An Integration Theorem for Representations of the Tangent Algebroid

Representation Theory of Lie Groupoids

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CHAPTER 1

Introduction

1.1. Motivation

Since their early days, the sciences of mathematics and physics have been tightly interwoven. Two physical subjects which particularly challenged new mathematical developments are the theory of general relativity and quantum mechanics. The theory of general relativity is founded upon concepts of differential geometry and uses objects with a *local* structure, such as smooth manifolds, vector fields, flows and differential operators. In contrast, quantum mechanics often requires an understanding of *global* properties. In this realm, a researcher may look at vector fields as unbounded linear operators on a Hilbert space, and at flows as exponentials of such operators defined by functional calculus. Both general relativity and quantum mechanics have been very fruitful over the years, but are still not unified in a common theory. Thus any new connection between their mathematical foundations can be of scientific interest.

One mathematical area which connects local and global theories is the representation theory of Lie groups and Lie algebras. Representations of Lie groups by bounded operators have been soundly studied during the last century. They carry deep information about the global group structure. As commonly known, a Lie group can be differentiated to a Lie algebra, namely its tangent space at the identity. Differentiation of a Lie group representation gives us a representation of the Lie algebra, which only carries local information. For representations on finite-dimensional spaces, this process can be reversed, integrating Lie algebra representations to Lie group representations. In principle, this is also possible for infinite-dimensional representations, but in that case, a bounded Lie group representation may differentiate to an *unbounded* Lie algebra representation. Integration then requires deeper knowledge of functional analytic properties, in particular, elements of the Lie algebra should be represented by essentially skew-adjoint operators. The necessary conditions were, among others, studied by Edward Nelson in the fifties, using analytic vectors and properties of the Laplace operator.

The representation theory of Lie groups and Lie algebras has given birth to many new ideas in mathematical physics, but it also has its limits. More recent publications have introduced the more general notions of Lie groupoids and Lie algebroids. Whereas a Lie group can be interpreted as composable arrows from a single point to itself from the perspective of category theory, a Lie groupoid consists of arrows between the infinitely many different points of a smooth manifold. Whereas Lie algebras have a basis of finitely many vector fields, Lie algebroids are infinite-dimensional and require an understanding of multiplication with smooth functions instead of just scalars.

The original goal of this dissertation was to develop a representation theory for Lie groupoids and Lie algebroids, resembling the classical theory of Lie groups and Lie algebras. In particular, I aimed to show that Lie algebroid representations can be integrated to Lie groupoid representations, because this direction is much more difficult than differentiation. There is hope that such a new representation theory could, similar to the classical Lie theory, inspire new developments in mathematics and theoretical physics.

1.2. Methods

For a Lie groupoid G, the simplest form of representation is not a representation defined on G itself, but a homomorphism $\pi : C^*(G) \to \mathbb{B}(H)$ from the groupoid C^*-algebra $C^*(G)$ into the bounded operators on a Hilbert space H. One reason for this is the fact that smoothness is, in a sense, a requirement which is too strong for groupoid homomorphisms. One can define a smooth theory of Lie groupoid representations on smooth vector bundles, but this is different from the measurable theory that will be used in this thesis, and also less rich.

Thus the common way to differentiate a Lie groupoid representation is to start with a representation $\pi : C^*(G) \to \mathbb{B}(H)$ and define an unbounded representation $R = \mathrm{diff}(\pi) : \mathrm{Diff}^R(G) \to \mathcal{O}(H)$ from the right-invariant differential operators on G into the unbounded operators on H by the formula $R(D)(\pi(f)v) = \pi(D(f))v$ on the domain $\{\pi(f)v \mid f \in C_c^\infty(G),\, v \in H\}$. This procedure was known before the start of my project and does not require the investigation of groupoid homomorphisms.

That changes for the integration process which I developed. Like in the classical Lie theory, I use exponential maps for the integration. Unfortunately, there is no obvious way to define the exponential of an arbitrary differential operator in our given context. Instead, I will consider exponentials of Lie algebroid sections, which are bisections in the corresponding Lie groupoid. This will allow us to define a map on the groupoid itself from a representation of its invariant differential operators.

To properly do this, more theory is required. As mentioned before, a natural way to define groupoid representations is to use a vector bundle instead of a vector space as target. The smooth theory is too limited here, but it turns out that the same idea can be used in a surprisingly general context. This is thanks to the theory on *measurable fields of Hilbert spaces* developed by Jacques Dixmier (see [**5**]). Using his theorems and adaptations to our given context, we will see how each Hilbert space relevant to us is isomorphic to a measurable field of Hilbert spaces. In this sense, we can decompose both the target space and unitary operators on it into fibres.

Using this decomposition technique and exponential maps, I was able to prove integration theorems for Lie algebroid representations in a particular context. Beyond the explicit definitions and computations, another important part in the process is Stone's theorem on one-parameter unitary groups (see [**23**]), which complements the used exponentials and allows us to investigate the domain of certain unbounded operators in a concise way.

While the overall process was successful, major hindrances occurred at certain steps. The most prominent example of this is the exponential map for Lie algebroids. Unlike for Lie algebras, exponentials are not always globally defined, and most importantly, a Baker-Campbell-Hausdorff formula does not apply for non-commuting sections of the algebroid.

This is the main reason why we focus on the case of pair groupoids and the tangent algebroid in the later chapters of this monograph. In this case, we can use commuting vector fields to avoid the need for a Baker-Campbell-Hausdorff formula. Using ideas from classical Lie theory as well as new methods from differential geometry and functional analysis, I have proven that representations of the tangent algebroid can actually be integrated under certain assumptions. In precise terms, I have proven the following theorem (in the main text, this is Corollary 7.3.5):

THEOREM 1.2.1. *Let M be a compact and simply connected smooth manifold. Let $\omega \in \Omega^m(M)$ be a volume form. Let $R : \mathrm{Diff}(M) \to \mathcal{O}(K)$ be a representation on a separable Hilbert space K such that $R|_{C^\infty(M)}$ is injective.*

Suppose that R is integrable. Then there is a representation $\pi = \mathrm{int}(R) : C^*(M \times M) \to \mathbb{B}(K)$ such that $\overline{\mathrm{diff}(\pi)(D)} = \overline{R(D)}$ for all $D \in \mathrm{Diff}_1(M)$.

The assumption that R is integrable is a necessary technical condition, which will be formally defined in the last chapter.

A short summary of the complete content is given in the next section.

1.3. Short Summary

This dissertation investigates representations of Lie groupoids and Lie algebroids and the connection between them. Lie groupoids and Lie algebroids are differential-geometric generalisations of Lie groups and Lie algebras. A Lie groupoid representation is a bounded *-homomorphism from the groupoid C^*-algebra of the groupoid into the bounded operators on a separable Hilbert space. A Lie algebroid representation is a unital *-homomorphism from the universal enveloping algebra of the Lie algebroid into the unbounded operators on a separable Hilbert space which has a common, invariant, dense domain. Similar to Lie groups, any Lie groupoid G can be differentiated to a Lie algebroid $A(G)$. In this case, the right-invariant differential operators on G (those differential operators which commute with the multiplication maps $r_g : h \mapsto hg$) are a universal enveloping algebra for $A(G)$ and carry a natural involution defined using the divergence for vector fields. I use an algebraic definition of differential operators, which does not involve charts. All of these notions are formally introduced in the first three chapters of this thesis.

In Chapter 4 I give a proof of the known fact that every non-degenerate Lie groupoid representation can be differentiated to a representation of its Lie algebroid on the same Hilbert space. I also show that in this derived representation, symmetric differential operators of order 1 act by essentially self-adjoint unbounded operators.

Chapter 5 covers measurable fields of Hilbert spaces, which are infinite-dimensional generalisations of vector bundles. I show how every Hilbert space which is the target of a Lie algebroid representation is isomorphic to the section space of a measurable field of Hilbert spaces. Any measurable field of Hilbert spaces H on a space M defines a groupoid of unitary maps $U(H)$ over M. I use this to define a third type of representation, which is a groupoid homomorphism from a Lie groupoid G to the unitary groupoid $U(H)$ of a measurable field of Hilbert spaces over the same base space. I show that each local groupoid homomorphism which is defined on a neighbourhood of the identities can be extended to a global homomorphism if the groupoid has simply connected fibres.

Chapters 6 and 7 serve the construction of an integration theorem for Lie algebroid representations. In Chapter 6 I show that exponentials of vector fields act by decomposable unitary operators and use this to integrate representations of the Euclidean tangent bundle $T\mathbb{R}^m$ to representations of the pair groupoid $\mathbb{R}^m \times \mathbb{R}^m$. I also demonstrate how to integrate groupoid homomorphisms to representations of the groupoid C^*-algebra. Then I show that the combination of both integration steps is actually inverse to differentiation using explicit computations.

Chapter 7 uses techniques introduced in Chapter 6 to prove a generalised integration theorem. This theorem states that every integrable representation of a tangent algebroid TM, where M is a compact, simply connected smooth manifold, can be uniquely integrated to a representation of the pair groupoid $M \times M$. A necessary and sufficient condition for integrability is that all symmetric differential operators of order 1 act by essentially self-adjoint operators and that exponentials of locally commuting vector fields fulfil a local group relation. I show that integration and differentiation are again inverse to each other in this scenario. Finally I investigate a few other conditions for integrability.

CHAPTER 2

Lie Groupoids and Lie Algebroids

2.1. An Elementary Example

Before we investigate the more theory-heavy concepts which are involved in our later main theorems, I would like to showcase a rather elementary example and define a few basic terms in the process. Maybe the simplest way to represent differential operators is to let them act on the function space they are defined on. So if we have a vector field X on the real numbers $\mathbb{R}$, we can view X as a linear operator $C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$. We can also regard the compactly supported smooth functions $C_c^\infty(\mathbb{R})$ as a subspace of the square-integrable functions $L^2(\mathbb{R})$. This then assigns operator theoretic properties to our vector field, for example, it must have an adjoint.

Having a focus on differential geometry, we should apply the same idea to arbitrary smooth manifolds instead of just the real line. In this case, we are lacking a canonical measure on our space, which is why we have to choose one. The convenient thing to do is to actually choose a volume form, which is slightly less general, because this allows us to immediately use Stokes' Theorem and related results.

In the following definition we give a shorter name to a manifold with a volume form on it and define the respective divergence of vector fields.

Definition 2.1.1. A *volumetric manifold* is a pair (M, ω), where M is an oriented smooth manifold and ω is a positive volume form on M, i.e. a smooth differential form $\omega \in \Omega^m(M)$, where $m = \dim M$, such that $\omega(e_1, \dots, e_m) > 0$ for every positively oriented smooth local frame $(e_1, \dots, e_m)$ of TM. If ω is clear from context or not relevant in the notation, I may call M itself a volumetric manifold.

Given a volumetric manifold (M, ω), for all smooth vector fields $X \in \mathfrak{X}(M)$ we define the divergence $\operatorname{div}(X) := f$ to be the unique smooth function $f \in C^\infty(M)$ such that $\mathcal{L}_X\omega = f \cdot \omega$. Here, $\mathcal{L}_X$ is the Lie derivative by X.

A vector field $X \in C^\infty(M)$ is called *solenoidal* if $\operatorname{div}(X) \equiv 0$.

A few things can be noted about this definition: By continuity, ω being positive is equivalent to $\omega_p(v_1, \dots, v_m) > 0$ for a single positively-oriented basis $(v_1, \dots, v_m)$ of T_pM and all $p \in M$. Furthermore, the divergence is uniquely defined because ω is non-vanishing and $\operatorname{rk} \wedge^m(T_p^*M) = 1$, so that (ω) alone is a frame of $\wedge^m(T_p^*M) = 1$. There are many different definitions of a vector field's divergence in different contexts, but often they are equivalent. For compactly supported vector fields X, the Lie derivative $\mathcal{L}_X\omega$ and hence the divergence have compact support, too. This will be important for the functional analysis point of view.

Beyond the mathematical content, the somewhat unusual name *solenoidal* can be translated as *tube-shaped* and stems from the mathematical analysis of magnetic fields.

The main application of volume forms is integration on manifolds, so it is no surprise that they induce a canonical measure.

Proposition 2.1.2. *Let (M, ω) be a volumetric manifold. Let $\tau = \{U \subseteq M | U \text{ open}\}$ be the topology of M and $\mathcal{B} = \sigma(\tau)$ the induced Borel σ-algebra of M. Then there is a unique Radon measure $\mu = \mu_\omega : \sigma(\tau) \to \mathbb{R}_{\geq 0} \cup \infty$ such that $\int_M f \mathrm{d}\mu = \int_M f\omega$ for all $f \in C_c(M)$.*

Proof: Consider the map $I : C_c(M) \to \mathbb{R},\ f \mapsto \int_M f\omega$ (using the integral of a continuous differential form as defined in [**11**]). By Proposition 16.6, page 407 in [**11**], I is a linear map and positive in the sense that $I(f) \geq 0$ for all functions f with $f(p) \geq 0$ for all $p \in M$ (because in that case, $f\omega$ is another positively oriented volume form on $f^{-1}(\mathbb{R}_{>0}) \subseteq M$). So I is a positive linear functional.

Being a smooth manifold, M is locally compact, thus by the Riesz-Markov representation theorem, there is a unique Radon measure μ on $\mathcal{B}$ with $\int_M f \mathrm{d}\mu = I(f) = \int_M f\omega$ for all $f \in C_c(M)$. □

Using this canonical measure, we define L^p-spaces.

Definition 2.1.3. Let (M, ω) be a volumetric manifold. For every $p \in \mathbb{R}_{>0}$ we define

$$L^p(M, \omega) := L^p(M, \sigma(\tau), \mu_\omega) = \{f : M \to \mathbb{K} \mid f \text{ measurable, } \int_M |f|^p d\mu_\omega < \infty\}/\mathcal{N},$$

where $\mathcal{N} = \{f : M \to \mathbb{K} \mid f \text{ measurable, } \int_M |f|^p d\mu_\omega = 0\}$. Here, measurable always means Borel-Lebesgue-measurable. $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. By convention, I will consider real-valued functions in this section and complex-valued ones later; the computations are all very similar.

If ω is clear from context, I may also just write $L^p(M)$ for the same object.

Of course I am going to use the main results about L^p-spaces, namely that $L^p(M, \omega)$ is a Banach space with $\|f\|_p = (\int_M |f|^p d\mu_g)^{\frac{1}{p}}$ for all $p \geq 1$ and that $L^2(M, \omega)$ is a Hilbert space with $\langle f, g\rangle = \int_M \bar{f} g d\mu_\omega$.

Remark 2.1.4. Clearly every continuous function on M is measurable, and if it has compact support, then it is a representative of an L^p-element for every $p \in \mathbb{R}_{\geq 1}$ since μ_ω is σ-finite. Because there are no open null sets in M by the strict positivity of ω, the projection map $\pi : C_c^0(M) \to L^p(M, \omega),\ f \mapsto [f]$ is injective (the lower index c indicates compact support). Because of this I will sometimes just identify functions $f, g \in C_c^0(M)$ with elements of $L^p(M, \omega)$.

Let us now include a formal definition of (not necessarily bounded) linear operators, which is often confusingly omitted.

Definition 2.1.5. Let V be a topological $\mathbb{K}$-vector space. The set of operators on V is defined to be

$$\mathcal{O}(V) = \{(W, \tau) \mid W \subseteq V \text{ vector subspace, } \tau : W \to V\ \mathbb{K}\text{-linear}\},$$

with addition $(W, \tau) + (X, \sigma) = (W \cap X, \tau|_{W\cap X} + \sigma_{W\cap X})$ and composition $(W, \tau) \circ (X, \sigma) = (X \cap \sigma^{-1}(W), \tau \circ \sigma|_{X\cap\sigma^{-1}(W)})$. The neutral elements are $(V, 0)$ for the addition and (V, id_V) for the composition.

An operator $(W, \tau) \in \mathcal{O}(V)$ is, intuitively, called *densely defined* if W is dense in V, i.e. $\overline{W} = V$. For any $(W, \tau) \in \mathcal{O}(V)$ we define $\mathrm{dom}(W, \tau) := W$ and occasionally just write $\tau \in \mathcal{O}(V)$ for the same operator; the domain is determined by τ anyway.

While the addition and composition defined as above make perfect sense, one has to be aware that both may drastically decrease the domain. Even if two operators on a Hilbert space are densely defined, their sum need not be. As an example consider the spaces of polynomial functions and functions vanishing in a neighbourhood of the boundary, which are both dense in $L^2([0, 1])$, with their intersection being plain $\{0\}$. So the densely defined operators usually do not form a subspace. However, we will see that important sets of operators will have a *common* dense domain, which is enough to ensure the validity of important theorems.

Now, what actually is the adjoint of a vector field? We are not yet ready to investigate the intricate domain issues, but we can at least deduce a formula which works on the smooth function space. This would traditionally be done using partial integration, but Stokes' theorem makes the proof much more compact.

THEOREM 2.1.6. *Let (M, ω) be a volumetric manifold and $X \in \mathfrak{X}(M)$. Then for all $f, g \in C_c^\infty(M)$, we have:*

$$\langle f, \mathcal{L}_X(g) \rangle = -\langle \mathcal{L}_X(f), g \rangle - \langle f \operatorname{div} X, g \rangle$$

Here the scalar products are taken in the real-valued $L^2(M, \omega)$.

In particular, $\mathcal{L}_X \in \mathcal{O}(L^2(M))$ is formally skew-adjoint if and only if X is solenoidal.

Proof: Let $X \in \mathfrak{X}(M)$ and $f, g \in C_c^\infty(M)$ be arbitrary. Then by the Leibniz rule for the Lie derivative we have:

$$\mathcal{L}_X(fg\omega) = fg\mathcal{L}_X\omega + \mathcal{L}_X(f)g\omega + f\mathcal{L}_X(g)\omega$$

Note that $\mathcal{L}_X(fg\omega) \in \Omega^m(M)$ ($m = \dim M$) still has compact support. Because the exterior derivative of forms of maximal rank is zero, we have $\mathcal{L}_X(fg\omega) = d \circ i_X(fg\omega)$. So by Stokes' theorem (as proven in [**11**], page 411, Theorem 16.11), it has integral

$$\int_M \mathcal{L}_X(fg\omega) = \int_M d \circ i_X(fg\omega) = \int_{\partial M} \iota_{\partial M}^* i_X(fg\omega) = 0,$$

using that $\partial M = \emptyset$. Hence using $\mathcal{L}_X\omega = \operatorname{div}(X)\omega$ we get

$$0 = \int_M fg\mathcal{L}_X\omega + \mathcal{L}_X(f)g\omega + f\mathcal{L}_X(g)\omega = \langle f, g \operatorname{div}(X) \rangle + \langle \mathcal{L}_X(f), g \rangle + \langle f, \mathcal{L}_X(g) \rangle$$

or equivalently:

$$\langle f, \mathcal{L}_X(g) \rangle = -\langle f, g \operatorname{div}(X) \rangle - \langle \mathcal{L}_X(f), g \rangle \qquad \square$$

It is crucial for this theorem that M has no boundary. Otherwise, the closed unit interval with linear and quadratic functions and the usual derivative would already give a counterexample.

Another important property of certain unbounded operators is that they have a *large enough* domain. The following theorem shows that this is always true in cases relevant for us. It uses important results from measure theory and functional analysis and combines them to get a suitable version of the statement for the context of volumetric manifolds.

PROPOSITION 2.1.7. *Let (M, ω) be a volumetric manifold. Then $C_c^\infty(M) \subseteq L^2(M, \omega)$ is dense.*

Proof: We know that (M, μ_ω) is a locally compact Hausdorff space with a regular Borel measure by basic manifold theory and Proposition 2.1.2. So by [**7**], page 78, Proposition 5.7, the compactly supported *continuous* functions $C_c(M)$ are dense in $L^2(M)$. This still holds for smooth functions. The proof goes as follows:

Concretely, we need to show that for every $f \in L^2(M)$ and $\epsilon > 0$ there is $h \in C_c^\infty(M)$ with $\|f - h\|_{L^2} < \epsilon$. So let $\epsilon > 0$ and $f \in L^2(M)$ be arbitrary. As mentioned before, $C_c(M) \subseteq L^2(M)$ is dense, so there is $g \in C_c(M)$ with $\|f - g\|_{L^2}^2 < \frac{\epsilon}{2}$. Set $K := \operatorname{supp} g$, which is compact.

Note that the restrictions of smooth global functions $C^\infty(M)|_K$ form a point-separating, non-vanishing subalgebra of $C(K)$, so by the Stone-Weierstraß Theorem, they are dense in $C(K)$, using the supremum norm. If $\mu(K) = 0$, then $g \equiv 0$ is already smooth, so assume that $\mu(K) > 0$, and choose $h \in C^\infty(M)$ with $\|(g - h)|_K\|_\infty < (\frac{\epsilon}{4\mu(K)})^{\frac{1}{2}}$. Using that $\mu = \mu_\omega$ is (outer) regular, we find an open precompact set $U \subseteq M$ with $K \subseteq U$ and $\mu(U \setminus K) < \frac{\epsilon}{4\|h|_V\|_\infty} \leq \frac{\epsilon}{4\|h|_U\|_\infty^2}$. Here, V is some open subset containing K, which is used to get a constant value at the right side; shrinking the set further to U can only decrease the supremum norm. The supremum norm is finite since U is precompact.

Now since U is open and $K \subseteq U$ is compact, there is a smooth bump function $b \in C_c^\infty(M)$ with $0 \le b \le 1$, $b|_K \equiv 1$ and $\mathrm{supp}(b) \subseteq U$ (e.g. use a partition of unity subordinate to $\{M \setminus K, U\}$). Set $k := bh \in C_c^\infty(M)$. Within this construction we compute:

$$\begin{aligned} \|f-k\|_{L^2}^2 &\le (\|f-g\|_{L^2} + \|g-k\|_{L^2})^2 \\ &\le \|f-g\|_{L^2}^2 + \|g-k\|_{L^2}^2 \\ &< \frac{\epsilon}{2} + \int_K (g-k)^2 d\mu + \int_{M\setminus K} (g-k)^2 d\mu \\ &= \frac{\epsilon}{2} + \int_K (g-h)^2 d\mu + \int_{U\setminus K} b^2h^2 d\mu \\ &\le \frac{\epsilon}{2} + \mu(K)\|(g-h)|_K\|_\infty^2 + \mu(U\setminus K)\|h|_U\|_\infty^2 \\ &< \frac{\epsilon}{2} + \mu(K)\left(\frac{\epsilon}{4\mu(K)}\right)^{\frac{1}{2}\cdot 2} + \frac{\epsilon}{4\|h|_U\|_\infty^2}\|h|_U\|_\infty^2 = \epsilon \end{aligned}$$

As $\epsilon > 0$ was arbitrary (in particular, we could also use ϵ^2), this finishes the proof. □

Knowing that Lie operators are defined on a dense domain allows us to build their adjoint. By Theorem 2.1.6, we can partly compute it.

Corollary 2.1.8. *Let M be a volumetric manifold and $X \in \mathfrak{X}(M)$. Then the adjoint $\mathcal{L}_X^* \in \mathcal{O}(L^2M)$ of $\mathcal{L}_X$ exists, its domain $\mathrm{dom}(\mathcal{L}_X^*)$ contains $C_c^\infty(M) \subseteq L^2M$, and*

$$\mathcal{L}_X^*|_{C_c^\infty(M)} = -\mathcal{L}_X - m_{\mathrm{div}\,X},$$

where $m_h : C_c^\infty(M) \to C_c^\infty(M)$, $f \mapsto h \cdot f$ denotes the multiplication operator for $h \in C^\infty(M)$.

Proof: The existence of an adjoint is guaranteed by the mere fact that the domain $\mathrm{dom}(\mathcal{L}_X) = C_c^\infty(M) \subseteq L^2M$ is dense (Proposition 2.1.7). By definition,

$$\mathrm{dom}(\mathcal{L}_X^*) = \{f \in L^2M | \exists h \in L^2M\ \forall g \in C_c^\infty(M) : \langle f, \mathcal{L}_X g\rangle = \langle h, g\rangle\}.$$

For $f \in \mathrm{dom}(\mathcal{L}_X^*)$, the value of the adjoint $\mathcal{L}_X^*(f)$ is just h.

Let $f \in C^\infty(M)$. Then we know by Theorem 2.1.6 that for all $g \in C_c^\infty(M) \subseteq L^2M$ we have $\langle f, \mathcal{L}_X g\rangle = \langle -\mathcal{L}_X(f) - f\,\mathrm{div}\,X, g\rangle$, so $f \in \mathrm{dom}(\mathcal{L}_X^*)$, and

$$\mathcal{L}_X^*(f) = -\mathcal{L}_X(f) - f\,\mathrm{div}\,X = (-\mathcal{L}_X - m_{\mathrm{div}\,X})(f).$$

Since $f \in C^\infty(M)$ was arbitrary, this implies indeed that $C_c^\infty(M) \subseteq \mathrm{dom}(\mathcal{L}_X^*)$ and $\mathcal{L}_X^*|_{C_c^\infty(M)} = -\mathcal{L}_X - m_{\mathrm{div}\,X}$. □

Now that we know about the adjoint, general theorems from functional analysis ensure that Lie operators are also closable:

Corollary 2.1.9. *Let M be a volumetric manifold and $X \in \mathfrak{X}(M)$. Then $\mathcal{L}_X$ is a closable operator.*

Proof: By Corollary 2.1.8, $C_c^\infty(M) \subseteq \mathrm{dom}(\mathcal{L}_X^*)$ holds for the domain of the adjoint. So by Proposition 2.1.7, the adjoint is densely defined. Hence by [**25**], Theorem 3, page 196, $\mathcal{L}_X$ has a closed linear extension, i.e. it is closable. □

We will see later how these humble formulas give rise to a *-algebra structure on the algebra of differential operators, with deep implications for representation theory. The divergence operator will be our trusted companion throughout the chapters of this thesis. For now, let us settle with our first results and finish the elementary example. The next sections will formally introduce our main character.

2.2. Lie Groupoids and their Sections

Readers of this work are probably familiar with the notion of Lie groupoids as well as with the fact that Lie groupoids can be *differentiated* to Lie algebroids, which can, under special circumstances, be *integrated* to obtain a Lie groupoid again. Although these structures are not new to the mathematical world, I will give a short introduction to Lie groupoids here to be self-contained. With the use of category-theoretic language, the definition looks as follows:

Definition 2.2.1. A Lie groupoid is a category $\mathcal{G}$ in which all arrows are isomorphisms, where the classes of objects $M = \mathcal{G}^0$ and of morphisms $G = \mathcal{G}^1$ are smooth manifolds such that:

(1) The source and target maps $s, t : G \to M$ are smooth surjective submersions. s and t are defined by $s(g) = x$ and $t(g) = y$ for any morphism $g \in \mathcal{G}(x, y)$, $x, y \in M$.
(2) The *inclusion* map $\iota : M \to G,\ x \to \mathrm{id}_x$ is a smooth embedding.
(3) The composition $\mathrm{mult} : \mathcal{G}^2 := \{(h, g) \in G^2 | t(g) = s(h)\} \to G$ is smooth.

Given such a structure, M is called the *base* of the groupoid, and G is often itself referred to as a groupoid, using the rest of the structure implicitly.

Lie groupoids with base M, morphism space G and source and target maps s, t are commonly denoted $s, t : M \rightrightarrows G$ or just $M \rightrightarrows G$. As for Lie algebroids, there are two basic examples of Lie groupoids, which can be thought of as corner cases, the first one being just a Lie group.

Example 2.2.2. Any Lie group G corresponds a Lie groupoid over the single-point base $\{0\}$.

Proof: It is a category theoretic fact that groups can be seen as categories with one object and only invertible morphisms. Call this corresponding category $\mathcal{G}$ and its object 0. The source, target and inclusion maps are all trivially smooth because their range or domain are singletons. The smoothness of the composition $\mathrm{mult} : \mathcal{G}^2 = G^2 \to G$ is a defining property of Lie groups. □

The other standard example is the pair groupoid over a given manifold.

Example 2.2.3. Let M be any smooth manifold. The *pair groupoid* $M \times M \rightrightarrows M$ of M is defined by source $s(m', m) = m$, target $t(m', m) = m'$ and multiplication $(z, y) * (y, x) = (z, x)$. The inclusion is $\iota : M \to M \times M,\ x \mapsto (x, x)$. Clearly $s, t, *$ and ι are all smooth, and s and t are surjective submersions, so the pair groupoid is a groupoid not only by name.

Unlike morphisms of Lie algebroids, Lie groupoid morphisms are quite obvious to define:

Definition 2.2.4. Let $\mathcal{G} = (G \rightrightarrows M)$ and $\mathcal{H} = (H \rightrightarrows N)$ be Lie groupoids. A morphism of Lie groupoids from $\mathcal{G}$ to $\mathcal{H}$ is a covariant functor $F : \mathcal{G} \to \mathcal{H}$ which is smooth on morphisms and on objects, i.e. a pair of smooth maps (F, f) of $f : M \to N$ with $f \circ s = s' \circ F$, $f \circ t = t' \circ F$ for the respective source and target maps and $F : G \to H$ with $F(g_2 g_1) = F(g_2)F(g_1)$ for all $(g_2, g_1) \in \mathcal{G}^2$.

The main difference between Lie groups and Lie groupoids is that not any two elements of a groupoid can be composed like in the group case. In particular, left and right translation can be defined, but not on the whole of G. We have to use the fibres of source and target instead.

Definition 2.2.5. Let $G \rightrightarrows M$ be a Lie groupoid. For any $x \in M$, define $G^x := t^{-1}(\{x\}) = \{g \in G \mid t(g) = x\}$ (the *target fibre* over x) and $G_x = s^{-1}(\{x\})$ (the *source fibre* over x).

On these level sets, left and right multiplication by a fixed $g \in G$ are defined:

$$l_g : G^{s(g)} \to G^{t(g)},\ h \mapsto gh$$

and

$$r_g : G_{t(g)} \to G_{s(g)},\ h \mapsto hg.$$

Note that both G^x and G_x are submanifolds of G by [**11**], Theorem 5.12, page 105 for any $x \in M$. Furthermore, the maps l_g and r_g are all diffeomorphisms with inverse $(l_g)^{-1} = l_{g^{-1}}$ and $(r_g)^{-1} = r_{g^{-1}}$, respectively.

The nature of Lie groupoids cuts down the number of tools we can use for theories, compared to Lie groups. To preserve a few of them in an altered form, we need to introduce the notion of bisections. They can be thought of as generalised elements of the groupoid and allow to define global translation maps and an exponential function again.

Definition 2.2.6. (Compare [**15**], 3.2, page 15)

Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. A *bisection* of G is a submanifold $S \subseteq G$ such that both $s|_S : S \to M$ and $t|_S : S \to M$ are diffeomorphisms. The set of bisections on G is denoted $\Gamma(G)$.

We define a product $\circ : \Gamma(G) \times \Gamma(G) \to \Gamma(G)$ by

$$S \circ T := \mathrm{mult}((S \times T) \cap \mathcal{G}^2) = \{gh \mid g \in S,\, h \in T,\, t(h) = s(g)\}$$

for all $S, T \in \Gamma(G)$, where mult denotes the composition in G.

There are a few basic properties that a well-schooled mathematician will assume at the sight of such a definition. I list the following, and more later on, without going into the proof.

Proposition 2.2.7. $\Gamma(G)$ *with the product defined above is a group.*

For the pair groupoid, bisections are simply given by diffeomorphisms of the base, as illustrated in the following short example.

Example 2.2.8. Let M be a smooth manifold. The bisections on its pair groupoid are given by diffeomorphisms of M, namely

$$\Gamma(M \times M) = \{S_\phi \mid \phi \in \mathrm{Diffeo}(M)\},$$

where $S_\phi := \{(\phi(x), x) \mid x \in M\}$ is the graph of a diffeomorphism ϕ, up to the order of the pair.

Proof: Set $G = M \times M$ and consider the map $\chi : \Gamma(G) \to \mathrm{Diffeo}(M),\ S \mapsto t|_S \circ s|_S^{-1}$. For any $R \in \Gamma(G)$ we have

$$S_{\chi(R)} = S_{t|_R \circ s|_R^{-1}} = \{(t \circ s|_R^{-1}(x), x) \mid x \in M\} = \{(y, x) \mid (y, x) \in S\} = S$$

because $s|_R^{-1}(x) = (y, x)$ for the unique $y \in M$ such that $(y, x) \in R$.

The other way around, we have

$$\chi(S_\phi)(x) = t|_{S_\phi} \circ s|_{S_\phi}^{-1}(x) = t(\phi(x), x) = \phi(x)$$

for all $\phi \in \mathrm{Diffeo}(M)$ and $x \in M$, i.e. $\chi(S_\phi) = \phi$. □

The name *section* suggests that bisections can also be viewed as maps from the base M to the arrow space G, which I will indeed do frequently, by writing $S(p) := (s|_S)^{-1}(p)$ for $S \in \Gamma(G)$ and $p \in M$. By definition, we always have $s(S(p)) = p$, and it is easily seen that $(S \circ T)(p) = S(t(T(p)) \cdot T(p)$ for two bisections $S, T \in \Gamma(G)$.

2.3. Differentiation of Lie Groupoids to Lie Algebroids

In this section I am going to do a quick recap of how to obtain a Lie algebroid from a Lie groupoid, similar to the investigation of Lie groups and their Lie algebras. This process is commonly called *differentiation*.

In our general definition, the base M of a Lie groupoid $G \rightrightarrows M$ does not have to be a subset of G. However, the canonical inclusion $\iota : M \to G$ was assumed to be an embedding, so that in any case its image $\iota(M) \subseteq G$ is a smooth submanifold, which can be identified with M via ι. To make the notation more comfortable, we will hence rightfully assume that $M \subseteq G$ and $\iota(x) = x$ for $x \in M$.

REMARK 2.3.1. Let $G \rightrightarrows M$ be a Lie algebroid, where $M \subseteq G$. Then $TM \subseteq TG|_M$ is a smooth vector subbundle, which allows us to take a quotient.

Furthermore, note that the source and target maps fulfil $s|_M = \mathrm{id}_M = t|_M$, hence $(Ts - Tt)|_{TM} = 0$, so that $Ts - Tt$ induces a well-defined map on the quotient by TM.

This allows for the following definition:

DEFINITION 2.3.2. ([**15**], page 58)
Let $G \rightrightarrows M$ be a Lie groupoid. The normal bundle $A = \mathrm{Lie}(G) = \nu(G, M)$ of G is defined as $A := TG|_M/TM$, which is a vector bundle over M. Furthermore, we define the *anchor*

$$\rho = \overline{(Ts - Tt)} : A \to TM, \; v_x + T_xM \mapsto (Ts - Tt)(v_x) \in T_xM,$$

which is a smooth vector bundle homomorphism.

So the vector bundle structure and the anchor of the Lie algebroid corresponding to $G \rightrightarrows M$ are quite easy to define. The definition of the Lie bracket requires some more effort. The idea is that we relate sections of the normal bundle to left-invariant vector fields on G and use their commutator bracket.

Let us look at the definition first. A priori there is no important reason to prefer left- or right-invariance.

DEFINITION 2.3.3. Let $G \rightrightarrows M$ be a Lie groupoid. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if it is tangent to the target fibres, i.e. $X(g) \in T(G^{t(g)})$ for all $g \in G$, and

$$T_h l_g(X_h) = X_{gh} \in T_{gh} G^{t(g)}$$

holds for all $g, h \in G$ with $t(h) = s(g)$. The set of left-invariant smooth vectors fields on G is denoted by $\mathfrak{X}^L(G)$.

Likewise, X is called right-invariant if $X(g) \in T(G_{s(g)})$ for all $g \in G$ (X is tangent to the source fibres) and

$$T_h r_g(X_h) = X_{hg} \in T_{hg} G_{s(g)}$$

holds for all $g, h \in G$ with $s(h) = t(g)$. The set of right-invariant smooth vector fields on G is denoted by $\mathfrak{X}^R(G)$.

It is easily proven that invariant vector fields are closed under the commutator.

LEMMA 2.3.4. *$\mathfrak{X}^L(G)$ and $\mathfrak{X}^R(G)$ are Lie subalgebras of $\mathfrak{X}(G)$ with its Lie bracket.*

Proof: Consider left-invariant vector fields first. For $X, Y \in \mathfrak{X}^L(G)$, the defining property is that $Tl_g X|_{G^{s(g)}} = X \circ l_g$ and $Tl_g Y|_{G^{s(g)}} = Y \circ l_g$ for all $g \in G$, hence also

$$Tl_g[X, Y]|_{G^{s(g)}} = [X, Y] \circ l_g$$

by the naturality of the Lie bracket as stated in [**11**], Proposition 8.30, page 188. In particular, $[X, Y](g) = [X, Y] \circ l_g(s(g)) = Tl_g[X, Y](s(g)) \in TG^{t(g)}$ for all $g \in G$. So indeed $[X, Y] \in \mathfrak{X}^L(G)$.

For right-invariant vector fields, the proof is similar. $\square$

The next proposition will flesh out the correspondence of normal sections to left-invariant vector fields.

PROPOSITION 2.3.5. *Let $G \rightrightarrows M$ be a Lie groupoid with $M \subseteq G$ and normal bundle $A = \nu(G, M)$. Then the map*

$$\tau : \mathfrak{X}^L(G) \to \Gamma(A), \; X \mapsto \overline{X|_M}$$

mapping left-invariant vector fields to the pointwise equivalence class of their restriction to M is an isomorphism of $C^\infty(M)$-modules. The module structure on $\mathfrak{X}^L(G)$ is defined by $f \cdot X(g) := f(s(g))X(g)$ for $f \in C^\infty(M)$, $g \in G$.

Proof: τ is clearly $\mathbb{R}$-linear, and for $f \in C^\infty(M), p \in M$ we have $\tau(fX)(p) = \overline{f(s(p))X(p)} = f(p)\overline{X(p)} = f\tau(X)(p)$, so it is a $C^\infty(M)$-module homomorphism. We will now construct the inverse.

First, let $b \in \mathfrak{X}(M) = \Gamma(TM)$. Let $g \in G$ and $p = s(g) \in M$. Then for any $f \in C^\infty(G)$ we have $f \circ l_g|_{M \cap G^p} \equiv f(g)$, hence $T_p l_g b(p)(f) = b(p)(f \circ l_g) = 0$. So

$$\sigma : \Gamma(A) \to \mathfrak{X}(G), \; \sigma(\bar{a})(g) := T_{s(g)} l_g a(s(g))$$

is a well-defined map, which is $C^\infty(M)$-linear because all the maps Tl_g are $\mathbb{R}$-linear. I also write $X_{\bar{a}} := \sigma(\bar{a})$ for this. Let $a \in \Gamma(A)$. Let $g, h \in G$ with $t(h) = s(g)$ and set $p := s(h)$. Then we have by definition

$$X_{\bar{a}}(gh) = T_p l_{gh} a(p) = T_p(l_g \circ l_h)a(p) = T_h l_g \circ T_p l_h a(p) = T_h l_g X_{\bar{a}}(h),$$

so $X_{\bar{a}}$ is indeed left-invariant.

Checking the compositions, we have of course

$$\tau \circ \sigma(\bar{a})(p) = \overline{\sigma(\bar{a})(p)} = \overline{T_p l_p a(p)} = \overline{a(p)} = \bar{a}(p)$$

for all $\bar{a} \in \Gamma(A)$ and $p \in M$, as well as

$$\sigma \circ \tau(X)(g) = \sigma(\overline{X|_M})(g) = T_{s(g)} l_g X(s(g)) = X(gs(g)) = X(g)$$

for all $X \in \mathfrak{X}^L(G)$ and $g \in G$, so τ and σ are indeed inverse to each other. □

This isomorphism allows us to define the Lie bracket of a Lie groupoid's algebroid as follows:

DEFINITION 2.3.6. Let $G \rightrightarrows M$ be a Lie groupoid with $M \subseteq G$ and normal bundle $A = \nu(G, M)$. Then the Lie bracket on $\Gamma(A)$ is defined by

$$[a, b] := \tau([\sigma a, \sigma b]) = \overline{[X_a, X_b]|_M}$$

for all $a, b \in \Gamma(A)$.

It remains to check that the resulting structure is really a Lie algebroid. I prove a short lemma in advance.

LEMMA 2.3.7. *Let $G \rightrightarrows M$ be a Lie groupoid with normal bundle $A = \nu(G, M)$. Let $a \in \Gamma(A)$. Then $TtX_a(g) = 0 \in T_{t(g)}M$ holds for all $g \in G$.*

Proof: By abuse of notation, we use the same glyph a for an element of $\Gamma(A)$ and a representative $a \in \mathfrak{X}(G)|_M$ of it.

Let $g \in G$ and $f \in C^\infty(M)$ be arbitrary. Then we have

$$TtX_a(g)(f) = X_a(g)(f \circ t) = T_{s(g)} l_g a(s(g))(f \circ t) = a(s(g))(f \circ t \circ l_g) = 0$$

because $t \circ l_g \equiv t(g)$ is constant. □

Now we can proceed to the main theorem of this section, which is not very hard to prove at this point.

THEOREM 2.3.8. *Let $G \rightrightarrows M$ be a Lie groupoid with $M \subseteq G$. Then the triple $(A, \rho, [\cdot,\cdot])$, where $A = \nu(G, M)$, $\rho = \overline{Ts - Tt}$ and $[\cdot,\cdot]$ is as in Definition* 2.3.6*, is a Lie algebroid over M.*

Proof: The vector bundle structure is clear. We know that τ and σ from the proposition before are module isomorphisms and that the commutator bracket makes $\mathfrak{X}^L(G)$ a Lie algebra. This implies that the bracket on $\Gamma(A)$ is also a Lie algebra bracket, i.e. bilinear and satisfying the Jacobi identity. ρ was checked to be a smooth vector bundle homomorphism. This leaves only the Leibniz identity to be proven.

To do so, let $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$ be arbitrary. Then by our previous lemma we have

$$\begin{aligned}
[a, fb] &= \overline{[X_a, X_{fb}]|_M} = \overline{[X_a, f \circ sX_b]|_M} \\
&= \overline{(f \circ s[X_a, X_b] + X_a(f \circ s)X_b)|_M} \\
&= f[a,b] + \overline{TsX_a(f)X_b|_M} \\
&= f[a,b] + \overline{(Ts - Tt)X_a(f)X_b|_M} \\
&= f[a,b] + \overline{(Ts - Tt)}(a)(f)b = f[a,b] + \rho(a)(f)b
\end{aligned}$$

as required. Here, $TsX_a(f)$ is defined by $TsX_a(f)(g) = T_g sX_a(g)(f)$ for $g \in G$. □

While the topic has been extensively studied by others, I will humbly stick to this short recap on the differentiation of Lie groupoids. I will also not investigate the backwards theory, the integration of Lie algebroids here. Only so much be said: In certain cases, it is possible to find a Lie groupoid that differentiates back to a given Lie algebroid.

2.4. Haar Systems and the Groupoid C*-Algebra

The recurring main topic of this thesis are groupoid and algebroid representations and how both are connected. However, we still have not defined them appropriately. While there are many different possible definitions, one of them is especially fruitful due to its intrinsic connections to functional analysis. In that definition, we investigate not representations of the morphism space itself, but of the function space $L^I(G)$ and its enveloping C^*-algebra, the groupoid C^*-algebra $C^*(G)$. Both of these algebras have their classical counterparts in Lie group theory.

Hence I will use the pages of this section to introduce and investigate the groupoid C^*-algebra of a Lie groupoid, which does carry an interesting structure in the form of being a C^*-algebra as the name implies. To do this, we need a set of measures first. Concretely, I will next introduce Haar systems, which are a generalisation of Haar measures on groups.

For that definition, we need to recall the concept of the support of a measure.

DEFINITION 2.4.1. Let (X, τ) be a Hausdorff space and μ a Radon measure on the Borel σ-algebra $\sigma(\tau)$. The support of μ is defined as

$$\operatorname{supp}(\mu) := X \setminus \left(\bigcup_{U \in \tau, \mu(U) = 0} U \right),$$

the complement of the largest open null set.

Note that for any measurable $f : X \to \mathbb{R}$ we have $\int_X f \,\mathrm{d}\mu = \int_{\operatorname{supp}\mu} f \,\mathrm{d}\mu$. This is of particular use for the definition of a Haar system, where the support of each individual measure is assumed to be a target fibre.

DEFINITION 2.4.2. Let $G \rightrightarrows M$ be a Lie groupoid. A *Haar system* on G is a family of Radon measures $(\lambda^x)_{x \in M}$ on the Borel σ-algebra of G such that

(1) $\operatorname{supp} \lambda^x = G^x = t^{-1}(\{x\}) \subseteq G$ for all $x \in M$,

(2) the map $M \to \mathbb{R},\ x \mapsto \int_G f \,\mathrm{d}\lambda^x$ is continuous for all $f \in C_c(G)$ and
(3) $\int_{G^{s(g)}} f \circ l_g \,\mathrm{d}\lambda^{s(g)} = \int_{G^{t(g)}} f \,\mathrm{d}\lambda^{t(g)}$ for all $f \in C_c(G)$ and $g \in G$.

$(\lambda^x)_{x\in M}$ is called smooth if for all $x \in M$, the measure λ^x is smooth and the map $M \to \mathbb{R},\ x \mapsto \int_G f \,\mathrm{d}\lambda^x$ is smooth for all $f \in C_c^\infty(G)$.

Although the multiplication function l_g is generally not defined on the whole of G, I may also write the integral as $\int_G f \circ l_g \mathrm{d}\lambda^{s(g)} := \int_{G^{s(g)}} f \circ l_g \mathrm{d}\lambda^{s(g)}$, which is justified by $\operatorname{supp} \lambda^{s(g)} = G^{s(g)}$, so that we would expect $\int_{G\setminus G^{s(g)}} f \circ l_g \mathrm{d}\lambda^{s(g)} = 0$ anyway.

The first place where Haar systems come to use is the definition of the convolution on a Lie groupoid, which works similarly to the usual L^p-product, but yields an independent value for every target (or source) fibre.

Definition 2.4.3. Let $G \rightrightarrows M$ be a Lie groupoid with a smooth Haar system $(\lambda^x)_{x\in M}$. We define the *convolution*

$$* : C_c(G) \times C_c(G) \to C_c(G),\ f * g(x) := \int_{G^{s(x)}} f(xy)g(y^{-1})\mathrm{d}\lambda^{s(x)}(y)$$

and the *involution*

$$* : C_c(G) \to C_c(G),\ f^*(x) := \overline{f(x^{-1})}.$$

In this case, $C_c(G)$ is the algebra of compactly supported continuous functions $G \to \mathbb{C}$.

By [**22**], 1.1.Proposition, page 48, $C_c(G)$ equipped with these operations and the inductive limit topology is a topological $*$-algebra (because the constant function 1 from G to $S^1 \subset \mathbb{C}$ is a continuous 2-cocycle). To recall, the inductive limit topology is defined as follows:

Definition 2.4.4. Let X be a locally compact Hausdorff space. The inductive limit topology or *topology of uniform convergence on compact sets* on $C_c(X)$ is the topology of the inductive limit vector space

$$(C_c(X), \tau) = \operatorname{colim}_{K\in \mathrm{Comp}(X)}(C_K(X), \tau_{p_K}),$$

where $C_K(X) = \{f \in C(X) \mid \operatorname{supp} f \subseteq K\}$ for a compact set $K \subseteq X$. $\mathrm{Comp}(X)$ is the category of compact subsets of X with inclusions as morphisms and τ_{p_K} is the topology generated by the norm given as $p_K(f) := \sup_{x\in K} |f(x)|$, $f \in C_K(X)$.

This gives rise to a new definition of Lie groupoid representations, or more precisely, of $C_c(G)$:

Definition 2.4.5. Let $G \rightrightarrows M$ be a Lie groupoid. A representation of $C_c(G)$ is a pair $(\mathcal{H}, L)$, where $\mathcal{H}$ is some Hilbert space and L is a $*$-homomorphism $L : C_c(G) \to \mathbb{B}(\mathcal{H})$, such that $L|_{C_K}$ is bounded for every compact subset $K \subseteq G$ (again, $C_K = \{f \in C(G) \mid \operatorname{supp} f \subseteq K\}$).

$(\mathcal{H}, L)$ is called non-degenerate if the closed linear span

$$\overline{\operatorname{span}\{L(f)v \mid f \in C_c(G),\, v \in \mathcal{H}\}}$$

is equal to the whole of $\mathcal{H}$.

As usually, I may also refer to L alone as a representation of $C_c(G)$ and omit the Hilbert space from the notation.

A logical next step in enriching the structure of $C_c(G)$ and defining the groupoid C^*-algebra is the definition of a norm on it. It turns out that there are a few (pre)norms to choose from. Firstly we have the source, target and maximal integral norms, defined as follows:

Definition 2.4.6. Let $G \rightrightarrows M$ be a Lie groupoid with smooth Haar system $(\lambda^x)_{x\in M}$, where M is compact. The I-norm or integral norm $\|\cdot\|_I$ on $C_c(G)$ is defined by

$$\|f\|_{I,t} := \sup_{u\in M} \int_{G^u} |f| \mathrm{d}\lambda^u,$$

$$\|f\|_{I,s} := \sup_{u\in M} \int_{G_u} |f| \mathrm{d}\lambda_u,$$

where $\lambda_u := \lambda^u \circ \mathrm{inv}^{-1}$ is the image measure of λ^u under the inversion in G, and

$$\|f\|_I := \max\{\|f\|_{I,t}, \|f\|_{I,s}\}$$

.

Obviously it should be checked that $\|\cdot\|_I$ *is* a norm.

Proposition 2.4.7. *$\|\cdot\|_I$ as defined above is a norm.*

Proof: $\|\cdot\|_I$ is always finite because M is compact. Namely for any $f \in C_c(G)$, $\|f\|_I$ is the maximum of the map $u \mapsto \int |f| \mathrm{d}\lambda^u$, which is continuous by the second property in the definition of Haar systems.

For every $u \in M$, the values $\int_{G^u} |f|\lambda^u$ and $\int_{G_u} |f|\lambda_u$ are the usual L^1-norms of f under the respective measure and hence fulfil the triangle inequality as well as $\|\lambda f\| = |\lambda|\|f\|$ for all scalars $\lambda \in \mathbb{C}$. These two properties are inherited by $\|\cdot\|_I$.

Furthermore, if $\|f\|_I = 0$ for some $f \in C_c(G)$, then $\int_{G^u} |f| \mathrm{d}\lambda^u = 0$ for all $u \in M$. Because f is continuous and λ^u is Borel with $\operatorname{supp} \lambda^u = G^u$, this implies that $f|_{G^u} \equiv 0$. But every element of G has some target, so $\bigcup_{u\in M} G^u = G$, hence $f = f|_G = 0$. ☐

The maximal integral norm gives rise to a space of functions on the Lie groupoid. To obtain a Banach algebra, we complete and get:

Definition 2.4.8. Let $G \rightrightarrows M$ be a Lie groupoid with a Haar system λ. The space $L^I(G)$ is defined as

$$L^I(G) := \overline{(C_c(G), \|\cdot\|_I)},$$

the topological completion with respect to the maximal integral norm.

It can be shown that $L^I(G)$ is indeed a Banach $*$-algebra, but I will not include the detailed computations here. For the special case of G just being a unimodular Lie group, this is indeed isomorphic to the usual function space $L^1(G, \lambda)$. For the more general case, it will serve as a useful generalisation.

Beyond the integral norms, there is another way to define a norm on functions over a Lie groupoid. In fact, this is a standard construction which is meant to give us a C^*-algebra.

Definition 2.4.9. Let $G \rightrightarrows M$ be a Lie groupoid with smooth Haar system $(\lambda^u)_{u\in M}$. The *maximal C^*-norm* of $C_c(G)$, $\|\cdot\|$, is defined by

$$\|f\| := \sup\{\|L(f)\| \mid L \text{ is a bounded representation of } \mathrm{C_c(G)}\}.$$

The speciality in this context is the interaction between the maximal C^*-norm and the integral norm. By Theorem 2.42, page 28 in [**17**], the maximal C^* norm fulfils $\|f\| \leq \|f\|_I$ for all $f \in C^\infty(M)$, in particular, it is finite. Knowing that, as the supremum of any set of norms, $\|\cdot\|$ is indeed a norm. Since $\mathbb{B}(\mathcal{H})$ is a C^*-algebra for every Hilbert space $\mathcal{H}$, we have

$$\|f^*f\| = \sup_L \|L(f)^*L(f)\| = \sup_L \|L(f)^2\| = \|f^2\|,$$

where L ranges over the bounded representations of C_cG.

Hence the following construction indeed gives a C^*-algebra, which can be used to define groupoid representations.

DEFINITION 2.4.10. Let $G \rightrightarrows M$ be a Lie groupoid with smooth Haar system $\lambda = (\lambda^u)_{u\in M}$. The *groupoid C^*-algebra* of G with respect to λ is

$$C^*(G,\lambda) := \overline{(C_c(G), \|\cdot\|)},$$

the topological completion of $C_c(G)$ with the maximal C^*-norm.

A representation of $C^*(G)$ is a pair $(\mathcal{H},\pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi : C^*(G) \to \mathbb{B}(\mathcal{H})$ is a $*$-homomorphism. Likewise, a representation of $L^I(G)$ is a pair $(\mathcal{H},\pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi : L^I(G) \to \mathbb{B}(\mathcal{H})$ is a $*$-homomorphism. Usually, we also assume that π is non-degenerate in both cases. It is called continuous if π is continuous with respect to the norm topology of $\mathbb{B}(H)$.

Continuity is sometimes assumed implicitly.

CHAPTER 3

Differential Operators

In the upcoming chapter, I am going to introduce and investigate differential operators on smooth manifolds. This is of course not a new topic, and most readers of this thesis probably have encountered differential operators in one form or another. I still wanted to include this section because I had a different focus than the most common sources on the same topic. Many authors prefer to give a hands-on definition of differential operators using coordinate vector fields. This works most easily on open subsets of the Euclidean space; the generalisation to manifolds is sometimes done and sometimes neglected.

The hands-on approach is useful when the text in question is about solving certain differential equations, but this is not my focus here. Instead, I am going to consider the set of all (linear, partial, smooth) differential operators on a given manifold as an algebraic object. I will give proves for the fact that they form a filtered algebra in a natural way and investigate (formal) adjoints.

Given this context and personal experience with algebraic geometry, I decided to use a definition for differential operators which originates from commutative algebra and is sometimes attributed to Grothendieck. Two modern references for this approach are [**13**] and [**10**]. The algebraic definitions are applicable to an arbitrary commutative ring R with a given sub-ring $S \subseteq R$, but I will specialise to the case of $R = C^\infty(M)$ and $S = \mathbb{R}$ here.

3.1. Definitions and Basic Properties

We start with a purely algebraic definition for differential operators over a smooth vector bundle.

Definition 3.1.1. (compare [**13**], Definition 1.2.1.9, page 7 or [**10**], page 17)

Let M be a smooth manifold (over $\mathbb{R}$), let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $E \to M$, $F \to M$ be smooth $\mathbb{K}$-vector bundles. A *differential operator* from E to F is a local, $\mathbb{K}$-linear map $D : \Gamma(E) \to \Gamma(F)$ for which there is a natural number $k \in \mathbb{N}_0$ such that for all $f_0, \dots, f_k \in C^\infty(M, \mathbb{K})$ the equation $[m_{f_0}, [\dots [m_{f_k}, D] \dots]] = 0$ holds. Here $m_f : \Gamma(E) \to \Gamma(E),\ s \mapsto fs$ (or likewise $m_f : \Gamma(F) \to \Gamma(F)$) denotes the multiplication operator given by any $f \in C^\infty(M, \mathbb{K})$. The space of differential operators from E to F is denoted as $\mathrm{Diff}(E, F)$.

The order $\operatorname{ord} D$ of a differential operator $D \in \mathrm{Diff}(E, F)$ is the smallest natural number $k \in \mathbb{N}_0$ fulfilling the above property. The space of differential operators of order k or less is denoted by $\mathrm{Diff}_k(E, F) \subseteq \mathrm{Diff}(E, F)$. If $E = F$ we write $\mathrm{Diff}(E) := \mathrm{Diff}(E, E)$ and $\mathrm{Diff}_k(E) := \mathrm{Diff}_k(E, E)$.

A (real) function-valued differential operator on M is defined as an operator from $\bigwedge^0 T^*M$ to itself, and the respective space is denoted as $\mathrm{Diff}(M) := \mathrm{Diff}(\bigwedge^0 T^*M)$, and likewise $\mathrm{Diff}_k(M) := \mathrm{Diff}_k(\bigwedge^0 T^*M)$ for all $k \in \mathbb{N}_0$. The space $\mathrm{Diff}^{\mathbb{C}}(M)$ of complex differential operators on M is defined likewise, using $\mathbb{C} \otimes \bigwedge^0 T^*M$ instead of $\bigwedge^0 T^*M$.

The real and complex form-valued differential operators are $\mathrm{Diff}(\Omega M) := \mathrm{Diff}(\bigwedge T^*M)$ and $\mathrm{Diff}^{\mathbb{C}}(\Omega M) := \mathrm{Diff}(\mathbb{C} \otimes \bigwedge T^*M)$, respectively.

By convention and because it fits better into the propositions to come, the order of the zero map is defined to be $\operatorname{ord} 0 := -\infty$ instead of 0. The commutator $[m_f, D]$ may also be abbreviated as $[f, D]$.

Remark 3.1.2. Because $C^\infty(M,\mathbb{C}) = C^\infty(M) \oplus \mathrm{i}C^\infty(M)$, it suffices to check the multiple commutator equality $[m_{f_0},[\dots[m_{f_k},D]\dots]] = 0$ only for real-valued $f_i \in C^\infty(M)$ even in the case of complex vector bundles as soon as $\mathbb{C}$-linearity is given.

Since $D \in \mathrm{Diff}_0(E,F)$ is equivalent to $0 = [m_f,D](s) = fD(s) - D(fs)$ for all $f \in C^\infty(M)$ and $s \in E$, the differential operators of order 0 are precisely the $C^\infty(M,\mathbb{K})$-linear maps: $\mathrm{Diff}_0(E,F) = \mathrm{Hom}_{C^\infty(M)}(\Gamma(E),\Gamma(F)) \equiv \Gamma(E^* \otimes F)$. Here we use that $C^\infty(M)$-linearity implies locality.

Of course $\bigwedge^0 T^*M$ is a trivial bundle of rank 1, so $\Gamma(\bigwedge^0 T^*M) = \Omega^0(M) \cong C^\infty(M)$, which justifies the name function-valued. This identification will be used frequently without further explanation.

In general, the locality of differential operators is just an additional assumption in the definition. For the function-valued operators however, it already follows from the other properties and could be left out. This is shown in the following proposition.

Proposition 3.1.3. *Let M be a smooth manifold and $D : C^\infty(M) \to C^\infty(M)$ an $\mathbb{R}$-linear map for which there is $k \in \mathbb{N}_0$ such that $[m_{f_0},[\dots[m_{f_k},D]\dots]] = 0$ holds for all $f_0,\dots,f_k \in C^\infty(M)$. Then D is a local map i.e. if two functions $f,g \in C^\infty(M)$ coincide on an open subset $U \subseteq M$, then $D(f)|_U = D(g)|_U$ for any differential operator D on E.*

Proof: By linearity of D it suffices to check that $D(f)|_U = 0$ for $f \in C^\infty(M)$ with $f|_U = 0$. To do so, let $p \in U$ be arbitrary. Choose an open neighbourhood $V \subseteq U$ of p with a smooth bump function $h \in C^\infty(M)$ fulfilling $h|_V \equiv 1$ and $h|_{M\setminus U} \equiv 0$. Then we have $h^k f = 0$, hence

$$\begin{aligned} 0 = [h,[\dots[h,[f,D]]\dots]] &= h^k[f,D] - [f,D]h^k \\ &= h^k fD - h^k Dm_f - fDm_{h^k} + Dm_{fh^k} = -h^k Dm_f - fDm_{h^k}, \end{aligned}$$

where we use k copies of h in the first line. In particular, we have $0 = (h^k Dm_f(1) + fDm_{h^k}(1))(p) = D(f)(p)$, because $f(p) = 0$. So since $p \in U$ was arbitrary, $D(f)|_U = 0$. ☐

We have already identified elements of $\mathrm{Diff}_0(E,F)$ with $C^\infty(M)$-linear maps $\Gamma(E) \to \Gamma(F)$, which was not hard to do. With slightly more effort, we can also give a characterisation of differential operators of order 1.

Proposition 3.1.4. *Let M be a smooth manifold and $D \in \mathrm{Diff}(M)$. Then $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if $D \in \mathrm{Diff}_1(M)$ and $D(1) = 0$.*

Proof: Suppose first that D is a derivation. Then we have

$$D(1) = D(1 \cdot 1) = 1 \cdot D(1) + D(1) \cdot 1 = 2D(1),$$

hence $D(1) = 0$. Also, for all $f,g,h \in C^\infty(M)$ we have

$$\begin{aligned} [f,[g,D]](h) &= fgD(h) - fD(gh) - gD(fh) + D(fgh) \\ &= fgD(h) - fD(g)h - fgD(h) - gD(fh) + gD(fh) + fhD(g) = 0, \end{aligned}$$

so $D \in \mathrm{Diff}_1(M)$.

Now let $D \in \mathrm{Diff}_1(M)$ with $D(1) = 0$ be arbitrary. Then for any $f,g \in C^\infty(M)$ we have

$$0 = [f,[g,D]](1) = fgD(1) - fD(g) - gD(f) + D(fg) = -fD(g) - gD(f) + D(fg)$$

i.e. $D(fg) = fD(g) - gD(f)$. This means that D is a derivation. ☐

Lemma 3.1.5. *Let $E,F \to M$ be smooth $\mathbb{K}$-vector bundles, $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Then for every $k \in \mathbb{N}_0$, $\mathrm{Diff}_k(E,F)$ is a $\mathbb{K}$-vector space. $\mathrm{Diff}(E,F)$ is also a $\mathbb{K}$-vector space.*

Proof: $\mathrm{Diff}(E, F)$ is contained in the $\mathbb{K}$-vector space of $\mathbb{K}$-linear maps $\mathrm{Hom}_{\mathbb{K}}(\Gamma(E), \Gamma(F))$, so it suffices to show that $\mathrm{Diff}_k(E, F)$ is closed under addition and multiplication for every $k \in \mathbb{N}_0 \cup \{\infty\}$, where $\mathrm{Diff}_\infty(E, F) := \mathrm{Diff}(E, F)$.

Let $c \in \mathbb{K}$ and $P, Q \in \mathrm{Diff}_k(E, F)$, $k < \infty$. Then $cP + Q$ is again local because $(cP + Q)(e)|_U = cP(e)|_U + Q(e)|_U$ for all open $U \subseteq M$ and $e \in \Gamma(E)$. In addition we have for all $f_0, \ldots, f_k \in C^\infty(M)$:

$$[f_0, [\ldots [f_k, cP + Q] \ldots]] = c[f_0, [\ldots [f_k, P] \ldots]] + [\ldots [f_k, Q] \ldots]] = 0$$

Hence $cP + Q \in \mathrm{Diff}_k(E, F)$. If $P, Q \in \mathrm{Diff}(E, F)$ are arbitrary, then still

$$cP + Q \in \mathrm{Diff}_{\max\{\mathrm{ord}\, P, \mathrm{ord}\, Q\}}(E, F) \subseteq \mathrm{Diff}(E, F)$$

by what was just shown. So indeed all sets $\mathrm{Diff}_k(E, F)$, $k \in \mathbb{N}_0 \cup \{\infty\}$, are subspaces of $\mathrm{Hom}_{\mathbb{K}}(\Gamma(E), \Gamma(F))$. ☐

At this point it may be worth noticing that $\mathrm{Diff}_k(E, F) \subseteq \mathrm{Diff}_l(E, F)$ for $k \leq l$ and that $\mathrm{Diff}(E, F) = \bigcup_{k \in \mathbb{N}_0} \mathrm{Diff}_k(E, F)$, which follows directly from the definition of differential operators.

An advantage in the language of differential operators compared to vector fields is that differential operators can not only be added and scaled, but also composed to get new differential operators. The proof is a little more complex than the last lemma, but still quite straightforward.

Lemma 3.1.6. *(compare* [**10**]*, page 19)*
Let $E, F, G \to M$ be a smooth $\mathbb{K}$-vector bundles. Let $P \in \mathrm{Diff}_k(E, F)$ and $Q \in \mathrm{Diff}_l(F, G)$. Then $Q \circ P \in \mathrm{Diff}_{k+l}(E, G)$.

Proof: First of all, notice that concatenations of local maps are local: For $\sigma, \tau \in \Gamma(E)$ with $\sigma|_U = \tau|_U$ we have $P(\sigma)|_U = P(\tau)|_U$ since P is local, hence $QP(\sigma)|_U = QP(\tau)|_U$ since Q is also local.

The main part of the lemma is proven by induction. Let $f_1, \ldots, f_{k+l+1} \in C^\infty(M)$. Let $t \in \{0, \ldots, k + l\}$ and assume that

$$[f_t, [\ldots [f_1, QP] \ldots]] = \sum_{i=1}^{n_t} [g_1^i, [\ldots [g_{r_i}^i, Q] \ldots]][h_1^i, [\ldots [h_{s_i}^i, P] \ldots]]$$

is a sum where the summands are products of r_i- and s_i-fold brackets such that $r_i + s_i = t$ for all i. Then for $t + 1$ we have

$$\begin{aligned}
[f_{t+1}, [f_t, [\ldots, f_1, QP] \ldots]]] &= \sum_{i=1}^{n_t} [f_{t+1}, [g_1^i, [\ldots [g_{r_i}^i, Q] \ldots]] \circ [h_1, [\ldots [h_{s_i}^i, P] \ldots]]]] \\
&= \sum_{i=1}^{n_t} [f_{t+1}, [g_1^i, [\ldots [g_{r_i}^i, Q] \ldots]]] \circ [h_1^i, [\ldots [h_{s_i}^i, P] \ldots]]] \\
&+ [g_1^i, [\ldots [g_{r_i}^i, Q] \ldots]] \circ [f_{t+1}, [h_1^i, [\ldots [h_{s_i}^i, P] \ldots]]]] \\
&= \sum_{i=1}^{n_{t+1}} [\tilde{g}_1^i, [\ldots [\tilde{g}_{\tilde{r}_i}^i, Q] \ldots]][\tilde{h}_1^i, [\ldots [\tilde{h}_{\tilde{s}_i}, P] \ldots]],
\end{aligned}$$

where

(1) $n_{t+1} = 2n_t$
(2) $\tilde{r}_i = r_i + 1$, $\tilde{s}_i = s_i$, $\tilde{g}_1^i = f_{t+1}$, $\tilde{g}_p^i = g_{p-1}^i$ for $p > 1$ and $\tilde{h}_p^i = h_p^i$ while $i \in \{1, \ldots, n_t\}$
(3) $\tilde{r}_i = r_{i-n_t}$, $\tilde{s}_i = s_{i-n_t} + 1$, $\tilde{g}_p^i = g_p^{i-n_t}$, $\tilde{h}_1^i = f_{t+1}$ and $\tilde{h}_p^i = h_{p-1}^{i-n_t}$ for $p > 1$ while $i \in \{n_t + 1, \ldots, 2n_t\}$.

Of course we have $\tilde{r}_i + \tilde{s}_i = r_j + s_j + 1 = t + 1$, so the induction step is complete.

The start of the induction is trivial for $t = 0$: $QP = QP$ suffices to fulfil the requirements, so the induction is complete.

By this induction, we know that $[f_{k+l+1}, [\dots[f_1, QP]\dots]] = \sum_{i=1}^n R_i$, where for each i, R_i is a product $R_i = [g_1, [\dots[g_r, Q]\dots]][h_1, [\dots[h_s, P]\dots]]$ with $r + s = k + l + 1$. In any case we have either $r \geq l + 1$ or $s \geq k + 1$. So because $Q \in \mathrm{Diff}_l(F, G)$ and $P \in \mathrm{Diff}_k(E, F)$, one of the factors must be zero, and hence $R_i = 0$. So in summary, $[f_{k+l+1}, [\dots[f_1, QP]\dots]] = \sum_{i=1}^n R_i = 0$. Since the functions f_i were arbitrary, this means that $QP \in \mathrm{Diff}_{k+l}(E, G)$. $\square$

As one may have expected, the last two lemmas can be combined to obtain an algebra of differential operators.

PROPOSITION 3.1.7. *Let $E \to M$ be a smooth $\mathbb{K}$-vector bundle. Then the set $\mathrm{Diff}(E)$ is an (associative) $\mathbb{K}$-algebra. It is a* filtered algebra *with the filtration*

$$\mathrm{Diff}(E) = \bigcup_{n\in\mathbb{N}_0} \mathrm{Diff}_n(E)$$

by order.

Proof: We use the vector space structure from Lemma 3.1.5 and composition as multiplication, which is well-defined by Lemma 3.1.6 since every differential operator has an order. It was proven there that $\mathrm{Diff}_k(E) \circ \mathrm{Diff}_l(E) \subseteq \mathrm{Diff}_{k+l}(E)$. Associativity is clear for all kinds of compositions, so we only have to check the law of distributivity.

Let $P, Q, R \in \mathrm{Diff}(E)$ and $c, d \in \mathbb{K}$. Then we have $P(Q + R) = PQ + PR$ because P is additive, $(P + Q)R = PR + QR$ holds for any kind of maps into a vector space, and $(cP)(dQ) = (cd)(PQ)$ hold because P is $\mathbb{K}$-multiplicative. $\square$

DEFINITION 3.1.8. Let $E \to M$ be a smooth vector bundle. The graded algebra of the filtered algebra $\mathrm{Diff}(E)$ is defined as follows: For every $n \in \mathbb{N}_0$, set

$$\mathcal{A}_n = \mathcal{A}_n(E) := \mathrm{Diff}_n(E)/\mathrm{Diff}_{n-1}(E),$$

with $\mathrm{Diff}_{-1}(E) := \{0\}$, with the inherited vector space structure. Define then $\mathcal{A}(E) := \bigoplus_{n\in\mathbb{N}_0} \mathcal{A}_n$. $\mathcal{A}_n(E)$ gets the algebra structure defined by

$$(P + \mathrm{Diff}_{k-1}(E)) \cdot (Q + \mathrm{Diff}_{l-1}(E)) := P \circ Q + \mathrm{Diff}_{k+l-1} \in \mathcal{A}_{k+l}$$

for all $(P + \mathrm{Diff}_{k-1}(E)) \in \mathcal{A}_k$, $(Q + \mathrm{Diff}_{l-1}(E)) \in \mathcal{A}_l$, which is well-defined because

$$(P + \mathrm{Diff}_{k-l}(E)) \circ \mathrm{Diff}_{l-1}(E) \subseteq \mathrm{Diff}_{k+l-1}(E)$$

and

$$\mathrm{Diff}_{k-1}(E) \circ (Q + \mathrm{Diff}_{l-1}(E)) \subseteq \mathrm{Diff}_{k+l-1}(E).$$

Associativity and distributivity are inherited from $\mathrm{Diff}(E)$.

DEFINITION 3.1.9. Let E, M as before. The *canonical involution* $^* : \mathcal{A}(E) \to \mathcal{A}(E)$ is defined by $[D]^* := (-1)^k[D]$ for homogeneous $[D] \in \mathcal{A}_k(E)$, extended additively.

Obviously, * is $\mathbb{R}$-linear on every component and we have $([D]^*)^* = (-1)^{2k}[D] = [D]$, so the canonical involution is indeed a self-inverse linear map. In addition we have $([P][Q])^* = (-1)^{k+l}[P][Q] = [P]^*[Q]^*$ for all $[P] \in \mathcal{A}_k$, $[Q] \in \mathcal{A}_l$, so it is even an algebra homomorphism.

3.2. Function-Valued Differential Operators and *-Structures

An important kind of mathematical object in the investigation of operators between Hilbert spaces are *-algebras. Most basically, the usual association of an adjoint makes $\mathbb{B}(\mathcal{H})$ a *-algebra ($\mathcal{H}$ is any Hilbert space here). Our goal to characterise representations of algebras of differential operators on Hilbert spaces makes it necessary to investigate

*-structures on $\mathrm{Diff}(M)$ first. The results from this section are closely related to the basic divergence formula proven in Theorem 2.1.6.

Definition 3.2.1. Let $E \to M$ be a smooth $\mathbb{K}$-vector bundle, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{D} \subseteq \mathrm{Diff}(E)$ a unital subalgebra. A *-structure on $\mathcal{D}$ is an involutive unital local algebra antihomomorphism $^* : \mathcal{D} \to \mathcal{D}$, $D \mapsto D^*$. More precisely, this means that * is a map which fulfils

(1) $(P+Q)^* = P^* + Q^*$
(2) $(PQ)^* = Q^* P^*$
(3) $1^* = 1$ ($1 = m_{\mathrm{const}_1} \in \mathrm{Diff}(E)$ is the multiplication operator given by the constant function 1).
(4) $(D^*)^* = D$
(5) $(cD)^* = \bar{c} D^*$ and
(6) $P^*|_U = Q^*|_U$ if $P|_U = Q|_U$

for all $P, Q, D \in \mathcal{D}$ and $c \in \mathbb{K}$.

A *-structure * on $\mathcal{D}$ is called *filtered* if it fulfils $D - (-1)^k D^* \in \mathrm{Diff}_{k-1}$ for all $D \in \mathrm{Diff}_k(E)$, $k \in \mathbb{N}_0$ and *weakly filtered* if this is true for $k \in \{0,1\}$, i.e. $(D + D^*) \in \mathrm{Diff}_0(E)$ for all $D \in \mathcal{D} \cap \mathrm{Diff}^1(E)$ and $D^* = D$ for $D \in \mathrm{Diff}_0(E) \cap \mathcal{D}$.

The name *filtered* for this kind of *-structure was chosen because filtered *-structures are exactly those which descend to a *-structure on the graded algebra $\mathcal{A}(M)$ given by the filtration. The proof requires a lemma on the generation of $\mathrm{Diff}(M)$ and is given a few statements later.

The main point of this section is to show a correspondence between *-structures on $\mathrm{Diff}(M)$ and volume forms on M, given some technical requirements. Because this correspondence is only obvious for Lie operators (we will define $\mathcal{L}_X^* = -\mathcal{L}_X - \mathrm{div}(X)$ as in Theorem 2.1.6), we have to show first that $\mathrm{Diff}(M)$ is actually generated by Lie operators and smooth functions. Then we can derive the *-structure on general operators from its value on the generators.

First we introduce some notation for multi-indices:

Definition 3.2.2. A (natural) multi-index is an n-tuple $I = (i_1, \dots, i_n) \in \mathbb{N}_0^n$ for some $n \in \mathbb{N}$. For such multi-indices $I, J \in N_0^n$ I write:

(1) $|I| := \sum_{k=1}^n i_k$,
(2) $J \leq I$ if $j_k \leq i_k$ for all $k \in \{1, \dots, n\}$,
(3) $I - J := (i_1 - j_1, \dots, i_n - j_n)$ (for $J \leq I$), and
(4) $\binom{I}{J} := \prod_{k=1}^n \binom{i_k}{j_k}$.

Let now M be a smooth manifold and (U, ϕ) a smooth chart of M. Then for any multi-index $I \in \mathbb{N}_0^m$, where $m = \dim M$, I write

(1) $\phi^I := \prod_{k=1}^m (\phi^k)^{i_k} \in C^\infty(M)$ and
(2) $\partial_I^\phi := \mathcal{L}_{\partial_1^\phi}^{i_1} \circ \cdots \circ \mathcal{L}_{\partial_m^\phi}^{i_m} \in \mathrm{Diff}(U)$, with the special case of
(3) $\partial_I := \partial_I^{\mathrm{id}_{\mathbb{R}}}$ on $\mathbb{R}^m$.

In the next step, we quickly prove that applying a differential operator preserves roots of functions, provided they have multiplicity greater than the order of the operator. More precisely:

Lemma 3.2.3. *Let M be smooth manifold and $D \in \mathrm{Diff}_k(M)$. Let $p \in M$ and $f_0, \dots, f_k \in C^\infty(M)$ with $f_i(p) = 0$ for all i. Then also $D(\prod_{i=0}^k f_i)(p) = 0$.*

Proof: By the defining property of differential operators and the assumption $f_i(p) = 0$ we have

$$\begin{aligned} 0 = [f_0, [\dots [f_k, D] \dots]](1)(p) &= \sum_{I \subseteq \{0,\dots,k\}} (-1)^{k+1-|I|} \prod_{i \in I} f_i(p) D(\prod_{j \in I^C} f_j)(p) \\ &= (-1)^{k+1} D(\prod_{i=0}^{k} f_i)(p), \end{aligned}$$

which is the required result up to sign. Here we use the notation $I^C = \{0, \dots, k\} \setminus I$ for the complement of the set I. □

This can be used to derive a coordinate representation of arbitrary differential operators.

LEMMA 3.2.4. *Let M be smooth manifold of dimension $\dim M = m$ and $D \in \mathrm{Diff}_k(M)$. Let (U, ϕ) be a smooth chart of M. Then locally over U we have:*

$$D|_U = \sum_{I,J \in \mathbb{N}^m,\, |I| \leq k,\, J \leq I} (-1)^{|I-J|} \frac{\binom{I}{J}}{|I|!} \phi^{I-J} D(\phi^J) \partial_I^\phi$$

Proof: This lemma is a consequence of Taylor's theorem. With the version stated in [**11**], page 648, Theorem C.15, we first deduce a manifold version of Taylor's theorem. Namely let $f \in C^\infty(M)$, then $f|_U = f\phi^{-1}\phi$, and $f\phi^{-1} \in C^\infty(\phi U)$ has domain $\phi(U) \subseteq \mathbb{R}^m$. So let $a \in U$ and $\tilde{V} \subseteq \tilde{U} := \phi(U)$ be a convex subset with $b := \phi(a) \in \tilde{V}$. Set $V := \phi^{-1}\tilde{V} \subseteq U$. By the version of Taylor's theorem in [**11**], we have for all $x \in V$ (using $y := \phi(x)$):

$$\begin{aligned} f(x) = f\phi^{-1}(y) =& \sum_{n=0}^{k} \frac{1}{n!} \sum_{I \in \mathbb{N}_0^m, |I|=n} \partial_I (f \circ \phi)(b)(y-b)^I \\ &+ \frac{1}{k!} \sum_{|I|=k+1} (y-b)^I \int_0^1 (1-t)^k \partial_I (f\phi^{-1})(b + t(y-b)) \mathrm{d}t \\ =& \sum_{n=0}^{k} \frac{1}{n!} \sum_{|I|=n} \partial_I^\phi f(a)(\phi x - \phi a)^I \\ &+ \frac{1}{k!} \sum_{|I|=k+1} (\phi x - \phi a)^I \int_0^1 (1-t)^k \partial_I^\phi f(\phi a + t(\phi x - \phi a)) \mathrm{d}t \end{aligned}$$

The two summands are called the (k-th order) Taylor polynomial

$$P_k^\phi = \sum_{|I| \leq k} \frac{1}{|I|!} \partial_I^\phi f(a)(\phi - \phi a)^I$$

and remainder term

$$R_k^\phi = \frac{1}{k!} \sum_{|I|=k+1} (\phi - \phi a)^I \int_0^1 (1-t)^k \partial_I^\phi f(\phi a + t(\phi - \phi a)) \mathrm{d}t$$

of f under ϕ.

Note that each summand of the remainder term is a product of $k+1$ smooth functions on M vanishing at a, namely $f_0 = (\phi^i - \phi^i a) \int_0^1 (1-t)^k \partial_I^\phi f(\phi a + t(\phi - \phi a)) \mathrm{d}t$ for some index i and multi-index I and $f_l = \phi^j - \phi^j a$ for some index j if $l > 0$. So by Lemma 3.2.3, $D(R_k^\phi)(p) = 0$.

Hence using once more the locality of D as well as some combinatorics, we get:

$$
\begin{aligned}
D(f)(a) = D(P_k^\phi)(a) &= \sum_{|I|\leq n} \frac{\partial_I^\phi f(a)}{|I|!} D((\phi - \phi a)^I)(a) \\
&= \sum_{|I|\leq k} \frac{\partial_I^\phi f(a)}{|I|!} D\left(\sum_{J\leq I} (-1)^{|I-J|} \binom{I}{J} \phi^{I-J}(a)\phi^J\right)(a) \\
&= \left(\sum_{|I|\leq k, J\leq I} (-1)^{|I-J|} \frac{\binom{I}{J}}{|I|!} \phi^{I-J} D(\phi^J)\partial_I^\phi\right)(f)(a)
\end{aligned}
$$

Since $f \in C^\infty(M)$ and $a \in U$ were arbitrary, this implies the desired result. $\square$

Using this coordinate representation, it is not hard to see how differential operators are generated by low orders.

LEMMA 3.2.5. *Let M be a smooth manifold. Then $\operatorname{Diff}_1(M) = \{m_f + \mathcal{L}_X | f \in C^\infty(M), X \in \mathfrak{X}(M)\}$ and $\operatorname{Diff}(M)$ is locally generated by $\operatorname{Diff}_1(M)$, i.e. every point $p \in M$ has an open neighbourhood $U \subseteq M$ such that $\operatorname{Diff}(U)$ is generated as an $\mathbb{R}$-algebra by $\operatorname{Diff}_1(U)$.*

More precisely: Each $p \in M$ has an open neighbourhood U such that for each operator $D \in \operatorname{Diff}_k(U)$, $k \geq 1$ there are $n \in \mathbb{N}$ and $D_{1,1}, \ldots, D_{n,k} \in \operatorname{Diff}_1(U)$ such that $D = \sum_{i=1}^n \prod_{j=1}^k D_{i,j}$ (with at most k factors of order 1 in each summand).

Proof: We prove the second part first, which is a direct consequence of the previous lemma. Namely let $p \in M$. Choose a smooth chart (U, ϕ) around p. Let $D \in \operatorname{Diff}_k(U)$, $k \in \mathbb{N}_0$ be arbitrary. (U, ϕ) is also a smooth chart of the manifold U, so by Lemma 3.2.4 we have

$$
\begin{aligned}
D &= \sum_{|I|\leq k, J\leq I} (-1)^{|I-J|} \frac{\binom{I}{J}}{|I|!} \phi^{I-J} D(\phi^J)\partial_I^\phi \\
&= \sum_{|I|\leq k, J\leq I} (-1)^{|I-J|} \frac{\binom{I}{J}}{|I|!} m_{\phi^{I-J}} m_{D(\phi^J)} \mathcal{L}_{\partial_1^\phi}^{i_1} \ldots \mathcal{L}_{\partial_m^\phi}^{i_m},
\end{aligned}
$$

which is a sum of products of multiplication operators and Lie operators, which are both clearly contained in $\operatorname{Diff}_1(U)$. This proves the local generation part. Note that we have at most k factors from $\operatorname{Diff}_1(U)$ in each summand, namely $(-1)^{|I-J|} \frac{\binom{I}{J}}{|I|!} m_{\phi^{I-J}} m_{D(\phi^J)} \mathcal{L}_{\partial_j^\phi}$, where j is chosen minimally such that $i_j \neq 0$, and $i_1 + \cdots + i_j - 1 + \cdots + i_m = |I| - 1 \leq k - 1$ partial derivatives; for $|I| = 0$, simply take the single factor $\frac{\binom{I}{J}}{|I|!} m_{\phi^{I-J}} m_{D(\phi^J)}$.

Note that for the case of $k = 1$, the summands are only products of multiplication operators and up to one Lie operator. So because $m_f m_g = m_{fg}$ and $m_f \mathcal{L}_X = \mathcal{L}_{fX}$ holds for all $f, g \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, using bump functions we deduce that any $D \in \operatorname{Diff}_1(M)$ is a locally finite sum

$$
D = \sum_{i\in I} m_{f_i} + \mathcal{L}_{X_i} = m_f + \mathcal{L}_X
$$

for $f := \sum_{i\in I} f_i$ and $X := \sum_{i\in I} X_i$. $\square$

The fact that differential operators are generated by vector fields and functions has direct consequences for the graded algebra of function-valued differential operators.

PROPOSITION 3.2.6. *For every smooth manifold M, the graded algebra $\mathcal{A}(M) := \mathcal{A}(\bigwedge^0 T^*M)$ is commutative.*

Proof: For all $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ we have $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} \in \mathrm{Diff}_1(M)$, hence $\mathcal{L}_X\mathcal{L}_Y + \mathrm{Diff}_1(M) = \mathcal{L}_Y\mathcal{L}_X + \mathrm{Diff}_1(M)$, $\mathcal{L}_X m_f + \mathrm{Diff}_0(M) = m_f\mathcal{L}_X + m_{X(f)} + \mathrm{Diff}_0(M) = m_f\mathcal{L}_X + \mathrm{Diff}_0(M)$ and $m_f m_g = m_g m_f$ by default.

Now we know by the previous lemma that the whole of $\mathrm{Diff}(M)$ is generated by operators of the form $\mathcal{L}_X$ and m_f for $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$. This implies directly that $\mathcal{A}(M)$ is generated by the equivalence classes of such in $\mathcal{A}_1$ and $\mathcal{A}_0$, respectively. But if the generators commute, all elements have to commute. □

This proposition tells us that the canonical involution on $\mathcal{A}(M)$ is an algebra antihomomorphism (which is the same as a homomorphism in the commutative case), which we would expect from a *-structure.

It is now time to show the background of the name *filtered* for certain *-structures.

Theorem 3.2.7. *For any smooth manifold M with a *-structure $*$ on $\mathrm{Diff}(M)$, the following statements are equivalent:*

(1) $*$ is weakly filtered.
(2) $*$ is filtered.
(3) $*$ descends to the canonical involution on $\mathcal{A}(M)$, i.e. $[D^*] = [D]^*$ holds for all $D \in \mathrm{Diff}(M)$.

Proof: I will show the implications (1) ⇒ (3), (3) ⇒ (2) and (2) ⇒ (1):

(1) ⇒ (3): For any operator $D \in \mathrm{Diff}_1(M)$ we know that $D - D^* \in \mathrm{Diff}_0(M)$, so $D^* = -D - m_f$ for some $f \in C^\infty(M)$. Hence in the graded algebra we get $[D^*] = [-D - m_f] = -[D] = [D]^*$. Also, $[m_f^*] = [m_f] = [m_f]^*$ is clear for all $f \in C^\infty(M)$.

Now let $D \in \mathrm{Diff}_k(M)$, $k > 1$. Then by Lemma 3.2.5, there is an open cover $\mathcal{U}$ of M with operators $D_{U,1,1}, \ldots, D_{U,n,k} \in \mathrm{Diff}_1(U)$ such that $D|_U = \sum_{i=1}^n D_{U,i,1} \ldots D_{U,i,k}$ for all $U \in \mathcal{U}$. So for all $U \in \mathcal{U}$ we have

$$\begin{aligned}
[D^*]|_U &= \left[\left(\sum_{i=1}^n D_{U,i,1} \ldots D_{U,i,k}\right)^*\right] = \left[\sum_{i=1}^n D^*_{U,i,k} \ldots D^*_{U,i,1}\right] \\
&= \sum_{i=1}^n [D^*_{U,i,k}] \ldots [D^*_{U,i,1}] = \sum_{i=1}^n [D_{U,i,k}]^* \ldots [D_{U,i,1}]^* \\
&= \left[\sum_{i=1}^n D_{U,i,1} \ldots D_{U,i,k}\right]^* = [D|_U]^* = [D]^*|_U,
\end{aligned}$$

hence $[D^*] = [D]^*$ on the whole of M.

(3) ⇒ (2): Let $D \in \mathrm{Diff}_k(M)$. Then by assumption we have $[D - (-1)^k D^*] = [D] - (-1)^k[D]^* = [D] - (-1)^{2k}[D] = [0]$, i.e. $D - (-1)^k D^* \in \mathrm{Diff}_{k-1}(M)$. Since D and k were arbitrary, $*$ is filtered.

(2) ⇒ (1): Trivial; (2) implies (1) a fortiori. □

We will soon proceed with the main theorem of this section. The proof requires one more rather technical assumption, which is described in the next definition. But first, let us look at an example:

Example 3.2.8. Consider the smooth manifold $M = \mathbb{R}^m$ and the real differential operators on it. The vector fields on the real plane have a global $C^\infty(M)$-basis $\{\partial_1, \partial_2, \ldots, \partial_m\}$. So by the generation lemma 3.2.5, the linear independence of $\partial_\alpha, \partial_\beta$ for $\alpha \neq \beta \in \mathbb{N}_0^m$ and the fact that $[\partial_i, \partial_j] = 0$, we know that $\mathrm{Diff}(\mathbb{R}^m)$ is a free $C^\infty(\mathbb{R}^m)$-module, namely $\mathrm{Diff}(\mathbb{R}^m) = \{p(\partial_1, \ldots, \partial_m) | p \in C^\infty(\mathbb{R}^m)[T_1, \ldots, T_m]\}$ (however, we do not always have $p(\partial_1, \ldots, \partial_m)q(\partial_1, \ldots, \partial_m) = (pq)(\partial_1, \ldots, \partial_m)$, so this association is not a homomorphism). Hence for every m-tuple $(P_1, \ldots, P_m) \in \mathrm{Diff}(\mathbb{R}^m)^m$ of differential operators, we can define an $\mathbb{R}$-linear map by

$$^* : \mathrm{Diff}(\mathbb{R}^m) \to \mathrm{Diff}(\mathbb{R}^m), \ \sum_{|\alpha|\leq k} a_\alpha \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m} \mapsto \sum_{|\alpha|\leq k} P_m^{\alpha_m} \dots P_1^{\alpha_1} m_{a_\alpha}.$$

This formula is of course a necessity if we want to have an involution with $\partial_i^* = P_i$ and $m_a^* = m_a$ for $a \in C^\infty(M)$. This is also local: If $D = \sum_\alpha a_\alpha \partial^\alpha$, $D' = \sum_\alpha a'_\alpha \partial^\alpha$ and $D|_U = D'|_U$, then $a_\alpha|_U = a'_\alpha|_U$ for all α, and hence $D^*|_U = (D')^*|_U$ because the P_i are local themselves.

This map * is involutive exactly if $(\partial_i^*)^* = P_i^* = \partial_i$ for all $i \in \{1, \dots, m\}$. A further characterisation of this property in the general case is hard because $C^\infty(M)$ does contain zero divisors.

In the case that we are specifically looking for weakly filtered *-structures, we have to assume that $\partial_i^* + \partial_i \in \mathrm{Diff}_0(\mathbb{R}^m)$ and $m_h^* - m_h \in \mathrm{Diff}_{-1}(\mathbb{R}^m) = \{0\}$ for $h \in C^\infty(M)$, i.e. $\partial_i^* = P_i = -\partial_i - g_i$ for all i and $m_h^* = m_h$ for certain $g_i \in C^\infty(\mathbb{R}^m)$ and all $h \in C^\infty(\mathbb{R}^m)$. In that case, $P_i^* = (-\partial_i - g_i)^* = -(-\partial_i - g_i) - g_i = \partial_i$.

Given such P_i, a necessary condition to generate an actual *-structure is that $P_iP_j = (\partial_i\partial_j)^* = (\partial_j\partial_i)^* = P_jP_i$, i.e. $[P_i, P_j] = 0$ for all i. This condition computes as

$$\begin{aligned} 0 = [P_i, P_j] &= [\partial_i + g_i, \partial_j + g_j] = [\partial_i, \partial_j] + [g_i, \partial_j] + [\partial_i, g_j] + [g_i, g_j] \\ &= g_i\partial_j - \partial_j g_i + \partial_i g_j - g_j \partial_i \\ &= g_i\partial_j - g_i\partial_j - \partial_j(g_i) + g_j\partial_i + \partial_i(g_j) - g_j\partial_i = \partial_i(g_j) - \partial_j(g_i). \end{aligned}$$

It turns out that this is also sufficient for * to be anti-multiplicative. Proving this requires a series of computations. First, let $i \in \{1, \dots, m\}$, $k \in \mathbb{N}_0$ and $f \in C^\infty(M)$. I will just write f for the multiplication operator $m_f \in \mathrm{Diff}_0(M)$. With this we have

$$\begin{aligned} \partial_i^k m_f &= \partial_i^{k-1}\partial_i m_f = \partial_i^{k-1}(f\partial_i + \partial_i(f)) \\ &= \partial_i^{k-2}\partial_i(f\partial_i + \partial_i(f)) = \partial_i^{k-2}(f\partial_i^2 + \partial_i(f)\partial_i + \partial_i^2(f) + \partial_i(f)\partial_i \\ &= \partial_i^{k-2}(\partial_i^2(f) + 2\partial_i(f)\partial_i + f\partial_i^2) = \dots = \sum_{t=0}^{k} \binom{k}{t} \partial_i^t(f)\partial_i^{k-t}. \end{aligned}$$

Applying this to a multi-index $\alpha \in \mathbb{N}_0^m$ we get

$$\begin{aligned} \partial_\alpha f &= \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m} f \\ &= \partial_1^{\alpha_1} \dots \partial_{m-1}^{\alpha_{m-1}} \sum_{i_m=0}^{\alpha_m} \binom{\alpha_m}{i_m} \partial_m^{i_m}(f)\partial_m^{\alpha_m - i_m} = \dots \\ &= \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_m=0}^{\alpha_m} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_m}{i_m} \partial_1^{i_1} \dots \partial_m^{i_m}(f) \partial_1^{\alpha_1 - i_1} \dots \partial_m^{\alpha_m - i_m} \\ &= \sum_{I \leq \alpha} \binom{\alpha}{I} \partial_I(f)\partial_{\alpha - I}. \end{aligned}$$

Now I introduce the notation $(-\partial - g)^\alpha := (-\partial_1 - g_1)^{\alpha_1} \dots (-\partial_m - g_m)^{\alpha_m}$. This is justified in the sense that $(-\partial - g)^\alpha(-\partial - g)^\beta = (-\partial - g)^{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{N}_0^m$ because $[-\partial_i - g_i, -\partial_j - g_j] = 0$ by assumption. We do another computation similar to what has been done before: First, we have $(-\partial_i - g_i)f = -\partial_i(f) - f\partial_i - fg_i = f(-\partial_i - g_i) - \partial_i(f)$, i.e. $f(-\partial_i - g_i) = (-\partial_i - g_i)f + \partial_i(f)$ for all $i \in \{1, \dots, m\}$ and $f \in C^\infty(M)$. It follows that

$$\begin{aligned} f(-\partial_i - g_i)^k &= (\partial_i(f) + (-\partial_i - g_i)f)(-\partial_i - g_i)^{k-1} \\ &= (\partial_i^2(f) + 2(-\partial_i - g_i)\partial_i(f) + (-\partial_i - g_i)^2 f)(-\partial_i - g_i)^k \\ &= \cdots = \sum_{j=0}^{k} \binom{k}{j} (-\partial_i - g_i)^j \partial_i^{k-j}(f) \end{aligned}$$

and hence

$$f(-\partial - g)^\alpha = \sum_{I \leq \alpha} \binom{\alpha}{I} (-\partial - g)^I \partial_{\alpha - I}(f) = \sum_{I \leq \alpha} \binom{\alpha}{I} (-\partial - g)^{\alpha - I} \partial_I(f).$$

Consider any $P = \sum_{|\alpha| \leq k} a_\alpha \partial_\alpha$, $Q = \sum_{|\beta| \leq k} b_\beta \partial_\beta$. Using both sides we can finally compute

$$\begin{aligned} (PQ)^* &= \left(\Big(\sum_{|\alpha| \leq k} a_\alpha \partial_\alpha \Big) \Big(\sum_{|\beta| \leq l} b_\beta \partial_\beta \Big) \right)^* \\ &= \left(\sum_{\alpha, \beta} a\alpha \partial_\alpha b_\beta \partial_\beta \right)^* = \left(\sum_{\alpha, \beta, I \leq \alpha} \binom{\alpha}{I} a_\alpha \partial_I(b_\beta) \partial_{\alpha - I} \partial_\beta \right)^* \\ &= \sum_{\alpha, \beta, I \leq \alpha} \binom{\alpha}{I} (-\partial - g)^{\alpha + \beta - I} a_\alpha \partial_I(b_\beta) \\ &= \sum_{\alpha, \beta} (-\partial - g)^\beta \sum_{I \leq \alpha} (-\partial - g)^{\alpha - I} \partial_I(b_\beta) a_\alpha \\ &= \sum_{\alpha, \beta} (-\partial - g)^\beta b_\beta (-\partial - g)^\alpha a_\alpha = Q^* P^* \end{aligned}$$

The condition that $\partial_i^* = -\partial_i - g_i$ for some $g_i \in C^\infty(M)$ is also sufficient for the resulting *-structure to be weakly filtered because in the graded algebra we have

$$\begin{aligned} \left[(\sum_{i=1}^{m} f_i \partial_i)^* \right] &= \sum_i [-\partial_i - g_i][f_i] \\ &= \sum_i -[\partial_i][f_i] = \sum_i [\partial_i]^* [f_i]^* = [\sum_i f_i \partial_i]^* \in \mathcal{A}_1(\mathbb{R}^m) \end{aligned}$$

for all $f_1, \dots, f_m \in C^\infty(M)$, i.e. $(\sum_i f_i \partial_i)^* \in (\sum_i f_i \partial_i) + \mathrm{Diff}_0(\mathbb{R}^m)$.

So indeed for every m-tuple $(g_1, \dots, g_m)$ of smooth functions with $\partial_i(g_j) = \partial_j(g_i)$ for all $i, j \in \{1, \dots, m\}$, there is a weakly filtered *-structure $* = *_{g_1, \dots, g_m}$ on $\mathrm{Diff}(\mathbb{R}^m)$ given by

$$\left(\sum_{|\alpha| \leq k} a_\alpha \partial_\alpha \right)^* = \sum_{|\alpha| \leq k} (-\partial - g)^\alpha a_\alpha,$$

and every possible weakly filtered *-structure on $\mathbb{R}^m$ is given in this way.

Now, what is the technical property mentioned before? The definition looks as follows.

Definition 3.2.9. A *-structure * on $\mathrm{Diff}(M)$ is called *commutative derivation preserving* if for all $P, Q \in \mathrm{Diff}_1(M)$ with $P(1) = Q(1) = 0$ and $[P, Q] = 0$, also

$$PQ^*(1) = QP^*(1)$$

is fulfilled.

The name *commutative derivation preserving* for such a *-structure is to be understood in the following sense: If $*$ is also filtered, then for all commuting derivations $P, Q \in \mathrm{Diff}(M)$, $PQ^* - QP^* = PQ^* - (PQ^*)^*$ is a derivation again.

Before the main theorem of this section, let us consider the example again and check when the extra property applies.

EXAMPLE 3.2.10. Consider the previous example 3.2.8 with the *-structure given by

$$\left(\sum_{|\alpha|\leq k} a_\alpha \partial_\alpha\right)^* = \sum_{|\alpha|\leq k} (-\partial - g)^\alpha m_\alpha$$

again.

Then since clearly ∂_i and ∂_j are commuting derivations for all $i, j \in \{1, \ldots, m\}$, it is necessary for $*_{f,g}$ to be commutative derivation preserving that $\partial_i(g_j) = -\partial_i \partial_j^*(1) = -\partial_j \partial_i^*(1) = \partial_j(g_i)$.

This is the same condition we already had found necessary for the map of the previous example to be anti-multiplicative.

We will now prove an important theorem for characterising *-structures on $\mathrm{Diff}(M)$ for the special case where $M \subseteq \mathbb{R}^m$ is a simply connected open subset. In short, they are all given by volume forms. Given an abstract *-structure *, we will construct a volume form such that the formal adjoint under * and the adjoint in the L^2-space of the volume form are formally equal. This comes with a caveat: I will not investigate the domain of adjoints in the following theorem, all results only hold on the space of smooth functions.

PROPOSITION 3.2.11. *Let M be a smooth manifold of dimension m which is diffeomorphic to a simply connected open subset of $\mathbb{R}^m$. Let * be a weakly filtered, commutative derivation preserving *-structure on* $\mathrm{Diff}(M)$*.*

Then there is a smooth volume form ω on M such that

$$\mathrm{ins} : \mathrm{Diff}(M) \to \mathcal{O}(L^2(M, \omega)),\ D \mapsto (C^\infty(M), D),$$

*is a unital *-homomorphism, using the *-structure $(C^\infty(M), D)^* = (C^\infty(M), (D^*)|_{C^\infty M})$ in $\mathcal{O}(L^2(M, \omega)$, the restriction of the usual adjoint to the smooth functions.*

Furthermore, any other $\eta \in \Omega^m(M)$ such that $\mathrm{ins} : \mathrm{Diff}(M) \to \mathcal{O}(L^2(M, \eta))$ *is a unital *-homomorphism is given as $\eta = c\omega$ for a constant $c \in \mathbb{R} \setminus \{0\}$.*

Proof: For each $X \in \mathfrak{X}(M)$, define $\mathrm{div}(X) := -(\mathcal{L}_X + \mathcal{L}_X^*)(1) = -\mathcal{L}_X^*(1)$. Then for all $f \in C^\infty(M)$ we have

$$\begin{aligned}\mathrm{div}(fX) &= -\mathcal{L}_{fX}^*(1) = -(m_f \circ \mathcal{L}_X)^*(1) = -\mathcal{L}_X^* \circ m_f(1) = -\mathcal{L}_X^*(f)\\ &= -f\mathcal{L}_X^*(1) + \mathcal{L}_X(f) = f\,\mathrm{div}(X) + X(f),\end{aligned}$$

using that $(\mathcal{L}_X + \mathcal{L}_X^*)(f) = f(\mathcal{L}_X + \mathcal{L}_X^*)(1) = f\mathcal{L}_X^*(1)$ and that m_f is an order 0 operator and hence self-adjoint under the given *-structure. Because $\mathcal{L}_{X+Y} = \mathcal{L}_X + \mathcal{L}_Y$ holds for all $X, Y \in \mathfrak{X}(M)$ and * is additive, div is also additive.

By assumption, there is a smooth global chart $\phi : M \to \tilde{U} \subseteq \mathbb{R}^m$. Using this, define

$$\theta_\phi := \sum_{i=1}^m \mathrm{div}(\partial_i^\phi)\mathrm{d}\phi^i \in \Omega^1(M).$$

For all $i, j \in \{1, \ldots, m\}$ we have

$$0 = \mathcal{L}_{[\partial_i^\phi, \partial_j^\phi]} = [\mathcal{L}_{\partial_i^\phi}, \mathcal{L}_{\partial_j^\phi}],$$

as well as $\mathcal{L}_{\partial_i^\phi}(1) = 0 = \mathcal{L}_{\partial_j^\phi}(1)$. So because $*$ was assumed to be commutative derivation preserving, we get

$$\partial_i^\phi \,\mathrm{div}\, \partial_j^\phi = \mathcal{L}_{\partial_i^\phi} \mathcal{L}_{\partial_j^\phi}^*(1) = \mathcal{L}_{\partial_j^\phi} \mathcal{L}_{\partial_i^\phi}^*(1) = \partial_j^\phi \,\mathrm{div}\, \partial_i^\phi.$$

Hence we have

$$\begin{aligned}
\mathrm{d}\theta_\phi &= \sum_{i=1}^m \mathrm{d}\operatorname{div}(\partial_i^\phi)\mathrm{d}\phi^i \\
&= \sum_{i,j=1}^m \partial_j^\phi \operatorname{div}(\partial_i^\phi)\mathrm{d}\phi^j\mathrm{d}\phi^i \\
&= \sum_{i<j}(\partial_i^\phi \operatorname{div}\partial_j^\phi - \partial_j^\phi \operatorname{div}\partial_i^\phi)\mathrm{d}\phi^i\mathrm{d}\phi^j = 0
\end{aligned}$$

Now because M is simply connected, we have $\mathrm{H}^1_\mathrm{d}(M) = \{0\}$ for the degree 1 de Rham cohomology, so $\mathrm{d}\theta = 0$ implies that $\theta = \mathrm{d}g_\phi$ for some $g_\phi \in C^\infty(U)$. Set $f_\phi := \exp \circ g_\phi$, then $\mathrm{d}f_\phi = \exp' \circ g_\phi \mathrm{d}g_\phi = f_\phi \mathrm{d}g_\phi = f_\phi\theta$, i.e. $\partial_i^\phi(f_\phi) = f_\phi \operatorname{div}(\partial_i^\phi)$ holds for all $i \in \{1, \dots, m\}$.

Set $\omega = \omega_\phi := f_\phi \mathrm{d}\phi^1 \dots \mathrm{d}\phi^m$. Then we have

$$\begin{aligned}
\mathcal{L}_{\partial_i^\phi}\omega_\phi &= \mathcal{L}_{\partial_i^\phi}(f_\phi)\mathrm{d}\phi^1 \dots \mathrm{d}\phi^m + f_\phi \mathcal{L}_{\partial_i^\phi}(\mathrm{d}\phi^1 \dots \mathrm{d}\phi^m) \\
&= \partial_i^\phi(f_\phi)\mathrm{d}\phi^1 \dots \mathrm{d}\phi^m \\
&= f_\phi \operatorname{div}(\partial_i^\phi)\mathrm{d}\phi^1 \dots \mathrm{d}\phi^m = \operatorname{div}(\partial_i^\phi)\omega_\phi
\end{aligned}$$

for all $i \in \{1, \dots, m\}$. It follows that for all $f \in C^\infty(M)$,

$$\begin{aligned}
\mathcal{L}_{f\partial_i^\phi}\omega_\phi &= f\mathcal{L}_{\partial_i^\phi}\omega_\phi + (-1)^{i+1} f_\phi \partial_j^\phi(f)\mathrm{d}\phi^j\mathrm{d}\phi^1 \dots \widehat{\mathrm{d}\phi^i} \dots \mathrm{d}\phi^m \\
&= f\mathcal{L}_{\partial_i^\phi}\omega_\phi + \partial_i^\phi(f) f_\phi \mathrm{d}\phi^1 \dots \mathrm{d}_m^\phi \\
&= (f \operatorname{div}(\partial_i^\phi) + \partial_i^\phi(f))\omega_\phi = \operatorname{div}(f\partial_i^\phi)\omega_\phi,
\end{aligned}$$

so because both $\mathcal{L}$ and div are additive, $\mathcal{L}_X\omega_\phi = \operatorname{div}(X)\omega_\phi$ holds for all $X \in \mathfrak{X}(U)$. This means that $\operatorname{div} = \operatorname{div}_\omega$ is the usual divergence defined on the volumetric manifold (M, ω_ϕ). Note also that ω_ϕ is non-vanishing because $\exp > 0$.

Now we have to show that this form ω really preserves the *-structure. So let $X \in \mathfrak{X}(M)$ be arbitrary. Then by the previous results we know that $-\mathcal{L}_X^*(1) = \operatorname{div}_\omega(X)$ holds for the usual divergence defined by the volume form ω on M. Because the *-structure on $\operatorname{Diff}^C(M)$ is relatively skew-adjoint, $\mathcal{L}_X + \mathcal{L}_X^*$ is $C^\infty(M)$-linear, so for any $f \in C^\infty(M)$ we have

$$f \operatorname{div}(X) = f(\mathcal{L}_X + \mathcal{L}_X^*)(1) = (\mathcal{L}_X + \mathcal{L}_X^*)(f),$$

hence

$$\operatorname{ins}(\mathcal{L}_X^*) = \operatorname{ins}(-\mathcal{L}_X - m_{\operatorname{div}(X)}) = \operatorname{ins}(-\mathcal{L}_X - m_{\operatorname{div}_\omega(X)}) = \operatorname{ins}(\mathcal{L}_X)^*,$$

because the latter two terms coincide on the dense subspace $C^\infty(M) \subseteq L^2(M)$ by Corollary 2.1.8, as well as $\operatorname{ins}(m_f^*) = \operatorname{ins}(m_f) = \operatorname{ins}(m_f)^*$. So since $\operatorname{Diff}(M)$ is generated by $\operatorname{Diff}_1(M) = \{m_f + \mathcal{L}_X | f \in C^\infty(M), X \in \mathfrak{X}(M)\}$ as stated in Lemma 3.2.5, this implies $\operatorname{ins}(D)^* = \operatorname{ins}(D^*)$ for all $D \in \operatorname{Diff}(M)$. The map ins is clearly $\mathbb{R}$-linear, so it is indeed a $*$-homomorphism. It is unital because multiplication with the constant function 1 is just the identity.

Now suppose that η is another volume form on M such that $\operatorname{ins} : \operatorname{Diff}(M) \to L^2M$ is a unital *-homomorphism. Then we have

$$\operatorname{div}_\eta X = -\mathcal{L}_X^*(1) = \operatorname{div}_\omega X$$

for all $X \in C^\infty(M)$. Because ω is non-vanishing, there must be a smooth non-vanishing function $f \in C^\infty(M)$ with $\eta = f\omega$. It follows that

$$\operatorname{div}_\eta(X)\eta = \mathcal{L}_X(f\omega) = X(f)\omega + f\mathcal{L}_X\omega = (X(f) + f \operatorname{div}_\omega(X)\omega) = X(f)\omega + \operatorname{div}_\eta(X)\eta,$$

hence $X(f) = 0$ for all $X \in \mathfrak{X}(M)$ (ω is non-vanishing). Because M is connected, this means that f is just a constant $f \equiv c \in \mathbb{R}$, giving $\eta = c\omega$. $c \neq 0$ must hold because f is non-vanishing. $\square$

We can also deduce an analogous result for the whole of $M = \mathbb{R}^m$.

Corollary 3.2.12. *Every weakly filtered *-structure on* $\mathrm{Diff}(\mathbb{R}^m)$ *is given by a volume form.*

Proof: This statement is not a corollary of Theorem 3.2.11 in the strict sense that it follows from its statement, but rather a slight variation of its proof.

By Example 3.2.8 we know that any weakly filtered *-structure * on $\mathrm{Diff}(\mathbb{R}^m)$ is given by $* = *_{g_1,\dots,g_m}$ for certain $g_i \in C^\infty(\mathbb{R}^m)$ with $\partial_i(g_j) = \partial_j(g_i)$ for all i, j. Notice that, with the notation of Theorem 3.2.11, we have $\mathrm{div}(\partial_i) = g_i$ for all i. Furthermore, $(\mathbb{R}^m, \mathrm{id})$ is a smooth chart of $\mathbb{R}^m$ with simply connected domain. So by the proof of Theorem 3.2.11, the volume form $\omega = f\mathrm{d}x_1 \dots \mathrm{d}x_m \in \Omega^m(\mathbb{R}^m)$, where $f = \exp \circ g$, $g \in C^\infty(\mathbb{R}^m)$ with $\mathrm{d}g = \sum_{i=1}^m g_i \mathrm{d}x_i$ fulfils $\mathrm{div}_\omega(\partial_i) = g_i$. Here, the theorem does not use the commutative derivation preservation for all derivations, but only for the coordinate vector field of the given charts, which is fulfilled here by $\partial_i\partial_j^*(1) = -\partial_i(g_j) = -\partial_j(g_i) = \partial_j\partial_i^*(1)$.

So because $\mathrm{Diff}(\mathbb{R}^m)$ is generated by $\partial_1, \dots, \partial_m$, it follows that $\mathcal{L}_X^* = -\mathcal{L}_X - \mathrm{div}_\omega(X)$ for all $X \in \mathfrak{X}(\mathbb{R}^m)$. $\square$

As it turns out, there is another generalisation that we can make. The proof idea was recommended to me by my supervisor and uses results from cohomology theory.

Theorem 3.2.13. *Let M be a compact and simply connected smooth manifold of dimension m and let * be a weakly filtered *-structure on* $\mathrm{Diff}(M)$*. Then there is a volume form $\omega \in \Omega^m(M)$ such that*

$$\mathrm{ins} : \mathrm{Diff}(M) \to \mathcal{O}(L^2(M, \omega)),\ D \mapsto (C^\infty(M), D),$$

*is a unital *-homomorphism, and this is unique up to multiplication with a non-zero real number.*

Proof: Because M is compact, there exists a finite *good cover* $\mathcal{U} = (U_1, \dots, U_n)$ for M; that is, open sets $U_1, \dots, U_n \subseteq M$ with $M = \bigcup_{i=1}^n U_i$ such that for all $I \subseteq \{1, \dots, n\}$, the intersection $\bigcap_{i\in I} U_i$ is diffeomorphic to $\mathbb{R}^m$. The existence of such a cover is proven in [**2**], Theorem 5.1, page 42.

In particular, the sets U_i themselves are diffeomorphic to $\mathbb{R}^m$. By definition, *-structures are local maps. Being filtered is also a local property (because the map $\mathrm{id}\pm^* : \mathrm{Diff}^k(M) \to \mathrm{Diff}^k(M)$ is local for all $k \in \mathbb{N}$), hence * restricts to a filtered *-structure on every open subset of M. By Corollary 3.2.12 (and using a pullback by a diffeomorphism), there is for every $i \in \{1, \dots, n\}$ a volume form $\omega_i \in U_i$ such that $^*|_{U_i} =^*_{\omega_i}$ is the *-structure given by ω_i; that is, $X^* = -X - \mathrm{div}_{\omega_i} X$ holds for all $X \in \mathfrak{X}(U_i)$.

As shown in Proposition 3.2.11, for every non-empty intersection $U_{ij} = U_i \cap U_j$, there must be a scalar $c_{ij} \in \mathbb{R}\setminus\{0\}$ such that $\omega_i|_{U_{ij}} = c_{ij}\omega_j|_{U_{ij}}$. Because M is is simply connected, it is orientable. Hence we can choose an orientation on M and assume every ω_i to be positively oriented, which implies that $c_{ij} > 0$ for all $i, j \in \{1, \dots, n\}$. For all i, j, set $d_{ij} := \log(c_{ij}) \in \mathbb{R}$. I claim that the family $d = (d_{ij})_{i,j\in\{1,\dots,n\},U_i\cap U_j\neq\emptyset}$ is a 1-coboundary in the Čech complex associated to the cover $\mathcal{U}$.

To prove this, consider any three indices $i, j, k \in \{1, \dots, n\}$ for which the respective intersection $U = U_{ijk} = U_i \cap U_j \cap U_k$ is non-empty. Then we find that

$$\begin{aligned}\omega_i|_U &= (\omega_i|_{U_{ij}})|_U = (c_{ij}\omega_j|_{U_{ij}})|_U = (c_{ij}\omega_j|_{U_{jk}})|_U \\ &= (c_{ij}c_{jk}\omega_k|_{U_{jk}})|_U = c_{ij}c_{jk}c_{ki}\omega_i|_U\end{aligned}$$

and hence $c_{ij}c_{jk}c_{ki} = 1$. It follows that $d_{ij} - d_{ik} + d_{jk} = \log(c_{ij}) + \log(c_{ki}) + \log(c_{jk}) = \log(c_{ij}c_{jk}c_{ki}) = \log(1) = 0$ because $c_{ki} = \frac{1}{c_{ik}}$. Thus d is a 1-*cocycle*. The rest follows from homotopy. Namely, M was assumed to be simply connected (so it has a trivial fundamental group $\pi_1(M) = \{0\}$). Hence by Corollary 15.4.2, page 140 in [**1**], it also has a trivial grade-1 de Rham cohomology: $H^1_{dR}(M) = \{0\}$. But by the de Rham theorem, there is an isomorphism $\{0\} \cong H^1_{dR}(M) \cong H^1_C(M, \mathcal{U})$ (using that U is a good cover) between the de Rham and Čech cohomology. Thus every 1-cocycle here is a coboundary, in particular, this is true for d.

Hence there is a set of real numbers $\{b_1, \dots, b_n\}$ such that for all $i, j \in \{1, \dots, n\}$ where U_{ij} is non-empty, $d_{ij} = b_i - b_j$. We set $a_i := \exp(b_i)$ and find that $a_i a_j^{-1} = c_{ij}$ for all i, j with intersection.

In the last step, we use that the cover is finite to guarantee existence of a product. For each $i \in \{1, \dots, n\}$, define:

$$\eta_i := \left(\prod_{j \in \{1,\dots,n\} \setminus \{i\}} a_j \right) \omega_i \in \Omega^m(U_i)$$

Consider any $i, j \in \{1, \dots, n\}$ such that $U_{ij} \neq \emptyset$. Then we have:

$$\begin{aligned} \eta_i|_{U_{ij}} &= \left(\prod_{k \in \{1,\dots,n\} \setminus \{i\}} a_k \right) \omega_i|_{U_{ij}} = \left(\prod_{k \in \{1,\dots,n\} \setminus \{i\}} a_k \right) a_i a_j^{-1} \omega_j|_{U_{ij}} \\ &= \left(\prod_{k \in \{1,\dots,n\} \setminus \{j\}} a_k \right) \omega_j|_{U_{ij}} = \eta_j|_{U_{ij}} \end{aligned}$$

Thus we can define $\omega \in \Omega^m(M)$ by $\omega|_{U_i} = \eta_i$ for all $i \in \{1, \dots, n\}$. By construction, we have $X^*|_{U_i} = -X|_{U_i} - \mathrm{div}_{\eta_i}(X|_{U_i}) = (-X - \mathrm{div}_\omega(X))\,|_{U_i}$ for all $i \in \{1, \dots, n\}$ and $X \in \mathfrak{X}(M)$, thus the given *-structure is indeed induced by the volume form ω, as discussed before in the proof of Proposition 3.2.11. The proof of uniqueness is also analogous to the one in Proposition 3.2.11. □

The previous Proposition 3.2.11 uses the technical assumption of a *-structure being *commutative derivation preserving*. This assumption already looks a bit arbitrary, an impression which is amplified by the observation that it is not required on $\mathbb{R}^m$ or in Theorem 3.2.13.

As it turns out, we can actually prove that every *-structure given by a volume form fulfils this extra property, so that it is a necessary assumption. The proof relies on computations within the Euclidean space and on the relation of diffeomorphisms with the divergence operator. We start with the following lemma:

LEMMA 3.2.14. *Let (M, ω) be a volumetric manifold, N another smooth manifold and let $\phi : M \to N$ be a diffeomorphism. Let $X \in \mathfrak{X}(M)$ be a vector field. Set $\phi_* X := T\phi \circ X \circ \phi^{-1} \in \mathfrak{X}(N)$. Then*

$$\mathcal{L}_{\phi_* X}((\phi^{-1})^* \omega) = (\phi^{-1})^* \mathcal{L}_X \omega.$$

Proof: Abbreviate $\eta := (\phi^{-1})^* \omega$. First notice that for $Y_2, \dots, Y_m \in \mathfrak{X}(N)$ we have

$$\begin{aligned} i_{\phi_* X} \eta(Y_2, \dots, Y_m) &= (\phi^{-1})^* \omega(T\phi X \phi^{-1}, Y_2, \dots, Y_m) \\ &= \omega(X, T\phi^{-1} Y_2 \phi, \dots, T\phi^{-1} Y_m \phi) = i_X \omega(\phi_* Y_2, \dots, \phi_* Y_m) \\ &= (\phi^{-1})^* (i_X \omega)(Y_2, \dots, Y_m), \end{aligned}$$

hence $i_{\phi_* X}\eta = (\phi^{-1})^*(i_X\omega)$. So because ω and η are forms of maximal degree, we get

$$\begin{aligned}\mathcal{L}_{\phi_* X}((\phi^{-1})^*\omega) &= \mathrm{d}i_{\phi_* X}((\phi^{-1})^*\omega) = \mathrm{d}((\phi^{-1})^* i_X\omega)\\ &= (\phi^{-1})^*(\mathrm{d}i_X\omega) = (\phi^{-1})^*\mathcal{L}_X\omega.\end{aligned}$$ □

Using this, we can prove a formula about the divergence under the pushforward by a diffeomorphism.

LEMMA 3.2.15. *Let (M,ω) and (N,η) be two volumetric manifolds with a diffeomorphism $\phi : M \to N$. Let $X, Y \in \mathfrak{X}(M)$. Then*

$$\phi_* X(\operatorname{div}_\eta(\phi_* Y)) = \left(\frac{XY(f\phi) - Y(f\phi)X(f\phi}{(f\phi)^2}) + X \operatorname{div}_\omega Y\right) \circ \phi^{-1}$$

holds for a smooth function $f \in C^\infty(N)$.

Proof: Let $m = \dim M$. Because $\operatorname{rk} \bigwedge^m T^*M = 1$, there must be a unique smooth, non-vanishing function $f \in C^\infty(M)$ with $\eta = f(\phi^{-1})^*\omega$. Then by the previous lemma we have

$$\begin{aligned}\mathcal{L}_{\phi_* X}\eta &= \mathcal{L}_{\phi_* X}(f)(\phi^{-1})^*\omega + f\mathcal{L}_{\phi_* X}(\phi^{-1})^*\omega\\ &= X(f\phi)\phi^{-1}(\phi^{-1})^*\omega + f(\phi^{-1})^*\mathcal{L}_X\omega\\ &= X(f\phi)\phi^{-1}(\phi^{-1})^*\omega + f\operatorname{div}_\omega(X)\phi^{-1}(\phi^{-1})^*\omega\\ &= \left(\frac{X(f\phi)\phi^{-1}}{f} + \operatorname{div}_\omega(X)\phi^{-1}\right)\eta,\end{aligned}$$

which implies that

$$\operatorname{div}_\eta(\phi_* X) = \left(\frac{X(f\phi)}{f} + \operatorname{div}_\omega(X)\right) \circ \phi^{-1}.$$

Using the quotient rule we compute finally that

$$\begin{aligned}\phi_* X(\operatorname{div}_\eta(\phi_* Y)) &= X\left(\frac{Y(f\phi)}{f\phi} + \operatorname{div}_\omega(Y)\right)\phi^{-1}\\ &= \left(\frac{XY(f\phi) - Y(f\phi)X(f\phi)}{(f\phi)^2} + X \operatorname{div}_\omega Y\right) \circ \phi^{-1}.\end{aligned}$$ □

In the Euclidean space, we can directly compute divergences and deduce the following fact about divergences of commuting vector fields:

PROPOSITION 3.2.16. *Let $U \subseteq \mathbb{R}^m$ be an open subset, $m \in \mathbb{N}$. Consider the standard volumetric manifold $(U, \mathrm{d}x_1 \ldots \mathrm{d}x_m)$. Let $X, Y \in \mathfrak{X}(U)$ with $[X, Y] = 0$. Then we have*

$$X(\operatorname{div} Y) = Y(\operatorname{div} X).$$

Proof: X and Y can be represented as $X = \sum_{i=1}^m f_i\partial_i$ and $Y = \sum_{i=1}^m g_i\partial_i$ for some smooth $f_i, g_i \in C^\infty(U)$. Then we have

$$0 = [X, Y] = \sum_{i,j=1}^m [f_i\partial_i, g_j\partial_j] = \sum_{i,j=1}^m (f_i\partial_i(g_j) - g_i\partial_i(f_j))\partial_j$$

by assumption, i.e. $\sum_{i=1}^m f_i\partial_i(g_j) = \sum_{i=1}^m g_i\partial_i(f_j)$ holds for all $j \in \{1,\dots,m\}$. Hence we compute:

$$\begin{aligned}
X(\operatorname{div} Y) &= X(\sum_{i=1}^m \partial_i(g_i)) = \sum_{i,j=1}^m f_i\partial_i\partial_j(g_j)\\
&= \sum_{i,j=1}^m f_i\partial_j\partial_i(g_j) = \sum_{i,j=1}^m \partial_j(f_i\partial_i(g_j)) - \partial_j(f_i)\partial_i(g_j)\\
&= \sum_{j=1}^m \partial_j(\sum_{i=1}^m \partial_j(\sum_{i=1}^m f_i\partial_i(g_j)) - \sum_{i=1}^m \partial_j(g_i)\partial_i(f_j)\\
&= \sum_{j=1}^m \partial_j(\sum_{i=1}^m \partial_j(\sum_{i=1}^m g_i\partial_i(f_j)) - \sum_{i=1}^m \partial_j(g_i)\partial_i(f_j)\\
&= \sum_{i,j=1}^m g_i\partial_j\partial_i(f_j) = \sum_{i,j=1}^m g_i\partial_i\partial_j(f_j) = Y(\operatorname{div} X)
\end{aligned}$$ ☐

For arbitrary manifolds, an equivalent result follows using pushforwards.

THEOREM 3.2.17. *Let (M,ω) be any volumetric manifold. Let $X, Y \in \mathfrak{X}(M)$ with $[X,Y] = 0$. Then we have*

$$X(\operatorname{div}_\omega Y) = Y(\operatorname{div}_\omega X).$$

Proof: Set $m := \dim M$. Let $p \in M$ be arbitrary. Let (U,ϕ) be a smooth chart of M around p, with range $\tilde{U} := \phi(U)$. Let $\eta := \mathrm{d}x_1 \dots \mathrm{d}x_m \in \Omega^m(\tilde{U})$. Set $\tilde{X} := \phi_* X|_U = T\phi X \phi^{-1}$ and $\tilde{Y} := \phi_* Y|_U$. We know by [**11**], Corollary 8.31, page 189 that $[\tilde{X},\tilde{Y}] = \phi_*[X,Y]|_U = 0$.

Hence by Proposition 3.2.16 we have

$$\tilde{X}(\operatorname{div}_\eta \tilde{Y}) = \tilde{Y}(\operatorname{div}_\eta \tilde{X}).$$

By Lemma 3.2.15 we get

$$\begin{aligned}
X(\operatorname{div}_\omega Y)|_U &= \phi_*^{-1}\tilde{X}(\operatorname{div}_\omega \phi_*^{-1}\tilde{Y})\\
&= \left(\frac{\tilde{X}\tilde{Y}(f\phi) - \tilde{Y}(f\phi)\tilde{X}(f\phi)}{(f\phi)^2} + \tilde{X}(\operatorname{div}_\eta \tilde{Y})\right) \circ \phi^{-1}\\
&= \left(\frac{\tilde{Y}\tilde{X}(f\phi) - \tilde{X}(f\phi)\tilde{Y}(f\phi)}{(f\phi)^2} + \tilde{Y}(\operatorname{div}_\eta \tilde{X})\right) \circ \phi^{-1}\\
&= Y(\operatorname{div}_\omega X)|_U,
\end{aligned}$$

in particular $X(\operatorname{div}_\omega Y)(p) = Y(\operatorname{div}_\omega X)(p)$. So because $p \in M$ was arbitrary, we have proven $X(\operatorname{div}_\omega Y) = Y(\operatorname{div}_\omega X)$. ☐

As promised, we can now prove that volume forms actually define *-structures with the desired properties.

THEOREM 3.2.18. *Let (M,ω) be a volumetric manifold. Then there is a unique *-structure * on* $\operatorname{Diff}(M)$ *such that*

$$\operatorname{ins} : \operatorname{Diff}(M) \to \mathcal{O}(L^2(M,\omega)),\ D \mapsto (C^\infty M, D),$$

*is a unital *-homomorphism.*

*This *-structure is also weakly filtered and commutative derivation preserving.*

Proof: Any *-structure $*$ on $\operatorname{Diff}(M)$ for which the insertion map ins is a *-homomorphism has to fulfil

$$\mathcal{L}_X^*(f) = \operatorname{ins}(\mathcal{L}_X^*)(f) = \operatorname{ins}(\mathcal{L}_X)^*(f) = -\mathcal{L}_X(f) - \operatorname{div}_\omega(X)f$$

for all $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, i.e. $\mathcal{L}_X^* = -\mathcal{L}_X - \operatorname{div}_\omega(X)$. Also, $m_f^* = m_f$ must hold for smooth functions $f \in C^\infty(M)$ because this holds for multiplication operators on $L^2(M, \omega)$. These two properties already determine uniquely any possible *-structure because $\operatorname{Diff}(M)$ is generated by $\{m_f + \mathcal{L}_X | f \in C^\infty(M), X \in \mathfrak{X}(M)\}$, as proven in Lemma 3.2.5.

Now we still have to prove that this formula actually gives a well-defined *-structure with the required properties. For this, let (U, ϕ) be any smooth chart of M. By Lemma 3.2.4 and the fact that these coordinate differential operators are linearly independent (over $C^\infty(M)$) we know that $\{\partial_I^\phi | I \in \mathbb{N}_0^m\}$ is a $C^\infty(M)$-basis of $\operatorname{Diff}(U)$. Hence

$$^* : \operatorname{Diff}(U) \to \operatorname{Diff}(U), \ \sum_I f_I \partial_I^\phi \mapsto \sum_I (\partial_I^\phi)^* m_{f_I},$$

where $(\partial_I^\phi)^* := (-\partial_m^\phi - \operatorname{div}(\partial_m^\phi))^{i_m} \dots (-\partial_1^\phi - \operatorname{div}(\partial_1^\phi))^{i_1}$, is a well-defined map. Because summation and differential operators are $\mathbb{R}$-linear and local, so is this map $*$. Note also that these definitions make sure that $1^* = (\partial_{(0,\dots,0)}^\phi)^* = 1$ as well as $m_f^* = 1^* m_f = m_f$ for general $f \in C^\infty(M)$.

Let $i, j \in \{1, \dots, m\}$. Then by Theorem 3.2.17 we have

$$\begin{aligned}(\operatorname{div}\partial_i^\phi + \partial_i^\phi)(\operatorname{div}\partial_j^\phi + \partial_j^\phi) &= \partial_i^\phi \partial_j^\phi + \operatorname{div}\partial_i^\phi \partial_j^\phi + \partial_i^\phi m_{\operatorname{div}\partial_j^\phi} + \operatorname{div}\partial_i^\phi \operatorname{div}\partial_j^\phi \\ &= \partial_i^\phi \partial_j^\phi + \operatorname{div}\partial_i^\phi \partial_j^\phi + \operatorname{div}\partial_j^\phi \partial_i^\phi + \partial_i^\phi(\operatorname{div}\partial_j^\phi) + \operatorname{div}\partial_i^\phi \operatorname{div}\partial_j^\phi \\ &= \partial_j^\phi \partial_i^\phi + \operatorname{div}\partial_j^\phi \partial_i^\phi + \operatorname{div}\partial_i^\phi \partial_j^\phi + \partial_j^\phi(\operatorname{div}\partial_i^\phi) + \operatorname{div}\partial_j^\phi \operatorname{div}\partial_i^\phi \\ &= (\operatorname{div}\partial_j^\phi + \partial_j^\phi)(\operatorname{div}\partial_i^\phi + \partial_i^\phi)\end{aligned}$$

Now let $I, J \in \mathbb{N}_0^m$ be two multi-indices. Then using this commutation property we compute by the definition above:

$$\begin{aligned}(\partial_I^\phi \partial_J^\phi)^* =& (\partial_{I+J}^\phi)^* \\ =& (-\partial_m^\phi - \operatorname{div}(\partial_m^\phi))^{i_m+j_m} \dots (-\partial_1^\phi - \operatorname{div}(\partial_1^\phi))^{i_1+j_1} \\ =& (-\partial_m^\phi - \operatorname{div}(\partial_m^\phi))^{j_m} \dots (-\partial_1^\phi - \operatorname{div}(\partial_1^\phi))^{j_1} \\ & \circ (-\partial_m^\phi - \operatorname{div}(\partial_m^\phi))^{i_m} \dots (-\partial_1^\phi - \operatorname{div}(\partial_1^\phi))^{i_1} \\ =& (\partial_J^\phi)^* (\partial_I^\phi)^*\end{aligned}$$

Let now $g \in C^\infty(M)$ and $i \in \{1, \dots, m\}$. Then we have:

$$\begin{aligned}(\partial_i^\phi m_g)^* &= (g\partial_i^\phi + \partial_i^\phi(g))^* \\ &= (\partial_i^\phi)^* m_g + \partial_i^\phi(g) = (-\partial_i^\phi - \operatorname{div}\partial_i^\phi) m_g + \partial_i^\phi(g) \\ &= -g\partial_i^\phi - \partial_i^\phi(g) - g \operatorname{div}\partial_i^\phi + \partial_i^\phi(g) \\ &= g(-\partial_i^\phi - \operatorname{div}\partial_i^\phi) = m_g (\partial_i^\phi)^*\end{aligned}$$

An induction argument hence implies that $(\partial_I^\phi m_g)^* = m_g (\partial_I^\phi)^*$ holds for all $I \in \mathbb{N}_0^m$.

Consider now general $I, J \in \mathbb{N}_0^m$ and $f, g \in C^\infty(U)$. In the next step we compute:

$$\begin{aligned}\left(f\partial_I^\phi g\partial_J^\phi\right)^* &= \left(f\sum_{K\le I}\partial_K^\phi(g)\partial_{I-K}^\phi\partial_J^\phi\right)^* = \sum_{K\le I}\left(\partial_{I-K}^\phi\partial_J^\phi\right)^*\partial_K^\phi(g)m_f\\ &= (\partial_J^\phi)^*\left(\sum_{K\le I}(\partial_{I-K}^\phi)^*\partial_K^\phi(g)\right)m_f\\ &= (\partial_J^\phi)^*\left(\sum_{K\le I}\partial_K^\phi(g)\partial_{I-K}^\phi\right)^* m_f\\ &= (\partial_J^\phi)^*(\partial_I^\phi m_g)^* m_f\\ &= (\partial_J^\phi)^* m_g(\partial_I^\phi)^* m_f = (g\partial_J^\phi)^*(f\partial_I^\phi)^*\end{aligned}$$

Finally we have the most general case of arbitrary $P, Q \in \mathrm{Diff}(U)$. P and Q can be expressed as locally finite sums $P = \sum_I f_I\partial_I^\phi$ and $Q = \sum_J g_J\partial_J^\phi$ for suitable $f_I, g_J \in C^\infty(M)$, $I, J \in \mathbb{N}_0^m$. So it easily follows from linearity and the previous computations that

$$(PQ)^* = \sum_{I,J}(f_I\partial_I^\phi g_I\partial_J^\phi)^* = \sum_{I,J}(g_J\partial_J^\phi)^*(f_I\partial_I^\phi)^* = Q^*P^*$$

just as required.

This anti-multiplicativity makes it easy to show that $*$ is an involution: Because $\mathrm{Diff}(U)$ is generated by multipliers m_f, $f \in C^\infty(M)$ and coordinate operators ∂_i^ϕ, $i \in \{1, \dots, m\}$, it suffices to note that $(m_f^*)^* = m_f^* = m_f$ as well as

$$((\partial_i^\phi)^*)^* = (-\partial_i^\phi - \operatorname{div}\partial_i^\phi)^* = \partial_i^\phi + \operatorname{div}\partial_i^\phi - \operatorname{div}\partial_i^\phi = \partial_i^\phi.$$

So indeed we have defined a *-structure on $\mathrm{Diff}(U)$.

This map * was constructed to fulfil $\mathcal{L}_X^* = -\mathcal{L}_X - \operatorname{div}(X)$ for all $X \in \mathfrak{X}(M)$, which is indeed the case, namely:

$$\begin{aligned}\mathcal{L}_X^* &= \left(\sum_{i=1}^m X(\phi^i)\partial_i^\phi\right)^* = \sum_{i=1}^m(-\partial_i^\phi - \operatorname{div}\partial_i^\phi)X(\phi^i)\\ &= \sum_{i=1}^m -\partial_i^\phi(X(\phi^i)) - X(\phi^i)\partial_i^\phi - X(\phi^i)\operatorname{div}\partial_i^\phi\\ &= \sum_{i=1} m - X(\phi^i)\partial_i^\phi - \operatorname{div}(X(\phi^i)\partial_i^\phi) = -\mathcal{L}_X - \operatorname{div}(X)\end{aligned}$$

It is now time to extend these local *-structures to the whole of M. If (V, ψ) is another smooth chart of M with its corresponding *-structure $\star$ on $\mathrm{Diff}(V)$ defined as before, then by locality, $*|_{U\cap V}$ and $\star|_{U\cap V}$ are both *-structures on $\mathrm{Diff}(U \cap V)$ with $m_f^* = m_f = m_f^\star$ and $\mathcal{L}_X^* = -\mathcal{L}_X - \operatorname{div}(X) = \mathcal{L}_X^\star$ for $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$. By the remark in the beginning of the proof, *-structures with these properties are unique, hence $\star|_{U\cap V} = *|_{U\cap V}$. Hence we can define $* : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ by $D^*|_U = (D|_U)^*$ for any smooth chart domain $U \subseteq M$.

$\mathbb{R}$-linearity, anti-multiplicativity, involutivity and locality itself are all local properties, so $^* : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ inherits them from the locally defined *-structures and is hence itself a *-structure on $\mathrm{Diff}(M)$. It is also clear from the local definitions that $\mathcal{L}_X^* = -\mathcal{L}_X - \operatorname{div}(X)$ for $X \in \mathfrak{X}(M)$ and $m_f^* = m_f$ for $f \in C^\infty(M)$. Because $\mathrm{Diff}(M)$ is generated by $\mathrm{Diff}_1(M)$, this suffices to assure that $\mathrm{ins} : \mathrm{Diff}(M) \to \mathcal{O}(L^2M)$ is indeed a unital *-homomorphism under this *-structure. It was shown to be unique in the beginning.

It is only left to prove that * defined above is also weakly filtered and commutative derivation preserving. The weakly filtered property is clear by sight since $\mathrm{Diff}_0(M) =$

$\{m_f | f \in C^\infty(M)\}$ and $m_f = m_f^*$ for every $f \in C^\infty(M)$ as well as $\mathrm{Diff}_1(M) = \{\mathcal{L}_X + m_f | X \in \mathfrak{X}(M), f \in C^\infty(M)\}$ and $\mathcal{L}_X + m_f + (\mathcal{L}_X + m_f)^* = 2m_f - m_{\mathrm{div}(X)} \in \mathrm{Diff}_0(M)$ for all operators $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$.

The operators $P, Q \in \mathrm{Diff}_1(M)$ with $P(1) = Q(1) = 0$ and $[P, Q] = 0$ are exactly the Lie operators $\mathcal{L}_X, \mathcal{L}_Y$ for which $[X, Y] = 0$. Theorem 3.2.17 directly implies that

$$\mathcal{L}_X \mathcal{L}_Y^*(1) = -X(\mathrm{div}\, Y) = -Y(\mathrm{div}\, X) = \mathcal{L}_Y \mathcal{L}_X^*(1)$$

holds for any such operators. Hence * is also commutative derivation preserving. ☐

This implies that being commutative derivation preserving is automatic for *-structures on $\mathbb{R}^m$.

COROLLARY 3.2.19. *Every weakly filtered *-structure on* $\mathrm{Diff}(\mathbb{R}^m)$ *is commutative derivation preserving.*

Proof: By Corollary 3.2.12, any weakly filtered *-structure * on $\mathrm{Diff}(\mathbb{R}^m)$ is determined by $\mathcal{L}_X^* = -\mathcal{L}_X - \mathrm{div}_\omega(X)$ for some volume form $\omega \in \Omega^m(\mathbb{R}^m)$. This is exactly the *-structure such that $\mathrm{ins} : \mathrm{Diff}(\mathbb{R}^m) \to \mathcal{O}(L^2(\mathbb{R}^m, \omega))$ is a unital *-homomorphism from Theorem 3.2.18, which is commutative derivation preserving by the theorem. ☐

The results of this section can be extended to *invariant* differential operators on Lie groupoids. This will be done in the last section of this chapter. Before investigating these invariant operators, the next section will show how they are useful.

3.3. The Enveloping Algebra of a Lie Algebroid

An important example of Lie algebras are associative $\mathbb{R}$-algebras with the commutator $[x, y] = xy - yx$. It turns out after only a little bit of theory that, in fact, every Lie algebra can be viewed as a subalgebra of one of that kind. An associative algebra in which a Lie algebra can be embedded is then called an enveloping algebra of that Lie algebra, and the *smallest* of all enveloping algebras is called universal. The universal enveloping algebra can be proven to exist by a general construction or found in the form of more natural examples.

In the context of Lie groups, one finds a close connection between the group C^*-algebra and the universal enveloping algebra of its Lie algebra, in particular, regarding their respective representations. This is the reason to be interested in enveloping algebras and may make the reader wonder whether the concept can be generalised to Lie algebroids, as we have generalised the group C^*-algebra to a groupoid C^*-algebra. Again, the answer is yes. The formal definition of enveloping algebras of Lie algebroids is now to follow, using the commutator product mentioned before.

DEFINITION 3.3.1. (compare [**19**], pages 14–15, or [**8**], pages 7–8)
Let $A \to M$ be a Lie algebroid with anchor ρ. An enveloping algebra of A is a triple (E, i_1, i_2), where E is an associative unital $\mathbb{R}$-algebra, $i_1 : C^\infty(M) \to E$ is a unital algebra homomorphism and $i_2 : \Gamma(A) \to E$ is a Lie algebra homomorphism, such that

$$i_1(f) i_2(a) = i_2(fa)$$

and

$$[i_2(a), i_1(f)] = i_1(\rho(a)(f))$$

hold for all $f \in C^\infty(M)$ and $a \in \Gamma(A)$.

A homomorphism of enveloping algebras from (E, i_1, i_2) to (F, j_1, j_2) is a unital algebra homomorphism $\phi : E \to F$ such that $\phi i_1 = j_1$ and $\phi i_2 = j_2$.

Since the identity $\mathrm{id}_E : E \to E$ is trivially a homomorphism of enveloping algebras and $\phi i_t = j_t$, $\psi j_t = k_t$ implies $\psi\phi i_t = \psi j_t = k_t$, $t \in \{1, 2\}$, enveloping algebras of a given Lie algebroid form a category with the usual concatenation of maps as composition.

So what does it mean for an enveloping algebra to be *small*? The answer is given by a basic concept of category theory.

Definition 3.3.2. Let $A \to M$ be a Lie algebroid. An enveloping algebra (E, i_1, i_2) of A is called *universal* if it is initial in the category of enveloping algebras of A, i.e. if for any other enveloping algebra (F, j_1, j_2) of A there is a unique unital algebra homomorphism $\phi : E \to F$ such that $\phi i_1 = j_1$ and $\phi i_2 = j_2$.

When we make up a new definition it is always good to know that it is not a void one. So next I am going to prove that every Lie algebroid has a universal enveloping algebra by an abstract construction. First we need the following lemma:

Lemma 3.3.3. *Let $A \to M$ be a Lie algebroid. On the $C^\infty(M)$-module $V = C^\infty(M) \oplus \Gamma(A)$ define an operation $[\cdot, \cdot]$ by*

$$[f + a, g + b] := \rho(a)(g) - \rho(b)(f) + [a, b]$$

for $f, g \in C^\infty(M)$, $a, b \in \Gamma(A)$. Then V with this operation is a Lie algebra.

Proof: $\mathbb{R}$-bilinearity is clear by (bi-)linearity of vector fields, ρ and the bracket of $\Gamma(A)$. The bracket is also skew-symmetric since $[f + a, f + a] = \rho(a)f - \rho(a)f + [a, a] = 0$ holds for arbitrary $f \in C^\infty(M)$, $a \in \Gamma(A)$. Checking the Jacobi identity is a mere computation. Namely for $f_1, f_2, f_3 \in C^\infty(M)$ and $a_1, a_2, a_3 \in \Gamma(A)$ we have:

$$\begin{aligned}[f_1 + a_1, [f_2 + a_2, f_3 + a_3]] &= [f_1 + a_1, \rho(a_2)f_3 - \rho(a_3)f_2 + [a_2, a_3]] \\ &= -\rho([a_2, a_3])f_1 + \rho(a_1)(\rho(a_2)f_3) - \rho(a_1)(\rho(a_3)f_2) + [a_1, [a_2, a_3]]\end{aligned}$$

Doing the same with cyclic permutations of the indices and using the Jacobi identity for $\Gamma(A)$, we get

$$\begin{aligned}&[f_1 + a_1, [f_2 + a_2, f_3 + a_3]] + [f_2 + a_2, [f_3 + a_3, f_1 + a_1]] + [f_3 + a_3, [f_1 + a_1, f_2 + a_2]] \\ &\quad = -\rho([a_2, a_3])f_1 + \rho(a_1)(\rho(a_2)f_3) - \rho(a_1)(\rho(a_3)f_2) + [a_1, [a_2, a_3]] \\ &\qquad -\rho([a_3, a_1])f_2 + \rho(a_2)(\rho(a_3)f_1) - \rho(a_2)(\rho(a_1)f_3) + [a_2, [a_3, a_1]] \\ &\qquad -\rho([a_1, a_2])f_3 + \rho(a_3)(\rho(a_1)f_2) - \rho(a_3)(\rho(a_2)f_1) + [a_3, [a_1, a_2]] \\ &\quad = -\rho([a_2, a_3])f_1 + [\rho(a_1), \rho(a_2)]f_3 - \rho([a_3, a_1])f_2 \\ &\qquad + [\rho(a_2), \rho(a_3)]f_1 - \rho([a_1, a_2])f_3 + [\rho(a_3), \rho(a_1)]f_2 = 0\end{aligned}$$

since $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism. □

After this, our theorem still requires a bit of effort, but the construction is rather intuitive.

Theorem 3.3.4. *Let $A \to M$ be a Lie algebroid. Then A has a universal enveloping algebra, and this is unique up to isomorphism.*

Proof: Since universal enveloping algebras were defined as initial objects, the uniqueness part follows from general category theory. The existence part is proven by an explicit construction.

Consider first the $C^\infty(M)$-module $V = C^\infty(M) \oplus \Gamma(A)$. This is a Lie algebra by Lemma 3.3.3, using the bracket defined there. Let $W := T(V) = \sum_{n=0}^\infty T^n_{\mathbb{R}}(V)$ be the $\mathbb{R}$-tensor algebra of V. Define

$$\begin{aligned}I := \langle &\{fg - f \otimes g | f, g \in C^\infty(M)\} \cup \{\mathrm{const}_r \oplus 0 - r | r \in \mathbb{R}\} \\ &\cup \{fa - f \otimes a | f \in C^\infty(M), a \in \Gamma(A)\} \cup \{x \otimes y - y \otimes x - [x, y] | x, y \in C^\infty(M) \oplus \Gamma(A)\}\rangle\end{aligned}$$

to be the ideal generated by the four sets written above. Here, $\mathrm{const}_r \in C^\infty(M)$ is the constantly r-valued function. Set $U = U(A) := W/I$. Intuitively, two maps are defined: $i_1 : C^\infty(M) \to U,\ f \mapsto f \oplus 0 + I$ and $i_2 : \Gamma(A) \to U,\ a \mapsto 0 \oplus a + I$.

Now we have to show that (U, i_1, i_2) is indeed a universal enveloping algebra for A. I will first prove that it is an enveloping algebra at all. The fact that i_1 and i_2 are homomorphisms follows right from the definition of I, namely we have $i_1(f)i_1(g) = f \otimes g + I = fg + I$,

$i_1(\mathrm{const}_1) = \mathrm{const}_1 + I = 1 + I$ and $[i_2(a), i_2(b)] = ab - ba + I = [0 \oplus a, 0 \oplus b] + I = [a, b] + I$ for all $f, g \in C^\infty(M)$, $a, b \in \Gamma(A)$. In addition we get $i_1(f)i_2(a) = f \otimes a + I = fa + I = i_2(fa)$ and $[i_2(a), i_1(f)] = a \otimes f - f \otimes a + I = [a, f] + I = \rho(a)(f) + I = i_1(\rho(a)(f))$. So (U, i_1, i_2) is an enveloping algebra of A.

Let (Z, k_1, k_2) be another enveloping algebra of A. Suppose $\tau : (U, i_1, i_2) \to (Z, k_1, k_2)$ is a homomorphism of enveloping algebras. Then for all $f \in C^\infty(M)$, $a \in \Gamma(A)$ we have $\tau(f \oplus a + I) = \tau(i_1(f) + i_2(f)) = k_1(f) + k_2(f)$. Since $W = T(V)$ is generated by elements of the form $f \oplus a$ as a unital algebra, U is generated by elements of the form $f \oplus a + I$, hence τ is completely determined be the formula above. So there can be at most one homomorphism of enveloping algebras $\tau : (U, i_1, i_2) \to (Z, k_1, k_2)$.

In fact the above formula can as well be used to define a homomorphism. We just have to check that it is well-defined and has the required properties. So define $\tilde{\tau} : W \to Z$ by $\tilde{\tau}(f \oplus a) := k_1(f) + k_2(a)$, extended to a unital algebra homomorphism, which is possible since k_1 and k_2 are $\mathbb{R}$-linear. Let $r \in \mathbb{R}$, $f, g \in C^\infty(M)$ and $a, b \in \Gamma(A)$, with $x := f \oplus a$, $y := g \oplus b$. Then we have:

(1) $\tilde{\tau}(\mathrm{const}_r) = k_1(\mathrm{const}_r) = rk_1(\mathrm{const}_1) = r \cdot 1 = r\tilde{\tau}(1) = \tilde{\tau}(r)$,
(2) $\tilde{\tau}(f \otimes g) = \tilde{\tau}(f)\tilde{\tau}(g) = k_1(f)k_1(g) = k_1(fg) = \tilde{\tau}(fg)$,
(3) $\tilde{\tau}(f \otimes a) = k_1(f)k_2(a) = k_2(fa) = \tilde{\tau}(fa)$,

and finally,

$$\begin{aligned}
\tilde{\tau}(x \otimes y - y \otimes x) =& (k_1(f) + k_2(a))(k_1(g) + k_2(b)) - (k_1(g) + k_2(b))(k_1(f) + k_2(a)) \\
=& k_1(f)k_1(g) - k_1(g)k_1(f) + k_2(a)k_1(g) - k_1(g)k_2(a) - k_2(b)k_1(f) \\
&+ k_1(f)k_2(b) + k_2(a)k_2(b) - k_2(b)k_2(a) \\
=& k_1(fg - gf) + [k_2(a), k_1(g)] - [k_2(b), k_1(f)] + [k_2(a), k_2(b)] \\
=& k_1(\rho(a)(g)) - k_1(\rho(b)(f)) + k_2[a, b] \\
=& \tilde{\tau}([f \oplus a, g \oplus b]) = \tilde{\tau}([x, y]).
\end{aligned}$$

So $\tilde{\tau}|_I \equiv 0$, hence $\tau : U \to Z$, $u + I \mapsto \tilde{\tau}(u)$ is a well-defined unital algebra homomorphism.

By construction we also have $\tau \circ i_1(f) = \tau(f + I) = k_1(f)$ and $\tau \circ i_2(a) = \tau(a + I) = k_2(a)$, hence τ is even a homomorphism of enveloping algebras.

We have hereby shown that there is a unique enveloping algebra homomorphism $\tau : (U, i_1, i_2) \to (Z, k_1, k_2)$, where the latter one was arbitrary. Hence (U, i_1, i_2) is indeed initial, i.e. a universal enveloping algebra. □

While the abstractly constructed universal enveloping algebra is well-suited to satisfy our curiosity, it stays of limited use in further proofs. What will really inspire further theories is the fact that the left-invariant (or right-invariant) differential operators on a Lie groupoid are an explicit realisation of the universal enveloping algebra. The actual proof is a bit tricky and was completed by other authors already, whom I will refer to instead of repeating the whole process.

What I will present are concrete definitions and examples regarding the construction of an enveloping algebra structure on $\mathrm{Diff}^L(G)$. Firstly, what even is a left-invariant operator in the formal sense?

Definition 3.3.5. Let $G \rightrightarrows M$ be a Lie groupoid. A differential operator $D \in \mathrm{Diff}(G)$ is called *tangent to the target fibres* if for all $p \in M$ and all $f_1, f_2 \in C^\infty(G)$ with $f_1|_{G^p} = f_2|_{G^p}$, also $D(f_1)|_{G^p} = D(f_2)|_{G^p}$ (so D restricts to a well-defined differential operator $D_p = D|_{G^p} : C^\infty(G^p) \to C^\infty(G^p)$ for all $p \in M$).

A differential operator $D \in \mathrm{Diff}(G)$ is called left-invariant if it is tangent to the target fibres and for all $g \in G$ and $f \in C^\infty(G)$, $D_{s(g)}(f \circ l_g) = D_{t(g)}(f) \circ l_g$. The set of left-invariant differential operators on G is denoted $\mathrm{Diff}^L(G) \subseteq \mathrm{Diff}(G)$.

Analogously, a differential operator $D \in \mathrm{Diff}(G)$ is called *tangent to the source fibres* if for all $p \in M$ and all $f_1, f_2 \in C^\infty(G)$ with $f_1|_{G_p} = f_2|_{G_p}$, also $D(f_1)|_{G_p} = D(f_2)|_{G_p}$. D is called right-invariant if it is tangent to the source fibres and $D_p(f \circ r_g) = D_{s(g)}(f) \circ r_g$ for all $f \in C^\infty(G)$ and $g \in G$, where $p = t(g)$ and $D_p = D|_{G_p}$. The set of right-invariant differential operators on G is denoted $\mathrm{Diff}^R(G)$.

From now on, I may use the same symbol for the differential operator $D \in \mathrm{Diff}^L(G)$ and its restrictions D_p to G^p, and likewise for $D \in \mathrm{Diff}^R(G)$ and $D|_{G_p}$.

The following three propositions can be proven in complete analogy for right-invariant differential operators. Choosing one or the other is a matter of readability in a given context or simply of taste.

As for general differential operators, the most basic example are multiplication operators. In this case the functions we multiply with have to be left-invariant themselves. One way to obtain such functions is to concatenate functions on the base manifold with the source map.

Example 3.3.6. Let $G \rightrightarrows M$ be a Lie groupoid and $\alpha \in C^\infty(M)$. Then the multiplication operator $m_\alpha := m_{\alpha\circ s} : C^\infty(G) \to C^\infty(G),\ h \mapsto fh$ is a left-invariant differential operator.

Proof: For $f_1, f_2 \in C^\infty(M)$ with $f_1(g) = f_2(g)$ for some $g \in G$ we have $m_\alpha(f_1)(g) = \alpha(sg)f_1(g) = \alpha(g)f_2(g) = m_\alpha(h_2)(g)$, hence m_f is clearly tangent to the target fibres.

For arbitrary $f \in C^\infty(G)$ and $g, h \in G$ with $s(g) = t(h)$ we have

$$m_\alpha(f \circ l_g)(h) = \alpha(s(h))f(gh) = \alpha(s(gh))f(gh) = m_\alpha(f) \circ l_g(h)$$

because $s(gh) = s(h)$, hence m_α is indeed left-invariant. □

Of course the left-invariant differential operators should be closed under addition and multiplication. This is quickly proven in the following lemma.

Lemma 3.3.7. $\mathrm{Diff}^L(G) \subseteq \mathrm{Diff}(G)$ *is a unital subalgebra.*

Proof: Let $D, E \in \mathrm{Diff}^L(G)$. Let $p \in M$ and $g \in G$. Then for any $f, f_1, f_2 \in C^\infty(G)$ with $f_1|_{G^p} = f_2|_{G^p}$ we have $E(f_1)|_{G^p} = E(f_2)|_{G^p}$ because E is tangent to the target fibres, hence $D(E(f_1))|_{G^p} = D(E(f_2))|_{G^p}$ because D is tangent to the target fibres, so DE is tangent to the target fibres, too.

Also, we have $DE(f \circ l_g) = D(E(f) \circ l_g) = DE(f) \circ l_g$, so $DE \in \mathrm{Diff}^L(G)$. The fact that $\mathrm{id}_{C^\infty(G)} \in \mathrm{Diff}^L(G)$ is tautological. □

To every vector field $X \in \mathfrak{X}(G)$ we can associate a differential operator $\mathcal{L}_X \in \mathrm{Diff}(G)$, which we have called Lie operator. It seems very natural to suggest that left-invariance of both X and $\mathcal{L}_X$ are equivalent properties. The proof is more of a routine task, but still worthy of some attention because it has applications in the later constructions.

Lemma 3.3.8. *For any left-invariant vector field* $X \in \mathfrak{X}^L(G)$*,* $\mathcal{L}_X \in \mathrm{Diff}^L(G)$ *is a left-invariant differential operator.*

Proof: Let $X \in \mathfrak{X}^L(G)$ be left-invariant. Let $p \in M$. Let $f_1, f_2 \in C^\infty(G)$ with $f_1|_{G^p} = f_2|_{G^p}$. Then by assumption we have $X_g \in TG^p$ for all $g \in G^p$, so $\mathcal{L}_X(f_1)(g) = X_g(f_1) = X_g(f_1|_{G^p}) = X_g(f_2) = \mathcal{L}_X(f_2)(g)$. Hence $\mathcal{L}_X$ is tangent to the target fibres.

Now let $f \in C^\infty(G)$ and $g \in G$. Then for $h \in G^{s(g)}$ we have $\mathcal{L}_X(f \circ l_g) = T_h l_g X_h(f) = X_{gh}(f) = \mathcal{L}_X(f)(gh) = \mathcal{L}_X(f) \circ l_g(h)$. So indeed $\mathcal{L}_X \in \mathrm{Diff}^L(G)$. □

Both of the previous lemmas are true in a completely similar form for right-invariance.

Let us detail how the invariant differential operators are an enveloping algebra. This step is not very hard to prove.

PROPOSITION 3.3.9. *Let* $G \rightrightarrows M$ *be a Lie groupoid (with* $M \subseteq G$*). Let* $A = \mathrm{Lie}(G)$ *be the corresponding Lie algebroid. Define the maps* $i_1 : C^\infty(M) \to \mathrm{Diff}^L(G)$ $\alpha \mapsto m_\alpha$ *and* $i_2 : \Gamma(A) \to \mathrm{Diff}^L(G),\ a \mapsto \mathcal{L}_{X_a}$*. Then* $(\mathrm{Diff}^L(G), i_1, i_2)$ *is an enveloping algebra of* A*.*

Proof: First of all, the maps i_1 and i_2 are well-defined by Example 3.3.6 and Lemma 3.3.8. The set $\mathrm{Diff}^L(G)$ was shown to be a unital (sub-) algebra in Lemma 3.3.7 and inherits associativity from $\mathrm{Diff}(G)$.

i_1 is a unital algebra homomorphism by $i_1(\alpha)i_2(\alpha) = m_\alpha \circ m_\beta = m_{(\alpha\circ s)(\beta\circ s)} = m_{(\alpha\beta)\circ s} = m_{\alpha\beta} = i_1(\alpha\beta)$ for $\alpha, \beta \in C^\infty(M)$ and $i_1(\mathrm{const}_1) = m_{\mathrm{const}_1} = \mathrm{id} \in \mathrm{Diff}^L(G)$. i_2 is a Lie algebra homomorphism because $i_2([a,b]) = \mathcal{L}_{X_{[a,b]}} = \mathcal{L}_{\sigma\tau[\sigma a,\sigma b]} = \mathcal{L}_{[X_a,X_b]} = [\mathcal{L}_{X_a}, \mathcal{L}_{X_b}] = [i_2(a), i_2(b)]$ for $a, b \in \Gamma(A)$. Here, τ and σ map sections of A to left-invariant vector fields and back as in Proposition 2.3.5.

Furthermore we have $i_1(\alpha)i_2(a) = m_{\alpha\circ s}\mathcal{L}_{X_a} = \mathcal{L}_{\alpha\circ s X_a} = \mathcal{L}_{X_{\alpha a}} = i_2(\alpha a)$ for all $\alpha \in C^\infty(M)$, $a \in \Gamma(A)$. Here we use that σ is a homomorphism of $C^\infty(M)$-modules.

Finally, let $\alpha \in C^\infty(M)$, $\bar{a} \in \Gamma(A)$ and $g \in G$ be arbitrary and $p = s(g)$. Then we have $T_p t(a(p)) = T_p t(X_a(p)) = 0$ by Lemma 2.3.7, hence

$$\begin{aligned} X_{\bar{a}}(\alpha \circ s)(g) &= T_{s(g)} l_g a(s(g))(\alpha \circ s) \\ &= a(p)(\alpha \circ s \circ l_g) = a(p)(\alpha \circ s) = T_p s(a(p))(\alpha) \\ &= (T_p s - T_p t)(a(p))(\alpha) = \rho(\bar{a})(\alpha)(p) = \rho(\bar{a})(\alpha) \circ s(g). \end{aligned}$$

This implies that for $a \in \Gamma(A)$,

$$\begin{aligned} [i_2(a), i_1(\alpha)] &= \mathcal{L}_{X_a} m_{\alpha\circ s} - m_{\alpha\circ s}\mathcal{L}_{X_a} \\ &= m_{\alpha\circ s}\mathcal{L}_{X_a} + m_{X_a(\alpha\circ s)} - m_{\alpha\circ s}\mathcal{L}_{X_a} \\ &= m_{\rho(a)(\alpha)\circ s} = i_1(\rho(a)(\alpha)) \end{aligned}$$

just as required.

So $(\mathrm{Diff}^L(G), i_1, i_2)$ is indeed an enveloping algebra for A. □

For universality, I refer to the following theorem:

THEOREM 3.3.10 ([**19**], page 133). *Let* G *be a Lie groupoid with Lie algebroid* A*. Then the enveloping algebra* $(\mathrm{Diff}^L(G), i_1, i_2)$ *of* A *as in the previous theorem is universal.*

3.4. The *-Algebra of Invariant Differential Operators

Without going into too much detail as to why this is the case at this point, it turns out that a very important tool for our investigations of groupoid representations in the next chapter is a suitable *-algebra structure on the algebra of right-invariant differential operators (which was already mentioned to be isomorphic to the universal enveloping algebra and can hence be used as a perfect substitute).

Constructing this *-structure is what we will do now, starting with the definition of an orientation on a Lie groupoid.

DEFINITION 3.4.1. Let $G \rightrightarrows M$ be a Lie groupoid. A (target fibre) orientation on the Lie groupoid G is a family of orientations $(O_x)_{x\in G}$ on the vector spaces $T_x G^{tx}$ that is continuous in the following sense: For every $x \in G$, there is a smooth local frame $(X_1, \ldots, X_n)$ of $(Tt)^{-1}(TM)$ over a neighbourhood $U \subseteq G$ of x such that $[(X_1(y), \ldots, X_n(y))] \in O_y$, i.e. the ordered basis $(X_1(y), \ldots, X_n(y))$ is positively oriented for all $y \in U$.

An oriented Lie groupoid is a pair $(G \rightrightarrows M, (O_x)_{x\in G})$ of a Lie groupoid and a fixed orientation.

As usual, preferring target over source fibres is an arbitrary choice. There is a similar definition for source fibres. I will state it now, only to shortly prove that both are equivalent.

DEFINITION 3.4.2. Let $G \rightrightarrows M$ be a Lie groupoid. A source fibre orientation on G is a family of orientations $(O_x)_{x\in G}$ on the vector spaces T_xG_{sx} such that for all $x \in G$, there is a smooth local frame $(X_1, \dots, X_n)$ of $(Ts)^{-1}(TM)$ over a neighbourhood $U \subseteq G$ of x such that $(X_1(y), \dots, X_n(y)) \in O_y$ is positively oriented for all $y \in U$.

Proving the equivalence is an easy task using the inverse map $\mathrm{inv} : G \to G$. The following proof and others to follow rely on the fact that this map is a diffeomorphism.

PROPOSITION 3.4.3. *Let $G \rightrightarrows M$ be a Lie groupoid. Then the following statements hold:*

- *If $O = (O_x)_{x\in G}$ is a target fibre orientation on G, then*
$$T\mathrm{inv}(O) := (T_{x^{-1}}\mathrm{inv}(O_{x^{-1}}))_{x\in G}$$
is a source fibre orientation on G.
- *If $P = (P_x)_{x\in G}$ is a source fibre orientation on G, then*
$$T\mathrm{inv}(P) := (T_{x^{-1}}\mathrm{inv}(P_{x^{-1}}))_{x\in G}$$
is a target fibre orientation on G.
- *These associations are inverse to each other, i.e. $T\mathrm{inv}T\mathrm{inv}(O) = O$ and*
$$T\mathrm{inv}T\mathrm{inv}(P) = P.$$

Proof: I only prove one direction here because the other one works completely analogously. So let $(O_z)_{z\in G}$ be a target fibre orientation on G.

Choose $x \in G$. First notice that the inversion map $\mathrm{inv} : G^{tx^{-1}} = G^{sx} \to G_{sx}$ is a diffeomorphism (with the inversion map itself as inverse). So in particular, $T_{x^{-1}}\mathrm{inv} : T_{x^{-1}}G^{tx^{-1}} \to T_xG_{sx}$ is a linear isomorphism. Hence $T_{x^{-1}}\mathrm{inv}O_{x^{-1}}$ is a well-defined orientation on the vector space T_xG_{sx}.

To show continuity, use the defining property of $(O_z)_{z\in G}$ to find a smooth local frame $(X_1, \dots, X_n)$ of $(Tt)^{-1}(TM)$ over a neighbourhood $U \subseteq G$ of x^{-1} such that $(X_1(y), \dots, X_n(y))$ is positively oriented for all $y \in U$. Then because inv is a diffeomorphism,
$$(T\mathrm{inv}X_1, \dots, T\mathrm{inv}X_n)$$
is a smooth local frame of $T\mathrm{inv}(Tt)^{-1}(TM) = T(t \circ \mathrm{inv})^{-1}(TM) = Ts^{-1}TM$ over U^{-1}, which is a neighbourhood of x. By the choice before we have $[(X_1(y), \dots, X_n(y))] \in O_y$ for all $y \in U$, hence $[(T_y\mathrm{inv}X_1(y), \dots, T_y\mathrm{inv}X_n(y))] \in T_y\mathrm{inv}O_y$ (and $\mathrm{inv} : U \to U^{-1}$ is surjective), which makes $(T\mathrm{inv}X_1, \dots, T\mathrm{inv}X_n)$ positively oriented in every fibre.

The inverse part of the proposition is computed in one line:
$$(T\mathrm{inv}T\mathrm{inv}(O))_x = T_{x^{-1}}\mathrm{inv}(T\mathrm{inv}O)_{x^{-1}} = T_{x^{-1}}\mathrm{inv}T_x\mathrm{inv}O_x = T_x\mathrm{inv}^2O_x = T_x\mathrm{id}O_x = O_x$$
holds for all $x \in G$, so $T\mathrm{inv}T\mathrm{inv}(O) = O$. For P, the proof is likewise. □

The reason to define an orientation on a manifold is usually that it allows us to integrate top-degree differential forms. However, the main object we are interested in for the investigation of groupoid algebras are rather smooth functions. As in the raw manifold case, we can integrate them, too, once we have chosen a fixed volume form to work with. We will do this fibrewise, as explained in the following definition.

DEFINITION 3.4.4. Let $(M \rightrightarrows G, (O_x)_{x\in G})$ be an oriented Lie groupoid. Let $n = \mathrm{rk}\, Tt^{-1}TM$ be the dimension of the target fibres (which is constant because the target map is a submersion). A (target) volume form on G is a non-vanishing differential form $\omega \in \Omega^n(G)$ with the following properties:

(1) $\omega(x) \in T^*_xG^{tx}$ for all $x \in G$.
(2) For all $x \in G$, the restricted form $\omega|_{G^{tx}} \in \Omega^n(G^{tx})$ is positive with respect to $(O_y)_{y\in G^{tx}}$.

(3) ω is left-invariant, i.e. for all $x \in G$ we have $l_x^*(\omega|_{G^{tx}}) = \omega|_{G^{sx}}$.

A pair of an oriented Lie groupoid and a volume form on it I will call a *volumetric groupoid*, resembling Definition 2.1.1.

The orientation is usually omitted from the notation; this is justified by more than readability because it could be reconstructed from the volume form by declaring an ordered basis $(v_1, \ldots, v_n)$ of $T_x G^{tx}$ to be positively oriented if and only if $\omega_x(v_1, \ldots, v_n) > 0$.

Again, we could have gone for source target forms instead, which are defined likewise.

Definition 3.4.5. Let $(M \rightrightarrows G, (O_x)_{x \in G})$ be an oriented Lie groupoid with target fibre dimension n. A source volume form on G is a non-vanishing differential form $\omega \in \Omega^n(G)$ with the following properties:

(1) $\omega(x) \in T_x^* G_{sx}$ for all $x \in G$.
(2) For all $x \in G$, $\omega|_{G_{sx}}$ is positive with respect to $(T_{x^{-1}}\mathrm{inv}(O_{x^{-1}}))_{x \in G}$
(3) ω is right-invariant, i.e. $r_x^* \omega|_{G_{sx}} = \omega|_{G_{tx}}$ for all $x \in G$.

Just like orientations, source and target volume forms are equivalent in the following sense:

Proposition 3.4.6. *Let $(M \rightrightarrows G, (O_x)_{x \in G})$ be an oriented Lie groupoid. Then the following statements hold:*

- *If ω is a target volume form on G, then* $\mathrm{inv}^*\omega$ *is a source volume form on G.*
- *If η is a source volume form on G, then* $\mathrm{inv}^*\eta$ *is a target volume form on G.*
- *These associations are inverse to each other, i.e.* $\mathrm{inv}^*\mathrm{inv}^*\omega = \omega$ *and* $\mathrm{inv}^*\mathrm{inv}^*\eta = \eta$.

For the rest of this chapter I will mostly stick to using source volume forms because they work together with right-invariant differential operators. Choosing them instead of left-invariant ones is more compatible with conventions of operator algebra theory.

In previous chapters we have worked with Haar systems on Lie groupoids; in fact most of our groupoid algebra theory is reliant on their usage. So it would clearly be beneficial to connect volume forms into that theory, just like in the manifold case, where every volume form gives a smooth measure. For groupoids we need to assure that the resulting measure is right-invariant in addition. But this is almost trivial because right-invariance (respectively left-invariance) has already been included in the definition of volume forms.

What takes a bit more effort is the verification that the resulting system of measures is really smooth. To connect the different styles of definitions in groupoid and manifold theory we have to prove a handy lemma on rather elementary analysis first. The process investigated there could be called *partial integration*, but with a different meaning than usual.

Lemma 3.4.7. *Let $k, m \in \mathbb{N}_0$, $n = m + k$. Let $W \subseteq \mathbb{R}^n$ be open and precompact and $f \in C^\infty(\overline{W})$ a smooth function (i.e. the restriction of a smooth function on a larger domain). Denote by* $\mathrm{pr}_1 : \mathbb{R}^n \to \mathbb{R}^m$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ *and* $\mathrm{pr}_2 : \mathbb{R}^n \to \mathbb{R}^k$, $(x_1, \ldots, x_n) \mapsto (x_{m+1}, \ldots, x_n)$ *the two projections and set $U := \mathrm{pr}_1(W)$ and $V := \mathrm{pr}_2(W)$. Then the map*

$$F : U \to \mathbb{R},\ (x_1, \ldots, x_m) \mapsto \int_{\mathrm{pr}_2(\mathrm{pr}_1^{-1}(\{(x_1, \ldots, x_m)\}))} f(x_1, \ldots, x_n) \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n$$

is smooth and bounded.

Proof: F is bounded by the mere fact that $\lambda^k(\mathrm{pr}_2(\mathrm{pr}_1^{-1}(\{(x_1, \ldots, x_m)\}))) \leq c_k d^k$, where λ^k is the k-dimensional Lebesgue measure, $d = \sup_{x,y \in W} \|x - y\|$ is the diameter of W and $c_k = \frac{\pi^{n/2}}{\Gamma(\frac{k}{2}+1)}$ is the proportion constant in the hyperball volume formula. By this we have $|F| \leq c_k d^k \sup |f| < \infty$; f is bounded on W because continuous on $\overline{W}$.

The more intricate part of this lemma is showing smoothness. To do this, we first assume that $W = (a_1, b_1) \times \cdots \times (a_n, b_n)$ is just a hyperrectangle. Then we have the simple relations $U = (a_1, b_1) \times \cdots \times (a_m, b_m)$ and $V = (a_{m+1}, b_{m+1}) \times \cdots \times (a_n, b_n) = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(\{(x_1, \ldots, x_m)\})$ for all $(x_1, \ldots, x_m) \in U$. Hence we can write

$$\begin{aligned} F(x_1, \ldots, x_m) &= \int_{(a_{m+1}, b_{m+1}) \times \cdots \times (a_n, b_n)} f(x_1, \ldots, x_n) \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n \\ &= \int_{a_{m+1}}^{b_{m+1}} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) \mathrm{d}x_n \ldots \mathrm{d}x_{m+1} \end{aligned}$$

So by an elementary analysis theorem on commutability of integration and differentiation along intervals (applied multiple times), every partial derivative of F exists and is given by

$$\begin{aligned} \frac{\partial F}{\partial x_i}(x_1, \ldots, x_m) &= \int_{a_{m+1}}^{b_{m+1}} \cdots \int_{a_n}^{b_n} \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \mathrm{d}x_n \ldots \mathrm{d}x_{m+1} \\ &= \int_U \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n. \end{aligned}$$

From here, smoothness of F follows by an inductive argument because all partial derivatives are smooth functions on $\overline{W}$ again.

For the general case, note the following two things: Firstly, the topology on $\mathbb{R}^n$ is separable and generated by open hyperrectangles, and secondly, every union of hyperrectangles can be written as a *disjoint* union of possibly smaller ones. Hence we can write $W = \bigcup_{i \in \mathbb{N}} R_i$ for hyperrectangles $R_i \subseteq W$ with $R_i \cap R_j = \emptyset$ for $i \neq j$.

Now we can use the Dominated Convergence Theorem and the rectangular case to deduce that the partial derivatives of F exist and are given by

$$\begin{aligned} \frac{\partial F}{\partial x_i}(x_1, \ldots, x_m) &= \frac{\partial}{\partial x_i} \sum_{j \in \mathbb{N}} \int_{\mathrm{pr}_2 \mathrm{pr}_1|_{R_j}^{-1}\{(x_1, \ldots, x_m)\}} f(x_1, \ldots, x_n) \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n \\ &= \sum_{j \in \mathbb{N}} \frac{\partial}{\partial x_i} \int_{\mathrm{pr}_2 \mathrm{pr}_1|_{R_j}^{-1}\{(x_1, \ldots, x_m)\}} f(x_1, \ldots, x_n) \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n \\ &= \sum_{j \in \mathbb{N}} \int_{\mathrm{pr}_2 \mathrm{pr}_1|_{R_j}^{-1}\{(x_1, \ldots, x_m)\}} \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n \\ &= \int_{\mathrm{pr}_2 \mathrm{pr}_1|_{W}^{-1}\{(x_1, \ldots, x_m)\}} \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) \mathrm{d}x_{m+1} \ldots \mathrm{d}x_n. \end{aligned}$$

□

All the technicalities aside, the intuitive interpretation of the process in the previous lemma is as follows: We consider a body of finite volume inside the Euclidean space, then we take parallel slices of it and measure the area of each of these slices. We would intuitively expect the function that assigns the area of each slice to its number or distance to the first slice to be continuous, even smooth. Of course the visual imagination falls short in higher dimensions, but this expectation was formally proven to be true in the lemma.

The one point to remember from this intuitive way of thinking is that this slicing process is essentially the same as assigning the target fibre to each base point of a groupoid, just that the space is not necessarily Euclidean any more. But with a few simple pullbacks by charts, the smoothness of a measure system obtained from a volume form becomes a breeze.

Lemma 3.4.8. *Let $(G \rightrightarrows M, \omega)$ be a volumetric groupoid. Then the family $(\mu_{\omega|_{G^p}})_{p \in M}$ of integral measures induced by ω is a smooth Haar system.*

Proof: In the first part of the proof I will show that the family of measures defined here is really left-invariant. To do so, first introduce the notation $\mu^p := \mu_{\omega|_{G^p}}$ for all $p \in M$. Then notice that the volume form ω was assumed to be non-vanishing, hence each induced measure has full support $\operatorname{supp} \mu^p = G^p$.

Now let $f \in C_c(G)$ and $x \in G$ be arbitrary. Then we have

$$\begin{aligned}\int_{G^{sx}} f \circ l_x \mathrm{d}\mu^{sx} &= \int_{G^{sx}} f \circ l_x \omega|_{G^{sx}} = \int_{G^{sx}} l_x^* f l_x^* \omega|_{G^{tx}} \\ &= \int_{G^{sx}} l_x^*(f\omega|_{G^{tx}}) = \int_{G^{tx}} f\omega|_{G^{tx}}\end{aligned}$$

by [**11**], Proposition 16.6 (d). Here we know that the diffeomorphism l_x must be orientation-preserving because both $\int_{G^{tx}} \omega|_{G^{tx}}$ and $\int_{G^{sx}} \omega|_{G^{sx}} = \int_{G^{sx}} l_x^*\omega|_{G^{tx}}$ are greater than 0 by positivity of the restrictions of ω. So the induced family of measures is really left-invariant.

Now I am going to show smoothness. To do so, let $f \in C_c^\infty(G)$ be arbitrary. Choose $x \in G$ and set $p = tx$. Denote the dimensions as $n = \dim G$, $m = \dim M$ and $k = \dim G^p = \dim G - \dim M$. By the definition of Lie groupoids, $t : G \to M$ is a smooth surjective submersion, so the triple (G, M, t) is a fibred manifold. Hence there must be a fibred chart (V, ψ) around x, i.e. a smooth chart (V, ψ) around x together with a smooth chart (U, ϕ) of M around p such that $U = t(V)$ and $\phi \circ t = \mathrm{pr}_1 \circ \psi$. Here we use $\mathrm{pr}_1 : \mathbb{R}^n \to \mathbb{R}^m,\ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$ and $\mathrm{pr}_2 : \mathbb{R}^n \to \mathbb{R}^m\ (x_1, \ldots, x_n) \mapsto (x_{m+1}, \ldots, x_n)$.

For every $p \in U$ we have $\mathrm{pr}_1 \circ \psi|_{G^p} = \phi \circ t|_{G^p} \equiv p$, hence $\psi_p := \mathrm{pr}_2 \circ \psi|_{G^p}$ is a smooth map with range $\tilde{V}^p = \mathrm{pr}_2 \circ \psi(G^p) = \mathrm{pr}_2(\mathrm{pr}_1|_{\psi(V)}^{-1}(\phi(p))$ (which is open by elementary analysis) and inverse $\psi_p^{-1} : \tilde{V}^p \to V^p = V \cap G^p,\ z \mapsto \psi^{-1}(\phi(p), z)$.

Note that for every $y \in V$ with $t(y) = p$ and all $i \in \{1, \ldots, k\}$ we have $\partial_i^{\psi_p}(y) = \frac{\mathrm{d}}{\mathrm{d}r}\psi_p^{-1}(\psi_p(y) + re_i) = \frac{\mathrm{d}}{\mathrm{d}r}\psi^{-1}(\phi(p), \mathrm{pr}_2\psi(y) + re_i) = \frac{\mathrm{d}}{\mathrm{d}r}\psi^{-1}(\psi(y) + re_{i+m}) = \partial_{i+m}^{\psi}(y)$, i.e. $\partial^\psi|_{G^p} = \partial^{\psi_p}$ for all $p \in M$ (in particular, these vector fields are tangent to the target fibres). By the very definition of ϕ_p, we also have $\mathrm{d}\psi^{i+m}|_{G^p} = \mathrm{d}\psi_p^i$ for all i and p.

Now for our volume form ω we know that

$$\omega|_{V^p} = \omega|_{V^p}(\partial_1^{\psi_p}, \ldots, \partial_k^{\psi_p})\mathrm{d}\psi_p^1 \wedge \cdots \wedge \mathrm{d}\psi_p^k = (\omega(\partial_{m+1}^{\psi}, \ldots, \partial_n^{\psi})\mathrm{d}\psi^{m+1} \wedge \cdots \wedge \mathrm{d}\psi^n)|_{V^p}$$

using the formula for arbitrary top-degree forms and charts. Hence

$$\omega|_V = \omega(\partial_{m+1}^{\psi}, \ldots, \partial_n^{\psi})\mathrm{d}\psi^{m+1} \wedge \cdots \wedge \mathrm{d}\psi^n$$

holds for the whole of $V = \bigcup_{p \in U} V_p$.

We use this formula to compute the integrals of $f \in C_c^\infty(M)$. Namely for each $p \in U$ we have:

$$\begin{aligned}\int_{V^p} f\mathrm{d}\mu^p &= \int_{V^p} f\omega|_{G^p} = \int_{V^p} f\omega|_{V^p}(\partial_1^{\psi_p}, \ldots, \partial_k^{\psi_p})\mathrm{d}\psi_p^1 \wedge \cdots \wedge \mathrm{d}\psi_p^k \\ &= \int_{\tilde{V}^p} f\omega(\partial_{m+1}^{\psi}, \ldots, \partial_n^{\psi}) \circ \psi_p^{-1}(x_1, \ldots, x_m)\mathrm{d}x_1 \ldots \mathrm{d}x_k \\ &= \int_{\tilde{V}^p} f\omega(\partial_{m+1}^{\psi}, \ldots, \partial_n^{\psi}) \circ \psi^{-1}(\phi(p), x_{m+1}, \ldots, x_n)\mathrm{d}x_{m+1} \ldots \mathrm{d}x_n \\ &= \int_{\mathrm{pr}_2\mathrm{pr}_1^{-1}\tilde{V}(\phi(p))} f\omega(\partial_{m+1}^{\psi}, \ldots, \partial_n^{\psi}) \circ \psi^{-1}(\phi(p), x_{m+1}, \ldots, x_n)\mathrm{d}x_{m+1} \ldots \mathrm{d}x_n\end{aligned}$$

This depends smoothly on p by Lemma 3.4.7 and because f, $\omega(\partial_{m+1}^{\psi}, \ldots, \partial_n^{\psi})$, ψ and ϕ are smooth. We may assume without loss of generality that V and $\tilde{V} := \psi(V)$ are precompact and that $f\omega(\partial_{m+1}^{\psi}, \ldots, \partial_n^{\psi}) \circ \psi^{-1} \in C^\infty(\overline{\psi(V)})$ by shrinking the chart domain if necessary.

To finish the proof, choose a finite open cover $(V_i)_{i \in I}$ of $\operatorname{supp} f$ by domains of fibred charts as before and a partition of unity $(h_i)_{i \in I}$ subordinate to it. Then $p \mapsto \int_{G^p} f\mathrm{d}\mu^p = \sum_{i \in I} \int_{V_i^p} h_i f\mathrm{d}\mu^p$ must be smooth as the sum of smooth functions. □

Now that I have proven that volumetric groupoids are a special case of Lie groupoids with smooth Haar system, I may use all the definitions and theorems made for them. For example, the convolution algebra $C^*(G)$ of a volumetric groupoid $(G \rightrightarrows M, \omega)$ is defined by the induced Haar system $(\mu_{\omega|_{G^p}})_{p \in M}$.

As said before, the current goal is to define a suitable $*$-structure on $\mathrm{Diff}^R(G)$. We can do this using the tools forged in the very first chapter of the thesis, applying them to each fibre individually. Firstly we do so with the divergence function.

Definition 3.4.9. Let $G \rightrightarrows M$ be a volumetric groupoid with volume form ω. Denote by $\omega^{-1} := \mathrm{inv}^*\eta$ the equivalent source volume form. Denote by $\lambda^p := \mu_{\omega|_{G^p}}$ the induced integral measure on each target fibre, and by $\lambda_p = \lambda^p \circ \mathrm{inv}^{-1}$ the equivalent measure on the source fibre. Let $X \in \mathfrak{X}^R(G)$ be a right-invariant vector field. The *right-divergence* of X with respect to ω is defined as

$$\mathrm{div}^R_\lambda(X) : G \to \mathbb{R},\ g \mapsto \mathrm{div}_{\omega^{-1}|_{G_{sg}}}(X|_{G_{sg}})$$

This function is uniquely determined by the property that

$$\int_{G_p} f \mathrm{d}\mathcal{L}_X \lambda_p = \int_{G_p} f\, \mathrm{div}(X) \mathrm{d}\lambda_p$$

holds for all $f \in C_c^\infty(G)$ and all $p \in G$.

Both indices of the divergence may be omitted if the Haar system and laterality of choice are clear from context.

When we define an adjoint of a right-invariant differential operator using the divergence function, we want this adjoint to be right-invariant, too. The following lemma will be used to show that it really is.

Lemma 3.4.10. *Let G, ω as before and $X \in \mathfrak{X}^R(G)$. Then* $\mathrm{div}(X) \circ r_g = \mathrm{div}(X)|_{G_{tg}}$ *holds for all $g \in G$.*

Proof: We do a short computation involving the Lie derivative of the volume form. Namely we get

$$\begin{aligned}
\mathrm{div}(X) \circ r_g \omega|_{G_{tg}} &= \mathrm{div}(X) \circ r_g r_g^* \omega \\
&= r_g^*(\mathrm{div}(X)\omega) = r_g^*(\mathcal{L}_{X|_{G_{sg}}} \omega|_{G_s g}) \\
&= \mathcal{L}_{Tr_g X|_{G_{sg}}} r_g^* \omega|_{G_{sg}} = \mathcal{L}_{X|_{G_{tg}}} \omega|_{G_{tg}} \\
&= (\mathrm{div}\, X\omega)|_{G_{tg}} = \mathrm{div}(X)|_{G_{tg}} \omega|_{G_{tg}}
\end{aligned}$$

using the right-invariance of both X and ω. So because ω is non-vanishing, it follows that $\mathrm{div}(X) \circ r_g = \mathrm{div}(X)|_{G_{tg}}$. □

Definition 3.4.11. Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid A. The divergence of a section $a \in \Gamma(A)$ is defined as $\mathrm{div}(a) := \mathrm{div}^R(X_a)|_M \in C^\infty(M)$, where $X_a \in \mathfrak{X}^R(G)$ is the right-invariant vector field corresponding to a.

Since $\mathrm{div}(X_a)(g) = \mathrm{div}^R(X_a) \circ r_g(tg) = \mathrm{div}^R(X_a)(tg) = \mathrm{div}(a) \circ t(g)$, we have $\mathrm{div}^R(X_a) = \mathrm{div}(a) \circ t$.

Proposition 3.4.12. *For any $a \in \Gamma(A)$ and $f \in C^\infty(M)$, we have* $\mathrm{div}(fa) = f\,\mathrm{div}(a) + \rho(a)(f)$.

Proof: Use the convention where $\rho = Tt$ and compute using $\mathrm{div}(fX) = f\,\mathrm{div}(X) + X(f)$ for normal divergence. □

So how is the formal adjoint actually defined? The formula is very simple.

Definition 3.4.13. Let $X \in \mathfrak{X}^R(G)$. The formal adjoint of its Lie operator $\mathcal{L}_X$ is defined by $\mathcal{L}_X^* := -\mathcal{L}_X - m_{\mathrm{div}(X)}$.

As mentioned before, this formal adjoint has to be right-invariant. Using the lemma proven before, the proof is a two-liner.

Proposition 3.4.14. *For all $X \in \mathfrak{X}^R(G)$ we have $\mathcal{L}_X^* \in \mathrm{Diff}^R(G)$.*

Proof: Clearly we have $\mathcal{L}_X \in \mathrm{Diff}^R(G)$. $m_{\mathrm{div}(X)} \in \mathrm{Diff}^R(G)$ follows by Lemma 3.4.10, hence $\mathcal{L}_X^* = -\mathcal{L}_X - m_{\mathrm{div}(X)} \in \mathrm{Diff}^R(G)$. □

Just as its name says, the groupoid C^*-algebra $C^*(G)$ was designed to be a C^*-algebra. However, in some cases it is more intuitive to think of it as a Hilbert module over itself, using the following definition:

DEFINITION 3.4.15. Let $G \rightrightarrows M$ be a volumetric groupoid. For $f, g \in C^*(G)$, define

$$\langle f, g\rangle := f^* * g \in C^*(G).$$

This is a general construction for arbitrary C^*-algebras and makes $C^*(G)$ a Hilbert $C^*(G)$-module. The advantage is that computations in this notation look more similar to the basic case of $L^2(M)$ discussed in the first chapter.

It should be checked that our formal adjoint is actually an adjoint (in the purely algebraic sense for the beginning). This is achieved in the next theorem.

THEOREM 3.4.16. *Let $(G \rightrightarrows M, \omega)$ be a volumetric groupoid and $X \in \mathfrak{X}^R(G)$. Then for all $f, g \in C_c^\infty(G)$ we have:*

$$\langle \mathcal{L}_X^*(f), g\rangle = \langle f, \mathcal{L}_X(g)\rangle$$

Proof: Choose $x \in G$. Set $p = s(x)$, $\omega_p := (\mathrm{inv}^*\omega)|_{G_p}$ and $\lambda_p := \mu_{\omega_p}$. Note that we have

$$\begin{aligned}
\langle f, g\rangle(x) = f^* * g(x) &= \int_{G^p} f^*(xy)g(y^{-1})\mathrm{d}\lambda^p \\
&= \int_{G_p} f^*(xy^{-1})g(y)\mathrm{d}\lambda_p = \int_{G_p} f(yx^{-1})g(y)\mathrm{d}\lambda_p \\
&= \int_{G_p} f \circ r_{x^{-1}}(y)g(y)\mathrm{d}\lambda_p = \langle f|_{G_p} r_{x^{-1}}, g|_{G_p}\rangle_{\lambda_p},
\end{aligned}$$

with the L^2-product under the measure λ_p in the last line.

Hence for the case we want to prove, we get

$$\begin{aligned}
\langle \mathcal{L}_X^* f, g\rangle(x) &= \langle \mathcal{L}_X^*|_{G_p}(f|_{G_p}) \circ r_{x^{-1}}, g|_{G_p}\rangle \\
&= \langle (-X(f|_{G_p}) - \mathrm{div}(X)|_{G_p} f|_{G_p}) \circ r_{x^{-1}}, g|_{G_p}\rangle \\
&= \langle -X(f|_{G_p} \circ r_{x^{-1}}) - \mathrm{div}_{\omega_p}(X) f|_{G_p} \circ r_{x^{-1}}, g|_{G_p}\rangle \\
&= \langle f|_{G_p} \circ r_{x^{-1}}, X(g|_{G_p})\rangle
\end{aligned}$$

by Theorem 2.1.6. □

The natural way to define the adjoint of higher order right-invariant differential operators is to use the expected formula $(\mathcal{L}_{X_1} \dots \mathcal{L}_{X_n})^* = \mathcal{L}_{X_n}^* \dots \mathcal{L}_{X_1}^*$. To do so, however, we need to prove that $\mathrm{Diff}^R(G)$ is generated by order 1 and 0 elements, i.e. by $C_c^\infty(G) \cup \mathfrak{X}^R(G)$. We show this relying on the fact that $\mathrm{Diff}^R(G)$ is a universal enveloping algebra of the Lie algebroid corresponding to G. Looking at the following lemma, readers will notice that not much more is needed to finish the proof of this fact.

LEMMA 3.4.17. *Let A be a Lie algebroid and (E, i_1, i_2) be a universal enveloping algebra of A. Then E is generated as an algebra by $i_1(C^\infty(M)) \cup i_2(\Gamma(A))$.*

Proof: Let

$$E_0 := \left\{ \sum_{k=1}^n x_1^k \dots x_{m_k}^k \mid n, m_k \in \mathbb{N}, x_j^k \in C^\infty(M)) \cup i_2(\Gamma(A) \right\} \subseteq E$$

be the algebra generated by $i_1(C^\infty(M)) \cup i_2(\Gamma(A))$. By definition, i_1 and i_2 have their images inside of E_0. E_0 is unital because $i_1(\mathrm{const}_1) = 1_E$. i_1 and i_2 still fulfil their enveloping algebra axioms as in Definition 3.3.1, so (E_0, i_1, i_2) is an enveloping algebra for A. We will quickly show that it is even universal.

Namely let (F, j_1, j_2) be another enveloping algebra of A. Since (E, i_1, i_2) is universal, there is a unique unital algebra homomorphism $\tilde{\phi} : E \to F$ with $\tilde{\phi} \circ i_k = j_k$, $k \in \{1, 2\}$. Denote by $\iota : E_0 \to E$ the inclusion (which is a unital algebra homomorphism), then $\phi := \tilde{\phi} \circ \iota : E_0 \to F$ is a unital algebra homomorphism with $\phi \circ i_k = \tilde{\phi} \circ i_k = j_k$, i.e. an enveloping algebra homomorphism.

If $\psi : E_0 \to F$ is any enveloping algebra homomorphism, then by $\psi \circ i_k = j_k = \phi \circ i_k$ we have $\psi|_{i_1(C^\infty(M)) \cup i_2(\Gamma(A))} = \phi|_{i_1(C^\infty(M)) \cup i_2(\Gamma(A))}$. But two algebra homomorphisms coinciding on a set are equal on the algebra generated by it, so $\psi = \phi$. Hence there is exactly one unital algebra homomorphism $E_0 \to F$, and because F was arbitrary, this means that E_0 is universal.

In particular, there is an enveloping algebra isomorphism $f : E_0 \to E$ by Theorem 3.3.4. But there is only one enveloping algebra homomorphism $E_0 \to E$, and the inclusion ι is one such homomorphism, hence $\iota = f$ is an isomorphism, which means that $E_0 = E$. □

The theorem mentioned in the last chapter actually uses left-invariant operators instead of right-invariant ones. So we need to assure that both are isomorphic. This is more of a routine task.

Lemma 3.4.18. *Let G be a Lie groupoid with Lie algebroid A. Then the pullback by inversion*

$$\mathrm{inv}^* : \mathrm{Diff}^L(G) \to \mathrm{Diff}^R(G),\ (\mathrm{inv}^* D)(f) = D(f \circ \mathrm{inv}) \circ \mathrm{inv}^{-1}$$

is an isomorphism of enveloping algebras.

Proof: It is clear that inv^* is a unital algebra isomorphism since this holds for any pullback of differential operators by a diffeomorphism. So we just have to show that it is compatible with the enveloping algebra inclusions.

Namely our enveloping algebras are given as $(\mathrm{Diff}^L(G), i_1^L, i_2^L)$ and $(\mathrm{Diff}^R(G), i_1^R, i_2^R)$, where $i_1^L(f) = m_{f\circ s}$, $i_1^R(f) = m_{f \circ t}$, $i_2^L(a) = \mathcal{L}_{\sigma^L(a)}$, $i_2^R(a) = \mathcal{L}_{\sigma^R(a)}$, $\sigma^L(a)(g) = T_{sg} l_g a(sg)$, $\sigma^R(a)(g) = T_{tg} r_g a(tg)$.

$\mathrm{inv}^* \circ i_1^L = i_1^R$ holds by

$$\begin{aligned}\mathrm{inv}^*(m_{f\circ s})(h) &= (f \circ s \cdot h \circ \mathrm{inv}) \circ \mathrm{inv} = (f \circ t \circ \mathrm{inv} \cdot h \circ \mathrm{inv}) \circ \mathrm{inv}\\ &= f \circ t \cdot h = m_{f\circ t}(h) = i_1^R(f)(h)\end{aligned}$$

for all $f \in C^\infty(M)$, $h \in C^\infty(G)$.

For the second inclusion map, we compute that

$$\begin{aligned} i_2^L(a)(f \circ \mathrm{inv})(g) &= T_{sg} l_g a_{sg}(f \circ \mathrm{inv})\\ &= T_g \mathrm{inv} T_{sg} l_g a_{sg}(f) = T_{sg}(\mathrm{inv} \circ l_g) a_{sg}(f)\\ &= T_{sg}(r_{g^{-1}} \circ \mathrm{inv}) a_{sg}(f) = T_{tg^{-1}} r_{g^{-1}} a_{tg^{-1}}(f) = i_2^R(a)(f)(g^{-1})\end{aligned}$$

for all $a \in \Gamma(A)$, $f \in C^\infty(G)$, $g \in G$. Hence $\mathrm{inv}^* i_2^L(a)(f) = i_2^L(f \circ \mathrm{inv}) \circ \mathrm{inv} = i_2^R(a)(f)$ for all f, so indeed $\mathrm{inv}^* \circ i_2^L = i_2^R$. □

The theorem on generation sets we wanted to prove is now just the composition of the previous propositions.

Theorem 3.4.19. *Let $G \rightrightarrows M$ be a Lie groupoid. Then $\mathrm{Diff}^R(G)$ is generated by $\{m_f | f \in C^\infty(M)\} \cup \{X_a | a \in \Gamma(A)\}$ as an algebra, where X_a denotes the right-invariant vector field corresponding to a.*

Proof: We know by Theorem 3.3.10 that $\mathrm{Diff}^L(G)$ is a universal enveloping algebra for A, hence by Lemma 3.4.17 it is generated by $i_1^L(C^\infty(M)) \cup i_2^L(\Gamma(A))$. Using the enveloping algebra isomorphism $\mathrm{inv}^* : \mathrm{Diff}^L(G) \to \mathrm{Diff}^R(G)$ from Lemma 3.4.18, we deduce that $\mathrm{Diff}^R(G)$ is generated by

$$\begin{aligned}\mathrm{inv}^*(i_1^L(C^\infty(M)) \cup i_2^L(\Gamma(A))) &= i_1^R(C^\infty(M)) \cup i_2^R(\Gamma(A)) \\ &= \{m_f | f \in C^\infty(M)\} \cup \{X_a | a \in \Gamma(A)\}. \qquad \square\end{aligned}$$

With this generation lemma, we can define star structures on right-invariant differential operators. We can rely on several propositions from the first chapter in the process.

THEOREM 3.4.20. *Let $(G \rightrightarrows M, \omega)$ be a volumetric groupoid. Then there is a unique star structure (an involutive local algebra anti-endomorphism) $*$ on $\mathrm{Diff}^R(G)$ such that $\mathcal{L}_{X_a}^* = -\mathcal{L}_{X_a} - m_{\mathrm{div}(X_a)}$ and $m_f^* = m_f$ for all $a \in \Gamma(A)$ and $f \in C^\infty(M)$.*

Proof: We know by Theorem 3.4.19 that $\mathrm{Diff}^R(G)$ is generated as an algebra by $\{m_f | f \in C^\infty(M)\} \cup \{X_a | a \in \Gamma(A)\}$. So clearly there can be at most one star structure with the required properties because it is completely determined by its values on the generating set.

It is now to show that such a star structure exists. This can be done with relative ease using previous work. Namely let $D \in \mathrm{Diff}^R(G)$. For every $p \in M$, define $D_p^* := (D|_{G_p})^* \in \mathrm{Diff}(G_p)$, using the unique star structure on $\mathrm{Diff}(G_p)$ with $m_f^* = m_f$ and $\mathcal{L}_X^* = -\mathcal{L}_X - m_{\mathrm{div}(X)}$ for all $f \in C^\infty(G_p)$ and $X \in \mathfrak{X}(G_p)$ constructed in Theorem 3.2.18. Then define $D^* : C^\infty(G) \to C^\infty(G)$ by $D^*(f)(x) := D_{sx}^*(f)(x)$.

Notice first that D^* is a map which is tangent to the source fibres by its very definition. $* : \mathrm{Diff}^R(G) \to \{F : C^\infty(M) \to C^\infty(M)\}$ is easily seen to be additive by $(P+Q)^*|_{G_p} = ((P+Q)|_{G_p})^* = (P|_{G_p} + Q|_{G_p})^* = P^*|_{G_p} + Q^*|_{G_p}$ for all $p \in M$. The proofs for anti-multiplicativity (also with constants) and locality are completely analogous, so * is a local algebra anti-homomorphism.

We still have to show that $D^* \in \mathrm{Diff}^R(G)$. To do this, use Theorem 3.4.19 one more time to write D in the form $D = \sum_{k=1}^m D_1^k \dots D_{n_k}^k$ for certain $D_i^j \in \{m_f | f \in C^\infty(M)\} \cup \{X_a | a \in \Gamma(A)\}$. Consequently we have $D^*|_{G_p} = (D|_{G_p})^* = \sum_{k=1}^m (D_{n_k}^k|_{G_p})^* \dots (D_1^k|_{G_p})^*$ for all $p \in M$.

Let $k \in \{1, \dots, m\}$ and $i \in \{1, \dots, n_k\}$ be arbitrary. Then there are two possibilities for D_i^k: If $D_i^k = m_f = m_{f\circ s}$ for some $f \in C^\infty(M)$, then $(D_i^k|_{G_p})^* = m_{f(p)}^* = m_{f(p)} = m_f|_{G_p}$. Otherwise we have $D_i^k = \mathcal{L}_X$ for some $X \in \mathfrak{X}^R(G)$. Then $(D_i^k|_{G_p})^* = -\mathcal{L}_{X|_{G_p}} - \mathrm{div}_{\omega|_{G_p}}(X|_{G_p}) = \mathcal{L}_X^*|_{G_p}$, where $\mathcal{L}_X^*$ is defined as in Definition 3.4.13 (this coincides with the definition made in this proof). So in either case we have $(D_i^k|_{G_p})^* = (D_i^k)^*|_{G_p}$ for an operator $(D_i^k)^* \in \mathrm{Diff}^R(G)$. Because $p \in M$ was arbitrary, we get that $D^* = \sum_{k=1}^m (D_{n_k}^k)^* \dots (D_1^k)^* \in \mathrm{Diff}^R(G)$ as the concatenation of right-invariant differential operators. $\square$

The last theorem in this section shows that the star structure defined before is more than an independent algebraic thing: It *is* the involution which gives us adjoints of operators on a Hilbert module, up to closure.

THEOREM 3.4.21. *Let $(G \rightrightarrows M, \omega)$ be a volumetric groupoid. Let $^* : \mathrm{Diff}^R(G) \to \mathrm{Diff}^R(G)$ be the induced star structure as in Theorem 3.4.20. Then $\iota : \mathrm{Diff}^R(G) \to \mathcal{O}(C^*(G))$, $D \mapsto (C_c^\infty(G), D)$ is an injective *-homomorphism.*

Proof: Evaluation of a linear map and closing an operator are always linear and multiplicative, and hence so is ι. ι is injective because given $P, Q \in \mathrm{Diff}^R(G)$, $P(f) = Q(f)$ for all $f \in C_c^\infty(G)$ implies $P(f) = Q(f)$ even for all $f \in C^\infty(G)$ because of locality, i.e. $P = Q$.

The more interesting part of this theorem is to show that $\langle \bar{D^*}(f), g\rangle = \langle f, \bar{D}(g)\rangle$ for all $f \in \mathrm{dom}(\bar{D^*}), g \in \mathrm{dom}(\bar{D})$. As always when working with closures of densely defined operators, it suffices to show that $\langle D^*(f), g\rangle = \langle f, D(g)\rangle$ for all $f, g \in C_c^\infty(M)$. This was already proven for the case that D is a Lie operator in Theorem 3.4.16. For the order 0 case, assume that $D = m_f$ for some $f \in C^\infty(M)$, then for all $g, h \in C_c^\infty(G)$ and $x \in G$ we have

$$\begin{aligned}
\langle m_f g, h\rangle(x) &= (m_f g)^* * h(x) \\
&= \int_{G^{sx}} (f \circ tg)^*(xy^{-1})h(y)\mathrm{d}\lambda^{sx}(y) \\
&= \int_{G^{sx}} f \circ t(y^{-1})g(yx^{-1})h(y)\mathrm{d}\lambda^{sx}(y) \\
&= \int_{G^{sx}} g(yx^{-1})f \circ t(y)h(y)\mathrm{d}\lambda^{sx}(y) = \langle g, m_f h\rangle(x),
\end{aligned}$$

so $\langle D^* g, h\rangle = \langle Dg, h\rangle = \langle g, Dh\rangle$. So since $\mathrm{Diff}^R(G)$ is generated by those order 0 and 1 elements, it follows that indeed $\langle D^* g, h\rangle = \langle g, Dh\rangle$ for all $g, h \in C_c^\infty(G)$ as required. $\square$

Strictly speaking, we have only shown that $\iota(D^*) \subseteq \iota(D)^*$ holds for the adjoints in the previous theorem; the domain may still be different. A more elaborate investigation of this distinction will follow in the next chapter.

For now, we have proven propositions for volumetric groupoids with compact base that are conceptually identical to those proven in the first chapter only for the manifold case, i.e. for pair groupoids. I could go and formulate everything in the exact same grammar or repeat the process for left-invariance, but our work so far is enough to understand the new concepts which lately arose. We can hence progress to the next section, where we start investigating actual groupoid representations on Hilbert modules.

CHAPTER 4

A Differentiation Theorem

The goal of this chapter will be to *differentiate* representations of a Lie groupoid to representations of its Lie algebroid. This was shown before to be possible in a slightly different context by Ralf Meyer in [**16**]. More precisely, we start with a representation of the groupoid C^*-algebra $C^*(G)$ and want to construct a representation of the universal enveloping algebra $U(A)$ of the corresponding Lie algebroid in a reasonable, *natural* way. After that, we may also go the other direction and *integrate* representations of $U(A)$.

The terms *differentiation* and *integration* should not be over-interpreted in this context. The origin of these is mainly the classical Lie theory, where the tangent map at the origin, also called differential, of a smooth Lie group representation turns out to be a representation of the corresponding Lie algebra. When we use this kind of theory in the upcoming section, it will mostly be implicitly by the use of important theorems. The actual mathematics there will rather consist of investigations on volume forms and algebraic properties.

4.1. From the Groupoid Algebra to Differential Operators

Given a representation $\pi : C^*(G) \to \mathbb{B}(E)$ of a groupoid G on a Hilbert module E, we are going to define a representation $K_\pi : \mathrm{Diff}^R(G) \to \mathcal{O}(E)$ using the formula $K_\pi(D)(\pi(f)e) = \pi(D(f))e$. One necessary condition for this to work out is that

$$\pi(D(f * g))e = K_\pi(D)(\pi(f * g)e) = K_\pi(D)(\pi(f)\pi(g)e) = \pi(D(f))\pi(g)e = \pi(D(f) * g)e$$

holds for all $D \in \mathrm{Diff}^R(G)$, $f, g \in C_c^\infty(G)$ and $e \in E$; if π is non-degenerate, this means that we should expect $D(f) * g = D(f * g)$ to hold. It will turn out that we do not really need to prove this in advance of the next big theorem because it is covered by a more general computation. However, proving this equation on a more elementary level provides us a bit of additional understanding in the area of groupoid representations. In particular, this part does not need a volume form or an orientation to work with. This is enough reason to go for it anyway.

First of all, let us state the definition of a representation:

Definition 4.1.1. Let A be a unital *-algebra. An (unbounded) representation of A is a pair (H, R), where H is a Hilbert C^*-module over any C^*-algebra and $R : A \to \mathcal{O}(H)$ is a map from A to the operators on H with the following properties:

(1) Common dense domain: There is a dense right-ideal $\operatorname{dom} R \subseteq H$ (called the domain of R) such that for all $D \in A$, the corresponding operator has domain $\operatorname{dom} R(D) = \operatorname{dom} R$.
(2) Invariance of domain: For each $D \in A$, we have $R(D)(\operatorname{dom} R) \subseteq \operatorname{dom} R$.
(3) Homomorphy: For all $P, Q \in A$ and $\lambda \in \mathbb{C}$, we have $R(\lambda P + Q) = \lambda R(P) + Q$ and $R(PQ) = R(P)R(Q)$ (these are defined on $\operatorname{dom} R$ by properties 1 and 2). Furthermore, $R(1) = \mathrm{id}_H|_{\operatorname{dom} R}$.
(4) Formal preservation of adjoints: For each $D \in A$ and all $v, w \in \operatorname{dom} R \subseteq H$, we have $\langle R(D)v, w\rangle = \langle v, R(D^*)w\rangle$ (i.e., $R(D^*) \subseteq R(D)^*$).

Of course, the *-algebra which we will usually use is the algebra of invariant differential operators $\mathrm{Diff}^R(G)$. The Hilbert C^*-module will often just be a separable Hilbert space; in that case the domain is simply a dense subspace.

The upcoming proof is not really hard, but requires one more lemma to be certain about which computational steps involving vector fields and measures are valid.

LEMMA 4.1.2. *Let M and N be smooth manifolds with smooth measures μ on M and ν on N. Let $X \in \mathfrak{X}(M)$ and let $F \in C^\infty(M \times N)$ be a smooth function. Define $X(F) \in C^\infty(M \times N)$ by $X(F)(x,y) := X(F(\cdot,y))(x) = T\pi_y X(F)(x,y)$. Suppose that for all $x \in M$, $F^x = F(x,\cdot) \in C^\infty(M)$ and $X(F)^x = X(F)(x,\cdot)$ are ν-integrable.*

Then we have

$$X\left(\int_M F(\cdot,y)\mathrm{d}\mu(y)\right) = \int_M X(F)(\cdot,y)\mathrm{d}\mu(y)$$

for all $x \in M$.

Proof: Let (U,ϕ) be a smooth chart of M and (V,ψ) be a smooth chart of N. By assumption, ν is a smooth measure, so there is a smooth $v \in C^\infty(\psi(V))$ such that $\nu|_V = v\psi^{-1} \cdot \mu_{\mathrm{d}\psi^1\wedge\cdots\wedge\mathrm{d}\psi^m}$, i.e. $\int_V f\mathrm{d}\nu = \int_{\psi U} f \circ \psi^{-1}(x)v(x)\mathrm{d}x_1\dots\mathrm{d}x_n$ for all integrable $f \in C^\infty(V)$ (here, $n = \dim N$).

So for any $i \in \{1,\dots,m\}$, we can use Leibniz' integral rule for real-variable functions (several times when we are working in higher dimensions) to compute:

$$\begin{aligned}
\partial_i^\phi\left(\int_V F(\cdot,y)\mathrm{d}\nu(y)\right) &= \left(\frac{\partial}{\partial\tilde{x}_i}\left(\int_V F(\cdot,y)\mathrm{d}\nu(y)\circ\phi^{-1}(\tilde{x})\right)\right)\circ\phi \\
&= \left(\frac{\partial}{\partial\tilde{x}_i}\int_{\psi V} F(\phi^{-1}\tilde{x},\psi^{-1}\tilde{y})v(\tilde{y})\mathrm{d}\tilde{y}\right)\circ\phi \\
&= \int_{\psi V}\frac{\partial}{\partial\tilde{x}_i}F(\phi^{-1}\tilde{x},\psi^{-1}\tilde{y})\circ\phi\; v(\tilde{y})\mathrm{d}\tilde{y} \\
&= \int_{\psi V}\partial_i^\phi F(\cdot,\psi^{-1}\tilde{y})v(\tilde{y})\mathrm{d}\tilde{y} = \int_V \partial_i^\phi F(\cdot,y)\mathrm{d}\nu(y)
\end{aligned}$$

So because integration is $\mathbb{R}$-linear:

$$\begin{aligned}
X\left(\int_V F(\cdot,y)\mathrm{d}\nu(y)\right)\Big|_U &= \sum_{i=1}^m X(\phi^i)\partial_i^\phi\int_V F(\cdot,y)\mathrm{d}\nu(y) \\
&= \sum_{i=1}^m X(\phi^i)\int_V \partial_i^\phi F(\cdot,y)\mathrm{d}\nu(y) \\
&= \int_V\sum_{i=1}^m X(\phi^i)\partial_i^\phi F(\cdot,y)\mathrm{d}\nu(y) = \int_V X(F(\cdot,y))\mathrm{d}\nu(y)\Big|_U
\end{aligned}$$

Since this holds for any smooth chart (U,ϕ) of M, it follows that

$$X\left(\int_V F(\cdot,y)\mathrm{d}\nu(y)\right) = \int_V X(F(\cdot,y))\mathrm{d}\nu(y).$$

Now choose a locally finite cover $(V_i,\psi_i)_{i\in I}$ of N by smooth charts and a smooth partition of unity $(h_i)_{i\in I}$ subordinate to it. Then because $\mathcal{L}_X$ is an $\mathbb{R}$-linear, local map, we have

$$\begin{aligned}
X\left(\int_N F(\cdot,y)\mathrm{d}\nu(y)\right) &= X\left(\sum_{i\in I}\int_{V_i} h_i(y)F(\cdot,y)\mathrm{d}\nu(y)\right) \\
&= \sum_{i\in I}\int_{V_i} h_i(y)X(F(\cdot,y))\mathrm{d}\nu(y) = \int_N X(F(\cdot,y))\mathrm{d}\nu(y)
\end{aligned}$$

as required. $\square$

Note that compactly supported smooth functions are always integrable with respect to smooth measures, so if $F(x,\cdot) \in C_c^\infty(N)$ for all $x \in M$, then the lemma can always be applied. This is of particular use in the next proof.

PROPOSITION 4.1.3. *Let $G \rightrightarrows M$ be a Lie groupoid with a smooth Haar system $(\lambda^p)_{p\in M}$. Let $X \in \mathfrak{X}^R(G)$ be a right-invariant vector field and $f, g \in C_c^\infty(G)$. Then $X(f*g) = X(f)*g$.*

Proof: Let $p \in M$. Then since X is tangent to the source fibres, we can write $X(f*g)(p) = X|_{G_p}(f*g|_{G_p})$. Because all considered functions have compact support, we can apply Lemma 4.1.2. Using that and the right-invariance of X, we get:

$$\begin{aligned}
X(f*g)|_{G_p} &= X|_{G_p}(f*g|_{G_p}) = X|_{G_p}\left(\int_{G^p} f\circ r_y g(y^{-1})\mathrm{d}\lambda^p(y)\right)\\
&= \int_{G^p} X|_{G_p}(g(y^{-1})f\circ r_y)\mathrm{d}\lambda^p(y)\\
&= \int_{G^p} Tr_y X_{G_p}(f)g(y^{-1})\mathrm{d}\lambda^p(y)\\
&= \int_{G^p} X_{G_p}(f)\circ r_y g(y^{-1})\mathrm{d}\lambda^p(y)\\
&= X|_{G_p}(f|_{G_p}) * g|_{G_p} = (X(f)*g)|_{G_p}
\end{aligned}$$

So because $p \in M$ was arbitrary, this implies that indeed $X(f*g) = X(f)*g$. □

For simplicity we have stuck to vector fields instead of general differential operators, but the proof could easily be expanded using the fact that differential operators can always be written as a locally finite polynomial of vector fields. This could either be done at the stage of Lemma 4.1.2 or after Proposition 4.1.3 with right-invariant generators, but shall not be further regarded in this thesis.

At this point it is certainly no wonder that the equivalent statement for left-invariant vector fields is also true. Whether justified by formal category theory or steadily built-up intuition, another proof seems unnecessary to write down here.

LEMMA 4.1.4. *Let $G \rightrightarrows M$ be a Lie groupoid with a smooth Haar system $(\lambda^p)_{p\in M}$. Let $X \in \mathfrak{X}^L(G)$ be a left-invariant vector field and $f, g \in C_c^\infty(G)$. Then $X(f*g) = f*X(g)$.*

So let us focus on our main theorem for this section now. This is a known result (compare [**16**]). Because an understanding of the proof in our given context will be important later, I decided to include another proof here. As most preliminary work has been done, we can step right into the theorem.

THEOREM 4.1.5. *Let $(G \rightrightarrows M, \omega)$ be a volumetric groupoid with Lie algebroid A. Let $\pi : L^I(G) \to \mathbb{B}(E)$ be a non-degenerate $*$-representation on a right Hilbert module E over any C^*-algebra B. Set $E^\infty := \{\sum_{i=1}^n \pi(f_i)x_i | n \in \mathbb{N}_0, f_i \in C_c^\infty(G), x_i \in E\}$. Define $K_\pi : \mathrm{Diff}^R(G) \to \mathcal{O}(E)$ by $\mathrm{dom}\, K_\pi(D) = E^\infty$ and*

$$K_\pi(D)(\sum_{i=1}^n \pi(f_i)x_i) := \sum_{i=1}^n \pi(D(f_i))x_i$$

for all $D \in \mathrm{Diff}^R(G)$, $f \in C_c^\infty(G)$, $x \in E$.

Then K_π is a well-defined representation of $\mathrm{Diff}^R(G)$.

Proof: First of all, notice that $E^\infty \subseteq E$ is a right-ideal because $\pi(f)(x)\beta = \pi(f)(x\beta)$ and sums are included by definition. It is dense by the assumption that π is non-degenerate.

For the next step, let $n, m \in \mathbb{N}_0$, $f_i, g_j \in C_c^\infty(G)$ and $x_i, y_j \in E$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, with $\sum_{i=1}^n \pi(f_i)x_i = \sum_{j=1}^m \pi(g_j)y_j$. We have to show that the definition yields the same value for both terms, i.e.

$$\sum_{i=1}^n \pi(D(f_i))x_i = \sum_{j=1}^m \pi(D(g_j))y_j$$

for any $D \in \mathrm{Diff}^R(G)$.

To do so, we make a computation on inner products, using the shorthand $hz := \pi(h)(z)$ for all $h \in C_c^\infty(G)$ and $z \in E$. So for all such h and z we get with Theorem 3.4.21:

$$\begin{aligned}\langle \sum_i D(f_i)x_i, hz\rangle &= \sum_i \langle x_i, D(f_i)^* * hz\rangle = \sum_i \langle x_i, \langle D(f_i), h\rangle z\rangle \\ &= \sum_i \langle x_i, \langle f_i, D^*(h)\rangle z\rangle = \langle \sum_i f_i x_i, D^*(h)z\rangle\end{aligned}$$

Doing the same for $\sum_j g_j y_j$, we deduce that

$$\langle \sum_j D(g_j)y_j, hz\rangle = \langle \sum_j g_j y_j, D^*(h)z\rangle = \langle \sum_i f_i x_i, D^*(h)z\rangle = \langle \sum_i D(f_i)x_i, hz\rangle.$$

π was assumed to be non-degenerate and continuous, and $C_c^\infty(G) \subseteq L^I(G)$ is dense, so we know that $\{hz | h \in C_c^\infty(G), z \in E\} \subseteq E$ is dense. Hence it follows by the previous equation that $\sum_j D(g_j)y_j = \sum_i D(f_i)x_i$ as required.

It is easy to show that $K_\pi(D)$ is actually a B-linear map for any $D \in \mathrm{Diff}^R(G)$. Namely for such D and $f \in C_c^\infty(G)$, $\beta \in B$ we have:

$$\begin{aligned}K_\pi(D)((\pi(f)(x))\beta) &= K_\pi(D)(\pi(f)(x\beta)) = \pi(D(f))(x\beta) \\ &= \pi(D(f))(x)\beta = K_\pi(D)(fx)\beta\end{aligned}$$

Alternatively, this also follows from the fact that an adjoint for $K_\pi(D)$ exists by the previous computation.

We have already proven that

$$\langle K_\pi(D)fx, hz\rangle = \langle D(f)x, hz\rangle = \langle fx, D^*(h)z\rangle = \langle fx, K_\pi(D^*)hz\rangle$$

for all $f, h \in C_c^\infty(G)$, $D \in \mathrm{Diff}^R(G)$ and $z \in E$. By the non-degeneracy, this implies that indeed $\langle K_\pi(D^*)v, w\rangle = \langle K_\pi(D)w\rangle$ for all $v, w \in E^\infty$.

It is clear from the definition that $K_\pi(D)(E^\infty) \subseteq E^\infty$. Thus it is only left to prove that K_π is a unital algebra homomorphism, which is achieved quickly. Because $K_\pi(D)$ is always linear, it suffices to check this on elements of the single-summand form $fx \in E^\infty$, $f \in C_c^\infty(G), x \in E$. So let $P, Q \in \mathrm{Diff}^R(G)$. Then we have $K_\pi(P+Q)(fx) = (P+Q)(f)x = (P(f)+Q(f))x = P(f)x+Q(f)x = K_\pi(P)(fx)+K_\pi(Q)(fx)$ and $K_\pi(PQ)(fx) = PQ(f)x = K_\pi P(Q(f)x) = K_\pi P K_\pi Q(fx)$. The latter equation shows that in particular $\lambda K_\pi(Q) = K_\pi(\lambda Q)$ for all $\lambda \in \mathbb{C}$ by the usual inclusion of $\mathbb{C}$ as multiplication operators by constant functions. Of course K_π is unital by $K_\pi(\mathrm{id}_{C^\infty(G)})(fx) = \mathrm{id}_{C^\infty(G)}(f)x = fx$, which finishes the proof. ☐

The representation obtained in the previous theorem will also be called $\mathrm{diff}(\pi) := K_\pi$.

4.2. Properties of the Derivative

In this section I wish to investigate additional properties of the representation $\mathrm{diff}(\pi)$ derived from a representation $\pi : C^*G \to \mathbb{B}(H)$. In particular we will see that vector fields act by essentially self-adjoint operators in such a derived representation, with a few technical caveats.

The following investigations require the use of *elliptic* differential operators. I will not give a formal introduction here. Just recall this much: A differential operator $D = \sum_{I\in\mathbb{N}^m,\ |I|\le k} f_I \partial_I^\phi \in \mathrm{Diff}_k(M)$ given by coordinate vector fields of a chart ϕ and their products has a principal symbol $\sigma(D) \in \Gamma(T^*M)$, defined as $\sigma(D)(\xi) = \sigma_k(D) = \sum_{I\in\mathbb{N}^m,\ |I|=k} f_I \xi(\phi)^I$, for $\xi(\phi)^I := \xi(\phi_1)^{i_1} \dots \xi(\phi_i)^{i_m}$ if $I = (i_1, \dots, i_m)$. One can show that this is independent of chart choice, using that the commutator $[P, Q]$ of operators $P \in \mathrm{Diff}_k(M)$, $Q \in \mathrm{Diff}_l(M)$ has at most order $k+l-1$, not $k+l$.

A differential operator $D \in \mathrm{Diff}(M)$ is elliptic if its principal symbol is non-zero at every point of the manifold. By extension, a right-invariant differential operator $D \in \mathrm{Diff}^R(G)$

on a Lie groupoid G is called elliptic if for every $p \in M = G^{(0)}$, the restriction $D|_{G^p}$ to the target fibre is elliptic.

The following lemma also mentions pseudodifferential operators, but only uses them as a technical term from Vassout's work. The important part here is that every differential operator is also a pseudodifferential operator.

LEMMA 4.2.1. *Let G be a compact volumetric groupoid. Let $D \in \mathrm{Diff}^R(G)$ be a right-invariant differential operator with $D = D^*$ (formally in $\mathrm{Diff}^R(G)$) which is elliptic. Then the densely defined unbounded operator $(C^\infty(G), D) \in \mathcal{O}(C^*G)$ is regular and essentially self-adjoint.*

Proof: If D has order 0, then it is bounded and we are done. So assume without loss of generality that $k := \operatorname{ord} D > 0$.

D is assumed to be right-invariant, i.e. the restriction $D|_{G_p} : C^\infty(G_p) \to C^\infty(G_p)$ is well-defined for all $p \in M$ and fulfils $D|_{G_p} \circ r_g^* = r_g^* \circ D|_{G_q}$ for all $g \in G$ with $tg = q$, $sg = p$. This means that D is a G-operator in the sense of [**24**], page 169. As D is smooth like all differential operators considered in this thesis, D is of class C^∞ in terms of that paper. As a differential operator of order k, D is in particular a pseudodifferential operator of constant order k, and of course $\mathrm{Re}(k) = k > 0$. Because G is compact, D is compactly supported. Note that $C^\infty(G) = C^{\infty,\infty}(G)$ in Vassout's notation, because G is compact.

So all the assumptions of Proposition 21, page 175 in [**24**] apply, hence by that Proposition, the adjoint of $(C^\infty(G), D)$ is:

$$(C^\infty(G), D)^* = \overline{(C^\infty(G), D^*)} = \overline{(C^\infty(G), D)}$$

Note that in the cited proposition, the musical natural sign simply denotes the *formal* adjoint $D^* \in \mathrm{Diff}^R(G)$ of D, without regards for the domain.

$(C^\infty(G), D)$ is regular by Proposition 21, page 175, [**24**] (if E is a Hilbert space, this is automatic). $\square$

This is an interesting result from Vassout, but more important is the corresponding result induced in a representation.

PROPOSITION 4.2.2 (compare Proposition 4, page 166 in [**24**]). *Let $\pi : L^I(G) \to \mathbb{B}(E)$ be a non-degenerate representation on a Hilbert module E and $R = \mathrm{diff}(\pi) = K_\pi$ the induced representation as in Theorem 4.1.5. Let $D \in \mathrm{Diff}^R(G)$ be any right-invariant differential operator such that $(C^\infty(G), D) \in \mathcal{O}(C^*G)$ is regular and essentially self-adjoint. Then $R(D) \in \mathcal{O}(E)$ is regular and essentially self-adjoint.*

Proof: The first thing to notice is that the complete tensor product $C^*G \otimes_\pi E$ is isomorphic to E. Namely consider the linear map $U : C^*G \otimes_\pi E \to E$, defined by $f \otimes e \mapsto \pi(f)e$. This fulfils

$$\begin{aligned}\langle U(f \otimes v), U(g \otimes w)\rangle &= \langle \pi(f)v, \pi(g)w\rangle = \langle v, \pi(f^*)\pi(g)w\rangle \\ &= \langle v, \pi(f^* * g)w\rangle = \langle v, \pi(\langle f, g\rangle)w\rangle = \langle f \otimes v, g \otimes w\rangle_\pi\end{aligned}$$

for all $f, g \in C^*G$, $v, w \in E$, so U is isometric, in particular bounded and injective. Surjectivity follows from π being non-degenerate. U is C^*G-linear by $gU(f \otimes v) = \pi(g)\pi(f)v = \pi(g * f)v = U(g(f \otimes v))$.

Having this noted down, we immediately see that the differentiated representation of D is given by

$$\begin{aligned}U^{-1}R(D)U(f \otimes v) &= U^{-1}R(D)(\pi(f)v) = U^{-1}\pi(D(f))v \\ &= \pi(D(f)) \otimes v = D \otimes 1(f \otimes v)\end{aligned}$$

for $f \in C^\infty G$, $v \in E$, so that $R(D) = U(D \otimes 1)U^{-1}$.

Thus $R(D)$ is regular, respectively essentially self-adjoint if and only if $D \otimes 1$ is. But indeed by Proposition 4, page 166 in [**24**], we know that $D \otimes 1$ is regular and:

$$(D \otimes 1)^* = (D^* \otimes 1) = (\overline{D} \otimes 1) = \overline{(\overline{D} \otimes 1)} = \overline{D \otimes 1}$$

where $\overline{(\overline{D} \otimes 1)} = \overline{D \otimes 1}$ because $\operatorname{dom} D \otimes_{\text{alg}} E$ is a core for both $(\overline{D} \otimes 1)$ and $D \otimes 1$ by (3) of the Proposition, and $\overline{D}|_{\operatorname{dom} D} = D|_{\operatorname{dom} D} = D$.

So $D \otimes 1$ is essentially self-adjoint, as well as $R(D)$. $\square$

This proposition shows in particular, in combination with Lemma 4.2.1, that $R(D)$ is essentially self-adjoint for every elliptic $D \in \operatorname{Diff}^R(G)$ and $R = \operatorname{diff}(\pi)$.

Next I will show that not only elliptic operators, but also any symmetric differential operator of order 1 acts by something self-adjoint. Proving this is not a trivial task and requires deep results from functional analysis, in this case a theorem from [**21**]. That theorem works on the premise that certain unbounded operators are bounded in modified Sobolev type norms. In the following lemma, we will see that this is satisfied.

LEMMA 4.2.3. *Let G be a compact volumetric groupoid, $\pi : C^*(G) \to \mathbb{B}(H)$ a representation on a Hilbert space and $R = \operatorname{diff}(\pi)$. Let $N \in \operatorname{Diff}^R(G)$ be a strictly positive symmetric elliptic differential operator of order $k \in \mathbb{N} \setminus \{0\}$ such that $R(N)$ is also positive. Let $A \in \operatorname{Diff}^R_k(G)$ be any differential operator of order at most k. Then for every $s \in \mathbb{R}_{>0}$, the operator $R(A) : H^s(R(N)) \to H^{s-k}(R(N))$ viewed as an operator between Sobolev spaces is bounded (the Sobolev space $H^s(R(N))$ is the closure of H equipped with the norm $\|v\|_{H^s(R(N))} = \|R(N)^{\frac{s}{k}}(v)\|_H$). That is, there exists a constant $c = c_n \in \mathbb{R}$ such that for all $v \in \operatorname{dom} R$:*

$$\|R(N)^{\frac{s-k}{k}} R(A)v\|_H \leq c\|R(N)^{\frac{s}{k}} v\|_H$$

Proof: By Lemma 4.2.1, $N \in \mathcal{O}(C^*G)$ is essentially self-adjoint and regular. It is also positive, allowing us to define a positive operator $N^x := \overline{N}^x \in \mathcal{O}(C^*G)$ for all $x \in \mathbb{R}_{>0}$ by functional calculus. By Theorem 41, page 184 in [**24**] and the following discussion this operator is actually another pseudodifferential operator of order xk with principal symbol $\sigma(N^x) = \sigma_2(N)^x$, thus it is elliptic. So we can use $N^x + \mathrm{i}$ to define the norm on H^{xk} as defined in definition 32, page 180, ibid., because this is still elliptic of order xk. For all $v, w \in C^\infty G$ we have

$$\begin{aligned}
\langle v, w\rangle_{H^{xk}(N^x+\mathrm{i})} &= \langle (N^x + \mathrm{i})v, (N^x + \mathrm{i})w\rangle + \langle v, w\rangle \\
&= \langle N^x v, N^x w\rangle - \langle v, w\rangle + \langle v, w\rangle + \mathrm{i}\langle v, N^x w\rangle + \bar{\mathrm{i}}\langle N^x v, w\rangle \\
&= \langle N^x v, N^x w\rangle + \mathrm{i}\langle v, N^x w\rangle - \mathrm{i}\langle v, N^x w\rangle = \langle N^x v, N^x w\rangle
\end{aligned}$$

because N^x is still symmetric. In particular, $\|v\|_{H^{xk}} = \|N^x v\|_{C^*G}$ (the norm may be different for another choice of N, but compatible).

Let $s \in \mathbb{R}_{>0}$ be arbitrary and consider the pseudodifferential operator $D = N^{\frac{s-k}{k}} A N^{-\frac{s}{k}}$. This has order at most $k\frac{s-k}{k} + k - k\frac{s}{k} = s - k + k - s = 0$, so it is bounded on $C^*(G)$ by Theorem 18, page 173, [**24**]. Thus $D \otimes 1$ is bounded on $C^*(G) \otimes_\pi H$ as mentioned on page 166 ibid.. Put $c = \|D \otimes 1\|$.

As proven in Proposition 4.2.2, $R(N)$ is essentially self-adjoint like N, allowing us to define $R(N)^x$. Furthermore, for the canonical unitary map $U : C^*G \otimes_\pi H \to H$, we have $R(B) = U(B \otimes 1)U^{-1}$ for all $B \in \operatorname{Diff}^R(G)$, as shown there. Hence we find for all

$f \in C^\infty(G)$, $v \in H$:

$$\begin{aligned}
\|R(N)^{\frac{s-k}{k}} R(A)\pi(f)v\| &= \|(N\otimes 1)^{\frac{s-k}{k}}(A\otimes 1)(f\otimes v)\| \\
&= \|(N^{\frac{s-k}{k}}A)(f)\otimes v\| = \|(N^{\frac{s-k}{k}}AN^{-\frac{s}{k}}N^{\frac{s}{k}})(f)\otimes v\| \\
&= \|(N^{\frac{s-k}{k}}AN^{-\frac{s}{k}}\otimes 1)(N^{\frac{s}{k}}(f)\otimes v)\| \\
&\leq c\|N^{\frac{s}{k}}(f)\otimes v\| = c\|R(N)^{\frac{s}{k}}(\pi(f)v)\|
\end{aligned}$$

Within the preceding computation, we implicitly used Proposition 4, page 166, [**24**] to get $(N\otimes 1)^x = N^x \otimes 1$.

The Sobolev space H^s here is defined as the completion of $\operatorname{dom}(R(N)^{\frac{s}{k}})$ with respect to the norm $\|v\|_s = \|R(N)^{\frac{s}{k}}v\|$, so it is clear that we get a bounded operator between the respective spaces. □

Our tool for the upcoming proof is a symmetric, elliptic, positive differential operator, which can be viewed as a generalisation of the Laplace operator on $\mathbb{R}^m$. But up to now, we cannot be sure that such an operator even exists in more general cases. Luckily, the proof of existence is not very hard.

Lemma 4.2.4. *Let $G \rightrightarrows M$ be a volumetric groupoid and $R : \operatorname{Diff}^R(G) \to \mathcal{O}(H)$ a representation. Then there exists a non-negative, symmetric, elliptic differential operator $L \in \operatorname{Diff}^R_2(G)$ of order 2 such that $R(L)$ is non-negative.*

Proof: Let A be the Lie algebroid of G. Let $(U_i)_{i\in I}$ be a countable cover of M by domains of local frames for A and $(h_i)_{i\in I}$ a smooth partition of unity subordinate to this cover. Set $k = \operatorname{rk} A$. For each $i \in I$, choose a frame $(a_1^i, \dots, a_k^i)$ of A over U_i. For each $i \in I$, $j \in \{1, \dots, m\}$, let $X_{ij} = X_{h_i a_j^i} \in \mathfrak{X}^R(G)$ be the right-invariant vector field corresponding to $h_i a_j^i \in \Gamma(A)$. For each $i \in I$, $j \in \{1,\dots,m\}$, define $A_{ij} := X_{ij} + \frac{1}{2}\operatorname{div}^R(X_{ij}) \in \operatorname{Diff}^R(G)$, which is skew-symmetric.

Define $L := -\sum_{i\in I}\sum_{j=1}^m A_{ij}^2$, which is well-defined because $(h_i)_i$ is locally finite. Now let $x \in G$ and $\xi \in T_x^*G_{sx} \setminus \{0\}$ be arbitrary. Then the principal symbol $\sigma(L)$ fulfils

$$\begin{aligned}
-\sigma(L)(x)(\xi) &= \sum_{i,j}\xi((X_{ij})(x))^2 = \sum_{ij} h_i^2(tx)\xi(T_{tx}r_x a_j^i(tx))^2 \\
&\geq \sum_{j=1}^k h_l^2(tx)\xi(T_{tx}r_x a_j^l(tx))^2 > 0,
\end{aligned}$$

because $(a_1^l(tx), \dots, a_k^l(tx))$ is an ordered basis for $A_{tx} \cong T_{tx}G_{tx}$ and thus

$$(T_{tx}r_x a_1^l(tx), \dots, T_{tx}r_x a_k^l(tx))$$

is one for $T_xG_{sx} \cong (T_x^*G_{sx})^*$. Here $l \in I$ is any index such that $tx \in U_l$ and $h_l(tx) > 0$, which must exist by the properties of a partition of unity. Thus L is elliptic.

Furthermore, L is symmetric because the A_{ij} are skew-symmetric: Within $\operatorname{Diff}^R(G)$ we have $L^* = -\sum_{ij}(-A_{ij}^*)^2 = -\sum_{ij} A_{ij}^2$. Clearly L has order 2.

Next we will show that L is non-negative. To do so, recall that an element b of a C^*-algebra B is non-negative if and only if $\phi(b) \in \mathbb{R}$ and $\phi(b) \geq 0$ for every state $\phi : B \to \mathbb{C}$, and that b^*b is always non-negative.

So let $f \in G_c^\infty(G)$ be arbitrary and let $\phi : C^*G \to \mathbb{C}$ be a state. Then we have:

$$\begin{aligned}
\phi(\langle Lf, f\rangle) &= -\sum_{ij}\phi\left(\langle A_{ij}^2 f, f\rangle\right) = \sum_{ij}\phi(\langle A_{ij}f, A_{ij}f\rangle) \\
&= \sum_{ij}\phi((A_{ij}f)^* * (A_{ij}f)) \geq 0
\end{aligned}$$

Since ϕ was arbitrary, this shows $\langle Lf, f\rangle \geq 0$. As f was arbitrary, $L \geq 0$. Here, convergence of the series $\sum_{ij} \phi(\langle A_{ij}f, A_{ij}f\rangle)$ is guaranteed by the fact that the partial sums are non-negative and dominated by $\phi(\langle Lf, f\rangle)$.

Likewise we see that $R(L) \in \mathcal{O}(H)$ is non-negative. Namely if $v \in \operatorname{dom} R$, then:

$$\langle R(L)v, v\rangle = \sum_{ij} \langle R(A_{ij})v, R(A_{ij})v\rangle \geq 0 \qquad \square$$

Using this generalized Laplacian, we can prove:

THEOREM 4.2.5. *Let G be a compact volumetric groupoid. Let $\pi : C^*G \to \mathbb{B}(H)$ be a representation on a Hilbert space and $R = \operatorname{diff}(\pi)$. Let $A \in \operatorname{Diff}^R_1(G)$ be a symmetric differential operator of order 1 (i.e., $A = \mathrm{i}(X + \frac{1}{2}\operatorname{div} X)$ for some $X \in \mathfrak{X}^R(G)$). Then $R(A) \in \mathcal{O}(H)$ is essentially self-adjoint.*

Proof: Let $L \in \operatorname{Diff}^R_2(G)$ be a non-negative, symmetric and elliptic operator of order 2 such that $R(L)$ is non-negative, which exists by the previous lemma. Set $N = L + 1$, which is still elliptic and symmetric and fulfils $N \geq 1$ and $R(N) \geq 1$.

By Lemma 4.2.3, we find a constant $c = \|R(A)\|_{\mathbb{B}(H^2,H^0)} \in \mathbb{R}$ (where $H^s = H^s(R(N))$ with norm $\|v\|_s = \|R(N)^{\frac{s}{2}}v\|_H$) such that $\|R(A)v\| \leq c\|R(N)v\|$ for all $v \in \operatorname{dom} R$.

Furthermore, notice that the commutator $[N, A]$ has at most order 2. Thus by the same lemma, there is a constant $d = \|R([N, A])\|_{\mathbb{B}(H^1,H^{-1})}$ such that

$$\|R(N)^{-\frac{1}{2}}R([N, A])v\| \leq d\|R(N)^{\frac{1}{2}}v\|$$

for all $v \in \operatorname{dom} R$.

$H^{-1} \cong (H^1)'$ is also the dual space of H^1, so we have for all $v \in C^\infty(G) \setminus \{0\}$:

$$\begin{aligned} d\|R(N)^{\frac{1}{2}}v\| &= d\|v\|_{H^1} \geq \|R[N, A]v\|_{\mathbb{B}(H^1,\mathbb{C})} \\ &= \sup_{w\in H^1\setminus\{0\}} \frac{1}{\|w\|_{H^1}} |\langle R[N, A]v, w\rangle| \geq \frac{1}{\|R(N)^{\frac{1}{2}}v\|} |\langle R[N, A]v, v\rangle| \end{aligned}$$

Hence

$$|\langle R(A)v, R(N)v\rangle - \langle R(N)v, R(A)v\rangle| = |\langle R([N, A])v, v\rangle| \leq d\|R(N)^{\frac{1}{2}}v\|^2$$

for all $v \in \operatorname{dom} R([A, N]) = \operatorname{dom} R$.

By Proposition 4.2.2, $R(N) \in \mathcal{O}(H)$ is essentially self-adjoint. So $\overline{R(N)}$ is self-adjoint and $\operatorname{dom} R = \operatorname{dom} R(A) \subseteq \operatorname{dom} \overline{N}$ is a core for $\overline{N}$. Thus by the inequalities we have shown before, we can apply Theorem X.37, page 197 in [**21**], which states that $R(A)$ is essentially self-adjoint on its domain. $\square$

Designing and proving our integration theorems is what we are about to do next, as promised from the beginning of this thesis. However, it turns out that besides the technical proofs, there is still one fundamental idea lacking to do this. The following chapter will fill in this gap.

CHAPTER 5

Measurable Fields of Hilbert Spaces

In the classical theory of Lie group representations, one defines an integrated form P of a Lie algebra representation R by the association $P(\exp(v)) = \mathrm{e}^{R(v)}$ (for sufficiently small vectors v in the Lie algebra), where we have the Lie group exponential on the left hand and functional calculus on the right one. Our main approach is to apply this idea to a representation R of differential operators to get a representation P of the morphism space of a Lie groupoid (which will be formally defined later).

However, there is one key difference to the Lie group case: The exponential $\exp(X)$ of a vector field is not an element of the groupoid itself, but a bisection. We obtain a groupoid element $\exp(X)(p) \in G$ by inserting a point p of the base space as second argument. Hence it would be natural for our desired generalisation to do the same on the right hand side, where functional calculus is used, to obtain *fibrewise* operators $(\mathrm{e}^{R(X)})(p)$. In general, this formula does not make sense for a unitary operator on a Hilbert space. But there is a certain context in which it does, namely when the operator is a so called *decomposable* operator over a whole field of Hilbert spaces. These concepts and how to work with them shall be introduced in this chapter. The main source for this will be Dixmier's *Von Neumann Algebras*, but secondary sources were also used.

5.1. Definitions and Basic Properties

Let us first define our next object of interest. We will usually assume that our base space is equipped with a suitable measure, but formally, only a notion of measurable sets is required, which means having a measurable space.

Definition 5.1.1. [**12**], page 1, Definition 1, compare also [**5**], page 164

Let (X, S) be a measurable space. A measurable field of separable Hilbert spaces (or shortly, a Hilbert field) on (X, S) is a pair (H, M) of a family $H = (H_x)_{x\in X}$ of Hilbert spaces H_x and a subset $M \subseteq P = \prod_{x\in X} H_x$ such that there is a countable subset $M_0 \subseteq M$ which satisfies the following properties:

(1) For all $g \in P$, we have $g \in M$ if and only if for all $f \in M_0$, the function $X \to \mathbb{R}$, $x \mapsto \langle g(x), f(x)\rangle$ is S-Lebesgue-measurable.
(2) For all $x \in X$, the linear span of $\{f(x)|f \in M_0\}$ is dense in H_x.

Such a countable subset is called a *fundamental* sequence.

I may also just write $H \to X$ for a Hilbert field on X, implicitly referring to the same structure.

Notice that in this definition, a measurable section is only formally defined; there is no canonical measure on the union $\bigcup_{p\in X} H_p$ which would allow us to define a measurable map. However, these measurable sections usually behave like we would expect measurable functions to do.

For example, using that limits of measurable functions are again measurable, one can quickly obtain the following lemma, which is a slight generalisation of the first defining property.

Lemma 5.1.2. *Let $H \to X$ be a Hilbert field on X. Then for all $f, g \in M$, the map $X \to \mathbb{R}$, $x \mapsto \langle f(x), g(x)\rangle$ is measurable.*

It is equally easy to prove that multiplying with a measurable function keeps measurability:

PROPOSITION 5.1.3. *Let X, H as before. Let σ be a measurable section in H and let $f : X \to \mathbb{C}$ be a measurable function. Then $f \cdot \sigma$ (defined as usual by $(f\sigma)(p) = f(p)\sigma(p)$) is again measurable.*

One of the most important structures associated to a given Hilbert field is its section space. This is also the point where an actual measure comes into play over merely measurable spaces. As I am most interested in Hilbert spaces for this thesis, I only define the space of square-integrable sections, which looks as follows:

DEFINITION 5.1.4. (Compare [**5**], page 168)

Let (M, ν) be a measure space and $H \to M$ a Hilbert field with respect to the ν-measurable sets in M. A measurable section σ of H is called square-integrable if the integral $\int_M \|\sigma(x)\|_{H_x}^2 \mathrm{d}\nu(x)$ is finite. The set of all square-integrable functions is denoted by $\mathcal{L}^2(H, \nu)$. Two section $\sigma, \tau \in \mathcal{L}^2(H, \nu)$ are called equivalent (written $\sigma \sim \tau$) if $\sigma(p) - \tau(p) = 0 \in H_p$ for ν-almost all $p \in M$. This defines an equivalence relation $\sim$, and the space

$$L^2(H, \nu) := \mathcal{L}^2(H, \nu)/\sim,$$

consisting of equivalence classes of square-integrable sections.

Furthermore, for $[\sigma], [\tau] \in L^2(H, \nu)$, we define $\langle [\sigma], [\tau] \rangle := \int_M \langle \sigma(p), \tau(p) \rangle_{H_p} \mathrm{d}\nu(p) \in \mathbb{C}$.

This definition is very similar to the standard definition of the function space $L^2(M, \nu)$. Indeed, square-integrable sections behave largely like square-integrable functions and the intuition working with both can remain the same. The only difference is often that an extra norm or inner product is written under the integral formulas.

As for L^2-functions, I will not further differentiate between sections $\sigma \in \mathcal{L}^2(H, \nu)$ and their equivalence classes in $L^2(H, \nu)$. Furthermore, I may just write $L^2(H)$ for the same object if the measure is clear from context.

The space $L^2(H)$ has exactly the structure we expect it to have:

PROPOSITION 5.1.5. *([**5**] page 162 and Corollary, page 172) Keep the previous notation. Then $L^2(H, \nu)$ with the bracket $\langle \cdot, \cdot \rangle$ described before is a Hilbert space. If M is locally compact, second countable and σ-compact and ν is a Radon measure, then $L^2(H, \nu)$ is separable.*

The precise properties of the space and measure required for this to work are not of great interest for this thesis. From now on it will suffice to remember that the section space is well-behaved in all cases which are considered here (i.e. where M is a smooth manifold and ν is chosen in an abstract way by Dixmier's own theory).

Let us now investigate what it means for an operator $A : L^2(H_1, \nu) \to L^2(H_2, \nu)$ to be decomposable. To do so, we need to look at measurable fields of operators, in the following sense:

DEFINITION 5.1.6. ([**5**], Definition 1, page 179) Let (M, ν) be a measure space. Let $H_1 \to M$, $H_2 \to M$ be two ν-Hilbert fields (i.e. Hilbert fields with respect to the ν-measurable sets). Let $A = (A_p)_{p \in M}$ be a family of bounded operators $A_p : H_1(p) \to H_2(p)$. This family is called a *measurable* field of operators if for all measurable sections σ in H_1, the family $(A_p(\sigma(p)))_{p \in M}$ is a measurable section in H_2.

A is said to be *essentially bounded* if it is measurable and the essential supremum

$$\operatorname{ess\,sup}_{p \in M, \nu} \|A_p\| < \infty$$

is finite.

Dixmier goes on in his book to show:

PROPOSITION 5.1.7. *Let H_1, H_2 as before. Let $(A_p)_{p\in M}$ be an essentially bounded field of operators between H_1 and H_2. Then the operator $\int_M^{\oplus} A_p \mathrm{d}\nu(p) : L^2(H_1) \to L^2(H_2)$ defined by*

$$\int_M^{\oplus} A_p \mathrm{d}\nu(p)(\sigma)(q) = A_q(\sigma(q)) \in H_2(q)$$

is a well-defined bounded operator with $\| \int_M^{\oplus} A_p \mathrm{d}\nu(p)\| = \operatorname{ess\,sup}_{p\in M} \|A_p\|$.

As usual for measurable function spaces, this definition is not sensitive to variations on null sets; if (B_p) is another essentially bounded field of operators with $A_p = B_p$ almost everywhere, then $\int_M^{\oplus} A_p \mathrm{d}\nu(p) = \int_M^{\oplus} B_p \mathrm{d}\nu(p)$.

Using this structure, we get the following definition of decomposable operators:

DEFINITION 5.1.8. ([**5**], Definition 2, page 182) Keep the previous notation. Let $A : L^2(H_1) \to L^2(H_2)$ be a bounded operator. Then A is said to be *decomposable* if and only if there is an essentially bounded field of operators $(A_p)_{p\in M}$ between H_1 and H_2 such that $A = \int_M^{\oplus} A_p \mathrm{d}\nu(p)$.

A simple but important example of decomposable operators are the multiplication operators, which map each section to its pointwise product with an essentially bounded function:

DEFINITION 5.1.9. Let $H \to M$ be a Hilbert bundle on a measure space M. Let $f \in L^\infty(M)$ be essentially bounded. This defines a bounded operator $T_f : L^2(M) \to L^2(M), T_f(\sigma)(p) := f(p)\sigma(p)$.

Again, these definitions fulfil many expectable properties, which can be read about in more detail in Dixmier's book. I will come back to decomposable operators later, for now it suffices to keep the definitions in mind.

One more preliminary thing about Hilbert fields that we have to consider are pullbacks, which will allow us to consider fields of operators which do not necessarily always keep the base point constant, i.e. we can have mappings $A_p : H_p \to H_q$ for $q \neq p \in M$. The definition is part of the following simple proposition:

PROPOSITION 5.1.10. *Let M and N be measurable spaces with a measurable map $f : M \to N$. Let H be a Hilbert field on N. Let S_0 be a fundamental sequence for H. Then there is a unique Hilbert field structure on $f^*H := (H_{f(p)})_{p\in M}$ such that $\sigma \circ f$ is measurable for all $\sigma \in S_0$.*

Proof: By construction, we have two important properties: Firstly, consider arbitrary elements $\sigma, \tau \in S_0$. Then the inner product $\langle \sigma \circ f, \tau \circ f\rangle = \langle \sigma, \tau\rangle \circ f$ is measurable because $\langle \sigma, \tau\rangle$ and f are.

Secondly, consider any $p \in M$. Then we have

$$\operatorname{span}_{\mathbb{C}}\left((f^*S_0)(p)\right) = \left\{\sum_{i=1}^{n} \lambda_i \sigma_i(fp) | \lambda_i \in \mathbb{C}, \sigma_i \in S_0\right\} \subseteq H_{f(p)},$$

which is dense in $H_{f(p)} = (f^*H)_p$ because S_0 is a fundamental sequence. Thus f^*S_0 is a total (pointwise dense) sequence in f^*H.

So by Proposition 4, page 167 in [**5**], there is a unique Hilbert field structure on f^*H such that all elements of f^*S_0 are measurable. □

It is routine to deduce elementary properties of the pullback. For example, the pulback $f^*\sigma$ of an arbitrary measurable section σ in H is a measurable section in f^*H. This defines a canonical map from the sections of a Hilbert field to the sections of its pullback. However, this is not bounded in every case, and in particular not unitary. But under mild conditions, we can multiply it with an error correction term to obtain such a unitary transformation.

PROPOSITION 5.1.11. *Let $H \to M$ be a Hilbert field over a measure space (M, μ) and $f : M \to M$ be a bi-measurable map such that $f_*\mu \ll \mu$ and $\mu \ll f_*\mu$. Then there is a unitary map $U_f : L^2(H, \mu) \to L^2(f^*H, \mu)$ defined by $(U_f\sigma)(p) := \left(\frac{\mathrm{d}\mu}{\mathrm{d}f_*\mu}(f(p))\right)^{\frac{1}{2}} \sigma(f(p)) \in H_{f(p)} = (f^*H)_p$.*

This map fulfils $U_f \circ T_g = T_{g\circ f} \circ U_f$ for all $g \in L^\infty(M, \mu)$.

Proof: Put $h = \left(\frac{\mathrm{d}\mu}{\mathrm{d}f_*\mu}\right)^{\frac{1}{2}}$. Let us prove that U_f preserves scalar products. Namely for $\sigma, \tau \in L^2H$, we have:

$$\begin{aligned}
\langle U_f\sigma, U_f\tau\rangle &= \int_M \langle h(fp)\sigma(f(p)), h(fp)\tau(f(p))\rangle \mathrm{d}\mu(p) \\
&= \int_M \langle h(p)\sigma(p), h(p)\tau(p)\rangle \mathrm{d}f_*\mu(p) \\
&= \int_M \langle \sigma(p), \tau(p)\rangle \frac{\mathrm{d}\mu}{\mathrm{d}f_*\mu}(p)\mathrm{d}f_*\mu(p) \\
&= \int_M \langle \sigma(p), \tau(p)\rangle \mathrm{d}\mu(p) = \langle \sigma, \tau\rangle
\end{aligned}$$

as required. In particular, this shows that $U_f\sigma \in L^2(f^*H, \mu)$ for $\sigma \in L^2(H, \mu)$ since $\|\sigma\| = \|U_f\sigma\|$.

Linearity is clear by

$$\begin{aligned}
U_f(\lambda\sigma + \tau)(p) &= h(fp)(\lambda\sigma + \tau)(f(p)) \\
&= \lambda h(fp)\sigma(f(p)) + h(fp)\tau(f(p)) = (\lambda U_f\sigma + U_f\tau)(p).
\end{aligned}$$

U_f is bijective with the inverse $U_f^{-1} : L^2(f^*H) \to L^2H,\ U_f^{-1}(\tau)(p) = h(p)^{-1}\tau(f^{-1}(p))$.

Another easy computation proves:

$$U_f \circ T_g(\sigma)(p) = h(fp)T_g(\sigma)(f(p)) = g(f(p))h(fp)\sigma(f(p)) = T_{g\circ f}U_f\sigma(p) \qquad \square$$

This will be an important ingredient for our integration theorem in the next chapter. For now, let us advance to a slightly more elaborate part of the Hilbert field theory.

5.2. Disintegration of Hilbert Spaces

As described at the beginning of this chapter, Hilbert fields and sections on them behave quite nicely in the context of Lie groupoid representation theory. However, assuming such a section space as domain of a representation seems rather arbitrary. A more natural assumption seems to be that the domain is (a dense subset of) an arbitrary separable Hilbert space. Luckily it turns out that a representation of differential operators already contains enough information in it to ensure that its domain is indeed isomorphic to a Hilbert field's section space, if only the measure is chosen correctly. In this section I will show how this isomorphy can be constructed, using a powerful theorem from Dixmier's work:

THEOREM 5.2.1 ([**5**] Theorem 1, page 233). *Let H be a separable Hilbert space, let $y \subseteq \mathbb{B}(H)$ be a commutative C^*-algebra within the bounded operators. Let $Z = \hat{y}$ be the Gelfand spectrum (i.e. the space of non-zero $*$-homomorphisms from y to $\mathbb{C}$) of y, and let ν be a basic measure on Z. Suppose that the identity id_H is contained in the weak closure of y within $\mathbb{B}(H)$.*

Then there exists a ν-Hilbert field $\mathcal{H} \to Z$ which has a unitary transformation $\phi : L^2(\mathcal{H}) \to H$ such that $\phi \circ T_{\hat{A}} \circ \phi^{-1} = A$ for all $A \in y$, where $\hat{A} \in C_0(\hat{y})$ is the Gelfand transform $(x \mapsto x(A))$, and $T_f : L^2(\mathcal{H}) \to L^2(\mathcal{H})$ is the multiplication operator by f for any $f \in C_0M$.

This theorem uses the notion of a *basic* measure. In this section I will *not* define this term, but merely use that it is guaranteed by one more proposition by Dixmier. Curious readers are welcome to look up the precise meaning in his book.

So our work in this section is to build a bridge between the Lie groupoid representation theory and the functional analysis notation of Dixmier to make the application of the theorem possible. In precise terms, this means constructing the y and ν of the above theorem. We start with the following lemma, which shows that representations of differential operators are automatically bounded on smooth functions vanishing at infinity.

LEMMA 5.2.2. *Let G be a Lie groupoid, H any Hilbert space, $R : \mathrm{Diff}^R(G) \to \mathcal{O}(H)$ a representation. Then for any $f \in C_b^\infty(M, \mathbb{R})$, the operator $R(m_f)$ is bounded with $\|R(m_f)\| \le \|f\|_\infty$.*

Proof: This proof uses an idea from Lemma 5.2, page 17, [**4**] in a different context.

Using that f is bounded, choose $c \in \mathbb{R}$ with $c > \|f\|_\infty = \sup_{p\in M} |f(p)|$. Then the function $g := (c^2 - f^2)^{\frac{1}{2}}$ is well-defined and smooth, and $f^2 + g^2 = c^2$. We have

$$\langle R(m_f)v, w\rangle = \langle v, R(m_f^*)w\rangle = \langle v, R(m_f)w\rangle$$

for all $v, w \in \operatorname{dom} R(m_f)$, hence :

$$\begin{aligned}\|R(m_f)v\|^2 &= \langle R(f)v, R(f)v\rangle = \langle R(f^2), v\rangle \\ &= \langle R(f^2 + g^2)v, v\rangle - \langle R(g^2)v, v\rangle \\ &= \langle c^2 v, v\rangle - \|R(g)v\|^2 = c^2\|v\|^2 - \|R(g)v\|^2 \le c^2\|v\|^2\end{aligned}$$

From this it follows that $\|R(m_f)v\| \le c\|v\|$, and because $c > \|f\|_\infty$ was arbitrary, $\|R(m_f)v\| \le \|f\|_\infty\|v\|$. This holds for all $v \in \operatorname{dom} R(m_f)$, so indeed $\|R(m_f)\| \le \|f\|_\infty$. □

Using that these multiplication operators are bounded, we can extend a representation of differential operators to continuous functions. We use the space

$$C_0(M, \mathbb{K}) = \{f \in C(M, \mathbb{K}) \mid \forall\epsilon > 0 \exists K \subseteq M \text{ compact} : \|f|_{M\setminus K}\|_\infty < \epsilon\} \subseteq C_b(M, \mathbb{K})$$

of functions vanishing at infinity, where $C(M, \mathbb{K})$ denotes the continuous functions from M to $\mathbb{K}$ and $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$. When the second argument is not specified, I mean $C_0(M) = C_0(M, \mathbb{C})$. The details of this extension process are explained in the following statement.

PROPOSITION 5.2.3. *R induces a bounded representation $r : C_0(M) \to \mathbb{B}(H)$, i.e. there is a unique bounded homomorphism of $\mathbb{C}$-algebras $r : C_0(M) \to \mathbb{B}(H)$ with $r(f)|_{\mathrm{dom}(R(m_f))} = R(m_f)$ for all $f \in C_0^\infty(M, \mathbb{R})$. r has norm 1 and $r(f)^* = r(\bar{f})$ for every $f \in C_0(M)$.*

Proof: Let $f \in C_b^\infty(M, \mathbb{R})$. Denote by E the common dense invariant domain of R. Then by Lemma 5.2.2, the operator $R(m_f) : E \to H$ is bounded with $\|R(m_f)\| \le \|f\|$. Because $\operatorname{dom} R(m_f)$ is dense by assumption, there is a unique bounded extension $A_f : H \to H$ of $R(m_f)$. If $g \in C_b^\infty(M)$ is another function, then $A_{fg}|_E = R(fg) = R(f)R(g) = A_f|_E A_g|_E = A_f A_g|_E$, hence (as $E \subseteq H$ is dense) $A_{fg} = A_f A_g$. Likewise, we get that $A_f + A_g = A_{f+g}$. This implies that the map

$$r_0 : C_b^\infty(M, \mathbb{R}) \to \mathbb{B}(H),\ f \to A_f$$

is a bounded homomorphism of $\mathbb{R}$-algebras with $\|r_0\| \le 1$. By putting $r_0(f + \mathrm{i}g) = r_0(f) + \mathrm{i}r_0(g)$ for each $h = f + \mathrm{i}g \in C_b^\infty(M, \mathbb{C})$, this extends to a homomorphism of $\mathbb{C}$-algebras which still fulfils $\|r_0\| \le 1$. $C_0^\infty(M) \subseteq C_0(M)$ is dense in the supremum norm, so as before, there is a unique continuous extension $r : C_0(M) \to \mathbb{B}(H)$ of $r_0|_{C_0^\infty(M)}$. This is again multiplicative by $r(\lim_i f_i)r(\lim_j g_j) = \lim_i \lim_j r(f_i)r(f_j) = \lim_i \lim_j r(f_i f_j) = r(\lim_i \lim_j f_i f_j) = r((\lim_i f_i)(\lim_j f_j))$ for $f_i, g_j \in C_b^\infty$. We have $r(f)|_E = r_0(f)|_E = A_f|_E = R(m_f)$ for $f \in C_b^\infty(M)$ by the very definition.

For self-adjointness, let $f = \lim_i f_i \in C_0(M, \mathbb{R})$, where $f_i \in C_0^\infty(M, \mathbb{R})$. Let $v, w \in E \subseteq H$. Then

$$\begin{aligned}\langle r(f)v, w\rangle &= \lim_i \langle r(f_i)v, w\rangle = \lim_i \langle R(m_{f_i})v, w\rangle \\ &= \lim_i \langle v, R(m_{f_i}^*)w\rangle = \lim_i \langle v, R(m_{f_i} w\rangle \\ &= \lim_i \langle v, r(f_i)w\rangle = \langle v, r(f)w\rangle\end{aligned}$$

by continuity of r and the scalar product maps $\langle \cdot v, w\rangle$ and $\langle v, \cdot w\rangle$. As $r(f)$ is bounded and E is dense in H, this suffices to show that it is self-adjoint. It follows that for $h = f + \mathrm{i}g \in C_0(M, \mathbb{C})$ we have $r(\bar{h}) = r(f - \mathrm{i}g) = r(f) - \mathrm{i}r(g) = r(f)^* + \mathrm{i}r(g)^* = r(f + \mathrm{i}g)^* = r(h)^*$ as required. $\square$

The image of this homomorphism is going to be our C^*-algebra. To make sure that this has our original space M as spectrum, we need that r is injective, which is implied by R being *faithful*. If r is not injective, the spectrum is a closed subset of M and we can proceed with that. A short proof of our desired fact is given below.

LEMMA 5.2.4. *Let $R : \mathrm{Diff}^R(G) \to \mathcal{O}(H)$ be a representation and $r : C_0(M) \to \mathbb{B}(H)$ the induced one. Suppose that $R|_{C^\infty M}$ is injective. Then r is injective. In particular, this is true if R is faithful, i.e. if $R(P)v = R(Q)v$ for all $v \in \operatorname{dom} R$ implies $P = Q$.*

Proof: It suffices to show that r has trivial kernel. We show this by contradiction: Suppose that there is a function $f \in C_0(M) \setminus \{0\}$ such that $r(f) = 0$. By continuity, the pre-image $f^{-1}(\mathbb{C} \setminus \{0\})$ is open, and it is non-empty since $f \neq 0$. So because M is locally compact, it contains a non-empty compact subset, which implies we can choose a non-zero bump function $h \in C_c^\infty(M) \setminus \{0\}$ with $\operatorname{supp} h \subseteq f^{-1}(\mathbb{C} \setminus \{0\})$. By this choice, the pointwise fraction $\frac{h}{f} \in C_0 M$ is a well-defined continuous function. Since $r(f) = 0$ by assumption, we have

$$0 = r(f) r(\frac{h}{f}) = r(h) = R(m_h),$$

which implies $m_h = 0 \in \mathrm{Diff}^R(G)$ since $R|_{C^\infty M}$ is injective, which is a contradiction to $h \neq 0$.

Hence the assumption must be wrong and we actually have $\ker r = \{0\}$, which means that r is injective. $\square$

So $r : C_0 M \to \mathbb{B}(H)$ is an injective homomorphism of C^*-algebras. Such homomorphisms are automatically isometries, which implies that $r : C_0 M \to r(C_0 M) \subseteq \mathbb{B}(H)$ is an isomorphism of *-algebras. In particular, $y := r(C_0 M) \subseteq \mathbb{B}(H)$ is a C^*-algebra isomorphic to $C_0 M$.

The next thing that we still need to show is that the weak closure of y contains the identity (in modern terms, this just means that r is non-degenerate).

LEMMA 5.2.5. *Let $y = r(C_0 M) \subseteq \mathbb{B}(H)$ as above. Then the weak closure of y contains the identity.*

Proof: Choose a function $f \in C_0^\infty(M, \mathbb{R})$ such that $f(x) > 0$ for all $x \in M$ (this is always possible, but for non-compact M, any such f will converge to 0 near infinity). Then the fraction $\frac{1}{f} \in C^\infty(M)$ is a well-defined (though not necessarily bounded) function, and for all $v \in E = \operatorname{dom} R$, we have:

$$r(f) R(\frac{1}{f}) v = R(1) v = v$$

Since $R(\frac{1}{f})v \in E$ and $r(f)v = R(m_f)v \in E$ for all $v \in E$ by invariance of R, this implies that $r(f)E = E$, which is dense in H by assumption.

Now choose a sequence of compactly supported functions $f_i \in C_c^\infty(M)$ with $0 \leq f_i \leq 1$ such that $(f_i)_i$ converges uniformly on compact subsets to 1 (e.g. by $f_j|_{K_i} \equiv 1$ for $j \geq i$ and an exceeding sequence $K_i \subseteq M$ of compact subsets). Then the product $f_i \cdot f$ converges uniformly to f by the following argument: Let $\epsilon > 0$ be arbitrary. Then since f vanishes at infinity, there is a compact set $K \subseteq M$ such that $f|_{M\setminus K} < \epsilon$. Since $f_i \to 1$ on compact sets, there is $i \in \mathbb{N}$ such that for $j \geq i$, $\|f_j|_K - 1\|_\infty < \frac{\epsilon}{\|f\|_\infty}$. Hence $\|f_j f - f\|_\infty = \max(\|f_j f|_K - f|_K\|, \|f_j f|_{M\setminus K} - f|_{M\setminus K}\|) < \max(\frac{\epsilon}{\|f\|}\|f\|, \|f|_{M\setminus K}\|) = \max(\epsilon, \epsilon) = \epsilon$.

Now let $w \in E$ be arbitrary. By the previous argument, there is another vector $v \in E$ such that $w = r(f)v$. So since $r : C_0M \to \mathbb{B}(H)$ is continuous, we find that $\lim_{i\to\infty} r(f_i)w = \lim_{i\to\infty} r(f_i f)v = r(f)v = w$. Since $E \subseteq H$ is dense, this implies that $\lim_i r(f_i)w = w$ even for all $w \in H$, which means that $r(f_i)$ converges weakly to id_H as i goes to infinity. This means that id_H is in the weak closure of $y = r(C_0M)$. $\square$

We can now apply the following lemma to obtain a basic measure:

Lemma 5.2.6 ([**5**], Proposition 4, page 130). *Let H be a separable Hilbert space. Let $Z \subseteq \mathbb{B}(H)$ be an abelian von Neumann algebra, and $y \subseteq Z$ a C^*-algebra. Then there is a bounded basic Radon measure ν on $\hat{y}$.*

Namely, if $y = r(C_0M) \subseteq \mathbb{B}(H)$ as before, then the weak closure Z of y is an abelian von Neumann algebra with $y \subseteq Z$. Hence indeed, a basic measure exists on $\hat{y}$.

In the upcoming theorem, we will have to use a few more technical notions from Dixmier's work. I summarize them in the following definition:

Definition 5.2.7. Let H be a Hilbert space and $A \subseteq \mathbb{B}(H)$.

- The commutant of A is $A' := \{T \in \mathbb{B}(H) | \forall S \in A : TS = ST\}$ ([**5**], page 1).
- An element $x \in H$ is called *cyclic* with respect to A if $Ax = \{Tx | T \in A\} \subseteq H$ is dense ([**5**] Definition 3, page 5). Dixmier only uses this when A is a $*$-subalgebra.
- Suppose that A is a separable commutative C^*-algebra and let $x \in H$ be arbitrary. Then the *spectral measure* ν_x is the unique measure on $\hat{A}$ (the space of characters on $A \to \mathbb{C}$) such that $\int_{\hat{A}} f \mathrm{d}\nu_x = \langle \hat{f}x, x\rangle$, where $\hat{f} \in A$ is the Gelfand transform of f, defined by $f(\chi) = \chi(\hat{f})$ for $\chi \in \hat{A}$ ([**5**], page 360).

Combining the different parts explained in this section, we get the following statement in the context of our representation theory:

Theorem 5.2.8. *Let $(G \rightrightarrows M, \omega)$ be a volumetric groupoid, H a Hilbert space and let $R : \mathrm{Diff}^R(G) \to \mathcal{O}(H)$ be a representation. Suppose that $R|_{C^\infty M}$ is injective or M is compact. Then there are a bounded Radon measure ν on M and a ν-Hilbert field $\mathcal{H} \to M$ together with a unitary map $\phi : L^2(\mathcal{H}) \to H$ such that $\phi^{-1} \circ \overline{R(m_f)} \circ \phi = T_f$ for all $f \in C_0^\infty(M)$.*

Furthermore, if $R|_{C^\infty M}$ is injective, then there is an element $\sigma \in L^2\mathcal{H}$ which is cyclic for the commutant of $\{T_f \mid f \in C_0M\} \subseteq \mathbb{B}(L^2\mathcal{H})$ such that $\nu = \nu_\sigma$, in the sense that $\int_M f \mathrm{d}\nu = \langle T_f\sigma, \sigma\rangle$ for $f \in C_0M$ (I implicitly use that $(\{T_f \mid f \in C_0M\})^\wedge \cong M$ here).

Proof: Let $r : C_0(M) \to \mathbb{B}(H)$ be the induced representation as in Lemma 5.2.3. As before, set $y := r(C_0(M)) \subseteq \mathbb{B}(H)$, which is a commutative C^*-algebra. By [**5**], Proposition 4, page 130 and the previous arguments, there is a bounded basic Radon measure ν' on $\hat{y}$. More precisely, we have $\nu' = \nu_x$ for an element $x \in y$ which is cyclic for the commutant y', as detailed in said proposition and its prerequisites.

As shown before, the weak closure of y contains the identity, hence by [**5**] Theorem 1, page 233, there are a ν'-Hilbert field $\mathcal{H}' \to \hat{y}$, together with a unitary $\psi' : L^2(\mathcal{H}') \to H$ compatible with the Gelfand transform as described before.

If $R|_{C^\infty M}$ is injective, then $r : C_0M \to y$ is an isometric *-isomorphism by Lemma 5.2.4. In this case, set $A = M$.

Even if $R|_{C^\infty M}$ is not injective, the kernel $\ker r = r^{-1}(\{0\}) \subseteq C_0M$ is a (closed) ideal. If M is compact, then $C_0M = C(M)$, and all ideals $I \trianglelefteq C(M)$ for a compact space M are of the form $I = I_B = \{f \in C(M) | f|_B = 0\}$ for a closed subset $B \subseteq M$. This is an example from elementary algebra. So let $A \subseteq M$ be closed such that $\ker r = I_A$. Then the induced homomorphism $\tilde{r} : C_0M/I_A \to y,\ f + I_A \mapsto r(f)$ is a well-defined isomorphism. Note that $C_0M/I_A = CM/I_A \cong C(A)$. We write $r_A : C_0(A) \to y$ for the induced isomorphism. Note that $r_A(f|_A) = r(f)$ for all $f \in C_0 = M$. Since A is closed in the compact set M, it is again compact and fulfils $C(A) = C_0(A)$.

In either case, the pullback $r_A^* : \hat{y} \to \widehat{C_0A},\ \chi \mapsto \chi \circ r_A$ is a homeomorphism, as is the canonical map $\mathrm{ev} : A \to \widehat{C_0A},\ p \mapsto \mathrm{ev}_p,\ \mathrm{ev}_p(f) = f(p)$. So let $\nu := (\mathrm{ev}^{-1} \circ r_A^*)_*(\nu')$ be the image measure under these maps. Because both are homeomorphisms, this ν is a Radon measure on A, and we have $\nu(A) = \nu'(\hat{y}) < \infty$, so ν is still bounded.

We pull back the Hilbert field $\mathcal{H}'$ by the same homeomorphisms, defining $\mathcal{H} \to_\nu A$ by $H_p := H'_{(r_A^*)^{-1} \circ \mathrm{ev}(p)}$, and the set of measurable sections as pullbacks of those on $\mathcal{H}'$, which is indeed a Hilbert field over ν with $L^2(\mathcal{H}, \nu) = ((r_A^*)^{-1} \circ \mathrm{ev})^* L^2(\mathcal{H}', \nu')$ almost by definition.

For the isomorphism, define $\phi := \psi \circ ((r_A^*)^{-1} \circ \mathrm{ev})^*)^{-1} : L^2\mathcal{H} \to H$, which is unitary as a composition of unitaries, where $((r_A^*)^{-1} \circ \mathrm{ev})^*$ is unitary because $\int_A ((r_A^*)^{-1} \circ \mathrm{ev})^* f \mathrm{d}\nu = \int_{\hat{y}} f \mathrm{d}\nu'$ for $f \in L^1(\hat{y})$ by definition of the image measure.

Now let us check the desired property: Let $f \in C_0^\infty(M)$ be arbitrary. Then because $\overline{R(m_f)} = r(f) = r_A(f|_A)$, we have

$$\begin{aligned}\phi^{-1} \circ \overline{R(m_f)} \circ \phi &= ((r_A^*)^{-1} \circ \mathrm{ev})^* \circ \psi^{-1} \circ \overline{R(m_f)} \circ \psi \circ (((r_A^*)^{-1} \circ \mathrm{ev})^*)^{-1} \\ &= \mathrm{ev}^* \circ ((r_A^*)^*)^{-1} \circ T_{\widehat{r_A(f|_A)}} \circ (r_A^*)^* \circ (\mathrm{ev}^*)^{-1},\end{aligned}$$

hence for $\sigma \in L^2H$ and $p \in A$:

$$\begin{aligned}(\phi^{-1} \circ \overline{R(m_f)} \circ \phi)(\sigma)(p) &= \left(T_{\widehat{r_A(f|_A)}}(\sigma \circ \mathrm{ev}^{-1} \circ r_A^*)\right) \circ (r_A^{-1})^* \circ \mathrm{ev}(p) \\ &= \widehat{r_A(f|_A)}((r_A^{-1})^* \mathrm{ev}_p) \cdot \sigma(p) = (\mathrm{ev}_p \circ r_A^{-1})(r_A(f|_A)) \cdot \sigma(p) \\ &= \mathrm{ev}_p(r_A^{-1} r_A(f|_A))\sigma(p) = \mathrm{ev}_p(f|_A)\sigma(p) \\ &= f(p)\sigma(p) = T_{f|_A}(\sigma)(p)\end{aligned}$$

To get a Hilbert field on the whole of M even in the case where $A \neq M$, we simply extend by 0. Formally, we set $\nu^+ := (\iota_A)_*\nu$ for the inclusion map $\iota_A : A \to M$ and define $\mathcal{H}^+ \to M$ by $H_p^+ := H_p$ for $p \in H$ and $H_p^+ = 0$ for $p \in M \setminus A$. If M is compact, ν^+ is a Radon measure because ι_A is proper. We see immediately that the restriction map $\psi : L^2(\mathcal{H}^+, \nu^+) \to L^2(\mathcal{H}, \nu),\ \sigma \mapsto \sigma|_A$ is an isomorphism in this case. It fulfils $\psi^{-1} \circ T_{f|_A} \circ \psi = T_f$ for all $f \in C_0M$ because sections in $L^2(\mathcal{H}^+)$ are already 0 outside of A by definition of $\mathcal{H}^+$. Hence we get that

$$(\phi \circ \psi)^{-1} \circ \overline{R(m_f)} \circ \phi \circ \psi = \psi^{-1} \circ T_{f|_A} \circ \psi = T_f$$

for all $f \in C_0M$.

Thus the first statement of this theorem is fulfilled with the measure ν^+, the Hilbert field $\mathcal{H}^+$ and the unitary map $\psi \circ \phi$.

Now consider again the case where $R|_{C^\infty M}$ (and thus r) is injective. As mentioned before, we have $\nu' = \nu_x$ for some $x \in H$, which is cyclic with respect to y'. Using our previous construction, we find that

$$\int_M f \mathrm{d}\nu = \int_{\hat{y}} f \circ \mathrm{ev}^{-1} \circ r^* \mathrm{d}\nu_x = \langle (f \circ \mathrm{ev}^{-1} \circ r^*)^\wedge(x), x\rangle = \langle r(f)x, x\rangle_H$$

because

$$\chi((f \circ \mathrm{ev}^{-1} \circ r^*)^\wedge) = f \circ \mathrm{ev}^{-1} \circ r^*(\chi) = f \circ \mathrm{ev}^{-1}(\chi \circ r) = \chi \circ r(f) = \chi(r(f))$$

for all $\chi \in \hat{y}$. We have proven before that $\phi^{-1}r(f)\phi = T_f$ and that ϕ is unitary, thus

$$\int_M f \mathrm{d}\nu = \langle \phi T_f \phi^{-1} x, x\rangle = \langle T_f \phi^{-1} x, \phi^{-1} x\rangle = \int_M f \mathrm{d}\nu_{\phi^{-1}x}$$

for $f \in C_0^\infty M$, which extends to $f \in C_0 M$ by continuity. So because Radon measures are determined by their values on continuous functions, we find that $\nu = \nu_\sigma$ for $\sigma = \phi^{-1}x$. x is cyclic with respect to y', i.e. $y'x \subseteq H$ is dense. So because ϕ is unitary, $\phi^{-1}y'\phi\sigma = \phi^{-1}y'x \subseteq L^2\mathcal{H}$ is also dense. For any $T \in y'$ and $f \in C_0 M$, we know by the definition of the commutant that $Tr(f) = r(f)T$, thus $\phi^{-1}T\phi T_f = \phi^{-1}T\phi\phi^{-1}r(f)\phi = \phi^{-1}Tr(f)\phi = \phi^{-1}r(f)t\phi = T_f\phi^{-1}T\phi$. As this holds for all f, we find that $\phi^{-1}T\phi \in \{T_f | f \in C_0 M\}'$. This in turn is true for all T, thus $\phi^{-1}y'\phi \subseteq \{T_f | f \in C_0 M\}'$. In particular, $\{T_f | f \in C_0 M\}'\sigma \supseteq \phi^{-1}y'\phi\sigma$ must be dense in $L^2\mathcal{H}$, thus σ is cyclic with respect to $\{T_f | f \in C_0 M\}'$. □

This shows that in the context of Lie algebroid representation theory, any Hilbert space which is the domain of a representation can be assumed without loss of generality to be the space of square-integrable sections on a measurable field of Hilbert spaces over the base manifold.

The rest of this section will build on the more technical second statement from the above theorem to show that the constructed measure ν is actually equivalent to any non-zero smooth measure on M, in the case of the pair groupoid. To do so, we need one result which will only be proven in the next chapter of this thesis. I still decided to include these proofs here because it is thematically fitting.

LEMMA 5.2.9. *Let $G = M \times M$ be a pair groupoid with a volume form $\omega \in \Omega(M)$ and a Hilbert field $H \to M$ over a spectral Radon measure $\nu = \nu_x$ on M, for an element $x \in L^2H$ which is cyclic with respect to $\{T_f | f \in C_0 M\}'$. Suppose that there is a representation $R : \mathrm{Diff}(M) \to L^2(H)$ with $\overline{R(m_f)} = T_f$ for all $f \in C_0^\infty(M)$.*

Consider the group of diffeomorphisms generated by compactly supported vector fields, $\mathrm{Diff}_e(M) := \{\theta_1^X \mid X \in \mathfrak{X}_c(M)\}$, denoting by θ_1^X the time-1 flow of X.

Then ν is $\mathrm{Diff}_e(M)$-quasi-invariant, that is, $\nu \circ \phi^{-1} \ll \nu$ for all $\phi \in \mathrm{Diff}_e(M)$.

Proof: Let $\phi \in \mathrm{Diff}_e(M)$. Then by Lemma 6.1.3, there is a unitary operator $U : L^2H \to L^2H$ such that $T_{f\circ\phi} = UT_fU^*$ for all $f \in C_c^\infty(M)$. In particular, we have

$$\begin{aligned}\int_M f \mathrm{d}\nu \circ \phi^{-1} &= \int_M f \circ \phi \mathrm{d}\nu = \langle T_{f\circ\phi} x, x\rangle \\ &= \langle UT_fU^* x, x\rangle = \langle T_f U^* x, U^* x\rangle = \int_M f \mathrm{d}\nu_{U^*x}\end{aligned}$$

for all such f. Here, ν_{U^*x} is the spectral measure with respect to U^*x.

x was assumed to be cyclic with respect to $y' = \{T_f \mid f \in C_0(M)\}' \subseteq \mathbb{B}(L^2H)$, i.e. $y'x := \{Tx \mid T \in y'\} \subseteq L^2H$ is dense. Since U is unitary, this implies that $U^*y'x = \{U^*TU(U^*x) \mid T \in L^2H$ is also dense. Note that $U^*TUT_f = U^*TT_{f\circ\phi}U = U^*T_{f\circ\phi}TU = T_fU^*TU$ for all $f \in C_c^\infty(M)$. Because $C_0(M) \to \mathbb{B}(L^2H)$, $f \to T_f$ is continuous and $C_c^\infty(M) \subseteq C_0(M)$ is dense, this implies that even $U^*TUT_f = T_fU^*TU$ for all $f \in C_0M$, hence $U^*TU \in y'$. Thus $y'U^*x \subseteq L^2H$ is dense, too, so U^*x is cyclic for y'. Hence ν_{U^*x} is basic by Proposition 2, page 129 [**5**] and thus equivalent to the basic measure $\nu = \nu_x$ as detailed on page 127, [**5**], in particular, $\nu_{U^*x} \ll \nu$.

As discussed in the beginning, we have $\int_M f \mathrm{d}\nu \circ \phi^{-1} = \int_M f \mathrm{d}\nu_{U^*x}$ for all $f \in C_c(M)$. So because both $\nu \circ \phi^{-1}$ and ν_{U^*x} are Radon measures and M is locally compact, this implies that $\nu \circ \phi^{-1} = \nu_{U^*x} \ll \nu$ (this uniqueness property is a part of the Riesz-Markov representation theorem and follows from outer regularity of the measures). □

This lemma can, in particular, be applied to the measure constructed in Theorem 5.2.8, using the induced representation $\tilde{R} : \mathrm{Diff}(M) \to \mathcal{O}(L^2\mathcal{H}, \nu),\ D \mapsto \phi^{-1}R(D)\phi$ – this is actually the only case where it will be used. Let us proceed.

Lemma 5.2.10. *Keep the notation from the previous lemma. Let $\phi : U \to \mathbb{R}^m$ be a chart of M. Then the image measure $\nu \circ \phi^{-1}$ on $\mathbb{R}^m$ is $\mathbb{R}^m$-quasi-invariant, that is, translations preserve null sets.*

Proof: Put $\mu = \nu \circ \phi^{-1}$. Let $A \subseteq \mathbb{R}^m$ be any Borel set. Notice that $\mathbb{R}^m = \bigcup_{I \in \mathbb{Z}^m} I + [0,1]^m$, so $\mu(A) \leq \sum_{I \in \mathbb{Z}^m} \mu(A_I)$, where $A_I = A \cap I + [0,1]^m$.

Now assume that A is a μ-null set. Let $c \in \mathbb{R}^m$ be arbitrary. Choose any $I \in \mathbb{Z}^m$. Then we know that $\mu(A_I) \leq \mu(A) = 0$. We construct a vector field X_c as follows: Set $r = \|I\| + 2 + \|c\|$, so that $tc + x \in B_r(0)$ for all $t \in [-1,1]$ and $x \in I + [0,1]^m$, where $B_d(0) = \{x \in \mathbb{R}^m \mid \|x\| \leq d\}$. Choose a smooth bump function $h \in C_c^\infty(\mathbb{R}^m)$ with $h|_{B_{5r}(0)} \equiv 1$. Define $X_c = h \cdot c\partial = h \sum_{i=1}^m c_i \partial_i$, which is compactly supported with $X_c|_{B_{2r}(0)} \equiv c\partial|_{B_{2r}(0)}$. So for each $(t,p) \in [0,1] \times B_{2r}(0)$ we have $\theta^{c\partial}(t,p) = p + tc \in B_{5r}(0)$ and hence $\theta^{X_c}(t,p) = \theta^{c\partial}(t,p) = p + tc$. Let $\psi = \theta_1^{X_c}$ be the flow at time 1. Then in particular, $\psi(p) = p + c$ for all $p \in A_I$.

Consider the pullback vector field $\phi^* X \in X_c(U)$. Its flow at time 1 is $\phi^{-1} \circ \psi \circ \phi$. Having compact support, this vector field extends to a global vector field $Y \in X_c(M)$, with the same flow inside of U. Extending by the identity, we find that $\tilde{\psi} = \phi^{-1} \circ \psi \circ \phi \in \mathrm{Diff}_e(M)$. Thus by Lemma 5.2.9, ν is quasi-invariant with respect to $\tilde{\psi}$.

We know that $0 = \mu(A_I) = \nu(\phi^{-1}(A_I))$, hence this implies that also $\mu(A_I + c) = \mu(\psi(A_I)) = \nu(\phi^{-1}\psi\phi\phi^{-1}(A_I)) = 0$. Consequently, $\mu(A + c) \leq \sum_{I \in \mathbb{Z}^m} \mu(A_I + c) = 0$. Since A was an arbitrary null set, μ is quasi-invariant with respect to translation by c. Because $c \in \mathbb{R}$ was arbitrary, this implies that μ is $\mathbb{R}$-quasi-invariant. ☐

We can now use a powerful theorem on quasi-invariance of measures to conclude with the desired result as follows:

Proposition 5.2.11. *Let (M, ω) be a volumetric manifold, H a Hilbert space and let $R : \mathrm{Diff}^R(M \times M) \to \mathcal{O}(H)$ be a representation. Suppose that $R|_{C^\infty M}$ is injective. Then the measure ν stipulated by Theorem 5.2.8 is equivalent to μ_ω.*

Proof: Let $\phi : U \to \mathbb{R}^m$ be a chart of M. By Lemma 5.2.10, the image measure $\mu_\phi = \nu \circ \phi^{-1}$ is $\mathbb{R}^m$-quasi-invariant. By [**3**], page 20, Proposition 11, this implies that μ_ϕ is equivalent to the left Haar measure on $\mathbb{R}^m$, i.e. the Lebesgue measure (on the Borel σ-algebra). So there is a Borel measurable function $h_\phi = \frac{\mu_\phi}{\lambda} : \mathbb{R}^m \to \mathbb{R}_{>0}$ (the Radon-Nikodým derivative with respect to the Lebesgue measure λ) such that $\mu_\phi = h_\phi \lambda$. Thus we find that $\nu|_U = \mu_\phi \circ \phi = h_\phi \circ \phi \lambda \circ \phi$.

The measure $\lambda \circ \phi$ is given by the volume form $\phi^* \mathrm{d}x \in \Omega^m(U)$ and thus equivalent to $\mu_\omega|_U$. Furthermore, $h_\phi \circ \phi$ is still Borel measurable (and non-zero) because this property is preserved by concatenations with diffeomorphisms. Thus there is a new Borel measurable function $h_U = h_\phi \circ \phi \cdot \frac{\lambda \circ \phi}{\mu_\omega|_U}$ with $\nu|_U = h_U \mu_\omega|_U$.

To conclude, we cover M by a countable family $(U_i)_{i \in \mathbb{N}}$ of subsets diffeomorphic to $\mathbb{R}^m$ and define $h : M \to \mathbb{R}_{>0},\ h(p) = h_i(p)$ for $p \in U_i$. This is well-defined for almost all $p \in M$ because Radon-Nikodým derivatives are unique up to deviations on null sets; the set $\{p \in M \mid \exists (i,j) \in \mathbb{N}^2 : p \in U_i \cap U_j \text{ and } h_i(p) \neq h_j(p)\}$ must still be a null set. h is locally Borel measurable and thus Borel measurable, non-zero almost everywhere and fulfils $\nu = h\mu_\omega$ by construction. ☐

In (pair) groupoid representation theory, we actually need equivalence not of the measures on the base space, but of their respective product measures. However, this can quickly be shown to follow:

LEMMA 5.2.12. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Let μ_1 and μ_2 be two σ-finite measures on $\mathcal{A}$. Likewise, let ν_1 and ν_2 be two σ-finite measure on $\mathcal{B}$.*

If $\mu_1 \ll \mu_2$ and $\nu_1 \ll \nu_2$, then the respective product measures fulfil $\mu_1 \times \nu_1 \ll \mu_2 \times \nu_2$.

Proof: By assumption, we are only using σ-finite measures. So by the Radon-Nikodým Theorem, we find densities $m = \frac{\mu_1}{\mu_2}$ and $n = \frac{\nu_1}{\nu_2}$ which are measurable with respect to $\mathcal{A}$ respectively $\mathcal{B}$ and the Borel σ-algebra on $\mathbb{R}$. Hence their product $mn : X \times Y \to \mathbb{R}_{\geq 0}$, $(x, y) \mapsto m(x)n(y) = m \circ \mathrm{pr}_A \cdot n \circ \mathrm{pr}_B$ is $\mathcal{A} \otimes \mathcal{B}$-Borel measurable, because the projections $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ are always measurable with respect to the product σ-algebra.

Now for any measurable function $f : X \times Y \to \mathbb{R}_{\geq 0}$, we know by Fubini's theorem that

$$\begin{aligned}
\int_{X \times Y} f mn \,\mathrm{d}\mu_2 \times \nu_2 &= \int_X \int_Y f(x, y) m(x) n(y) \,\mathrm{d}\nu_2(y) \,\mathrm{d}\mu_2(x) \\
&= \int_X \int_Y f(x, y) \mathrm{d}\nu_1(y) m(x) \mathrm{d}\mu_2(x) \\
&= \int_X \int_Y f(x, y) \mathrm{d}\nu_1(y) \mathrm{d}\mu_1(x) = \int_{X \times Y} f \,\mathrm{d}\mu_1 \times \nu_1,
\end{aligned}$$

and thus $\mu_1 \times \nu_1 = mn \cdot \mu_2 \times \nu_2$ is a usable density function, which means that $\mu_1 \times \nu_1 \ll \mu_2 \times \nu_2$. ☐

We conclude with one more proposition:

PROPOSITION 5.2.13. *Let (M, ω) be a volumetric manifold, H a Hilbert space and let $R : \mathrm{Diff}^R(M \times M) \to \mathcal{O}(H)$ be a representation. Suppose that $R|_{C^\infty M}$ is injective. Let ν be the measure stipulated by Theorem 5.2.8. Then $\mu_\omega \times \nu$ is equivalent to $\nu \times \mu_\omega$ on $M \times M$.*

Proof: By Proposition 5.2.11, ν is equivalent to μ_ω. Set $\mu_1 = \nu_2 = \mu$ and $\mu_2 = \nu_1 = \nu$ to find that $\mu \times \nu \ll \nu \times \mu$, by Lemma 5.2.12. This also works the other way around, so both product measures are indeed equivalent. ☐

In the next section, I will discuss quasi-invariance of measures on groupoids more formally. We will see how it applies to our given setting.

5.3. Measurable Homomorphisms of Groupoids

At this point, we have worked with representations of the groupoid C^*-algebra and of (invariant) differential operators, and even proven a differentiation theorem. What we have not yet investigated are representations of a Lie groupoid itself. As explained at the beginning of this chapter, those are an important step in the integration theory. To allow their usage in the next chapter, this section will serve as an introduction to representations of groupoids.

The concept is actually quite simple: A representation of a Lie groupoid will be a homomorphism between it and the unitary groupoid of a Hilbert field. A homomorphism of groupoids is, at first glance, just a map P fulfilling $P(gh) = P(g)P(h)$ for all g, h in the groupoid which are composable; in category theory, this is called a functor. However, in the context of Hilbert fields, we need a *measurable* theory, where such homomorphisms are possibly undefined on a null set within the groupoid.

To accurately and consistently account for such null sets, we first need to define a measure on the morphism space. Before, we have already worked with measures on both the fibres and the base space of a groupoid. One can quite easily combine the two to get the following definition.

DEFINITION 5.3.1. Let $G \rightrightarrows M$ be a locally compact groupoid with a Haar system α. Let ν be a Radon measure on M. In this situation, define the following measures:

- ν^1 (also denoted $\nu \circ \alpha$) on G by
$$\int_G f \mathrm{d}\nu^1 := \int_M \int_{G^p} f(g) \mathrm{d}\alpha^p(g) \mathrm{d}\nu(p)$$
for all $f \in C_c G$
- ν^2 on $G^{(2)} = \{(g,h) \in G \times G | sg = th\}$ by
$$\int_{G^{(2)}} f \mathrm{d}\nu^2 = \int_M \int_{G^p} \int_{G^{sg}} f(g,h) \mathrm{d}\alpha^{sg}(h) \mathrm{d}\alpha^p(g) \mathrm{d}\nu(p) = \int_G \int_{G^{sg}} f(g,h) \mathrm{d}\alpha^{sg}(h) \mathrm{d}\nu^1(g)$$
for all $f \in C_c G^{(2)}$

Notably, there are more ways to canonically define measures on G and its higher products when given a Haar system and a measure on the base space. For example, we can define $(\nu^1)' = \nu \circ \tilde{\alpha}$ by $\int_G f \mathrm{d}\nu \circ \tilde{\alpha} = \int_M \int_{G_p} f(g) \mathrm{d}\alpha_p(g) \mathrm{d}\nu(p)$, where $\alpha_p = \mathrm{inv}_* \alpha^p$ for the inversion map $\mathrm{inv} : G \to G,\ g \mapsto g^{-1}$.

A further discussion of this can be found in [**4**], page 7. For now it suffices to consider the measures above because all the methods yield equivalent measures (i.e. the same null sets) in relevant situations. A measure ν for which $\nu \circ \alpha$ and $\nu \circ \tilde{\alpha}$ have the same null sets is called *quasi-invariant*.

As we are using pair groupoids so often, it seems natural to look at them for an example of these freshly defined measures.

Example 5.3.2. Let M be a smooth manifold with a volume form ω and a bounded Radon measure ν. Consider the corresponding volumetric pair groupoid $(M \times M \rightrightarrows M, \mathrm{pr}_2^* \omega)$. Let $f : G^{(2)} \to \mathbb{R}$ be ν^2-measurable. Denote the corresponding Haar system by $\alpha = (\alpha^p)_{p \in M} = (\mu_{\mathrm{pr}_2|^*_{G^p}\omega})_{p \in M}$. Then we have
$$\begin{aligned} \int_{G^{(2)}} f \mathrm{d}\nu^2 &= \int_M \int_{G^p} \int_{G^{sg}} f(g,h) \mathrm{d}\alpha^{sg}(h) \mathrm{d}\alpha^p(g) \mathrm{d}\nu(p) \\ &= \int_M \int_{G^p} \int_M f(g,(sg,q)) \mathrm{d}\mu_\omega(q) \mathrm{d}\alpha^p(g) \mathrm{d}\nu(p) \\ &= \int_M \int_M \int_M f((p,w),(w,q)) \mathrm{d}\mu_\omega(q) \mathrm{d}\mu_\omega(w) \mathrm{d}\nu(p) \end{aligned}$$
because $\int_M \phi^* \omega = \int_N \omega$ for any diffeomorphism $\phi : M \to N$ between smooth manifolds M and N with a top-degree form ω on N.

Furthermore, we find for $f : G \to \mathbb{R}$ that
$$\int_G f \mathrm{d}\nu \circ \alpha = \int_M \int_{G^p} f(g) \mathrm{d}\alpha^p(g) \mathrm{d}\nu(p) = \int_M \int_M f(p,q) \mathrm{d}\mu_\omega(q) \mathrm{d}\nu(p) = \int_G f \mathrm{d}\nu \times \mu_\omega$$
and
$$\int_G f \mathrm{d}\nu \circ \tilde{\alpha} = \int_M \int_{G_p} f(g) \mathrm{d}\alpha_p(g) \mathrm{d}\nu(p) = \int_M \int_M f(p,q) \mathrm{d}\mu_\omega(p) \mathrm{d}\nu(q) = \int_G f \mathrm{d}\mu_\omega \times \nu,$$
hence ν is quasi-invariant if and only if $\nu \times \mu_\omega$ is equivalent to $\mu_\omega \times \nu$.

So as it turns out, the canonical measure on the pair groupoid is quite easy to use for integration.

Having seen this, we can now formally define *essential* homomorphisms of groupoids, where the word essential is meant to point to the fact that null sets are neglected, like with the essential supremum, for example.

Definition 5.3.3. Let $G \rightrightarrows M$ be a locally compact groupoid with Haar system α and a measure ν on M. Let H be any groupoid over M.

An essential homomorphism of first type from G to H is a pair (D, R), where $D \subseteq G$ is a co-null set (i.e. $\nu^1(G \setminus D) = 0$) and $R : D \to H$ is a map which fulfils:

- $s \circ R = s|_D$ and $t \circ R = t|_D$.

- There is a set $N \subseteq G^{(2)}$ with $\nu^2(N) = 0$ such that for all $g, h \in D$ with $sg = th$, $gh \in D$ and $(g, h) \notin N$, we have $R(gh) = R(g)R(h)$.

An essential homomorphism of second type from G to H is a pair (D, R), where again $D \subseteq G$ is co-null and $R : D \to H$ fulfils:

- $s \circ R = s|_D$ and $t \circ R = t|_D$.
- $R(gh) = R(g)R(h)$ for all $g, h \in D$ such that $sg = th$ and $gh \in D$.

Now that we have seen this concept, let us immediately also look at the *local* variant. There our homomorphisms are only defined in a neighbourhood of the diagonal in the groupoid. Besides that, the idea stays the same, resulting in the following definition. Note that the special case $W = G$ is precisely the definition of a global essential homomorphism.

Definition 5.3.4. Let G, H, M, ν as before.

A local essential homomorphism of first type is a pair (D, R), where $D = W \setminus N$ for an open subset $W \subseteq G$ with $M \subseteq W$ such that $W \cap G^p$ is path-connected for all $p \in M$ and a null set $N \subseteq G$, and $R : D \to H$ is such that there exists a null set $N_2 \subseteq G^{(2)}$ with $R(gh) = R(g)R(h)$ for all $g, h \in D$ with $sg = th$, $gh \in D$ and $(g, h) \notin N_2$.

A local essential homomorphism of second type is a pair (D, R), where $D = W \setminus N$ for an open subset $W \subseteq G$ with $M \subseteq W$ such that $W \cap G^p$ is path-connected for all $p \in M$ and a null set $N \subseteq G$, and $R : D \to H$ is such that $R(gh) = R(g)R(h)$ for all $g, h \in D$ with $gh \in D$ and $sg = th$.

But why do we need two different definitions each time? As it turns out, the measurable theory of groupoid morphisms spawns a variety of rather inconvenient detail questions. The *first type* definition is, in my opinion, the most intuitive one: Not only in the domain of definition but also in the homomorphy check, a variation on null sets is possible. This is also what we will get at the end of some of the upcoming theorems. In contrast, the second definition is slightly shorter and easier to work with, and as such best suited as a proposition's premise.

A priori, the second definition is a stronger requirement: There may be essential homomorphisms of first type which are not essential homomorphisms of second type. However, after a short time of getting used to these objects, it turns out that both definitions are *mostly* equivalent: Every first type homomorphism is also a second type homomorphism, but on a neglectably smaller domain. A comparable result in a different context was proven in Theorem 3.2, page 328 of [**20**].

This equivalence is what I want to show next. As usual, we need another lemma first.

Lemma 5.3.5. *Let $N \subseteq G$ be a null set, i.e. $\nu^1(N) = 0$. Then for ν^1-almost all $g \in G$, the set $N_g := (G^{sg} \cap N) \cup (g^{-1}(G^{tg} \cap N))$ is a null set, i.e. $\alpha^{sg}(N_g) = 0$.*

Proof: We know by assumption that

$$0 = \nu^1(N) = \int_M \alpha^p(N \cap G^p)\mathrm{d}\nu(p),$$

hence there is a ν-null set $P \subseteq M$ such that for all $p \in M \setminus P$, $\alpha^p(N \cap G^p) = 0$. Furthermore, by the invariance of a Haar system, we have $\alpha^{sg}(g^{-1}(G^{tg} \cap N)) = \alpha^{tg}(G^{tg} \cap N)$. So for all $g \in G$ with $sg \notin P$ and $tg \notin P$, we have $\alpha^{sg}(N_g) \leq \alpha^{sg}(N \cap G^{sg}) + \alpha^{tg}(N \cap G^{tg}) = 0$. It is left to show that $Q := \{g \in G | sg \in P \text{ or } tg \in P\}$ is a ν^1-null set.

Obviously we have $Q = Q_1 \cup Q_2$ for $Q_1 = t^{-1}P$ and $Q_2 = s^{-1}P$. We have $\nu^1(Q_1) = \int_M \alpha^p(G^p \cap Q_1)\mathrm{d}\nu(p) = \int_M \alpha^p(G^p)\chi_P(p)\mathrm{d}\nu(p) = 0$ since $\nu(P) = 0$. Furthermore, denote by $\tilde{\nu}^1 = \nu \circ \tilde{\alpha}$ the measure on G induced by the right-invariant Haar system $\tilde{\alpha}$ corresponding to α. Then $\tilde{\nu}^1(Q_2) = \int_M \alpha_p(G_p \cap Q_2)\mathrm{d}\nu(p) = \int_M \alpha_p(G_p)\chi_P(p)\mathrm{d}\nu(p) = 0$ since $\nu(P) = 0$. Since ν is quasi-invariant by assumption, ν^1 is absolutely continuous with respect to $\tilde{\nu}^1$ (and vice versa), hence also $\nu^1(Q_2) = 0$. This implies $\nu^1(Q) \leq \nu^1(Q_1) + \nu^1(Q_2) = 0$, which finishes the proof. $\square$

Using this lemma, the proof of the actual fact is not hard, albeit a bit technical. The main idea is to show that a certain set has positive measure and hence cannot be empty, involving a few set-theoretic computations.

PROPOSITION 5.3.6. *Let $G \rightrightarrows M$, α, ν, H, r as before. Let (D, R) be an essential local homomorphism of first type from G to H. Then there is a subset $\tilde{D} \subseteq D$ with $\nu^1(D \setminus \tilde{D}) = 0$ such that $(\tilde{D}, R|_{\tilde{D}})$ is a homomorphism of second type.*

In particular, if (D, R) is global, then $(\tilde{D}, R|_{\tilde{D}})$ is a global homomorphism of second type.

Proof: Let $D = W \setminus Z$ for $W \subseteq G$ open with $M \subseteq W$ and $Z \subseteq G$ a null set. Let $N \subseteq G^{(2)}$ be a set with $\nu^2(N) = 0$ and $R(gh) = R(g)R(h)$ for all composable $g, h \in W$ with $gh \in W \setminus Z$ and $(g, h) \notin N$. Then by the definition of ν^2, we have

$$0 = \nu^2(N) = \int_{G^{(2)}} \chi_N \mathrm{d}\nu^2 = \int_G \int_{G^{sg}} \chi_N(g, h) \mathrm{d}\alpha^{sg}(h) \mathrm{d}\nu^1(g),$$

where χ_N is the characteristic function of N, so by elementary measure theory and $\chi_N \geq 0$, there is a set $N_1 \subseteq G$ with $\nu^1(N_1) = 0$ such that $\int_{G^{sg}} \chi_N(g, h) \mathrm{d}\alpha^{sg}(h) = 0$ for all $g \in G \setminus N_1$. Hence for all $g \in G \setminus N_1$ and α^{sg}-almost all $h \in G^{sg}$, $(g, h) \notin N$.

Put $C := G \setminus Z$. Since $l_g : G^{sg} \to G^{tg}$ is a bijection for all $g \in G$, we have the following set theoretic computation:

$$\begin{aligned} G^{sg} \setminus g^{-1}(G^{tg} \cap C) &= g^{-1}g(G^{sg} \setminus g^{-1}(G^{tg} \cap C)) \\ &= g^{-1}(gG^{sg} \setminus gg^{-1}(G^{tg} \cap C)) = g^{-1}(G^{tg} \setminus (G^{tg} \cap C)) = g^{-1}(G^{tg} \setminus C), \end{aligned}$$

hence

$$\begin{aligned} G^{sg} \setminus (G^{sg} \cap C \cap g^{-1}(G^{tg} \cap C)) &= (G^{sg} \setminus C) \cup (G^{sg} \setminus (g^{-1}(G^{tg} \cap C))) \\ &= (G^{sg} \setminus C) \cup g^{-1}(G^{tg} \setminus C) = (G^{sg} \cap (G \setminus C)) \cup (g^{-1}(G^{tg} \cap (G \setminus C))) \\ &= (G^{sg} \cap Z) \cup (g^{-1}(G^{tg} \cap Z)) \end{aligned}$$

By Lemma 5.3.5, there is a null set $N_2 \subseteq G$ such that $(G^{sg} \cap Z) \cup (g^{-1}(G^{tg} \cap Z))$ is an α^{sg}-null set for all $g \in G \setminus N_2$, i.e. $C_g := G^{sg} \cap C \cap g^{-1}(G^{tg} \cap C)$ is co-null by the computation before.

Now for all $g \in D$, put $D_g := G^{sg} \cap D \cap g^{-1}(G^{tg} \cap D)$. We have $D = W \setminus Z = W \cap C$, so $D_g = G^{sg} \cap W \cap C \cap g^{-1}(G^{tg} \cap W \cap C) = (W \cap G^{sg} \cap g^{-1}(G^{tg} \cap W) \cap C_g$. For all $g \in W$, we obviously have $sg = g^{-1}g \in g^{-1}(G^{tg} \cap W)$ and $sg \in W \cap G^{sg}$ since $M \subseteq W$. So $(W \cap G^{sg} \cap g^{-1}(G^{tg} \cap W) \subseteq G^{sg}$ is non-empty as well as open, hence it has strictly positive measure. Hence also $\alpha^{sg}(D_g) > 0$.

Set $\tilde{D} := D \setminus (N_1 \cup N_2)$, let $g \in \tilde{D} \cap W = W \setminus (N_1 \cup N_2 \cup Z)$ and define $R_h(g) := R(gh)R(h)^{-1}$ for all $h \in D_g = G^{sg} \cap D \cap g^{-1}(G^{tg} \cap D)$. Then as shown in the first part, for almost all $h \in G^{sg}$ we have $(g, h) \notin N$, so for almost all $h \in D_g \subseteq G^{sg}$, we have

$$R_h(g) = R(gh)R(h)^{-1} = R(g)R(h)R(h)^{-1} = R(g)$$

by the defining property of a first-type homomorphism.

Now let $g, h \in \tilde{D} \cap W$ be arbitrary with $sg = th$ and $gh \in \tilde{D} \cap W$. Consider the set $P := \{h^{-1}x | x \in G^{sg},\ (g, x) \in N\} \cup \{y \in G^{sh} | (gh, y) \in N\} \cup \{y \in G^{sh} | (h, y) \in N\}$. Because $g, h, gh \in G \setminus N_1$ and because α is left-invariant, P is still an α^{sh}-null set. Then for almost all $y \in D_h \setminus P \subseteq G^{sh}$, we have

$$\begin{aligned} R(g)R(h) &= R(g)R_y(h) = R(g)R(hy)R(y)^{-1} \\ &= R(ghy)R(y)^{-1} = R(gh)R(y)R(y)^{-1} = R(gh) \end{aligned}$$

because $(h, y) \notin N$, $(g, hy) \notin N$ and $(gh, y) \notin N$. $D_h \setminus P$ still has strictly positive measure, so it is non-empty in particular, which implies that indeed $R(g)R(h) = R(gh)$. This shows that $(\tilde{D}, R|_{\tilde{D}})$ is an essential homomorphism of second type, with the same open neighbourhood of the diagonal W as for the original (D, R). $\square$

So it is now clear that the difference between the two definitions is really just a technical one (which does not mean it will not be of use). Having that settled, we can now proceed to the more advanced part of this section. We will soon prove that, like ordinary Lie group homomorphisms, a local essential homomorphism from a Lie groupoid to another groupoid can be extended to a global one if the source fibres are simply connected.

The key idea for this theorem will be, as for Lie groups, to take a path from the identity to any point in the groupoid, cut it in small pieces and define the global homomorphism by products of these small pieces. Of course more detail will follow soon, but this extension theorem requires a few more preliminaries to handle the technical details.

Let us start proving the necessary lemmas now. The assumptions of the first one may seem a bit arbitrary, but the important part here is that they are fulfilled almost everywhere if N is a null set.

LEMMA 5.3.7. *Let $p \in M$. Let $N \subseteq G$ be a set. For all $i \in \mathbb{N}$, let $h_i \in G^p$ such that $\alpha^{sh_i}(G^{sh_i} \cap (N \cup N^{-1})) = 0$. Let $g \in G^p$ and let $U \subseteq G^p$ be an open neighbourhood of g. Then there is an element $h \in U$ such that $h_i^{-1}h \notin N$ and $h^{-1}h_i \notin N$ for all $i \in \mathbb{N}$.*

Furthermore, if $\alpha^p(s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\} \cap G^p) = 0$, then h can be chosen with the former property such that $\alpha^{sh}(G^{sh} \cap (N \cup N^{-1})) = 0$.

Proof: We have $\alpha^p(h_i(G^{sh_i} \cap (N \cup N^{-1})) = \alpha^{sh_i}(G^{sh_i} \cap (N \cup N^{-1})) = 0$ for each i, so the countable union $\bigcup_{i\in\mathbb{N}} h_i(G^{sh_i} \cap (N \cup N^{-1}))$ is still a null set. Because α^p has full support as part of a Haar system and U is non-empty and open, we have $\alpha^p(U) > 0$. Hence $U \setminus \bigcup_{i\in\mathbb{N}} h_i(G^{sh_i} \cap (N \cup N^{-1}))$ has strictly positive measure, which means it is non-empty. So choose an element $h \in U \setminus \bigcup_{i\in\mathbb{N}} h_i(G^{sh_i} \cap (N \cup N^{-1}))$.

Then for each $i \in \mathbb{N}$, we have $h \notin h_i(G^{sh_i} \cap N)$, so $h_i^{-1}h \notin G^{sh} \cap N)$. Clearly $t(h_i^{-1}h) = sh_i$, so $h_i^{-1}h \in G^{sh_i} \setminus N$. Furthermore, we have $h \notin h_i(G^{sh_i} \cap N^{-1})$, hence $h_i^{-1}h \notin G^{sh_i} \cap N^{-1}$ (but still $h_i^{-1}h \in G^{sh_i}$), hence $h^{-1}h_i = (h_i^{-1}h)^{-1} \notin N = (N^{-1})^{-1}$. This proves the first required property.

Now assume that $\alpha^p(s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\} \cap G^p) = 0$. Then the set $U \setminus (s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\} \cup \bigcup_{i\in\mathbb{N}} h_i(G^{sh_i} \cap (N \cup N^{-1})))$ still has the same positive measure as U, hence is non-empty. So choose h from this set. Then in addition to $h_i^{-1}h \notin N$ and $h^{-1}h_i \notin N$ as before, we have $h \notin s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\}$, i.e. $sh \in \{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) = 0\}$, i.e. $\alpha^{sh}(G^{sh} \cap (N \cup N^{-1})) = 0$. $\square$

This lemma is designed to carefully avoid certain null sets in our extension theorem, which is necessary because essential homomorphisms are not defined everywhere. The second part of the lemma allows us to use it for an induction process.

The second lemma will now build on the first one to give us sequences of points in groupoid fibres, which not only avoid a null set themselves, but also are arranged such that their products avoid it.

LEMMA 5.3.8. *Let $W \subseteq G$ be an open set with $M \subseteq W$. Let $N \subseteq G$ be any set such that $N \cap M = \emptyset$. Let $p \in M$ be such that G^p is path-connected, $\alpha^p(s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\} \cap G^p) = 0$, $\alpha^p(G^p \cap (N \cup N^{-1})) = 0$. Let $g \in G^p$ such that $g \notin N \cup N^{-1}$ and $\alpha^{sg}(G^{sg} \cap (N \cup N^{-1})) = 0$.*

Then there are finitely many elements $h_0 = p, h_1, \ldots, h_{n-1}, h_n = g$ such that $h_i^{-1}h_{i+1} \in W$ for all $i \in \{0, \ldots, n-1\}$ and $h_i^{-1}h_j \notin N$ for all $i, j \in \{0, \ldots, n\}$.

Proof: Let $\gamma : [0, 1] \to G^p$ be a path from p to g. Then since γ and the multiplication in G are continuous and $[0, 1]$ is compact, there is a positive $\delta > 0$ such that for all $a, b \in [0, 1]$ with $|a - b| < \delta$, $\gamma(a)^{-1}\gamma(b) \in W$. Choose $n \in \mathbb{N}$ with $n > \frac{1}{\delta}$ and set $g_i := \gamma(\frac{i}{n})$ for all

$i \in \{0, \dots, n\}$. Then we have $g_i^{-1} g_{i+1} \in W$ for all $i \in \{0, \dots, n-1\}$. Since the multiplication is continuous and W is open, we can choose an open neighbourhood U_i of g_i for each $i \in \{0, \dots, n\}$ such that $U_i^{-1} U_{i+1} \subseteq W$ for all $i \in \{0, \dots, n-1\}$.

Now we use Lemma 5.3.7 for an induction argument: Set $h_0 = g_0 = p \notin N$ and $h_n = g_n = g$. By assumption we have $\alpha^p(s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\} \cap G^p) = 0$, $\alpha^{sg}(G^{sg} \cap (N \cup N^{-1})) = 0$ and $\alpha^{sh_0}(G^{sh_0} \cap (N \cup N^{-1})) = 0$, which gives the beginning of the induction.

For each $i \in \{1, \dots, n-1\}$, choose, using the lemma, an element $h_i \in U_i$ such that $\alpha^{sh_i}(G^{sh_i} \cap (N \cup N^{-1})) = 0$, $h_j^{-1} h_i \notin N$ and $h_i^{-1} h_j \notin N$ for all $j \in \{0, \dots, i-1\} \cup \{n\}$, which is possible since the requirements of the lemma are fulfilled in each step.

By this process we have made sure that for each $i \in \{1, \dots, n-1\}$ and each $j \in \{0, \dots, i-1, i+1, \dots, n\}$ we have $h_i^{-1} h_j \notin N$ and $h_j^{-1} h_i \notin N$. Furthermore, we have $h_i^{-1} h_i = s(h_i) \notin N$ for all i, $h_0^{-1} h_n = g \notin N$ and $h_n^{-1} h_0 = g^{-1} \notin N$. So indeed $h_i^{-1} h_j \notin N$ for all $i, j \in \{0, \dots, n\}$.

By the choice of the U_i, we have made sure that $h_i^{-1} h_{i+1} \in W$ for all $i \in \{0, \dots, n-1\}$ since $h_i \in U_i$. $\square$

Whenever it gets hard to remember all the technicalities, we just need to keep in mind that our helpful lemmas give us certain paths through the groupoid which are *good* in the sense that they avoid the undefined zones.

We need just one more lemma of this kind. This time, it is less about measures and more about topology and metrics.

LEMMA 5.3.9. *Let $G \rightrightarrows M$ be a Lie groupoid with a metric d such that the topology of G is induced by d. Let $p \in M$, K be a compact metric space and let $\gamma : K \to G^p$ be a continuous map. Let $W \subseteq G$ be open with $s \circ \gamma(K) \subseteq M \cap W$.*

Then there exists an $\epsilon > 0$ such that for all $x, y \in K$ and $g, h \in G^p$ with $d_K(x, y) < \epsilon$, $d(\gamma(x), g) < \epsilon$ and $d(\gamma(y), h) < \epsilon$, we have $g^{-1} h \in W$.

Proof: To begin with, on $G \times G$ (and a fortiori on $G^{(2)} \subseteq G \times G$), we use the product metric d^2 with $d^2((x, g), (y, h)) := d(x, y) + d(g, h)$, which induces the usual product topology. On K^2 we use d_K^2 defined by $d_K^2((a, b), (x, y)) = d_K(a, x) + d_K(b, y)$. Other metrics would work, too.

γ is continuous with a compact domain, hence it is uniformly continuous. The same holds for $\gamma_2 : K^2 \to G^{(2)}$, $(a, b) \mapsto (\gamma(a)^{-1}, \gamma(b))$. Let $\Delta = \{(a, a) | a \in K\} \subseteq K^2$ be the diagonal. We have $\mathrm{mult} \circ \gamma_2(a, a) = \gamma(a)^{-1} \gamma(a) = s \circ \gamma(a) \in W$ for all $a \in K$, hence $\gamma_2(\Delta) \subseteq \mathrm{mult}^{-1} W$, where $\mathrm{mult} : G^{(2)} \to G$ is the multiplication map. Because W is open and mult is continuous, $\mathrm{mult}^{-1} W$ is also open.

Because $\mathrm{mult}^{-1} W$ is open and $\gamma_2(\Delta) \subseteq \mathrm{mult}^{-1} W$, for every $a \in K$, there is an $\epsilon_a > 0$ such that the d^2-ball $U_{2\epsilon_a}(\gamma_2(a, a)) \subseteq \mathrm{mult}^{-1} W$. Because Δ and hence $\gamma_2(\Delta)$ is compact, we can choose finitely many $a_1, \dots, a_n \in K$ such that we have a finite subcover $\gamma_2(\Delta) \subseteq \bigcup_{i=1}^n U_{\epsilon_{a_i}}(\gamma_2(a_i, a_i))$. Set $\epsilon_0 := \min_{i=1}^n \epsilon_{a_i} > 0$. Then if $a \in K$ is arbitrary, and $x \in U_{\epsilon_0}(\gamma_2(a, a))$, we have $d^2(\gamma_2(a, a), \gamma_2(a_i, a_i)) < \epsilon_0$ for some $i \in \{1, \dots, n\}$, hence $d^2(x, \gamma_2(a_i, a_i)) < 2\epsilon_0 \leq 2\epsilon_{a_i}$, hence $x \in \mathrm{mult}^{-1} W$.

Choose a compact neighbourhood L of $\gamma(K)$ within G^p. Because $\gamma(K)$ is compact, there exists an $\epsilon_1 > 0$ such that $U_{\epsilon_1}(g) \subseteq L$ for all $g \in \gamma(K)$. Because L is compact, the restricted inversion map $\mathrm{inv}|_L$ is uniformly continuous, hence there is a $\delta > 0$ such that for all $g, h \in L$ with $d(g, h) < \delta$, $d(g^{-1}, h^{-1}) < \frac{\epsilon_0}{3}$.

And because γ is uniformly continuous, there is a $\delta_2 > 0$ such that for all $x, y \in K$ with $d_K(x, y) < \delta_2$, $d(\gamma(x), \gamma(y)) < \frac{\epsilon_0}{3}$.

Using all of the above, set $\epsilon := \min\{\frac{\epsilon_0}{3}, \delta, \delta_1, \epsilon_1\}$. Let $x, y \in K$ with $d_K(x, y) < \epsilon$ and $g, h \in G^p$ with $d(\gamma(x), g), d(\gamma(y), h) < \epsilon$. Then because $d(g, \gamma(x)) < \epsilon_1$, we have $g \in L$. So because $d(g, \gamma(x)) < \delta$, we have $d(g^{-1}, \gamma(x)^{-1}) < \frac{\epsilon_0}{3}$. Because $d_K(x, y) < \delta_1$, we have

$d(\gamma(x), \gamma(y)) < \frac{\epsilon}{3}$. By definition we also have $d(h, \gamma(y)) < \frac{\epsilon_0}{3}$. Hence we get:

$$\begin{aligned} d^2((g^{-1}, h), \gamma_2(x, x)) &= d^2((g^{-1}, h), (\gamma(x)^{-1}, \gamma(x))) \\ &= d(g^{-1}, \gamma(x)^{-1}) + d(h, \gamma(x)) \\ &< \frac{\epsilon_0}{3} + d(h, \gamma(y)) + d(\gamma(x), \gamma(y)) < 3\frac{\epsilon_0}{3} = \epsilon_0 \end{aligned}$$

This shows that $(g^{-1}, h) \in U_{\epsilon_0}(\gamma_2(x, x)) \subseteq \mathrm{mult}^{-1}W \subseteq G^{(2)}$, so $\mathrm{mult}((g^{-1}, h)) = g^{-1}h \in W$. ☐

So without further ado, let us look at the main theorem of this section.

THEOREM 5.3.10. *Let $G \rightrightarrows M$ be a Lie groupoid with simply connected target fibres. Let ν be a quasi-invariant measure on M and $\alpha = (\alpha^x)_{x \in M}$ a Haar system on G. Let H be a groupoid. Let (D, R) be a local essential homomorphism of second type from G to H. Then R extends to a global essential homomorphism of second type, i.e. there is a global essential homomorphism $(\tilde{D}, \tilde{R})$ from G to H such that $\tilde{R}|_{\tilde{D} \cap D} = R|_{\tilde{D} \cap D}$. Any two such extensions are equal up to changes on a null set.*

Proof: All smooth (and second-countable, which is contained in my standard definition) manifolds are metrizable. So choose, once for the whole theorem, a metric d on G which induces the topology of G. This will be used later.

Let $W \subseteq G$ be open with $M \subseteq W$ and $N \subseteq G$ a ν^1-null set such that $D = W \setminus N$. Because ν is quasi-invariant, the set N^{-1} is still a null set, as is $N \cup N^{-1}$. Hence for ν-almost all $p \in M$ we have $m_p := \alpha^p(G^p \cap (N \cup N^{-1})) = 0$. Hence we have $\tilde{\nu}^1(s^{-1}\{p \in M | m_p \neq 0\}) = \int_M \alpha_p(G_p \cap s^{-1}\{q \in M | m_q \neq 0\}) \mathrm{d}\nu(p) = \int_M \alpha_p(s^{-1}\{q \in M | m_q \neq 0, q = p\}) \mathrm{d}\nu(p) = \int_M \alpha_p(G_p) \cdot (1 - \delta_{m_p,0}) \mathrm{d}\nu(p) = 0$ as $(1 - \delta_{m_p,0}) = 0$ ν-almost everywhere. It may occur that $\alpha_p(G_p) = \infty$, but the computation is still valid in this case. Since ν is quasi-invariant, null sets under ν^1 and $\tilde{\nu}^1$ are the same, so $0 = \nu^1(s^{-1}\{p \in M | m_p \neq 0\}) = \int_M \alpha^p(s^{-1}\{q \in M | m_q \neq 0\} \cap G^p) \mathrm{d}\nu(p)$. Hence for ν-almost all $p \in M$, we have $0 = \alpha^p(s^{-1}\{q \in M | m_q \neq 0\} \cap G^p) = \alpha^p(s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\} \cap G^p)$.

Since $N \cup N^{-1}$ is a null set, we also have $\alpha^p(N \cup N^{-1}) = 0$ for almost all $p \in M$. Let $p \in M$ be a point such that this is fulfilled as well as $0 = \alpha^p(s^{-1}\{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \neq 0\} \cap G^p)$. In a more worded but equivalent sense this means that for almost all $g \in G^p$, we have $\alpha^{sg}(G^{sg} \cap (N \cup N^{-1})) = 0$. So choose $g \in G^p$ such that this is true and such that $g \notin N \cup N^{-1}$.

G^p is (even simply) connected, so choose a path $\gamma : [0, 1] \to G^p$ from p to g. By Lemma 5.3.9, there is an $n \in \mathbb{N}$ such that $g^{-1}h \in W$ for all $x, y \in [0, 1]$, $g, h \in G^p$ with $|x - y|, d(g, \gamma(x)), d(h, \gamma(y)) \leq \frac{3}{n}$. Set $g_0 = p = \gamma(0)$ and $g_n = g = \gamma(1)$. By inductive use of Lemma 5.3.7 as in Lemma 5.3.8, choose $g_i \in U_{\frac{1}{n}}(\gamma(\frac{i}{n}))$ for all $i \in \{1, \ldots, n-1\}$ such that $\alpha^{sg_i}(G^{sg_i} \cap (N \cup N^{-1})) = 0$ and $g_i^{-1}g_j \notin N$ for all $i, j \in \{0, \ldots, n\}$. Then by the choice of n and the construction, we have $g_i^{-1}g_{i+1} \in W \setminus N$, and clearly $s(g_i^{-1}g_{i+1}) = s(g_{i+1}) = t(g_{i+1}^{-1}) = t(g_{i+1}^{-1}g_{i+2})$ for all i, hence we can define:

$$R_{\gamma,n,g_1,\ldots,g_{n-1}}(g) := R(g_0^{-1}g_1)R(g_1^{-1}g_2)\ldots R(g_{n-1}^{-1}g_n)$$

I will now show, step by step, that this definition is in fact independent of the choice of the g_i, of n and of γ. This is the most tedious part of the proof.

Step 1: Let $h_i \in U_{\frac{1}{n}}(\gamma(\frac{i}{n}))$ be another choice of elements for each $i \in \{1, \ldots, n-1\}$ such that $h_i^{-1}h_j \notin N$ and $\alpha^{sh_i}(G^{sh_i} \cap (N \cup N^{-1})) = 0$. Using the same induction by Lemma 5.3.7 again, this time with the starting elements $g_0, \ldots, g_n, h_0, \ldots, h_n, a_0 \ldots, a_{i-1}$ we can find $a_i \in U_{\frac{1}{n}}(\gamma(\frac{i}{n}))$ for $i \in \{1, \ldots, n-1\}$ such that $a_i^{-1}a_j, a_i^{-1}g_j, g_i^{-1}a_j, a_i^{-1}h_j, h_i^{-1}a_j \notin N$ for all $i, j \in \{0, \ldots, n\}$. Then for all i we have $g_i^{-1}a_{i+1} \in W \setminus N$, $a_{i+1}^{-1}g_{i+1} \in W \setminus N$ and

$g_i^{-1}a_{i+1}a_{i+1}^{-1}g_{i+1} = g_i^{-1}g_{i+1} \in W \setminus N$, hence

$$R(g_i^{-1}a_{i+1})R(a_{i+1}^{-1}g_{i+1}) = R(g_i^{-1}g_{i+1}).$$

Likewise, we have $a_i^{-1}g_i, g_i^{-1}a_{i+1}, a_i^{-1}a_{i+1} \in W \setminus N$, hence

$$R(a_i^{-1}g_i)R(g_i^{-1}a_{i+1}) = R(a_i^{-1}a_{i+1}).$$

So we compute:

$$\begin{aligned}
R_{\gamma,n,g_1,\dots,g_{n-1}}(g) &= R(g_0^{-1}g_1)\dots R(g_{n-1}^{-1}g_n)\\
&= R(g_0^{-1}a_1)R(a_1^{-1}g_1)R(g_1^{-1}a_2)R(a_2^{-1}g_2)\dots R(g_{n-1}^{-1}a_n)R(a_n^{-1}g_n)\\
&= R(g_0^{-1}a_1)R(a_1^{-1}a_2)\dots R(a_{n-1}^{-1}a_n)R(a_n^{-1}g_n)\\
&= R(a_0^{-1}a_1)\dots R(a_{n-1}^{-1}a_n) = R_{\gamma,n,a_1,\dots,a_{n-1}}(g),
\end{aligned}$$

using the former two lines and the fact that $a_0 = g_0 = p$ and $a_n = g_n = g$, so that $R(a_n^{-1}g_n) = R(sg) = 1_{sg} \in H_{sg}$.

The $n-1$-tuples $(g_1, \dots, g_{n-1})$ and $(h_1, \dots, h_{n-1})$ are completely interchangeable here, so we get the analogous result for the h_i:

$$R_{\gamma,n,h_1,\dots,h_{n-1}}(g) = R_{\gamma,n,a_1,\dots,a_{n-1}}(g)$$

Consequently, we have equality between the two choices and denote:

$$R_{\gamma,n}(g) := R_{\gamma,n,g_1,\dots,g_{n-1}}(g) = R_{\gamma,n,h_1,\dots,h_{n-1}}(g)$$

Step 2: Let $m \in \mathbb{N}$ be another natural number such that $g^{-1}h \in W$ for all $x, y \in [0,1]$, $g, h \in G^p$ with $|x-y|, d(g,\gamma(x)), d(h,\gamma(y)) \le \frac{3}{m}$. Using our beloved Lemma 5.3.7, choose $a_i \in U_{\frac{1}{nm}}(\gamma(\frac{i}{nm}))$ for all $i \in \{1, \dots, nm-1\}$ such that $a_i^{-1}a_j \notin N$ for all $i, j \in \{0, \dots, nm\}$, where $a_0 = p$ and $a_{nm} = g$. Then because $U_{\frac{1}{nm}}(\gamma(\frac{i}{nm})) \subseteq U_{\frac{1}{n}}(\gamma(\frac{i}{nm}))$ for all $i \in \{0, \dots, nm\}$ and $|\frac{i}{nm} - \frac{i+j}{nm}| = \frac{j}{nm} \le \frac{1}{n}$ for all $j \in \{1, \dots, m\}$, we have $a_i^{-1}a_{i+j} \in W \setminus N$ for $i \le nm - j$ by the choice of n. Likewise, we have $a_i^{-1}a_{i+j} \in W \setminus N$ for all $j \in \{1, \dots, n\}$ and all $i \in \{0, \dots, nm-j\}$ by the choice of m. This implies that $R(a_i^{-1}a_{i+1})R(a_{i+1}^{-1}a_{i+j}) = R(a_i^{-1}a_{i+j})$ for all $j \in \{1, \dots, \max(n,m)\}$, so inductively $R(a_i^{-1}a_{i+1})\dots R(a_{i+m-1}^{-1}a_{i+m}) = R(a_i^{-1}a_{i+m})$ for $i \in \{0, \dots, (n-1)m\}$ and $R(a_i^{-1}a_{i+1})\dots R(a_{i+n-1}^{-1}a_{i+n}) = R(a_i^{-1}a_{i+n})$ for $i \in \{0, \dots, n(m-1)\}$.

Set $g_i := a_{im}$ for all $i \in \{0, \dots, n\}$ and $h_i := a_{in}$ for all $i \in \{0, \dots, m\}$. Then as noted before, $g_i \in U_{\frac{1}{n}}(\gamma(\frac{i}{n}))$ and $g_i^{-1}g_j \notin N$ a fortiori since this is true for the a_i, and likewise for the h_i. By Step 1 of the proof we can use these g_i and h_i to define $R_{\gamma,n}(g)$ and $R_{\gamma,m}(g)$, respectively. By this fact and the previous explanation, we deduce:

$$\begin{aligned}
R_{\gamma,n}(g) &= R(g_0^{-1}g_1)\dots R(g_{n-1}^{-1}g_n) = R(a_0^{-1}a_m)\dots R(a_{(n-1)m}^{-1}a_{nm})\\
&= R(a_0^{-1}a_1)\dots R(a_{m-1}^{-1}a_m)\dots R(a_{(n-1)m}^{-1}a_{(n-1)m+1})\dots R(a_{nm-1}^{-1}a_{nm})\\
&= R(a_0^{-1}a_n)\dots R(a_{n(m-1)}^{-1}a_{nm}) = R(h_0^{-1}h_1)\dots R(h_{m-1}^{-1}h_m) = R_{\gamma,m}(g),
\end{aligned}$$

which shows independence of the choice of n. Hence we denote, for any large enough $n \in \mathbb{N}$:

$$R_\gamma(g) := R_{\gamma,n}(g)$$

Step 3: Let $\gamma_0, \gamma_1 : [0,1] \to G^p$ be two paths from p to g. Since G^p is simply connected, there is a continuous homotopy $\eta : [0,1]^2 \to G^p$ from γ_0 to γ_1. So we have $\gamma_i = \eta(i, \cdot)$ for both i and $\eta(\cdot, 0) \equiv p$, $\eta(\cdot, 1) \equiv g$. By Lemma 5.3.9, choose an $n \in \mathbb{N}$ such that for all $x, y \in [0,1]^2$ and all $g, h \in G^p$ with $\|x-y\|, d(\eta(x), g), d(\eta(y), h) \le \frac{13}{n}$ we have $g^{-1}h \in W$.

Using this n, define $\eta_i := \eta(\frac{i}{n}, \cdot)$ for all $i \in \{0, \dots, n\}$. Furthermore for all $i \in \{0, \dots, n-1\}$ and all $j \in \{0, \dots, n+1\}$, define

$$\eta_{ij} : [0,1] \to G_x,\ \eta_{ij}(r) := \begin{cases} \eta_{i+1}(r),\ r \le \frac{j-1}{n} \\ \eta(\frac{i+j-nr}{n}, r),\ \frac{j-1}{n} < r \le \frac{j}{n} \\ \eta_i(r),\ \frac{j}{n} \le r \end{cases}$$

Note that η_{ij} is still a continuous path from p to g, $\eta_{ij}(\frac{j}{n}) = \eta_i(\frac{j}{n})$, $\eta_{i0} = \eta_i$ and $\eta_{i,n+1} = \eta_{i+1}$ for all i, j. Furthermore we have $\eta_{00} = \eta_0 = \gamma_0$ and $\eta_{n-1,n+1} = \eta_n = \gamma_1$.

For all $i \in \{0, \dots, n\}$ and all $j \in \{-1, 0\}$ set $g_{i,j} := p$, and $g_{i,j} := g$ for $j \in \{n, n+1, n+2\}$. Using Lemma 5.3.7 once more, choose $g_{i,j} \in U_{\frac{1}{n}}(\eta(\frac{i}{n}, \frac{j}{n}))$ for all $i \in \{0, \dots, n\}$ and all $j \in \{1, \dots, n-1\}$, such that $g_{i,j}^{-1} g_{k,l} \notin N$ for all $i, k \in \{0, \dots, n\}$ and all $j, l \in \{-1, \dots, n+2\}$. Then by the choice of n and the fact that $\|(\frac{i+1}{n}, \frac{j-1}{n}) - (\frac{i}{n}, \frac{j+1}{n})\| = \frac{1}{n}\|(1, -2)\| \le \frac{3}{n}$, we have $g_{i+1,j-1}^{-1} g_{i,j+1} \in W \setminus N$ for all $i \in \{0, \dots, n-1\}$ and all $j \in \{0, \dots, n+1\}$. By likewise arguing, we also have $g_{i+1,j-1}^{-1} g_{i,j}, g_{i,j}^{-1} g_{i,j+1}, g_{i+1,j-1}^{-1} g_{i+1,j}, g_{i+1,j}^{-1} g_{i,j+1} \in W \setminus N$. So using that R is a local essential homomorphism on $W \setminus N$, we find that

$$R(g_{i+1,j-1}^{-1} g_{i,j}) R(g_{i,j}^{-1} g_{i,j+1}) = R(g_{i+1,j-1}^{-1} g_{i,j+1}) = R(g_{i+1,j-1}^{-1} g_{i+1,j}) R(g_{i+1,j}^{-1} g_{i,j+1}).$$

For all i, j and all $x, y \in [0,1]$ we have $\eta_{i,j}(x) = \eta(\tilde{x})$ and $\eta_{i,j}(y) = \eta(\tilde{y})$ for some $\tilde{x}, , \tilde{y} \in [0,1]^2$ with $\|\tilde{x} - \tilde{y}\| \le 2|x - y|$, namely

$$\tilde{x} = \begin{cases} (i+1, x),\ x \le \frac{j-1}{n} \\ (\frac{i+j-nr}{n}, x),\ \frac{j-1}{n} < x \le \frac{j}{n} \\ (i, x),\ \frac{j}{n} < x \end{cases}$$

and likewise for y. Verification of the equation above is just case distinction with the three possible steps for x and y. This assures that $R_{\eta_{i,j}}(g) = R_{\eta_{i,j},n}(g)$ and that the latter is defined; that n is large enough with respect to d and $\eta_{i,j}$.

Choose $i \in \{0, \dots, n-1\}$ and $j \in \{0, \dots, n\}$. We have $g_{i+1,k} \in U_{\frac{1}{n}}(\eta(\frac{i+1}{n}, \frac{k}{n})) = U_{\frac{1}{n}}(\eta_{i,j}(\frac{k}{n}))$ for all $k \in \{1, \dots, j-1\}$ and $g_{i,k} \in U_{\frac{1}{n}}(\eta(\frac{i}{n}, \frac{k}{n})) = U_{\frac{1}{n}}(\eta_{i,j}(\frac{k}{n}))$ for $k \in \{j, \dots, n-1\}$, hence by the previous two steps,

$$\begin{aligned} R_{\eta_{i,j}}(g) &= R_{\eta_{i,j},n,g_{i+1,1},\dots,g_{i+1,j-1},g_{i,j},\dots,g_{i,n-1}}(g) \\ &= R(g_{i+1,0}^{-1} g_{i+1,1}) \dots R(g_{i+1,j-2}^{-1} g_{i+1,j-1}) R(g_{i+1,j-1}^{-1} g_{i,j}) R(g_{i,j}^{-1} g_{i,j+1}) \\ &\cdot R(g_{i,j+1}^{-1} g_{i,j+2}) \dots R(g_{i,n-1}^{-1} g_{i,n}) \\ &= A R(g_{i+1,j-1}^{-1} g_{i,j}) R(g_{i,j}^{-1} g_{i,j+1}) B \end{aligned}$$

for $A = R(g_{i+1,0}^{-1} g_{i+1,1}) \dots R(g_{i+1,j-2}^{-1} g_{i+1,j-1})$ and $B = R(g_{i,j+1}^{-1} g_{i,j+2}) \dots R(g_{i,n-1}^{-1} g_{i,n})$. Likewise, we find that:

$$R_{\eta_{i,j+1}}(g) = A R(g_{i+1,j-1}^{-1} g_{i+1,j}) R(g_{i+1,j}^{-1} g_{i,j+1}) B$$

As shown before, we have

$$R(g_{i+1,j-1}^{-1} g_{i,j}) R(g_{i,j}^{-1} g_{i,j+1}) = R(g_{i+1,j-1}^{-1} g_{i+1,j}) R(g_{i+1,j}^{-1} g_{i,j+1}),$$

so indeed $R_{\eta_{i,j}}(g) = R_{\eta_{i,j+1}}(g)$. By finite induction, this implies that $R_{\eta_i}(g) = R_{\eta_{i,0}}(g) = R_{\eta_{i,n+1}}(g) = R_{\eta_{i+1}}(g)$. This is true for all $i \in \{0, \dots, n-1\}$, hence by one more induction, $R_{\gamma_0}(g) = R_{\eta_0}(g) = R_{\eta_n}(g) = R_{\gamma_1}(g)$, which is the required result. Hence, we may define

$$\tilde{R}(g) := R_\gamma(g)$$

for any path γ from p to g within G^p.

Step 4: Domain of definition: Define $Z := \{q \in M | \alpha^q(G^q \cap (N \cup N^{-1})) \ne 0\}$. All of the above construction was done for an arbitrary $p \in M$ such that $\alpha^p(G^p \cap (N \cup N^{-1})) = 0$

(i.e. $p \in M \setminus Z$) and $\alpha^p(G^p \cap s^{-1}Z) = 0$, and an arbitrary $g \in G$ with $tg = p$, $g \notin N \cup N^{-1}$ and $\alpha^{sg}(G^{sg} \cap (N \cup N^{-1})) = 0$. So set $N^{\pm} := N \cup N^{-1}$ and define:

$$\tilde{D} := \{g \in G \setminus (N^{\pm}) | 0 = \alpha^{tg}(G^{tg} \cap (N^{\pm})) = \alpha^{sg}(G^{sg} \cap (N^{\pm})) = \alpha^{tg}(G^{tg} \cap s^{-1}Z)\},$$

which is co-null within G because N and $s^{-1}Z$ are null sets. This was already discussed in more detail before. Hence the construction from before gives us a well-defined map $\tilde{R} : \tilde{D} \to H$.

Step 5: We will show homomorphy now. Let $g, h \in \tilde{D}$ be arbitrary with $sg = th$ and $gh \in \tilde{D}$. Set $p = tg$, $q = th = sg$. Let $\gamma_g : [0,1] \to G^p$ be a path from p to g and $\gamma_h : [0,1] \to G^q$ a path from q to h. Define $\gamma_{[0,1]} \to G^p$ by

$$\gamma(r) := \begin{cases} \gamma_g(2r), & r \leq \frac{1}{2} \\ g\gamma_h(2r-1), & r > \frac{1}{2} \end{cases}$$

for all $r \in [0,1]$, which is a continuous path from p to gh by $\gamma_g(2\cdot\frac{1}{2}) = g = gq = g\gamma_h(2\cdot\frac{1}{2}-1)$. Let $n \in \mathbb{N}$ be large enough to define $R_{\gamma_g,n}(g)$, $R_{\gamma_h,n}(h)$ and $R_{\gamma,n}(gh)$. Then a fortiori, also $R_{\gamma,2n}(gh)$ is defined.

Let K be a compact neighbourhood of $\gamma_h([0,1]) \subseteq G^q$, which exists because G^q is locally compact and $\gamma_h([0,1])$ is compact. The map $l_g : G^q \to G^p,\ a \mapsto ga$ is continuous, hence $l_g|_K$ is uniformly continuous. So choose an $\epsilon > 0$ such that $d(ga, gb) < \frac{1}{2n}$ for all $a, b \in K$ with $d(a,b) < \epsilon$. Without loss of generality, choose this ϵ such that also $U_\epsilon(\gamma_h(\frac{i}{n})) \subseteq K$ for all $i \in \{0, \ldots, n\}$ and such that $\epsilon < \frac{1}{2n}$.

Set $g_0 = p$ and $g_n = g$. We have $g, gh \in \tilde{D}$ by assumption, so $p, g, g^{-1}, gh, h^{-1}g^{-1} \notin N$, $\alpha^{sg}(G^{sg} \cap (N \cup N^{-1})) = 0 = \alpha^{s(gh)}(G^{s(gh)} \cap (N \cup N^{-1}))$, $\alpha^p(G^p \cap (N \cup N^{-1})) = 0$ and $\alpha^p(G^p \cap s^{-1}Z) = 0$. Hence using Lemma 5.3.7, we can choose, inductively, $g_i \in U_\epsilon(\gamma_g(\frac{i}{n})) \subseteq G^p$ for all $i \in \{1, \ldots, n-1\}$ such that $g_i^{-1}g_j \notin N$, $g_i^{-1}gh \notin N$ and $(gh)^{-1}g_j = h^{-1}g^{-1}g_j \notin N$ for all $i, j \in \{0, \ldots, n\}$, as well as $\alpha^{sg_i}(G^{sg_i} \cap (N \cup N^{-1})) = 0$.

Furthermore, by $h \in \tilde{D}$ we have $q, h, h^{-1} \notin N$, $\alpha^{sh}(G^{sh} \cap (N \cup N^{-1})) = 0$, $\alpha^q(G^q \cap (N \cup N^{-1})) = 0$ and $\alpha^q(G^q \cap s^{-1}Z) = 0$. We also have $\alpha^{s(g^{-1}g_j}(G^{s(g^{-1}g_j)} \cap (N \cup N^{-1})) = \alpha^{sg_j}(G^{sg_j} \cap (N \cup N^{-1})) = 0$ for all $j \in \{0, \ldots, n\}$. So set $h_0 = q$, $h_n = h$ and by using Lemma 5.3.7 again, choose, inductively, $h_i \in U_\epsilon(\gamma_h(\frac{i}{n})$ for all $i \in \{1, \ldots, n-1\}$ such that $\alpha^{sh_i}(G^{sh_i} \cap (N \cup N^{-1})) = 0$, $h_i^{-1}h_j \notin N$, $h_i^{-1}g^{-1}g_j \notin N$ and $g_i^{-1}gh_j \notin N$ for all $i, j \in \{0, \ldots, n\}$.

Now for each $i \in \{0, \ldots, n\}$, set $c_i := g_i$, and for each $i \in \{n+1, \ldots, 2n\}$, set $c_i := gh_{i-n}$. Then for each $i \in \{0, \ldots, n\}$ we have

$$c_i = g_i \in U_\epsilon(\gamma_g(\frac{i}{n})) = U_\epsilon(\gamma(\frac{i}{2n})) \subseteq U_{\frac{1}{2n}}(\gamma(\frac{i}{2n})),$$

and for each $i \in \{n+1, \ldots, 2n\}$ we have

$$c_i = gh_{i-n} \in gU_\epsilon(\gamma_h(\frac{i-n}{n})) \subseteq U_{\frac{1}{2n}}(g\gamma_h(\frac{i-n}{n})) = U_{\frac{1}{2n}}(g\gamma_h(2\frac{i}{2n}-1)) = U_{\frac{1}{2n}}(\gamma(\frac{i}{2n})).$$

We also have $c_0 = p$, $c_n = g$ and $c_{2n} = gh$. We need to check that $c_i^{-1}c_j \notin N$ for all $i, j \in \{0, \ldots, 2n\}$. The specific choices of the g_i and h_i were to make this sure. Namely there are the following cases:

If $i, j \leq n$, then $c_i^{-1}c_j = g_i^{-1}g_j \notin N$ by the choice of the g_i. If $i, j > n$, then $c_i^{-1}c_j = (gh_{i-n})^{-1}gh_{j-n} = h_{i-n}^{-1}h_{j-n} \notin N$ by the choice of the h_i. If $i \leq n$ and $n < j < 2n$, then $c_i^{-1}c_j = g_i^{-1}gh_{j-n} \notin N$ and $c_j^{-1}c_i = (gh_{j-n})^{-1}g_j = h_{j-n}^{-1}g^{-1}g_j^{-1} \notin N$ by the choice of the h_j. If $i \leq n$ and $j = 2n$, then $c_i^{-1}c_j = g_i^{-1}gh \notin N$ and $c_j^{-1}c_i = h^{-1}g^{-1}g_i \notin N$ by the choice of the g_i. These are all possible cases, so indeed $c_i^{-1}c_j \notin N$ for all $i, j \in \{0, \ldots, 2n\}$.

Lastly, we can be sure that $\alpha^{sc_i}(G^{sc_i} \cap (N \cup N^{-1})) = 0$ because $sc_i = sg_i$ or $sc_i = sh_{i-n}$ and the g_i, h_i we chosen in this way.

Hence $R_{\gamma,2n,c_1,\dots,c_{2n-1}}$ is defined and by the steps 1 to 3 we know that

$$\tilde{R}(gh) = R_{\gamma,2n,c_1,\dots,c_{2n-1}}(gh).$$

Because $\epsilon < \frac{1}{n}$, $g_i^{-1}g_j \notin N$, $\alpha^{sg_i}(G^{sg_i} \cap (N \cup N^{-1})) = 0 = \alpha^{sh_i}(G^{sh_i} \cap (N \cup N^{-1}))$ and $h_i^{-1}h_j \notin N$ for all $i,j \in \{0,\dots,n\}$ by the choices made before, we also have $\tilde{R}(g) = R_{\gamma_g,n,g_1,\dots,g_{n-1}}(g)$ and $\tilde{R}(h) = R_{\gamma_h,n,h_1,\dots,h_{n-1}}(h)$.

Hence we finally compute:

$$\begin{aligned}\tilde{R}(gh) &= R_{\gamma,2n,c_1,\dots,c_{2n-1}} = R(c_0^{-1}c_1)\dots R(c_{n-1}^{-1}c_n)R(c_n^{-1}c_{n+1})\dots R(c_{2n-1}^{-1}c_{2n}) \\ &= R(g_0^{-1}g_1)\dots R(g_{n-1}^{-1}g_n)R(g^{-1}gh_1)R(h_1^{-1}g^{-1}gh_2)\dots R(h_{n-1}^{-1}g^{-1}gh_n) \\ &= R(g_0^{-1}g_1)\dots R(g_{n-1}^{-1}g_n)R(h_0^{-1}h_1)\dots R(h_{n-1}^{-1}h_n) \\ &= R_{\gamma_g,n,g_1,\dots,g_{n-1}}(g)R_{\gamma_h,n,h_1,\dots,h_{n-1}}(h) = \tilde{R}(g)\tilde{R}(h)\end{aligned}$$

Because $g,h \in \tilde{D}$ were arbitrary, this shows that $(\tilde{D},\tilde{R})$ is indeed an essential homomorphism of second type.

Step 6: Let us now show that $\tilde{R}|_{\tilde{D}\cap D} = R|_{\tilde{D}\cap D}$. To do this, let $g \in \tilde{D} \cap D = \tilde{D} \cap W$ be arbitrary. Set $p = tg$. By assumption, $G^p \cap W$ is connected and $p \in G^p \cap W$, so choose a path $\gamma : [0,1] \to G^p \cap W$ from p to g. Choose $n \in \mathbb{N}$ large enough and $g_i \in W \cap U_{\frac{1}{n}}(\gamma(\frac{i}{n}))$ for $i \in \{1,\dots,n-1\}$ suitable to define $R_{\gamma,n,g_1,\dots,g_{n-1}}(g)$, which is possible since W is open and $\gamma(\frac{i}{n}) \in W$. Then for all $i \in \{0,\dots,n-1\}$ we have $g_0^{-1}g_ig_i^{-1}g_{i+1} = g_0^{-1}g_{i+1} = g_{i+1} \in W \setminus N$, so because R is a local essential homomorphism on $W \setminus N$, we have $R(g_0^{-1}g_ig_i^{-1}g_{i+1}) = R(g_0^{-1}g_{i+1})$, and hence by induction:

$$\tilde{R}(g) = R_{\gamma,n,g_1,\dots,g_{n-1}}(g) = R(g_0^{-1}g_1)\dots R(g_{n-1}^{-1}g_n) = R(g_0^{-1}g_n) = R(g)$$

Step 7: Now, assume that $(E,P) : G \dashrightarrow H$ is any global essential homomorphism of second type such that $P|_{E\cap D} = R|_{E\cap D}$. I claim that there is another set $F \subseteq E \cap \tilde{D}$ which is co-null in G such that $P|_F = \tilde{R}|_F$, so that in this sense, the extension $\tilde{R}$ is unique up to null sets.

Namely consider the set $L := (G \setminus E) \cup (G \setminus \tilde{D})$, which is still a null set as union of null sets. Define $Z_L := \{q \in M | \alpha^q(G^q \cap (L \cup L^{-1})) \neq 0\}$ and

$$F := \{g \in G\setminus(L\cup L^{-1}) | 0 = \alpha^{tg}(G^{tg}\cap(L\cup L^{-1})) = \alpha^{sg}(G^{sg}\cap(L\cup L^{-1})) = \alpha^{tg}(G^{tg}\cap s^{-1}Z_L)\}$$

like in step 4 with L instead of N, which is co-null as discussed before. By construction, $F \subseteq E \cap \tilde{D}$.

Let $g \in F$ be arbitrary. Put $p = tg$. Since G^p is connected, choose a path $\gamma : [0,1] \to G^p$ from p to g. Use Lemma 5.3.9 to choose an $n \in \mathbb{N}$ such that for all $x,y \in [0,1]$, $g,h \in G^p$ with $|x-y|, d(\gamma(x),g), d(\gamma(y),h) < \frac{4}{n}$, $g^{-1}h \in W$. We have $\alpha^p(G^p \cap s^{-1}Z_L) = 0$ by construction of F, so we can once more apply Lemma 5.3.7 inductively to obtain $g_i \in U_{\frac{1}{n}}(\gamma(\frac{i}{n}))$, $i \in \{1,\dots,n-1\}$, such that $g_i^{-1}g_j \notin L$ for all $i,j \in \{0,\dots,n\}$, where $g_0 = p$, $g_n = g$. By the choice of n, we have $g_i^{-1}g_{i+1} \in W$. Using this precise setting, the following intuitive computation is actually valid:

$$\begin{aligned}\tilde{R}(g) &= \tilde{R}(g_0^{-1}g_1g_1^{-1}g_2\dots g_{n-1}^{-1}g_n) = \tilde{R}(g_0^{-1}g_1)\dots\tilde{R}(g_{n-1}^{-1}g_n) \\ &= R(g_0^{-1}g_1)\dots R(g_{n-1}^{-1}g_n) = P(g_0^{-1}g_1)\dots P(g_{n-1}^{-1}g_n) \\ &= P(g_0^{-1}g_1\dots g_{n-1}^{-1}g_n) = P(g)\end{aligned}$$

In the first line we use that $g_i^{-1}g_j \in \tilde{D}$ and that $\tilde{R}$ is an essential homomorphism. In the second line we use that $g_i^{-1}g_j \in D \cap \tilde{D} \cap E$ and that all three maps are equal on this intersection. In the third line we use that $g_i^{-1}g_j \in E$ and the fact that P is an homomorphism.

Since $g \in F$ was arbitrary, this shows that $P|_F = \tilde{R}|_F$ as required. $\square$

In a moment of humour, I called the theorem that we have just proven the “Sausage Theorem”. The reason for this name is the following idea: In the classical Lie group case, we just choose a path from the identity to an element, follow it step by step and are done. In contrast, with essential homomorphisms we need to keep a whole neighbourhood around the path to be able to avoid the unwanted null sets. In my imagination, this neighbourhood looks like a tube, or perhaps, a *sausage*. The name may not be the most fitting one, but it brought me a spark of joy, which I hope to share with my readers through this anecdote. Time will show whether it sticks. For now, let us finish this chapter and prepare for the first part of the actual integration theory in the next one.

CHAPTER 6

An Integration Theorem for the Euclidean Space

The last of the more independent foundations have been laid out in the last chapter and it is now time to begin proving the first of our integration theorems for representations of a Lie algebroid. This first one will be suited for the pair groupoid over the Euclidean space $\mathbb{R}^m$, which has a particularly convenient structure: Namely, we always have the coordinate vector fields as a global, complete, commuting frame for the tangent algebroid, and the exponential map is easy to write down and understand, as well as its inverse.

6.1. From Differential Operators to the Groupoid

The integration process will take two distinct steps: First we go from representations of the differential operator algebra to representations of the groupoid itself, in the sense of essential global homomorphisms to a unitary groupoid. This step is the hardest one and the reason for most of our technical foundations. In the next section, I will then show how to integrate further to a representation of the groupoid algebra, which is a bit more routine.

Proving the first integration theorem requires, beyond the basis laid before, a variety of specialised lemmas. The main ingredient will be a certain set of unitary operators obtained by taking the exponential of coordinate vector fields under functional calculus. The precise properties of these exponentials need to be understood first. The goal for now shall be to show that these exponential operators are decomposable, which is true under a few technical caveats.

Let us start with a simple lemma from the realm of manifold analysis:

Lemma 6.1.1. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ a complete vector field with flow θ. Then for all $t \in \mathbb{R}$ and all $f \in C^\infty(M)$:*

$$X(f) \circ \theta_t = X(f \circ \theta_t)$$

Proof: Let $p \in M$ be arbitrary. By definition of flows, we have $X(q) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\theta_t(q)$ for all $q \in M$, so

$$\begin{aligned} X(f) \circ \theta_t(p) &= X(\theta_t(p))(f) = \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \theta_s(\theta_t(p))\Big)(f) = \Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f \circ \theta_s \circ \theta_t(p)\Big) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f \circ \theta_{s+t}(p) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f \circ \theta_t \circ \theta_s(p) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \theta_s(p)(f \circ \theta_t) = X(p)(f \circ \theta_t) = X(f \circ \theta_t)(p) \end{aligned}$$ $\square$

The second lemma details a few properties carried by the exponential of a Lie algebroid section. Any reader not familiar with those exponentials is welcome to think only of the pair groupoid case, where the exponential of a vector field is given by its flow, as this will be our main environment anyway. Only for now I will keep the slightly more general context for possible future applications.

Lemma 6.1.2. *Let $G \rightrightarrows M$ a Lie groupoid with induced Lie algebroid*

$$A = \bigcup_{p \in M} T_p(G_p) \to M$$

and its anchor $\rho = Tt|_A$. *Let* $a \in \Gamma(A)$ *have compact support. Then* a *is* complete, *i.e.* $\exp(ra) \in \Gamma_M(G)$ *is defined globally for all* $r \in \mathbb{R}$.

Furthermore, for all $r \in \mathbb{R}$ *denote* $\phi_r := t \circ \exp(ra) : M \to M$ *and let* θ *be the maximal flow of* $\rho(a) \in \mathfrak{X}(M)$. *Then* $\phi_r = \theta_r$.

Proof: By Proposition 3.6.1., page 133 in [**14**], there are $\epsilon_p > 0$ and $U_p \subseteq M$ open for all $p \in M$, such that $\exp(ra) \in \Gamma_{U_p}(G)$ is defined for all $|r| < \epsilon_p$. Since $\operatorname{supp} a$ is compact, choose finitely many $p_1, \dots, p_n \in \operatorname{supp} a$ such that $\operatorname{supp} a \subset U := \bigcup_{i=1}^n U_{p_i}$. Set $\epsilon := \min_{i=1}^n \epsilon_{p_i} > 0$. Then for $|r| < \epsilon$, $\exp(ra)(p)$ is defined for all $p \in U$. For $p \in M \setminus \operatorname{supp} a$, set $\exp(ra)(p) := p \in G_p$. Using this definition, $\exp(ra) : M \to G,\ p \mapsto \exp(ra)(p)$ is a well-defined smooth map since $\exp(ra)|_{U \setminus \operatorname{supp} a}(p) = \exp(0)(p) = p$ for all p, so that both definitions are compatible.

We have $\phi_x \circ \phi_y = t\exp(xa)t\exp(ya) = t\exp(xa) \star \exp(ya) = t\exp(x+y)(a)$, so ϕ is a flow, and $\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} \phi_r(p) = Tt \frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} \exp(ra)(p) = Tt(a) =: \rho(a)$, which proves that ϕ generates $\rho(a)$. Using this, we know that $\phi(U) = \theta^{\rho(a)|_U}(U) = U$ and $\phi|_U : U \to U$ is a (compactly supported) diffeomorphism. ϕ is an extension of this by the identity, hence a diffeomorphism, hence $\exp(ra) \in \Gamma_M(G)$ as required. We can now also define $\exp(ra) := \exp(\frac{r}{n}a)^n$ for every $r \in \mathbb{R}$ and any sufficiently large $n \in \mathbb{N}$, which is well-defined and smooth because exp is a local homomorphism. □

Combining these two lemmas, we can prove the following more interesting proposition. The proof uses a simple idea from elementary analysis: If a function has 0 as its derivative, then it is constant. This old idea applies nicely to our unitary operators, though in the more advanced environment of Banach algebras.

Proposition 6.1.3. *Let* $(G \rightrightarrows M, \omega)$ *be a volumetric groupoid with source* s *and target* t *and let* $A \to M$ *be the corresponding Lie algebroid with anchor* ρ. *Let* H *be a Hilbert space,* $R : \mathrm{Diff}^R(G) \to \mathcal{O}(H)$ *a representation and* $r : C_0(M) \to \mathbb{B}(H)$ *a continuous representation such that* $r(f) = \overline{R(m_f)}$ *for all* $f \in C_c^\infty(M)$ *(where* $m_f(h) = f \circ t \cdot h$*). Let* $a \in \Gamma(A)$ *be a complete section and let* $g \in C_c^\infty(M)$ *be such that* $\overline{R(m_g + \mathcal{L}_a) \in \mathcal{O}(H)}$ *is essentially skew-adjoint. For* $x \in \mathbb{R}$, *define the unitary operator* $U_x := \mathrm{e}^{x\overline{R(m_g+\mathcal{L}_a)}} \in \mathbb{B}(H)$. *Let* $f \in C_c^\infty(M)$.

Then for all $x \in \mathbb{R}$, *we have*

$$U_x^* r(f \circ \phi_x) U_x = r(f) \in \mathbb{B}(H),$$

where $\phi_x := t \circ \exp(xa)$ *for the exponential map* $\exp : \Gamma(A) \dashrightarrow \Gamma(G)$.

Proof: Let θ be the maximal flow of $\rho(a)$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}x} f \circ \phi_x = \frac{\mathrm{d}}{\mathrm{d}x} f \circ \theta_x = \frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} f \circ \theta_y \circ \theta_x = (\frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} \theta_y)(f) \circ \theta_x = \rho(a)(f) \circ \phi_x$$

by Lemma 6.1.2, so because r is continuous and linear, $\frac{\mathrm{d}}{\mathrm{d}x} r(f \circ \phi_x) = r(\rho(a)(f) \circ \phi_x)$.

Furthermore, $\overline{R(m_g + \mathcal{L}_a)} \in \mathcal{O}(H)$ is self-adjoint by assumption, hence

$$U_x := \mathrm{e}^{x\overline{R(m_g+\mathcal{L}_a)}}$$

is unitary (bounded, in particular). The product rule for differentiation applies and yields for every $v \in \operatorname{dom} R$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} U_x^* r(f \circ \phi_x) U_x v = &\frac{\mathrm{d}}{\mathrm{d}x} (\mathrm{e}^{-x\overline{R(m_g+\mathcal{L}_a)}} r(f \circ \phi_x) \mathrm{e}^{x\overline{R(m_g+\mathcal{L}_a)}} v) \\
= &- \mathrm{e}^{-x\overline{R(m_g+\mathcal{L}_a)}} \overline{R(m_g + \mathcal{L}_a)} r(f \circ \phi_x) \mathrm{e}^{x\overline{R(m_g+\mathcal{L}_a)}} v \\
&+ \mathrm{e}^{-x\overline{R(m_g+\mathcal{L}_a)}} r(\rho(a)(f) \circ \phi_x) \mathrm{e}^{x\overline{R(m_g+\mathcal{L}_a)}} v \\
&+ \mathrm{e}^{-x\overline{R(m_g+\mathcal{L}_a)}} r(f \circ \phi_x) \mathrm{e}^{x\overline{R(m_g+\mathcal{L}_a)}} \overline{R(m_g + \mathcal{L}_a)} v \\
= &U_x^* \left(-\overline{R(m_g + \mathcal{L}_a)} r(f \circ \phi_x) + r(\rho(a)(f) \circ \phi_x) + r(f \circ \phi_x) \overline{R(m_g + \mathcal{L}_a)} \right) U_x v,
\end{aligned}$$

using that $\overline{R(m_g + \mathcal{L}_a)}$ commutes with exponentials of itself. Proceeding with the inner term, we get

$$\begin{aligned}
&- \left(\overline{R(m_g + \mathcal{L}_a)} r(f \circ \phi_x) + r(\rho(a)(f) \circ \phi_x) + r(f \circ \phi_x) \overline{R(m_g + \mathcal{L}_a)} \right) w \\
&= R(-(m_g + \mathcal{L}_a) m_{f \circ \phi_x} + m_{\rho(a)(f) \circ \phi_x} + m_{f \circ \phi_x} (m_g + \mathcal{L}_a)) w \\
&= R(-m_{g \cdot f \circ \phi_x} - m_{f \circ \phi_x} \mathcal{L}_a - m_{\rho(a)(f \circ \phi_x)} + m_{\rho(a)(f) \circ \phi_x} + m_{f \circ \phi_x \cdot g} + m_{f \circ \phi_x} \mathcal{L}_a) w \\
&= r(\rho(a)(f) \circ \theta_x - \rho(a)(f \circ \theta_x)) w = 0
\end{aligned}$$

for all $w \in \operatorname{dom} R$, using Lemma 6.1.1. Hence $\frac{\mathrm{d}}{\mathrm{d}x} U_x^* r(f \circ \phi_x) U_x \equiv 0$, so

$$U_x^* r(f \circ \phi_x) U_x = U_0^* r(f \circ \phi_0) U_0 = r(f)$$

for all $x \in \mathbb{R}$. □

As mentioned before, we aim to prove that the operator given as the functional calculus exponential of a coordinate vector field is decomposable. For technical reasons, we have to make the following adaptations: Half of the divergence of the vector field has to be added to ensure the vector field is actually formally skew-adjoint. The resulting operator has to be closed to make it actually skew-adjoint, and then we need to pull back by a flow map to keep the fibre constant. This results in the following decomposability theorem, which is now relatively easy to prove using the previous lemmas.

THEOREM 6.1.4. *Let $(G \rightrightarrows M, \omega)$ be a volumetric groupoid with Lie algebroid A and $H \to M$ a Hilbert bundle. Let ν be a Borel measure on M. Let $H \to M$ be a Hilbert field (with respect to the Borel σ-algebra of M). Set $K := L^2(H, \nu)$. Let $R : \operatorname{Diff}^R(G) \to \mathcal{O}(K)$ be a $*$-representation such that $r : C_0(M) \to \mathbb{B}(K),\ f \mapsto T_f$ is a continuous representation with $r(f) = R(m_f)$ for all $f \in C_c^\infty(M)$.*

Let $a \in \Gamma(A)$ and $g \in C_c^\infty(M)$ be such that $R(\mathcal{L}_a + m_g)$ is essentially skew-adjoint. Set $U_x = e^{x\overline{R(\mathcal{L}_a + m_g)}}$, $\phi_x := t \circ \exp(xa)$ and $K_x := L^2((\phi_x^{-1})^* H, \nu)$ for all $x \in \mathbb{R}$. Then $(\phi_x^{-1})^* \circ U_x : K \to K_x$ is a decomposable operator.

Proof: It is a known fact that $C_c^\infty(M)$ is not norm-dense in $L^\infty(M, \nu)$, but weakly *-dense. For example, the case where M is an open subset of $\mathbb{R}^2$ is proven in [**9**], Theorem 7. So for each $f \in L^\infty(M, \nu)$, we find a sequence $(f_i)_{i \in \mathbb{N}} \subset C_c^\infty(M)$ such that for all $g \in L^1(M, \nu)$, $\lim_{i \to \infty} \int_M f_i g \,\mathrm{d}\nu = \int_M f g \,\mathrm{d}\nu$.

So let $f \in L^\infty(M, \nu)$ be arbitrary and choose a uniformly bounded sequence $(f_i)_{i \in \mathbb{N}} \subset C_c^\infty(M)$ which converges to f in the weak *-topology. Let $\xi \in K$ be arbitrary. Then the function $M \to \mathbb{R}$, $\langle \xi, \xi \rangle : p \mapsto \langle \xi(p), \xi(p) \rangle_{H_p}$ is in $L^1(M, \nu)$. Hence we find that

$$\begin{aligned}\|f_i\xi - f\xi\|_{L^2}^2 &= \int_M \langle f_i\xi - f\xi, f_i\xi - f\xi \rangle \, \mathrm{d}\nu \\ &= \int_M (\bar{f}_i f_i + \bar{f} f - \bar{f}_i f - \bar{f} f_i)\langle \xi, \xi \rangle \, \mathrm{d}\nu \\ &= \int_M (f_i - f)\overline{(f_i - f)}\langle \xi, \xi \rangle \, \mathrm{d}\nu \\ &\leq \|f_i - f\|_\infty \int_M |f_i - f| \langle \xi, \xi \rangle \, \mathrm{d}\nu \to 0\end{aligned}$$

and thus $f_i\xi \to f\xi$ in $L^2(H, \nu)$. We know that $|f_i - f| \to 0$ *-weakly because $f_i - f \to 0$ *-weakly and $|\cdot| : L^\infty(M) \to L^\infty(M)$ is continuous.

Hence by Proposition 6.1.3, we have

$$\begin{aligned}(T_f(\phi_x^{-1})^* U_x)(\xi) &= f(\phi_x^{-1})^* U_x(\xi) = \lim_{i \in I} f_i(\phi_x^{-1})^* U_x(\xi) \\ &= \lim_{i \in I} (T_{f_i \circ \phi \circ \phi^{-1}}(\phi_x^{-1})^* U_x)(\xi) = \lim_{i \in I} (\phi_x^{-1})^* T_{f_i \circ \phi} U_x(\xi) \\ &= \lim_{i \in I} (\phi_x^{-1})^* U_x T_{f_i}(\xi) = (\phi_x^{-1})^* U_x T_f(\xi),\end{aligned}$$

in the weak *-topology. So since ξ was arbitrary, $T_f(\phi_x^{-1})^* U_x = (\phi_x^{-1})^* U_x T_f$. Hence by [**5**], Theorem 1, page 187, the continuous linear operator $(\phi_x^{-1})^* U_x$ is decomposable. □

Before we can prove our first integration theorem, we need one more lemma, which is only a short computation on Radon-Nikodým derivatives. It plays a technical role in the upcoming main proof.

LEMMA 6.1.5. *Let M be a smooth manifold and ν a measure on M. Let $\phi : \mathbb{R}^m \times M \to M$ be a flow (i.e. $\phi_x \circ \phi_y = \phi_{x+y}$ for all $x, y \in \mathbb{R}^m$) such that $(\phi_x)_*\nu \ll (\phi_y)_*\nu$ for all $x, y \in \mathbb{R}$. Then we have:*

$$\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \cdot \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-y})_*\nu} \circ \phi_x = \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x-y})_*\nu}$$

Proof: Let $f : M \to \mathbb{R}$ be ν-integrable and $x, y \in \mathbb{R}^m$. Then we have:

$$\begin{aligned}&\int_M f \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-y})_*\nu} \circ \phi_x \mathrm{d}(\phi_{-x-y})_*\nu \\ &\quad = \int_M \left(f \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \right) \circ \phi_{-x} \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-y})_*\nu} \mathrm{d}(\phi_x)_*(\phi_{-x-y})_*\nu \\ &\quad = \int_M \left(f \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \right) \circ \phi_{-x} \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-y})_*\nu} \mathrm{d}(\phi_{-y})_*\nu \\ &\quad = \int_M \left(f \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \right) \circ \phi_{-x} \mathrm{d}\nu = \int_M f \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \mathrm{d}(\phi_{-x})_*\nu = \int_M f \mathrm{d}\nu\end{aligned}$$

Since f was arbitrary, this implies that (almost everywhere):

$$\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-y})_*\nu} \circ \phi_x = \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x-y})_*\nu}$$ □

Having this settled, let us formulate and prove the first integration theorem.

THEOREM 6.1.6. *Let $M = \mathbb{R}^m$ and let $\omega \in \Omega^m(M)$ be a volume form. Let $(G = M \times M \rightrightarrows M, \mathrm{pr}_2^*\omega)$ be the corresponding volumetric pair groupoid. Let $H \to M$ be a ν-Hilbert field for a quasi-invariant measure ν and $K = L^2(H, \nu)$. Let $R : (\mathrm{Diff}(M), *_\omega) \cong (\mathrm{Diff}^R(M \times M), *_{\mathrm{pr}_2^*\omega}) \to \mathcal{O}(K)$ be a representation such that $\overline{R(m_f)} = r(f) = T_f$ for all $f \in C_c^\infty(M)$, where $r : C_0(M) \to \mathbb{B}(K)$ is a bounded representation. Let* $\mathrm{Lap} = \sum_{i=1}^m (\partial_i + \frac{1}{2}\,\mathrm{div}_\omega(\partial_i))^2 \in \mathrm{Diff}(M)$ *be the Laplacian with respect to ω.*

If $R(\mathrm{Lap}) \in \mathcal{O}(K)$ is essentially self-adjoint, then there is a local essential homomorphism of first type $P : G \dashrightarrow U(H)$ with

$$P(p,q) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}\right)^{\frac{1}{2}}(p)\left(\phi^*_{p-q}\mathrm{e}^{\bar{R}\left((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial)\right)}\right)(q)$$

for almost all $(p,q) \in G$ close enough to the diagonal, where $\phi_x(p) = p + x$.

Proof: Consider the operators $A_i := R(\mathcal{L}_{\partial_i} + \frac{1}{2}m_{\mathrm{div}_\omega \partial_i}) \in \mathcal{O}(K)$, $i \in \{1,\ldots,m\}$, and for $x \in \mathbb{R}^m$, $xA := \sum_{i=1}^m x_iA_i$. We know that $[\partial_i, \partial_j] = 0$ within $\mathrm{Diff}(M)$ (by Schwarz's Theorem), so by Theorem 3.2.17 also $[\mathcal{L}_{\partial_i} + \frac{1}{2}m_{\mathrm{div}_\omega \partial_i}, \mathcal{L}_{\partial_j} + \frac{1}{2}m_{\mathrm{div}_\omega \partial_j}] = 0$ and hence $[A_i, A_j] = 0$ for all $i,j \in \{1,\ldots,m\}$. The differential operators $\tilde{\partial}_i = \mathcal{L}_{\partial_i} + \frac{1}{2}m_{\mathrm{div}_\omega \partial_i}$ are formally skew-adjoint by construction, so A_i is skew-symmetric in $\mathcal{O}(K)$, as is xA for all x. Skew-symmetric operators are closable, so let $xB = \overline{(\sum_{i=1}^m x_iA_i)}$ be the closure and $U_x = \mathrm{e}^{xB} \in \mathcal{O}(K)$ (this definition is possible because xB is skew-adjoint, as detailed in the next paragraph). The first step now is to show that this is well-defined and that $U_xU_y = U_{x+y}$ for all sufficiently small $x, y \in \mathbb{R}^m$.

Let $L = R(\mathrm{Lap}) = \sum_{i=1}^m A_i^2 \in \mathcal{O}(K)$, $\xi := |A_1| + \cdots + |A_m| \in |\mathcal{O}(K)|$ and $\alpha := |L| + |\mathrm{id}_K|\mathcal{O}(K)|$ in the set of absolute values of operators (see the second chapter in [**18**] for details on this). By [**18**], Lemma 6.2, page 588, α analytically dominates ξ (this lemma can be applied because the Lie algebra generated by $A_1,\ldots,A_m$ is finite-dimensional since the operators commute on their domain). Let $E := \bigcap_{n\in\mathbb{N}} \mathrm{dom}(\overline{L}^n)$ be the common domain of all powers of the Laplacian's closure (note that $\mathrm{dom}\, R \subseteq E$ because R is invariant). Then by [**18**], Lemma 5.2, page 14, we know two things: Firstly, the operators xA are all essentially self-adjoint, hence U_x is well-defined and unitary. Secondly, there are an $s > 0$ and a dense subset $E^\omega \subset E$ such that for all $v \in E^\omega$, $\|\mathrm{e}^{s\sum_{i=1}^m |B_i|_E|}v\| < \infty$. More explicitly:

$$\sum_{n=0}^\infty \frac{s^n}{n!} \sum_{1\le i_1,\ldots,i_n\le m} \|B_{i_1}\ldots B_{i_n}(v)\| < \infty$$

The above formula contains compositions of the B_i in different orderings. But the expectation is that the order does not matter here, because we have started with commuting vector fields. Indeed, we know that the operators A_i are skew-symmetric. Hence their closures B_i are also skew-symmetric, which then implies that the Lie bracket $[B_i, B_j] = B_iB_j - B_jB_i$ is again skew-symmetric. In particular, it is closable. Hence we have $0|_{\mathrm{dom}\, R} = [A_i, A_j] \subseteq [B_i, B_j] \subseteq \overline{[B_i, B_j]}$. Because the right side is closed and $\mathrm{dom}\, R \subseteq K$ is dense, this implies that $[B_i, B_j] = 0$. Namely let $v \in \mathrm{dom}[B_i, B_j] \subseteq \mathrm{dom}\,\overline{[B_i, B_j]}$. As $\mathrm{dom}\, R$ is dense, choose a sequence $(v_i) \subset \mathrm{dom}\, R$ which converges to v. We have $[B_i, B_j](v_i) = [A_i, A_j](v_i) = 0$ for all i, so in particular, $\overline{[B_i, B_j]}(v_i) = [B_i, B_j](v_i)$ converges to 0. Since $\overline{[B_i, B_j]}$ is closed, this implies $\overline{[B_i, B_j]}(v) = 0$.

In particular, we have $B^IB^J(v) = B^{I+J}(v)$ for all $I, J \in \mathbb{N}^m$ and $v \in E^\omega$. Using this commutation and the absolute convergence from Nelson's Lemma, we can write our exponentials as power series and see that for $x, y \in \mathbb{R}^m$ with $\|x\|_\infty + \|y\|_\infty \le s$ and $v \in E^\omega$, we have:

$$
\begin{aligned}
U_xU_y(v) &= \sum_{n=0}^{\infty}\frac{1}{n!}(xB)^n\left(\sum_{k=0}^{\infty}\frac{1}{k!}(yB)^k(v)\right)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{I\in\mathbb{N}^m,|I|=n}\binom{n}{I}x^IB^I\left(\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{J\in\mathbb{N}^m,|J|=k}\binom{n}{J}y^JB^J(v)\right)\\
&= \sum_{I\in\mathbb{N}^m}\sum_{J\in\mathbb{N}^m}\frac{1}{I!J!}x^IB^Iy^JB^J(v) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{|I|+|J|=n}\frac{n!}{I!J!}x^IB^Iy^JB^J(v)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k+l=n}\frac{n!}{k!l!}(xB)^k(yB)^l(v) = \sum_{n=0}^{\infty}\frac{1}{n!}((x+y)B)^n(v) = U_{x+y}(v)
\end{aligned}
$$

For this computation, we used that the operators B_i are closed and the partial sums are absolutely convergent by Nelson's Lemma and $|x^I| = \prod_{k=1}^m |x_k^{i_k}| \leq s^{|I|}$ to obtain the third line. Because U_xU_y and U_{x+y} are bounded and E^ω is dense in K, we conclude that $U_xU_y = U_{x+y}$ for $\|x\|_\infty + \|y\|_\infty \leq s$. In the proof of this fact I used details from Nelson's theory to generalize his results.

For $x \in \mathbb{R}^m$, let $\phi_x := t \circ \exp(x\partial) = \theta^{\partial_1}_{x_1} \circ \cdots \circ \theta^{\partial_m}_{x_m}$ with the respective maximal domain inside M, where θ^{∂_i} is the flow of ∂_i. In this case, we have explicitly: $\phi_x(p) = p + x$. Note that $\phi_{x+y} = \phi_x\phi_y = \phi_y\phi_x$ for all $x, y \in \mathbb{R}^m$ since $[\partial_i, \partial_j] = 0$.

Let $\Phi_x := ((\phi_x)^{-1})^* : L^2H \to L^2(((\phi_x)^{-1})^*H)$. The operator Φ_xU_x is decomposable by Lemma 6.1.4, so there is a section $\xi \in L^\infty(\mathbb{B}(H, (\phi_{-x})^*H))$ such that $\Phi_xU_x = \int_M^\oplus \xi \mathrm{d}\nu$. For all $p \in M$ and $\|x\|_\infty < s$, set $U_x(p) := \xi(p)$.

The next step is to show that for all $x, y \in \mathbb{R}^m$ with $\|x\| + \|y\|$ we have $U_x(\phi_{-y}p)U_y(p) = U_{x+y}(p)$ for almost all $p \in M$. To do so, define $\Psi_y := (\phi_y)^{-1})^* : L^2((\phi_{-x})^*H) \to L^2((\phi_{-y})^*(\phi_{-x})^*H) = L^2((\phi_{-x}\phi_{-y})^*H)$. Then for all $\sigma \in L^2H$, $p \in M$,

$$\Psi_y\Phi_x(\sigma)(p) = \sigma(\phi_{-y}\phi_{-x}(p)) = \sigma(\phi_{-x-y}(p)) = \Phi_{x+y}\sigma(p),$$

hence $\Psi_y\Phi_x = \Phi_{x+y}$.

It follows that

$$\Phi_{x+y}U_{x+y} = \Phi_{x+y}U_xU_y = \Psi_y\Phi_xU_xU_y = \Psi_y\Phi_xU_x(\Phi_y)^{-1}\Phi_yU_y$$

Let ξ be a section with $\int_M^\oplus \xi\mathrm{d}\nu = \Phi_xU_x$ and χ a section with $\int_M^\oplus \chi\mathrm{d}\nu = \Phi_yU_y$. Let $\sigma \in L^2H$ be arbitrary. Then we know by the previous results that for ν-almost all $p \in M$:

$$
\begin{aligned}
U_{x+y}(p)(\sigma(p)) &= \Phi_{x+y}U_{x+y}(\sigma)(p) = \Psi_y\Phi_xU_x(\Phi_y)^{-1}\Phi_yU_y(\sigma)(p)\\
&= (\int_M^\oplus \xi\mathrm{d}\nu)(\Phi_y)^{-1}(\int_M^\oplus \chi\mathrm{d}\nu)(\sigma)(\phi_{-y}p)\\
&= \xi(\phi_{-y}(p))\left((\Phi_y)^{-1}(\int_M^\oplus \chi\mathrm{d}\nu)(\sigma)(\phi_{-y}p)\right)\\
&= \xi(\phi_{-y}(p))(\int_M^\oplus \chi\mathrm{d}\nu)(\sigma)(p)\\
&= \xi(\phi_{-y}(p))\chi(p)(\sigma(p)) = U_x(\phi_{-y}(p))U_y(p)(\sigma(p))
\end{aligned}
$$

Since $\{\sigma(p)|\sigma \in L^2H\}$ is dense in H_p ν-almost everywhere, this implies that

$$U_{x+y}(p) = U_x(\phi_{-y}(p))U_y(p)$$

for almost all $p \in M$, just as required.

There is yet another small problem to solve here: The operator Φ_xU_x is decomposable, but no longer unitary. To get this property back, we have to multiply with a Radon-Nikodým term. Namely by Lemma 5.1.11, the operator $\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu} \circ \phi_{-x}\right)^{\frac{1}{2}} \Phi_x : L^2H \to L^2(\phi_{-x}^*H)$ is indeed unitary. So $\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu}\circ\phi_{-x}\right)^{\frac{1}{2}}\Phi_x U_x$ is unitary as the product of the unitary operators U_x and $\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu}\circ\phi_{-x}\right)^{\frac{1}{2}}\Phi_x$, and decomposable as the product of the decomposable operators $T_{\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu}\circ\phi_{-x}\right)^{\frac{1}{2}}}$ and $\Phi_x U_x$. Hence the operator $\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu}\circ\phi_{-x}\right)^{\frac{1}{2}}\Phi_x U_x\right)(q): H_q\to H_{q-x}$ is well-defined and unitary for almost all $q\in M$ by Proposition 3, page 182 in [**5**].

For each $x\in\mathbb{R}^m$ with $\|x\|<\frac{s}{2}$, let $N_x\subseteq M$ be a null set such that

$$\left(\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu}\circ\phi_{-x}\right)^{\frac{1}{2}}\Phi_x U_x\right)(q)$$

is unitary for all $p\in M\setminus N_x$. Define $N':=\bigcup_{x\in\mathbb{R}^m,\|x\|<\frac{s}{2}}\{(q-x,q)|q\in N_x\}$, which is a null set in G.

For any $(p,q)\in M\times M$ with $\|p-q\|_\infty<\frac{s}{2}$, define

$$P(p,q)=\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}\right)^{\frac{1}{2}}(p)(\phi^*_{p-q}U_{q-p})(q)$$

Define $W:=\{(p,q)\in M^2|\|p-q\|_\infty<\frac{s}{2}\}$, which is an open neighbourhood of the diagonal, and $N=W\cap N'$, so that $P(p,q)$ is unitary for all $(p,q)\in W\setminus N$.

I claim that P is a first-type local essential homomorphism. To show this, let first $x,y\in\mathbb{R}^m$ be arbitrary with $\|x\|,\|y\|<\frac{s}{2}$. Then by the previous argument and Lemma 6.1.5, there is a null set $N_{x,y}\subseteq M$ such that $\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x})_*\nu}(p)\cdot\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-y})_*\nu}\circ\phi_x(p)=\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{-x-y})_*\nu}(p)$ and $\Phi_xU_x(p-y)\Phi_yU_y(p)=\Phi_{x+y}U_{x+y}(p)$ for all $p\in M\setminus N_{x,y}$. Define

$$N_2:=\{(r+x+y,r+x)\mid x,y\in\mathbb{R}^m,\|x\|,\|y\|<\frac{s}{2},r\in N_{x,y}\},$$

which is a null set in G.

Now let $g=(p,q),h=(q,r)\in W\setminus N$ such that $gh=(p,r)\in W\setminus N$. Set $x=p-q$ and $y=q-r$, so that $g=\exp(x\partial)(q)$ and $h=\exp(y\partial)(r)$. Then if also $(g,h)\notin N_2$, we have $r\notin N_{x,y}$ and hence:

$$\begin{aligned}
P(g)P(h)&=\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_x)_*\nu}\right)^{\frac{1}{2}}(p)(\phi^*_xU_{-x})(q)\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_y)_*\nu}\right)^{\frac{1}{2}}(q)(\phi^*_yU_{-y})(r)\\
&=\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_x)_*\nu}(p)\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_y)_*\nu}(p-x)\right)^{\frac{1}{2}}(\phi^*_xU_{-x})(q)(\phi^*_yU_{-y})(r)\\
&=\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_x)_*\nu}(p)\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_y)_*\nu}\circ\phi_{-x}(p)\right)^{\frac{1}{2}}\Phi_{-x}U_{-x}(r+y)\Phi_{-y}U_{-y}(r)\\
&=\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{x+y})_*\nu}(p)\right)^{\frac{1}{2}}\Phi_{-x-y}U_{-x-y}(r)\\
&=\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-r})_*\nu}(p)\right)^{\frac{1}{2}}\phi^*_{p-r}U_{r-p}(r)=P(p,r)
\end{aligned}$$

As g,h were arbitrary, $P:W\setminus N\to U(H)$ is indeed a local essential homomorphism of first type. □

After this laborious proof, let us enjoy the freshly proven theorem's fruits by combining it with one of our previous results to quickly get an even more pleasant result. We get the following corollary:

Corollary 6.1.7. *Under the assumptions of Theorem 6.1.6, there is even a* global *essential homomorphism of second type $P : G \dashrightarrow U(H)$ with*

$$P(p,q) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}\right)^{\frac{1}{2}}(p)\left(\phi_{p-q}^* \mathrm{e}^{\bar{R}\left((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial)\right)}\right)(q)$$

for almost all $(p,q) \in G$ close enough to the diagonal.

Proof: Let P_0 be the local essential homomorphism of first type obtained from Theorem 6.1.6 with domain $W \setminus N$ for a null set $N \subseteq G$ and a neighbourhood of the base space $W \subseteq G$. By Lemma 5.3.6, there is another null set $N_2 \subseteq G$ such that $P_0|_{W\setminus N_2}$ is a local essential homomorphism of second type. This extends to a global one by the Sausage Theorem 5.3.10. □

Summarizing this section, we started with a representation $R : \mathrm{Diff}^R(M \times M) \to \mathcal{O}(L^2H)$ of the differential operator algebra over $M = \mathbb{R}^m$. We assumed that it maps a Laplacian element to a self-adjoint operator. A representation with this property we can now rightfully call *integrable*, for we have proven it to integrate to a representation (an essential global homomorphism of second type) $P : M \times M \dashrightarrow U(H)$ of the pair groupoid. This first integration map shall be denoted int_1, setting $\mathrm{int}_1(R) := P$ in the notation of Theorems 6.1.6 and 6.1.7.

6.2. From the Groupoid to its Groupoid Algebra

Now that we have completed the first step of the integration for the Euclidean space, this section will consider the second step, being the integration from groupoid representations to groupoid algebra representations. The theorems from this section are thankfully a bit easier to prove and also a bit more general than those before. Most of the relevant computations have been done by other authors before in different contexts, yet it is useful to lay them down here in a condensed and adapted manner.

To begin with, recall the I-norm on a groupoid with Haar system λ: For $f : G \to \mathbb{C}$, $\|f\|_I = \max\{\sup_{p\in M}\int_{G^p}|f(p)|\mathrm{d}\lambda^p, \sup_{p\in M}\int_{G_p}|f(p)|\mathrm{d}\lambda_p\}$, where $\lambda_p = \lambda^p \circ \mathrm{inv}^{-1}$. Recall also that we have measures $\nu\circ\lambda$ and $\nu\circ\tilde{\lambda}$ on G defined by $\int_G f\mathrm{d}\nu\circ\lambda = \int_M\int_{G^p} f\mathrm{d}\lambda^p\mathrm{d}\nu(p)$ and $\int_G f\mathrm{d}\nu\circ\tilde{\lambda} = \int_M\int_{G_p} f\mathrm{d}\lambda_p\mathrm{d}\nu(p)$. We will mainly consider the groupoid algebra $L^I(G)$, which is the completion of $C_c(G)$ with respect to this I-norm.

The first thing to prove is an inequality between different norms:

Lemma 6.2.1. *Let $G \rightrightarrows M$ be a groupoid with Haar system λ and ν a quasi-invariant measure on M. Let $H \to M$ be a ν-Hilbert field. Denote by $\mu = \nu\circ\lambda$ the measure on G induced by ν and λ. Then for all $f \in L^I(G)$ and all $\sigma \in L^2(H)$, we have*

$$\int_G |f(x)|\|\sigma\circ s(x)\|^2\mathrm{d}\mu^{-1}(x) \leq \|f\|_I\|\sigma\|^2$$

and

$$\int_G |f(x)|\|\sigma\circ t(x)\|^2\mathrm{d}\mu(x) \leq \|f\|_I\|\sigma\|^2.$$

Proof: We have

$$\begin{aligned}\int_G |f(x)|\|\sigma\circ s(x)\|^2\mathrm{d}\mu^{-1}(x) &= \int_M\int_{G_p}|f(x)|\|\sigma(p)\|^2\mathrm{d}\lambda_p(x)\mathrm{d}\nu(p)\\ &\leq \sup_{p\in M}\int_{G_p}|f(x)|\mathrm{d}\lambda_p(x)\cdot\int_M\|\sigma(p)\|^2\mathrm{d}\nu(p)\\ &= \|f\|_{I,s}\|\sigma\|^2 \leq \|f\|_I\|\sigma\|^2\end{aligned}$$

and likewise:

$$\begin{aligned}\int_G |f(x)|\|\sigma\circ t(x)\|^2 \mathrm{d}\mu(x) &= \int_M\int_{G^p} |f(x)|\|\sigma(p)\|^2 \mathrm{d}\lambda^p(x)\mathrm{d}\nu(p)\\ &\leq \sup_{p\in M}\int_{G^p}|f(x)|\mathrm{d}\lambda^p(x)\cdot\int_M\|\sigma(p)\|^2\mathrm{d}\nu(p)\\ &= \|f\|_{I,t}\|\sigma\|^2 \leq \|f\|_I\|\sigma\|^2 \qquad\square\end{aligned}$$

Using this inequality, we can already face our main theorem for this section.

THEOREM 6.2.2. *Let $G \rightrightarrows M$ be a Lie groupoid with a Haar system λ and a quasi-invariant measure ν on M. Let $H \to M$ be a ν-Hilbert field and let $P : G \supset D \to U(H)$ be an essential homomorphism of second type. Suppose that for all measurable sections $\sigma, \tau \in \mathcal{M}(H)$, the map $G \to \mathbb{C},\ x \mapsto \langle P(x)\sigma(sx), \tau(tx)\rangle$ is measurable.*

For $f \in C_c(G)$ and $\sigma \in L^2(H) \cap L^\infty(H)$, define:

$$\pi(f)(\sigma) \in L^2(H),\ \pi(f)(\sigma)(p) := \int_{G^p} \Delta^{-\frac{1}{2}}(x)f(x)P(x)(\sigma(sx))\mathrm{d}\lambda^p(x),$$

where $\Delta := \frac{\mathrm{d}\nu\circ\lambda}{\mathrm{d}\nu\circ\bar\lambda}$. This is well-defined with $\|\pi(f)(\sigma)\| \leq \|f\|_I\|\sigma\|$, and π extends to a $*$-homomorphism $\pi : L^I(G) \to \mathbb{B}(L^2H)$.

Proof: First, let us check that the written integral is well-defined. Namely for all $p \in M$ such that $G^p \cap D$ is co-null (which is true for almost all $p \in M$) and all $\tau \in L^\infty(H)$, we know that $G^p \to \mathbb{C},\ x \mapsto \langle \Delta^{-\frac{1}{2}}f(x)P(x)\sigma(sx), \tau(tx)\rangle$ is measurable by assumption, and absolutely integrable because

$$\begin{aligned}\int_{G^p} |\Delta^{-\frac{1}{2}}\langle f(x)P(x)\sigma(sx),\tau(tx)\rangle|\mathrm{d}\lambda^p(x) &\leq \int_{G^p}|\Delta^{-\frac{1}{2}}f(x)|\|P(x)\sigma(sx)\|\|\tau(tx)\|\mathrm{d}\lambda^p(x)\\ &= \int_{G^p}|\Delta^{-\frac{1}{2}}f(x)|\|\sigma(sx)\|\|\tau(tx)\|\mathrm{d}\lambda^p(x)\\ &\leq \|\Delta^{-\frac{1}{2}}|_{\operatorname{supp} f}\|_\infty\|f\|_\infty\|\sigma\|_\infty\|\tau\|_\infty\lambda^p(\operatorname{supp} f\cap G^p)\end{aligned}$$

For a dense subset of vectors $\tau_p \in H_p$ we can choose $\tau \in L^\infty(H)$ such that $\tau(p) = \tau_p$ and $\|\tau\|_\infty \leq \|\tau\|$. This implies that $H_p \to \mathbb{C},\ v \mapsto \int_{G^p}\langle f(x)P(x)\sigma(sx), v\rangle\mathrm{d}\lambda^p(x)$ is a bounded linear functional, hence by the Riesz representation theorem there is indeed a vector $w \in H_p$ with $\langle w, v\rangle = \int_{G^p}\langle\Delta^{-\frac{1}{2}}(x)f(x)P(x)\sigma(sx), v\rangle\mathrm{d}\lambda^p(x)$ for all v, which is denoted, as usually, by $\int_{G^p}\Delta^{-\frac{1}{2}}f(x)P(x)\sigma(sx)\mathrm{d}\lambda^p(x) = w$.

So far we have proven that $\pi(f)(\sigma)$ is indeed a well-defined section of H. It is measurable by the following argument: If again $\tau \in \mathcal{M}(H)$ is a measurable section, then $G \to \mathbb{C},\ x \mapsto \langle P(x)\sigma(sx), \tau(tx)\rangle$ is measurable by assumption, so $x \mapsto \Delta^{-\frac{1}{2}}(x)f(x)\langle P(x)\sigma(sx), \tau(tx)\rangle$ is of course also measurable, and by the properties of a Haar system, $M \to \mathbb{C},\ p \mapsto \int_{G^p}\Delta^{-\frac{1}{2}}(x)f(x)\langle P(x)\sigma(sx),\tau(tx)\rangle\mathrm{d}\lambda^p(x) = \langle\int_{G^p}\Delta^{-\frac{1}{2}}(x)f(x)P(x)\sigma(sx)\mathrm{d}\lambda^p(x), \tau(p)\rangle$ must be measurable, too. Since this is true for sufficiently many sections τ, following the defining properties of Hilbert fields yields that $\pi(f)(\sigma)$ is indeed measurable.

Next, for the norm: Let f, σ as before, $\tau \in L^2H$. Using the Cauchy-Schwartz inequality to obtain the fifth line, we compute:

$$\begin{aligned}|\langle\pi(f)(\sigma),\tau\rangle| &= \int_M |\langle\pi(f)(\sigma)(p),\tau(p)\rangle|\mathrm{d}\nu(p)\\ &\leq \int_M\int_{G^p}\Delta^{-\frac{1}{2}}(x)|f(x)|\,|\langle P(x)\sigma(sx),\tau(p)\rangle|\mathrm{d}\lambda^p(x)\mathrm{d}\nu(p)\\ &\leq \int_G \Delta^{-\frac{1}{2}}(x)|f(x)|\|\sigma(sx)\|\|\tau(tx)\|\mathrm{d}\nu\circ\lambda(x)\\ &= \langle\Delta^{-\frac{1}{2}}f^{\frac{1}{2}}\|\sigma\circ s\|, f^{\frac{1}{2}}\|\tau\circ t\|\rangle_{L^2(G,\nu\circ\lambda)}\end{aligned}$$

$$\begin{aligned}
&\leq \|\Delta^{-\frac{1}{2}} f^{\frac{1}{2}} \|\sigma \circ s\|\|_2 \cdot \|f^{\frac{1}{2}} \|\tau \circ t\|\|_2 \\
&= \Big(\int_G \Delta^{-1}(x)|f(x)|\|\sigma \circ s(x)\|^2 \mathrm{d}\nu \circ \lambda(x)\Big)^{\frac{1}{2}} \\
&\cdot \Big(\int_G |f(x)|\|\tau \circ t(x)\|^2 \mathrm{d}\nu \circ \lambda(x)\Big)^{\frac{1}{2}} \\
&= \Big(\int_G |f(x)|\|\sigma \circ s(x)\|^2 \mathrm{d}\nu \circ \tilde{\lambda}(x)\Big)^{\frac{1}{2}} \\
&\cdot \Big(\int_G |f(x)|\|\tau \circ t(x)\|^2 \mathrm{d}\nu \circ \lambda(x)\Big)^{\frac{1}{2}} \\
&\leq \Big(\|f\|_I \|\sigma\|^2 \|f\|_I \|\tau\|^2\Big)^{\frac{1}{2}} = \|f\|_I \|\sigma\|\|\tau\|,
\end{aligned}$$

where the last line is by Lemma 6.2.1. In particular,

$$\|\pi(f)\sigma\|^2 = |\langle \pi(f)\sigma, \pi(f)\sigma\rangle| \leq \|f\|_I \|\sigma\|\|\pi(f)\sigma\|,$$

i.e. $\|\pi(f)\sigma\| \leq \|f\|_I \|\sigma\|$, just as required.

Because the integral map and the operators $P(x)$, $x \in G$, are linear, the mapping $\sigma \mapsto \pi(f)(\sigma)$ is also linear. Consequently, as $L^2H \cap L^\infty H \subseteq L^2H$ is dense, $\pi(f)$ extends to a bounded operator $L^2(H) \to L^2(H)$ with $\|\pi(f)\| \leq \|f\|_I$.

The map $L^I(G) \to \mathbb{B}(L^2H),\ f \mapsto \pi(f)$ is also linear because the integral map has this property, and we have just shown that it is bounded by 1. Let us now show the next important property: Compatibility with the $*$-structure. To do this, let $f, g \in C_c^\infty(G)$ be arbitrary. Then for all $\sigma \in L^2H \cap L^\infty H$ and almost all $p \in M$, we have

$$\begin{aligned}
\pi(f)\pi(g)(\sigma)(p) &= \int_{G^p} \Delta^{-\frac{1}{2}}(x) f(x) P(x) \pi(g)(\sigma)(sx) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \Delta^{-\frac{1}{2}}(x) f(x) P(x) \int_{G^{sx}} \Delta^{-\frac{1}{2}}(y) g(y) P(y) \sigma(sy) \mathrm{d}\lambda^{sx}(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \int_{G^{sx}} \Delta^{-\frac{1}{2}}(x) \Delta^{-\frac{1}{2}}(y) f(x) g(y) P(xy) \sigma(sy) \mathrm{d}\lambda^{sx}(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \int_{G^{tx}} \Delta^{-\frac{1}{2}}(x) \Delta^{-\frac{1}{2}}(x^{-1}y) f(x) g(x^{-1}y) P(y) \sigma(s(x^{-1}y)) \mathrm{d}\lambda^{tx}(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \int_{G^p} \Delta^{-\frac{1}{2}}(x) \Delta^{-\frac{1}{2}}(x^{-1}y) f(x) g(x^{-1}y) P(y) \sigma(sy) \mathrm{d}\lambda^p(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \int_{G^p} \Delta^{-\frac{1}{2}}(x) \Delta^{-\frac{1}{2}}(x^{-1}y) f(x) g(x^{-1}y) P(y) \sigma(sy) \mathrm{d}\lambda^p(x) \mathrm{d}\lambda^p(y) \\
&= \int_{G^p} \int_{G^p} \Delta^{-\frac{1}{2}}(y) \Delta^{-\frac{1}{2}}(y^{-1}x) f(y) g(y^{-1}x) P(x) \sigma(sx) \mathrm{d}\lambda^p(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \int_{G^{tx}} \Delta^{-\frac{1}{2}}(y) \Delta^{-\frac{1}{2}}(y^{-1}x) f(y) g(y^{-1}x) P(x) \sigma(sx) \mathrm{d}\lambda^{tx}(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \int_{G^{sx}} \Delta^{-\frac{1}{2}}(xy) \Delta^{-\frac{1}{2}}(y^{-1}) f(xy) g(y^{-1}) P(x) \sigma(sx) \mathrm{d}\lambda^{sx}(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \int_{G^{sx}} \Delta^{-\frac{1}{2}}(x) f(xy) g(y^{-1}) P(x) \sigma(sx) \mathrm{d}\lambda^{sx}(y) \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \Delta^{-\frac{1}{2}}(x) f * g(x) P(x) \sigma(sx) \mathrm{d}\lambda^p(x) = \pi(f * g)(\sigma)(p),
\end{aligned}$$

using that $\Delta(xy) = \Delta(x)\Delta(y)$ for almost all composable $x, y \in G$ by Theorem 3.15, page 37 in [**17**].

This implies $\pi(f * g) = \pi(f)\pi(g)$ since $L^\infty H \cap L^2 H \subseteq L^2 H$ is dense. Since π is continuous and $C_c^\infty G \subseteq L^I G$ is dense, this implies that $\pi(f * g) = \pi(f) * \pi(g)$ for all $f, g \in L^I G$, hence π is multiplicative.

It is left to check that $\pi(f^*) = \pi(f)^*$ for all f. Again by Theorem 3.15, page 37 in [**17**], we know that $\Delta(x^{-1}) = \Delta(x)^{-1}$ for almost all $x \in G$, so we easily compute:

$$\begin{aligned}
\langle \pi(f^*)\sigma, \tau\rangle(p) &= \int_{G^p} \Delta^{-\frac{1}{2}}(x)\bar{f}(x^{-1})\langle P(x)\sigma(sx), \tau(tx)\rangle \mathrm{d}\lambda^p(x) \\
&= \int_{G^p} \Delta^{-\frac{1}{2}}(x^{-1})\bar{f}(x^{-1})\langle \sigma(sx), P(x^{-1})\tau(tx)\rangle \mathrm{d}\lambda^p(x) \\
&= \int_{G_p} \Delta^{-\frac{1}{2}}(x)^{-1}\bar{f}(x)\langle \sigma(tx), P(x)\tau(sx)\rangle \mathrm{d}\lambda_p(x),
\end{aligned}$$

and hence

$$\begin{aligned}
\langle \pi(f^*)\sigma, \tau\rangle &= \int_M \langle \pi(f^*)\sigma, \tau\rangle(p)\mathrm{d}\nu(p) \\
&= \int_M \int_{G_p} \Delta^{-\frac{1}{2}}(x)^{-1}\bar{f}(x)\langle \sigma(tx), P(x)\tau(sx)\rangle \mathrm{d}\lambda_p(x)\mathrm{d}\nu(p) \\
&= \int_G \Delta\Delta^{-\frac{1}{2}}\bar{f}\langle \sigma\circ t, P\circ\tau\circ s\rangle \mathrm{d}(\nu\circ\tilde{\lambda}) \\
&= \int_G \Delta^{-\frac{1}{2}}\bar{f}\langle \sigma\circ t, P\circ\tau\circ s\rangle \mathrm{d}(\nu\circ\lambda) \\
&= \int_M \int_{G^p} \Delta^{-\frac{1}{2}}(x)\bar{f}(x)\langle \sigma(tx), P(x)\tau(sx)\rangle \mathrm{d}\lambda^p(x)\mathrm{d}\nu(p) \\
&= \int_M \langle \sigma, \pi(f)\tau\rangle(p)\mathrm{d}\nu(p) = \langle \sigma, \pi(f)\tau\rangle;
\end{aligned}$$

as before, continuity finishes the proof. □

The representation π obtained from P in the previous theorem will be denoted by $\mathrm{int}_2(P)$ from now on, adding to the maps diff from Theorem 4.1.5 and int_1 from Corollary 6.1.7.

Of these three maps, int_2 is in a sense the most well-behaved because we can directly prove that it is actually a bijection. To do this, I rely on a powerful theorem from [**4**], which in turn was based on ideas by Renault. Unfortunately that theorem is not directly applicable at this point because it was stated in a slightly different context. But as we will see, the gaps can be bridged with just a few more lemmas.

Firstly, let us have another look at an important property of groupoid C^*-algebras: Almost by construction, their representations correspond to those of the groupoid algebra $L^I(G)$.

Lemma 6.2.3. *Let $G \rightrightarrows M$ be a Lie groupoid with Haar system λ and let K be a Hilbert space. Then the representations of the groupoid algebra $L^I(G)$ and those of the groupoid C^*-algebra $C^*(G)$ on K are in bijection.*

More precisely: Let $\pi : L^I(G) \to \mathbb{B}(K)$ be a non-degenerate bounded representation. Then there is a representation $\tilde{\pi} : C^*(G) \to \mathbb{B}(K)$ with $\pi|_{C_cG} = \tilde{\pi}|_{C_cG}$. Likewise, if $\tau : C^*(G) \to \mathbb{B}(K)$ is a non-degenerate bounded representation, then there is one $\tilde{\tau} : L^I(G) \to \mathbb{B}(K)$ with $\tau|_{C_cG} = \tilde{\tau}|_{C_cG}$, and these constructions are inverse to each other.

Proof: Start with a representation $\pi : L^I(G) \to \mathbb{B}(K)$. Let $f \in C_cG \subseteq L^I(G)$ be arbitrary. By the definition of the groupoid C^*-algebra, the norm of f as an element of C^*G is $\|f\|_{C^*} = \sup_\tau \|\tau(f)\|$, where τ ranges over all non-degenerate representations of C_cG with involution. But $\pi|_{C_cG}$ *is* such a representation. So in particular:

$$\|f\|_{C^*} = \sup_\tau \|\tau(f)\| \geq \|\pi(f)\|$$

Hence $\pi|_{C_cG} : C^*G \supset C_cG \to \mathbb{B}(K)$ is bounded (by 1) and defined on a dense subspace, and thus extends to a bounded linear map $\tilde{\pi} : C^*G \to \mathbb{B}(K)$. Compatibility with convolution and involution is given on C_cG by definition of π and extends to C^*G by continuity.

Likewise, let $\tau : C^*G \to \mathbb{B}(K)$ be a representation. Then for all $f \in C_cG$ we have

$$\|\tau(f)\| \leq \|\tau\|\|f\|_{C^*} \leq \|\tau\|\|f\|_I$$

by Theorem 2.42., page 28 in [**17**]. Actually, $\|\tau\| = 1$ for C^*-algebra representations τ, but this is not important here.

So $\tau|_{C_cG} : L^I(G) \supset C_cG \to \mathbb{B}(K)$ is bounded on a dense subset and hence extends to a bounded representation of $L^I(G)$; compatibility with the *-structure is inherited from τ as before.

The mappings are inverse to each other because $\tilde{\tilde{\pi}}|_{C_cG} = \pi|_{C_cG}$, both sides are continuous, and C_cG is dense in both $L^I(G)$ and $C^*(G)$. □

In the next step, we consider another way to encode a groupoid representation. We will see that an essential homomorphism from a groupoid to a unitary groupoid is equivalent to a unitary map between certain L^2-spaces. Namely:

Lemma 6.2.4. *Let G be as before and let ν be a quasi-invariant measure on $M = G^{(0)}$. Let H be a ν-Hilbert field and $P : G \dashrightarrow U(H)$ an essential homomorphism of second type. Then the map*

$$U_P : L^2(s^*H, \nu \circ \tilde{\lambda}) \to L^2(t^*H, \nu \circ \lambda),\ U_P(\sigma)(x) := \Delta^{-\frac{1}{2}}(x)P(x)(\sigma(x))$$

is unitary (here, $\Delta = \frac{\nu\circ\lambda}{\nu\circ\tilde{\lambda}}$).

Proof: First of all, the map U_P is well-defined in terms of sections because P is defined almost everywhere, and for all $\sigma \in L^2(s^*H, \nu \circ \tilde{\lambda})$ and almost all $x \in G$, we have $\sigma(x) \in (s^*H)_x = H_{sx}$, so that $P(x)\sigma(x)$ is defined and an element of $H_{tx} = (t^*H)_x$. For the norms and scalar products, we compute:

$$\begin{aligned}\langle U_P\sigma, U_P\tau\rangle &= \int_G \langle \Delta^{-\frac{1}{2}}(x)P(x)\sigma(x), \Delta^{-\frac{1}{2}}(x)P(x)\tau(x)\rangle \mathrm{d}\nu\circ\lambda(x)\\ &= \int_G \langle P(x)\sigma(x), P(x)\tau(x)\rangle \mathrm{d}\nu\circ\tilde{\lambda}(x)\\ &= \int_G \langle \sigma(x), \tau(x)\rangle \mathrm{d}\nu\circ\tilde{\lambda}(x) = \langle\sigma,\tau\rangle,\end{aligned}$$

so U_P is isometric. It has an inverse defined by $U_P^{-1}(\sigma)(x) = \Delta^{\frac{1}{2}}(x)P(x)^{-1}(\sigma(x))$, which can be easily computed. So U_P is in particular surjective, and hence unitary. □

As discussed in [**4**], page 11, the mapping $P \mapsto U_P$ is even a bijection between (equivalence classes given by changes on null sets of) essential homomorphisms $P : G \to U(H)$ and unitary maps $U : L^2(s^*H, \nu\circ\tilde{\lambda}) \to L^2(t^*H, \nu\circ\lambda)$ which intertwine with multiplication operators.

To grant compatibility with the foreign theorem, we need yet another equivalence: I will quickly demonstrate a way to identify the L^2-spaces from before with certain tensor products.

Lemma 6.2.5. *Let $G \rightrightarrows M$ be a Lie groupoid with Haar system λ and quasi-invariant measure ν and a ν-Hilbert field H. Consider the non-degenerate $*$-homomorphism $\phi : L^\infty(M,\nu) \to \mathbb{B}(L^2H),\ f \mapsto (T_f : \sigma \mapsto f\cdot\sigma)$. For $A \in \{L^2(G,s,\tilde{\lambda}), L^2(G,t,\lambda)\}$ (which are Hilbert-$L^\infty M$ modules), denote by $A \otimes_\phi L^2(H,\nu)$ the complete tensor product, i.e. the topological completion of the algebraic tensor product $A \otimes L^2(H,\nu)$ with respect to the inner product given by $\langle a_1\otimes\sigma_1, a_2\otimes\sigma_2\rangle = \langle\sigma_1, \phi(\langle a_1, a_2\rangle)\sigma_2\rangle$.*

Then there are two unique continuous linear maps

$$\alpha : L^2(G,s,\tilde{\lambda}) \otimes_\phi L^2(H,\nu) \to L^2(s^*H, \nu\circ\tilde{\lambda})$$

with $\alpha(f \otimes \sigma) = f \cdot (\sigma \circ s)$ *and*

$$\beta : L^2(G,t,\lambda) \otimes_\phi L^2(H,\nu) \to L^2(t^*H, \nu \circ \lambda)$$

with $\beta(g \otimes \sigma) = g \cdot (\sigma \circ t)$ *for all* $f \in L^2(G,s,\tilde{\lambda})$*,* $g \in L^2(G,t,\lambda)$ *and* $\sigma \in L^2(H,\nu)$*, and those maps are unitary.*

Proof: It is clear that the given formulas uniquely determine continuous linear maps because pure tensors densely span the complete tensor product. One property which is a bit more complicated to prove in all details is the fact that α and β are surjective. This fact is discussed in [**4**], Proposition 2.6, page 5 and on page 11 there. The idea is that the span of pure tensors is dense in the inductive limit topology.

What I want to prove in detail here is isometry, because the necessary computation is enlightening for the following proposition. Namely, let $f, g \in L^2(G,s,\tilde{\lambda})$ and $\sigma, \tau \in L^2(H,\nu)$ be arbitrary. Then we have

$$\begin{aligned}
\langle \alpha(f \otimes \sigma), \alpha(g \otimes \tau)\rangle &= \int_G \langle f\sigma \circ s, g\tau \circ s\rangle \mathrm{d}\nu \circ \tilde{\lambda} \\
&= \int_M \int_{G_p} \overline{f(x)} g(x) \mathrm{d}\lambda_p(x) \langle \sigma(p), \tau(p)\rangle \mathrm{d}\nu(p) \\
&= \int_M \langle f, g\rangle(p) \langle \sigma(p), \tau(p)\rangle \mathrm{d}\nu(p) \\
&= \int_M \langle \sigma(p), \langle f, g\rangle(p)\tau(p)\rangle \mathrm{d}\nu(p) \\
&= \langle \sigma, \phi(\langle f, g\rangle)\tau\rangle = \langle f \otimes \sigma, g \otimes \tau\rangle.
\end{aligned}$$

Likewise, if $f, g \in L^2(G,t,\lambda)$, then

$$\begin{aligned}
\langle \beta(f \otimes \sigma), \beta(g \otimes \tau)\rangle &= \int_M \int_{G^p} \overline{f(x)} g(x) \mathrm{d}\lambda^p(x) \langle \sigma(p), \tau(p)\rangle \mathrm{d}\nu(p) \\
&= \int_M \langle \sigma(p), \langle f, g\rangle(p)\tau(p)\rangle \mathrm{d}\nu(p) = \langle f \otimes \sigma, g \otimes \tau\rangle. \qquad \square
\end{aligned}$$

Let us now take these ingredients and combine them to show that the integration map $\int_2$ is bijective (up to equivalence classes).

PROPOSITION 6.2.6. *Let* $G \rightrightarrows M$ *be a Lie groupoid with Haar system* λ*, a quasi-invariant measure* ν *on* M *and a* ν*-Hilbert field* H*.*

The integration map int_2 *is* essentially bijective*, in the following sense: Given two essential homomorphisms* $P, Q : G \dashrightarrow U(H)$ *such that* $\mathrm{int}_2(P) = \mathrm{int}_2(Q)$*, we have* $P(x) = Q(x)$ *for almost all* $x \in G$*, and every representation* $\pi : L^I(G) \to \mathbb{B}(L^2 H)$ *fulfils* $\pi = \mathrm{int}_2(P)$ *for some representation* $P : G \dashrightarrow U(H)$*.*

Proof: This theorem can be viewed as a version of [**4**], Theorem 3.23., page 10. Indeed, the more advanced details of the proof can be found there and shall not be repeated in this thesis. Our work here lies in bridging the gaps between the slightly different definitions to make that theorem applicable in our scenario.

Namely by Theorem 3.23, there is a bijection between representations $\pi : C^*(G) \to \mathbb{B}(L^2H, \nu)$ and representations (ϕ, U) of (G, λ) on the Hilbert $\mathbb{C}$-module $L^2(H,\nu)$. Given a representation (ϕ, U) of (G,λ), denote its integrated form as of that theorem by $\mathrm{int}_3(\phi, U)$. If we start with an essential homomorphism $P : G \dashrightarrow U(H)$, then by Lemma 6.2.4, it induces a unique unitary $U_P : L^2(s^*H, \nu \circ \tilde{\lambda}) \to L^2(t^*H, \nu \circ \lambda)$. Using the isomorphisms from Lemma 6.2.5, we get a unitary $U = \beta^{-1} \circ U_P \circ \alpha : L^2(G,s,\tilde{\lambda}) \otimes_\phi L^2(H,\nu) \to L^2(G,t,\lambda) \otimes_\phi L^2(H,\nu)$. This operator intertwines the C_0G-actions, namely if $h \in C_cG$, $f \in L^2(G, s\tilde{\lambda})$, $\sigma \in L^2(H,\nu)$, then $U(h \cdot (f \otimes \sigma)) = \beta^{-1} U_P(hf\sigma \circ s) = \beta^{-1}(hU_P(f\sigma \circ s)) = h \cdot \beta^{-1}(U_P(f\sigma \circ s)) = h \cdot U(f \otimes \sigma)$ since $U_P(hf\sigma \circ s)(x) = h(x)\Delta^{-\frac{1}{2}}(x)P(x)(f(x)\sigma(sx)) = (hU_P(f\sigma \circ s))(x)$ and $\beta((hf) \otimes \sigma)) = hf\sigma = h\beta(f \otimes \sigma)$.

So (ϕ, U) is the data of a representation as in Definition 3.12, [**4**]. The property from Definition 3.18 is equivalent to $P(xy) = P(x)P(y)$ for almost all $x, y \in G^{(2)}$ as discussed on page 11 of the same paper, so (ϕ, U) is indeed a representation as in Definition 3.18, implying that it integrates to $\mathrm{int}_3(\phi, U) : C^*G \to \mathbb{B}(L^2H, \nu)$. By our Lemma 6.2.3, this is equivalent to a representation $\mathrm{int}_4(P) := \big(\mathrm{int}_3(\phi, \beta^{-1}U_P\alpha)\big)^\sim : L^I(G) \to \mathbb{B}(L^2H, \nu)$. As discussed in this section and in the aforementioned paper, this whole mapping int_4 is a bijection as a composition of bijections.

What is left to show now is that $\mathrm{int}_4 = \mathrm{int}_2$ from our previous definition. To do that, fix $f \in C_cG \subseteq L^I(G)$. Denote $L = \mathrm{int}_3(\phi, U)$. As defined in [**4**], Definition 4.1, we have $L(f) = T^*_{h_1}U(M_f \otimes \mathrm{id}_K)T_{h_2}$ for any $h_1, h_2 \in C_cG$ with $h_i|_{\operatorname{supp} f} \equiv 1$, using the notation from there and $K = L^2(H, \nu)$. To compute this expression explicitly, first note that for all $g \in L^2(G, t, \lambda)$ and $\tau \in L^2(H, \nu)$ and almost all $p \in M$ we have

$$\begin{aligned} T^*_{h_1}\beta^{-1}(g \cdot \tau \circ t)(p) &= T^*_{h_1}(g \otimes \tau)(p) = \langle h_1, g\rangle(p) \cdot \tau(p) \\ &= \int_{G^p} \overline{h_1(x)}g(x)\mathrm{d}\lambda^p(x)\tau(p) \\ &= \int_{G^p} \overline{h_1(x)}g(x)\tau \circ t(x)\mathrm{d}\lambda^p(x) = \langle h_1, g\tau \circ t\rangle(p). \end{aligned}$$

This implies $T^*_{h_1}\beta^{-1}(\chi) = \langle h_1, \chi\rangle$ for all $\chi \in L^2(t^*H, \nu \circ \lambda)$. Hence we get for all $\sigma \in L^2(s^*H, \nu \circ \tilde{\lambda})$

$$\begin{aligned} L(f)(\sigma) &= T^*_{h_1}U(M_f \otimes \mathrm{id}_K)T_{h_2}(\sigma) = T^*_{h_1}U(M_f \otimes \mathrm{id}_K)(h_2 \otimes \sigma) \\ &= T^*_{h_1}\beta^{-1}U_P\alpha((fh_2) \otimes \sigma) = T^*_{h_1}\beta^{-1}U_P(fh_2\sigma \circ s) \\ &= T^*_{h_1}\beta^{-1}(fh_2\Delta^{-\frac{1}{2}}P(\sigma \circ s)) = \langle h_1, fh_2\Delta^{-\frac{1}{2}}P(\sigma \circ s)\rangle. \end{aligned}$$

Using that $h_i|_{\operatorname{supp} f} \equiv 1$ by assumption, this yields for almost all $p \in M$

$$\begin{aligned} L(f)(\sigma)(p) &= \int_{G^p} \overline{h_1(x)}f(x)h_2(x)\Delta^{-\frac{1}{2}}(x)P(x)\sigma(sx)\mathrm{d}\lambda^p(x) \\ &= \int_{G^p} f(x)\Delta^{-\frac{1}{2}}(x)P(x)\sigma(sx)\mathrm{d}\lambda^p(x) = \mathrm{int}_2(P)(f)(\sigma)(p). \end{aligned}$$

So because p and σ were arbitrary, $\mathrm{int}_4(P)(f) = \mathrm{int}_3(\phi, U)(f) = L(f) = \mathrm{int}_2(P)(f)$ just as required. This proves that $\mathrm{int}_2 = \mathrm{int}_4$ is a bijection (up to changes on null sets). ☐

The notation int_3 and int_4 is now obsolete and shall not be further used. What remains important are the mappings diff, int_1 and int_2. Further properties of these are to be discussed in the next section.

6.3. Differentiation versus Integration

At this point, we have successfully defined a representation of the groupoid algebra in dependence of a differential operator representation. As discussed more than once, this is meant to be an inverse to the differentiation map defined in Theorem 4.1.5. This chapter exists to prove that this is actually the case (for the pair groupoid $G = \mathbb{R}^m \times \mathbb{R}^m$).

The proof idea is relatively straightforward: We compose the differentiation and integration maps using explicit formulas and reduce the expression using partial integration (in the manifold sense, involving divergence). As usual, things are a bit more complicated in the details. Firstly, the partial integration I want to show is only possible with *left*-invariant vector fields, not right-invariant. This is why we will make use of the following short lemma to connect the two.

LEMMA 6.3.1. *Let $G \rightrightarrows M$ be a volumetric groupoid with algebroid A. Let $a \in \Gamma(A)$ be a section and let $a^L \in \mathfrak{X}^L(G)$, $a^R \in \mathfrak{X}^R(G)$ be the corresponding left- and right-invariant* *vector fields. Let $f \in C^\infty(G)$. Then:*

$$(a^R(f^*))^* = a^L(f)$$

Proof: First notice that the transition from right- to left-invariant vector fields is given by $\mathfrak{X}^R G \to \mathfrak{X}^L G,\ X \mapsto \mathrm{inv}_* X = TX \circ X \circ \mathrm{inv}$, where $\mathrm{inv} : G \to G,\ x \mapsto x^{-1}$ is the inversion map. So we have for all $x \in G$:

$$\begin{aligned}(a^R(f^*))^*(x) &= a^R(f^*)(x^{-1}) = a^R_{x^{-1}}(f \circ \mathrm{inv}) = T_{x^{-1}}\mathrm{inv}(a^R_{x^{-1}})(f) \\ &= T\mathrm{inv} \circ a^R(x^{-1})(f) = T\mathrm{inv} a^R \mathrm{inv}(x)(f) = \mathrm{inv}_* a^R(f)(x) = a^L(f)(x),\end{aligned}$$

hence $a^R(f^*)^* = a^L(f)$ as required. □

Let us save this information for later. Before we can go to the main computation of this section, we need a series of lemmas related to measures and Radon-Nikodým derivatives. The first one may be known from elementary measure theory, but it is at least worth reminding of:

LEMMA 6.3.2. *Let M be a measurable space with two measures μ and ν and $\phi : M \to M$ be bimeasurable such that $\mu \ll \nu$. Then $\phi_*\mu \ll \phi_*\nu$ and $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = \frac{\mathrm{d}\phi_*\mu}{\mathrm{d}\phi_*\nu} \circ \phi$, where $\phi_*\mu$ is the pushforward measure, defined by $\int_M \sigma \mathrm{d}\phi_*\mu = \int_M \sigma \circ \phi \mathrm{d}\mu$.*

Proof: Let $f \in L^1(M, \phi_*\mu)$. Then we have:

$$\int_M f \mathrm{d}\phi_*\mu = \int_M f \circ \phi \mathrm{d}\mu = \int_M f \circ \phi \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \mathrm{d}\nu = \int_M f \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \circ \phi^{-1} \mathrm{d}\phi_*\nu,$$

hence $\phi_*\mu \ll \phi_*\nu$ with $\frac{\mathrm{d}\phi_*\mu}{\mathrm{d}\phi_*\nu} = \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \circ \phi^{-1}$. □

With this small computation tool, we get another result which is more specific to our representation theory and more interesting.

LEMMA 6.3.3. *Let $M = \mathbb{R}^m$ with volume form ω, let $G = M \times M$, let ν be a quasi-invariant measure on M, let $\lambda = \lambda_\omega$ be the Haar system induced by ω on G. Let $\Delta = \frac{\mathrm{d}\nu \circ \lambda}{\mathrm{d}\nu \circ \tilde{\lambda}}$ be the modular function of (G, λ). Let $\phi_x : M \to M, y \mapsto x + y$ for all $x \in M$. Then for almost all $(p, q) \in G$, we have:*

$$\Delta(p, q) = \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}(p) \cdot \frac{\mathrm{d}(\phi_{p-q})_*\omega}{\mathrm{d}\omega}(p)$$

Proof: To ease the notation, I will sometimes just write $\frac{\nu}{\mu}$ for the Radon-Nikodým derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ of two measures. Firstly, we have:

$$\begin{aligned}\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}(p) \cdot \frac{\mathrm{d}(\phi_{p-q})_*\nu}{\mathrm{d}\nu}(p) &= \frac{\omega}{\nu}(p) \frac{(\phi_{p-q})_*\nu}{(\phi_{p-q})_*\omega}(p) \\ &= \frac{\omega}{\nu}(p) \frac{\nu}{\omega} \circ \phi_{q-p}(p) = \frac{\omega}{\nu}(p) \frac{\nu}{\omega}(q)\end{aligned}$$

It follows that for every $\nu \circ \tilde{\lambda}$-integrable $f : G \to \mathbb{R}$, we have

$$\begin{aligned}\int_G f(x) \frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{tx-sx})_*\omega}(tx) \frac{\mathrm{d}(\phi_{tx-sx})_*\nu}{\mathrm{d}\nu}(tx) \mathrm{d}\nu \circ \lambda(x) &= \int_M \int_M f(p, q) \frac{\omega}{\nu}(p) \frac{\nu}{\omega}(q) \mathrm{d}\omega(q) \mathrm{d}\nu(p) \\ &= \int_M \int_M f(p, q) \frac{\omega}{\nu}(p) \mathrm{d}\nu(q) \mathrm{d}\nu(p) = \int_M \int_M f(p, q) \mathrm{d}\nu(q) \mathrm{d}\omega(p) \\ &= \int_M \int_{G_q} f(p, q) \mathrm{d}\lambda_q(p) \mathrm{d}\nu(q) = \int_G f \mathrm{d}\nu \circ \tilde{\lambda},\end{aligned}$$

and hence for almost all $x = (p, q) \in G$:

$$\begin{aligned}(\Delta(p,q))^{-1} &= \frac{\mathrm{d}\nu \circ \tilde{\lambda}}{\mathrm{d}\nu \circ \lambda}(x) \\ &= \frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{tx-sx})_*\omega}(tx)\frac{\mathrm{d}(\phi_{tx-sx})_*\nu}{\mathrm{d}\nu}(tx) = \frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}(p)\frac{\mathrm{d}(\phi_{p-q})_*\nu}{\mathrm{d}\nu}(p)\end{aligned}$$

We conclude:

$$\Delta(p,q) = \left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}(p)\frac{\mathrm{d}(\phi_{p-q})_*\nu}{\mathrm{d}\nu}(p)\right)^{-1} = \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}(p) \cdot \frac{\mathrm{d}(\phi_{p-q})_*\omega}{\mathrm{d}\omega}(p) \qquad \square$$

This lemma will make sure that the possibly non-smooth modular function cancels out of the upcoming computations to allow for differentiation.

A main ingredient of the theory is the divergence of a vector field, which defines our formal adjoints. The next three lemmas give us more tools to compute it and relate it to other parts of the theory.

Firstly, I have not yet formally stated the following elementary property of the divergence:

Lemma 6.3.4. *Let (M, ω) be a volumetric manifold and $X \in \mathfrak{X}(M)$. Let $f \in C^\infty(M)$ with $f^{-1}(0) = \emptyset$. Then:*

$$\operatorname{div}_{f\omega}(X) = \frac{X(f)}{f} + \operatorname{div}_\omega(X)$$

Proof: We compute:

$$\begin{aligned}\left(\frac{X(f)}{f} + \operatorname{div}_\omega(X)\right) f\omega &= X(f)\omega + f\operatorname{div}_\omega(X)\omega \\ &= \mathcal{L}_X(f)\omega + f\mathcal{L}_X\omega = \mathcal{L}_X(f\omega) = \operatorname{div}_{f\omega}(X)f\omega,\end{aligned}$$

which gives the result by $f\omega \neq 0$. $\square$

Using this first lemma on the divergence, we can compute the divergence of coordinate vector fields.

Lemma 6.3.5. *Let $M = \mathbb{R}^m$ and let $\omega \in \Omega^m(M)$ be a volume form. Then the divergence of the coordinate vector fields is*

$$\operatorname{div}_\omega(\partial_i) = \partial_i\Big(\frac{\omega}{\mathrm{d}x_1 \dots \mathrm{d}x_m}\Big) \cdot \frac{\mathrm{d}x_1 \dots \mathrm{d}x_m}{\omega}$$

for all $i \in \{1, \dots, m\}$.

Proof: Let $\eta = \mathrm{d}x_1 \dots \mathrm{d}x_m$ be the standard volume form of $\mathbb{R}^m$. Then we have $\mathcal{L}_{\partial_i}\eta = \mathrm{d}i_{\partial_i}\eta = \mathrm{d}((-1)^{i+1}\mathrm{d}x_1 \dots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \dots \mathrm{d}x_m) = \partial_i(1)\mathrm{d}x_1 \dots \mathrm{d}x_m = 0$, hence $\operatorname{div}_\eta(\partial_i) = 0$. By definition of the quotient, we have $\omega = \frac{\omega}{\mathrm{d}x_1 \dots \mathrm{d}x_m}\eta$. Using Lemma 6.3.4, we deduce that

$$\operatorname{div}_\omega(\partial_i) = \frac{\partial_i(\frac{\omega}{\mathrm{d}x_1 \dots \mathrm{d}x_m})}{\frac{\omega}{\mathrm{d}x_1 \dots \mathrm{d}x_m}} + \operatorname{div}_\omega(\partial_i) = \partial_i\left(\frac{\omega}{\mathrm{d}x_1 \dots \mathrm{d}x_m}\right)\frac{\mathrm{d}x_1 \dots \mathrm{d}x_m}{\omega}. \qquad \square$$

And with this expression for the coordinate divergence, we can get another, somewhat unintuitive result: The partial derivatives of a much-used Radon-Nikodým derivative can be expressed in terms of this divergence. Namely:

Lemma 6.3.6. *Let $M = \mathbb{R}^m$ with a volume form ω, let ϕ_x be the time-1 flow of $x\partial = \sum_{i=1}^m x_i\partial_i$ for $x \in \mathbb{R}^m$. Then for all $p, q \in M$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}q_i}\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}\right)^{\frac{1}{2}}(p) = -\frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}\right)^{\frac{1}{2}}(p)\operatorname{div}_\omega(\partial_i)(q)$$

Proof: To ease the notation, I just write $\frac{\nu}{\mu} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ for Radon-Nikodým derivatives where appropriate. Denote the Lebesgue measure on $\mathbb{R}^m$ by λ. λ is translation invariant, so we have $(\phi_x)_*\lambda = \lambda$ for all $x \in \mathbb{R}^m$. Hence by Lemma 6.3.5, we get the following equation:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}q_i}\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}(p) &= \frac{\mathrm{d}}{\mathrm{d}q_i}\left(\frac{\omega}{\lambda}\frac{(\phi_{p-q})_*\lambda}{(\phi_{p-q})_*\omega}\right)(p) = \frac{\omega}{\lambda}(p)\frac{\mathrm{d}}{\mathrm{d}q_i}\frac{\lambda}{\omega}(q) \\
&= \frac{\omega}{\lambda}(p)\partial_i(\frac{\lambda}{\omega})(q) = \frac{\omega}{\lambda}(p)\partial_i(\left(\frac{\omega}{\lambda}\right)^{-1})(q) \\
&= -\frac{\omega}{\lambda}(p)\partial_i\left(\frac{\omega}{\lambda}\right)(q)\left(\frac{\omega}{\lambda}\right)^{-2}(q) \\
&= -\frac{\omega}{\lambda}(p)\operatorname{div}_\omega(\partial_i)(q)\frac{\omega}{\lambda}(q)\left(\frac{\lambda}{\omega}\right)^2(q) \\
&= -\frac{\omega}{\lambda}(p)\frac{\lambda}{\omega}(q)\operatorname{div}_\omega(\partial_i)(q) = -\frac{\omega}{\lambda}(p)\frac{(\phi_{p-q})_*\lambda}{(\phi_{p-q})_*\omega}(\phi_{p-q}q)\operatorname{div}_\omega(\partial_i)(q) \\
&= -\frac{\omega}{(\phi_{p-q})_*\omega}(p)\operatorname{div}_\omega(\partial_i)(q)
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}q_i}\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}\right)^{\frac{1}{2}}(p) &= \frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}\right)^{-\frac{1}{2}}(p)\frac{\mathrm{d}}{\mathrm{d}q_i}\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}(p) \\
&= -\frac{1}{2}\left(\frac{(\phi_{p-q})_*\omega}{\omega})\right)^{\frac{1}{2}}(p)\frac{\omega}{(\phi_{p-q})_*\omega}(p)\operatorname{div}_\omega(\partial_i)(q) \\
&= -\frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}\right)^{\frac{1}{2}}(p)\operatorname{div}_\omega(\partial_i)(q)
\end{aligned}$$

as required. □

With this set of lemmas we are prepared to prove the following proposition. As mentioned before, it involves left-invariance, partial integration and the computational tools constructed before. Beyond that, it mainly depends on a page-long series of equations, which does not need many comments.

Proposition 6.3.7. *Let $M = \mathbb{R}^m$, $\omega \in \Omega^m(M)$ be a volume form, $G = M \times M$, $H \to M$ a ν-Hilbert field and $R : \mathrm{Diff}^R(G) \cong \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ a representation with respect to the star structure induced by ω such that $R(\sum_i(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)^2)$ is essentially self-adjoint. Denote by λ the Haar system induced by ω.*

Let $P : G \dashrightarrow U(H)$ be the representation obtained from R by Corollary 6.1.7. Let $\pi : L^1(G) \to \mathbb{B}(L^2H)$ be the representation obtained from P by Theorem 6.2.2.

Let $W \subseteq G$ be such that

$$P(p,q) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}\right)^{\frac{1}{2}}(p)\left(\phi_{p-q}^*\mathrm{e}^{\bar{R}((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial))}\right)(q)$$

for all $(p,q) \in W$. For $X \in \mathfrak{X}(M)$, denote by X^L the left-invariant vector field corresponding to X. For $p \in M$, denote by $\phi_p : M \to M$, $q \mapsto p+q$ the flow of $p\partial$. Then for all $f \in C_c^\infty(G)$ with $\operatorname{supp} f \subseteq W$, all $\sigma \in \operatorname{dom} R$ and all $i \in \{1,\dots,m\}$, we have

$$\pi(\partial_i^L f)(\sigma) = \pi(f)R(-\partial_i - \operatorname{div}\partial_i)\sigma.$$

Proof: As defined in the aforementioned theorems, there is a neighbourhood W of the diagonal in G such that for almost all $x = (p,q) \in W$ we have $P(p,q) = P(\exp((p-q)\partial)(q)) =$

$U_{q-p}(q) : H_q \to H_p$, where $U_{q-p} : L^2H \to L^2(\phi^*_{p-q}H)$ is the unitary decomposable operator defined as

$$U_{q-p} = \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}^{\frac{1}{2}} \circ \phi_{p-q} \cdot \phi^*_{p-q}\mathrm{e}^{R((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial))},$$

fulfilling

$$\begin{aligned}P(p,q)(\sigma(q)) &= \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}^{\frac{1}{2}} \circ \phi_{p-q}(q) \cdot \phi^*_{p-q}\mathrm{e}^{R((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial))}(\sigma)(q)\\ &= \frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}^{\frac{1}{2}}(p) \cdot \mathrm{e}^{R((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial))}(\sigma)(p).\end{aligned}$$

Using this and Lemma 6.3.3, for almost all $p \in M$ we compute as follows:

$$\begin{aligned}
\pi(\partial_i^L f)(\sigma)(p) &= \int_{G^p} \Delta^{-\frac{1}{2}}(x)\partial_i^L f(x)P(x)\sigma(sx)\mathrm{d}\lambda^p(x)\\
&= \int_M \Delta^{-\frac{1}{2}}(p,q)\partial_i^L f(p,q)P(p,q)\sigma(q)\mathrm{d}\mu_\omega(q)\\
&= \int_M \Delta^{-\frac{1}{2}}(p,q)\partial_i(f(p,\cdot))(q)P(p,q)\sigma(q)\mathrm{d}\mu_\omega(q)\\
&= \int_M \Delta^{-\frac{1}{2}}(p,q)\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}^{\frac{1}{2}}(p)\partial_i(f(p,\cdot))(q)\mathrm{e}^{\bar{R}((q-p)(\partial+\frac{1}{2}\operatorname{div}_\omega\partial))}(\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&= \int_M \frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)\partial_i(f(p,\cdot))(q)\mathrm{e}^{\bar{R}((q-p)(\partial+\frac{1}{2}\operatorname{div}_\omega\partial))}(\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&= -\int_M f(p,q)(\partial_i + \operatorname{div}_\omega\partial_i)\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-\cdot})_*\omega}^{\frac{1}{2}}(p)e^{(\cdot-p)\bar{R}(\partial+\frac{1}{2}\operatorname{div}\partial)}(\sigma)(p)\right)(q)\mathrm{d}\mu_\omega(q)\\
&= -\int_M f(p,q)\frac{\mathrm{d}}{\mathrm{d}q_i}\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)e^{(q-p)\bar{R}(\partial+\frac{1}{2}\partial)}(\sigma)(p)\right)\mathrm{d}\mu_\omega(q)\\
&\quad-\int_M f(p,q)\operatorname{div}_\omega\partial_i(q)\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)e^{(q-p)\bar{R}(\partial+\frac{1}{2}\partial)}(\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&= -\int_M f(p,q)\frac{\mathrm{d}}{\mathrm{d}q_i}\left(\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)\right)\cdot e^{(q-p)\bar{R}(\partial+\frac{1}{2}\partial)}(\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&\quad-\int_M f(p,q)\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)\frac{\mathrm{d}}{\mathrm{d}q_i}\left(e^{(q-p)\bar{R}(\partial+\frac{1}{2}\partial)}(\sigma)(p)\right)\mathrm{d}\mu_\omega(q)\\
&\quad-\int_M f(p,q)\operatorname{div}_\omega\partial_i(q)\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)e^{(q-p)\bar{R}(\partial+\frac{1}{2}\partial)}(\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&= -\int_M -\frac{1}{2}f(p,q)\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)\operatorname{div}_\omega(\partial_i)(q)e^{(q-p)\bar{R}(\partial+\frac{1}{2}\partial)}(\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&\quad-\int_M f(p,q)\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)\mathrm{e}^{\bar{R}((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial))}(R(\partial_i+\frac{1}{2}\operatorname{div}\partial_i)\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&\quad-\int_M f(p,q)\operatorname{div}_\omega\partial_i(q)\frac{\mathrm{d}\omega}{\mathrm{d}(\phi_{p-q})_*\omega}^{\frac{1}{2}}(p)e^{(q-p)\bar{R}(\partial+\frac{1}{2}\partial)}(\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&= -\int_M f(p,q)\Delta^{-\frac{1}{2}}(p,q)\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}^{\frac{1}{2}}(p)\mathrm{e}^{\bar{R}((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial))}(R(\partial_i+\frac{1}{2}\operatorname{div}\partial_i)\sigma)(p)\mathrm{d}\mu_\omega(q)\\
&\quad-\int_M \frac{1}{2}\operatorname{div}_\omega(\partial_i)(q)f(p,q)\Delta^{-\frac{1}{2}}(p,q)\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_*\nu}^{\frac{1}{2}}(p)\mathrm{e}^{\bar{R}((q-p)(\partial+\frac{1}{2}\operatorname{div}\partial))}(\sigma)(p)\mathrm{d}\mu_\omega(q)
\end{aligned}$$

$$
\begin{aligned}
&= -\int_M f(p,q)\Delta^{-\frac{1}{2}}(p,q)P(p,q)((R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)\sigma)(q))\mathrm{d}\mu_\omega(q)\\
&- \int_M \frac{1}{2}\operatorname{div}_\omega(\partial_i)(q) f(p,q)\Delta^{-\frac{1}{2}}(p,q)P(p,q)(\sigma(q))\mathrm{d}\mu_\omega(q)\\
&= -\int_M f(p,q)\Delta^{-\frac{1}{2}}(p,q)P(p,q)((R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)\sigma)(q) + \frac{1}{2}\operatorname{div}_\omega(\partial_i)(q)\sigma(q))\mathrm{d}\mu_\omega(q)\\
&= -\int_M f(p,q)\Delta^{-\frac{1}{2}}(p,q)P(p,q)((R(\partial_i + \operatorname{div}\partial_i)(\sigma)(q))\mathrm{d}\mu_\omega(q)\\
&= \pi(f)(R(-\partial_i - \operatorname{div}\partial_i)(\sigma)(p) \qquad \square
\end{aligned}
$$

If we denote the differential operator representation returned from this π by R', the previous proposition is almost enough to show that $R'(\partial_i) = R(\partial_i)$. Heuristically, we could deduce from here that $\pi(f)R'(\partial_i)^* = \pi(\partial_i^L f) = \pi(f)R(\partial_i^*) = \pi(f)R(\partial_i)^*$, and thus $R'(\partial_i) = R(\partial_i)$ by non-degeneracy of π. However, this computation is not valid in the realm of unbounded operators. The one remaining problem is the question of domain, which needs to be looked at independently.

Before we do this, let us investigate smooth functions instead of coordinate vector fields; we will need results for both. Luckily, the computation is much easier in this case, and the domain is also not an issue as both sides are bounded, so that their closures have the whole Hilbert space as domain.

Lemma 6.3.8. *Keep the notation from the previous proposition. In addition to that, let R' be the representation derived from π using Theorem 4.1.5. Then for all $g \in C^\infty(M)$, we have $\overline{R(g)} = \overline{R'(g)}$.*

Proof: Let $f \in C_c(G)$ be arbitrary. On the one hand, we have

$$
\begin{aligned}
\pi(f)\overline{R'(g)} &= (R'(g)^*\pi(f)^*)^* = (R'(g)\pi(f^*))^*\\
&= \pi(g\circ t\cdot f^*)^* = \pi((g\circ t\cdot f^*)^*) = \pi(g\circ s\cdot f),
\end{aligned}
$$

because

$$(g\circ t\cdot f^*)^*(x) = (g\circ t\cdot f^*)(x^{-1}) = g\circ s(x)\cdot f(x)$$

and $\pi(f)$, $R'(g)$ are bounded.

On the other hand, we compute for all $\sigma \in L^2H$, $p \in M$, that

$$
\begin{aligned}
\pi(f)\overline{R(g)}\sigma(p) &= \pi(f)(g\sigma)(p)\\
&= \int_{G^p} \Delta^{-\frac{1}{2}}(x)f(x)P(x)g(sx)\sigma(sx)\mathrm{d}\lambda^p(x)\\
&= \int_{G^p} \Delta^{-\frac{1}{2}}(x)(g\circ s\cdot f)(x)P(x)\sigma(sx)\mathrm{d}\lambda^p(x) = \pi(g\circ s\cdot f)(\sigma)(p),
\end{aligned}
$$

and consequently,

$$\pi(f)\overline{R(g)} = \pi(g\circ s\cdot f) = \pi(f)\overline{R'(g)}.$$

Because π is non-degenerate, this implies that $\overline{R(g)} = \overline{R'(g)}$. $\square$

We still want to show that our integration map is a right-inverse to the differentiation map, and there still is an open question of domain. Besides that, the other thing to show is that it is also a left-inverse. That direction is actually a bit easier in different aspects, as we do not need to worry about domains of unbounded operators too much. So instead of pondering restlessly about the domain question, let us look at this direction first.

The main point in the upcoming proofs is the fact that our integrated representation yields unitary operators, which fulfil a certain differential equation; the original operators fulfil the same one, hence they must be equal.

So under a more detailed look, what unitaries am I talking about?

Lemma 6.3.9. *Let $G \rightrightarrows M$ be a Lie groupoid with Haar system λ, ν a quasi-invariant measure on M and $P : G \dashrightarrow U(H)$ a representation of G on a ν-Hilbert field H. Then for every bisection $\alpha \in \Gamma(G)$, the map*

$$P(\alpha) : L^2H \to L^2H,\ P(\alpha)(\sigma)(p) := \left(\frac{\phi_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(\alpha(\phi^{-1}p))(\sigma(\phi^{-1}p)),$$

where $\phi = t \circ (s|_\alpha)^{-1}$, is unitary. By convention, I write $\alpha(p) := (s|_\alpha)^{-1}(p)$ here.

Proof: $P(\alpha)$ is linear because $P(\alpha(\phi^{-1}p))$ is linear for all $p \in M$, as is the evaluation at a point. Regarding isometry, let $\sigma, \tau \in L^2H$ be arbitrary. Then we compute:

$$\begin{aligned}
\langle P(\alpha)\sigma, P(\alpha)\tau\rangle &= \int_M \langle (P(\alpha)\sigma)(p), (P(\alpha)\tau)(p)\rangle \mathrm{d}\nu(p) \\
&= \int_M \langle \left(\frac{\phi_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(\alpha(\phi^{-1}p))(\sigma\phi^{-1}p), \left(\frac{\phi_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(\alpha\phi^{-1}p)(\tau\phi^{-1}p)\rangle \mathrm{d}\nu(p) \\
&= \int_M \left(\frac{\nu}{\phi_*^{-1}\nu}\right)(\phi^{-1}p)\langle P(\alpha\phi^{-1}p)(\sigma\phi^{-1}p), P(\alpha\phi^{-1}p)(\tau\phi^{-1}p)\rangle \mathrm{d}\nu(p) \\
&= \int_M \frac{\nu}{\phi_*^{-1}\nu}(p)\langle P(\alpha p)(\sigma p), P(\alpha p)(\tau p)\rangle \mathrm{d}\phi_*^{-1}\nu(p) \\
&= \int_M \langle P(\alpha p)(\sigma p), P(\alpha p)(\tau p)\rangle \mathrm{d}\nu(p) \\
&= \int_M \langle \sigma(p), \tau(p)\rangle \mathrm{d}\nu(p) = \langle \sigma, \tau\rangle,
\end{aligned}$$

using that $P(\alpha p)$ is unitary for almost all $p \in M$. □

So in short, instead of inserting elements of the groupoid to get fibrewise unitaries, we can insert bisections to get global ones. The advantage of those is that, in a controlled context, we can compute values more explicitly, as in the following lemma.

Lemma 6.3.10. *Let $M = \mathbb{R}^m$ with a volume form ω and ν a quasi-invariant measure on M (with respect to the volumetric groupoid M^2). Let $H \to M$ be a ν-Hilbert field. Let $P : M^2 \dashrightarrow U(H)$ be a representation and $\pi = \mathrm{int}_2(P)$. Then we have:*

$$P(\exp(x\partial))(\pi(f)\sigma) = \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}\pi(f \circ l_{\exp -x\partial})(\sigma)$$

for all $f \in C_c^\infty(M^2)$ and $\sigma \in L^2H$, where $\phi_x(y) = y + x = t \circ \exp(x\partial)(y)$.

Proof: By definition, we have:

$$P(\exp x\partial)(\pi(f)\sigma)(p) = \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p) \cdot P(p, p-x)(\pi(f)\sigma(p-x))$$

Computing the right part first gives us (for almost all p):

$$\begin{aligned}
P(p, p-x)(\pi(f)\sigma(p-x)) &= P(p, p-x)\int_M \Delta^{-\frac{1}{2}}(p-x, q)f(p-x, q)P(p-x, q)\sigma(q)\mathrm{d}\omega(q) \\
&= \int_M \Delta^{-\frac{1}{2}}(p-x, q)f(p-x, q)P(p, p-x)P(p-x, q)\sigma(q)\mathrm{d}\omega(q) \\
&= \int_M \Delta^{-\frac{1}{2}}(p-x, q)f(p-x, q)P(p, q)\sigma(q)\mathrm{d}\omega(q) \\
&= \Delta^{-\frac{1}{2}}(p-x, p)\int_M \Delta^{-\frac{1}{2}}(p, q)f \circ l_{\exp -x\partial}(p, q)P(p, q)\sigma(q)\mathrm{d}\omega(q) \\
&= \Delta^{-\frac{1}{2}}(p-x, p)\pi(f \circ l_{\exp -x\partial})(\sigma)(p)
\end{aligned}$$

Now by virtue of Lemma 6.3.3, we have almost everywhere:

$$\left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)\Delta^{-\frac{1}{2}}(p-x,p) = \left(\frac{(\phi_{p-(p-x)})_*\nu}{\nu}\right)^{\frac{1}{2}}(p)\Delta^{\frac{1}{2}}(p,p-x)$$
$$= \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}(p)$$

Putting these computations together yields:

$$\begin{aligned}
P(\exp x\partial)(\pi(f)\sigma)(p) &= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)\cdot P(p,p-x)(\pi(f)\sigma(p-x)) \\
&= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)\Delta^{-\frac{1}{2}}(p-x,p)\pi(f\circ l_{\exp -x\partial})(\sigma)(p) \\
&= \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}(p)\pi(f\circ l_{\exp -x\partial})(\sigma)(p) \\
&= \left(\left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}\pi(f\circ l_{\exp -x\partial})(\sigma)\right)(p)
\end{aligned}$$
□

With such an explicit formula, it is no wonder that we can also explicitly explain how to derive the derivations of our unitary. Using both formulas together characterizes the unitaries in terms of a differential equation.

LEMMA 6.3.11. *Keep the previous notation. Then the partial derivatives of this expression are*

$$\frac{\partial}{\partial x_i}P(\exp x\partial)(\pi(f)\sigma) = \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}\pi((-\partial_i - \frac{1}{2}\operatorname{div}_\omega\partial_i)^R(f)\circ l_{\exp -x\partial})(\sigma)$$

for all $i\in\{1,\dots,m\}$, *where* $D^R\in\operatorname{Diff}^R(M^2)$ *denotes the right-invariant differential operator corresponding to any* $D\in\operatorname{Diff}(M)$.

In particular:

$$\frac{\partial}{\partial x_i}P(\exp x\partial)(\pi(f)\sigma) = P(\exp(x\partial))(\pi((-\partial_i - \frac{1}{2}\operatorname{div}_\omega\partial_i)^R(f))\sigma)$$

Proof: We differentiate both factors and use the product rule for Banach algebras. On the one hand, we can compute that

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}(p) =& \frac{\partial}{\partial x_i}\left(\left(\frac{\omega}{(\phi_{p-(p-x)})_*\omega}\right)^{\frac{1}{2}}(p)\right)^{-1} \\
=& -\left(\frac{\omega}{(\phi_{p-(p-x)})_*\omega}\right)^{-2*\frac{1}{2}}(p)\cdot(-\frac{1}{2}) \\
&\cdot\left(\frac{\omega}{(\phi_{p-(p-x)})_*\omega}\right)^{\frac{1}{2}}(p)\cdot\operatorname{div}_\omega(\partial_i)(p-x)\cdot(-1) \\
=& -\frac{1}{2}\left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}(p)\cdot\operatorname{div}_\omega(\partial_i)(p-x) \\
=& \left(-\frac{1}{2}\left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}}\operatorname{div}_\omega(\partial_i)\circ\phi_{-x}\right)(p)
\end{aligned}$$

using Lemma 6.3.6 and the chain rule.

On the other hand, we easily see for all $(p,q) \in M^2$ that

$$\begin{aligned}\frac{\partial}{\partial x_i} f \circ l_{\exp -x\partial}(p,q) = \frac{\partial}{\partial x_i} f(p-x,q) &= -\partial_i(f(\cdot,q))(p-x) \\ &= -\partial_i^R(f)(p-x,q) = -\partial_i^R(f) \circ l_{\exp -x\partial}(p,q)\end{aligned}$$

and hence

$$\frac{\partial}{\partial x_i}\pi(f \circ l_{\exp -x\partial})(\sigma) = \pi(\frac{\partial}{\partial x_i} f \circ l_{\exp -x\partial})(\sigma) = \pi(-\partial_i^R(f) \circ l_{\exp -x\partial})(\sigma)$$

by continuity of π.

Combining these and using the fact that

$$t \circ l_{\exp -x\partial}(p,q) = t(p-x,q) = p - x = \phi_{-x} \circ t(p,q)$$

for $p, q \in M$, we compute as follows:

$$\begin{aligned}\frac{\partial}{\partial x_i} P(\exp x\partial)(\pi(f)\sigma) =& \frac{\partial}{\partial x_i}\left(\left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(f \circ l_{\exp -x\partial})(\sigma)\right) \\ =& -\frac{1}{2}\left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \operatorname{div}_\omega(\partial_i) \circ \phi_{-x}\pi(f \circ l_{\exp -x\partial})(\sigma) \\ &+ \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(-\partial_i^R(f) \circ l_{\exp -x\partial})(\sigma) \\ =& \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(-\frac{1}{2}\operatorname{div}_\omega(\partial_i) \circ t \circ l_{\exp -x\partial} \cdot f \circ l_{\exp -x\partial})(\sigma) \\ &+ \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(-\partial_i^R(f) \circ l_{\exp -x\partial})(\sigma) \\ =& \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi((-\partial_i - \frac{1}{2}\operatorname{div}_\omega \partial_i)^R(f) \circ l_{\exp -x\partial})(\sigma) \\ =& P(\exp(x\partial))(\pi((-\partial_i - \frac{1}{2}\operatorname{div}_\omega \partial_i)^R(f))\sigma) \qquad \square\end{aligned}$$

Furthermore, we can also compute explicitly the value of the unitaries induced by the integrated form of a representation. Recall that, in this chapter, a representation R of $\mathrm{Diff}(\mathbb{R}^m)$ is called integrable if the Laplacian $L = \sum_{i=1}^m(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)^2$ gets mapped to an essentially self-adjoint operator $R(L)$. In the next chapter, we will use a slightly more complicated definition in a more general setting.

Lemma 6.3.12. *Let M, ω, ν, H be as before and let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ be an integrable representation. Let $Q := \mathrm{int}_1(R)$. Then for all $\sigma \in L^2H$ we have*

$$Q(\exp x\partial)(\sigma) = \mathrm{e}^{\overline{R(-x\tilde{\partial})}}(\sigma),$$

where $x\tilde{\partial} = \sum_{i=1}^m x_i(\partial_i + \frac{1}{2}\operatorname{div}_\omega(\partial_i))$.

Proof: Using our definitions from Lemma 6.3.9 and Theorem 6.1.6, we see that for almost all $p \in M$,

$$\begin{aligned}Q(\exp x\partial)(\sigma)(p) &= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)Q(p,p-x)(\sigma(p-x)) \\ &= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)\left(\frac{\nu}{(\phi_x)_*\nu}\right)^{\frac{1}{2}}(p)\mathrm{e}^{\bar{R}(-x\tilde{\partial})}(\sigma)(p) = \mathrm{e}^{\bar{R}(-x\tilde{\partial})}(\sigma)(p). \qquad \square\end{aligned}$$

Now let us compare both sides:

LEMMA 6.3.13. *Let M, ω, ν, H as before. Let $P : M^2 \dashrightarrow U(H)$ be a representation. Set $Q = \mathrm{int}_1 \circ \mathrm{diff} \circ \mathrm{int}_2(P) : M^2 \dashrightarrow U(H)$. Then for all $x \in \mathbb{R}^m$, we have:*

$$Q(\exp x\partial) = P(\exp x\partial)$$

Proof: Set $\pi = \mathrm{int}_2(P)$, $R = \mathrm{diff}(\pi)$. Let $\tau = \pi(f)\sigma \in \mathrm{dom}\, R = \pi(C_c^\infty(M^2))L^2H$ be an arbitrary element in the domain of the differentiated representation. Firstly, we have $Q(\exp 0\partial)\tau = \tau = P(\exp 0\partial)\tau$. Furthermore, by Lemma 6.3.12 we have for all $i \in \{1, \ldots, m\}$:

$$\frac{\partial}{\partial x_i} Q(\exp x\partial)\tau = \frac{\partial}{\partial x_i} \mathrm{e}^{\bar{R}(-x\tilde{\partial})}(\tau) = \mathrm{e}^{\bar{R}(-x\tilde{\partial})}(R(-\tilde{\partial}_i)\tau) = Q(\exp x\partial)(R(-\tilde{\partial}_i)\tau)$$

Additionally, by Lemma 6.3.11 we know that

$$\begin{aligned}\frac{\partial}{\partial x_i} P(\exp x\partial)\tau &= \frac{\partial}{\partial x_i} P(\exp x\partial)(\pi(f)\sigma) = P(\exp(x\partial))(\pi((-\partial_i - \frac{1}{2}\operatorname{div}_\omega \partial_i)^R(f))\sigma) \\ &= P(\exp x\partial)(R(-\tilde{\partial}_i\pi(f)\sigma) = P(\exp x\partial)(R(-\tilde{\partial}_i)\tau).\end{aligned}$$

By elementary calculus, this implies that $Q(\exp x\partial)\tau - P(\exp x\partial)\tau = Q(\exp 0\partial)\tau - P(\exp 0\partial)\tau = 0$ for all $x \in \mathbb{R}^m$, i.e. $Q(\exp x\partial)\tau = P(\exp x\partial)\tau$. Since $\mathrm{dom}\, R \subseteq L^2H$ is dense and $Q(\exp x\partial)$ and $P(\exp x\partial)$ are bounded, this implies that $Q(\exp x\partial) = P(\exp x\partial)$ as required. □

PROPOSITION 6.3.14. *Let $M = \mathbb{R}^m$ with a volume form ω and a quasi-invariant measure ν. Let $H \to M$ be a ν-Hilbert field and $P : M^2 \dashrightarrow U(H)$ a representation, i.e. an essential homomorphism of second type. Then* $\mathrm{int}_1 \circ \mathrm{diff} \circ \mathrm{int}_2(P) = P$ *(almost everywhere).*

Proof: Put $Q := \mathrm{int}_1 \circ \mathrm{diff} \circ \mathrm{int}_2(P)$. Let $(\sigma_i)_{i\in\mathbb{N}}$ be a generating sequence for H. By Lemma 6.3.13, for all $x \in \mathbb{R}^m$ and $i \in \mathbb{N}$ there is ν-null set $N_i \subseteq M$ such that $Q(\exp x\partial)(\sigma_i)(p) = P(\exp x\partial)(\sigma_i)(p)$ for all $p \in M \setminus N_i$. Put $N = \bigcup_{i\in\mathbb{N}} N_i$, which is still a null set as a countable union of such. Then for all $x \in \mathbb{R}^m$ and all $p \in M \setminus N$, $Q(\exp x\partial)(\sigma_i)(p) = P(\exp x\partial)(\sigma_i)(p)$ holds for all $i \in \mathbb{N}$. In particular, these equations hold for almost all $(x, p) \in M \times M$. The map $M^2 \to M^2$, $(q, p) \mapsto (p - q, p)$ is bimeasurable, hence we get that for almost all $(p, q) \in M^2$ the equation

$$\begin{aligned}P(p, q)\sigma_i(q) &= \left(\frac{(\phi_{p-q})_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(\exp(p-q)\partial)(\sigma_i)(p) \\ &= \left(\frac{(\phi_{p-q})_*\nu}{\nu}\right)^{\frac{1}{2}}(p)Q(\exp(p-q)\partial)(\sigma_i)(p) = Q(p, q)\sigma_i(q)\end{aligned}$$

holds for all $i \in \mathbb{N}$. Since $\mathrm{span}_\mathbb{C}\{\sigma_i(q) | i \in \mathbb{N}\}$ is dense in H_q for all q, this implies that $P(p, q) = Q(p, q)$ for almost all $p, q \in M^2$, just as required. □

This was not quite the identity we wanted! Luckily, we have already proven that the second integration map int_2 is a bijection. So it is clear that our last lemma readily implies:

PROPOSITION 6.3.15. *Let $M = \mathbb{R}^m$ with a volume form ω and a quasi-invariant measure ν. Let $H \to M$ be a ν-Hilbert field and let $\pi : L^I(M \times M) \to \mathbb{B}(L^2H)$ be a representation. Then:*

$$\mathrm{int}_2 \circ \mathrm{int}_1 \circ \mathrm{diff}(\pi) = \pi$$

Proof: By Proposition 6.3.14, we know that $\mathrm{int}_1 \circ \mathrm{diff} \circ \mathrm{int}_2 = \mathrm{id}$ (on equivalence classes of homomorphisms, where $P \sim Q$ if $P(x) = Q(x)$ for almost all x). By Proposition 6.2.6, int_2 is a bijection, hence $\mathrm{int}_2 \circ \mathrm{int}_1 \circ \mathrm{diff} = \mathrm{int}_2 \circ (\mathrm{int}_1 \circ \mathrm{diff} \circ \mathrm{int}_2) \circ \mathrm{int}_2^{-1} = \mathrm{int}_2 \circ \mathrm{int}_2^{-1} = \mathrm{id}$. □

As it turns out, the unitaries investigated for proving that the integration map is a left inverse to differentiation are also useful for the other direction. The question of domain which was left unanswered before can now be approached using Stone's theorem on one-parameter unitary groups. Formally, we can formulate the following lemma:

LEMMA 6.3.16. *Let $M = \mathbb{R}^m$ and let ω be a volume form on M. Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ be an integrable representation. Let $P = \mathrm{int}_1(R) : M^2 \dashrightarrow U(H)$, $\pi = \mathrm{int}_2(P) : L^I(M^2) \to \mathbb{B}(L^2H)$ and $R' = \mathrm{diff}(\pi) : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$. Then for all $i \in \{1, \dots, m\}$ we have:*

$$R'(\partial_i + \frac{1}{2}\operatorname{div}\partial_i) \subseteq \overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)}$$

Proof: By definition, R is integrable if and only if $R(L)$ is essentially self-adjoint, where $L = \sum_{i=1}^m (\partial_i + \frac{1}{2}\operatorname{div}\partial_i)^2$. As discussed in the proof of Theorem 6.1.6, this implies that $R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)$ is essentially skew-adjoint, i.e., $\overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)} = -R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)^*$. By Lemma 6.3.12, we have

$$P(\exp x\partial) = \mathrm{e}^{\overline{R(-x\tilde{\partial})}}$$

for all $x \in \mathbb{R}^m$, in particular,

$$P(\exp x\partial_i) = \mathrm{e}^{-x\overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)}}$$

holds for all $i \in \{1, \dots, m\}$ and $x \in \mathbb{R}$.

Thus by one part of Stone's theorem on one-parameter unitary groups (Theorem D, page 647 in [**23**]), we know that

$$v \in \operatorname{dom}\overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)}$$

if and only if the map $x \mapsto P(\exp x\partial_i)v$ is differentiable at 0, and in that case,

$$\overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)}v = -\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0} P(\exp x\partial_i)v.$$

If $v = \pi(f)\sigma \in \operatorname{dom} R'$, then, as Lemma 6.3.11 states, we have

$$\frac{\partial}{\partial x_i}P(\exp x\partial)(\pi(f)\sigma) = P(\exp(x\partial))(\pi((-\partial_i - \frac{1}{2}\operatorname{div}_\omega\partial_i)^R(f))\sigma)$$

for all $x \in \mathbb{R}^m$. In particular, the map $\mathbb{R} \to L^2H,\ x \mapsto P(\exp x\partial_i)v$ is differentiable with

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0} P(\exp x\partial_i)v = \pi((-\partial_i - \frac{1}{2}\operatorname{div}\partial_i)^R(f))\sigma = -R'(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)(v),$$

thus $v \in \operatorname{dom}\overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)}$ and $\overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)}(v) = R'(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)(v)$. As $v \in \operatorname{dom} R'$ was arbitrary, this shows that $R'(\partial_i + \frac{1}{2}\operatorname{div}\partial_i) \subseteq \overline{R(\partial_i + \frac{1}{2}\operatorname{div}\partial_i)}$. ☐

A side effect of the last lemma is that the computation in Proposition 6.3.7 is not actually necessary for our final theory. As the reader may guess, this only occurred to me in hindsight. The proof technique with Stone's theorem appears to be much more elegant than partial integration and will be central in the next chapter. I still wanted to showcase the more explicit computation to illustrate the original thought process.

From the inclusion of the images of coordinates, we can deduce further inclusions for more general vector fields. As we are dealing with inclusions of unbounded operators, we need to remember a few computational rules which apply here. These formulas are not new, but also not hard to prove. First about sums and products of adjoints:

LEMMA 6.3.17. *Let H be a Hilbert space and $A, B \in \mathcal{O}(H)$ be densely defined. Then we have*

$$A^* + B^* \subseteq (A + B)^*$$

if $\operatorname{dom} A \cap \operatorname{dom} B$ *is dense and*

$$B^*A^* \subseteq (AB)^*$$

if $\operatorname{dom}(AB)$ *is dense in H.*

Proof: Let $v \in \operatorname{dom}(A^*) \cap \operatorname{dom}(B^*) = \operatorname{dom}(A^* + B^*)$. Then by definition, the maps $\operatorname{dom} A \to \mathbb{C},\ w \mapsto \langle v, Aw\rangle$ and $\operatorname{dom} B \to \mathbb{C},\ w \mapsto \langle v, Bw\rangle$ are continuous. In particular, their restrictions onto $\operatorname{dom} A \cap \operatorname{dom} B = \operatorname{dom}(A + B)$ are continuous. Hence the map $\operatorname{dom}(A+B) \to \mathbb{C},\ w \mapsto \langle v, (A+B)w\rangle = \langle v, Aw\rangle + \langle v, Bw\rangle$ must also be continuous. Hence $v \in \operatorname{dom}(A+B)^*$. Furthermore, we have $\langle v, (A+B)w\rangle = \langle A^*v+B^*v, w\rangle = \langle (A^*+B^*)v, w\rangle$ for all $w \in \operatorname{dom}(A + B)$, so that $(A + B)^*v = (A^* + B^*)v$ as required.

Let $v \in \operatorname{dom}(B^*A^*)$, i.e., $v \in \operatorname{dom}(A^*)$ and $A^*v \in \operatorname{dom} B^*$. Then for $w \in \operatorname{dom}(AB)$, we have $\langle v, ABw\rangle = \langle A^*v, Bw\rangle = \langle B^*A^*v, w\rangle$, so $v \in \operatorname{dom}(AB)^*$ and $(AB)^*v = B^*A^*v$. □

Furthermore, I need the following technicalities about inclusions and closures of operators:

Lemma 6.3.18. *Let H_1, H_2, H_3 be vector spaces. Let $A : H_1 \to H_2$ and $B : H_2 \to H_3$ be linear operators (defined on the whole spaces) and $T, S \in \mathcal{O}(H_2)$. Then $B(T + S)A = BTA + BSA : H_1 \supset \operatorname{dom}(B(T + S)A) \to H_3$.*

If $T \subseteq S$, then $BTA \subseteq BSA$.

If A is bounded and T is closable, then TA is closable, and $\overline{TA} \subseteq \overline{T}A$.

Proof: We have the following equivalence for $v \in H_1$:

$$\begin{aligned} v \in \operatorname{dom} B(T + S)A &\Leftrightarrow Av \in \operatorname{dom}(T + S) = \operatorname{dom} T \cap \operatorname{dom} S \\ &\Leftrightarrow Av \in \operatorname{dom} T \wedge Av \in \operatorname{dom} S \\ &\Leftrightarrow v \in \operatorname{dom} TA \cap \operatorname{dom} SA \\ &= \operatorname{dom} BTA \cap \operatorname{dom} BSA = \operatorname{dom}(BTA + BSA) \end{aligned}$$

So $\operatorname{dom}(BTA + BSA) = \operatorname{dom}(B(T + S)A) = \operatorname{dom} BTA \cap \operatorname{dom} BSA$. Obviously, for every $v \in \operatorname{dom} BTA \cap \operatorname{dom} BSA$, we have $B(T + S)Av = B(TAv + SAv) = BTAv + BSAv = (BTA + BSA)v$ so that actually $B(T + S)A = BTA + BSA$.

Now suppose that $T \subseteq S$ and let $v \in \operatorname{dom} BTA$. Then $Av \in \operatorname{dom} T \subseteq \operatorname{dom} S$, so $v \in \operatorname{dom} SA = \operatorname{dom} BSA$, and $BSA(v) = BS(Av) = BT(Av) = BTA(v)$. Thus $BTA \subseteq BSA$.

Now suppose that A is bounded and T is closable. Let $v \in H_1$ and $(v_i) \subset \operatorname{dom}(TA) \subseteq H_1$ be an arbitrary sequence with $v_i \to v$ such that TAv_i is convergent. Then since A is bounded, $(Av_i) \subset \operatorname{dom} T$ is convergent (to Av), hence because T is closable, the limit TAv_i must be the same for all such sequences v_i:

$$\lim_i (TA)v_i = \lim_i T(Av_i) = \overline{T}(Av)$$

Thus TA is closable. We have shown that $Av \in \operatorname{dom} \overline{T}$ and $\overline{TA}(v) = \overline{T}Av$, thus $\overline{TA} \subseteq \overline{T}A$. □

We can now use these lemmas to prove:

Proposition 6.3.19. *Let $M = \mathbb{R}^m$ and ω be a volume form on M. Let $R : \operatorname{Diff}(M) \to \mathcal{O}(L^2H)$ be an integrable representation. Let $P = \operatorname{int}_1(R) : M^2 \dashrightarrow U(H)$, $\pi = \operatorname{int}_2(P) : L^I(M^2) \to \mathbb{B}(L^2H)$ and $R' = \operatorname{diff}(\pi) : \operatorname{Diff}(M) \to \mathcal{O}(L^2H)$. Then for all $X \in \mathfrak{X}(M)$, we have:*

$$R'(X) \subseteq R(X^*)^*$$

Proof: For each $Y \in \mathfrak{X}(M)$, set $\tilde{Y} = Y + \frac{1}{2}\operatorname{div} Y$, as usual. Let $X \in \mathfrak{X}(M)$ be arbitrary. Then since $(\partial_1, \dots, \partial_m)$ is a global frame, there are $f_1, \dots, f_m, g \in C^\infty(M)$ such that $X = g + \sum_{i=1}^m f_i\tilde{\partial}_i$. As discussed before, $R(\tilde{\partial}_i)$ is essentially skew-adjoint for integrable R, thus by Lemma 6.3.16, we know that

$$R'(\tilde{\partial}_i) \subseteq \overline{R(\tilde{\partial}_i)} = -R(\tilde{\partial}_i)^* = R(\tilde{\partial}_i{}^*)^*$$

holds for all $i \in \{1, \dots, m\}$.

Consequently, we can compute using Lemma 6.3.17 and Lemma 6.3.18:

$$\begin{aligned} R'(X) &= R'(g) + \sum_{i=1}^{m} R'(f_i)R'(\tilde{\partial}_i) \subseteq R(g^*)^* + \sum_{i=1}^{m} R(f_i^*)^* R(\tilde{\partial}_i{}^*)^* \\ &\subseteq \left(R(g^*) + \sum_{i=1}^{m} R(\tilde{\partial}_i{}^*) R(f_i^*) \right)^* = R\left(\left(g + \sum_{i=1}^{m} f_i \partial_i \right)^* \right)^* = R(X^*)^* \end{aligned} \qquad \square$$

Notice how we used the double adjoint (once formally within $\mathrm{Diff}(M)$, once within $\mathcal{O}(L^2H)$) to ensure a large enough domain. The main point here is that sums of adjoints are more well-behaved than sums of closures of operators.

A natural follow-up hypothesis is that we actually have $\overline{R'(X)} = \overline{R(X)}$ for all $X \in \mathfrak{X}(M)$. This property can indeed be proven under certain circumstances (we will do it for the context where M is a compact manifold in the next chapter), but the proof will use that derivatives of groupoid representations are integrable. This is something which we actually have not proven; keep in mind that Proposition 4.2.1 does not apply to $\mathbb{R}^m \times \mathbb{R}^m$ because this is not compact. Thus we will stick with the double adjoint formula for now. Further characterisations of domain and integrability will follow in the next chapter in a more general context.

At this point we have decent knowledge about the differentiation and integration of Lie groupoid representations for the basic case where the groupoid is $\mathbb{R}^m \times \mathbb{R}^m$. We have constructed three mappings and shown that they are inverse to each other (up to an open question of domain). This information is condensed in the diagram below:

$$\begin{tikzcd}
 & & & \mathbb{B}(L^2H) & & & \\
 & & & \pi \arrow[u] \arrow[ddrr, "\mathrm{diff}"'] & & & \\
 & & & L^I(G) \arrow[u, dash] & & & \\
U(H) & P \arrow[l] \arrow[uurr, "\mathrm{int}_2"'] \arrow[r, dash] & G & & \mathrm{Diff}^R(G) \arrow[r, dash] & R \arrow[r] \arrow[llll, bend left, "\mathrm{int}_1"'] & \mathcal{O}(L^2H)
\end{tikzcd}$$

$$\pi(f)(\sigma)(p) = \int_{G^p} \Delta^{-\frac{1}{2}}(x) f(x) P(x) \sigma(sx) \, \mathrm{d}\lambda^p$$

$$R(D)(\pi(f)\sigma) = \pi(D(f))\sigma$$

$$P(p,q) = \left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_{p-q})_* \nu} \right)^{\frac{1}{2}} (p) \left(\phi_{p-q}^* \mathrm{e}^{\bar{R}((q-p)(\partial + \frac{1}{2} \operatorname{div} \partial))} \right)(q)$$

The most important parts in the definition of int_1 are color-coded: We have in purple the m-tuple of partial derivatives $\partial = (\partial_1, \dots, \partial_m)$, in orange its divergence with respect to a volume form ω, in teal the pullback by the flow of ∂ which makes our unitary decomposable and in blue the Radon-Nikodým derivative necessary for norm-correction.

In the above diagram, the maps diff and int_2 are well-defined for general volumetric groupoids, while the important integration map int_1 is only defined for the Euclidean pair groupoid $\mathbb{R}^m \times \mathbb{R}^m$ at the moment. Isn't that result a bit meagre? While the Euclidean environment is a very important first step, we started out wondering about arbitrary Lie groupoids, not just real numbers. To engage this limitation, there is still one more chapter to this thesis, which will generalize our integration theorem to give a more convincing result.

CHAPTER 7

An Integration Theorem for Smooth Manifolds

In this final mathematical chapter we will use ideas from the previous chapter to prove a more general integration theorem, which works for tangent bundles of compact manifolds instead of just for $\mathbb{R}^m$. With a differential-geometric background, the natural idea here is to cover the manifold by charts and apply the Euclidean integration theory locally. The special challenge of this context is that certain properties of unbounded operators, like being self-adjoint, do not work well together with restrictions to open subsets. In particular, for a given representation $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ and a differential operator $D \in \mathrm{Diff}(M)$ such that $R(D)$ is essentially self-adjoint, a restriction $R(D|_U) \in \mathcal{O}(L^2H|_U)$ can be defined, but is usually not essentially self-adjoint; an example for this is the Laplacian on $\mathbb{R}$, which is not essentially self-adjoint on $C_c^\infty((0,1))$.

This is why we will always consider global vector fields $X \in \mathfrak{X}(M)$ and assume that $R(X + \frac{1}{2}\operatorname{div} X) \in \mathcal{O}(L^2H)$ is essentially skew-adjoint. Because global commuting frames do not exist for general manifolds, I will only assume that certain vector fields commute on an open subset. Suitable vector fields can be constructed using local charts. With these caveats, the process is relatively straightforward: Starting with a representation $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$, we first construct local representations $P_U : U \times U \dashrightarrow U(L^2H|_U)$ on open subsets U. We cover M by such local representations and show that the overlaps are identical. Then we stitch the pieces together to get a global integration theorem. Finally we show compatibility with differentiation and investigate integrability conditions.

7.1. Construction of Local Representations

To get started, it is very helpful to define an *integration frame* for an algebroid representation. In short, this is simply a local frame of commuting vector fields which act by essentially skew-adjoint operators and whose exponentials commute locally. This object type is exactly what I will use throughout this chapter to define local integrals of representations, hence the name. The precise formulation is as follows:

Definition 7.1.1. Let (M, ω) be a volumetric manifold with a quasi-invariant measure ν. Let $R : \mathrm{Diff}(M) \to \mathcal{O}(K)$ be a representation on a Hilbert space K (e.g., $K = L^2H$ for a ν-Hilbert field H).

An *integration frame* for R on an open subset $U \subseteq M$ is a tuple $(X_1, \dots, X_m)$ of vector fields $X_i \in \mathfrak{X}(M)$ such that:

(1) $xB := \sum_{i=1}^m x_i \overline{R(X_i + \frac{1}{2}\operatorname{div}_\omega(X_i))}$ is essentially skew-adjoint for all $x \in \mathbb{R}^m$,
(2) $[X_i, X_j]|_U \equiv 0$ for all $i, j \in \{1, \dots, m\}$,
(3) $(X_1(p), \dots, X_m(p))$ is an (ordered) basis for T_pM for all $p \in U$ and
(4) there exists an $\epsilon > 0$ such that for all $x, y \in U_\epsilon(0) \subseteq \mathbb{R}^m$ and all $h \in C_b^\infty(M)$ with $h|_{M \setminus U} \equiv 0$: $\overline{R(h)}\mathrm{e}^{\overline{xB}}\mathrm{e}^{\overline{yB}} = \overline{R(h)}\mathrm{e}^{\overline{(x+y)B}}$.

The fourth property in particular may seem a bit technical, but it is required to show that our local construction defines a homomorphism. I will show later that it is fulfilled in relevant cases. Note that for a Hilbert field representation, $\overline{R(h)} = T_h$ is the multiplication operator by h. This is the only case we will deal with in the following constructions, but it is useful for the final theorem and its corollary to formulate the definition in a slightly more general context.

In the previous chapter (Theorem 6.1.4), we have seen that the exponential of a vector field in a representation is a decomposable operator. The first thing that we need to prove now is that the pointwise decomposition of this operator inherits the group relation from the global unitaries. More precisely:

LEMMA 7.1.2. *Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ be a representation. Let $(X_1, \ldots, X_m)$ be an integration frame on an open subset $U \subseteq M$.*

For $p \in M$ and $x \in \mathbb{R}^m$ sufficiently small, let ϕ_x be the flow of $xX = \sum_{i=1}^m x_i X_i$ at time 1 and define

$$U_x(p) := \left(\phi_x^* \mathrm{e}^{-xB}\right)(p) : H_p \to H_{\phi_x p}$$

Then there is an open subset $W \subseteq \mathbb{R}^m \times \mathbb{R}^m \times M$ with $\{0\} \times \{0\} \times U \subset W$ such that for almost all $(x, y, p) \in W$ we have:

- *$U_x(\phi_y p) U_y(p) = U_{x+y}(p)$ and*
- *$\phi_x \phi_y(p) = \phi_{x+y}(p)$*

Proof: Let $K \subseteq U$ be an arbitrary compact subset. Choose a bump function $h \in C_c^\infty(U)$ with $h|_K \equiv 1$ (which is possible since K is compact and U is open). Define $K_1 := \operatorname{supp} h \subseteq U$. By continuity of the flow maps, there is an $\epsilon_1 > 0$ such that for all $x, y, z \in \mathbb{R}^m$ with $|x|, |y|, |z| \le \epsilon_1$, $\phi_x \circ \phi_y \circ \phi_z(K_1) \subseteq U$. Also by continuity, the set $K_2 := \{\phi_x\phi_y\phi_z(p) | p \in K_1, |x|, |y|, |z| \le \epsilon_1\} \subseteq U$ is again compact. So we can choose another $\epsilon_2 > 0$ such that $\phi_x\phi_y\phi_z(K_2) \subseteq U$ for all $|x|, |y|, |z| \le 3\epsilon_2$. Then for all $|x|, |y|, |z| \le \epsilon$ and all $p \in K_2$, we know that $\phi_x\phi_y\phi_z(p) = \theta_x^{X|_U}\theta_y^{X|_U}\theta_z^{X|_U}(p) = \theta_{x+y+z}^{X|_U}(p) = \phi_{x+y+z}(p)$ because $[X_i|_U, X_j|_U] = 0$. Here, $\theta_x^{X|_U}$ denotes the time-1-flow of $x_1X_1|_U + \cdots + x_mX_m|_U$, and these flows commute wherever defined since the vector fields commute.

Set $\epsilon = \epsilon_K := \min\{\epsilon_1, \epsilon_2, s\} > 0$. Let $x, y \in \mathbb{R}^m$ with $|x|, |y| \le \epsilon$ and $p \in K$ be arbitrary. Consider any section $\sigma \in L^2H$. Notice that we have $\operatorname{supp} h \circ \phi_{-y} \circ \phi_{-x} = \phi_x\phi_y(\operatorname{supp} h) \subseteq K_2 \subseteq U$, so we know by the definition of integration frames that $T_{h\circ\phi_{-y}\circ\phi_{-x}}\mathrm{e}^{-xB}\mathrm{e}^{-yB} = T_{h\circ\phi_{-y}\circ\phi_{-x}}\mathrm{e}^{(-x-y)B}$. Thus we compute:

$$\begin{aligned}
U_x(\phi_y p)U_y(p)\sigma(p) &= h(p)U_x(\phi_y p)U_y(p)\sigma(p) = U_x(\phi_y p)U_y(p)(T_h\sigma)(p) \\
&= U_x(\phi_y p)\left(\mathrm{e}^{-yB}(T_h\sigma)(\phi_y p)\right) = \left(\mathrm{e}^{-xB}\mathrm{e}^{-yB}(T_h\sigma)\right)(\phi_x\phi_y p) \\
&= \left(T_{h\circ\phi_{-y}\circ\phi_{-x}}\mathrm{e}^{-xB}\mathrm{e}^{-yB}\sigma\right)(\phi_x\phi_y p) \\
&= \left(T_{h\circ\phi_{-y}\circ\phi_{-x}}\mathrm{e}^{(-x-y)B}\sigma\right)(\phi_x\phi_y p) \\
&= h(\phi_{-y}\phi_{-x}\phi_x\phi_y(p))\left(\mathrm{e}^{(-x-y)B}\sigma\right)(\phi_x\phi_y p) = h(p)\left(\mathrm{e}^{(-x-y)B}\sigma\right)(\phi_x\phi_y p) \\
&= h(p)\left(\mathrm{e}^{(-x-y)B}\sigma\right)(\phi_{x+y} p) = U_{x+y}(\sigma(p))
\end{aligned}$$

Since σ was arbitrary, this implies $U_x(\phi_y p)U_y(p) = U_{x+y}(p)$ almost everywhere.

Define $W_K := U_\epsilon \times U_\epsilon \times K^\circ \subseteq \mathbb{R}^m \times \mathbb{R}^m \times U$, where K° denotes the interior of K. W_K is open with $\{0\} \times \{0\} \times K^\circ \subseteq W_K$, and a fortiori we know by the previous arguments that $U_x(\phi_y p)U_y(p) = U_{x+y}(p)$ and $\phi_x\phi_y p = \phi_{x+y} p$ for almost all $(x, y, p) \in W_K$.

This procedure can be done for any compact $K \subseteq U$, so take the union $W := \bigcup_{K\subseteq U \text{ compact}} W_K \subseteq \mathbb{R}^m \times \mathbb{R}^m \times U$, which is still open and fulfils $U_x(\phi_y p)U_y(p) = U_{x+y}(p)$ as well as $\phi_x\phi_y p = \phi_{x+y} p$ for almost all $(x, y, p) \in W$. Furthermore, because U is locally compact and open, it can be covered by precompact open subsets, thus we have

$$\{0\} \times \{0\} \times U = \bigcup_{K\subseteq U \text{ compact}} \{0\} \times \{0\} \times K^\circ \subseteq W,$$

which finishes the proof. □

The formulation with an open set $W \subseteq \mathbb{R}^m \times \mathbb{R}^m \times M$ is just a formally correct and relatively short way to describe that the bound for x and y in the lemma depends on p, unlike for complete globally commuting vector fields.

Before we really use this result, we need to introduce a minor technical trick to guarantee that fibres are connected in our local integration.

LEMMA 7.1.3. *Let M be a smooth manifold. Let $W \subseteq M \times M$ be an open neighbourhood of* $\operatorname{diag}(M) = \{(p,p) | p \in M\}$*. Then there is another open set $Z \subseteq W$ with* $\operatorname{diag}(M) \subseteq Z$*, such that $Z^p = (\{p\} \times M) \cap Z$ is connected for all $p \in M$.*

Proof: Consider any $p \in M$. Choose a chart $\phi : U \to \tilde{U} \subseteq \mathbb{R}^m$ with $p \in U$ and $\phi(p) = 0 \in \tilde{U}$. Since W is open, there exists an $\epsilon > 0$ such that $Z_p := (\phi^{-1}U_\epsilon(0))^2 \subseteq W$. $Z_p \cong U_\epsilon(0)^2$ is connected and open. Let $q \in M$ be another arbitrary point. Put $U_p = \phi^{-1}U_\epsilon(0)$. Then we find for the intersection:

$$(\{q\} \times M) \cap Z_p = (\{q\} \times M) \cap U_p^2 = \begin{cases} \{q\} \times U_p, \ q \in U_p \\ \emptyset, \ q \notin U_p \end{cases}$$

So in either case, it is connected. If it is non-empty, it contains (q,q).

Define $Z := \bigcup_{p \in M} Z_p$. Let $q \in M$ be arbitrary. Then by the previous discussion, we have:

$$\begin{aligned} Z^q = (\{q\} \times M) \cap Z &= \bigcup_{p \in M} (\{q\} \times M) \cap Z_p \\ &= \bigcup_{p \in M,\ q \in U_p} \{q\} \times U_p \end{aligned}$$

Since $\{q\} \times U_p$ is connected and $(q,q) \in \{q\} \times U_p$ for all p with $q \in U_p$, this union is again connected. Because $(p,p) \in Z_p$ for all $p \in M$, we also have $\operatorname{diag}(M) \subseteq Z$. □

Using these two lemmas, we can already construct a local representation and prove the main result of this section:

PROPOSITION 7.1.4. *Let $R : \operatorname{Diff}(M) \to \mathcal{O}(L^2(H,\nu))$ be a representation. Let*

$$(X_1, \ldots, X_m)$$

be an integration frame on $U \subseteq M$.

Then on every precompact simply connected open set $V \subseteq \bar{V} \subseteq U$ there is a local essential homomorphism $P = P_X : V \times V \dashrightarrow U(H|_V)$ with

$$P(\exp(xX)(p)) = \left(\frac{\nu}{(\phi_x)_*\nu}\right)^{\frac{1}{2}} (\phi_x p) \left(\phi_x^* \mathrm{e}^{-xB}\right)(p)$$

for almost all $p \in V$ and almost all sufficiently small $x \in \mathbb{R}^m$, where ϕ_x is the time-1 flow of xX.

Proof: By Proposition 5.1.11 and Theorem 6.1.4, the operator $\left(\frac{\nu}{(\phi_x)_*\nu}\right)^{\frac{1}{2}} \circ \phi_x \cdot \phi_x^* \mathrm{e}^{-xB}$ is unitary and decomposable (for x small enough).

For $p \in M$ and $x \in \mathbb{R}^m$ such that e^{-xB} is defined, put $U_x(p) := \left(\phi_x^* \mathrm{e}^{-xB}\right)(p)$. By Lemma 7.1.2, there are an open subset $W \subseteq \mathbb{R}^{2m} \times U$ with $\{0\} \times U \subseteq W$ and a null set $N \subseteq W$ (with respect to $\lambda \times \lambda \times \nu$, λ being the Lebesgue measure), such that $U_x(\phi_y p) U_y(p) = U_{x+y}(p)$ for all $(x,y,p) \in W \setminus N$ (in particular, the terms are defined), and also $\phi_{x+y}(p) = \phi_x \phi_y(p)$.

Because $\bar{V} \subseteq U$ is compact, there exists an $\epsilon_1 > 0$ such that $U_\epsilon(0)^2 \times V \subseteq W$. Because $(X_1|_U, \ldots, X_m|_U)$ is a smooth commuting frame and $\bar{V}$ is compact, there is another $\epsilon_2 > 0$

such that the map $x \mapsto (\phi_x(p)) = \exp(xX)(p)$ is a diffeomorphism for all $p \in V$ and $|x| < \epsilon_2$. Put $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and define:

$$W' := \{(\phi_x(p), p) | x \in U_\epsilon(0),\ p \in V\} \cap V^2$$

By the choice of ϵ_2, $\exp(xX)|_{U_\epsilon(0)\times V} : U_\epsilon(0) \times V \to \{(\phi_x(p), p) | x \in U_\epsilon(0),\ p \in V\}$ is a diffeomorphism, so that $W' \subseteq V^2$ is open. Clearly for all $p \in V$, $(p, p) = (\phi_0 p, p) \in W'$, so that W' contains the diagonal.

By Lemma 6.1.5, there is another null set $N' \subseteq W \subseteq \mathbb{R}^m \times \mathbb{R}^m \times U$ such that $\frac{\nu}{(\phi_{x+y})_*\nu} \circ \phi_{x+y}(p) = \frac{\nu}{(\phi_x)_*\nu} \circ \phi_{x+y}(p) \cdot \frac{\nu}{(\phi_y)_*\nu}(\phi_y p)$ for all $(x, y, p) \in W \setminus N'$. So put $N_2 := \{((\phi_x\phi_y p, \phi_y p), (\phi_y p, p)) | (x, y, p) \in N \cup N'\} \subseteq (M \times M)^{(2)}$, which is a null set since N and N' are null sets and $(x, p) \mapsto \phi_x p$ is smooth.

Define

$$P : W' \to U(V),\ \exp(xX)(p) \mapsto \left(\frac{\nu}{(\phi_x)_*\nu}\right)^{\frac{1}{2}} (\phi_x p) U_x(p)$$

Now let $g, h \in W'$ be arbitrary such that $th = sg$ and $(g, h) \notin N_2$. Then because $\exp(X) : (x, p) \mapsto (\phi_x p, p) = \exp(xX)(p)$ is locally bijective, there are unique $x, y \in U_\epsilon(0) \subseteq \mathbb{R}^m$, $p \in V$, such that $g = (\phi_x\phi_y p, \phi_y p)$ and $h = (\phi_y p, p)$. In particular, the above definition of P is well-posed. Since $(g, h) \notin N_2$, we know that $(x, y, p) \notin N \cup N'$. By construction, we have $(x, y, p) \in U_\epsilon^2 \times V \subseteq W$, so that indeed

$$\begin{aligned}
P(gh) = P(\exp((x+y)X)(p)) &= \frac{\nu}{(\phi_{x+y})_*\nu}(\phi_{x+y} p) U_{x+y}(p) \\
&= \frac{\nu}{(\phi_x)_*\nu} \circ \phi_{x+y}(p) \cdot \frac{\nu}{(\phi_y)_*\nu}(\phi_y p) U_x(\phi_x p) U_y(p) \\
&= P(\exp(xX)\phi_y p) P(\exp(yX)p) = P(g)P(h)
\end{aligned}$$

By Lemma 7.1.3, choose another open set $Z \subseteq W'$ with $\operatorname{diag}(V) \subseteq W'$ such that $Z \cap (\{p\} \times V)$ is connected for all $p \in V$. Then we a fortiori have $P(gh) = P(g)P(h)$ for all $g, h \in Z$ with $gh \in Z$ and $(g, h) \notin N_2$. So $P|_Z V \times V \dashrightarrow U(H|_V)$ is an essential local homomorphism of first type. By Proposition 5.3.6, $P|_D$ is a homomorphism of second type for some co-null set $D \subseteq Z$, which finishes the proof. $\square$

A priori, this construction depends on the chosen integration frame and can yield a different result for another choice. We will show in the next section that this is actually not the case.

7.2. Showing Independence of Chart Choice

Proving that the local integrals of our representation are independent of chart choice is not at all a trivial task. Generally speaking, two charts on the same domain are connected by a transformation matrix of smooth functions. In many areas of differential geometry, independence of chart choice is shown using explicit computations with this transformation matrix. Now the challenge in our scenario is that there is no obvious relation between the exponential of a vector field and the exponential of its product with a given smooth function. For non-commuting vector fields, we cannot even say much about the exponential of their sum; the Baker-Campbell-Hausdorff formula still applies in functional calculus, but not for groupoid exponentials.

We will solve this problem by essentially outsourcing the computations from the unitary operators back to the differential operator algebra. Namely we will immediately differentiate our local integral again and investigate the relation of this derived representation with the original. We will find that this derivative does not significantly depend on the chosen integration frame, thus the local integral cannot either.

Our first goal is to consider different definitions of the exponential of a vector field and show that they are equivalent. To do so, let us start with a computation involving Radon-Nikodým derivatives and divergence.

Recall that for a diffeomorphism $\phi : M \to M$, $\phi^*\omega \in \Omega^m(M)$ is the pullback of a differential form, while $\phi_*\omega = \phi_*\mu_\omega$ is the pushforward measure of the measure induced by ω. We find that $\phi^*\omega = \phi_*^{-1}\omega$ and formulate the following lemma:

LEMMA 7.2.1. *Let (M,ω) be a volumetric manifold. Let $X \in \mathfrak{X}(M)$ be a vector field. Denote by $\theta : D \to M$, $D \subseteq \mathbb{R} \times M$ open, the flow of X. Then for all $(x,p) \in D$, we have:*

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\omega}{(\theta_x)_*\omega} \circ \theta_x(p) = \frac{\theta_x^*\omega}{\omega}(p) \cdot \mathrm{div}_\omega(X) \circ \theta_x(p)$$

Proof: Usually I use the definition $\mathcal{L}_X = \mathrm{d} \circ i_X + i_X \circ \mathrm{d}$ for the Lie derivative, but it is also possible to define $\mathcal{L}_X\eta = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \theta_t^*\eta$, for any tensor field η. For differential forms $\eta \in \Omega(M)$, these expressions are equivalent by Theorem 14.35, page 372 in [**11**]. Hence we find:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{\omega}{(\theta_x)_*\omega} \circ \theta_x \cdot \omega &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{(\theta_{-x})_*\omega}{\omega} \cdot \omega = \frac{\mathrm{d}}{\mathrm{d}x}(\theta_{-x})_*\omega \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\theta_x^*\omega = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \theta_x^*\theta_t^*\omega = \theta_x^* \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \theta_t^*\omega \\
&= \theta_x^*\mathcal{L}_X\omega = \theta_x^*(\mathrm{div}_\omega(X) \cdot \omega) \\
&= \mathrm{div}_\omega(X) \circ \theta_x \cdot \theta_x^*\omega = \frac{\theta_x^*\omega}{\omega} \cdot \mathrm{div}_\omega(X) \circ \theta_x \cdot \omega,
\end{aligned}$$

and thus, since $\omega(p) \neq 0$ for all $p \in M$, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\omega}{(\theta_x)_*\omega} \circ \theta_x = \frac{\theta_x^*\omega}{\omega} \cdot \mathrm{div}_\omega(X) \circ \theta_x$ as required.

More precisely, for every precompact open $U \subseteq M$, by continuity of θ we find an $\epsilon > 0$ such that $U_\epsilon(0) \times U \subseteq D$, and the above equations are true for $(x,p) \in U_\epsilon(0) \times U \subseteq D$. Since M can be covered by precompact open subsets, the result follows. ☐

Notice that the pullbacks in the preceding lemma are only defined in a neighbourhood of each point, which suffices. In a representation of the groupoid algebra, any vector field acts by left-multiplication – we have used this construction before. With our lemma, we can compute the derivative of this map.

LEMMA 7.2.2. *Let (M,ω) be volumetric, with a quasi-invariant measure ν on M and a Hilbert field $H \to M$. Let $\pi : C^*(M^2) \to \mathbb{B}(L^2H)$ be a representation. Let $X \in \mathfrak{X}(M)$ be a vector field (not necessarily complete), and let $\theta : D \subseteq \mathbb{R} \times M$ be its flow. Consider any $f \in C_c^\infty(M^2)$. Choose $\epsilon > 0$ such that $U_{2\epsilon} \times t(\operatorname{supp} f) \subseteq D$. Then for all $x \in (-\epsilon,\epsilon)$, we have:*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(f \circ l_{\exp(-xX)}) = \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi((-X - \frac{1}{2}\mathrm{div}_\omega(X))^R(f) \circ l_{\exp(-xX)})$$

Proof: Using Lemma 7.2.1 and the chain rule, we compute first:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\omega}{(\theta_{-x})_*\omega} \circ \theta_{-x}\right)^{\frac{1}{2}} \\
&= -\frac{1}{2}\left(\frac{(\theta_{-x})_*\omega}{\omega} \circ \theta_x\right)^{\frac{1}{2}} \cdot \frac{(\theta_{-x})^*\omega}{\omega} \cdot \mathrm{div}_\omega(X) \circ \theta_{-x} \\
&= -\frac{1}{2}\left(\frac{\omega}{(\theta_x)_*\omega}\right)^{\frac{1}{2}} \cdot \frac{(\theta_x)_*\omega}{\omega} \cdot \mathrm{div}(X) \circ \theta_{-x} \\
&= -\frac{1}{2}\left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \mathrm{div}(X) \circ \theta_{-x}
\end{aligned}$$

On the other hand, we find for all $x \in (-\epsilon, \epsilon)$ that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f \circ l_{\exp(-xX)}(p,q) &= \frac{\mathrm{d}}{\mathrm{d}x} f(\theta_{-x}p, q) = -X(f(\cdot, q))(\theta_{-x}p) \\ &= -X^R(f)(\theta_{-x}p, q) = -X^R(f) \circ l_{\exp(-xX)}(p,q)\end{aligned}$$

for all $p \in \theta(U_\epsilon(0), t(\operatorname{supp} h)) \subseteq t(\operatorname{supp} f \circ l_{\exp(-xX)})$, $q \in M$, so that

$$\frac{\mathrm{d}}{\mathrm{d}x} f \circ l_{\exp(-xX)} = -X^R(f) \circ l_{\exp(-xX)}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} \pi(f \circ l_{\exp(-xX)}) = \pi(-X^R(f) \circ l_{\exp(-xX)})$$

because π is continuous.

Hence by the product rule we get:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(f \circ l_{\exp(-xX)}) \\ &= -\frac{1}{2}\left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \operatorname{div}(X) \circ \theta_{-x} \pi(f \circ l_{\exp(-xX)}) + \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(-X^R(f) \circ l_{\exp(-xX)}) \\ &= -\frac{1}{2}\left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(\operatorname{div}(X) \circ \theta_{-x} \circ t f \circ l_{\exp(-xX)}) + \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(-X^R(f) \circ l_{\exp(-xX)}) \\ &= \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi((-X - \frac{1}{2}\operatorname{div}(X))^R(f) \circ l_{\exp(-xX)})\end{aligned}$$

□

Given a groupoid representation $P : M^2 \dashrightarrow U(H)$, any vector field $X \in \mathfrak{X}(M)$, even if it is not complete, defines an exponential $P(\exp(xX))$ acting on *certain* sections in L^2H. The next lemma is mostly about the precise definition of this, but also shows that this mapping behaves like a unitary with group properties.

Lemma 7.2.3. *Let (M, ω) be volumetric, ν quasi-invariant on M, $H \to M$ a Hilbert field, $P : M^2 \dashrightarrow U(H)$ a representation. Let $X \in \mathfrak{X}(M)$ be a vector field (not necessarily complete). Let $\theta : D \subseteq \mathbb{R} \times M \to M$ be the flow of X. Let $K \subseteq M$ be compact and $\sigma \in L^2(H)$ such that the essential support* $\operatorname{ess\,supp}(\sigma) \subseteq K$ *is contained in K. Choose $\epsilon > 0$ such that $U_{2\epsilon} \times K \subseteq D$. For all $x \in (-\epsilon, \epsilon)$, define $P(\exp(xX)\sigma)$ by*

$$\left(P(\exp(xX)\sigma)\right)(p) := \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}} (p) P(p, \theta_{-x}p)\sigma(\theta_{-x}p)$$

for $p \in \theta([-\epsilon, \epsilon] \times K)$ (where $\theta_{-x}p$ is defined) and $P(\exp(xX)\sigma)(p) = 0$ for

$$p \in M \setminus \theta([-\epsilon, \epsilon] \times K)$$

(where $\sigma(\theta_{-x}p) = 0$ if it is defined).

Then $P(\exp(xX)\sigma) \in L^2H$, and $\|P(\exp(xX)\sigma)\| = \|\sigma\|$. Furthermore, we have $P(\exp xX)P(\exp yX)\sigma = P(\exp(x+y)X)\sigma$ for $x, y \in \mathbb{R}$ with $|x| + |y| < \epsilon$ and

$$\langle P(\exp(xX))\sigma, \tau\rangle = \langle \sigma, P(\exp(-xX))\tau\rangle$$

for $\tau \in L^2H$ with $\operatorname{ess\,supp}\sigma \subseteq K$.

Proof: Let $\sigma, \tau \in L^2H$ with $\operatorname{ess\,supp}\sigma, \operatorname{ess\,supp}\tau \subseteq K$. Then we have

$$\begin{aligned}
\langle P(\exp xX)\sigma, P(\exp xX)\tau\rangle &= \int_M \langle (P(\exp xX)\sigma)(p), (P(\exp xX)\tau)(p)\rangle \nu(p) \\
&= \int_M \frac{(\theta_x)_*\nu}{\nu}(p)\langle P(p,\theta_{-x}p)\sigma(\theta_{-x}p), P(p,\theta_{-x}p)\tau(\theta_{-x}p)\rangle\nu(p) \\
&= \int_M \frac{(\theta_x)_*\nu}{\nu}(p)\langle \sigma(\theta_{-x}p), \tau(\theta_{-x}p)\rangle\nu(p) \\
&= \int_M \langle \sigma(\theta_{-x}p), \tau(\theta_{-x}p)\rangle(\theta_x)_*\nu(p) \\
&= \int_M \langle \sigma(p), \tau(p)\rangle\nu(p) = \langle\sigma,\tau\rangle
\end{aligned}$$

because $P(p, \theta_{-x}p)$ is unitary. In particular:

$$\|P(\exp xX)\sigma)\| = \langle P(\exp xX)\sigma, P(\exp xX)\sigma\rangle^{\frac{1}{2}} = \langle\sigma,\sigma\rangle^{\frac{1}{2}} = \|\sigma\|$$

The next property also follows from a straightforward computation. Namely, we have for all $x, y \in \mathbb{R}$ with $|x| + |y| < \epsilon$ and almost all $p \in M$:

$$\begin{aligned}
P(\exp xX)P(\exp yX)(\sigma)(p) &= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(p,\theta_{-x}p)(P(\exp yX)\sigma)(\theta_{-x}p) \\
&= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(p,\theta_{-x}p)\left(\frac{(\theta_y)_*\nu}{\nu}\right)^{\frac{1}{2}}(\theta_{-x}p)P(\theta_{-x}p,\theta_{-y}\theta_{-x}p)\sigma(\theta_{-y}\theta_{-x}p) \\
&= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)\left(\frac{(\theta_x)_*(\theta_y)_*\nu}{(\theta_x)_*\nu}\right)^{\frac{1}{2}}(p)P(p,\theta_{-y}\theta_{-x}p)\sigma(\theta_{-y}\theta_{-x}p) \\
&= \left(\frac{(\theta_{x+y})_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(p,\theta_{-(x+y)}p)\sigma(\theta_{-(x+y)}p) = P(\exp(x+y)X)(\sigma)(p)
\end{aligned}$$

Thus $P(\exp xX)P(\exp yX)\sigma = P(\exp(x+y)X)\sigma$.

Combining these two results, we see that

$$\begin{aligned}
\langle P(\exp xX)\sigma, \tau\rangle &= \langle P(\exp(-xX))P(\exp xX)\sigma, P(\exp(-xX))\tau\rangle \\
&= \langle\sigma, P(\exp(-xX))\tau\rangle
\end{aligned}$$

holds for all $x \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. □

So in a sense, we have once more defined a unitary operator. It is important to keep in mind though that $P(\exp xX)$ is not globally defined on all sections of L^2H for any fixed $x \in \mathbb{R}$, but the possible values for x depend on the support of the section.

We will show soon that the definition by left-action and the exponential definition coincide. To do this, we need a short lemma which characterises the modular function of a pair groupoid.

LEMMA 7.2.4. *Let (M, ω) be a volumetric manifold and ν a quasi-invariant measure on M with respect to $M \times M$. Let $\Delta = \frac{\nu\circ\lambda}{\nu\circ\tilde{\lambda}}$ be the corresponding modular function.*

Then for almost all $(p, q) \in M^2$ we have:

$$\Delta(p,q) = \frac{\nu}{\omega}(p)\frac{\omega}{\nu}(q)$$

Proof: In the following, I will omit the d from the integral notation. Put $G = M \times M$. Notice that we have:

$$\begin{aligned}
\int_G \frac{\nu}{\omega}\circ s\frac{\omega}{\nu}\circ t\nu\circ\lambda &= \int_M\int_M \frac{\nu}{\omega}(q)\frac{\omega}{\nu}(p)\omega(q)\nu(p) \\
&= \int_M\int_M \nu(q)\omega(p) = \int_M\int_M \omega(p)\nu(q) = \int_G \nu\circ\tilde{\lambda}
\end{aligned}$$

Thus we know that $\frac{\nu}{\omega}\circ s\frac{\omega}{\nu}\circ t = \Delta^{-1}$ almost everywhere, i.e. $\Delta(p,q) = \Delta^{-1}(q,p) = \frac{\nu}{\omega}(p)\frac{\omega}{\nu}(q)$ for almost all $(p,q) \in G$. □

The desired intermediate result is now achievable by a straightforward computation.

LEMMA 7.2.5. *Let (M,ω) be volumetric, ν quasi-invariant on M, $H \to M$ a Hilbert field, $P : M^2 \dashrightarrow U(H)$ a representation, $\pi = \operatorname{int}_2(P) : C^*(M^2) \to \mathbb{B}(L^2H)$. Let $X \in \mathfrak{X}(M)$ be a vector field (not necessarily complete). Let $\theta : D \subseteq \mathbb{R}\times M \to M$ be the maximal flow of X. Let $f \in C_c^\infty(M^2)$. Choose $\epsilon > 0$ such that $U_{2\epsilon} \times t(\operatorname{supp} f) \subseteq D$. Then for all $x \in (-\epsilon,\epsilon)$, we have:*

$$P(\exp(xX))(\pi(f)\sigma) = \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(f\circ l_{\exp(-xX)})\sigma$$

Proof: Let $p \in M$ be arbitrary. If $p \notin \theta(\overline{U_\epsilon(0)} \times t(\operatorname{supp} f))$, then we know that

$$P(\exp(xX))(\pi(f)\sigma)(p) = 0$$

by definition since $\operatorname{ess\,supp}(\pi(f)\sigma) \subseteq t(\operatorname{supp} f)$. Likewise we know that

$$\operatorname{ess\,supp}(\pi(f\circ l_{\exp(-xX)})\sigma) \subseteq (t\operatorname{supp} f\circ \exp(-xX)) \subseteq \theta(U_\epsilon(0)\times t(\operatorname{supp} f)),$$

so that also $\left(\left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(f\circ l_{\exp(-xX)})\sigma\right)(p) = 0$ on the other side.

Thus look at the case where $p \in \theta(\overline{U_\epsilon(0)} \times t(\operatorname{supp} f))$. By our definition, we have for almost all such p:

$$\begin{aligned}
P(\exp xX)(\pi(f)\sigma)(p) &= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(p,\theta_{-x}p)\left((\pi(f)\sigma)(\theta_{-x}p)\right)\\
&= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)P(p,\theta_{-x}p)\int_M \Delta^{-\frac{1}{2}}(\theta_{-x}p,q)f(\theta_{-x}p,q)P(\theta_{-x}p,q)\sigma(q)\omega(q)\\
&= \int_M \left(\frac{\omega}{\nu}(\theta_{-x}p)\frac{\nu}{\omega}(q)\frac{(\theta_x)_*\nu}{\nu}(p)\right)^{\frac{1}{2}} f\circ l_{\exp(-xX)}(p,q)P(p,\theta_{-x}p)P(\theta_{-x}p,q)\sigma(q)\omega(q)\\
&= \int_M \left(\frac{(\theta_x)_*\omega}{(\theta_x)_*\nu}(p)\frac{\nu}{\omega}(q)\frac{(\theta_x)_*\nu}{\nu}(p)\right)^{\frac{1}{2}} f\circ l_{\exp(-xX)}(p,q)P(p,q)\sigma(q)\omega(q)\\
&= \int_M \left(\frac{(\theta_x)_*\omega}{\omega}(p)\frac{\nu}{\omega}(q)\frac{\omega}{\nu}(p)\right)^{\frac{1}{2}} f\circ l_{\exp(-xX)}(p,q)P(p,q)\sigma(q)\omega(q)\\
&= \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}}(p)\int_M \Delta^{-\frac{1}{2}}(p,q)f\circ l_{\exp(-xX)}(p,q)P(p,q)\sigma(q)\omega(q)\\
&= \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}}(p)(\pi(f\circ l_{\exp(-xX)})\sigma)(p)
\end{aligned}$$

Combining both parts, we see that indeed

$$P(\exp(xX))(\pi(f)\sigma) = \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(f\circ l_{\exp(-xX)})\sigma$$

just as required. □

Again, x can only be chosen in dependence of the support of our section. This is undesirable, because we want to think of the mapping we defined as a unitary operator and link it to the functional calculus exponential, which is defined globally. The first step in filling the gap is to consider the natural projection and inclusion maps between local and global section of our Hilbert field.

DEFINITION 7.2.6. Let $H \to M$ be a ν-Hilbert field and let $U \subseteq M$ be measurable. Define $\mathrm{pr}_U : L^2H \to L^2(H|_U)$, $\sigma \mapsto \sigma|_U$ and $\iota_U : L^2(H|_U) \to L^2H$, $\sigma \mapsto \bar{\sigma}$, where $\bar{\sigma}(p) = \sigma(p)$ for $p \in U$ and $\bar{\sigma}(p) = 0$ for $p \in M \setminus U$.

These maps are bounded with $\|\mathrm{pr}_U\| = 1 = \|\iota_U\|$. With their assistance, we can relate the first couple of exponentials, first only for the elements of our integration frame.

LEMMA 7.2.7. *Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2(H,\nu))$ be a representation. Let $(X_1, \ldots, X_m)$ be an integration frame for R on an open set $U \subseteq M$ and let $V \subseteq \bar{V} \subseteq U$ be precompact.*

Let $P : V \times V \supset W \dashrightarrow U(H|_V)$ be the local integrated representation defined in that proposition. Let $X \in \{X_1, \ldots, X_m\}$. Then for all $\sigma \in L^2_c(H|_V)$ and all sufficiently small $x \in \mathbb{R}$, we have:

$$\iota_V P(\exp(xX|_V))\sigma = \mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}(\iota_V\sigma)$$

Proof: Denote $\overline{\sigma} := \iota_V\sigma \in L^2H$ for $\sigma \in L^2(H|_V)$. Choose a bump function $h \in C^\infty_c(M)$ with $h|_{\operatorname{supp}\sigma} \equiv 1$ and $\operatorname{supp} h \subseteq V$. Choose $\epsilon > 0$ such that $U_{2\epsilon} \times \operatorname{supp} h \subseteq D$, where D is the common flow domain of $X_1|_V, \ldots, X_m|_V$, and also $\exp(X) = (\theta, \mathrm{id}_M)((-2\epsilon, 2\epsilon) \times \operatorname{supp} h) \subseteq W$, where θ is the flow of X. Then for all $x \in (-\epsilon, \epsilon)$ and almost all $p \in \theta([-\epsilon, \epsilon] \times \operatorname{supp} h)$, we have

$$\begin{aligned}
P(\exp(xX|_V))(\sigma)(p) &= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p) P(\exp(xX)(\theta_{-x}p))\sigma(\theta_{-x}p)\\
&= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}}(p)\left(\frac{\nu}{(\theta_x)_*\nu}\right)^{\frac{1}{2}}(\theta_x\theta_{-x}p)\left(\theta_x^*\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}\right)(\theta_{-x}p)\bar{\sigma}(\theta_{-x}p)\\
&= \mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}(\bar{\sigma})(\theta_x\theta_{-x}p) = \mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}(\bar{\sigma})(p),
\end{aligned}$$

assuming without loss of generality that ϵ is small enough that the formula for $P(\exp(xX)q)$ applies like in Proposition 7.1.4.

Now consider $p \in M \setminus \theta([-\epsilon, \epsilon] \times \operatorname{supp} h)$. Then we have on the right hand side

$$\begin{aligned}
\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}(\bar{\sigma})(p) &= \mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}(h\bar{\sigma})(p)\\
&= h \circ \theta_{-x}(p)\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}(\bar{\sigma})(p) = 0
\end{aligned}$$

since $\theta_{-x}(p) \notin \operatorname{supp} h$ by the choice of p. On the left hand side, if $p \notin V$, we have $\iota_V P(\exp(xX|_V))(\sigma)(p) = 0$ by definition of the extension to L^2H. If $p \in V \setminus \theta([-\epsilon, \epsilon] \times \operatorname{supp} h)$, then

$$\iota_V P(\exp(xX|_V))(\sigma)(p) = P(\exp(xX|_V))(\sigma)(p) = 0$$

as defined in Lemma 7.2.3.

So in all of the cases, we have indeed:

$$\iota_V P(\exp(xX|_V))\sigma(p) = \mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div} X)}}(\iota_V\sigma)(p) \qquad \square$$

This result is very interesting because we also expect that $P(\exp xX) = \mathrm{e}^{-x\overline{R'(X+\frac{1}{2}\operatorname{div} X)}}$, where $R' = \operatorname{diff}\operatorname{int}_2(P)$, so we have a clear connection between R and R'. Going beyond intuition, we have to carefully investigate the domain of each of the unbounded operators involved. For example, $R'(X+\frac{1}{2}\operatorname{div} X)$ need not be essentially skew-adjoint, and the domain of R' need not be contained in the domain of R. We will deal with these intricacies using Stone's theorem on one-parameter unitary groups and arrive at the following formulation:

LEMMA 7.2.8. *Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2(H,\nu))$ be a representation. Let $X_1, \ldots, X_m \in \mathfrak{X}(M)$ be an integration frame on an open set $U \subseteq M$ and let $V \subseteq \bar{V} \subseteq U$ be precompact and simply connected.*

Let $P_0 : V \times V \supset W \dashrightarrow U(H|_V)$ be the integrated representation defined in Proposition 7.1.4 and $P : V \times V \dashrightarrow U(H|_V)$ the extension of P_0 as in Theorem 5.3.10. Let $\pi = \operatorname{int}_2(P)$

and $R' = \operatorname{diff}(\pi)$. *Let* $X \in \{X_1, \ldots, X_m\}$. *Then for all* $f \in C_c^\infty(V^2)$ *and* $\sigma \in L^2(H|_V)$, *we have* $\iota_V \pi(f)\sigma \in \operatorname{dom} \overline{R(X + \frac{1}{2}\operatorname{div} X)}$ *and:*

$$\iota_V R'((X + \frac{1}{2}\operatorname{div} X)|_V)\pi(f)\sigma = \overline{R(X + \frac{1}{2}\operatorname{div} X)}\iota_V \pi(f)\sigma \in L^2 H$$

Proof: Remember that by the requirements for integration frames, $B = \overline{R(X + \frac{1}{2}\operatorname{div} X)}$ is skew-adjoint.

Denote the flow of X by θ. By Lemma 7.2.5 and Lemma 7.2.7, we know that there is an $\epsilon > 0$ such that for all $x \in (-\epsilon, \epsilon)$:

$$\begin{aligned}\iota_V \left(\left(\frac{(\theta_x)_* \omega}{\omega} \right)^{\frac{1}{2}} \pi(f \circ l_{\exp(-xX)})\sigma \right) &= \iota_V P(\exp(xX|_V))(\pi(f)\sigma) \\ &= \mathrm{e}^{-x\overline{R(X + \frac{1}{2}\operatorname{div} X)}}(\iota_V(\pi(f)\sigma))\end{aligned}$$

Thus by Lemma 7.2.2 (and $\frac{\mathrm{d}}{\mathrm{d}x}0 = 0$), we find that the map

$$x \mapsto \mathrm{e}^{-x\overline{R(X + \frac{1}{2}\operatorname{div} X)}}(\iota_V \pi(f)\sigma)$$

is differentiable around 0 with:

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0} \mathrm{e}^{-x\overline{R(X + \frac{1}{2}\operatorname{div} X)}}(\iota_V \pi(f)\sigma) &= \frac{\mathrm{d}}{\mathrm{d}x}\iota_V \left(\left(\frac{(\theta_x)_* \omega}{\omega} \right)^{\frac{1}{2}} \pi(f \circ l_{\exp(-xX)})\sigma \right) \\ &= \iota_V \left(\left(\frac{(\theta_0)_* \omega}{\omega} \right)^{\frac{1}{2}} \pi((-X - \frac{1}{2}\operatorname{div}_\omega(X))^R(f) \circ l_{\exp(-0X)})\sigma \right) \\ &= \iota_V \left(\pi((-X - \frac{1}{2}\operatorname{div}_\omega(X))^R(f))\sigma \right) \\ &= \iota_V \left(R'((-X - \frac{1}{2}\operatorname{div} X)|_V)\pi(f)\sigma \right)\end{aligned}$$

Hence by one part of Stone's Theorem on One-Parameter Unitary Groups (Theorem D, page 647, [**23**]), we find that $\iota_V \pi(f)\sigma \in \operatorname{dom} \overline{-R(X + \frac{1}{2}\operatorname{div} X)}$ and

$$\iota_V(R'((-X - \frac{1}{2}\operatorname{div} X)|_V)\pi(f)\sigma) = \overline{-R(X + \frac{1}{2}\operatorname{div} X)}(\iota_V \pi(f)\sigma)$$

The result follows by multiplying with -1. □

So in short, we have proven that $\iota_V \circ R'((X + \frac{1}{2}\operatorname{div} X)|_V) \subseteq \overline{R(X + \frac{1}{2}\operatorname{div} X)} \circ \iota_V$ for vector fields X in our integration frame. To go on with our computations, we first need an analogous result for smooth functions. Luckily, this is easy to prove.

LEMMA 7.2.9. *Let* $R : \operatorname{Diff}(M) \to \mathcal{O}(L^2(H, \nu))$ *be a representation. Let* $(X_1, \ldots, X_m)$ *be an integration frame on an open set* $U \subseteq M$ *and let* $V \subseteq \bar{V} \subseteq U$ *be precompact and simply connected.*

Let $P : V \times V \dashrightarrow U(H|_V)$ *be the (extension of the) integrated representation defined in Proposition 7.1.4. Let* $\pi = \operatorname{int}_2(P)$ *and* $R' = \operatorname{diff}(\pi)$.

Then for all $f \in C_b^\infty(M)$, *we have*

$$\overline{R'(f|_V)} = \operatorname{pr}_V \circ \overline{R(f)} \circ \iota_V$$

Proof: By Theorem 5.2.8 R can be assumed to fulfil $\overline{R(f)} = T_f : \tau \mapsto f\tau$. On the other hand, we know that

$$\begin{aligned}
R'(f|_V)\pi(g)\sigma &= \pi(m_f^R(g))\sigma(p) = \pi(f|_V \circ t \cdot g)\sigma(p) \\
&= \int_V f(p)\Delta^{-\frac{1}{2}}(p,q)g(p,q)P(p,q)\sigma(q)\omega(q) = f(p)\pi(g)\sigma(p) \\
&= f(p)\overline{\pi(g)\sigma}(p) = T_f\overline{\pi(g)\sigma}(p) \\
&= \mathrm{pr}_V \circ \overline{R(f)} \circ \iota_V(\pi(g)\sigma)(p)
\end{aligned}$$

for all $\sigma \in L^2(H|_V)$, $g \in C_c^\infty(V^2)$, $p \in V$, so that $\overline{R'(f|_V)} = T_f = \mathrm{pr}_V \circ \overline{R(f)} \circ \iota_V$ since $\{\pi(g)v | g \in C_c^\infty(V^2), v \in L^2(H|_V)\}$ is dense in $L^2(H|_V)$ and both sides are bounded. □

The next lemma will deal with frame vector fields again, combining the previous results. The point in taking an extra adjoint in the formula is that sums of adjoints are more well-behaved than sums of closures of operators. In the end, the difference will not matter much because we are dealing with essentially skew-adjoint operators.

LEMMA 7.2.10. *Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2(H,\nu))$ be a representation. Let $(X_1,\dots,X_m)$ be an integration frame (in particular, $R(X_i + \frac{1}{2}\operatorname{div} X_i)$ are essentially self-adjoint) on an open set $U \subseteq M$ and let $V \subseteq \bar{V} \subseteq U$ be precompact and simply connected.*

Let $P : V \times V \dashrightarrow U(H|_V)$ be the integrated representation defined in that proposition (with extension). Let $\pi = \mathrm{int}_2(P)$ and $R' = \mathrm{diff}(\pi)$. Then for all $X \in \{X_1,\dots,X_m\}$, we have:

$$\iota_V \circ R'(X|_V) \subseteq R(X^*)^* \circ \iota_V$$

Proof: By Lemma 7.2.8, we know that $\iota_V \circ R'((X + \frac{1}{2}\operatorname{div} X)|_V) \subseteq \overline{R(X + \frac{1}{2}\operatorname{div} X)} \circ \iota_V$. We also assumed that $\overline{R(X + \frac{1}{2}\operatorname{div} X)} = -R(X + \frac{1}{2}\operatorname{div} X)^*$.

By Lemma 7.2.9, we have $\iota_V R'(f|_V) \subseteq \overline{R(f)}\iota_V = R(f)^*\iota_V$ for all $f \in C^\infty(M)$.

Thus by Lemma 6.3.17 and Lemma 6.3.18, we have

$$\begin{aligned}
\iota_V \circ R'(X|_V) &= \iota_V(R'(X + \frac{1}{2}\operatorname{div} X) - R'(\frac{1}{2}\operatorname{div} X|_V)) \\
&= \iota_V R'(X + \frac{1}{2}\operatorname{div} X) - \iota_V R'(\frac{1}{2}\operatorname{div} X|_V) \\
&\subseteq -R(X + \frac{1}{2}\operatorname{div} X)^* \iota_V - \overline{R(\frac{1}{2}\operatorname{div}_X)}\iota_V \\
&= -R(X + \frac{1}{2}\operatorname{div} X)^* \iota_V - R(\frac{1}{2}\operatorname{div}_X)^* \iota_V \\
&= \left(-R(X + \frac{1}{2}\operatorname{div} X)^* - R(\frac{1}{2}\operatorname{div}_X)^*\right) \circ \iota_V \\
&\subseteq (-R(X + \frac{1}{2}\operatorname{div} X) - R(\frac{1}{2}\operatorname{div} X))^*\iota_V \\
&= R(-X - \operatorname{div} X)^*\iota_V = R(X^*)^* \circ \iota_V
\end{aligned}$$

□

With the previous results, the analogous formula for arbitrary vector fields follows quickly.

LEMMA 7.2.11. *Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2(H,\nu))$ be a representation. Let $(X_1,\dots,X_m)$ be an integration frame (in particular, $R(X_i + \frac{1}{2}\operatorname{div} X_i)$ are essentially self-adjoint) on an open set $U \subseteq M$ and let $V \subseteq \bar{V} \subseteq U$ be precompact and simply connected.*

Let $P : V \times V \dashrightarrow U(H|_V)$ be the integrated representation defined in Proposition 7.1.4. Let $\pi = \mathrm{int}_2(P)$ and $R' = \mathrm{diff}(\pi)$. Then for all $X \in \mathfrak{X}(M)$ we have:

$$\iota \circ R'(X|_V) \subseteq R(X^*)^* \circ \iota_V$$

Proof: Put $\iota = \iota_V : L^2(H|_V) \to L^2 H$ for this proof.

Since $(X_1|_U, \dots, X_m|_U)$ is a frame, there must be $f_1, \dots, f_m \in C_b^\infty(M)$ and $Y \in \mathfrak{X}(M)$ with $\operatorname{supp} Y \subseteq M \setminus \bar{V}$ such that $X = \sum_{i=1}^m f_i X_i + Y$.

Obviously we have $Y|_V = 0$, so $R'(Y|_V) = 0$. On the other hand, since $\operatorname{supp} Y \subseteq M \setminus \bar{V}$ (which is open), we can find a bump function $f \in C_c^\infty(M)$ with $\operatorname{supp} f \subseteq M \setminus \bar{V}$ and $f|_{\operatorname{supp} Y} \equiv 1$, so that $fY^* = f(-Y - \operatorname{div} Y) = -Y - \operatorname{div} Y = Y^*$ (since $\operatorname{supp}(\operatorname{div} Y) \subseteq \operatorname{supp} Y$). Hence

$$R(Y^*)^* = R(fY^*)^* = (R(f)R(Y^*))^* \supseteq R(Y^*)^* R(f)^* \supseteq R(Y)T_f$$

by Lemma 6.3.17, and since $T_f \iota = \iota T_{f|_V} = 0$,

$$R(Y^*)^* \iota \supseteq R(Y) T_f \iota = 0$$

by Lemma 6.3.18, thus $R(Y^*)^* \iota = 0 : L^2(H|_V) \to L^2 H$.

Thus we have:

$$\begin{aligned}
\iota R'(X|_V) &= \sum_{i=1}^m \iota R'(f_i|_V) R'(X_i|_V) + \iota R'(Y|_V) = \sum_{i=1}^m \iota T_{f_i|_V} R'(X_i|_V) \\
&= \sum_{i=1}^m T_{f_i} \iota R'(X_i|_V) \subseteq \sum_{i=1}^m T_{f_i} R(X_i^*)^* \iota \\
&= \sum_{i=1}^m R(f_i)^* R(X_i^*)^* \iota \subseteq \sum_{i=1}^m (R(X_i^*) R(f_i))^* \iota \\
&= \sum_{i=1}^m (R(X_i^*) R(f_i))^* \iota + R(Y^*)^* \iota \\
&\subseteq \left(\sum_{i=1}^m R(X_i^* m_{f_i}^* + R(Y^*)) \right)^* \iota = R((\sum_{i=1}^m f_i X_i + Y)^*)^* \iota = R(X^*)^* \iota \qquad \square
\end{aligned}$$

This result is indeed independent of our chosen integration frame. Now we need to transfer it back to the local integral $P|_U : U^2 \dashrightarrow U(H|_U)$. One more time, the tool we use are the one-parameter unitary groups $P(\exp xX)$ of vector fields $X \in \mathfrak{X}(M)$.

To get a proper differential equation in our upcoming computations, we need that a vector field commutes with exponentials of itself in a derived representation. More precisely:

LEMMA 7.2.12. *Let (M, ω) be a volumetric manifold and let $P : M \times M \dashrightarrow U(H)$ be a representation of the pair groupoid on a Hilbert field H. Let $\pi = \operatorname{int}_2(P) : C^*(M \times M) \to \mathbb{B}(L^2 H)$ and $R = \operatorname{diff}(\pi) : \operatorname{Diff}(M) \to \mathcal{O}(L^2 H)$. Let $X \in \mathfrak{X}(M)$ be a (not necessarily complete) vector field and let $f \in C_c^\infty(M)$ and $\sigma \in L^2 H$. Then for all sufficiently small $x \in \mathbb{R}$, we have:*

$$P(\exp xX) R(X + \frac{1}{2} \operatorname{div} X) \pi(f) \sigma = R(X + \frac{1}{2} \operatorname{div} X) P(\exp xX) \pi(f) \sigma$$

Proof: Set $v = \pi(f)\sigma \in \operatorname{dom} R$ and $\tilde{X} = X + \frac{1}{2} \operatorname{div} X$. As described in Lemma 7.2.3, we can choose an $\epsilon > 0$ such that $P(\exp xX)v$ is defined for all $x \in (-\epsilon, \epsilon)$. By Lemma 7.2.2 and Lemma 7.2.5, we know that $\frac{\mathrm{d}}{\mathrm{d}x} P(\exp xX) v = P(\exp xX) R(-\tilde{X}) v$; in particular, the derivative exists (shrink ϵ if necessary).

Consider any other $w \in \operatorname{dom} R$. Since $P(\exp(x+y)X)v = P(\exp xX)P(\exp yX)v$ for $|x|+|y|<\epsilon$ by Lemma 7.2.3, we get for all $x \in (-\epsilon,\epsilon)$:

$$\begin{aligned}
\langle P(\exp xX)R(\tilde{X})v, w\rangle &= \langle -\frac{\mathrm{d}}{\mathrm{d}x}P(\exp xX)v, w\rangle = -\frac{\mathrm{d}}{\mathrm{d}x}\langle P(\exp xX)v, w\rangle \\
&= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle P(\exp tX)P(\exp xX)v, w\rangle \\
&= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle P(\exp xX)v, P(\exp -tX)w\rangle \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle P(\exp xX)v, P(\exp tX)w\rangle \\
&= \langle P(\exp xX)v, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}P(\exp tX)w\rangle = \langle P(\exp xX)v, -R(\tilde{X})w\rangle \\
&= \langle -R(\tilde{X})^* P(\exp xX)v, w\rangle = \langle R(-\tilde{X}^*)P(\exp xX)v, w\rangle \\
&= \langle R(\tilde{X})P(\exp xX)v, w\rangle
\end{aligned}$$

In this computation, t has to lie in a neighbourhood of 0 which depends on w, but the final equation does not depend on w. In the second last line we use that $P(\exp xX)v \in \operatorname{dom} R$ by the formula in Lemma 7.2.5. So because $\operatorname{dom} R \subseteq L^2H$ is dense and $w \in \operatorname{dom} R$ was arbitrary, we find that $P(\exp xX)R(\tilde{X})v = R(\tilde{X})P(\exp xX)v$. □

With this lemma, we finally get an integration-frame-independent equality for operators.

LEMMA 7.2.13. *Let $R : \operatorname{Diff}(M) \to \mathcal{O}(L^2(H,\nu))$ be a representation. Let $(X_1,\ldots,X_m)$ be an integration frame on an open set $U \subseteq M$ and let $V \subseteq \bar{V} \subseteq U$ be precompact and simply connected. Let $P : V \times V \dashrightarrow U(H|_V)$ be the integrated representation. Let $\pi = \operatorname{int}_2(P)$ and $R' = \operatorname{diff}(\pi)$.*

Let $X \in \mathfrak{X}(M)$ be any vector field such that $R(X + \frac{1}{2}\operatorname{div} X)$ is essentially skew-adjoint. Then for every $h \in C_c^\infty(V)$, we have:

$$P(\exp xX)T_h = \operatorname{pr}_V \overline{\mathrm{e}^{-xR(X+\frac{1}{2}\operatorname{div} X)}}\iota_V T_h$$

for all sufficiently small $x \in \mathbb{R}$.

Proof: First of all, choose $\epsilon > 0$ such that $(-2\epsilon, 2\epsilon) \times \operatorname{supp} h \subseteq \operatorname{dom}\theta$, where θ is the flow of $X|_V$, and such that $\overline{\mathrm{e}^{xR(X+\frac{1}{2}\operatorname{div} X)}}$ is defined for all $|x| < \epsilon$. Consider any $f \in C_c^\infty(V)$ and $\sigma \in L^2(H|_V)$ and put $v = \pi(f)\sigma$. Then by Lemma 7.2.2 and Lemma 7.2.5, we have for all $|x| < \epsilon$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}P(\exp xX)T_h v &= \left(\frac{(\theta_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi((-X - \frac{1}{2}\operatorname{div}(X))^R(h \circ tf) \circ l_{\exp(-xX)})\sigma \\
&= P(\exp xX)R'(-X|_V - \frac{1}{2}\operatorname{div} X|_V)T_h v
\end{aligned}$$

By Lemmas 7.2.12, 7.2.9 and 7.2.11, this is:

$$\begin{aligned}
P(\exp xX)R'(-X|_V - \frac{1}{2}\operatorname{div} X|_V)T_h v &= R'(-X|_V - \frac{1}{2}\operatorname{div} X|_V)P(\exp xX)T_h v \\
&= \operatorname{pr}_V(R(-X^*)^* - R(\frac{1}{2}\operatorname{div} X)^*)\iota_V P(\exp xX)T_h v \\
&= \operatorname{pr}_V R(X + \frac{1}{2}\operatorname{div} X)^* \iota_V P(\exp xX)T_h v
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{pr}_V \mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v &= \mathrm{pr}_V \mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\overline{R(-X-\frac{1}{2}\operatorname{div}X)}\iota_V T_h v\\ &= \mathrm{pr}_V \overline{R(-X-\frac{1}{2}\operatorname{div}X)}\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v\\ &= \mathrm{pr}_V R(X+\frac{1}{2}\operatorname{div}X)^*\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v\end{aligned}$$

since $R(X+\frac{1}{2}\operatorname{div}X)$ is essentially skew-adjoint.

Extending it by 0, consider h an element of $C_c^\infty(M)$ where appropriate. We know that $\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}T_h = T_{h\circ\theta_{-x}}\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}$ by Proposition 6.1.3, where θ is the flow of X. Because this flow is continuous, $h\circ\theta_{-x}$ still has compact support inside of V for sufficiently small x. This implies that $\iota_V\mathrm{pr}_V T_{h\circ\theta_{-x}} = T_{h\circ\theta_{-x}}$. Thus we get:

$$\begin{aligned}&\mathrm{pr}_V R(X+\frac{1}{2}\operatorname{div}X)^*\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v\\ &= \mathrm{pr}_V R(X+\frac{1}{2}\operatorname{div}X)^* T_{h\circ\theta_{-x}}\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V v\\ &= \mathrm{pr}_V R(X+\frac{1}{2}\operatorname{div}X)^*\iota_V\mathrm{pr}_V T_{h\circ\theta_{-x}}\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V v\\ &= \mathrm{pr}_V R(X+\frac{1}{2}\operatorname{div}X)^*\iota_V\mathrm{pr}_V\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v\end{aligned}$$

So the functions $F_1 : x\mapsto P(\exp xX)T_h v$ and $F_2 : x\mapsto \mathrm{pr}_V\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v$ both solve the same differential equation $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \mathrm{pr}_V R(X+\frac{1}{2}\operatorname{div}X)^*\iota_V F(x)$. We also have $P(\exp 0X)T_h v = T_h v = \mathrm{pr}_V\mathrm{e}^{-0\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v$.

Hence there must be an $\epsilon_2\in(0,\epsilon]$ such that

$$P(\exp xX)T_h v = \mathrm{pr}_V\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h v$$

for all $x\in(-\epsilon_2,\epsilon_2)$. This statement can be found in the Picard-Lindelöf theorem, for example. To be able to apply that theorem, we use that the operator $B := \mathrm{pr}_V R(X+\frac{1}{2}\operatorname{div}X)^*\iota_V$ is actually bounded when considering the right domain. Namely put, for any given v, $E := \overline{\operatorname{span}(\{F_1(x),F_2(x)\mid x\in\mathbb{R}\})}\subseteq L^2(H|_V)$. For each $x\in\mathbb{R}$ we know that

$$\|BF_1(x)\| = \|P(\exp xX)R'(-X|_V-\frac{1}{2}\operatorname{div}X|_V)T_h v\| = \|R'(X|_V+\frac{1}{2}\operatorname{div}X|_V)T_h v\| =: c_1$$

and

$$\|BF_2(x)\| = \|\mathrm{pr}_V\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}R(X+\frac{1}{2}\operatorname{div}X)\iota_V T_h v\| \le \|R(X+\frac{1}{2}\operatorname{div}X)\iota_V T_h v\| =: c_2$$

by the previous computations, using that $P(\exp xX)$ preserves the norm by Lemma 7.2.3, $\mathrm{e}^{x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}$ is unitary, and $\|\mathrm{pr}_V\|\le 1$. It follows that $\|B|_E\|\le\max\{c_1,c_2\}$.

Because v can be an arbitrary element of $\operatorname{dom}R' = \pi(C_c^\infty(V^2)L^2(H|_V)$, $\operatorname{dom}R'\subseteq L^2(H|_V)$ is dense and $P(\exp xX)T_h$, $\mathrm{pr}_V\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h$ are both bounded, this implies that indeed:

$$P(\exp xX)T_h = \mathrm{pr}_V\mathrm{e}^{-x\overline{R(X+\frac{1}{2}\operatorname{div}X)}}\iota_V T_h \qquad \square$$

One important point of the preceding equality is that both sides are bounded operators now, so we do not need to worry about different domains any more. That makes the final result of this section relatively easy to show. The involved computation only gets a little bit longer because $P(\exp xX)$ was only explored for a single vector field X in certain lemmas, not for a frame $X = (X_1,\dots,X_m)$. The same intuition applies nonetheless.

LEMMA 7.2.14. *Let* $(X_1,\dots,X_m)$ *and* $(Y_1,\dots,Y_m)$ *be integration frames for a representation* $R:\mathrm{Diff}(M)\to\mathcal{O}(L^2H)$ *on open sets* $U\subseteq M$ *and* $V\subseteq M$. *Let* $U_0\subseteq\overline{U_0}\subseteq U$ *and* $V_0\subseteq\overline{V_0}\subseteq V$ *be precompact simply connected open subsets. Let* $P=P_X: V_0^2 \dashrightarrow U(H|_{V_0})$ *and* $Q=P_Y: U_0^2 \dashrightarrow U(H|_{U_0})$ *be the respective groupoid representations obtained by Proposition 7.1.4.*

Then on $W=U_0\cap V_0$, *we have* $P|_{W^2}=Q|_{W^2}$ *almost everywhere.*

Proof: Let $K\subseteq W$ be compact. Let ϕ_x be the time-1 flow of xX. Because the map $\phi:(x,p)\mapsto\phi_x(p)$ is continuous in a neighbourhood of $\{0\}\times K\subseteq\mathbb{R}^m\times W$, we can choose an $\epsilon>0$ such that $\phi_x(p)\in W$ for $x\in U_{2\epsilon}(0)$ and $p\in K$. Choose $h\in C_c^\infty(M)$ with $\operatorname{supp}h\subseteq W$ and $h(\phi_x p)=1$ for all $x\in U_\epsilon(0)$ and $p\in K$ (which is possible since $\{\phi_x p|x\in\overline{U_\epsilon(0)},p\in K\}$ is still compact).

Then by Lemma 7.2.13 and because $[X_i,X_j]|_W=0=[Y_i,Y_j]$, we know that

$$\begin{aligned}\mathrm{pr}_W P(\exp x_iX_i)T_h\iota_W &= \mathrm{pr}_W\mathrm{pr}_{U_0}\overline{\mathrm{e}^{-x_iR(X_i+\frac12\operatorname{div}X_i)}}\iota_{U_0}T_h\iota_W\\ &=\mathrm{pr}_W\mathrm{pr}_{V_0}\overline{\mathrm{e}^{-x_iR(X_i+\frac12\operatorname{div}X_i)}}\iota_{V_0}T_h\iota_W\\ &=\mathrm{pr}_WQ(\exp x_iX_i)T_h\iota_W\end{aligned}$$

for $|x_i|<\epsilon$ (shrink ϵ if necessary), where we simply write $P(\exp xX)$ for $P(\exp(xX|_{U_0}))$. Hence for all $\sigma\in L^2H$ and almost all $x\in U_\epsilon(0)$ and $p\in K$:

$$\begin{aligned}&\left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac12}(p)h\circ\phi_{-x}(p)P(p,\phi_{-x}p)\sigma(\phi_{-x}p)=(P(\exp xX)T_h\sigma|_{U_0})(p)\\ &=P(\exp x_1X_1)\dots P(\exp x_mX_m)T_h\sigma|_{U_0}(p)\\ &=\mathrm{pr}_WP(\exp x_1X_1)T_{h\circ\phi_{(0,x_2,\dots,x_m)}}\iota_W\dots\mathrm{pr}_WP(\exp x_mX_m)\iota_W\sigma|_W(p)\\ &=\mathrm{pr}_WQ(\exp x_1X_1)T_{h\circ\phi_{(0,x_2,\dots,x_m)}}\iota_W\dots\mathrm{pr}_WP(\exp x_mX_m)\iota_W\sigma|_W(p)\\ &=\mathrm{pr}_WQ(\exp x_1X_1)\iota_W\mathrm{pr}_WP(\exp x_2X_2)T_{h\circ\phi_{(0,0,x_3,\dots,x_m)}}\iota_W\\ &\quad\dots\mathrm{pr}_WP(\exp x_mX_m)\iota_W\sigma|_W(p)\\ &=\dots=\mathrm{pr}_WQ(\exp x_1X_1)\iota_W\dots\mathrm{pr}_WQ(\exp x_mX_m)T_h\iota_W\sigma|_W(p)\\ &=(Q(\exp xX)T_h\sigma|_{V_0})(p)=\left(\frac{(\phi_x)_*\nu}{\nu}\right)^{\frac12}(p)h\circ\phi_{-x}(p)Q(p,\phi_{-x}p)\sigma(\phi_{-x}p)\end{aligned}$$

Within the preceding computation, I used that by $[X_i,X_j]|_U=0$ we have $\exp(xX)(p)=\exp(x_1X_1)\dots\exp(x_mX_m)(p)$ for all $p\in\operatorname{supp}h$ and all sufficiently small x. The point here is that $R(xX+\frac12\operatorname{div}xX)$ was not assumed to be essentially skew-adjoint; our integrability assumption only states that the sum of closures $\sum_{i=1}^m\overline{x_iR(X_i+\frac12\operatorname{div}X_i)}$ is essentially skew-adjoint. This nuance has to do with the theory of analytic vectors. However, $x_iR(X_i+\frac12\operatorname{div}X_i)$ is already essentially skew-adjoint for each $i\in\{1,\dots,m\}$.

Since $h\circ\phi_{-x}p=1$ and σ was arbitrary, it follows that $P(p,\phi_{-x}p)=Q(p,\phi_{-x}p)$ for almost all $x\in U_\epsilon(0)$ and $p\in K$. Because $(X_1|_U,\dots,X_m|_U)$ is a frame, the set $E_K:=\{(p,\phi_{-x}p)|p\in K^\circ,\ x\in U_\epsilon(0)\}\subseteq W^2$ is open. Furthermore, $\operatorname{diag}(K^\circ)\subseteq E_K$ since $\phi_0=\mathrm{id}$ and we have just shown that $P(g)=Q(g)$ for almost all $g\in E_K$ (since the exponential map is a local diffeomorphism and thus bimeasurable).

Do this process for all compact $K\subseteq W$ and put $E=\bigcup_{K\subseteq W\text{ compact}}W_K$. As a union of open sets, $E\subseteq W^2$ is still open. Since W is locally compact, $\operatorname{diag}W\subseteq E$. And as proven, $P(g)=Q(g)$ for almost all $g\in E$. It follows by the uniqueness part of Theorem 5.3.10 that $P|_W=Q|_W$ almost everywhere. ☐

7.3. Combining the Pieces

Let us now combine the newly constructed local representations to a global one. To make this possible, we need to cover our manifold of choice by integration frames first. We need to be a bit careful with the size of each frame domain, because our local integration theorem only works on a subset of the original domain. In the following definition I give a special name to such covers to make keeping track of these technicalities easier.

DEFINITION 7.3.1. Let (M, ω) be a volumetric manifold with a quasi-invariant measure ν. Let $R : \mathrm{Diff}(M) \to \mathcal{O}(K)$ be a representation on a Hilbert space K (e.g., $K = L^2 H$ for a Hilbert field H).

An *integration gallery* for R is a finite or countable family $(V_i, X_1^i, \dots, X_m^i)_{i \in I}$ of $X_j^i \in \mathfrak{X}(M)$ and precompact, simply connected, open $V_i \subseteq M$ such that $M = \bigcup_{i \in I} V_i$ and such that for every $i \in I$, there is an open subset $U_i \subseteq M$ with $\overline{V_i} \subseteq U_i$ on which $(X_1^i, \dots, X_m^i)$ is an integration frame for R.

The representation R is called *integrable* if there exists an integration frame for R and for all $X \in \mathfrak{X}(M)$, $R(X + \frac{1}{2} \operatorname{div} X)$ is essentially skew-adjoint.

The name *integration gallery* was inspired by the image of an art gallery where a collection of framed paintings is stored. In our case, we have a collection of integration frames.

Up to now, the usual assumption about adjoints in our representation was that $R(X + \frac{1}{2} \operatorname{div} X)$ is essentially skew-adjoint for certain vector fields X. A more general condition is when $R(D)^* = \overline{R(D^*)}$ holds for all differential operators D. For higher orders, this may not always be true, but at least for order-1 operators, this follows from skew-adjointness of vector fields, as the following lemma shows.

LEMMA 7.3.2. *Let $R : \mathrm{Diff}^R(G) \to \mathcal{O}(L^2 H)$ be a representation. If $R(X + \frac{1}{2} \operatorname{div} X)$ is essentially skew-adjoint for all $X \in \mathfrak{X}^R(G)$, then even $\overline{R(D^*)} = R(D)^*$ for all right-invariant differential operators $D \in \mathrm{Diff}_1^R(G)$ of order* 1.

Proof: Let $D \in \mathrm{Diff}^R(G)$ be an arbitrary (real-valued) differential operator. Then there are a smooth function $f \in C^\infty(M)$ and a right-invariant vector field $X \in \mathfrak{X}^R(G)$ such that $D = \mathcal{L}_{\tilde{X}} + m_{f \circ t}$, where $\tilde{X} = X + \frac{1}{2} \operatorname{div}^R(X)$. By abuse of notation, we simply write $D = \tilde{X} + f$. By assumption, we have $R(\tilde{X})^* = -\overline{R(\tilde{X})} = \overline{R(\tilde{X}^*)}$. Recall that $R(f)$ is bounded, in particular, $R(f)^* = T_f = \overline{R(f)}$ has domain $L^2 H$. Thus we compute:

$$\begin{aligned} R(D)^* &= R(\tilde{X} + f)^* - R(f)^* + R(f)^* \subseteq (R(\tilde{X} + f) - R(f))^* + R(f)^* \\ &= \overline{R(-\tilde{X})} + \overline{R(f)} = \overline{R(-\tilde{X}) + R(f)} = \overline{R((\tilde{X} + f)^*)} = \overline{R(D^*)} \end{aligned}$$

Here $\overline{R(-\tilde{X})} + \overline{R(f)} = \overline{R(-\tilde{X}) + R(f)}$ follows from $R(f)$ being bounded. We also have $R(D^*) \subseteq R(D)^*$ by the defining properties of a representation, and $R(D)^*$ is closed. Thus $R(D)^* = \overline{R(D^*)}$. □

This property is already very useful because higher-order differential operators can be represented by products of order-1 operators, and because all of our integration theory only concerns order-1 operators anyway.

There is one last technicality to alleviate before the proof of our main theorem. To prove that the global construction is again a homomorphism, we have to consider products of groupoid elements from different frame domains. Luckily there is a trick to deal with this: The following lemma shows that we can refine our original cover in such a way that the intersections of the refinement are always contained in a single set of the original cover, so that homomorphy is guaranteed by local homomorphy. Let us have a look at the precise mathematics.

LEMMA 7.3.3. *Let M be a compact smooth manifold. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of M. Then there exists a* barely intersecting refinement *of $(U_i)_{i\in I}$, that is, an open refinement $(V_j)_{j\in J}$ of $\mathcal{U}$ with an open set $W \subseteq \bigcup_{j\in J} V_j \times V_j \subseteq M \times M$ such that $\Delta(M) = \{(p,p)|p \in M\} \subseteq W$ with the following properties for all $(p,q) \in W$:*

(1) If $q \in V_j$ for some $j \in J$, then there is an $i \in I$ with $V_j \subseteq U_i$ and $p \in U_i$.
(2) If $p \in V_j$ for some $j \in J$, then there is an $i \in I$ with $V_j \subseteq U_i$ and $q \in U_i$.
(3) $W \cap G^p$ is path-connected.

Proof: Every smooth manifold is metrizable, so choose a metric d on M which induces the topology of M. Define the metric d^2 on $M \times M$ by $d^2((p,q),(x,y)) = d(p,x) + d(q,y)$. For every point $p \in M$, there is an $i = i_p \in I$ with $p \in U_i$, and because U_i is open, an $\epsilon_p > 0$ such that $U_\epsilon(p) = \{q \in M|d(p,q) < \epsilon\} \subseteq U_i$. Put $V_p := U_{\frac{\epsilon}{4}}(p) \subseteq U_{i_p}$. Using this, we get an open cover $(V_p)_{p\in M}$ of M (indexed by M) which is indeed a refinement of $\mathcal{U}$ by construction.

Now because M is compact, there is a finite subset $J \subseteq M$ such that $(V_j)_{j\in J}$ is still an open cover of M (and a refinement of $\mathcal{U}$). Put $\epsilon := \min_{j\in J} \frac{\epsilon_j}{4} > 0$ and define:

$$W := U_\epsilon(\Delta(M)) = \{(p,q) \in M \times M|\exists x \in M : d^2((p,q),(x,x)) = d(p,x) + d(q,x) < \epsilon\}$$

It is left to show that W has the required properties. So let $(p,q) \in W$ be arbitrary. Assume that $q \in V_j$ and put $i = i_j \in I$. Choose $x \in M$ such that $d(p,x)+d(q,x) < \epsilon$. Then $d(p,q) \le d(p,x) + d(x,q) < \epsilon \le \frac{\epsilon_j}{4}$. Also by definition of V_j and $q \in V_j$ we have $d(j,q) < \frac{\epsilon_i}{4}$. Hence $d(j,p) \le d(j,q) + d(p,q) < \frac{\epsilon_i}{2}$. So because $U_{\epsilon_j}(j) \subseteq U_i$, we have $p \in U_i$.

Likewise, if $p \in V_j$, then again $d(p,q) < \frac{\epsilon_j}{4}$ and $d(j,p) < \frac{\epsilon_j}{4}$, so $d(j,q) < d(p,q)+d(j,p) < \frac{\epsilon_j}{2}$, hence $q \in U_{i_j}$. ☐

As it turned out, the proof of this lemma was simpler than the formulation of the idea. This is certainly not the case for our upcoming integration theorem. But thanks to all of our previous results, we can confidently state and prove:

THEOREM 7.3.4. *Let M be a compact and simply connected smooth manifold. Let $\omega \in \Omega^m(M)$ be a volume form. Let ν be a quasi-invariant measure on M and $H \to M$ a ν-Hilbert field. Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ be a representation in which smooth functions act by multiplication operators ($R(m_f) = T_f$) for $f \in C^\infty M)$).*

Suppose that there exists an integration gallery for R. Then there is a representation $P = \mathrm{int}_1(R) : G = M \times M \dashrightarrow U(H)$ such that $R' = \mathrm{diff}\,\mathrm{int}_2(P)$ fulfils $R'(X) \subseteq R(X^*)^*$ for all $X \in \mathfrak{X}(M)$.

If R is integrable, then $\overline{R'(D)} = \overline{R(D)}$ for all $D \in \mathrm{Diff}_1(M)$.

Proof: Let $(V_i, X_1^i, \dots, X_m^i)_{i\in I}$ be an integration gallery for R. Consider any $i \in I$. By assumption, there is another open set $U_i \subseteq M$ with $\overline{V_i} \subseteq U_i$ such that $(X_1^i, \dots, X_m^i)$ is an integration frame on U_i. $V_i \subseteq \overline{V_i} \subseteq U_i$ is precompact, so by Proposition 7.1.4, there is a local essential homomorphism of second type $P_i^0 : V_i^2 \dashrightarrow U(H|_{V_i})$. By Theorem 5.3.10, this extends to a global essential homomorphism of second type $P_i : V_i^2 \dashrightarrow U(H|_{V_i})$.

For every $i,j \in I$, Lemma 7.2.14 implies that $P_i|_{V_i\cap V_j} = P_j|_{V_i\cap V_j}$. By virtue of Lemma 7.3.3, we may choose a barely intersecting refinement $((W_j)_{j\in J}, W)$ of $(V_i)_{i\in I}$. Let $\alpha : J \to I$ be the refinement map (the map satisfying $V_j \subseteq U_{\alpha(j)}$ for all $j \in J$). For each pair $(i,j) \in I$, let $A_{ij} \subseteq (U_i \cap U_j)^2$ be the null set of $x \in (U_i \cap U_j)^2$ such that $Q_i(x) \neq Q_j(x)$ (or one of the two is undefined). Set $A := \bigcup_{i,j\in I} A_{ij} \subseteq M^2$, which is still a null set because I (and hence I^2) is countable. We use this to define a map

$$P : W \setminus A \to U(H)$$

by setting $P(x) := P_i(x)$ for any $i \in I$ such that $x \in V_i^2$, which is well-defined since $P_i(x) = P_j(x)$ for all $x \in (V_i \cap V_j)^2 \setminus A$ by the construction of A. I claim that P is a local essential homomorphism of second type.

First, let us check that the source and target of $P(x)$ are actually the right ones. Namely if $(p,q) = x \in W_j^2 \subseteq V_{\alpha(j)}^2$, then for $i = \alpha(j)$ we have $P(x) = P_i(x) : H_{sx} = (H|_{V_i})_{sx} \to (H|_{V_i})_{tx} = H_{tx}$, and this is a unitary operator just as required.

Now let $x, y \in W \setminus A$ be two composable elements with $xy \in W$. Since $W \subseteq \bigcup_j W_j^2$, there are $i, j \in J$ such that $x \in W_i^2$, $y \in W_j^2$. Let $p = tx$, $q = sx = ty$ and $r = sy$. Then we have $p, q \in W_i \subseteq V_{\alpha(i)}$. By the property of our chosen subcover as described in the lemma, this also implies that $r \in V_{\alpha(i)}$, so that $y = (q,r) \in V_k^2$ and $xy = (p,r) \in V_k^2$ as well as $x = (p,q) \in V_k^2$ for $k = \alpha(i) \in I$. So because P_k is an essential homomorphism of second type on V_k^2, this shows that

$$P(xy) = P_k(xy) = P_k(x)P_k(y) = P(x)P(y)$$

just as required. Hence P is a local essential homomorphism of second type.

By the Sausage Theorem 5.3.10, P extends to a global essential homomorphism of second type $\tilde{P} : M \times M \to U(H)$. Via Theorem 6.2.2, let $\pi = \mathrm{int}_2(P) : L^I(M \times M) \to \mathbb{B}(L^2H)$ be the integrated form of P.

It is left to show that actually $\mathrm{diff}(\pi) = R$. We can obtain this result relatively quickly using that we already know it is fulfilled locally.

Choose a smooth partition of unity $(h_i)_{i \in I}$ subordinate to $(V_i)_{i \in I}$. Let $i \in I$ and $X \in \mathfrak{X}(M)$ be arbitrary such that $R(X + \frac{1}{2}\operatorname{div} X)$ is essentially skew-adjoint. Denote the flow of X by $\theta : \mathbb{R}^m \times M \to M$ (X is complete since M is compact). We know that $\operatorname{supp} h_i \subseteq V_i$ is compact, so we can choose $\epsilon_i > 0$ such that $\theta([-2\epsilon_i, 2\epsilon_i] \times \operatorname{supp} h_i) \subseteq V_i$. For now, put $h = h_i$. Then for all $x \in (-\epsilon, \epsilon)$, $P_i(\exp xX|_{V_i})T_h : L^2(H|_{V_i}) \to L^2(H|_{V_i})$ is defined (and bounded), and for all $\sigma \in L^2H$ and almost all $p \in V_i$:

$$\begin{aligned}
\left(\iota_{V_i} P_i(\exp xX|_{V_i}) T_h \mathrm{pr}_{V_i}\sigma\right)(p) &= P_i(\exp xX|_{V_i})(h\sigma|_{V_i}) \\
&= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}} (p) P_i(p, \theta_{-x}p) h(\theta_{-x}p)\sigma(\theta_{-x}p) \\
&= \left(\frac{(\theta_x)_*\nu}{\nu}\right)^{\frac{1}{2}} (p) P(p, \theta_{-x}p) h(\theta_{-x}p)\sigma(\theta_{-x}p) \\
&= \left(P(\exp xX) T_h(\sigma)\right)(p)
\end{aligned}$$

For $p \in M \setminus V_i$, both sides are 0 because $\operatorname{supp} h \circ \theta_x \subseteq V_i$. Hence $\iota_{V_i} P_i(\exp xX|_{V_i}) T_h \mathrm{pr}_{V_i} = P(\exp xX) T_h$.

Since M is compact, we can assume I to be finite without loss of generality. So define $\epsilon = \min_{i \in I} \epsilon_i > 0$. It follows that for all $x \in (-\epsilon, \epsilon)$:

$$\begin{aligned}
P(\exp xX) &= \sum_{i \in I} P(\exp xX) T_{h_i} = \sum_i \iota_{V_i} P_i(\exp xX|_{V_i}) T_{h_i} \mathrm{pr}_{V_i} \\
&= \sum_i \iota_{V_i} \mathrm{pr}_{V_i} \mathrm{e}^{\overline{-xR(X + \frac{1}{2}\operatorname{div} X)}} \iota_{V_i} \mathrm{pr}_{V_i} T_{h_i} \\
&= \sum_{i \in I} \mathrm{e}^{\overline{-xR(X + \frac{1}{2}\operatorname{div} X)}} T_{h_i} = \mathrm{e}^{\overline{-xR(X + \frac{1}{2}\operatorname{div} X)}}
\end{aligned}$$

by Lemma 7.2.13 and the fact that $\operatorname{supp} h_i \circ \theta_x \subseteq V_i$.

Denote $R' := \mathrm{diff}(\pi) = \mathrm{diff}\,\mathrm{int}_2(P)$. Then we know by Lemma 7.2.2 and Lemma 7.2.5 that

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\overline{-xR(X + \frac{1}{2}\operatorname{div} X)}} v = \frac{\mathrm{d}}{\mathrm{d}x} P(\exp xX)v = P(\exp xX) R'(-X - \frac{1}{2}\operatorname{div} X)v$$

for all $v \in \operatorname{dom} R'$ and $x \in (-\epsilon, \epsilon)$; in particular, the derivative exists. Hence by Stone's theorem (Theorem D, page 647, [**23**]), we find that $v \in \operatorname{dom} \overline{-xR(X + \frac{1}{2}\operatorname{div} X)}$ and $\overline{-xR(X + \frac{1}{2}\operatorname{div} X)}v = R'(-X - \frac{1}{2}\operatorname{div} X)v$. Thus $R'(X + \frac{1}{2}\operatorname{div} X) \subseteq \overline{R(X + \frac{1}{2}\operatorname{div} X)} = -R(X + \frac{1}{2}\operatorname{div} X)^*$.

We have already discussed multiple times that $\overline{R'(f)} = T_f = \overline{R(f)}$. It follows that $R'(X) = R'(X + \frac{1}{2}\operatorname{div} X) - R(\frac{1}{2}\operatorname{div} X) \subseteq -R(X + \frac{1}{2}\operatorname{div} X)^* - R(\frac{1}{2}\operatorname{div} X)^* \subseteq (R(-X - \frac{1}{2}\operatorname{div} X) - R(\frac{1}{2}\operatorname{div} X))^* = R(X^*)^*$.

Now let $X \in \mathfrak{X}(M)$ be an arbitrary vector field. Then there are $f_{ij} \in C^\infty(M) = C_c^\infty(M)$ such that $X = \sum_{i\in I}\sum_{j=1}^m f_{ij} X_j^i$ since $(X_1^i, \dots, X_m^i)$ is a frame on V_i and the V_i cover M. Thus we find:

$$\begin{aligned} R'(X) &= \sum_{i\in I}\sum_{j=1}^m R'(f_{ij})R'(X_j^i) \subseteq \sum_{i,j} R(f_{ij}^*)^* R((X_j^i)^*)^* \\ &\subseteq \left(\sum_{i,j} R((X_j^i)^*)R(f_{ij}^*)\right)^* = \left(R((\sum_{i,j} f_{ij}X_j^i)^*)\right)^* = R(X^*)^* \end{aligned}$$

Lastly, suppose that R is integrable, so that $R(X + \frac{1}{2}\operatorname{div} X)$ is essentially skew-adjoint for *all* $X \in \mathfrak{X}(M)$. By Theorem 4.2.5, the same holds for $R' = \operatorname{diff}\operatorname{int}_2(P)$. Thus we have $\overline{R(D^*)} = R(D)^*$ and $\overline{R'(D^*)} = R'(D)^*$ for all $D \in \operatorname{Diff}_1(M)$ by Lemma 7.3.2. So let $D = X + f \in \operatorname{Diff}_1(M)$, for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. We have proven that $R'(X) \subseteq R(X^*)^*$, so $R'(D) = R'(X) + R'(f) \subseteq R(X^*)^* + T_f = R(X^*)^* + R(f^*)^* \subseteq R(D^*)^* = \overline{R(D)}$, and thus $\overline{R'(D)} \subseteq \overline{R(D)}$. On the other hand, because $A \subseteq B$ implies $B^* \subseteq A^*$ for any unbounded densely defined operators A, B, we see that also $\overline{R(D)} = (R(D)^*)^* \subseteq R'(D^*)^* = \overline{R'(D)}$, using D^* instead of D in the previous equation. Hence $\overline{R'(D)} = \overline{R(D)}$. ☐

Because two representations R' and R as well as their integrals are completely determined by their values on $\operatorname{Diff}_1(M)$, it is justified to call them equivalent if $\overline{R(D)} = \overline{R'(D)}$ for all $D \in \operatorname{Diff}_1(M)$. In this sense, we have constructed a right-sided inverse to the differentiation map, mapping equivalence classes of integrable algebroid representations to representations of the groupoid C^*-algebra. As before in the Euclidean case, we should also consider the other direction. But before we do this, let us remember that a measurable field of Hilbert spaces is a technical tool. Thanks to our previous endeavours, we can formulate the same theorem for a representation on a more general Hilbert space. The proof mainly relies on the fact that unitary intertwiners commute with closure and adjoint operations.

Corollary 7.3.5. *Let M be a compact and simply connected smooth manifold. Let $\omega \in \Omega^m(M)$ be a volume form. Let $R : \operatorname{Diff}(M) \to \mathcal{O}(K)$ be a representation on a separable Hilbert space K such that $R|_{C^\infty M}$ is injective.*

Suppose that R is integrable. Then there is a representation $\pi = \operatorname{int}(R) : C^*(M \times M) \to \mathbb{B}(K)$ such that $\overline{\operatorname{diff}(\pi)(D)} = \overline{R(D)}$ for all $D \in \operatorname{Diff}_1(M)$.

Proof: By Theorem 5.2.8, there are a bounded Radon measure ν and a ν-Hilbert field H on M, together with a unitary map $\alpha : L^2H \to K$ such that $\alpha^{-1}\overline{R(f)}\alpha = T_f$ for all $f \in C_0^\infty(M) = C^\infty(M)$. By Proposition 5.2.13 and Example 5.3.2, this measure ν is quasi-invariant. Define

$$R_H : \operatorname{Diff}(M) \to \mathcal{O}(L^2H),\ D \mapsto \alpha^{-1}R(D)\alpha,$$

which is another representation with domain $\operatorname{dom} R_H = \alpha^{-1}(\operatorname{dom} R)$. R was assumed to be integrable, and we quickly see that this implies integrability of R_H. Namely for $X \in \mathfrak{X}(M)$ and $\tilde{X} = X + \frac{1}{2}\operatorname{div} X$, we have

$$\begin{aligned} R_H(\tilde{X}) &= \left(\alpha^{-1}R(\tilde{X})\alpha\right)^* = \alpha^{-1}R(\tilde{X})^*\alpha \\ &= \alpha^{-1}(-\overline{R(\tilde{X})})\alpha = -\overline{\alpha^{-1}R(\tilde{X})\alpha} = -\overline{R_H(\tilde{X})} \end{aligned}$$

because α is unitary. Consider any integration frame $(X_1, \dots, X_m)$ for R on $U \subseteq M$. Then for any $h \in C^\infty M$ supported in U or its closure and $x, y \in \mathbb{R}^m$ sufficiently small, we have

$$\begin{aligned}\overline{R_H(h)}\mathrm{e}^{\overline{R_H(xX)}}\mathrm{e}^{\overline{R_H(yX)}} &= \alpha^{-1}\overline{R(h)}\mathrm{e}^{\overline{R(xX)}}\mathrm{e}^{\overline{R(yX)}}\alpha \\ &= \alpha^{-1}\overline{R(h)}\mathrm{e}^{\overline{R((x+y)X))}}\alpha = \overline{R_H(h)}\mathrm{e}^{\overline{R_H((x+y)X))}},\end{aligned}$$

so that $(X_1, \dots, X_m)$ is an integration frame for R_H on the same subset U. Thus R_H is again integrable.

Hence by Theorem 7.3.4, there is a representation $P : M \times M \dashrightarrow U(H)$ such that $\overline{\operatorname{diff}\operatorname{int}_2(P)(D)} = \overline{R_H(D)}$ for all $D \in \operatorname{Diff}_1(M)$. Set $\pi_H = \operatorname{int}_2(P) : C^*(M \times M) \to \mathbb{B}(L^2 H)$ and define $\pi : C^*(M\times M) \to \mathbb{B}(K),\ f \mapsto \alpha\pi_H(f)\alpha^{-1}$, which is a representation because π_H is one. Set $R_1 = \operatorname{diff}(\pi)$ and consider also $R_2 : \operatorname{Diff}(M) \to \mathcal{O}(K),\ D \mapsto \alpha \operatorname{diff}(\pi_H)(D)\alpha^{-1}$. Then we find that

$$\begin{aligned}\operatorname{dom} R_1 &= \pi(C^\infty M^2)K = \alpha\pi_H(C^\infty M^2)\alpha^{-1}K \\ &= \alpha\pi_H(C^\infty M^2)L^2 H = \alpha \operatorname{dom}\operatorname{diff}(\pi_H) = \operatorname{dom} R_2\end{aligned}$$

holds for the domains of these representations. Furthermore, for all $D \in \operatorname{Diff}(M)$, $f \in C^\infty M$, $v \in K$ we have:

$$\begin{aligned}R_1(D)(\pi(f)v) &= \pi(Df)v = \alpha\pi_H(Df)\alpha^{-1}v \\ &= \alpha \operatorname{diff}(\pi_H)(D)(\pi_H(f)\alpha^{-1}v) \\ &= \alpha \operatorname{diff}(\pi_H)(D)\alpha^{-1}(\pi(f)v) = R_2(D)(\pi(f)v)\end{aligned}$$

Hence $R_1(D) = R_2(D)$ (including the domain). Consequently, for all $D \in \operatorname{Diff}_1(M)$ we find that

$$\begin{aligned}\overline{\operatorname{diff}(\pi)(D)} = \overline{R_1(D)} = \overline{R_2(D)} &= \overline{\alpha \operatorname{diff}(\pi_H)(D)\alpha^{-1}} \\ &= \alpha\overline{\operatorname{diff}(\pi_H)(D)}\alpha^{-1} = \alpha\overline{R_H(D)}\alpha^{-1} = \overline{\alpha R_H(D)\alpha^{-1}} = \overline{R(D)}\end{aligned}$$

just as required. □

The Hilbert space K in the previous corollary is isomorphic to the sections of a ν-Hilbert field H even if $R|_{C^\infty M}$ is not injective, but then the measure ν will not be quasi-invariant due to the extension by 0. It is also not possible to apply the integration theory directly to the closed subspace of M that we find in Theorem 5.2.8 because it may not be a smooth submanifold. Thus we will settle for this result.

Let us now show that our integration map is also a left-sided inverse to differentiation. This is relatively easy using the one-parameter groups generated by exponentials of vector fields, as detailed in the next theorem.

THEOREM 7.3.6. *Let (M, ω) be a compact simply connected volumetric manifold with quasi-invariant measure ν and let $P : M \times M \dashrightarrow U(H)$ be a representation on a ν-Hilbert field H. Suppose that there exists an integration gallery for $R = \operatorname{diff}\operatorname{int}_2(P)$. Then $\operatorname{int}_1(R) = P$ almost everywhere.*

Proof: Denote $P' := \operatorname{int}_1(R)$. Consider any vector field $X \in \mathfrak{X}(M)$ such that $R(X + \frac{1}{2}\operatorname{div} X)$ is essentially skew-adjoint. Let $v \in \operatorname{dom} R$. Then we know by Lemma 7.2.2 and Lemma 7.2.5 that

$$\frac{\mathrm{d}}{\mathrm{d}x} P(\exp xX)v = P(\exp xX)R(-X - \frac{1}{2}\operatorname{div} X)v$$

for all $x \in \mathbb{R}$. So because $R(X + \frac{1}{2}\operatorname{div} X)$ is essentially skew-adjoint, $P(\exp xX) = \mathrm{e}^{\overline{-xR(X+\frac{1}{2}\operatorname{div} X)}}$. Furthermore, we have found within the proof of Theorem 7.3.4 an $\epsilon > 0$ such that for $x \in (-\epsilon, \epsilon)$:

$$P'(\exp xX) = \mathrm{e}^{\overline{-xR(X+\frac{1}{2}\operatorname{div} X)}} = P(\exp xX)$$

Now let $(V_i, X_1^i, \dots, X_m^i)_{i\in I}$ be an integration gallery for R. Let $i \in I$. For $x \in \mathbb{R}^m$, denote by ϕ_x^i the time-1 flow of $xX^i = x_1X_1^i + \dots + x_mX_m^i$. By definition, $(X_1^i, \dots, X_m^i)$ is an integration frame on an open subset U_i which contains the closure of V_i; in particular, $[X_k^i, X_l^i]|_{U_i} = 0$. So because flows of commuting vector fields commute, we find an $\epsilon > 0$ such that

$$P(\exp xX^i)\sigma(p) = P(\exp x_1X_1^i)\dots P(\exp x_mX_m^i)\sigma(p)$$

for all $\sigma \in L^2H$ and almost all $x \in U_\epsilon(0) \subseteq \mathbb{R}^m$, $p \in V_i$, and likewise for P'. For these x, p we get:

$$\begin{aligned}
P(p, \phi_{-x}^i p)\sigma(\phi_{-x}p) &= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{-\frac{1}{2}}(p)(P(\exp xX^i)\sigma)(p) \\
&= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{-\frac{1}{2}}(p)P(\exp x_1X_1^i)\dots P(\exp x_mX_m^i)\sigma(p) \\
&= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{-\frac{1}{2}}(p)P'(\exp x_1X_1^i)\dots P'(\exp x_mX_m^i)\sigma(p) \\
&= \left(\frac{(\phi_x)_*\nu}{\nu}\right)^{-\frac{1}{2}}(p)(P'(\exp xX^i)\sigma)(p) = P'(p, \phi_{-x}^i p)\sigma(\phi_{-x}p)
\end{aligned}$$

Since σ was arbitrary, $P(p, \phi_{-x}^i p) = P'(p, \phi_{-x}^i p)$. Because $(X_1^i, \dots, X_m^i)$ is a frame on U_i, the set $W_i := \{(p, \phi_{-x}^i p) | p \in V_i, x \in U_\epsilon(0)\}$ is a neighbourhood of $\operatorname{diag}(V_i)$ in M^2. Doing this for all i, we get a neighbourhood $W = \bigcup_{i\in I} W_i$ of $\operatorname{diag}(M) = \bigcup_{i\in I}\operatorname{diag}(V_i)$ with $P(g) = P'(g)$ for almost all $g \in W$. The uniqueness part of Theorem 5.3.10 now shows that $P = P'$ almost everywhere. ☐

We already know that the integration map int_2 from groupoid homomorphisms to groupoid algebra representations is a bijection, up to equivalence classes, where two groupoid homomorphisms are equivalent if they only differ on a null set. So we have once more established a commuting triangle of maps between representation types, this time for the pair groupoid $G = M \times M$ of a simply connected compact manifold M:

$$\begin{tikzcd}
 & & & \mathbb{B}(L^2H) & & & \\
 & & & \pi \arrow[u] \arrow[d, dash] \arrow[ddrr, "\mathrm{diff}"', "R(D)(\pi(f)\sigma)=\pi(D(f))\sigma"] & & & \\
 & & & L^I(G) & & & \\
U(H) & P \arrow[l] \arrow[r, dash] \arrow[uurr, "\mathrm{int}_2"', "{\pi(f)(\sigma)(p)= \int_{G^p}\Delta^{-\frac{1}{2}}(x)f(x)P(x)\sigma(sx)\mathrm{d}\lambda^p}"] & G & & \mathrm{Diff}^R(G) \arrow[r, dash] & R \arrow[r] \arrow[llll, bend left, "\mathrm{int}_1", "{P(\exp(xX)p)=\left(\frac{\mathrm{d}\nu}{\mathrm{d}(\phi_x)_*\nu}\right)^{\frac{1}{2}}(\phi_x p)\left(\phi_x^*\overline{\mathrm{e}^{R(-x(X+\frac{1}{2}\operatorname{div}X))}}\right)(p)}"'] & \mathcal{O}(L^2H)
\end{tikzcd}$$

However, this only works if the derivative of a groupoid representation is actually integrable. The main challenge is to show that vector fields act by essentially skew-adjoint operators, which we have shown to be true at least for compact manifolds. Using this result, integrability is relatively quick to show, and we will do so in the next section.

7.4. Integrability of the Derivative

To show integrability of the derivative, we need to construct integration frames. For the tangent bundle, we can always find commuting local frames using charts, and self-adjointness was investigated before. The remaining thing to show now is that exponentials of suitable vector fields commute locally. This is formally described and proven in the following lemma.

LEMMA 7.4.1. *Let (M,ω) be a compact volumetric manifold, $\pi : C^*(M\times M) \to \mathbb{B}(L^2H)$ a representation of the pair groupoid on a Hilbert field and $R = \operatorname{diff}(\pi)$. Let $X_1,\ldots,X_m \in \mathfrak{X}(M)$ be vector fields and $U \subseteq M$ open such that $[X_i,X_j]|_U = 0$ for all $i,j \in \{1,\ldots,m\}$.*

Then for every compact set $K \subseteq U$ there exists an $\epsilon > 0$ such that for all $h \in C^\infty(M)$ with $\operatorname{supp} h \subseteq K$ and all $x,y \in U_\epsilon(0) \subseteq \mathbb{R}^m$:

$$T_h \mathrm{e}^{\overline{R(xX+\frac{1}{2}\operatorname{div}(xX))}} \mathrm{e}^{\overline{R(yX+\frac{1}{2}\operatorname{div}(yX))}} = T_h \mathrm{e}^{\overline{R((x+y)X+\frac{1}{2}\operatorname{div}((x+y)X))}}$$

Proof: First of all, we know by Theorem 4.2.5 that $R(xX + \frac{1}{2}\operatorname{div}(xX))$ is essentially skew-adjoint for all $x \in \mathbb{R}$, so that $U_x := \mathrm{e}^{\overline{R(-xX-\frac{1}{2}\operatorname{div}(xX))}}$ is well-defined. Furthermore, by Proposition 6.2.6 we know that $\pi = \operatorname{int}_2(P)$ for some representation $P : M\times M \dashrightarrow U(H)$. Using this, we see by Lemma 7.2.2 and Lemma 7.2.5 that $P(\exp(xX)) = U_x$; this was discussed before in the proof of Theorem 7.3.6. Thus for any $f \in C_c^\infty(M^2)$, $v \in L^2H$ we find that

$$U_x\pi(f)v = \left(\frac{(\phi_x)_*\omega}{\omega}\right)^{\frac{1}{2}} \pi(f\circ l_{\exp(-xX)})v$$

for all $x \in \mathbb{R}^m$ (using that M is compact). Here ϕ_x is the time-1 flow of xX.

For each $i \in \{1,\ldots,m\}$, let θ^i be the flow of $X_i|_U$. Then because $[X_i,X_j]|_U = 0$, there is an open set $D \subseteq \mathbb{R}^m\times U$ with $\{0\}\times U \subseteq D$ such that $\phi_x(p) = \theta^1_{x_1}\circ\cdots\circ\theta^m_{x_m}(p) \in U$ for all $(x,p) \in D$; D is the common domain of the commuting flows $\theta^1,\ldots,\theta^m$. Because $\theta^1,\ldots,\theta^m$ are continuous and commute, we even find an open subset $D_2 \subseteq \mathbb{R}^m\times\mathbb{R}^m\times U$ with $\{0\}\times\{0\}\times U \subseteq D_2$ such that $\phi_{x+y}(p) = \theta^1_{x_1+y_1}\cdots\theta^m_{x_m+y_m} = \phi_x\phi_y(p)$ for all $(x,y,p) \in D_2$. Because K is compact and D_2 is open, we can choose $\epsilon > 0$ such that $\overline{U_\epsilon(0)}^2 \times K \subseteq D_2$.

Let $f \in C_c(M\times M)$ be arbitrary. Consider any $x,y \in U_\epsilon(0) \subseteq \mathbb{R}^m$ and $(p,q) \in G = M\times M$. If $p \notin K$, then we have $h(p) = 0$ and thus:

$$((h\circ t)\cdot(f\circ l_{\exp((x+y)X)}))(p,q) = 0 = ((h\circ t)\cdot(f\circ l_{\exp(yX)}\circ l_{\exp(xX)}))(p,q)$$

Remember that by the construction of ϵ, we have $\{y\}\times\{x\}\times K \subseteq D_2$. Thus if $p \in K$, we have $\phi_y\phi_x p = \phi_{x+y}p$ and hence:

$$\begin{aligned} f\circ l_{\exp yX}\circ l_{\exp xX}(p,q) &= f\circ l_{\exp yX}(\phi_x p,q) = f(\phi_y\phi_x p,q)\\ &= f(\phi_{x+y}p,q) = f\circ l_{\exp(x+y)X}(p,q)\end{aligned}$$

Combining both cases, we see that $h\circ tf\circ l_{\exp yX}\circ l_{\exp xX} = h\circ tf\circ l_{\exp(x+y)X}$, and likewise $h\circ tf\circ l_{\exp -yX}\circ l_{\exp -xX} = h\circ tf\circ l_{\exp -(x+y)X}$

Besides this, consider the pull-back $\phi_x^*\phi_y^*\omega$. For every $p \in K$ and all $v_1,\ldots,v_m \in T_pM$, we have

$$\begin{aligned}\phi_x^*\phi_y^*\omega(p)(v_1,\ldots,v_m) &= \phi_y^*\omega(\phi_x p)(T_p\phi_x v_1,\ldots,T_p\phi_x v_m)\\ &= \omega(\phi_y\phi_x p)(T_p(\phi_y\phi_x)v_1,\ldots,T_p(\phi_y\phi_x)v_m)\\ &= \omega(\phi_{x+y}p)(T_p\phi_{x+y}v_1,\ldots,T_p\phi_{x+y}v_m) = \phi_{x+y}^*\omega(p)(v_1,\ldots,v_m)\end{aligned}$$

because $\phi_y\phi_x = \phi_{x+y}$ in a neighbourhood of p. Since $(\phi_x)_*\omega = \phi_{-x}^*\omega$, we also see that $(\phi_{x+y})_*\omega = (\phi_x)_*(\phi_y)_*\omega$.

Combining the previous results, we compute for any $\sigma \in L^2H$ and $p \in K$:

$$
\begin{aligned}
&T_h \mathrm{e}^{\overline{R(xX+\frac{1}{2}\operatorname{div}(xX))}} \mathrm{e}^{\overline{R(yX+\frac{1}{2}\operatorname{div}(yX))}}(\pi(f)\sigma)(p) = T_h U_x U_y (\pi(f)\sigma)(p) \\
=&T_h U_x \left(\left(\frac{(\phi_y)_*\omega}{\omega} \right)^{\frac{1}{2}} \pi(f \circ l_{\exp(-yX)})(\sigma) \right)(p) \\
=&T_h \left(\frac{(\phi_x)_*\omega}{\omega} \right)^{\frac{1}{2}} \pi((\left(\frac{(\phi_y)_*\omega}{\omega} \right)^{\frac{1}{2}} \circ t \circ l_{\exp(-xX)}) \cdot (f \circ l_{\exp(-yX)} \circ l_{\exp(-xX)}))(\sigma)(p) \\
=&\left(\frac{(\phi_x)_*\omega}{\omega} \right)^{\frac{1}{2}} \left(\frac{(\phi_y)_*\omega}{\omega} \right)^{\frac{1}{2}} \circ \phi_{-x} \pi((h \circ t) \cdot (f \circ l_{\exp(-yX)} \circ l_{\exp(-xX)}))(\sigma)(p) \\
=&\left(\frac{(\phi_x)_*\omega}{\omega} \right)^{\frac{1}{2}} \left(\frac{(\phi_x)_*(\phi_y)_*\omega}{(\phi_x)_*\omega} \right)^{\frac{1}{2}} \pi((h \circ t) \cdot (f \circ l_{\exp(-(x+y)X)}))(\sigma)(p) \\
=&\left(\left(\frac{(\phi_{x+y})_*\omega}{\omega} \right)^{\frac{1}{2}} h\pi(f \circ l_{\exp(-(x+y)X)})(\sigma) \right)(p) = T_h \mathrm{e}^{\overline{R((x+y)X+\frac{1}{2}\operatorname{div}((x+y)X))}}(\pi(f)\sigma)(p)
\end{aligned}
$$

Because p, f and σ were arbitrary, it follows that $T_h U_x U_y v = T_h U_{x+y} v$ for all $v \in \operatorname{dom} R = \pi(C_c^\infty(M^2))L^2H$. Because both $T_h U_x U_y$ and $T_h U_{x+y}$ are bounded and $\operatorname{dom} R \subseteq L^2H$ is dense, it follows that $T_h U_x U_y = T_h U_{x+y}$. x and y were also chosen arbitrarily within $U_\epsilon(0) \subseteq \mathbb{R}^m$, so this is the desired result. $\square$

To summarize the previous proof, we mostly rely on the fact that functional calculus exponentials on the operator side are given by left-actions of groupoid exponentials, for which domain is not an issue.

Using this result and Theorem 4.2.5, the construction of an integration gallery is straightforward.

Proposition 7.4.2. *Let (M, ω) be a compact volumetric manifold, $\pi : C^*(M^2) \to \mathbb{B}(L^2H)$ a representation and $R = \operatorname{diff}(\pi)$. Then R is integrable.*

Proof: We know by Theorem 4.2.5 that $R(X + \frac{1}{2}\operatorname{div} X)$ is essentially skew-adjoint for all $X \in \mathfrak{X}(M)$. So we only need to construct an integration gallery for R, which is easy using our previous knowledge.

Namely let $p \in M$ be arbitrary. Choose a chart (U, ϕ) around p. Choose a compact neighbourhood $K \subset U$ of p and a bump function $h \in C_c^\infty(M)$ with $h|_K \equiv 1$ and $\operatorname{supp} h \subseteq U$. Put $X_j^p = h\partial_j^\phi \in \mathfrak{X}(M)$ for each $j \in \{1, \dots, m\}$. Let $U_2 \subseteq K$ be another, smaller neighbourhood of p. Then since $h|_{U_2} \equiv 1$ and $[\partial_i^\phi, \partial_j^\phi] = 0$, we know that $[X_i^p, X_j^p]|_{U_2} = 0$ for all $i, j \in \{1, \dots, m\}$. Choose another open neighbourhood $U_3^p = U_3$ of p with $\overline{U_3} \subseteq U_2$. $\overline{U_3}$ is compact as a closed subset of K, so by Lemma 7.4.1, we find an $\epsilon > 0$ such that

$$T_f \mathrm{e}^{\overline{R(xX+\frac{1}{2}\operatorname{div}(xX))}} \mathrm{e}^{\overline{R(yX+\frac{1}{2}\operatorname{div}(yX))}} = T_f \mathrm{e}^{\overline{R((x+y)X+\frac{1}{2}\operatorname{div}((x+y)X))}}$$

for all $x, y \in U_\epsilon(0)$ and all $f \in C^\infty(M)$ with $f|_{M \setminus U_3} \equiv 0$.

Since $h|_{U_3} \equiv 1$ and $(\partial_1^\phi, \dots, \partial_m^\phi)$ is a frame for TU, $(X_1^p(p), \dots, X_m^p(p))$ is a basis for T_pM for each $p \in U_3$, and we know that $[X_i^p, X_j^p]|_{U_3} = 0$ a fortiori for all $i, j \in \{1, \dots, m\}$ since this is even true on U_2. As mentioned, the vector fields act by essentially skew-adjoint operators. So in summary, $(X_1^p, \dots, X_m^p)$ is an integration frame on U_3.

Let $V_p \subseteq U_3$ be a simply connected open neighbourhood of p such that $\overline{V_p} \subseteq U_3$, e.g. $V_p = \phi^{-1}(U_\delta(p))$ for sufficiently small $\delta > 0$. V_p is precompact since K (or even M) is compact. Since $p \in V_p$, the family $(V_p)_{p \in M}$ is an open cover of M. As M is compact, we can choose a finite set $I \subseteq M$ such that $(V_p)_{p \in I}$ is still a cover of M. By construction, for each $p \in I$, $(X_1^p, \dots, X_m^p)$ is an integration frame for R on the larger neighbourhood U_3^p. Hence $(V_p, X_1^p, \dots, X_m^p)_{p \in I}$ is an integration gallery for R, in particular R is integrable. $\square$

Now that we have a complete result for the case we specialized on, we could close this chapter and with it, the thesis. However, our integrability assumption is still a bit technical. We know that it is fulfilled for derivatives, but when we start with an algebroid representation from a different context, the necessary assumption might be hard to prove. Thus it is worthwhile to try and find other conditions which imply integrability and are more handy. This is what I will engage in during the remaining section of this chapter.

7.5. Integrability and Analytic Vectors

In the construction of an integration theorem for the Euclidean space, we have used analytic vectors to deduce skew-adjointness of vector fields and exponential relations from the mere fact that a Laplace operator acts by an essentially self-adjoint operator. I originally planned to do the same for the more general integration theorem. This failed because Nelson's theory only applies for finite-dimensional Lie algebras and the Lie algebra generated by non-commuting vector fields is usually infinite-dimensional. However, the case of globally commuting vector fields is still an interesting example. In this section, I take the opportunity to investigate analytic vectors once more and show how we can get an integration frame from an essentially self-adjoint Laplacian.

As many times before, we will start showing a few results which seem rather obvious, but should be handled with care in the context of unbounded operators. The first one is a Lemma on sums of skew-symmetric operators.

Lemma 7.5.1. *Let H be a Hilbert space. Let $A, B \in \mathcal{O}(H)$ be densely defined operators. If A is (skew-)symmetric, then its closure $\bar{A}$ is (skew-)symmetric, too. If A and B are both (skew-)symmetric, then so is $A + B : \operatorname{dom} A \cap \operatorname{dom} B \to H$.*

Proof: Let us first look at the symmetric case.

Let $v, w \in \operatorname{dom} \bar{A}$. Then there are sequences $(v_i), (w_i) \subset \operatorname{dom} A$ such that $v_i \to v$, $w_i \to w$ and such that $Av_i \to \bar{A}v$, $Aw_i \to \bar{A}w$. For these we have

$$\langle \bar{A}v, w\rangle = \lim_i \langle Av_i, w_i\rangle = \lim_i \langle v_i, Aw_i\rangle = \langle v, \bar{A}w\rangle$$

because the inner product is continuous and A is symmetric.

Let now $v, w \in \operatorname{dom} A \cap \operatorname{dom} B = \operatorname{dom}(A + B)$. Then we have

$$\langle (A + B)v, w\rangle = \langle Av, w\rangle + \langle Bv, w\rangle = \langle v, Aw\rangle + \langle v, Bw\rangle = \langle v, (A + B)w\rangle$$

because the inner product is bilinear and A and B are symmetric.

For skew-symmetric operators, the result follows by insertion of a minus sign in the above equations or using that multiplying with i makes them symmetric. □

The second lemma is about unbounded commutators.

Lemma 7.5.2. *Let H be a Hilbert space and let $A, B \in \mathcal{O}(H)$ be any symmetric linear operators. Then their commutator $[A, B] = AB - BA : \operatorname{dom}(AB) \cap \operatorname{dom}(BA) \to H$ is skew-symmetric.*

Proof: Let $v, w \in \operatorname{dom}(BA) \cap \operatorname{dom}(AB)$ be arbitrary. Then we have:

$$\begin{aligned} \langle [A, B]v, w\rangle &= \langle ABv - BAv, w\rangle = \langle ABv, w\rangle - \langle BAv, w\rangle \\ &= \langle Bv, Aw\rangle - \langle Av, BW\rangle = \langle v, BAw\rangle - \langle v, ABw\rangle \\ &= \langle v, BAw - ABw\rangle = -\langle v, [A, B]w\rangle \end{aligned}$$ □

The third lemma shows that a closed operator is 0 on its domain if it contains the zero operator.

Lemma 7.5.3. *Let X be a Banach space and let $A, B \in \mathcal{O}(X)$ be densely defined operators with $A \subseteq B$. If B is closable and $Av = 0$ for all $v \in \operatorname{dom} A$, then $Bw = 0$ for all $w \in \operatorname{dom} B$.*

Proof: Let $v \in \operatorname{dom} B$ be arbitrary. As A is densely defined, choose a net $(v_i) \subset \operatorname{dom} A$ with $v_i \to v$. Since A is 0 on its domain and $A \subseteq B$, we have $Bv_i = Av_i = 0$, and hence $\lim_i Bv_i = 0$; in particular, the sequence converges. Because B is closable and $v_i \to v$, this implies that $Bv = \lim_i Bv = \lim_i Bv_i = 0$. □

Next we give a condition for closability of products.

LEMMA 7.5.4. *Let H be a Hilbert space, $A, B \in \mathcal{O}(H)$ densely defined linear operators such that $\operatorname{dom}(B^*A^*) \subseteq H$ is dense. Then AB is a closable operator.*

Proof: Let $v \in H$ be arbitrary. Let $(v_i) \subset \operatorname{dom} AB$ be any sequence with $v_i \to v$ such that (ABv_i) converges. Then we have

$$\begin{aligned} \langle \lim_i ABv_i, w\rangle &= \lim_i \langle ABv_i, w\rangle = \lim_i \langle Bv_i, A^*w\rangle \\ &= \lim_i \langle v_i, B^*A^*w\rangle = \langle v, B^*A^*w\rangle \end{aligned}$$

for all $w \in \operatorname{dom}(B^*A^*)$. Since $\operatorname{dom}(B^*A^*)$ is dense, this uniquely determines $\lim_i ABv_i$. So AB is closable. □

We use the previous facts to show that if vector fields commute locally, then their closures in a representation also commute locally, in the following formal sense:

LEMMA 7.5.5. *Let M be a smooth manifold with a volume form ω and $U \subseteq M$ open. Let $H \to M$ be a ν-Hilbert field, where ν is quasi-invariant. Let $R : \operatorname{Diff}(M) \to \mathcal{O}(L^2H)$ be a representation. Choose $m \in \mathbb{N}$ and let $X_1, \dots, X_m \in \mathfrak{X}(M)$ be vector fields such that $[X_i, X_j]|_U \equiv 0$ for all $i, j \in \{1, \dots, m\}$.*

Let $h \in C^\infty(M)$ with $h|_{M\setminus U} \equiv 0$ and denote by $T_h \in \mathbb{B}(L^2H)$ its multiplication operator. Let $B_i := \overline{R(X_i + \frac{1}{2}\operatorname{div} X_i)}$ be the closures of the operators given by coordinate vector fields. Choose $k \in \mathbb{N}$ and $i_1, \dots, i_k \in \{1, \dots, m\}$. Then for every permutation $\sigma \in S_k$ and all $v \in \operatorname{dom}(B_{i_1} \dots B_{i_k}) \cap \operatorname{dom}(B_{i_{\sigma 1}} \dots B_{i_{\sigma k}})$, we have:

$$T_h B_{i_1} \dots B_{i_k} v = T_h B_{i_{\sigma 1}} \dots B_{i_{\sigma k}} v$$

Proof: For each $i \in \{1, \dots, m\}$, put $A_i := R(X_i + \frac{1}{2}\operatorname{div} X_i) \in \mathcal{O}(H)$, which is densely defined on $\operatorname{dom} R$ and symmetric. For all $i, j \in \{1, \dots, m\}$, we have $[X_i, X_j]|_U = 0$ by assumption. By Lemma 3.2.17, we thus also have $X_i(\operatorname{div} X_j)|_U = X_j(\operatorname{div} X_i)|_U$, and hence $[X_i + \frac{1}{2}\operatorname{div} X_i, X_j + \frac{1}{2}\operatorname{div} X_j]|_U = \Big([X_i, X_j] + \frac{1}{2}X_i(\operatorname{div} X_j) - \frac{1}{2}X_j(\operatorname{div} X_i)\Big)|_U = 0$. By an induction argument, this implies that even for all $i_1, \dots, i_k \in \{1, \dots, m\}$ and $\sigma \in S_k$, we have $\tilde{X}_{i_1} \dots \tilde{X}_{i_k}|_U = \tilde{X}_{i_{\sigma 1}} \dots \tilde{X}_{i_{\sigma k}}|_U$, where $\tilde{X} = \mathcal{L}_X + \frac{1}{2}m_{\operatorname{div} X} \in \operatorname{Diff}(M)$.

So let $h \in C^\infty(M)$ be any smooth function with $h(p) = 0$ for all $p \in M\setminus U$. Then because differential operators are local, we have $h(\tilde{X_{i_1}} \dots \tilde{X_{i_k}} - \tilde{X_{i_{\sigma 1}}} \dots \tilde{X_{i_{\sigma k}}}) = 0 \in \operatorname{Diff}(M)$.

Consequently, we get that

$$T_h(A_{i_1} \dots A_{i_k} - A_{i_{\sigma 1}} \dots A_{i_{\sigma k}}) = R(h(\tilde{X}_{i_1} \dots \tilde{X}_{i_k} - \tilde{X}_{i_{\sigma 1}} \dots \tilde{X}_{i_{\sigma k}})) = 0$$

on its domain $\operatorname{dom} R$, which is dense.

Recall the notation $B_i = \overline{A_i}$. Because R has an invariant domain, we know that

$$\operatorname{dom} R \subseteq \operatorname{dom}(B_{i_1} \dots B_{i_k}) \cap \operatorname{dom}(B_{i_{\sigma 1}} \dots B_{i_{\sigma k}}),$$

so that

$$T_h(A_{i_1} \dots A_{i_k} - A_{i_{\sigma 1}} \dots A_{i_{\sigma k}}) \subseteq T_h(B_{i_1} \dots B_{i_k} - B_{i_{\sigma 1}} \dots B_{i_{\sigma k}}).$$

Also by the domain invariance of R, we know that

$$T_h \operatorname{dom} R \subseteq \operatorname{dom} R \subseteq \operatorname{dom}(B_{i_1} \dots B_{i_k} - B_{i_{\sigma 1}} \dots B_{i_{\sigma k}})^*.$$

Hence, as $T_h = T_h^*$:

$$\operatorname{dom} R \subseteq \operatorname{dom}((B_{i_1} \dots B_{i_k} - B_{i_{\sigma 1}} \dots B_{i_{\sigma k}})^* T_h^*)$$

In particular, this domain is dense. Lemma 7.5.4 thus implies that

$$T_h(B_{i_1}\dots B_{i_k} - B_{i_{\sigma 1}}\dots B_{i_{\sigma k}})$$

is a closable operator. Now Lemma 7.5.3 applies and shows that

$$T_h(B_{i_1}\dots B_{i_k} - B_{i_{\sigma 1}}\dots B_{i_{\sigma k}}) = 0$$

on its domain because

$$0|_{\operatorname{dom} R} = T_h(A_{i_1}\dots A_{i_k} - A_{i_{\sigma 1}}\dots A_{i_{\sigma k}}) \subseteq T_h(B_{i_1}\dots B_{i_k} - B_{i_{\sigma 1}}\dots B_{i_{\sigma k}}).$$

So indeed, for all

$$v \in \operatorname{dom}(T_h(B_{i_1}\dots B_{i_k} - B_{i_{\sigma 1}}\dots B_{i_{\sigma k}})) = \operatorname{dom}(B_{i_1}\dots B_{i_k}) \cap \operatorname{dom}(B_{i_{\sigma 1}}\dots B_{i_{\sigma k}}),$$

we see that $0 = T_h(B_{i_1}\dots B_{i_k} - B_{i_{\sigma 1}}\dots B_{i_{\sigma k}})v$, i.e., $T_h B_{i_1}\dots B_{i_k} v = T_h B_{i_{\sigma 1}}\dots B_{i_{\sigma k}} v$. □

We get to a more elaborate proposition now. Using the existence of common analytic vectors, we will show that certain exponentials commute locally. Namely:

PROPOSITION 7.5.6. *Let M be a smooth manifold with a volume form ω and $U \subseteq M$ open. Let $H \to M$ be a ν-Hilbert field, where ν is quasi-invariant. Let $R : \operatorname{Diff}(M) \to \mathcal{O}(L^2 H)$ be a representation. Choose $m \in \mathbb{N}$ and let $X_1, \dots, X_m \in \mathfrak{X}(M)$ be vector fields such that $[X_i, X_j]|_U \equiv 0$ for all $i, j \in \{1, \dots, m\}$.*

Set $B_i := \overline{R(X_i + \frac{1}{2}\operatorname{div}(X_i))} \in \mathcal{O}(L^2 H)$. For $x \in \mathbb{R}^m$, put $xB := \sum_{i=1}^m x_i B_i$. Set

$$E := \bigcap_{k\in\mathbb{N},\ 1\le i_1,\dots,i_k\le m} \operatorname{dom}(B_{i_1}\dots B_{i_k}) \supseteq \operatorname{dom} R$$

Suppose that there are $s \in \mathbb{R}_{>0}$ and a set $E^\omega \subseteq E$ which is dense in $L^2 H$ such that for all $v \in E^\omega$:

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \sum_{1\le i_1,\dots,i_n\le m} \|B_{i_1}\dots B_{i_n}(v)\| < \infty$$

Then for all $h \in C_b^\infty(M)$ with $h|_{M\setminus U} \equiv 0$ and all $x, y \in \mathbb{R}^m$ with $\|x\|_\infty, \|y\|_\infty \le \frac{s}{2}$, we have:

$$T_h \mathrm{e}^{\overline{xB}} \mathrm{e}^{\overline{yB}} = T_h \mathrm{e}^{\overline{(x+y)B}}$$

Proof: For all $x, y \in \mathbb{R}$ with $\|x\|_\infty, \|y\|_\infty \le \frac{s}{2}$ and $v \in E^\omega$, we know by our assumption that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n!}\|((x+y)B)^n v\| &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\| \sum_{1\le i_1,\dots,i_n\le m} (x_{i_1}+y_{i_1})\dots(x_{i_m}+y_{i_m}) B_{i_1}\dots B_{i_m} v \right\| \\
&\le \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1\le i_1,\dots,i_n\le m} |(x_{i_1}+y_{i_1})\dots(x_{i_m}+y_{i_m})|\, \|B_{i_1}\dots B_{i_m} v\| \\
&\le \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1\le i_1,\dots,i_n\le m} (|x_{i_1}|+|y_{i_1}|)\dots(|x_{i_m}|+|y_{i_m}|)\, \|B_{i_1}\dots B_{i_m} v\| \\
&\le \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1\le i_1,\dots,i_n\le m} s^n\, \|B_{i_1}\dots B_{i_m} v\| < \infty,
\end{aligned}$$

because $|x_i|, |y_i| < \frac{s}{2}$ for all i. In particular, $(x+y)B$ is essentially self-adjoint by Nelson's theorem, which ensures that the exponentials in this lemma exist to begin with.

Similarly, we get the following estimate:

$$\sum_{k,l=0}^{\infty} \frac{1}{k!l!}\|(xB)^k(yB)^l v\| = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+l=n} \frac{n!}{k!l!}\|(xB)^k(xB)^l v\|$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+l=n} \binom{n}{k} \sum_{1\leq i_1,\dots,i_k\leq m} \sum_{1\leq j_1,\dots,j_l\leq m} |x_{i_1}\dots x_{i_k} y_{j_1}\dots y_{j_l}| \|B_{i_1}\dots B_{i_k} B_{j_1}\dots B_{j_l} v\|$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+l=n} \binom{n}{k} \sum_{1\leq i_1,\dots,i_k\leq m} \sum_{1\leq j_1,\dots,j_l\leq m} \left(\frac{s}{2}\right)^n \|B_{i_1}\dots B_{i_k} B_{j_1}\dots B_{j_l} v\|$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{s}{2}\right)^n \sum_{k+l=n} \binom{n}{k} \sum_{1\leq i_1,\dots,i_n\leq m} \|B_{i_1}\dots B_{i_n} v\|$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{s}{2}\right)^n 2^n \sum_{1\leq i_1,\dots,i_n\leq m} \|B_{i_1}\dots B_{i_n} v\| = \sum_{n=0}^{\infty} \frac{s^n}{n!} \sum_{1\leq i_1,\dots,i_n\leq m} \|B_{i_1}\dots B_{i_n} v\| < \infty$$

So the two series $\sum_{n=0}^{\infty} \frac{1}{n}((x+y)B)^n v$ and $\sum_{k,l=0}^{\infty} \frac{1}{k!l!}(xB)^k(yB)^l v$ are absolutely convergent. Hence we can use them to express the corresponding exponentials. That is, we have

$$\mathrm{e}^{\overline{(x+y)B}} v = \sum_{n=0}^{\infty} \frac{1}{n}((x+y)B)^n v$$

and

$$\mathrm{e}^{\overline{xB}}\mathrm{e}^{\overline{yB}} v = \sum_{k,l=0}^{\infty} \frac{1}{k!l!}(xB)^k(yB)^l v$$

for all $v \in E^\omega$.

Now let $h \in C_b^\infty(M)$ with $h|_{M\setminus U} \equiv 0$ be arbitrary. Recall that the multiplication operator $T_h : L^2H \to L^2H$, $\sigma \mapsto h\sigma$ is bounded with $\|T_h\| = \|h\|_\infty$, and in particular, continuous. Furthermore, we know by Lemma 7.5.5 that

$$\begin{aligned} T_h yB(xB)^n v &= \sum_{1\leq j,i_1,\dots,i_n\leq m} y_j x_{i_1}\dots x_{i_n} T_h B_j B_{i_1}\dots B_{i_n} v \\ &= \sum_{1\leq j,i_1,\dots,i_n\leq m} y_j x_{i_1}\dots x_{i_n} T_h B_{i_1}\dots B_{i_n} B_j v = T_h(xB)^n yBv, \end{aligned}$$

so that a version of the binomial theorem applies and yields

$$T_h(xB+yB)^n v = \sum_{k+l=n} \frac{n}{k!l!} T_h(xB)^k(yB)^l v.$$

Thus we have:

$$\begin{aligned} T_h \mathrm{e}^{\overline{xB}}\mathrm{e}^{\overline{yB}} v &= \sum_{k,l=0}^{\infty} \frac{1}{k!l!} T_h(xB)^k(yB)^l v \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k+l=n} \frac{n!}{k!l!} T_h(xB)^k(yB)^l v \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} T_h(xB+yB)^n v = T_h \mathrm{e}^{\overline{(x+y)B}} v \end{aligned}$$

Because this is true for all $v \in E^\omega$, $E^\omega \subseteq H$ is dense, and both $T_h \mathrm{e}^{\overline{xB}}\mathrm{e}^{\overline{yB}}$ and $T_h \mathrm{e}^{\overline{(x+y)B}}$ are bounded, this implies that indeed $T_h \mathrm{e}^{\overline{xB}}\mathrm{e}^{\overline{yB}} = T_h \mathrm{e}^{\overline{(x+y)B}}$. $\square$

An experienced reader may notice at this point that the bump function h in the previous proposition may be supported on the whole subset U where the X_i commute, unlike in previous cases where we needed a compact subset of U as support. This correlates with the intuition that common analytic vectors are something we cannot expect in general.

But for our special case, we can now show that our regarded frame is an integration frame.

LEMMA 7.5.7. *Let M be a smooth manifold with a volume form ω and $U \subseteq M$ open. Let $H \to M$ be a ν-Hilbert field, where ν is quasi-invariant. Let $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ be a representation. Choose $m \in \mathbb{N}$ and let $X_1, \ldots, X_m \in \mathfrak{X}(M)$ be vector fields such that $[X_i, X_j]|_U \equiv 0$ for all $i, j \in \{1, \ldots, m\}$ and such that $(X_1|_U, \ldots, X_m|_U)$ is a local frame of TM.*

Set $B_i := \overline{R(X_i + \frac{1}{2}\operatorname{div}(X_i))} \in \mathcal{O}(L^2H)$. For $x \in \mathbb{R}^m$, put $xB := \sum_{i=1}^m x_i B_i$. Set

$$E := \bigcap_{k\in\mathbb{N},\ 1\le i_1,\ldots,i_k\le m} \operatorname{dom}(B_{i_1}\ldots B_{i_k}) \supseteq \operatorname{dom} R$$

Suppose that there are $s \in \mathbb{R}_{>0}$ and a set $E^\omega \subseteq E$ which is dense in L^2H such that for all $v \in E^\omega$:

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \sum_{1\le i_1,\ldots,i_n\le m} \|B_{i_1}\ldots B_{i_n}(v)\| < \infty$$

Then $(X_1, \ldots, X_m)$ is an integration frame for R on U.

Proof: The facts that $[X_i, X_j]|_U \equiv 0$ for all $i, j \in \{1, \ldots, m\}$ and $(X_1(p), \ldots, X_m(p))$ is an (ordered) basis for T_pM for all $p \in U$ were included in our assumptions. Let $x \in \mathbb{R}^m$ be arbitrary. Choose $c \in \mathbb{R} > 0$ such that $\|\frac{x}{c}\| < 1$. Then from our assumption it follows instantly that every $v \in E^\omega$ is a fortiori an analytic vector for $\frac{x}{c}B$, thus $\frac{x}{c}B$ must be essentially skew-adjoint by Nelson's theorem. Hence $xB = c\frac{x}{c}B$ is essentially skew-adjoint. $T_h\overline{e^{xB}}\,\overline{e^{yB}} = T_h\overline{e^{(x+y)B}}$ was shown in Lemma 7.5.6 to be true for $\|x\|_\infty, \|y\|_\infty \le \frac{s}{2}$. Since all norms on $\mathbb{R}^m$ are equivalent, the result follows. $\square$

Using a self-adjoint Laplacian, we can also formulate the following concise proposition, which is in line with our Euclidean integration theory.

PROPOSITION 7.5.8. *Let (M, ω) be a volumetric manifold and $R : \mathrm{Diff}(M) \to \mathcal{O}(L^2H)$ be a representation. Suppose that there exists a global commuting frame $(X_1, \ldots, X_m)$ of TM, and that $R(\sum_{i=1}^m X_i + \frac{1}{2}\operatorname{div}(X_i))$ is essentially self-adjoint. Then $(X_1, \ldots, X_m)$ is an integration frame on M.*

Proof: Like in the previous lemma, let $B_i = \overline{R(X_i + \frac{1}{2}\operatorname{div} X)}$ and

$$E := \bigcap_{k\in\mathbb{N},\ 1\le i_1,\ldots,i_k\le m} \operatorname{dom}(B_{i_1}\ldots B_{i_k}) \supseteq \operatorname{dom} R.$$

The operators $A_i = R(X_i + \frac{1}{2}\operatorname{div} X_i)$ commute, thus the Lie algebra generated by them is finite-dimensional. Put $\Delta = \sum_{i=1}^m R(X_i + \frac{1}{2}\operatorname{div} X_i)^2$. By Lemma 6.2 in [**18**], $|\Delta| + |I|$ analytically dominates $\sum_{i=1}^m |A_i|$, where I is the identity on $\operatorname{dom} R$. Thus by Lemma 5.2, page [**18**], there are an $s > 0$ and a subset $E^\omega \subseteq E$ such that for all $v \in E^\omega$,

$$\|e^{s\sum_{i=1}^m |B_i|_E|}v\| = \sum_{n=0}^{\infty} \frac{s^n}{n!} \sum_{1\le i_1,\ldots,i_n\le m} \|B_{i_1}\ldots B_{i_n}(v)\| < \infty,$$

and $E^\omega \subseteq L^2H$ is dense. Thus Lemma 7.5.7 applies, showing that $(X_1, \ldots, X_m)$ is an integration frame. $\square$

CHAPTER 8

Conclusion

8.1. Discussion of Results

Behind us lies a large amount of dense mathematical statements, including new terms and theories developed on our way. Before the end of this book, let us take a moment and reflect on our achievements. In the beginning of the thesis, I have already given a short summary of all the contents and I will not repeat it here. Instead I will use the opportunity to highlight a few of our most important findings and discuss their extent as well as their limits.

A personal favourite of mine is the Sausage Theorem 5.3.10, which allows us to complete local groupoid homomorphisms to global ones. One reason for this status is the fact that it can be applied in quite a general context: The result is independent of representation theory and may be applied in other areas. The target of the homomorphism which we want to extend does not even need any additional structure like topology or measures. Another advantage is that the statement resembles a result from classical Lie theory and is easy to understand despite its technical depth. On the flip side, our theorem is only applicable for a domain of definition which has simply connected fibres as well as suitable measures and a useful topology. The formal definition of a local groupoid homomorphism is, involving null sets, also quite technical. Still the Sausage Theorem has proven useful many times and is an important cornerstone of the Lie groupoid representation theory.

Regarding the actual representation theory, we have three types of representations and three theorems mapping between them. First there is the differentiation theorem 4.1.5, which differentiates representations of the groupoid algebra to representations of the algebroid. Secondly, we have Theorem 6.2.2, which associates a groupoid algebra representation to a groupoid representation. Likely the most important theorem in this monograph is the integration theorem 7.3.4, which maps algebroid to groupoid representations.

Between integration and differentiation, there is a notable gap in the amount of formal requirements: Every non-degenerate representation of the groupoid C^*-algebra can be differentiated to a representation of the algebroid. This works for an arbitrary Lie groupoid and even if the domain of the representation is a Hilbert C^*-module instead of an ordinary Hilbert space. In contrast, our integration theorem requires that the domain is a Hilbert space, because the disintegration into measurable fields of Hilbert spaces is only possible for those. And while many intermediate results are true for general Lie groupoids (such as Theorem 6.1.4, in the end we restricted to the case of pair groupoids. In addition to that, we need simply connected fibres to extend local homomorphisms.

These limitations may be viewed as inspirations for further research. The most promising direction may be the generalisation to more general Lie groupoids. With the current knowledge, this will not go very far because the exponential map for Lie algebroids does not have the desired properties. However, with a better understanding of this map, also in the context of functional analysis, it might still be possible to prove analogous results using methods similar to those presented in this dissertation. Most of the constructions work in principle, and the challenge is to show that the map constructed is a homomorphism.

Alleviation of the other restrictions seems less likely. I do not know how to disintegrate Hilbert modules. Maybe methods of their representation theory such as the local-global principle from [**16**] can be used to obtain some results. Having simply connected fibres is also not a requirement that we can dismiss, because it is necessary in the extension process of local homomorphisms just like in classical Lie theory.

Let us summarize. Due to technical obstructions, we have not completed our most ambitious original goal of building an integration theory for all Lie algebroid representations. However, we were able to obtain a satisfying result for the more confined case of the tangent algebroid. We have constructed an integration theorem for tangent algebroid representations on Hilbert spaces, shown that it is inverse to differentiation and defined understandable integrability conditions. With these results I close this thesis. Any motivated researcher who wishes to do so is hereby invited to continue investigations on the topic.

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