The Spectra on Lie Groups and Its Application to twisted L^2-Invariants

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To Xiaobei Jin and Zhenbao Han,
with deepest love and gratitude.

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Danksagung

> Each great work, in mathematics as in other forms of creativity, has its own uniqueness. But I believe it is accurate to say that no mathematician of our time has successfully completed so long and so arduous a climb, solo... As long as group representation theory continues with remarkable vitality, his spirit will hover here. And that will continue for a long, long time.
>
> George Mostow, on Harish-Chandra

The thesis marks the end of my four years in Göttingen. They were not always pretty days, as were possibly every other doctoral studies. But I hold dear every memory I shared with this place, more so with the people I acquainted and befriended here.

My deepest gratitude goes to my advisor, Thomas Schick, without whose support and encouragement I could hardly imagine myself to have come this far. His enthusiasm, passion and vigour for mathematics, as well as for other aspects of life, will always be a place from where I seek inspiration in the future. It would be beyond words and spaces to express every bit for which I am indebted to him, so these merriest memories will be kept to myself.

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2023 is the centenary of Harish-Chandra's birth and the 40th anniversary of his premature departure. The jewel of his work radiates with an unwavering brilliance, illuminating the path of knowledge for generations to come. I hope that this thesis would spell my reverences for him.

Z.C. Han
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Chapter 0

Introduction

This introduction provides historical backgrounds and motivation for spectral analysis on Lie groups and gives an overview of five main theorems we are about to prove in this thesis. We also discuss some open problems and future directions.

Motivation

The theory of the heat equation was first formulated by Joseph Fourier in 1807 in his submission of manuscript of *Théorie analytique de la Porpagation de la Chaleur dans les Solides* to the French Academy. He used the form of trigonometric series to present its solutions. In hindsight, this, alongside many other developments in the 19th century, can be seen as evidence of the power of Fourier analysis and its earlier success in the theory of differential equation.

To put in simple modern mathematics terms, we consider the heat equation on $\mathbb{R}^n$. Given a function $f : \mathbb{R}^n \to \mathbb{C}$, one would like to know the answer to which function u that solves the following equation:

$$\begin{cases} u(x,0) = f(x) \\ \frac{\partial}{\partial t} u(x,t) = -\Delta_x u(x,t) \end{cases} \tag{1}$$

with $-\Delta_x = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ the heat diffusion in all directions. If we now apply the classical Fourier transform:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$$

the heat equation under Fourier transform becomes: $\frac{\partial}{\partial t}\widehat{u}(x,t) = \sum_{i=1}^n \xi_i^2 \widehat{u}(\xi,t)$, and then the equation becomes a simple ordinary differential equation from freshman calculus. To retrieve our solution u in the phase space, we need also apply inverse Fourier transform $\check{u}$.

The earlier developments of trigonometric series was regarded by, amongst others, Euler and Lagrange with suspicion, due to the problem of convergence. For instance, an attempt to solve the above equation with initial condition $f = 1$ the constant function produces a sum of trigonometric series, yet this series is hardly convergent in any sense. For this reason, Fourier's work was not immediately accepted by the mathematical community. It was not until the work of Dirichlet and Riemann in the 19th century that the theory of Fourier series was put on a rigorous foundation.

Heat kernel and Plancherel formula

The heat equation later sees its many application to geometry in the 20th century. Its vastness is beyond the account of this thesis, but we want to stress in particular that the spectrum of heat operator carries important geometric and topological information. We shall further restrict our attention to the spectral theory of manifolds which admit extra symmetries. Such is the main focus of this thesis.

In a broader sense our approach to the heat problem follows the spirit of the original approach by Fourier – namely we also apply Fourier transform to the function in question and solve the equation in the frequency space. This is made possible by the *Plancherel theorem*, which gives the Fourier expansion of a function on a large class of Lie groups by the characters of its irreducible unitary representation. This is the main focus of the first part of the thesis. In Chapter 2 we collect some key elements of the representation theory and Plancherel formula that goes beyond the original class of Harish-Chandra.

We also share the discretion of Euler and Lagrange. To justify a legit use of Plancherel formula, we need to establish first the fact that the Schwartz kernel of heat operators is rapidly decaying. This entails a two-stage effort: First one is to establish a representation-theoretic formula of Laplacian on a Lie group. The first result is classical in the case of symmetric cases and is new for the differential form Laplacian on the Lie groups. Here comes the first main result of the thesis:

Theorem A (Proposition 1.5 & Corollary 1.7)**.** *Let G be a Lie group. Then its differential form Laplacian admits an expression in adjoint and coadjoint representation. In the case G is a reductive Lie group, this can be reduced to an expression in adjoint representation.*

Having established this, we move on to Chapter 4 to study the Laplacian on the homogeneous spaces via the spectacles of representation theory, and treat all relevant differential-geometric quantities as G-modules and G-equivariant maps. We also discuss the relation between various Laplacians. This concludes the first step.

For the second step, we notice that the Laplacians on Lie groups often entail a first-order perturbation, as opposed to the classical case of symmetric spaces. We hence developed some perturbation estimates for such operators, following the results first by Langlands and Robinson, which establish the kernel estimates for the unperturbed elliptic operators (see Theorem A.4), and then the work of Hille and Phillips, which expanded the perturbed operator as absolutely convergent formal series (see Theorem 5.3). Combining these elements, together with some elements from the case of symmetric spaces we obtain the second main result:

Theorem B (Lemma 5.5 & Theorem 5.8)**.** *Let G be a reductive or nilpotent Lie group. Then the Schwartz kernels associated with its Laplacians are rapidly decaying.*

This establishes the eligibility of applying Fourier decomposition to the heat kernel associated with differential form Laplacians on the Lie group G.

As an application, we apply the Plancherel formula to $\widetilde{SL_2(\mathbb{R})}$, the universal cover of $SL(2,\mathbb{R})$ which is well-known to be a nonlinear group, and later to $\mathbb{H}_3$ the Heisenberg group. We then compute the spectrum of the Laplacian, at all degrees, with the rescaling in metric in sight:

Theorem C (Theorem 6.2, Theorem 6.5 & Section 8.1)**.** *The spectra of the Laplacians on $\widetilde{SL_2(\mathbb{R})}$, with a rescaling in the K-direction, are explicitly given. A closed formula for the heat kernels is also given. A similar formula is also given for the Heisenberg group $\mathbb{H}_3$ with a rescaling of the Z-direction.*

Dirac operators and their spectra

Another main focus of this thesis is on the Dirac operators. The study of the Dirac equation dates back to Paul Dirac in 1922 during his investigation of the hydrogen spectral series. The influence of the Dirac operator was first felt by the mathematics community in the early 1950s, and saw its first triumph in the groundbreaking work of Atiyah and Singer, first announced in 1963.

Stated in modern mathematics terms, we want to study the spectral behavior of the Dirac operator induced by the metric connection on general semisimple Lie groups and nilpotent Lie groups. The strategy resembles that of the heat kernel, namely first by understanding the representation-theoretic expression of the Dirac operator, and then by establishing the Schwartz kernel estimates. The spinor bundle structure in this case is more complicated, as we have sacrificed the bi-invariance enjoyed by the metric on the symmetric spaces.

Following the original approach of Moscovici and Stanton in the case of symmetric spaces, we first lay the ground of constructing the spinor bundle and spin module over the Clifford algebra of Lie groups. It is understood as an intermediate step as a $\mathfrak{k}$ -representation in Proposition 3.10, paving the road eventually to a closed formula for the Dirac operator acting on the invariant spinors:

Theorem C (Proposition 3.10, Section 4.4)**.** *Given a semisimple Lie group G with prescribed metric B^θ, the Dirac operator acting on the spinor bundle $\$_{\mathfrak{g}}$ is given by:*

$$\not{D}_\sigma = \sum_{a \in I_{\mathfrak{g}}} R_{X_a} \otimes c\ell(X_a) + \frac{1}{2} \sum_{i \in I_{\mathfrak{k}}} c\ell(X_i) \sigma_{\mathfrak{g}}(X_i)$$

with the decomposition of $\sigma_{\mathfrak{g}}$ into irreducible $\mathfrak{k}$-representation completely known.

The rest of the proof is essentially a Doppelgänger of the heat kernel proof. we establish the rapid decay property of the spinor Laplacian in this case:

Theorem B (Theorem 5.8)**.** *If G is a reductive Lie group, then the Schwartz kernel of the Dirac operator is of Schwartz class.*

Again we compute the spectra in the case of $\widetilde{SL_2(\mathbb{R})}$ and $\mathbb{H}_3$:

Theorem D (Section 6.3 & Section 8.2)**.** *The spectra of the Dirac operator on $\widetilde{SL_2(\mathbb{R})}$ and $\mathbb{H}_3$ are explicitly given.*

Applications

The last theme of the thesis is the application of the results heretofore to compute L^2-invariants. From a topological point of view, the L^2-invariants are topological or homeomorphism invariants describing various properties of the L^2-cohomology. We though approach the problem from a more analytic point of view. A first result is the computation of Novikov-Shubin invariants in the case of $\widetilde{SL_2(\mathbb{R})}$:

Theorem B' (Theorem 7.18)**.** *Let $\Gamma \subseteq \widetilde{SL_2(\mathbb{R})}$ a uniform lattice, then the Novikov-Shubin invariants are $\alpha_0(\Gamma) = \alpha_3(\Gamma) = \infty^+$ and $\alpha_1(\Gamma) = \alpha_2(\Gamma) = 1$.*

Later we move on to define the twisted L^2-invariants, aiming to encapsulate similar information in L^2-cohomology with coefficients. The definition is given in Section 7.1. We then compute the twisted L^2-Betti numbers of symmetric spaces, following the strategy of Olbrich:

Theorem E (Theorem 7.8)**.** *Given V_ρ a finite-dimensional G-representation with an admissible metric, and Γ a uniform lattice in G. The (twisted) L^2-Betti numbers $b_p^{(2)}(\Gamma; V_\rho)$, the Novikov-Shubin invariants $\alpha_p(\Gamma; V_\rho)$ and the L^2-torsion $\rho_{\mathrm{an}}^{(2)}(\Gamma; V_\rho)$ are explicitly computed.*

We note that the twisted L^2-Betti numbers and L^2-torsion have been known for some time. We computed the twisted Novikov-Shubin invariants in this case, stating the result for the first time. This addresses a question of Lück (see Remark 7.11). Moreover, the constants of previous computations are known to have several minor errors, we have corrected them in this thesis in Remark 7.16.

Relation with other works and future directions

We conclude this introduction by discussing the relations between the results in this thesis and some problems considered in other literature.

The spectral problem of (locally) homogeneous spaces have been earlier studied extensively by Kassel and Kobayashi in [KK20]. There they considered the metric as the Killing form, which is pseudo-Riemannian and bi-G-invariant, and have studied the spectra of the respective Laplacians for various homogeneous spaces beyond the case of symmetric spaces and group manifolds. This entails a detailed study of the restriction problem of representations of G to closed normal subgroups. We on the other hand will focus on the Riemannian metric on G, and meanwhile have sacrificed the bi-invariance. It would be an interesting problem to understand how our current methods on the Laplacian of $\widetilde{SL_2(\mathbb{R})}$ can be extended to a general reductive Lie group of class $\widetilde{\mathcal{H}}$.

The Dirac operators and their index theory have been recently studied in [PPST21] in the context of G-manifolds, where G is a linear reductive group. In their appendix A, computations on the Dirac operators on G were performed independently, which is similar to ours in Section 4.4. We remark that they have omitted the case when the dimensions of $\sigma_{\mathfrak{p}}$ and $\sigma_{\mathfrak{k}} \otimes \sigma_{\mathfrak{p}}$ do not agree. We have in addition given a extra expression for Dirac operators acting on the invariant spinors via representation-theoretic methods. This might be of use for future studies in pertinent areas.

There is abundant literature on the Gaussian upper bounds for the scalar heat kernel on arbitrary manifolds. In the case of heat kernels associated with vector bundles, much less is known. With the methods discussed in Chapter 5 we hope the estimates can be extended to more general elliptic operators on homogeneous spaces, and to establish the rapid decay of their respective kernels, which is a necessary step before one deploys the Plancherel formula to solve spectral problems. We also remark that some of the very crude bounds obtained in this chapter can be readily refined with a careful study of various bounds on the weighted norms and bounded perturbations, as was already partially done in [Rob91] and [HP74]. This can be a potential object for future studies.

Lastly we remind that the twisted L^2-invariants has been studied extensively in the last few years, with a special focus on how the L^2-torsion varies with a twisting in representations of the underlying lattice Γ. We refer the reader to [Lüc18] for relevant discussions from the topological angle, and to [Liu17] and [FL19] for its computation and significance in the case of 3-manifolds. The computations in Chapter 7 can be seen as 'lattice' analogues of the L^2-torsion function, as all our Γ-representations come from the restriction of G-representations. Also we stress more the analytic aspects of the theory. In particular, our computations on the Novikov-Shubin invariants give affirmative answers to [Lüc18, Quesiton 0.2] in the case Γ is a uniform lattice in a linear reductive Lie group G, and V is a G-module that is regarded as a Γ-module.

Overview and prerequisites

Overall we try to keep the content self-contained. Nonetheless, some basic familiarity with certain subjects will enhance understanding of pertinent chapters. Below we list an overview of the chapters and their prerequisites.

1. In Chapter 1 we derives a formula of Hodge Laplacians on $\mathfrak{g}$-modules. This requires little prior knowledge, except for possibly some very basic knowledge of differential forms and Lie algebra cohomology;
2. In Chapter 2 we list all the background on the representation theory of general reductive groups. All necessary notions are defined along the way, though a prior knowledge of the representation theory of linear semisimple Lie groups with finite center will be helpful for the understanding of some subtle points. We encourage the readers to briefly read [Kna86, Chapter 2], in order to get a flavour of the subject matter;
3. Chapter 3 and Chapter 4 should be read side by side: One takes care of the algebraic aspects or Clifford modules and the other tends its geometric implications. Chapter 3 requires little knowledge besides the classical representation theory of Weyl, whereas in Chapter 4 all quantities in differential geometry are transcribed to the language of $\mathfrak{g}$-modules;
4. Chapter 5 deals with the estimates of heat kernels, so some awareness in kernel method are preferred. All the necessary details supplemented in Appendix A.1;
5. In Chapter 6 and Chapter 8 we compute the spectra of various operators on the universal cover of $SL(2,\mathbb{R})$, as well as the Heisenberg group, and everything is spelled out as concretely as possible, hence no prior knowledge is required;
6. Lastly in Chapter 7 we compute various twisted L^2-invariants of symmetric spaces. As we completely neglected the topological aspects of the theory, one should consult [Lüc02] for the iceberg beneath the surface. Also in the computation of twisted L^2-torsions some solid knowledge of the Weyl group is desired, lest one gets lost in the forest of roots and weights. At the back of the thesis we include index lists of glossaries and symbols. We hope this will be of use to the reader.

Chapter 1

Laplacian on differential forms of Lie groups

In this section, we derive a formula for the Hodge Laplacian acting on p-forms of a general Lie group G, and derive its action on the L^2-completion of p-forms, $L^2\Omega^p(G)$. As Proposition 1.5 manifests, the final expression depends only on the adjoint representation and coadjoint action of G, as well as the unitary spectrum. The highlight of this section is Corollary 1.7, where a formula for the Hodge Laplacian on the differential forms of G is derived, expressing solely in terms of adjoint representations. To prove such statement, we explicit the extra existing symmetries in the reductive Lie algebra.

Recall first some basics on the Lie algebra cohomology associated with a Lie group G with the respective Lie algebra $\mathfrak{g}$. Relevant details can be found in [**BW00**, Chapter I]. In particular, fix a $\mathfrak{g}$-module (V, τ) over a field $F = \mathbb{R}$ or $\mathbb{C}$, where τ denotes the $\mathfrak{g}$-module structure of V.

Denote $C^q = C^q(\mathfrak{g}; V) = \mathrm{Hom}_F(\Lambda^q\mathfrak{g}, V)$, with differentials $d : C^q \to C^{q+1}$:

$$\begin{aligned} df(X_0, \cdots, X_q) = &\sum_i (-1)^i \tau(X_i) f(X_0, \cdots, \widehat{X_i}, \cdots, X_q) \\ &+ \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_q) \end{aligned} \tag{1.1}$$

As usual the $\widehat{\cdot}$ stands for omission of the argument. Then $d^2 = 0$ and we denote $H^*(\mathfrak{g}; V)$ the cohomology of the complex. We denote the first sum as $d_\circ$ and the second sum as $d_\wedge$ respectively, that is: $d = d_\circ + d_\wedge$.

Recall also the definition of the exterior multiplication and the contraction operator on $C^*(\mathfrak{g}; \mathbb{C}) \cong \wedge\mathfrak{g}^*$ from classical differential geometry:

$$\begin{aligned} \varepsilon(\omega) &: \omega^1 \wedge \cdots \wedge \omega^p \mapsto \omega \wedge \omega^1 \wedge \cdots \wedge \omega^p \\ \iota(X) &: \omega^1 \wedge \cdots \wedge \omega^p \mapsto \sum_{i=1}^p (-1)^{i+1} \omega^i(X) \omega^1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega^p \end{aligned} \tag{1.2}$$

with $\omega \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. Then we can rewrite $d_\circ$ and $d_\wedge$ using the basis $\{X_0, \cdots, X_n\}$ of $\mathfrak{g}$ with the dual basis $\{\omega^0, \ldots, \omega^n\}$ of $\mathfrak{g}^*$ [**BW00**, I.1.1(7)]:

$$d_\circ = \sum_i \varepsilon(\omega^i) \otimes \tau(X_i) \qquad d_\wedge = \frac{1}{2} \sum_i \varepsilon(\omega^i) \cdot \mathcal{L}_{X_i} \tag{1.3}$$

where $\mathcal{L}_X$ is the Lie derivative, which we can alternatively interpret as $(\tau \otimes \mathrm{coad})(X)$:

$$\mathcal{L}_X f(X_0, \cdots, X_q) = \tau(X) f(X_1, \cdots, X_q) + \sum_i f(X_1, \cdots, [X_i, X], \cdots, X_q) \tag{1.4}$$

Note here as we taken the expression based to $\mathbb{Z}$-pair between vectors and covectors, hence the expressions are formed independent of choices of the metric. Recall lastly the Lie derivative is related with the contraction by Cartan's magic formula:

$$\mathcal{L}_X = d \circ \iota(X) + \iota(X) \circ d \tag{1.5}$$

Fix a Riemannian metric on G, then it induces a natural inner product on $\bigwedge^* \mathfrak{g}^*$. Denote the overall inner product on $C^*(\mathfrak{g}; V)$ as $\langle -, - \rangle$. If we assume the Riemannian metric to be left G-invariant, this indeed induces a left $\mathfrak{g}$-module V with a left G-invariant positive non-degenerate scalar product $\langle -, - \rangle_V$, that is $\langle Xu, v\rangle_V + \langle u, Xv\rangle_V = 0$ for all $u, v \in V$. Often V is taken to be $L^2(G)$ or the tempered series π that occur in the decomposition of G.

The metric also allows one to define an adjoint operator to d. To simplify the notation, we follow the convention of [BW00, §II.1.4]: Denote ω^i the dual basis of $\mathfrak{g}^*$ with respect to the perfect pairing $\omega_i(X_j) = \delta_{ij}$. and put $\omega^J = \omega^{j_1} \wedge \omega^{j_2} \wedge \cdots \wedge \omega^{j_q}$ for a multi-index $J = \{j_1, \cdots, j_q\}$and $J_m = \{1, \cdots, m\}$.

Moreover, we denote the structural constants associated with this Lie algebra as $C^{\gamma}_{\alpha,\beta}$, where:

$$[X_\alpha, X_\beta] = \sum_\gamma C^{\gamma}_{\alpha,\beta} X_\gamma \tag{1.6}$$

and it is immediate from (B.3) that:

$$\operatorname{coad} X_\alpha(\omega^\beta) = \sum_\gamma C^{\beta}_{\gamma,\alpha} \omega^\gamma \tag{1.7}$$

Whenever the index is unspecified, it means that we sum over the whole basis of $\mathfrak{g}$.

If $\eta \in C^q(\mathfrak{g}; V)$ one writes $\eta_J = \eta(X_{j_1}, \cdots, X_{j_q})$ and $\eta = \sum_J \eta_J \cdot \omega^J$. In this way (1.1) can be rewritten for $\eta \in C^q(\mathfrak{g}; V)$ as

$$(d_\circ \eta)_J = \sum_{1 \le u \le q+1} (-1)^{u-1} \tau(X_{j_u}) \cdot \eta_{J(u)} \qquad \text{for } J \subseteq J_m, |J| = q+1$$

where $J(u_1, u_2, \ldots, u_n)$ denotes the J with the u_i entries removed for $i = 1, \ldots, n$. Meanwhile,

$$(d_\wedge \eta)_J = \sum_{1 \le \alpha < \beta \le q+1} (-1)^{\alpha+\beta} \eta_{[j_\alpha, j_\beta] \cup J(j_\alpha, j_\beta)} = \sum_{1 \le \alpha < \beta \le q+1} \sum_j (-1)^{\alpha+\beta} C^{j}_{j_\alpha, j_\beta} \eta_{j \cup J(\alpha,\beta)} \tag{1.8}$$

Definition 1.1. Let $\delta : C^q(\mathfrak{g}; V) \to C^{q-1}(\mathfrak{g}; V)$ be the linear operator adjoint to d:

$$\langle \delta\eta, \mu \rangle = \langle \eta, d\mu \rangle \quad \text{for all } \eta \in C^q(\mathfrak{g}; V), \mu \in C^{q-1}(\mathfrak{g}; V)$$

The summands $d_\circ$ and $d_\wedge$ admit adjoints $\delta_\circ$ and $\delta_\wedge$ respectively. We express them succinctly with contraction operators:

$$\delta_\circ = \sum_i \iota(X_i) \otimes \tau^*(X_i) \qquad \delta_\wedge = \frac{1}{2} \sum_i \iota(X_i) \cdot \mathcal{L}^*_{X_i} \tag{1.9}$$

where $\mathcal{L}^*$ is the dual of the Lie derivative, which we saw in the previous discussion. $\mathcal{L}^*$ acts on $\wedge^* \mathfrak{g} \otimes V$ by $(\operatorname{coad}^* \otimes \tau^*)(X)$ with coad^* the dual coadjoint representation as defined below in Definition 1.3. Express now the adjoint operators in this explicit bases:

Proposition 1.2. *The operator $\delta_\circ$ admits the following expression:*

$$(\delta_\circ \eta)_J = \sum_j \tau(X_j)^* \eta_{\{j\} \cup J} \tag{1.10}$$

where $\tau(X)^*$ is the adjoint of $\tau(X)$ with respect to $\langle -, - \rangle_V$. The operator $\delta_\wedge$ admits the following expression:

$$(\delta_\wedge \eta)_J = \sum_{\alpha < \beta} \sum_{1 \le u \le q-1} (-1)^{u-1} C^{j_u}_{\alpha,\beta} \eta_{\{\alpha,\beta\} \cup J(u)} \tag{1.11}$$

for $\eta \in C^q(\mathfrak{g}; V)$ and $|J| = q - 1$.

Proof. The first statement is a direct adaptation of the argument in [**BW00**, Proposition II.2.3]. As for the second, it suffices to verify the identity $\langle \delta_\wedge \eta, \nu\rangle = \langle \eta, d_\wedge \nu\rangle$ for $\eta \in C^q(\mathfrak{g}; V)$ and $\nu \in C^{q-1}(\mathfrak{g}; V)$. Note:

$$\begin{aligned}\langle \eta, d_\wedge \nu\rangle &= \sum_{|J|=q} \langle \eta_J, (d_\wedge \nu)_J\rangle = \sum_{|J|=q} \langle \eta_J, \sum_{1\le\alpha<\beta\le q} \sum_j (-1)^{\alpha+\beta} C^j_{j_\alpha, j_\beta} \nu_{j\cup J(\alpha,\beta)}\rangle \\ \langle \delta_\wedge \eta, \nu\rangle &= \sum_{|J'|=q-1} \langle (\delta_\wedge \eta)_{J'}, \nu_{J'}\rangle = \sum_{|J'|=q-1} \langle \sum_{\alpha<\beta} \sum_{1\le u\le q-1} (-1)^{u-1} C^{j'_u}_{\alpha,\beta} \eta_{\{\alpha,\beta\}\cup J'(u)}, \nu_{J'}\rangle\end{aligned} \tag{1.12}$$

For fixed index set J of η, in the second expression, the index of ν associated with η_J are:

$$\bigcup_{\alpha,\beta} \left\{ J' \mid |J'| = q-1 \quad \{\alpha,\beta\}\cup J'(u) = J \quad \text{for some } 1 \le u \le q-1 \right\}$$

For fixed J, this is:

$$\begin{aligned}&\bigcup_{1\le u\le q-1} \bigcup_{\alpha,\beta} \bigcup_{1\le\alpha'<\beta'\le q} \left\{ J' \mid j'_{\alpha'} = \alpha, j'_{\beta'} = \beta, \{\alpha,\beta\}\cup J'(u) = J \right\} \\ =& \bigcup_{1\le u\le q-1} \bigcup_{1\le\alpha'<\beta'\le q} \left\{ J' \mid J'(u) = J(\alpha',\beta') \right\} \\ =& \bigcup_{j} \bigcup_{1\le\alpha,\beta\le q} \left\{ J' \mid J' = j\cup J(\alpha,\beta) \right\}\end{aligned}$$

which is precisely the index set in the first expression. Because $C^j_{\alpha,\beta} = -C^j_{\beta,\alpha}$ and $C^*_{\alpha,\alpha} = 0$ by the property of the Lie bracket, together with the following fact:

$$\eta_J = (-1)^{\alpha-1+\beta-2} \eta_{j_\alpha, j_\beta, j_1, \cdots, \widehat{j_\alpha}, \cdots, \widehat{j_\beta}, \cdots, j_q} = (-1)^{\alpha+\beta-1} \eta_{\{j_\alpha, j_\beta\}\cup J(j_\alpha, j_\beta)}$$

the claim is therefore proven. □

Next we express the Hodge Laplacian $\Delta := \delta \circ d + d \circ \delta$ in four parts:

$$\Delta = (d_\circ + d_\wedge)(\delta_\circ + \delta_\wedge) + (\delta_\circ + \delta_\wedge)(d_\circ + d_\wedge) = \Delta_\circ + \Delta_\wedge + \Delta_{\circ,\wedge} + \Delta_{\wedge.\circ} \tag{1.13}$$

where:

$$\Delta_\circ = d_\circ \delta_\circ + \delta_\circ d_\circ \qquad\qquad \Delta_\wedge = d_\wedge \delta_\wedge + \delta_\wedge d_\wedge \tag{1.14}$$

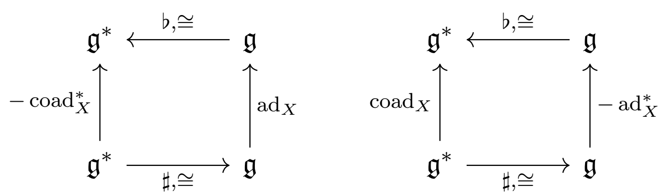
$$\Delta_{\circ,\wedge} = d_\circ \delta_\wedge + \delta_\wedge d_\circ \qquad\qquad \Delta_{\wedge,\circ} = d_\wedge \delta_\circ + \delta_\circ d_\wedge \tag{1.15}$$

The next goal is to express each of the four parts in reasonably computable terms. We begin by defining the dual representation of the adjoint representation on $\bigwedge^* \mathfrak{g}^*$ with respect to the prescribed inner product:

Definition 1.3 (coad*-**representation**)**.** Given a Lie algebra $\mathfrak{g}$ with the prescribed non-degenerate bilinear form $\langle -, -\rangle_{\mathfrak{g}}$, which induces a scalar product $(-,-)_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$. We define the **coadjoint* representation** such that for all $X \in \mathfrak{g}$:

$$((\operatorname{coad}^* X)l, l')_{\mathfrak{g}^*} = (l, (\operatorname{coad} X)l')_{\mathfrak{g}^*} \tag{1.16}$$

Remark 1.4. With a fixed choice of positive-definite bilinear form on $\mathfrak{g}$ (hence on $\mathfrak{g}^*$), the following diagrams commute



for all $X \in \mathfrak{g}$. Here $\flat$ and $\sharp$ are the musical isomorphisms and the diagram can be checked easily by invoking bases.

From now onwards we assume that X_is form an orthonormal basis of $\mathfrak{g}$ with respect to the prescribed inner product. For convenience, we denote $\overline{\Omega}_G = \sum_i X_i^2 \in U(\mathfrak{g}_{\mathbb{C}})$. Moreover, we keep our expressions simple at the cost of slight abusing the following notation:

$$\operatorname{coad}(X_i)(\eta_J) := (\operatorname{coad}(X_i)(\eta_J\omega^J))_J = (\eta_J \operatorname{coad}_{X_i}(\omega^J))_J = \sum_{1\leq u\leq q}\sum_{j\in I_{\mathfrak{g}}}(-1)^{u-1}C_{j,i}^{j_u}\eta_{j\cup J(u)} \tag{1.17}$$

where the last identity comes from the expression (1.7).

Proposition 1.5. *Given a unitary $\mathfrak{g}$-module (V,τ) and the chain complex $C^q(\mathfrak{g};V)$ as above, then the Laplacian admits the following representations:*

1. *$\Delta_\circ$ acts on $C^q(\mathfrak{g};V)$ as:*

$$(\Delta_\circ\eta)_J = -\sum_j \tau(X_j)^2\eta_J + \sum_a \tau(X_a)\sum_{1\leq u\leq q}(-1)^u C_{j,j_u}^a \eta_{\{j\}\cup J(u)} \tag{1.18}$$

2. *$\Delta_\wedge$ acts on $C^q(\mathfrak{g};V)$ as:*

$$(\Delta_\wedge\eta)_J = \sum_j\sum_{\alpha<\beta} C_{\alpha,\beta}^j\left(\sum_{1\leq\gamma\leq q}(-1)^\gamma C_{\alpha,\beta}^{j_\gamma}\eta_{j\cup J(\gamma)} + \sum_{1\leq u<v\leq q}(-1)^{u+v}C_{j_u,j_v}^j\eta_{\{\alpha,\beta\}\cup J(u,v)}\right) \tag{1.19}$$

3. *$\Delta_{\circ,\wedge}$ acts on $C^q(\mathfrak{g};V)$ as $\sum_k \tau(X_k)^* \operatorname{coad}(X_k)$;*
4. *$\Delta_{\wedge,\circ}$ acts on $C^q(\mathfrak{g};V)$ as $-\sum_k \tau(X_k)\operatorname{coad}^*(X_k)$;*

In particular, the 0-th Laplacian on $C^0(\mathfrak{g};V)$ is seen to take the form:

$$\Delta_0 = -\tau(\overline{\Omega}_G) \tag{1.20}$$

Proof. Given $\eta \in C^q(\mathfrak{g};V)$ and $|J| = q$.

1. For the first statement, we follow the argument of [**BW00**, Theorem II.2.5(i)], and using Proposition 1.2:

$$(\Delta_\circ\eta)_J = \sum_j \tau(X_j)^*\tau(X_j)\eta_J + \sum_{1\leq u\leq q}\sum_j(-1)^{u-1}[\tau(X_{j_u}),\tau(X_j)^*]\eta_{j\cup J(u)}$$

Here the first term is $-\sum_k \tau(X_k)^2\cdot\eta_J$ as $\tau(X)^* = -\tau(X)$; whereas in the second term is:

$$[\tau(X_{j_u}),\tau(X_j)^*] = -\sum_a C_{j,j_u}^a(\tau(X_a))$$

Hence:

$$\sum_{1\leq u\leq q}\sum_j(-1)^{u-1}[\tau(X_{j_u}),\tau(X_j)^*]\eta_{j\cup J(u)} = \sum_a -\tau(X_a)\left(\sum_{\substack{j\\1\leq u\leq q}}(-1)^{u-1}C_{j,j_u}^a\eta_{j\cup J(u)}\right)$$

hence the (1.18) is verified. In the case of $C^0(\mathfrak{g};V)$ one sees $\delta_\circ$ and $\delta_\wedge d_\wedge$ all vanish on 0-forms, we see only the τ-terms survive:

$$\Delta_0 f = \Delta_{0,\circ}f = \delta_{0,\circ}d_{0,\circ} = \sum_i \tau^*(X_i)\tau(X_i) = -\sum_i\tau(X_i^2) = -\tau(\overline{\Omega}_G)$$

2. To prove the second claim:

$$\begin{aligned}(d_\wedge \delta_\wedge \eta)_J &= \sum_{1\le u<v\le q} \sum_j (-1)^{u+v} C^j_{j_u,j_v} (\delta_\wedge \eta)_{j\cup J(u,v)} \\ &= \sum_{1\le u<v\le q} \sum_j (-1)^{u+v} C^j_{j_u,j_v} \sum_{\alpha,\beta} \Bigg(C^j_{\alpha,\beta} \eta_{\{\alpha,\beta\}\cup J(u,v)} \\ &\qquad + \sum_{\substack{1\le\gamma\le q\\ \gamma\ne u,v}} C^{j_\gamma}_{\alpha<\beta} (-1)^{\gamma[u,v]} \eta_{\{\alpha,\beta,j\}\cup J(u,v,\gamma)} \Bigg)\end{aligned}$$

z where we abbreviate notations with: (for $u<v$)

$$\gamma[u,v] = \begin{cases} \gamma & \text{if } \gamma<u \\ \gamma-1 & \text{if } u<\gamma<v \\ \gamma-2 & \text{if } \gamma>v \end{cases} \equiv \begin{cases} \gamma & \text{if } \gamma<u \text{ or } \gamma>v \\ \gamma-1 & \text{if } u<\gamma<v \end{cases} \pmod 2 \tag{1.21}$$

On the other hand,

$$\begin{aligned}(\delta_\wedge d_\wedge \eta)_J &= \sum_{\alpha<\beta} \sum_{1\le\gamma\le q} (-1)^{\gamma-1} C^{j_\gamma}_{\alpha,\beta} (d_\wedge \eta)_J \\ &= \sum_{\alpha<\beta} \sum_{1\le\gamma\le q} (-1)^{\gamma-1} C^{j_\gamma}_{\alpha,\beta} \sum_j \Bigg((-1)^{1+2} C^j_{\alpha,\beta} \eta_{j\cup J(\gamma)} \\ &\qquad + \sum_{\substack{1\le v\le q\\ v\ne\gamma}} (-1)^{1+v[\gamma]} C^j_{\alpha,j_v} \eta_{\{j,\beta\}\cup J(v,\gamma)} + \sum_{\substack{1\le v\le q\\ v\ne\gamma}} (-1)^{2+v[\gamma]} \eta_{\{j,\beta\}\cup J(v,\gamma)} \\ &\qquad + \sum_{\substack{1\le u<v\le q\\ u,v\ne\gamma}} (-1)^{u[\gamma]+v[\gamma]} C^j_{j_u,j_v} \eta_{\{j,\alpha,\beta\}\cup J(u,v,\gamma)} \Bigg) \\ &= \sum_{\alpha<\beta} \sum_{1\le\gamma\le q} (-1)^{\gamma-1} C^{j_\gamma}_{\alpha,\beta} \sum_j \Bigg(-C^j_{\alpha,\beta} \eta_{j\cup J(\gamma)} + \\ &\qquad + \sum_{\substack{1\le u<v\le q\\ u,v\ne\gamma}} (-1)^{u[\gamma]+v[\gamma]} C^j_{j_u,j_v} \eta_{\{j,\alpha,\beta\}\cup J(u,v,\gamma)} \Bigg)\end{aligned}$$

Note that the middle terms cancel with each other for symmetry reasons. Again we abbreviate the notation as:

$$v[u] = \begin{cases} v & \text{if } v<u \\ v-1 & \text{if } v<u \end{cases} \tag{1.22}$$

Note that $\eta_{j,\alpha,\beta} = \eta_{\alpha,\beta,j}$ by anti-commutativity of the exterior algebra. Also for all u, v, γ pairwise different,

$$u[\gamma] + v[\gamma] + \gamma \equiv u + v + \gamma[u,v] \pmod 2$$

so the last sums of $\delta_\wedge d_\wedge \eta$ and $d_\wedge \delta_\wedge \eta$ cancel each other term-wise. Summing up, we have:

$$(\Delta_\wedge \eta)_J = \sum_j \sum_{\alpha<\beta} C^j_{\alpha,\beta} \Bigg(\sum_{1\le\gamma\le q} (-1)^\gamma C^{j_\gamma}_{\alpha,\beta} \eta_{j\cup J(\gamma)} + \sum_{1\le u<v\le q} (-1)^{u+v} C^j_{j_u,j_v} \eta_{\{\alpha,\beta\}\cup J(u,v)} \Bigg) \tag{1.19}$$

3. To prove third part:

$$
\begin{aligned}
(\delta_\wedge d_\circ \eta)_J &= \sum_{\alpha<\beta} \sum_{1\le u\le q-1} (-1)^{u-1} C^{j_u}_{\alpha,\beta} (d_\circ \eta)_{\{\alpha,\beta\}\cup J(u)} \\
&= \sum_{\alpha<\beta} \sum_{1\le u\le q-1} (-1)^{u-1} C^{j_u}_{\alpha,\beta} \Bigg(\tau(X_\alpha)\eta_{\beta\cup J(u)} - \tau(X_\beta)\eta_{\alpha\cup J(u)} \\
&\qquad - \sum_{\substack{1\le v\le q \\ v\neq u}} (-1)^{v[u]} \tau(X_{j_v}) \eta_{\{\alpha,\beta\}\cup J(u,v)} \Bigg)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(d_\circ \delta_\wedge \eta)_J &= \sum_{1\le v\le q+1} (-1)^{v-1} \tau(X_{j_v}) (\delta_\wedge \eta)_{J(v)} \\
&= \sum_{1\le v\le q+1} \sum_{\alpha<\beta} \sum_{\substack{1\le u\le q \\ u\neq v}} (-1)^{v-1} \tau(X_{j_v}) C^{j_u}_{\alpha,\beta} \cdot (-1)^{u[v]} \eta_{\{\alpha,\beta\}\cup J(u,v)}
\end{aligned}
$$

We see the last summand in $d_\circ \delta_\wedge \eta$ cancels with $\delta_\wedge d_\circ$ as

$$
(-1)^{v+u[v]} = -(-1)^{u+v[u]} \qquad \text{for all } u \neq v
$$

Hence

$$
\begin{aligned}
(\Delta_{\circ,\wedge}\eta)_J &= \sum_{1\le u\le q} \sum_{\alpha<\beta} (-1)^{u-1} C^{j_u}_{\alpha,\beta} \left(\tau(X_\alpha)\eta_{\beta\cup J(u)} - \tau(X_\beta)\eta_{\alpha\cup J(u)} \right) \\
&= \sum_\gamma (\tau(X_\gamma) \operatorname{coad}(X_\gamma))(\eta_J)
\end{aligned}
\tag{1.23}
$$

where the last identity comes from our convention (1.17). Hence the third statement is proven.

4. Lastly,

$$
\begin{aligned}
(\delta_\circ d_\wedge \eta)_J &= \sum_j \tau(X_j)^* (d_\wedge \eta)_{\{j\}\cup J} \\
&= \sum_j \Bigg(\sum_{1\le\gamma\le q} (-1)^\gamma \tau(X_j)^* \sum_k C^k_{j,j_\gamma} \eta_{k\cup J(\gamma)} \\
&\qquad + \sum_{1\le\alpha<\beta\le q} \tau(X_j)^* \sum_{k\neq j} C^k_{j_\alpha,j_\beta} \eta_{\{k,j\}\cup J(\alpha,\beta)} \Bigg)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(d_\wedge \delta_\circ \eta)_J &= \sum_{1\le\alpha<\beta\le q} \sum_k (-1)^{\alpha+\beta} C^k_{j_\alpha,j_\beta} (\delta_\circ \eta)_{\{k\}\cup J(\alpha,\beta)} \\
&= \sum_{1\le\alpha<\beta\le q} (-1)^{\alpha+\beta} \sum_{j\neq k} C^k_{j_\alpha,j_\beta} \tau(C_j)^* \eta_{\{j,k\}\cup J(\alpha,\beta)}
\end{aligned}
$$

Again the second summand of $d_\wedge \delta_\circ \eta$ cancels with $\delta_\circ d_\wedge \eta$ as $\eta_{\{k,j\}\cup J(\alpha,\beta)} = -\eta_{\{j,k\}\cup J(\alpha,\beta)}$. Hence again from (1.17),

$$
\begin{aligned}
(\Delta_{\wedge,\circ})_J &= \sum_j \sum_{1\le\gamma\le q} (-1)^\gamma \tau(X_j)^* \sum_k C^k_{j,j_\gamma} \eta_{k\cup J(\gamma)} \\
&= \sum_j (\tau(X_j)^* \operatorname{coad}^*(X_j))(\eta_J)
\end{aligned}
$$

and the proposition is proven. $\square$

Remark 1.6. For later purposes, we note that the operator

$$(\square_\circ \eta)_J := (\Delta_\circ \eta + \sum_j \tau(X_j)^2 \eta)_J = \sum_a \tau(X_a) \sum_{1\le u\le q} (-1)^u C^a_{j,j_u} \eta_{\{j\}\cup J(u)}$$

acts on the $C^q(\mathfrak{g}; V) \cong V \otimes \wedge^p \mathfrak{g}^*$ as derivations on the exterior algebra part, in the following sense:

$$\square_\circ(\eta_J \omega^J) = \square_\circ(\eta_J \omega^{J_1}) \wedge \omega^{J_2} + \omega^{J_1} \wedge \square_\circ(\eta_J \omega^{J_2}) \tag{1.24}$$

with for any $\omega^J = \omega^{J_1} \wedge \omega^{J_2}$. This can be proved by compare the term-wise the expressions on both sides:

$$\begin{aligned}
&\square_\circ(\eta_J \omega^{J_1}) \wedge \omega^{J_2} = \omega^{J_1} \wedge \square_\circ(\eta_J \omega^{J_2}) \\
=&\sum_a \tau(X_a)(\eta_J)\Big(\sum_{1\le u\le |J_1|} (-1)^u C^a_{j,j_u} \omega^j \wedge \omega^{J_1(u)} \wedge \omega^{J_2} \\
&+ \sum_{|J_1|+1\le u \le |J_2|} (-1)^{u-|J_1|} C^a_{j,j_u} \omega^{J_1} \wedge \omega^j \wedge \omega^{J_2(u-|J_1|)} \Big) \\
=&\square_\circ(\eta_J \omega^J)
\end{aligned}$$

where the last identity follow from $(-1)^{u-|J_1|}\omega^{J_1} \wedge \omega^j = (-1)^u \omega^j \wedge \omega^{|J_1|}$. Hence the claim is proven.

If we assume further that $\mathfrak{g}$ is reductive, one can define the Killing form $B(X,Y) := \mathrm{tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ on $[\mathfrak{g}, \mathfrak{g}]$. It extends over $Z_\mathfrak{g}$ to be a non-degenerate symmetric bilinear form on $\mathfrak{g}$, that is invariant under any automorphism of $\mathfrak{g}$, which we also denote as B. Moreover, we fix a Cartan involution θ of $\mathfrak{g}$ and denote the ± 1-eigenspaces under θ as $\mathfrak{p}$ and $\mathfrak{k}$ respectively. In this case the previous $\overline{\Omega}_G$ admits a nicer expression:

$$\overline{\Omega}_G = \Omega_G - 2\Omega_K \in U(\mathfrak{g}_\mathbb{C}) \tag{1.25}$$

Recall Ω_G is the Casimir element in $Z(\mathfrak{g}_\mathbb{C})$ induced by G-bi-invariant bilinear form B:

$$\Omega_G = \sum_i X_i X^i$$

for a basis $\{X_i\}$ of $\mathfrak{g}$ with the dual basis $\{X^i\}$ with respect to B. Here Ω_K is defined in a similar way by replacing the basis of $\mathfrak{g}$ by that of $\mathfrak{k}$. If we fix a pseudo-orthonormal basis $\{X_i\}$ with respect to the Killing form B, then $\overline{\Omega}_G = \sum_i X_i^2$ is just the sum of squares if we take the underlying metric on G to be:

$$B^\theta(X,Y) := B(X, \theta Y) \quad \text{for all } X, Y \in \mathfrak{g} \tag{1.26}$$

Notation: For convenience we take from now on a pseudo-orthonormal bases

$$\{X_1, \cdots, X_m, Y_1, \cdots, Y_{n-m}\}$$

for the Killing form B, such that $\{X_i\}$ to be the orthonormal basis of $\mathfrak{k}$ and $\{Y_\alpha\}$ that of $\mathfrak{p}$ with respect to B^θ. This is always possible as θ fixes X_is and acts by -1 on Y_is. Before the proof of the corollary, we remark one last identity between the adjoint and coadjoint representation:

$$\mathrm{coad}\,|_\mathfrak{k} = \mathrm{coad}^*\,|_\mathfrak{k} \qquad \mathrm{coad}\,|_\mathfrak{p} = -\,\mathrm{coad}^*\,|_\mathfrak{p} \tag{1.27}$$

Corollary 1.7 (Generalized Kuga's Lemma). Assume $\mathfrak{g}$ is reductive with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and fix an orthonormal basis as above. We introduce a new bi-grading on $C^k(\mathfrak{g}; V) = \bigoplus_{p+q=k} C^{p,q}(\mathfrak{g}; V)$ with elements $\eta \in C^{p,q}(\mathfrak{g}; V)$ of the following form:

$$\eta : X_{i_1}, \ldots, X_{i_p}, Y_{j_1}, \ldots, Y_{j_q} \to V$$

Then $\Delta_\circ$ and $\Delta_\wedge$ take the following simplified form:

1. $\Delta_\circ = -\sum_j \tau(X_j)^2 + \Box_\circ$ acts on $C^{p,0}(\mathfrak{g};V)$ and $C^{0,q}(\mathfrak{g};V)$ by

$$\Box_\circ\Big|_{C^{p,0}(\mathfrak{g};V)} \cong +\tau(X_j)\,\mathrm{coad}^*(X_j) \qquad \Box_\circ\Big|_{C^{0,q}(\mathfrak{g};V)} \cong -\tau(X_j)\,\mathrm{coad}^*(X_j) \tag{1.28}$$

   This extends to an action on $C^{p,q}(\mathfrak{g};V)$ by derivation, as a consequence of Remark 1.6.
2. $\Delta_\wedge$ acts on $C^n(\mathfrak{g};V)$ by $\sum_j \mathrm{coad}^*(X_j)\,\mathrm{coad}(X_j)$.

In particular, if we abbreviate the action by:

$$\begin{aligned}
A_\mathfrak{g} &= -\tau(\overline{\Omega}_G) : C^{p,q}(\mathfrak{g};V) \longrightarrow C^{p,q}(\mathfrak{g};V)\\
B_\mathfrak{k} &= \sum_{k\in I_\mathfrak{k}} \tau(X_k)\,\mathrm{coad}^*(X_k) : C^{p,q}(\mathfrak{g};V) \longrightarrow C^{p,q}(\mathfrak{g};V)\\
C_\mathfrak{p} &= \sum_{\alpha\in I_\mathfrak{p}} \tau(Y_\alpha)\,\mathrm{coad}^*(Y_\alpha) : C^{p,q}(\mathfrak{g};V) \longrightarrow C^{p-1,q+1}(\mathfrak{g};V)\oplus C^{p+1,q-1}(\mathfrak{g};V)\\
D_\mathfrak{g} &= \frac{1}{2}\,\mathrm{coad}^*(\Omega_G) : C^{p,q}(\mathfrak{g};V) \longrightarrow C^{p,q}(\mathfrak{g};V)
\end{aligned} \tag{1.29}$$

then each component of Δ_1 acts on 1-forms $C^1(\mathfrak{g};V)\cong(V\otimes\mathfrak{k}^*)\oplus(V\otimes\mathfrak{p}^*)$ via the following operator-valued matrices:

$$\begin{aligned}
\Delta_\circ|_{C^1(\mathfrak{g};V)} &= \begin{pmatrix} A_\mathfrak{g}+B_\mathfrak{k} & -C_\mathfrak{p}\\ C_\mathfrak{p} & A_\mathfrak{g}-B_\mathfrak{k}\end{pmatrix} \qquad \Delta_\wedge|_{C^1(\mathfrak{g};V)} = \begin{pmatrix} D_\mathfrak{g} & \\ & D_\mathfrak{g}\end{pmatrix}\\
\Delta_{\circ,\wedge}|_{C^1(\mathfrak{g};V)} &= \begin{pmatrix} -B_\mathfrak{k} & C_\mathfrak{p}\\ C_\mathfrak{p} & -B_\mathfrak{k}\end{pmatrix} \qquad \Delta_{\wedge,\circ}|_{C^1(\mathfrak{g};V)} = \begin{pmatrix} -B_\mathfrak{k} & -C_\mathfrak{p}\\ -C_\mathfrak{p} & -B_\mathfrak{k}\end{pmatrix}
\end{aligned} \tag{1.30}$$

Summing all terms up, the 1-th Laplacian on $C^1(\mathfrak{g};V)$ is seen to take the form:

$$\Delta_1 = \begin{pmatrix} A_\mathfrak{g}-B_\mathfrak{k}+D_\mathfrak{g} & -C_\mathfrak{p}\\ C_\mathfrak{p} & A_\mathfrak{g}-3B_\mathfrak{k}+D_\mathfrak{g}\end{pmatrix} \tag{1.31}$$

Remark 1.8. The above corollary can be seen as an extension of Kuga's Lemma from the K-invariant cases to K-equivariant cases. In the classical setting [**BW00**, Theorem II.2.5], lots of terms vanish due to K-invariance, and the Laplacian collapses into the Casimir element $\Omega_G \in Z(\mathfrak{g})$.

Throughout this proof we fix the Latin subscript for bases of $\mathfrak{k}$ or $\mathfrak{g}$ and reserve the Greek subscript for bases of $\mathfrak{p}$. We also abbreviate the index set of orthonormal bases of $\mathfrak{p}$ and $\mathfrak{k}$ by $I_\mathfrak{p}$ and $I_\mathfrak{k}$ respectively.

Remark 1.9. Such a choice of metric is indispensable for the following computations, as we often exploit the extra symmetries rendered by the Killing form to simplify the expressions. Recall that the Killing form gives extra symmetries in the structural constants: For semisimple Lie algebra $\mathfrak{g}$:

$$B([X,Y],Z) = B(X,[Y,Z]) \qquad \forall X,Y,Z\in\mathfrak{g}$$

Furthermore, as B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$, by fixing $X_i, X_j, X_k \in \mathfrak{k}$ and $Y_\alpha, Y_\beta \in \mathfrak{p}$ pseudo-orthonormal bases of the Killing form, one readily verify the following identity:

$$C^\beta_{\alpha,i} = C^\alpha_{i,\beta} = -C^i_{\beta,\alpha} \qquad C^k_{i,j} = C^i_{j,k} \qquad \text{and all } C^\alpha_{i,j} = C^i_{\alpha,j} = 0 \text{ due to orthogonality}$$

whereas the rest of the structural constants are all zero by orthogonality. We abbreviate the index set of orthonormal bases of $\mathfrak{p}$ and $\mathfrak{k}$ by $I_\mathfrak{p}$ and $I_\mathfrak{k}$ respectively. If the index set was not specified, then it means the index runs over all $I_\mathfrak{g}$.

Proof. Starting from (1.18). Assuming $J \subseteq I_{\mathfrak{k}}$, the coadjoint action for fixed a can be rewritten as:

$$\begin{aligned}
(\square_\circ \eta)_J &= \sum_{a\in I_{\mathfrak{g}}} \tau(X_a) \sum_{\substack{j\\1\le u\le q}} (-1)^{u-1} C^a_{j,j_u}\eta_{j\cup J(u)} \\
&= -\sum_{a\in I_{\mathfrak{k}}}\Big(\tau(X_a)(-1)^u \sum_{\substack{1\le u\le q\\ j\in I_{\mathfrak{k}}}} C^a_{j,j_u}\eta_{j\cup J(u)}\Big) - \sum_{\alpha\in I_{\mathfrak{p}}}\Big(\tau(X_\alpha)(-1)^u \sum_{\substack{1\le u\le q\\ \beta\in I_{\mathfrak{p}}}} C^\alpha_{\beta,j_u}\eta_{j\cup J(u)}\Big) \\
&= \sum_{a\in I_{\mathfrak{k}}}\Big(\tau(X_a)(-1)^u \sum_{\substack{1\le u\le q\\ j\in I_{\mathfrak{k}}}} C^j_{a,j_u}\eta_{j\cup J(u)}\Big) + \sum_{\alpha\in I_{\mathfrak{p}}}\Big(\tau(X_\alpha)(-1)^u \sum_{\substack{1\le u\le q\\ \beta\in I_{\mathfrak{p}}}} C^\beta_{\alpha,j_u}\eta_{j\cup J(u)}\Big) \\
&= \sum_{a\in I_{\mathfrak{g}}} \tau(X_a)\,\mathrm{coad}(X_a)(\eta_J),
\end{aligned} \tag{1.32}$$

where in the last identity we used $\mathrm{coad}\,X_\alpha(\omega^\beta) = \sum_\gamma C^\beta_{\gamma,\alpha}\omega^\gamma$, as well as our convention (1.17), and then by exploiting the identities of the structural constants above. Similarly, for $J \subseteq I_{\mathfrak{p}}$ we deduce that $\Delta_\circ$ acts on it as $-\sum_a \tau(X_a)\,\mathrm{coad}^*(X_a)$.

Note now $\square_\circ$ extends to general $C^{p,q}(\mathfrak{g}; V)$ by applying the derivation property of $\square_\circ$ in Remark 1.6 to the following decomposition

$$\wedge^k \mathfrak{g}^* = \sum_{\substack{0\le p,q\le k\\ p+q=k}} \wedge^p \mathfrak{k}^* \otimes \wedge^q \mathfrak{p}^*. \tag{1.33}$$

Arguing similarly, we prove the second statement for $\Delta_\wedge$ now. It suffices to verify that (1.19) and $\frac{1}{2}\sum_j \mathrm{coad}^*(X_j)\,\mathrm{coad}(X_j)$ give the same expression:

$$\begin{aligned}
&\frac{1}{2}\sum_{j\in I_{\mathfrak{g}}} \mathrm{coad}^*(X_j)\,\mathrm{coad}(X_j)\eta_J \\
&= \frac{1}{2}\sum_{1\le\gamma\le q}\sum_{\beta}(-1)^{\gamma-1}C^{j_\gamma}_{j,\beta}(\mathrm{coad}^*(X_j)\eta_{\beta\cup J(\gamma)}) \\
&= \sum_{\alpha<\beta}\Bigg(\sum_{1\le\gamma\le q} C^{j_\gamma}_{j,\beta}C^\alpha_{j,\beta}\eta_{\alpha\cup J(\gamma)} - \sum_{\substack{1\le u,v\le q\\ u\ne v}} (-1)^{u[v]} C^{j_\gamma}_{j,\beta}C^\alpha_{j,j_u}\eta_{\{\alpha,\beta\}\cup J(u,v)}\Bigg)
\end{aligned} \tag{1.34}$$

It is clear that the first sum of (1.19) and that of this expression agrees term-wise. Also keep in mind that the ad and coad*-actions give the same set of structural constants, since they are isomorphism by taking to the dual space. To finish the proof of the corollary it suffices to establish the following identity between structural constants:

$$\sum_j C^{j_u}_{j,\beta}C^\alpha_{j,j_u} = \sum_j C^j_{j_u,j_v}C^j_{\alpha,\beta} \tag{1.35}$$

for each fixed α, β, j_u, j_v. First by Ad-invariance of the Killing form $B([Z_a, Z_b], Z_c) = B(Z_a, [Z_b, Z_c])$ for the chosen bases $Z \in \{X_i, \cdots Y_\alpha\}$ above, and Jacobi identity:

$$\begin{aligned}
\sum_{\alpha\in I_{\mathfrak{p}}} C^\alpha_{a,b}C^\alpha_{c,d} - \sum_{j\in I_{\mathfrak{k}}} C^j_{a,b}C^j_{c,d} &= B([Z_a, Z_b], [Z_c, Z_d]) \\
&= B(Z_a, [Z_b, [Z_c, Z_d]]) \\
&= -B(Z_a, [Z_c, [Z_d, Z_b]]) - B(Z_a, [Z_d, [Z_b, Z_c]]) \\
&= -B([Z_a, Z_c], [Z_d, Z_b]) - B([Z_a, Z_d], [Z_b, Z_c]) \\
&= \sum_{j\in I_{\mathfrak{k}}}\Big(C^j_{a,c}C^j_{d,b} + C^j_{b,d}C^j_{b,c}\Big) - \sum_{j\in I_{\mathfrak{p}}}\Big(C^j_{a,c}C^j_{d,b} + C^j_{a,d}C^j_{b,c}\Big)
\end{aligned} \tag{1.36}$$

By the symmetry $C^i_{\alpha,\beta} = -C^i_{\beta,\alpha}$, we can identify the two summand above together. Also by the fact that the two sums are mutually exclusive, i.e.,

$$\sum_{j\in I_{\mathfrak{p}}} C^j_{a,b}C^j_{c,d} \neq 0 \text{ implies } \sum_{j\in I_{\mathfrak{k}}} C^j_{a,b}C^j_{c,d} = 0 \qquad \sum_{j\in I_{\mathfrak{k}}} C^j_{a,b}C^j_{c,d} \neq 0 \text{ implies } \sum_{j\in I_{\mathfrak{p}}} C^j_{a,b}C^j_{c,d} = 0$$

one concludes in term of structural constants, that for $a,b,c,d \in I_{\mathfrak{p}} \sqcup I_{\mathfrak{k}}$:

$$\begin{aligned}\sum_{\substack{a<b\\ j\in I_{\mathfrak{p}}}} C^j_{a,b}C^j_{c,d}\eta_{\{a,b\}\cup J(c,d)} &= -\sum_{\substack{a<b\\ j\in I_{\mathfrak{p}}}} (C^j_{a,d}C^j_{b,c} + C^j_{a,c}C^j_{d,b})\eta_{\{a,b\}\cup J(c,d)} \\ &= -\sum_{\substack{a,b\\ j\in I_{\mathfrak{p}}}} (-1)^{a[b]}C^j_{a,d}C^j_{b,c}\eta_{\{a,b\}\cup J(c,d)}\end{aligned} \tag{1.37}$$

and same identity holds replacing $I_{\mathfrak{p}}$ by $I_{\mathfrak{k}}$ in the identity. Sub this into right hand side and again by the mutual exclusion of two sums, it suffices to show:

$$\sum_{j\in I_{\mathfrak{p}}} C^{j_u}_{j,\beta}C^{\alpha}_{j,j_u} = -\sum_{j\in I_{\mathfrak{p}}} C^j_{j_u,\alpha}C^j_{\beta,j_v} \qquad \sum_{j\in I_{\mathfrak{k}}} C^{j_u}_{j,\beta}C^{\alpha}_{j,j_u} = -\sum_{j\in I_{\mathfrak{k}}} C^j_{j_v,\alpha}C^j_{\beta,j_v}$$

This deserves a case-by-case analysis. Consider first the case $j \in I_{\mathfrak{k}}$, this forces α, β to be both in $I_{\mathfrak{k}}$ or in $I_{\mathfrak{p}}$, for otherwise $C^j_{\alpha,\beta}$ vanishes. In this case:

$$\sum_{j\in I_{\mathfrak{k}}} C^j_{\alpha,\beta}C^j_{j_u,j_v} = 0 \text{ and } \sum_{j\in I_{\mathfrak{k}}} C^{j_v}_{j,\beta}C^{\alpha}_{j,j_u} = \pm\sum_{j\in I_{\mathfrak{k}}} C^j_{j_v,\beta}C^j_{\alpha,j_u} = \pm\sum_{j\in I_{\mathfrak{k}}} C^j_{j_v,j_u}C^j_{\alpha,\beta} = 0$$

where the second last equality derives from Remark 1.9. The exact parity of the sum is irrelevant here so we kept them implicit. This indeed implies j_u, j_v are both in the same index set, as are α, β for the same reason. But then again from Remark 1.9

$$C^j_{\alpha,j_u} = \begin{cases} -C^{\alpha}_{j,j_u} & \text{if } \alpha,\beta \in I_{\mathfrak{k}} \\ C^{\alpha}_{j,j_u} & \text{if } \alpha,\beta \in I_{\mathfrak{p}} \end{cases} \quad C^j_{j_v,\beta} = \begin{cases} -C^{j_v}_{j,\beta} & \text{if } \alpha,\beta \in I_{\mathfrak{k}} \\ C^{j_v}_{j,\beta} & \text{if } \alpha,\beta \in I_{\mathfrak{p}} \end{cases}$$

We can derive other cases in similar dichotomy and therefore the equality (1.35) is established. To conclude the expansion (1.19) from (1.34), one shows the following:

$$\sum_{\alpha,\beta}\sum_{\substack{1\le u,v\le q\\ u\neq v}} (-1)^{u[v]}C^{j_\gamma}_{j,\beta}C^{\alpha}_{j,j_u}\eta_{\{\alpha,\beta\}\cup J(u,v)} = \sum_{\alpha,\beta}\sum_{1\le u<v\le q} (-1)^{u[v]+v[u]}C^j_{\alpha,\beta}C^j_{j_u,j_v}\eta_{\{\alpha,\beta\}\cup J(u,v)}$$

with the fact that $(-1)^{u[v]+v[u]} = -(-1)^{u+v}$. This concludes the proof of the corollary. □

The proof of Proposition 1.5 and Corollary 1.7 can be seemingly tedious and error-prone, but they are essential to the computations of the spectra in Theorem 6.2 and Theorem 8.1. We remark that at low-dimensions a direct computation of $\mathfrak{g}$-chain complex can already be daunting. We also remark the above computations were vindicated by testing on several nilpotent and semisimple groups against a program written by Tim Höpfner. The code of this program was included in his thesis [**Hoe23**, p.87 - 101].

Chapter 2

Representation theory of real reductive Lie groups

This chapter serves as a succinct introduction to representation theory on a fairly general class of real reductive groups, introduced by Wolf in [**Wol74**]. It also collects all the necessary data in Plancherel theorem which plays a role in the computation of Chapter 6.

Harish-Chandra's theory of discrete series was also extended to this class. Most results are similar in flavour with the original treatment, and are in fact critically dependent on the latter. We will classify all tempered representations of groups of class $\widetilde{\mathcal{H}}$, and will give detailed information on pertinent results such as infinitesimal character, K-type and Plancherel measure of respective representations.

As usual, we denote the whole unitary dual of G as $\widehat{G}$.

2.1 Reductive Lie groups and structural theorems

In this section we collect basic data of reductive Lie groups. A more detailed introduction can be found in [**Kna96**] and [**Kna86**].

Definition 2.1 (reductive groups of class $\widetilde{\mathcal{H}}$). Let G be a reductive Lie group, i.e., its Lie algebra admits the following decomposition:

$$\mathfrak{g} = Z_{\mathfrak{g}} \oplus \mathfrak{g}_0 \quad \text{where } Z_{\mathfrak{g}} \text{ is central and } \mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}] \text{ semisimple}$$

A reductive Lie group is defined to be of class $\widetilde{\mathcal{H}}$ if it satisfies the following properties:

1. If $g \in G$ then $\mathrm{ad}(g)$ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$;
2. It contains a closed normal abelian subgroup Z that centralizes the identity component G^0 of G;
3. ZG^0 has finite index in G;
4. $Z \cap G^0$ is cocompact in the center Z_{G^0} of G^0.

In particular this contains all those reductive Lie groups with finitely many connected components, i.e., if $|G/G^0| < \infty$ with $Z = Z_{G^0}$ satisfies the condition in the definition above. Our groups of interest in particular include those nonlinear Lie groups which are universal covers of reductive Lie groups of Harish-Chandra's class, e.g., $\widetilde{SL_2(\mathbb{R})}$, $\widetilde{U(n,n)}$ and $\widetilde{Sp_n(\mathbb{R})}$.

The construction of the relative discrete series in Section 2.2 includes a two-step construction: First we construct the respective objects on the connected component G^0, and then lift them to an intermediate group $G^\dagger$, and then to the whole group G. This resembles the treatment the way general discrete series on Levi components of semisimple Lie groups were constructed (see e.g., [**Kna86**, Chapter XII.8]). It also means a digression into disconnected groups is necessary, even though our targets are simply connected Lie groups. Parallel to that development we define the intermediate group $G^\dagger$ as:

$$G^\dagger := \left\{ g \in G \,\middle|\, \mathrm{Ad}(g) \text{ is an inner automorphism on } G^0 \right\}$$

Note that $G^\dagger = Z_G(G^0)G^0$ with $Z_G(G^0)/Z$ compact and $G^\dagger/ZG^0$ finite. Meanwhile, as the underlying Lie algebra $\mathfrak{g}$ is still reductive, the settings of structural theorems and root systems carry seamlessly over here. We retain the notations of [Kna86]. For a more comprehensive treatment, [Kna96, Chapter VII] serves as a perfect text.

By definition G/Z has only finite many components, hence we can choose a maximally compact subgroup by general theory. This motivates the following definition:

Definition 2.2 (relative maximal compact subgroup). Define $K \subseteq G$ to be a relative maximal compact subgroup if K/Z is a maximal compact subgroup of G/Z. In particular, when G is of Harish-Chandra's class [KV95, Defnition 4.29], this implies K is itself maximal compact in G.

Note that K contains $Z_G(G^0)$ as $Z \cap G^0$ is cocompact in Z_{G^0} and $K/Z_G(G^0)$ is a maximal compact subgroup of $G/Z_G(G^0)$.

As in the semisimple case, K can be realized as fixed point of the unique involutive automorphism of G [Wol74, Lemma 4.1.2]. Fix a Cartan involution θ of $\mathfrak{g}$ and denote the ± 1-eigenspaces under θ as $\mathfrak{p}$ and $\mathfrak{k}$ respectively. As in the semisimple case, θ exponentiates to an involutive automorphism Θ of G [Kna96, Proposition 7.21], and we take $K = G^\Theta$ to be its fixed point set. Note the Killing form B then gives a bilinear form on $\mathfrak{g}$ which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{k}$.

Moreover, any two Cartan involutions are G^0-conjugate and every Cartan subgroup of G is stable under Cartan involution, so it suffices for our purpose to fix one Cartan involution θ throughout, as all Cartan subgroups of G are G^0-conjugate to one of the θ-stable Cartan subgroups.

Let $\mathfrak{h}$ a θ-stable maximal abelian subalgebra of $\mathfrak{g}$ then $H := Z_G(\mathfrak{h})$ is a Θ-stable Cartan subgroup of G, which is itself a reductive Lie group [Kna96, Proposition 7.25]. Denote $H^0 = H \cap G^0$ the respective Cartan subgroup of G^0. In general only H^0 is commutative.

Definition 2.3 (root system). Fix a Cartan subalgebra $\mathfrak{h}$ of the reductive Lie algebra $\mathfrak{g}$, that is, if its complexification $\mathfrak{h}_\mathbb{C}$ is the centralizer of itself in the complex Lie algebra $\mathfrak{g}_\mathbb{C}$, i.e., $\mathfrak{h}_\mathbb{C} = Z_{\mathfrak{g}_\mathbb{C}}(\mathfrak{h}_\mathbb{C})$. (See [Kna96, Proposition 2.7]) Therefore $\mathfrak{h}$ must be of the form:

$$\mathfrak{h} = Z_\mathfrak{g} \oplus (\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]) \tag{2.1}$$

Here the second summand is a Cartan subalgebra of the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. We can then define the root system and Weyl group of reductive Lie groups in the same way as in the classical case:

1. The root system $\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C}) = \Delta([\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}]; \mathfrak{h}_\mathbb{C} \cap [\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}])$ contains the roots of $[\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}]$ with respect to its Cartan subalgebra $\mathfrak{h}_\mathbb{C} \cap [\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}]$, which can be extended by taking zero on $Z_{\mathfrak{g}_\mathbb{C}}$ to $\mathfrak{h}$.
2. The set of positive roots and the set of simple roots are denoted as $\Delta^+(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C})$ and $\Pi(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C})$ respectively. Denote $\delta_G := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha \in \mathfrak{h}_\mathbb{C}^*$ as the half-sum of positive roots.
3. The algebraic Weyl group $W(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C})$ is the Weyl group of $\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C})$ as an abstract root system, i.e. is generated by reflections in the members of Δ, it consists the members of the Weyl group of its semisimple subalgebra. This extends to a symmetry group on $\mathfrak{g}_\mathbb{C}$ by defining as acting by identity on $Z_{\mathfrak{g}_\mathbb{C}}$.
4. The analytic Weyl group $W(G; H)$ is defined to be $N_G(\mathfrak{h})/Z_G(\mathfrak{h})$.

Remark 2.4. In the case when $\mathfrak{h}$ is θ-stable, we may further identify $W(G; H) = N_K(\mathfrak{h})/Z_K(\mathfrak{h})$ [Kna96, Corollary 7.91ff]. Also by the property $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$ in Definition 2.1 we might assume that $W(G; H)$ is a subgroup of $W(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C})$.

Next we consider the Iwasawa decomposition of $\mathfrak{g}$ similar to that of semisimple Lie algebras, and its restricted root decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Again $\mathfrak{a}$ admits the following decomposition:

$$\mathfrak{a} = \mathfrak{p} \cap Z_\mathfrak{g} \oplus (\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}])$$

Any two maximal abelian subspaces of $\mathfrak{p}$ are $\mathrm{Ad}(K)$-conjugate. One might even conjugate by semisimple elements of K [Kna96, Proposition 7.29]. Relative to $\mathfrak{a}$ we construct the set of restricted roots of $\mathfrak{g}$, denoted as $\Sigma(\mathfrak{g}; \mathfrak{a})$. For each $\lambda \in \Sigma(\mathfrak{g}; \mathfrak{a})$, the respective root space $\mathfrak{g}_\lambda$ is defined as:

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X \quad \text{for all } H \in \mathfrak{a}\}$$

Apparently such a restricted root is obtained by extending the restricted roots on $\mathfrak{g}_0$ trivially on the center. Denote $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ and choose the positivity of roots in a similar way as in the semisimple cases, and denote $\Sigma^+(\mathfrak{g};\mathfrak{a})$ the same way as in the semisimple cases, with the nilpotent Lie subalgebra $\mathfrak{n} = \oplus_{\lambda\in\Sigma^+}\mathfrak{g}_\lambda$ the sum of all positive restricted root spaces of $\mathfrak{g}$. We have an Iwasawa decomposition with exactly the same property as in the semisimple cases [**Kna96**, Proposition 7.30].

Reflections in the restricted roots generate the Weyl group $W(\Sigma)$, which coincides with the analytically defined Weyl group $W(G;A) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.[**Kna96**, Proposition 7.32] We often use the second notation.

Definition 2.5 (**parabolic subalgebra/subgroup**)**.** If we write the Iwasawa decomposition as $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$ and take $\mathfrak{m}_{\mathfrak{p}} = Z_{\mathfrak{k}}(\mathfrak{a}_{\mathfrak{p}})$, then define $\mathfrak{q}_{\mathfrak{p}} := \mathfrak{m}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}}$ and its G-conjugates to be minimal parabolic subalgebras of G. A parabolic subalgebra of $\mathfrak{g}$ is a Lie subalgebra that contains some conjugate of $\mathfrak{q}_{\mathfrak{p}}$.

For a fixed minimal parabolic subalgebra $\mathfrak{q}_{\mathfrak{p}}$, all parabolic subalgebras containing it are parameterized by the set of subsets of simple restricted roots:

Proposition 2.6 ([**Kna96**, Proposition 7.76])**.** *Let Σ^+ be the set of positive restricted roots of $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$, with the positivity determined by $\mathfrak{n}_{\mathfrak{p}}$. Let Π be the set of simple restricted roots in Σ^+. Then there is a one-to-one correspondence between:*

$$\left\{\mathfrak{q} \subseteq \mathfrak{g} \mid \mathfrak{q} \supseteq \mathfrak{q}_{\mathfrak{p}} \text{ parabolic}\right\} \longleftrightarrow \left\{\Pi' \subseteq \Pi\right\} \tag{2.2}$$

with the correspondence being $\lambda \in \Pi'$ if and only if $\mathfrak{g}_\lambda \subseteq \mathfrak{m}$. Furthermore, no two of these parabolic subgroups are G-conjugate.

Definition 2.7 (**Langlands decomposition**)**.** Similar to the Iwasawa decomposition, there is the Langlands decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ on each parabolic subgroup. Denote $\Sigma_{\pi'} = \Sigma^+ \cup \{\beta \in \Sigma | \beta \in \mathrm{Span}(\Pi')\}$. We define the decomposition accordingly:

$$\mathfrak{a} = \bigcap_{\beta\in\Sigma_{\pi'}\cap\Sigma_{\pi'}^-} \ker\beta \subseteq \mathfrak{a}_{\mathfrak{p}} \qquad \mathfrak{a}_M = \mathfrak{a}^\perp \subseteq \mathfrak{a}_{\mathfrak{p}} \tag{2.3}$$

$$\mathfrak{m} = \mathfrak{a}_M \oplus \mathfrak{m}_{\mathfrak{p}} \oplus \bigoplus_{\beta\in\Sigma_{\pi'}\cap-\Sigma_{\pi'}} \mathfrak{g}_\beta \qquad \mathfrak{n} = \bigoplus_{\beta\in\Sigma_{\pi'}\setminus-\Sigma_{\pi'}} \mathfrak{g}_\beta \tag{2.4}$$

Relations between these subalgebras are characterized by the following properties: [**Kna96**, Proposition 7.78]:

1. $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$;
2. $\mathfrak{m}, \mathfrak{a}, \mathfrak{n}$ are mutually orthogonal with respect to the Killing form B;
3. $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{q} \cap \theta\mathfrak{q} = Z_{\mathfrak{g}}(\mathfrak{a})$.

At the group level the situation is slightly deviated from the original treatment of Harish-Chandra due to relative compactness. Nonetheless, the conclusions are similar. The following are taken from [**Wol74**, Section 5].

Fix a Cartan involution θ of G and K the respective relative compact subgroup throughout. Given now a Θ-stable Cartan subgroup H, then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ by θ-parity, and $H = T \times A$ with $T = H \cap K$ the Cartan subgroup of M and $A = \exp(\mathfrak{a})$.

Definition 2.8. A root $\alpha \in \Delta$ is defined to be **real**, **imaginary** or **complex** if its respective value on $\mathfrak{h}$ is real, imaginary or neither. Denote the set of real, imaginary, complex roots to be $\Delta_R, \Delta_I, \Delta_C$ respectively. We also define an imaginary root α (hence fixed by θ) to be **compact** (resp. **noncompact**) if the respective root space $\mathfrak{g}_\alpha$ lies in $\mathfrak{k}$ (resp. $\mathfrak{p}$). Denote the set of compact and noncompact roots to be Δ_n and Δ_K, with $\Delta_I = \Delta_n \sqcup \Delta_K$.

Definition 2.9. A θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called **maximally compact** (resp. **maximally noncompact**) if its compact dimension (resp. noncompact dimension) is as large as possible. A Θ-stable Cartan subgroup is called maximally compact (resp. maximally noncompact) of its corresponding Cartan subalgebra is maximally compact (resp. maximally noncompact).

The maximally noncompact $\mathfrak{h}$ is characterized by having no noncompact imaginary roots and the maximally compact $\mathfrak{h}$ is characterized by having no real roots. [**Kna96**, Proposition 6.70]

$Z_G(A)$ has a unique splitting into $Z_G(A) = M \times A$. The significance of class $\widetilde{\mathcal{H}}$ is that the Levi subgroup M inherits the class, i.e.: if G is of class $\widetilde{\mathcal{H}}$, M is also of class $\widetilde{\mathcal{H}}$ [**Wol18**, Proposition 4.1.4].

Definition 2.10 (cuspidal parabolic subgroup). If we denote further $N = \exp(\mathfrak{n})$ with $Q = N_G(N)$, then Q is a parabolic subgroup, i.e., it contains a minimal parabolic subgroup $Q_{\mathfrak{p}}$ corresponding to $\mathfrak{q}_{\mathfrak{p}}$. Moreover, it has a unipotent radical N and reductive subgroup MA, with Langlands decomposition $Q = MAN$ similar to the classical case [**Wol18**, Lemma 4.2.1].

A parabolic subgroup $Q \subset G$ is said to be cuspidal if $[(M_Q A_Q)^0, (M_Q A_Q)^0]$ has a relative compact Cartan subgroup H_Q, i.e., if $H_Q \subseteq K$.

Given a Cartan subgroup H, we can construct a Q using the above recipe. This is a bijection between G-conjugacy classes of Cartan subgroups and G-conjugacy class of cuspidal parabolic subgroups. Also Q is cuspidal if and only if its reductive part has relative discrete series [**Wol18**, Proposition 4.1.2] (c.f. definition 2.12) As we will see in Theorem 2.23 such Levi subgroups are the building blocks of tempered representations of G.

2.2 Relative discrete series

This section is an excerpt from [**Wol18**], classifying all the tempered representations. Readers are assumed to be familiar with the classical results of [**Kna86**]. Before the description we first cover a few general facts about G:

Theorem 2.11 ([**Wol74**, §3])**.** *Let G be a reductive Lie group of class $\widetilde{\mathcal{H}}$ and K its relative maximal compact subgroup. then G has the following properties:*

1. *Every irreducible unitary representations of G is admissible, i.e., for all $[\kappa] \in \widehat{K}$ and $[\pi] \in \widehat{G}$: the multiplicity of κ in π is finite:*

$$m(\kappa, \pi|_K) \leq n_G \dim(\kappa) < \infty$$

   for a constant integer $n_G \geq 1$ depending only on G. We can further choose $n_G \leq |G/ZG^0|$.

2. *Every G of class $\widetilde{\mathcal{H}}$ are CCR groups (after canonical commutation relation), i.e., for each $[\pi] \in \widehat{G}$ an $f \in L^1(G)$, then $\pi(f) := \int_G f(g)\pi(g)dg$ defines a compact operator on H_π. In particular, they are groups of Type I and the Schur's Lemma applies.*

Proof. The admissibility of $\pi \in \widehat{G}$ was proven by Harish-Chandra (see [**Kna86**, Theorem 8.1]) for connected reductive groups. Hence it suffices to extend it form G^0 to G. As Z centralizes G_0, every $[\pi_1] \in \widehat{ZG^0}$ can be written as the form $\pi_1 = \xi \otimes \pi_0$ where $\pi_0 \in \widehat{G}$ and $\xi \in \widehat{Z}$ a character. Then one can bound the multiplicity $m(\kappa, \pi|_K)$ by sum of their restriction to finite-index subgroups ZK^0 and ZG^0 respectively, which both are finite, as for all $\kappa_1 = \xi' \otimes \kappa_0 \in \widehat{ZK^0}$:

$$m(\kappa_1, \pi_1|_{ZK^0}) \leq m(\kappa_0, \pi_0|_{ZG^0}) \leq n \dim(\kappa_0) = n \dim(\kappa_1)$$

with the bound on n_G directly from [**CM82**]. The second statement is a direct consequence of the first. ☐

The reader may refer to [**BdlH19**, 6.D. & 6.E.] for detailed exposition on groups of CCR class and Type I. We need the mere fact that one may apply the abstract Plancherel Theorem B.1 in our case.

Now by Mackey's machinery of induction, the left regular representation of G decomposes as:

$$\mathrm{ind}_{\{1\}}^G(\mathrm{Id}) = \mathrm{ind}_Z^G(\int_{\widehat{Z}} \ell_\xi \,\mathrm{d}\xi) = \int_{\widehat{Z}} \mathrm{ind}_Z^G(\xi)\,\mathrm{d}\xi \tag{2.5}$$

where $\ell_\xi := \mathrm{ind}_Z^G(\xi)$ the ξ-equivariant representations of $L^2(G)$:

$$L^2(G/Z;\xi) := \left\{ f \in G \to \mathbb{C} \,\middle|\, f(gz) = \xi(z)^{-1} f(g) \text{ and } \int_{G/Z} |f(g)|^2 \,\mathrm{d}\bar{g} < \infty \right\} \tag{2.6}$$

This inspires the following definition:

Definition 2.12 (relative discrete series). Let Z be a closed normal abelian subgroup of G and $\xi \in \widehat{Z}$. Denote:

$$\widehat{G}_\xi = \left\{ [\pi] \in \widehat{G} \,\middle|\, \xi \text{ is a subrepresentation of } \pi|_Z \right\}$$

We call such $\pi \in \widehat{G}_\xi$ ξ-discrete if π is a subrepresentation of ℓ_ξ. All ξ-discrete series form the set $\widehat{G}_{\xi-disc}$, and define a G-representation is in the discrete series $\widehat{G}_d$ (relative to Z) if $\pi \in \widehat{G}_d := \bigcup_{\xi \in \widehat{Z}} \widehat{G}_{\xi-disc}$.

Remark 2.13. If Z is central in G, then we have a surjective map:

$$\mathrm{res}_Z^G : \widehat{G} \to \widehat{Z} \qquad \pi \mapsto \pi|_Z$$

by trivially extending characters of Z to characters of G, as Z is assumed to be central. In particular the union of the relative discrete series above is disjoint. We remark this construction does not depend on the fact that G is reductive. In particular, when G is a nilpotent Lie group, the characterization of relative series of G has been known to exist if and only if the symmetric polynomial generated by the center $S(\mathfrak{z}) \subseteq U(\mathfrak{g}_\mathbb{C})$ is the whole center of the universal enveloping algebra $Z(\mathfrak{g})$ [MW73, Theorem 3]. Note here we have slightly abused the notation by identifying $\mathbb{C}[\mathfrak{g}^*]^G \cong S(\mathfrak{g}) \cong Z(\mathfrak{g})$. Such identity holds for arbitrary connected Lie groups G [CG90, Corollary 3.3.3].

To construct the whole family of discrete series, we begin with constructing such on G^0:

Theorem 2.14 (relative discrete series on connected reductive Lie groups). *Let G^0 be the connected component of G of class $\widetilde{\mathcal{H}}$ and Z the centralizer of G^0. Then:*

1. *G^0 has a relative (to Z) discrete series if and only if $G^0/(Z \cap G^0)$ has a compact Cartan subgroup.*
2. *Choose a Cartan subgroup H^0 of G^0 such that $H^0/Z \cap G^0$ is compact. Denote L the lattice in $i\mathfrak{h}^*$ satisfying:*

$$L := \{\lambda \in i\mathfrak{h}^* \mid e^\lambda \text{ defines a character on } H^0\} \tag{2.7}$$

Consider the subset L' that is nonsingular relative to $\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C})$, i.e., $\langle \lambda, \alpha \rangle \neq 0$ for all roots $\alpha \in \Delta$. Then for every $\lambda \in L'$, there is a unique class $[\pi_\lambda^0] \in (\widehat{G^0})_d$. Moreover, two classes $[\pi_\lambda^0]$ and $[\pi_{\lambda'}^0]$ are equivalent if the respective characters lie in the same $W(G^0; H^0)$-orbit. We call these λ the Harish-Chandra parameters of H_π. They are also the infinitesimal character of the corresponding π.

3. *The construction exhausts all relative discrete series of G^0.*

This was proven in [Wol74, §3]. The essence of the proof relies on Mackey central extension:

$$1 \longrightarrow S \longrightarrow G[\xi] \longrightarrow ZG^0/Z \longrightarrow 1 \tag{2.8}$$

where $S \cong U(1) \subseteq \mathbb{C}$ the circle group, $\xi \in \widehat{Z}$ a central character and:

$$G[\xi] = \{S \times ZG^0\}/\{(\xi(z)^{-1}, z) \mid z \in Z\} \tag{2.9}$$

Note that $G[\xi]$ is a connected reductive group, with Lie algebra $\mathfrak{g} \oplus (\mathfrak{g}/\mathfrak{z})$ that has compact center, to which we can apply the theory of Harish-Chandra. Now $\widehat{G}[\xi]$ is mapped to $\widehat{ZG^0}$ in a fiber-preserving way:

Theorem 2.15 (**[Wol18, Theorem 3.3.2]**)**.** *Denote* Pr *the canonical projection of* $S \times ZG^0 \to G[\xi]$ *restricted to* ZG^0*. Then the following map* $\widehat{\Pr}$ *is a bijection:*

$$\widehat{\Pr} : \widehat{G[\xi]}_{1_S} \to \widehat{ZG^0}_\xi \qquad [\pi^0] \mapsto [\pi^0 \circ \Pr]$$

which maps $\widehat{G[\xi]}_{1_S-disc}$ *onto* $\widehat{ZG^0}_{\xi-disc}$*, and carries the Plancherel measure of* $\widehat{G[\xi]}_{1_S-disc}$ *to that of* $\widehat{ZG^0}_\xi$*.*

Now Theorem 2.14 follows readily from this and the classical result of Harish-Chandra. (see e.g. [**Kna86**, Theorem 9.20 & 12.21])

Remark 2.16 (Method of ascent)**.** To sketch its proof, one first reduces to the case when $Z \subseteq G^0$ by noting there is an isomorphism between $G^0[\xi|_{ZG^0}]$ and $G[\xi]$. One can further reduce to the case when G^0 is simply connected by passing to possible central cover. But in this case G^0 can be written as $V \times G_{ss}$ where G_{ss} is semisimple and $V \cong \mathbb{R}^n$ by [**Kna96**, Proposition 7.27(f)]. Hence the character splits into two parts $\xi \in \widehat{Z} = \nu \otimes \delta \in \widehat{V} \times \widehat{D}$ where $D \subseteq Z(G_{ss})$ is a subgroup of finite index. Again $G_{ss}[\sigma] \cong G^0[\xi|_{ZG^0}]$. Then this proof is reduced to the case $G^0 = G_{ss}$ simply connected semisimple, with $Z \subseteq G$ discrete. Note that $S \times G^0 \to G[\xi] \to G^0[\xi]$ is a Lie group covering and $\widehat{G^0[\xi]} \cong \widehat{G^0}_\xi$ via the map $\widehat{\Pr}$. Such a method of ascent will be a recurring theme in the upcoming proofs.

Next we lift the discrete series of the connected reductive group G^0 to $G^\dagger$, where all the central characters are accounted, and then to G, where the datum takes disconnectedness into account. Recall $G^\dagger = Z_G(G^0)G^0$ and ZZ_{G^0} has finite index in $Z_G(G^0)$ as G is of class $\widetilde{\mathcal{H}}$. Hence every representation of $\chi \in Z_G(G^0)$ is a summand of induced representations $\operatorname{ind}_{ZZ_{G^0}}^{Z_G(G^0)}(\tau \otimes \xi)$ from some character $\tau \otimes \xi$ on ZZ_{G^0}, and hence have representation dimension bounded by the index. The following summarizes the transition of unitary duals from G^0 to G [**Wol18**, §3]:

Theorem 2.17 ($\widehat{G^\dagger}$ **and** $\widehat{G}_d$)**.** *Recall* $G^\dagger = Z_G(G^0)G^0$*. Then* $\widehat{G^\dagger}$ *is a disjoint union of the following sets:*

$$\bigsqcup_\xi \left\{ [\chi \otimes \pi^0] \;\middle|\; [\chi] \in \widehat{Z_G(G^0)}_\xi, [\pi^0] \in \widehat{G^0}_\xi \text{ for } \xi \in \widehat{Z_{G^0}} \right\} \tag{2.10}$$

and $\pi = \operatorname{ind}_{G^\dagger}^G(\xi \otimes \pi^0) \in \widehat{G}$*. Moreover,*

$$\pi \in \widehat{G}_d \iff [\chi \otimes \pi^0] \in \widehat{G^\dagger}_d \iff [\pi^0] \in \widehat{G^0}_d \tag{2.11}$$

The following proof is modelled after [**Kna96**, Chapter XII.8]

Proof. First note that $[\chi \otimes \pi^0] \in \widehat{G^\dagger}_\xi$, and $\chi \otimes \pi^0|_{G^0} = \dim \chi \cdot \pi^0$ as $Z_G(G^0)$ acts trivially on $\widehat{G^0}$ and by Theorem 2.11 we apply Schur's lemma to G^0. On the other hand, each $[\gamma] \in \widehat{G^\dagger}$ contains $[\chi \otimes \pi^0]$ for obvious reasons, but then $[\gamma] = [\chi \otimes \pi^0]$ by Schur's lemma again.

To prove the second statement, note first that the second equivalence is an easy consequence of the first statement. To prove the first equivalence, one first observes that $\pi_d \in \widehat{G}_d$ are subrepresentation of $\operatorname{ind}_{G^\dagger}^G(\chi \otimes \pi^0)$ for some $\chi \otimes \pi^0 \in \widehat{G^\dagger}_d$, as $|G/G^\dagger| < \infty$. So to prove the assertion it suffices to prove the induced representation itself is irreducible. Again choose H and H^0 the relative compact Cartan subgroups. Take $W(G;H)$ as a subgroup of $W(\mathfrak{g}_\mathbb{C};\mathfrak{h}_\mathbb{C})$ by Remark 2.4 and choose a set of representatives $x_1, \dots, x_n \in G$ of the Weyl group $W(G;H)$. Next recall $G^\dagger = Z_G(G^0)G^0$ fixes $W(G^0;H^0)$ as a group, so it suffices to write $W(G;H)$ as $W(G^0;H^0)$-cosets:

$$W(G;H) = \bigcup_{x_i \in G/G^\dagger} (x_i H) W(G^0;H^0)$$

Now specify $\pi^0 = \pi^0_\lambda \in \widehat{G^0}_d$ for some nonsingular analytically integral $\lambda \in i\mathfrak{h}'$ by Theorem 2.14. Then $[\pi^0_\lambda \circ \operatorname{ad}(x_j)] = [\pi^0_{\operatorname{Ad}_{x_j}(\lambda)}] =: [\pi^0_{\lambda_j}]$ by comparing the global characters of G^0 [**Wol18**, Theorem 3.4.4]. Hence:

$$\pi|_{G^\dagger} = \sum_{x_j \in G/G^\dagger} \chi \otimes \pi^0_{\lambda_j} \tag{2.12}$$

with the terms on right side pairwise inequivalent. Therefore any bounded linear operator commuting with $\pi|_{G^\dagger}$ must be scalar on each $\pi^0_{\lambda_j}$. If this bounded linear operator commutes with G, then all these scalar must match. Therefore we have proven $[\pi]$ is irreducible. □

Next we give a complete description of the relative discrete series of G by carefully tracing the representation parameters along the theorems [**Wol18**, Theorem 3.5.7]:

Theorem 2.18 (**relative discrete series of groups of class $\widetilde{\mathcal{H}}$**). *Let H be a relative compact Cartan subgroup as above. Let $\lambda \in i\mathfrak{h}^*$ be a nonsingular relative to $\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C})$ and analytically integral on H^0 as in Theorem 2.14. Let $[\chi] \in \widehat{Z_G(G^0)}_\xi$ such that $\xi = e^{\lambda - \delta_G}|_{Z_{G^0}}$. Denote $[\pi^0_\lambda] \in \widehat{G^0}_d$ as before. Then every discrete series representations of G is of the form $[\pi_{\chi,\lambda}] := [\mathrm{ind}^G_{G^\dagger}(\chi \otimes \pi^0_\lambda)]$, with $[\pi_{\chi,\lambda}] = [\pi_{\chi',\lambda'}]$ if and only if they lie in the same $W(G;H)$-orbit.*

The realization of such relative series representations is a matter of separate interest, and will not be pursued in full generality in this thesis, except for the case of $\widetilde{SL_2(\mathbb{R})}$.

2.3 Tempered series and representation data

Similar to the original classification of irreducible tempered representations on linear reductive groups with compact center due to Knapp and Zuckerman [**KZ82**, Theorem 14.2] using non-degenerate cuspidal data, a similar classification gives rise to, according to Wolf, H-series representations here for reductive groups of class $\widetilde{\mathcal{H}}$. They behave similarly as Knapp-Zuckerman's original data, and in particular forms a disjoint union of irreducible tempered representations.

The bulk of the original work of Knapp and Zuckerman deals with the reducibility of induced series at the wall of Weyl Chamber. This matter will not be pursued here.

Definition 2.19 (**H-series representation**). Given a cuspidal parabolic subgroup $P = MAN$ as in Definition 2.10. We define the H-series to be the following unitarily induced representations: Given $\eta \in \widehat{M}_d$ and $\nu \in \mathfrak{a}^*$,

$$\pi_{\eta,i\nu} = \mathrm{ind}^G_{MA}(\eta \otimes e^{i\nu}) = \{f \in L^2(G; V_\eta) \mid f(gman) = e^{-(\delta_P + i\nu)(\log a)}\eta(m)^{-1}f(g)\} \tag{2.13}$$

with the L^2-norm taken over K/Z and G has a natural left regular action. i.e., for all $f \in \pi_{\eta,i\nu}$:

$$\|f\|^2 := \int_{K/Z} |f(k)|^2 \,\mathrm{d}kZ < \infty \qquad \pi_{\eta,i\nu}(g)f(g') = f(g^{-1}g') \tag{2.14}$$

where $\delta_P = \frac{1}{2}\sum_{\lambda \in \Sigma^+(\mathfrak{g};\mathfrak{a})}(\dim \mathfrak{g}_\lambda)\lambda$.

In the case H is compact, this recovers the relative discrete series $\widehat{G}_d$. Such H-series are precisely the set on which the Plancherel measure of $\widehat{G}$ is supported on. That is, they are tempered. Before we state the main result, we refine the description of $\widehat{M}_d$ more explicitly like $\widehat{G}_d$.

Recall $\mathfrak{t}$ defines a Cartan subalgebra of $\mathfrak{m}$ in the decomposition of $\mathfrak{h}$ with the corresponding Cartan subgroup T relative compact. Hence we can parametrize $\widehat{M}_d$ in the same way as we parametrize $\widehat{G}_d$ in Theorem 2.18: Denote:

$$L'_\mathfrak{t} = \{\tau \in i\mathfrak{t}^* \mid e^\tau \in \widehat{T^0} \text{ and } \tau \text{ nonsingular relative to } \Delta(\mathfrak{m}_\mathbb{C}; \mathfrak{t}_\mathbb{C})\} \tag{2.15}$$

with $[\chi] \in \widehat{Z_M(M^0)}_\xi$ with $\xi = e^{\tau - \delta_\mathfrak{t}}|_{Z^0_M}$, and we write $[\pi_{\eta,\sigma}]$ as:

$$\left\{[\pi_{\chi,\tau,\sigma}] \,\middle|\, \tau \in L'_\mathfrak{t}, [\chi] \in \widehat{Z_M(M^0)}_\xi \text{ such that } \xi = e^{\tau - \delta_\mathfrak{t}}|_{Z^0_M}\right\} \tag{2.16}$$

and note that $[\pi_{\chi,\tau,\sigma}] = [\pi_{\chi',\tau',\sigma'}]$ if and only if the two tuples (χ, τ, σ) lies in the same $W(G;H)$-orbit via the following remark and [**Wol18**, Theorem 4.5.3]:

Remark 2.20. Such a $(\mathfrak{m}_{\mathbb{C}}; \mathfrak{t}_{\mathbb{C}})$-root system can be extended in a unique way to a $(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$-root system: If $\sum_{\mathfrak{a}}^{+}$ is a positive restricted $\mathfrak{a}$-root system on $\mathfrak{g}$ and $\Delta^{+}(\mathfrak{m}_{\mathbb{C}}; \mathfrak{t}_{\mathbb{C}})$ a root system on $\mathfrak{m}_{\mathbb{C}}$, then there exists a unique positive $\mathfrak{h}_{\mathbb{C}}$-root system on $\mathfrak{g}_{\mathbb{C}}$, with $\Delta^{+}(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$ such that: [**Wol18**, Lemma 4.1.5]

$$\Sigma_{\mathfrak{a}}^{+} = \{\alpha|_{\mathfrak{a}} : \gamma \in \Delta_G^{+} \quad \gamma|_{\mathfrak{a}} \neq 0\} \qquad \Delta^{+}(\mathfrak{m}_{\mathbb{C}}; \mathfrak{t}_{\mathbb{C}}) = \{\alpha|_{\mathfrak{t}} : \gamma \in \Delta_G^{+} \quad \gamma|_{\mathfrak{a}} = 0\} \tag{2.17}$$

We conclude this section with a computation of Casimir eigenvalues. Recall $\Omega_G = \sum_i X_i X^i \in U(\mathfrak{g}_{\mathbb{C}})$ with X_i a basis of $\mathfrak{g}$, and X^i its dual basis with respect to the Killing form B. The center of universal enveloping algebra $Z(\mathfrak{g}_{\mathbb{C}})$ acts on each irreducible $\mathfrak{g}$-module by scalars. Using Harish-Chandra isomorphism [**Kna86**, Theorem 8.18] one can identify $Z(\mathfrak{g}_{\mathbb{C}})$ with $W(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$-invariant polynomials on $\mathfrak{h}$. In particular, any infinitesimal character χ_π can be identified with an element $\lambda_\pi \in \mathfrak{h}_{\mathbb{C}}^{*}$; and $\Omega_G \in Z(\mathfrak{g}_{\mathbb{C}})$ the Casimir element with an element in $P[\mathfrak{h}]^W$. We may explicitly compute the Casimir eigenvalue $\chi_\pi(\Omega)$ as:

Proposition 2.21. [**Kna86**, Proposition 8.22] *Fix the parabolic data $P = MAN$ of G and decompose $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ based on θ-parity again. Let π_σ be an irreducible unitary representation V_σ of M with the infinitesimal character λ_σ, such that $\sigma \in \mathfrak{t}_{\mathbb{C}}$. Now if $\nu \in \mathfrak{a}_{\mathbb{C}}^{*}$, then the tempered series $\pi_{\sigma,\nu}$ has infinitesimal character $\lambda_\sigma + \nu \in (\mathfrak{a} \oplus \mathfrak{t})_{\mathbb{C}}^{*}$. Its Casimir eigenvalue is*

$$\Omega_{\sigma,\nu} = \chi_{\sigma,\nu}(\Omega) = \|\sigma + i\nu\|^2 - \|\delta_G\|^2 = -\|\nu\|^2 + \|\lambda_\sigma\|^2 - \|\delta_G\|^2 . \tag{2.18}$$

In particular, the case $M = G$ corresponds to the (relative) discrete series of G and $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}$ the compact Cartan subalgebra, and consequently the Casimir eigenvalue is $\|\sigma\|^2 - \|\delta_G\|^2$.

2.4 Normalization of invariant measure

Note that the normalization of Plancherel measure is dependent on normalization of the measure on the group G. As the matter will play a major role in later explicit computations of torsion constants, we shall explain in more detail.

The normalization of the Plancherel measure is related with that of the Haar measure such that, for all $f \in C_c^\infty(G)$, the following equality holds:

$$f(e_G) = \int_{\widehat{G}} \Theta_\pi(f)\, \mathrm{d}\mu(\pi) \tag{2.19}$$

where the definition of $\Theta_\pi(f)$ uses the Haar measure. Often one normalizes the measure such that the eventual formula admits much cleaner form.

In hindsight, the desired normalization needs to be compatible with some prescribed bilinear forms on the subgroups and quotients of the Lie group. In the reductive cases this is the Killing form B, with respect to which we perform the integration. In the case one choose another bi-invariant bilinear form, the scalar difference should therefore be heeded. In nilpotent groups, there is a canonical measure on each coadjoint orbit of some $l \in \mathfrak{n}$, which is induced from the symplectic form induced by $B_l(X,Y) := l([X,Y])$.

We follow [**War72**, 8.1.2] in the case of reductive Lie groups:

1. First one restricts to the Lie algebra case: Take the measure $\mathrm{d}\mathfrak{g}$ on $\mathfrak{g}$ to be induced by B^θ. This also give canonical measure on each Cartan subalgebra $\mathfrak{j}$. This measure is independent of choice of Cartan involution. The general reductive case is dealt similarly;
2. Next write the regular set of $\mathfrak{g}$ as a disjoint union of conjugacy class of each Cartan subalgebra $\mathfrak{j}$, where the complement of this union in $\mathfrak{g}$ has measure zero. This leads to a normalization of a G-invariant measure $d_{G/J}$ for each coset such that the Weyl integration formula holds [**Kna86**, Lemma 11.4]. Such a normalization will be handy when we invert the invariant orbital integrals later.

3. As for the Cartan subgroup $J \in \mathrm{Car}(G)$, we take $J = J_K J_{\mathfrak{p}}$ with respect to the Cartan involution into ± 1-part, and take $d_{J_{\mathfrak{p}}}$ the Haar measure on $J_{\mathfrak{p}}$ induced by exponentiating the corresponding measure on $\mathfrak{h}_{\mathfrak{p}}$. One normalizes the Haar measure on J_K such that $d_H = d_{J_K} d_{J_{\mathfrak{p}}}$ is the product measure under above decomposition. With all these normalizations in force, the Plancherel density is specified to the last detail.

Now the normalization can be extended to the case of $\widetilde{\mathcal{H}}$ by taking into account the following factors [**HW86b**, §1]: Assume first Z has been enlarged so that $Z \cap G^0 = Z_{G^0}$. Recall $G[\xi]$ in (2.9), which is a reductive group of Harish-Chandra's class. So we follow [**War72**, 8.1.2], i.e., such that $B^\theta(X, Y) := B(X, \theta Y)$ induced by the Killing form defines the Riemannian metric hence the Haar measure on $G[\xi]$. Next we see how to lift the measure to the whole G. The pathway is the same as in Remark 2.16:

1. First normalize the Haar measure on $S \subseteq G[\xi]$ such that $\mathrm{vol}(S) = 1$. Then the measure on $G[\xi]$ admits the splitting
$$\int_{\widetilde{G}[\xi]} \phi(x)\,\mathrm{d}x = \int_{G[\xi]/S} \int_S \phi(xs)\,\mathrm{d}s\,\mathrm{d}xS. \tag{2.20}$$
We assume the isomorphism $G[\xi]/S \to ZG^0/Z$ measure-preserving;
2. Fix a Haar measure on $Z_G(G^0)$ that it has measure 1 if it is compact, and use counting measure if it is an infinite discrete group;
3. Fix the measure on Z and ZG^0 such that it is a product measure of $Z_G(G^0)/Z \times Z$ and $ZG^0/Z \times Z$ respectively. Now the Haar measure on G can be normalized to be a product measure of the counting measure G/ZG^0 and the measure on ZG^0.
4. Finally we normalize the measure on $\widehat{Z}$ so that $f(x) = |Z_G(G^0)/Z| \int_{\widehat{Z}} f_\xi(x)\,\mathrm{d}\xi$ for f_ξ the Fourier transform on Z according to ξ.

Remark 2.22 (different normalizations)**.** The aforementioned normalization differs from Harish-Chandra's original calculation [**HC75**, Section 7, Page 115] in the Haar measure on K: there the measure on K is normalized to have volume one. This accounts for the difference in the Plancherel density by some compact root product. We will return to this matter when discussing specific cases.

In this thesis we have adopted two version: First in this section when discussing the Plancherel formula of class $\widetilde{\mathcal{H}}$, in which K is a noncompact subgroup, whence following the above convention is more convenient. Later in the discussion of twisted L^2-invariant, where K in question is compact, whence we normalize the Haar measure on K to be one for cleaner exposition. See e.g. Remark 7.9. We will not attempt to align these two, as each serves their purposes in a better way.

2.5 Global Plancherel formula

With almost all the ingredients in place we state the full formula as stated in [**HW86b**, Theorem 6.17]:

Theorem 2.23 (Plancherel formula of real reductive groups of class $\widetilde{\mathcal{H}}$). *Let G be a linear connected reductive group of class $\widetilde{\mathcal{H}}$, and let* $\mathrm{Car}(G)$ *be a complete set of non-G-conjugate Θ-stable Cartan subgroups, with H a fundamental Cartan subgroup of G. Fix Q_J, M_J, A_J, N_J as in Definition 2.10 for each $J \in \mathrm{Car}(G)$ and take $L_J := L_{\mathfrak{t}_J}$ and L'_J as in (2.15). For $\xi \in \widehat{Z}$, write:*
$$L_\xi = \{\tau \in L \mid e^{\tau - \delta_{\mathfrak{t}}} = \xi|_{Z \cap M^0}\} \tag{2.21}$$
and let $L'_{J,\xi} = L_{J,\xi} \cap L'_J$. Then there exists a unique family of explicitly computable meromorphic functions $m^J(\lambda, \xi, \nu)$ on $\mathfrak{a}^*$ such that for all functions $f \in C_c^\infty(G)$:
$$\begin{aligned} f(e_G) =& \mathbf{c}_G \cdot \frac{|W(G^0, H \cap G^0)|}{|G/Z_G(G^0)G^0|(2\pi)^{r+\mathrm{frk}(G)}} \sum_{J \in \mathrm{Car}(G)} \\ & \int_{\chi \in \widehat{Z_{M_J}(M_J^0)}} \sum_{\tau \in L'_{J,\chi}} \Big\{ \dim \chi \times \int_{\mathrm{i}\mathfrak{j}_{\mathfrak{p}}{}^*} \Theta^J_{\chi,\tau,\nu}(f) m^J(\chi : \tau : \nu)\,\mathrm{d}\nu\,\mathrm{d}\chi \Big\} \end{aligned} \tag{2.22}$$

where $\mathbf{c}_G := |\pi_1((ZG^0/Z)_{\mathbb{C}})|$ *the fundamental group of the complexified group of* ZG^0/Z. *It is known to be finite and independent of* ξ *[**HW86b**, Lemma 1.18]; r the number of positive roots, and* $\mathrm{frk}(G) = \mathrm{rank}_{\mathbb{C}}(G) - \mathrm{rank}_{\mathbb{C}}(K)$ *the fundamental rank of* G. *The Plancherel density* $m^J(\chi : \tau : \nu)$ *is a meromorphic function in* $\nu \in \mathfrak{j}_{\mathfrak{p}\mathbb{C}}^*$ *which admits the following expression:*

$$m^J(\chi : \tau : \nu) := c(G;J) \left| \prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}};\mathfrak{j}_{\mathbb{C}})} \langle \alpha, \tau + i\nu\rangle \prod_{\alpha\in\Delta_R^+(\mathfrak{g};\mathfrak{j})} \overline{p_\alpha}(\chi:\nu) \right| \tag{2.23}$$

with:

(a) $c(G;J)$ *an explicit expression given in [**HW86b**, Theorem 4.18]:*

$$c(G;J)^{-1} := c_{\mathfrak{g}} |W(G;J)| [J_K : J_K \cap M_H^\dagger] \prod_{\alpha\in R_J} \|\alpha\| \tag{2.24}$$

here $c_{\mathfrak{g}}$ *is a constant depending on the two-systems of* G[1]; R_J *a set of noncompact strongly orthogonal roots of* $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$ *that determines the Cartan subgroup* $J \in \mathrm{Car}(G)$. *and recall* $M^\dagger = Z_M(M^0)M$ *appearing in Theorem 2.17.*

(b) $\overline{p_\alpha}(\chi:\nu)$ *is the average of elementary functions* $\overline{p_\alpha}(\chi:\nu) := (\dim\chi)^{-1}\,\mathrm{tr}\,p_\alpha(\chi:\nu)$ *where:*

$$p_\alpha(\chi:\nu) := \sinh \pi\nu_\alpha \cdot \mathbb{I}_k \cdot \left(\cosh \pi\nu_\alpha \cdot \mathbb{I}_k - \frac{\chi_\alpha(\gamma_\alpha)}{2}[\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})]\right)^{-1} \tag{2.25}$$

with $\mathbb{I}_k$ *the* $k\times k$*-identity matrix,* $\nu_\alpha = \frac{2\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}$; γ_α *the generator of a suitably normalized* J_K *[**HW86b**, §2], and* χ_α *a character on it corresponding to* $\delta_{\Delta_\alpha^+}$:

$$\Delta_\alpha^+ := \{\beta \in \Delta^+(\mathfrak{g}_{\mathbb{C}};\mathfrak{j}_{\mathbb{C}}) \mid \beta|_{\mathfrak{j}_{\mathfrak{p}}} \textit{ is a multiple of } \alpha\} \tag{2.26}$$

Moreover, m^J *satisfies the following properties: [**Wol18**, Theorem 5.1.1]*

1. *The* $m^J(\chi:\tau:\nu)$ *are* $W(G;J)$*-invariant:* $(w^*m^J)(\chi:\tau:\nu) := m^J(w^*\chi : w^*\nu : w^*\sigma) = m^J(\chi : \tau:\nu)$;
2. *If* $\tau \neq L_{j,\xi}$*, then* $m^J(\chi:\tau:\nu) = 0$.

Remark 2.24 (fundamental Plancherel density)**.** For later use, we see in particular in the case $J = H$ the maximal compact (cuspidal) Cartan subgroup, the Plancherel density simplified significantly: first $\prod_{\alpha\in\Delta_R^+(\mathfrak{g};\mathfrak{h})}$ is dropped altogether, as H has no real roots [**Kna86**, Proposition 11.16]. Consequently, the density function admits the following form again for $\nu \in \mathfrak{h}_{\mathbb{C}}^*$:

$$m^H(\chi:\tau:\nu) = c(G;H) \left| \prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})} \langle\alpha, \tau+\nu\rangle \right| \tag{2.27}$$

where $c(G;H)$ is computed in this case similarly to [**HC75**, Lemma 27.3]. It depends only on the normalization of Haar measure. The above expression differs from the original formula of Harish-Chandra by a factor of:

$$\prod_{\alpha\in\Delta^+(\mathfrak{k}_{\mathbb{C}};\mathfrak{t}_{\mathbb{C}})} \langle\alpha, \delta_K\rangle \tag{2.28}$$

this arises from the different normalizations on K, as mentioned in Remark 2.22. For the other version, we set an expression as (7.38).

[1] A **two system** is a decomposition of the root system into irreducible components of type A_1 or C_2, invented for regrouping the characters for Plancherel formula of higher rank. Interested readers can refer to [**Kna86**, P.504].

To keep a lean exposition, we will not define most of the terms in full details, but rather seeing their incarnation in the explicit formula of $G = \widetilde{SL_2(\mathbb{R})}$. In this case, G has two Cartan subgroups up to conjugacy: the fundamental Cartan subgroup H_1 is isomorphic to the subgroup $K \cong \mathbb{R}^1$, where the maximally noncompact one H_2 is the inverse image of the $\mathbb{R}$-split component A the group of diagonal matrices of determinant 1 in $SL_2(\mathbb{R})$, under the canonical projection $\widetilde{SL_2(\mathbb{R})} \longrightarrow SL_2(\mathbb{R})$, and is isomorphic to $Z_G \times A$. Moreover, denote $\{\beta\} = \Delta^+(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_{1,\mathbb{C}})$ and $\{\alpha\} = \Delta^+(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_{2,\mathbb{C}})$. We collect the data here:

1. If $G = \widetilde{SL_2(\mathbb{R})}$, then $G = G^0$ connected, with $G[\xi]/S = SL_2(\mathbb{R})$ with $r = 1$, and $ZG^0/Z \cong PSL(2, \mathbb{R})$, hence:
$$\pi_1(PSL(2,\mathbb{R})) = \pi_1(SO(3,1)) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$$
and $\mathrm{frk}(G) = 0$, and we see $|W(G : H_1)| = 2$;
2. Same as the $SL_2(\mathbb{R})$ case, $\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_{1,\mathbb{C}}) = \Delta_I(\mathfrak{g}; \mathfrak{h}_1)$ consists solely of imaginary roots, whereas $\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_{2,\mathbb{C}}) = \Delta_R(\mathfrak{g}; \mathfrak{h}_2) = \{E_\alpha, E_{-\alpha}\}$ consisting solely of real roots.
3. $M_{H_1} = M_1 = M_1^0 = G$, where $Z_{M_1}(M_1^0) = Z_G \cong \mathbb{Z}$, also $M_2 = M_{H_2} = Z_G$ hence $Z_{M_2}(M_2^0) = Z_{Z_G}(e_G) \cong \mathbb{Z}$, hence by our normalization $\int_{\chi \in \widehat{Z_{M_1}(M_1^0)}} \mathrm{d}\chi = \int_0^1 \mathrm{d}\sigma$ parameterized as Lebesgue measure on $[0, 1)$;
4. For any $\chi \in \widehat{Z_{M_1}(M_1^0)}$, identify L_1 with $\{t \in i\mathfrak{t}^* | e^t \in \widehat{K}\} \cong i\mathbb{R}$, hence:
$$L'_{1,\chi} = \{t \in i\mathbb{R} \mid t \neq 0; e^t = \chi|_{Z_G}\} = \{\tau \in \mathbb{R}_{\neq 0} \mid i\tau \equiv d\chi(\log \gamma_Z) \mod \mathbb{Z}\} \tag{2.29}$$
where γ_Z the generator of Z_G. Similarly, we obtain that $L'_{2,\chi}$ are singletons for each $\chi \in Z_{M_2}(M_2^0)$;
5. the constants $c(G; J)$ are very simple in our case: $R_{H_1} = \emptyset$ and $R_{H_2} = \{\alpha\}$; $c_\mathfrak{g} = 1$ and $M_{H_2}^\dagger = M_{H_2}$, with $(H_2)_K = Z_G$, hence $[(H_2)_K : (H_2)_K \cap M_{H_2}^\dagger] = 1$. Lastly $|W(G; H_1)| = 1$ and $|W(G; H_2)| = 2$. Hence the terms in (2.24) can explicitly computed here as:
$$\begin{aligned} c(G; H_1) &= |W(G; H_1)|^{-1} = 1 \\ c(G; H_2) &= \left(|W(G; H_2)| \cdot \prod_{\alpha \in R_{H_2}} \|\alpha\| \right)^{-1} = \frac{1}{2} \end{aligned} \tag{2.30}$$
6. Next we compute the product terms in m^J. For m^{H_2}, $\prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_{2,\mathbb{C}})} \langle \alpha, \tau + i\nu \rangle = \langle \alpha, \tau \rangle$ because α are all real-valued in this case, hence are orthogonal to $i\nu$. For similar reasons $\prod_{\beta \in \Delta^+(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_{1,\mathbb{C}})} \langle \beta, \tau + i\nu \rangle = \langle \beta, i\nu \rangle \in \mathbb{R}$.
7. Lastly we compute the $\bar{p}^{H_i}(\chi : \nu) = p^{H_i}(\chi : \nu)$. There is no real root in H_1, hence $p^{H_1} = 1$. On the other hand, for $\gamma_\alpha := \exp(\pi(E_\alpha - E_{-\alpha}))$. If we identify $\chi_\sigma \in \widehat{Z(\mathfrak{a})}$ with $\sigma \in [0, 1)$ as above, then $\chi_\sigma(\gamma_\alpha) = e^{2\pi i \sigma}$. Note that $\nu_\alpha = 2\nu$:
$$p^{H_2}(\chi_\sigma, \nu) = \frac{\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha + \frac{\chi_\sigma(\gamma_\alpha) + \chi_\sigma(\gamma_\alpha^{-1})}{2}} = \frac{\sinh 2\pi\nu}{\cosh 2\pi\nu + \cos 2\pi\sigma} = \mathrm{Re}\tanh(\pi(\sigma + i\nu)) \tag{2.31}$$
by identifying $\{\chi(\gamma_\alpha) \mid \chi \in \widehat{Z_G}\}$ with $\mathbb{R}/\mathbb{Z}$. The last identity comes from:
$$\begin{aligned} \tanh(x + iy) &= \frac{2\sinh(x+iy)\cosh(x-iy)}{2\cosh(x+iy)\cosh(x-iy)} \\ &= \frac{\sinh(2x) + \sinh(2iy)}{\cosh(2x) + \cosh(2iy)} \\ &= \frac{\sinh(2x)}{\cosh(2x) + \cos(2y)} + i\frac{\sin(2y)}{\cosh(2x) + \cos(2y)} \end{aligned}$$

Now we incorporate all data into (2.22) and note:

$$\begin{aligned}2\pi \cdot f(e_{\widetilde{SL_2(\mathbb{R})}}) &= 2\int_0^1 \sum_{\substack{\tau\in\mathbb{Z}\\ \tau\neq 0}} \Theta_{\sigma,\tau}(f)\tau \, \mathrm{d}\sigma + 2\int_0^1 \int_{-\infty}^{\infty} \Theta_{\sigma,\nu}(f)|\sigma \mathrm{Re}\tanh(\pi(\sigma+i\nu))| \, \mathrm{d}\nu \, \mathrm{d}\sigma \\ &= 2\int_{\mathbb{R}} \Theta_\tau(f)\tau \, \mathrm{d}\tau + 2\int_0^1 \int_0^{\infty} \Theta_{\sigma,\nu}(f)\mathrm{Re}\tanh(\pi(\sigma+i\nu))\sigma \, \mathrm{d}\nu \, \mathrm{d}\sigma\end{aligned} \tag{2.32}$$

and we have retrieved the original formula due to Pukánzky [**Puk64**, Introduction]. There are two minor differences here: First in the original paper of Pukánzky the discrete series are parameterized by lowest K-type, which is $\chi + \frac{1}{2}$ here. Hence there is a change of variable here; Secondly our convention differs from the original in the parametrization of the Lebesgue measure, The original convention parametrizes K/Z to be of measure 1, as opposed to π here. This accounts for a factor of π in the final expression.

2.6 Schwartz spaces

This section is a topic of separate interest. Our applications require the Plancherel formula can be applied to a wider class of functions than compact support ones, owing to the fact that the heat kernel on noncompact manifolds is a Schwartz function but always with noncompact support. Readers who are not interested in the technical details can skip this section or take the following statement for granted:

Theorem 2.25 ([**HW86a**, Theorem 7.6])**.** *Let G be reductive Lie group of class $\widetilde{\mathcal{H}}$. Assume further that Z/Z^0 is finitely generated. Then for $f \in \mathcal{S}(G)$, the* ***Plancherel formula of real reductive groups of class*** $\widetilde{\mathcal{H}}$ *holds.*

Meticulous readers, on the other hand, should not be obscured by the simplicity of the statement. Proving Fourier transform is an automorphism on the Schwartz functions is a result of fundamental importance in classical harmonic analysis. This, when extended to the realm of noncommutative harmonic analysis, posed problems of equal if not greater significance. In Harish-Chandra's original treatment of the Plancherel formula, the theory of Schwartz distribution is also needed in the derivation of the Plancherel densities in fully explicit form. See [**Var73**] for a clear exposition of the method of Harish-Chandra, namely his theory of cusp forms.

In the meantime, Sally and Warner [**SW73**] took up the task of directly inverting the orbit integral F_f^B, bypassing the formidable machinery of Eisenstein integrals and intertwining operators. This method was later furthered by Herb [**Her79**], and is the method adopted by Herb and Wolf eventually in [**HW86b**, **HW86a**] to prove the Plancherel formula for general semisimple Lie groups. In this approach it is not known *a priori* that the characters, or packets of characters are of Schwartz class. For our purposes though, such an approach is sufficient.

The main idea is again to use Remark 2.16, the method of ascent: Construct first $\mathcal{S}(G/Z,\xi)$ the Schwartz space on G/Z, this is essentially the original Schwartz space of Harish-Chandra. Then we lift the corresponding function to ZG^0/Z and finally to the whole G. We begin with a brief introduction of relative Schwartz functions on G/Z mirroring the original treatment of Harish-Chandra class. In a nutshell these are functions suitably generalizing the idea of rapid decay of Schwartz function on $\mathbb{R}^n$ against any polynomial function. In the class $\widetilde{\mathcal{H}}$ there are typically Euclidean factors in K, so we need a slight modification in bounding the spherical function. One might compare them with the Schwartz function of Harish-Chandra class [**Kna86**, VII.8 &XII.4].

First we recall the definition of spherical functions. In representation-theoretical terms, these are matrix coefficients of the K-invariant vectors in the irreducible representations. Then every $g \in G$ can be factored into:

$$g = \kappa(g)\exp H(g)n(g) \text{ where } H(x) \in \mathfrak{a}, \kappa(g) \in K, n(g) \in N \tag{2.33}$$

by Iwasawa decomposition Definition 2.5. Define the spherical function on G as:

$$\phi_\nu^{G/Z}(g) := \int_{K/Z} e^{-(\nu+\delta_{\mathfrak{p}})H(kg)} \, \mathrm{d}(kZ) \qquad \nu \in \mathbb{C}^{\dim A_P} \tag{2.34}$$

where $\delta_{\mathfrak{p}} = \delta_{P_{\mathfrak{p}}}$ corresponds to the minimal parabolic subgroup. Recall $G/Z_G(G^0)$ is of Harish-Chandra's class. These $\phi_0^{G/Z}(g)$ are matrix coefficients of the induced representation $\mathrm{ind}_P^{G/Z}(1 \otimes e^{\nu})$. In this case the classical theory applies [Kna86, VII.8] and $\phi_\nu^{G/Z}(g)$ has the following properties directly from those of $G/Z_G(G^0)$:

1. It is K-bi-invariant: $\phi_\nu^{G/Z}(kgk') = \phi_v^{G/Z}(g)$ for all $k, k' \in K$;
2. It is symmetric and $W(G;A)$-invariant: $\phi_{wv}^{G/Z}(g) = \phi_v^{G/Z}(g) = \phi_v^{G/Z}(g^{-1})$ for all $w \in W(G;A)$ and $g \in G$.
3. It is dominated by the spherical vector $\phi_0^{G/Z}$: i.e., for $a \in \exp \overline{\mathfrak{a}^+}$, if ν is real and dominant with respect to $\Sigma^+(\mathfrak{g};\mathfrak{a})$, then:
$$\phi_\nu^{G/Z}(a) \le e^{\nu \log a} \phi_0^{G/Z}(a)$$
4. $\phi_0^{G/Z}$ is tempered: For a suitable constant $d \ge 0$, the following estimate holds for all $a \in \exp \overline{\mathfrak{a}^+}$:
$$\phi_0^{G/Z}(a) \le C e^{-\delta_{\mathfrak{p}} \log a} (1 + \log a)^d \tag{2.35}$$
hence $\phi_0^{G/Z} \in L^{2+\epsilon}(G/Z)$ for any $\epsilon > 0$, as $K \exp \overline{\mathfrak{a}^+} K = G$ and we use the integration formula [Kna86, Proposition 5.28] to yield this result.

To obtain a finer estimate of $\phi_0^{G/Z}$ for later use, define a function by extending the norm on $\mathfrak{p}$, like the classical case: The Euclidean norm induced by the Killing form is positive definite on $\mathfrak{p}$ and hence defines a Euclidean norm on $\mathfrak{p}$. Next we extend this to a function $|\cdot|_{\mathfrak{p}} : G \to \mathbb{R}_{\ge 0}$ by defining it for $g = k \exp X \in K \exp \mathfrak{p}$ to be $|g|_{\mathfrak{p}} = B(X,X)_{\mathfrak{p}}^{1/2}$. Then one readily checks this defines a K-bi-invariant seminorm on G with:

$$c_1 |a|_{\mathfrak{p}} \le \rho_{\mathfrak{p}} \log a \le c_2 |a|_{\mathfrak{p}} \qquad \text{for } a \in \exp \overline{\mathfrak{a}^+} \tag{2.36}$$

From this one derives a finer estimate of $\phi_0^{G/Z}(g)$:

$$\frac{\phi_0^{G/Z}(g)}{(1+|g|)^r} \text{ is in } L^2(G/Z) \text{ for } r \text{ sufficient large} \tag{2.37}$$

By identifying the action of $U(\mathfrak{g}_{\mathbb{C}})$ on $C^\infty(G)$ from the left (resp. from the right) with the algebra of left-invariant (resp. right-invariant) differential operators. We denote their actions as D_L and D_R respectively. .

Definition 2.26 (relative Schwartz space). Let G be a Lie group of class $\widetilde{\mathcal{H}}$. For $\xi \in \widehat{Z}$, define:

$$C^\infty(G/Z;\xi) := \left\{ f \in C^\infty(G) \,\middle|\, f(xz) = \xi(z)^{-1} f(x) \right\} \tag{2.38}$$

For $D, E \in U(\mathfrak{g}_{\mathbb{C}})$ and $r \in \mathbb{R}$, define seminorms $v_{D,E,r}(\cdot)$ on $C^\infty(G/Z;\xi)$:

$$v_{D,E,r}(F) = \sup_{x \in G} \left| (1 + |x|_{\mathfrak{p}})^r \phi_0^{G/Z}(x)^{-1} D_L E_R F(x) \right| \tag{2.39}$$

Define the relative Schwartz space $\mathcal{S}(G/Z;\xi)$ as functions $F \in C^\infty(G/Z;\xi)$ such that $|F|_{D,E,r} < \infty$ for all $D, E \in U(\mathfrak{g}_{\mathbb{C}})$ and for all $r > 0$.

To extend the notion of Schwartz spaces from G/Z to G, we need to replace the growth function $|\cdot|_{\mathfrak{p}}$ by functions that encapsulates the distance along the Z direction. **From now onwards we assume $G = G^0$ is connected.** The following lemma isolates the central direction from K:

Lemma 2.27 ([HW90, Proposition 2.1])**.** *K has a unique maximal compact subgroup K_1 with decomposition:*

$$K = K_1 \times V \tag{2.40}$$

where V is a closed normal vector subgroup of K. If we further take $Z = Z_G \cap V$, then Z is cocompact in both V and Z_G.

Now $(v, k_1, X) \mapsto v \cdot k_1 \cdot \exp(X)$ defines a diffeomorphism of $V \times K_1 \times \mathfrak{p}$ onto G by [HW86a, Lemma 6.3] and we decomposition $g \in G$ as in (2.33):

$$g = v(g) \cdot \kappa_1(g) \exp(X(g)) \in VK_1 \cdot \exp(\mathfrak{p}) \tag{2.41}$$

Then we extend $|\cdot|_{\mathfrak{p}}$ to a function that captures the V-direction growth:

$$|\cdot|_{\mathfrak{p}\mathfrak{z}} : G \to \mathbb{R}_{\geq 0} \qquad |g|_{\mathfrak{p}\mathfrak{z}} := \|v(g)\| + |X|_{\mathfrak{p}}$$

where $\|\cdot\|$ is the induced norm on $\mathfrak{k}$ by B^θ. Then $|\cdot|_{\mathfrak{p}\mathfrak{z}}$ is K_1-bi-invariant. Now the Schwartz space $\mathcal{S}(G)$ on the whole group G can be defined in a similar fashion as that on G/Z:

Definition 2.28 (Schwartz space). Let G be a connected reductive Lie group of class $\widetilde{\mathcal{H}}$. Let $X^\alpha, X^\beta \in U(\mathfrak{g}_{\mathbb{C}})$ and $r \in \mathbb{R}$. We define a seminorm:

$$v_{\alpha,\beta,r}(F) := \sup_{x \in G} \left| (1 + |x|_{\mathfrak{p}\mathfrak{z}})^r \phi_0^{G/Z}(x)^{-1} L(X^\alpha) R(X^\beta) F(x) \right| \tag{2.42}$$

The Schwartz space $\mathcal{S}(G)$ is the space of all functions $F \in C^\infty(G)$ such that $|F|_{D,E,r} < \infty$ for all $D, E \in U(\mathfrak{g}_{\mathbb{C}})$ and for all $r > 0$.

The seminorms $v_{\alpha,\beta,r}$ make $\mathcal{S}(G)$ into a complete locally convex topological vector space [HW86a, p.81]. The following properties of $\mathcal{S}(G/Z;\xi)$ (resp. $\mathcal{S}(G)$) manifests that it is a natural object of study:

1. $\mathcal{S}(G/Z;\xi)$ (resp. $\mathcal{S}(G)$) is a dense subspace of $L^2(G/Z;\xi)$ (resp. $L^2(G)$) and the inclusion is continuous; [HW86a, Theorem 2.7 &6.11]
2. $C_c^\infty(G/Z;\xi)$ (resp. $C_c^\infty(G)$) is dense in $\mathcal{S}(G/Z;\xi)$ (resp. $\mathcal{S}(G)$); [HW86a, Theorem 2.8 &6.13]
3. $\mathcal{S}(ZG^0/Z;\xi)$ (resp. $\mathcal{S}(G)$) forms a topological algebra under convolution, with left and right regular representations of ZG^0 (resp. G) on it are differentiable. [HW86a, Theorem 2.12ff &6.14ff]
4. For each Cartan subgroup J, each (J,χ)-summand in (2.22), interpreted as a distribution on $f \in C_c^\infty(G/Z;\xi)$, extends continuously to $\mathcal{S}(G/Z;\xi)$, i.e., they are tempered distributions. [HW86a, Lemma 3.5].
5. Fix $\xi \in \widehat{Z}$, the integral along Z direction defines a continuous map on the Schwartz spaces [HW86a, Theorem 7.2]:

$$\mathcal{S}(G) \to \mathcal{S}(G/Z;\xi) \qquad f(g) \mapsto f_\xi(g) := \int_Z f(gz)\xi(z)\,\mathrm{d}Z \tag{2.43}$$

   In fact $v_{D,E,r}(f_\xi) \leq (\int_Z (1 + |z|_{\mathfrak{p}\mathfrak{z}})^{-d}\,\mathrm{d}z) \cdot v_{D,E,r}(f)$ for each pair (D, E, r), for sufficiently large d such that the integral is finite.

We now sketch the key ingredients in the proof of Theorem 2.25. Recall again Remark 2.16 the method of ascent. The tempered estimates are first established at $\mathcal{S}(G/Z;\xi)$-level, where all estimates are essentially lifting respective results from the group $G/Z_G(G^0)$ of Harish-Chandra class. As Z is cocompact in $Z_G(G^0)$, such a lifting has little effect on estimates of growth.

Next we measure the decay of tempered ξ-distribution on G, that is, continuous linear functional on $\mathcal{S}(G/Z;\xi)$. Given a central $Z(\mathfrak{g}_{\mathbb{C}})$-finite distribution T, which by Harish-Chandra's regularity theorem [War72, Theorem 8.3.3.1] is a locally integrable function, that is:

$$T(f) = \int_G f(g) F_T(g)\,\mathrm{d}g \tag{2.44}$$

for some $F_T \in L^1_{\mathrm{loc}}(G)$ that is real analytic on the regular set G'. Recall the regular elements of a Cartan subgroup H are those elements with $e^\lambda(H) \neq 1$, and G' contains all the G-conjugates of these regular elements, for all $H \in \mathrm{Car}(G)$. Then T is proven to be tempered if and only if

$$\sup_{g \in G'} (1 + |x|_{\mathfrak{p}})^{-m} |D(g)|^{1/2} |F_T(g)| < \infty \tag{2.45}$$

with

$$D(g) := \det(\mathrm{Ad}(g^{-1}) - 1)|_{\mathfrak{n}} \tag{2.46}$$

the square of Weyl denominator. Alternatively one can interpret it as the analytic function associated with $t^{\mathrm{rank}(G)}$ in the expansion of polynomial $\det((t+1)-\mathrm{Ad}(g))$ with respect to t.[2] This was established [**HW86a**, Theorem 2.12] extending the finite-center case [**War72**, Theorem 8.3.6.1 & 8.3.8.2].

Now the (relative) Plancherel formula on G/Z, interpreted as a distribution on functions of G/Z, can be verified to be a tempered distribution by estimating each term contributing to the formula separately, in particular those terms which are eventually lifted to the global Plancherel formula on ZG^0/Z (and hence G) admit the following estimates:

1. the class functions associated to the character $\Theta^J_{\chi,\tau,\nu}$, which we denote as $\Theta^J_{\chi,\tau,\nu}(x)$ as well, have bounded numerator [3] [**HW86a**, Lemma 3.1], i.e.:

$$\sup_{g\in G'} |D(g)|^{1/2} \cdot |\Theta^J_{\chi,\tau,\nu}(g)| \leq c \quad \text{for all } \chi \in \widehat{Z_M(M^0)}, \tau \in L'_\chi, \nu \in \mathfrak{a}^* \tag{2.47}$$

2. $m^J(\chi : \tau : \nu)$ in (2.23) are bounded by $C(1 + \|\tau\|^2 + \|\nu\|^2)^m$ for some integer m [**HW86a**, Lemma 3.3];
3. The distribution character $\Psi(H : \nu)(f)$ can be subsequently bounded by spherical functions and large-order derivatives of f by exploiting the fact that Ω^n_G acts on the characters by scalars [**HW86a**, Lemma 3.5]. This proves that the integral contributed by one family of characters associated with a fixed pair $(J : \chi)$ in (2.22) is a tempered distribution. As the Plancherel theorem is a finite sum of these integrals, the result establishes the Plancherel theorem for tempered function on $G[\xi]/S$.

Lastly the estimate of $f \in \mathcal{S}(ZG^0)$ was obtained by directly integrating each $\Psi(H : \nu)(f)$ along $\widehat{Z}$, whereas the bounds along the $\widehat{Z}$-direction are estimated by [**HW86a**, Theorem 6.13 & 7.2]. The extension from ZG^0 to G is trivial as $[G : ZG^0] < \infty$.

[2] One should heed a difference in the conventions of notations between [**Kna86**, Theorem 10.33] and [**War72**, Theorem 8.2.3.9]. We follow the latter.

[3] The characters can be written as rational functions on G' by Harish-Chandra's regularity theorem. The numerator here refers to the numerator of these rational functions.

Chapter 3

Clifford algebra, spin group and representation

In this chapter we discuss the algebraic aspects of Clifford algebras and spin representations. The novelty of this chapter is to extend the Dirac method of [**Var73**] on equirank symmetric spaces to the Clifford algebra of a complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to the positive definite form B^{θ}, with the help of an explicit construction of the spinor bundle based on the work of Huang and Pandzic [**HP06**].

We begin by recalling some basic facts of Clifford algebras and their representations. Let V be an $\mathbb{R}$-vector space of dimension n, equipped with a positive definite scalar product $(-,-)_V$. Choose an orthonormal basis $\{u_1, \ldots, u_n\}$ of V with respect to $(-,-)_V$. Then the Clifford algebra $\mathrm{C}\ell(V)$ is isomorphic to the standard real Clifford algebra $\mathrm{C}\ell_n$ associated with $\mathbb{R}^n$:

Definition 3.1 (**Clifford algebra**)**.** Define $\mathrm{C}\ell(V)$ as an associative algebra over $\mathbb{R}$ with unity, of dimension 2^n, generated as a vector space by the following basis:

$$\{u_{i_1} u_{i_2} \cdots u_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

satisfying the Clifford relations:

$$u_i^2 = -1, \qquad u_i u_j = -u_j u_i \text{ if } i \neq j \tag{3.1}$$

Define the complex Clifford algebra as its complexification: $\mathbb{C}\ell(V) = \mathrm{C}\ell(V) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{C}\ell(V \otimes_{\mathbb{R}} \mathbb{C})$, with $\mathbb{C}$-bilinear form $(-,-)_{V_{\mathbb{C}}}$ induced by $(-,-)_V$.

Define the spin subalgebra as the Lie subalgebra of $\mathrm{C}\ell(V)$ generated by elements of degree 2:

$$\mathfrak{spin}(V) = \sum_{i \neq j} \mathbb{R} u_i u_j \tag{3.2}$$

The following map defines an isomorphism: [**LM90**, Proposition I.6.2]:

$$\varphi : \mathfrak{so}(V) \to \mathfrak{spin}(V) \qquad \varphi(E_{ij} - E_{ji}) := -\frac{1}{2} u_i u_j \tag{3.3}$$

in which we identify $\mathfrak{so}(V)$ with the anti-symmetric matrices acting on V, with $E_{ij} - E_{ji} \in \mathfrak{so}(V)$ the matrix sending u_i to u_j and u_j to $-u_i$. Of course one could identify V with the elements in $\mathrm{C}\ell(V)$ of degree 1, whence the isomorphism φ has a compatible action with the Lie bracket $[-,-]_{\mathrm{C}\ell}$:

$$[\varphi(x), u_j]_{\mathrm{C}\ell} = x u_j \qquad \text{for all } x \in \mathfrak{so}(V) \tag{3.4}$$

Define the left Clifford multiplication of $\mathrm{C}\ell(V)$ on itself as:

$$c\ell : \mathrm{C}\ell(V) \to \mathrm{End}_{\mathbb{R}}(\mathrm{C}\ell(V)) \qquad c\ell(x)y := xy \tag{3.5}$$

It defines a representation of $\mathrm{C}\ell(V)$, and consequently determines the action of $\mathfrak{spin}(V)$ on $\mathrm{C}\ell(V)$.

Remark 3.2 (the Lie algebra structure)**.** The Clifford multiplication induces a natural Lie bracket operation $[x,y]_{\mathrm{C}\ell} = xy - yx$ for $x, y \in \mathrm{C}\ell(V)$. This makes $\mathrm{C}\ell(V)$ a Lie algebra. Now in the case of symmetric spaces, $V = \mathfrak{p}$ occurs as a subspace in the Cartan decomposition. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In this case we choose the inner product $(-,-)_{\mathfrak{p}}$ to be the restriction of the Killing form B to $\mathfrak{p}$, then the adjoint map ad defines a Lie algebra homomorphism: $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$. We see hence $\varphi \circ \mathrm{ad}$ defines a Lie algebra homomorphism $\mathfrak{k} \to c\ell(\mathfrak{p})$. In particular, the natural Lie bracket on $\mathfrak{g}$ and the Lie bracket induced by Clifford multiplication on $\mathrm{C}\ell(\mathfrak{p})$ agree.

For general reductive homogeneous spaces on the other hand, one faces the dilemma between the positivity of the quadratic form and the compatibility of the inner product. In the case discussed by [**HP06**] it is the positive definiteness of the Clifford algebra that is sacrificed.

Unless otherwise stated, we always denote $[-,-]$ for the Lie bracket on $\mathfrak{g}$.

Definition 3.3 (**Spinors, spin representation**)**.** Given V with inner product $\langle -,- \rangle$, we define the space of spinors for V as a finite-dimensional vector space $\mathcal{S}$ over $\mathbb{C}$ together with an $\mathbb{R}$-linear map $c\ell : V \to \mathrm{End}(\mathcal{S})$ satisfying:

1. $c\ell(v)^2 = -\langle v, v\rangle \cdot \mathrm{id}$ for all $v \in V$
2. There is no $c\ell(V)$-invariant subspace of $\mathcal{S}$, i.e., any $W \subsetneq \mathcal{S}$ such that $\gamma(v)W \subseteq W$ for all v implies $W = \{0\}$.

We further construct explicitly the space of spinors as an exterior algebra generated by the maximal isotropic subspace of $V_{\mathbb{C}}$ with respect to the $\mathbb{C}$-bilinear form. Later we shall use this model as a realization of the $\mathfrak{k}$-representations.

Remark 3.4 (Exterior algebra model)**.** Extend the inner product $\langle -,- \rangle$ on V to a $\mathbb{C}$-bilinear symmetric form on $V_{\mathbb{C}}$, which we denote also as $\langle -,- \rangle$. Choose a maximal isotropic subspace $\mathcal{Z}$ of $V_{\mathbb{C}}$, i.e., $\mathcal{Z}$ is a complex subspace of maximal dimension such that:

$$\mathcal{Z} = \{ z \in V_{\mathbb{C}} \mid \langle z, w\rangle = 0 \text{ for all } w \in \mathcal{Z} \ \} \tag{3.6}$$

We also denote $\mathcal{Z}^*$ the complementary isotropic subspace of $\mathcal{Z}$, i.e., for each $z \in \mathcal{Z}$ there is a $\bar{z} \in \mathcal{Z}^*$ such that $\langle z, \bar{z}\rangle = 1$. Starting from an orthonormal basis $u_1, \ldots, u_{\dim V}$, we can set the basis of $\mathcal{Z}$ and $\mathcal{Z}^*$ respectively as:

$$z_j = \frac{1}{\sqrt{2}}(u_{2j-1} + iu_{2j}) \qquad \bar{z}_j = \frac{1}{\sqrt{2}}(u_{2j-1} - iu_{2j}) \quad \text{for } 1 \le j \le \lfloor \dim V/2 \rfloor \tag{3.7}$$

When V is even dimensional, we realize the spin module as $\mathcal{S}_V := \bigwedge^* \mathcal{Z}$. The Clifford multiplication realized on such models acts via exterior multiplication ε and contraction ι with respect to $\langle -,- \rangle$:

$$\begin{aligned} c\ell(z) &:= \varepsilon(z) : z_1 \wedge \cdots \wedge z_k \mapsto z \wedge z_1 \wedge \cdots \wedge z_k \\ c\ell(\bar{z}) &:= \iota(\bar{z}) : z_1 \wedge \cdots \wedge z_k \mapsto \sum_i (-1)^i 2\langle \bar{z}, z_i\rangle z_1 \wedge \cdots \wedge \widehat{z_i} \cdots \wedge z_k \end{aligned} \tag{3.8}$$

for $z \in \mathcal{Z}$ and $\bar{z} \in \mathcal{Z}^*$. In the case V is odd-dimensional, fix a vector $u_{2n+1} \in V_{\mathbb{C}}$ of unit length, orthogonal to both $\overline{V} = \mathcal{Z} \oplus \mathcal{Z}^*$, and realize $\mathcal{S}_{\overline{V}}$ as in the even dimensional case. Now $\mathcal{S}_{\overline{V}} = \wedge^* \mathcal{Z}$ can be made into an $\mathbb{C}\ell(V)$-module, by allowing u_{2n+1} to act on the even-degree forms $\mathcal{S}^+ = \wedge^{\mathrm{even}} \mathcal{Z}$ by i or $-i$, this forces u_{2n+1} acts on the odd-degree forms $\mathcal{S}^- = \wedge^{\mathrm{odd}} \mathcal{Z}$ by $-i$ or i respectively. Consequently we have two different $\mathbb{C}\ell(V)$-module structure on $\mathcal{S}$ that are equivalent as $\mathbb{C}\ell(\overline{V})$-modules. To distinguish their structures, we denote, for $\varpi \in \wedge^* \mathcal{Z}$, the symbols ${}_+\varpi$ and ${}_-\varpi$ to indicate the $c\ell(u_{2n+1})$-action on it by i and $-i$ respectively.

Besides using the action of u_{2n+1} to determine the Clifford module structure, one can use the volume element $\omega_{\mathbb{C}} = u_1 \cdot \cdots \cdot u_{2n+1}$ to determine its structure as well. It is a rather classical fact [**LM90**, Proposition I.5.9] that the Clifford module structure is determined by whether the volume element acts by $+\mathbb{I}$ or $-\mathbb{I}$, in the case V is odd-dimensional.

To make this action unitary, one should rescale the basis $u_1 \wedge \cdots \wedge u_r$ by $\frac{1}{2^r}$ to remove the 2-factors generated by the $c\ell(\overline{B})$ action, i.e.:

$$\langle u_1 \wedge \cdots \wedge u_k, u_1' \wedge \cdots \wedge u_k' \rangle_{\wedge^*(U)} = \det(2\langle u_i, u_j' \rangle) \tag{3.9}$$

the determinant of the matrix with (i,j)-entry $2\langle u_i, u_j' \rangle$. Then the adjoint of any $c\ell(v)$ on $\mathcal{S}$ is $-c\ell(\bar{v})$ for all $v \in V_{\mathbb{C}}$, and the vectors in its real form V act on $\mathcal{S}$ as skew-symmetric operators [**HP06**, Proposition 2.3.10].

Remark 3.5 (weights of spinor representation)**.** The $\mathcal{S}^+$ and $\mathcal{S}^-$ correspond respectively to the irreducible representations of $\mathfrak{so}(V)$ with highest weights $\frac{1}{2}(e_1 + e_2 + \cdots + e_m)$ and $\frac{1}{2}(e_1 + e_2 + \cdots - e_m)$ in the even-n case, and both to $\frac{1}{2}(e_1 + e_2 + \cdots + e_m)$ in the odd-n case. In fact, if we fix the root vector e_j to be dual of $h_j := E_{2j-1,2j} - E_{2j,2j-1}$ for $1 \leq j \leq \lfloor n/2 \rfloor = m$, then the weights of $\mathcal{S}$ with respect to $(\mathfrak{so}(V); \sum_j \mathbb{R}h_j)$ are precisely the linear forms [**Kna96**, p.343]:

$$\frac{1}{2}(e_1 + \cdots + e_m) - e_{j_1} - \cdots - e_{j_k} \tag{3.10}$$

with $1 \leq j_1 < \cdots < j_k \leq m$, and each weight occurs with multiplicity one. If we take the above realization of the spin module into consideration, we can see even more explicitly that $\varphi(h_i) = \bar{z}_j z_j + 1 = u_{2j}u_{2j-1}$ for $1 \leq j \leq m$, with the above weights corresponding to $z_S := z_{j_1} \wedge \ldots z_{j_k}$ for $S = \{j_1, \ldots, j_k\}$.

Definition 3.6 (Spin group)**.** The group of units $\mathrm{C}\ell(\mathfrak{p})^\times$ is a Lie group with Lie algebra $\mathfrak{cl}^\times \equiv (\mathrm{C}\ell(\mathfrak{p}); [-,-]_{\mathrm{C}\ell})$. Define the spin group $\mathrm{Spin}(\mathfrak{p})$ to be the analytic subgroup of $\mathrm{C}\ell(\mathfrak{p})^\times$ corresponds to the Lie subalgebra $\mathfrak{spin}(\mathfrak{p})$.

Given $s \in \mathrm{Spin}(\mathfrak{p})$, the adjoint map under Clifford multiplication gives a double covering, of $\mathfrak{so}(\mathfrak{p})$:

$$\varphi : \mathrm{Spin}(\mathfrak{p}) \to \mathrm{SO}(\mathfrak{p}) \qquad \varphi(s)Y = sYs^{-1}$$

This map derives at unit 1 to give the φ in (3.3). We denote both with the same letter.

We now resolve the issue mentioned in Remark 3.2 in the case of semisimple Lie groups. For our applications in sight, the $\mathfrak{so}(\mathfrak{p})$ is often armed with an additional $\mathfrak{l}$-module structure for some Lie algebra $\mathfrak{l}$. This gives a $\mathfrak{l}$-module structure of $\mathcal{S}$. The following discussion was inspired by more general discussions of spin representations for quadratic Lie algebras in [**HP06**, Section 2.3.3], but there the description is aimed at more general Clifford modules where the underlying quadratic forms are not bound to be positive definite, while the description of the root data can be quite complicated to describe at times. We therefore narrow down our description with a hopefully cleaner exposition.

We consider solely three cases, namely $V = (\mathfrak{k}, -B)$, $V = (\mathfrak{p}, B)$ and lastly $V = (\mathfrak{g}, B^\theta)$. In all cases we take $\mathfrak{l} = \mathfrak{k}$ with the adjoint action on V. Clearly V is a $\mathfrak{k}$-module and $\sigma = c\ell \circ \varphi \circ \mathrm{ad}$ defines a $\mathfrak{k}$-module structure on $\mathcal{S}_V$. Sometimes we use the subscript to stress on which space we construct the space of spinors.

Fix $\{X_i\}$ and $\{Y_\alpha\}$ an orthonormal basis of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to B^θ, like Remark 1.9. Again $\{u_i\}$ are used to denote the orthonormal basis of V, if the vector space is intended to be indeterminate.

Lemma 3.7. Given $\sigma = c\ell \circ \varphi \circ \mathrm{ad}$ the above representation of $\mathfrak{k}$ on $\mathcal{S}_V$. Then for all $Z \in \mathfrak{k}$:

$$\sigma_V(Z) = \begin{cases} \frac{1}{4} \sum_{i,j \in I_{\mathfrak{k}}} \langle Z, [X_i, X_j] \rangle c\ell(X_i X_j) & \text{if } V = \mathfrak{k} \\ -\frac{1}{4} \sum_{\alpha,\beta \in I_{\mathfrak{p}}} \langle Z, [Y_\alpha, Y_\beta] \rangle c\ell(Y_\alpha Y_\beta) & \text{if } V = \mathfrak{p} \\ \frac{1}{4} \Big(\sum_{i,j \in I_{\mathfrak{k}}} \langle Z, [X_i, X_j] \rangle c\ell(X_i X_j) - \sum_{\alpha,\beta \in I_{\mathfrak{p}}} \langle Z, [Y_\alpha, Y_\beta] \rangle c\ell(Y_\alpha Y_\beta) \Big) & \text{if } V = \mathfrak{g} \end{cases} \tag{3.11}$$

Moreover, in every case:

$$\sigma_V(Z) = -\frac{1}{4} \sum_{j \in I_V} c\ell([Z, u_j]) c\ell(u_j) \tag{3.12}$$

where j runs through the index set of the basis of V.

Proof. we can write the matrix elements of $\operatorname{ad} Z$ for $Z \in \mathfrak{k}$ as, for $i, j \in I_{\mathfrak{k}}$:

$$(\operatorname{ad} Z)_{ij} = \langle [Z, u_j], u_i \rangle = -\langle Z, [u_i, u_j] \rangle \tag{3.13}$$

by the ad-invariance of B in Remark 1.9. Similarly one has $(\operatorname{ad} Z)_{\alpha\beta} = \langle Z, [Y_\alpha, Y_\beta] \rangle$. Hence we can write $\varphi \circ \operatorname{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{g}) \to \mathfrak{spin}(\mathfrak{g})$ explicitly in the basis: For all $Z \in \mathfrak{k}$,

$$\begin{aligned}
\varphi \circ \operatorname{ad}(Z) &= \sum_{ij \in I_{\mathfrak{k}}} (\operatorname{ad} Z)_{ij} E_{ij} + \sum_{\alpha\beta \in I_{\mathfrak{p}}} (\operatorname{ad} Z)_{\alpha\beta} E_{\alpha\beta} \\
&= -\sum_{i<j} \langle Z, [X_i, X_j] \rangle \varphi(E_{ij} - E_{ji}) + \sum_{\alpha<\beta} \langle Z, [Y_\alpha, Y_\beta] \rangle \varphi(E_{\alpha\beta} - E_{\beta\alpha}) \\
&= \frac{1}{2} \left(\sum_{i<j} \langle Z, [X_i, X_j] \rangle X_i X_j - \sum_{\alpha<\beta} \langle Z, [Y_\alpha, Y_\beta] \rangle Y_\alpha Y_\beta \right) \\
&= -\frac{1}{4} \sum_{i,j \in I_{\mathfrak{g}}} \langle [Z, u_j], u_i \rangle u_i u_j = -\frac{1}{4} \sum_{j \in I_{\mathfrak{g}}} [Z, u_j] u_j
\end{aligned} \tag{3.14}$$

where the last identity follows from $\sum_i \langle [Z, u_j], u_i \rangle u_i = [Z, u_j]$. The case $V = \mathfrak{k}$ and $\mathfrak{p}$ are handled similarly. □

Remark 3.8. In fact one can formulate a similar identity as the lemma above, by replacing the inner product $\langle -, - \rangle$ here by the Killing form B, and by replacing u_i by an arbitrary basis b_i of V. Denote its dual basis with respect to B by d_i: $B(b_i, d_j) = \delta_{ij}$. Since $[X, d_j] = \sum_i B([X, d_j], d_i) b_i$ for all $X \in \mathfrak{g}$, we have:

$$\begin{aligned}
\phi \circ \operatorname{ad}(Z) = -\frac{1}{4} \sum_j [Z, u_j] u_j &= \frac{1}{4} \sum_j [Z, d_j] b_j = \frac{1}{4} \sum_{i,j} B(Z, [d_i, d_j]) b_i b_j \\
&= \frac{1}{2} \sum_{i<j} B(Z, [d_i, d_j]) (b_i b_j + \langle b_i, b_j \rangle)
\end{aligned} \tag{3.15}$$

by expressing u_j and $[Z, d_j]$ in bases, and using the invariance of the Killing form and lastly the Clifford relations $b_i b_j + b_j b_i = -2\langle b_i, b_j \rangle$. This identity taken in this form will be handy in the computation of Casimir eigenvalues.

Recall the discussion from Remark 3.5: We already know $\mathcal{S}_V$ is by definition an irreducible $\mathfrak{so}(V)$-module and all its weights are of multiplicity one. We now want to express its $\mathfrak{k}_{\mathbb{C}}$-weights instead of $\mathfrak{so}(V)$-weights. We need to fix some auxillary root data, reminiscent of Remark 2.20:

As always we fix a Cartan involution θ of $\mathfrak{g}$ with $\mathfrak{k} = \mathfrak{g}^\theta$ its fixed set. Let $\mathfrak{h}$ the maximally compact θ-stable Cartan subalgebra of $\mathfrak{g}$, and decompose $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ such that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, and $\mathfrak{a} \subseteq \mathfrak{p}$, with a corresponding root system $\Delta(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$. Again there is no real root in this system, as $\mathfrak{h}$ is maximally compact.

The involution θ extends $\mathbb{C}$-linearly to $\mathfrak{g}_{\mathbb{C}}$, which is denoted again as θ. Its fixed point set is the complexified compact Lie algebra $\mathfrak{k}_{\mathbb{C}}$. Extend B also to an $\mathbb{C}$-linear quadratic form $B_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$. Denote $\mathfrak{g}_{\alpha^\vee} \subseteq \mathfrak{g}_{\mathbb{C}}$ the dual space of the root space $\mathfrak{g}_\alpha$ with respect to $B_{\mathbb{C}}$. The relations between different root spaces are summarized by the following diagram:

$$\begin{array}{ccc}
\mathfrak{g}_\alpha & \xrightarrow{B_{\mathbb{C}}} & \mathfrak{g}_{\alpha^\vee} \\
{\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\
\mathfrak{g}_{-\alpha} & \xrightarrow{B_{\mathbb{C}}} & \mathfrak{g}_{-\alpha^\vee}
\end{array} \tag{3.16}$$

where the horizontal map $B_{\mathbb{C}}$ denotes the non-degenerate pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\alpha^\vee}$ and is easily seen, by the invariance of $B_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$, as sending α-spaces to $-\alpha$-spaces. θ acts on $\Delta(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$ by simply restricting the action to $\mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$, which indeed acts by -1 on $\mathfrak{a}_{\mathbb{C}}$ and 1 on $\mathfrak{t}_{\mathbb{C}}$.

Remark 3.9. We now fix a root system $\Delta(\mathfrak{k}_{\mathbb{C}};\mathfrak{t}_{\mathbb{C}})$ such that it is compatible with the $\mathfrak{g}$-roots: i.e., every positive $\mathfrak{k}$-root is a restriction of some positive $\mathfrak{g}$-root to $\mathfrak{t}$, and θ preserves positivity of $\mathfrak{k}$-roots.

To construct it explicitly, choose a hyperbolic element $H \in \mathfrak{t}_{\mathbb{C}}$, i.e., $\operatorname{ad} H$ acts on $\mathfrak{g}$ with real eigenvalues. In our case this always exists, as one can always choose an elliptic element of $iH \in \mathfrak{t}$ which has only imaginary value. [1] We claim H only commutes with $\mathfrak{h}$.

To verify the claim, suppose there is a root vector $X_\alpha \in \mathfrak{g}_\alpha$ with $[X_\alpha, H] = 0$, then $X_\alpha + \theta X_\alpha \in \mathfrak{k}$ which also commutes with H. But it is linearly independent from $\mathfrak{h}$, so contradicting the fact $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{k}$.

Hence $Z_{\mathfrak{g}}(H) = \mathfrak{h}$ determines a minimal θ-stable parabolic subalgebra of $\mathfrak{g}$, by taking the Levi subalgebra to be the centralizer $Z_{\mathfrak{g}}(H) = Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a}$. Its nilradical contains all the root spaces $\mathfrak{g}_\alpha$ such that $\alpha(H) > 0$. In particular, this defines the polarity on $\Delta(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$ as H is $\mathfrak{g}_{\mathbb{C}}$-regular. The positive roots are therefore those roots α defined by $\alpha(H) > 0$.

Now $\{\alpha_i|_{\mathfrak{t}} : \alpha_i \in \Delta(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})\}$ defines a root system by removing repetitions. It also defines a $\mathfrak{k}$-root to be positive if the corresponding $\alpha_i(H) > 0$. Hence we have a compatible positive root system $\Delta^+(\mathfrak{k}_{\mathbb{C}};\mathfrak{t}_{\mathbb{C}})$ with that of $\mathfrak{g}$. This comes from the fact that $\theta H = H$ and θ commutes with the bracket. Now we can take these roots as $\mathfrak{t}$-weights, and group them into two classes:

1. The roots α: They are either imaginary compact roots, which are $\mathfrak{t}$-weights in their own right, or are complex roots such that $\alpha|_{\mathfrak{t}} = \theta\alpha|_{\mathfrak{t}}$ for every pair $(\alpha, \theta\alpha)$ of $\mathfrak{t}$-weights. The pair have $\mathfrak{t}$-weight $\alpha|_{\mathfrak{t}}$ with the weight vector $X_\alpha + \theta X_\alpha \in \mathfrak{k}$ for $X_\alpha \in \mathfrak{g}_\alpha$;
2. The roots β: They are again either imaginary noncompact or occur in pairs $(\beta, \theta\beta)$ with corresponding weights vectors $X_\beta - \theta X_\beta \in \mathfrak{p}$ of $\mathfrak{t}$-weights $\beta|_{\mathfrak{t}}$ for $X_\beta \in \mathfrak{g}_\beta$.

Here each α and β occur with multiplicity one. In fact, any two pairs of complex roots cannot contribute to the same $\mathfrak{t}$-weight, i.e., if $(\alpha, \theta\alpha) \neq \pm(\beta, \theta\beta)$ then $\alpha|_{\mathfrak{t}} \neq \pm\beta|_{\mathfrak{t}}$. Moreover, the set of spaces of the α_is together with $\mathfrak{t}$ form a root space decomposition of $\mathfrak{k}$ relative to $\mathfrak{t}$. For details of the bracket relations, see [**KV95**, p. 257].

We now construct the maximal isotropic subspaces of V, which leads to a natural description of the spin module. Recall that we denote by $\mathcal{Z}_V$ the maximal isotropic subspace of V. We also fix a basis element $u_\alpha \in \mathfrak{g}_\alpha$ for each root α:

$$\mathcal{Z}_V = \begin{cases} \mathbb{C}\{u_\alpha \mid \alpha \in \Delta_K^+\} \oplus \mathbb{C}\{u_\beta + \theta u_\beta \mid \beta \in \Delta_C^+\} \oplus \mathcal{Z}_{\mathfrak{t}} & \text{if } V = \mathfrak{k} \\ \mathbb{C}\{u_\alpha \mid \alpha \in \Delta_n^+\} \oplus \mathbb{C}\{u_\beta - \theta u_\beta \mid \beta \in \Delta_C^+\} \oplus \mathcal{Z}_{\mathfrak{a}} & \text{if } V = \mathfrak{p} \\ \mathbb{C}\{u_\alpha \mid \alpha \in \Delta_I^+\} \oplus \mathbb{C}\{u_\beta \pm \theta u_\beta \mid \beta \in \Delta_C^+\} \oplus \mathcal{Z}_{\mathfrak{h}} & \text{if } V = \mathfrak{g} \end{cases} \tag{3.17}$$

with Clifford multiplication realized as in (3.8). Recall that $\Delta_C^+, \Delta_n^+, \Delta_I^+$ denotes the (positive) complex, noncompact imaginary and imaginary roots respectively. Note that $\theta\beta$ and β generate the same space for each $\beta \in \Delta_C^+$, so the second summand generates a space of dimension only half of the size of the positive noncompact roots. The spin module $\mathcal{S}_V$ is the exterior algebra bundle $\wedge^* \mathcal{Z}_V$. One should also normalize the above bases by $2^{\deg/2}$ to make the Clifford multiplication unitary (see Remark 3.4). As this part is solely about algebraic aspects, we drop the normalization constants for cleaner exposition.

To compute explicitly the possible $\mathfrak{t}$-weights occurring in the spin representation $\mathcal{S}_V$ we invoke Remark 3.8: Fix one root vector $v_i \in \mathfrak{g}_{\alpha_i}$ for each $\alpha_i \in \Delta^+(\mathfrak{k}_{\mathbb{C}};\mathfrak{t}_{\mathbb{C}})$, then their dual elements v_i^* with respect to B lie in $\mathfrak{g}_{-\alpha_i}$. Similarly, let w_j be the weight vector corresponding to weight $\beta_j \in \Delta_n$, with dual element w_j^*. The sets $\{v_i, v_i^*\}$ and $\{w_j, w_j^*\}$ constitute a basis of $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ respectively:

$$\sigma_{\mathfrak{k}}(X) = -\frac{1}{4}\sum_i \big([X, v_i]v_i^* + [X, v_i^*]v_i\big) = -\frac{1}{4}\sum_i (\alpha_i(X)v_i v_i^* + \alpha_i(X)v_i^* v_i) = \frac{1}{2}\sum_i \alpha_i(X)(v_i^* v_i + 1) \tag{3.18}$$

[1] This is because otherwise $\operatorname{ad} iH$ contains only complex eigenvalue, then applying Cayley transform to it would generate a Cartan subgroup with larger compact dimension, contradicting the fact that $\mathfrak{t} \oplus \mathfrak{a}$ is the maximal compact Cartan subgroup.

We observe that the end expression $v_i^* v_i$ is independent of the scaling of v_i, hence we can renormalize v_i and v_i^* to be unit length, and identify them with z_i and $-\bar{z}_i$ in our construction of spin modules. From Remark 3.5 we see immediately that each summand corresponds to $\frac{1}{2}\alpha_i(X)h_i$ for h_i in the Cartan subalgebra of $\mathfrak{so}(\mathfrak{k})$. Immediately we see the $\mathfrak{t}$-weights of $S_{\mathfrak{k}}$, for $I \subseteq \mathcal{Z}_{\mathfrak{k}}$, are:

$$\frac{1}{2}\sum_{k\in I}\alpha_k - \frac{1}{2}\sum_{k\notin I}\alpha_k \tag{3.19}$$

which corresponds to the weight vector z_I. In particular, we see $\frac{1}{2}\sum_{j=1}^{\dim \mathfrak{k}}\alpha_i = \delta_K$ the half root sum of $\mathfrak{k}$ is at the same time the highest $\mathfrak{t}$-weight of $\mathcal{S}_{\mathfrak{k}}$.

For $V = \mathfrak{p}$ we repeat the same steps, and obtain similar results by parity, with now the weights expressed in noncompact roots β_js instead of α_js. With these preparations we can now prove the following proposition, which is the main result of this section:

Proposition 3.10. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and let $\sigma_V : \mathfrak{k} \to \mathbb{C}\ell(V)$ be the corresponding spin representation of $\mathfrak{k}$ on the space of spinors associated with V for $V = \mathfrak{k}, \mathfrak{p}, \mathfrak{g}$. Form the root systems as above, and let W^1 be a subset of Weyl group G:*

$$W^1 = \{w \in W \mid w\Delta^+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) \textit{ is compatible with } \Delta^+(\mathfrak{k}_{\mathbb{C}};\mathfrak{t}_{\mathbb{C}})\} \tag{3.20}$$

Recall compatibility means positivity of the root is preserved by restriction to $\mathfrak{t}$ and by Cartan involution, i.e.:

1. *For each $\alpha \in \Delta_K^+$, there exists a $\beta \in \Delta_G^+$, such that $\alpha = \beta|_{\mathfrak{t}}$;*
2. *If $\alpha \in \Delta_G^+$, then $\theta\alpha \in \Delta_G^+$.*

Then the spin representation decomposes into

$$\sigma_{\mathfrak{k}} = 2^{\lfloor \dim \mathfrak{t}/2 \rfloor}\tau_{\delta_K} \quad \sigma_{\mathfrak{p}} = \bigoplus_{w\in W^1} 2^{\lfloor \dim \mathfrak{a}/2 \rfloor}\tau_{w\delta_G - \delta_K} \quad \sigma_{\mathfrak{g}} = \bigoplus_{w\in W^1} 2^{\lfloor \dim \mathfrak{h}/2 \rfloor}\tau_{w\delta_G - \delta_K} \otimes \tau_{\delta_K} \tag{3.21}$$

into irreducible representations τ_λ of K with the highest weight λ. In particular, the representation is independent of choices of the Clifford module structure, and:

$$\sigma_V(\Omega_K) = \begin{cases} 3|\delta_K|^2 \cdot \mathrm{id} & \text{if } V = \mathfrak{k} \\ (|\delta_G|^2 - |\delta_K|^2)\cdot \mathrm{id} & \text{if } V = \mathfrak{p} \end{cases} \tag{3.22}$$

Remark 3.11. We stress the fact that in the expression above $\delta_G \in \mathfrak{h}^*$ vanish on $\mathfrak{a}$, and can actually be taken as a linear functional on $\mathfrak{t}^*$. This is because we know the imaginary roots vanish on $\mathfrak{a}$, whereas the sum of each pair of complex roots also vanishes on $\mathfrak{a}$. For the same reason we can also define $\delta_G - \delta_K$ as a linear functional on $\mathfrak{t}$. Also their weights are easy to be seen to be integral, hence are well-defined K-weights.

The proof is essentially a collection of the results we have insofar discussed. The original proof of [**BW00**, § II.6] on $V = \mathfrak{p}$ focused on the root computations and we will work it out in the other cases with help of the explicit model.

Proof of Proposition 3.10. First by Remark 4.17, one verifies $\mathcal{R}_{ijkl} := B([X_i, X_j], [X_k, X_l])$ for $X \in \mathfrak{g}$ satisfies the Bianchi identities: $\mathcal{R}_{ijkl} = \mathcal{R}_{klij} = -\mathcal{R}_{jikl}$ follows readily from the invariance property of B; whereas $\mathcal{R}_{ijkl} + \mathcal{R}_{kijl} + \mathcal{R}_{jkil} = 0$, we have implicitly proven in (1.36), which is obtained by further using the Jacobi identity. Hence $\sigma_{\mathfrak{k}}(\Omega_K)$ and $\sigma_{\mathfrak{p}}(\Omega_K)$ are easily proven to be scalar operators by using (4.56) and Lemma 3.7:

$$\begin{aligned} -\sigma_{\mathfrak{k}}(\Omega_K) &= \sum_{i\in I_{\mathfrak{k}}}\sigma_{\mathfrak{k}}(X_i)^2 = (\frac{1}{4})^2 \sum_{ijklm\in I_{\mathfrak{k}}} B(X_i,[X_j,X_k])B(X_i,[X_l,X_m])c\ell_{jklm} \\ &= \frac{1}{16}\sum_{jklm\in I_{\mathfrak{k}}}\mathcal{R}_{jklm}c\ell_{jklm} = \frac{1}{8}\left(\sum_{ij}\mathcal{R}_{ijji}\right)\cdot \mathrm{id} \end{aligned} \tag{3.23}$$

The same argument proves $\sigma_{\mathfrak{p}}(\Omega_K)$ is also a scalar operator $\sum_{\alpha\beta\in I_{\mathfrak{p}}} \mathcal{R}_{\alpha\beta\beta\alpha} \cdot \mathrm{id}$, a fact which can be otherwise found in [BW00, Lemma 6.9].

In the case of $\mathcal{S}_{\mathfrak{k}}$, the construction (3.17) shows the following is a highest-$\mathfrak{t}$-weight vector:

$$z_{\mathrm{top},\mathfrak{k}} := u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m} \wedge (u_{\beta_1} + \theta u_{\beta_1}) \wedge \cdots \wedge (u_{\beta_k} + \theta u_{\beta_k}) \tag{3.24}$$

where α_is enumerate the compact imaginary roots and β_js enumerate the complex roots pair $(\beta, \theta\beta)$ of $\mathfrak{g}_{\mathbb{C}}$. Moreover, each u_α contributes a $\mathfrak{t}$-weight $\alpha_i \in \mathfrak{t}$, whereas each $u_{\beta_i} + \theta u_{\beta_i}$ a weight of $\beta_i|_{\mathfrak{t}} = \theta\beta_i|_{\mathfrak{t}}$, therefore its weight is $\frac{1}{2}\sum_{\Delta_K^+} \alpha_i + \frac{1}{4}\sum_{\Delta_C^+} \beta_i = \delta_K$. Hence σ_K has a infinitesimal character σ_K, with corresponding Casimir eigenvalue using Proposition 2.21:

$$\chi_{\sigma_{\mathfrak{k}}}(\Omega_K) = |2\delta_K|^2 - |\delta_K|^2 = 3|\delta_K|^2.$$

But now we notice that $\sigma_{\mathfrak{k}}(\Omega_K)$ acts on the spin module by scalar multiplication, therefore $\sigma_{\mathfrak{k}}(\Omega_K) = 3|\delta_K|^2$. For the same reason we see the following $z_{\mathrm{top},\mathfrak{p}}$ is a highest-$\mathfrak{t}$-weight vector of $\mathcal{S}_{\mathfrak{p}}$:

$$z_{\mathrm{top},\mathfrak{p}} := u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m} \wedge (u_{\beta_1} - \theta u_{\beta_1}) \wedge \cdots \wedge (u_{\beta_k} - \theta u_{\beta_k})$$

again α_is enumerate noncompact imaginary roots and β_js enumerate complex root pairs $(\beta, \theta\beta)$ of $\mathfrak{g}_{\mathbb{C}}$. Moreover, $z_{\mathrm{top},\mathfrak{p}}$ has $\mathfrak{t}$-weight $\frac{1}{2}\sum_{\Delta_n^+} \alpha_i + \frac{1}{4}\sum_{\Delta_C^+} \beta_i = \delta_G - \delta_K$ by taking note of Remark 3.11, and therefore $\tau_{\delta_G - \delta_k}$ the irreducible representation with highest weight $\delta_G - \delta_K$ is contained in $\sigma_{\mathfrak{p}}$. Arguing like the compact case, we get:

$$\chi_{\sigma_{\mathfrak{p}}}(\Omega_K) = |\delta_G - \delta_K + \delta_K|^2 - |\delta_K|^2 = |\delta_G|^2 - |\delta_K|^2 \tag{3.25}$$

As a next step, let us take a look of the highest weights occurring in $\sigma_{\mathfrak{k}}$. Suppose τ_λ occurs in $\sigma_{\mathfrak{k}}$. Then $\lambda = \delta_K - \sum_{\alpha \in Q_K} \alpha$ for a subset $Q_K \subseteq \Delta^+(\mathfrak{k}_{\mathbb{C}}; \mathfrak{t}_{\mathbb{C}})$. Now $\sigma_{\mathfrak{k}}(\Omega_K)$ must agree with $3|\delta_K|^2$, implying:

$$|\delta_K - \sum_{\alpha \in Q_K} \alpha + \delta_K|^2 = |\lambda + \delta_K|^2 - |\delta_K|^2 = 3|\delta_K|^2 \tag{3.26}$$

This indeed implies $|\lambda + \delta_K| = |2\delta_K|$. Hence $\lambda + \delta_K$ lies inside the orbit of $2\delta_K$ under the $W(\mathfrak{k}_{\mathbb{C}}; \mathfrak{t}_{\mathbb{C}})$-action. As λ is a dominant weight, therefore $\lambda + \delta_K = 2\delta_K$. Hence δ_K is the unique highest weight occurring in $\sigma_{\mathfrak{k}}$.

Again the case of $\sigma_{\mathfrak{p}}$ is dealt with in a similar fashion. The only complication comes from the permutation by the Weyl group $W(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$. Again assume τ_λ occurs in $\sigma_{\mathfrak{p}}$ as a $\mathfrak{k}$-representation. Then its highest weight vector must be a wedge product of root vectors in our model, hence:

$$\lambda = (\delta_G - \delta_K - \sum_{\alpha \in Q} \alpha)|_{\mathfrak{t}}$$

for some subset $Q \subseteq \Delta^+(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$, hence $\lambda + \delta_K = (\delta_G - \sum_{\alpha \in Q} \alpha)|_{\mathfrak{t}}$ is the infinitesimal character of τ_λ. On the other hand, (3.25) implies $|\lambda + \delta_K|^2 = |\delta_G|^2$. Now:

$$\left|\delta_G - \sum_{\alpha \in Q} \alpha\right|^2 \geq \left|\delta_G - \sum_{\alpha \in Q} \alpha|_{\mathfrak{t}}\right|^2 = |\delta_G|^2 \tag{3.27}$$

The first inequality is because every root on a stable Cartan subalgebra is real-valued on $\mathfrak{a}$ and imaginary valued on $\mathfrak{t}$ [Kna96, Corollary 6.49] and because $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathfrak{t}_{\mathbb{C}}^*$ are orthogonal. Moreover, by Remark 3.11 we see that $(\delta_G - \sum_Q \alpha)|_{\mathfrak{t}} = \delta_G - \sum_Q \alpha|_{\mathfrak{t}}$. The last equality is a consequence of (3.25), since $|\lambda + \delta_K|^2 = |\delta_G|^2$.

Next we observe the $\mathfrak{t}_{\mathbb{C}}$-weights of $\mathcal{S}_{\mathfrak{p}}$ are also $\mathfrak{h}_{\mathbb{C}}$-weights by our construction. Moreover, since all the highest weights of $S_{\mathfrak{p}}$ are integral sums of roots, with equal length as the highest weight $\delta_G - \delta_K$, then by a classical result of Kostant [BW00, Scholium II.6.8], this implies that:

$$\lambda + \delta_K = \delta_G - \sum_{\alpha \in Q} \alpha = w(\delta_G) \tag{3.28}$$

for some $w \in W$. Now one verifies easily all the noncompact and complex root vectors are compatible with Δ_K^+. Hence $w(\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C}))$ must be compatible with $\Delta(\mathfrak{k}_\mathbb{C}; \mathfrak{t}_\mathbb{C})$, i.e. $w \in W^1$.

Lastly we address the question of multiplicity. In both $V = \mathfrak{k}$ or $\mathfrak{p}$-cases, for $z_i \in \mathcal{Z}_\mathfrak{h}$, we note the fact that taking wedge product between $z_{i_1} \wedge \cdots \wedge z_{i_l}$ and $z_{\text{top},V}$ does not alter its $\mathfrak{t}$-weights. Moreover, this exhausts all $\mathfrak{t}$-weight vectors of weights δ_K and $\delta_G - \delta_K$ in the respective cases. Therefore the multiplicity in each case is $2^{\lfloor \dim \mathfrak{t}/2 \rfloor}$ and $2^{\lfloor \dim \mathfrak{a}/2 \rfloor}$ respectively.

For $V = \mathfrak{g}$, $\sigma_\mathfrak{g}(\Omega_K)$ is no longer a scalar. Nonetheless, as one already witnessed in Lemma 3.7 that $\sigma_\mathfrak{g}$ should behave like the tensor product of $\sigma_\mathfrak{k}$ and $\sigma_\mathfrak{p}$, with the only possible complication that there might be one extra isotropic dimension when considering the spinor modules of the direct sum of two vector spaces. Such a phenomenon is illustrated in Example 3.12 below. Hence we prove the claim by showing the following direct sum decomposition

$$\bigoplus_{w \in W^1} 2^{\lfloor \dim \mathfrak{h}/2 \rfloor} V_{w\delta_G - \delta_K} \otimes V_{\delta_K} \subset \mathcal{S}_\mathfrak{g}$$

is an inclusion of $\mathfrak{k}_\mathbb{C}$-submodule into $\sigma_\mathfrak{g}$ and their dimensions agree. Now each $\tau_{w\sigma_G - \delta_K} \otimes \delta_K$ corresponds to the tensor product of two $\mathfrak{k}_\mathbb{C}$-representations, with highest weights:

$$z_{\text{top},\mathfrak{k}} \wedge z_{\text{top},\mathfrak{p}} \wedge z'_{\gamma_1} \wedge \cdots \wedge z'_{\gamma_l}$$

with $z'_{\gamma_i} \in \mathcal{Z}_\mathfrak{h}$ the isotropic vectors in $\mathfrak{h}_\mathbb{C}$. Again taking wedge product with these z'_{γ_i}s does not alter the weights. Moreover all these vectors are linearly independent in $\mathcal{S}_\mathfrak{g}$, so indeed the direct sum defines a $\mathfrak{k}_\mathbb{C}$-submodule.

Now to see that their dimension agree, recall $\dim \mathcal{S}_\mathfrak{g} = 2^{\lfloor \dim \mathfrak{g}/2 \rfloor}$. We verify the equality of dimensions case-by-case: First assume $\mathfrak{g}$ is even-dimensional, and both $\mathfrak{k}$ and $\mathfrak{p}$ are odd-dimensional. Then the dimensions of the Cartan subalgebra and of the Lie algebra must have the same parity, as all the positive root vectors α can be paired with their negative roots $-\alpha$ and hence have even dimension. This implies in this case $\mathfrak{h}$ is even dimensional, whereas $\mathfrak{t}$ and $\mathfrak{a}$ are odd dimensional. Now:

$$\begin{aligned}
&\dim \left(\bigoplus_{w \in W^1} 2^{\lfloor \dim \mathfrak{h}/2 \rfloor} \tau_{w\delta_G - \delta_K} \otimes \tau_{\delta_K} \right) \\
&= \dim \left(\bigoplus_{w \in W^1} 2^{\lfloor \dim \mathfrak{t}/2 \rfloor} \tau_{w\delta_G - \delta_K} \right) \cdot \dim \left(2^{\lfloor \dim \mathfrak{a}/2 \rfloor} \tau_{\delta_K} \right) \cdot 2 \\
&= \dim \sigma_\mathfrak{p} \cdot \dim \sigma_\mathfrak{k} \cdot 2 = \dim 2^{\lfloor \dim \mathfrak{p}/2 \rfloor + \lfloor \dim \mathfrak{k}/2 \rfloor + 1} = \dim 2^{\lfloor \dim \mathfrak{g}/2 \rfloor} = \dim \mathcal{S}_\mathfrak{g}
\end{aligned} \tag{3.29}$$

The cases when $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}$ admit other parity conditions can be checked in a similar fashion, and the identity will hold again by comparing the dimensions of the corresponding modules of spinors. This concludes the proof of the proposition. □

We end this section with a construction of a concrete spin module on $SL_3(\mathbb{R})$ as a non-equirank example to navigate the readers through heavy discussions of the root data.

Example 3.12 ($\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$). *The reader is reminded to distinguish the natural complex structure in a complex semisimple Lie algebra from the complexification. In particular, the complex linear Lie groups are treated as a real groups. Refer to [Kna96, § VI.2] for more details.*

Fix the Cartan involution $\theta(X) = -X^{\text{tr}}$ the negative transpose. Therefore we identify $\mathfrak{k}_\mathbb{C} = \mathfrak{k} \otimes_\mathbb{R} \mathbb{C}$ the subalgebra of traceless antisymmetric matrices, and $\mathfrak{p}_\mathbb{C}$ the traceless complex symmetric matrices. The Killing form $B(X,Y) = \frac{1}{4}\operatorname{tr}(XY)$ is G-invariant and θ-invariant on $\mathfrak{g}$, extending to a $\mathbb{C}$-linear form $B_\mathbb{C}(X,Y) = \frac{1}{4}\operatorname{tr}(XY)$ on $\mathfrak{sl}_3(\mathbb{C})$. This can be identified with the natural Killing form on $\mathfrak{sl}_3(\mathbb{C})$.

Consider first the root space decomposition of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ with respect to the maximally compact Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{a} \subseteq \mathfrak{p}$ spanned by H_1, and $\mathfrak{t} \subseteq \mathfrak{k}$ spanned by H_2 with:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.30}$$

Fix the vectors $f_1 \in \mathfrak{a}^*$ and $f_2 \in i\mathfrak{t}^*$ such that $f_1(H_1) = 3$ and $f_2(H_2) = i$. Fix f_1 as the positive restricted root of $\mathfrak{a}$. One readily checks $-iH_2$ defines a hyperbolic element of $\mathfrak{t}$, and therefore by our procedure defines a system of positivity roots on $\mathfrak{g}$, namely:

$$\Delta^+(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C}) = \{f_1 + f_2, f_2 - f_1, 2f_2\} \quad \Delta^+(\mathfrak{k}_\mathbb{C}; \mathfrak{t}_\mathbb{C}) = \{f_2\} \tag{3.31}$$

with $2f_2$ the unique noncompact imaginary root. Their corresponding vectors $E_{\alpha\in\Delta}$ are:

$$\begin{aligned} E_{2f_2} &= \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & E_{f_1+f_2} &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & E_{f_2-f_1} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 1 & 0 \end{pmatrix} \\ E_{-2f_2} &= \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & E_{f_1-f_2} &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} & E_{-f_1-f_2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & -1 & 0 \end{pmatrix} \end{aligned} \tag{3.32}$$

We see θ acts -1 on f_1, and $+1$ on f_2, hence interchanges $E_{f_1+f_2}$ and $E_{f_2-f_1}$ and fixes E_{2f_2}. Therefore it preserves the positive system. Moreover, $\mathfrak{k}_\mathbb{C}$ is spanned by the following elements $X_0 = H_2$ and X_+, X_- as follows:

$$X_+ = E_{f_1+f_2} + \theta E_{f_1+f_2} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ -i & -1 & 0 \end{pmatrix} \quad X_- = E_{f_1-f_2} + \theta E_{f_1-f_2} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & -1 \\ -i & 1 & 0 \end{pmatrix} \tag{3.33}$$

and the complement $\mathfrak{p}_\mathbb{C}$ is spanned by $Y_0 = H_1$, $Y_{\pm 2} := E_{\pm 2f_2}$ and

$$Y_1 = E_{f_1+f_2} - \theta E_{f_1+f_2} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix} \quad Y_{-1} = E_{f_1-f_2} - \theta E_{f_1-f_2} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & -1 \\ i & -1 & 0 \end{pmatrix} \tag{3.34}$$

The Xs and Ys define an orthonormal basis of $\mathfrak{g}_\mathbb{C}$. Note that Δ_n^+ can be chosen to be $\{2f_2\}$, and hence the corresponding spaces of spinors are spanned by:

$$\text{basis of } \mathcal{Z}_V = \begin{cases} \{X_+\} & \text{if } V = \mathfrak{k} \\ \{E_{2f_2}, Y_1\} & \text{if } V = \mathfrak{p} \\ \{E_{2f_2}, X_+, Y_1, H_1 + iH_2\} & \text{if } V = \mathfrak{g} \end{cases} \tag{3.35}$$

We see in particular that H_1 and H_2, like $u_{2m+1} \in V$ from Remark 3.4, are complementary vectors in their modules of spinors $\mathcal{S}_\mathfrak{p}$ and $\mathcal{S}_\mathfrak{k}$ respectively. But on the other hand they form an isotropic subspace $H_1 + iH_2$ in $\mathcal{S}_\mathfrak{g}$.

The algebraic Weyl group in this case is isomorphic to S_3, generated by reflections along the hyperplane of s_α for $\alpha \in \Delta$. Hence W^1 contains only the identity. Here we collect the data of all highest weights with corresponding weight vectors in the following table:

V	z_{top}	highest weight	dimension	multiplicity
$\mathfrak{k}$	X_+	$\frac{1}{2}f_2$	2	1
$\mathfrak{p}$	$E_{2f_2} \wedge Y_1$	$\frac{1}{2}(2f_2 + f_2)$	4	1
$\mathfrak{g}$	$E_{2f_2} \wedge X_+ \wedge Y_1 \wedge (H_1 + iH_2)$, $E_{2f_2} \wedge X_+ \wedge Y_1$	$\frac{1}{2}(2f_2 + f_2 + f_2)$ $\frac{1}{2}(2f_2 + f_2 - f_2)$	3, 5	2

Table 3.1: highest weights of spin modules

Following the notations of Remark 3.4, the highest weight vector z_{top} in each case is represented by the vectors with corresponding highest weights from (3.19). Nonetheless, the action of $c\ell(H_2)$ on X_+ via i or $-i$ corresponds to different Clifford module structures of $\mathbb{C}\ell(\mathfrak{k})$, and these nonequivalent $\mathbb{C}\ell\mathfrak{k}$-modules are equivalent as $\mathfrak{spin}(\mathfrak{k})_\mathbb{C}$-module structures. Similarly for the action of $c\ell(H_1)$ on $E_{2f_2} \wedge Y_1$.

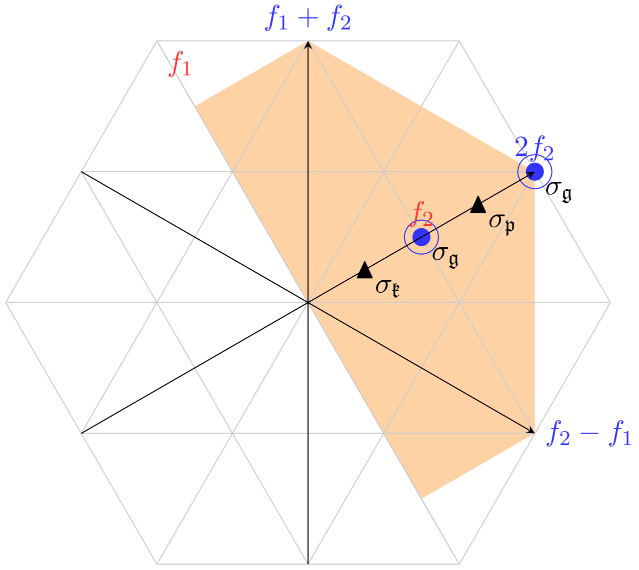


Figure 3.1: Root system and highest weights of spinor representations σ_V

We conclude this case by computing the multiplicities of each highest weight. As $\dim \mathfrak{a} = \dim \mathfrak{t} = 1$, *whence there is a unique* $\mathfrak{k}$*-representation for each fixed compatible positive root system.*

Lastly we illustrate the weights and the root data in the diagram.

Note that the two rounded dots denote the highest weights occurring in $\mathcal{S}_{\mathfrak{g}}$, *each of multiplicity two.*

Chapter 4

Homogeneous vector bundles and Dirac operators

In this chapter we consider homogeneous vector bundles in a very general setting, transferring all differential geometric quantities to their representation-theoretic counterparts via exploiting the homogeneity structure. In the end we develop a version of the Bochner identity that will be used in the estimation of the heat kernel in Chapter 5.

4.1 Homogeneous vector bundle and connection Laplacian

For this section only, let G be a connected unimodular Lie group and let L be a closed compact subgroup of G. Although the following discussion is intended for general Lie groups, the reader should keep in mind our main focus still lies in the reductive and nilpotent Lie groups. These two classes are somewhat orthogonal: The reductive Lie groups have abundant compact subgroups, and they play a critical role in studying the representation theory of G; while in the case of nilpotent Lie groups, all compact subgroups lie in the center of G by Lemma B.8. In that case, the following discussion is mostly adapted to the group manifold case.

We first give the broadest notion of homogeneous vector bundles over homogeneous spaces. This part is an adaptation of results in [**KN96**, Chapter X, Section 2 & 3].

Definition 4.1 (homogeneous vector bundle). A vector bundle E over $X = G/L$ is called a homogenous vector bundle if there is a G-action on E from the left, which induces a linear map such that $g \circ E_x = E_{g\cdot x}$ for all $x \in G/L$ and $g \in G$.

Given (ρ, V) a finite dimensional representation of L, let L acts on $G \times V$ with $(g, v) \circ h = (gh, \rho(h)^{-1}v)$. This gives an associated vector bundle:

$$E := G \times_\rho V = (G \times V)/L \tag{4.1}$$

with the bundle projection being

$$p : E \to G/L \qquad [g, v] \mapsto gL$$

where $[g, v] := (g, v) \circ L$ the L-orbit in $G \times V$. The homogeneous vector bundle is equipped with a canonical left G-action $g_0 \circ [g, v] = [g_0 g, v]$. Every homogeneous vector bundle can be constructed in this way. Next let $\langle -, - \rangle_V$ be a Hermitian product on V which makes (ρ, V) a unitary representation of L. Then there is a Hermitian structure on E that is homogeneous, by identifying V_{eL} with V in a canonical manner: $([g, v], [g, w])_{gL} = \langle v, w \rangle_V$.

The G-action on G/L induces an action on its vector fields. Given $\mathcal{Y}$ a vector field of G/L, G acts by $\langle g_1 \circ \mathcal{Y} \rangle|_{gL} := (L_{g_1}\mathcal{Y})|_{g_1^{-1}gL}$. This in particular gives a homomorphism of Lie algebras from $\mathfrak{g}$ into the vector fields of G/L given by associating each $Y \in \mathfrak{g}$ a vector field Y_L:

$$Y_L \circ f(gL) = \frac{\partial}{\partial t}\Big|_{t=0} f\big(\exp(-tY) \cdot gL\big) \tag{4.2}$$

for $f \in C^\infty(G/L)$ and G acts on Y_L by left translations: $g \cdot Y_L := (\mathrm{Ad}_g Y)_L$ for $g \in G$ and $Y \in \mathfrak{g}$.

Next the smooth sections of $\Gamma(E) := C^\infty(G/L, E)$ can be identified with the L-invariant part of the bundle $[C^\infty(G) \otimes V]^L$ of $C^\infty(G, E) \cong C^\infty(G) \otimes V$, with L acting on $C^\infty(G) \otimes V$ on the right as $R_h \otimes \rho(h)$ for $h \in L$, with R the right regular representation of G. $\Gamma(E)$ is equipped also with a natural left G-action:

$$(g \circ f)(x) := g \cdot_V (f(g^{-1}x)) \quad \text{for } f \in \Gamma(E) \text{ and } x \in G/L \tag{4.3}$$

Here $\cdot_V$ denotes the module action of G on V. Alternatively one considers the smooth vector subspace of an induced G-module:

$$C^\infty(G;\rho) := \mathrm{ind}_L^G(\rho)^\infty := \{F : G \to V_\rho \mid f \in C^\infty, F(gl) = \rho(l^{-1})F(g) \text{ for all } l \in L\} \tag{4.4}$$

which can be identified with $\Gamma(E)$ as G-modules via the canonical G-equivariant isomorphism:

$$\mathcal{A} : \Gamma(E) \longrightarrow C^\infty(G;\rho) \qquad F(g) := (\mathcal{A}f)(g) = \rho(g^{-1})(f(gL)) \tag{4.5}$$

with G acting on $C^\infty(G;\rho)$ by left regular representation: $(L_g F)(g') := F(g^{-1}g')$. With this G-action on $\Gamma(E)$ and $C^\infty(G;\rho)$ is an intertwining operator between the two spaces i.e.:

$$\mathcal{A}(g \circ f) = L_g \circ \mathcal{A}(f) \tag{4.6}$$

Hence we can extend this isomorphism to its L^2-completion $L^2(X;E)$, which we denote by the same symbol:

$$\mathcal{A} : L^2(G/L, E) \cong [L^2(G) \otimes V]^L \xrightarrow{\cong} L^2(G;\rho)$$

Conversely $\mathcal{A}^{-1}$ maps $f \in \Gamma(E)$ to $F \in C^\infty(G;\rho)$ by $\mathcal{A}^{-1}(F)(gL) = [g, F(g)]$. Later we shall see how things like connection and curvature in differential geometry carry over via $\mathcal{A}$ to linear maps and homomorphisms on $C^\infty(G;\rho)$. This renders a more representation-theoretical flavored description of all terms.

From now on we follow the convention by using capital letters F as elements of $C^\infty(G;\rho)$ and f of $\Gamma(E)$.

Let ρ_1, ρ_2 be two finite-dimensional unitary representations of L, with the underlying vector spaces V_1, V_2 respectively, and let $[U(\mathfrak{g}_\mathbb{C}) \otimes \mathrm{Hom}(V_1, V_2)]^L$ the L-invariant subspace under the L-action on $\sum_i D_i \otimes C_i \in U(\mathfrak{g}_\mathbb{C}) \otimes \mathrm{Hom}(V_1, V_2)$ by:

$$l \circ (\sum_i D_i \otimes C_i) := \sum_i \mathrm{Ad}_l(D_i) \otimes (\rho_1(l) \circ C_i \circ \rho_2(l)^{-1}) \tag{4.7}$$

Again the space of G-invariant differential operators $\mathcal{D} : C^\infty(G/L; E_1) \to C^\infty(G/L; E_2)$ is under the isomorphism $\mathcal{A}$ identified with the L-invariant subspace $D \in [U(\mathfrak{g}_\mathbb{C}) \otimes V]^L$ by (4.5), with:

$$\sum_i X_i \otimes C_i \mapsto \sum_i R_{X_i} \otimes C_i$$

The formal adjoint of $\mathcal{D}$ is defined in the natural way

$$\mathcal{D}^* = \sum_i X_i^* \otimes C_i^* \tag{4.8}$$

with $C_i^* \in \mathrm{Hom}(E_2, E_1)$ the adjoint of C_i, and X_i^* the image of X_i under canonical anti-involution of $U(\mathfrak{g}_\mathbb{C})$ induced by taking $X + iY \in \mathfrak{g}_\mathbb{C}$ to $-X + iY$. The corresponding differential operator $\mathcal{D}^*$ is a densely defined operator, hence $\mathcal{D} : \mathrm{Dom}\,\mathcal{D} \subseteq L^2(G/L; E_1) \to L^2(G/L; E_2)$ is a closable operator. Let π be a unitary representation of G, with the underlying Hilbert space H_π, and denote H_π^∞ its C^∞-vectors. Accordingly we define the operator

$$\pi(D) : [H_\pi \otimes V_1]^L \to [H_\pi \otimes V_2]^L \qquad \pi(D) := \sum_i \pi(X_i) \otimes C_i \tag{4.9}$$

If $\widehat{G}$ is discrete, one should think of $\pi(D)$ as the restriction of D to one summand $H_\pi \in \widehat{G}$. If D is an elliptic operator, then the minimal and maximal domain of $\pi(D)$ coincide [Mos82, Corollary 1.2], so the closure of $\pi(D)$ has no ambiguity, with $\pi(D)^* = \pi(D^*) = \sum_i \pi(X_i^*) \otimes C_i^*$.

Now we restrict our attention to the two types of operators:

Bochner Laplacian: $\Delta_{\nabla^E} := \nabla^{E,*}\nabla^E$. By taking $\rho = \mathrm{CoAd}$, this differs from the Hodge Laplacian defined earlier in Chapter 1 by a curvature term.

Dirac operator: $\mathscr{D}$ on spinor bundles over X, which we will define in section 4.2. We will see that the Dirac operator associated with the Levi-Civita connection differs from the Dirac operator seen in many articles of representation theory.

We focus on the Bochner Laplacian in this section and leave the discussion of the Dirac Operator to Section 4.2. Suppose $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ as a vector space direct sum. Also assume $\mathfrak{p}$ is closed under the $\mathrm{Ad}\, L$-action.

Definition 4.2. Given an inner product $(-,-)$ on $\mathfrak{p}$ with $(\mathfrak{p}, \mathrm{Ad})$ an orthogonal representation of L. This gives a left G-invariant metric on G/L. Define ∇^E a G-invariant metric connection on E by:

$$\nabla^E : \Gamma(E) \to \Gamma(T^*(G/L) \otimes E) \qquad (\nabla^E f)(\mathcal{Y}) = \nabla^E_{\mathcal{Y}} f$$

with G-equivariant condition: for all $f \in \Gamma(E)$ and $\mathcal{Y}$ vector fields on G/L:

$$g \circ (\nabla^E_{\mathcal{Y}} f) = \nabla^E_{g \circ \mathcal{Y}}(g \circ f) \tag{4.10}$$

More verbosely, G acts on general connections ∇^E, admits the following local expression around $g'L$ as:

$$(g \circ \nabla^E f)(\mathcal{Y})\Big|_{g'L} := g \circ \left[\nabla^E f\Big(g^{-1} \circ \mathcal{Y}\Big)\Big|_{g^{-1}g'L}\right] \tag{4.11}$$

Then the G-equivariant condition is equivalent to stating $g \circ (\nabla^E_{g^{-1} \circ \mathcal{Y}} f) = \nabla^E_{\mathcal{Y}}(g \circ f)$.

Remark 4.3. Each G-invariant metric connection is uniquely determined by the following linear map: [Sle87, p. 287]

$$\gamma^V : \mathfrak{g} \to \mathfrak{so}(V) \tag{4.12}$$

satisfying:

1. $\gamma^V(Z) = \rho(Z)$ for $Z \in \mathfrak{l}$;
2. $\gamma^V(\mathrm{Ad}_h Y) = \rho(h) \circ \gamma^V(Y) \circ \rho(h)^{-1}$ for $h \in L$ and $Y \in \mathfrak{g}$.

Then ∇^E is retrieved by defining for $Y \in \mathfrak{g}$ and $f \in \Gamma(E)$:

$$\nabla^E_{Y_L} f = Y \circ f - \Lambda(Y) f \tag{4.13}$$

with $Y\circ$ the action of $\mathfrak{g}$ on $C^\infty(X; E)$ induced by G in (4.3):

$$(Y \circ f)(gL) = \frac{\partial}{\partial t}\Big|_{t=0} \left(e^{tY} \cdot_V f(e^{-tY} gL)\right) \tag{4.14}$$

and Y_L as in (4.2) and $\Lambda : \mathfrak{g} \to \Gamma(\mathfrak{so}(E))$ given at the origin by:

$$\Lambda(Y)|_{eL} : (e, v) \mapsto (e, \gamma^V(Y) v) \qquad \text{for } Y \in \mathfrak{g} \text{ and } v \in V \tag{4.15}$$

with $\Lambda(Y)$ at $x = gL$ given by

$$\Lambda(Y)|_{gL} = g \circ \Lambda(\mathrm{Ad}_{g^{-1}} Y)|_{eL} \cdot g^{-1} \circ \tag{4.16}$$

Under the isomorphism $\mathcal{A}$ in (4.5), ∇^E can be interpreted as a linear map acting on the representation side $F = \mathcal{A} f \in C^\infty(G; \rho)$ with:

$$(\mathcal{A}(\nabla^E_{Y_L} f))(g) = (L_Y F)(g) - \gamma^V(\mathrm{Ad}_{g^{-1}} Y) F(g) \tag{4.17}$$

This derives directly from (4.13), with the Λ-term identified with γ^V by combining the equivariance formula $(\nabla^E_{Y_L} f)(gL) = (\nabla^E_{g^{-1} \circ Y_L}(g^{-1} \circ f))(eL)$ and:

$$\Lambda(\mathrm{Ad}_g Y)|_{eL} f(eL) = \gamma^V(\mathrm{Ad}_g Y) f(eL) = \gamma^V(\mathrm{Ad}_g Y) F(e)$$

It is neater to interpret $\nabla^\rho F := \mathcal{A}(\nabla^E f)$ as a map:

$$\nabla^\rho F : G \to \mathrm{Hom}(\mathfrak{g}; V) \qquad \nabla^\rho F(g)(X) := \mathcal{A}(\nabla^E_{-g\circ X_L} f)(g) \tag{4.18}$$

Then the above covariance can be written as:

$$(\nabla^\rho F)(g) : Y \mapsto (R_Y F)(g) + \gamma^V(Y)(F(g)) \tag{4.19}$$

by $g \cdot Y_L := (\mathrm{Ad}_g Y)_L$ and rewriting $-L_{\mathrm{Ad}_g Y} F(g) = R_Y F(g)$. Note that $\mathfrak{l} \subseteq \ker(\nabla^\rho F)(g)$ for all F and g by the covariance of F and γ^V.

Definition 4.4 (**Bochner Laplacian**). Define the Bochner Laplacian as $\Delta_{\nabla^E} := \nabla^{E,*}\nabla^E$ where $\nabla^{E,*} : C^\infty(T^*M \otimes E) \to C^\infty(E)$ is the formal adjoint of ∇^E with respect to the prescribed metric $(-,-)_E$ on E. It is also known as connection Laplacian in some literature.

The connection Laplacian is completely determined by the choice of the metric connections ∇^E on E and ∇ on the tangent space $T(G/L)$. Indeed, by choosing a local orthonormal tangent frame field $(\mathcal{Y}_1, \dots, \mathcal{Y}_n)$ on G/L, the Bochner Laplacian can also be written locally as:

$$\Delta_{\nabla^E} = \sum_j \nabla^{E,2}_{\mathcal{Y}_j,\mathcal{Y}_j} = \sum_j \nabla^E_{\mathcal{Y}_j}\nabla^E_{\mathcal{Y}_j} + \nabla^E_{\nabla_{\mathcal{Y}_j}\mathcal{Y}_j} \tag{4.20}$$

with ∇ the connection on TX. Transcribing using isomorphism $\mathcal{A}$ in (4.5), the corresponding connection Laplacian $\Delta_{\nabla^\rho} := \mathcal{A}^{-1}(\Delta_{\nabla^E})$ acting on $F \in C^\infty(G;\rho)$ admits the following expression, by choosing an orthonormal basis $Y_1, \dots, Y_n$ of $\mathfrak{p}$, we have [Sle87, (0.5.1)]:

$$-\Delta_{\nabla^\rho} F = \sum_i (R_{Y_i} + \gamma^V(Y_i))^2 \cdot F - R_{\gamma(Y_i)\cdot Y_i} F - \gamma^V(\gamma(Y_i)Y_i)F \tag{4.21}$$

with R the right regular representation of G and of $\mathfrak{g}$ on $C^\infty(G;\rho)$, γ^V and γ are the linear map that determines the G-invariant metric connection on E and TX respectively, as remarked in Remark 4.3. This expression holds for all G-invariant metric connections on X and E, even those with torsions.

Use Remark 4.3 above, we define reductive connections on $T(G/L)$ and E respectively:

Definition 4.5. Define a connection ∇^E to be reductive if the corresponding γ^V vanishes on $\mathfrak{p}$. In this case $\nabla^\rho F(g)$ defined in (4.18) admits the simple form $\nabla^\rho F(Y) \equiv R_Y F$ for $Y \in \mathfrak{p}$ and $\nabla^\rho F(Y) = 0$ for $Y \in \mathfrak{l}$, with g implicit in the expression.

In fact one can again treat the curvature form on E and the torsion tensor on $T(G/L)$ as linear maps as follows:

$$\mathcal{R}^\rho(X,Y)(\mathcal{A}f) := \mathcal{A}(\mathcal{R}^E(X_L,Y_L)f) \quad T^\rho(X,Y) := T(X_L,Y_L)(eL) \tag{4.22}$$

for $Y, Z \in \mathfrak{g}$ and $f \in \Gamma(E)$, and $\mathcal{R}^E$ the curvature form on E, and T torsion tensor on $T(G/L)$ respectively. Note that T and $\mathcal{R}^\rho$ are G-invariant tensors, hence it suffices to consider their value at the identity: [Sle87, (0.4.3)]

$$\begin{aligned} \mathcal{R}^\rho(Y,Z)(e_G) &= R_{P^{\mathfrak{g}}_{\mathfrak{l}}[Y,Z]} + [\gamma^V(X), \gamma^V(Y)] - \gamma^V(P^{\mathfrak{g}}_{\mathfrak{p}}[Y,Z]) \\ T(Y,Z) &= -P^{\mathfrak{g}}_{\mathfrak{p}}[Y,Z] + \gamma(Y)(P^{\mathfrak{g}}_{\mathfrak{p}}Z) - \gamma(Z)(P^{\mathfrak{g}}_{\mathfrak{p}}Y) \end{aligned} \tag{4.23}$$

again P denotes the orthogonal projection onto the space of the subscript.

Definition 4.6. Given a left G-invariant metric on a homogeneous space G/L, the Levi-Civita connection ∇ on $T(G/L)$ is defined to be the unique G-invariant metric connection on G/L such that the corresponding torsion $T(Y,Z) = 0$ for all $Y, Z \in \mathfrak{g}$.

Remark 4.7. In view of Remark 4.3 the Levi-Civita connection corresponds to the following map γ_0: [1]

$$\gamma_0(Y)(Z) = \nabla_Y Z = \frac{1}{2} P^{\mathfrak{g}}_{\mathfrak{p}}([Y, Z] - \operatorname{ad}^*_Y Z - \operatorname{ad}^*_Z Y) \tag{4.24}$$

Recall the Koszul formula on $\mathfrak{g}$:

$$\langle \nabla_Y Z, W\rangle = \frac{1}{2}\left(\langle [Y, Z], W\rangle - \langle [Y, W], Z\rangle - \langle [Z, W], Y\rangle\right)$$

It is straightforward to verify that this indeed defines a linear map, for $W \in \mathfrak{p}$ and $Y, Z \in \mathfrak{g}$:

$$\begin{aligned}\langle \gamma_0(Y)(Z), W\rangle &= \frac{1}{2}\langle P^{\mathfrak{g}}_{\mathfrak{p}}([Y, Z] - \operatorname{ad}^*_Y Z - \operatorname{ad}^*_Z Y), W\rangle \\ &= \frac{1}{2}\left(\langle [Y, Z], W\rangle - \langle Z, [Y, W]\rangle - \langle Y, [Z, W]\rangle\right)\end{aligned}$$

We remark that γ_0 is not in general G-equivariant. Next $\langle \nabla_Y Z, W\rangle_{\mathfrak{p}} = -\langle Z, \nabla_Y W\rangle_{\mathfrak{p}}$ as $Y\langle Z, W\rangle_{\mathfrak{p}} = 0$ when treating Z, W as left G-invariant vector fields. Hence we have verified that γ_0 is indeed a metric connection.

Next by (4.23) we see that $T(Y, Z) = 0$ for all $Y, Z \in \mathfrak{p}$, showing γ_0 indeed is the map associated with the Levi-Civita connection by the fundamental theorem of Riemannian geometry [**LM90**, Theorem II.4.17], i.e., the uniqueness of such connection for a fixed metric.

4.2 Dirac operators and Bochner identities

In this section we focus on the Dirac operators on the homogeneous spaces. The settings are exactly as in the previous section. We follow the discussions in [**Sle87**, §2 & 3] and partly [**MS89**, §3].

The reader of this section is assumed to be familiar with the basic notions of Clifford algebras and spin representation. We focus on the geometric aspects in this section. For relevant algebraic details, refer to Chapter 3.

Again assume $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$, with a choice of orthonormal basis $\{Y_1, \ldots, Y_n\}$ of $\mathfrak{p}$ with respect to the prescribed metric $(-,-)_{\mathfrak{p}}$.

Definition 4.8 (G**-spin**)**.** A homogeneous space G/L is said to be G-spin if the adjoint representation of $\operatorname{Ad} : L \to SO(\mathfrak{p})$ lifts to the spin cover $\operatorname{Spin}(\mathfrak{p})$ of $SO(\mathfrak{p})$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \operatorname{Spin}(\mathfrak{p}) \\ & \overset{\widetilde{\operatorname{Ad}}}{\nearrow} & \downarrow \varphi^{-1} \\ L & \xrightarrow[\operatorname{Ad}]{} & SO(\mathfrak{p}) \end{array} \tag{4.25}$$

We assume from now onwards that G/L is always G-spin. This gives the spin representation of L on the space of spinors $\mathcal{S}$ by $\sigma = c\ell \circ \widetilde{\operatorname{Ad}}$. At Lie algebra level, this is:

$$\mathfrak{l} \xrightarrow{\operatorname{ad}} \mathfrak{so}(\mathfrak{p}) \xrightarrow{c\ell\circ\varphi} \operatorname{End}(\mathcal{S}) \tag{4.26}$$

Recall $c\ell$ is the left Clifford multiplication on S and $\varphi : \mathfrak{spin}(\mathfrak{p}) \cong \mathfrak{so}(\mathfrak{p})$ as in Chapter 3. One explicitly writes, for $T \in \mathrm{SO}(\mathfrak{p})$:

$$c\ell \circ \varphi(T) = -\frac{1}{4}\sum_{i=1}^{\dim \mathfrak{p}} c\ell(T(u_i))c\ell(u_i) \tag{4.27}$$

One readily verifies this is the spin representation of $\mathfrak{spin}(\mathfrak{p})$ if we realize $\mathcal{S}$ in the concrete way as in Chapter 3. In the case $\dim \mathfrak{p}$ is even, we denote $\sigma^{\pm}$ as the representations on $\mathcal{S}^{\pm}$ respectively. The spin representation of L intertwines with the Clifford multiplication in the following way:

$$c\ell(Y)\sigma(h) = \sigma(h)c\ell(\operatorname{Ad}_{h^{-1}} Y) \quad \text{for } h \in L \text{ and } Y \in \mathfrak{p} \tag{4.28}$$

[1] In original paper [**Sle83**, (0.4.3)] it is wrongly claimed that $\gamma_0 = \frac{1}{2}P^{\mathfrak{g}}_{\mathfrak{p}} \circ \operatorname{ad}$. In fact $\gamma = \frac{1}{2}\operatorname{ad}$ in the case $\mathfrak{p} = \mathfrak{g}$ can almost never be a nontrivial metric connection due to the fact that semisimple $\mathfrak{g}$ have no finite-dimensional unitary representations from Weyl's unitary trick [**Kna86**, Corollary 2.3], therefore $\operatorname{ad}(\mathfrak{g}) \not\subseteq \mathfrak{so}(\mathfrak{g})$

Now let us see the effect of representations on corresponding bundles. By taking the representation $\rho = \mathrm{Ad}$ and $\rho = \sigma$, we form vector bundles $T(G/L)$ and the irreducible complex spinor bundle, which we denote as $\$_{\mathbb{C}}$. .

In this case each G-invariant metric connection ∇ on TX lifts to a unique connection ∇^S in the following sense:

Proposition 4.9 ([Sle83, Proposition I.3])**.** *Given a G-invariant metric connection ∇ on the tangent bundle $T(G/L)$, there is a unique metric connection ∇^S on the bundle of spinors $\$$ which acts on $\$$ as module derivation, i.e., it satisfies the following Leibniz rule:*

$$\nabla^S_Y(c\ell(Z)s) = c\ell(\nabla_Y Z)s + c\ell(Z)\nabla^S_Y s \tag{4.29}$$

for all Y, Z vector fields on G/L and all smooth section s of $\$$. In virtue of Remark 4.3, given γ the homomorphism that determines ∇ on $T(G/L)$, the respective lift $\gamma^S : \mathfrak{g} \to \mathfrak{so}(\$)$ is defined by:

$$\gamma^S(Y) := (c\ell \circ \varphi)\gamma(Y) \tag{4.30}$$

Proof. This is straightforward to verify. First verify that γ^S satisfies the cocycle condition:

$$\begin{aligned}\gamma^S(\mathrm{Ad}_h Y) &= (c\ell \circ \varphi)(\mathrm{Ad}_h \circ \gamma(Y) \circ \mathrm{Ad}_h^{-1}) \\ &= -\frac{1}{4}\sum_{i=1}^{\dim \mathfrak{p}} \sigma(h)c\ell(\gamma(Y)\,\mathrm{Ad}_h^{-1} Y_i)\sigma(h)^{-1}c\ell(Y_i) \\ &= \sigma(h)\gamma^S(Y)\sigma(h)^{-1}\end{aligned} \tag{4.31}$$

Hence this indeed defines a G-invariant metric connection on S. To see this lift of connection is unique, let ∇' be another G-invariant metric connection on G/L associated with γ', which lifts to a connection on ∇'^S with γ'^S, then the difference of two connections is tensorial, in fact they differ only by 1-forms:

$$\nabla - \nabla' =: \alpha \in \Omega^1(G/L; \mathfrak{so}(T(G/L))) \qquad \nabla^S - \nabla'^S =: \beta \in \Omega^1(G/L; \mathfrak{so}(\$)) \tag{4.32}$$

so $c\ell(\alpha(X)Y) = [\beta(X), c\ell(Y)]$. Now $c\ell(\alpha(X)Y) = [c\ell \circ \varphi(\alpha(X)), c\ell(Y)]$. Now from the fact Clifford multiplication generates $\mathfrak{so}(\$)$ and $\mathfrak{so}(\$)$ has zero center, we conclude that $\beta(X) = (c\ell \circ \varphi)\alpha(X)$. Hence such a lifting is unique and is characterized by $c\ell \circ \varphi$ above. □

Consider the following data:

1. **Representations:** Given (V, ρ) a unitary representation of L and a inner product $(-,-)_{\mathfrak{p}}$. The spin representation of L is (S, σ);
2. **Connection data:** Given ∇^V a G-invariant connection of $E = G \times_\rho V$ and ∇ on $T(G/L) = G \times_{\mathrm{Ad}} \mathfrak{p}$. Note here we only assume the connection is G-invariant, but not Levi-Civita. Let ∇^S the unique lifting of ∇ to its spinor bundle, with the associated homomorphism denoted by γ^V, γ and γ^S respectively.

Definition 4.10 (**Twisted Dirac operator**)**.** Define a twisted Dirac operator $\not{D}_V$ associated with the above data as the following composite map:

$$\not{D}_V := \sum_i c\ell(\mathcal{Y}_i)\nabla^{S\otimes V}_{\mathcal{Y}_i} : \quad \Gamma(S \otimes V) \overset{\nabla^{S\otimes V}}{\longrightarrow} \Gamma(\mathfrak{p} \otimes S \otimes V) \overset{c\ell \otimes \mathbb{I}_V}{\longrightarrow} \Gamma(S \otimes V) \tag{4.33}$$

where $\{\mathcal{Y}_i\}$ is an orthonormal frame field on $T(G/L)$, with $\mathcal{Y}_i(eL) = Y_i$ such that $\{Y_i\}$ defines a orthonormal basis of $\mathfrak{p}$. $\not{D}_V$ is a first-order G-invariant elliptic differential operator. When $V \cong \mathbb{C}$ the trivial representation, then we denote $\not{D}_V = \not{D}$ the untwisted Dirac operator.

Here $\nabla^{S\otimes V}$ is the tensor product connection on $S \otimes V$, which corresponds to the map $\gamma^{S\otimes V} : \mathfrak{g} \to \mathfrak{so}(S \otimes V)$. Then the following generalized Bochner identity is due to Rawnsley: [Sle87, (1.3.3)]

$$\not{D}_V^2 = \Delta_{\nabla^{S\otimes V}} - \frac{1}{2}\sum_{i,j} c\ell(X_i)c\ell(X_j)\nabla^{S\otimes V}_{T(X_i,X_j)} + \frac{1}{2}\sum_{i,j} c\ell(X_i)c\ell(X_j)\mathcal{R}^{S\otimes V}(X_i, X_j) \tag{4.34}$$

with the curvature and torsion terms described in the last section. In fact using (4.23), we get an expression in terms of γ's and the right regular action on $C^\infty(G;\sigma\otimes\rho)$. We summarize the discussions here:

Theorem 4.11 ([**Sle87**, (1.3.5)])**.** *Under the isomorphism $\mathcal{A}$ in (4.5), the Dirac operator $\not{D}_V$ can be written as $\not{D}_{\sigma\otimes\rho}F := \mathcal{A}(\not{D}_V f) = \sum_i c\ell(Y_i)(\nabla^{\sigma\otimes\rho}F)(Y_i)$. Moreover, the general Bochner identity transforms under $\mathcal{A}$ to the following form:*

$$\not{D}^2_{\sigma\otimes\rho}F = \Delta_{\nabla^{\sigma\otimes\rho}} - \frac{1}{2}\sum_{i,j} c\ell(Y_i)c\ell(Y_j)\nabla^{\sigma\otimes\rho}F(T(Y_i,Y_j)) + \frac{1}{2}\sum_{i,j} c\ell(Y_i)c\ell(Y_j)\mathcal{R}^{\sigma\otimes\rho}(Y_i,Y_j)F \tag{4.35}$$

with $\mathcal{R}^{\sigma\otimes\rho}$ and T admit expressions in the right regular action and γ as in (4.23) and $\nabla^{\sigma\otimes\rho}F$ defined in (4.18).

Remark 4.12. In proving (4.35) one makes critical use of the Clifford commutation relations, i.e.:

$$c\ell(X_i)c\ell(X_j) = -c\ell(X_j)c\ell(X_i) \qquad c\ell(X_i)c\ell(X_i) = -2B^\theta(X_i,X_i) = -2 \tag{4.36}$$

for the X_i, X_j. These identities occur when the underlying bilinear form is positive definite.

We now move to the second theme of this section: namely the Bochner identity on a homogeneous vector bundle, tailored for our purposes. More specifically, the Weitzenböck formula relates the Hodge Laplacian in Chapter 1 with the connection Laplacian defined above, with a difference term expressed in curvature. We shall refrain from discussing the general Dirac bundle in detail due to the irrelevance to ensuing discussion, merely referring intent readers to [**LM90**, Section II.5] or [**BGV92**, Section 3.5] for relevant details. We are satisfied with citing the following formula [**BGV92**, (3.16)]:

$$\Delta = \Delta_{\nabla^{\Lambda T^*M}} - \sum_{ijkl}\mathcal{R}_{ijkl}\cdot\varepsilon^k\iota^l\varepsilon^i\iota^j \tag{4.37}$$

with Δ as Hodge Laplacian acting on the whole exterior bundle $\Lambda(T^*M) = \oplus_n\Lambda^nT^*M$ and $\Delta_{\nabla^{\Omega T^*M}}$ the connection Laplacian associated with the Levi-Civita connection $\nabla^{\Omega T^*M}$ extending that on 1-forms, with $\mathcal{R}_{ijkl}$ the $(0,4)$-curvature tensor defined from Riemannian curvature:

$$\mathcal{R}_{ijkl} := \langle\mathcal{R}^{TM}(X_k,X_l)X_j,X_i\rangle$$

with respect to orthonormal bases, with the ε and ι the (left) exterior multiplication and contraction operator acting on Λ^*V as defined in (1.2). This identity holds for general Riemannian manifolds. A representation-theoretic formula can be hoped in the case of general homogeneous spaces, but is too complicated involving some case-by-case discussion, owing to the fact that most terms are plagued by projection operators as in (4.23). Nonetheless, we discuss here two extreme examples: The first when $L = K$ the maximal compact subgroup, in which case G/K is a symmetric space; and the second when $L = e_G$ the trivial subgroup, in which case $G/L = G$ is the group manifold.

We begin with the group manifold case, assuming only that G is a connected Lie group. Choose a positive definite left-G-invariant bilinear form on $\mathfrak{g}$, which indeed defines a Riemannian metric on G. This gives rise to the Levi-Civita connection which corresponds to a γ_0 as in (4.24). The Riemannian curvature in this case admits a neat formula either by read off from (4.23) or from general facts about group manifolds (see e.g. [**CE08**, Proposition 3.18]):

$$\begin{aligned}\mathcal{R}_{ijkl} &= \langle\nabla_{Y_k}Y_j,\nabla_{Y_l}Y_i\rangle - \langle\nabla_{Y_l}Y_j,\nabla_{Y_k}Y_i\rangle - \langle\nabla_{[Y_k,Y_l]}Y_j,Y_i\rangle \\ &= \Gamma^m_{kj}\Gamma^m_{li} - \Gamma^m_{lj}\Gamma^m_{ki} - C^m_{kl}\Gamma^i_{mj}\end{aligned} \tag{4.38}$$

by expressing the covariant derivative in terms in structural constants of the underlying Lie algebra using (4.24)

$$\nabla_{Y_i}Y_j = \Gamma^m_{ij}Y_m = \frac{1}{2}\sum_m(C^m_{ij} + C^j_{im} + C^i_{jm})Y_m \tag{4.39}$$

where Γ_{ij}^m is the Christoffel symbol.

Furthermore the action of $\mathfrak{g}$ induces an action on the exterior algebra $\wedge^*$ coad with a corresponding linear map $\gamma^{\wedge^*\mathfrak{g}^*}$, which we for brevity denote as coad and $\gamma^\wedge$. Note $\nabla_Y\omega(X) = \omega(\nabla_Y X)$. Applying the expression of γ_0 (4.24) to $\omega \in \mathfrak{g}^* = \wedge^1\mathfrak{g}^*$ we obtain:

$$\gamma^\wedge(Y)(\omega) = \nabla_Y^{T^*M}\omega = -\frac{1}{2}\left(\mathrm{coad}_Y^*\,\omega - \mathrm{coad}_Y\,\omega - \mathrm{coad}_{\sharp\omega}(\flat Y)\right) \tag{4.40}$$

by noting ad* and coad as heeded by Remark 1.4, as well as the fact $\nabla_Y\sharp\omega = \sharp\nabla_Y\omega$. Recall that $\sharp$ and $\flat$ are musical isomorphisms. It is straightforward to extend the above map to a map on $\mathfrak{so}(\wedge^*\mathfrak{g}^*)$ by the Leibniz rule:

$$\nabla^{T^*M}(\omega\wedge\eta) = \nabla^{T^*M}(\omega)\wedge\eta + (-1)^{\deg\omega}\omega\wedge\nabla^{T^*M}(\eta) \tag{4.41}$$

Remark 4.13. It is interesting to note that the first terms in (4.40) can be understood as representations of $\mathfrak{g}$; whereas the third can be interpreted as a linear map

$$\mathrm{coad}^\flat : \wedge^*\mathfrak{g}^*\otimes\mathfrak{g} \to \wedge^*\mathfrak{g}^* \qquad \mathrm{coad}^\flat(Y)(\omega) := \mathrm{coad}_{\sharp\omega}(\flat Y) \tag{4.42}$$

that extend the map on 1-forms by the Leibniz rule:

$$\mathrm{coad}_{\sharp(\omega\wedge\eta)}(\flat Y) = \mathrm{coad}_{\sharp\omega}(\flat Y)\wedge\eta + (-1)^{\deg\omega}\omega\wedge\mathrm{coad}_{\sharp\eta}(\flat Y)$$

Note that this map does not define any $\mathfrak{g}$-module structure:

$$\mathrm{coad}_{X_k}[X_i,X_j] = \sum_{m,l} C_{kl}^m C_{ij}^m X_l \neq \sum_{l,m}(C_{il}^m C_{km}^j - C_{jl}^m C_{km}^i)X_l = [\mathrm{coad}^\flat(X_i),\mathrm{coad}^\flat(X_j)](\flat X_k)$$

It is the compensation one has to pay to obtain a Levi-Civita connection on $\mathfrak{g}$ in the absence of a bi-G-invariant Riemannian metric on G.

Hence using (4.21) the connection Laplacian $\Delta_{\nabla^{\mathrm{ad}}}$ gives the following expression:

$$\Delta_{\nabla^{\wedge^*\mathrm{ad}}} = \sum_i (R_{Y_i} + \gamma^\wedge(Y_i))^2 \tag{4.43}$$

with the Levi-Civita connection implies $\gamma_0(Y_i)Y_i = 0$ the remaining terms vanish. Putting all terms together:

$$\Delta = \sum_i (R_{Y_i} + \gamma^\wedge(Y_i))^2 - \sum_{ijkl}\mathcal{R}_{ijkl}\cdot\varepsilon^k\iota^l\varepsilon^i\iota^j \tag{4.44}$$

Remark 4.14 (Ricci curvature)**.** One special case of this identity is when we restrict both sides to 1-forms on the reductive Lie group G, where the curvature terms tensorial in two variables, therefore equal to:

$$\sum_{ijkl}\mathcal{R}_{ijkl}\cdot\varepsilon^k\iota^l\varepsilon^i\iota^j = \sum_{ij}\mathcal{R}_{ijij}\cdot\varepsilon^i\iota^j\varepsilon^i\iota^j = \mathrm{Ric} \tag{4.45}$$

In particular, this shows that the Ricci curvature is bounded from below, as the curvature tensor $\mathcal{R}_{ijij}$ can be expressed as a finite sum of products of structural constants. We will use this fact when estimating heat kernel later.

Remark 4.15 (sectional curvature)**.** Recall that the sectional curvature $\mathcal{K}(X,Y)$ is defined to be:

$$\mathcal{K}(X,Y) := \frac{\langle\mathcal{R}^{TM}(X,Y)Y,X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} \tag{4.46}$$

The sectional curvature determines the curvature tensor completely [GHL04, Theorem 3.8]. In the case of the a symmetric space G/K, this is closely related with the Killing form. In particular, if we choose the metric on $\mathfrak{p}$ to be $B_\mathfrak{p}$, its sectional curvature is given by:

$$\mathcal{K}(X,Y) = -\|[X,Y]\|^2 \tag{4.47}$$

for any two orthonormal vector field $X, Y \in \mathfrak{p}$. As a particular example of interest, we consider the case of $\mathbb{H}^n = \mathrm{Spin}(n,1)/\mathrm{Spin}(n)$, where the Killing form on $\mathfrak{so}(n,1)$ is given by: $B(X,Y) = (n-1)\,\mathrm{tr}(XY)$. We see consequently the sectional curvature of $\mathbb{H}^n$ is $-2(n-1)$, which is constant. This is a special case of symmetric spaces of noncompact type. In order to model it as a space of constant negative sectional curvature -1, we need to rescale the metric as:

$$\langle X, Y\rangle_{\mathbb{H}^n} = -\frac{1}{2(n-1)}B(X,Y) = -\frac{1}{2}\,\mathrm{tr}(XY) \tag{4.48}$$

we will be using this fact in Remark 7.12 the calibration of L^2-torsion constants.

Let us end this section with a remark about the fact that both the Bochner and the Hodge Laplacian are essentially self-adjoint [**LM90**, p.155], so there is no ambiguity of domain extension. Thereafter the extension of the unbounded operator to a unique self-adjoint operator on suitable L^2-spaces is assumed.

4.3 Examples

Having developed the machinery in full generality, we see now how the vast existing literature can be subsumed into this framework, for which we give a few examples.

1. Miatello computed the Plancherel decomposition of the connection Laplacians associated with ∇^E [**Mia80**, Theorem 4.1]. In this case, we take $K = L$ the maximal compact subgroup of a connected reductive Lie group G of Harish-Chandra class, and $\mathfrak{p}$ corresponds to the subspace of $\mathfrak{g}$ on which the Cartan involution acts by -1. The G-invariant metric connection on E is defined to be such that for all $Y \in \mathfrak{p}$:

$$\nabla^E_{Y_K}(f)\Big|_{eK} = \frac{d}{dt}\Big|_{t=0} f(\exp(tY)\cdot K) \tag{4.49}$$

Using Remark 4.3, one verifies that this defines a corresponding $\Lambda(\mathfrak{p})|_{eK}$ which acts trivially on the fiber, that is $\gamma^V(\mathfrak{p}) = 0$. Hence ∇^E is reductive. Moreover, a direct computation of (4.23) shows the connection is Levi-Civita when $E = T(G/K)$ the tangent bundle; and we apply (4.24) to see $\gamma = \gamma_0|_{\mathfrak{p}}$ equals to $\frac{1}{2}P^{\mathfrak{g}}_{\mathfrak{p}} \circ \mathrm{ad}|_{\mathfrak{p}}$ in this case, by the fact $\mathrm{ad}^*|_{\mathfrak{p}} = -\mathrm{ad}|_{\mathfrak{p}}$ with our choice of the metric:

$$\gamma_0(Y)(Z) = \frac{1}{2}([Y,Z] - \mathrm{ad}^*_Y(Z) - \mathrm{ad}^*_Z(Y)) = \frac{1}{2}[Y,Z] \tag{4.50}$$

for all $Y, Z \in \mathfrak{p}$. Substituting both into (4.21):

$$\Delta_{\nabla^\rho} F = -\sum_{Y_i \in \mathfrak{p}} R_{Y_i^2} F = -R(\Omega_G - \Omega_K)F = -R(\Omega_G) - \rho(\Omega_K)\cdot \mathbb{I} \tag{4.51}$$

Recall that R is the right regular representation of G on $L^2(G)$, and that $-R(\Omega_K)F = \rho(\Omega_K)\cdot F$ as we are concerning ourselves here solely with the K-invariant sections of $R \otimes \rho$. The formula hence agrees with [**Mia80**, Proposition1.1].

It might be enlightening to compare with the expression of group manifold in Remark 4.13 to see the difference. For one the γ_0 in this case is a genuine $\mathfrak{g}$-representation, this allows one to deal with the operator acting on the invariant forms, on which it acts by $\rho(\Omega)$. For another the connection chosen on V frees us of complications created by curvature $R^{S\otimes V}$. We hence obtain a very simple formula.

2. Moscovici and Stanton formulated the Plancherel decomposition [**MS89**, Section 4] on the(locally) symmetric spaces. Again $L = K$ and G is a connected real semisimple Lie group. They chose ∇ on G/K to be Levi-Civita, this induces a unique connection on ∇^S that satisfies the Leibniz rule (4.29) on the Dirac bundle as well as on the spinor bundle. Their choice of connection again forces $\nabla^\rho F(g)(Y) = R_Y F(g)$ for all $Y \in \mathfrak{p}$ (and extends by 0 to all values on $\mathfrak{g}$ since $\nabla^\rho F$ vanishes on $\mathfrak{k}$.) Hence this forces $\gamma^V(\mathfrak{p})$ to be trivial and hence reductive. Note in such case a lifting of the orthogonal

representation of $\mathfrak{k}$ to the spin representation of $\mathfrak{k}$ is always possible, by our discussions in chapter 3 or [**Kna96**, Problem V.16 - V.27]. Hence by (4.35), one arrives at the form [**MS89**, (3.3)]:

$$\not{D}^2_{\sigma\otimes\rho}F = \Big(\Delta_{\nabla^{\sigma\otimes\rho}} + \frac{1}{2}\sum_{\alpha,\beta} c\ell(Y_\alpha)c\ell(Y_\beta)\mathcal{R}^{\sigma\otimes\rho}(Y_\alpha, Y_\beta)\Big)F \tag{4.52}$$

First we compute the $\gamma^{S\otimes V}$ to determine $\Delta_{\nabla^{\sigma\otimes\rho}}$. Note that $\gamma^{S\otimes V}$ is not reductive in contrast with the aforementioned scenario. Nonetheless, ∇^ρ is reductive and from (4.30), we see for $Y \in \mathfrak{g}$

$$\gamma^{S\otimes V}(Y) = \gamma^S(Y) = (c\ell\circ\varphi)(\gamma(Y)) = \frac{1}{2}(c\ell\circ\varphi)(P^{\mathfrak{g}}_{\mathfrak{p}}\circ \mathrm{ad}(Y)) \in \mathfrak{so}(\mathfrak{p})$$

We see this map vanishes on $Y \in \mathfrak{p}$ and $\gamma^{S\otimes V}(Z) \cong \frac{1}{2}\sigma(Z)$ for $Z \in \mathfrak{k}$. In particular, $\gamma^{S\otimes V}(\mathfrak{p}) = 0$ and $\Delta_{\nabla^{\sigma\otimes\rho}} = -\sum_{Y_\alpha\in\mathfrak{p}} R_{Y_\alpha^2}$ as in the non-twisted case. To write down the formula in more representation theoretic terms, first recall the expression of $\mathcal{R}^{\sigma\otimes\rho}$ in (4.23) and simplify the term as:

$$\begin{aligned}
&\Delta_{\nabla^{\sigma\otimes\rho}} + \frac{1}{2}\sum_{\alpha,\beta} c\ell(Y_\alpha)c\ell(Y_\beta)\mathcal{R}^{\sigma\otimes\rho}(Y_\alpha, Y_\beta)\\
&= -R(\Omega_G) + R(\Omega_K) + \frac{1}{2}\sum_{\alpha,\beta} c\ell(Y_\alpha)c\ell(Y_\beta)R_{P^{\mathfrak{g}}_{\mathfrak{k}}[Y_\alpha,Y_\beta]}
\end{aligned}$$

Now $[Y_\alpha, Y_\beta] \subseteq \mathfrak{k}$ for $Y \in \mathfrak{p}$, whence the projection operator can be dropped. Let Z_k be an orthonormal basis of $\mathfrak{k}$ with respect to $-B$ the negative of the Killing form which is positive definite on $\mathfrak{k}$. First note that $[Y_\alpha, Y_\beta] = \sum_k B([Y_\alpha, Y_\beta], Z_k)Z_k$, and combine with the computation in Lemma 3.7:

$$\begin{aligned}
\frac{1}{2}\sum_{\alpha,\beta} c\ell(Y_\alpha)c\ell(Y_\beta)R_{[Y_\alpha,Y_\beta]} &= -\frac{1}{2}\sum_{\alpha,\beta,k} R_{Z_k}B([Y_\alpha, Y_\beta], Z_k)c\ell(Y_\alpha)c\ell(Y_\beta)\\
&= -\sum_{\beta,k} R_{Z_k}c\ell([Z_k, Y_\beta]_{\mathfrak{g}})c\ell(Y_\beta)\\
&= -2\sum_k R_{Z_k}\sigma_{\mathfrak{p}}(Z_k)
\end{aligned} \tag{4.53}$$

where the last identity comes from the fact $[\varphi\circ \mathrm{ad}(Z), Y]_{\mathbb{C}\ell} = [Z, Y]_{\mathfrak{g}}$ for all $Z \in \mathfrak{k}$ and $Y \in \mathfrak{p}$ and also Lemma 3.7. Grouping expressions together, we see:

$$\begin{aligned}
\not{D}^2_{\sigma\otimes\rho} &= -R(\Omega_G) + R(\Omega_K) + 2\sum_{Z_k\in\mathfrak{k}} R(Z_k)\sigma_{\mathfrak{p}}(Z_k)\\
&= -R(\Omega_G) + R\otimes\sigma_{\mathfrak{p}}(\Omega_K) - \sigma_{\mathfrak{p}}(\Omega_K),
\end{aligned} \tag{4.54}$$

where the second equality is a formal computation of tensor product of representations. The last term $\sigma_{\mathfrak{p}}(\Omega_K) = (|\delta_G|^2 - |\delta_K|^2)\cdot \mathrm{id}$ is computed explicitly in Proposition 3.10. Also the dependence on ρ of this operator is hidden in the right regular action (on $C^\infty(G, \sigma\otimes\rho)$). This in particular gives the identity proved in [**AS79**, (A.8)]. They proved it for the case in which the (complex) rank of G and K are the same, but we see here readily it holds for general reductive group cases. If we further simplify by taking $\rho = \mathrm{id}_{\mathbb{C}}$ trivial, this indeed gives the formula by Parthasarathy [**Par72**, Proposition 3.1] in his geometric construction of the discrete series.

3. Let us also see how the matter is significantly simplified when we assume $G = K$ a compact Lie group. In this case we choose the metric on K to be $-B$ the negative of the Killing form which is positive definite and bi-invariant. The curvature and Ricci curvature admits much simpler forms: [**CE08**, Corollary 3.19]

- $\nabla_X Y = \frac{1}{2}[X, Y]$ whereas $\gamma_0 \equiv \frac{1}{2}\,\mathrm{ad}$;
- $B(\mathcal{R}(X, Y)Z, W) = \frac{1}{4}\left(B([X, W], [Y, Z]) - B([X, Z], [Y, W])\right)$;
- $-B(\mathcal{R}(X, Y)Y, X) = \frac{1}{4}\,\|[X, Y]\|^2$

- $\mathcal{R}_{ijkl} = -B(\mathcal{R}(X_k, X_l)X_j, X_i) = -\frac{1}{4}B([X_i, X_j], [X_k, X_l]);$

whereas the last identity can be derived from the second, with the following identity extracted from (1.36):

$$B([X_i, X_j], [X_k, X_l]) + B([X_i, X_k], [X_l, X_j]) + B([X_i, X_l], [X_j, X_k]) = 0 \tag{4.55}$$

An expression for Ricci curvature follows from the next identity [**BW00**, Lemma II.6.4]:

Lemma 4.16. If $\{\mathcal{R}_{ijkl}\}_{i,j,k,l}$ is a set of complex numbers satisfying the Bianchi identities, that is $\mathcal{R}_{ijkl} = \mathcal{R}_{klij} = -\mathcal{R}_{jikl}$, and $\mathcal{R}_{ijkl} + \mathcal{R}_{kijl} + \mathcal{R}_{jkil} = 0$, then:

$$\sum_{ijkl} \mathcal{R}_{ijkl} c\ell(X_i X_j X_k X_l) = 2\left(\sum_{ij} \mathcal{R}_{ijji}\right) \cdot \mathrm{id} \tag{4.56}$$

Remark 4.17. The statement of this lemma is independent of the geometric structures, requiring merely the fact that $c\ell(V)$ satisfies the Clifford relations, and that $\mathcal{R}_{ijkl}$s are scalars that satisfy some formal relations.

Next one uses Lemma 3.7 to simplify the Clifford multiplication by using (3.23):

$$\begin{aligned}\sum_{ijkl\in I_\mathfrak{k}} \mathcal{R}_{ijkl} c\ell(X_i X_j X_k X_l) &= -\sum_{aijkl\in I_\mathfrak{k}} \langle [X_a, X_i], X_j\rangle\langle [X_a, X_k], X_l\rangle c\ell(X_i X_j X_k X_l) \\ &= \sum_a 16\sigma_\mathfrak{k}(-X_a^2) = 16\sigma_\mathfrak{k}(\Omega_K)\end{aligned} \tag{4.57}$$

Recall $\sigma_\mathfrak{k} = c\ell \circ \varphi \circ \mathrm{ad} : \mathfrak{k} \to \mathrm{End}(\mathcal{S}_\mathfrak{k})$ is the spin representation of $\mathfrak{k}$ on the space of spinors associated with $\mathfrak{k}$. Combining the two identities above, one sees $\sum_{ij} \mathcal{R}_{ijji} = 8\sigma_\mathfrak{k}(\Omega_K)$. This tells in particular that $\sigma_\mathfrak{k}(\Omega_K)$ acts as a scalar operator on $\mathcal{S}_\mathfrak{k}$. Moreover, by the computation of Casimir eigenvalues in Proposition 3.10, we find the scalar of its infinitesimal character $\chi_{\sigma_\mathfrak{k}}$ can be explicitly computed as:

$$\sum_{ij} \mathcal{R}_{ij\in I_\mathfrak{k}} = 8\chi_{\sigma_\mathfrak{k}}(\Omega_K) = 24|\delta_K|^2 \tag{4.58}$$

We now derive a formula for $\not{D}_\sigma$. Consider first $\gamma^S = \frac{1}{2}c\ell \circ \varphi \circ \mathrm{ad} = \mathfrak{k} \to \mathrm{End}(\mathcal{S}_\mathfrak{k})$:

$$\gamma^S(X_i) = \frac{1}{2}c\ell \circ \varphi\left(\sum_{jk}(\mathrm{ad}(X_i))_{jk}E_{jk}\right) = \frac{1}{8}\sum_{jk} C_{ij}^k c\ell(X_j X_k) = \frac{1}{2}\sigma_\mathfrak{k}(X_i) \tag{4.59}$$

as in Lemma 3.7. The same lemma produces the following expression for Dirac operator on a compact Lie group:

$$\not{D}_\sigma(F) = \sum_i c\ell(X_i)(R_{X_i} + \gamma^S(X_i)) = \sum_i R_{X_i} c\ell(X_i) + \frac{1}{2}c\ell(X_i)\sigma_\mathfrak{k}(X_i) \tag{4.60}$$

Lastly the Bochner identity can be derived by mimicking the argument in (4.53):

$$\not{D}_\sigma^2 = R(\Omega_K) + (R\otimes\sigma_\mathfrak{k})(\Omega_K) + \frac{1}{2}\sigma_\mathfrak{k}(\Omega_K) \tag{4.61}$$

which agrees with [**Sle85**, (1.3.13)]. There, another Bochner identity for the reductive connection $\nabla = \nabla^0$ is also computed:

$$D_{\sigma_0}^2 = 2R(\Omega_K) - (R\otimes\sigma_\mathfrak{k})(\Omega_K) + \sigma_\mathfrak{k}(\Omega_K w)$$

4.4 Dirac operators on semisimple Lie groups

The rest of this chapter is devoted to applying the above discussions to semisimple Lie groups, by treating them as homogeneous spaces of the form G/e_G. The Killing form in this case renders extra symmetries between adjoint and coadjoint representations, a feature which we have already been exploiting in the **Generalized Kuga's Lemma**. In comparison with the symmetric spaces, the difficulties to surmount are twofold:

Problem 1 Here the Levi-Civita connection on TM induces a corresponding connection on T^*M, therefore we see that the $\gamma^V = \gamma^{\wedge^*\mathfrak{g}^*}$ in this case can almost never be trivial;

Problem 2 The Killing form is no longer positive definite on $\mathfrak{g}$, in fact there is no positive definite bi-G-invariant non-degenerate bilinear form on $\mathfrak{g}$ that serves the purpose, as the Killing form is, up to isomorphism, the unique non-degenerate bi-G-invariant form. One can hence no longer expect a nice expression as (4.54).

We begin with expressing the untwisted Dirac operator $\not{D}$ on the real spinor bundles. Following the convention of the **Generalized Kuga's Lemma**, we fix a (pseudo)-orthonormal basis $\{X_i\}$ of $\mathfrak{k}$ and $\{Y_\alpha\}$ of $\mathfrak{p}$ with respect to the Killing form, which is also an orthonormal basis with respect to our left G-invariant metric B^θ. We also use $\{X_\alpha\}$ occasionally to describe the basis of $\mathfrak{g}$.

One immediate consequence is $\mathrm{ad}^*(\mathfrak{p}) \equiv \mathrm{ad}(\mathfrak{p})$ and $\mathrm{ad}^*(\mathfrak{k}) \equiv -\mathrm{ad}(\mathfrak{k})$. For instance, if $Z \in \mathfrak{p}$:

$$B^\theta(\mathrm{ad}^*_{X_i} Z, Z') = B^\theta(Z, [X_i, Z']) = B^\theta([Z, X_i], Z') = -B^\theta(\mathrm{ad}_{X_i} Z, Z')$$
$$B^\theta(\mathrm{ad}^*_{Y_\alpha} Z, Z') = B^\theta(Z, [Y_\alpha, Z']) = -B^\theta([Z, Y_\alpha], Z') = B^\theta(\mathrm{ad}_{Y_\alpha} Z, Z')$$

Similar identities for $Z \in \mathfrak{k}$ can be checked similarly. Moreover, the linear map $\mathrm{ad}^\flat(X)Y := \mathrm{ad}^*_Y X$ in this case can be identified with $\pm\,\mathrm{ad}$ using the identities above such that $\mathrm{ad}^\flat(\mathfrak{g})|_\mathfrak{p} = -\,\mathrm{ad}(\mathfrak{g})|_\mathfrak{p}$ and $\mathrm{ad}^\flat(\mathfrak{g})|_\mathfrak{k} = \mathrm{ad}(\mathfrak{g})|_\mathfrak{k}$, in the sense that $\mathrm{ad}^\flat(Z)(Y_\alpha) = [Z, Y_\alpha]$ and $\mathrm{ad}^\flat(Z)(X_i) = -[Z, X_i]$. Summing up, we can take γ_0 as a 2×2-block matrix:

$$\gamma_0 : (X, Y) \mapsto \frac{1}{2}\begin{pmatrix} \mathrm{ad}_X & \mathrm{ad}_Y \\ -\,\mathrm{ad}_Y & 3\,\mathrm{ad}_X \end{pmatrix} = \frac{1}{2}\,\mathrm{ad}_X + (\mathrm{ad}_X|_\mathfrak{p}) + \frac{1}{2}\begin{pmatrix} & \mathrm{ad}_Y \\ -\,\mathrm{ad}_Y & \end{pmatrix} \tag{4.62}$$

in which $(X, Y) \in \mathfrak{k} \oplus \mathfrak{p}$ and γ_0 takes matrix forms with respect to this basis. This is exactly a special case of (1.31) by taking τ to be trivial and by identifying coad^* with $-\,\mathrm{ad}$ using Remark 1.4.

Also we use the exterior algebra model of the spin module as in Remark 3.4. Denote the decomposition of $\mathcal{Z}_\mathfrak{g} := \mathcal{Z}_\mathfrak{p} \oplus \mathcal{Z}_\mathfrak{p}^\perp$ with respect to the natural inner product on the spinor module. In most cases $\mathcal{Z}_\mathfrak{p}^\perp \cong \mathcal{Z}_\mathfrak{k}$, with the exception when $\mathfrak{g}$ is even-dimensional and $\mathfrak{p}$ is odd-dimensional, in which case $\mathcal{Z}_\mathfrak{p}^\perp = \mathcal{Z}_\mathfrak{k} \oplus \mathbb{C}$ with the extra dimension due to the new isotropic dimension. Refer to Example 3.12 for such phenomenon. For this reason we introduce a bi-grading (p, q) on $\mathcal{S}_\mathfrak{g}$ similar as in the **Generalized Kuga's Lemma**, i.e.,

$$\mathcal{S}_\mathfrak{g} = \bigoplus_{p,q} \mathcal{S}_{p,q} = \bigoplus_{p,q} \bigwedge^{p+q}(\mathcal{Z}_\mathfrak{p}^\perp \oplus \mathcal{Z}_\mathfrak{p}) = \bigoplus_{p,q} \bigwedge^p \mathcal{Z}_\mathfrak{p}^\perp \otimes \bigwedge^q \mathcal{Z}_\mathfrak{p} \tag{4.63}$$

Now we want to figure out the spin connection on $\mathcal{S}_\mathfrak{g}$. Recall from (4.30) that it is completely determined by γ^S. Precomposing ad with $c\ell \circ \varphi$ yields:

$$\gamma^S(X) = (c\ell \circ \varphi)(\gamma_0(X)) = \frac{1}{2}(c\ell \circ \varphi)(\mathrm{ad}_X - \mathrm{ad}^*_X - \mathrm{ad}^\flat(X)) \tag{4.64}$$

If we express the ad_X for fixed $X \in \mathfrak{g}$ as a matrix with respect to the chosen basis, and denote E_{cb} the elementary matrix with 1 on the (c, b)-entry and zeroes elsewhere, then $(\mathrm{ad}_{X_a})_{cb} = \langle [X_a, X_b], X_c\rangle E_{cb}$. Recall the spin cover map φ sends $E_{ij} - E_{ji}$ to $-\frac{1}{2}c\ell(X_i)c\ell(X_j)$, hence γ^S admits the following expression by combining Lemma 3.7 with the expression of γ_0 in Remark 4.13. Again we abbreviate

$c\ell(X_aX_b) = c\ell_{ab}$, and follow the convention of chapter 1 that uses the Latin scripts $i, j, \ldots$ for indexes of $\mathfrak{k}$ and the Greek scripts $\alpha, \beta, \ldots$ for indexes of $\mathfrak{p}$. We have now:

$$\begin{aligned}\gamma^S(X_i) &= \frac{1}{2}\sum_{j<k} C_{ik}^{j} c\ell \circ \varphi(E_{jk} - E_{kj}) + \frac{3}{2}\sum_{\alpha<\beta} C_{i\beta}^{\alpha} c\ell \circ \varphi(E_{\alpha\beta} - E_{\beta\alpha}) \\ &= \frac{1}{8}\left(\sum_{j,k\in I_{\mathfrak{k}}} C_{jk}^{i} c\ell_{jk} - \sum_{\alpha,\beta\in I_{\mathfrak{p}}} C_{\alpha\beta}^{i} c\ell_{\alpha\beta}\right) - \frac{1}{4}\sum_{\alpha,\beta} C_{\alpha\beta}^{i} c\ell_{\alpha\beta} \\ &= \frac{1}{2}\sigma_{\mathfrak{g}}(X_i) + \mathbb{I}_{\mathcal{S}_{*,0}} \otimes \sigma_{\mathfrak{p}}(X_i)\end{aligned} \tag{4.65}$$

for all $X_i \in \mathfrak{k}$. In the last step we abbreviate the sum expression as $\sigma_{\mathfrak{p}}(X_i)$ with the help of Lemma 3.7, and note that it acts trivially on the subspace of $\mathcal{S}_{\mathfrak{g}}$ generated by the orthogonal complement of $\mathcal{S}_{\mathfrak{p}}$. Hence using the bi-grading we have defined above, it acts by identity on $\wedge^* \mathcal{Z}_{\mathfrak{p}}^{\perp} \cong \mathcal{S}_{*,0}$. On the other hand, for $Y_\alpha \in \mathfrak{p}$, we have a similar expression:

$$\begin{aligned}\gamma^S(Y_\alpha) &= \frac{1}{2}\sum_{i\in I_{\mathfrak{k}},\beta\in I_{\mathfrak{p}}} \Big(-\operatorname{ad}(Y_\alpha)_{\beta i} E_{\beta i} + \operatorname{ad}(Y_\alpha)_{i\beta} E_{i\beta}\Big) \\ &= \frac{1}{2}\sum_{i,\beta} C_{\alpha\beta}^{i} \varphi(E_{i\beta} - E_{\beta i}) = -\frac{1}{4}\sum_{i,\beta} C_{\alpha\beta}^{i} c\ell_{i\beta}\end{aligned} \tag{4.66}$$

Combining both expressions we get a formula for $\not{D}_\sigma$ from $(\nabla^\sigma F)(X) = R_X + \gamma^S(X)$:

$$\not{D}_\sigma = \sum_i c\ell(X_i)(R_{X_i} + \gamma^S(X_i)) + c\ell(Y_\alpha)\sum_\alpha (R_{Y_\alpha} + \gamma^S(Y_\alpha)) = \sum_{a\in I_{\mathfrak{g}}} R_{X_a} \otimes c\ell(X_a) + \urcorner_\sigma \tag{4.67}$$

Where the operator $\urcorner_\sigma$ acts trivially on $L^2(G)$-component of $L^2(G) \otimes \mathcal{S}_{\mathfrak{g}}$ and takes the following form:

$$\begin{aligned}\urcorner_\sigma &= \sum_{i\in I_{\mathfrak{k}}} c\ell(X_i)\gamma^S(X_i) + \sum_{\alpha\in I_{\mathfrak{p}}} c\ell(Y_\alpha)\gamma^S(Y_\alpha) \\ &= \frac{1}{2}\sum_{i\in I_{\mathfrak{k}}} c\ell(X_i)\sigma_{\mathfrak{g}}(X_i) - \frac{1}{4}\sum_{i,\alpha,\beta} C_{\alpha\beta}^{i} c\ell_{i\alpha\beta} + \frac{1}{4}\sum_{i,\alpha,\beta} C_{\alpha\beta}^{i} c\ell_{i\alpha\beta} \\ &= \frac{1}{2}\sum_{i\in I_{\mathfrak{k}}} c\ell(X_i)\sigma_{\mathfrak{g}}(X_i)\end{aligned} \tag{4.68}$$

This is just a combination of all the computations that we have conducted in this section.

Chapter 5

Heat kernel decomposition

This chapter serves to bridge the gap between the Plancherel formulae in the first half of the thesis and their applications to specific elliptic operators on homogeneous spaces for our purposes. In the case of homogeneous spaces, the heat kernel is locally constant, therefore the problem reduces to computing its value at the origin, and here is where we apply Theorem 2.25 to decompose the function as integral of characters.

Apart from the case of symmetric spaces, scarce literature has been known to the author that tend this matter carefully before applying the Plancherel formula. As revealed in our ensuing discussions, all one needs is to give a Gaussian estimate of the pertinent heat kernel, i.e., the heat kernel is bounded by a Gaussian function on $\mathbb{R}^n$, with implied constants to be decided.

The upper Gaussian bounds for scalar Laplacians of Riemannian manifolds are well-known for a breadth of manifolds. In [**LY86**, §3], Li and Yau established it for all Riemannian manifolds with a lower bound on Ricci curvature. In the case of operators on vector bundles, the case becomes more intricate. One exception is the symmetric space, on which the kernels of Hodge and Dirac Laplacian differ from the scalar Laplacian by a mere scalar operator, as manifested in (4.51) and (4.54).

We now extend the estimate to those kernels which can expressed as a bounded perturbation of the scalar Laplacians of G. The bound we obtain in Lemma 5.5 is by no means optimal but serves right our purposes. For this reason we include in the first section a detailed account of the estimate of heat kernel upper bounds for general Lie groups.

We refer the reader to [**Rob91**] for a detailed account of the heat kernel estimates on Lie groups. In Appendix A.1 we include a short introduction to this topic including all relevant definitions and theorems.

We also follow the convention of notations therein. In particular, we remind the readers that $|g|$ denotes the distance between g and e_G, induced by a left-invariant metric on G, and $\dim G = d$. This in turn fixes a left G-invariant Haar measure $\mathrm{d}g$ on G throughout this chapter.

5.1 Elliptic operators on representations: a primer

In this section we summarize the results of [**Rob91**, Chapter III] to develop a Gaussian upper bound for the Schwartz kernel of elliptic operators. This is done with great generality: We establish the estimates in Lemma 5.1 for an arbitrary Lie group G with arbitrary strongly continuous representations. This is an easy consequence of the analyticity of the kernel function on G, as discussed in Appendix A.1.

For $1 \leq p \leq \infty$, define the weighted L^p-space L^p_ρ for some $\rho \geq 0$ as $L^p_\rho := L^p(G; e^{\rho|g|}\,\mathrm{d}g)$ and let L^ρ be the left regular representation of G on it. These are Banach algebras under convolution, with respect to the weighted p-norms defined as [**KMB12**, Section I]:

$$\|\varphi\|_{L^p_\rho} := \left(\int |\varphi(g)e^{\rho|g|}|^p \,\mathrm{d}g\right)^{1/p} \qquad \|\varphi\|_{L^\infty_\rho} := \operatorname*{ess.\,sup}_{g\in G} e^{-\rho|g|}|\varphi(g)| \tag{5.1}$$

We now derive a pointwise Gaussian bound by implementing Theorem A.1 by taking the representation in question to be $(L^\rho, L^p_\rho(G))$ for suitably chosen ρ:

Lemma 5.1 ([Rob91, Corollary III.2.5]). Given a left G-invariant strongly elliptic operator D on a Lie group G, form the kernel k_t for $t > 0$ as in Theorem A.4. Then for each $\rho \geq 0$ there exist $a, b, c > 0$ and $\omega \geq 0$ such that:

$$\left|\partial_g^\alpha \partial_t^\ell k_t(g)\right| \leq ab^{|\alpha|}c^\ell |\alpha|!\ell!(1 + t^{-(\ell + \frac{|\alpha|+d+1}{m})})e^{\omega t}e^{-\rho|g|} \tag{5.2}$$

for all $g > G$ and $t > 0$. Here ∂_g (resp. ∂_t) denotes the differential in the direction of group (resp. time).

To prove the claim we recall the proof strategy of the third statement of Theorem A.4, which estimates the norms of k_t as suitable operator norms of $e^{-t\overline{D}}$, and the norms of its derivatives as operator norms between weighted L^p-spaces.

Proof. For simplicity we prove the statement only for G to be unimodular and remark the other case in the end of the proof. First create a function space to capture the growth of $k_t \cdot e^{-\rho|g|}$. By Theorem A.1 $e^{-tL^\rho(\overline{D})}$ is the corresponding kernel and its derivatives admits the following integral formula:

$$X^\alpha D^\ell e^{-tL^\rho(\overline{D})}\varphi(e_G) = \int_G \left(\partial_g^\alpha \partial_t^\ell k_t\right)(g)e^{\rho|g|} \cdot (\varphi(g^{-1})e^{-\rho|g|})\,dg. \tag{5.3}$$

Here we denote $X^\alpha = L^\rho(X^\alpha)$ as the action in question is clear from the context. Immediately we have:

$$\sup_{g\in G} |(\partial_g^\alpha \partial_t^\ell k_t)(g)|e^{\rho|g|} = \sup_{\|\varphi\|_{L^1_\rho}\leq 1} \left\{|X^\alpha D^\ell e^{-tL^\rho(\overline{D})}\varphi(e_G)|\right\}$$

Now mimicking the proof of Langlands' theorem on kernels one consider $C^k_\rho(G) := C^k(L^1_\rho(G))$ the C^k-vectors of the weighted L^1-space, with respective norms. Consider their restriction to a bounded open neighborhood U of e_G, where the L^1-norm and the weighted L^1-norm are equivalent. Then applying the Sobolev embedding Lemma A.3, we get a constant $c_{\rho,U} > 0$ such that:

$$|X^\alpha D^\ell e^{-tL^\rho(\overline{D})}\varphi(e_G)| \leq c_{\rho,U} \left\|X^\alpha D^\ell e^{-tL^\rho(\overline{D})}\varphi\right\|_{C^{d+1}_\rho(U)} \leq c_{\rho,U} \left\|D^\ell e^{-tL^\rho(\overline{D})}\varphi\right\|_{C^{|\alpha|+d+1}_\rho(U)}$$

Now we see $|\partial_g^\alpha \partial_t^\ell k_t(g)|$ is bounded for all $g \in G$ and $t > 0$ by the operator norm of $D^\ell e^{-tL^\rho(\overline{D})}$ on $C^{|\alpha|+d+1}_\rho(U)$ with an implied constant $c_{\rho,U}e^{-\rho|g|}$. Next the chain rule gives:

$$D^\ell e^{-tL^\rho(\overline{D})} = e^{-t_1L^\rho(\overline{D})}(De^{-\frac{t_2}{\ell}L^\rho(\overline{D})})^\ell e^{-t_3L^\rho(\overline{D})}$$

for every triplet $t_1, t_2, t_3 > 0$ with $t_1 + t_2 + t_3 = t$. Denote $\beta := |\alpha| + d + 1$, we therefore want to estimate the following:

$$e^{\rho|g|}\left|(\partial_g^\alpha \partial_t^\ell k_t(g))\right| \leq c_{\rho,U} \left\|e^{-t_1L^\rho(\overline{D})}\right\|_{L^1_\rho\to C^\beta_\rho} \left(\left\|De^{-\frac{t_2}{\ell}L^\rho(\overline{D})}\right\|_{L^1_\rho\to L^1_\rho}\right)^\ell \left\|e^{-t_3L^\rho(\overline{D})}\right\|_{L^1_\rho\to L^1_\rho}$$

with $\|\cdot\|_{X\to Y}$ denotes the operator norm between Banach spaces X and Y. We bound the norms in three time intervals via different methods:

1. For $t_1 \in (0, 1]$, we use the small time estimate (A.4). Applying to L^1_ρ-norm to estimate the C^β-norm:
$$\left\|e^{-t_1L^\rho(\overline{D})}\right\|_{L^1_\rho\to C^\beta_\rho} \leq c'c^\beta\beta!t^{-\frac{\beta}{m}} \tag{5.4}$$
for some $c > 0$. This contributes the factor $a'b^{|\alpha|}|\alpha|!t^{-\beta/m}$ in the final expression;
2. For $t_2 \in (0, \ell]$, recall Theorem A.4 that the convolution kernel of $De^{-tL^\rho(\overline{D})}$ is $\frac{\partial}{\partial t}k_t$:
$$\left\|De^{-\frac{t_2}{\ell}L^\rho(\overline{D})}\varphi\right\|_{L^1_\rho\to L^1_\rho} \leq \left\|\frac{\partial}{\partial t}k_{\frac{t_2}{\ell}}\right\|_{L^\infty_\rho} \leq C'\frac{\ell}{t_2} \tag{5.5}$$
The last inequality comes from the fact k_t forms a holomorphic family of continuous functions with respect to t, so its first derivative $\frac{t_2}{\ell}\left\|\frac{\partial}{\partial t}K_{\frac{t_2}{\ell}}\right\|$ is bounded. Hence the second factor contributes a term $\ell^\ell c_2^\ell t^{-\ell} \sim \ell!c^\ell t^{-\ell}$ to the final expression by Stirling's approximation;

3. Lastly one uses the continuity bound (A.12): $\left\|e^{-t_3 L^\rho(\overline{D})}\varphi\right\|_{L^1_\rho} \le C'' e^{\omega t_3} \|\varphi\|_{L^1_\rho}$ and Consequently $\left\|e^{-t_3 L^\rho(\overline{D})}\right\|_{L^1_\rho \to L^1_\rho} \le C'' e^{\omega t_3}$ for some $C'' > 0$. Hence the third factor contributes a term $e^{\omega t_3}$ to the final expression.

Summing up all the contributions we see the upper bound of the derivative is indeed as claimed.

We end the proof with a comment on generalizing the proof to the non-unimodular case. All the key ingredients are the same, but one needs to take into account the difference between the right and the left-invariant Haar measure, and also to measure the growth of the modular function $\Delta(g)$ resulting from changing one measure to the other. The modular function is an analytic homomorphism and hence we can bound it pointwise by $e^{\omega'|g|}$ for some $\omega' > 0$ and $C \ge 1$. Replacing ρ in the argument by $\rho + \omega'$ one then gets the correct estimate. □

Remark 5.2. Analyzing the proof of the theorem closely, one notices that this Gaussian bound is in fact crude and has much room for improvement. The use of the Sobolev inequalities in many parts compromises the regularity and therefore the bound. Meanwhile, little is known about the behavior of the kernel with regard to the weight factor ρ, further compromising the bound. To remedy these one can in fact get a much better bound by exploiting both facts, via a detailed analysis of the growth of $\left\|e^{-tL^\rho(\overline{\Delta})}\right\|$ and its derivatives with respect to the ρ, and bypass the Sobolev inequalities with Nash inequalities, and produces the following much better bound for a cone region [Rob91, Theorem 4.1]:

$$\left|(\partial_g^\alpha \partial_t^\ell k_t(g))\right|_{t=z} \sim \mathcal{O}\left(|z|^{-\ell + \frac{|\alpha|+d}{m}} e^{\omega|z|} e^{\omega t} e^{(-b\frac{|g|^m}{|z|})^{\frac{1}{m-1}}}\right)$$

with the same implied constant as (5.2). We see immediately this captures more precisely the derivative bound at small time. Nonetheless, as the first estimate is already sufficient for our purposes, we refrain from further including the proof of this and are satisfied with referring the readers to [Rob91, Section III.4b] for the methods and details.

5.2 Bounded perturbation of elliptic operators and estimates

In the last section we have given an upper bound estimate for the heat kernel on weighted L^p-spaces. This alone would establish the Schwartz estimate for functions on G. Nonetheless, as we have encountered in (4.21), the case of vector bundles often entails a perturbation by first-order bounded operators. This complicates the matter in two ways: In the first case one may in fact adopt the Li-Yau estimate [LY86, §3 & §4], which is not available for the latter case. Secondly, the perturbation by bounded operators resists a simple decomposition of Laplacians on sections into the scalar Laplacian and a scalar operator, so one can not hope to estimate the Hodge Laplacians via that of scalar Laplacians. We resolve both issues by recycling some elements from the case of symmetric spaces, and use a perturbation approximation argument due to Hille and Phillips.

In our working case, the property that $\Delta + \rho(X)$ defines a strongly continuous semigroup for the bounded operator $\rho(X)$ can be derived using the resolvent method (see for instance [HP74, Theorem 13.2.1]), but its derivatives' estimates are more intricate to track, as they entail Cauchy transforms at different levels. For this reason we take a detour by adopting a slightly more complicated approximation method to the semigroup, ending up with estimating these approximations. The following theorem was partly from [HP74, Section 13.4]:

Theorem 5.3 (Bounded perturbation of heat semigroup)**.** *Let A be the infinitesimal generator of a semigroup e^{-tA}, and B a linear operator such that* $\mathrm{Dom}(A) = \mathrm{Dom}(B)$*. Moreover, we assume e^{-tA} and Be^{-tA} satisfy the following estimates:*

- *For some $M, \omega > 0$, $\|e^{-tA}\| \le Me^{t\omega}$ for all $t > 0$.*
- *For some $\alpha < 1$ and $c > 0$, $\|Be^{-tA}\| \le ct^{-\alpha}$ for all $t \in (0, 1]$.*

Then $A+B$ also generates a strongly continuous semigroup $e^{-t(A+B)}$ which admits the ***Dyson-Phillips expansion****, which is an absolutely convergent series for all $t>0$:*

$$e^{-t(A+B)}f = \sum_{k=0}^{\infty} B\,\mathrm{Per}^k(e^{-tA}f) \tag{5.6}$$

with $\mathrm{Per}^i(u(t))$ *is defined recursively as:*

$$\mathrm{Per}^0(u(t)) = u \quad \mathrm{Per}^k(u(t)) = \int_0^t e^{t-t_{k-1}}\,\mathrm{Per}^{k-1}(u(t_{k-1}))dt_{k-1} \tag{5.7}$$

To prove this theorem, we first establish the fact that $e^{-t(A+B)}$ indeed generates a strongly continuous semigroup. Namely we want to prove that the Dyson-Phillips expansion defines an absolutely convergent series. Take $\phi(t) = \left\|e^{-tA}\right\|$ and $\psi(t) = \left\|Be^{-tA}\right\|$. Both functions are non-negative and measurable, and they satisfy two conditions:

Property 1 $\int_0^1 \phi(t) + \psi(t)\,\mathrm{d}t < \infty$;

Property 2 $\psi(t)$ satisfies the inequality: $\|\psi(t+s)\| \le \psi(t)\phi(s)$ for all $t, s > 0$. This indeed comes directly from the semigroup property of e^{-tA}.

These two properties alone give some quantitative bounds of ϕ and ψ:

- The finite sub-multiplicative function $\phi(t)$ is bounded on each interval of the form $(\epsilon, 1/\epsilon)$ by [HP74, Theorem 7.4.1]. Together with property 2, we see $\psi(t)$ is also bounded there;
- $\lim_{t\to\infty} t^{-1}\log\phi(t)$ exists by the fact ϕ is sub-multiplicative [HP74, Theorem 7.6.1]. Denote this limit by ω_0. This also gives an upper bound of ψ by property 2:

$$\limsup_{t\to\infty} t^{-1}\log\psi(t) \le \omega_0 \tag{5.8}$$

- For any $\omega > \omega_0$ we have $\int_0^\infty e^{-t\omega}(\phi(t)+\psi(t))\,\mathrm{d}t =: M_\omega < \infty$. This is immediate from the first property and (5.8). Using the property 2 with the growth estimate of $\phi(t)$,

$$0 \le 4e^{-t\omega}\psi(t) \le (e^{-\omega(t-t_1)}\phi(t-t_1) + e^{-t_1\omega}\psi(t_1))^2$$

  by the sub-additivity inequality. Hence:

$$\begin{aligned} t(e^{-t\omega}\psi(t))^{1/2} &= 2\int_0^{t/2} (e^{-t\omega}\psi(t))^{1/2}\,\mathrm{d}t_1 \\ &\le \int_0^{t/2} e^{-\omega(t-t_1)}\psi(t-t_1) + e^{-t_1\omega}\phi(t_1)\,\mathrm{d}t_1 \le M_\omega \end{aligned}$$

  Hence we obtain an upper bound for $\psi(t)$ for $t>0$:

$$\psi(t) \le e^{t\omega}t^{-2}M_\omega^2 \tag{5.9}$$

Now the norm of $\mathrm{Per}^k(e^{-tA}f)$ in the Dyson-Phillips series can be shown to be bounded by the convolution product $\phi * \psi^{*k}$, which can be estimated as follows:

Lemma 5.4 ([HP74, Lemma 13.4.3])**.** Suppose ψ_0 and ψ_1 are two nonnegative measurable functions satisfying property 1 and 2 as stated above. Then the series $\theta(t) := \sum_{k=0}^{\infty}(\psi_0 * \psi_1^{*k})(t)$ converges uniformly with respect to t in the interval of $(\varepsilon, 1/\varepsilon)$ for $\varepsilon \in (0,1)$. Moreover, if $\omega > \omega_0 > 0$ is such that $\int_0^\infty e^{-t\omega}\psi_1(t)\,\mathrm{d}t \le 1$, then $\int_0^\infty e^{-t\omega}\theta(t)\,\mathrm{d}t < \infty$.

Proof. By the first quantitative bound above we see both $\phi(t)$ and $\psi(t)$ remain bounded on each interval of the form $(\epsilon, 1/\epsilon)$. Choose $\omega_1 > \omega_0$ so that:

$$\int_0^\infty e^{-\omega_1 \xi}(\phi(t) + \psi_0(t))\,\mathrm{d}t \le 1 \qquad \int_0^\infty e^{-\omega_1 \xi}(\phi(t) + \psi_1(t))\,\mathrm{d}t \le \frac{1}{16}$$

Now from (5.9) we get $\psi_0(t) \le t^{-2}e^{t\omega_1}$ and $\psi_1(t) \le \frac{1}{16}t^{-2}e^{t\omega_1}$. By induction we will establish:

$$\psi_0 * \psi_1^{*n}(t) \le 2^{-n}t^{-2}e^{t\omega_1} \tag{5.10}$$

It suffices to estimate the constant for the induction step using the quantitative bounds we established before the lemma:

$$\begin{aligned}
&(\psi_0 * \psi_1^{*n})(t)\\
=&e^{t\omega_1}\left(\int_0^{t/2} + \int_{t/2}^{t}\left(e^{-(t-s)\omega_1}\psi_0 * \psi_1^{*(n-1)}(t-s)\right)\cdot\left(e^{-s\omega_1}\psi_1(s)\right)\mathrm{d}s\right)\\
\le&e^{t\omega_1}\left(2^{-(n-1)}2^2t^{-2}\int_0^{t/2} e^{-s\omega_1}\psi_1(s)\,\mathrm{d}s + \frac{1}{16}2^2t^{-2}\int_0^{t/2} e^{-s\omega_1}(\psi_0 * \psi_1^{*(n-1)})(s)\,\mathrm{d}s\right)\\
\le&2^{-n}t^{-k}e^{-t\omega_1}
\end{aligned}$$

with the two inequalities coming from the induction hypothesis. Having established (5.10) we see θ_t is an absolutely convergent series, as it is majorized by the uniformly convergent series $\sum_n 2^{-n}t^{-2}e^{t\omega_1}$. This proves the first claim. For the second claim, we note the following:

$$\int_0^\infty e^{-t\omega}(\psi_0 * \psi_1^{*n})(t)\,\mathrm{d}t = \left(\int_0^\infty e^{-t\omega}\psi_0(t)\,\mathrm{d}t\right)\cdot\left(\int_0^\infty e^{-t\omega}\psi_1(t)\,\mathrm{d}t\right)^n$$

Hence we can write $\int_0^\infty e^{-t\omega}\theta(t)\,\mathrm{d}t = \left(\int_0^\infty e^{-t\omega}\psi_0(t)\,\mathrm{d}t\right)\cdot\left(1 - \int_0^\infty e^{-t\omega}\psi_1(t)\,\mathrm{d}t\right)^{-1}$, which is finite by our assumption. ∎

Now the proof of the theorem is immediate from all these estimates:

Proof of Theorem 5.3. We begin by proving that the Dyson-Phillips series converges uniformly in the strong operator topology for $t > 0$ by bounding it using ϕ and ψ:

$$\left\|\mathrm{Per}^n(e^{-tA}f)\right\| \le \phi * \psi^{*n}(t) \qquad \left\|B\,\mathrm{Per}^n(e^{-tA}f)\right\| \le \psi^{*n}(t)$$

First note that the functions $\phi = \psi_0$ and $\psi = \psi_1$ satisfy the estimates in Lemma 5.4, by the assumption we made in Theorem 5.3. Also by our assumption on e^{-tA}, the case $n = 0$ is trivially true. The case for general n is shown by an easy induction with the following inequality:

$$\left\|\mathrm{Per}^{k+1}(e^{-tA})\right\| \le \int_0^t \left\|e^{-(t-s)A}\right\|\cdot\left\|B\,\mathrm{Per}^k(e^{-sA})\right\|\mathrm{d}s \le \phi * \psi^{*(k+1)}(t)$$

and similarly for $B\,\mathrm{Per}^{k+1}(e^{-tA})$:

$$\left\|B\,\mathrm{Per}^{k+1}(e^{-tA}f)\right\| \le \int_0^t \left\|Be^{-(t-s)A}\right\|\cdot\left\|B\,\mathrm{Per}^k(e^{-sA})\right\|\mathrm{d}s \le \psi^{*(k+2)}(t)$$

Now $\sum_{k=0}^\infty \left\|\mathrm{Per}^k(e^{-tA})\right\|$ is bounded by $\theta(t)$, a fact we have proven in the above lemma to be uniformly convergent in t on any interval $(\varepsilon, 1/\varepsilon)$ for $0 < \varepsilon < 1$. These show $e^{-t(A+B)}$ defines a strongly continuous heat semigroup. ∎

5.3 Elliptic operators on homogeneous spaces

Following the discussions and notations in Section 4.1, we establish the rapid decay estimate of the kernels via the Dyson-Phillips expansion that was developed in the last section. We reiterate the fact that the Plancherel decompositions in Theorem 2.25 and Theorem B.6 are only valid for Schwartz functions, with the notion of Schwartz functions depending on the group.

Consider now a G-invariant strongly elliptic operator $\mathcal{D}$ on the homogeneous vector bundle $G \times_\rho V \to G/L$. Via the isomorphism $\mathcal{A}$ from (4.5) we identify $\mathcal{D}$ with an element $D_\rho \in U(\mathfrak{g}_\mathbb{C}) \otimes \mathrm{Hom}(V_1, V_2)$ as in Section 4.1. Its kernel $K_t^\rho := \mathcal{A}(K_t^\mathcal{D}) \in L^2(G; \mathrm{End}(V))$ can be treated as a function $k_t^\rho : G \to \mathbb{C}$ via $k_t^\rho(g_1^{-1} g_2) := K_t^\rho(g_1, g_2)$, which satisfies the following covariance property:

$$k_t^\rho(g) = \rho(a) k_t^\rho(a^{-1} g b) \rho(b)^{-1} \qquad \text{for } x \in G, a, b \in L \tag{5.11}$$

We now estimate its growth against bounded perturbations:

Lemma 5.5 (Perturbed kernel estimates)**.** Given a left G-invariant strongly elliptic operator D of order m, with its kernel k_t^D as in Theorem A.4. Then for any differential operator B with highest order $\ell < m$, the perturbed semigroup $e^{-t(A+B)}$ is strongly continuous in t, with k_t^{D+B} its kernel. The derivatives of k_t^{D+B} satisfy the following estimate: For each $\rho > 0$ there exist $a, b, c > 0$ and $\omega \geq 0$ such that:

$$\int_G e^{\rho|g|} \left| \partial_g^\alpha k_t^{D+B} \right| \mathrm{d}g < \infty \tag{5.12}$$

for all fixed $t > 0$.

Proof. The strong continuity of the perturbed heat semigroup $e^{-t(A+B)}$ is straightforward from the Theorem 5.3. Derivatives of the perturbed heat kernel can be estimated by bounding its weighted L^∞-norm, similar to the proof of Lemma 5.1

$$\int_G \left| \partial_g^\alpha k_t^{D+B} \right| \mathrm{d}g = \sup_{\|\varphi\|_{L_\rho^\infty} \leq 1} \left\{ \left| X^\alpha e^{-tL^\rho(\overline{D+B})} \varphi(e_G) \right| \right\} \leq \sup_{\|\varphi\|_{L_\rho^\infty} \leq 1} \left\{ \left\| e^{-tL^\rho(\overline{D+B})} \varphi \right\|_{C_{\rho,\infty}^{|\alpha|}} \right\} \tag{5.13}$$

Recall that $L_\rho^\infty(G)$ is the weighted L^∞-space with norm $\|\varphi\|_{L_\rho^\infty} = \sup_{g \in G} e^{-\rho|g|} |\varphi(g)|$. Denote $C_{\rho,\infty}^k = C^k(L_\rho^\infty(G))$ its C^k vectors. As the Dyson-Phillips series is absolutely convergent, it suffices to estimate the C^k-norm of each term. Now we apply Theorem 5.3 further to the $C^{|\alpha|}$-norms of each term by verifying the norm assumptions:

- $\left\| e^{-tL^\rho(\overline{D})} \right\|_{L_\rho^\infty \to C_{\rho,\infty}^\alpha} \leq M e^{t\omega}$ for some $M, \omega > 0$ by the continuity bound (A.12);
- For small $t \in (0, 1]$ the small time estimate (A.4) gives:

  $$\left\| B e^{-tL^\rho(\overline{D})} \right\|_{L_\rho^\infty \to L_\rho^\infty} \leq \left\| e^{-tL^\rho(\overline{D})} \right\|_{L_\rho^\infty \to C_{\infty,\rho}^\ell} \leq a b^\ell \ell! t^{-\ell/m} \sim C_\ell t^{-\ell/m} \tag{5.14}$$

  with $\ell/m < 1$ by our assumption.

Now repeating the arguments in Lemma 5.4 we see the $C_{\rho,\infty}^{|\alpha|}$-norm is indeed finite:

$$\left\| e^{-tL^\rho(\overline{D+B})} \varphi \right\|_{C_{\rho,\infty}^{|\alpha|}} \leq \sum_{k=0}^\infty \left\| \mathrm{Per}^k(e^{-t\overline{D}} \varphi) \right\|_{C_{\rho,\infty}^{|\alpha|}} \leq \theta_{B,A}(t) \tag{5.15}$$

with the absolutely convergent series $\theta_{B,A}(t)$ being finite for each fixed t:

$$\theta_{B,A}(t) = \sum_{k=0}^\infty \left\| e^{-t\overline{D}} \right\|_{L_\rho^\infty \to C_{\rho,\infty}^{|\alpha|}} * \left\| B e^{-t\overline{D}} \right\|_{L_\rho^\infty \to L_\rho^\infty}^k (t) \tag{5.16}$$

which, as Lemma 5.4 proved, is majorized by $\sum_n 2^{-n} t^{-2} e^{-t\omega_1}$ for some positive $\omega_1 > \omega > 0$ bounded from below. This proves that $e^{\rho|g|} \left| \partial_g^\alpha k_t^{D+B} \right| \in L^1(G)$ for all $t > 0$. $\square$

This estimate shows that $e^{\rho|g|}L_{X^I}k_t^{D+B} \in L^1(G)$ for any $\rho > 0$. Here L_{X^I} is the differentiation from the left side. In particular, given a for any polynomial of degree $|I|$, this implies that $p_I L_{X^I}k_t^{D+B} \in L^1(G)$. In fact, $L_{X^J}p_I$ remains polynomial therefore bounded for any $|J| \geq 0$, therefore:

$$L_{X^J}(p_I L_{X^I}k_t^{D+B}) \in L^1(G) \tag{5.17}$$

To establish the rapidly decaying property of the heat kernel on nilpotent Lie groups, we only need the following variant of the Sobolev lemma adapted to Lie groups: [**Pou72**, Lemma 5.1]

Lemma 5.6. Fix an integer $s > \dim G$. Then for each compact neighborhood $\mathcal{B}(e_G)$ of e_G there exists a constant C such that:

$$f(e_G) \leq C \sum_{|I|\leq s} \int_{\mathcal{B}(e_G)} |L_{X^I}f(y)|\,\mathrm{d}y \tag{5.18}$$

for all $f \in C^\infty(G)$. This follows directly from the Sobolev lemma in $\mathbb{R}^n$ by choosing local coordinates.

Therefore the supremum of $p_I L_{X^I}k_t^{D+B}$ for an arbitrary polynomial p_I remains finite:

$$\begin{aligned} |p_I L_{X^J}k_t^{D+B}(x)| &\leq C \sum_{|J|\leq s} \int_B \left|p_I(yx)L_{X^J}k_t^{D+B}(yx)\right|\mathrm{d}y \\ &\leq C \sum_{|J|\leq s} \left\|L_{X^J}p_I L_{X^I}k_t^{D+B}\right\|_{L^1(G)} < \infty \end{aligned} \tag{5.19}$$

In the case of nilpotent Lie groups N, the heat kernel on differential forms is different from $\tau(\sum_j X_j^2)$ by a first-order differential operator. This is established by inspecting Proposition 1.5. Hence the supremum indeed remains bounded under differentiation with polynomial coefficients. In view of Definition B.5, we conclude that $k_t^{D+B} \in \mathcal{S}(N)$ is a Schwartz function. Hence we have proven that the kernel $k_t^{\Delta_*(N)}$ of differential form Laplacians on the nilpotent Lie groups is of Schwartz class.

If G is a reductive Lie group, one needs also to test its growth against the spherical vectors $\phi_0^G(x)$. Recall that the Definition 2.28 requires the kernel to be rapidly decaying against $(1+|x|_{\mathfrak{p}_\mathfrak{z}})\phi_0^G(x)^{-1}$ with $\phi_0^G(x) = \langle \pi_0(x)v, v\rangle$ for some spherical vector v. We prove here a stronger estimate for the analogs of seminorms defined in Definition 2.28:

$$v^p_{\alpha,\beta,r}(F) := \sup_{x\in G}\left|(1+|x|_{\mathfrak{p}_\mathfrak{z}})^r \phi_0^G(x)^{-2/p} L(X^\alpha)R(X^\beta)F(x)\right| < \infty \tag{5.20}$$

for all $m \geq 0$ and $\alpha, \beta \in \mathbb{N}_0^n$. In a similar fashion we denote:

$$\mathcal{S}^p(G) := \left\{F \in C^\infty(G) \mid v^p_{\alpha,\beta,r}(F) < \infty \text{ for all } v^p_{\alpha,\beta,r} \mid \alpha,\beta \in \mathbb{N}_0^n, r > 0\right\}.$$

In particular we see $v^2_{\alpha,\beta,r} = v_{\alpha,\beta,r}$ and $\mathcal{S}(G) = \mathcal{S}^2(G)$. Again we dominate the growth of $\phi_0^G(x)$ by appropriately choosing $e^\rho|g|$. From (2.35) we see that:

$$-\log \phi_0^G(x) \leq \gamma d(e_G, x) \quad \text{ for all } x \in G \tag{5.21}$$

for some constant $\gamma > 0$. Hence if we choose the constant $\rho > \gamma > 0$ in Lemma 5.5 this indeed implies $\phi_0^G(x)^{-2/p}L(X^\alpha)k_t^{D+B} \in L^1(G)$ for all $t > 0$ and $p > 0$. Apply (5.18) again:

$$\begin{aligned} |\phi_0^G(x)^{-2/p}L(X^\alpha)k_t^{D+B}(x)| &\leq C \sum_{|\beta|\leq s} \int_B \left|L_{X^\alpha}\phi_0^G(yx)^{-2/p}L_{X^\beta}k_t^{D+B}(yx)\right|\mathrm{d}y \\ &\leq C \sum_{|J|\leq s} \left\|L_{X^\alpha}\phi_0^G(yx)^{-2/p}L(X^\beta)k_t^{D+B}\right\|_{L^1(G)} < \infty \end{aligned} \tag{5.22}$$

Then we finished proving the fact that $k_t^{D+B} \in \mathcal{S}^p(G)$ for all p, if we can establish a relation between $L_{X^I}R_{X^J}k_t$ and $L_{X^{I'}}k_t$.

To prove this we use the fact that k_t is $\mathrm{Ad}(K)$-invariant, i.e.: $k_t(k^{-1}gk) = k_t(g)$ for all $k \in K$. This follows readily from the fact that K acts on the vector bundle by isometries. One could observe this by verifying the identity $B^\theta(\mathrm{ad}_X Y, Z) = B^\theta(Y, \mathrm{ad}_X Z)$ for all $X \in \mathfrak{k}$ and $Y, Z \in \mathfrak{g}$. Consequently

$$k_t^\wedge(k^{-1}gk) = K_t^\wedge(k^{-1}k, k^{-1}gk) = K_t^\wedge(e_G, g) = k_t^\wedge(g) \tag{5.23}$$

for the kernel of the Hodge Laplacian on the differential forms $k_t^\wedge$. Other k_t^ρ can be estimated in a similar way.

We now consider the following construction for shifting the differential operators from one side to the other. The following is essentially modelled after an argument of Harish-Chandra [HC84, §11, Lemma 17]:

Lemma 5.7. For each pair $(D, D') \in U(\mathfrak{g}_\mathbb{C})^2$, one can choose a finite number of $D_i \in U(\mathfrak{g}_\mathbb{C})(1 \leq i \leq p)$ with the following property: If $f : G \to V$ is an $\mathrm{Ad}(K)$-invariant C^∞-function with V a unitary K-bimodule, then:

$$|L_D R_{D'} f(g)| \leq \sum_{j=1}^{p} |L_{D_j} f(g)| \quad |L_D R_{D'} f(g)| \leq \sum_{j=1}^{p} |R_{D_j} f(g)| \tag{5.24}$$

Proof. Decompose the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ via the Iwasawa decomposition as in Proposition 2.6, where $\mathfrak{a} = \mathfrak{a}_\mathfrak{p}$ and $\mathfrak{n} = \mathfrak{n}_\mathfrak{p}$. Recall $\mathfrak{n} = \sum_{i=1}^{l} \mathfrak{g}_{\alpha_i}$ is the space containing all the simple positive $(\mathfrak{g}; \mathfrak{a}_\mathfrak{p})$-roots $\{\alpha_i\}_{i=1}^{l}$. Applying the Poincaré-Birkhoff-Witt theorem, we decompose the universal enveloping algebra as:

$$U(\mathfrak{g}_\mathbb{C}) = U(\mathfrak{k}_\mathbb{C}) U(\mathfrak{a}_\mathbb{C}) U(\mathfrak{n}_\mathbb{C}) \tag{5.25}$$

Denote $U_d(\mathfrak{g}_\mathbb{C}) \subseteq U(\mathfrak{g}_\mathbb{C})$ containing all elements of order $\leq d$. Fix now an integer $d \geq 0$ such that $D, D' \in U_d(\mathfrak{g}_\mathbb{C})$. We can choose a basis $\{D_\mathfrak{k} D_\mathfrak{a} D_\mathfrak{n}\}$ of $U_d(\mathfrak{g}_\mathbb{C})$ where $D_\mathfrak{k} \in U(\mathfrak{k}_\mathbb{C})$, $D_\mathfrak{a} \in U(\mathfrak{a}_\mathbb{C})$ and $D_\mathfrak{n} \in U(\mathfrak{n}_\mathbb{C})$. Denote this basis as $\mathcal{B}_d$. Moreover, recall that the natural action of G on $U(\mathfrak{g}_\mathbb{C})$ extends the adjoint action on $\mathfrak{g}_\mathbb{C} \subseteq U_1(\mathfrak{g}_\mathbb{C})$. Then:

$$\mathrm{Ad}(a) D_\mathfrak{n} = \exp\left(\sum_{1 \leq i \leq l} m_i \alpha_i(\log a)\right) D_\mathfrak{n} \tag{5.26}$$

for all $a \in A$, where m_i are non-negative integers. This equality is obtained by extending the map $\mathrm{Ad}(a)\mathfrak{g}_\alpha = e^{\alpha(\log a)}\mathfrak{g}_\alpha$ via the fact that $\mathfrak{g}_\alpha$ are $\mathfrak{a}$-roots. For the same reason we can expand the $\mathrm{Ad}(K)$ action based on the action on the basis:

$$\mathrm{Ad}(k) D = \sum_{b \in \mathcal{B}_d} a_b(k) b \qquad \mathrm{Ad}(k) D' = \sum_{b \in \mathcal{B}_d} a'_b(k) b \tag{5.27}$$

for $k \in K$. Here a_bs and a'_bs are sum of exponential functions, and hence are continuous functions in K. Since V is a unitary K-bimodule, hence $\|k_1 v k_2\|_V = \|v\|_V$ for all $k_1, k_2 \in K$ and $v \in V$. Next by the KAK-decomposition (see [Kna86, Theore 5.20] and [Her92, Lemma 2.5] for the case K is compact and noncompact respectively), we write $G = KA^+K$ where $A^+ = \exp(\overline{\mathfrak{a}^+})$ where $\mathfrak{a}^+$ contains all $H \in \mathfrak{a}$ such that $\alpha(H) \geq 0$ for all restricted roots α. Denote:

$$c = \sup_{k \in K} \max_{b \in B} (|a_b(k)|, |a'_b(k)|) \tag{5.28}$$

This constant is clearly finite if K is compact. In the case K is noncompact, we can appeal to Lemma 2.27 and then write a_b and a'_b as continuous functions that are Z-invariant instead (see [Her92, Lemma 2.7]). In all cases, c is finite. Next we estimate growths in the A-direction under derivations. Write $g = k_1 a k_2 \in KA^+K$, then:

$$\|L_D R_{D'} f(g)\|_V \leq \left\| L_{(\mathrm{Ad}_{k_1^{-1}} D)} R_{(\mathrm{Ad}_{k_2} D')} f(a) \right\|_V \leq \sum_{b, b' \in \mathcal{B}_d} c^2 \|L_b R_{b'} f(a)\|_V \tag{5.29}$$

So it suffices to estimate the norm for each $L_b R_{b'} f(a)$. Write each $b \in \mathcal{B}_d$ in the form of $b = D_{\mathfrak{k}}^b D_{\mathfrak{a}}^b D_{\mathfrak{n}}^b$. Use the fact f is $\mathrm{Ad}(K)$-invariant, we see $L_{D_{\mathfrak{k}}} f = R_{D_{\mathfrak{k}}} f$ for any $D_{\mathfrak{k}} \in U(\mathfrak{k}_{\mathbb{C}})$. Now since $\mathfrak{a}$ is abelian, and recall the action of A on $\mathfrak{n}$, one can hence shift the actions on the left to the right one by one:

$$L_b R_{b'} f(a) = L_{D_{\mathfrak{k}}^b D_{\mathfrak{a}}^b D_{\mathfrak{n}}^b} R_{b'} f(a) = R_{D_{\mathfrak{k}}^b D_{\mathfrak{a}}^b \mathrm{Ad}_{a^{-1}}(D_{\mathfrak{n}}^b) b'} f(a) \tag{5.30}$$

Where $R_{D_1 D_2} f = R_{D_2}(R_{D_1} f)$. Next,

$$\left\| R_{D_{\mathfrak{k}}^b D_{\mathfrak{a}}^b \mathrm{Ad}_{a^{-1}}(D_{\mathfrak{n}}^b) b'} f(a) \right\|_V \leq \left\| R_{D_{\mathfrak{k}}^b D_{\mathfrak{a}}^b D_{\mathfrak{n}}^b b'} f(a) \right\|_V \tag{5.31}$$

by combining (5.26) and the fact $\alpha_i(\log a) \geq 0$ since α_i are positive roots. At last one notices that the following finite set

$$\{D_{\mathfrak{k}}^b D_{\mathfrak{a}}^b D_{\mathfrak{n}}^b b' \mid b, b' \in \mathcal{B}_d\}$$

spans a finite-dimensional subspace in $U(\mathfrak{g}_{\mathbb{C}})$. We denote the basis of this subspace as $g_j : 1 \leq j \leq p$. Hence we can choose a uniform bound c_2 such for each element in the above set can hence be written as sums $\sum_{1 \leq j \leq p} \gamma_j g_j$ with $|\gamma_j| \leq C'$ for some uniformly chosen bound C'. This completes the proof, as we can now estimate the derivatives from both sides by combining (5.29) and (5.31) into the above estimate:

$$\|L_D R_{D'} f(g)\|_V \leq \sum_{b, b' \in \mathcal{B}_d} c^2 \left\| R_{D_{\mathfrak{k}}^b D_{\mathfrak{a}}^b D_{\mathfrak{n}}^b b'} f(a) \right\|_V \leq \sum_{1 \leq j \leq p} |\mathcal{B}_d|^2 C' c^2 \left\| R_{g_j} f(g) \right\|$$

note we have again exploited the fact f is $\mathrm{Ad}(K)$-invariant in deducing the last inequality. This concludes the proof. □

Now if we apply the lemma to (5.22) together with the fact $k_t^{\wedge}(g)$ is $\mathrm{Ad}(k)$-invariant, we have finally established the fact that $k_t^{\wedge} \in \mathcal{S}^p(G)$ for all $1 \leq p \leq \infty$.

Now the kernel of the spinor Laplacian as well as the Hodge Laplacian are both of Schwartz class, from the expression (4.21) and (4.35). We summarize the above discussions in the following theorem:

Theorem 5.8 (Schwartz kernel). *Let $t > 0$. Then:*

1. *For any reductive Lie group G of class $\widetilde{\mathcal{H}}$, the kernels of the spinor and the Hodge Laplacian k_t^{ρ} associated with any finite-dimensional representation ρ are L^p-Schwartz functions on G, in the sense of Harish-Chandra, for all $p > 0$.*
2. *For any nilpotent Lie group N, the kernel of the Hodge Laplacian k_t^{ρ}, associated with any finite-dimensional representation ρ is an L^p-Schwartz function on N for all $p > 0$.*

One now applies the Plancherel decompositions in the form of Theorem B.6 and Theorem 2.25 to the heat kernel k_t^D accordingly. Previously the only question was the eligibility of applying the Plancherel theorem, which was resolved by the above discussions.

Remark 5.9. The proof of similar statements in the case of symmetric spaces can be carried out similarly. In several steps, the arguments are considerably simplified:

1. The discussion of bounded perturbation in such context is rather irrelevant, as both the Dirac Laplacian $\not{D}_{\rho}^2$ and the Hodge Laplacians Δ_{ρ} differ from $\sum_i R(X_i^2)$ by a scalar operator. Hence a version of Lemma 5.5 can be readily arrived at;
2. Lemma 5.7 was also replaced by a classical lemma by [HC84, Lemma 17], which asserts a similar result on K-finite functions, regardless of the twisting by finite-dimensional representation ρ. Then the two-sided estimates follow immediately from (5.22).

Chapter 6

Local spectra of $\widetilde{SL_2(\mathbb{R})}$

This chapter is devoted to computing the spectra of various differential operators on the $\widetilde{SL_2(\mathbb{R})}$. The main result is Theorem 6.2, where we compute the spectrum of Δ_i. We give the first example of the spectral decomposition of $L^2 \bigwedge^p(G)$, where G is a reductive Lie group of class $\widetilde{\mathcal{H}}$. In comparison with the case of symmetric space case, our case is much more intricate. There the eigenvalues of the Laplacian localized at each irreducible unitary representation π are given by Casimir eigenvalues and hence are very computable scalars. Nonetheless in our case, as the **Generalized Kuga's Lemma** reveals, one demands much more information about the representations. Our example on $G = \widetilde{SL_2(\mathbb{R})}$ in particular, requires a complete description of the $(\mathfrak{g}, K)$-module structure.

Throughout this chapter we would like to fix the Cartan involution θ on the Lie algebra $\mathfrak{g}$ as well on G. This in turn fixes a maximal compact subgroup K of G, with Lie subalgebra $\mathfrak{k}$.

6.1 Representation theory of $\widetilde{SL_2(\mathbb{R})}$: A short introduction

The classification of the unitary representations of $\widetilde{SL_2(\mathbb{R})}$ is due to Pukanszky [**Puk64**], the method of which dates back to Bargmann [**Bar47**]. Consider the following explicit basis of $\mathfrak{sl}_2$:

$$X_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.1}$$

To comply with our notational convention in Chapter 1, we use X_0 to denote the (unique) basis vector of $\mathfrak{k}$ and Y_is the basis vectors of $\mathfrak{p}$. They are related by the following commutation relations:

$$[X_0, Y_1] = Y_2 \quad [Y_1, Y_2] = -X_0 \quad [X_0, Y_2] = -Y_1$$

We form annihilation operators and creation operators by complexifying the basis

$$X_+ := \frac{1}{\sqrt{2}}(Y_1 + iY_2) \qquad X_- := \frac{1}{\sqrt{2}}(Y_1 - iY_2) \tag{6.2}$$

and form the root system $\Delta(\mathfrak{g}_\mathbb{C}; \mathfrak{h}_\mathbb{C}) = \{\pm\alpha\}$ with $\delta_G = \frac{1}{2}\alpha \in i\mathfrak{h}^*$ the half sum of roots. We normalize the metric such that $\alpha(X_0) = 1$, i.e.: we fix the following $\mathbb{C}$-linear symmetric form on $\mathfrak{g}_\mathbb{C}$

$$\langle X, Y \rangle = \frac{1}{8} B^\theta(X, Y) = \frac{1}{2} \operatorname{tr}(X\theta(Y)) \quad \text{for } X, Y \in \mathfrak{sl}_2(\mathbb{C}). \tag{6.3}$$

Here we choose the Cartan involution $\theta : X \mapsto -\overline{X}^t$ to be the negative conjugate transpose. This defines a Riemannian metric on $\mathfrak{sl}_2(\mathbb{R})$, and this metric makes X_0, Y_1, Y_2 an orthonormal basis, and consequently α into unit vectors in $\mathfrak{g}^*$.

Then the unitary representations are classified into three classes, in the same fashion as Bargmann's classification of the unitary representations of $SL_2(\mathbb{R})$. They are distinguished from each other by their Casimir eigenvalues $\chi_\pi(\Omega_G)$:

1. **Relative discrete series $D_k^{\pm} : k > 0$:** These are representations with underlying Hilbert spaces $H_k := H_{D_k}$ with infinitesimal characters χ_k and Casimir eigenvalue $\chi_{\pi_k}(\Omega_G) = k^2 - \frac{1}{4}$. In this case the X_is act on the orthonormal basis $\{v_m \mid m \in k + \frac{1}{2} + \mathbb{N}\}$ and give the $(\mathfrak{g}, K)$-module structures on the K-finite subspaces:

$$
\begin{aligned}
D_k^{\pm}(iX_+)v_m &= \omega_m \cdot \sqrt{\frac{-\chi_\pi(\Omega_G) + (m+1)m}{2}} \cdot v_{m+1} \\
D_k^{\pm}(iX_-)v_m &= \omega_m^{-1} \cdot \sqrt{\frac{-\chi_\pi(\Omega_G) + (m-1)m}{2}} \cdot v_{m-1} \\
D_k^{\pm}(iX_0)v_m &= m \cdot v_m
\end{aligned}
\tag{6.4}
$$

with the normalization constants ω_m equal to i for D_m^+ and equal to $-i$ for D_m^-. To correspond it back to our general theory in Section 2.2: In this case $G = G^0$ is connected, the χ-parameters is dropped, and $\widehat{G}_d$ is directly parametrized by L' as in Theorem 2.14. Also $H = H^0 \cong \mathbb{R}$ is simply connected, hence every element $k \in i\mathfrak{h}^*$ defines a character on H^0. Hence we parametrize L'_H by $k \in \mathbb{R}\backslash\{0\}$. We identify the positive half line with D_k^+ and the negative with D_k^-. Note that $D_0^+ \equiv D_0^-$ corresponds to the limits of the relative discrete series. Moreover, $\delta_G \in i\mathfrak{h}^*$ can be identified with $\frac{1}{2}$, and Proposition 2.21 yields the corresponding eigenvalue of $\chi_k(\Omega_G) = \|k\|^2 - \|\delta_G\|^2 = k^2 - \frac{1}{4}$ as claimed.

2. **Principal series $P_{\tau,i\nu} = \mathrm{ind}_{MA}^G(\tau \otimes e^{i\nu})$ for $\nu \in \mathbb{R}_+, \tau \in [0,1)$:** These are G-representations with infinitesimal characters $\chi_{\tau,i\nu} = i\nu \in i\mathfrak{a}^*$ and the Casimir eigenvalue $\chi_{\pi_{\tau,i\nu}}(\Omega_G) = -\nu^2 - \frac{1}{4}$. Then $\mathfrak{g}$ acts on their orthonormal bases $\{v_m \mid m \equiv \tau \mod \mathbb{Z}\}$ with the following $(\mathfrak{g}, K)$-module structure:

$$
\begin{aligned}
P_{\tau,i\nu}(iX_+)v_m &= \omega_m \cdot \sqrt{\frac{-\chi_\pi(\Omega_G) + (m+1)m}{2}} \cdot v_{m+1} \\
P_{\tau,i\nu}(iX_-)v_m &= \omega_m^{-1} \cdot \sqrt{\frac{-\chi_\pi(\Omega_G) + (m-1)m}{2}} \cdot v_{m-1} \\
P_{\tau,i\nu}(iX_0)v_m &= m \cdot v_m
\end{aligned}
$$

with normalization constants $\omega_m = \frac{(m+\frac{1}{2})-i\nu}{\sqrt{-\chi_\pi(\Omega_G)+m(m+1)}}$. Again in conformity with the discussions in Section 2.3, $\widehat{M_d} \cong \widehat{\mathbb{Z}} \cong S^1$ in this case, and our choice of measure on M makes it a unit circle and can be identified with $\mathbb{R}/\mathbb{Z}$. The infinitesimal character $\chi_{\tau,i\nu}(\Omega_G)$ is $\|i\nu\|^2 - \|\delta_G\|^2 = -\nu^2 - \frac{1}{4}$.

3. **Complementary series $C_{\tau,\nu} = \mathrm{ind}_{MA}^G(\tau \otimes e^{\nu})$ for:**

$$
\left\{(\tau, \nu) \in [0,1) \times [0, \frac{1}{2}) \mid |\tau - \frac{1}{2}| < \nu\right\}
$$

These G-representations have the infinitesimal character $\chi_\tau(\Omega_G) = \nu \in \mathfrak{a}^*$ and the Casimir eigenvalue $\chi_{\tau,\nu}(\Omega_G) = \nu^2 - \frac{1}{4}$. Again their underlying $(\mathfrak{g}, K)$-module structures are given by the $\mathfrak{g}$-action on a corresponding basis $\{v_m \mid m \equiv \tau \mod \mathbb{Z}\}$:

$$
\begin{aligned}
C_{\tau,\nu}(iX_+)v_m &= \omega_m \cdot \sqrt{\frac{-\chi_\pi(\Omega_G) + (m+1)m}{2}} \cdot v_{m+1} \\
C_{\tau,\nu}(iX_-)v_m &= \omega_m^{-1} \cdot \sqrt{\frac{-\chi_\pi(\Omega_G) + (m-1)m}{2}} \cdot v_{m-1} \\
C_{\tau,\nu}(iX_0)v_m &= m \cdot v_m
\end{aligned}
$$

with again normalization constant $\omega_m = i$.

Remark 6.1. We parameterize the discrete series by their Harish-Chandra parameters instead of their lowest K-type, which is otherwise known as the Blattner parameters. The latter is adopted by Pukanszky, which differ from ours by an offset of $\frac{1}{2}$.

6.2 Spectra of Hodge Laplacian of $\widetilde{SL_2(\mathbb{R})}$

Define the Riemannian metric $\langle -,-\rangle = B^\theta$ on the $\widetilde{SL_2(\mathbb{R})}$ based on the Killing form B, that is:

$$\langle X, Y\rangle = \frac{1}{8}B(X, \theta Y) \tag{6.5}$$

so that $\langle -,-\rangle$ is positive definite on $\mathfrak{g}$. This normalization agrees with the last section. Define the dual (orthonormal) basis of X_0, Y_1, Y_2 to be $\omega_0, \omega_1, \omega_2$ on $\mathfrak{g}^*$ respectively. This gives dual basis ω_+, ω_- with respect to the complexified bilinear form $B^\theta_{\mathbb{C}}$, of X_+, X_- defined in (6.2):

$$\omega_+ = \frac{1}{\sqrt{2}}(\omega_1 - i\omega_2) \qquad \omega_- = \frac{1}{\sqrt{2}}(\omega_1 + i\omega_2)$$

We begin by computing the Laplacian on L^2-functions. Recall from Proposition 1.5 that $-\Delta_0$ can be identified with $\overline{\Omega}_G = X_0^2 + Y_1^2 + Y_2^2 \in U(\mathfrak{g})$ acting on $L^2(G)$.

Applying Theorem 2.25, we obtain the following expression:

$$k_{0,t}(e_G) = \int_{\pi\in\widehat{G}} \operatorname{tr}(e^{t\pi(\overline{\Omega}_G)})\, \mathrm{d}\pi \tag{6.6}$$

To evaluate the sum $\operatorname{tr}(e^{-t\pi(\overline{\Omega}_G)})$ on each representation H_π, one notes that for each admissible representation π, the K-finite vectors of H_π form a dense subspace of H_π. It suffices to evaluate the $\operatorname{tr}(e^{-t\pi(\overline{\Omega}_G)})$ on the K-finite vectors in $C^0(\mathfrak{g}; V)$, which is itself a $(\mathfrak{g}; K)$-module that we denote as $C^0(\mathfrak{g}; V)_K \cong V_K$. For each $\sigma \in \widehat{K}$, the respective K-type component $V_{\pi,\sigma}$ of V_π is isomorphic to $\sigma^{\oplus[\pi:\sigma]}$ as $\mathfrak{k}$-modules. Here $[\pi:\sigma]$ denotes the multiplicity of σ in H_π.

$$\begin{aligned}
\operatorname{tr}(e^{t\pi(\overline{\Omega}_G)})\Big|_{C^0(\mathfrak{g};V)_K} &= \operatorname{tr}(e^{t\pi(\overline{\Omega}_G)})\Big|_{\bigoplus_{\sigma\in\widehat{K}}\sigma^{\oplus[\pi:\sigma]}} \\
&= \sum_{\sigma\in\widehat{K}} \operatorname{tr} e^{t(\Omega_G - 2\Omega_K|_{V_{\pi,\sigma}})} \\
&= \sum_{\sigma\in\widehat{K}} [\pi:\sigma] \dim\sigma \cdot e^{t(\chi_\pi(\Omega_G) - 2\chi_\sigma(\Omega_K))}
\end{aligned} \tag{6.7}$$

The last equality follows because Ω_G and Ω_K evaluate on $V_{\pi,\sigma}$ as scalar matrices, with respective infinitesimal characters on $\chi_\pi \in \mathfrak{h}^*_{\mathbb{C}}$ and $\chi_\sigma \in \mathfrak{t}^*_{\mathbb{C}}$. With a slight abuse of notation one identifies $\Omega_G \in Z(\mathfrak{g}_{\mathbb{C}})$ with its image under the Harish-Chandra isomorphism in $S(\mathfrak{h})^W$, and similarly for Ω_K. Then $\overline{\Omega}_G$ acts on $V_{\pi,\sigma}$ as scalar operators, of value $\chi_\pi(\Omega_G) - 2\chi_\sigma(\Omega_K)$. We compute their values on each representation:

1. On the relative discrete series $D^\pm_k$, the $\sigma = m \in \widehat{K}$-isotypic subspace contains a unique unit vector $v_{k,m}$ with $\pi(\overline{\Omega}_G)v_{k,m} = (k^2 - \frac{1}{4} - 2m^2)\cdot v_{k,m}$ where $|m| \geq k + \frac{1}{2}$;
2. On the principal series $P_{\tau,i\nu}$, similar situation applies, and $\pi(\overline{\Omega}_G)\cdot v_{(\tau,i\nu),m} = (-2m^2 - \frac{1}{4} - \nu^2)\cdot v_{(\tau,i\nu),m}$, for $m - \tau \in \mathbb{Z}$;
3. On complementary series $C_{\tau,\nu}$, $\pi(\overline{\Omega}_G)\cdot v_{(\tau,\nu),m} = (\nu^2 - \frac{1}{4} - 2m^2)\cdot v_{(\tau,\nu),m}$ for $m - \tau \in \mathbb{Z}$.

In all cases, $m(\pi:\tau) = \dim\tau = 1$ for all $\tau \in \widehat{K}$, a special feature of $\mathfrak{sl}_2$ due to the fact that K is abelian. This concludes the computation of the function on $\widetilde{SL_2(\mathbb{R})}$.

To see the computation of the spectrum of Δ_1 on 1-forms, we proceed similarly and first form a decomposition into K-finite vectors. Given a $(\mathfrak{g}, K)$-module V, we denote its σ-type subspace as V^σ, for $\sigma \in \widehat{K}$.

Now we take $V = H_\pi$ for those $\pi \in \widehat{G}$. Note $C^*(\mathfrak{g}; V) = H_\pi \otimes \mathfrak{g}^*$ in such case. Generically each K-types block $[H_\pi \otimes \mathfrak{g}^*_{\mathbb{C}}]_\sigma$ in $H_\pi \otimes \mathfrak{g}^*_{\mathbb{C}}$ is spanned by the following set of vectors:

$$\{v_m \otimes \omega_0, v_{m+1}\otimes\omega_+, v_{m-1}\otimes\omega_- \mid m = \sigma \in \widehat{K}\}$$

By Corollary 1.7 or by direct computation, we see Δ_1 on $f\otimes d\omega_i\in[H_\pi\otimes\mathfrak{g}^*_{\mathbb{C}}]_K\cong C(\mathfrak{g};H_{\pi,K})$ as:

$$\begin{aligned}\Delta_1(f\otimes\omega_+)&=(-\pi(\overline{\Omega}_G)+1)f\otimes\omega_+-3i\pi(X_0)f\otimes\omega_+-i\pi(X^-)f\otimes\omega_0\\ \Delta_1(f\otimes\omega_-)&=(-\pi(\overline{\Omega}_G)+1)f\otimes\omega_-+3i\pi(X_0)f\otimes\omega_-+i\pi(X^-)f\otimes\omega_0\\ \Delta_1(f\otimes\omega_-)&=(-\pi(\overline{\Omega}_G)+1)f\otimes\omega_--i\pi(X_+)f\otimes\omega_++i\pi(X^-)f\otimes\omega_-\end{aligned}\tag{6.8}$$

Rewrite Δ_1 with respect to the ordered bases $\{v_m\otimes\omega_0,v_{m+1}\otimes\omega_+,v_{m-1}\otimes\omega_-\}$, which complies with (1.31):

$$\begin{pmatrix}A_{\mathfrak{g}}-B_{\mathfrak{k}} & -C_{\mathfrak{p}}\\ C_{\mathfrak{p}} & A_{\mathfrak{g}}-3B_{\mathfrak{k}}\end{pmatrix}=\begin{pmatrix}-\pi(\overline{\Omega}_G)+1 & -iX_- & iX_+\\ -iX_+ & -\pi(\overline{\Omega}_G)-3iX_0+1 & \\ iX_- & & -\pi(\overline{\Omega}_G)+3iX_0+1\end{pmatrix}\tag{6.9}$$

Now combining (6.9) with our explicit realization at the beginning of this section, each $\pi(\Delta_1)$ can be block-diagonalized into infinitely many concrete generic matrices as below:

$$\begin{aligned}&\begin{pmatrix}2m^2-\Omega+1 & -\omega_m^{-1}\sqrt{\frac{-\Omega+(m+1)m}{2}} & \omega_m\sqrt{\frac{-\Omega+(m-1)m}{2}}\\ -\omega_m\sqrt{\frac{-\Omega+(m+1)m}{2}} & 2(m+1)^2-\Omega-3(m+1)+1 & \\ \omega_m^{-1}\cdot\sqrt{\frac{-\Omega+(m-1)m}{2}} & & 2(m-1)^2-\Omega+3(m-1)+1\end{pmatrix}\\ =&(2m^2-\Omega)\cdot\mathbb{I}_3+\begin{pmatrix}1 & -\omega_m^{-1}\sqrt{\frac{-\Omega+(m+1)m}{2}} & \omega_m\sqrt{\frac{-\Omega+(m-1)m}{2}}\\ -\omega_m\sqrt{\frac{-\Omega+(m+1)m}{2}} & m & \\ \omega_m^{-1}\cdot\sqrt{\frac{-\Omega+(m-1)m}{2}} & & -m\end{pmatrix}\end{aligned}\tag{6.10}$$

Here $\chi_\pi(\Omega_G)$ is abbreviated as Ω. Note that $2m^2-\Omega_G$ is actually $\chi_\pi(\overline{\Omega}_G)$ evaluated on v_m, and for the time being we denote this value as Ξ_m. We compute the characteristic polynomial of the rightmost matrix to be:

$$\begin{aligned}p(X)&=(1-X)(m-X)(-m-X)-(m-X)\frac{-\Omega+(m-1)m}{2}+(m+X)\frac{-\Omega+(m+1)m}{2}\\ &=m^2X-m^2-X^3-X^2+2m^2+X(-\Omega+m^2)\\ &=(\Xi_m+X-X^2)X\end{aligned}\tag{6.11}$$

Hence, given a generic block $[H_{\pi,K}\otimes\mathfrak{g}^*]_m$ of K-type m, it produces a set of eigenvalues,

$$\left\{\Xi_m,\Xi_m+\frac{1}{2}\pm\sqrt{\Xi_m+\frac{1}{4}}\right\}\tag{6.12}$$

each of multiplicity one. We see in the following what are the respective K-types and their corresponding eigenvalues are:

1. In the relative discrete series $\{D_k^\pm:k>0\}$, one should heed those low K-types m of $H_{\pi_k,K}$, in which $\dim[H_{\pi_k,K}\otimes\mathfrak{g}^*]_m<3$. This occurs in particular when $m<k+\frac{1}{2}+2$. In these cases some vectors in the aforementioned bases will vanish, resulting in smaller blocks. Abbreviate the lowest K-type $k+\frac{1}{2}$ as l, the eigenvalues are as follows:

1.1. $\{v_{k,l}\otimes\omega_+\}$ forms a 1×1-block of K-type $l-1$ under the Δ_1-action, with an eigenvalue $(\pi(\overline{\Omega}_G)-3iX_0+1)\cdot v_{k,l}=2l^2-l(l-1)-3l+1=(l-1)^2$;

1.2. $\{v_{k,l}\otimes\omega_0,v_{k,l+1}\otimes\omega_+\}$ forms a 2×2-block of K-type $l+1$. The corresponding matrix with respect to this basis is:

$$\begin{pmatrix}l^2+l+1 & i\sqrt{l}\\ -i\sqrt{l} & l^2+2l\end{pmatrix}=(l^2+l)\cdot\mathbb{I}_2+\begin{pmatrix}1 & i\sqrt{l}\\ -i\sqrt{l} & l\end{pmatrix}\tag{6.13}$$

with character polynomial of the remainder matrix being $p(X)=X(X-l-1)$. Hence the eigenvalues are l^2+l and $l^2+l+l+1=(l+1)^2$, each of multiplicity one;

1.3. For $m \geq l+2$, the action of Δ_1 on $[H_{\pi_k,K} \otimes \mathfrak{g}^*]_m$ can be realized as the generic 3×3-matrix in the form of (6.10). Hence the corresponding eigenvalues is computed as $\Xi_m, \Xi_m + \frac{1}{2} \pm \sqrt{\Xi_m + \frac{1}{4}}$, each are multiplicity one;

2. The case of D_k for $k \leq 0$ is dealt in exactly the same way as above;
3. In the principal series, each block matrix occurring in the decomposition gives a 3×3-matrix, which follows directly from Frobenius reciprocity:

$$[\mathrm{ind}_M^G(\tau \otimes i\nu) \otimes \mathfrak{g}^* : \tau_m] = [\tau \otimes \mathrm{res}_M^G(\mathfrak{g}^*) : \mathrm{res}_{K\cap M}^K(\tau_m)] = 3$$

Now we mimic the computation for discrete series to derive that the eigenvalue are again $\Xi_m, \Xi_m + \frac{1}{2} \pm \sqrt{\Xi_m + \frac{1}{4}}$, each of multiplicity one, with Ξ_m the eigenvalue of $\pi_{\tau,i\nu}(\overline{\Omega}_G)$ with eigenvector v_m: $\pi_{\tau,i\nu}(\overline{\Omega}_G)v_m = (2m^2 - \Omega_G)v_m$;
4. Lastly for the complementary series, we argue in the same way for the principal series and note that the corresponding family of eigenvalues are in the form

$$\left\{ \Xi_m, \Xi_m + \frac{1}{2} \pm \sqrt{\Xi_m + \frac{1}{4}} \;\middle|\; \Xi_m = 2m^2 + \nu^2 - \frac{1}{4} \quad m \equiv \tau \mod \mathbb{Z} \right\} \tag{6.14}$$

for each representation $C_{\tau,\nu}$ in the range of $|\tau - \frac{1}{2}| < \nu$ and $0 \leq \nu < \frac{1}{2}$.

This concludes the computations.

Theorem 6.2 (**Spectra of Hodge Laplacians on $\widetilde{SL_2(\mathbb{R})}$**). *Let $G = \widetilde{SL_2(\mathbb{R})}$ be the universal cover of the 2×2-special linear group over $\mathbb{R}$. Let B^θ be the positive-definite bilinear form defined by twisting the Killing form B on $\mathfrak{g}$ via a Cartan involution θ, i.e.: $B^\theta(X,Y) := B(X, \theta Y)$. This defines a Riemannian metric on G. To stress the relation between Δ_0 and Δ_1, we abbreviate the following values:*

- $\Xi_{k,m} := -k^2 + \frac{1}{4} + 2m^2$ *the eigenvalue of Δ_0 associated with the vector of K-type m of $D_k^\pm$;*
- $\Xi_{(\tau,i\nu),m} := 2m^2 + \frac{1}{4} + \nu^2$ *the eigenvalue of Δ_0 associated with the vector of K-type m of $P_{\tau,i\nu}$;*
- $\Xi_{(\tau,\nu),m} := 2m^2 + \frac{1}{4} - \nu^2$ *the eigenvalue of Δ_0 associated with the vector of K-type m of $C_{\tau,\nu}$.*

Then the corresponding Laplacians Δ_0 and Δ_1 have the following spectrum, indexed by two parameters:

1. The Laplacian Δ_0 localized over each representation takes the following values:

$$\mathrm{Spec}_\pi(\Delta_0) = \begin{cases} \{\Xi_{k,m} \mid m \in \mathbb{N} + \frac{1}{2} + k\} & \text{if } \pi = D_k^+, \quad k \geq 0 \\ \{\Xi_{k,m} \mid m \in \mathbb{Z}_{\leq 0} - \frac{1}{2} - k\} & \text{if } \pi = D_k^-, \quad k \geq 0 \\ \{\Xi_{(\tau,i\nu),m} \mid m - \tau \in \mathbb{Z}\} & \text{if } \pi = P_{\tau,i\nu}, \quad \tau \in [0,1), \nu \in \mathbb{R}_+ \\ \{\Xi_{(\tau,\nu),m} \mid m - \tau \in \mathbb{Z}\} & \text{if } \pi = C_{\tau,\nu}, \quad \tau \in [0,1), \nu \in \mathbb{R}_+ \end{cases} \tag{6.15}$$

Each of multiplicity one.

2. The Laplacian Δ_1 localized over the relative discrete series has spectrum:

$$\begin{aligned} \mathrm{Spec}_{D_k^\pm}(\Delta_1) = \Big\{ &(k - \frac{1}{2})^2, (k+1)^2 - \frac{1}{4}, (k + \frac{3}{2})^2, \Xi_{k,m}, \\ &\Xi_{k,m} + \frac{1}{2} \pm \sqrt{\Xi_{k,m} + \frac{1}{4}} \;\Big|\; m \in \mathbb{N}_{\geq 0} + \frac{1}{2} + k \Big\} \end{aligned} \tag{6.16}$$

for all $k > 0$, each of multiplicity one; whereas for the principal series $\pi = P_{\tau,i\nu}, \tau \in [0,1), \nu \in \mathbb{R}_+$:

$$\mathrm{Spec}_{P_{\tau,i\nu}}(\Delta_1) = \left\{ \Xi_{(\tau,i\nu),m}, \Xi_{(\tau,i\nu),m} + \frac{1}{2} \pm \sqrt{\Xi_{(\tau,i\nu),m} + \frac{1}{4}} \;\middle|\; m - \tau \in \mathbb{Z} \right\} \tag{6.17}$$

and lastly for the complementary series $\pi = C_{\tau,\nu}$, with $\{(\tau,\nu) \in [0,1) \times [0,\frac{1}{2}) \mid |\tau - \frac{1}{2}| < \nu\}$:

$$\operatorname{Spec}_{C_{\tau,\nu}}(\Delta_1) = \left\{ \Xi_{(\tau,\nu),m}, \Xi_{(\tau,\nu),m} + \frac{1}{2} \pm \sqrt{\Xi_{(\tau,\nu),m} + \frac{1}{4}} \;\middle|\; m - \tau \in \mathbb{Z} \right\} \tag{6.18}$$

Each also of multiplicity one.

3. Combining the computation with the explicit Plancherel formula (2.32), the heat kernel $e^{-t\Delta_0}$ in this case admits an explicit form for its kernel as below:

$$\begin{aligned} k_{0,t}(e_G) = &\int_0^\infty \sum_{m \in k + \frac{1}{2} + \mathbb{N}} \exp\left[-t \cdot \Xi_{k,m}\right] k \,\mathrm{d}k \\ &+ \int_0^1 \int_0^\infty \sum_{m \in \tau + \mathbb{Z}} \exp\left[-t \cdot \Xi_{(\tau,i\nu),m}\right] \operatorname{Re}\tanh(\tau, i\nu) \,\mathrm{d}\nu \,\mathrm{d}\tau \end{aligned} \tag{6.19}$$

4. Lastly $e^{-t\Delta_1}$ admits the following form for kernel:

$$\begin{aligned} k_{1,t}(e_G) = &\int_0^\infty \Bigg[e^{-t(k-\frac{1}{2})^2} + e^{-t(k+1)^2 + \frac{1}{4}} + e^{-t(k+\frac{3}{2})^2} + \\ &\sum_{m \in k + \frac{1}{2} + \mathbb{N}} \left(e^{-t\Xi_{k,m}} + e^{-t\left(\Xi_{k,m} + \frac{1}{2} \pm \sqrt{\Xi_{k,m} + \frac{1}{4}}\right)} \right) \Bigg] k \,\mathrm{d}k \\ &+ \int_0^1 \int_0^\infty \sum_{m \in \tau + \mathbb{Z}} \Bigg(e^{-t\Xi_{(\tau,i\nu),m}} + \\ &+ e^{-t\left(\Xi_{(\tau,i\nu),m} + \frac{1}{2} \pm \sqrt{\Xi_{(\tau,i\nu),m} + \frac{1}{4}}\right)} \Bigg) \operatorname{Re}\tanh(\tau + i\nu) \,\mathrm{d}\nu \,\mathrm{d}\tau \end{aligned} \tag{6.20}$$

In Section 6.5 we illustrate the spectra of the Laplacians against the representation parameters. We conclude this section with the remark that the Hodge Laplacians of $\widetilde{SL_2(\mathbb{R})}$ is hence completely known by a simple application of Poincaré duality.

6.3 Spectra of the Dirac operator of $\widetilde{SL_2(\mathbb{R})}$

We now set off to compute the spectra of the Dirac operators restricted to each unitary representations. Based on (4.67), it suffices to compute the operator $\sum_a R(X_a)c\ell(X_a)$ and the operator $\urcorner_\sigma$ on invariant forms separately. We begin with $\urcorner_\sigma$. Again we construct the spin module $S_\mathfrak{g}$ for the Clifford module $\mathbb{C}\ell(\mathfrak{g})$. Recall the notations of Remark 3.4.

Following the bases of (6.1) and (3.17) we choose two isotropic subspaces $\mathcal{Z}_\mathfrak{p} = \mathbb{C}X_+$, $\mathcal{Z}_\mathfrak{k} = \mathbb{C}X_0$ and $\mathcal{Z}_\mathfrak{g} = \mathcal{Z}_\mathfrak{p} \oplus \mathcal{Z}_\mathfrak{k}$. Following the discussion in Remark 3.4, X_0 is the complementary vector in both $\mathfrak{g}$ and $\mathfrak{k}$, and hence it gives the two module structures on $\mathcal{S}_\mathfrak{g}$ and $\mathcal{S}_\mathfrak{k}$. The corresponding module structures are illustrated by the following diagrams:

$$c\ell(X_0), i \circlearrowright 1 \underset{c\ell(X_-),-2}{\overset{c\ell(X_+),1}{\rightleftarrows}} X_+ \circlearrowleft c\ell(X_0), -i \qquad c\ell(X_0), -i \circlearrowright X_0 \underset{c\ell(X_-),-2}{\overset{c\ell(X_+),1}{\rightleftarrows}} X_+ \wedge X_0 \circlearrowleft c\ell(X_0), i \tag{6.21}$$

Where $c\ell(X_i), c$ denotes the action of X_i on the vector with a scalar c. We see these diagrams give two inequivalent $\mathbb{C}\ell(\mathfrak{g})$-modules. They are however equivalent as $\mathbb{C}\ell(\mathfrak{p})$-modules. To compute $\urcorner_\sigma$, we use the formula (4.68):

$$\urcorner_\sigma = \frac{1}{2} c\ell(X_0)\sigma_\mathfrak{g}(X_0) = \frac{1}{4} c\ell(X_0 X_1 X_2) = \frac{1}{4} c\ell(\omega_\mathbb{C}) \tag{6.22}$$

In our case $\sigma_\mathfrak{p} \cong \sigma_\mathfrak{g} : X_0 \mapsto \frac{1}{4} c\ell(X_1 X_2 - X_2 X_1) = \frac{1}{2} c\ell(X_1 X_2)$ by Lemma 3.7. Recall from Remark 3.4 that $\omega_\mathbb{C} = X_0 X_1 X_2$ acts on the Clifford module by ± 1, with the sign depending on the Clifford module structure. In both cases $\urcorner_\sigma$ is a scalar operator.

Remark 6.3. To corroborate the formula we derived in previous chapters, one may alternatively verify directly the formula for γ_0 in (4.24). The Levi-Civita connection ∇ on the invariant vector fields chosen as in (6.1) acts as follows:

$$\begin{aligned} \nabla_{X_0} Y_1 &= \frac{3}{2} Y_2 & \nabla_{X_0} Y_2 &= -\frac{3}{2} Y_1 & \nabla_{Y_1} Y_2 &= -\frac{1}{2} X_0 \\ \nabla_{Y_1} X_0 &= \frac{1}{2} Y_2 & \nabla_{Y_2} X_0 &= -\frac{1}{2} Y_1 & \nabla_{Y_2} Y_1 &= \frac{1}{2} X_0 \end{aligned} \tag{6.23}$$

whereas the spin connection on invariant vector field can be computed explicitly using the general formula for spinor bundles [LM90, Theorem II.4.14]: $\nabla^S_X = \sum_{Y_i} (\nabla_{Y_i} X) \cdot Y_i$ with Y_i ranging over the orthonormal basis. From this one can derive the same result.

Now the rest of the computation is essentially a replication of what we have done in the previous section. We fix a $\mathbb{C}\ell(\mathfrak{g})$-module structure on $\mathcal{S}_\mathfrak{g}$ by taking $c\ell(X_0) = i \cdot \mathrm{id}$ (and consequently $c\ell(\omega_\mathbb{C}) = \mathrm{id}$). Moreover,

$$R(X_1) \otimes c\ell(X_1) + R(X_2) \otimes c\ell(X_2) = R(X_+) \otimes c\ell(X_-) + R(X_-) \otimes c\ell(X_+). \tag{6.24}$$

In view of Remark 3.4 we renormalize the vectors accordingly, and take the ordered orthonormal basis $\{1, \frac{1}{\sqrt{2}} X_+\}$ of $\mathcal{S}_\mathfrak{g}$, and see:

$$D\!\!\!/_\sigma = \begin{pmatrix} iR_{X_0} & -\sqrt{2} R_{X_+} \\ \sqrt{2} R_{X_-} & -iR_{X_0} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \tag{6.25}$$

Similarly, $\pi(D\!\!\!/_\sigma)$ is given by a matrix of the same form replacing R_{X_i} by $\pi(X_i)$. Next for each $\pi \in \widehat{G}$, we perform the computation as in Section 6.1 and compute the spectrum. We start from the relative discrete series again. Following the dichotomy and notation of last section, the generic matrices are:

$$\begin{pmatrix} -m & i\omega_{\pi,m} \sqrt{-\Omega + (m+1)m} \\ -i\omega_{m,\pi}^{-1} \sqrt{-\Omega + (m+1)m} & m+1 \end{pmatrix} + \begin{pmatrix} 1/4 & \\ & 1/4 \end{pmatrix} \tag{6.26}$$

with respect to $v_m \otimes {}_+ 1$ and $v_{m+1} \otimes {}_+ X_+$ respectively. The spectrum for such block is:

$$\frac{1}{2} \pm \sqrt{\frac{1}{4} - \Omega + 2m(m+1)} + \frac{1}{4} = \frac{3}{4} \pm \sqrt{\frac{1}{4} - \Omega + 2m(m+1)} \tag{6.27}$$

In addition, the lowest K-types $v_{k+\frac{1}{2}}$ of each relative discrete series $D_k^\pm$ forms a single 1×1-block with $v_{k+\frac{1}{2}} \otimes {}_+ X_+$ as basis. This gives a single $\frac{3}{4} - k$ for each $k \in \mathbb{R}_{\neq 0}$.

The results of this section have been computed in a similar fashion in [BNPW16]. Our computation deflected from theirs by a factor of $\frac{1}{2}$. It seems that they used different metric for the computation. As there the metric on spinors was not specified, we stick to our normalization, given explicitly in Remark 3.4. In their paper one can find plots of the spectrum on the Dirac operators. We note that the geometric Dirac operators have a unilateral shift from their result by $-\frac{1}{4}$. Finally we summarize the results for ease of references:

Proposition 6.4. *Let $G = \widetilde{SL_2(\mathbb{R})}$ be the universal cover of the group of 2×2-unimodular matrices, endowed with the metric as in Theorem 6.2. Then the Dirac operator $D\!\!\!/_\sigma$ associated with the Levi-Civita connection admits the following localized spectra:*

1. *For D_k the relative discrete series with $k \in \mathbb{R}_{\neq 0}$:*

$$\mathrm{Spec}_{D_k}(D\!\!\!/_\sigma) = \left\{ \frac{3}{4} - |k|, \frac{3}{4} \pm \sqrt{2m(m+1) - k^2} \;\middle|\; m \geq |k| + \frac{1}{2} \right\} \tag{6.28}$$

2. *For principal series $P_{\tau,i\nu}$ with $\tau \in [0,1)$ and $\nu \in \mathbb{R}_+$:*

$$\mathrm{Spec}_{P_{\tau,i\nu}}(D\!\!\!/_\sigma) = \left\{ \frac{3}{4} \pm \sqrt{2m(m+1) + \nu^2} \;\middle|\; m - \tau \in \mathbb{Z} \right\} \tag{6.29}$$

3. *For complementary series $C_{\tau,\nu}$, with $\{(\tau,\nu) \in [0,1) \times [0,\frac{1}{2}) \mid |\tau - \frac{1}{2}| < \nu\}$:*

$$\mathrm{Spec}_{C_{\tau,i\nu}}(D\!\!\!/_\sigma) = \left\{ \frac{3}{4} \pm \sqrt{2m(m+1) - \nu^2} \;\middle|\; m - \tau \in \mathbb{Z} \right\} \tag{6.30}$$

6.4 Adiabatic limit of the spectrum

If we rescale the metric in the direction of $\mathfrak{k}$ such that the new inner product $\langle -,-\rangle_\kappa$ takes value:

$$\langle -,-\rangle|_{\mathfrak{p}} = \frac{1}{8}B^\theta \quad \langle -,-\rangle_\kappa|_{\mathfrak{k}} = \frac{\kappa^2}{8}B^\theta \tag{6.31}$$

then we form a new orthonormal basis $X_+, X_-, \frac{1}{\kappa}X_0$, with $X_\pm$ and X_0 as Section 6.1. Denote by $\Delta_{p,\kappa}$ the Hodge Laplacian on p-forms under the new metric.

To investigate the spectrum of the Hodge Laplacian under such change of metric, we follow the steps of Section 8.1: First the function Laplacian under such rescaling:

$$\Delta_{0,\kappa} = \frac{X_0^2}{\kappa^2} + X_1^2 + X_2^2 \in U(\mathfrak{g}_{\mathbb{C}}) \tag{6.32}$$

which acts a vector $v_m \in \pi_\lambda$ of weight m as:

$$\pi(\Delta_{0,\kappa})v_m = \left(\frac{m^2}{\kappa^2} + 1 - \chi_\lambda(\Omega_G)\right)v_m \tag{6.33}$$

For $\Delta_{1,\kappa}$, we again obtain a block matrix. In this case we compute it directly, as the Corollary 1.7 is formulated using the orthonormal basis with respect to B. Also choose dual vectors $\omega_{0,\kappa}$ and $\omega_\pm$ accordingly. The matrix is given explicitly as:

$$\begin{pmatrix} -\pi(\Delta_{0,\kappa}) + \kappa^2 & -i\kappa X_- & i\kappa X_+ \\ -i\kappa X_+ & -\pi(\Delta_{0,\kappa}) - (\kappa + \frac{2}{\kappa})iX_0 + \frac{1}{\kappa^2} & \\ i\kappa X_- & & -\pi(\Delta_{0,\kappa}) + (\kappa + \frac{2}{\kappa})iX_0 + \frac{1}{\kappa^2} \end{pmatrix} \tag{6.34}$$

with respect to the ordered orthonormal basis $\{v_m \otimes \omega_{0,\kappa}, v_{m+1} \otimes \omega_+, v_{m-1} \otimes \omega_-\}$, the generic block can be written in the following form:

$$\begin{aligned} &\begin{pmatrix} m^2 + \frac{m^2}{\kappa^2} - \Omega + \kappa^2 & -\omega_m^{-1}\sqrt{\frac{-\Omega+(m+1)m}{2}} & \omega_m\sqrt{\frac{-\Omega+(m-1)m}{2}} \\ -\omega_m\sqrt{\frac{-\Omega+(m+1)m}{2}} & (m+1)m + \frac{m^2}{\kappa^2} - \Omega & \\ \omega_m^{-1}\cdot\sqrt{\frac{-\Omega+(m-1)m}{2}} & & (m-1)m + \frac{m^2}{\kappa^2} - \Omega \end{pmatrix} \\ = (2m^2 - \Omega)\cdot\mathbb{I}_3 + &\begin{pmatrix} 1 & -\omega_m^{-1}\sqrt{\frac{-\Omega+(m+1)m}{2}} & \omega_m\sqrt{\frac{-\Omega+(m-1)m}{2}} \\ -\omega_m\sqrt{\frac{-\Omega+(m+1)m}{2}} & m & \\ \omega_m^{-1}\cdot\sqrt{\frac{-\Omega+(m-1)m}{2}} & & (m+1)m + \frac{m^2}{\kappa^2} - \Omega \end{pmatrix} \end{aligned} \tag{6.35}$$

In particular the formula specializes to (6.10) when $\kappa = 1$. We omit the exact computations on each representation, as the computation performed Section 6.2 carries verbatim over here. Instead we state the theorem in a way to stress its relation to the function spectrum. Denote:

$$\Xi_{\pi,\kappa,m} := \frac{m^2}{\kappa^2} + m^2 - \chi_\pi(\Omega_G) \tag{6.36}$$

If the subscript π is clear from the context, we denote it by $\Xi_{\kappa,m}$ instead.

Theorem 6.5 (Spectrum of the rescaled Laplacian)**.** *Endow $\widetilde{SL_2}(\mathbb{R})$ with a metric as (6.31), then the spectrum of the Hodge Laplacian $\Delta_{0,\kappa}$ on each type of representation is given by:*

1. $\mathrm{Spec}_{D_k^\pm}(\Delta_{0,\kappa}) = \{\Xi_{\kappa,m} \mid m \in \mathbb{N}+\frac{1}{2}+|k|\}$ *for the discrete series $D_k^\pm$, $k > 0$, with $\chi_{D_k}(\Omega_G) = k^2 - \frac{1}{4}$;*
2. $\mathrm{Spec}_{\pi,\nu}(\Delta_{0,\kappa}) = \{\Xi_{\kappa,m} \mid m - \tau \in \mathbb{Z}\}$ *for the principal series $P_{\tau,i\nu}$, $\tau \in [0,1), \nu \in \mathbb{R}_+$, with $\chi_{P_{\tau,i\nu}}(\Omega_G) = -\nu^2 - \frac{1}{4}$*
3. $\mathrm{Spec}_{\pi,\nu}(\Delta_{0,\kappa}) = \{\Xi_{\kappa,m} \mid m - \tau \in \mathbb{Z}\}$ *for the complementary series $P_{\tau,\nu}$, $\tau \in [0,1), \nu \in \mathbb{R}_+$.*

As for $\Delta_{1,\kappa}$, except for $\Xi_{D_k,\kappa,|k|+\frac{1}{2}}$ and $\Xi_{D_k,\kappa,|k|+\frac{3}{2}}$, the rest of the 1-spectrum are all of the form:

$$\mathrm{Spec}_\pi(\Delta_{1,\kappa}) = \Xi_{\pi,\kappa,m} + \frac{\kappa^2}{2} \pm \sqrt{\frac{\kappa^4}{4} + \kappa^2 \cdot \Xi_{\pi,\kappa,m}} \tag{6.37}$$

Whereas in the exceptional cases:

- *For each $\Xi_{D_k,\kappa,|k|+1/2}$ in the 0-spectrum, there is a unique eigenvalue $\frac{(k-1)^2}{\kappa^2}$ in the 1-spectrum;*
- *For each $\Xi_{D_k,\kappa,|k|+\frac{3}{2}}$ in the 0-spectrum, there are two eigenvalues:*

$$\left\{ \frac{k^2}{\kappa^2} + \left(\frac{1}{\kappa^2}+1\right)k + \left(\frac{1}{4\kappa^2}+\frac{1}{2}\right), \quad \left(\frac{2k+1}{\kappa}+\kappa\right)^2 \right\} \tag{6.38}$$

  Each of multiplicity one.

There the eigenvalues of $\Delta_{2,\kappa}$ and $\Delta_{3,\kappa}$ are the same as $\Delta_{1,\kappa}$ and $\Delta_{0,\kappa}$ respectively, by an easy application of Poincaré duality.

We conclude this chapter by remarking that the complementary series $C_{\tau,\nu}$ here are not tempered series, hence do not appear in the expressions $k_{i,t}(e_G)$. We only compute them for possible applications to the case of compact quotients of $\widetilde{SL_2(\mathbb{R})}$.

6.5 Illustrations of Spectra

This is a gallery of spectra of Hodge Laplacian on $\widetilde{SL_2}(\mathbb{R})$, localized on each (tempered) representation. We use the same color for spectra created by the same K-type of each representation to stress their potential relation. The right upper box in the first figure magnifies the behavior of the discrete series spectra at small value.

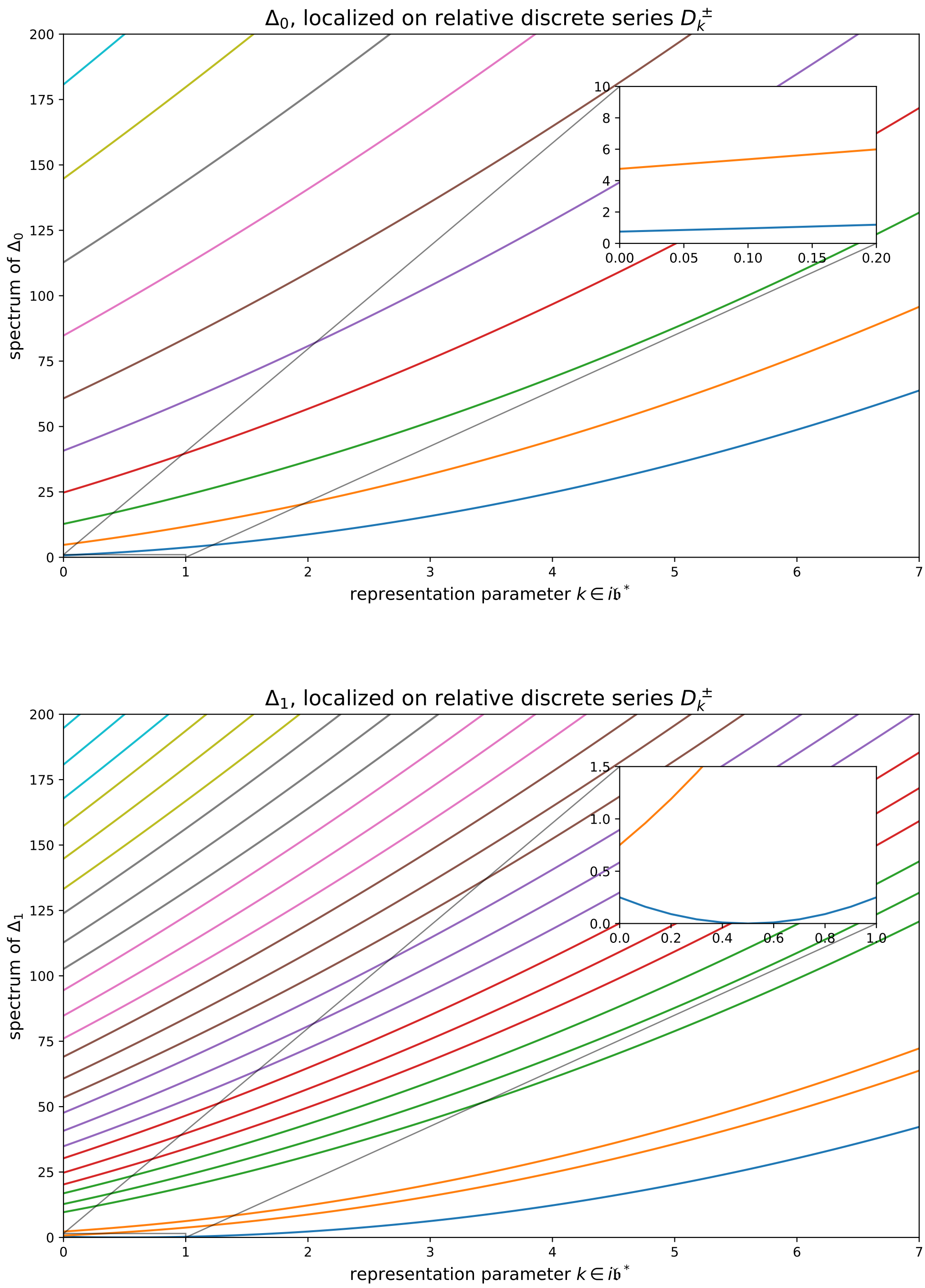


Figure 6.1: Hodge Laplacian on the relative discrete series of $\widetilde{SL_2}(\mathbb{R})$

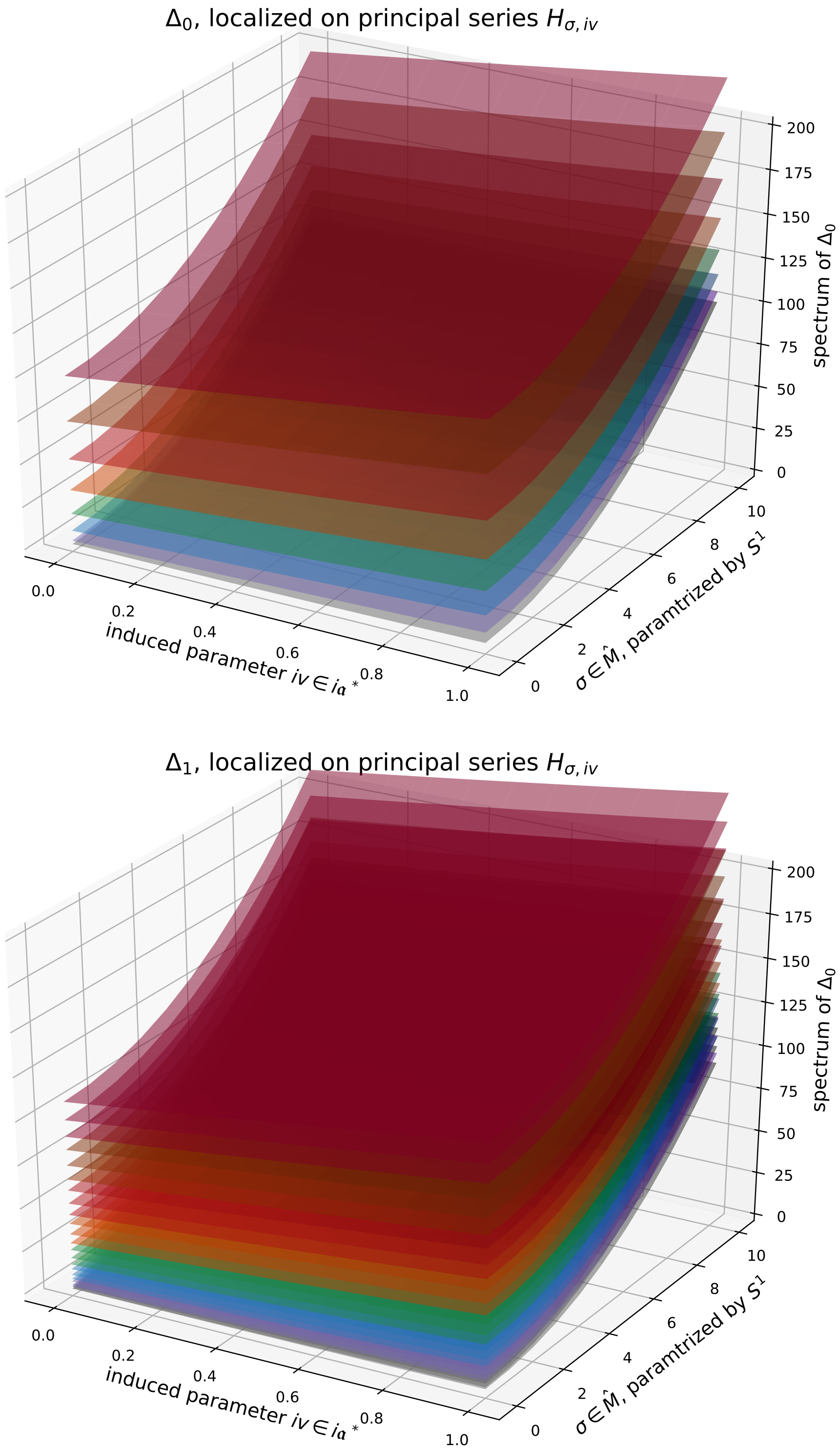


Figure 6.2: Hodge Laplacian on the principal series of $\widetilde{SL_2(\mathbb{R})}$

Chapter 7

Applications to twisted L^2–invariants

In this chapter we apply the spectral data to the computation of various topological invariants. A major theme of this chapter is the computation of various (twisted) L^2-invariants via analytical methods. To maintain our discussion self-sufficient, we include a quick introduction solely from the analytical viewpoint, almost completely omitting their topological counterparts. To keep the discussion even leaner, we will exploit the nice geometry of homogeneous spaces, defining away certain analytical technicalities.

Towards the second part of the chapter, we shall compute the L^2-invariants for noncompact symmetric spaces, and for $\widetilde{SL_2}(\mathbb{R})$. The first application is classical and is seen in many places, e.g. [**Olb02**] for the untwisted case, and [**MP13**] and [**BV13**] for the twisted L^2-torsion. Our computation is meant to extend the methods to the twisted Novikov-Shubin invariants in this case. We will correct some minor errors in the formulae of the existing literature in Remark 7.16.

The second application is more novel. The results have been stated correctly earlier by [**LL95**] in their paper of computing L^2-invariants of 3-manifolds, but without proof.

We will stick to the notation of Chapter 5 to denote all the objects occurring in the universal cover and homogeneous spaces without tilde. In seldom cases objects from locally homogeneous spaces are defined, and they are decorated with Γ to stress their habitations in the quotient.

7.1 Analytic L^2-invariants of homogeneous spaces: A primer

We assume throughout this section some familiarities with the L^2-cohomology and the Hilbert modules over group von Neumann algebras. The introduction in [**Lüc02**, Chapter 1.1] would be more than sufficient.

Given Γ a discrete group, the **group von Neumann algebra** $\mathcal{N}(\Gamma)$ is defined to be the algebra of G-equivariant bounded operators on $\ell^2(\Gamma)$. There is a dimension function for arbitrary $\mathcal{N}(\Gamma)$-modules, which, when restricts to the finitely generated Hilbert modules, agrees with the von Neumann dimension defined via the canonical trace (see [**Lüc02**, Theorem 6.7]). We denote this dimension by $\dim_{\mathcal{N}(\Gamma)}$.

In this section we give an introduction of twisted L^2-invariants of a uniform lattice Γ in a Lie group G, that is Γ is a discrete subgroup of G and $\Gamma\backslash G$ is compact. Given a finite dimensional representation $\rho : G \to GL(V)$, its restriction to Γ gives a finite dimensional representation of Γ.

Given a closed compact Lie subgroup L of G as in Section 4.1, we assume furthermore that the G/L is simply connected. This is the case whenever G is locally compact, G/G^0 is compact. In this case the maximal compact subgroup K has the same homotopy type as G [**Ant12**, Corollary 1.4]. Another example includes those simply connected Lie groups such as $SU(2)$, $SL_2(\mathbb{C})$ etc.

Let D be a strongly elliptic G-invariant differential operator on G/L with coefficients in V. Following the discussion in Appendix A.1 its closure generates a holomorphic semigroup e^{-tD} with the corresponding kernel $K_t^D(x, y)$. Again by exploiting the L-equivariance (4.7), we may take its counterpart $K_t^\rho(x, y)$ as a bi-L-equivariant function k_t^ρ on G:

$$k_t^\rho(g) : G \to \mathbb{C} \qquad k_t^\rho(g) = \rho(a)k_t^\rho(a^{-1}gb)\rho(b)^{-1} \tag{7.1}$$

for all $g \in G$ and $a, b \in L$. From Theorem A.4 k_t^ρ is defined at e_G for all $t > 0$.

Suppose now the homogeneous vector bundle $G \times_L V \to G/L$ is furnished with a Γ-invariant metric $\langle -, - \rangle$. Using the construction of the homogeneous vector bundle in Section 4.1, we first construct a G-invariant metric and metric connection on $G \times_L V$, and its restriction to Γ gives automatically such a metric. Assuming further D to be positive self-adjoint, we denote:

$$\operatorname{tr}_{\mathcal{N}(\Gamma)} e^{-tD} := \int_{\mathcal{F}} K_t^D(x, x)\,\mathrm{d}x = \operatorname{vol}(\mathcal{F}) \cdot k_t^\rho(e_G) \tag{7.2}$$

where $\mathcal{F}$ the fundamental domain of the Γ-action on G/L. This is essentially [**Ati76**, Proposition 4.16]. Readers with knowledge in von Neumann algebra should not have problem identifying this with the von Neumann trace.

In fact, as the operator e^{-tD} commutes with the spectral projections, we can refine it using the Borel functional calculus. This is a family of spectral projection $\{E_\lambda^D\}_{\lambda \in \mathbb{R}}$ attached to D, which is uniquely determined by the property $D = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}E_\lambda^D$ (See [**Lüc02**, Section 1.4.1] for a quick survey). Now define the **spectral density function** $F^D : [0, \infty) \to [0, \infty]$ to be the function:

$$F^D(\sqrt{\lambda}) = \operatorname{tr}_{\mathcal{N}(\Gamma)}(E_\lambda^D D E_\lambda^D) \tag{7.3}$$

which measures the size of subspaces with spectrum $\leq \lambda$.

Again we assume $\mathfrak{p} = \mathfrak{g}/\mathfrak{l}$ is closed under the Ad L-action. Hence the G-action on the bundle induces a G-action on the differential forms $C_{(2)}^i(G/L; V) \cong [L^2(G) \otimes \wedge^i(\mathfrak{g}/\mathfrak{l}) \otimes V]^L$, the isomorphism given by $\mathcal{A}$ in (4.5). This endows the chain complex with a $\mathcal{N}(\Gamma)$-module structure. We remark that F^D is upper semi-continuous and monotonically nondecreasing for all $\lambda \geq 0$.

The Hodge Laplacian $\Delta_{p,V}$ on the chain complex $C_{(2)}^p(G/L; V)$ defines a strongly elliptic operator on the Hilbert $\mathcal{N}(\Gamma)$-module $C_{(2)}^p(G/L; V)$. We can define the analytic L^2-invariants now:

Definition 7.1. Define **analytic L^2-Betti numbers** $b_{(2)}^p(\Gamma; V)$ of $\Gamma \backslash G/L$ with coefficients in V as the following limit:

$$b_{(2)}^p(\Gamma; V) := \lim_{t \to \infty} \operatorname{tr}_{\mathcal{N}(\Gamma)}(e^{-t\Delta_{p,V}}) = F^{\Delta_p}(0) = \operatorname{vol}(\mathcal{F}) \cdot \lim_{t \to \infty} k_t^{p,V}(e_G) \in [0, \infty] \tag{7.4}$$

where $\mathcal{F}$ is the fundamental domain of $\Gamma \backslash G/L$ in G/L.

Next we define the analytic Novikov-Shubin invariant of G/L in greater generality. First define a quantity that measures the density of the spectrum of D as $\lambda \to 0$:

$$\alpha_D(\Gamma; V) := \liminf_{\lambda \to 0^+} \frac{\ln(F^D(\lambda) - F^D(0))}{\ln(\lambda)} \in [0, \infty] \tag{7.5}$$

Now denote $\Delta'_{p,V} := \Delta_{p,V}|_{\ker \Delta_{p,V}^\perp}$, restricting $\Delta_{p,V}$ to the orthogonal complement $(\ker \Delta_{p,V})^\perp$ of the closed L^2-forms in $C_{(2)}^p(G/L; V)$; as well as $\Delta_{p,V}^c := (\delta_{p+1,V} d_{p,V})'$ the restriction to all coclosed forms in $(\ker \Delta_{p,V})^\perp$.

Definition 7.2. The **analytic pth-Novikov-Shubin invariant** $\alpha_p(\Gamma; V)$ is defined to be:

$$\alpha_p(\Gamma; V) := \begin{cases} \sup\left\{\beta \,\middle|\, \operatorname{tr}_{\mathcal{N}(\Gamma)} e^{-t\Delta_p^c} = \mathcal{O}(t^{-\beta/2}) \text{ as } t \to \infty\right\} & \text{if } 0 \in \operatorname{spec} \Delta_p \\ \infty^+ \text{ if otherwise} \end{cases} \tag{7.6}$$

and define $\alpha_p^\Delta(\Gamma; V)$ by replacing $\Delta_{p,V}^c$ in the definition by $\Delta'_{p,V}$. It is also worth noting that two expressions of $\alpha_p(\Gamma; V) = \alpha_{\Delta_p^c}(\Gamma; V)$ are equal. This follows also readily from the fact we can interpret the trace of the heat kernel $\operatorname{tr}_{\mathcal{N}\Gamma} e^{-tD}$ as the Laplace transform of the spectral density function [**Lüc02**, Theorem 3.136]. As a consequence $\alpha_p(\Gamma; V) = \alpha_{\Delta_p^c}(\Gamma; V)$ and $\alpha_p^\Delta(\Gamma; V) = \alpha_{\Delta'_p}(\Gamma; V)$.

The ∞^+-convention is added to make $\alpha_p(\Gamma; V)$ more compatible with their topological counterparts. Refer to [Lüc02, Definition 2.8 §Theorem 2.55] for relevant discussion when $V = \mathbb{C}$. We remark that α_*^Δ and α_* are related by the following identity:

$$\alpha_p^\Delta(\Gamma; V) = \frac{1}{2} \min \{\alpha_p(\Gamma; V), \alpha_{p-1}(\Gamma; V)\} \tag{7.7}$$

This follows from an easy adaptation of the proof in the untwisted case: The arguments are purely functional analytic, for which reason the arguments of [Lüc02, Lemma 2.11 & 2.66] carry over.

Before defining the twisted L^2-torsion, we recall the definition that D is called of determinant class if the associate spectral density function dF^D satisfies $\int_{0^+}^\infty \ln(\lambda)\, \mathrm{d}F^D > -\infty$. This allows the definition of the **Fuglede-Kadison determinant**:

$$\det{}_{\mathcal{N}(\Gamma)}(D) := \exp\left(\int_{0^+}^\infty \ln(\lambda)\, \mathrm{d}F^D\right) \in [0, \infty) \tag{7.8}$$

note that the information of the underlying lattice Γ is encoded in the spectral density function F^D.

Remark 7.3. We remark the fact that D is of determinant class if the respective $\alpha_D(\Gamma; V)$ is positive. To prove this, we assume, without loss of generality, that $F^D(0) = 0$. Suppose $\alpha_p(\Gamma; V)$ is finite, then $F^D(\lambda) \leq C_t e^{t\lambda}$ for all $t \geq 0$. Hence a simple integration by parts gives:

$$\int_{0^+}^\infty \ln(\lambda)\, \mathrm{d}F^D = \ln(\lambda) F^D(\lambda)\big|_0^\infty - \int_0^\infty \frac{F^D(\lambda)}{\lambda}\, \mathrm{d}\lambda \tag{7.9}$$

Note that the first term is positive as $\ln(\lambda)F^D(\lambda)$ is monotonically nondecreasing, whereas the second term is finite as $e^{t\lambda}/\lambda$ is integrable over $[0, \infty)$.

We are now ready to define the twisted L^2-torsion:

Definition 7.4. Suppose $F^{\Delta_p^c}$ or $F^{\Delta_p'}$ is of determinant class for all p. Then we define the **analytic L^2-torsion with coefficients in V** as:

$$\rho_{\mathrm{an}}^{(2)}(\Gamma; V) := \frac{1}{2} \sum_{p=0}^n (-1)^p \log \det{}_{\mathcal{N}(\Gamma)}(\Delta_{p,V}^c) = \frac{1}{2} \sum_{p=0}^n (-1)^{p+1} p \log \det{}_{\mathcal{N}(\Gamma)}(\Delta_{p,V}') \tag{7.10}$$

Here each $\log\det_{\mathcal{N}(\Gamma)}(D)$ admits an integral expression:

$$\log \det{}_{\mathcal{N}(\Gamma)}(D) := \frac{d}{ds}\Big|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^\epsilon \mathrm{tr}_{\mathcal{N}(\Gamma)}\, e^{-tD} t^{s-1} dt\right) + \int_\epsilon^\infty \mathrm{tr}_{\mathcal{N}(\Gamma)}\, e^{-tD} t^{-1} dt \tag{7.11}$$

for $D = \Delta_p^c$ or Δ_p'.

The first integral could be interpreted as a function in s which is analytic on $\mathrm{Re}(s) > \frac{\dim(G/L)}{2}$ and admits a meromorphic extension to the whole complex plane with no pole at $s = 0$. In fact the above definition should be interpreted as a Mellin transform, which is a well-known technique for number theorist in transferring between theta series and zeta series, but is less known to topologists. We recall below some salient fact from Zagier's excellent appendix [Zag06]:

Lemma 7.5. Let $\varphi(t)$ be a function of rapid decay at infinity, which admits an asymptotic expansion for large t

$$\varphi(t) \sim \sum_{j=1}^\infty a_j t^{-\alpha_j} \quad \text{as } t \to \infty \tag{7.12}$$

such that $\alpha_j \to \infty$ as $j \to \infty$, then the Mellin transform:

$$\mathcal{M}(\varphi)(s) := \int_0^\infty \varphi(t) t^{s-1}\, \mathrm{d}t \tag{7.13}$$

is defined in a half-plane $\mathrm{Re}(s) > \min_j \mathrm{Re}(\alpha_j)$ analytically which has a meromorphic extension to the complex plane with simple poles of residue a_j at $s = -\alpha_j$ and no other poles.

By this lemma we see immediately for D with $\alpha_D(\Gamma; V) > 0$ that the Mellin transform exists, with the residues lying on the negative real half-line $s \leq -\alpha_D(\Gamma; V)$. In this case, the $\log\det_{\mathcal{N}(\Gamma)}(D)$ admits a simplified expression:

$$\log\det_{\mathcal{N}(\Gamma)}(D) = \frac{d}{ds}\Big|_{s=0}\left(\frac{1}{\Gamma(s)}\int_0^\infty \operatorname{tr}_{\mathcal{N}(\Gamma)} e^{-tD} t^{s-1} dt\right) \tag{7.14}$$

In our application this is indeed the case.

7.2 Twisted analytic L^2-invariants of symmetric spaces of noncompact type

We now compute the twisted L^2-invariants of symmetric spaces of noncompact type. The highlight of this section is Theorem 7.8.

Briefly recall the content from Section 4.3: Let G/K be a symmetric space of noncompact type, with G a real linear semisimple Lie group of noncompact type and K its maximal compact subgroup. The Killing form $B(-,-)$ restricted to $T_{e_G K} G/K \cong \mathfrak{p}$ is positive definite. Moreover, we twist the bundle with a finite-dimensional G-representation ρ, together with an **admissible metric** $\langle -,- \rangle$ satisfying:

1. $\langle \rho(X)v, w\rangle = -\langle v, \rho(X)w\rangle$ for all $X \in \mathfrak{k}$;
2. $\langle \rho(X)v, w\rangle = \langle v, \rho(X)w\rangle$ for all $X \in \mathfrak{p}$.

By Remark 4.3 this defines a G-invariant metric connection. As we remarked in Section 4.3, this connection is also reductive. The isomorphism $\mathcal{A}$ from (4.5) identifies the sections of p-forms with $C^\infty(G; \wedge^p \operatorname{coad} \otimes \rho)$. Under this isomorphism, the connection Laplacian admits a simple expression (4.51) and the twisted Laplacian Δ_p can be derived from **Kuga's Lemma** [**BW00**, Theorem II.2.5(iii)]:

$$\Delta_p^\rho f = -R(\Omega_G)f + \rho(\Omega_G)f \quad f \in C^\infty(G; \wedge^p \operatorname{coad} \otimes \rho) \tag{7.15}$$

where $\Delta_p^\rho = \mathcal{A}(\Delta_{p,V})$. In particular, the twisted Laplacian differs from the right regular representation by a scalar operator. Now following the argument of Chapter 5 yields immediately the heat kernel $k_t^\rho(x)$ associated with any finite-dimensional representation ρ is of L^p-Schwartz function on G for all $p > 0$. The steps here are much simplified: There is no first-order perturbation here, and since we are acting on K-invariant vectors, we see from (7.15) that:

$$\begin{aligned}\Delta_p^\rho f &= -R(\overline{\Omega}_G)f + 2R(\Omega_K)f + \rho(\Omega_G)f \\ &= -R(\overline{\Omega}_G)f + (\wedge^p \operatorname{coad} \otimes \rho)(\Omega_K)f + \rho(\Omega_G)f\end{aligned}$$

where the second and third can be decomposed into finitely many scalar operators as both $\wedge^p \operatorname{coad} \otimes \rho$ and ρ are finite-dimensional. Hence we may directly apply Lemma 5.1 to obtain asymptotic estimates of $-R(\overline{\Omega}_G)f$. Now directly applying (5.22) proves the claim. Now one applies the Plancherel decomposition Theorem 2.25:

$$k_t^{p,\rho}(e_G) = \int_{\pi\in\widehat{G}} \dim \pi \Theta_\pi(k_t^{p,\rho})\, d\pi = \int_{\pi\in\widehat{G}} e^{-t(\rho(\Omega_G)-\pi(\Omega_G))} \dim[H_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\rho]^K\, d\pi \tag{7.16}$$

We first compute the L^2-Betti numbers $b_{(2)}^p(\Gamma; V)$. All the cohomology classes are represented by L^2-forms, hence from a classical result [**Kna86**, Proposition 9.6] are contributed solely by the discrete series $\widehat{G}_d$. A closer scrutiny of the formula above gives:

$$k_t^{p,\rho}(e_G) = \sum_{\pi\in\widehat{G}_d} e^{-t(\rho(\Omega_G)-\pi(\Omega_G))} d_\pi \dim_{\mathbb{C}} \operatorname{Hom}_K(H_\pi, \Lambda^p\mathfrak{p} \otimes V_\rho) + \int_{\pi\in\widehat{G}\setminus\widehat{G}_d} \Theta_\pi(k_t^{p,\rho})\, d\pi \tag{7.17}$$

where discrete sum is the only part contributing to the L^2-eigenforms, with $d_\pi = d\pi(D_\pi)$ the **formal degree** of a discrete series D_π. Hence one has:

$$\dim_{\mathbb{C}}\begin{pmatrix} L^2\text{-eigenforms of } \Delta_{p,V} \\ \text{with eigenvalue } \lambda \end{pmatrix} = \sum_{\substack{\pi\in\widehat{G}_d \\ \pi(\Omega_G)=\rho(\Omega_G)-\lambda}} d_\pi \cdot \dim_{\mathbb{C}} \mathrm{Hom}_K(H_\pi, \Lambda^p\mathfrak{p}\otimes V_\rho^*) \tag{7.18}$$

where V_ρ^* is the contragredient representation of V_ρ. Note that this sum is finite, from the fact that each K-type μ occurs only in finitely many discrete series [Kna86, Corollary 12.22]. In fact, in principle one can compute the dimension for each λ explicitly, by means of **Blattner's formula** [Wal92, 6.5.4], though in practice (especially in higher-rank cases) this would be un-manageable for humans and can only be done by appealing to softwares such as ATLAS, which on the other hand has size limitations.

The eigenvalue 0-case on the other hand, can be handled via cohomological means. Here the cochain complex is constructed as in Chapter 1, by restricting the $\mathfrak{g}$-modules $C^q(\mathfrak{g},\mathfrak{k};V)$ to the $\mathfrak{k}$-invariant subspaces $\mathrm{Hom}_{\mathfrak{k}}(\Lambda^p(\mathfrak{p}),V)$. We refer the readers to [BW00, Section I.1.2] for more details. We recollect the results here for convenience:

Proposition 7.6 ([BW00, Proposition II.3.1, Corollary II.3.2])**.** *Assume that $\sigma(\Omega) = s\cdot \mathrm{id}_{H_\pi}$ and $\rho(\Omega) = r\cdot \mathrm{id}_{V_\rho}$, (This is in particular the case when H_π is an irreducible admissible representation and V_ρ is furnished with a $\mathfrak{g}$-invariant metric), then:*

1. *If $r\neq s$, then $H^q(\mathfrak{g},\mathfrak{k};H_\pi\otimes V_\rho)=0$ for all q's;*
2. *If $r=s$, then all cochains $C^q(\mathfrak{g},\mathfrak{k};V)$ are closed, harmonic, and we have:*

$$H^q(\mathfrak{g},\mathfrak{k};H_\pi\otimes V_\rho) = C^q(\mathfrak{g},\mathfrak{k};H_\pi\otimes V_\rho^*) = \mathrm{Hom}_{\mathfrak{k}}(\Lambda^p\mathfrak{p}\otimes V_\rho, H_\pi) \qquad \textit{for all } q\text{'s}$$

So in our case it suffices to consider those $\pi\in\widehat{G}_d$ with non-trivial $(\mathfrak{g};K)$-cohomology. To this end we give a more general result regarding all cuspidal representations, that we will use again to compute the Novikov-Shubin invariants later. Recall the construction of H-series in Section 2.3. Fix a cuspidal parabolic subgroup $P=MAN$ and associate a tempered series $\pi_{\eta,i\nu}$ for some $\eta\in\widehat{M}_d$ with the infinitesimal character χ_η and $\nu\in\mathfrak{a}^*$. Denote the representation space as $H^{\eta,i\nu}$. We consider the subset:

$$W^P := \{w\in W(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) \mid w^{-1}(\alpha)>0 \text{ for all } \alpha\in\Delta(\mathfrak{m}_{\mathbb{C}};\mathfrak{t}_{\mathbb{C}})\} \tag{7.19}$$

This is a set of representatives of the right coset $W(\mathfrak{m}_{\mathbb{C}};\mathfrak{t}_{\mathbb{C}})\backslash W$.

Theorem 7.7. [BW00, Theorem III.5.1] *Let $(\pi_{\eta,i\nu},H^{\eta,i\nu})$ be the standard induced representation. Set $\mathrm{frk}(G)=\mathrm{rank}_{\mathbb{C}}G-\mathrm{rank}_{\mathbb{C}}K$ and $n=\dim_{\mathbb{R}}G-\dim_{\mathbb{R}}K$. Then $H^*(\mathfrak{g},K;H_K^{\eta,i\nu}\otimes V_\rho)\neq\{0\}$ only if P is fundamental, $\nu=0$, and there exists a $w\in W^P$ of length $\dim N_P/2$ such that:*

$$-\delta_G+\rho = w\cdot(\lambda_\rho+i\nu) \tag{7.20}$$

i.e., the infinitesimal character of V_ρ and $\pi_{\eta,i\nu}$ are lying in the same W^P-orbit. Then:

$$\dim H^p(\mathfrak{g},K;H_K^{\eta,0}\otimes V_\rho) = \begin{cases} \binom{\mathrm{frk}(G)}{p-\frac{n-\mathrm{frk}(G)}{2}} & \textit{if } p\in[\frac{n-\mathrm{frk}(G)}{2},\frac{n+\mathrm{frk}(G)}{2}] \\ 0 & \textit{if otherwise.}\end{cases} \tag{7.21}$$

We conclude this section by stating the main result of this chapter. Some terms in the theorem will be explained in the course of the proof in the following section.

Theorem 7.8 (Twisted L^2-invariant of symmetric spaces)**.** *Set $n:=\dim G-\dim K$ and $\mathrm{frk}(G):=\mathrm{rank}_{\mathbb{C}}(G)-\mathrm{rank}_{\mathbb{C}}(K)$ the fundamental rank of G. Let V_ρ be a finite-dimensional G-representation with an admissible metric. Then we fix the associated homogeneous vector bundle $G\times_K V_\rho\to G/K$ and Γ a uniform lattice of G. Denote $\mathfrak{g}^c$ the (unique) compact form associated with $\mathfrak{g}_{\mathbb{C}}$ with the corresponding compact Lie group G^c with K as its fixed points of the Cartan involution. Then the twisted L^2-invariants of G/K are given by:*

I $b_p^{(2)}(\Gamma; V_\rho) \neq 0$ *if and only if* $\mathrm{frk}(G) = 0$ *and* $p = \frac{n}{2}$. *In this case,*

$$b_{\frac{n}{2}}^{(2)}(\Gamma; V_\rho) = (-1)^{\frac{n}{2}} \chi(\Gamma; V_\rho) = \frac{\mathrm{vol}(\Gamma\backslash G/K)}{\mathrm{vol}(G^c/K)} \chi(G^c/K) \cdot \dim V_\rho \tag{7.22}$$

II $\alpha_p(\Gamma, V_\rho) \neq \infty^+$ *if and only if* $m = \mathrm{frk}(G) > 0$ *and* $p \in [\frac{n-m}{2} + 1, \frac{n+m}{2}]$. *Within this range,*

$$\alpha_p(\Gamma, V_\rho) = m \tag{7.23}$$

III $\rho_{\mathrm{an}}^{(2)}(\Gamma, V_\rho) \neq 0$ *if and only if* $\mathrm{frk}(G) = 1$.

IV *Suppose further* $\mathrm{frk}(G) = 1$, *then* $G = G_0 \times G_1$, *where* G_0 *is a equirank semisimple Lie group, i.e., when* $\mathrm{frk}(G_0) = 0$ *and* $G_1 = \mathrm{Spin}(2p+1, 2q+1)$ *with* $p > q \in \mathbb{N}_{\geq 0}$ *or* $G = SL(3, \mathbb{R})$. *Moreover, its maximal compact subgroup* K *and the lattice can be written in a product form:* $K_0 \times K_1$ *and the lattice* $\Gamma = \Gamma_0 \times \Gamma_1$ *with* $\Gamma_i \subseteq G_i$ *a uniform lattice for* $i = 0, 1$. *In this case, the* G*-representation* ρ *can be written as* $\rho \cong \rho^{G_0} \times \rho^{G_1}$ *with* ρ^{G_i} *a representation of* G_i. *The* L^2*-torsion of* G/K *is correlated to the volume of* $\Gamma\backslash G/K$ *by a constant* $C^{(2)}(G; V_\rho)$ *depending on* G *and* V_ρ:

$$\rho_{\mathrm{an}}^{(2)}(\Gamma; V_\rho) = \mathrm{vol}(\Gamma\backslash G/K) \cdot \dim V_\rho \cdot C^{(2)}(G; V_\rho) \tag{7.24}$$

where $C^{(2)}(G; V_\rho)$ *in this case admits the following form:*

$$C^{(2)}(G; V_\rho) := (-1)^{\frac{\dim G - \dim K}{2}} \cdot \frac{\chi(G_0^c)}{\mathrm{vol}(G_0^c/K_0)} \cdot C^{(2)}(G_1; V_{\rho^{G^1}}) \tag{7.25}$$

with G_i^c *the analytic group corresponding to the compact real form of* $\mathfrak{g}_{i,\mathbb{C}}$. *Moreover, if we fix the Haar measure on* G *as in Remark 7.9, then* $C^{(2)}(G_1; V_{\rho^{G^1}})$ *is explicitly given by:*

(1) *For the* $\mathrm{Spin}(2p+1, 2q+1)$*-representation* ρ^{G^1} *with highest weight* $\Lambda_\rho = \sum_{i=1}^{p+q+1} \rho_i e_i$ *as in* (7.51):

$$C^{(2)}(\mathrm{Spin}(2p+1, 2q+1); V_{\rho_{G_1}}) = \frac{2\pi(-1)^{p+q}}{\mathrm{vol}(G^c/K)} \binom{p+q}{p} \sum_{k=0}^{n} \int_0^{c_{\rho,k}} \prod_{j=0, j\neq k}^{n} \frac{\nu^2 - c_{\rho,s_j}^2}{c_{\rho,s_k}^2 - c_{\rho,s_j}^2} \, \mathrm{d}\nu \tag{7.26}$$

with $c_{\rho,s_j} = \rho_{j+1} + n - k$ *and* $\|\cdot\|$ *the norm induced by the Killing form on* $\mathfrak{a}^*$.

(2) *For the* $SL(3, \mathbb{R})$*-representation* ρ^{G^1} *with highest weight* $\Lambda_\rho = \rho_1\omega_1 + \rho_2\omega_2$ *parametrized by fundamental weights:*

$$C^{(2)}(SL(3, \mathbb{R})/SO(3)) = \frac{\pi}{2 \cdot \mathrm{vol}(G^c/K)} D_{\rho_1,\rho_2} \tag{7.27}$$

where $D_{\rho_1,\rho_2} = \frac{C_{\rho,1}C_{\rho,3}}{A_{\rho,2}} + \frac{C_{\rho,2}C_{\rho,3}}{A_{\rho,1}}$ *if* $\rho_1 \geq \rho_2$, *and equals to* $\frac{C_{\rho,1}C_{\rho,3}}{A_{\rho,2}} - \frac{C_{\rho,2}C_{\rho,3}}{A_{\rho,3}}$ *otherwise, with* $A_{\rho,i}s$ *and* $C_{\rho,i}s$ *linear functions in* ρ_1, ρ_2 *as given by* (7.63).

7.3 Proof of Theorem 7.8

Having made the necessary preparations, let us compute the L^2-Betti numbers first:

Proof of Theorem 7.8, part I. Apply Theorem 7.7 to the equirank case, i.e., when $m = 0$, then we see immediately that the fundamental series are the discrete series, and the cohomology is only non-vanishing in the middle degree $\frac{d}{2} = \frac{\dim G/K}{2}$. In this case,

$$\dim H^{\frac{d}{2}}(\mathfrak{g}, K; H_K^{\eta,0} \otimes V_\rho) = \binom{m}{0} = 1$$

Moreover, $W^P = W^G = 1$ as $W(\mathfrak{m}_\mathbb{C};\mathfrak{t}_\mathbb{C}) = W$ in this case. In particular, we have recovered [**BW00**, Theorem II.5.3]. So now it suffices to evaluate the number of representations $\pi \in \widehat{G}_d$ such that π has the same infinitesimal character as ρ. This is again due to Harish-Chandra (see [**Kna86**, Theorem 9.20]). There are $W(G,H)/W(K,H)$-many such discrete series.

Now the formal degree d_π in (7.18) can be interpreted via the Weyl dimension formula as

$$d_\pi = \frac{\mathrm{vol}(\exp(i\mathfrak{p}))}{\chi(\exp(i\mathfrak{p}))} \dim V_\rho \tag{7.28}$$

(see e.g. the argument in [**Olb02**, Corollary 5.2]) where $\exp(i\mathfrak{p})$ is the compact dual space of G/K, corresponding to the compact real form G_c associated with $G_\mathbb{C}$, quotient by K. Note that $\exp(i\mathfrak{p})$ is a compact symmetric space, hence has finite volume, with its prescribed metric induced by the Killing form.

Summing up, we see the L^2-Betti number $b^p_{(2)}(\Gamma;V) \neq 0$ only when $p = \frac{d}{2}$ where $d = \dim G - \dim K$ and $\mathrm{rank}_\mathbb{C}\, G = \mathrm{rank}_\mathbb{C}\, K$. It equals to

$$b^{d/2}_{(2)}(\Gamma;V_\rho) = \mathrm{vol}(\mathcal{F}) \cdot \frac{\chi(\exp(i\mathfrak{p}))}{\mathrm{vol}(\exp(i\mathfrak{p}))} \cdot \dim V_\rho \tag{7.29}$$

If we take G^c as the compact real form of $G_\mathbb{C}$, then $\exp(i\mathfrak{p})$ is the compact dual space of G^c/K. □

Let us now move on to compute the twisted Novikov-Shubin invariants. To this end we need a more refined formula than (7.16). Before deploying the Plancherel formula, it is vital to discuss the normalization of the Haar measure again:

Remark 7.9 (Normalization of Haar measure)**.** As briefly mentioned in Remark 2.22, in the following discussions we normalize the K-volume to be one, whereas the measure on $\mathfrak{p}$ and therefore on $\mathfrak{a}_\mathfrak{j}$ remains the same as induced by the Killing form. This normalization was also taken up by [**BV13**], with minor alterations.

Müller and Pfaff follow largely the same normalization, with a minor difference in the measure on $\mathfrak{a}_H$ to make the restricted root of length one. This accounts for the difference in the constant $c_{G/K}$ appearing in the Plancherel density (7.38). Overall our normalization here differs from that by Harish-Chandra in two aspects:

1. The Haar measure in [**HC75**, Lemma 37.2] was normalized by a factor of $2^{\frac{d-\mathrm{frk}(G)}{2}}$;
2. Our normalization of the metric on $i\mathfrak{p}$ differed from the original by a factor of $(2\pi)^{\mathrm{frk}(G)}$ (see the proof of [**HC75**, Lemma 37.1]), this accounts for another factor of $(2\pi)^{\mathrm{frk}(G)}$ in our case.

Now it suffices to consider only the tempered series of G. In view of Section 2.3, we can rewrite $\widehat{G}_{\mathrm{temp}}$ as a disjoint union of H-series representations, parametrized by the cuspidal parabolic data as follows:

Lemma 7.10. Let G be a linear connected reductive Lie group of Harish-Chandra class. Following the notation of Section 2.5, the heat kernel $k_t^{p,\rho}(e_G)$ admits the following Fourier expansion:

$$\frac{1}{\mathrm{vol}(\mathcal{F})} k_t^{p,\rho}(e_G) = \sum_{J\in \mathrm{Car}(G)} \sum_{\eta \in \widehat{M}_d} \int_{\mathfrak{j}_\mathfrak{p}{}^*} e^{-tc_{\eta,\rho,\nu}} \dim[H^{\eta,i\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho]^K m^J(\eta : i\nu) d\nu \tag{7.30}$$

with constants

$$c_{\eta,\rho,\nu} := \|\nu\|^2 + \chi_\rho(\Omega_G) - \chi_\eta(\Omega_M) + \|\delta_{\mathfrak{j}_P}\|^2 \tag{7.31}$$

Where $\delta_{\mathfrak{j}_P} = \frac{1}{2}\sum_{\beta\in\Delta^+(\mathfrak{g}:\mathfrak{j}_P)} \beta$ is the half-sum of restricted positive $\mathfrak{j}_P$-roots. Moreover, the kernel $k_t^{p,\rho,c}$ associated with $\Delta^c_{p,\rho}$ admits a similar formula:

$$\frac{1}{\mathrm{vol}(\mathcal{F})} k_t^{p,\rho,c}(e_G) = \sum_{J\in \mathrm{Car}(G)} \sum_{\eta \in \widehat{M}_d} \int_{\mathfrak{j}_P^*} e^{-tc_{\eta,\rho,\nu}} \dim[\mathrm{Im}\,\mathrm{Proj}^c_{p,(\lambda,i\nu)}] m^J(\eta : i\nu) d\nu \tag{7.32}$$

where

$$\mathrm{Proj}^c_{p,\pi} : [H^*_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho]^K \to \ker\left(d^p_\pi : [H^*_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho]^K \to [H^*_\pi \otimes \Lambda^{p+1}\mathfrak{p}^* \otimes V_\rho]^K\right)^\perp$$

the canonical projection onto the orthogonal complement of the kernel.

Proof. We abbreviate the discrete series parameters of M as η to simplify the exposition. The first form (7.30) is straightforward from (2.22) by noting $G = Z_G(G^0)G^0$ and by using the induced (infinitesimal) character formula from Proposition 2.21. Denote $\lambda_\pi \in \mathfrak{h}_{\mathbb{C}}^*$ for the infinitesimal character of π:

$$\begin{aligned}\chi_{\eta,i\nu}(\Omega_G) - \chi_\rho(\Omega_G) &= -\|\nu\|^2 + \|\lambda_\eta\|^2 - \|\delta_G\|^2 - \chi_\rho(\Omega_G) \\ &= \|\lambda_\eta\|^2 - \|\delta_{M_J}\|^2 - \|\delta_{\mathfrak{j}_\mathfrak{p}}\|^2 - \|\nu\|^2 - \chi_\rho(\Omega_G) \\ &= \chi_\eta(\Omega_M) - \|\delta_{\mathfrak{j}_\mathfrak{p}}\|^2 - \|\nu\|^2 - \chi_\rho(\Omega_G) = -c_{\eta,\rho,\nu}\end{aligned} \tag{7.33}$$

The second identity comes from the orthogonality of $\mathfrak{j}_K$ (the Cartan subalgebra of $\mathfrak{m}_J$) and $\mathfrak{j}_P$. For the second formula we first note that the following diagram commutes:

$$\begin{array}{ccc} (\ker d^{p,V})^\perp & \xrightarrow{\quad\Delta_p^c\quad} & (\ker d^{p,V})^\perp \\ \mathcal{A}\big\uparrow\!\!\downarrow & & \big\uparrow\!\!\downarrow\mathcal{A} \\ (L^2(G) \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho)^K & \xrightarrow[-R(\Omega_G)+\rho(\Omega_G)]{} & (L^2(G) \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho)^K \end{array} \tag{7.34}$$

from which:

$$\frac{1}{\mathrm{vol}(\mathcal{F})} k_t^{p,\rho,c}(e_G) = \int_{\widehat{G}} \Theta_\pi(k_t^{p,\rho,c}) d\pi = \int_{\widehat{G}} \Theta_\pi(k_t^{p,\rho}) \operatorname{tr}\mathrm{Proj}_{p,\pi}^c\, m^J(\eta : i\nu) d\mu(\pi) \tag{7.35}$$

and hence the formula (7.32). $\square$

Proof of Theorem 7.8, part II. The Novikov-Shubin invariants can be computed by inspecting the dimension of $[H_\pi^* \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho]^K$ for π a principal series representation, following the dichotomy below:

Case (1): If $H^p(\mathfrak{g}, K, H^{\eta,i\nu} \otimes V_\rho) = 0$ for all $\nu \in \mathfrak{j}_P^*$. This happens in particular when P is not fundamental. In view of Proposition 7.6, this happens when $H^p(\mathfrak{g}, K, H^{\eta,i\nu} \otimes V_\rho) = [H^{\eta,i\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho]^K = 0$, or when $\chi_{\eta,i\nu}(\Omega_G) \neq \chi_\rho(\Omega_G)$.

In the first case, the corresponding integral does not contribute to (7.32), so let us suppose for the moment that $\chi_{\eta,i\nu}(\Omega_G) \neq \chi_\rho(\Omega_G)$ for all ν. But now $\chi_{\eta,i\nu}(\Omega_G) = \|\nu\|^2 + \|\delta_{\mathfrak{j}_P}\|^2 - \chi_\eta(\Omega_M)$ is an polynomial in ν. Hence $(\chi_{\eta,i\nu} - \chi_\rho)(\Omega_G)$ is a polynomial in ν which is positive at some value, and is nonzero for all $\nu \in \mathbb{R}$. Hence,

$$\inf_{\nu\in\mathfrak{a}^*} (\chi_\rho(\Omega_G) - \chi_{\eta,i\nu}(\Omega_G)) > \epsilon > 0 \quad \text{for all } \nu \in \mathfrak{j}_P^*$$

for some $\epsilon > 0$. The respective integrand $e^{-t(\rho(\Omega_G)-\pi(\Omega_G))} \to 0$ decays exponentially fast as $t \to \infty$, which does not contribute to the Novikov-Shubin invariant, implying the respective Novikov-Shubin invariant is ∞^+, if the vanishing result holds for all $J \in \mathrm{Car}(G)$.

Case (2): If $H^p(\mathfrak{g}, K, H^{\eta,i\nu} \otimes V_\rho) \neq 0$ for some $\nu \in \mathfrak{a}^*$. In view of Theorem 7.7 this happens when $\nu = 0, p \in [\frac{n-\mathrm{frk}(G)}{2}, \frac{n+\mathrm{frk}(G)}{2}]$ and η_ρ lies in the W^p-orbit of $\lambda_{\eta,i\nu} \in \mathfrak{h}^*$. Denote this orbit set as Ξ_ρ.

Applying Frobenius reciprocity formula one gets

$$\dim[H^{\eta,i\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho]^K = \dim[H^{\eta,0} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\rho]^K = \dim H^p(\mathfrak{g}, K, H^{\eta,0} \otimes V_\rho) \neq 0,$$

whereas the second equality comes from the Proposition 7.6. So it suffices to compute the cohomology $H^*(\mathfrak{g}, K, H^{\eta,0} \otimes V_\rho)$. But this is the content of Theorem 7.7. Now:

$$\dim[H^{\eta,i\nu} \otimes \Lambda^p\mathfrak{p} \otimes V_\rho]^K = \dim[\mathrm{Im}\,\mathrm{Proj}_{p,(\lambda,i\nu)}^c] + \dim[\mathrm{Im}\,\mathrm{Proj}_{p-1,(\lambda,i\nu)}^c]$$

Hence we can compute $\dim[\mathrm{Im}\,\mathrm{Proj}_{p,(\lambda,i\nu)}^c]$ degree by degree, which yields:

$$\dim[\mathrm{Im}\,\mathrm{Proj}_{p,(\eta,i\nu)}^c] = \begin{cases} \binom{\mathrm{frk}(G)-1}{p-\frac{n-\mathrm{frk}(G)}{2}} & \text{if } p \in [\frac{n-\mathrm{frk}(G)}{2}, \frac{n+\mathrm{frk}(G)}{2}], \eta \in \Xi_\rho \\ 0 & \text{if otherwise} \end{cases} \tag{7.36}$$

First abbreviate $m = \mathrm{frk}(G)$ for readability. To compute the twisted Novikov-Shubin invariant, it suffices to compute the following integral, for $p \in [\frac{n-m}{2}, \frac{n+m}{2}]$:

$$\frac{1}{\mathrm{vol}(\mathcal{F})} k_t^{p,\rho,c}(e_G) \sim_{t\to\infty} \sum_{\eta\in\Xi_\rho} \int_{\mathfrak{a}^*} e^{-tc_{\eta,\rho,\nu}} \binom{m-1}{p-\frac{n-m}{2}} m^H(\eta : i\nu) d\nu + \mathcal{O}(e^{-ct}) \tag{7.37}$$

where H is the maximal compact subgroup of G, corresponding to the fundamental parabolic subgroup with its Levi factor MA. If H is itself compact, i.e., when G and K are of equal rank and $M = G$, we see the above integral is a discrete set, therefore it creates a spectral gap around 0. This forces the Novikov-Shubin invariants to be ∞^+.

For the non-equirank case, we compute $c_{\eta,\rho,\nu}$ first. Note that the non-vanishing cohomology implies in particular $\chi_\rho(\Omega_G) = \chi_{\eta,0}(\Omega_G)$, hence $c_{\eta,\rho,\nu} = \|\nu\|^2$ by (7.31). Moreover, in the case P is fundamental, the fundamental Plancherel density admits the following form [**HC75**, Lemma 27.3]:

$$m^H(\eta : i\nu) = c_{G/K}(-1)^{\dim \mathfrak{n}_H/2} \prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C};\mathfrak{h}_\mathbb{C})} \frac{\langle \alpha, \lambda_\eta + i\nu\rangle}{\langle \alpha, \delta_G\rangle} \tag{7.38}$$

with $c_{G/K}$ only depending on the volume form of G/K. One could compare this with Remark 2.24 for the effect different normalizations have on the expression. We will defer the determination of this constant $c_{G/K}$ to (7.67). For the time being it suffices to note that m^H is a polynomial in ν of dimension $\dim \mathfrak{n}_H$, which is even. To prove this, we recall the Langlands decomposition (2.3). The nilradical $\mathfrak{n}_H$ admits a restricted root decomposition:

$$\mathfrak{n}_H = \bigoplus_{\alpha\in\Delta^+(\mathfrak{g};\mathfrak{a})} \mathfrak{g}_\alpha \tag{7.39}$$

where the $\mathfrak{a}$-invariant subspaces are precisely those positive root spaces $\mathfrak{g}_\alpha$ where $\alpha|_\mathfrak{a} \neq 0$. These are precisely the root spaces corresponding to the complex roots in the root decomposition, as appeared in Remark 3.9, whereas the imaginary roots α all vanish on $\mathfrak{a}$. But we see in the remark that the complex root spaces come in pairs, with θ interchanges each complex roots with its complex conjugate. Also for each $\alpha \in \Delta_C^+$, $\langle \alpha, i\nu\rangle \neq 0$ because ν are $\mathfrak{a}_\mathbb{C}$-regular. Lastly we notice that m^H is a even polynomial, as $m^H(\eta : i\nu) = m^H(\eta : -i\nu)$, again from the fact the ν-factors are even. Summing up, we see $\dim \mathfrak{n}$ is even and $m^H(\eta : i\nu)$ is a polynomial of even degree.

Now we decompose

$$m^H(\eta : \nu) = \sum_{k=0}^{\dim \mathfrak{n}_H/2} p_{\eta,2k}(\nu) \tag{7.40}$$

into homogeneous polynomials of even degree. We can now express the whole integral in spherical coordinates by noting that $\dim \mathfrak{a} = m = \mathrm{frk}(G)$ for the fundamental parabolic subgroup. For each $\eta \in \Xi_\rho$ we have

$$\begin{aligned} \int_{\mathfrak{a}^*} e^{-t\|\nu\|^2} m^H(\eta : i\nu) d\nu &= \sum_{k=0}^{\dim \mathfrak{n}_H/2} C_{\eta,k} \int_0^\infty e^{tr^2} r^{m-1+2k} dr \\ &= \sum_{k=0}^{\dim \mathfrak{n}_H/2} t^{-(\frac{m}{2}+k)} C_{\eta,k} \int_0^\infty e^{-y^2} y^{m-1+2k} dy \end{aligned} \tag{7.41}$$

with $C_{\eta,k} := \int_{\|\nu\|=1} p_{\eta,2k} d\nu$ a constant. Hence the above sum is dominated by the $t^{-(\frac{m}{2}+k')}$-term as $t \to \infty$, where k' is the lowest degree with $C_{\eta,k'} \neq 0$. Consequently we have proven that the heat kernel (7.41) grows in the magnitude of $\mathcal{O}(t^{-\frac{m}{2}})$ as $t \to \infty$, if $C_{\eta,k'} \neq 0$. This implies the correspondinbg Novikov-Shubin invariant $\alpha_p(\Gamma, V_\rho) = \frac{m}{2} + k'$.

So it left to prove the claim $C_{\eta,0} \neq 0$, which implies $\alpha_p(\Gamma, V_\rho) = m/2$. This is equivalent to proving $p_{\eta,0} = m^H(\eta : 0) \neq 0$. We note the following nonvanishing result:

$$\prod_{\beta\in\Delta^+(\mathfrak{g}^*;\mathfrak{h}^*)} \frac{\langle\beta,\lambda_\eta\rangle}{\langle\beta,\delta_G\rangle} \neq 0 \quad \text{if } H^*(\mathfrak{g};H_{\eta,0}\otimes V_\rho)\neq 0 \tag{7.42}$$

This nonvanishing result again follows from Theorem 7.7, and the fact that the infinitesimal character λ_{ρ^*} of the dual representation ρ^* of ρ, and $\lambda_{\eta,0}$ lie in the same W^P-orbit, i.e., there exists an $w\in W^P$ such that $\lambda_{\eta,0}=w\lambda_{\rho^*}$. Also since the highest weights of ρ and ρ^* are related by $\Lambda_\rho = w_0\Lambda_{\rho^*}$, with w_0 the longest element of W. By noting the highest weight Λ_ρ of a finite-dimensional representation ρ of G is related with its infinitesimal character λ_ρ by $\lambda_\rho=\Lambda_\rho+\rho_G$, and by using the fact that W acts on $\mathfrak{h}^*_{\mathbb{C}}$ isometrically:

$$\lambda_{\eta,0}=w\lambda_{\rho^*}=-w_0w(\lambda_\rho-\delta_G)+\delta_G \tag{7.43}$$

Now suppose, to the contrary, that the product (7.42) equals to zero. Then by changing the system of positive roots from $\Delta^+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$ to $w_0w\Delta^+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$, we see there exists some positive root β such that β is orthogonal to $-w_0w(\lambda_\rho-\delta_G)+\delta_G$. But as this is the infinitesimal character of ρ^*, it would contradict the fact that $\rho^*-\delta_G$ is a dominant integral weight. Hence we have proven the claim, since we already assumed $\lambda\in\Xi_\rho$ has non-vanishing cohomology. Summing up, the Novikov-Shubin invariant in this case is:

$$\alpha^p(\Gamma;V_\rho)=m \quad \text{for } p\in\left[\frac{d-m}{2},\frac{d+m}{2}-1\right] \tag{7.44}$$

and equals to ∞^+ in all other cases. In particular, we see any uniform lattice Γ of G is of determinant class for any admissible G-representation ρ. □

Lastly let us compute the twisted L^2-torsion is given as claimed in Theorem 7.8, part III & IV. This has been done in several places: First by Olbrich in [Olb02] for the untwisted case, and then by Müller and Pfaff in [MP13, section 6] and Bergeron and Venkatesh in [BV13, section 5] independently. In fact, much of the proof strategy dates back to [MS91] in evaluating the Reidemeister torsion. Hence we only highlight the key steps with special attention to certain subtle points here:

Step I Reduction to fundamental rank-one case: First observe that for $\mathrm{frk}(G)\neq 1$, the twisted L^2-torsion vanishes $\rho^{(2)}_{\mathrm{an}}(\Gamma;V_\rho)=0$. For $\mathrm{frk}(G)=0$, the dimension of the symmetric space is even, and this is a classical consequence of the Poincaré's duality. For $\mathrm{frk}(G)\geq 2$, it follows from a computation of the representation ring, that the even-degree and odd-degree sums cancel with each other (see e.g., [MP13, Proposition 4.1(ii)]). So it suffices to consider only the fundamental rank-one integral, constituting those fundamental series representations which contribute non-trivially to $\rho^{(2)}_{\mathrm{an}}(\Gamma;V_\rho)$. This establishes Theorem 7.8, part III.

Step II Reduction to simple fundamental rank-one case: Next apply the product formula to reduce the computation to the case G is simple Lie group of fundamental rank one. Here one proves a product formula for L^2-torsion (see [MP13, Proposition 5.3]) and it then suffices to consider the simple factors of real rank-one. This reduces the computation of torsion integrals to the case where $G=\mathrm{Spin}(2p+1,2q+1)$ with $p>q\in\mathbb{N}_{\geq 0}$, or $G=SL(3,\mathbb{R})$, by the classification of simple Lie groups.

Step III Computing K-invariant dimensions: The integral is now simplified in a significant manner. First an easy application of the Poincaré-Euler principle allows us to evaluate the alternating sum of $(\mathfrak{g},K)$-cohomology directly. Moreover, a classical result applying the Hochschild-Serre cohomology instead of that of the original cochains, thereby reducing the question to the $(\mathfrak{m},K\cap M)$-module $H^\eta\otimes V_\rho|_{\mathfrak{m}}$ (e.g. [BW00, Theorem III.3.3]). As the representation in question is tempered, whence $H^\eta\in\widehat{M^0}_d$ is a discrete series and we have reduced the question to the discrete series case. One ends up finding the unique non-vanishing of the cohomology in the middle degree. One now sees the desired contribution of the $[\wedge^*\mathfrak{p}^*\otimes V_\rho\otimes H_{\eta,i\nu}]^K$ to the integral.

Step IV Assembling the integral: Let us now investigate the constants appearing in (7.30). For convenience, we take α_0 to be the unique restricted noncompact root and we parametrize the fundamental series by $\pi(\eta : i\nu\cdot\alpha_0)$ for $\nu\in\mathbb{R}$. By taking the $\frac{1}{2}$-factor in Definition 7.4 into account, we abbreviate the alternating sum:

$$k_t^\rho(x) := \frac{1}{2}\sum_{p=0}^{n}(-1)^p \cdot p \cdot k_t^{\rho,p}(x) \tag{7.45}$$

which admits the following form [MP13, Proposition 6.6]:

$$k_t^\rho(e_G) = i^{\dim \mathfrak{p}_\mathfrak{m}} \frac{|W_\mathfrak{m}|}{|W_{K_M}|} \sum_{w\in W^P} (-1)^{\ell(w)+1} \int_{\mathbb{R}} e^{-t((\nu^2+c_{\rho,w}^2)\|\alpha_0\|^2)} m^H(\eta_{\rho,w}^\vee : i\nu) d\nu \tag{7.46}$$

Here W^P is again a set of representatives for $W_M\backslash W_G$, and $c_{\rho,w}$ is the value of the infinitesimal character $w\cdot\lambda_\rho$ restricted to $\mathfrak{a}_H$. Then it can be written as:

$$c_{\rho,w}\alpha_0 := w\cdot\lambda_\rho|_{\mathfrak{a}_H} \tag{7.47}$$

and $\eta_{\rho,w}^\vee$ is the contragredient representation of $\eta_{w\rho}$. Note $w\cdot\lambda_\rho|_\mathfrak{t}$ is the infinitesimal character of $\eta_{w\rho}$. The norm $\|\cdot\|$ is induced by the Killing form, following Remark 7.9. Lastly, $m^H(\eta : i\nu)$ is as (7.38).

Step V A trick of Mellin transform: The Plancherel density on the right can be written under Mellin transform as a sum of products. To this end one uses an identity from Mellin transform: ([Fri86, Lemma 2 & 3]) When $m^H(i\nu)$ is an even polynomial of degree $2k$, the integral $\int_\mathbb{R} e^{-t\nu^2} m^H(i\nu)\,\mathrm{d}\nu = t^{-\frac{1}{2}}Q(t)$ for some explicit polynomial Q of degree k, and the corresponding Mellin transform exists against e^{-tc^2} for all $c > 0$:

$$\mathcal{M}(e^{-tc^2}t^{-1/2}Q(t))(0) = -2\pi\int_0^c m^H(\nu)\,\mathrm{d}\nu \tag{7.48}$$

In the case $c = 0$ the Mellin transform vanishes identically. To justify this one needs to break up the Mellin transform into two parts as in (7.11) and handle them separately [Zag06, Example 1]. As a result, we can rewrite the integral as:

$$\begin{aligned} C^{(2)}(G;V_\rho) &:= \frac{1}{\dim V_\rho}\mathcal{M}(k_t^\rho(e_G))(0) \\ &= \pi\|\alpha_0\|\cdot C_G' \sum_{w\in W^P}(-1)^{\ell(w)+1}\int_0^{|c_{\rho,w}|} m^H(\eta_{\rho,w}^\vee : \nu)\frac{1}{\dim V_\rho}\,\mathrm{d}\nu \end{aligned} \tag{7.49}$$

where $C_G' := i^{\dim\mathfrak{p}_\mathfrak{m}}\frac{|W_\mathfrak{m}|}{|W_{K_M}|}$, with $M = M_H$. We will reveal the reason for renormalizing the constant by $\dim V_\rho$ in the ensuing computation. It is left to compute the constants in the integral in each of the cases mentioned above.

Step VI Integral of simple real rank-one case: We follow the computation as in [MP13, Proposition 6.7 & 6.8]: Abbreviate $p + q = n$.

To express W^P and the corresponding values $c_{\rho,w}$ and $\ell(w)$, we parametrize the weight lattices of each group. In the case $G = \mathrm{Spin}(2p+1, 2q+1)$, we fix the simple roots with respect to the maximally split Cartan subalgebra $\mathfrak{h}$, abbreviating the classical notations as α_i to simplify notations:

$$\begin{aligned} \Pi(\mathfrak{g}_\mathbb{C};\mathfrak{h}_\mathbb{C}) &= \{\alpha_1,\dots\alpha_{n+1}\} := \{e_1-e_2,\dots,e_n-e_{n+1},e_n+e_{n+1}\} \\ \Pi(\mathfrak{m}_\mathbb{C};\mathfrak{t}_\mathbb{C}) &= \{\alpha_2,\dots,\alpha_{n+1}\} = \{e_2-e_3,\dots,e_n-e_{n+1}\} \end{aligned} \tag{7.50}$$

In this case $e_1 = \alpha_0$. Now the highest weights Λ_ρ are parametrized by the following integral dominant weights:

$$\Lambda_\rho = \sum_{i=1}^{n+1}\rho_i e_i = (\rho_1,\dots,\rho_{n+1}) \in \mathbb{Z}\left[\frac{1}{2}\right]^{n+1} \quad \text{where } \rho_1 \geq \cdots \geq |\rho_{n+1}| \tag{7.51}$$

The M^0-representation η is parameterized by highest weight $\Lambda_\eta = (\eta_2,\dots,\eta_{n+1})$, with the same integrality and dominance condition as above. In such parametrization, θ and w_0 act by reversing the sign of ρ_{n+1} and η_{n+1} respectively.

Now the computation of $\{(c_{\rho,w}, \eta^\vee_{\rho,w}, \ell(w)) \mid w \in W^P\}$ can be done as in the real rank-one case [BW00, Section VI.3.1], since we are dealing with a complexified root system here. Take s_{α_i} to be the simple reflection with respect to α_i, then:

$$W^P = \{s_0 = 1, s_1, \ldots, s_n, t_n, s'_0, \ldots, s'_{n-1}\} \tag{7.52}$$

Here $s_i = s_{\alpha_1} \cdots s_{\alpha_i}$ for $1 \le i \le n$, $t_k := s_{k-1}s_{\alpha_{n+1}}$ and $s'_i := w_{M,0} s w_{G,0}$ with w_0 again denoting the longest element in the corresponding group. Now using the fact $\ell(ww_0) = \ell(w_0) - \ell(w)$ for all $w \in W$, and the length of longest element is exactly the number of positive roots [ABB08, Proposition 1.77] we compute:

$$\ell(s_i) = i \quad \ell(t_n) = n \quad \ell(s'_i) = (n+1)n - (n-1)n - i = 2n - i$$

Moreover, the longest elements $w_{G,0}$ and $w_{M,0}$ correspond to -1 on $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{t}^*_{\mathbb{C}}$ respectively. Now one notices s_{α_i} swap e_i and e_{i+1}-coordinates for $i \le n$ and $s_{\alpha_{n+1}}$ is the reflection between e_n and $-e_{n+1}$. One gets:

$$s_i(\rho_1, \ldots, \rho_{n+1}) = (\rho_j, \rho_1, \ldots, \rho_{j-1}, \rho_{j+1}, \ldots, \rho_{n+1})$$

and $s_i(\delta_G) = s_i(n, n-1, \ldots, 1, 0) = (n-i, n, \ldots, n-i+1, n-i-1, \ldots, 1, 0)$. Hence:

$$c_{\rho,s_k} = \rho_{k+1} + n - k \tag{7.53}$$

Next the infinitesimal character and the highest weight of η_{ρ,s_i} are respectively given by:

$$\begin{aligned} \lambda_{\eta_{\rho,s_k}} &= s_k(\lambda_\rho)|_{\mathfrak{t}} = \sum_{i=2}^{k+1} c_{\rho,s_{i-2}} e_i + \sum_{i=k+2}^{n+1} c_{\rho,s_{i-1}} e_i \\ \Lambda_{\eta_{\rho,s_k}} &= \lambda_{\eta_{\rho,s_k}} - \delta_M = \sum_{i=2}^{k+1} (\rho_{i-1}+1) e_i + \sum_{i=k+2}^{n+1} \rho_i e_i \end{aligned} \tag{7.54}$$

In the same way, one can compute:

$$\Lambda_{\eta_{\rho,s'_i}} = w_{G,0}\Lambda_{\eta_{\rho,s_i}} \quad \Lambda_{\eta_{\rho,t_n}} = w_{G,0}\Lambda_{\eta_{\rho,s_n}} \quad c_{\rho,s'_i} = -c_{\rho,s_i} \quad c_{\rho,t_n} = -c_{\rho,s_n}$$

Therefore we note the invariance of the integrand under the w_0-action:

$$m^H(\eta_{\rho,w} : \nu) = m^H(\eta_{\rho,w_0 w} : \nu) = m^H_{\eta^\vee_{\rho,w}:\nu} \quad \text{for all } w \in W, \nu \in \mathfrak{a}^*_{\mathbb{C}},$$

from which we rewrite $\sum_{w \in W^P}$ in (7.49) now as $2\sum_{w \in \{s_0,\ldots,s_n\}}$ as the s'_i-summand can be identified with s_i-summand. Lastly we compute the Plancherel density $m^H(\eta^\vee_{\rho,w} : \nu)$ under this parametrization. Recall again $m^H(\eta^\vee_{\rho,w} : \nu) = m^H(\eta_{\rho,w} : \nu)$ for all $w \in W^P$ and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. Now we compute $m^H(\eta_{\rho,s_k} : \nu)$ for $k \le n$:

$$m^H(\eta_{\rho,s_k} : \nu) = (-1)^{\dim \mathfrak{n}_H/2} c_{G/K} \prod_{\beta \in \Delta^+(\mathfrak{g}^*;\mathfrak{h}^*)} \frac{\langle \beta, \lambda_{\pi_{\eta,i\nu}} \rangle}{\langle \beta, \delta_G \rangle} = (-1)^n c_{G/K} P_{\eta_{\rho,s_k}}(\nu) \tag{7.55}$$

where $\dim \mathfrak{n}_H = 2n$ from an easy computation of the dimension:

$$\dim \mathfrak{n}_H = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{m} - \dim \mathfrak{a}) = \frac{1}{2}(\frac{(2n+2)(2n+1) - 2n(2n-1)}{2} - 1) = 2n \tag{7.56}$$

Whence the product $P_{\eta_{\rho,s_k}}(\nu)$ admits a cleaner formula:

$$\begin{aligned}
P_{\eta_{\rho,s_l}}(\nu\cdot e_1) =& \prod_{j=1}^{n-1}\prod_{q=j+1}^{n}\frac{\langle \nu\cdot e_1+\sum_{i=2}^{l+1}c_{\rho,s_{i-2}}e_i+\sum_{i=l+2}^{n}c_{\rho,s_{i-1}}e_i, e_j-e_q\rangle}{\langle\sum_{k=1}^{n}(n-k)e_k, e_j-e_q\rangle}\\
&\cdot\prod_{j=1}^{n-1}\prod_{q=j+1}^{n}\frac{\langle \nu\cdot e_1+\sum_{i=2}^{l+1}c_{\rho,s_{i-2}}e_i+\sum_{i=l+2}^{n}c_{\rho,s_{i-1}}e_i, e_j-e_q\rangle}{\langle\sum_{k=1}^{n}(n-k)e_k, e_j-e_q\rangle}\\
=&\left(\prod_{\substack{0\le i\le n\\ i\ne l}}(\nu^2-c_{\rho,s_i}^2)\right)\left(\prod_{\substack{0\le j<i\le n\\ i,j\ne l}}(c_{\rho,s_j}^2-c_{\rho,s_i}^2)\right)\left(\prod_{1\le j<i\le n+1}((n-j)^2-(n-i)^2)\right)^{-1}\\
=&(-1)^l\cdot\prod_{0\le j<i\le n}\frac{c_{\rho,s_j}^2-c_{\rho,s_i}^2}{(n-j-1)^2-(n-i-1)^2}\prod_{\substack{0\le j\le n\\ j\ne l}}\frac{\nu^2-c_{\rho,s_j}^2}{c_{\rho,s_k}^2-c_{\rho,s_j}^2}\\
=&(-1)^l\dim(V_\rho)\prod_{j=0,j\ne l}^{n}\frac{\nu^2-c_{\rho,s_j}^2}{c_{\rho,s_l}^2-c_{\rho,s_j}^2}
\end{aligned}\tag{7.57}$$

where the last step is a direct consequence of the Weyl dimension formula. Hence $(-1)^l$ in this expression cancels with $\ell(w)+1$ in (7.49). Summing up the discussion into (7.49):

$$C^{(2)}(\mathrm{Spin}(2p+1,2q+1);V_\rho)=2\pi\|e_1\|C'_Gc_{G/K}\sum_{k=0}^{n}\int_0^{c_{\rho,k}}\prod_{j=0,j\ne k}^{n}\frac{\nu^2-c_{\rho,s_j}^2}{c_{\rho,s_k}^2-c_{\rho,s_j}^2}\,\mathrm{d}\nu \tag{7.58}$$

The last step is to compute the integral. The following trick is from [**BV13**, 5.9.1]: Denote the product by $\prod_{\rho,k}(\nu)$ and denote $Q_{\rho,k}(\nu)=\sum_{j=0}^{k}\prod_{\rho,k}(\nu)$. This is an even polynomial of degree $\le 2n$ which on the strictly decreasing sequence

$$c_{\rho,s_0}>\cdots>c_{\rho,s_n}\ge -c_{\rho,s_n}>\cdots>-c_{\rho,s_0}$$

takes value 1 on the first and last $k+1$ entries, and 0 on the rest. This means its derivative $\frac{\partial}{\partial\nu}Q_{\rho,k}(\nu)$ has a root in every interval partitioned by the above sequence. But $\frac{\partial}{\partial\nu}Q_{\rho,k}$ is a polynomial in ν of degree $\le 2n-1$, which means $Q_{\rho,k}$ is either constantly equal to 1, or is strictly increasing between $c_{\rho,s_{k+1}}$ and c_{ρ,s_k}. By the same logic $Q_{\rho,n}\equiv 1$. Hence:

$$\sum_{k=0}^{n}\int_0^{c_{\rho,s_k}}\Pi_{\rho,k}(\nu)\,\mathrm{d}\nu=\sum_{k=0}^{n}\int_{c_{\rho,s_{k+1}}}^{c_{\rho,s_k}}Q_{\rho,k}(\nu)\,\mathrm{d}\nu>0 \tag{7.59}$$

by setting $c_{\rho,s_{n+1}}:=0$. This concludes the computation in the case of $\mathrm{Spin}(2p+1,2q+1)$.

The case of $G=SL(3,\mathbb{R})$ is similar, yet the computation is much simpler. Form the root space decomposition with respect to the split Cartan subgroup here. Its roots e_i-e_js correspond to the fundamental Cartan subgroup in Example 3.12 via a $\mathfrak{g}_\mathbb{C}$ inner-automorphism:

$$e_1-e_2\mapsto 2f_2\quad e_1-e_3\mapsto f_1+f_2\quad e_2-e_3\mapsto f_1-f_2 \tag{7.60}$$

In this case $\Delta^+(\mathfrak{m}_\mathbb{C};\mathfrak{t}_\mathbb{C})=\{2f_2\}$ is mapped to $\{e_1-e_2\}$ and the highest weights are parameterized by an integral combination of fundamental weights:

$$\Lambda_\rho=\rho_1\omega_1+\rho_2\omega_2\quad\text{where }\omega_1:=\frac{1}{3}f_1+f_2\quad \omega_2=\frac{2}{3}f_1 \tag{7.61}$$

In this case $f_1=\alpha_0$ is the unique restricted root. W^P in this case contains $1, s_{e_2-e_3}$ and $s_{e_1-e_2}s_{e_2-e_3}$, of respective length $0,1,2$. Following the computation of [**MP13**, Proposition 6.8] and arguing like in the previous case produces:

$$\sum_{w\in W^P}(-1)^{\ell(\omega)}\int_0^{|c_{\rho,w}|}P_{\eta_{\rho,\omega}}(\nu)\,\mathrm{d}\nu=\sum_{k=1}^{3}(-1)^{k-1}\int_0^{C_{\rho,k}}A_{\rho,k}\left(\frac{9\nu^2}{4}-A_{\rho,k}^2\right)\mathrm{d}\nu \tag{7.62}$$

where

$$A_{\rho,1} := (\rho_1+1)/2 \qquad A_{\rho,2} := (\rho_1+\rho_2+2)/2 \qquad A_{\rho,3} := (\rho_2+1)/2 \tag{7.63}$$

$$C_{\rho,1} := (\rho_1+2\rho_2+3)/3 \qquad C_{\rho,2} := (\rho_1-\rho_2)/3 \qquad C_{\rho,3} := (2\rho_1+\rho_2+3)/3 \tag{7.64}$$

Lastly $\dim \mathfrak{p}_{\mathfrak{m}} = 2$. Substituting these into (7.49) again, and dropping the ρ-sign of the constants, we see (7.62) is equal to:

$$\begin{cases} 2A_1A_3C_1C_3 + 2A_2C_2A_3C_3 & \text{if } \rho_1 \geq \rho_2 \\ 2A_1A_3C_1C_3 - 2A_2C_2A_1C_1 & \text{if } \rho_1 \leq \rho_2 \end{cases} \tag{7.65}$$

Lastly by the Weyl dimension formula we get $\dim V_\rho = \frac{1}{2}(\rho_1+1)(\rho_2+1)(\rho_1+\rho_2+2) = 4A_1A_2A_3$, and expanding the constants, we see:

$$C^{(2)}(SL(3,\mathbb{R});V_\rho) = \begin{cases} -\frac{C'_G c_{G/K}\|f_1\|\pi}{2}\left(\frac{C_1C_3}{A_2} + \frac{C_2C_3}{A_1}\right) & \text{if } \rho_1 \geq \rho_2 \\ -\frac{C'_G c_{G/K}\|f_1\|\pi}{2}\left(\frac{C_1C_3}{A_2} - \frac{C_2C_3}{A_3}\right) & \text{if } \rho_1 \leq \rho_2 \end{cases} \tag{7.66}$$

In particular we see in both cases the rational functions $\frac{C_1C_3}{A_2} \pm \frac{C_2C_3}{A_1}$ are positive, by our corresponding choices of the highest weight $\Lambda_\rho = (\rho_1, \rho_2)$.

Step VII Normalizations of the Haar measure and Plancherel density constants: Now almost every term is constant in (7.38). We see that the rest of the implicit constants cancels out nicely. As mentioned before this depends solely on the normalization of the Haar measure on G. We follow the discussion of [BV13, §5.9] and [Olb02, Lemma 5.1] here:

$$c_{G/K} = \frac{\sqrt{-1}^{\dim \mathfrak{n}_H}}{|W_A|(2\pi)^{d+\mathrm{frk}(G)}} \frac{\prod_{\alpha\in\Delta^+_{\mathfrak{g}}}\langle\alpha,\delta_G\rangle}{\prod_{\alpha\in\Delta^+_{\mathfrak{k}}}\langle\alpha,\delta_K\rangle} = \frac{\sqrt{-1}^{\dim \mathfrak{n}_H}}{|W_{A_H}|(2\pi)^{\mathrm{frk}(G)}} \frac{\mathrm{vol}(\exp(i\mathfrak{a}_H))}{\mathrm{vol}(\exp(i\mathfrak{p}))} \tag{7.67}$$

where $W_{A_H} := \{k \in K \mid \mathrm{Ad}(k)\mathfrak{a}_H \subseteq \mathfrak{a}_H\}/K_M$ and the volume in both cases is induced by the Killing form on $\mathfrak{k}$. The last equality comes from [HC75, Lemma 37.3 & 37.4]:

$$\prod_{\alpha\in\Delta^+_{\mathfrak{k}}}\langle\alpha,\delta_K\rangle = (2\pi)^{\frac{\dim\mathfrak{k}-\dim\mathfrak{t}}{2}}\frac{\mathrm{vol}(T)}{\mathrm{vol}(K)} \tag{7.68}$$

This holds for any connected Lie group. In particular, applying it to the compact real form (G^d, H^d) and to (K,T) yields the final equality. Recall again T is the Cartan subgroup of K.

In view of Remark 7.9 the same Killing form on $\mathfrak{g}$ complexified to $\mathfrak{g}_{\mathbb{C}}$ gives the volume form on $i\mathfrak{a}_H \subseteq i\mathfrak{p}$. This makes $i\mathfrak{a}_H$ a circle of radius $\|\alpha_0\|^{-1}$. Recall that α_0 is the unique noncompact restricted root. Consequently one has $\mathrm{vol}(\exp(i\mathfrak{a}_H)) = \frac{2\pi}{\|\alpha_0\|}$.

Lastly the volume form here is given by the Riemannian metric by restricting B^θ on $\mathfrak{k}$, and should not be mistaken with the (normalized) Haar measure dk we have been using throughout.

To compute $|W_{A_H}|$ we use [MP13, Lemma 6.1]: $\frac{|W_{K_M}|}{|W_{\mathfrak{k}_{\mathfrak{m}}}|} = \frac{2}{|W_{A_H}|}$. Recall on the other hand that $C'_G = i^{\dim\mathfrak{p}_{\mathfrak{m}}}\frac{|W_{\mathfrak{m}}|}{|W_{K_M}|}$. Take the product $C'_G c_{G/K}$ to get rid of the Weyl group terms:

$$\begin{aligned} C'_G c_{G/K} &= \sqrt{-1}^{\dim\mathfrak{n}_H+\dim\mathfrak{p}_{\mathfrak{m}}}\frac{|W_{\mathfrak{m}}|}{|W_{K_M}|}\cdot\frac{2\pi\cdot(2\pi)^{-\mathrm{frk}(G)}}{|W_{A_H}|\cdot\|\alpha_0\|\,\mathrm{vol}(\exp(i\mathfrak{p}))} \\ &= \frac{|W_{\mathfrak{m}}|}{|W_{\mathfrak{k}_{\mathfrak{m}}}|}\cdot\frac{\sqrt{-1}^{\dim\mathfrak{n}_H+\dim\mathfrak{p}_{\mathfrak{m}}}}{2\|\alpha_0\|\,\mathrm{vol}(\exp(i\mathfrak{p}))} \end{aligned} \tag{7.69}$$

In the case of $G = \mathrm{Spin}(2p+1, 2q+1)$, $\dim\mathfrak{p}_{\mathfrak{m}} = 4pq$, and $\dim\mathfrak{n}_H = 2(p+q)$. Moreover, $\mathfrak{m}_{\mathbb{C}} = \mathfrak{so}(2p+2q,\mathbb{C})$, of type D_{p+q}, and $\mathfrak{k}_{\mathfrak{m}} = \mathfrak{so}(2p,\mathbb{C})\oplus\mathfrak{so}(2q,\mathbb{C})$, of type $D_p \oplus D_q$ from [Kna96, Appendix C.1], $\frac{|W_{\mathfrak{m}}|}{|W_{\mathfrak{k}_{\mathfrak{m}}}|} = 2\binom{p+q}{p}$.

In the case of $G = SL(3,\mathbb{R})$, $\dim \mathfrak{p}_\mathfrak{m} = \dim \mathfrak{n}_H = 2$ and $\frac{|W_\mathfrak{m}|}{|W_{\mathfrak{k}_\mathfrak{m}}|} = |W_{A_2}| = 2$. Recall now $\alpha_0 = f_1$ in the case of $G = SL(3,\mathbb{R})$, and is e_1 in the other case, hence:

$$c'_G c_{G/K} = \begin{cases} \frac{(-1)^{p+q}}{\|e_1\|}\binom{p+q}{p} \cdot \mathrm{vol}(\exp(i\mathfrak{p}))^{-1} & \text{if } G = \mathrm{Spin}(2p+1, 2q+1) \\ \frac{1}{\|f_1\|}\mathrm{vol}(\exp(i\mathfrak{p}))^{-1} & \text{if } G = SL(3,\mathbb{R}) \end{cases} \tag{7.70}$$

Putting this expression into (7.58) and (7.66) we see that $\|\alpha_0\|$ cancels. We then get the final expression by noting $\rho^{(2)}_{\mathrm{an}}(\Gamma; V_\rho) = \mathrm{vol}(\Gamma\backslash G/K) \cdot C^{(2)}(G; V_\rho)$ in both cases.

This concludes the computation of L^2-torsions, the result as claimed in Theorem 7.8, part IV. It also concludes the proof of the whole theorem.

Now summarizing (7.29), (7.44), and lastly (7.58), (7.66) with the product formula, we have proven Theorem 7.8.

The above computation can be easily extended to the case of nonuniform lattices, asserting again the same constants for each L^2-invariant. The application of the equivariance rule establishes (7.2) and one can compute the von Neumann trace in exactly the same way. As we do not have any particular application in mind, and the definition of L^2-invariants over noncompact manifolds of finite volume would impose more analytical problems, we are satisfied with stating the theorem as above. We conclude this section by a few remarks:

Remark 7.11 (Independence on the twisting)**.** Theorem 7.8 partially addresses a question asked by Lück [**Lüc18**, Question 0.1 - 0.3] that whether the twisted L^2-invariants depend on the twisting by representations ρ of Γ. There Lück also proved the independence of the L^2-invariants on the twisting in the case ρ is a unitary representation of Γ [**Lüc18**, Theorem 4.1]. We note that in our case this is almost never the case. An easy way to see this is by inspecting part IV of the theorem, where the torsion constants $C^{(2)}(G; V_\rho)$ do always depend on the representation ρ in a nontrivial way. In the unitary case, such a $C^{(2)}(G; V)$ would be one.

The independence of the Novikov-Shubin invariants is a new result, which is not found in the existing literature.

Remark 7.12 (Descent to trivial representation)**.** As an example, as well as corroboration to our computation, let us see what happens when ρ is the trivial representation. In the first case, ρ^{G^1} is the trivial representation of $\mathrm{Spin}(2p+1, 2q+1)$. $c_{\rho, s_k} = p + q - k$, and the integral is easily seen to be dependent only on the sum $p+q$. Hence up to the volume form of $\mathrm{Spin}(k)/\mathrm{Spin}(l)$, this problem has been reduced to the odd hyperbolic case $\mathrm{Spin}(2k+1, 1)/\mathrm{Spin}(2k+1)$. We discuss this example below.

In the second case, one readily computes that $A_{\mathrm{triv},1} = A_{\mathrm{triv},3} = \frac{1}{2}$, $C_{\mathrm{triv},2} = 0$, and $C_{\mathrm{triv},1} = C_{\mathrm{triv},3} = A_{\mathrm{triv},2} = 1$ in (7.63). therefore: $D_{\mathrm{triv}} = 1$ and $C^{(2)}(SL(3,\mathbb{R}), \mathrm{triv}) \cdot \mathrm{vol}(G^c/K) = \frac{\pi}{2}$. The volume of $SU(3)/SO(3)$ is $\frac{\pi^3}{\sqrt{2}}$ by Macdonald's formula [**Mac80**, (1)]. Hence $C^{(2)}(SL(3,\mathbb{R})/SO(3), \mathrm{triv}) = \frac{1}{\sqrt{2}\pi^2}$. [**BV13**, p.423] also computed the numerical results of $\rho^{(2)}_{\mathrm{an}}(SL(2,\mathbb{Z}))$ when one endows $SL_3(\mathbb{R})$ with the standard measure such that $SL_3(\mathbb{Z})$ has covolume $\zeta(2)\zeta(3)$.

Example 7.13 (Hyperbolic space)**.** *We are particularly interested in the case of hyperbolic spaces with* $G = \mathrm{Spin}(2p+1, 1)$ *with* $q = 0$*. To find the right constant one first needs to renormalize the current metric* B *by a constant* $\frac{1}{2p}$*. Among all the quantities used in defining* $C^{(2)}$*, we see that only* $\mathrm{vol}(G^c/K)$ *is sensitive to the rescaling of the metric.*

In the new normalization, G^c/K *becomes a unit sphere* $\mathbb{S}^{2p+1} \cong SO(2p+2)/SO(2p+1)$*. Its volume* $\mathrm{vol}(\mathbb{S}^n)$ *is* $\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$*. In particular,* $\mathrm{vol}(\mathbb{S}^{2p+1}) = 2\frac{\pi^{p+1}}{\Gamma(p+1)} = 2\frac{\pi^{p+1}}{p!}$*. Hence we have recovered the result of Hess and Schick [**HS98**, Corollary 4] for the torsion constant of the hyperbolic manifolds:*

$$C^{(2)}(\mathrm{Spin}(2p+1,1), V_{\mathrm{triv}}) = (-1)^p \frac{(p!)}{\pi^p} \sum_{k=0}^{p-1} \int_0^{p-k} \prod_{j=0, j\neq k}^{p} \frac{\nu^2 - (p-j)^2}{(j-k)(2p-k-j)} \,\mathrm{d}\nu \tag{7.71}$$

To see how exactly this is done, we rewrite the denominator in the integrand:

$$\Pi_k^{-1} := \left(\prod_{j=0, j\neq k}^{n} (j-k)(2p-k-j)\right)^{-1} = (-1)^k \frac{(2p-2k)}{(p-k)(2p-k)!\ k!} = (-1)^k \binom{2p}{k} \frac{2}{(2p)!}$$

Now the coefficients can be aligned by regrouping the terms:

$$\Pi_k^{-1} \frac{p!}{\pi^p} = (-1)^k \binom{2p}{k} \left(\frac{4^p p!}{(2p)!\sqrt{\pi}}\right) \frac{2\pi}{(4\pi)^{p+\frac{1}{2}}} = 2\pi(-1)^k \binom{2p}{k} \frac{(4\pi)^{-(p+\frac{1}{2})}}{\Gamma(p+\frac{1}{2})}$$

*We see this expression indeed agrees with the one derived by Lott [**Lot92**, Proposition 15]. Note that 2π is the constant that arises from Mellin transform. In the case of hyperbolic 3-manifolds, take $p = 1$ with only one integral with $p = j = 1$ and $k = 0$:*

$$C^{(2)}(\mathrm{Spin}(3,1), V_{\mathrm{triv}}) = -\frac{1}{\pi} \int_0^1 \nu^2 \,\mathrm{d}\nu = -\frac{1}{3\pi} \tag{7.72}$$

that is, $C^{(2)}_{\mathbb{H}^3} = \frac{-1}{3\pi}$.

Remark 7.14 (Signficance of $c_{\rho,w}$)**.** As we have deliberately kept the tedious notation of (7.47), the constants c_{ρ,s_k} in (7.58) and $C_{\rho,s_k} = C_{\rho,k}$ in (7.66) signify that the lowest spectral value of each family of fundamental series $\{H_{\eta,i\nu} : \lambda_\eta = s_k\lambda_\rho\}$ can contribute to the eventual computation of L^2-torsion. If we inspect the proof of (twisted) Novikov-Shubin invariants, we will notice these constants account for the actual bottom of the Hodge Laplacian spectrum in the middle degree. Such bounds are critical to performing the spectral inversion of the Laplacian – a key step in proving the convergence of the analytic torsion to the L^2-torsion. This is the case in both aforementioned literatures: They have both massaged the representations ρ to avoid the case of $c_{\rho,w} = 0$. One undesirable yet essential case here is when ρ is invariant under the Cartan involution, i.e. when θ fixes Λ_ρ. In this case we have at least one family for which $c_{\rho,w} = 0$, as our computation above manifests. This case in particular contains the case when $\rho = \mathrm{triv}$ is the trivial representation.

Remark 7.15. In the paper of Müller and Pfaff [**MP13**] an extra assumption regarding the highest weight of V_ρ is made: They assumed θ does not fix the highest weight. They used it for two reasons:

1. This gives an easy proof to establish the spectral gap for the twisted bundle [**MP13**, Lemma 5.1] and consequently the exponential decay of the heat kernel which leads to well-definedness of the L^2-torsion: We have worked around this issue by measuring the explicit decay rate of the heat kernel at each level by computing the Novikov-Shubin invariants;
2. They also used this critically to establish the lower spectral bound [**MP13**, Proposition 7.4]. As we have deliberately kept the tedious notation (7.47), the constants c_{ρ,s_k} and $C_{\rho,s_k} = C_{\rho,k}$, appearing in above computations signify the lowest spectral value that each family of fundamental series $H_{\eta_{\rho,w=s_k},i\nu}$ can contribute to the computation of L^2-torsion. One needs such bound to perform the spectral inversion in order to prove the convergence of the analytic torsion to L^2-torsion in [**MP13**, Lemma 8.1]

Since the goal of our computations is merely to compute the twisted L^2-invariants, we have dropped this assumption in our statement.

Bergeron and Venkatesh addressed the well-definedness of the L^2-torsion by proving some crude bounds on the heat kernel [**BV13**, Lemma 3.8 & (4.5.1)]. This is sufficient for their purposes, as only an upper bound is needed to establish the convergence of the integral. We have, on the other hand, gives an explicit estimate of the leading exponent of the heat kernel asymptotics, namely the Novikov-Shubin invariants.

Finally we would like to point out a few minor errors in the cited literature:

Remark 7.16. In [**BV13**, §5.7] the constant $c(S)$ was falsely claimed to be $\frac{\pi}{2}\frac{|W_{\mathfrak{m}}|}{|W_{\mathfrak{k}_{\mathfrak{m}}}|}c_{G/K}$, whereas the correct constant would be $\pi\frac{|W_{\mathfrak{m}}|}{|W_{K_M}|}c_{G/K}$, Compare this with [**MP13**, Proposition 6.6]. The error is promulgated to [**BV13**, (5.9.6)]. The correct $c(S)$ should be $2\pi|C'_G c_{G/K}|$ in the case of $\mathrm{Spin}(2p+1, 2q+1)$, and $\frac{\pi|C'_G c_{G/K}|}{2}$ in the $SL(3,\mathbb{R})$-case.

The numbers of [**HS98**] seems at first glance to agree with our results, but in their definition, the L^2-torsion is scaled by a factor of 2, so in the end their result should be scaled by $\frac{1}{2}$ to be numerically correct. The rest of the numerical results of [**HS98**] agrees with our computation.

Lastly, we point out the constant in [**Olb02**, Proposition 1.4] should be $\frac{\pi}{2}$ instead of $\frac{2\pi}{3}$, as in Remark 7.12. This results from a minor error in the root product: The integrand in Q_X of his proposition 5.3 should be $\frac{9\nu^2-1}{8}$ instead of ν^2.

Remark 7.17. We further refer the reader to [**MP13**, Proposition 6.8] for a study of the asymptotic behavior of $\rho^{(2)}_{\mathrm{an}}(\Gamma, V_\rho)$ when the representation ρ is not fixed by the Cartan involution, i.e., θ acts on the highest weight Λ_ρ nontrivially. We remark that the case Λ_ρ is fixed by θ can also be derived easily from the formulas above, in particular, in the case of $G = SL(3,\mathbb{R})$, such representations are parametrized by $\Lambda_\rho = (\rho_1, \rho_1)$, and D_{ρ_1,ρ_1} can be easily computed to be $\rho_1 + 1$.

7.4 Novikov-Shubin invariants of $\widetilde{SL_2(\mathbb{R})}$

We conclude the discussion of this chapter by taking a look at a distinct example, $\widetilde{SL_2(\mathbb{R})}$. As opposed to the approach towards symmetric spaces, we work directly with the explicit kernel computed in Theorem 6.2. In this case the L^2-Betti number and L^2-torsion both vanishes, due to existence of an $\mathbb{S}^1$-action on $\Gamma\backslash\widetilde{SL_2(\mathbb{R})}$ with no fixed point [**Lüc02**, Theorem 3.105]: To be more precise, given $\Gamma \subseteq G$ a uniform lattice, then $\mathbb{R} \cong K \subseteq G$ gives $(\Gamma \cap K)\backslash K \cong \mathbb{S}^1$. Consequently $U(1)$ acts on the quotient by rotation, and hence gives a fixed-point free $\mathbb{S}^1$-action. Hence the L^2-Betti numbers and L^2-torsion of $\widetilde{SL_2(\mathbb{R})}$ vanish altogether.

Alternatively, one can also readily observe there is no L^2-Betti numbers as there exists no square integrable representations of $\widetilde{SL_2(\mathbb{R})}$ contributing to the L^2-cohomology.

So it is left to compute the Novikov-Shubin invariants. We note that in [**LL95**, Theorem 4.11] the Novikov-Shubin invariants of $\widetilde{SL_2(\mathbb{R})}$ were correctly stated without any proof. We give the first proof available in the literature.

Theorem 7.18. *For $\Gamma \subseteq \widetilde{SL_2(\mathbb{R})}$ a uniform lattice, the Novikov-Shubin invariants are given by $\alpha_0(\Gamma) = \alpha_3(\Gamma) = \infty^+$ and $\alpha_1(\Gamma) = \alpha_2(\Gamma) = 1$*

Proof. By Poincaré duality, it suffices to compute α_0^Δ and α_1^Δ. The kernel in this case is homogeneous, hence $\mathrm{tr}_\Gamma(e^{-\Delta_1^c}) = \mathrm{vol}(\Gamma\backslash G)\cdot K_{1,t}(e_G)$. For α_0^Δ, we note from Theorem 6.2 that the spectra of all representations are bounded below by $1/4$. Consequently, $\alpha_0^\Delta = \infty^+$ by definition.

As for $K_{1,t}(e_G)$, recall the expression of the heat kernel:

$$
\begin{aligned}
K_{1,t}(e_G) = 2\int_0^\infty \Bigg[& e^{-t(k-\frac{1}{2})^2} + e^{-t[(k+1)^2+\frac{1}{4}]} + e^{-t(k+\frac{3}{2})^2} + \\
& \sum_{m\in k+\frac{1}{2}+\mathbb{N}} \left(e^{-t\Xi_{k,m}} + e^{-t\left(\Xi_{k,m}+\frac{1}{2}\pm\sqrt{\Xi_{k,m}+\frac{1}{4}}\right)} \right)\Bigg] k\,\mathrm{d}k \\
+ \int_0^1\int_0^\infty & \sum_{m\in\tau+\mathbb{Z}} \Bigg(e^{-t\Xi_{(\tau,i\nu),m}} + \\
& + e^{-t\left(\Xi_{(\tau,i\nu),m}+\frac{1}{2}\pm\sqrt{\Xi_{(\tau,i\nu),m}+\frac{1}{4}}\right)}\Bigg) \mathrm{Re}\tanh(\tau+i\nu)\,\mathrm{d}\nu\,\mathrm{d}\tau
\end{aligned}
\tag{6.20}
$$

The integral associated with the principal series are again bounded below by 1/4, since

$$\Xi_{(\tau,i\nu),m} + \frac{1}{2} + \sqrt{\Xi_{(\tau,i\nu),m} + \frac{1}{4}} \tag{7.73}$$

is a strictly increasing function in both $|m|$ and $|\nu|$-direction. The infinimal value at $m = \tau \in (0,1]$ is bounded away from zero. Hence the only contribution to the integral comes from the relative discrete series. Here all the expressions except for $e^{-(k-\frac{1}{2})^2}$ are again bounded away from zeros, therefore decaying exponentially fast as $t \to \infty$. Summing up:

$$K_{1,t}(e_G) = 2\int_0^\infty e^{-t(k-\frac{1}{2})^2} k \,\mathrm{d}k + o(e^{-ct}) \tag{7.74}$$

for some $c > 0$. Here the c can be aptly chosen to be 1/4. For the last remaining integral, we note:

$$\int_0^\infty e^{-tk^m} k^j \,\mathrm{d}k = \Gamma\left(\frac{j+1}{m}\right) t^{-(j+1)/m} \tag{7.75}$$

again by observing the integral as a Mellin transform. Now:

$$\int_0^\infty e^{-t(k-\frac{1}{2})^2} k \,\mathrm{d}k = \int_{-\frac{1}{2}}^\infty e^{-tk^2}(k+\frac{1}{2}) \,\mathrm{d}k \sim \mathcal{M}(e^{-tk^2})(1) = \mathcal{O}(t^{-\frac{1}{2}}) \tag{7.76}$$

as $t \to \infty$. Consequently, $\alpha_1^\Delta(\Gamma) = \frac{1}{2}$. To finish the computation of α_is, we use the formula (7.7) to relate $\alpha_i^\Delta(\Gamma)$ and $\alpha_i(\Gamma)$. □

7.5 Illustration of torsion constants for $SL(3,\mathbb{R})/SO(3)$

As customary, we plot the rational function, D_{ρ_1,ρ_2}, defined to be such that:

$$C^{(2)}(e_{SL(3,\mathbb{R})}) = \frac{\pi}{2 \cdot \mathrm{vol}(G^c/K)} D_{\rho_1,\rho_2} \quad \Lambda_\rho = \rho_1 f_1 + \rho_2 f_2 \tag{7.77}$$

Recall the definition of D_{ρ_1,ρ_2} from Figure 7.1. The weight lattice is parametrized by fundamental weights. We see immediately it grows linearly with the size of ρ_1, ρ_2. In the extreme case when it depends only on one fundamental weight, the gradient is $4/9$. Only the blue dots are true weights of G-representations, whereas the contour lines are merely a guide to the eye.

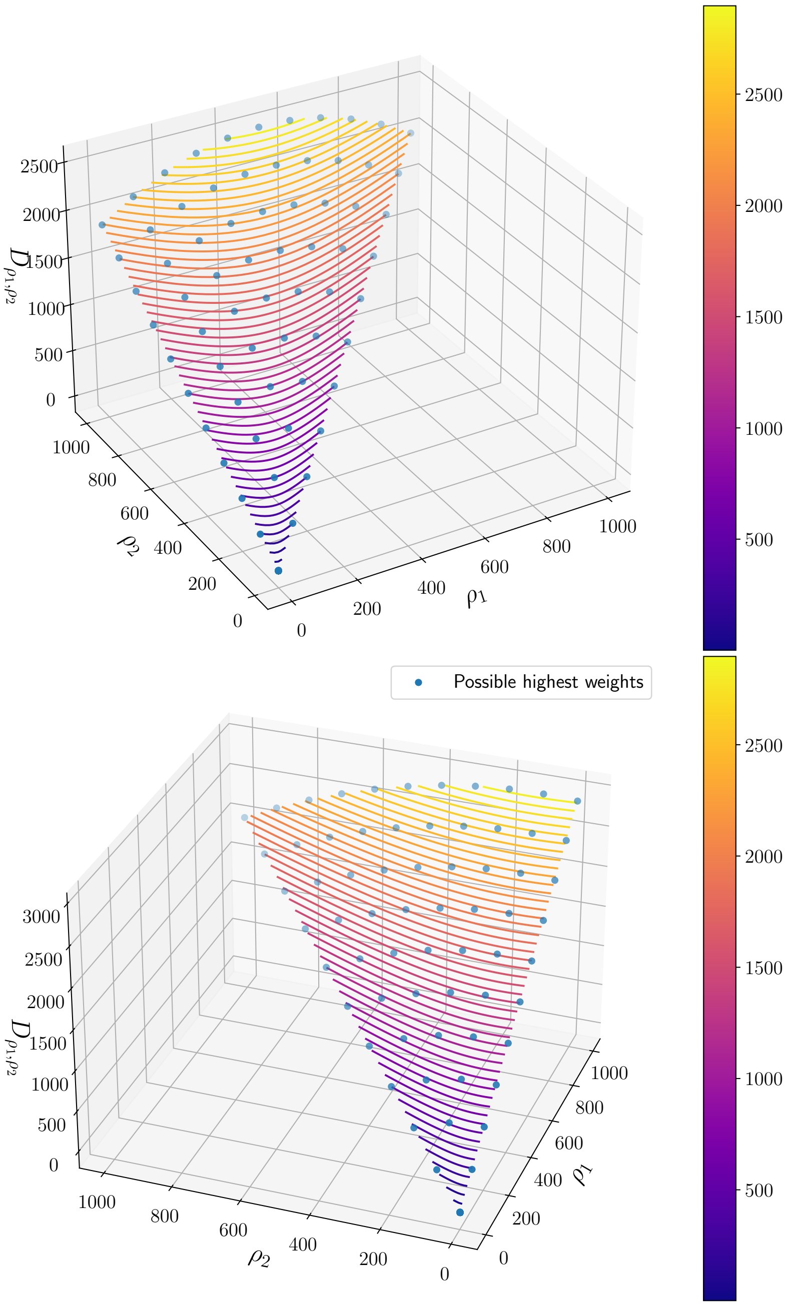


Figure 7.1: $SL(3,\mathbf{R})$-Torsion constant D_{ρ_1,ρ_2} for representation $\Lambda_\rho = \rho_1\omega_1 + \rho_2\omega_2$

Chapter 8

Other computations of the spectrum

In the next two sections we compute the spectra of Hodge Laplace operators and Dirac operators on the Heisenberg group $\mathbb{H}_3$. Most of the computations have been carried out partially elsewhere, see e.g. [Lot92] or [DS84]. On the other hand, we include this chapter to emphasize its resemblance with the computation we carried out in the case of $\widetilde{SL_2(\mathbb{R})}$.

8.1 Hodge Laplacian spectrum of rescaled Heisenberg group

We note that the results of this section are due to John Lott [Lot92, Section VII.C]. For the sake of completeness, we include the computation here. The proof strategy resembles that in Chapter 6.

Fix $\kappa \in (0, \infty)$. Given the Heisenberg group $N := \mathbb{H}_3$ consisting of all 3×3-upper triangular real matrices with ones on the diagonal, we fix the following basis of its Lie algebra $\mathfrak{n} := \mathfrak{h}_3$:

$$X = E_{12} \quad Y = E_{23} \quad Z = \frac{1}{\kappa} E_{13} \tag{8.1}$$

with E_{ij} is the matrix with (i, j)-entry equal to one and zero elsewhere. The Lie bracket is given by:

$$[X, Z] = [Y, Z] = 0 \quad [X, Y] = \kappa Z \tag{8.2}$$

with the Riemannian metric defined to be such that $\{X, Y, Z\}$ forms an orthonormal basis. Form their dual basis $\omega_X, \omega_Y, \omega_Z \in \mathfrak{h}_3^*$ respectively. Hence the function Laplacian is represented by:

$$-\Delta_0 = X^2 + Y^2 + Z^2 \in U(\mathfrak{h}_3) \tag{8.3}$$

and by the discussion in the Theorem 5.8 we see this generates a heat kernel $e^{-t\Delta_0}$ of Schwartz class.

For the 1-form Laplacian, we use the formulae from Proposition 1.5 instead of Corollary 1.7 and directly compute:

$$\begin{aligned}
\Delta_1(f \otimes \omega_X) &= R(\Delta_0) f \otimes \omega_X + R([Y, X^*]) f \otimes \omega_Y + R([Z, X^*]) f \otimes \omega_Z + R(\kappa Y) f \otimes \omega_Z \\
&= R(\Delta_0) f \otimes \omega_X + R(\kappa Z) f \otimes \omega_Y + R(\kappa Y) f \otimes \omega_Z \\
\Delta_1(f \otimes \omega_Y) &= R(\Delta_0) f \otimes \omega_Y + R([X, Y^*]) f \otimes \omega_X + R([Z, Y^*]) f \otimes \omega_Z - R(\kappa X) f \otimes \omega_Z \\
&= R(\Delta_0) f \otimes \omega_Y + R(-\kappa Z) f \otimes \omega_X - R(\kappa X) f \otimes \omega_Z \\
\Delta_1(f \otimes \omega_Z) &= R(\Delta_0 + \kappa^2) f \otimes \omega_Z + R([X, Z^*] + \kappa Y^*) f \otimes \omega_X + R([Y, Z^*] - \kappa X^*) f \otimes \omega_Y \\
&= R(\Delta_0 + \kappa^2) f \otimes \omega_Z - R(\kappa Y) f \otimes \omega_X + R(\kappa X) f \otimes \omega_Y
\end{aligned} \tag{8.4}$$

for the right regular representation R. Here $R(X^*)$ is a shorthand for the adjoint of $R(X)$, and hence $R(X^*) = -R(X)$ as the representation is unitary, whereas the coadjoint and adjoint representations are written out explicitly.

To proceed one needs again decompose the tensor product of representations into blocks, like (6.10). First we express the unitary representation of $\mathbb{H}_3$ in terms of the usual holomorphic model. The structure of these modules is similar to that of $\widetilde{SL_2(\mathbb{R})}$ in Section 6.1, but the formulation is simpler.

First we consider the Fock space $\mathcal{F}_{|\lambda|}(\mathbb{C}^1)$ as the space of single-variable holomorphic functions f that are finite under the norm induced by the inner product:

$$\langle f, g\rangle_{|\lambda|} = \frac{|\lambda|}{\pi}\int_{\mathbb{C}} |F(\lambda)|^2 e^{-|\lambda|\xi^2}\,\mathrm{d}\xi < \infty \tag{8.5}$$

Note that we choose a realization which is easier to present, but not easily seen to be the original induced representation $\pi_\lambda = \mathrm{ind}_{\mathbb{R}X\oplus\mathbb{R}Z}^{\mathbb{H}_3}(\lambda) \cong L^2(\mathbb{R})$. It is unitarily isomorphic to $L^2(\mathbb{R})$ by the following unitary intertwining operator known as **Segal-Bargmann transform** $\mathcal{SB}_\lambda : L^2(\mathbb{R}) \to \mathcal{F}(\mathbb{C})$:

$$\begin{aligned}(\mathcal{SB}_\lambda F)(z) &= e^{-\pi\lambda z^2/2}\sqrt{\frac{1}{\lambda}}\int_{\mathbb{R}} e^{2\pi\lambda t z}e^{-\pi\lambda t^2}\,\mathrm{d}t\\ (\mathcal{SB}_\lambda^{-1} f)(t) &= \sqrt{\frac{\lambda}{2}}\int_{\mathbb{R}} e^{-\pi i\lambda y t}e^{-\frac{1}{2}\pi\lambda(y^2+t^2)} f(t+iy)\,\mathrm{d}y\end{aligned} \tag{8.6}$$

Under this action $\mathcal{SB}_\lambda^{-1}$ takes the monomial z^n to appropriate multiples of Hermite functions. We now define the $\mathfrak{h}_3$-action by first complexifying the basis with:

$$B = \frac{1}{\sqrt{2}}(X - iY) \qquad \overline{B} = \frac{1}{\sqrt{2}}(X + iY) \tag{8.7}$$

They act on $\mathcal{F}_{|\lambda|}(\mathbb{C})$ as the creation and annihilation operators respectively. We define a $\mathbb{H}_3$-module structure on $\mathcal{F}_\lambda(\mathbb{C})$ by:

1. For $\lambda > 0, \pi_\lambda(B) = \lambda z\cdot$, $\pi_\lambda(\overline{B}) = \partial_z$, $\pi_\lambda(\kappa Z) = i\lambda$;
2. For $\lambda < 0, \pi_\lambda(B) = \partial_z$, $\pi_\lambda(\overline{B}) = -\lambda z\cdot$, $\pi_\lambda(\kappa Z) = i\lambda$.

These indeed define a unitary representation, and $\mathcal{SB}_\lambda$ consequently defines an intertwining operator between $\mathcal{F}_\lambda(\mathbb{C})$ and $L^2(\mathbb{R})$ (see e.g. [CG90, Example 2.2.6] for concrete realization.)

Hence the tempered series $\pi_\lambda \in \widehat{\mathbb{H}_3}$ corresponds to the generic coadjoint orbits $\mathcal{O}_\lambda := N\cdot\lambda$ for $\lambda \in \mathfrak{z}^*$ in Theorem B.6. From now on we fixes $\lambda > 0$. The negative case can be dealt similarly. We fix an orthonormal basis $\{v_m\}_{m\in\mathbb{N}_{\geq 0}}$ of $\mathcal{F}_\lambda(\mathbb{C})$ in Hermite polynomials:

$$v_m = i^{|m|}\sqrt{\frac{\lambda^{|m|}}{m!}}z^m \tag{8.8}$$

Then the above action is given by:

$$\begin{aligned}\pi_\lambda(B)v_m &= -\sqrt{\lambda(m+1)}\cdot v_{m+1}\\ \pi_\lambda(\overline{B})v_m &= \sqrt{\lambda m}\cdot v_{m-1}\\ \pi_\lambda(\kappa Z)v_m &= -i\lambda\cdot v_m\end{aligned} \tag{8.9}$$

In particular, $\pi(\Delta_0)$ acts on $\{v_m | m \in \mathbb{N}_{\geq 0}\}$ with corresponding eigenvalues $\{\Xi_m | m \in \mathbb{N}_{\geq 0}\}$:

$$\left\{\Xi_m := \frac{\lambda^2}{\kappa^2} + 2\lambda m + \lambda \mid m \in \mathbb{N}_{\geq 0}\right\} \tag{8.10}$$

with corresponding eigenvector v_m. To compute the 1-form spectrum, we first bring them into block form. Write the dual basis of B and $\overline{B}$ in $\mathfrak{h}_3^*$ as ω_+ and ω_-:

$$\omega_+ = \frac{1}{\sqrt{2}}(\omega_X + i\omega_Y) \quad \omega_- = \frac{1}{\sqrt{2}}(\omega_X - i\omega_Y). \tag{8.11}$$

Now (8.4) can be written in the complexified basis as:

$$\begin{pmatrix}\pi(\Delta_0) - i\pi(\kappa Z) & & -\kappa i\pi(B)\\ & \pi(\Delta_0) + i\pi(\kappa Z) & \kappa i\pi(\overline{B})\\ -\kappa i\pi(\overline{B}) & \kappa i\pi(B) & \pi(\Delta_0) + \kappa^2\end{pmatrix} \tag{8.12}$$

The matrix above is block-diagonal in an ordered basis $\{v_{m+1}\otimes\omega_+, v_{m-1}\otimes\omega_-, v_m\otimes\omega_Z\}$. It behaves similarly to the relative discrete series of $\widetilde{SL_2(\mathbb{R})}$:

1. The basis element $\{v_0 \otimes \omega_+\}$ contributes a single block, with eigenvalue $\Xi_1 - \lambda = \frac{\lambda^2}{\kappa^2} + \lambda - \lambda = \frac{\lambda^2}{\kappa^2}$;
2. The basis $\{v_1 \otimes \omega_+, v_0 \otimes \omega_Z\}$ contribute a 2×2-block:

$$\begin{pmatrix} -\lambda & \kappa i\sqrt{\lambda} \\ -\kappa i\sqrt{\lambda} & \kappa^2 \end{pmatrix} + \Xi_0 \cdot \mathbb{I}_2 \tag{8.13}$$

   The matrix has characteristic polynomial $p(X) = X^2 - \kappa^2 X - \lambda X$. The whole block contributes eigenvalues $\frac{\lambda^2}{\kappa^2} + \lambda + \lambda + \kappa^2 = (\frac{\lambda}{\kappa} + \kappa)^2$ and Ξ_0, each of multiplicity one;
3. For each $m \geq 1$, we obtain the following generic block:

$$\begin{pmatrix} \lambda & & \kappa i\sqrt{\lambda(m+1)} \\ & -\lambda & \kappa i\sqrt{\lambda m} \\ -\kappa i\sqrt{\lambda(m+1)} & -\kappa i\sqrt{\lambda m} & \kappa^2 \end{pmatrix} + \Xi_m \cdot \mathbb{I}_3 \tag{8.14}$$

   where the left matrix has characteristic polynomial $p(X) = X(X^2 - \kappa^2 X - \kappa^2 \Xi_m)$. Hence there are three eigenvalues:

$$\left\{ \Xi_m, \Xi_m + \frac{\kappa^2}{2} + \sqrt{\frac{\kappa^4}{4} + \kappa^2 \Xi_m} \right\} \tag{8.15}$$

   each of multiplicity one.

We collect the results here:

Theorem 8.1. *Given the Heisenberg group $\mathbb{H}_3$, realized as the upper 3×3-triangular matrices with ones on the diagonal, we fix the metric on it such that the following mutually orthogonal vectors have respective norms*

$$\|E_{12}\| = \|E_{23}\| = 1 \quad \|E_{13}\| = \kappa^2 \tag{8.16}$$

for some $\kappa > 0$. For each representation π_λ for $\lambda \in \mathbb{R}_{\neq 0}$, abbreviate $\Xi_{\lambda,m} := \frac{\lambda^2}{\kappa^2} + 2|\lambda|m + |\lambda|$ as the eigenvalue of Δ_0 acting on the eigenvector $v_m \in H_{\pi_\lambda}$. Then the localized spectrum $\pi_\lambda(H_\pi)$ of Δ_1 over each representation is given by:

1. *$\frac{\lambda^2}{\kappa^2}$, of multiplicity 1;*
2. *$(\frac{|\lambda|}{\kappa} + \kappa)^2$ and $\frac{\lambda^2}{\kappa^2} + |\lambda|$ of multiplicity 1;*
3. *$\Xi_{\lambda,m}$ and $\Xi_{\lambda,m} + \frac{\kappa^2}{2} \pm \sqrt{\frac{\kappa^4}{4} + \kappa^2 \Xi_{\lambda,m}}$ of multiplicity 1, for $m \geq 1$.*

We finish this section by remarking that the Novikov-shubin invariants in this case are given by:

$$\alpha_0^\Delta(\mathbb{H}_3) = \alpha_3^\Delta(\mathbb{H}_3) = 4 \quad \alpha_1^\Delta(\mathbb{H}_3) = \alpha_2^\Delta(\mathbb{H}_3) = 2 \tag{8.17}$$

which was first computed by Lott [**Lot92**, Proposition 53]. In Tim Höpfner's thesis, an investigation was conducted to study the relation between the Novikov-Shubin invariants and the exponential rescaling of the metric in Z-direction. We refer to [**Hoe23**, Theorem 5.9] for details.

8.2 Dirac spectrum of Heisenberg group

This section is devoted to computing the localized spectrum of the Dirac operator on the Heisenberg group. In the computation of the eta invariants of $\mathbb{H}_{2n+1}(\mathbb{R})/\mathbb{H}_{2n+1}(\mathbb{Z})$, Deninger and Singhof computed the spectra of the Dirac operator by dropping the zero-order operator on invariant spinors [**DS84**, Proposition 4.1]. We will instead compute the spectra of the full Dirac operator here, localized on each irreducible unitary representation.

For simplicity, we fix $\kappa = 1$ in this section. The general case is dealt with similarly.

As opposed to the semisimple case, where owing to the discussion in Section 4.4 admits a much simpler formula for invariant spinors, the spinor bundles of nilpotent Lie groups lack such a general framework. Hence we need to restart over from Theorem 4.11 to obtain a Bochner formula tailored to this case. Take ρ to be trivial representation.

We first compute the Levi-Civita connection on $\mathbb{H}_3$. By (4.24) one easily computes:

$$\begin{aligned} \nabla_X Y = \frac{1}{2} Z \quad \nabla_X Z = -\frac{1}{2} Y \quad \nabla_Y Z = \frac{1}{2} X \\ \nabla_Y X = -\frac{1}{2} Z \quad \nabla_Z X = -\frac{1}{2} Y \quad \nabla_Z Y = \frac{1}{2} X \end{aligned} \tag{8.18}$$

Consequently the $\gamma^\wedge : \mathfrak{n} \to \mathfrak{so}(\mathfrak{n})$, with respect to the ordered basis $\{X, Y, Z\}$ maps:

$$X \mapsto -\frac{1}{2}(E_{23} - E_{32}) \quad Y \mapsto \frac{1}{2}(E_{13} - E_{31}) \quad Z \mapsto \frac{1}{2}(E_{12} - E_{21}) \tag{8.19}$$

Next apply the map φ from (3.3) to it, and together with (4.30) one sees:

$$\gamma^S : X \mapsto \frac{1}{4} c\ell(YZ) \quad Y \mapsto -\frac{1}{4} c\ell(XZ) \quad Z \mapsto -\frac{1}{4} c\ell(XY) \tag{8.20}$$

Now let us find out the spin module $\mathcal{S}_\mathfrak{n}$ of $\mathbb{C}\ell(\mathfrak{h}_3)$. The construction will resemble that on $\mathfrak{sl}_2$ in Section 6.3: After all, the two vector spaces are isometric by forgetting their structures as modules over algebras. Form the complexified B and $\overline{B}$ as in the previous section. Take the isotropic space $\mathcal{Z}_\mathfrak{n} = \mathbb{C}B \oplus \mathbb{C}Z$, with the module structure given by:

$$c\ell(Z), i \circlearrowright 1 \underset{c\ell(\overline{B}),-2}{\overset{c\ell(B),1}{\rightleftarrows}} B \circlearrowleft c\ell(Z), -i \qquad c\ell(Z), -i \circlearrowright Z \underset{c\ell(\overline{B}),-2}{\overset{c\ell(B),1}{\rightleftarrows}} B \wedge Z \circlearrowleft c\ell(Z), i \tag{8.21}$$

where the above two spaces of $\wedge^* \mathcal{Z}_\mathfrak{n}$ form two non-isomorphic $\mathbb{C}\ell(\mathfrak{n})$-modules.

From now on we fix $c\ell(Z)$ acting on 1 by i, and on Z by $-i$. This consequently makes $c\ell(XYZ) = c\ell(\omega_\mathbb{C}) = 1$. Now the rest of the computation follows in the same vein as Section 6.3. We write the ordered orthonormal basis of $\mathcal{S}_\mathfrak{n}$ as $\{\omega_1, \omega_2\} = \{1, \frac{1}{\sqrt{2}} B\}$ lest confusion:

$$\begin{aligned} \not{D}_\sigma(f \otimes \omega_1) &= c\ell(X)\nabla^S_X(f \otimes \omega_1) + c\ell(Y)\nabla^S_Y(f \otimes \omega_1) + c\ell(Z)\nabla^S_Z(f \otimes \omega_1) \\ &= \sum_{W \in \{X,Y,Z\}} R(W) f \otimes c\ell(W)\omega_1 + \frac{1}{4} f \otimes c\ell(XYZ - YXZ - ZXY)\omega_1 \\ &= R(B) f \otimes c\ell(\overline{B})\omega_1 + R(\overline{B}) f \otimes c\ell(B)\omega_1 + R(Z) f \otimes c\ell(Z)\omega_1 + \frac{1}{4} f \otimes c\ell(\omega_\mathbb{C})\omega_1 \\ &= \sqrt{2} R(\overline{B}) f \otimes \omega_2 + i R(Z) f \otimes \omega_1 + \frac{1}{4} f \otimes \omega_1 \end{aligned} \tag{8.22}$$

Similarly for $\not{D}_\sigma(f \otimes \omega_2)$, one takes similar action:

$$\begin{aligned} \not{D}_\sigma(f \otimes \omega_2) &= R(B) f \otimes c\ell(\overline{B})\omega_1 + R(\overline{B}) f \otimes c\ell(B)\omega_1 + \frac{1}{4} f \otimes c\ell(\omega_\mathbb{C})\omega_2 \\ &= -\sqrt{2} R(B) f \otimes \omega_1 - i R(Z) f \otimes \omega_2 + \frac{1}{4} f \otimes \omega_2 \end{aligned} \tag{8.23}$$

As a result, the Dirac operator Δ_σ takes the form:

$$\not{D}_\sigma = \begin{pmatrix} iR_Z & -\sqrt{2} R_B \\ \sqrt{2} R_{\overline{B}} & -iR_Z \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \tag{8.24}$$

In particular, this formula gives an ad-hoc Bochner identity in this case:

$$\not{D}_\sigma^2 = \begin{pmatrix} R(\Delta_0) & \\ & R(\Delta_0) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} iR_Z & -\sqrt{2} R_B \\ \sqrt{2} R_{\overline{B}} & -iR_Z \end{pmatrix} + \frac{1}{16} \mathbb{I} \tag{8.25}$$

In particular, we see it differs from $R(\Delta_0)$ also by an lower-order differential operator satisfying the assumptions of Lemma 5.5. Hence by following the argument of Lemma 5.6, we see the heat kernel associated with $\not{D}_\sigma$ is indeed of Schwartz class. One may now apply the Plancherel formula of Example B.7 and compute the eigenvalues of the Dirac operator localized at each unitary representation:

1. For the lowest Z-weight v_0 in each representation, $v_0 \otimes \omega_2$ forms one single block, with a corresponding eigenvalue $\lambda + \frac{1}{4}$;
2. The rest of the Z-weights occur in generic blocks of the form:

$$\begin{pmatrix} -\lambda & -\sqrt{2\lambda(m+1)} \\ \sqrt{2\lambda(m+1)} & \lambda \end{pmatrix} + \frac{1}{4} \cdot \mathbb{I}_2 \tag{8.26}$$

   with corresponding eigenvalue $\frac{1}{4} \pm \sqrt{\lambda^2 + 2\lambda(m+1)}$.

We recapitulate the discussions in this chapter for ease of reference:

Proposition 8.2. *Given the Heisenberg group $\mathbb{H}_3$ with the standard metric. Then the Dirac operator $D\!\!\!/_\sigma$ acting on each irreducible unitary representation $\pi_\lambda : \lambda \neq 0$ gives the following family of spectrum:*

1. $\lambda + \frac{1}{4}$, *of multiplicity* 1;
2. *For each* $m \geq 1$, $\frac{1}{4} \pm \sqrt{\lambda^2 + 2\lambda(m+1)}$, *of multiplicity* 1

Moreover, the Dirac operator induced by the reductive connection ∇^0 on $\mathcal{S}_{\mathfrak{h}_3}$ has symmetric spectrum with respect to 0.

Appendix A

Lie groups, kernels and abstract representation theory

In this appendix we collect some essential information for the discussion of heat kernel asymptotics in Chapter 5. This includes a development of the kernel method for arbitrary strong elliptic operators on unimodular Lie groups, originally developed by Langlands and Nelson concurrently.

A.1 Convolution kernels and Langlands' dissertation

In this section we prove existence, universality and analyticity of kernels of the semigroups generated by strong elliptic operators associated with any invariant strongly elliptic operators on any Lie group G. The culmination is the exponential decay of the kernels of such operator, as functions of G. The result was attributed to Langlands in his laconic dissertation, though a much exhaustive discussion can be found in [**Rob91**].

For this section only we consider the cases when G might not be unimodular. By stating function spaces of G such as $L^p(G)$, we stick to the convention of the left Haar measure.

Recall some basics of topologies on function spaces. Given $\mathscr{X}$ a Banach space, the dual space $\mathscr{X}^*$ of all bounded linear functionals is, with operator norm, also a Banach spaces. A predual $\mathscr{X}_*$ of $\mathscr{X}$ is a Banach space such that $\mathscr{X}$ is the dual of $\mathscr{X}_*$.

By a standard result of Yosida [**BR87**, Corollary 3.1.8] we define a continuous representation (H_π, π) of π to be a **strongly continuous** if for each $g \in G$ and $v \in H$, $\pi(g)v$ defines a continuous map on the dual representation H_π^*, this is equivalent to the following map to be continuous in norm for each $v \in H_\pi$:

$$\mathrm{ev}_v : G \to H_\pi \qquad g \mapsto \pi(g)v \tag{A.1}$$

We define the representation π to be a **weak* continuous** if each $\pi(g)v$ defines a continuous map on the predual $(H_\pi)_*$, equivalently this means $\pi(g)v$ defines a continuous map on $(H_\pi)_*$.

The major examples of strongly continuous representations are $L^p(G; V) = L^2(G) \otimes V^*$ for $p \in [1, \infty)$ and V a finite-dimensional representation. The example of a weak* continuous representation are $L^\infty(G; V)$ and again V a finite-dimensional representation.

Let $(X_1, \dots, X_d)$ a basis of $\mathfrak{g}$. If we take a left-invariant Riemannian metric on $\mathfrak{g}$, this defines a metric distance d on G, and we denote $|g|$ as the distance $d(e_G, g)$ and the Haar measure as $\mathrm{d}g$. Denote $D_i = \pi(X_i)$. A differential operator affiliated with π and G is defined as a polynomial in D_i. Define the symbol of a differential operator $\sum_I c_I D^I$ for $I \in \mathbb{C}$ to be $\xi \mapsto \sum_I c_I \xi^I$ and the principal symbol the highest order term, i.e., $P_m(\xi) = \sum_{|I|=m} c_I D^I$.

We say a differential operator $D = \sum_I c_I D^I$ is elliptic if $P_m(\xi) \neq 0$ if $\xi \in \mathbb{R}^n \backslash \{0\}$. It is **strongly elliptic** if:

$$\mathrm{Re}\left((-1)^{\frac{m}{2}} \sum_{|I|=m} c_I \xi^I\right) > 0 \tag{A.2}$$

for all $\xi \in \mathbb{R}^d \backslash \{0\}$.

The first major result of Langlands is the following:

Theorem A.1 ([Rob91, Theorem I.5.1]). *Let D be a strongly elliptic operator on the weak* or strongly continuous representation (H_π, π). Then it is closable and its closure $\overline{D}$ generates a holomorphic semigroup, denoted as $e^{-t\overline{D}}$ on respective representations satisfying:*

1. *$e^{-t\pi(\overline{D})}$ maps its domain into the smooth vectors H_π^∞ of H_π for all $t > 0$;*
2. *The map $z \mapsto e^{-z\pi(\overline{D})}$ defines a holomorphic map in the sector $\{z \in \mathbb{C} : |\arg z| < C\}$ for some $C \in (0, \pi/2]$ depending on the elliptic operator.*
3. *If the principal coefficients $\{c_\alpha \mid |\alpha| = m\}$ are all real, then $e^{-z\pi(\overline{D})}$ is holomorphic in the open right half-plane, that is C can be chosen to be $\frac{\pi}{2}$.*

One also needs the following small-time kernel estimate, which stands as a byproduct in proving the analytic vectors of a continuous representation of a Lie group is dense in the representation space. We denote $C^n(H_\pi)$ as the C^n-vectors in the representation H_π:

$$C^n(H_\pi) = \bigcap_{1 \le i_1, \ldots, i_n \le d} \mathrm{Dom}(\pi(X_{i_1} \ldots X_{i_n})) \quad \text{with norm } \|v\|_{C^n(H_\pi)} = \sup_{|I| \le n} \left\| \pi(X^I) v \right\|_{H^\pi} \tag{A.3}$$

and the analytic vectors of H_π are defined to be those $v \in H_\pi$ such that $\sum_{n \ge 1} \frac{t^n}{n!} \|v\|_{C^n(H_\pi)} < \infty$ for some $t > 0$.

Theorem A.2 ([Rob91, Theorem II.2.2]). *Let D be a strongly elliptic operator of order m with the corresponding semigroup $e^{-t\pi(\overline{D})}$ acting on the weak* or strongly continuous representation (H_π, π). Then there exists $k, l > 0$ such that:*

$$\left\| e^{-t\pi(\overline{D})} v \right\|_{C^n(H_\pi)} \le k l^n n! \, \|v\|_{H_\pi} t^{-\frac{n}{m}} \tag{A.4}$$

for all $v \in H_\pi$ and $t \in (0, 1]$. Consequently the analytic vectors $(H_\pi)_{\mathrm{an}}$ forms a dense subspace in H_π.

This result was proved independently by Langlands and Nelson, and its proof can be found in numerous sources. We therefore omit he proof but remind one important fact, that we consider the subspace the analytic vectors of $H_\pi(D)$, which equals to $\bigcup_{t>0} e^{-t\pi(\overline{D})} H_\pi$ and is dense in H_π. The above theorem amounts to prove $(H_\pi(D))_{\mathrm{an}}$ is a subspace of $(H_\pi)_{\mathrm{an}}$.

The above definitions give natural definition of Hölder spaces $C^p(G)$ and Sobolev spaces $W^{k,p}(G)$ by taking H_π to be $C(G)$ and $L^p(G)$ respectively. This gives the following adaptation of the Sobolev embedding lemma:

Lemma A.3. Let U be a bounded open subset of G_0, and $p \in [1, \infty)$. Then for $np < d = \dim G$, the following embedding is continuous:

$$W^{p,n}(U) \subseteq L^{\frac{dp}{d-np}} \tag{A.5}$$

Also for $0 \le m < n - \frac{n}{p}$ the Sobolev space $W^{p,n}$ can be continuously embedded into the Hölder space $C_m(U)$.

The proof descends from $\mathbb{R}^d$-case by choosing a small chart such that the exponential map is an isometry. General bounded U then follow suit by a covering argument.

We now start considering the measure associated with the heat semigroup. Let μ is a complex measure on G. Then the representation π is said to be **μ-measurable** if the operator $\pi(\mu)$ is an bounded operator:

$$\pi(\mu) v = \int_G (\pi(g) v) \mu(\mathrm{d}g) \tag{A.6}$$

on the domain $\mathrm{Dom}(\pi(\mu))$ which contains all $v \in H_\pi$ such that the map $g \mapsto \pi(g)v$ is μ-measurable.

Define now a **convolution semigroup** in G as a family of complex measures $\{\mu_t\}_{t>0}$ with the following properties:

1. $\mu_{s+t}(\mathrm{d}g) = \int_G \mu_t(h^{-1}\,\mathrm{d}g)\mu_s(\mathrm{d}h)$;
2. the map $t \mapsto \mu_t$ is weakly continuous;
3. for each open neighborhood U of e_G, $\lim_{t\to 0}\int_U \mu_t(\mathrm{d}g) = 1$.

The main result in this section is the following result from Langlands' thesis:

Theorem A.4 ([**Rob91**, Theorem III.2.1])**.** *Let G and $X_1, \ldots, X_d$ as above and D be a strongly elliptic operator. Then:*

1. *there exists a convolution semigroup μ_t such that each strong continuous or weakly-$*$ continuous representation (H_π, π) of G is μ-measurable and:*
$$e^{-t\pi(\overline{D})} = \pi(\mu_t) = \int_G \pi(g)\mu_t(\mathrm{d}g) \tag{A.7}$$
with $e^{-t\pi(\overline{D})}$ the holomorphic semigroup generated by the closure of D in H_π as in the theorem above.
2. *The measure μ_t is absolutely continuous with respect to the left Haar measure $\mathrm{d}g$ and hence there exists a unique $k_t \in L^1(G)$ such that for every measurable $M \subset G$,*
$$\int_M \mu_t(\mathrm{d}g) = \int_M k_t(g)\,\mathrm{d}g \tag{A.8}$$
3. *The kernel $(t, g) \mapsto k_t(g)$ defines an analytic function on $\mathbb{R}_+ \times G$ such that if we take D^* to be the formal adjoint with k_t^* the corresponding kernel, then $k_t^*(g) = \Delta(g)^{-1}\overline{k_t(g^{-1})}$ with $\Delta(g)$ the modular function of G.*

We outline the proof here to collect the essential elements that we recycle in the proof of Lemma 5.1.

1. Assume for simplicity the group is unimodular. Then the right regular representation acts on $L^1(G)$ by isometry between the metric $|g|$ that was chosen above, and the Euclidean metric chosen on $\mathfrak{g}$ accordingly (c.f. [**Rob91**, p. 13]). Now suppose T is a bounded operator on $L^1(G)$ which commutes with $R(G)$, then T is a multiplier, i.e., there is a complex measure μ such that $L(G)$ is μ-measurable and $T = \mu*$ a convolution operator [**Rob91**, Lemma 2.2]. Now $e^{-t\overline{D}} := e^{-tL(\overline{D})}$ is a G-invariant operator and hence commutes with right translations, hence we can form a family of complex measures $\mu = \{\mu\}_t$ such that $e^{-t\overline{D}} = \mu_t*$. This establish the first statement for $L^1(G)$.

To extend it to arbitrary representations, we first establish it on $C_0(G)$. Given $\phi \in L^1(G)$ and $\psi \in C_0(G)$, Young's inequality gives $\|\phi * \psi\| \le \|\phi\|_1 \cdot \|\psi\|_\infty$, therefore we have $L^1(G) \to B(C_0(G))$ by convolution. Moreover, because convolution preserves regularity by $X(\phi * \psi) = (X\phi) * \psi$, so if $\phi \in C^n(G) \cap L^1(G)$, then $\phi * \psi \in C^n(G) \cap C_0(G)$. Hence if D is a strongly elliptic operator of order m, then $\phi * \psi \in \mathrm{Dom}(D)$ for $\phi \in C^m \cap L^1$, and $D(\phi * \psi) = (D\phi * \psi)$. But now if we consider $e^{-t\overline{D}}$ the semigroup generated by $\overline{D}$ on $L^1(G)$, we see $e^{-t\overline{D}}\phi \in C^\infty(G) \cap L^1(G)$ and it solves the following differential equation:
$$\frac{\partial}{\partial t}(e^{-t\overline{D}}\phi * \psi) = D(e^{-t\overline{D}}\phi) * \psi = D(e^{-t\overline{D}}\phi * \psi) \tag{A.9}$$
with the first identity of a consequence of that $f = e^{-t\overline{D}}\phi$ being the solution on $L^1(G)$ of $(\frac{\partial}{\partial t} + D)f = 0$. Now $e^{-t\overline{D}}\phi*\psi$ solves the heat equation with the corresponding kernel $e^{-t\overline{D}'}$, but now by the uniqueness of the solution of the abstract Cauchy problem we have $e^{-t\overline{D}}\phi * \psi = e^{-t\overline{D}'}(\phi * \psi)$. Consequently from the result we established on $L^1(G)$ of $e^{-t\overline{D}}$ we have $e^{-t\overline{D}'} = \mu_t * (\phi * \psi)$, that is the function $\phi * \psi$ is μ_t-measurable for each $t > 0$, where the semigroup acts by left convolution. Therefore we extend the result to $C_0(G)$.

Lastly the result is extended to general representations by extending the arguments to the weighted L^1-space with the weighing factor $\lambda(g) := \|\pi(g)\| + \|\pi(g^{-1})\|$:
$$L^1_\pi(G) = \left\{\psi \in L^1(G) \mid \int_G |\phi(g)|\lambda(g)\,\mathrm{d}g < \infty\right\} \tag{A.10}$$
using similar methods.

2. For the second statement, we apply the first statement of the theorem to the representation $(L^1(G), L_g)$ to produce a convolution semigroup $e^{-t\overline{D}}$ with the corresponding measure μ_t. Then by Theorem A.1 $e^{-t\overline{D}}\phi \subseteq C^\infty$ for $\phi \in L^1$. In particular, they are continuous function which by Sobolev embedding Lemma A.3 have for each bounded open neighborhood U of e_G:

$$\left|e^{-t\overline{D}}\phi(h)\right| \leq c_U \left\|e^{-t\overline{D}}\phi\right\|_{W^{1,d+1}(G)} \tag{A.11}$$

Next by kernel estimate (A.4), one has $\left\|e^{-t\overline{D}}\phi\right\|_{W^{1,d+1}} \leq ct^{-\frac{d+1}{m}} \|\phi\|_1$ for $t \in (0,1]$, with suitable $c > 0$. On the other hand, for large t, the fact $e^{-t\overline{D}}$ being a holomorphic family of bounded operators for $t > 0$ allows we choose $\omega = \inf_{t>0} \frac{\log\|e^{-t\overline{D}}\|}{t}$, with the norm taken as operator norm, and consequently

$$\left\|e^{-t\overline{D}}\phi\right\|_{L^1(G)} \leq Me^{\omega t} \|\phi\|_{L^1(G)} \tag{A.12}$$

for all $t > 0$. Combining together, we have a universal bound of the supremum of $e^{-t\overline{D}}$ for all $t > 0, \phi \in L^1(G)$:

$$|e^{-t\overline{D}}\phi(h)| \leq c'_U t^{-\frac{d+1}{m}} e^{\omega' t} \|\phi\|_1 \tag{A.13}$$

with suitable constant $c'_U, \omega' > 0$. Consequently $e^{-t\overline{D}} : L^1(G) \to C(\Omega)$ defines a bounded map for all $t > 0$. By Riesz representation theorem there is an essentially unique bounded measurable function $K_{t,h} : g \mapsto K_t(h,g)$ such that for each $t > 0, h \in \Omega$:

$$e^{-t\overline{D}}\phi(h) = \int_G K_t(h,g)\phi(g)\,\mathrm{d}g$$

for all $\phi \in L^1(G)$. But by the first statement of the theorem $e^{-t\overline{D}}\psi(h) = \int_G \psi(g)\mu_t(h\,\mathrm{d}g^{-1})$ for each $\psi \in C_c(G)$. Hence the two measures agree on $C_c(G)$. Moreover, by taking $h = e$ and by using the transformation relation $\mu_t(\mathrm{d}g^{-1}) = \Delta(g^{-1})\mu_t(\mathrm{d}g)$, we define:

$$k_t(g) := \Delta(g^{-1})K_t(e_G, g^{-1}) \tag{A.14}$$

then $\mu_t(\mathrm{d}g) = k_t(g)$. But now the left translation on $C_0(G)$ identifies $e^{-t\overline{D}}\phi = K_t * \phi(g)$ for $\phi \in C_0(G)$, hence $\left\|e^{-t\overline{D}}\right\| = \|k_t\|_{L^1}$, and consequently $e^{-t\overline{D}}$ being bounded establishes $k_t \in L^1(G)$.

3. To prove the analyticity of K_t, we begin by arguing like in the second statement to get an exponential bound with respect to time. Again by Sobolev embedding Lemma A.3 that for each bounded open neighborhood U of e_G and for all $t > 0$, there is a $c, \omega > 0$ such that:

$$\left\|e^{-t\overline{D}}\phi\right\|_{C^1(U)} \leq c_U\phi \left\|e^{-t\overline{D}}\right\|_{W^{1,d+2}} \leq c'_U t^{\frac{d+2}{m}} e^{\omega' t} \|\phi\|_{W^{1,d+2}} \tag{A.15}$$

for all $\phi \in L^1(G)$. But now as a integral kernel, the essential supremum of $K_t(h,g)$ can be identified with the operator norm of $e^{-t\overline{D}}$, but as $e^{-t\overline{D}}$

$$\operatorname*{ess.sup}_{g\in G} |K_t(h,g) - K_t(e_G,g)| \leq \sup\left\{e^{-t\overline{D}}\phi(h) - e^{-t\overline{D}}\phi(e_G)\middle| \|\phi\|_{L^1(G)} \leq 1\right\} \leq c''_\Omega |h| t^{\frac{d+2}{m}} e^{\omega' t}$$

with the last inequality an application of Duhamel's principle to (A.15). Dually for right multiplication, one replace $e^{-t\overline{D}}$ by $\Delta(g^{-1})e^{-t\overline{D}}\Delta(g)$ and obtains the same estimate for $\Delta(h^{-1}g)K_t(h,g)$. hence the essential spectrum of $k_t(h^{-1}) - k_t(e_G)$ goes to 0 as $h \to e_G$, and we can consequently assume k_t to be continuous (up to measure equivalence). Now some easy formal computation yields $k_{s+t} = k_t * k_s = e^{-t\overline{D}}k_s$ and left regular G-action on $L^1(G)$ can be lifted to the G act on the family $\{e^{-t\overline{D}}k_s\}$. Again we exploit that the small time bound (A.4) gives the estimate:

$$\left\|e^{-t\overline{D}}k_s\right\|_{C^n(G)} \leq kl^n n! t^{-\frac{n}{m}} \|k_s\|_{L^\infty(G)}$$

for $t \in (0,1]$. Now the analyticity in the g-direction is immediate as $\sum_{n\geq 1} \frac{r^n}{n!} \left\| e^{-t\overline{D}} k_s \right\|_{C^n}$ clearly converges by taking $r \leq \frac{1}{l}$ in the inequality. One the other hand, the analyticity at t-direction also can be readily derived from the holomorphy of $e^{-z\overline{D}}$ in the Theorem A.1. To prove the joint analyticity, one needs a real analytic version of Hartog's theorem (see [**BR87**, Page 109]), with the additional assumption that one may uniformly bound the derivatives in all directions, that is one can choose $c_1, c_2 > 0$ such that $\sup_{s\in U} |\partial_i^k \phi(s)| \leq ab^k k!$ for all direction $i = 1, \ldots, d$. But this uniform bound is direct from the above two estimates, hence we see $K : (t, g) \mapsto K_t(g^{-1}h)$ is analytic.

Lastly the adjointness of the kernel follows easily by mimicking the classical argument in $\mathbb{R}^n$ with the extra $\Delta(g)^{-1}$-factor comes from right translation. This concludes the proof of the theorem. □

Remark A.5. The significance of this result lies in the fact that the complex measure μ depends on G and its chosen basis X, also the symbol, but not on the particular representation. In our applications we bound eventually the kernels by an Gaussian bound, which in particular only depending on the height of g, and therefore the bound can be derived as a quantity depending solely on G.

Appendix B

Representation theory of nilpotent Lie groups

We collect some basics about representation theory, with a quick introduction to the representation theory of nilpotent Lie groups and the orbit method.

B.1 Fourier transform on groups

In this section we give a quick introduction of the abstract Plancherel formula, sacrificing generality for simplicity. Assume G to be a second countable locally compact unimodular group of **type I**. For the definition of Type I groups, we refer to [BdlH19, Chapter 6] for details. Also refer their for a more detailed introduction for general measurable fields of representations. Denote $\widehat{G}$ as the equivalence classes of irreducible unitary representations.

Theorem B.1. [Dix82, Theorem B.2.32] *Let G be a group as above. We fix a left G-invariant Haar measure* $\mathrm{d}g$. *Then there exists a positive σ-finite measure μ on $\widehat{G}$ Let λ be the left-regular representations of G. Then there exists a positive measure μ on $\widehat{G}$ and an isomorphism W:*

$$W : L^2(G) \cong \int_{\widehat{G}} H_\pi \otimes H_\pi^* \,\mathrm{d}\mu(\pi) \qquad \lambda \cong \int_{\widehat{G}} \pi \otimes \mathrm{id}_{H_\pi^*} \,\mathrm{d}\mu(\pi) \tag{B.1}$$

where H^* is the dual Hilbert space of H. There is a complete analog for the right regular representations.

B.2 Nilpotent Lie groups and Plancherel formula

This section contains some basics regarding the orbit method on nilpotent Lie groups and a short introduction to the Plancherel formula in this case. For a detailed account one is to refer to [CG90].

Definition B.2 (**coadjoint representation**)**.** Given a Lie group G, define the coadjoint representation G on $\mathfrak{g}^*$ as following: Given $g \in G$,

$$\operatorname{CoAd} g(l)(Y) := l(\operatorname{Ad}_{g^{-1}} Y) \qquad \forall Y \in \mathfrak{g}, l \in \mathfrak{g}^*. \tag{B.2}$$

Its derivative at e_G gives the corresponding representation of $\mathfrak{g}$ on $\mathfrak{g}^*$:

$$\operatorname{coad} X(l)(Y) := l([Y, X]) = l(\operatorname{ad}_{-X} Y) \qquad \forall X, Y \in \mathfrak{g} \quad l \in \mathfrak{g}^* \tag{B.3}$$

Theorem B.3 (**Baker-Campbell-Hausdorff Formula**)**.** *Let G be a connected Lie group, with exponential map* $\exp : \mathfrak{g} \to G$, *then $X * Y = \log(\exp X \cdot \exp y)$ defines an analytic function near $X = Y = 0$ on G with a universal power series expression, independent of choice of basis:*

$$X * Y = \sum_{n>0} \frac{(-1)^n}{n} \sum_{\substack{p_i+q_i>0 \\ \forall 1 \le i \le n}} \frac{\left(\sum_{i=1}^n (p_i + q_i)\right)^{-1}}{p_1! q_1! \cdots p_n! q_n!} \times (\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \cdots (\operatorname{ad} X)^{p_n} (\operatorname{ad} Y)^{q_n - 1} Y \tag{B.4}$$

Similarly, the adjoint action of $\exp X$ *on* Y *admits the following formula:*

$$\mathrm{Ad}_{e^X} Y = e^{\mathrm{ad}_X Y} = \sum_{n=0}^{\infty} \frac{[(X)^n, Y]}{n!} \tag{B.5}$$

where $[(X)^n, Y] = [X, \cdot [X, Y]]$ *by applying* X n *times. In particular, when* $G = N$ *a nilpotent Lie group, this expression is finite.*

Definition B.4. Given a filtration of subalgebras $\mathfrak{n}_1 \subseteq \cdots \subseteq \mathfrak{n}_k = \mathfrak{n}$ of a nilpotent Lie algebra $\mathfrak{n}$ with $\dim \mathfrak{n}_j = m_j$:

1. A basis $\{X_1, \cdots X_n\}$ of $\mathfrak{n}$ is called a **weak Malcev basis** if for each m, the $\mathbb{R}$-vector space spanned by $\mathfrak{h}_j = \{X_1, \cdots X_m\}$ is a subalgebra of $\mathfrak{n}$, with $\mathfrak{h}_{m_j} = \mathfrak{n}_j$ for all j;
2. If furthermore the filtration can be chosen to consist of ideals of $\mathfrak{n}$, we can pick X_j with each $\mathfrak{h}_m$ to be ideals of $\mathfrak{n}$. We call such a basis **strong Malcev basis**.

As paralleled with the discussion in semisimple Lie group of Wolf's class, we also give a definition of Schwartz space on the nilpotent Lie groups. In contrast to the reductive group case in Definition 2.28, the Schwartz functions on nilpotent groups behave much more similarly as the ordinary Schwartz functions on $\mathbb{R}^n$. In fact by choosing a polynomial coordinate map

$$\phi : \mathbb{R}^n \to G \qquad \phi(t_1, \ldots, t_n) = \exp(t_1 X_1) \cdot \ldots \exp(t_n X_n) = \exp(t_1 X_1 * \cdots * t_n X_n) \tag{B.6}$$

with a choice of weak Malcev basis $\{X_1, \ldots, X_n\}$ on N. Then ϕ is a polynomial diffeomorphism with a polynomial inverse [CG90, Proposition 1.2.8]. We then define the Schwartz space on N with its topology induced by the seminorms on $\mathbb{R}^n$ under such map:

Definition B.5 (Schwartz space, nilpotent Lie group). Define the Schwartz space $\mathcal{S}(N)$ on a nilpotent Lie group N to be the locally convex topological vector space that contains all $f \in C^\infty$ functions such that Df remains bounded for all polynomial-coefficient differential operators:

$$\|p_I L_{X^I} f\|_{L^\infty(N)} < \infty \quad \text{for all polynomials } p_I \text{ on } N \tag{B.7}$$

In fact one may define the Schwartz space with the help of any coordinate map ϕ as above, and:

$$\left\|x^I D^J (f \circ \phi)\right\|_{L^\infty(\mathbb{R}^n)} < \infty \qquad \text{for all } D^J = \frac{\partial^{j_1}}{\partial^{j_1} x_1} \cdots \frac{\partial^{j_n}}{\partial^{j_n} x_n}, \quad x^I = x^{i_1} \cdots x^{i_m} \tag{B.8}$$

As opposed to the reductive Lie group case, the existence of polynomial coordinate maps allows us to transfer the problem to the Schwartz function on $\mathbb{R}^n$. Moreover, one no longer stipulates the $U((\mathfrak{n}))$ action on both sides, as both of them in this case give equivalent topology [CG90, Corollary A.2.3].

The following theorem is due to Kirillov [CG90, Theorem 4.3.9]:

Theorem B.6 (Plancherel inversion theorem). *Let* $\{X_1, \ldots, X_n\}$ *be a strong Malcev basis for a nilpotent Lie algebra* $\mathfrak{n}$ *with dual basis* $\{l_1, \ldots l_n\}$. *Define* U *to be the set of generic coadjoint orbits with the index set* $S = \{i_1 < \cdots < i_{2k}\}$ *for* $\mathfrak{r}_l \backslash \mathfrak{n}$ *and* T *the complement of* S. *Then we define the* ***Pfaffian*** *associated to* l *by:*

$$|\mathrm{Pf}(l)|^2 = \det B_l, \quad \textit{where} \quad (B_l)_{jk} = \ell([X_{i_j}, X_{i_k}])$$

as above. Then for $f \in \mathcal{S}(G)$, *the function evaluated at* e_N *is given by an absolutely convergent integral:*

$$f(e_N) = \int_{U \cap V_T} \mathrm{tr}\, \pi_l(f)\ |\mathrm{Pf}(l)|\, \mathrm{d}l$$

with dl *the Lebesgue measure on* $V_T = \mathbb{R} - \mathrm{span}\{l_i : i \in T\}$ *such that the cube determined by* T *has measure* 1.

For convenience, we record the formula when $N = \mathbb{H}_3$ is the Heisenberg group here:

Example B.7. *Given $\mathfrak{h}_3$ the Heisenberg algebra, with a Malcev basis $\{Z, Y, X\}$ such that $[X, Y] = Z$. We partition the basis into $S = \{2, 3\}$ and $T = \{1\}$. Then the generic orbits $U = \{l : l(Z) \neq 0\}$ and $V_T = \mathbb{R}l_1$ in the Plancherel formula. The Pfaffian is given by $\mathrm{Pf}(l) = l(Z)$. Hence by taking $\mathrm{d}l$ as the Lebesgue measure on $\mathbb{R}$, the Plancherel formula of the Heisenberg group is given by:*

$$f(e_G) = \int_{\mathbb{R}} \mathrm{tr}\, \pi_l(f)|l|\, \mathrm{d}l \tag{B.9}$$

with Θ_{π_l} the character of π_l for $l \in Z^* \cong \mathbb{R}$.

Another important feature of the representations of nilpotent Lie groups is that every finite-dimensional representation is unipotent. That is **Engel's theorem** [**CG90**, Theorem 1.1.9]. One immediate consequence of this is that the compact subgroups are meager in nilpotent Lie groups. In fact, they all lie in the center:

Lemma B.8. Let N be a connected nilpotent Lie group and $K \subseteq N$ a maximal compact subgroup. Then K is central.

Proof. Consider the adjoint representation of N on $\mathfrak{n}$. By Engel's theorem, we can choose a suitable basis such that G is the group of upper triangular matrices U. That is $\mathrm{Ad}(K) \subseteq U$ is a compact subgroup. But in U every non-identity element generates a closed non-compact subgroup. Hence K lies in the kernel of the adjoint representation, i.e., K is central. □

Main Index

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