algebra morphism is a non-degenerate $*$-homomorphism $f\colon A \to B$ such that the following commutes:

$$\begin{tikzcd} & C_0(X) \arrow[dl, "\theta_A"'] \arrow[dr, "\theta_B"] & \\ M(A) \arrow[rr, "M(f)"] & & M(B). \end{tikzcd}$$

This of course just means that f preserves the $C_0(X)$-action.

Example 4.2. The simplest $C_0(X)$-algebra is of course $C_0(X)$ itself, with θ being the inclusion of $C_0(X)$ into its multiplier algebra.

Example 4.3. Let E be a bundle of C*-algebras on X, for example, the bundle of compact operators associated to a principal $PU(\mathcal{H})$-bundle. Then the sections of E that vanish at infinity, $\Gamma_0(E)$, is a $C_0(X)$-algebra. The action is of $C_0(X)$ on $\Gamma_0(E)$ is $(\varphi \cdot \sigma)(x) = \varphi(x) \cdot \sigma(x)$, with $\varphi \in C_0(X)$, $\sigma \in \Gamma_0(E)$.

The algebra of sections of a bundle is more than just an example of a $C_0(X)$-algebra. Indeed, there is a close connection between a $C_0(X)$-algebra and a bundle of algebras on X. The most well-known example of this is the Serre-Swan theorem, which states that taking the algebra of sections gives a correspondence between the complex vector bundles on X and the finitely generated projective $C_0(X)$-modules. Furthering this relationship, we can talk about the fiber of a $C_0(X)$-algebra:

Definition 4.4. The fiber of a $C_0(X)$-algebra A at $x \in X$ is defined as $A_x = A/I_xA$, where $I_x = \{f \in C_0(X) : f(x) = 0\}$. If $\varphi\colon A \to B$ is a map of $C_0(X)$-algebras and $x \in X$, then $\varphi_x\colon A_x \to B_x$ denotes the associated map on fibers.

Example 4.5. As expected, the fiber of $\Gamma_0(E)$ at $x \in X$ is E_x, where E is a bundle of C*-algebras over X.

The following definition of a pullback of C*-algebras comes from [RW85, §1].

Definition 4.6. Let $f\colon X \to Y$ be a continuous map between locally compact spaces and let A be a $C_0(Y)$-algebra. The pullback of A along f is

$$f^*A := C_0(X) \otimes_{C_0(Y)} A,$$

where tensor product is defined as the quotient of $C_0(X) \otimes A$ by the ideal generated by elements of the form

$$(\varphi \cdot \psi) \otimes a - \varphi \otimes (\psi \cdot a),$$

where $\varphi \in C_0(X)$, $\psi \in C_0(Y)$, and $a \in A$.

Two results justify calling this construction the pullback. The first, found in [RW85, Prop 1.3], is relevant to us because algebras of sections occur in the definition of K-theory.

Proposition 4.7. *Let E be a bundle of C*-algebras over a locally compact space Y and let $f\colon X \to Y$ be a continuous map. Then the $C_0(X)$-algebras $f^*\Gamma_0(E)$ and $\Gamma_0(f^*E)$ are isomorphic.*

The next also matches our intuition from bundles; if $E \to Y$ is a bundle and $f\colon X \to Y$, then the fiber of f^*E at $x \in X$ is the fiber of E at $f(x)$.

Proposition 4.8. *Let A be a $C_0(Y)$-algebra and let $f\colon X \to Y$ be a continuous map. If $x \in X$, then $(f^*A)_x$ is isomorphic to $A_{f(x)}$.*

Proof. The isomorphism is given by

$$\Phi\colon (f^*A)_x = \frac{C_0(X) \otimes_{C_0(Y)} A}{I_x(C_0(X) \otimes_{C_0(Y)} A)} \longrightarrow \frac{A}{I_{f(x)}A} = A_x$$
$$[\varphi \otimes a] \mapsto [\varphi(x)a].$$

If $\psi \in C_0(Y)$, then

$$\Phi([\varphi \cdot (\psi \circ f) \otimes a]) = [\varphi(x)\psi(f(x))a] = [\varphi(x)(\psi \cdot a)] = \Phi([\varphi \otimes (\psi \cdot a)]).$$

The second equality comes from the fact that $\psi - \psi(f(x)) \in I_{f(x)}$, where $\psi(f(x))$ is viewed as a constant function. Moreover, if $\varphi \in I_x$ then

$$\Phi\big([\varphi \otimes a]\big) = [\varphi(x)a] = 0.$$

Therefore, Φ is a well-defined function. Its inverse is given by $[a] \mapsto [1 \otimes a]$. $\square$

Definition 4.9. Let $\mathcal{G}$ be a locally compact topological groupoid. A $\mathcal{G}$-algebra is a $C_0(\mathcal{G}_0)$-algebra A together with an isomorphism $\alpha\colon s^*A \xrightarrow{\cong} t^*A$ of $C_0(\mathcal{G}_1)$-algebras satisfying $\alpha_{\gamma_1 \circ \gamma_2} = \alpha_{\gamma_1} \circ \alpha_{\gamma_2}$ for all $(\gamma_1, \gamma_2) \in \mathcal{G}_2$.

Remark 4.10. This definition fits into a general framework for defining a groupoid-equivariant object. For example, a $\mathcal{G}$-equivariant vector bundle consists of a vector bundle $E \to \mathcal{G}_0$ together with an isomorphism $s^*E \cong t^*E$ of bundles over $\mathcal{G}_1$ that satisfies an associativity condition over $\mathcal{G}_2$.

Naturally, groupoid equivariant C*-algebras are the objects for which we define groupoid equivariant KK-theory. The cycles for $\mathcal{G}$-equivariant KK-theory are $\mathcal{G}$-equivariant Kasparov bimodules.

Definition 4.11. Let B be a $\mathcal{G}$-algebra. A $\mathcal{G}$-equivariant Hilbert B-module is a Hilbert B-module $\mathcal{E}$ together with a unitary $V \in \mathcal{L}(s^*\mathcal{E}, t^*\mathcal{E})$ such that $V_{\gamma_1 \circ \gamma_2} = V_{\gamma_1} \circ V_{\gamma_2}$ for all $(\gamma_1, \gamma_2) \in \mathcal{G}_2$.

Definition 4.12. Let A and B be two $\mathcal{G}$-algebras. A Kasparov A-B-bimodule is a triple $(\mathcal{E}, \pi, T)$ consisting of a $\mathcal{G}$-equivariant Hilbert B-module $\mathcal{E}$, a $\mathcal{G}$-equivariant representation $\pi\colon A \to \mathcal{L}(\mathcal{E})$, and a $\mathcal{G}$-equivariant operator T on $\mathcal{L}(\mathcal{E})$ such that

1. $\pi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{E})$,
2. $\pi(a)(T - T^*) \in \mathcal{K}(\mathcal{E})$,
3. $[T, \pi(a)] \in \mathcal{K}(\mathcal{E})$, and
4. $[\pi \otimes 1](a')(V(s^*T)V^* - t^*T) \in t^*\mathcal{K}(\mathcal{E})$

for all $a \in \mathcal{A}$ and $a' \in t^*A$, with V being the unitary map in the definition of a $\mathcal{G}$-equivariant Hilbert module.

If one considers graded $\mathcal{G}$-algebras, as is the case when the Clifford algebra is introduced, then one must use graded Hilbert modules and graded homomorphisms. Then, T is required to be an operator of degree 1.

Definition 4.13. Two Kasparov A-B-bimodules $(\mathcal{E}_0, \pi_0, T_0)$ and $(\mathcal{E}_1, \pi_1, T_1)$ are unitarily equivalent if there exists a $\mathcal{G}$-equivariant unitary operator $U\colon E_0 \to E_1$ of degree 0 such that $U\pi_0(a) = \pi_1(a)U$ for all $a \in A$ and $UT_0 = T_1U$.

Let $E^{\mathcal{G}}(A, B)$ be the semi-group of unitary equivalence classes of $\mathcal{G}$-equivariant Kasparov A-B-bimodules. For any $\mathcal{G}$-algebra B, let IB be the $\mathcal{G}$-algebra of paths $[0, 1] \to B$ and $\mathrm{ev}_t\colon IB \to B$ be defined via evaluation at t. For each $t \in [0, 1]$, there is an 'evaluation' semigroup homomorphism

$$\mathrm{ev}_{t*}\colon E^{\mathcal{G}}(A, IB) \longrightarrow E^{\mathcal{G}}(A, B).$$

See the definitions of KK-theory in [Bla98, 17.2.2] or [Gar22, 2.6.4] for precise details of this construction.

Definition 4.14. A homotopy between two Kasparov A-B-bimodules $(\mathcal{E}_0, \pi_0, T_0)$ and $(\mathcal{E}_1, \pi_1, T_1)$ is a $\mathcal{G}$-equivariant Kasparov A-IB-bimodule $(\mathcal{E}, \pi, T)$ such that, for each $i \in \{0, 1\}$, we have

$$\mathrm{ev}_{i*}\big[(\mathcal{E}, \pi, T)\big] = \big[(\mathcal{E}_i, \pi_i, T_i)\big] \in E^{\mathcal{G}}(A, B).$$

Definition 4.15. $KK_0^{\mathcal{G}}(A, B)$ is the homotopy equivalence classes of $E^{\mathcal{G}}(A, B)$. Moreover,

$$KK_1^{\mathcal{G}}(A, B) := KK_0^{\mathcal{G}}(A, B \otimes \mathbb{C}\ell(1)),$$

where $\mathbb{C}\ell(1)$ is the complex Clifford algebra with one generator.

The following result is a summary of the main properties of groupoid equivariant KK-theory:

Proposition 4.16. *Groupoid equivariant KK-theory has the following properties:*

1. *If $\alpha\colon A \to A'$ is a morphism of $\mathcal{G}$-algebras, then there is a morphism*

   $$\alpha^*\colon KK^{\mathcal{G}}(A', B) \to KK^{\mathcal{G}}(A, B),$$

   that is, $KK^{\mathcal{G}}(-, B)$ is a contravariant functor.

2. *If $\beta\colon B \to B'$ is a morphism of $\mathcal{G}$-algebras, then there is a morphism*

   $$\beta_*\colon KK^{\mathcal{G}}(A, B) \to KK^{\mathcal{G}}(A, B'),$$

   that is, $KK^{\mathcal{G}}(A, -)$ is a covariant functor.

3. *A generalised map Φ from $\mathcal{G}$ to $\mathcal{H}$ induces a map*

   $$\Phi^*\colon KK^{\mathcal{H}}(A, B) \to KK^{\mathcal{G}}(\Phi^*A, \Phi^*B),$$

where the pullbacks $\Phi^* A$ and $\Phi^* B$ are defined by Le Gall [LG99, §3].

4. *There are exterior product maps*

$$\mathrm{ext}_D\colon KK^{\mathcal{G}}(A, B) \to KK^{\mathcal{G}}(A \otimes D, B \otimes D)$$

$$\mathrm{ext}'_D\colon KK^{\mathcal{G}}(A, B) \to KK^{\mathcal{G}}(D \otimes A, D \otimes B)$$

5. *There is an associative Kasparov product, natural with respect to all the previous maps:*

$$KK^{\mathcal{G}}(A, D) \times KK^{\mathcal{G}}(D, B) \to KK^{\mathcal{G}}(A, B).$$

We denote the product by $(x, y) \mapsto x \otimes_D y$.

4.2 Twisted K-Theory as KK-Theory

To make use of groupoid equivariant KK-theory, it is useful to be able to express our already-defined twisted equivariant K-groups as $(G \ltimes X)$-equivariant KK-groups. To this end, we shall show that, when X is compact, there is an isomorphism

$$KK^{G \ltimes X}\big(C_0(X), \Gamma_0(X, P_{\mathcal{K}})\big) \cong KK^G\big(\mathbb{C}, \Gamma_0(X, P_{\mathcal{K}})\big). \tag{4.1}$$

Let us start by mentioning the following result, which tells us that the Hilbert modules of a C*-algebra of sections are always sections of some bundle of Banach algebras.

Proposition 4.17. [TXLG04, Prop A.4]. *Let $\mathcal{A} \to X$ be a bundle of C*-algebras over X and $\mathcal{E}$ a Hilbert $\Gamma_0(X, \mathcal{A})$-module. Then there exists a bundle of Banach algebras $\tilde{\mathcal{E}}$ such that $\mathcal{E} \cong \Gamma_0(X, \tilde{\mathcal{E}})$.*

The authors of the cited paper use the term "field of C*-algebras" instead of "bundle" and distinguish between continuous and upper semi-continuous fields of C*-algebras. A field of C*-algebras $\mathcal{A}$ is continuous (resp. upper semi-continuous) if the norm function $\mathcal{A} \to \mathbb{R}_{\geq 0}$ is continuous (resp. upper semi-continuous). In this thesis, when we say "bundle of C*-algebras" or "bundle of Banach algebras", upper semi-continuity will suffice.

The isomorphism (4.1) will follow from the following result, whose proof relies heavily on the results of [TXLG04].

Theorem 4.18. *Let X be a compact G-space and let A be a $C_0(X)$-algebra with G-action (equivalently a C*-algebra with $(G \ltimes X)$-action). There is an isomorphism*

$$KK^{G \ltimes X}(C_0(X), A) \cong KK^G(\mathbb{C}, A).$$

Proof. By [TXLG04, Prop A.3], A is isomorphic to $\Gamma_0(X, E)$ for some bundle E of C*-algebras over X. Thus we can assume that $A = \Gamma_0(X, E)$.

Let $(\mathcal{E}, \pi, T)$ be a $(G \ltimes X)$-equivariant Kasparov $(C_0(X), A)$-module. By Proposition 4.17, there exists a bundle $\tilde{E}$ of Banach algebras such that $\mathcal{E} \cong \Gamma_0(X, \tilde{\mathcal{E}})$. Then, since an isomorphism $s^*\mathcal{E} \cong t^*\mathcal{E}$ satisfying the associativity property is equivalent to a G-action on $\tilde{\mathcal{E}}$, it is clear that $\mathcal{E}$ being $(G \ltimes X)$-equivariant is equivalent to $\tilde{\mathcal{E}}$ being a G-equivariant bundle. Thus we can forget π so that $(\mathcal{E}, T)$ is a G-equivariant Kasparov $(\mathbb{C}, A)$-module. This defines a map

$$KK^{G \ltimes X}(C_0(X), A) \to KK^G(\mathbb{C}, A).$$

For the other direction, start with a G-equivariant Kasparov $(\mathbb{C}, A)$-module $(\mathcal{E}, T)$. Again, we can assume that $\mathcal{E}$ is the algebra of sections of a Banach bundle over X, and $\pi\colon C_0(X) \to \mathcal{L}(\mathcal{E})$ can be defined using the canonical left action of $C_0(X)$ on $\mathcal{E}$. Then, using the same reasoning as the previous paragraph, $(\mathcal{E}, \pi, T)$ is a $(G \ltimes X)$-equivariant Kasparov $(C_0(X), A)$-module.

These constructions induce maps in both directions. For them to be inverse to each other, we need to show that if $(\mathcal{E}, \pi, T)$ is a $(G \ltimes X)$-equivariant Kasparov $(C_0(X), A)$-module, then $\pi\colon C_0(X) \to \mathcal{L}(\mathcal{E})$ is the map induced by the canonical $C_0(X)$-structure on the space of sections $\mathcal{E}$. Consider the following diagram:

$$\begin{tikzcd}
& C_0(X) \arrow[r, hook, "L_1"] \arrow[dd, "\pi"] & \mathcal{Z}M(C_0(X)) \arrow[dd, "M(\pi)"] \\
C_0(X) \arrow[ur, "="] \arrow[dr, "\varphi"'] & & \\
& \mathcal{Z}\mathcal{L}(\mathcal{E}) \arrow[r, hook, "L_2"'] & \mathcal{Z}M(\mathcal{L}(\mathcal{E}))
\end{tikzcd}$$

Let us explain the maps: L_1 is the inclusion of $C_0(X)$ into its multiplier algebra, so that $L_1(f)(g) = fg$. The map $\varphi\colon C_0(X) \to \mathcal{Z}\mathcal{L}(\mathcal{E})$ is the one defining the $C_0(X)$-algebra structure of $\mathcal{E}$. L_2 is the inclusion of $\mathcal{L}(\mathcal{E})$ into its multiplier algebra, restricted to the centre. The composition $L_2 \circ \varphi$ is then the induced $C_0(X)$-algebra structure on $\mathcal{L}(\mathcal{E})$. The map $M(\pi)$ is the map that π induces on the multiplier algebras. For $M(\pi)$ to exist, π must be non-degenerate, but this is one of the requirements for Kasparov modules. The condition of π being a $C_0(X)$-algebra map precisely means that the outer triangle commutes. So, the commutativity of the outer triangle and the square gives

$$L_2 \circ \varphi = M(\pi) \circ L_1 = L_2 \circ \pi.$$

Thus, since L_2, is injective, we have $\pi = \varphi$, as required. □

Applying this theorem to $A = \Gamma(X, P_{\mathcal{K}})$ gives us isomorphism (4.1). As a final remark, we note that this proof is much less technical than the similar result [TXLG04, Proposition 6.10]. This is because they are working in the more general setting of twisted K-theory of differentiable stacks, which requires more advanced techniques.

4.3 Thom Isomorphism

In this section, we outline Moutuou's construction of the Thom isomorphism [Mou13, Theorem 8.1] for our case of interest; twisted equivariant K-theory. We reiterate that this is not an original result, but is included to make the thesis more self-contained and to state it in the context of algebraic topology. Moutuou's Thom isomorphism works in a more general context, namely for "real" twisted K-theory of groupoids, which includes our special case.

Theorem 4.19. *Let X be a G-space, $P \to X$ a stable G-equivariant principal $PU(\mathcal{H})$-bundle, and $\pi\colon V \to X$ a G-equivariant vector bundle of rank n. Then, there is an isomorphism*

$$K_G^*(X, P + \mathbb{C}l(V)) \cong K_G^*(V, \pi^*P), \tag{4.2}$$

where $\mathbb{C}l(V)$ is the Clifford bundle of V. Moreover, if $\pi\colon V \to X$ is K-oriented, then this reduces to an isomorphism

$$K_G^*(X, P) \cong K_G^{*+n}(V, \pi^*P). \tag{4.3}$$

In the non-equivariant setting, the Thom isomorphism in twisted K-theory is well established [CW08, Kar08]. One could consider the Thom isomorphism as one of the motivations for defining twisted K-theory. Just as local coefficients are needed to define the Thom isomorphism in ordinary cohomology for non-oriented vector bundles, twisted K-theory is needed for bundles that are not K-oriented.

Proof. The proof comes down to constructing a KK-equivalence

$$\tau_{V,P} \in KK^G\big(\Gamma_0(V, \pi^*P_k), \Gamma_0(X, P_K \otimes \mathbb{C}l(V))\big).$$

The Kasparov product then gives us the desired isomorphism:

$$\begin{aligned} K_G^*(V, \pi^*P) &\cong KK_*^G\big(\mathbb{C}, \Gamma_0(V, \pi^*P_{\mathcal{K}})\big) \\ &\xrightarrow{\otimes\tau_{V,P}} KK^G\big(\mathbb{C}, \Gamma_0(X, \pi^*P_K \otimes \mathbb{C}l(V))\big) \\ &\cong K_G^*(X, P + \mathbb{C}l(V)). \end{aligned}$$

Start by constructing the KK-equivalence in the (partially) untwisted case, that is, when P is trivial but V may not be K-oriented. Any G-vector bundle $\pi\colon V \to X$ is classified up to isomorphism by a generalised map from $G \ltimes X$ to $O(n)$. Indeed, any such vector bundle has an associated principal $O(n)$-bundle, which, by passing to a trivialising open cover, gives a generalised morphism from $G \ltimes X$ to $O(n)$.

Let $\Phi\colon G \ltimes X \to O(n)$ be the generalised morphism associated with V. Consider the Bott element $\alpha \in KK^{O(n)}\big(C_0(\mathbb{R}^n), \mathbb{C}l_n\big)$ introduced by Kasparov in [Kas81, §5]. Note that Kasparov's Bott element is actually in Spin^c-equivariant K-theory, but is constructed by first constructing an $O(n)$-equivariant Bott element. The desired KK-equivalence in the partially untwisted case is the pullback of the Bott element under Φ,

$$\tau_V := \Phi^*\alpha \in KK^{G\ltimes X}\big(C_0(V), \Gamma_0(X, \mathbb{C}l(V))\big).$$

For this to make sense, one needs to understand how to pullback a C*-algebra along a generalised morphism; this process was first described by Le Gall [LG99, §3], but Moutuou also recalls the construction in his paper. We indeed have that

$$\Phi^* C_0(\mathbb{R}^n) \cong C_0(V) \quad \text{and} \quad \Phi^*\mathbb{C}l_n \cong \Gamma_0(X, \mathbb{C}l(V)).$$

An instructive demonstration of the first isomorphism can be found in Example 4.2.5 of [Gar22].

From here, one can incorporate the twist P by taking the exterior product with $\Gamma_0(X, P_{\mathcal{K}})$. Let $\tilde{\tau}_{V,P}$ be the image of τ_V under the map

$$\begin{aligned} &KK_*^{G\ltimes X}\big(C_0(V), \Gamma_0(X, \mathbb{C}l(V))\big) \xrightarrow{\mathrm{ext}_{\Gamma_0(X,P_{\mathcal{K}})}} \\ &\qquad KK_*^{G\ltimes X}\big(C_0(V)\otimes_{C_0(X)} \Gamma_0(X, P_{\mathcal{K}}), \Gamma_0(X, \mathbb{C}l(V)) \otimes_{C_0(X)} \Gamma_0(X, P_{\mathcal{K}})\big). \end{aligned}$$

By Proposition 4.7, which says that the pullback of an algebra of sections is the algebra of sections of the pullback bundle, we have

$$C_0(V)\otimes_{C_0(X)} \Gamma_0(X, P_{\mathcal{K}}) \cong \Gamma_0(V, \pi^* P_{\mathcal{K}}).$$

Moreover,

$$\Gamma_0(X, \mathbb{C}l(V)) \otimes_{C_0(X)} \Gamma_0(X, P_{\mathcal{K}}) \cong \Gamma_0(X, P_{\mathcal{K}} \otimes \mathbb{C}l(V)).$$

Therefore, $\tilde{\tau}_{V,P} \in KK^{G\ltimes X}\big(\Gamma_0(V, \pi^* P_k), \Gamma_0(X, P_K \otimes \mathbb{C}l(V))\big)$. To obtain $\tau_{V,P}$, we use that there is a natural forgetful functor $KK^{G\ltimes X} \to KK^G$, see the remark in [Kas88, §2.19]. On the level of cycles this map forgets about the $C_0(X)$-action. Applying this functor to $\tilde{\tau}_{V,P}$ gives the desired KK-equivalence $\tau_{V,P}$.

When $\pi\colon V \to X$ is K-oriented, the bundle $\mathbb{C}l(V)$ is trivial [Mou13, Prop 7.9]. Using Proposition 4.7 again, we have $\Gamma_0(X, \mathbb{C}l(V)) \cong C_0(X)\hat{\otimes}\mathbb{C}l_n$, since $\mathbb{C}l(V)$ is a pullback of a Clifford bundle over a point. Therefore,

$$\begin{aligned}K^*_G(X, P + \mathbb{C}l(V)) &\cong KK^G_*\big(\mathbb{C}, \Gamma_0(X, P \otimes \mathbb{C}l(V))\big)\\ &\cong KK^G_*\big(\mathbb{C}, \Gamma_0(X, P)\hat{\otimes}\mathbb{C}l_n\big)\\ &\cong KK^G_{*+n}\big(\mathbb{C}, \Gamma_0(X, P)\big)\\ &\cong K^{*+n}_G(X, P)\end{aligned}$$

So, isomorphism (4.2) implies isomorphism (4.3). This completes the proof. □

4.4 Pushforwards for Principal S^1-Bundles

With the Thom isomorphism in twisted equivariant K-theory established, it is straightforward to define the pushforward along K-oriented maps $f\colon X \to Y$. One uses the standard Pontryagin-Thom construction: choose an embedding of X into $\mathbb{R}^N$, then factor f through $\mathbb{R}^N \times Y$ and use the Thom isomorphism for the normal bundle together with the Pontryagin-Thom collapse map. In this section, we restrict our attention to G-equivariant principal S^1-bundles. These are all K-oriented, and the pushforward can be constructed relatively explicitly.

Let $p\colon E \to X$ be a G-equivariant principal S^1-bundle. In this case, the pushforward can be constructed very explicitly. We can always factor p through the associated complex line bundle:

$$\begin{tikzcd} & E \times_{S^1} \mathbb{C} \arrow[dr, "\pi"] & \\ E \arrow[ur, hook, "i"] \arrow[rr, "p"] & & X \end{tikzcd}$$

The pushforward $p_!$ is defined by pushing forward along the embedding i and then using the Thom isomorphism for π. The second part is clear, since we have established the Thom isomorphism in twisted equivariant K-theory. Thus we focus on the pushforward along the embedding $i\colon E \to E \times_{S^1} \mathbb{C}$.

Lemma 4.20. *Let $p\colon E \to X$ be a G-equivariant principal S^1-bundle. The normal bundle of the embedding $i\colon E \to E \times_{S^1} \mathbb{C}$ is isomorphic to $E \times \mathbb{R}$.*

Proof. For notational ease, let $\tilde{E} = E \times_{S^1} \mathbb{C}$. We write elements of $T\tilde{E}$ as pairs (e, u) with $e \in \tilde{E}$ $u \in T_e\tilde{E}$. Choose the following inner product on $\tilde{E}$:

$$\big\langle [e, z_1], [e, z_2] \big\rangle := \mathrm{Re}\big(\bar{z}_1 z_2\big).$$

This comes from the standard inner product on $\mathbb{R}^2$ after the canonical identification

$\mathbb{C} = \mathbb{R}^2$. This inner product satisfies the following properties

$$i(E) = \big\{ e \in E \times_{S^1} \mathbb{C} : \|e\| = 1 \big\} \tag{4.4}$$

$$i_*(T_v E) = \big\{ (e,u) \in (T_v \tilde{E})|_{i(E)} : \langle e, u \rangle = 0 \big\} \tag{4.5}$$

We explain property (4.5): $T_v E$ and $T_v \tilde{E}$ denote the vertical sub-bundle of the tangent bundles on E and $\tilde{E}$, respectively. The product $\langle e, u \rangle$ makes sense after identifying $(T_v \tilde{E})_e$ with $\tilde{E}_{p(e)}$. This is possible because there is a canonical isomorphism $T_v \tilde{E} \cong \pi^* \tilde{E}$. Property (4.4) is straightforward to show directly. For (4.5), it is just a matter of checking the condition locally.

These conditions are motivated by the standard embedding $S^1 \hookrightarrow \mathbb{C}$. The image of S^1 is precisely the unit complex numbers and the tangent vectors to $u \in S^1$ are precisely those orthogonal to u.

From here, we can write the map directly. Let $P\colon T\tilde{E} \to T_v \tilde{E}$ be a choice of projection map onto the vertical sub-bundle. Consider the map

$$T\tilde{E}|_{i(E)} \to E \times \mathbb{R}, \quad (e,u) \mapsto \big(e, \langle e, Pu \rangle\big), \tag{4.6}$$

again making the identification $(T_v \tilde{E})_e = \tilde{E}_{\pi(e)}$ for the inner product to make sense. An element $(e,u) \in i_*(TE)$ is mapped to $(e,0)$, because of (4.5), and thus (4.6) descends to a map on the normal bundle $T\tilde{E}|_{i(E)}/i_*(TE)$. This is an isomorphism; the inverse sends (e,t) to (e,te), where $te \in E_{\pi(e)} \cong (T_v \tilde{E})_e$, and hence lives in the correct space, $T_e \tilde{E}$. □

With this done, we define the pushforward along $i\colon E \to E \times_{S^1} \mathbb{C}$. Using the tubular neighbourhood theorem, choose a neighbourhood $U \supseteq i(E)$ such that $U \cong N_i$, where $N_i \cong E \times \mathbb{R}$ is the normal bundle. We can in fact choose $U = E \times_{S^1} \mathbb{C}^*$, because $E \times \mathbb{R} \cong E \times_{S^1} \mathbb{C}^*$ via a map $(e,t) \mapsto [e, \gamma(t)]$, where $\gamma\colon \mathbb{R} \to (0,\infty)$ is any homeomorphism. Then, the pushforward is the composition

$$i_!\colon K_G^*(E) \cong K_G^{*+1}(E \times \mathbb{R}) \cong K_G^{*+1}(E \times_{S^1} \mathbb{C}^*) \to K_G^{*+1}(E \times_{S^1} \mathbb{C}).$$

The final map is the extension map; it can be described by extending compactly supported sections on $E \times_{S^1} \mathbb{C}^*$ to $E \times_{E^1} \mathbb{C}$ by mapping $[e,0]$ to 0.

Therefore, the final pushforward map is the composition

$$p_!\colon K_G^*(E) \xrightarrow{i_!} K_G^{*+1}(E \times_{S^1} \mathbb{C}) \xrightarrow{\pi_!} K_G^{*+1}(X),$$

where $\pi_!$ is the inverse of the Thom isomorphism $K_G(X) \cong K_G(E \times_{S^1} \mathbb{C})$. We have written the pushforward for untwisted K-theory, but it works just the same for twisted K-theory; we have shown that no further orientability conditions are required to produce a Thom isomorphism in twisted equivariant K-theory, and since this pushforward is defined using two Thom isomorphisms, everything works in this setting as well.

The pushforward is a composition of several maps, all of which are isomorphisms except for one. Hence, the pushforward is an isomorphism if and only if the extension map $K_G^*(E \times_{S^1} \mathbb{C}^*) \to K_G^*(E \times_{S^1} \mathbb{C})$ in the definition of $i_!$ is an isomorphism. The following results will describe some situations where this is the case:

Theorem 4.21. *Let X be a G-space and let P be a G-equivariant twist on X. The pushforward*

$$K_G^i(X \times S^1, \mathrm{pr}_1^* P) \to K_G^{i-1}(X, P)$$

is an isomorphism if and only if $K_G^i(X, P) = 0$, where G acts trivially on the S^1 factor of $X \times S^1$.

Proof. The normal bundle of the embedding of $X \times S^1$ into $X \times \mathbb{C}$ is trivial and homeomorphic to $X \times \mathbb{C}^*$, that is, we have $U = X \times \mathbb{C}^*$, using the above notation. Thus we need to show that

$$K_G^{i-1}(X \times \mathbb{C}^*, \mathrm{pr}_1^* P) \to K_G^{i-1}(X \times \mathbb{C}, \mathrm{pr}_1^* P) \tag{4.7}$$

is an isomorphism if and only if $K_G^i(X, P) = 0$.

There is the following short exact sequence of C*-algebras:

$$0 \to \Gamma_0(X \times \mathbb{C}^*, P_{\mathcal{K}} \times \mathbb{C}^*) \to \Gamma_0(X \times \mathbb{C}, P_{\mathcal{K}} \times \mathbb{C}) \to \Gamma_0(X, P) \to 0$$

The first map is defined by extending sections by defining $\sigma(x, 0) = 0$, where the zero is the zero in the fiber. This defines a continuous section because σ vanishes at infinity. The second map sends a section σ to $\sigma(-, 0)$. We therefore have the following long exact sequence of K-theory groups:

$$\begin{tikzcd}
K_G^0(X \times \mathbb{C}^*, \mathrm{pr}_1^* P) \arrow[r] & K_G^0(X \times \mathbb{C}, \mathrm{pr}_1^* P) \arrow[r] & K_G^0(X, P) \arrow[d] \\
K_G^1(X, P) \arrow[u] & K_G^1(X \times \mathbb{C}, \mathrm{pr}_1^* P) \arrow[l] & K_G^1(X \times \mathbb{C}^*, \mathrm{pr}_1^* P) \arrow[l]
\end{tikzcd} \tag{4.8}$$

We can show that $K_G^{i-1}(X \times \mathbb{C}, \mathrm{pr}_1^* P) \to K_G^{i-1}(X, P)$ is the zero map. This map is induced by pulling back along the map $s \colon X \to X \times \mathbb{C}$, $x \mapsto (x, 0)$. On the level of C*-algebras, noting that $\mathrm{pr}_1^* P = P \times \mathbb{C}$, this gives

$$\Gamma_0\big(X \times \mathbb{C}, P_{\mathcal{K}} \times \mathbb{C}\big) \to \Gamma_0\big(X, P_{\mathcal{K}}\big), \tag{4.9}$$

$$\sigma \mapsto \big(x \mapsto \mathrm{pr}_1 \circ \sigma(x, 0)\big).$$

Observe that $\Gamma_0(X \times \mathbb{C}, P_{\mathcal{K}} \times \mathbb{C})$ is isomorphic to the sections of $P_{\mathcal{K}} \times D^2 \to X \times D^2$ that vanish on $X \times S^1$. Denote such sections by $\Gamma_0(X \times D^2, X \times S^1; P_{\mathcal{K}} \times D^2)$. We now have the following map

$$\Gamma_0\big(X \times D^2, X \times S^1; P_{\mathcal{K}} \times D^2\big) \to \Gamma_0\big(X, P_{\mathcal{K}}\big),$$

$$\sigma \mapsto \mathrm{pr}_1 \circ \sigma(x, 0).$$

This is homotopic to the constant map via

$$F_t \colon \sigma \mapsto \big(x \mapsto \mathrm{pr}_1 \circ \sigma(x, t)\big),$$

noting that $\sigma(x, 1) = 0$ since σ vanishes on $X \times S^1$. Therefore, we can conclude that (4.9) induces the zero map on K-theory.

Thus, exactness of the sequence implies that $K^i_G(X, P) = 0$ if and only if the map $K^{i-1}_G(X \times \mathbb{C}^*, \mathrm{pr}_1^* P) \to K^{i-1}_G(X \times \mathbb{C}, \mathrm{pr}_1^* P)$ is an isomorphism. This completes the proof. □

We consider two specific examples. Let $E_k := S^1$ be the S^1-space with the kth power action, that is, with action $S^1 \to S^1 \subseteq \mathrm{Aut}(S^1)$, $z \mapsto z^k$. When $k > 0$, we have

$$K^0_{S^1}(E_k) \cong K^0_{S^1}(S^1/\mathbb{Z}_k) \cong R(\mathbb{Z}_k) \quad \text{and} \quad K^1_{S^1}(E_k) = 0.$$

When $k = 0$, the action is trivial. Since

$$H^3_{S^1}(E_0) = H^3(S^1 \times BS^1) \cong H^1(S^1) \otimes H^2(BS^1) \cong \mathbb{Z},$$

the isomorphism classes of twist are classified by $\mathbb{Z}$. If P_k is a twist classified by $k \in \mathbb{Z}$, then a Mayer-Vietoris argument shows that

$$K^0_{S^1}(E_0, P_k) \cong 0 \quad \text{and} \quad K^1_{S^1}(E_0, P_k) \cong R(\mathbb{Z}_k).$$

The explicit calculation is found in [FHT11, §1]. The first example, which is a corollary of the previous theorem, considers the bundle $E_k \times E_0 \to E_k$.

Corollary 4.22. *For any S^1-equivariant twist $P \to E_k$, the following pushforward map is an isomorphism:*

$$K^1_{S^1}(E_k \times E_0) \xrightarrow{\cong} K^0_{S^1}(E_k, P).$$

The other pushforward is the zero map, since $K^1_{S^1}(E_k, P) = 0$.

The second example considers the bundle $E_0 \times E_k \to E_0$. It is not a consequence of the theorem but the main idea of the proof is the same.

Theorem 4.23. *Let $P_k \to E_0$ be a twist classified by $k \in \mathbb{Z} \cong H^3_{S^1}(E_0;\mathbb{Z})$. The following pushforward along $p\colon E_0 \times E_k \to E_0$,*

$$K^0_{S^1}(E_0 \times E_k, p^*P_k) \xrightarrow{\cong} K^1_{S^1}(E_0, P_k),$$

is an isomorphism. The other pushforward is the zero map, since $K^0_{S^1}(E_0, P_k) = 0$.

Proof. We again need to show that

$$K^1_{S^1}(E_0 \times \mathbb{C}^*, \mathrm{pr}_1^* P_k) \to K^1_{S^1}(E_0 \times \mathbb{C}, \mathrm{pr}_1^* P_k)$$

is an isomorphism. Let $X = E_0$ in the long exact sequence (4.8) from the proof of Theorem 4.21:

$$\begin{tikzcd}
K^0_{S^1}\big(E_0 \times \mathbb{C}^*, \mathrm{pr}_1^* P_k\big) \arrow[r] & K^0_{S^1}\big(E_0 \times \mathbb{C}, \mathrm{pr}_1^* P_k\big) \arrow[r] & K^0_{S^1}\big(E_0, P_k\big) \arrow[d] \\
K^1_{S^1}\big(E_0, P_k\big) \arrow[u] & K^1_{S^1}\big(E_0 \times \mathbb{C}, \mathrm{pr}_1^* P_k\big) \arrow[l] & K^1_{S^1}\big(E_0 \times \mathbb{C}^*, \mathrm{pr}_1^* P_k\big) \arrow[l]
\end{tikzcd}$$

Since $K^0_{S^1}(E_0, P_k) = 0$, it suffices to show that $K^1_{S^1}(E_0 \times \mathbb{C}, \mathrm{pr}_1^* P_k) \to K^1_{S^1}(E_0, P_k)$ is the zero map.

Let $E_0 = U \cup V$ where U and V are small open neighbourhoods of each hemisphere so that U and V are contractible and $U \cap V$ is homotopy equivalent to two points. Then $E_0 \times \mathbb{C} = (U \times \mathbb{C}) \cup (V \times \mathbb{C})$ and we can consider the resulting Mayer-Vietoris sequences side by side:

$$\begin{tikzcd}[column sep=small]
K^0_{S^1}(E_0 \times \mathbb{C}, \mathrm{pr}_1^* P_k) \arrow[dr] \arrow[dd] & & K^0_{S^1}(U \cap V \times \mathbb{C}) \arrow[dr] \arrow[dd] & \\
& K^0_{S^1}(U \times \mathbb{C}) \oplus K^0_{S^1}(V \times \mathbb{C}) \arrow[ur] \arrow[d] & & K^1_{S^1}(E_0 \times \mathbb{C}, \mathrm{pr}_1^* P_k) \arrow[d] \\
K^0_{S^1}(E_0, P_k) \arrow[r] & K^0_{S^1}(U) \oplus K^0_{S^1}(V) \arrow[r] & K^0_{S^1}(U \cap V) \arrow[r] & K^1_{S^1}(E_0, P_k)
\end{tikzcd}$$

There is no room for the zeroes on either end, but the first arrows in each row are injective and the last arrows in each row are surjective. In the centre two columns we have used trivialisations of the twist to replace the twisted K-groups with untwisted ones. This changes the centre horizontal map in each row, but for the proof it does not matter precisely how.

Now, using the contractibility of U and V, it is sufficient to show that the map $K^0_{S^1}(\mathbb{C}) \to K^0_{S^1}(*)$ is zero. This map is the pullback of the constant map at $0 \in \mathbb{C}$. Since $\mathbb{C}$ is not compact, we have

$$K^0_{S^1}(\mathbb{C}) = \widetilde{K}^0_{S^1}\big(\mathbb{C}^+\big) \cong \widetilde{K}^0_{S^1}\big(S^2\big)$$

We thus show that the map $\widetilde{K}^0_{S^1}(S^2) \to \widetilde{K}^0_{S^1}(S^0)$ is the zero map. The composition $S^2 \vee S^0 \to S^2 \times S^0 \to S^2 \wedge S^0$ gives the exact sequence

$$\widetilde{K}^0_{S^1}(S^2) = \widetilde{K}^0_{S^1}(S^2 \wedge S^0) \longrightarrow \widetilde{K}^0_{S^1}(S^2 \times S^0) \longrightarrow \widetilde{K}^0_{S^1}(S^2) \oplus \widetilde{K}^0_{S^1}(S^0).$$

Further composing with the projection onto $\widetilde{K}^0_{S^1}(S^0)$ gives the map we are considering, and since it factors through this exact sequence, it is the zero map. $\square$

We also have the following consequence of Corollary 3.18.

Theorem 4.24. *Let $\pi\colon E \to X$ be a G-equivariant principal S^1-bundle and P a G-equivariant twist coming from a central extension of G. If the pushforward*

$$\pi^*\colon K^*_H(E, \pi^*P) \to K^{*-1}_H(X, P)$$

is an isomorphism for all cyclic subgroups $H \subseteq G$, then the pushforward

$$\pi^*\colon K^*_G(E, \pi^*P) \to K^{*-1}_G(X, P)$$

is an isomorphism.

Proof. As discussed, the G-equivariant pushforward is an isomorphism if and only if

$$K^*_G(E \times_{S^1} \mathbb{C}^*, \pi^*P) \to K^{*-1}_G(E \times_{S^1} \mathbb{C}, \pi^*P)$$

is an isomorphism. This is induced by pulling back along the collapse map, so by Corollary 3.18, it is an isomorphism when

$$K^*_H(E \times_{S^1} \mathbb{C}^*, \pi^*P) \to K^{*-1}_H(E \times_{S^1} \mathbb{C}, \pi^*P)$$

is an isomorphism for all cyclic subgroups $H \subseteq G$ $\square$

CHAPTER 5
EQUIVARIANT T-DUALITY

5.1 Equivariant T-Duality: The Setup

Let G be a compact group and X a locally compact G-space. Equivariant (topological) T-duality is a relationship between pairs (E, P) consisting of a G-equivariant principal S^1-bundle $E \to X$ together with a G-equivariant twist $P \to E$. In practice, there is a choice to be made about which model of twist to use, perhaps depending on how one wants to define twisted K-theory. Instead of making a choice, we will define equivariant T-duality and the T-duality transformation for twists satisfying a set of prescribed axioms, which are detailed in the appendix. When we later discuss the T-duality transformation in twisted equivariant K-theory, we shall use equivariant principal $PU(\mathcal{H})$-bundles.

Consider two G-equivariant pairs (E, P) and $(\hat{E}, \hat{P})$ fitting into the following diagram:

$$\begin{tikzcd}
 & p^*P \arrow[rr, "u"] \arrow[dl] \arrow[dr] & & \hat{p}^*\hat{P} \arrow[dl] \arrow[dr] & \\
P \arrow[dr] & & E \times_X \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & & \hat{P} \arrow[dl] \\
 & E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] & \\
 & & X & &
\end{tikzcd} \tag{5.1}$$

All of the maps in the diagram are G-equivariant. One can therefore apply the Borel construction to the entire diagram to get a non-equivariant diagram of circle bundles and twists over the Borel construction $X \times_G EG$. This leads to our definition of equivariant T-duality:

Definition 5.1. A G-equivariant T-duality triple is a triple $\big((E, P), (\hat{E}, \hat{P}), u\big)$ fitting into a diagram of the form (5.1) and such that the induced non-equivariant triple over $X \times_G EG$ is a T-duality triple.

An obvious question is whether every T-duality triple over $X \times_G EG$ comes from an equivariant triple. We will now prove that this is indeed the case. Let $S(E, \hat{E})$ denote the sphere bundle of $(E \times_{S^1} \mathbb{C}) \oplus (\hat{E} \times_{S^1} \mathbb{C})$. We can construct $S(E, \hat{E})$ as the gluing of two mapping cylinders:

$$S(E, \hat{E}) \cong \mathrm{cyl}(p) \cup_{E \times_X \hat{E}} \mathrm{cyl}(\hat{p}).$$

There are canonical inclusions $i\colon E \to S(E, \hat{E})$ and $i\colon \hat{E} \to S(E, \hat{E})$ that send E and $\hat{E}$ to the ends of their respective cylinders.

Proposition 5.2. *There is a bijection between the set of isomorphisms $p^*P \to \hat{p}^*\hat{P}$ and the set of isomorphism classes of twists T on $S(E, \hat{E})$ such that $T|_E \cong P$ and $T|_{\hat{E}} \cong \hat{P}$.*

Proof. Let $f\colon \operatorname{cyl}(p) \to E$ be the canonical map $[e, \hat{e}, t] \mapsto e$, and similarly define $\hat{f}\colon \operatorname{cyl}(\hat{p}) \to \hat{E}$. Let i_E, $i_{E\times_X\hat{E}}$, and $i_{\hat{E}}$ denote the inclusions of E, $E \times_X \hat{E}$, and $\hat{E}$ into $S(E, \hat{E})$, respectively.

The forward direction of the bijection is defined as follows. Note that

$$(f^*P)|_{E\times\hat{E}} = p^*P \quad \text{and} \quad (\hat{f}^*\hat{P})|_{E\times_X\hat{E}} = \hat{p}^*\hat{P}.$$

Therefore, given an isomorphism $u\colon p^*P \to \hat{p}^*\hat{P}$, we can glue together f^*P and $\hat{f}^*\hat{P}$ along $E \times_X \hat{E}$ to obtain a twist $T = f^*P \cup_u \hat{f}^*P$ that appropriately restricts to P and $\hat{P}$.

For the inverse map, given a twist $T \to S(E, \hat{E})$ that restricts to P and $\hat{P}$, we define $u\colon p^*P \to \hat{p}^*\hat{P}$ by the sequence of isomorphisms

$$p^*P \cong p^*(T|_E) \cong T|_{E\times_X\hat{E}} \cong \hat{p}^*(T|\hat{E}) \cong \hat{p}^*\hat{P}. \tag{5.2}$$

Here, we have used that $i_E \circ p \simeq i_{E\times_X\hat{E}} \simeq i_{\hat{E}} \circ \hat{p}$.

Let us confirm that these constructions are inverse to each other. Given an isomorphism $u\colon p^*P \to \hat{p}^*\hat{P}$, we first need to show that if $T = f^*P \cup_u \hat{f}^*\hat{P}$ the composition (5.2) is equal to u. In this case, $T|_{E\times_X\hat{E}} = p^*P \cup_u \hat{p}^*\hat{P}$ and (5.2) becomes

$$p^*P \cong p^*P \cup_u \hat{p}^*\hat{P} \cong \hat{p}^*\hat{P}$$

$$x \mapsto x \sim u(x) \mapsto u(x),$$

so this construction indeed returns u.

Now, let $T \to S(E, \hat{E})$ be a twist with $T|_E \cong P$ and $T|_{\hat{E}} \cong \hat{P}$. We show that $T \cong f^*P \cup_u \hat{f}^*\hat{P}$, where u is the isomorphism (5.2). Since $i_E \circ f \simeq \mathrm{id}$, we know that

$$T|_{\operatorname{cyl}(p)} \cong f^*(T|_E) \cong f^*P.$$

Similarly $T|_{\operatorname{cyl}(\hat{p})} \cong \hat{f}^*\hat{P}$. We need these two parts to glue together according to (5.2), which means that the composition $(f^*P)|_{E\times_X\hat{E}} \cong T|_{E\times_X\hat{E}} \cong (\hat{f}^*\hat{P})|_{E\times_X\hat{E}}$ is equal to (5.2). Both isomorphisms factor through $T|_{E\times_X\hat{E}}$, so we can split the maps into two parts; a map $p^*P \to T|_{E\times_X\hat{E}}$ and another $T|_{E\times_X\hat{E}} \to \hat{p}^*\hat{P}$. For the first, consider the following diagram:

$$
\begin{tikzcd}
p^*P \arrow[rr, "P\cong T|_E"] \arrow[dr, "p=f\circ i_{E\times_X \hat{E}}"'] & & p^*(T|_E) \arrow[rr, "i_E\circ p\simeq i_{E\times_X \hat{E}}"] \arrow[dr, dashed] & & T|_{E\times_X \hat{E}} \\
& (f^*P)|_{E\times_X \hat{E}} \arrow[rr, "P\cong T|_E"] & & (f^*T|_E)|_{E\times_X \hat{E}} \arrow[ur, "i_E\circ f\simeq \mathrm{id}"'] &
\end{tikzcd}
$$

The arrows are labelled with the relations that induce them. The upper path is the first two maps in (5.2). We need to show that it equals the lower path, which comes from the isomorphism $T|_{\mathrm{cyl}(p)} \cong f^*P$ restricted to $E \times_X \hat{E}$. The dotted arrow is induced by $p = f \circ i_{E\times_X \hat{E}}$ and produces a commutative square. Since

$$
i_E \circ f \simeq \mathrm{id} \implies i_E \circ p = i_E \circ f \circ i_{E\times_X \hat{E}} \simeq i_{E\times_X \hat{E}},
$$

the triangle also commutes. Therefore, the above diagram commutes. The same argument works for the isomorphism $T|_{E\times_X \hat{E}} \to \hat{p}^*\hat{P}$. This completes the proof. $\square$

This proposition can be used to formulate an equivalent definition of topological T-duality, where the twist isomorphism $p^*P \cong \hat{p}^*\hat{P}$ is replaced with a twist on $S(E, \hat{E})$. See [DS23] for further details.

Theorem 5.3. *The G-equivariant pairs (E, P) and $(\hat{E}, \hat{P})$ over X are G-equivariantly T-dual if and only if $(E \times_G EG, P \times_G EG)$ and $(\hat{E} \times_G EG, \hat{P} \times_G EG)$ are non-equivariantly T-dual over $X \times_G EG$.*

Proof. The forward direction follows by definition of equivariant T-duality. For the reverse direction, we show that every twist morphism $u\colon p^*P \times_G EG \to \hat{p}^*\hat{P} \times_G EG$ is induced from an equivariant morphism $p^*P \to \hat{p}^*\hat{P}$.

Consider the following diagram:

$$
\begin{tikzcd}
\left\{\begin{array}{c} G\text{-equivariant morphisms} \\ p^*P \to \hat{p}^*\hat{P} \end{array}\right\} \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\text{Borel}"] & \left\{\begin{array}{c} G\text{-equivariant twists on} \\ S(E,\hat{E}) \text{ that restricts to} \\ P \text{ and } \hat{P} \end{array}\right\} \arrow[d, "\text{Borel}"] \\
\left\{\begin{array}{c} \text{Morphisms} \\ p^*P \times_G EG \to \hat{p}^*\hat{P} \times_G EG \end{array}\right\} \arrow[r, leftrightarrow, "\cong"] & \left\{\begin{array}{c} \text{Twists on} \\ S(E\times_G EG, \hat{E}\times_G EG) \\ \text{that restrict to } P\times_G EG \\ \text{and } \hat{P}\times_G EG \end{array}\right\}
\end{tikzcd}
$$

Here, the horizontal maps are a result of Proposition 5.2 and the vertical maps are obtained by taking the Borel construction. Isomorphism classes of G-equivariant twists on a space X are in bijection with isomorphism classes of non-equivariant twists on

$X \times_G EG$; they are both in bijection with $H^3_G(X;\mathbb{Z})$. We thus conclude that the vertical map on the right-hand side is an isomorphism. So, the two horizontal maps and right-hand side vertical map are bijections, implying that the left-hand side map is a bijection. This completes the proof. $\square$

Corollary 5.4. *Every T-duality triple over $X \times_G EG$ comes from a G-equivariant T-duality triple over X.*

Proof. We need every pair over $X \times_G EG$ to be of the form $(E \times_G EG, P \times_G EG)$ for some equivariant pair (E, G). This is true because both equivariant S^1-bundles and bundles on $X \times_G EG$ are classified by $H^2_G(X)$ and a similar statement can be made for the twists. $\square$

The theorem also implies that the existence and uniqueness properties of T-duals carry over to the equivariant setting:

Corollary 5.5. *For each G-equivariant pair (E, P) over X, there is a unique T-dual $(\hat{E}, \hat{P})$ characterised by the relations*

$$\pi_!\big([P]\big) = c_1(E) \text{ and } \hat{\pi}_!\big([\hat{P}]\big) = c_1(\hat{E}).$$

These pushforward maps were defined in the previous chapter.

Example 5.6. The most trivial G-equivariant T-duality triple is the triple over a trivial G-space X consisting of trivial circle bundles and trivial twists.

Example 5.7. A G-equivariant S^1-bundle $E \to X$ equipped with a trivial twist is T-dual to the bundle $X \times S^1$, where G acts trivially on the S^1-factor, equipped with a twist classified by the equivariant Chern class of E,

$$c_1(E) \in H^2_G(X) \hookrightarrow H^3_G(X \times S^1).$$

This is summarised by saying that pairs with trivial twists are T-dual to pairs with trivial bundles.

Example 5.8. If (E, P) and $(\hat{E}, \hat{P})$ are T-dual, then $(E, P \otimes \pi^* Q)$ and $(\hat{E}, \hat{P} \otimes \hat{\pi}^* Q)$ are also T-dual, where Q is an equivariant twist on X.

More specific examples will be given in Section 5.7.

5.2 The T-Duality Transformation

The T-duality transformation can be defined for general twisted equivariant cohomology theories. Included in Appendix A is a minimal axiomatic description of equivariant twists and twisted equivariant cohomology theories for which the T-duality transformation can be defined. Later we will restrict our attention to twisted equivariant K-theory.

Definition 5.9. Let h_G be a twisted equivariant cohomology theory and consider a G-equivariant T-duality triple $\big((E,P),(\hat{E},\hat{P}),u\big)$. The T-duality transformation is the composition $T = \hat{p}_! \circ u^* \circ p^*$, that is,

$$h_G^*(E,P) \xrightarrow{p^*} h_G^*(E \times_X \hat{E}, p^*P) \xrightarrow{u^*} h_G^*(E \times_X \hat{E}, \hat{p}^*\hat{P}) \xrightarrow{\hat{p}_!} h_G^{*-1}(\hat{E},\hat{P}).$$

Note the pushforward gives a degree shift of -1.

Given a G-equivariant T-duality triple $\big((E,P),(\hat{E},\hat{P}),u\big)$ over X, one can use a G-equivariant function $f\colon Y \to X$ to pull back the triple to a G-equivariant T-duality triple over Y. Similarly, one can restrict along a group homomorphism $\alpha\colon H \to G$ to obtain a H-equivariant triple on X. The T-duality transformation is natural with respect to these constructions:

Proposition 5.10. *The T-duality transformation is natural with respect to group homomorphisms and continuous functions, that is, given a map $f\colon Y \to X$ and a group homomorphism $\alpha\colon H \to G$, the following diagrams commute:*

$$\begin{tikzcd} h_G(E,P) \arrow[r, "T"] \arrow[d, "f^*"'] & h_G(\hat{E},\hat{P}) \arrow[d, "f^*"] \\ h_G(f^*E, f^*P) \arrow[r, "T"] & h_G(f^*\hat{E}, f^*\hat{P}) \end{tikzcd} \qquad \begin{tikzcd} h_G(E,P) \arrow[r, "T"] \arrow[d, "\alpha^*"'] & h_G(\hat{E},\hat{P}) \arrow[d, "\alpha^*"] \\ h_H(E,P) \arrow[r, "T"] & h_H(\hat{E},\hat{P}) \end{tikzcd}$$

Proof. This follows directly from the naturality properties of p^*, u^* and $\hat{p}_!$. ☐

Example 5.11. In K-theory, the T-duality transformation for the trivial T-duality diagram described in Example 5.6 is the T-duality transformation of the diagram with forgotten G-action tensored with the identity on $R(G)$, that is,

$$\begin{tikzcd} K_G^i(X \times S^1) \arrow[r, "\cong"] \arrow[d, "T"'] & K^i(X \times S^1) \otimes R(G) \arrow[d, "T \otimes \mathrm{id}"] \\ K_G^{i-1}(X \times S^1) \arrow[r, "\cong"] & K^{i-1}(X \times S^1) \otimes R(G). \end{tikzcd}$$

In particular, this is an isomorphism because the T-duality transformation is an isomorphism for non-equivariant T-duality triples.

5.3 T-Admissibility

Following Bunke and Schick [BS05], we introduce a notion of T-admissibility for equivariant T-duality triples. Being T-admissible means that the T-duality transformation is an isomorphism for 0-cells in a G-CW-complex. This will in turn imply that the T-duality transformation is an isomorphism for all finite G-CW-complexes.

Definition 5.12. A twisted equivariant cohomology theory is G-T-admissible if for each closed subgroup $H \subseteq G$, the T-duality transformation is an isomorphism for all pairs over the one-point space with trivial H-action.

As a somewhat trivial example, Borel cohomology and Borel equivariant K-theory are T-admissible because they are defined via non-equivariant cohomology groups. In general, it is difficult to prove T-admissibility because the equivariant T-duality over a point is still highly non-trivial; G-equivariant T-duality over a point is the same as T-duality over BG.

We will show that if a G-equivariant cohomology theory is T-admissible, then the T-duality transformation is an isomorphism for all G-CW-complexes. This will be proven by induction on the number of cells. The following two results give the base case for this induction. They essentially state when $H \subseteq G$ is a subgroup, a H-equivariant pair over a point can be induced up to a G-equivariant pair over G/H and that every pair on G/H arises this way.

Lemma 5.13. *Let $H \subseteq G$ be a closed subgroup. The function*

$$\big(E_0, P_0\big) \longmapsto \big(E_0 \times_H G, \mathrm{Ind}_H^G(P_0)\big)$$

induces a bijection between isomorphism classes of H-equivariant pairs over a H-space X and the isomorphism classes of G-equivariant pairs over $X \times_H G$.

Proof. Let (E, P) be a G-equivariant pair over $X \times_H G$. If $E_0 \to X$ is a H-equivariant principal S^1-bundle then $E_0 \times_H G \to X \times_H G$ is a G-equivariant principle S^1-bundle, and this construction induces a bijection between isomorphism classes of H-equivariant bundles on X and G-equivariant bundles on $X \times_H G$. Therefore, we can assume that $E \cong E_0 \times_H G$ for a unique (up to isomorphism) H-equivariant S^1-bundle $E_0 \to X$. By the twist axioms, there is an induction construction

$$\mathrm{Ind}_H^G \colon \mathrm{Twist}_H(E_0) \to \mathrm{Twist}_G(E_0 \times_H G)$$

that induces a bijection on isomorphism classes. Therefore, $P \in \mathrm{Twist}_G(E)$ is isomorphic to $\mathrm{Ind}_H^G(P_0)$ for a unique (up to isomorphism) $P_0 \in \mathrm{Twist}_H(E_0)$. We conclude that (E, P) is isomorphic to $(E_0 \times_H G, \mathrm{Ind}_H^G(P_0))$. □

This lemma tells us that we can induce H-equivariant pairs over a H-space X to G-equivariant pairs over $X \times_H G$:

$$P \to E \to X \quad \rightsquigarrow \quad \mathrm{Ind}_H^G(P) \to E \times_H G \to X \times_H G.$$

In the special case where X is a point, Lemma 5.13 says that this is an equivalence between H-equivariant pairs over a point and G-equivariant pairs over G/H.

Theorem 5.14. *If an equivariant cohomology theory $h^*_-(-)$ is G-T-admissible and $H \subseteq G$ is a closed subgroup, then the T-duality transformation is an isomorphism for all pairs over G/H.*

Proof. Let (E, P) and $(\hat{E}, \hat{P})$ be G-equivariant T-dual pairs over G/H. Let (E_0, P_0) and $(\hat{E}_0, \hat{P}_0)$ be the corresponding H-equivariant pairs over a point. Consider the following diagram:

$$\begin{array}{ccc} h^*_G(E,P) & \xrightarrow{T} & h^{*-1}_G(\hat{E},\hat{P}) \\ \big\downarrow{\cong} & & \big\downarrow{\cong} \\ h^*_H(E_0,P_0) & \xrightarrow[\cong]{T} & h^{*-1}_H(\hat{E}_0,\hat{P}_0). \end{array}$$

The diagram commutes because the T-duality transformation commutes with the induction isomorphism. The bottom arrow is an isomorphism by T-admissibility. The vertical maps are isomorphisms by the induction axiom. Hence the top arrow is an isomorphism, as required. □

Theorem 5.15. *If a twisted G-equivariant cohomology theory is T-admissible then the T-duality transformation is an isomorphism for finite G-CW-complexes.*

Proof. This is proven in the same way as in [BS05], except now we use induction on the number of G-CW-cells. The base case is true by the previous lemma. Then one checks that the T-duality transformation is natural with respect to pullbacks and the boundary operator in the Mayer-Vietoris sequence. The induction step is proven by attaching a cell and using the 5-lemma on the resulting Mayer-Vietoris sequence. □

Example 5.16. $\mathbb{Z}$-equivariant K-theory offers a baby example of T-admissibility, since, for cohomological reasons, there are no non-trivial $\mathbb{Z}$-equivariant pairs over a point.

5.4 $\mathbb{Z}_n$-Equivariant K-Theory

The simplest groups to consider are the finite cyclic groups, yet even in this case it is non-trivial to show that the T-duality transformation is an isomorphism. In this section, we prove Theorem 5.20, which states that $\mathbb{Z}_n$-equivariant K-theory is T-admissible.

We must start by investigating the possible $\mathbb{Z}_n$-equivariant T-duality triples over a point and the relevant K-theory groups.

Since $H^2_{\mathbb{Z}_n}(*;\mathbb{Z}) \cong \mathbb{Z}_n$, there are n isomorphism classes of $\mathbb{Z}_n$-equivariant principal S^1-bundle over a point. For each $k \in \mathbb{Z}_n$, let $E_k \to *$ denote the corresponding bundle. Explicitly, E_k is isomorphic to S^1 with the action $\xi \cdot z = \xi^k z$, where $\xi \in \mathbb{Z}_n \subseteq S^1$ is the generator. The Gysin sequence for E_k gives

$$\cdots \to H^3(B\mathbb{Z}_n;\mathbb{Z}) \to H^3_{\mathbb{Z}_n}(E_k;\mathbb{Z}) \to H^2(B\mathbb{Z}_n;\mathbb{Z}) \xrightarrow{\cdot k} H^4(B\mathbb{Z}_n;\mathbb{Z}) \to \cdots,$$

which, because the odd cohomology groups of $\mathbb{Z}_n$ are trivial, allows us to conclude that

$$H^3_{\mathbb{Z}_n}(E_k;\mathbb{Z}) \cong \{l \in \mathbb{Z}_n \mid kl = 0\} \subseteq \mathbb{Z}_n.$$

Thus, every $\mathbb{Z}_n$-equivariant T-duality pair over a point is of the form (E_k, l) with $kl \equiv 0 \mod n$. The pair (E_k, l) is T-dual to (E_l, k), so our task is to show that the T-duality transformation gives an isomorphism $K^*_{\mathbb{Z}_n}(E_k, l) \cong K^{*-1}_{\mathbb{Z}_n}(E_l, k)$. First, let us calculate these K-theory groups.

Lemma 5.17. *Let $E_k = S^1$ with the $\mathbb{Z}_n$-action defined by $\mathbb{Z}_n \xrightarrow{\times k} \mathbb{Z}_n \subseteq S^1$. Let τ_ℓ be the $\mathbb{Z}_n$-equivariant twist on E_k classified by $\ell \in \ker(\mathbb{Z}_n \xrightarrow{\times k} \mathbb{Z}_n) \cong H^3_{\mathbb{Z}_n}(E_k;\mathbb{Z})$. Then,*

$$K^0_{\mathbb{Z}_n}(E_k, \tau_\ell) \cong R(\mathbb{Z}_{\gcd(n,k)})^{\xi^{\frac{\gcd(n,k)\ell}{n}}} \text{ and}$$
$$K^1_{\mathbb{Z}_n}(E_k, \tau_\ell) \cong R(\mathbb{Z}_{\gcd(n,k)})/\langle 1 - \xi^{\frac{\gcd(n,k)\ell}{n}} \rangle,$$

where ξ is the representation generating $R(\mathbb{Z}_{\gcd(n,k)})$.

We remark on why $\gcd(n,k)\ell/n$ is an integer. Indeed, the kernel of $\mathbb{Z}_n \xrightarrow{\times k} \mathbb{Z}_n$ is the subgroup generated by $n/\gcd(n,k)$ and so since ℓ is an element of this subgroup, it must be a multiple of $n/\gcd(n,k)$. This implies that $\gcd(n,k)\ell/n$ is an integer.

Proof. The groups are computed using a Mayer-Vietoris argument; the same technique is used in [FHT11, §1]. For convenience, write $d = \gcd(n,k)$. E_k has a $\mathbb{Z}_n$-CW-structure consisting of a 0-cell $e^0 \times \mathbb{Z}_n/\mathbb{Z}_d$ and a 1-cell $e^1 \times \mathbb{Z}_n/\mathbb{Z}_d$. Now and throughout the proof, we implicitly identify $\mathbb{Z}_d$ with the subgroup of $\mathbb{Z}_n$ generated by n/d.

Let U be a small open neighbourhood around the 0-cell and V an open set containing $E_k \setminus U$ disjoint from the 0-cell; see Figure 5.1. Then $U \simeq \mathbb{Z}_n/\mathbb{Z}_d \simeq V$ and $U \cap V$ is a disjoint union of two copies of $\mathbb{Z}_n/\mathbb{Z}_d$. A twist on E_k can be modelled as a $\mathbb{Z}_n$-equivariant line bundle on the intersection; this is equivalent to two choices of $\mathbb{Z}_d$-representation. Up to stable isomorphism (of Hitchin gerbes), we can choose one

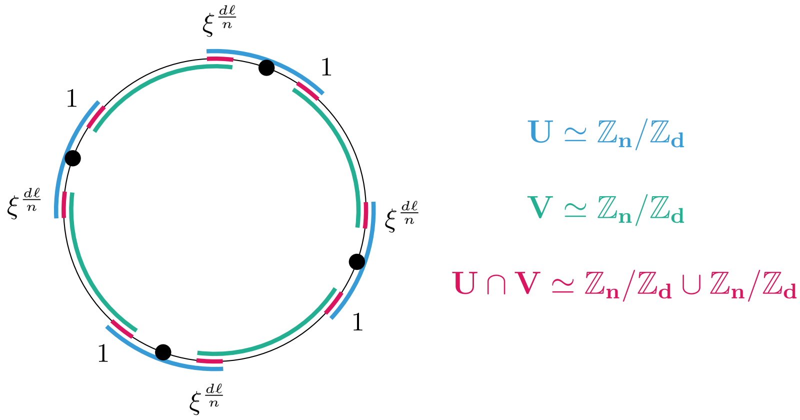

Figure 5.1: The open cover $\{U, V\}$ used to calculate $K^*_{\mathbb{Z}_n}(E_k, \tau_\ell)$.

of these representations to be trivial. To represent τ_ℓ, the remaining $\mathbb{Z}_d$-representation is chosen to be $\xi^{d\ell/n}$. The twist is depicted in Figure 5.1.

Noting that $K^0_{\mathbb{Z}_n}(\mathbb{Z}_n/\mathbb{Z}_d) = R(\mathbb{Z}_d)$ and $K^1_{\mathbb{Z}_n}(\mathbb{Z}_n/\mathbb{Z}_d) = 0$, the Mayer-Vietoris sequence for $E_k = U \cup V$ is

$$0 \to K^0_{\mathbb{Z}_n}(E_k, \tau_\ell) \to R(\mathbb{Z}_d)^2 \xrightarrow{\left(\begin{smallmatrix} 1 & -\xi^{d\ell/n} \\ 1 & -1 \end{smallmatrix}\right)} R(\mathbb{Z}_d)^2 \to K^1_{\mathbb{Z}_n}(E, \tau_\ell) \to 0.$$

The map in the center is a result of choosing trivialisations of τ_ℓ over U and V; more details are found in [FHT11, §1]. From the sequence, we conclude that

$$K^0_{\mathbb{Z}_n}(E_k, \tau_\ell) \cong \ker \left(\begin{smallmatrix} 1 & -\xi^{d\ell/n} \\ 1 & -1 \end{smallmatrix}\right) \cong R(\mathbb{Z}_d)^{\xi^{d\ell/n}} \text{ and}$$

$$K^1_{\mathbb{Z}_n}(E_k, \tau_\ell) \cong \operatorname{coker} \left(\begin{smallmatrix} 1 & -\xi^{d\ell/n} \\ 1 & -1 \end{smallmatrix}\right) \cong R(\mathbb{Z}_d)/\langle 1 - \xi^{d\ell/n} \rangle.$$

The final isomorphism is given by $[(p, q)] \mapsto [p - q]$. □

It will also be useful to know what the restriction maps are:

Lemma 5.18. *Consider again the assumptions made in the previous lemma and let m be an integer dividing n. Restricting the K-theory groups of E_k along the inclusion $\mathbb{Z}_m \hookrightarrow \mathbb{Z}_n$ induce the following diagrams:*

$$\begin{tikzcd} K^0_{\mathbb{Z}_n}(E_k, \tau_\ell) \arrow[r, "\cong"] \arrow[d] & R\big(\mathbb{Z}_{\gcd(n,k)}\big)^{\xi^{\frac{\gcd(n,k)\ell}{n}}} \arrow[d] \\ K^0_{\mathbb{Z}_m}(E_k, \tau_\ell) \arrow[r, "\cong"] & R\big(\mathbb{Z}_{\gcd(m,k)}\big)^{\eta^{\frac{\gcd(m,k)\ell}{m}}} \end{tikzcd} \tag{5.3}$$

$$
\begin{array}{ccc}
K^1_{\mathbb{Z}_n}(E_k,\tau_\ell) & \xrightarrow{\cong} & R\big(\mathbb{Z}_{\gcd(n,k)}\big)/\langle 1-\xi^{\frac{\gcd(n,k)\ell}{n}}\rangle \\
\big\downarrow & & \big\downarrow \\
K^1_{\mathbb{Z}_m}(E_k,\tau_\ell) & \xrightarrow{\cong} & R\big(\mathbb{Z}_{\gcd(m,k)}\big)/\langle 1-\eta^{\frac{\gcd(m,k)\ell}{m}}\rangle
\end{array}
\tag{5.4}
$$

The map on the right-hand side of (5.3) *is a restriction of* $R(\mathbb{Z}_{\gcd(n,k)}) \to R(\mathbb{Z}_{\gcd(m,k)})$, *which is in turn induced by the inclusion* $\mathbb{Z}_{\gcd(m,k)} \hookrightarrow \mathbb{Z}_{\gcd(n,k)}$. *The map sends* ξ *to* $\eta = \xi^{\frac{\gcd(n,k)}{\gcd(m,k)}}$. *The vertical map on the right-hand side of* (5.4) *is given by*

$$
[p(\xi)] \longmapsto \big[(1+a+a^2+\cdots+a^{\frac{n\gcd(m,k)}{m\gcd(n,k)}-1})p(\eta)\big],
$$

where $a = \eta^{\frac{\gcd(n,k)\ell}{n}}$. *Here, we write the elements of the representation ring as polynomials, with* $p(x)$ *denoting a polynomial in* x.

Proof. We continue using the notation established in the previous lemma's proof. In addition to writing $d = \gcd(n,k)$, we write $d' = \gcd(m,k)$. Let ξ be the generator of $R(\mathbb{Z}_d)$ and $\eta = \xi^{d/d'}$ the generator of $R(\mathbb{Z}_{d'})$. Elements of these rings will be written as polynomials $p(\xi)$ or $p(\eta)$.

Start by observing that $K_{\mathbb{Z}_m}(\mathbb{Z}_n/\mathbb{Z}_d) \cong R(\mathbb{Z}_{d'})^{\frac{nd'}{md}}$. This is a result of the orbit-stabiliser theorem: we have $\mathbb{Z}_m$ acting on n/d points with stabiliser $\mathbb{Z}_m \cap \mathbb{Z}_d = \mathbb{Z}_{d'}$ (identifying these groups with subgroups of $\mathbb{Z}_n$). Consider the Mayer-Vietoris sequence for the $\mathbb{Z}_m$-equivariant K-theory alongside the sequence considered in the previous proof:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & K^0_{\mathbb{Z}_n}(E_k,\tau_\ell) & \longrightarrow & R(\mathbb{Z}_d)^2 & \xrightarrow{\begin{pmatrix} 1 & -\xi^{d\ell/n} \\ 1 & -1 \end{pmatrix}} & R(\mathbb{Z}_d)^2 & \longrightarrow & K^1_{\mathbb{Z}_n}(E_k,\tau_\ell) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & K^0_{\mathbb{Z}_m}(E_k,\tau_\ell) & \longrightarrow & \big[R(\mathbb{Z}_{d'})^{\frac{nd'}{md}}\big]^2 & \xrightarrow{\Phi} & \big[R(\mathbb{Z}_{d'})^{\frac{nd'}{md}}\big]^2 & \longrightarrow & K^1_{\mathbb{Z}_m}(E_k,\tau_\ell) & \longrightarrow & 0
\end{array}
$$

The two vertical arrows in the center are given by

$$
\big(p(\xi), q(\xi)\big) \mapsto \big(p(\eta),\ldots,p(\eta),q(\eta),\ldots,q(\eta)\big).
$$

The map Φ can be defined as

$$
\begin{aligned}
&\Phi\big(p_1,\ldots,p_j,q_1,\ldots,q_j\big) \\
&= \big(p_1 - aq_1, p_2 - aq_2, \ldots, p_j - aq_j, p_1 - q_2, p_2 - q_3, \ldots, p_{j-1} - q_j, p_j - q_1\big),
\end{aligned}
$$

where, for brevity, $j = \frac{nd'}{md}$ and $a = \eta^{d\ell/n}$. Applying Lemma 5.17, we get

$$
K^0_{\mathbb{Z}_m}(E_k,\tau_\ell) = R(\mathbb{Z}_{d'})^{\eta^{d'\ell/m}} \quad \text{and} \quad K^1_{\mathbb{Z}_m}(E_k,\tau_\ell) = \frac{R(\mathbb{Z}_{d'})}{\langle 1-\eta^{d'\ell/m}\rangle}.
$$

We can identify these with $\ker\Phi$ and $\operatorname{coker}\Phi$ as follows:

$$\ker\Phi \xrightarrow{\cong} R(\mathbb{Z}_{d'})^{\eta^{d'\ell/m}}, \quad (p_1,\dots,p_j,q_1,\dots,q_j) \longmapsto p_1,$$

$$\operatorname{coker}\Phi \xrightarrow{\cong} \frac{R(\mathbb{Z}_{d'})}{\langle 1-\eta^{d'\ell/m}\rangle} \tag{5.4}$$

$$[(x_1,\dots,x_j,y_1,\dots,y_j)]$$
$$\longmapsto [(x_1+ax_j+a^2x_{j-1}\cdots+a^{j-1}x_2)-(ay_j+a^2y_{j-1}+\cdots+a^jy_1)].$$

To make sense of this: first note that if $\Phi(p_1,\dots,p_j,q_1,\dots,q_j)=0$, then

$$p_1 = aq_1 = ap_j = a^2q_j = \cdots = a^jp_1 \text{ and}$$

$$a^j = \left(\eta^{d\ell/n}\right)^{\frac{nd'}{md}} = \eta^{d'\ell/m},$$

so $p_1 \in R(\mathbb{Z}_{d'})^{\eta^{d'\ell/m}}$. Furthermore, (5.4) is well defined, since

$$\begin{aligned}
&\Phi(p_1,\dots,p_j,q_1,\dots,q_j)\\
&\quad = (p_1-aq_1,p_2-aq_2,\dots,p_j-aq_j,p_1-q_2,p_2-q_3,\dots,p_{j-1}-q_j,p_j-q_1)\\
&\qquad\longmapsto (p_1-aq_1)+a(p_j-aq_j)+\cdots a^{j-1}(p_2-aq_2)\\
&\qquad\qquad - [a(p_j-q_1)+a^2(p_{j-1}-q_j)+\cdots+a^j(p_1-q_2)]\\
&\qquad = (1-a^j)p_1\\
&\qquad = (1-\eta^{d'\ell/m})p_1.
\end{aligned}$$

It is straightforward to confirm that the inverse map is $[p]\mapsto[(p,0,\dots,0)]$.

Now, to determine the map on K^0-groups, consider the following:

$$\begin{tikzcd}
K^0_{\mathbb{Z}_n}(E_k,\tau_\ell) \arrow[r, "\cong"] \arrow[d] & \ker\begin{pmatrix}1 & -\xi^{d\ell/n}\\ 1 & -1\end{pmatrix} \arrow[r, "\cong"] \arrow[d] & R(\mathbb{Z}_d)^{\xi^{\frac{d\ell}{n}}} \arrow[d]\\
K^0_{\mathbb{Z}_m,\tau_\ell}(E_k) \arrow[r, "\cong"] & \ker\Phi \arrow[r, "\cong"] & R(\mathbb{Z}_{d'})^{\eta^{d'\ell/m}}
\end{tikzcd}$$

On the level of elements, the right-hand square is

$$\begin{tikzcd}
\big(p(\xi),p(\xi)\big) \arrow[r, leftrightarrow] \arrow[d, mapsto] & p(\xi) \arrow[d, mapsto]\\
\big(p(\eta),\dots,p(\eta)\big) \arrow[r, leftrightarrow] & p(\eta).
\end{tikzcd}$$

This proves the first part of the lemma. For the second part, the relevant diagram is:

$$\begin{array}{ccccc} K^1_{\mathbb{Z}_n}(E_k,\tau_\ell) & \xrightarrow{\cong} & \operatorname{coker}\begin{pmatrix} 1 & -\xi^{d\ell/n} \\ 1 & -1 \end{pmatrix} & \xrightarrow{\cong} & R(\mathbb{Z}_d)\big)/\langle 1-\xi^{d\ell/n}\rangle \\ \downarrow & & \downarrow & & \downarrow \\ K^1_{\mathbb{Z}_m}(E_k,\tau_\ell) & \xrightarrow{\cong} & \operatorname{coker}\Phi & \xrightarrow{\cong} & R(\mathbb{Z}_{d'})\big)/\langle 1-\eta^{d'\ell/m}\rangle \end{array}$$

On the level of elements, the right-hand side square is:

$$\begin{array}{ccc} \big(p(\xi),0\big) & \longleftrightarrow & p(\xi) \\ \downarrow & & \downarrow \\ \big(p(\eta),\ldots,p(\eta),0,\ldots,0\big) & \longleftrightarrow & \big[(1+a+a^2+\cdots+a^{\frac{nd'}{md}-1})p(\eta)\big] \end{array}$$

On the bottom row we have used the correspondence (5.4). This completes the proof. □

To prove that $\mathbb{Z}_n$-equivariant K-theory is T-admissible, we first prove an intermediary theorem that tells us that the T-duality transformation is an isomorphism for pairs with trivial bundle on one side and trivial twist on the other. The main theorem, Theorem 5.20, is proved by reducing to this case.

Theorem 5.19. *The T-duality transformation for the $\mathbb{Z}_n$-equivariant pairs $(E_k, 0)$ and (E_0, k) is an isomorphism.*

Proof. We prove this by reducing to the T-duality transformation for the trivial T-duality relation between $(E_0, 0)$ and itself. This is already known to be an isomorphism; see Example 5.11.

Let $d = \gcd(n, k)$ and consider the following diagram:

$$\begin{array}{ccc} K^*_{\mathbb{Z}_n}(E_k) & \xrightarrow{T} & K^{*-1}_{\mathbb{Z}_n}(E_0,\tau_k) \\ \downarrow & & \downarrow \\ K^*_{\mathbb{Z}_d}(E_k) & \xrightarrow{T} & K^{*-1}_{\mathbb{Z}_d}(E_0,\tau_k) \end{array} \tag{5.5}$$

The vertical arrows are the restriction along the inclusion $\mathbb{Z}_d \hookrightarrow \mathbb{Z}_n$. The induced $\mathbb{Z}_d$ action on E_k is trivial, as is the twist τ_k when viewed as $\mathbb{Z}_d$-equivariant. Therefore, the lower horizontal map is the T-duality transformation for the trivial T-duality triple, which is an isomorphism.

All the K-theory groups and restriction maps have been calculated in Lemma 5.17 and Lemma 5.18. We show that, in this case, the restriction maps are isomorphisms.

Start by considering (5.5) with $* = 0$. We have the following identifications:

$$\begin{tikzcd}
K^0_{\mathbb{Z}_n}(E_k) \arrow[r, "\cong"] \arrow[d] & R(\mathbb{Z}_d) \arrow[d, "\mathrm{id}"] & K^1_{\mathbb{Z}_n}(E_0, k) \arrow[r, "\cong"] \arrow[d] & \frac{R(\mathbb{Z}_n)}{\langle 1-\xi^k\rangle} \arrow[d] \\
K^0_{\mathbb{Z}_d}(E_k) \arrow[r, "\cong"] & R(\mathbb{Z}_d) & K^1_{\mathbb{Z}_d}(E_0, k) \arrow[r, "\cong"] & R(\mathbb{Z}_d).
\end{tikzcd}$$

The right-most vertical map is induced by $R(\mathbb{Z}_n) \to R(\mathbb{Z}_d)$, which is surjective with kernel the ideal $\langle 1-\xi^k\rangle$. Hence, all of the vertical maps are isomorphisms and diagram (5.5) implies that the T-duality transformation is an isomorphism for this case.

Now consider the other case. The identifications are as follows:

$$\begin{tikzcd}
K^1_{\mathbb{Z}_n}(E_k) \arrow[r, "\cong"] \arrow[d] & R(\mathbb{Z}_d) \arrow[d] & K^0_{\mathbb{Z}_n}(E_0, k) \arrow[r, "\cong"] \arrow[d] & R(\mathbb{Z}_n)^{\xi^k} \arrow[d] \\
K^1_{\mathbb{Z}_d}(E_k) \arrow[r, "\cong"] & R(\mathbb{Z}_d) & K^0_{\mathbb{Z}_d}(E_0, k) \arrow[r, "\cong"] & R(\mathbb{Z}_d).
\end{tikzcd}$$

In this situation, the vertical maps are seen to be injective maps onto $(n/d)R(\mathbb{Z}_d)$. The T-duality isomorphism $K^1_{\mathbb{Z}_d}(E_k) \cong K^0_{\mathbb{Z}_d}(E_0, k)$ restricts to an isomorphism of the subgroups $(n/d)K^1_{\mathbb{Z}_d}(E_k) \cong (n/d)K^1_{\mathbb{Z}_d}(E_0, k)$. Therefore, (5.5) implies that the T-duality transformation is an isomorphism in this case as well. $\square$

We are ready to prove the main theorem of this section.

Theorem 5.20. *$\mathbb{Z}_n$-equivariant K-theory is T-admissible.*

Proof. Consider the $\mathbb{Z}_n$-equivariant T-dual pairs (E_k, τ_ℓ) and (E_ℓ, τ_k) and for convenience let $d = \gcd(n, k)$ and $d' = \gcd(n, \ell)$. We must show that the corresponding T-duality transformation is an isomorphism. The idea is to consider the following commutative diagrams:

$$\begin{tikzcd}
K^0_{\mathbb{Z}_n}(E_k, \tau_\ell) \arrow[r, "T"] \arrow[d, "rest."'] & K^1_{\mathbb{Z}_n}(E_l \tau_\ell) \arrow[d, "rest."] & K^1_{\mathbb{Z}_n}(E_k, \tau_\ell) \arrow[r, "T"] \arrow[d, "rest."'] & K^0_{\mathbb{Z}_n}(E_l \tau_\ell) \arrow[d, "rest."] \\
K^0_{\mathbb{Z}_d}(E_k, \tau_\ell)^{\mathbb{Z}_n} \arrow[r, "T"] & K^1_{\mathbb{Z}_d}(E_l, \tau_\ell)^{\mathbb{Z}_n} & K^1_{\mathbb{Z}_d}(E_k, \tau_\ell)^{\mathbb{Z}_n} \arrow[r, "T"] & K^0_{\mathbb{Z}_d}(E_l, \tau_\ell)^{\mathbb{Z}_n}
\end{tikzcd} \tag{5.6}$$

The horizontal maps are T-duality transformations and the vertical maps are restrictions along the inclusion $\mathbb{Z}_d \hookrightarrow \mathbb{Z}_n$. We will calculate that on the left-hand side the restrictions are isomorphisms and on the right-hand side they are injective maps onto the subgroups $C \cdot K^1_{\mathbb{Z}_d}(E_k, \tau_\ell)^{\mathbb{Z}_n}$ and $C \cdot K^0_{\mathbb{Z}_d}(E_l, \tau_\ell)^{\mathbb{Z}_n}$, respectively, for a fixed integer C. Once we have done this, the theorem will be proved, since Theorem 5.19

implies that the lower T-duality transformations are isomorphisms and, for the right-hand diagram, this remains true then restricting to the aforementioned subgroups.

Now for the computation. By Lemma 5.17, the K-theory groups are

$$K^0_{\mathbb{Z}_d}(E_k, \tau_\ell) \cong R(\mathbb{Z}_d)^{\xi^\ell}, \;\; K^1_{\mathbb{Z}_d}(E_k, \tau_\ell) \cong \frac{R(\mathbb{Z}_d)}{\langle 1 - \xi^\ell \rangle}, \text{ and}$$

$$K^0_{\mathbb{Z}_d}(E_\ell, \tau_k) \cong R(\mathbb{Z}_{\gcd(d,\ell)}) \cong K^1_{\mathbb{Z}_d}(E_\ell, \tau_k).$$

One should remember that the notation E_k and τ_k refers to the $\mathbb{Z}_n$-action; as $\mathbb{Z}_d$-equivariant objects we would write E_0 and τ_0. Acting via the generator of $\mathbb{Z}_n$ induces a $\mathbb{Z}_d$-equivariant automorphism of E_k and pulling back along this map gives an automorphism of the K-theory groups. For this, it must be noted that the pullback of τ_ℓ is canonically isomorphic to τ_ℓ. Using the Mayer-Vietoris sequence – the same one used in the proof of Lemma 5.18 – one can calculate that the generator of $\mathbb{Z}_n$ acts via multiplication by $\xi^{d\ell/n} \in R(\mathbb{Z}_d)$ on $K^i_{\mathbb{Z}_d}(E_k, \tau_\ell)$ and via multiplication by $\zeta^{d'k/n} \in R(\mathbb{Z}_{\gcd(d,\ell)})$ on $K^i_{\mathbb{Z}_d}(E_\ell, \tau_k)$, where ξ and ζ are generators of their respective representation rings.

We use Lemma 5.18 to write out each of the restriction maps explicitly. The first we consider is:

$$\begin{array}{ccc} K^0_{\mathbb{Z}_n}(E_k, \tau_\ell) & \xrightarrow{\cong} & R(\mathbb{Z}_d)^{\xi^{d\ell/n}} \\ \downarrow & & \downarrow \\ K^0_{\mathbb{Z}_d}(E_k, \tau_\ell) & \xrightarrow{\cong} & R(\mathbb{Z}_d)^{\xi^\ell} \end{array}$$

This restriction map is just the inclusion. This becomes an isomorphism if we restrict the codomain to the $\xi^{d\ell/n}$-invariant subgroup. We have seen that this is identified with the $\mathbb{Z}_n$-invariant subgroup, so we have the isomorphism we desire. The next restriction to consider is:

$$\begin{array}{ccc} K^1_{\mathbb{Z}_n}(E_\ell, \tau_k) & \xrightarrow{\cong} & R(\mathbb{Z}_{d'})/\langle 1 - \xi^{d'k/n} \rangle \\ \downarrow & & \downarrow \\ K^1_{\mathbb{Z}_d}(E_\ell, \tau_k) & \xrightarrow{\cong} & R(\mathbb{Z}_{\gcd(d,\ell)}) \end{array}$$

This map is

$$[p(\xi)] \mapsto \big(1 + \eta^{d'k/\ell} + \cdots + (\eta^{d'k/\ell})^{\frac{n \gcd(d,\ell)}{dd'} - 1}\big) p(\eta).$$

In this case, $\frac{n \gcd(d,\ell)}{dd'}$ is the multiplicative order of $\eta^{d'k/\ell}$. Thus, this becomes an isomorphism if we restrict to the $\eta^{d'k/\ell}$-invariant subgroup. One sees this by noting that the map induces a bijection between the elements $1, \xi, \ldots, \xi^{\gcd(d', d'k/n) - 1}$ and the elements

$$\eta^j\big(1+\eta^{d'k/\ell}+\cdots+(\eta^{d'k/\ell})^{\frac{n\gcd(d,\ell)}{dd'}-1}\big)$$

for $j\in\{0,\ldots,\gcd(d',d'k/\ell)-1\}$, both of which form a $\mathbb{Z}$-linear basis for their respective groups. We have now established that the restriction maps in the left-hand side diagram of (5.6) are isomorphisms.

The next restriction map is:

$$\begin{array}{ccc} K^1_{\mathbb{Z}_n}(E_k,\tau_\ell) & \xrightarrow{\cong} & R(\mathbb{Z}_d)/\langle 1-\xi^{d\ell/n}\rangle \\ \downarrow & & \downarrow \\ K^1_{\mathbb{Z}_d}(E_k,\tau_\ell) & \xrightarrow{\cong} & R(\mathbb{Z}_d)/\langle 1-\eta^{\ell}\rangle \end{array}$$

The map is

$$\big[p(\xi)\big]\mapsto\Big[\big(1+\xi^{d\ell/n}+\cdots+(\xi^{d\ell/n})^{\frac{n}{d}-1}\big)p(\xi)\Big].$$

Since the multiplicative order of $\xi^{d\ell/n}$ is $\alpha:=\frac{\gcd(d,\ell)}{\gcd(d,\ell,d\ell/n)}$, we have

$$1+\xi^{d\ell/n}+\cdots+(\xi^{d\ell/n})^{\frac{n}{d}-1}=\frac{n}{d\alpha}\big(1+\xi^{d\ell/n}+\cdots+(\xi^{d\ell/n})^{\alpha-1}\big)$$

We see that the restriction map is an isomorphism onto the subgroup $\frac{n}{d\alpha}\cdot K^1_{\mathbb{Z}_d}(E_k,\tau_\ell)^{\xi^{d\ell/n}}$.

The final restriction map is:

$$\begin{array}{ccc} K^0_{\mathbb{Z}_n}(E_\ell,\tau_k) & \xrightarrow{\cong} & R(\mathbb{Z}_{d'})^{\xi^{d'k/n}} \\ \downarrow & & \downarrow \\ K^0_{\mathbb{Z}_d}(E_\ell,\tau_k) & \xrightarrow{\cong} & R(\mathbb{Z}_{\gcd(d,\ell)}) \end{array}$$

This map is again simply $p(\xi)\mapsto p(\eta)$. Note that

$$\begin{aligned} 1+\xi^{d'k/n}+\cdots+(\xi^{d'k/n})^{\beta-1} &\longmapsto 1+\eta^{d'k/n}+\cdots+(\eta^{d'k/n})^{\beta-1} \\ &\qquad=\frac{\beta}{\beta'}\big(1+\eta^{d'k/n}+\cdots+(\eta^{d'k/n})^{\beta'-1}\big), \end{aligned}$$

where, for notational convenience, we write $\beta=d'/\gcd(d',d'k/n)$ for the multiplicative order of $\xi^{d'k/n}$ and $\beta'=\gcd(d,\ell)/\gcd(d,\ell,d'k/n)$ for the multiplicative order of $\eta^{d'k/n}$. Thus, we can conclude that the restriction map is an injection onto $\frac{\beta}{\beta'}\cdot K^0_{\mathbb{Z}_d}(E_\ell,\tau_k)^{\eta^{d'k/n}}$.

Now, for our proof to work we need that $\frac{n}{d\alpha}$ and $\frac{\beta}{\beta'}$ are equal. Fortunately, this is true:

$$\begin{aligned}
\frac{n\beta'}{d\alpha\beta} &= \frac{n \cdot \gcd(d,\ell) \cdot \gcd(d', d'k/n) \cdot \gcd(d,\ell,d\ell/n)}{d \cdot \gcd(d,\ell,d'k/n) \cdot d' \cdot \gcd(d,\ell)} \\
&= \frac{n \cdot \gcd(d', d'k/n) \cdot \gcd(d,\ell,d\ell/n)}{d \cdot \gcd(d,\ell,d'k/n) \cdot d'} && \text{Cancel } \gcd(d,\ell). \\
&= \frac{\gcd(nd', d'k) \cdot \gcd(nd, nl, dl)}{d \cdot d' \cdot \gcd(nd, nl, d'k)} && \text{Insert } n \text{ into the gcds.} \\
&= \frac{d' \cdot \gcd(n,k) \cdot \gcd(nd, nl, dl)}{d \cdot d' \cdot \gcd(nd, nl, d'k)} && \text{Factor out } d \text{ and } d'. \\
&= \frac{\gcd(nd, nl, dl)}{\gcd(nd, nl, d'k)} && \text{Cancel } d \text{ and } d'. \\
&= \frac{\gcd(n^2, nk, nl, nl, kl)}{\gcd(n^2, nl, nk, k\ell)} && \text{Substitute values of } d, d'. \\
&= 1
\end{aligned}$$

Therefore $\frac{n}{d\alpha} = \frac{\beta}{\beta'}$; this is the constant C mentioned at the beginning of the proof. We have now shown that all the vertical maps in (5.6) are isomorphisms and so, as discussed at the beginning of the proof, we are done. $\square$

5.5 Rational Equivariant K-Theory for Finite Groups

Using the decomposition theorem from Chapter 2, we prove that the cyclic group case implies that the T-duality transformation is rationally an isomorphism for all finite groups.

Theorem 5.21. *Let G be a finite group and let (E, P) and $(\hat{E}, \hat{P})$ be G-equivariant T-dual pairs over a G-CW-complex X. Then the T-duality transformation on rational twisted equivariant K-theory,*

$$K_G^*(E,P)_{\mathbb{Q}} \xrightarrow{\cong} K_G^{*-1}(\hat{E},\hat{P})_{\mathbb{Q}},$$

is an isomorphism, that is, G-equivariant K-theory is rationally T-admissible.

We use the decomposition result for bundles, Corollary 2.11, which says that there is a decomposition map

$$K_G(E,P)_{\mathbb{Q}} \to \Big[\bigoplus_{g\in G} K_{\langle g\rangle}(E|_{X^g}, P|_{E|_{X^g}})_{\mathbb{Q}}\Big]^G$$

that is an isomorphism onto the elements of the right-hand side satisfying the following relation:

If $\langle h\rangle \subseteq \langle g\rangle$, then the elements in the $\langle h\rangle$- and $\langle g\rangle$-summand restrict (*)
to the same element in $K_{\langle h\rangle}(E|_{X^g}, P|_{E|_{X^g}})$.

Thus, we will decompose the K-theory groups and use the T-duality isomorphism for the cyclic group equivariant T-duality triples that we get from the restrictions. By Theorem 5.20, these are isomorphisms.

Proof. Consider the following diagram:

$$\begin{array}{ccc} K_G^*(E,P)_{\mathbb{Q}} & \longrightarrow & K_G^{*-1}(\hat{E},\hat{P})_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \left[\bigoplus_{g\in G} K_{\langle g\rangle}^*(E|_{X^g}, P|_{E|_{X^g}})_{\mathbb{Q}}\right]^G & \longrightarrow & \left[\bigoplus_{g\in G} K_{\langle g\rangle}^{*-1}(\hat{E}|_{X^g}, \hat{P}|_{E|_{X^g}})_{\mathbb{Q}}\right]^G \end{array}$$

The upper horizontal map is the T-duality transformation. The vertical maps are the decomposition maps; these are induced by the inclusions $E|_{X^g} \to E$. The lower horizontal map is induced by the T-duality transformations for the restrictions to X^g. Since the T-duality transformation is natural with respect to pullbacks and morphisms of twists, these restrict to the invariant subspace and the diagram commutes. For the same reason, the map also restricts to a map between the elements satisfying (*). By Theorem 5.20, the lower map is an isomorphism on this specified subspace, which implies that the upper map is an isomorphism. $\square$

5.6 The General Case: Compact Lie Groups

We are almost ready to prove the main result, which is that the T-duality transformation for twisted equivariant K-theory is an isomorphism for all compact Lie groups. Let us give the idea of the proof. We will show that the T-duality transformation being an isomorphism is equivalent to a certain pullback map being injective. There exist results that imply that a map in equivariant K-theory is injective if it is injective when restricted to all finite subgroups. This result will be combined with the fact that the T-duality transformation is rationally an isomorphism for all finite groups to get to our main result. As this isomorphism has only been proved rationally, it will be helpful to know that the involved groups are torsion-free:

Lemma 5.22. *Let G be a finite group acting on S^1 via a homomorphism $\varphi\colon G \to S^1$ and let $K = \ker(\varphi)$. Let P be a G-equivariant twist on S^1. Then there exists a*

1-*dimensional representation* ξ *of* K *such that*

$$K^0_G(S^1, P) \cong R(K)^{\xi} \quad \textit{and} \quad K^1_G(S^1, P) \cong R(K)/(1-\xi)R(K).$$

In particular, $K^*_G(S^1, P)$ *is torsion-free.*

Proof. Give S^1 the G-CW-structure $S^1 = (e^0 \times G/K) \cup (e^1 \times G/K)$. The twist P can be represented by a G-equivariant S^1-bundle on G/K, which is equivalent to a 1-dimensional (complex) representation of K. Let ξ be this representation. The Mayer-Vietoris sequence gives

$$0 \to K^0_G(X, P) \to R(K)^2 \to R(K)^2 \to K^1_G(X, P) \to 0,$$

where the middle map is $(x, y) \mapsto (x - \xi y, x - y)$. This implies the isomorphisms claimed in the lemma.

Being the subgroup of a torsion-free module, it is clear that $R(K)^{\xi}$ is torsion-free. For the second group, we observe that ξ acts via permutations on the irreducible representations of K, which form a basis of $R(K)$. By partitioning the irreducible representations into ξ-orbits, $R(K)$ is isomorphic to a direct sum of modules of the form $\mathbb{Z}[\xi]/(1-\xi^d)\mathbb{Z}[\xi]$, where d divides the multiplicative order of ξ. The quotient of such a module by the sub-module generated by$(1-\xi)$ is a free abelian group of rank 1. This altogether implies that $R(K)/(1-\xi)R(K)$ is torsion-free. $\square$

The following is a key part of the main proof:

Lemma 5.23. *Let G be a compact Lie group and $\big((E, Q), (\hat{E}, \hat{Q}), u\big)$ a G-equivariant T-duality triple over a point. Let $f\colon E \times \hat{E} \to *$ be the constant map. The map*

$$F^*\colon K^*_G(E, Q) \to K^*_G(f^*E, F^*Q) \tag{5.7}$$

is injective, where $F\colon f^*E \to E$ is the canonical map in the pullback square.

Proof. Let $\pi, \hat{\pi}, p$ and $\hat{p}$ be the maps denoted in the following diagram:

$$\begin{tikzcd} & E \times_X \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] \\ & * & \end{tikzcd}$$

Assume, for now, that G is finite. Since $f = \pi \circ p$, we can write (5.7) as the composition

$$K^*_G(E, Q) \xrightarrow{\Pi^*} K^*_G(\pi^*E, \Pi^*Q) \xrightarrow{P^*} K^*_G(f^*E, F^*Q), \tag{5.8}$$

where $\Pi\colon \pi^*E \to E$ is defined as part of the pullback square. The pullback of a principal bundle along its own map is trivial, so $\pi^*E \to E$ is the trivial bundle. The induced map on K-theory is therefore injective, for instance by looking at the Künneth exact sequence for C*-algebras [Bla98, 23.3.1].

For the second map, observe that P^* is the first map in the T-duality transformation for the pullback of $\big((E,Q),(\hat{E},\hat{Q}),u\big)$ along π. By Theorem 5.21, this T-duality transformation is rationally an isomorphism, so P^* is rationally injective. By Lemma 5.22, $K_G(E,Q)$ is torsion-free and so the image of Π^* is torsion-free. We can then conclude that, for finite groups, the composition (5.8) is injective.

Now we show that the finite group case implies the general case. It was proved by McClure that if $x \in K_G(X)$ restricts to zero in $K_H(X)$ for every finite subgroup H of G, then $x = 0$, see [McC86]. This was generalised to KK-theory by Uuye in [Uuy12] under the following assumptions:

- $KK_n^H(A,B)$ is a finitely generated $R(G)$-module for all $n \in \mathbb{Z}$ and all closed subgroups $H \subseteq G$.
- $KK_n^F(A,B)$ is a finitely generated group for all finite subgroups $F \subseteq G$.

Under these conditions, if an element $x \in KK^G(A,B)$ restricts to zero in $KK^H(A,B)$ for all finite subgroups H of G then $x = 0$. In our case, the relevant K-groups were calculated in Lemma 5.22. They are either a subgroup or a quotient of the representation ring of a finite group. These are all finitely generated, so Uuye's conditions are satisfied for the K-groups under consideration.

Now, let $x \in K_G(E,Q)$ be in the kernel of $K_G^*(E,Q) \to K_G^*(f^*E, F^*Q)$ and denote by $x_H \in K_H(E,Q)$ the restriction of x for $H \subseteq G$. If H is finite, we have already shown that $K_H(E,Q) \to K_H(f^*E, F^*Q)$ is injective and since x_H is in its kernel, we deduce that $x_H = 0$. Therefore, Uuye's theorem implies that $x = 0$, since $x_H = 0$ for all finite subgroups $H \subseteq G$. This completes the proof. $\square$

For each T-duality triple $\big((E,P),(\hat{E},\hat{P}),u\big)$, we call $\big((\hat{E},\hat{P}),(E,P),u^{-1}\big)$ the dual T-duality triple. The resulting T-duality transformation,

$$K_G^*(\hat{E},\hat{P}) \to K_G^{*-1}(E,P),$$

will be called the dual T-duality transformation.

We are now prepared for the proof of the main theorem:

Theorem 5.24. *Twisted G-equivariant K-theory is T-admissible when G is a compact Lie group.*

Proof. We use a method introduced by Bei Liu in his Göttingen PhD thesis [Liu14].

Consider a T-duality diagram:

$$
\begin{tikzcd}
& p^*P \arrow[rr, "u"] \arrow[dl] \arrow[dr] & & \hat{p}^*\hat{P} \arrow[dl] \arrow[dr] & \\
P \arrow[dr] & & E \times_X \hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & & \hat{P} \arrow[dl] \\
& E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] & \\
& & X & &
\end{tikzcd}
\tag{5.9}
$$

Let $f = \pi \circ p = \hat{\pi} \circ \hat{p}$ be the canonical map $E \times_X \hat{E} \to X$. Pulling back along f gives a T-duality diagram over $E \times_X \hat{E}$. This pulled-back diagram is the trivial T-duality diagram, that is, the S^1-bundles and twists are trivial. This is because the pull-back of a principal bundle along itself is trivial. For example, $f^*E = p^*\pi^*E$ and π^*E is the trivial principal S^1-bundle over E. Here, we mean it is trivial as an equivariant bundle; G acts trivially on the S^1-fiber. It is straightforward to see that the T-duality transformation for the trivial T-duality triple is an isomorphism; this was discussed in Example 5.11. Moreover, when X is a point, the inverse of this T-duality transformation is its dual T-duality transformation.

Let $F\colon f^*E \to E$ and $\hat{F}\colon f^*\hat{E} \to \hat{E}$ be the resulting maps between the S^1-bundles. By the naturality of the T-duality transformation, we have

$$
\begin{tikzcd}
K_G^*(E, P) \arrow[r, shift left, "T"] \arrow[d] & K_G^{*-1}(\hat{E}, \hat{P}) \arrow[l, shift left, "T"] \arrow[d] \\
K_G^*(f^*E, F^*P) \arrow[r, shift left, "T"] & K_G^{*-1}(f^*\hat{E}, \hat{F}^*\hat{P}). \arrow[l, shift left, "T"]
\end{tikzcd}
\tag{5.10}
$$

Let X be a point; this is indeed the only case required for T-admissibility. Then, we know that the lower T-duality transformations are inverse to each other. Moreover, Lemma 5.23 implies that the vertical maps are injective. We therefore have that the T-duality transformations in the top row of (5.10) are isomorphisms, as required. $\square$

By considering the T-dual pairs introduced in Example 5.7, we have the following corollary.

Corollary 5.25. *Let G be a compact Lie group, X a G-space and $E \to X$ a G-equivariant principal S^1-bundle with Chern class $c_1 \in H_G^2(X)$. There is an isomorphism*

$$K_G^*(E) \cong K_G^{*-1}(X \times S^1, P),$$

where G-acts trivially on the S^1-factor of $X \times S^1$ and P is a S^1-equivariant twist classified by $c_1 \in H_G^2(X) \hookrightarrow H_G^3(X \times S^1)$.

This result also includes the case where the twist comes from the group, by considering twists classified by a class in the image of $H^2_G(*) \to H^2_G(X)$.

5.7 Examples

The T-duality isomorphism can be a useful tool for calculating the K-theory of principal S^1-bundles. We finish the chapter with some example calculations. The first can be considered a worked example of the T-duality isomorphism, while the rest are a result of it.

Example 5.26. We walk through the case where $G = S^1$ and X is a point since everything can be made explicit in this case. Let E_k be S^1 with the kth power S^1 action, that is, with the S^1-action given by

$$S^1 \to S^1 \subseteq \mathrm{Aut}(S^1), \quad z \mapsto z^k.$$

This is the same E_k as Section 5.4, except the S^1-action there is restricted to a $\mathbb{Z}_n$-action. Note that $E_k \to *$ is an S^1-equivariant principal S^1-bundle over a point and that all such bundles are of this form, because

$$H^2_{S^1}(*) = H^2(BS^1) \cong \mathbb{Z},$$

and $E_k \to *$ is the bundle classified by $k \in \mathbb{Z}$. An application of the corresponding Gysin sequence shows that $H^3_{S^1}(E_k) = \mathbb{Z}$ when $k = 0$ and $H^3_{S^1}(E_k) = 0$ otherwise. We then have that the pairs (E_0, P_k) and $(E_k, 0)$ are T-dual to each other, where P_k is the twist on E_0 corresponding to the integer k. These are all of the possible T-duality triples. Via the Mayer-Vietoris sequence, the relevant twisted K-groups are

$$K^0_{S^1}(E_k) \cong R(\mathbb{Z}_k) \cong K^1_{S^1}(E_0, P_k) \quad \text{and} \quad K^1_{S^1}(E_k) = 0 = K^0_{S^1}(E_0, P_k).$$

There are thus only two non-trivial T-duality transformations,

$$K^1_{S^1}(E_0, P_k) \xrightarrow{T} K^0_{S^1}(E_k) \quad \text{and} \quad K^0_{S^1}(E_k) \xrightarrow{T} K^1_{S^1}(E_0, P_k).$$

All of the maps defining these transformations are contained in the following diagram:

$$\begin{tikzcd}
& K^1_{S^1}(E_0 \times E_k, \mathrm{pr}_1^* P_k) \arrow[r, "\cong"] & K^1_{S^1}(E_0 \times E_k) \arrow[dr, "(\mathrm{pr}_2)_!"] & \\
K^1_{S^1}(E_0, P_k) \arrow[ur, "\mathrm{pr}_1^*"] & & & K^0_{S^1}(E_k) \arrow[dl, "\mathrm{pr}_2^*"] \\
& K^0_{S^1}(E_0 \times E_k, \mathrm{pr}_1^* P_k) \arrow[ul, "(\mathrm{pr}_1)_!"] & K^0_{S^1}(E_0 \times E_k) \arrow[l, "\cong"] &
\end{tikzcd}$$

It turns out that, in this case, each of the individual maps in this diagram is an isomorphism. The pullbacks can be shown to be isomorphism using a Mayer-Vietoris argument and the pushforwards were shown to be isomorphisms in Chapter 4.

Example 5.27. Consider S^1 acting on S^2 by rotations. By a standard Mayer-Vietoris argument, $H^2_{S^1}(S^2) \cong \mathbb{Z}^2$. Let $E_{p,q}$ denote the S^1-equivariant principal S^1-bundle on S^2 classified by $(p,q) \in \mathbb{Z}^2$. This can be explicitly constructed as

$$E_{p,q} = (D^2 \times S^1_p) \cup_{S^1 \times S^1_0} (D^2 \times S^1_q) \to D^2 \cup_{S^1} D^2 = S^2,$$

where S^1_k denotes S^1 with the kth power action. The gluing maps come from the fact that $S^1 \times S^1_k$ is S^1-equivariantly homeomorphic to $S^1 \times S^1_0$ for all $k \in \mathbb{Z}$.

Another Mayer-Vietoris argument tells us that

$$H^3_{S^1}(E_{p,q}) = \begin{cases} \mathbb{Z}^2, & p = q = 0, \\ \mathbb{Z}, & p = 0, q \neq 0, \\ \mathbb{Z}, & p \neq 0, q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

If $p = q = 0$, then $E_{p,q}$ is the trivial bundle. Let $P_{k,\ell}$ be the twist on $E_{0,0}$ classified by $(k,\ell) \in \mathbb{Z}^2$. Then $(E_{0,0}, P_{p,q})$ is T-dual to $(E_{p,q}, 0)$. This is the situation described in Corollary 5.25.

Let P_k and Q_ℓ denote the twists on $E_{p,0}$ and $E_{0,q}$ classified by $k, l \in \mathbb{Z}$, respectively. Then $(E_{p,0}, P_q)$ is T-dual to $(E_{0,q}, Q_p)$. The corresponding T-duality isomorphism is

$$K^*_G(E_{p,0}, P_q) \cong K^{*-1}_G(E_{0,q}, Q_p).$$

Example 5.28. Let $E \to X$ be a (non-equivariant) principal S^1-bundle on X with Chern class $c_1 \in H^2(X)$. The bundle $E^{\otimes k} = E \otimes \cdots \otimes E$, which is classified by c_1^k, has a natural action of the symmetric group Σ_k. This makes $E^{\otimes k} \to X$ a Σ_k-equivariant principal S^1-bundle on X, where X is given the trivial Σ_k-action. Corollary 5.25 implies that

$$K^*_{\Sigma_k}(E^{\otimes k}) \cong K^{*-1}_{\Sigma_k}(X \times S^1, P) \cong K^{*-1}(X \times S^1, P) \otimes R(\Sigma_k),$$

where P is a twist classified by the image of c_1^k under

$$H^2(X) \hookrightarrow H^3(X \times S^1) \hookrightarrow H^3_{\Sigma_k}(X \times S^1).$$

If we further assume that $K^i(X) = 0$ for $i \in \{0, 1\}$, then a Mayer-Vietoris argument for $X \times S^1$ reveals that

$$K^{i-1}(X \times S^1, P) \cong K^{i-1}(X)^{L^k} \text{ and}$$

$$K^i(X \times S^1, P) \cong K^{i-1}(X)/L^k K^{i-1}(X),$$

where L_k is the line bundle classified by c_1^k (in other words the line bundle associated with $E^{\otimes k}$). So, in this case,

$$K^i_{\Sigma_k}(E^{\otimes k}) \cong K^{i-1}(X)^{L^k} \otimes R(\Sigma_k) \text{ and}$$

$$K^{i-1}_{\Sigma_k}(E^{\otimes k}) \cong \frac{K^{i-1}(X)}{L^k K^{i-1}(X)} \otimes R(\Sigma_k).$$

APPENDIX A

TWISTED EQUIVARIANT COHOMOLOGY

Here, we introduce an axiomatic definition of equivariant twists and twisted equivariant cohomology, following Bunke and Schick [BS05]. The T-duality transformation and the notion of T-admissibility will be defined for any twisted equivariant cohomology theory satisfying these axioms.

With an axiomatic approach, one does not need to choose a specific model for twists. In K-theory for example, there are many notions of twists, including principal $PU(\mathcal{H})$-bundles, Hitchin gerbes, bundle gerbes, and Čech cocycles. It can be difficult to translate between these directly. Instead of choosing a specific one, we rely on a set of axioms that are sufficient to prove the results we need. The same is true for twisted equivariant cohomology. Even though our main focus is K-theory, we are leaving room for other twisted cohomology theories to have a T-duality isomorphism.

A.1 Equivariant Twists

A model for G-equivariant twists consists of a presheaf of monoidal groupoids

$$(X, G) \mapsto \mathrm{Twist}_G(X)$$

on the category of spaces with group action. This must satisfy the following properties:

1. There is a natural monoidal transformation

$$\mathrm{Twist}_G(X) \to H^3_G(X;\mathbb{Z}), \quad P \mapsto [P],$$

   where the Borel cohomology group $H^3_G(X;\mathbb{Z}) := H^3(X \times_G EG;\mathbb{Z})$ is viewed as a monoidal category with only identity morphisms. This transformation classifies the isomorphism classes of $\mathrm{Twist}_G(X)$.

2. There is a natural "Borel construction", meaning there is a monoidal transformation

$$\mathrm{Twist}_G(X) \to \mathrm{Twist}(X \times_G EG). \quad P \mapsto P \times_G EG$$

   This transformation is natural with respect to the classification by $H^3_G(X;\mathbb{Z})$.

3. Let H is a subgroup of G and X a H-space. The map

$$\mathrm{Twist}_G(X \times_H G) \to \mathrm{Twist}_H(X)$$

defined by restricting along the inclusion $H \hookrightarrow G$ and pulling back along the H-equivariant map $X \to X \times_H G$ is an equivalence of groupoids. The inverse will be denoted $P \mapsto \mathrm{Ind}_H^G(P)$.

4. (Gluing twists) Let $X = U \cup V$ where U and V are G-invariant subspaces, $P_U \in \mathrm{Twist}_G(U)$, $P_V \in \mathrm{Twist}_G(V)$ and $u\colon P_U|_{U\cap V} \cong P_V|_{U\cap V}$ an isomorphism. Then, there exists $P \in \mathrm{Twist}_G(X)$ such that there are isomorphisms $\psi_U\colon P|_U \cong P_U$ and $\psi_V\colon P|_V \cong P_V$ satisfying $u = \psi_V \circ \psi_U^{-1}$. Moreover, this P is unique up to isomorphism.

The first four axioms are used to prove that T-admissibility implies that the T-duality transformation is an isomorphism. The gluing axiom is used to prove Proposition 5.2, which allowed us to prove that G-equivariant pairs over X are T-dual if and only if their Borel constructions are T-dual over $X \times_G EG$.

Example A.1. Our twists of choice when working with twisted K-theory are stable equivariant $PU(\mathcal{H})$-bundles. We expand on these twists in A.3.

Example A.2. In [TXLG04], the authors define twisted K-theory for differentiable stacks. The twists they use are S^1-central extensions over groupoids. So, S^1-central extensions over groupoid representatives of the global quotient stack $X/\!\!/G$ are a model of equivariant twists.

Example A.3. A Hitchin gerbe for a space X is an open cover $\{U_i\}$ together with a collection of line bundles $L_{ij} \to U_i \cap U_j$ and isomorphisms $\delta_{ijk}\colon L_{ij} \otimes L_{jk} \to L_{ik}$ satisfying a cocycle condition, see [Hit01]. By considering equivariant line bundles, we obtain the notion of equivariant Hitchin gerbes. These can be made quite explicit and so are useful in calculations, see for example [FHT11, §1].

A.2 Axioms for Twisted Equivariant Cohomology

Fix a model of twists $\mathrm{Twist}_G(X)$. A twisted equivariant cohomology theory consists of, for each $n \in \mathbb{Z}$, a functor

$$(X, G, P) \mapsto h^n_G(X, P)$$

on the category of spaces equipped with a group action and equivariant twist. In addition to functoriality with respect to spaces, groups, and twists, these functors must satisfy the following axioms:

1. (Homotopy invariance) If $f\colon X \to Y$ and $g\colon X \to Y$ are equivariantly homotopic, then
$$g^* = u^* \circ f^*\colon h_G(Y, P) \to h_G(X, g^*P)$$
for some isomorphism $u\colon f^*P \cong g^*P$.

2. (Pushforward) If $p\colon Y \to X$ is a G-equivariant h_G-oriented map, then there is an integration map
$$p_!\colon h^n_G(Y, p^*P) \to h^{n-d}_G(X, P).$$
This map has a degree shift of $d := \dim X - \dim Y$ and is natural with respect to pullbacks.

3. (Induction) If $H \subseteq G$ is a subgroup, then there is a natural isomorphism
$$h^n_H\big(E, P\big) \cong h^n_G\big(E \times_H G, \mathrm{Ind}^G_H(P)\big).$$

4. (Mayer-Vietoris) If $X = U \cup V$ is a decomposition of X into two open sets, then there is a natural long exact sequence:

$$\cdots \to h^{n-1}_G\big(U \cap V, P|_{U\cap V}\big) \to h^n_G(X, P) \to h^n_G(U, P|_U) \oplus h^n_G(V, P|_U) \to h^n_G\big(U \cap V, P|_{U\cap V}\big) \to \cdots$$

We remark that the category of spaces that a cohomology theory is defined on can be context-dependent. To define the pushforward, for example, one often restricts to the category of manifolds.

Example A.4. Given a twisted non-equivariant cohomology theory, one can obtain a twisted equivariant cohomology theory via the Borel construction, that is,
$$h_G(X, P) := h(X \times_G EG, P).$$
A particular example of this is twisted Borel K-theory.

Example A.5. We have discussed the Thom isomorphism in twisted equivariant K-theory and hence have a pushforward along K-oriented maps. Therefore twisted equivariant K-theory is a twisted equivariant cohomology theory.

A.3 Principal $PU(\mathcal{H})$-bundles

We use stable equivariant principal $PU(\mathcal{H})$-bundles in our definition of twisted equivariant K-theory. Here, we provide a definition and their basic properties.

Definition A.6. Let X be a G-space. A G-equivariant principal $PU(\mathcal{H})$-bundle on X is a principal $PU(\mathcal{H})$-bundle $P \to X$ together with an action of G on P such that

$$\pi(g \cdot p) = g \cdot \pi(p) \quad \text{and} \quad g \cdot (p \cdot u) = (g \cdot p) \cdot u,$$

where $g \in G$, $p \in P$, and $u \in PU(\mathcal{H})$.

Definition A.7. A stable homomorphism $f\colon G \to PU(\mathcal{H})$ is a homomorphism such that for the induced central extension $1 \to S^1 \to \widetilde{G} \to G \to 1$ defined by $\widetilde{G} := f^*U(\mathcal{H})$, the induced homomorphism $\widetilde{f}\colon \widetilde{G} \to U(\mathcal{H})$ contains all of the irreducible representations of $\widetilde{G}$ where the central S^1 acts via scalar multiplication, countably infinitely many times.

A homomorphism $G \to PU(\mathcal{H})$ is the same as a G-equivariant principal $PU(\mathcal{H})$-bundle over a point. A motivation for considering stable homomorphisms comes from the bijection

$$\mathrm{Hom}_{st}\big(G, PU(\mathcal{H})\big)/PU(\mathcal{H}) \longleftrightarrow \mathrm{Ext}(G, S^1)$$

between stable homomorphisms up to conjugation and S^1-central extensions of G [BEJU14, Prop 1.6]. In other words, we want G-equivariant twists over a point to correspond to S^1-extensions of G and this is only true if we restrict to the stable homomorphisms.

A further motivation is that the twisted G-equivariant K-theory of a point should be the ring of "twisted" representations of G corresponding to the twist, that is, representations of the resulting central extension $\widetilde{G}$ such that the central S^1 acts by scalar multiplication. This is not guaranteed if the twist is not stable, see [BEJU14, §4.3.4] and [LU14, §15.2].

We extend this now to the notion of stable equivariant principal $PU(\mathcal{H})$-bundle. These are, roughly speaking, G-equivariant principal bundles where, locally, isotropy groups act via stable homomorphisms. This definition comes from [BEJU14, Def 2.2].

Definition A.8. A stable G-equivariant principal bundle $\pi\colon P \to X$ is one such that for each $x \in X$ there exists a G-neighbourhood V of x and a G_x-contractible slice U of x such that $V \cong U \times_{G_x} G$ together with a local trivialisation

$$P|_V \cong (PU(\mathcal{H}) \times U) \times_{G_x} G,$$

where G_x acts on the $PU(\mathcal{H})$-factor via a stable homomorphism.

Let us show that stable equivariant principal $PU(\mathcal{H})$-bundles satisfy the twist axioms we have presented. The first result states that stable equivariant principal bundles satisfy Axioms 1 and 2.

Proposition A.9. *There is a bijection between the isomorphism classes of stable G-equivariant principal $PU(\mathcal{H})$-bundles on X and $H^3_G(X;\mathbb{Z})$. This classification factors through the Borel construction*

$$\mathrm{Proj}_G(X) \to \mathrm{Proj}(X \times_G EG) \cong H^3_G(X).$$

Proof. See [AS04, Prop 6.3] or, in the case that G is discrete, [BEJU14, Theorem 3.8]. □

The next result proves that stable equivariant $PU(\mathcal{H})$-bundles have an induction isomorphism, giving us Axiom 3.

Proposition A.10. *If $P \to X$ is a H-equivariant stable $PU(\mathcal{H})$-principal bundle then $P \times_H G \to X \times_H G$ is a G-equivariant stable $PU(\mathcal{H})$-principal bundle. Therefore, the assignment $P \mapsto P \times_H G$ induces a bijection between the H-equivariant stable bundles on X and the G-equivariant stable bundles on $X \times_H G$.*

Proof. It is clear enough that $P \times_H G \to X \times_H G$ is a G-equivariant $PU(\mathcal{H})$-principal bundle; we only show that it is stable. Consider $[x, g] \in X \times_H G$. The isotropy group at $[x, g]$ is H_x and this does not depend on the choice of x in the coset $[x, g]$. We must prove that there exists an open neighbourhood V of $[x, g]$ and a slice U such that

$$V \cong U \times_{H_x} G \quad \text{and} \quad (P \times_H G)|_V \cong (PU(\mathcal{H}) \times U) \times_{H_x} G,$$

where H_x acts on $PU(\mathcal{H})$ via a stable homomorphism. Since $P \to X$ is stable, there exists an open neighbourhood V' of x and a slice U' such that

$$V' \cong U' \times_{H_x} H \quad \text{and} \quad P|_{V'} \cong (PU(\mathcal{H}) \times U') \times_{H_x} H.$$

Let $V = V' \times_H G$ and let U be the image of U in $X \times_H G$ under the injection $y \mapsto [y, g]$. Then U is a slice at $[x, g]$ with

$$V = V' \times_H G \cong (U' \times_{H_x} H) \times_H G \cong U' \times_{H_x} G \cong U \times_{H_x} G,$$

and so

$$\begin{aligned}
(P \times_H G)|_V &= P|_{V'} \times_H G \\
&\cong (PU(\mathcal{H}) \times U') \times_{H_x} H \times_H G \\
&\cong (PU(\mathcal{H}) \times U') \times_{H_x} G \\
&\cong (PU(\mathcal{H}) \times U) \times_{H_x} G.
\end{aligned}$$

Therefore, $P \times_H G$ is a G-equivariant stable principal $PU(\mathcal{H})$-bundle. □

For Axiom 4, we use that bundles can be glued together and that stability is a local property, so that the gluing of two stable equivariant bundles is again stable. The outcome of this discussion is the following:

Proposition A.11. *Stable equivariant principal $PU(\mathcal{H})$-bundles satisfy the twist axioms introduced in Section A.1.*

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