# Invariants of Kähler manifolds and Generalized Seiberg-Witten equations

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*"When life gives you lemons, make lemmata."*

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# Contents

# Introduction

The Seiberg-Witten equations, born out of the fertile grounds of theoretical physics, is one of the main tools in the study of geometry of smooth four-manifolds through the lens of gauge theory. Their profound impact on both mathematics and physics became evident as they ushered in a new era of research in Donaldson theory, offering insights into the classification of smooth structures on four-manifolds. However, the scope of these equations has transcended their original formulation. In the pursuit of understanding the deeper geometrical properties of manifolds, researchers have sought to generalize the Seiberg-Witten equations to explore new avenues of investigation.

This dissertation embarks on a long (and sometimes quite burdensome, when it comes to a couple of computations in coordinates) journey, one that places Kähler structures on manifolds at its core and leverages the generalized Seiberg-Witten equations as a tool of investigation. What makes these equations all the more intriguing is the inclusion of hyperKähler manifolds into the picture, which carry a quaterionic structure on their tangent bundle.

The overall scheme is the following: We fix a four dimensional Kähler manifold $X$ (the main target of our investigation), and define a certain non-linear differential equation, which depends on a certain hyperKähler manifold $M$. The solution space of these equations, after taking the quotient by a gauge group action, will consist of finitely many points. We show that a suitable count of these points only depends on the Kähler structure[1] and voilà, we have cooked up an invariant!

We begin our story by establishing our main protagonists in the first chapter; after recalling some facts about Riemannian manifolds, we take a closer look at Kähler manifolds, which lie in the intersection of Riemannian, symplectic and complex geometry, and therefore bring into being a rich variety of interesting properties. Going from the complex world to the quaternionic, we introduce hyperKähler manifolds and their role in the generalization of spinors.

We then set the scenery for our tale in the second chapter, defining the generalized Seiberg-Witten equations and making a couple of basic observations.

One of the ingredients of these equations will be a fixed hyperKähler manifold $M$, and we will restrict ourselves to the simplest case with $\dim M = 4$. In the third chapter, we explore how these arise via the Gibbons-Hawking construction.

[1]Technically, it will be an invariant of a connected component in the space of all Kähler structures, but of course this will be explained later.

In the fourth chapter, we analyze the resulting solution space, computing its dimension and dealing with compactness and orientation issues. Ultimately, this will lead us to the of well-defined invariants of the underlying Kähler manifold $X$ by counting solutions. These invariants are characteristics of the solution space that remain consistent even when subjected to variations in the Kähler structure.

The final chapter is dedicated to the computation of these invariants, which can be realized as a combinatorical count of certain configurations of complex curves in the underlying Kähler manifold $X$.

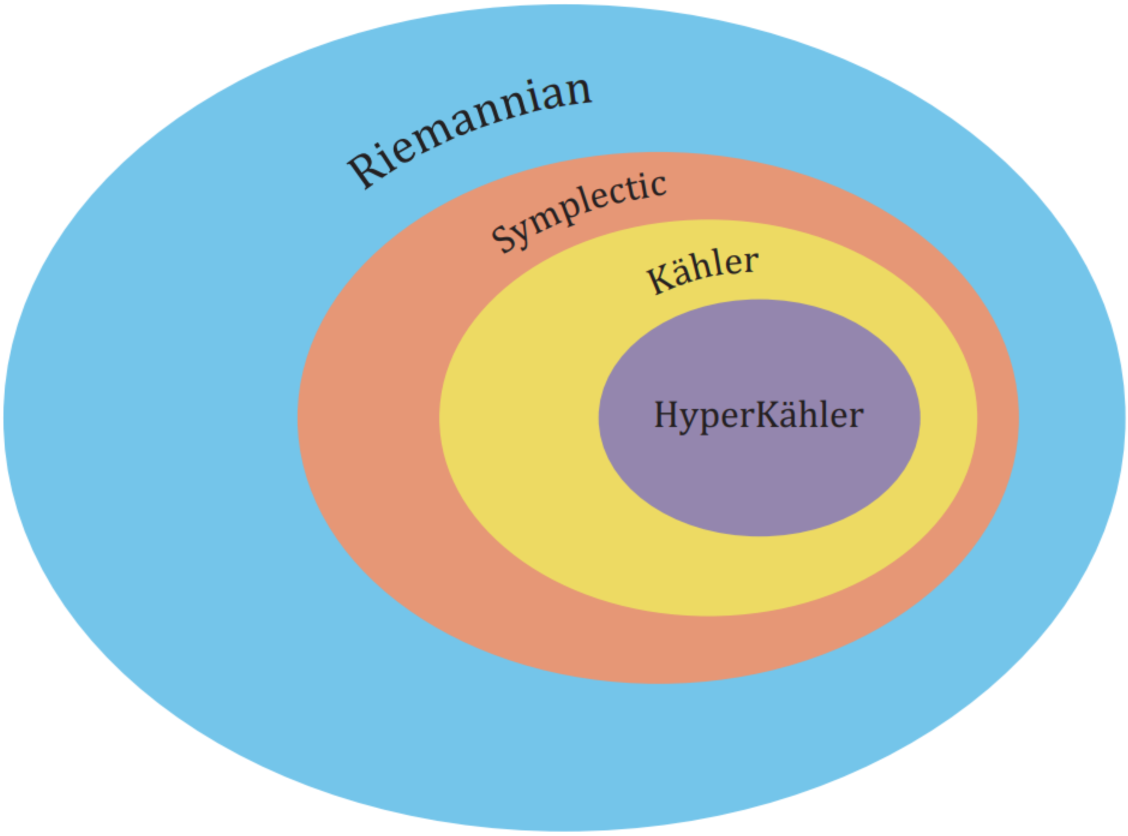
# 1 Preliminaries

*"The beginner ... should not be discouraged if ... he finds that he does not have the prerequisits for reading the prerequisits."*

- Paul Halmos

This chapter serves as an introduction to the basic concepts and tools necessary to understand the setup of the generalized Seiberg-Witten equations. It is by no means a complete overview and only touches on what is needed later on, therefore it assumes some familiarity with the topics of differential geometry, gauge theory and algebraic geometry.

In the following, if not stated otherwise, $\mathsf{X}$ will be a *smooth, compact, oriented, simply-connected four-dimensional manifold*, also referred to as the *source manifold*. For a comprehensive review of the basic notions regarding smooth manifolds, see [15]. We will work our way through the forest of structures one can impose on manifolds, going ever deeper, stopping once in a while to investigate the scenery.



## 1.1 Riemannian and Spin Geometry

In this chapter we explore the notion of a Riemannian metric on a smooth manifold. A Riemannian metric endows the manifold with a notion of distance and angle, and introduces the $L^2$ scalar product and the Hodge operator on differential forms. Additionally, we will introduce the concept of $\mathrm{Spin}/\mathrm{Spin}^{\mathbb{C}}$-structures on manifolds. For in-depth textbooks on Riemannian manifolds, see [13], [16] and [30].

**Definition 1.** A *Riemannian metric* $g$ is a smooth section of $S^2T^*X$, such that for every $x \in X$, $g_x : T_xX \times T_xX \to \mathbb{R}$ is positive-definite. In other words, it is a smoothly varying scalar product on the tangent spaces. We call $(X, g)$ a *Riemannian manifold*.

The Riemannian metric is frequently used to identify the tangent bundle $TX$ with the cotangent bundle $T^*X$ via

$$g^b : TX \to T^*X, v \mapsto g(v, \cdot). \tag{1}$$

In particular, $g$ also induces a scalar product on $T^*X$.

A Riemannian metric $g$ on a manifold $X$ with $\dim(X) = n$ induces a volume form $dvol_g$ in the following way: For $x \in X$ let $e_1, ..., e_n$ be an oriented orthonormal basis of $T_x^*X$, then $dvol_g(x)$ is given by

$$dvol_g(x) = e_1 \wedge ... \wedge e_n. \tag{2}$$

One checks that this definition is independent of the choice of basis and varies smoothly in $x$, therefore gives rise to a globally defined volume form $dvol_g \in \Omega^n(X, \mathbb{R})$.
Furthermore, $g$ induces a scalar product on the exterior algebra $\Lambda^\bullet T^*X := \bigoplus_{k=0}^n \Lambda^k T^*X$ in the following manner: For $x \in X$ let $e_1, ..., e_n$ be an oriented orthonormal basis of $T_x^*X$, then we define a scalar product on $\Lambda^\bullet T_x^*X$ by demanding that

$$\{e_{i_1} \wedge ... \wedge e_{i_k}, i_1 < ... < i_k, k = 0, ..., n\} \tag{3}$$

is an orthonormal basis[2]. This gives rise to a $L^2$ scalar product on the space of

[2] By convention, $\Lambda^0 T^*X$ is the trivial bundle $X \times \mathbb{R}$, and the orthonormal basis vector spanning this space is given by the constant function $1$.

differential forms $\Omega^\bullet(X,\mathbb{R}) := \Gamma(X, \Lambda^\bullet(X,\mathbb{R}))$ by integrating over the manifold:

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \langle \alpha_x, \beta_x \rangle \, \mathrm{dvol}_g(x) \text{ for } \alpha, \beta \in \Omega^\bullet(X,\mathbb{R}) \tag{4}$$

Recall the *Hodge star operator* "$\star$" on a Riemannian manifold $X$, which is defined by the property

$$\int_X \alpha \wedge \star\beta = \langle \alpha, \beta \rangle_{L^2} \,, \;\; \alpha, \beta \in \Omega^k(X,\mathbb{R}) \tag{5}$$

In particular, it is a linear map $\star : \Omega^k(X,\mathbb{R}) \to \Omega^{n-k}(X,\mathbb{R})$ satisfying $\star 1 = \mathrm{dvol}_g$. If $n = 4$, the Hodge star operator maps two-forms into two-forms, and satisfies $\star^2 = 1$. We can thus decompose

$$\Omega^2(X,\mathbb{R}) = \Omega^2_+(X,\mathbb{R}) \oplus \Omega^2_-(X,\mathbb{R}) \tag{6}$$

into the eigenspaces of the eigenvalues $\pm 1$ of $\star$, which we call $\Omega^2_\pm(X,\mathbb{R})$, or the *space of self-dual/anti-self-dual two forms* respectively.

Given two-dimensional submanifolds $C, C'$ of $X$ which meet transversally [3]

$$C \cdot C' = \sum_{x \in C \cap C'} \pm 1 \tag{7}$$

where the $\pm 1$ depends on the orientation at the intersection points. If the submanifolds do not meet transversally, one can deform one of them slightly to achieve transversality and then still define the intersection form. In particular, we can define the intersection of a two-dimensional manifold with itself. We can make this more precise: Denote by $\mathrm{PD}(C) \in H^2(X,\mathbb{Z})$ the Poincare dual of $C$. Then

$$C \cdot C' = \int_M \mathrm{PD}(C) \wedge \mathrm{PD}(C') \,. \tag{8}$$

Thus the intersection form can be computed using integer cohomology, and is therefore purely topological data.

Another important result which we will use later is the *Hodge decomposition*:

**Theorem 1** (Hodge Decomposition). *Let $X$ be compact. Denote the adjoint of the deRham differential $d$ with respect to the scalar product on $\Omega^\bullet(X,\mathbb{R})$ by $d^*$ and define the Laplacian $\Delta := dd^* + d^*d$. Further denote by $\mathcal{H}^k(X,\mathbb{R}) := \ker \Delta \subseteq \Omega^k(X,\mathbb{R})$ the*

[3] This means their intersection is a finite set of points, and their separate tangent spaces at that point together generate the tangent space of the ambient manifold at that point.

*space of harmonic* $k$*-forms. Then we have an orthogonal decomposition*

$$\Omega^k(X,\mathbb{R}) = d(\Omega^{k-1}(X,\mathbb{R})) \oplus \mathcal{H}^k(X,\mathbb{R}) \oplus d^*(\Omega^{k+1}(X,\mathbb{R})) \tag{9}$$

### 1.1.1 Spin and $\mathsf{Spin}^{\mathbb{C}}(4)$ Structures

We will look at yet another viewpoint of a Riemannian geometry using the language of principal bundles, which will lead us to the notion of $\mathsf{Spin}/\mathsf{Spin}^{\mathbb{C}}$ manifolds. For a general overview, see [3], [5], for the material on $\mathsf{Spin}^{\mathbb{C}}$-structures and their use in Seiberg-Witten theory, consult [22]. In the following we assume the reader is comfortable with the concepts of frame bundles, reductions of princial bundles and associated fibre bundle constructions.

Denote by $Gl(X)$ the bundle of frames of $X$, which is a $Gl(n)$-principal bundle[4], where $n = \dim X$. A choice of Riemannian metric (and orientation) is equivalent to a reduction of $Gl(X)$ to a $SO(n)$ frame bundle.

**Definition 2.** Let $SO(X) \to X$ be the *bundle of oriented orthonormal frames* of $X$.

If $\dim(X) = 4$, then the structure group of this principal bundle is $SO(4)$.

**Lemma 1.** *The group* $SO(4)$ *is isomorphic to* $\big(SU(2) \times SU(2)\big)\big/\mathbb{Z}_2$*, the quotient of* $SU(2) \times SU(2)$ *by the normal subgroup* $< (1,1), (-1,-1) >$.

We notice that $SU(2) \simeq Sp(1) = \{q \in \mathbb{H},\ |q| = 1\}$ is the group of unit quaternions and simply-connected, therefore it is also apparent that $SU(2)\times SU(2)$ is the $\mathsf{Spin}(4)$ group, the double covering of $SO(4)$.

**Definition 3.** A ***Spin****-structure* on a Riemmanian manifold $X$ of dimension $n$ is a $\mathsf{Spin}(n)$ reduction $\mathsf{Spin}(X) \to SO(X)$, with respect to the double covering

$$\mathsf{Spin}(n) \to SO(n). \tag{10}$$

Although the notion of spinors is absolutely vital for what is about to come, unfortunately not every four-dimensional manifold carries a Spin structure[5]. But not all is lost, one just has to adjust the group slightly:

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[4] Recall that a frame at a point $x \in X$ is a choice of linear isomorphism $\mathbb{R}^n \to T_xX$. The group $Gl(n)$ of linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ acts via precomposition.

[5] In fact, the obstruction to the existence of a Spin structure is the vanishing of the second Stiefel-Whitney class $\omega_2 \in H^2(X,\mathbb{Z}/2\mathbb{Z})$.

**Definition 4.** The group $\mathsf{Spin}^{\mathbb{C}}(4)$ is defined as the quotient of $S^1 \times \mathsf{Spin}(4)$ by the normal subgroup $< (1,1,1), (-1,-1,-1) >$, or in other words:

$$\mathsf{Spin}^{\mathbb{C}}(4) = \Big(S^1 \times SU(2) \times SU(2)\Big)\Big/\mathbb{Z}_2 \tag{11}$$

We have a natural homomorphism

$$\mathsf{Spin}^{\mathbb{C}}(4) \to SO(4),\ [\lambda, q_+, q_-] \mapsto [q_+, q_-] \tag{12}$$

and therefore we give the following definition analogous to a Spin structure:

**Definition 5.** A $\mathsf{Spin}^{\mathbb{C}}(4)$-*structure* on a Riemmanian manifold $X$ is a $\mathsf{Spin}^{\mathbb{C}}(4)$-reduction $\Sigma \to SO(X)$, with respect to the map above.

$\mathsf{Spin}^{\mathbb{C}}(4)$-structures are not unique if they exists. This is summarized in the next two lemmas:

**Lemma 2** ([22])**.** *A simply-connected four-dimensional Riemannian manifold always admits a* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structure.*

**Lemma 3** ([22])**.** *The additive group* $H^2(X, \mathbb{Z})$ *acts on the isomorphism classes of* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structures on* $X$ *in the following way: For* $\omega \in H^2(X,\mathbb{Z})$ *there exists a complex line bundle* $L$ *on* $X$ *admitting a hermitian form such that* $c_1(L) = \omega$*. Given a* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structure* $\Sigma \to SO(X)$ *on* $X$*, define a new* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structure* $L \bullet \Sigma$ *as the bundle* $\Sigma \times_{S^1} P_L$*, where* $P_L$ *is the* $S^1$*-bundle of unitary frames on* $L$*. This action is free and transitive.*

Here our convention is that

$$\Sigma \times_{S^1} P_L = \Big(\Sigma \times P_L\Big)/\sim \text{ with } (p,q) \sim (e^{it}.p, e^{-it}.q)$$
$$\text{and } p, q \text{ project to the same point in } X.$$

In other words, the set of isomorphism classes of $\mathsf{Spin}^{\mathbb{C}}(4)$ classes on $X$ is a $H^2(X,\mathbb{Z})$-*torsor*, meaning it is non-canonically isomorphic to $H^2(X,\mathbb{Z})$, in the sense that the "difference" of two $\mathsf{Spin}^{\mathbb{C}}(4)$ structures is an element in $H^2(X,\mathbb{Z})$[6]. In particular, usually there is no "canonical" $\mathsf{Spin}^{\mathbb{C}}(4)$-structure on a manifold.

The groups above have some interesting representations which give rise to associated vector bundles, which we will now investigate.

[6]Notice the similarity to affine spaces.

**Lemma 4.** *Let* $\Sigma$ *be a* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structure on* $X$*. We define the following representations of the group* $\mathsf{Spin}^{\mathbb{C}}(4)$*:*

*(1) The positive/negative spinor representations:*

$$\rho_{\pm} : \mathsf{Spin}^{\mathbb{C}}(4) \curvearrowright W^{\pm} \simeq \mathbb{H},\ [\lambda, q_+, q_-] \mapsto \left(h \mapsto q_{\pm} h \lambda\right)$$

*giving rise to the* ***positive/negative Spinor bundles*** $S^{\pm}(\Sigma) := \Sigma \times_{\rho_{\pm}} W^{\pm}$.

*(2) The determinant representation:*

$$\rho_{\det} : \mathsf{Spin}^{\mathbb{C}}(4) \to U(1) \curvearrowright \mathbb{C},\ [\lambda, q_+, q_-] \mapsto \left(z \mapsto \lambda^2 z\right)$$

*giving rise to the* ***determinant bundle*** $\det(\Sigma) := \Sigma \times_{\rho_{\det}} \mathbb{C}$.

*(3) The canonical* $SO(4)$*-representation:*

$$\rho_c : \mathsf{Spin}^{\mathbb{C}}(4) \to SO(4) \curvearrowright \mathbb{R}^4 \simeq \mathbb{H},\ [\lambda, q_+, q_-] \mapsto \left(h \mapsto q_+ h \overline{q_-}\right)$$

*giving rise to the* ***tangent bundle*** $TX = \Sigma \times_{\rho_c} \mathbb{R}^4$.

*(4) The self-dual/anti-self-dual two form representations:*

$$\Lambda_{\pm} : \mathsf{Spin}^{\mathbb{C}}(4) \to SO(3) \curvearrowright \Lambda^2_{\pm}\mathbb{R}^4 \simeq \mathrm{Im}(\mathbb{H}),\ [\lambda, q_+, q_-] \mapsto \left(h \mapsto q_{\pm} h \overline{q_{\pm}}\right)$$

*giving rise to the* ***bundle of self-dual/anti-self-dual two forms*** $\Lambda^2_{\pm} T^*X = \Sigma \times_{\Lambda_{\pm}} \Lambda^2_{\pm}\mathbb{R}^4$.

Notice that the last two reprentations factor through $SO(4)$, which is not surprising, as the associated bundles can be defined without a choice of $\mathsf{Spin}^{\mathbb{C}}(4)$-structure.

**Lemma 5.** *Let* $X$ *be as above.*

*(1) The map*

$$\mathsf{Spin}^{\mathbb{C}}(4) \to U(1) \times SO(4),\ [\lambda, q_+, q_-] \mapsto (\lambda^2, [q_+, q_-]) \tag{13}$$

*is 2:1.*

*(2) Given a* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structure* $\Sigma \to X$ *over* $X$*, we have an induced 2:1 map*

$$\Sigma \to \det(\Sigma) \times_X SO(X) \tag{14}$$

*(3) Given a connection* $a$ *on* $\det(\Sigma)$*, and a connection* $A_0$ *on* $SO(X)$*, there exists a unique lift of* $a \oplus A_0$ *to a connetion* $A$ *on* $\Sigma$*.*

It is also interesting on how the associated bundles change when we change the $\mathrm{Spin}^{\mathbb{C}}(4)$-structure:

**Lemma 6** ([22])**.** *Assume* $\Sigma$ *is a* $\mathrm{Spin}^{\mathbb{C}}(4)$*-structure on* $X$*, and* $S^{\pm}(\Sigma)$ *are the associated spinor bundles. Let* $L \to X$ *be a complex line bundle and* $L \bullet \Sigma$ *the corresponding twisted* $\mathrm{Spin}^{\mathbb{C}}(4)$*-structure. Then we have*

- $S^{\pm}(L \bullet \Sigma) = S^{\pm}(\Sigma) \otimes L$
- $\det(L \bullet \Sigma) = \det(\Sigma) \otimes L^2$.

## 1.2 Symplectic and Kähler Geometry

If Riemannian geometry is the study of a certain symmetric non-degenerate bilinear form on the tangent bundle, then symplectic geometry is the study of a certain anti-symmetric non-degenerate bilinear form on the tangent bundle. It turns out that these two concepts together are intricately related to complex geometry. For more details on symplectic geometry, see [18], for complex geometry refer to [12].

**Definition 6.** A *symplectic form* on a $X$ is a closed non-degenerate two form $\omega$, i.e. $d\omega = 0$, and for every point $x \in X$ the skew-symmetric pairing on the tangent space $T_xX$ defined by $\omega$ is non-degenerate. We then call $(X, \omega)$ a *symplectic manifold* and $\omega$ a *symplectic structure on* $X$.

**Definition 7.** An *almost complex structure* $J$ on a manifold $X$ is an endomorphism $J \in \mathrm{End}(TX)$, such that $J^2 = -\mathrm{id}$. We call an almost complex structure $J$ *compatible* with a symplectic form $\omega$ on $X$, if $\omega(Jv, Jw) = \omega(v, w)$ for all $x \in X$ and $v, w \in T_xX$, and $\omega(v, Jv) > 0$ for all non-zero tangent vectors $v \in TX$.

Notice that if $J$ is compatible with $\omega$, then the bilinear form $g(v, w) := \omega(v, Jw)$ defines a Riemannian metric on $X$.

**Definition 8.** We say that $(X, g, \omega, J)$ is a *compatible triple*[7], if $J$ is compatible with $\omega$ and $g(v, w) = \omega(v, Jw)$. A manifold $X$ endowed with such a structure is called an *almost Kähler manifold.*

[7] The triple $(g, \omega, J)$ satisfy the so called *2 out of 3 principle*, meaning that two of these structures always define the third. This stems from the fact that the pairwise intersection of $Gl(n, \mathbb{C})$, $Sp(2n)$ and $SO(2n)$ in $Gl(2n, \mathbb{R})$ is always $U(n)$.

**Theorem 2** ([18], Prop 4.1). *Let* $(X, g)$ *be a Riemmanian manifold, and let* $\omega$ *be a symplectic structure on* $X$. *Then there exists a canonical almost complex structure* $J$ *compatible with* $\omega$, *such that* $J$ *is skew-symmetric with respect to* $g$.

Notice that in theorem above, in general $g(v, w) \neq \omega(v, Jw) =: g_J(v, w)$, but the triple $(\omega, J, g_J)$ defines an almost Kähler structure on $X$.

Given an almost complex structure $J$ on a four manifold $X$, it is natural to ask the question whether we can cover $X$ by charts, such that in each chart $J$ corresponds to the canonical complex structure on $\mathbb{R}^4 \simeq \mathbb{C}^2$.

**Definition 9.** We call an almost complex structure $J$ on $X$ *integrable* if the above condition is satisfied.

**Definition 10.** Given an almost complex structure $J$ on $X$, we define the *Nijenhuis tensor*

$$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \tag{15}$$

for two vector field $X, Y : M \to TM$.

**Theorem 3** ([12]). *An almost complex structure is integrable if and only if* $N_J \equiv 0$.

**Theorem 4** ([12]). *Let* $\omega$ *be a non-degenerate two-form on* $X$ *compatible with* $J$, *and denote by* $\nabla^{LC}$ *the Levi-Civita connection associated to the Riemannian metric* $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$. *Then the following are equivalent:*

*(1)* $\nabla^{LC} J = 0$.

*(2)* $J$ *is integrable and* $\omega$ *is closed.*

**Definition 11.** A *Kähler manifold* $(X, g, J, \omega)$ is an almost Kähler manifold, such that the complex structure $J$ is integrable.

### 1.2.1 Canonical $\mathrm{Spin}^{\mathbb{C}}(4)$-structure and $U^{\mathbb{C}}(2)$-structures

Before delving into the details of complex geometry, lets first appreciate the interplay between symplectic manifolds and $\mathrm{Spin}^{\mathbb{C}}(4)$-structures; again in the language of principal bundles:

**Corollary 1** ([3]). *Let* $(X, g)$ *be a Riemannian* 4*-manifold with an almost complex structure* $J$. *Then*

$$U(X) := \bigcup_{x \in X} \{s_J := (s_1, s_2, Js_1, Js_2) \mid s_J \text{ is an orthonormal basis of } T_x X\} \tag{16}$$

*defines a* $\mathsf{U}(2)$*-reduction of* $\mathsf{SO}(X)$*.*

**Lemma 7** ([22])**.** *The map*

$$j : \mathsf{U}(2) \simeq \left(S^1 \times \mathsf{SU}(2)\right)_{\pm} \to \mathsf{Spin}^{\mathbb{C}}(4), [\lambda, q] \mapsto [\lambda, \bar{\lambda}, q] \tag{17}$$

*makes the following diagram commute, where the map* $i$ *is the natural inclusion.*

$$\begin{array}{ccc} \mathsf{U}(2) & \xrightarrow{j} & \mathsf{Spin}^{\mathbb{C}}(4) \\ & \searrow^{\iota} & \downarrow{\rho_c} \\ & & \mathsf{SO}(4) \end{array}$$

*Proof.* The natural inclusion is given by $\iota([\lambda, q]) = \left(\mathbb{H} \ni h \mapsto qh\lambda\right)$, which is equal to the composition $[\lambda, q] \mapsto [\lambda, \bar{\lambda}, q] \mapsto \left(\mathbb{H} \ni h \mapsto qh\lambda\right)$. □

This has an important consequence:

**Corollary 2.** *Let* $(X, g)$ *be a Riemmanian manifold. Assume that there exists a* $\mathsf{U}(2)$*-reducion* $\mathsf{U}(X)$ *of* $\mathsf{SO}(X)$*. Then* $X$ *carries a natural* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structure, given by* $\Sigma_{can} := \mathsf{U}(X) \times_j \mathsf{Spin}^{\mathbb{C}}(4)$*. In particular, a symplectic manifold always has a canonical* $\mathsf{Spin}^{\mathbb{C}}(4)$*-structure.*

Notice furthermore that by Lemma (3), up to isomorphism any $\mathsf{Spin}^{\mathbb{C}}$-structure is given by $L \bullet \Sigma_{can}$, which we can also construct in the following way:

**Definition 12.** The group $\mathsf{U}^{\mathbb{C}}(2)$ is defined as the quotient of $S^1 \times S^1 \times \mathsf{SU}(2)$ by the normal subgroup $< (1, , 1, 1), (-1, -1, -1) >$, or in other words:

$$\mathsf{U}^{\mathbb{C}}(2) = \left(S^1 \times S^1 \times \mathsf{SU}(2)\right) \big/ \mathbb{Z}_2 \tag{18}$$

This group should be understood as the little brother of the $\mathsf{Spin}^{\mathbb{C}}(4)$-group, the only difference being that we replaced the first $\mathsf{Sp}(1)$ factor by the smaller group $S^1$. In analogy to the map $\mathsf{Spin}^{\mathbb{C}}(4) \to \mathsf{SO}(4)$, we have the map

$$s : \mathsf{U}^{\mathbb{C}}(2) \to \mathsf{U}(2), [\lambda_1, \lambda_2, q] \mapsto [\lambda_2, q] \tag{19}$$

**Definition 13.** A $\mathsf{U}^{\mathbb{C}}(2)$-*structure* on a Riemmanian manifold $X$ is a $\mathsf{U}^{\mathbb{C}}(2)$-reduction $Q \to \mathsf{SO}(X)$ with respect to the composition

$$\mathsf{U}^{\mathbb{C}}(2) \xrightarrow{s} \mathsf{U}(2) \xrightarrow{\iota} \mathsf{SO}(4). \tag{20}$$

**Lemma 8.** *Assume there exists a* $\mathsf{U}(2)$*-reduction* $\mathsf{U}(X)$ *of* $\mathsf{SO}(X)$*.*

*(1)* $X$ *carries a natural* $U^{\mathbb{C}}(2)$*-structure, given by*

$$Q_{can} := (S^1 \times U(X))\big/\mathbb{Z}_2, \tag{21}$$

where we identify $(1, p) \sim (-1, (-1).p), p \in U(X)$.

*(2) In analogy to* $\mathrm{Spin}^{\mathbb{C}}(4)$*-structures: The group* $H^2(X, \mathbb{Z})$ *acts on the isomorphism classes of* $U^{\mathbb{C}}(2)$*-structures on* $X$ *in the following way: For* $\omega \in H^2(X, \mathbb{Z})$ *there exists a complex line bundle* $L$ *on* $X$ *admitting a hermitian form such that* $c_1(L) = \omega$*. Given a* $U^{\mathbb{C}}(2)$*-structure* $Q \to SO(X)$ *on* $X$*, define a new* $U^{\mathbb{C}}(2)$*-structure* $L \bullet Q$ *as the bundle* $Q \times_{S^1} P_L$*, where* $P_L$ *is the* $S^1$*-bundle of unitary frames on* $L$*. This action is free and transitive.*

*(3) We have* $L \bullet \Sigma_{can} = (L \bullet Q_{can}) \times_\kappa \mathrm{Spin}^{\mathbb{C}}(4)$*, where*

$$\kappa : U^{\mathbb{C}}(2) \to \mathrm{Spin}^{\mathbb{C}}(4) \tag{22}$$

*is the inclusion.*

We also get the analogue of Lemma (5):

**Lemma 9.** *Let* $U(X)$ *be* $U(2)$ *reduction of* $SO(X)$*.*

*(1) The map*

$$U^{\mathbb{C}}(2) \to U(1) \times U(2), \ [\lambda_1, \lambda_2, q_-] \mapsto (\lambda_1^2, [\lambda_2, q_-]) \tag{23}$$

*is 2:1.*

*(2) Given a* $U^{\mathbb{C}}(2)$*-structure* $Q \to X$ *over* $X$*, we have an induced 2:1 map*

$$Q \to \det(Q) \times_X U(X) \tag{24}$$

*(3) Given a connection* $a$ *on* $\det(Q)$*, and a connection* $A_0$ *on* $U(X)$*, there exists a unique lift of* $a \oplus A_0$ *to a connetion* $A$ *on* $Q$*.*

### 1.2.2 Complex Differential Forms

Next up, we investigate the complex geometry arising on the differential forms of an almost Kähler manifold.

Let $(X, g, \omega, J)$ be an almost Kähler manifold. Let $T_{\mathbb{C}}X := TX \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle. We can extend $J$ $\mathbb{C}$-linearly to an endomorphism of $T_{\mathbb{C}}X$, and since $J^2 = -1$, it has eigenvalues $\pm i$. We can therefore decompose

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X \tag{25}$$

where $J$ acts on $T^{1,0}X$ as multiplication by $i$ and by $-i$ on $T^{0,1}X$ respectively. This induces a decomposition of complex forms on $X$:

$$\Lambda^{p,q}X := \Lambda^p(T^{1,0}X)^* \otimes_{\mathbb{C}} \Lambda^q(T^{0,1}X)^*, \tag{26}$$

$$\Lambda^k_{\mathbb{C}}X := \Lambda^k(T_{\mathbb{C}}X)^* = \bigoplus_{p+q=k} \Lambda^{p,q}X\,. \tag{27}$$

We denote the spaces of sections with $\Omega^{p,q}(X)$ and $\Omega^k(X, \mathbb{C})$ respectively.
We also can extend the deRham differential $d : \Omega^k(X, \mathbb{R}) \to \Omega^{k+1}(X, \mathbb{R})$ $\mathbb{C}$-linearly to the deRham differential on complex forms.

**Definition 14.** We define the maps

$$\partial := pr_{p+1,q} \circ d : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)\,, \tag{28}$$

$$\overline{\partial} := pr_{p,q+1} \circ d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X) \tag{29}$$

In general, the deRham differential is a map

$$d : \Omega^{1,0}(X) \to \Omega^{2,0}(X) \oplus \Omega^{1,1}(X) \oplus \Omega^{0,2}(X),$$

where the first two components are given by $\partial$ and $\overline{\partial}$. But what about the third?

**Lemma 10.** *The map* $pr_{0,2} \circ d : \Omega^{1,0}(X) \to \Omega^{0,2}(X)$ *is given by the conjugate of the Nijenhuis tensor* $N_J$.

**Corollary 3.** *The complex structure* $J$ *is integrable if and only if*

$$pr_{0,2} \circ d \equiv 0 \text{ on } \Omega^{1,0}(X) \text{ or equivalently, } d = \partial + \overline{\partial}. \tag{30}$$

*In this case,* $\partial^2 = \overline{\partial}^2 = 0$ *and* $\partial\overline{\partial} = -\overline{\partial}\partial$. *Conversely, if* $\overline{\partial}^2 = 0$, *then* $J$ *is integrable.*

We also have a complex version of the Hodge decomposition:

**Theorem 5** (Complex Hodge decomposition)**.** *Let* $X$ *be compact. Let* $\partial^*$ *and* $\overline{\partial}^*$ *the adjoints of* $\partial$ *and* $\overline{\partial}$ *respectively, and define the Laplace operators* $\Delta_\partial = \partial\partial^* + \partial^*\partial$,

$\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ *and denote their kernels with* $\mathcal{H}^{\bullet,\bullet}_{\partial}(X)$ *and* $\mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X)$. *Then there are orthogonal decompositions*

$$\Omega^{p,q}(X) = \partial(\Omega^{p-1,q}(X)) \oplus \mathcal{H}^{p,q}_{\partial}(X) \oplus \partial^*(\Omega^{p+1,q}(X)) \tag{31}$$

*and*

$$\Omega^{p,q}(X) = \overline{\partial}(\Omega^{p,q-1}(X)) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}(X) \oplus \overline{\partial}^*(\Omega^{p,q+1}(X)) \tag{32}$$

**Theorem 6.** *On a Kähler manifold, we have*

$$\frac{1}{2}\Delta = \Delta_{\partial} = \Delta_{\overline{\partial}} \ . \tag{33}$$

*In particular*

$$\mathcal{H}^k(X, \mathbb{R}) \otimes \mathbb{C} \simeq \mathcal{H}^k_{\partial}(X) = \mathcal{H}^k_{\overline{\partial}}(X) \ . \tag{34}$$

In other words, complex harmonic forms agree with holomorphic forms. We will switch back and forth between those two notions without a lot of care in the following chapters.

### 1.2.3 Holomorphic Line Bundles

Let us briefly recall some results about holomorphic line bundles, following [28].

**Definition 15.** Let $L \to X$ be a complex one dimensional bundle, also called a *line bundle*. A *Dolbeault operator* is a $\mathbb{C}$-linear operator

$$\overline{\partial}_L : \Gamma(X, L) \to \Omega^{0,1}(X) \otimes \Gamma(X, L) \tag{35}$$

satisfying

(1) $\overline{\partial}_L^2 = 0$

(2) For any section $s \in \Gamma(X, L)$ and function $f^\infty(X, \mathbb{C})$, one has

$$\overline{\partial}_L(f \cdot s) = \overline{\partial}(f) \cdot s + f \cdot \overline{\partial}_L(s).$$

One calls $(L, \overline{\partial}_L)$ a *holomorphic line bundle*.

Let $(L, \overline{\partial}_L)$ be a holomorphic line bundle. Assume $h$ is a hermitian form on $L$, and let $P^L_{S^1}$ be the corresponding bundle of unit length vectors of $L$. We call a connection

$A$ on $P^L_{S^1}$ *holomorphic with respect to* $\overline{\partial}_L$ if the induced covariant derivative $\nabla^A$ on $L$ fulfills $\pi^{0,1} \circ \nabla^A = \overline{\partial}_L$.

**Lemma 11.** *Let* $(L, \overline{\partial}_L, h)$ *be a hermitian holomorphic line bundle over a Kähler manifold* $X$*. Then there is a unique holomorphic hermitian connection* $\nabla$ *on* $L$*, called the Chern connection.*

**Lemma 12.** *Let* $(L, h)$ *be a hermitian line bundle over a Kähler manifold X. Given a hermitian connection* $A$ *on* $P^L_{S^1}$*, the induced Cauchy-Riemann operator* $\overline{\partial}_A := \pi^{0,1} \circ \nabla^A$ *defines a holomorphic structure on* $L$ *if and only if* $F^{0,2}_A = 0$*, where* $F^{0,2}_A$ *denotes the* $\Omega^{0,2}(X)$*-part of the curvature* $F_A$*.*

Notice that if $X$ is a (real) four dimensional Kähler manifold, the $\overline{\partial}$ operator induces a holomorphic structure on $\Lambda^{p,q}(X)$.

**Definition 16.** We call the complex holomorphic line bundle

$$K_X := \Lambda^{2,0}(X) = \det_{\mathbb{C}}(\Lambda^{1,0}(X)) \tag{36}$$

the *canonical bundle of X.*

Notice, that holomorphic sections of $K_X$ are prescisely harmonic $(2, 0)$-forms.

**Definition 17.** Given a holomorphic line bundle $L$ over $X$, we denote the vector space of holomorphic sections of $L$ by $H^0(X, L)$, and define its *linear system*

$$|L| := \mathbb{P}\Big(H^0(X, L)\Big) \tag{37}$$

as the complex projectivization of this vector space.

### 1.2.4 Self-Dual Forms Revisited

To examine the interplay of self-dual forms with complex forms, it is worthwhile to look at the local picture and introduce a basis.

Let $e_1, e_1, e_3, e_4$ be an orthonormal basis of $(\mathbb{R}^4)^*$, and $dz_1 = e_1 + ie_2$, $dz_2 = e_3 + ie_4$ be the complex basis obtained by complexifying. Recall that the space of self-dual forms $\Lambda^2_+\mathbb{R}^4$ is spanned by the forms

$$\eta_1 = e_1 \wedge e_2 + e_3 \wedge e_4, \tag{38}$$

$$\eta_2 = e_1 \wedge e_3 - e_2 \wedge e_4, \tag{39}$$

$$\eta_3 = e_1 \wedge e_4 + e_2 \wedge e_3. \tag{40}$$

We identify $\operatorname{Im}\mathbb{H} \simeq \Lambda^2_+\mathbb{R}^4$ by $ai + bj + ck \mapsto a\eta_1 + b\eta_2 + c\eta_3$. Let us take a closer look at the decomposition

$$\Lambda^2_+(\mathbb{R}^4) \otimes \mathbb{C} \subseteq \Lambda^2(\mathbb{R}^4) \otimes \mathbb{C} \simeq \Lambda^{2,0}(\mathbb{R}^4) \oplus \Lambda^{1,1}(\mathbb{R}^4) \oplus \Lambda^{0,2}(\mathbb{R}^4). \tag{41}$$

**Lemma 13.** *We have an isomorphism*

$$\Lambda^2_+(\mathbb{R}^4) \otimes \mathbb{C} \simeq \Lambda^{2,0}(\mathbb{R}^4) \oplus \mathbb{R} \cdot \omega \oplus \ \Lambda^{0,2}(\mathbb{R}^4), \tag{42}$$

*where* $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ *is the canonical Kähler form on* $\mathbb{R}^4$.

*Real self-dual two-forms are of the form*

$$s \cdot \omega + \beta + \overline{\beta}, \ \textit{where } s \in C^\infty(X, \mathbb{R}), \beta \in \Lambda^{2,0}(\mathbb{R}^4). \tag{43}$$

*Proof.* The statement follows directly from expressing the complex forms in terms of the real ones:

$$\begin{aligned} dz_1 \wedge dz_2 &= (e_1 + ie_2) \wedge (e_3 + ie_4) = e_1 \wedge e_3 - e_2 \wedge e_4 + i(e_1 \wedge e_4 + e_2 \wedge e_3) \\ &= \eta_2 + i\eta_3. \end{aligned}$$

Similarly we obtain $d\bar{z}_1 \wedge d\bar{z}_2 = \eta_2 - i\eta_3$ and $\omega = \eta_1$. □

**Corollary 4.** *Let* $\mathcal{H}^2_+(X, \mathbb{R}) := \mathrm{pr}_{\Omega^2_+} \circ \mathcal{H}^2(X, \mathbb{R})$ *be the space of real self-dual, harmonic two forms. Then*

$$\mathcal{H}^2_+(X, \mathbb{R}) \simeq \mathbb{R} \cdot \omega \oplus \mathcal{H}^{2,0}_{\bar{\partial}}. \tag{44}$$

*Proof.* The symplectic form $\omega$ is self-dual and closed, therefore harmonic. The rest follows from the above Lemma and the Hodge decomposition. □

Using all the identifications above, we can now express an element in $\operatorname{Im}\mathbb{H}$ as

$$ai + (b + ci)j = ai + bj + ck \simeq a\eta_1 + b\eta_2 + c\eta_3 \tag{45}$$

$$\simeq a \cdot \omega + \frac{1}{2}(b - ic)dz_1 \wedge dz_2 + \frac{1}{2}(b + ic)d\bar{z}_1 \wedge d\bar{z}_2. \tag{46}$$

The identification $\mathfrak{sp}(1) \simeq \Lambda^2_+(\mathbb{R}^4)$ goes even deeper; in fact they are isomorphic as $Sp(1)$-representations.[8]

[8] Recall that the representation on $\Lambda^2_+\mathbb{R}^4$ is given by $(q_+, h) \mapsto q_+ h \overline{q_+} = q_+ h (q_+)^{-1}$, which is precisely the Adjoint representation of $Sp(1)$ on its Lie algebra.

## 1.3 HyperKähler Geometry

*"Where i-squared equals negative one, and j-squared equals negative one, and k-squared equals negative one, which equals I-J-K!!"*

- A Capella Science, *William Rowan Hamilton*

**Definition 18.** A *hyperKähler manifold* is a Riemannian manifold $(M, g)$[9] admitting three skew-symmetric endomorphisms $I_1$,$I_2$ and $I_3 \in \mathsf{End}(TM)$ fulfilling $I_1^2 = I_2^2 = I_3^2 = -1$ and $I_1 \cdot I_2 = I_3$, which are covariantly constant with respect to the Levi-Civita connection, i.e. $\nabla^{LC} I_l = 0$ for $l = 1, 2, 3$, called *complex structures*.

For three complex structures fulfilling the relations above, a sufficient condition to being covariantly constant is that the induced forms $\omega_l(\cdot, \cdot) = g(\cdot, I_l \cdot)$ are closed for $l = 1, 2, 3$ meaning they are symplectic forms with respect to the complex structure $I_l$ [10]. In particular, the complex structures are integrable and $M$ is a Kähler manifold with respect to any of them.
The complex structures define a scalar multiplication:

$$\mathcal{S} : \mathbb{H} \to \Gamma(M, \mathsf{End}(TM)), \tag{47}$$

$$h = (a + bi + cj + dk) \mapsto \mathcal{S}_h := a + bI_1 + cI_2 + dI_3 \tag{48}$$

inducing

$$I : \operatorname{Im}\mathbb{H} \simeq \mathfrak{sp}(1) \to \Gamma(M, \mathsf{End}(TM)), \quad \zeta \mapsto \mathcal{S}_\zeta\,. \tag{49}$$

Note that given $\zeta = ai + bj + ck \in \operatorname{Im}\mathbb{H}$ with $|\zeta|^2 = 1$, $I_\zeta$ is a covariantly constant complex structure, i.e. we have a two-sphere $S^2 \subseteq \operatorname{Im}\mathbb{H}$ of complex structures.
Define $\omega \in \mathfrak{sp}(1)^* \otimes \Omega^2(M)$ by $\omega(\zeta)(X, Y) = \omega_\zeta(X, Y) := g(X, I_\zeta Y)$. For any $\zeta \in \mathfrak{sp}(1)$ we have $\nabla^{LC}\omega(\zeta) = d\omega(\zeta) = 0$, in particular if $|\zeta|^2 = 1$ then $\omega(\zeta)$ is a Kähler form.
From now on $M$ is a hyperKähler manifold if not stated otherwise.

**Definition 19.** A *permuting action* of $\mathsf{Sp}(1)$ on $M$ is an isometric action satisfying

$$dL_q \circ I_\zeta \circ dL_{\overline{q}} = I_{q\zeta\overline{q}} = I_{\mathrm{Ad}_q(\zeta)}, \tag{50}$$

[9] We will refer to a hyperKähler manifold $M$ usually as the *target manifold* for reasons that will become apparent soon.
[10] Unlike the *Kähler* case, where this condition is not sufficient!

meaning the induced action on the two-sphere of complex structures is the standard action of $SO(3)$. Equivalently, an action is permuting if and only if

$$(L_q)^*\omega_\zeta = \omega_{q\zeta\overline{q}} = \omega_{Ad_q(\zeta)} \text{ for all } \zeta \in \mathrm{Im}\,\mathbb{H}, q \in Sp(1). \tag{51}$$

**Definition 20.** A group action of $S^1$ on a hyperKähler manifold $M$ is called *permuting* if there are complex structures $I_1, I_2$ and $I_3$, such that $I_3 = I_1 I_2$, and the group action fixes $I_1$, while acting in the 2-plane spanned by $I_2$ and $I_3$ by the standard action of $S^1$ on the 2-plane composed with the map $S^1 \to S^1$, $z \mapsto z^2$.

For example, given a permuting $Sp(1)$ action, the circle group fixing a complex structure $I_1$ inside $Sp(1)$ acts permuting in the definition above.

**Definition 21.** An action of a Lie group $G$ on $M$ is called *hyperKähler* if it leaves the metric and the symplectic forms invariant, i.e. $L_h^* g = g$ and $L_h^* \omega = \omega$ for all $h \in G$.

**Definition 22.** A *hyperKähler moment map* $\mu$ for a hyperKähler action $G \curvearrowright M$ is a $G$-equivariant map

$$\mu : M \to \mathfrak{g}^* \otimes \mathfrak{sp}^*(1), \tag{52}$$

which satisfies $d\mu = \iota_\mathfrak{g}\omega$. Here $G$-equivariance means

$$\mu \circ L_h = Ad^*_{h^{-1}} \otimes id_{\mathfrak{sp}^*(1)} \circ \mu \text{ for all } h \in G. \tag{53}$$

This is a generalization of moment maps on symplectic manifolds:

**Definition 23.** A *moment map* of a symplectic action $G \curvearrowright M$ on a symplectic manifold $(M, \omega)$ (where the action being symplectic means it preserves the symplectic form, i.e. $L_h^*\omega = \omega$) is a map $\mu : M \to \mathfrak{g}^*$ satisfying $d\mu = \iota_\mathfrak{g}\omega$.

A hyperKähler moment map is therefore a moment map for any Kähler form $\omega_\zeta$, $\zeta \in \mathfrak{sp}(1)$, $|\zeta|^2 = 1$. With the identification $\mathfrak{sp}(1)^* \simeq \mathrm{Im}(\mathbb{H})$ one usually writes

$$\mu = i \cdot \mu_1 + j \cdot \mu_2 + k \cdot \mu_3 = i \cdot \mu_I + j \cdot \mu_\mathbb{C}, \tag{54}$$

where $\mu_\mathbb{C} = \mu_2 + i\mu_3$ is complex valued.

**Lemma 14.** *If $M$ is a simply-connected hyperKähler manifold admitting a hyperKähler $G$-action, then this action admits a hyperKähler moment map $\mu$.*

*Proof.* The form $\iota_{\mathfrak{g}}\omega$ is closed, and since $M$ is simply-connected, also exact. □

It is also worthwhile to consider the interplay between a hyperKähler moment map and a permuting action:

**Lemma 15.** *Let $M$ be a hyperKähler manifold with a hyperKähler $G$-action admitting a hyperKähler moment map $\mu$.*

*(1) Assume there is a permuting $Sp(1)$-action on $M$. Then*

$$\mu(q.x) = q\mu(x)\overline{q} = Ad_q\mu(x) \text{ for all } q \in Sp(1), x \in M. \quad (55)$$

*(2) Assume there is a permuting $S^1$-action on $M$. Then*

$$\mu(\lambda.x) = i \cdot \mu_I(x) + j \cdot \mu_{\mathbb{C}}(x) \cdot \overline{\lambda}^2 \text{ for all } \lambda \in S^1, x \in M. \quad (56)$$

The last lemma is the key to relate spinors to self-dual two forms; In the standard example $M = \mathbb{H}$ with the permuting $Sp(1)$ acting from the left and the hyperKähler $S^1$ from the right, the moment map of the $S^1$-action is a map of representations $\mu : \mathbb{H} \to \Lambda^2_+\mathbb{R}^4$.

### 1.3.1 HyperKähler Reductions

One way of obtaining new hyperKähler manifolds from old ones is via the hyperKähler quotient construction, which is described by the following theorem:

**Theorem 7.** *([11]) Let $M$ be a hyperKähler manifold admitting a hyperKähler action of a Lie group $G$ with a hyperKähler moment map $\mu : M \to \mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$. Let $\lambda$ be a fixed element of the coadjoint action of $G$ on $\mathfrak{g}^* \otimes \mathfrak{sp}(1)^*$. Suppose $\lambda$ is a regular value of $\mu$, meaning $\mu^{-1}(\lambda)$ is a manifold. Assume further the orbit space $\mu^{-1}(\lambda)/G$ is a manifold (for example if $G$ acts freely and properly on $\mu^{-1}(\lambda)$). Then $\mu^{-1}(\lambda)/G$ is a hyperKähler manifold with hyperKähler structure induced from M via inclusions and projections.*

We give a short sketch of the construction:

The projection map $\pi : \mu^{-1}(\lambda) \to \mu^{-1}(\lambda)/G$ turns $\mu^{-1}(\lambda)$ into a $G$-principal bundle, which carries a connection defined by taking the horizontal subspaces to be the orthogonal complements (w.r.t. the induced Riemannian metric on $\mu^{-1}(\lambda) \subseteq M$) of the vertical subspaces (which are spanned by the fundamental vector fields of the $G$-action). We denote a horizontal lift of a tangent vector v in $T(\mu^{-1}(\lambda)/G)$ to

$T\mu^{-1}(\lambda)$ by $v^*$.

The complex structures are given by $Iv := \tilde{I}v^*$, where $\tilde{I}$ is the corresponding complex structure on $M$. Furthermore, the metric is given by $g(v,w)(x) = g_M(v^*,w^*)(y)$, where $g_M$ is the metric on $M$ restricted to $\mu^{-1}(\lambda)$, $x \in \mu^{-1}(\lambda)/G$ and $y \in \mu^{-1}(\lambda)$ any point projecting onto $x$. Consequently, the symplectic form is defined by $\rho(v,w)(x) := \omega_{|\mu^{-1}(\lambda)}(v^*,w^*)(y)$, where $\omega$ is the symplectic form on $M$. It turns out that to define the symplectic form one does not necessarily need to take horizontal lifts; any lift will do. This follows from the fact that for any fundamental vector field $K^M_\zeta$, $\zeta \in \mathfrak{g}$ and tangent vector $X \in T\mu^{-1}(\lambda) = \ker d\mu$ one has $\omega(K^M_\zeta, X) = \langle d\mu(X), \zeta\rangle = 0$, so changing a lift by a vertical vector and inserting it into the symplectic form does not change its value. In other words, $\rho$ is uniquely defined by the property $\pi^*\rho = \omega_{|\mu^{-1}(\lambda)}$.
Note that since $\mu^{-1}(\lambda)$ has dimension $\dim M - 3\dim G$, $\mu^{-1}(\lambda)/G$ has dimension $\dim M - 4\dim G$, which is a multiple of 4, as required for a hyperKähler manifold.

We will look at concrete examples later on, when discussing MEH spaces.

### 1.3.2 Clifford Multiplication

The tangent bundle of a hyperKähler manifold carries a quaternionic structure, as we have seen. We will not develop the whole theory of Clifford algebras here, only what is need for our purposes. For a more detailed account, see [19].

**Definition 24.** Let $(V,q)$ be a finite dimensional vector space with quadratic form $q$ over a field $k$. Then the *Clifford algebra* $Cl(V,q)$ is defined as the associative (but not necessarily commutative) algebra over $k$, which is generated by $V$ and a unit element $1_{Cl}$, satisfying

$$v \cdot v = -q(v)1_{Cl} \text{ for all } v \in V. \tag{57}$$

Note that one has an inclusion $i : V \to Cl(V,q)$.

**Proposition 1.** The Clifford algebra satisfies the following universal property:
Let $A$ be a unital associative algebra over $k$ and $j : V \to A$ be a map such that

$$j(v) \cdot j(v) = -q(v) \cdot 1_A \text{ for all } v \in V. \tag{58}$$

Then there is a unique algebra homomorphism $f : Cl(V,q) \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\ i\ } & \mathsf{Cl}(V,q) \\ & \searrow^{j} & \vdots\, f \\ & & A \end{array}$$

**Example 1.** The map $j_- : V \to \mathsf{Cl}(V,q), v \mapsto -v$ satisfies the Clifford property, and therefore there exists an algebra homomorphism $\kappa : \mathsf{Cl}(V,q) \to \mathsf{Cl}(V,q)$ with $\kappa(v) = -v$ for all $v \in V$.

**Definition 25.** Define $\mathsf{Cl}_0(V,q) = \ker(\mathrm{id} - \kappa) = \mathrm{im}(\mathrm{id} + \kappa)$, the *even part* and $\mathsf{Cl}_1(V,q) = \ker(\mathrm{id} + \kappa) = \mathrm{im}(\mathrm{id} - \kappa)$ to be the *odd part* of $\mathsf{Cl}(V,q)$. Note that $\mathsf{Cl}(V,q) = \mathsf{Cl}_0(V,q) \oplus \mathsf{Cl}_1(V,q)$.

Let $q_{st}$ be the standard quadratic form on $\mathbb{R}^n$.

**Example 2.** One has $\mathsf{Cl}(3) := \mathsf{Cl}(\mathbb{R}^3, q_{st}) = \mathbb{H} \oplus \mathbb{H}$ with generators $f_1, f_2, f_3$ satisfying

$$f_i^2 = -1,\ f_i \cdot f_k = -f_k \cdot f_i,\ (f_i \cdot f_k)^2 = -1,\ (f_1 \cdot f_2 \cdot f_3)^2 = -1$$

**Example 3.** One has $\mathsf{Cl}(4) := \mathsf{Cl}(\mathbb{R}^4, q_{st}) = M_2(\mathbb{H})$, the quaternionic $2\times 2$-matrices, with generators $e_1, e_2, e_3, e_4$. In particular one has $\mathsf{Cl}(4)_0 = \mathsf{Cl}(3)$, generated by $f_1 = e_1 e_4$, $f_2 = e_2 e_4$ and $f_3 = e_3 e_4$.

Note that $\mathsf{Spin}(4) = \mathsf{Sp}(1)_+ \times \mathsf{Sp}(1)_- \subseteq \mathsf{Cl}(4)$, with the inclusion given by

$$(q_+, q_-) \mapsto \begin{pmatrix} q_+ & 0 \\ 0 & q_- \end{pmatrix}.$$

**Lemma 16.** *The* $\mathsf{Spin}(4)$ *representations on* $W^\pm \simeq \mathbb{H}$ *can be extended to a Clifford module structure on* $W^\pm$. *Furthermore, the map*

$$\mathbb{R}^4 \times W^+ \to W^- \tag{59}$$

*given by the composition of the inclusion* $\mathbb{R}^4 \hookrightarrow \mathsf{Cl}(4)$ *and Clifford multiplication is* $\mathsf{Spin}(4)$*-equivariant.*

Given a hyperKähler manifold $M$, recall the map

$$I : \mathrm{Im}\,\mathbb{H} \to \Gamma(M, \mathsf{End}(TM)), \quad \zeta \mapsto \mathcal{S}_\zeta\,.$$

This map satisfies the property of Proposition (1), therefore it extends to a map

$$\mathsf{Cl}(3) \to \Gamma(M, \mathsf{End}(TM)) \tag{60}$$

or in other words, it endows $TM$ with the structure of a $Cl(4)_0 \simeq Cl(3)$-module. Therefore we can define a $Cl(4)$-module bundle

$$\widehat{TM} = Cl(4) \otimes_{Cl(4)_0} TM \simeq \underbrace{Cl(4)_0 \otimes_{Cl(4)_0} TM}_{=: TM^+} \oplus \underbrace{Cl(4)_1 \otimes_{Cl(4)_0} TM}_{=: TM^-} \tag{61}$$

Notice that the bundle $TM^+$ is isomorphic to $TM$ as a $Cl(4)_0$-module, and the bundle $TM^-$ is isomorphic to $TM$ as a vector bundle, but carries a different $Cl(4)_0$-module structure.

**Definition 26.** We define the *Clifford multiplication* as the composition

$$\mathbb{R}^4 \times \widehat{TM} \xrightarrow{i \times id} Cl(4) \times (TM^+ \oplus TM^-) \to TM^- \oplus TM^+ \tag{62}$$

where the second map is just given by multiplication, as $TM^+ \oplus TM^-$ is a $Cl(4)$-module. Usually we will be more interested in the restriction to the first factor, i.e. the map $cl : \mathbb{R}^4 \times TM^+ \to TM^-$.

We can make this purely algebraic construction a little more explicit:

**Proposition 2.** Denote by $W^\pm \simeq \mathbb{H}$ the representations of $Spin(4) \simeq Sp(1)_+ \times Sp(1)_-$ on $\mathbb{H}$ defined by $(q_+, q_-).v = q_\pm v$. A hyperKähler manifold $M$ with permuting action of $Sp(1)$ and a hyperKähler action of $S^1$ has an induced action on $TM$ of $Spin^{\mathbb{C}}(4)$ via differentials, i.e.

$$(\lambda, q_+, q_-).v = Tq_+ T\lambda \cdot v, \quad v \in TM. \tag{63}$$

where we introduced the notation $Tq := dL_q$ for the differential. Further let $E$ be the vector bundle $TM$, but equipped with the action of $Spin^{\mathbb{C}}(4)$ given by

$$(\lambda, q_+, q_-).v = I_{\overline{q}_+} Tq_+ T_\lambda \cdot v, \quad v \in TM \tag{64}$$

(which is well-defined because $Sp(1)_+$ acts permuting). Then we have the following isomorphism of equivariant vector bundles:

$$TM \otimes_{\mathbb{R}} \mathbb{C} \simeq E \otimes_{\mathbb{C}} W^+ \ , \ v \otimes z \mapsto v \otimes z - I_2 v \otimes jz$$

*Proof.* The complex tensor product is defined via the relation $I_1 v \otimes z = v \otimes i \cdot z$. It is sufficient to check the statement for $q = i, j, k \in Sp(1)_+$. For simplicity we omit the trivially acting $Sp(1)_-$ factor, and do the proof for $q = i$, the other proofs are similar.

$$(i,\lambda).(v \otimes z) = TiT\lambda \cdot v \otimes z \mapsto TiT\lambda \cdot v \otimes z - I_2 TiT\lambda \cdot v \otimes jz =$$
$$I_1(-I_1)TiT\lambda \cdot v \otimes z + TiT\lambda I_2 \cdot v \otimes =$$
$$(-I_1)TiT\lambda \cdot v \otimes zi + (-I_1)TiT\lambda I_2 \cdot v \otimes jzi =$$
$$(-I_1)T_iT\lambda \cdot v \otimes iz - (I_1)T_iT\lambda I_2 \cdot v \otimes ijz =$$
$$(i,\lambda).(v \otimes z - I_2 v \otimes jz)$$

□

Equivalently, the identification $I_1 v \otimes 1 = v \otimes i$ corresponds to the isomorphism $TM \simeq T^{1,0}_{I_1}M$, $v \mapsto \pi^{1,0}(v)$. Thus the isomorphism above is just given by

$$TM \otimes \mathbb{C} \simeq T^{1,0}_{I_1}M \otimes W^+,\ v \mapsto \pi^{1,0}(v) \otimes 1 - \pi^{1,0}(I_2 v) \otimes j.$$

**Corollary 5.** *If we denote by* $\mathbb{R}^4$ *the* $\mathrm{Spin}(4)$*-representation given by* $(q_+, q_-).v = q_- v \overline{q_+}$*, the well-known Clifford multiplication is given by* $\mathbb{R}^4 \otimes W^+ \to W^-$ , $h_1 \otimes h_2 \mapsto h_1 \cdot h_2$*. Thus we can define a Clifford multiplication* $\mathbb{R}^4 \otimes T_xM \otimes \mathbb{C} \simeq \mathbb{R}^4 \otimes E_x \otimes_{\mathbb{C}} W^+ \to E_x \otimes_{\mathbb{C}} W^-$*. Notice that the homomorphism defined above induces a homomorphism of the real parts of the representations, we therefore have a map*

$$Cl : \mathbb{R}^4 \otimes T_xM \simeq [\mathbb{R}^4 \otimes T_xM \otimes \mathbb{C}]_{\text{real part}} \to [E_x \otimes_{\mathbb{C}} W^-]_{\text{real part}}.$$

We denote $TM$ as $TM^+$ and the bundle $[E \otimes_{\mathbb{C}} W^-]_{\text{real part}}$ by $TM^-$. As a vector bundle it is isomorphic to $TM$, but the group action of $\mathrm{Spin}^{\mathbb{C}}(4)$ is given by $(\lambda, q_+, q_-).v = I_{q_-} I_{\overline{q_+}} T q_+ T\lambda \cdot v$. Clifford multiplication is then just given by the map

$$\mathbb{R}^4 \to \mathrm{End}(TM^+ \oplus TM^-)\ ,\ h \mapsto \begin{pmatrix} 0 & -I_{\overline{h}} \\ I_h & 0 \end{pmatrix}.$$

### 1.3.3 Clifford Multiplication in the Kähler Case

Recall that on a Kähler manifold $X$ there is a canonical $\mathrm{Spin}^{\mathbb{C}}(4)$- structure $\Sigma_{can}$ induced by the homomorphism:

$$\begin{array}{ccc} U(2) & \xrightarrow{\ j\ } & \mathrm{Spin}^{\mathbb{C}}(4) \\ & \searrow^{\iota} & \downarrow{\rho_c} \\ & & SO(4) \end{array}$$

where

$$\rho_c : [(\lambda, q_+, q_-)] \mapsto (x \mapsto q_- x \overline{q}_+), \ j : [(\lambda, q)] \mapsto [(\lambda, \overline{\lambda}, q)], \tag{65}$$

and $\iota : [(\lambda, q)] \mapsto (x \mapsto qx\lambda)$ is the inclusion of $U(2)$ in $SO(4)$.

Recall that the positive and negative spinor bundles for some $Spin^{\mathbb{C}}(4)$-structure $\Sigma$ are given by $S^{\pm}(\Sigma) = \Sigma \times_{\rho_\pm} \mathbb{C}^2$, where

$$\rho_\pm : [(\lambda, q_+, q_-)] \mapsto (x \mapsto q_\pm x\lambda). \tag{66}$$

We have for the canonical $Spin^{\mathbb{C}}(4)$-structure $\Sigma_{can}$:

$$S^\pm := S^\pm(\Sigma_{can}) = P_{U(2)} \times_{\rho_\pm \circ j} \mathbb{C}^2, \tag{67}$$

where $\rho_+ \circ j([(\lambda, q)]).x = \overline{\lambda} x \lambda$. Writting $\mathbb{C}^2 \simeq \mathbb{C} + j\mathbb{C}$, wee see that the representation is trivial on the first $\mathbb{C}$-factor, while acting via multiplication of $\overline{\lambda}^2 = \det([(\lambda, q)])^{-1}$ on the second factor. Thus $S^+$ splits into the direct sum of the trivial line bundle and the inverse of the complex determinant bundle:

$$S^+ = \Lambda^{0,0}(X, \mathbb{C}) \oplus \Lambda^{0,2}(X, \mathbb{C}). \tag{68}$$

Similarly,

$$S^- = \Lambda^{0,1}(X, \mathbb{C}). \tag{69}$$

Furthermore we have that $T^*X = P_{U(2)} \times_{(\rho_c^* \circ \iota)} \mathbb{R}^4$, where $\rho_c^* : SO(4) \curvearrowright \mathbb{R}^4$ is the dual of the standard representation.

The Clifford multiplication is then induced by the quaternionic multiplication $\mathbb{R}^4 \times \mathbb{H} \to \mathbb{H}$, identifying $\mathbb{H} \simeq \mathbb{C}^2$:

$$T^*X \times S^+ = P_{U(2)} \times_{(\rho^* \circ \iota) \oplus (\rho_+ \circ j)} \mathbb{R}^4 \oplus \mathbb{C}^2 \to P_{U(2)} \times_{\rho_- \circ j} \mathbb{C}^2 = S^-. \tag{70}$$

In more explicit terms, Clifford multiplication with a one form $\alpha$ is given by

$$c(\alpha)s = \sqrt{2}(\pi^{0,1}(\alpha) \wedge s - \pi^{0,1}(\alpha) \angle s) \tag{71}$$

where $\angle$ denotes the contraction with s:

$$\alpha \angle \alpha_1 \wedge \ldots \wedge \alpha_p = \sum_{i=1}^{p} (-1)^{i+1} h_X(\alpha_i, \alpha) \alpha_1 \wedge \ldots \wedge \widehat{\alpha_i} \wedge \ldots \wedge \alpha_p \tag{72}$$

$h_X$ denoting the hermitian product on $X$, defined to be complex-antilinear in the second component.

## 1.4 Interlude: Algebraic Geometry

As the realm of complex geometry lies very close to the one of algebraic geometry, we introduce some new language, following [12].

Denote the sheaf of holomorphic functions on a complex manifold $X$ by $\mathcal{O}_X$, the sheaf of invertible holomorphic functions by $\mathcal{O}_X^*$ and the sheaf of invertible (non-zero) meromorphic functions by $\mathcal{K}_X^*$.

**Definition 27.** An *analytic subvariety* $Y$ of a complex manifold $X$ is a closed subset $Y \subseteq X$ such that for any point $x \in X$ there exists a neighborhood $U$, such that $Y \cap U$ is given as the zero set of finitely many holomorphic functions $f_1, \ldots f_k \in \mathcal{O}(U)$. We say an analytic subvariety $Y$ is *irreducible* if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two proper analytic subvarieties. We define the *dimension* of an irreducible analytic subvariety to be the dimension of the subset $Y_{reg} \subseteq Y$, where $Y_{reg}$ is a complex submanifold of $X$ and is maximal in the subsets of $Y$ with respect to that property. We call an irreducible analytic subvariety of codimension 1 an *irreducible hypersurface*.

One can show that a hypersurface is locally always given by the zero set of a single holomorphic function.

**Definition 28.** A *divisor* $D$ on a complex compact manifold $X$ is a formal linear combination $D = \sum a_i[Y_i]$ with $Y_i \subseteq X$ irreducible hypersurfaces and $a_i \in \mathbb{Z}$ with only finitely many $a_i$ non-zero. A divisor $D$ is called *effective* if all $a_i \geq 0$, we then write $D \geq 0$. We call the group of all divisors $Div(X)$.

Assume that $Y \subseteq X$ is an irreducible hypersurface and that around a fixed point $x \in Y$, $Y$ is given by the zero set of an irreducible element $g \in \mathcal{O}_{X,x}$.

**Definition 29.** Let $f$ be a meromorphic function defined on a neighborhood of $x \in Y$. Then the *order* $\mathrm{ord}_{Y,x}(f) \in \mathbb{Z}$ of $f$ in $x$ with respect to $Y$ is given by the equality $f = g^{\mathrm{ord}_{Y,x}(f)} \cdot h$ with $h \in \mathcal{O}_{X,x}^*$. One can show that such a point always exists and the definition of the order is then independent of the choice of a point, thus we define the *order along an irreducible hypersurface* $Y$, given by $\mathrm{ord}_Y(f)$ for a global meromorphic function $f$. Notice that the order satisfies $\mathrm{ord}_Y(f_1 f_2) = \mathrm{ord}_Y(f_1) + \mathrm{ord}_Y(f_2)$. We associate to a global meromorphic function on $X$ the Divisor $(f) = \sum \mathrm{ord}_{Y_i}(f)[Y_i]$ where the sum is taken over all irreducible hypersurfaces of $X$. We call a divisor of this form a *principal divisor*. We can write every divisor $D$ as the difference of two effective divisors $D_0 - D_\infty$, respectively the *zero set* and *pole set* of the divisor $D$.

**Lemma 17.** *There is a natural isomorphism* $\mathrm{Div}(X) \simeq H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

*Proof.* An element $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is given by non-trivial meromorphic functions $f_i$ on open subsets $U_i$ covering $X$ such that on the overlaps $f_i \cdot f_j^{-1}$ is a holomorphic functions without zeros. Hence on these overlaps, one has for an irreducible hypersurface $Y \subseteq X$ that $\mathrm{ord}_Y(f_i) = \mathrm{ord}_Y(f_j)$, thus we can associate to $f$ the divisors $\sum \mathrm{ord}_{Y_i}(f)[Y_i]$. Using the additivity of the order, we actually see that this is a group homomorphism. Define the inverse map as follows: For $D = \sum a_i[Y_i]$ there exists an open covering of $X = \bigcup U_j$ such that $Y_i \cap U_j$ is defined by some $g_{ij} \in \mathcal{O}(U_j)$ which is unique up to elements in $\mathcal{O}^*(U_j)$. We then define $f_j = \prod_i g_{ij}^{a_i} \in \mathcal{K}_X^*(U_j)$. Since $g_{ij}$ and $g_{ik}$ define the same hypersurface in $U_j \cap U_k$, they differ by an element in $O_X^*$, thus the $f_j$ define an element $f$ in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. The constructions are inverse to each other. □

Denote by $\mathrm{Pic}(X)$ the group of holomorphic line bundles on $X$ up to isomorphism, where the group action is given by the tensor product.

**Lemma 18.** *The short exact sequence of sheaves*

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0 \tag{73}$$

*induces a group homomorphism*

$$\mathrm{Div}(X) \simeq H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X)\,. \tag{74}$$

*We denote the image of a divisor* $D$ *under this homomorphism by* $\mathcal{O}(D)$*. A divisor defines the trivial line bundle if and only if the divisor is principal, and two Divisors* $D_1$ *and* $D_2$ *define isomorphic line bundles if and only if their difference* $D_1 - D_2$ *is a principal divisor. We then say the divisors are linearly equivalent.*

This group homomorphism can be given more explictly:
For $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ locally given by $f_i$ on $U_i$, the functions $f_i \cdot f_j^{-1}$ are invertible holomorphic functions satisfying the cocycle condition and thus define a line bundle

$L$. It turns out that $f$ carries even more information, it defines a meromorphic section of $L$! The $f_i$ and $f_j$ are locally meromorphic functions such that their difference on overlaps is precisely given by $f_i \cdot f_j^{-1}$, which is the cocycle of the line bundle $L$. When $f$ is an effective divisor, meaning it is actually locally given by a holomorphic function, it defines a holomorphic section.
On the other hand, given a meromorphic section of a line bundle $L$ (i.e. locally defined meromorphic functions such that their difference on overlaps is the cocycle of $L$), we can associate to it a divisor. We thus have the following lemma:

**Lemma 19.**
- *Given a holomorphic line bundle $L$, there is a one-to-one correspondence between divisors $D$ such that $\mathcal{O}(D) = L$ and meromorphic sections of $L$ up to scalar multiples.*
- *Given a holomorphic line bundle $L$, there is a one-to-one correspondence between effective divisors $D$ such that $\mathcal{O}(D) = L$ and holomorphic sections of $L$ up to scalar multiples.*

The statement about scalar multiples follows from the fact that a divisor $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is determined up to global invertible holomorphic functions, which on a compact manifold are precisely the constant functions.

Notice that we can still make sense of the intersection form for divisors on a two dimensional complex manifold[11], although they are not necessarily smooth. More precisely, for two divisors $D, E$:

$$D \cdot E = \int_M c_1(\mathcal{O}(D)) \wedge c_1(\mathcal{O}(E)) \tag{75}$$

where $c_1$ denotes the first Chern class of the corresponding line bundle. We usually abreviate the self-intersection $D \cdot D$ by $D^2$.

We conclude by stating two theorems for complex two-dimensional projective varieties[12] which will come in handy later on.

**Theorem 8** (Hodge-Index Theorem). *Let $X$ be a complex projective variety. Let $D$ and $E$ be divisors on $X$ satisfying $D^2 > 0$ and $D \cdot E = 0$. Then $E^2 \leq 0$ with equality if and only if $E = 0$.*

[11]Hence, a real four dimensional manifold.

[12]A complex projective variety is a complex submanifold of $\mathbb{CP}^n$ inheriting the standard Kähler structure.

**Theorem 9** (Adjunction formula)**.** *Let* $X$ *be a complex projective variety. Let* $C$ *be a complex curve on* $X$*. Then*

$$C \cdot (K_X + C) = 2p_a(C) - 2 \tag{76}$$

*where* $p_a(C)$ *is the arithmetic genus of* $C$*.*

# 2 Generalized Seiberg-Witten Equations

Before diving into the technicalities of the generalized Seiberg-Witten equations, we will take a quick peek at the well-understood linear case.

## 2.1 Linear Seiberg-Witten Equations

Introduced in [25], the (linear) Seiberg-Witten equations revolutionized the game of finding invariants of smooth four-dimensional manifolds. We give a brief overview of the construction and a couple of results. We mainly follow [20] and [22].

Let $X$ be a smooth compact oriented four dimensional Riemannian manifold, and let $\Sigma \to X$ a $\mathsf{Spin}^{\mathbb{C}}(4)$-structure over $X$.

Let $S^{\pm}(\Sigma) = \Sigma \times_{\rho_\pm} \mathbb{H}$ be the associated positive and negative spinor bundles respectively. We notice that $\mathbb{H}$ is a hyperKähler manifold with the usual left multiplications by quaternions, and in the representation

$$\rho_+ : \mathsf{Spin}^{\mathbb{C}}(4) = (S^1 \times \mathsf{Sp}(1)_+ \times \mathsf{Sp}(1)_-)_\pm \curvearrowright \mathbb{H},\ [\lambda, q_+, q_-] \mapsto \Big(h \mapsto q_+ h \lambda\Big) \tag{77}$$

the $S^1$-action is hyperKähler (it is acting via right multiplication, which commutes with the complex structures acting from the left), $\mathsf{Sp}(1)_+$ acts permuting (via left multiplication) and $\mathsf{Sp}(1)_-$ acts trivially. By Lemma (14), the $S^1$-action admits a hyperKähler moment map $\mu : \mathbb{H} \to \operatorname{Im}\mathbb{H} \simeq \Lambda^2_+\mathbb{R}^4$ and by Lemma (15) it induces a map of bundles $S^+ \to \Lambda^2_+ T^*X$, which by abuse of notation we also denote by $\mu$.

Let $a$ be a connection on $\det(\Sigma)$ and denote by $F_a^+ \in \Omega^2_+(X, \mathbb{R})$ the self-dual part of its curvature. Let $A_0$ be the Levi-Civita connection on $SO(X)$. By Lemma (5), given a connection $a$ on $\det(\Sigma)$, we obtain a connection $A$ on $\Sigma$ as the lift of $a \oplus A_0$. Such a connection gives rise to the twisted Dirac operator

$$\not{D}^A : \Gamma(X, S^+) \xrightarrow{\nabla_A} \Gamma(X, T^*X \otimes S^+) \xrightarrow{\mathrm{cl}} \Gamma(X, S^-) \tag{78}$$

as the composition of the covariant derivative $\nabla_A$ and Clifford multiplication.

We throw all these ingredients into a big pot: Let $u \in \Gamma(X, S^+)$ be a spinor and $a$ a connection on $\det(\Sigma)$.

**Definition 30.** The **Seiberg-Witten equations** are defined to be

$$\not{D}^A u = 0 \tag{79}$$

$$F_a^+ = \mu \circ u \tag{80}$$

The **Gauge group** $\mathcal{G} := \mathrm{Map}(X, S^1)$ acts on spinors via

$$(g.u)(x) = g(x) \cdot u(x) \tag{81}$$

and on connections via gauge transformations:

$$g.a = g^*(a) = a + (g^{-1})^* \eta \tag{82}$$

where $\eta$ denotes the Maurer-Cartan form on $S^1$. The Seiberg-Witten equations are equivariant with respect to the action of the gauge group, therefore the gauge group acts on the space of solutions, and we can form the moduli space

$$\mathcal{M}_{SW}(\Sigma) := \{(u, a) \text{ is a Sol. to the SW equations}\}/\mathcal{G} \tag{83}$$

**Theorem 10** ([22])**.** *For a generic choice of Riemannian metric on $X$, the moduli space $\mathcal{M}_{SW}$ is a compact, orientable, finite dimensional manifold.*

By integrating certain universal differential forms on the moduli space one obtains a number, which in nice cases is independent of the choice of Riemannian metric, therefore one obtains an invariant of the underlying smooth manifold $X$.

In the general case, the moduli space is very hard to describe explicitly, but more can be said when we confine ourselves to the world of complex geometry.

Assume the source manifold $X$ is Kähler. Then, by Lemma (2), $X$ carries a natural $\mathrm{Spin}^{\mathbb{C}}(4)$-structure $\Sigma_{can}$ and any $\mathrm{Spin}^{\mathbb{C}}(4)$-structure is ismorphic to $L \bullet \Sigma_{can}$ for some complex line bundle $L \to X$. Using the results of Section (1.3.3), we observe that

$$S^+(L \bullet \Sigma) = L \oplus L \otimes \Lambda^{0,2} X \tag{84}$$

and thus a spinor splits as

$$u = (\alpha, \beta) \in \Omega^0(X, L) \oplus \Omega^{0,2}(X, L) \tag{85}$$

**Definition 31.** Define the *degree* of a line bundle $L \to X$ over a two-dimensional Kähler manifold with symplectic form $\omega$ as

$$\deg_\omega(L) = \int_X c_1(L) \wedge \omega \tag{86}$$

where $c_1(L)$ denotes the first Chern class of $L$.

**Theorem 11** ([25]). *Let $X$ be a Kähler manifold and $\Sigma = L \bullet \Sigma_{can}$ a $\mathrm{Spin}^{\mathbb{C}}(4)$-structure on $X$, where $L \to X$ is a complex line bundle.*

*(1) If* $\deg_\omega(L) \geq 0$*, then*

$$\mathcal{M}_{SW}(\Sigma) = |\, L \,| = \mathbb{P}(H^0(X, L)) \tag{87}$$

*(2) If* $\deg_\omega(L) < 0$*, then*

$$\mathcal{M}_{SW}(\Sigma) = |\, L \otimes K_X^{-1} \,| = \mathbb{P}(H^{0,2}(X, L)) \tag{88}$$

*where $H^0(X, L)$ denotes the vector space of holomorphic sections of $L$ and $H^{0,2}(X, L)$ the vector space of anti-holomorphic two forms with values in $L$.*

For reasons that will become clear later, we also consider the following:

**Definition 32.** Let $\phi \in \Omega^2_+(X, \mathbb{R})$.
The **perturbed Seiberg-Witten equations** are defined to be

$$\not{D}^A u = 0 \tag{89}$$

$$F_a^+ = \mu \circ u + \phi \tag{90}$$

As before, we form the *perturbed moduli space*

$$\mathcal{M}^\phi_{SW}(\Sigma) := \{(u, a) \text{ is a Sol. to the SW equations with perturbation } \phi\}/\mathcal{G} \tag{91}$$

**Theorem 12** ([22],3.2.13). *Let $X$ be a Kähler manifold and $\Sigma = L \bullet \Sigma_{can}$ a $\mathrm{Spin}^{\mathbb{C}}(4)$-structure on $X$, where $L \to X$ is a complex line bundle. Assume that $\phi \in \mathcal{H}^{2,0}_{\bar\partial}$, with the inclusion into $\Omega^2_+(X, \mathbb{R})$ given as discussed in corallary (4). Then*

$$\mathcal{M}^\phi_{SW}(\Sigma) = \left\{[\alpha] \in \mathbb{P}(H^0(X, L)),\ \beta := \phi/\alpha \textit{ is a holomorphic section of } \Omega^{2,0}(X, L^{-1})\right\} \tag{92}$$

In other words: The moduli space consits of all possible factorizations $\phi = \alpha \cdot \beta$, $\alpha \in H^0(X, L), \beta \in H^{2,0}(X, L^{-1})$ up to scalars. As the divisor $(\phi)$ has finitely many

irreducible components, this is a finite number of points! This simplifies matters a lot; for example the integral over the moduli space becomes a simple count of the number of (oriented) points[13].
According to Lemma (8), when considering the Seiberg-Witten equations on a Kähler manifold $X$, we could work with $U^{\mathbb{C}}(2)$-structures instead of $Spin^{\mathbb{C}}(4)$-structures. We then would not consider the whole permuting $Sp(1)$-action on $\mathbb{H}$, but instead restrict to a permuting $S^1$-action, where we consider $S^1$ as a subgroup of $Sp(1)$. In the following we would like to replace $\mathbb{H}$ with a hyperKähler manifold $M$, which admits a permuting $S^1$-action, but NOT a permuting $Sp(1)$-action in general, therefore restricting ourselves to a source almost-Kähler manifold $X$ is necessary.

## 2.2 Non-Linear Dirac Operator

One very important ingredient to the Seiberg-Witten equations is the Dirac operator, a differential operator on the spinor bundle involving covariant derivatives and Clifford multiplication. In the next section we describe how to generalize these concepts to the non-linear case, e.g. fiber bundles (not necessarily vector bundles) with hyperKähler manifolds as fibre.

### 2.2.1 Covariant Derivative

Let $Q$ be a $U^{\mathbb{C}}(2)$-structure over a Kähler manifold $X$ induced by $Q = L \bullet Q_{can}$ for some line bundle $L$ , and $M$ a hyperKähler manifold with a permuting $S^1$-action together with a $S^1$ hyperKähler-action. We form the associated fibre bundle

$$F := Q \times_{U^{\mathbb{C}}(2)} M, \tag{93}$$

where when writing $U^{\mathbb{C}}(2) \simeq (S^1 \times S^1 \times Sp(1))_{\pm}$, the first $S^1$ acts hyperKähler, the second $S^1$ acts permuting and $Sp(1)$ trivially. We identify sections of the fibre bundle $F$ with $U^{\mathbb{C}}(2)$-equivariant maps from $Q$ to $M$, i.e.

$$\Gamma(X, F) \simeq Map(Q, M)^{U^{\mathbb{C}}(2)} \tag{94}$$

We call these sections *generalized spinors*.

[13]To be more precise: We assign to each positively oriented point a $+1$ and to every negative oriented one a $-1$, and sum over all points.

[13]writing $U(2) \times S^1 = (S^1 \times SU(2))_{\pm} \times S^1$, the first $S^1$ acts permuting, $SU(2)$ acts trivially and the second $S^1$ acts hyperKähler.

**Definition 33.** Given a connection $A$ on $Q$ and a section $u \in \mathrm{Map}(Q, M)^{U^{\mathbb{C}}(2)}$, we define the *covariant derivative* of $u$ with respect to $A$ as

$$(\nabla^A u)(v) := du(v) + K^M_{A(v)} = du \circ pr_{horA}(v) \in \mathrm{Map}(TQ, TM)^{U^{\mathbb{C}}(2)}\ ,\ v \in TQ, \quad (95)$$

where $K^M$ is the fundamental vector field of the hyperKähler $S^1$-action.

Notice that $\nabla^A u$ vanishes on fundamental vector fields, thus the map is horizontal. Making use of the isomorphism $T_{\pi(p)}X \simeq (horA)_p$, where $(horA)_p$ is the horizontal subspace in $T_pQ$ defined by the connection $A$, we can regard the covariant derivative as a map

$$\nabla^A u \in \mathrm{Map}(Q, (\mathbb{R}^4)^* \otimes TM)^{U^{\mathbb{C}}(2)}, \quad (96)$$

where $\nabla^A u(p)(w) = Tu(\widetilde{p(w)})$ for $p \in Q, w \in \mathbb{R}^4$, with $\widetilde{p(w)}$ being the unique horizontal lift of $p(w)$ with respect to the connection $A$. Notice the subtle abuse of notation, we identify $p \in Q$ with its image under the map $Q \to SO(X)$, which gives us a frame $p : \mathbb{R}^4 \to T_{\pi(p)}X$.

Yet another perspective is to view the covariant derivative of a spinor as a section

$$\nabla^A u \in \mathrm{Map}(Q, (\mathbb{R}^4)^* \otimes u^*TM)^{U^{\mathbb{C}}(2)} \simeq \Gamma(X, T^*X \otimes \pi_! u^*TM), \quad (97)$$

where $\pi_! u^*TM$ is the pushdown of $u^*TM \to Q$ to a bundle on $X$ by taking of the quotient of $U^{\mathbb{C}}(2)$. Note also that the Riemannian metrics on $X$ and $M$ give rise to a metric on the bundle $T^*X \otimes \pi_! u^*TM$ over $X$, allowing us to integrate the expression $\nabla^A u$.

In the following we will be mostly interested in connections which are lifts of the Levi-Civita connection $A_0$ on $X$ and a connection $a$ on $L$.

**Definition 34.** Denote by $\mathcal{A}(Q)$ the space of connections on $Q$ and by $\mathcal{A}(L)$ the connections on the bundle of unitary frames of $L$. We further denote the Levi-Civita connection on $X$ by $A_0$ and define

$$\mathcal{A}_0 = \big\{A \in \mathcal{A}(Q);\ A \text{ is a lift of } A_0 \oplus a, a \in \mathcal{A}(L)\big\}. \quad (98)$$

The attentive reader will have noticed that, by Lemma (9), we can technically only lift connections on $\det(Q) = L^2$, but any connection on $L$ induces a connection on $L^2 = L \otimes L$.

### 2.2.2 Generalized Dirac Operator

We now have the tools to define the generalized non-linear Dirac Operator:

Given a generalized spinor $u \in \mathsf{Map}(Q, M)^{U^{\mathbb{C}}(2)}$ and a connection $A \in \mathcal{A}_0$ we define the Dirac operator as the composition of covariant derivative and Clifford multiplication, namely

$$\not{D}^A : u \ni \mathsf{Map}(Q, M)^{U^{\mathbb{C}}(2)} \xrightarrow{\nabla_A} \mathsf{Map}(Q, (\mathbb{R}^4)^* \otimes u^*TM)^{U^{\mathbb{C}}(2)} \xrightarrow{cl} \mathsf{Map}(Q, u^*TM^-)^{U^{\mathbb{C}}(2)} \tag{99}$$

For $u \in \mathsf{Map}(Q, M)^{U^{\mathbb{C}}(2)}$, we will have $\not{D}^A u \in \Gamma(X, \pi! u^*TM^-)$, where as before, $\pi! u^*TM^-$ denotes the quotient vector bundle $u^*TM^-/U^{\mathbb{C}}(2)$. We observe that the space in which the section $\not{D}^A u$ lives in depends on the spinor $u$, so the Dirac operator should be understood as a section of an infinite dimensional vector bundle

$$\mathcal{E}^- \to \mathsf{Map}(Q, M)^{U^{\mathbb{C}}(2)} \tag{100}$$

where the fibre $\mathcal{E}^-_u$ over $u \in \mathsf{Map}(Q, M)^{U^{\mathbb{C}}(2)}$ is given by $\mathsf{Map}(Q, u^*TM^-)^{U^{\mathbb{C}}(2)}$.

### 2.2.3 Weitzenböck Formula

A very useful formula for computations involving the Dirac operator defined above is the Weitzenböck formula. We will in fact only use a simplified form, assuming the base manifold is Kähler. Denote the Kähler form on $X$ by $\omega_X$ and the Kähler form on $M$ which is fixed by the permuting $S^1$ action by $\omega_I$. Further let $\mu_{\mathbb{C}} = \mu_j + i\mu_k$ be the complex moment map of the hyperKähler action of $S^1$ and $A \in \mathcal{A}_0$.

**Theorem 13** (Weitzenböck formula)**.**

$$\int_M \|\not{D}^A u\|^2 * 1 = \int_M |\nabla^A u|^2 * 1 + \omega_I(\nabla^A u, \nabla^A u) \wedge \omega_X + \langle \mu_{\mathbb{C}} \circ u, (F_a)^{2,0} \rangle * 1$$

*Proof.* See Appendix. □

## 2.3 Generalized Seiberg-Witten Equations

Let $X$ be a compact, simply connected, four dimensional Kähler manifold, which will be referred to as the *source manifold*, and $Q \to X$ a $U^{\mathbb{C}}(2)$-structure on $X$, given by $Q = L \bullet Q_{can}$ for some line bundle $L \to X$. Here $Q_{can}$ denotes the canonical $U^{\mathbb{C}}(2)$ structure on $X$.

Let $M$ be a hyperKähler manifold carrying a permuting $S^1$-action and a hyperKähler

$S^1$-action with hyperKähler moment map $\mu$, which will be referred to as the *target manifold*. Although the following exposition can be done in arbitrary dimensions, we will focus on the simplest case, namely $\dim M = 4$.
Recall that a connection $A \in \mathcal{A}_0$ is given by a lift $A = A_0 \oplus a$, where $A_0$ is the Levi-Civita connection and $a \in \mathcal{A}(L)$. Furthermore, call to mind the identification in corollary (4):

$$\mathbb{R} \cdot \omega \times \mathcal{H}^{2,0}_{\partial}(X) \simeq \mathcal{H}^2_+(X, \mathbb{R}), \tag{101}$$

$$(s \cdot \omega, \phi) \mapsto (s \cdot \omega + \phi - \overline{\phi}). \tag{102}$$

**Definition 35.** We define the **Seiberg-Witten map**

$$sw : Map(Q, M)^{U^{\mathbb{C}}(2)} \times \mathcal{A}_0 \to Map(Q, u^*TM^-)^{U^{\mathbb{C}}(2)} \times \Omega^2_+(X), \tag{103}$$

$$(u, a) \mapsto (\not{D}^A u,\ F^+_a - \mu \circ u) \tag{104}$$

and the **enhanced Seiberg-Witten map**

$$sw_+ : Map(Q, M)^{U^{\mathbb{C}}(2)} \times \mathcal{A}_0 \times \mathcal{H}^2_+(X, \mathbb{R}) \to Map(Q, u^*TM^-)^{U^{\mathbb{C}}(2)} \times \Omega^2_+(X), \tag{105}$$

$$(u, a, s \cdot \omega + \phi - \overline{\phi}) \mapsto (\not{D}^A u,\ F^+_a - \mu \circ u + s \cdot \omega + \phi - \overline{\phi}) \tag{106}$$

We call the parameter $(s, \phi) \in \mathbb{R} \times \mathcal{H}^{2,0}_{\partial}(X)$ a *perturbation* and for a fixed perturbation define the **perturbed Seiberg-Witten map**

$$sw_{s,\phi} : Map(Q, M)^{U^{\mathbb{C}}(2)} \times \mathcal{A}_0 \to Map(Q, u^*TM^-)^{U^{\mathbb{C}}(2)} \times \Omega^2_+(X), \tag{107}$$

$$(u, a) \mapsto (\not{D}^A u,\ F^+_a - \mu \circ u + s \cdot \omega + \phi - \overline{\phi}) \tag{108}$$

The target of the maps depends on the input (namely the bundle $u^*TM^-$ depends on the spinor $u$), and should be interpreted as sections of certain vector bundles.
Notice that $\mu \circ u \in Map(Q, \mathbb{R} \otimes \mathfrak{sp}(1)^*)^{U^{\mathbb{C}}(2)}$ so we use the identification $\mathfrak{sp}(1)^* \simeq \mathfrak{sp}(1) \simeq \Lambda^2_+\mathbb{R}^4$ using the Killing form on $\mathfrak{sp}(1)$ to identify it with a self-dual two form.
The **Gauge Group** $\mathcal{G} := Map(X, S^1) = Map(Q, S^1)^{S^1} \subseteq Maps(Q, U^{\mathbb{C}}(2))^{U^{\mathbb{C}}(2)}$ acts on pairs of generalized spinors and connections via

$$g.(u, a) = (g \cdot u, a + (g^{-1})^*\eta) \tag{109}$$

where $\eta$ denotes the Maurer-Cartan form on $S^1$.

**Lemma 20.** *The (perturbed) Seiberg-Witten map is equivariant with respect to the action of the gauge group.*

*Proof.* Since the gauge group is abelian, the moment map is invariant with respect to the group action. On the other hand, we have

$$F_{a+(g^{-1})^*\eta} = F_a \tag{110}$$

so the second equation $F_a^+ = \mu \circ u$ is invariant under the gauge action.

Furthermore, since $X$ is simply connected, there exists a function $f : X \to \mathbb{R}$ such that $g(x) = e^{if(x)}$ and $(g^{-1})^*\eta = g \cdot dg^{-1} = -df$, where we again identify $i \cdot \mathbb{R} \simeq \mathbb{R}$.

We compute

$$\begin{aligned}
\not{D}_{g.A}(g \cdot u) &= cl \circ \nabla_{A-df}(e^{if} \cdot u) = cl \circ \Big(T(e^{if} \cdot u) + K_A^M(e^{if} \cdot u) - K_{df}^M(e^{if} \cdot u)\Big) \\
&= cl \circ \Big(K_{df}^M(e^{if} \cdot u) + e^{if} \cdot Tu + K_A^M(e^{if} \cdot u) - K_{df}^M(e^{if} \cdot u)\Big) \\
&= cl \circ \Big(e^{if} \cdot Tu + K_A^M(e^{if} \cdot u)\Big) = e^{if} \cdot cl \circ \nabla_A u = e^{if} \cdot \not{D}_A u = g.\not{D}_A u
\end{aligned}$$

□

We can thus form the **Moduli Spaces**

$$\mathcal{M} := sw^{-1}(\{0\})/\mathcal{G} \tag{111}$$

$$\mathcal{M}^{s,\phi} := sw_{s,\phi}^{-1}(\{0\})/\mathcal{G} \tag{112}$$

which will be the main subject of our further investigations.

## 2.4 Simplifiying The Equations

Consider a permuting $S^1$-action on a hyperKähler manifold $M$, leaving the complex structure $I_1$ invariant while rotating $I_2$ and $I_3$. Consider the element $g = e^{i\frac{\pi}{2}} \in S^1$, which acts on the complex structures via $I_1 \mapsto I_1$, $I_2 \mapsto -I_2$, $I_3 \mapsto -I_3$, and induces the map $g : M \to M$, $x \mapsto e^{i\frac{\pi}{2}}.x$. Further we assume there is a hyperKähler action $S^1 \curvearrowright M$ with hyperKähler-moment map $\mu = \mu_I i + \mu_{\mathbb{C}} j$, where $\mu_{\mathbb{C}} = \mu_J + \mu_K i$. In the case where the base manifold is Kähler, the second equation $F_a^+ = \mu \circ u$ can be further decomposed into $(F_a)^{2,0} = \mu_{\mathbb{C}} \circ u$, $(F_a^+)^{1,1} = \mu_I \circ u$. We then have the following:

**Lemma 21.** *For all spinors $u$ and connections $A \in \mathcal{A}$ we have:*

*(a)* $|\nabla^A(g\circ u)|^2 = |\nabla^A u|^2$

*(b)* $\mu_I \circ g = \mu_I$

*(c)* $\mu_{\mathbb{C}} \circ g = -\mu_{\mathbb{C}}$

*(d)* $Lg^*\omega_{I_1} = \omega_{I_1}$

*Proof.* We use that the two group actions are isometric and commute, as well as the permuting/hyperKähler properties:

(a) $|\nabla^A(g\circ u)|^2 = |Tg\circ Tu\circ pr_{horA}|^2 = |Tu\circ pr_{horA}|^2 = |\nabla^A u|^2$, since $g$ is an isometry of $M$.

(b) For $\zeta \in i\cdot\mathbb{R}$, $Y \in \Gamma(M, TM)$ we have

$$\langle d(\mu_I\circ g)(Y), \zeta\rangle = \langle d\mu_I \circ dg(Y), \zeta\rangle = g(K^M_\zeta, I_1 dg(Y)) = \tag{113}$$

$$g_M(dg^{-1}K^M_\zeta, I_1 Y) = g_M(K^M_\zeta, I_1 Y) = \langle d\mu_I(Y), \zeta\rangle, \tag{114}$$

where we use that $d\mu_I = \iota_{i\mathbb{R}}\omega_I$ and that $g$ is an isometry and commutes with $I_1$, and that the two $S^1$ group actions commute. Thus $d\mu_I\circ g = d\mu_I$ and hence $\mu_I \circ g = \mu_I + c$. Using $g^4 = id$, we see that $\mu_I = \mu_I\circ g^4 = \mu_I + 4c$, so $c = 0$ and $\mu_I = \mu_I\circ g$.

(c) The proof for $\mu_{\mathbb{C}}$ is similar, using that $I_l dg = -dg I_l$, $l = 2, 3$.

(d) This follows directly, since $g$ leaves the complex structure $I_1$ invariant.

□

**Lemma 22.** *Assume the* $U^{\mathbb{C}}(2)$*-principal bundle* $Q$ *is given by* $Q = L \bullet Q_{can}$*, where* $Q_{can}$ *is the canonical* $U^{\mathbb{C}}(2)$ *bundle and* $L$ *a hermitian line bundle. Assume further that* $L$ *admits a holomorphic structure, i.e. there is a connection* $a$ *on* $P^L_{S^1}$ *with* $F^{2,0}_a = 0$*. Then for any connection* $b$ *on* $P^L_{S^1}$ *the (2,0)-part of its curvature* $F^{2,0}_b$ *lies in the orthogonal complement of* $\mathcal{H}^{2,0}_{\bar\partial}(X)$*.*

*Proof.* Any other connection $b$ on $P^l_{S^1}$ differs from $a$ by an imaginary one form $a' \in i\Omega^1(X)$. In particular, if $b = a + a'$, we have $F_b = F_a + da'$. Projecting onto the $(2,0)$-factor, we have $F^{2,0}_b = pr^{2,0}\circ da' = \partial\ (a')^{1,0}$, using that $X$ is Kähler and the deRham differential therefore splits into $d = \partial + \bar\partial$. We thus have shown that the $(2,0)$-part of the curvature of any connection on $P^L_{S^1}$ lies in $\partial\Omega^{1,0}(X)$, which by Hodge decomposition is orthogonal to $\mathcal{H}^{2,0}_{\partial}(X) = \mathcal{H}^{2,0}_{\bar\partial}(X)$ in $\Omega^{2,0}(X)$. □

We assume from now on that $L$ admits a holomorphic structure and deal with the general case later.

**Lemma 23.** *If* $L$ *admits a holomorphic structure and* $(u, a, is \cdot \omega, \phi)$ *is a solution to the perturbed Seiberg-Witten equations, we have:*

$$\not{D}^A u = 0 \tag{115}$$

$$F_a^{2,0} = F_a^{0,2} = 0 \tag{116}$$

$$\mu_{\mathbb{C}} \circ u = \phi \tag{117}$$

$$\mu_I \circ u = (F_a^+)^{1,1} + is \cdot \omega \tag{118}$$

*Proof.* We only need to show $F_a^{2,0} = 0$ since $F_a^{0,2} = \overline{F_a^{2,0}}$ and the other equations are a consequence of the decomposition $\Omega_+(X) = \Omega^0(X) \cdot \omega + \Omega^{2,0}(X)$.
Solutions to the perturbed Seiberg-Witten equations are minima of the functional

$$S(u, a, is \cdot \omega, \phi) = \tag{119}$$

$$\int_M ||\not{D}^A u||^2 + \frac{1}{2}||(F_a)^{2,0} - \mu_{\mathbb{C}} \circ u + \phi||^2 + ||(F_a^+)^{1,1} - \mu_I \circ u + is \cdot \omega||^2 = \tag{120}$$

$$\int_M ||\not{D}^A u||^2 + \frac{1}{2}||(F_a)^{2,0}||^2 + \frac{1}{2}||\mu_{\mathbb{C}} \circ u||^2 + \frac{1}{2}||\phi||^2 - \langle (F_a)^{2,0}, \mu_{\mathbb{C}} \circ u\rangle \tag{121}$$

$$- \langle \mu_{\mathbb{C}} \circ u, \phi\rangle + ||(F_a^+)^{1,1} - \mu_I \circ u + is \cdot \omega||^2 . \tag{122}$$

where we used that $\langle (F_a)^{2,0}, \phi\rangle = 0$ by the previous lemma.
Using the Weitzenböck formula (Theorem 13) we obtain

$$S(u, a, is \cdot \omega, \phi) = \tag{123}$$

$$\int_M |\nabla^A u|^2 + \omega_I(\nabla^A u, \nabla^A u) \wedge \omega_X + \frac{1}{2}||(F_a)^{2,0}||^2 + \frac{1}{2}||\mu_{\mathbb{C}} \circ u||^2 + \frac{1}{2}||\phi||^2 \tag{124}$$

$$- \langle \mu_{\mathbb{C}} \circ u, \phi\rangle + ||(F_a^+)^{1,1} - \mu_I \circ u + is \cdot \omega||^2 \tag{125}$$

Using $g = e^{i\frac{\pi}{2}}$ from the previous lemma, we see that every term in $S(u, a, is \cdot \omega, \phi)$ is invariant under the transformation

$$(u, a, is \cdot \omega, \phi) \mapsto (g \circ u, a, is \cdot \omega, -\phi). \tag{126}$$

Therefore $(g \circ u, a, is \cdot \omega, -\phi)$ also minimizes the functional and is a solution to the perturbed Seiberg-Witten equations. In particular we have $(F_a)^{2,0} = \mu_{\mathbb{C}} \circ g \circ u + \phi = -(\mu_{\mathbb{C}} \circ u - \phi)$, while the perturbed Seiberg-Witten equation for $(u, a, is \cdot \omega, \phi)$

yields $(F_a)^{2,0} = \mu_{\mathbb{C}} \circ u - \phi$, and thus $(F_a)^{2,0} = 0$.

□

To understand the implications for the first equation in the Kähler case, namely $\not{D}^A u = 0$, we follow [28]:

**Theorem 14.** *Let $Q \to X$ be a principal $G$-bundle over an almost complex manifold $(X, I)$ and let $(M, I_1)$ be another almost complex manifold with a left $G$-action which commutes with the almost complex structure $I_1$. Furthermore, let $A$ be a connection on $Q$. Then the associated bundle $F := Q \times_G M \xrightarrow{\pi} X$ carries an almost complex structure*

$$I(A) := I \oplus I_1 \text{ on } TF = \mathrm{hor}_A F \oplus VF \simeq \pi^*(TX) \oplus VF \tag{127}$$

If $I$ and $I_1$ are integrable, and $F_A^{0,2} = 0$, then $I(A)$ is integrable. We say a section $s : X \to \mathcal{M}$ is *holomorphic* if

$$Ts \circ I = I(A) \circ Ts \tag{128}$$

or equivalently, if $s$ is determined by the equivariant map $u_s \in Map(Q, M)^G$,

$$Tu_s|_{\mathrm{hor}_A TQ} \circ \tilde{I} = I_1 \circ Tu_s|_{\mathrm{hor}_A TQ} \tag{129}$$

where $\tilde{I}(\tilde{v}) := \widetilde{I(v)}$. Notice that in the case where $M$ is a vector space and $F$ is an associated vector bundle, this coincides with the usual definition of holomorphic sections.

Applying this to our set-up we immediately get:

**Corollary 6.** *If $(A, u)$ is a solution of the generalized Seiberg-Witten equations over a Kähler manifold $X$, then the connection A induces an integrable complex structure $I(A)$ on the bundle $F = Q \times_{U^{\mathbb{C}}(2)} M$.*

It turns out that the harmonic spinors are precisely the holomorphic sections:

**Theorem 15.**
- *If $(A, u)$ is a solution of the generalized Seiberg-Witten equations over a Kähler manifold $X$, then $u$ is holomorphic with respect to the complex structure $I(A)$.*
- *If $A \in \mathcal{A}$ is a connection with $F_A^{0,2} = 0$ and $u \in Map(Q, M)^{Spin^G(4)}$ is holomorphic with respect to $I(A)$, then $\not{D}^A u = 0$.*

*Proof.* See [28]. □

We quickly glance over the non-holomorphic case: Assume the line bundle $L$ over $X$ is not necessarily of type $(1,1)$, i.e. does not admit a holomorphic structure.
The fibre bundle $F = Q \times_G M$ still admits a (non-integrable) almost complex structure $J_A$ depending on a connection $A$, by splitting

$$TF = \mathrm{hor}\, A \oplus TM \tag{130}$$

and the almost complex structure on the vertical space is induced by the complex structure of $M$, while the almost complex structure on the horizontal space is induced by the isomorphism $\mathrm{hor}\, A \simeq TX$ and the complex structure on $X$.

**Proposition 3.** A section $u \in \Gamma(X, F)$ satisfies $\not{D}^A u = 0$ if and only if $u$ is a $J_A$-holomorphic map, i.e. if it satisfies $\bar{\partial}_{J_A}(u) = du + J_A \circ du \circ j = 0$.

*Proof.* See [10], [28]. □

**Corollary 7.** *If* $u \in \Gamma(X, M)$ *satisfies* $\not{D}^A u = 0$*, then* $\mu_{\mathbb{C}} \circ u$ *is a holomorphic form.*

*Proof.* Since $\mu_{\mathbb{C}} : M \to \mathbb{C}$ is holomorphic, the induced map $\mathcal{M} \to K_X$ is holomorphic.
So we have $d(\mu_{\mathbb{C}} \circ u) \circ j = d\mu_{\mathbb{C}} \circ du \circ j = J_A \circ d\mu_{\mathbb{C}} \circ du$.
Notice here that $J_A$ on $K_X$ does not depend on the $L$-component of the connection $A$. □

Therefore a solution $(u, a)$ to the gSW-equation satisfying the equation

$$F_a^{2,0} = \mu_{\mathbb{C}} \circ u + \phi \tag{131}$$

further implies that $F_a^{2,0}$ is holomorphic. Unfortunately, we cannot use the Weitzenböck formula technique here to obtain further constraints, as in the integrable case.

## 2.5 Linearization

Notice that for $u \in \mathrm{Map}(Q, M)^{U^{\mathbb{C}}(2)}$, the tangent space $T_u\mathrm{Map}(Q, M)^{U^{\mathbb{C}}(2)}$ can be identified with equivariant sections $\Gamma(Q, u^*TM)^{U^{\mathbb{C}}(2)} \simeq \Gamma(Q, u^*TM^+)^{U^{\mathbb{C}}(2)}$ while $\mathcal{A}_0$ is an affine space over $\Omega^1(X)$.
The linearization of the Dirac operator at some pair $(u, a)$ is therefore a map

$$\not{D}^{\mathrm{lin}}_{(u,a)} : \Gamma(Q, u^*TM^+)^{U^{\mathbb{C}}(2)} \times \Omega^1(X) \to \Gamma(Q, u^*TM^-)^{U^{\mathbb{C}}(2)} \tag{132}$$

It was shown in [24] that this is indeed a geometric linear Dirac operator in the usual sense:

On the other hand, $\mathbb{R}\cdot\omega$, $\mathcal{H}^{2,0}_{\bar\partial}(X)$ and $\Omega^2_+(X)$ are vector spaces, so we can identify the tangent space at a point with the space itself.

Let $u_t \in \Gamma(Q,M)^{U^{\mathbb{C}}(2)}$ be a curve with $u_0 = u$ representing a tangent vector $\psi = \frac{d}{dt}_{|t=0} u_t \in \Gamma(Q, u^*TM^+)^{U^{\mathbb{C}}(2)}$ and $\alpha \in \Omega^1(X)$. We then compute:

$$\frac{d}{dt}_{|t=0} \not{D}^{A+t\alpha} u_t = \frac{d}{dt}_{|t=0} cl \circ \nabla^{A+t\alpha} u_t = \frac{d}{dt}_{t=0} cl \circ (du_t + K^M_A + t\cdot K^M_\alpha) = \tag{133}$$

$$cl \circ ((d\psi + K^{TM}_A) + cl(K^M_\alpha) = (\not{D}^{lin}_{(u,a)})\psi + cl(K^M_\alpha) \tag{134}$$

In conclusion:
Linearizing the enhanced Seiberg-Witten map at a point $(u, a, s\cdot\omega, \phi)$ yields the map

$$D_{(u,a,s\cdot\omega,\phi)} sw : \Gamma(Q, u^*TM^+)^{U^{\mathbb{C}}(2)} \times \Omega^1(X) \times \mathcal{H}^2_+(X,\mathbb{R}) \to \Gamma(Q, u^*TM^-)^{U^{\mathbb{C}}(2)} \times \Omega^2_+(X),$$

given by

$$(\psi, \alpha, t\cdot\omega, \phi') \mapsto (\not{D}^{lin}_{(u,a)}\psi + cl(K^M_\alpha), d^+\alpha - d\mu\circ\psi + t\cdot\omega + \phi' - \overline{\phi'}) \,. \tag{135}$$

**Lemma 24.** *Assume that the set of fixed points set of the hyperKähler* $S^1$*-action on* $M$ *is a finite set of points, called the singularities. Then the linearization of the enhanced Seiberg-Witten map at a solution* $(u, a, s\cdot\omega, \phi)$*, where* $u$ *is non-constant, is a surjective map.*

*Proof.* Let $\zeta \in \Gamma(Q, u^*TM^-)^{U^{\mathbb{C}}(2)}$ be $L^2$-orthogonal to the image of

$$(\psi, \alpha) \mapsto \not{D}^{lin}_{(u,a)}\psi + cl(K^M_\alpha) \tag{136}$$

In particular for $\alpha = 0$ we have $\langle \not{D}^{lin}_{(u,a)}\psi, \zeta\rangle_{L^2} = 0$, so $\zeta \perp \operatorname{Im}\not{D}^{lin}$, implying $\zeta \in \operatorname{Coker}\not{D}^{lin} = \ker(\not{D}^{lin})^*$, where $(\not{D}^{lin})^*$ is the adjoint Dirac operator. By the unique continuation property of elliptic operators $\zeta$ is either constantly zero or there is no open set on which $\zeta$ vanishes identically.
Assume that $\zeta$ is not trivial. Since $u$ is holomorphic, there is no open set on which $u = a_i$ constantly for some fixed point of the $S^1$ actions $a_i \in M$. This means that we can choose $x$ in $X$ such that $\zeta$ does not vanish in a small neighborhood $U$ around $x$ and $u(p)$ is not a singularity for all $p \in U$. We claim that there is a locally defined

one-form $\alpha \in \Omega^1(U)$, such that $cl(K^M_\alpha) = \zeta$ on $U$. Indeed, Clifford multiplication induces a pointwise isomorphism

$$T^*_pX \simeq \pi_! u^* TM^-_p, a \mapsto cl(K^M_a)(u(p)) \tag{137}$$

using that the fundamental vector field $K^M_1$ does not vanish at $u(p)$, since $u(p)$ is not a fixed point. But extending this one-form by multiplying it with your favorite bump function non-vanishing on $U$ yields a one-form $\alpha_0$ such that

$$\langle \not{D}^{lin}\psi + cl(K^M_{\alpha_0}), \zeta\rangle_{L^2} = \langle cl(K^M_{\alpha_0}), \zeta\rangle_{L^2} > 0, \tag{138}$$

giving a contradiction. Therefore, $\zeta$ must vanish identically, establishing surjectivity onto the first factor.
The surjectivity onto the second factor follows from the fact that

$$\Omega^2_+(X) = d^+(\Omega^1(X)) \oplus \mathcal{H}^2_+(X)$$

□

In the following we compare the linearization of the generalized Seiberg-Witten map with the linearization of the linear Seiberg-Witten map.

Assume from now on that $\dim M = 4$.

Notice that the vector bundle $\pi_! u^*TM$ carries a Clifford structure inherited by the Clifford structure on $TM$. By the classification of Clifford bundles, $\pi_! u^*TM \simeq L_0 \otimes S^+$ for some complex line bundle $L_0$, where $S^+ = Q_{can} \times_{\rho_+} \mathbb{C}^2$ is the canonical spinor bundle. Writing $Q_{L_0} = Q_{can} \times_X P_{L_0}$, we can identify

$$\Gamma(Q, u^*TM)^{U^{\mathbb{C}}(2)} \simeq \Gamma(X, \pi_! u^*TM) \simeq Map(Q_{L_0}, \mathbb{C}^2)^{U^{\mathbb{C}}(2)} \tag{139}$$

Denoting the space of connections on $Q_{L_0}$, which are lifts of the Levi-Civita connection by $\mathcal{A}^{L_0}_0$, we also have the linear Seiberg-Witten map

$$sw_{L_0} : Map(Q_{L_0}, \mathbb{C}^2_+)^{Spin^{\mathbb{C}}(4)} \times \mathcal{A}^{L_0}_0 \to Map(Q_{L_0}, \mathbb{C}^2_-)^{Spin^{\mathbb{C}}(4)} \times i\Omega^2_+(X), \tag{140}$$

$$(\psi, A') \mapsto ((\not{D}^{lin})^{A'}\psi, F^+_{a'} - \mu^{L_0} \circ \psi) \tag{141}$$

and also the corresponding perturbed version. Naturally, the question arises as to how the linearization of the non-linear equations compare with the linearization of the linear equations. For this, we first notice that $\not{D}^{lin}_{(u,A)}$ is a geometric Dirac operator on $\pi_! u^*TM$. Therefore it differs from $(\not{D}^{lin})^{A'}$ only by a zero order perturbation. In the the other factor, the only term which is not of order zero is $d^+\alpha$, which agrees in both cases. Therefore we have:

**Corollary 8.** *The linearization of the non-linear Seiberg-Witten map and the linearization of the corresponding classical Seiberg-Witten map coincide up to a homotopy of order zero terms.*

This crucial fact will be important later when we speak about orientations of the spaces involved. The bundle $\pi_! u^* TM^+$ carries an interesting section given by the fundamental vector field of the hyperKähler $S^1$ action:

$$\mathcal{K}(x) := \left[p, K_i^M(u(p))\right] \in \Gamma(X, \pi_! u^* TM^+), p \in \pi^{-1}(x). \tag{142}$$

In other words, it is given by the equivariant section $Q \to u^*TM,\ p \mapsto K_i^M(u(p))$. This is well-defined, since $u$ and fundamental vector fields are equivariant.

We end this chapter with a small lemma, which at first glance seems to make strong assumptions on the target manifold $M$, but just you wait for the next chapter!

**Lemma 25.** *Assume $(u, A = a \oplus A_0)$ is a solution to the generalized Seiberg-Witten equations with perturbation $\phi$, and assume $M$ is covered by holomorphic charts $M_j \simeq \mathbb{C}^2$, which are equivariant with respect to the $S^1$ hyperKähler action on $M$ and the standard $S^1$ action on $\mathbb{C}^2$. Furthermore assume, that the complex symplectic form $\omega_{\mathbb{C}} = \omega_2 + i\omega_3$ in these coordinates is given by $dz_1 \wedge dz_2$. Then, there exists a connection $A'$ and a (real) gauge transformation $g$, such that $(g \cdot \mathcal{K}, g.A')$ is a solution to the linear Seiberg-Witten equations with perturbation $\phi$.*

*Proof.* First, we observe that

$$(\not{D}^{lin}_{(u,a)})\mathcal{K}(x) = \frac{d}{dt}_{|t=0} \not{D}^A e^{it}.u(p) = \frac{d}{dt}_{|t=0} \not{D}^A u(p.e^{-it}) = 0,$$

since $\not{D}^A u(p) = 0$ for all $p \in Q$. Since $\not{D}^{lin}$ is a geometric Dirac operator, there exists a connection $A'$ on $Q^{L_0}$ such that $\not{D}^{lin} = (\not{D}^{lin})^{A'}$. Next up we show $\mu_{\mathbb{C}}^{L_0} \circ \mathcal{K} = \mu_{\mathbb{C}} \circ u$. Locally, $u$ is given by maps

$$u_{|u^{-1}(M_j)} : u^{-1}(M_j) \to \mathbb{C}^2, u_{|u^{-1}(M_j)} = (\alpha_j, \beta_j)$$

and by our assumption that the hyperKähler action is the standard one,

$$\mathcal{K}_{|u^{-1}(M_j)} : u^{-1}(M_j) \to \mathbb{C}^2, \mathcal{K}_{|u^{-1}(M_j)} = (i \cdot \alpha_j, -i \cdot \beta_j)$$

Since $\omega_{\mathbb{C}} = dz_1 \wedge dz_2$ locally, the complex moment map is locally given by $\mu_{\mathbb{C}}^{L_0}(z_1, z_2) = z_1 \cdot z_2$, so we can conclude $\mu_{\mathbb{C}}^{L_0} \circ \mathcal{K} = \alpha_j \cdot \beta_j = \mu_{\mathbb{C}} \circ u$. We know that a solution to the generalized Seiberg-Witten equations with perturbation $\phi$ must satisfy $\mu_{\mathbb{C}} \circ u = \phi$. We conclude that $\mathcal{K}$ satisfies $(\not{D}^{lin})^{A'}\mathcal{K} = 0$ and $\mu_{\mathbb{C}}^{L_0} \circ \mathcal{K} = \phi$. From classical Seiberg-Witten theory one knows (see for example ([22])), that there exists a real gauge $g$, such that $(g \cdot \mathcal{K}, g.A')$ also satisfies the $(F^+_{g.A'})^{1,1} = \mu_I^{L_0} \circ g \cdot \mathcal{K} + is \cdot \omega$ equation and thus defines a solution to the linear Seiberg-Witten equation. □

# 3 Gibbons-Hawking Spaces

*“Wettschulden.”*

- Paul Middelanis

In this section we describe four dimensional hyperKähler manifolds obtained from the Gibbons-Hawking-Ansatz, which admit certain symmetries. In fact, these are ALL four dimensional hyperKähler manifolds suitable for our purposes:

**Theorem 16.** *Let $M$ be a simply-connected, complete four dimensional hyperKähler manifold. If there exists a hyperKähler $S^1$-action on $M$ admitting a hyperKähler moment map, then $M$ is obtained by the Gibbons-Hawking Ansatz.*

*Proof.* See [14]. □

We follow the exposition in [21].

**Example 4.** (Incomplete Gibbons-Hawking Space)

We begin with an open subset $U$ of $\mathbb{R}^3$ and a harmonic, positive function

$$V : U \to \mathbb{R}_{>0}, \tag{143}$$

i.e.

$$\Delta V = d * dV = \sum_{i=1}^{3} \partial_i^2 V \cdot vol = 0. \tag{144}$$

In particular, for $F = -2\pi * dV$, we have $dF = 0$. Assume now there is a $U(1)$-principal bundle $\pi : M \to U$ with a connection $\Theta$ whose curvature is $F$, i.e. a one-form $\Theta \in \Omega^1(M, \mathbb{R})$ with $d\Theta = \pi^* F$ (We identify $i\mathbb{R} \simeq \mathbb{R}$). Recall that $U(1)$-principal bundles over $U$ are classified by their first chern class $[c_1(M)] \in H^2(U, \mathbb{Z})$, and that for any connection on $M$ with curvature $F$, we have $[c_1(M)] = [\frac{F}{2\pi}]$, hence such a principal bundle exists exactly when $\frac{F}{2\pi}$ lies in the image $H^2(U, \mathbb{Z}) \to H^2_{dR}(U, \mathbb{R})$. Denote by $\partial_{S^1}$ the fundamental vector field on $M$ associated to $i \in \mathrm{Lie}(S^1) \simeq i\mathbb{R}$. Since $\Theta(\partial_{S^1}) = 1$, in a local trivialisation of $M$ over a patch $U_\alpha \subseteq U$ we have

$$\Theta_\alpha = \pi^* A_\alpha + d\chi_\alpha, \tag{145}$$

where $\chi_\alpha$ is the fibre coordinate and $A_\alpha$ is a one-form on $U_\alpha$ with $dA_\alpha = F$. For convenience, define $\tilde{\Theta} = \frac{\Theta}{2\pi}$. The manifold $M$ carries a hyperKähler-structure, defined as follows:

From now on we denote pulled back forms on $M$ by the same symbol, for example we abbreviate $dx_i := \pi^* dx_i$ for the form living on $M$. We do all the manipulations on $\mathbb{R}^3$ and then pull back to $M$, for example the Hodge Star operator "$*$" in the following will be the one on $\mathbb{R}^3$. Define three symplectic forms

$$\omega_i = \tilde{\Theta} \wedge dx_i + V * dx_i \tag{146}$$

These are indeed closed, since $d\omega_i = - * dV \wedge dx_i + dV \wedge * dx_i = 0$ and all these forms are pulled back from $\mathbb{R}^3$, where we have $\alpha \wedge * \beta = * \alpha \wedge \beta$ for one forms $\alpha, \beta$.

Define

$$\Omega_1 = \omega_2 + i\omega_3 \in \Omega^2(M, \mathbb{C}) \tag{147}$$

$$z_1 = x_2 + ix_3 \in \Omega^0(M, \mathbb{C}) \tag{148}$$

$$\alpha_1 = V^{-1}\tilde{\Theta} + idx_1 \in \Omega^1(M, \mathbb{C}). \tag{149}$$

$$\tag{150}$$

We calculate

$$\Omega_1 = \tilde{\Theta} \wedge dz_1 + V * d(x_2 + ix_3) \tag{151}$$

$$= \tilde{\Theta} \wedge dz_1 + V(dx_3 \wedge dx_1 + idx_1 \wedge dx_2) \tag{152}$$

$$= \tilde{\Theta} \wedge dz_1 + iVdx_1 \wedge dz_1 \tag{153}$$

$$= V\alpha_1 \wedge dz_1 \tag{154}$$

Thus, ker $\Omega_1$ is spanned by $\tilde{\partial}_2 + i\tilde{\partial}_3$ and $2\pi V \partial_{S^1} + i\tilde{\partial}_1$, where $\tilde{\partial}_i$ is the horizontal lift of $\partial_i$ with repect to $\Theta$. We use the following lemma:

**Lemma 26.** *Suppose M is a 2n-dimensional manifold and there is* $\Omega \in \Omega^2(M, \mathbb{C})$ *such that* $d\Omega = 0$ *and* $T_{\mathbb{C}}M =$ *ker* $\Omega \oplus$ *ker* $\overline{\Omega}$. *Then there is an integrable complex structure* $I$ *on* $M$.

*Proof.* Define $I_{\mathbb{C}}$ on $T_{\mathbb{C}}M$ via multiplication on ker $\Omega$ with $-i$ and multiplication with $i$ on ker $\overline{\Omega}$. Since $I_{\mathbb{C}}v = \overline{I_{\mathbb{C}}\overline{v}}$, $I_{\mathbb{C}}$ is the complexification of a real operator $I$ on $TM$. The integrability of the distribution $T^{0,1}M =$ ker $\Omega$ then follows then from $d\Omega = 0$. □

In our case, $\Omega_1$ defines a complex structure $I_1$. Furthermore,

$$\Omega_2 = V\alpha_2 \wedge dz_2, \text{ where } z_2 = z_3 + iz_1 \text{ and } \alpha_2 = V^{-1}\tilde{\Theta} + idx_2 \tag{155}$$

$$\Omega_3 = V\alpha_3 \wedge dz_3, \text{ where } z_3 = z_1 + iz_2 \text{ and } \alpha_3 = V^{-1}\tilde{\Theta} + idx_3 \tag{156}$$

define complex structures in the same manner.

For the global frame $(2\pi V\partial_\chi =: \partial_t, \tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3)$ of $TM$, the complex structures are given by:

$$I_1 : (\partial_t, \tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3) \mapsto (\tilde{\partial}_1, -\partial_t, \tilde{\partial}_3, -\tilde{\partial}_2) \tag{157}$$

$$I_2 : (\partial_t, \tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3) \mapsto (\tilde{\partial}_2, \tilde{\partial}_3, -\partial_t, -\tilde{\partial}_1) \tag{158}$$

$$I_3 : (\partial_t, \tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3) \mapsto (\tilde{\partial}_3, \tilde{\partial}_2, -\tilde{\partial}_1, -\partial_t), \tag{159}$$

$$\tag{160}$$

in particular

$$(\partial_t, \tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3) = (\partial_t, I_1\partial_t, I_2\partial_t, I_3\partial_t). \tag{161}$$

We can recover the metric from $I_1$ and $\omega_1$. Since

$$\omega_1 = V \operatorname{Re} \alpha_1 \wedge \operatorname{Im} \alpha_1 + V \operatorname{Re} dz_1 \wedge \operatorname{Im} dz_1, \tag{162}$$

the metric is given by

$$g = V((\operatorname{Re} \alpha_1)^2 + (\operatorname{Im} \alpha_1)^2) + V((\operatorname{Re} dz_1)^2 + (\operatorname{Im} dz_1))^2 \tag{163}$$

$$= V(V^{-2}\tilde{\Theta}^2 + dx_1^2) + V(dx_2^2 + dx_3^2) \tag{164}$$

$$= V^{-1}\tilde{\Theta}^2 + V||dx||^2 \tag{165}$$

Notice that the principal $U(1)$-action on the fibres is hyperKähler, since the action leaves the forms $\omega_i$ invariant. Its moment map is given by the projection

$$\mu : M \to U \subseteq \mathbb{R}^3. \tag{166}$$

**Example 5.** ($\mathbb{R}^4 \backslash \{0\}$ as a Gibbons-Hawking Space)

Consider $U = \mathbb{R}^3 \backslash \{0\}$ with spherical coordinates

$$x_1 = r \cdot \cos\theta \tag{167}$$

$$x_2 = r \cdot \sin\theta \cos\varphi \tag{168}$$

$$x_3 = r \cdot \sin\theta \sin\varphi \tag{169}$$

and the harmonic function

$$V = \frac{1}{4\pi r}. \tag{170}$$

Then we have $F = \frac{1}{2} \sin\theta \, d\theta \wedge d\varphi$, satisfying $\int_{S^2} \frac{F}{2\pi} = 1$. Thus the integrability condition is satisfied and the $U(1)$- principal bundle is the Hopf fibration

$$\mathbb{R}^4 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}, (z_1, z_2) \mapsto \left( \frac{|z_1|^2 - |z_2|^2}{2}, z_1 z_2 \right). \tag{171}$$

**Remark 1.** *We will make the transformation $x_1 \mapsto -x_1$, which will be useful for calculations later, changing the Hopf map to*

$$\mathbb{R}^4 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}, (z_1, z_2) \mapsto \left( \frac{|z_2|^2 - |z_1|^2}{2}, z_1 z_2 \right).$$

*Notice that this corresponds to changing the order of the variables $z_1$ and $z_2$ and the moment map $\mu_I = x_1$ changes sign under this transformation, which will be important later.*

Notice further that $x = (\frac{|z_2|^2 - |z_1|^2}{2}, z_1 z_2)$ satisfies $|x| = r = \frac{|z_1|^2 + |z_2|^2}{2}$, so we can write

$$|z_1| = (|x| - x_1)^{1/2}, \ |z_2| = (|x| + x_1)^{1/2}. \tag{172}$$

Write $z = x_2 + i x_3 = |z| \cdot e^{i\varphi}$. We have the following trivialisation of the Hopf bundle on $\mathbb{R}^3 \backslash \{(x_1, x_2, 0), x_2 \leq 0\}$, or equivalently on the domain where $-\pi < \varphi < \pi$:

$$\Phi : (x, e^{it}) \mapsto (z_1, z_2) = \left( (|x| - x_1)^{1/2} e^{i\varphi/2} e^{it}, (|x| + x_1)^{1/2} e^{i\varphi/2} e^{-it} \right) \tag{173}$$

A lengthy computation shows (see [17]) that this trivialisation is holomorphic with respect to all three complex structures and the metric is given by

$$g = \mathrm{Re}(dz_1)^2 + \mathrm{Im}(dz_1)^2 + \mathrm{Re}(dz_2)^2 + \mathrm{Im}(dz_2)^2, \tag{174}$$

which is just the standard hyperKähler structure on $\mathbb{H}\backslash\{0\}$! In particular the hyperKähler structure extends over the singularity at $x = 0$, thus we can add a single point to obtain a complete hyperKähler manifold.

We want to generalize the potential above to potentials of the form

$$V(x) = c + \sum_{i \in I} \frac{1}{4\pi||x - a_i||} \tag{175}$$

where $c \in \mathbb{R}_{>0}$ is a constant, and we assume that all singularities $a_i$ lie on the $x_2 = x_3 = 0$ line (This is needed to have a well-defined permuting $S^1$ action later on). Furthermore, we assume from now on that the singularities $\{a_i\}_{i \in I}$ are ordered, i.e. $a_i < a_{i+1}$, where we denote the point and its $x_1$ coordinate both by $a_i$.
We will also consider the case where the sum above is infinite and discuss when this is actually convergent and well-defined.
Near a singularity $a_0$, such a potential is of the form $V(x) = \frac{1}{4\pi||x-a_0||} + V_{reg}(x)$, where $V_{reg}(x)$ is a harmonic function which is also defined at $a$. We will show now that in such a case the hyperKähler structure can be extended over the singularity. From now on we use $2\pi V$ instead of $V$ as our potential to make computations easier. We start by constructing holomorphic functions with respect to the complex structure $I_1$.

**Lemma 27.** *Using polar coordinates on $\mathbb{R}^3$ around a singularity $a_0$ of the harmonic function $V$, fix a trivialisation of $M$ on a simply-connected set $U_0 \subseteq \mathbb{R}^3$, such that $a_0$ gets mapped to the origin. Then there exists a function $h \in C^\infty(U_0)$ which is invariant under rotation in the $\varphi$-direction in the spherical coordinates defined in equations (167-169), such that the connection form can locally be written as*

$$\Theta = d\chi + h \cdot d\varphi \in \Omega^1(M, \mathbb{R})^{S^1}, \tag{176}$$

*where $\chi$ is the circle coordinate and as before and $h \cdot d\varphi$ is pulled back to $M$.*

*Proof.* Using polar coordinates on $\mathbb{R}^3$ around a singularity $a$ of the harmonic function $V$, fix a trivialisation of $M$ on a simply-connected set $U \subseteq \mathbb{R}^3$, such that $a$ gets mapped to the origin. Since the harmonic function $V$ is invariant , we have

$$dV(\partial_\varphi) = 0 \tag{177}$$

Thus

$$F = -2\pi * dV = (f(x)dr + g(x)d\theta) \wedge d\varphi \text{ for some } f, g \in C^\infty(U). \tag{178}$$

In particular, it contains no $dr \wedge d\theta$-term, and furthermore the functions $f$ and $g$ do not depend on $\varphi$. Since $dF = 0$, we compute

$$0 = dF = d(f(x)dr + g(x)d\theta) \wedge d\varphi + (f(x)dr + g(x)d\theta) \wedge \underbrace{d^2\varphi}_{=0} \tag{179}$$

$$= d(f(x)dr + g(x)d\theta) \wedge d\varphi \tag{180}$$

and since the term $f(x)dr + g(x)d\theta$ does not depend on $\varphi$, it is closed, and because $U_0$ is simply connected, exact on $U_0$. Let $h(x)$ be a function with

$$dh = f(x)dr + g(x)d\theta, \tag{181}$$

then the connection form can be written as

$$\Theta = d\chi + h \cdot d\varphi \in \Omega^1(M, \mathbb{R})^{S^1} \tag{182}$$

□

Again, we fix a trivialisation around $a_i$ without the negative $x_2$ line, i.e. $-\pi < \varphi < \pi$ such that $a_i$ gets mapped to the origin.

We would like to construct holomorphic coordinates on $M$, which are compatible with the $S^1$-action, or in other worlds, a holomorphic coordinate function $\psi : M_{|U_0} \to \mathbb{C}$ of the form

$$\psi((x_1, x_2, x_3, \chi = e^{it})) = f((x_1, x_2, x_3)) \cdot e^{-it}. \tag{183}$$

Recall that in our trivialisation, $\Theta = d\chi + h \cdot d\varphi$, in particular $\Theta(\partial_1) = 0$, so $\partial_1 = \widetilde{\partial_1}$ is parallel. Thus the complex structure $I_1$ restricted to $(\partial_\chi, \partial_1)$ is given by (compare to (157))

$$\begin{pmatrix} 0 & -2\pi V \\ \frac{1}{2\pi}V^{-1} & 0 \end{pmatrix}. \tag{184}$$

Choosing polar coordinates on $\mathbb{C}$, the standard complex structure $I_{\mathbb{C}}$ is given on coordinate vector fields $(\partial_\varphi, \partial_r)$ by

$$\begin{pmatrix} 0 & r^{-1} \\ -r & 0 \end{pmatrix} \tag{185}$$

The differential $d\psi$ maps the span of $(\partial_\chi, \partial_1)$ to the span of $(\partial_\phi, \partial_r)$ , and restricted to those vectors $d\psi$ has the form

$$\begin{pmatrix} -1 & 0 \\ 0 & \frac{df}{dx_1} \end{pmatrix} \tag{186}$$

Thus $\psi$ can only be holomorphic if $d\psi$ commutes with the complex structures, i.e. $I_\mathbb{C} \cdot d\psi = d\psi \cdot I_1$. Plugging in the matrices above, this is exactly the case when

$$\frac{df}{dx_1} = f \cdot 2\pi V, \tag{187}$$

which is solved by:

$$f(x_1, x_2, x_3) = e^W, W = 2\pi \int_0^{x_1} V(q, x_2, x_3) dq \tag{188}$$

We now transform back to cartesian coordinates, such that

$$\Theta = d\chi + a_2 dx_2 + a_3 dx_3. \tag{189}$$

The relation of the curvature of $\Theta$ and $V$ gives

$$\frac{da_2}{dx_1} = -\frac{dV}{dx_3} \tag{190}$$

$$\frac{da_3}{dx_1} = \frac{dV}{x_2}, \tag{191}$$

therefore the horizontal lifts are given by $\tilde{\partial}_2 = \partial_2 - a_2\partial_\chi$ and $\tilde{\partial}_3 = \partial_3 - a_3\partial_\chi$. Since $I_1\tilde{\partial}_2 = \tilde{\partial}_3$, $\psi$ is holomorphic with respect to $I_1$ when $\frac{d\psi}{dx_2} + i\frac{d\psi}{dx_3} = (a_2 + ia_3)\frac{d\psi}{d\chi}$. Applying this to the function constructed above, we get

$$\frac{d\psi}{dx_2} + i\frac{d\psi}{dx_3} = e^W e^{-it}\left(\frac{dW}{dx_2} + i\frac{dW}{dx_3}\right) = \psi(a_3 - ia_2) \tag{192}$$

$$= -i\psi(a_2 + ia_3) = \frac{d\psi}{d\chi}(a_2 + ia_3) \tag{193}$$

So $\psi_1 = e^W e^{-it}$ is holomorphic with respect to $I_1$. Similarly, $\psi_2 = e^{-W} e^{it}$ is holomorphic aswell.

**Lemma 28.** *Following the construction above, the functions*

$$\psi_1 = s(z)e^{W}e^{-it}, \psi_2 = s(z)e^{-W}e^{it}$$

*are holomorphic with respect to* $I_1$*, where* $s$ *is a holomorphic function depending on* $z = x_2 + ix_3$*.*

*Proof.* Recall that z=$x_2 + ix_3$ is holomorphic with respect to the complex structure $I_1$, so the claim follows from the fact that compositions and products of holomorphic functions are holomorphic. □

We can now prove:

**Lemma 29.** *Let* $U_0$ *be a small punctured ball around* $0$ *in* $\mathbb{R}^3$*, and* $\widetilde{M}$ *the hyperKähler manifold over* $U_0$ *obtained by taking a potential of the from* $V = \frac{1}{4\pi r} + V_{reg}$*, where* $V_{reg}$ *is a harmonic function which can be extended over* $x = 0$*. Then* $\widetilde{M}$ *can be extended to a hyperKähler manifold* $M$ *by adding the single point at the origin.*

*Proof.* First we notice that the curvature splits as $F + F_{reg}$, where $F$ is the curvature associated to the $\frac{1}{4\pi r}$ term. We have

$$\int_{S^2} \frac{F}{2\pi} = 1 + \int_{S^2} \frac{F_{reg}}{2\pi} = 1 + 0 \tag{194}$$

where we integrate around a sphere centered at zero with a radius small enough to be contained in $U_0$. The equality above follows from the fact that $F_{reg}$ extends over the point $x = 0$, thus is a closed form on a simply-connected domain, and then applying Stokes theorem.
Therefore, $\widetilde{M}$ is the (to $\mathbb{R}^3\backslash\{0\}$ extended) Hopf bundle, and we can choose the trivialisation

$$\Phi : (x, e^{it}) \mapsto (z_1, z_2) = \left((|x| - x_1)^{1/2}e^{i\varphi/2}e^{it}, (|x| + x_1)^{1/2}e^{i\varphi/2}e^{-it}\right) \tag{195}$$

Now, applying the previous lemma with $s(z) = \sqrt{z}$, we calculate:

$$W(x) = 2\pi\int_0^{x_1} V(q, x_2, x_3)dq = \int_0^{x_1} \frac{1}{2}\frac{1}{\sqrt{|z|^2 + q^2}} + 2\pi V_{reg}(q, x_2, x_3)dq = \tag{196}$$

$$\frac{1}{2}(\log(x_1 + \sqrt{|z|^2 + x_1^2}) - \log(|z|)) + 2\pi\int_0^{x_1} V_{reg}(q, x_2, x_3)dq \tag{197}$$

so we obtain holomorphic maps

$$\tilde{z}_1 := s(z)e^{-W}e^{it} = \sqrt{|z|}e^{i\varphi/2} \cdot \sqrt{|z|}\frac{1}{\sqrt{|x| + x_1}}e^{-W_{reg}}e^{it} = e^{-W_{reg}} \cdot z_1 \tag{198}$$

$$\tilde{z}_2 := s(z)e^{W}e^{-it} = \sqrt{|z|}e^{i\varphi/2} \cdot \frac{1}{\sqrt{|z|}}\sqrt{|x| + x_1}e^{W_{reg}}e^{-it} = e^{W_{reg}} \cdot z_2 \tag{199}$$

where we used the notation $W_{reg} = 2\pi \int_0^{x_1} V_{reg}(q, x_2, x_3)dq$. Or in other words, we have holomorphic coordinates

$$\psi(z_1, z_2) = (\tilde{z}_1, \tilde{z}_2) = (z_1, z_2) + ((e^{-W_{reg}} - 1)z_1, (e^{W_{reg}} - 1)z_2) \tag{200}$$

which are the usual Hopf coordinates "perturbed" by a factor vanishing at $x = 0$. One then calculates that the metric and the Kähler forms are of the form $g_{\mathbb{H}} + \delta g$ and $\omega_i + \delta\omega_i$, where $g_{\mathbb{H}}$ and $\omega_i$ are the standard metric and Kähler forms on $\mathbb{H}$, and $\delta g$ and $\delta\omega_i$ vanish in $x = 0$ Thus the Kähler structure can be extended over the singularity. This involves very technical and lengthy calculations, so we refer to ([27]). □

Notice that the coordinates $(\tilde{z}_1, \tilde{z}_2)$ defined above are equivariant with respect to the standard left multiplication of $S^1$ on $\mathbb{H} \simeq \mathbb{C} \oplus j \cdot \mathbb{C}$ and the principal $S^1$-action on $M$. Furthermore, since in the standard coordinates the moment map is just the Hopf map, we obtain

$$\mu_{\mathbb{C}}(z_1, z_2) = z_1 \cdot z_2 = \tilde{z}_1 \cdot \tilde{z}_2 \tag{201}$$

$$\mu_I(z_1, z_2) = a_0 + \frac{|z_2|^2 - |z_1|^2}{2} = a_0 + \frac{|e^{-W_{reg}}\tilde{z}_2|^2 - |e^{W_{reg}}\tilde{z}_1|^2}{2} \tag{202}$$

How does the change of coordinates look like, when we define these coordinates around different centers?
By abuse of notation, we denote the singularities by $a_k = a_k \cdot (1, 0, 0) = a_k \cdot e_1$ (so $a_k$ is a point in $\mathbb{R}^3$ defined by the multiple $a_k \in \mathbb{R}$ of the standard vector $e_1$!).

Introducing

$$V'_{reg} = V - \frac{1}{4\pi||x - a_{k+1} \cdot e_1||} - \frac{1}{4\pi||x - a_k \cdot e_1||} \text{ and } W'_k = 2\pi \int_{a_{k+1}}^{a_k} V'_{reg} \tag{203}$$

we have:

**Lemma 30.**

$$\psi_{k+1}^{-1} \circ \psi_k : (e^{-W_{reg,k}} \cdot z_1, e^{W_{reg,k}} \cdot z_2) \mapsto (e^{-W_{reg,k}-W'_k} \cdot z_1^2 \cdot z_2, e^{W_{reg,k}+W'_k} \cdot \frac{1}{z_1}) \quad (204)$$

*Proof.* See Appendix B. □

This is precisely the standard change of charts of the bundle $T^*\mathbb{C}P^1$ (up to the perturbing factor $e^{u'_k}$)! This is not suprising as the Gibbons-Hawking space with two singularities is isomorphic to $T^*\mathbb{C}P^1$.

Let us look at some examples next.

**Example 6.** (Multi-Eguchi-Hanson Spaces) These are the hyperKähler manifolds obtained by taking potentials of the form

$$V(x) = \sum_{i=1}^{n} \frac{1}{4\pi||x - a_i||} \quad (205)$$

The case n=1 yields the model hyperKähler manifold $\mathbb{H}$. The unperturbed generalized Seiberg-Witten equations for these spaces were studied in [28] and [4].

**Example 7.** (Multi-Taub-NUT-Spaces) These are the hyperKähler manifold obtained by taking potentials of the form

$$V(x) = 1 + \sum_{i=1}^{n} \frac{1}{4\pi||x - a_i||} \quad (206)$$

Also allowing infinite sums, one has to be a bit careful with convergence issues, so we do not allow the sequence $a_i$ to have an accumulation point. If $a_i$ is unbounded, then $V(x)$ converges at a point $x \in \mathbb{R}^3\backslash\{a_i\}_{i\in\mathbb{Z}}$ if and only if the series $\sum_{i\in\mathbb{Z}} \frac{1}{||a_i||}$ converges. Therefore, choosing the $a_i$ to be equidistant is also problematic.

**Example 8.** (Ooguri-Vafa Space) To obtain a harmonic function with singularities at equidistant points, we try to renormalize the divergent potential

$$V(x) = \sum_{n\in\mathbb{Z}} \frac{1}{4\pi||x - n\epsilon \cdot e_1||} \quad (207)$$

where $e_1 = (1, 0, 0)^T$.

**Lemma 31.** *Let*

$$V_j = \frac{1}{4\pi} \sum_{n=-j}^{j} \left( \frac{1}{||x - n\epsilon \cdot e_1||} - a_{|n|} \right) \quad (208)$$

*where*

$$a_n = \begin{cases} 1/n\epsilon & n \neq 0 \\ 2(-\gamma + \log(2\epsilon))/\epsilon & n = 0 \end{cases} \tag{209}$$

*and* $\gamma$ *is Euler's constant. Denote the open unit disc in* $\mathbb{R}^2$ *as* $D$*. Then:*

- *The sequence* $\{V_j\}$ *converges uniformly on compact sets in* $\mathbb{R} \times D \backslash (\epsilon\mathbb{Z} \times \{0\})$ *to a harmonic function* $V_0$*.*
- *For* $z := (x_2, x_3)$*,* $V_0$ *has an expression valid when* $z \neq 0$*:*

$$V_0 = -\frac{1}{4\pi\epsilon}\log(|z|^2) + \sum_{m \in \mathbb{Z}\backslash\{0\}} \frac{1}{2\pi\epsilon}\cos(2\pi m x_1/\epsilon) \cdot K_0(2\pi|mz|/\epsilon) \tag{210}$$

*where* $K_0$ *is the modified Bessel function.*

*Proof.* See [9]. □

Although the explicit form of the potential $V_0$ looks quite scary, we have good news: For the following calculations it is sufficient to know that in a small neighborhood of a singularity $a_0$, the potential $V_0$ is of the form $V_0 = \frac{1}{4\pi\|x-a_0\|} + V_{reg}$, where $V_{reg}$ is a harmonic function which can be extended over the singularity. Also note that for convergence reasons we have to impose an upper bound on the $z$-coordinate.

## 3.1 The Permuting $S^1$-Action

From now on, we assume that $M$ is one of the hyperKähler manifolds defined in the previous examples. We need a permuting $S^1$ action on $M$ fixing the complex structure $I_1$ and rotating $I_2$ and $I_3$. The first idea that comes to mind would be to lift the action $S^1 \curvearrowright \mathbb{R}^3 \simeq \mathbb{R} \times \mathbb{C}, e^{it}.(x_1, z) = (x_1, e^{-i2t}z)$, $z = x_2 + ix_3$, to the principal bundle $\pi : M \to U \subseteq \mathbb{R}^3$, i.e. $e^{it}.p = P_{\gamma(t)}(p)$, where

$$\gamma(t) : [0, 2\pi] \to \mathbb{R}^3, \gamma(t) = (x_1, e^{-i2t}z) \tag{211}$$

and $P_{\gamma(t)}$ is the parallel transport from $P_{\gamma(0)}$ to $P_{\gamma(t)}$ along the path $\gamma(t)$ defined by the connection $\Theta$. But this is not a well-defined group action, since $\Theta$ has non-zero curvature, so $P_{\gamma(2\pi)}$ is not necessarily the identity map. Thus, we need a "correction" term: Recall the function $h$ defined in equation (181).

**Lemma 32.** *For each* $l \in \mathbb{Z}$ *define locally around a singularity the action* $S^1 \curvearrowright M\backslash\{z = 0\}, e^{it}.p = \rho(e^{i(lt - th(\pi(p)))}, P_{\gamma(t)}(p))$*, where* $l \in \mathbb{Z}$*,* $\rho$ *is the principal* $S^1$*-action* *and*

$$h(x) = \frac{1}{2\pi||z||}\int_{S^1_x}\Theta = \frac{1}{2\pi||z||}\int_{D^2_x} F \tag{212}$$

*where* $S^1_x$ *is the circle* $\{(x_1, e^{it}z); e^{it} \in S^1\} \subseteq \mathbb{R}^3$ , $x = (x_1, z)$ *and* $\pi : M \to \mathbb{R}^3$ *denotes the projection. This Lie group action fixes the complex structure* $I_1$ *and rotates* $I_2$ *and* $I_3$.

*Proof.* Using the function $h$ defined in Lemma (27), we compute, that indeed

$$\frac{1}{2\pi r}\int_{S^1_x}\Theta = \frac{1}{2\pi r}\int_{S^1_x} h(x)\cdot d\varphi = \frac{1}{2\pi r}2\pi r \cdot h(x) = h(x) \tag{213}$$

Using that parallel transport commutes with the principal action and satisfies $P_{\gamma(t)} \circ P_{\gamma(s)} = P_{\gamma(t+s)}$ , and that $h$ is constant along the circle $S^1_x$, we get

$$e^{it}.(e^{is}.p) = \rho(e^{i(l(t+s)-(t+s)2h(\pi(p)))}, P_{\gamma(t+s)}(p)) = e^{i(t+s)}.p$$

In the choosen trivialisation, the path $\gamma(t) = (r, \theta, \varphi - 2t)$ is lifted horizontally to the path $\gamma^*(t) = (\chi + 2ht, r, \theta, \varphi - 2t)$, because $\Theta(\dot{\gamma^*}) = \Theta(2h \cdot \partial_\chi - 2\partial_\phi) = 2h - 2h = 0$. In these coordinates the action is therefore just given by $e^{it}.(\chi, r, \theta, \varphi) = \rho(e^{i(lt-2ht)}, (\chi + 2ht, r, \theta, \varphi - 2t)) = (\chi + lt, r, \theta, \varphi - 2t)$, which also shows that the action is smooth. In particular we see that $e^{2\pi i}.p = p$, so the action is well-defined. To see that it fixes the complex structure $I_1$ and rotates $I_2$ and $I_3$, we recall that $\omega_i = \tilde{\Theta} \wedge dx_i + V dx_{i+1} \wedge dx_{i+2}$. For a fixed $e^{it} \in S^1$, the diffeomorphism $q(t) : p \mapsto e^{it}.p$ leaves $\Theta$ and $dx_1$ invariant, while rotating $dx_2$ and $dx_3$. To be more explicit,

$$q(t)^*\tilde{\Theta} = \tilde{\Theta} \tag{214}$$

$$q(t)^* dx_1 = dx_1 \tag{215}$$

$$q(t)^* dx_2 = \cos(2t)dx_2 + \sin(2t)dx_3 \tag{216}$$

$$q(t)^* dx_3 = \cos(2t)dx_3 - \sin(2t)dx_2, \tag{217}$$

where the first statement follows from the fact that locally, $\Theta = d\chi + h d\varphi$ and $h$ is independent of $\chi$ and $\varphi$, and the fundamental vectorfield for this action is given by $K^M_i = k \cdot \partial_\chi - 2\partial_\varphi$. Recalling the definition of the symplectic forms in equation

(146), plugging in the above yields:

$$q(t)^*\omega_1 = \omega_1 \tag{218}$$

$$q(t)^*\omega_2 = \cos(2t)\omega_2 + \sin(2t)\omega_3 \tag{219}$$

$$q(t)^*\omega_3 = \cos(2t)\omega_3 - \sin(2t)\omega_2 \tag{220}$$

which is precisely the condition for the $S^1$ action to be permuting. □

To extend the action over the points at $z = 0$, we apply the Riemann extension theorem:

**Theorem 17.** *(Riemann extension theorem) Let* $M$ *be a complex manifold, and* $A \subseteq M$ *the zero set of a holomorphic function* $g : M \to \mathbb{C}$.

*Let* $f : M\backslash A \to \mathbb{C}$ *be a bounded holomorphic function. Then* $f$ *has a unique holomorphic extension* $\widetilde{f} : M \to \mathbb{C}$.

**Lemma 33.** *The permuting action of Lemma (32) extends locally to a permuting group action on the subset* $z = 0$.

*Proof.* For a fixed $q \in S^1$, consider the map

$$q : M\backslash\{z = 0\} \to M\backslash\{z = 0\}, x \mapsto q.x \tag{221}$$

Since $z$ is a holomorphic map, we can apply the Riemann extension theorem to construct a unique holomorphic map $\tilde{q} : M \to M$. To be more precise, using a local chart $(\psi, U)$, the map $\psi \circ q : M\backslash\{z = 0\} \to \mathbb{C}^2$ can be extended componentwise to a map $\widetilde{\psi \circ q}$ and we define $\tilde{q} = \psi^{-1} \circ \widetilde{\psi \circ q}$. The uniqueness of the extension shows that this is indeed independent of the chart, hence glues to a global map.
For two elements $e^{it}, e^{is} \in S^1$ the maps $\widetilde{e^{it} \circ e^{is}}$ and $\widetilde{e^{i(t+s)}}$ coincide on the set $M\backslash\{z = 0\}$, so by the uniqueness of the extension they coincide on the whole of $M$. Similarly $\tilde{1} = \widetilde{e^{2\pi i}} = id_M$, so this is a well-defined group action. To show smoothness, we use the holomorphic charts constructed in Lemma (28):

$$\begin{aligned}
&e^{is}.(e^{-W_{reg}} \cdot z_1, e^{W_{reg}} \cdot z_2) = \\
&\left(e^{-W_{reg}}(|x| - x_1)^{1/2} e^{i(\varphi-2s)/2} e^{i(t+l\cdot s)}, e^{W_{reg}}(|x| + x_1)^{1/2} e^{i(\varphi-2s)/2} e^{-i(t+l\cdot s)}\right) = \\
&\left(e^{-W_{reg}}(|x| - x_1)^{1/2} e^{i\varphi/2} e^{i(t+(l-1)\cdot s)}, e^{W_{reg}}(|x| + x_1)^{1/2} e^{i\varphi/2} e^{-i(t+(l+1)\cdot s)}\right) = \\
&(e^{-W_{reg}} \cdot z_1 \cdot e^{i(l-1)s}, e^{W_{reg}} \cdot z_2 \cdot e^{-i(l+1)s})
\end{aligned}$$

which is clearly smooth.
The holomorphic extension on $\{z = 0\}$ is given by

$$e^{is}.x = \begin{cases} (e^{i(l-1)s} \cdot z_1, 0) & x_1 \leq 0 \\ (0, e^{-i(l+1)s} \cdot z_2) & x_1 \geq 0 \end{cases} \tag{222}$$

□

Notice that we obtain the usual permuting action on $\mathbb{H}$ when $V = \frac{1}{4\pi r}$, by choosing $l = 1$.

**Lemma 34.** *The locally defined permuting actions above patch together to a permuting action on the whole of* $M$.

*Proof.* In the coordinates around a singularity $a_i$ the permuting action is given by

$$e^{is}.(e^{-W_{reg}} \cdot z_1, e^{W_{reg}} \cdot z_2) = (e^{-W_{reg}} \cdot z_1 \cdot e^{i(l-1)s}, e^{W_{reg}} \cdot z_2 \cdot e^{-i(l+1)s}) \tag{223}$$

while around the next singularity $a_{i+1}$, it is given by

$$e^{is}.(e^{-W_{reg}} \cdot z_1, e^{W_{reg}} \cdot z_2) = (e^{-W_{reg}} \cdot z_1 \cdot e^{i(l'-1)s}, e^{W_{reg}} \cdot z_2 \cdot e^{-i(l'+1)s}) \tag{224}$$

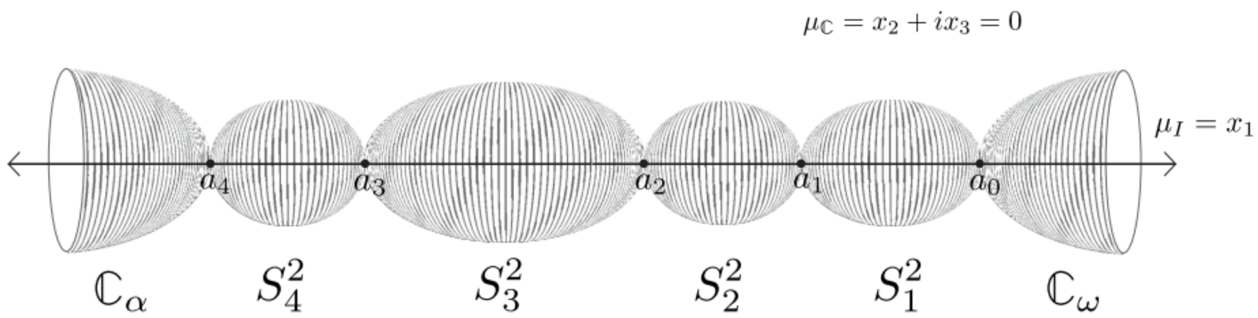
Changing charts to coordinates around the next singularity $a_{i+1}$ transforms the angular coordinate as $e^{it} \to e^{-it+\varphi}$(it is in fact the change of the trivialisation of the Hopf fibration $S^3 \to S^2$ from the north to south pole), and interchanges the $z_1$ and $z_2$ coordinate, so the locally defined actions agree if and only if $l + 1 = l' - 1$. In other words, $l'$ must differ from $l$ by 2. So, choosing a singularity $a_0$ and defining the permuting action there with an integer $k$ gives rise to the whole of $M$, where for every singularity to the right we "twist" by letting $l$ go to $l + 2$ and for every singularity to the left letting $l$ go to $l - 2$. □

## 3.2 The Niveau Set $\mu_{\mathbb{C}}^{-1}(\{0\})$

For a complete hyperKähler manifold obtained by the Gibbons-Hawking-Ansatz with a potential of the form $V = c + \sum_{i \in \mathbb{Z}} \frac{1}{4\pi \|x - a_i\|}$, recall that the moment map of the hyperKähler $S^1$-action is given by $\mu_I = x_1$ and $\mu_{\mathbb{C}} = x_2 + ix_3$. Over the points with $x_2 = x_3 = 0$, $M$ has the following structure:

**Lemma 35.**
- *For two adjacent singularities* $a_{k+1}$, $a_k$, *the set* $\mu_I^{-1}([a_{k+1}, a_k])$ *in* $\mu_{\mathbb{C}}^{-1}(\{0\})$ *is a complex submanifold of* $M$ *with respect to the complex structure* $I_1$, *which is biholomorpic to* $S^2$. *We denote this submanifold by* $S^2_{(k+1)}$.

- *If* $a_N$ *is the "leftmost" singularity, i.e. there is no* $a_i$ *with* $a_i < a_N$*, then the set* $\mu_I^{-1}((-\infty, a_N])$ *in* $\mu_{\mathbb{C}}^{-1}(\{0\})$ *is a complex submanifold of* $M$ *with respect to the complex structure* $I_1$*, which is biholomorpic to* $\mathbb{C}$*. We denote this submanifold with* $\mathbb{C}_\alpha$*.*
- *If* $a_0$ *is the "rightmost" singularity, i.e. there is no* $a_i$ *with* $a_i > a_0$*, then the set* $\mu_I^{-1}([a_0, \infty))$ *in* $\mu_{\mathbb{C}}^{-1}(\{0\})$ *is a complex submanifold of* $M$ *with respect to the complex structure* $I_1$*, which is biholomorpic to* $\mathbb{C}$*. We denote this submanifold with* $\mathbb{C}_\omega$*.*




*Proof.* To see that these are complex submanifolds, recall that $z = x_2 + ix_3$ is a holomorphic coordinate for the complex structure $I_1$ and the above sets are just given by $z = 0$. The only singular values of $z$ are the singularities, so we are left with finding suitable holomorphic charts around the singularities.
From Lemma (28) we have holomorphic coordinates away from the singularities of the form

$$(e^{-W_{reg}} \cdot z_1, e^{W_{reg}} \cdot z_2) = \left(e^{-W_{reg}}(|x| - x_1)^{1/2} e^{i\varphi/2} e^{it}, e^{W_{reg}}(|x| + x_1)^{1/2} e^{i\varphi/2} e^{-it}\right) \tag{225}$$

And the niveau set $z = 0$ consists of precisely the points where either $z_1 = 0$ (the points "right" of the singularity) or $z_2 = 0$ (the points "left" of the singularity).
In particular around $a_N$, the "leftmost" singularity (if it exists), the biholomorphism from $\mathbb{C}_\alpha$ to $\mathbb{C}$ is explicitly given by

$$\psi_\alpha : ((x_1, 0, 0), e^{it}) \mapsto e^{-W_{reg}} \cdot z_1 = e^{-W_{reg}} \cdot \sqrt{2(a_N - x_1)} e^{it} \tag{226}$$

For the "rightmost" singularity $a_0$, we have the map from $\mathbb{C}_\omega$ to $\mathbb{C}$ given by

$$\psi_\omega : ((x_1, 0, 0), e^{it}) \mapsto e^{W_{reg}} \cdot z_2 = e^{W_{reg}} \cdot \sqrt{2(x_1 - a_0)} e^{-it} \tag{227}$$

For singularities $a_k$ and $a_{k+1}$, and the two-sphere $S^2_{(k+1)}$ between them, we have the map

$$S^2_{(k+1)} \backslash \{a_{k+1}\} \to \mathbb{C}, \psi^N_{(k+1)} : ((x_1, 0, 0), e^{it}) \mapsto e^{W'_{reg}} \cdot \sqrt{2 \frac{x_1 - a_k}{a_{k+1} - x_1}} e^{-it} \tag{228}$$

corresponding to the trivialisation of $\mathbb{CP}^1 \backslash \{[1, 0]\} \to \mathbb{C}, [z_1, 1] \mapsto z_1$ where we use

$$\begin{aligned} W_{reg} &= \int_{a_k}^{x_1} V_{reg}(q, 0, 0) dq = \int_{a_k}^{x_1} \frac{1}{2(a_{k+1} - q)} + V'_{reg}(q, 0, 0) dq \\ &= \log((a_{k+1} - x_1)^{-1/2}) + \log((a_{k+1} - a_k)^{1/2}) + \int_{a_k}^{x_1} V'_{reg}(q, 0, 0) dq \\ &= \log((a_{k+1} - x_1)^{-1/2}) + W'_{reg} \end{aligned}$$

Similarly we get the map

$$\psi^S_{(k+1)} : S^2_{(k+1)} \backslash \{a_k\} \to \mathbb{C}, ((x_1, 0, 0), e^{it}) \mapsto e^{-W'_{reg}} \cdot \sqrt{2 \frac{a_{k+1} - x_1}{x_1 - a_k}} e^{it} \tag{229}$$

which corresponds to the trivialisation $\mathbb{CP}^1 \backslash \{[0, 1]\} \to \mathbb{C}, [1, z_2] \mapsto z_2$.
We obtain the induced biholomorphism

$$\psi_{(k+1)} : S^2_{(k+1)} \to \mathbb{CP}^1, p \mapsto [\psi^N_{(k+1)}(p), \psi^S_{(k+1)}(p)] \tag{230}$$

□

We will refer to these submanifolds as *generalized spheres* in the future.

Notice that with these explicit biholomorphisms we can directly read of the induces group actions: We define the permuting action by fixing a singularity and defining the local permuting action around it with the choice $l = 1$. If there is a "rightmost" singularity we set it to be $a_0$, otherwise, if there are infinitely many spheres we fix an arbitrary $a_n$.

**Remark 2.** *Notice that all possible choices to define the permuting action differ by a "twist" by an integer $l \in \mathbb{Z}$. We will see later later that this is equivalent to "twisting" the $U^{\mathbb{C}}(2)$-structure by $k$ copies of the line bundle $K_X$, when we glue $M$ into a $U^{\mathbb{C}}(2)$-bundle over a Kähler manifold $X$.*

**Corollary 9.** *Denote the* $S^1 \times S^1$ *group action by* $p \mapsto (e^{it}, e^{is}).p$*, where the first factor is the hyperKähler group action given by the principal action and the second factor is the permuting action, and assume there is a finite number of singularities* $n \in \mathbb{N}$ *given by* $a_0, ..., a_{n-1}$*. Then we have:*

- *For* $p \in \mathbb{C}_\alpha$*,* $\psi_\alpha((e^{it}, e^{is}).p) = e^{-it} e^{i(2n)s} \psi_\alpha(p)$*.*
- *For* $p \in \mathbb{C}_\omega$*,* $\psi_\omega((e^{it}, e^{is}).p) = e^{it} \psi_\omega(p)$*.*
- *For* $p \in S^2_{(k+1)} \backslash \{a_k\}, \psi^S_{(k+1)}((e^{it}, e^{is}).p) = e^{it} e^{-i2(k+1)s} \psi^S_{(k+1)}(p)$*.*

*In particular, we have for some* $U^{\mathbb{C}}(2)$*-structure* $Q = L \bullet Q_{can}$ *over a Kähler four manifold* $X$*:*

- $Q \times_{U^{\mathbb{C}}(2)} \mathbb{C}^\alpha = K_X^{\otimes n} \otimes L^{-1}$.
- $Q \times_{U^{\mathbb{C}}(2)} \mathbb{C}^\omega = L$
- $Q \times_{U^{\mathbb{C}}(2)} S^2_{(k+1)} = \mathbb{P}(K_X^{\otimes(k+1)} \otimes L^{-1} \oplus \underline{\mathbb{C}})$ , $k = 0, ..., n-2$

## 3.3 Multi-Eguichi-Hanson Spaces as HyperKähler Quotients

Here, we give an alternative construction of Multi-Eguichi-Hanson spaces as hyper-Kähler quotients due to Gibbons and Rychenkova([8]).

Let $\mathbb{H}^n$ be equipped with three complex structures defined by the standard left multiplication on the quaternions. Together with the flat metric, this turns $\mathbb{H}^n$ into a hyperKähler manifold. $Sp(1)$ acts permuting on $\mathbb{H}^n$ via left multiplication, and the action of $Sp(n)$ via right multiplication commutes with the complex structures, is therefore hyperKähler. This induces hyperKähler actions of the subgroups

$$SU(n) \subset U(n) \subset Sp(n) \; . \tag{231}$$

Let $\mathbb{T}^{n-1} \subset SU(n)$ be the maximal torus given by the embedding

$$\bigoplus_{s=1}^{n-1} e^{i\theta_s} \mapsto \left( e^{i\theta_1}, e^{i(\theta_2 - \theta_1)}, ..., e^{i(\theta_{n-1} - \theta_{n-2})}, e^{-i\theta_{n-1}} \right). \tag{232}$$

The moment map

$$\mu^{\mathbb{T}^{n-1}} : \mathbb{H}^n \to \operatorname{Im} \mathbb{H} \otimes \mathfrak{t}^{n-1} \tag{233}$$

of the induced action given by restricting the $\mathsf{Sp}(n)$ action is given by

$$\mu^{\mathbb{T}^{n-1}}(h) = \Big(\sigma(h_2) - \sigma(h_1), \dots, \sigma(h_n) - \sigma(h_{n-1})\Big), \; h = (h_1, \dots, h_n) \in \mathbb{H}^n, \tag{234}$$

where $\sigma$ is the moment map of the standard $S^1$-action on $\mathbb{H}$:

$$\sigma(h) = hi\overline{h}. \tag{235}$$

Writing $\operatorname{Im}\mathbb{H} = i \cdot \mathbb{R} + j \cdot \mathbb{C}$, we can choose a point

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n-1}) \in i \cdot \mathbb{R} \otimes \mathfrak{t}^{n-1} \subset \operatorname{Im}\mathbb{H} \otimes \mathfrak{t}^{n-1} \tag{236}$$

such that $(-i) \cdot \epsilon_s > 0$ for all $s$. The action of $\mathbb{T}^{n-1}$ on the level set

$$\left(\mu^{\mathbb{T}^{n-1}}\right)^{-1}(\epsilon) = \Big\{h \in \mathbb{H}^n \mid \sigma(h_{s+1}) - \sigma(h_s) = \epsilon_s, s = 1, \dots, n-1\Big\} \tag{237}$$

is free, so we can form the hyperKähler reduction

$$M_\epsilon = \left(\mu^{\mathbb{T}^{n-1}}\right)^{-1}(\epsilon) \Big/ \mathbb{T}^{n-1}. \tag{238}$$

**Theorem 18.** *The hyperKähler manifold* $M_\epsilon$ *is triholomorphic to the Multi-Eguichi-Hanson space with* $n$ *singularities, with distance between the* $s$*-singularity and the* $(s+1)$*-singularity given by* $n \cdot (-i) \cdot \epsilon_s$*.*

Triholomorphic here means isomorphic as hyperKähler manifolds, i.e. there exists a diffeomorphism between them preserving the metric and the complex structure.

We can make this more explicit:

The $S^1$ action on $\mathbb{H}^n$ via diagonal matrices induces a hyperKähler action on $M_\epsilon$ with $n$ fixed points given by

$$\left\{[h] \in M_\epsilon \mid \prod_{s=1}^{n} h_s = 0\right\} \tag{239}$$

where $[h]$ is the equivalence class of $h = (h_1, \dots, h_n) \in \left(\mu^{\mathbb{T}^{n-1}}\right)^{-1}(\epsilon)$. In other words, the singularities are given by the orbits of points where one of the entries is zero.

The moment map of this hyperKähler $S^1$-action is given by

$$\mu([h]) = \sum_{s=1}^{n} \sigma(h_s). \tag{240}$$

Furthermore, let $S^1 \subseteq \mathsf{Sp}(1)$ be the subgroup of the permuting $\mathsf{Sp}(1)$ preserving the complex structure $I_1$ on $\mathbb{H}^n$. This induces a permuting action on the quotient, since the singularities are invariant under this $S^1$-action.

# 4 Analysis of the Moduli Space

In this section, we solve the perturbed generalized Seiberg-Witten equations with a Multi-Eguichi-Hanson space (or short, MEH space) $M$ as the target manifold explicitly and analyze the resulting moduli space.

We assume that our source manifold is a Kähler manifold $X$ equipped with a $U^{\mathbb{C}}(2)$-structure given by $Q = L \bullet Q_{can}$ for some complex line bundle $L$, and that $L$ admits a holomorphic structure. As before, $u$ will be a section $u \in \Gamma(X, F)$, where $F = Q \times_{U^{\mathbb{C}}(2)} M$ is the associated fibre bundle using the permuting and hyperKähler actions on $M$, and $a$ denotes a connection on $L$.

## 4.1 The unperturbed Equations

The case when $\phi = 0$ was solved in [4]: Let $M$ be the MEH space with $n$ singularites $a_0, ..., a_{n-1}$ labeled from right to left. Let

$$\mathcal{M} = \left\{ \not{D}_A u = 0 \ , \ F_a^+ = \mu \circ u \right\} / \mathcal{G}$$

As in the classical Seiberg-Witten equations, the moduli spaces depends on the line bundle $L$, or more specifically on the number

$$\deg_{\omega_X}(L) := \int_M c_1(L) \wedge \omega_X \ . \tag{241}$$

**Theorem 19** ([4]). *A solution $(u, a)$ satisfies $\mu_{\mathbb{C}} \circ u = 0$, and the image of $u$ is contained in precisely one generalized sphere, depending on $\deg_{\omega_X}(L)$:*

- *If $\deg_\omega(L) > \frac{vol(M)}{8\pi} a_0$, then the image of $u$ is contained in the "right" copy of $\mathbb{C}$, and we have:*

$$\mathcal{M} \simeq \left\{ \textit{effective divisors } D, \textit{such that } c_1(\mathcal{O}(D)) = c_1(L) \right\} \tag{242}$$

- *If $\deg_\omega(L) < \frac{vol(M)}{8\pi} a_{n-1}$, then the image of $u$ is contained in the "left" copy of $\mathbb{C}$ and we have:*

$$\mathcal{M} \simeq \left\{ \textit{effective divisors } D, \textit{such that } c_1(\mathcal{O}(D)) = n \cdot c_1(K_X) - c_1(L) \right\} \tag{243}$$

- *If $\deg_\omega(L) \in \frac{vol(M)}{8\pi}(a_{k+1}, a_k)$ for some $0 < k < n-1$, then the image of $u$ is* *contained in the sphere* $S^2_{(k+1)}$ *and we have:*

$$\begin{aligned}\mathcal{M} \simeq \big\{ & \textit{divisors } D = D_0 - D_\infty, \\ & \textit{such that } D_0 \cap D_\infty = \emptyset \textit{ and } c_1(\mathcal{O}(D)) = c_1(L) - (k+1)c_1(K_X) \big\}\end{aligned}$$

In other words, the first two cases are similar to the classical Seiberg-Witten case, the moduli space given by the projectivization of the vector space of holomorphic sections of $L$ and $K_X^{\otimes n} \otimes L^{-1}$ respectively. The more interesting is the last case, where the space is given by certain meromorphic functions. Here the moduli space is usually non-compact in several ways. It might have infinitely many different components of increasing dimension, which themselves are not compact, thus it is hard to do the usual business where one obtains invariants by integrating over the moduli space.

One reason this problem occurs is because the Seiberg-Witten map defined in equation (103) is not surjective if we do not include the perturbation parameter $\phi$. Therefore, the equations do not cut out the moduli space transversally, and the actual dimension of the space is bigger then expected.

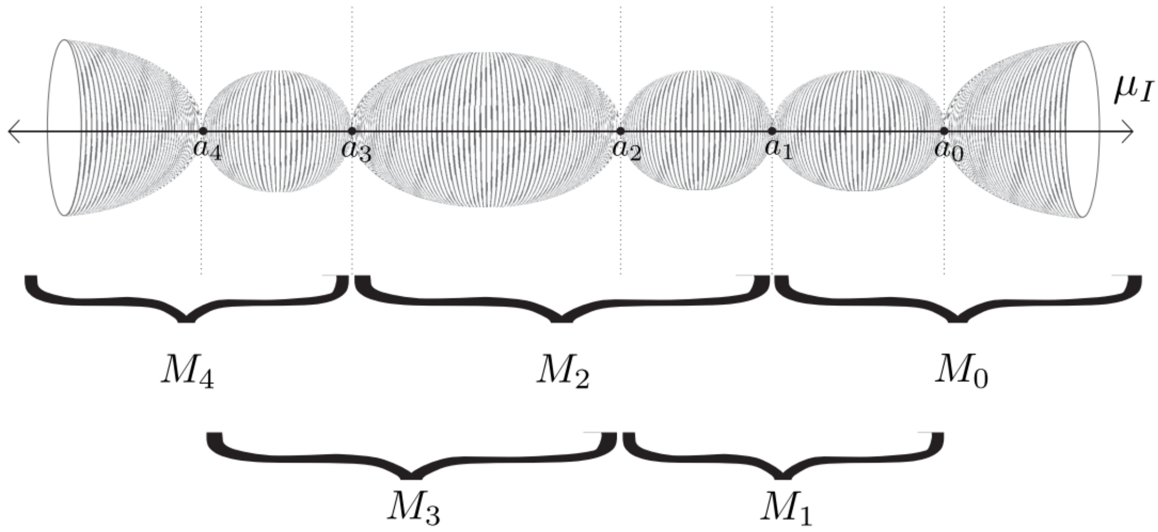
Since our eventual goal is to define invariants and need a compact moduli space for this, we will consider only perturbations $\phi \neq 0$, which allows us to use the regular value theorem, as we will see in the next chapters.

## 4.2 Solving the perturbed Equations

Let $(u, a, is \cdot \omega_X, \phi)$ be a solution to the generalized perturbed Seiberg-Witten equations with target hyperKähler manifold $M$, where $s \in \mathbb{R}$ and $\phi \neq 0$ is a holomorphic section of $K_X$. Then by theorem (15) the connection $a$ induces a holomorphic structure on $L$, and $u$ is a holomorphic map with respect to that holomorphic structure. We assume in this section that $M$ is a MEH space with $n$ singularities $a_0, ..., a_{n-1}$ labelled from right to left. We decompose $M$ into the following open sets:

$$\begin{aligned} M_0 &= \mu_I^{-1}((a_1, \infty)) \\ M_k &= \mu_I^{-1}((a_{k+1}, a_{k-1})) \text{ for } 0 < k < n-1 \\ M_{n-1} &= \mu_I^{-1}((-\infty, a_{n-1})) \end{aligned}$$

We may set $a_{-1} = \infty$, $a_n = \infty$ to make the notation consistent.




Each of these sets is biholomorphic to $\mathbb{C}^2 \overset{\psi_k}{\simeq} (M_k, I_1)$, and by Lemma (30), the coordinate changes are precisely the ones of $T^*\mathbb{C}P^1$, up to a perturbing factor:

$$\psi_{k+1}^{-1} \circ \psi_k : (z_1, z_2) \mapsto \left(e^{-W'_k} \cdot z_1^2 \cdot z_2 \ , \ e^{W'_k} \cdot \frac{1}{z_1}\right) \tag{244}$$

where

$$W'_k = 2\pi \int_{a_{k+1}}^{a_k} V'_{reg,k}(q, x_2, x_3) \, dq \ , \tag{245}$$

$$V'_{reg,k} = V - \left(\frac{1}{4\pi\|x - a_{k+1}e_1\|} + \frac{1}{4\pi\|x - a_k e_1\|}\right) , \tag{246}$$

$$x_2 + ix_3 = z_1 \cdot z_2 \tag{247}$$

Furthermore, the $S^1 \times S^1$ action $\rho$ on $M_k$ described in corollary (9) is precisely such that

$$Q \times_\rho M_k \simeq Q \times_{(\psi_k \circ \rho)} \mathbb{C}^2 \simeq K_X^{k+1} \otimes L^{-1} \oplus K_X^{-k} \otimes L \tag{248}$$

while the components of the moment map are given by equation (201), using the changed sign convention introduced in remark (1):

$$\mu_{\mathbb{C}}(z_1, z_2) = z_1 \cdot z_2 \tag{249}$$

$$\mu_I(z_1, z_2) = a_k + \frac{|e^{W_{reg,k}} z_2|^2 - |e^{-W_{reg,k}} z_1|^2}{2} \tag{250}$$

where

$$W_{\text{reg},k} = 2\pi \int_{a_{k+1}}^{x_1} V_{\text{reg},k}(q, x_2, x_3) dq \;, \tag{251}$$

$$V_{\text{reg},k} = V - \frac{1}{4\pi ||x - a_k e_1||} \;. \tag{252}$$

**Remark 3.** *Notice that the functions* $u_{\text{reg},k}$ *and* $u'_k$ *are defined on open subsets of the MEH space, but we will use them as functions on* $\mathbb{C}^2$ *by precomposing with a coordinate chart. So if we write* $u_{\text{reg},k}$ *we actually mean the function*

$$(z_1, z_2) \mapsto W_{\text{reg},k}\Big(\psi_k(z_1, z_2)\Big). \tag{253}$$

Given a generalized spinor $u : Q \to M$, let $Q_k^u \subseteq Q$ be defined by $Q_k^u = u^{-1}(M_k)$, and let $U_k^u := \pi(Q_k^u)$, where $\pi : Q \to X$ is the projection.

We now give an explicit description of the space of solutions.

**Lemma 36.** *The map* $u : Q \to M$ *determines a holomorphic section* $\alpha$ *of* $L$*, such that* $\frac{\phi^n}{\alpha}$ *is a holomorphic section of* $K_X^n \otimes L^{-1}$*, where* $K_X^n = \underbrace{K_X \otimes K_X \otimes \ldots \otimes K_X}_{n \text{ times}}$.

*Proof.* On $U_k^u$, the restriction $u_K := u_{|U_k^u}$ defines a local section of the bundle $K_X^{k+1} \otimes L^{-1} \oplus K_X^{-k} \otimes L$, i.e. $u_k = (\beta_k, \alpha_k)$ with $\beta_k \cdot \alpha_k = \phi$, so we may write $u_k = (\frac{\phi}{\alpha_k}, \alpha_k)$ on the set where $\alpha_k \neq 0$. Define a section $\alpha$ of $L$ locally by setting

$$\alpha_{|U_k^u} := e^{(-\sum_{l=0}^{k-1} W'_l)} \phi^k \cdot \alpha_k \tag{254}$$

Notice that $W'_l(p) = \int_{a_{l+1}}^{a_l} V'_{\text{reg},l}(q, \phi(p)) \; dq$ only depends on $\phi(p)$. This indeed glues to a global section, since for $\alpha_k \neq 0$ we have

$$\psi_{k+1}^{-1} \circ \psi_k(\frac{\phi}{\alpha_k}, \alpha_k) = (e^{-W'_k} \frac{\phi^2}{\alpha_k}, e^{W'_k} \frac{\alpha_k}{\phi}) = (\frac{\phi}{\alpha_{k+1}}, \alpha_{k+1}). \tag{255}$$

So we have on $U_{k+1}^u \cap U_k^u$:

$$e^{(-\sum_{l=0}^{k} W'_l)} \phi^{k+1} \cdot \alpha_{k+1} = e^{(-\sum_{l=0}^{k} W'_l)} \phi^{k+1} \cdot e^{W'_k} \frac{\alpha_k}{\phi} = e^{(-\sum_{l=0}^{k-1} W'_l)} \phi^k \cdot \alpha_k \tag{256}$$

establishing that $\alpha$ is a global section up to the set where $\alpha_k = 0$. Furthermore it is holomorphic, since the the map obtained by

$$\Big(\psi_0^{-1} \circ \psi_1\Big) \circ \ldots \circ \Big(\psi_{k-2}^{-1} \circ \psi_{k-1}\Big) \circ \Big(\psi_{k-1}^{-1} \circ \psi_k\Big)(\frac{\phi}{\alpha_k}, \alpha_k) \tag{257}$$

composed with the projection onto the second component yields $e^{(-\sum_{l=0}^{k-1} u_l')}\phi^k \cdot \alpha_k$, so we obtain $\alpha_{|U_k^u}$ as a composition of holomorphic maps[14]. By the Riemann extension theorem, this section can be extended over the set where $\alpha_k = 0$, by setting $\alpha = 0$. We can use the same argument (by using the change of coordinates into the "other" direction) to establish that $\frac{\phi^n}{\alpha}$ is a well-defined holomorphic section of $K_X^n \otimes L^{-1}$. $\square$

**Definition 36.** Let $f$ be a meromorphic section of some line bundle over $X$ given globally as a quotient $f = g/h$ for holomorphic sections $g, h$ which do not share a common non-invertible factor. Then, the *indeterminacy set of* $f$ is the set

$$N_f = \{x \in X : g(x) = h(x) = 0\}. \tag{258}$$

This is precisely the set where $f$ cannot locally be written as a function with values in $\mathbb{CP}^1$.

Notice that $\frac{\alpha}{\phi^l}$ has an empty indeterminacy set for all $l = 1, ..., n$, since locally we can write

$$\frac{\alpha}{\phi^l}(x) = \frac{e^{(-\sum_{l=0}^{k-1} W_l')}\phi^k \cdot \alpha_k}{\phi^l}(x) \tag{259}$$

depending on which $U_k^u$ the point $x$ lies in. But this is either always holomorphic ( if $l \leq k$) or $k < l$, in which case we know that $\frac{\phi}{\alpha_k}$ has no pole set in $U_k^u$, so also $\phi^{l-k+1} \cdot \frac{\phi}{\alpha_k}$ will not have one in $U_k^u$, and therefore $\frac{\alpha_k}{\phi^{l-k}}$ has no indeterminacy set. From now on we write $U_k$ instead of $U_k^u$, and keep in mind that the definition depends on the map $u$.

It turns out that we can reconstruct $u$ from $\alpha$. To do this, lets start with a technical lemma:

**Lemma 37.** *Let* $u$ *be a solution, which determines a section* $\alpha$ *of* $L$ *as described above. Using the hermitian metric* $h_0$ *on the line bundle* $L$ *and the induced metric on* $K_X$*, we have, setting* $N := \{x \in X : (\alpha \cdot \phi)(x) = 0\}$*, we can describe the sets* $U_k$ *above as follows:*

- $U_0 \backslash N = \{x \in X : e^{2(-W_0' - W_{\text{reg},1})}|\phi(x)|^3 < |\alpha(x)|^2\}$
- $U_k \backslash N =$
  $\{x \in X : e^{2(\sum_{l=0}^{k} -W_l' - W_{\text{reg},k+1})}|\phi(x)|^{2k+3} < |\alpha(x)|^2 < e^{2(\sum_{l=0}^{k-2} -W_l' - W_{\text{reg},k-1})}|\phi(x)|^{2k-1}\}$
  *for* $0 < k < n-1$
- $U_{n-1} \backslash N = \{x \in X : |\alpha(x)|^2 < e^{2(\sum_{l=0}^{n-3} -W_l' - W_{\text{reg},n-2})}|\phi(x)|^{2n-3}\}$

[14] The maps $\psi_j^{-1} \circ \psi_{j+1}$ are precisely the holomorphic changes of coordinates.

*Proof.* The first and third claim are similar to the second, which we prove in the following:

We make use of the decomposition

$$M_k = (M_k \cap M_{k+1}) \cup (M_k \cap M_{k-1}) \cup (M_k \cap N) \tag{260}$$

In $M_{k+1}$, a point $p$ belongs to $M_k$ if and only if $\mu_I(p) > a_{k+1}$. If we assume that $p = (\frac{\phi}{\alpha_{k+1}}, \alpha_{k+1}) = (e^{-\sum_{l=0}^{k} W'_l} \frac{\phi^{k+2}}{\alpha}, e^{\sum_{l=0}^{k} W'_l} \frac{\alpha}{\phi^{k+1}})$ and use the local expression for $\mu_I$, we obtain that $\mu_I(p) > a_{k+1}$ is equivalent to the condition

$$e^{2(\sum_{l=0}^{k} -W'_l - W_{reg,k+1})} |\phi(x)|^{2k+3} < |\alpha(x)|^2 \tag{261}$$

In a similar fashion we obtain the second inequality, using the local expression of $\mu_I$ in $M_{k-1}$. □

Given a holomorhpic structure $\bar{\partial}$ on $L$, and a holomorphic section $\alpha$ of $L$, such that $\frac{\phi^n}{\alpha}$ is holomorphic and all $\frac{\phi^k}{\alpha}$ have empty determinancy sets for $k = 1, ..., n$, how do we construct a solution to the perturbed gen. SW equations? The idea is to reverse the above construction:

**Lemma 38.** *Given a holomorphic section $\alpha$ of $L$, such that $\frac{\phi^n}{\alpha}$ is holomorphic and $\frac{\phi^k}{\alpha}$ has an empty indeterminacy set for all $k = 1, ..., n$, there is a holomorphic section $u_\alpha \in \Gamma(X, F)$, such that $u_\alpha$ determines the section $\alpha$ as described above.*
*Furthermore, if $u \in \Gamma(X, F)$ is a holomorphic section determining the holomorphic section $\alpha \in \Gamma(X, L)$, then $u = u_\alpha$.*

*Proof.* W.l.o.g. we assume $|\phi(x)| < 1$ for all $x \in X$.
If the section $\frac{\phi^n}{\alpha}$ is holomorphic it has no pole set, therefore $\alpha(x) = 0$ implies $\phi^n(x) = 0$ which implies $\phi(x) = 0$.

Away from $N := \{x \in X : (\alpha \cdot \phi)(x) = 0\} = \{x \in X : \phi(x) = 0\}$, we set

- $U'_0 \backslash N = \left\{x \in X : e^{-2W'_0} |\phi(x)|^3 < |\alpha(x)|^2\right\}$
- $U'_k \backslash N = \left\{x \in X : e^{2\sum_{l=0}^{k} -W'_l} |\phi(x)|^{2k+3} < |\alpha(x)|^2 < e^{2\sum_{l=0}^{k-2} -W'_l} |\phi(x)|^{2k-1}\right\}$ for $0 < k < n-1$
- $U'_{n-1} \backslash N = \left\{x \in X : |\alpha(x)|^2 < e^{2\sum_{l=0}^{n-3} -W'_l} |\phi(x)|^{2n-3}\right\}$

Since $e^{2\sum_{l=0}^{k} -W'_l} |\phi(x)|^{2k+3} < e^{2\sum_{l=0}^{k-1} -W'_l} |\phi(x)|^{2k+1}$ for each $x \in X \backslash N$ ($u'_l$ are positive functions), these define a partition of $\mathbb{R}_{>0}$ and therefore each $x \in X \backslash N$ is contained in one $U'_k \backslash N$. On $U'_k \backslash N$, we define $u_\alpha$ locally by $u_\alpha = (\frac{\phi}{\alpha_k}, \alpha_k)$ mapping into $M_k$, where $\alpha_k := e^{(\sum_{l=0}^{k-1} W'_l)} \frac{\alpha}{\phi^k}$. By the same computation as above, these local definitions agree on intersections $(U'_k \cap U'_{k+1})\backslash N$. By the Riemann extension theorem, we can extend this section over $N$, or more explicitly, for $x \in N$, we set $x \in U'_k$ if and only if $k \in \mathbb{N}_0$ is the smallest natural number s.t. $\frac{\phi^k}{\alpha}(x) < \infty$. This exists, since $\frac{\phi^n}{\alpha}$ is holomorphic. Also it is important here that $\frac{\phi^k}{\alpha}$ has an empty indeterminacy set. It is clear from the construction that $u_\alpha$ determines the holomorphic section $\alpha$ as described above. On the other hand, if $\alpha$ is determined by a holomorphic section $u$, then $u_\alpha$ and $u$ agree on the open sets $U'_k\backslash N$ (Notice that $U'_k\backslash N \subseteq U_k\backslash N$), and by the identity theorem for holomorphic functions they agree everywhere. □

The previous discussion also holds true in the smooth category:

**Corollary 10.** *Given a smooth section $\phi \in \Gamma(X, K_X)$, then there is a one to one-correspondence between maps $u \in \mathrm{Map}(Q, M)^{\mathrm{Spin}^{\mathbb{C}}(4)}$ satisfying $\mu_{\mathbb{C}} \circ u = \phi$ and sections $\alpha \in \Gamma(X, L)$, such that $\frac{\phi^n}{\alpha}$ is smooth and $\frac{\phi^k}{\alpha}$ has no indeterminacy set.*

*Proof.* Follow the steps of the previous proofs, and observe that $u$ and $u_\alpha$ agree everywhere on $X\backslash N$. Finally, a small compututation shows they actually agree on the whole of $X$. □

We now know how to deal with the map $u$, but we are still left with the equation

$$(F_a^+)^{1,1} + is \cdot \omega_X = \mu_I \circ u. \tag{262}$$

We follow the Ansatz one uses in the linear Seiberg-Witten theory, i.e. we extend the Gauge group from $\mathrm{Map}(X, S^1)$ to $\mathrm{Map}(X, \mathbb{C}\backslash\{0\})$ in order to solve equation (262). For this, recall that $L$ is a hermitian line bundle, so it comes equipped with a hermitian metric $h_0$. Any other metric $h_1$ differs from $h_0$ by a positive function, i.e. $h_1 = e^\lambda \cdot h_0$ for $\lambda \in C^\infty(X, \mathbb{R})$. To construct the holomorphic section $u_\alpha$ from $\alpha$, we used the metric $h_0$. Denote the holomorphic section using the metric $h_1 = e^\lambda \cdot h_0$ by $u_\alpha^\lambda$.

**Lemma 39.** *Let $\alpha$ be a holomorphic section of $L$ with respect to the holomorphic structure $\overline{\partial}_A$ induced by a connection $A$. Then there exists a unique hermitian metric $h_1 = e^\lambda h_0$, such that the Chern connection $B$ uniquely defined by the triple $(L, h_1, \overline{\partial}_A)$ fulfills*

$$(F_B^+)^{1,1} + is \cdot \omega_X = \mu_I \circ u_\alpha^\lambda \tag{263}$$

*Proof.* A quick calculation (see [22]) shows that if $B$ is the unique holomorphic connection determined by $(L, h_1, \overline{\partial} := \overline{\partial}_A)$, then

$$F_B = F_A + \overline{\partial}\partial\lambda \tag{264}$$

so $h_l$ fulfills the above equation if and only $\lambda$ satisfies

$$(F_{2B}^{+})^{1,1} + is \cdot \omega_X = 2(F_A^{+})^{1,1} + 2(\bar{\partial}\partial\lambda^{+})^{1,1} + is \cdot \omega_X = i \cdot \mu_I \circ u_\alpha^\lambda \cdot \omega_X \tag{265}$$

which, since $\Omega_-^2(M, \mathbb{C}) = \omega_X^\perp \cap \Omega^{1,1}(M, \mathbb{C}) \subseteq \Omega^{1,1}(M, \mathbb{C})$, is equivalent to

$$2F_A \wedge \omega_X + 2\bar{\partial}\partial\lambda \wedge \omega_X + is \cdot \omega_X \wedge \omega_X = i \cdot \mu_I \circ u_\alpha^\lambda \cdot \omega_X \wedge \omega_X \,. \tag{266}$$

We can now make use of the relation $\bar{\partial}\partial\lambda \wedge \omega_X = -2i\Delta(\lambda)\ d\mathrm{vol}$ to rewrite the above equation as

$$\left(\Delta(\lambda) + \frac{1}{2}\mu_I \circ u_\alpha^\lambda\right) d\mathrm{vol} = \frac{1}{i} F_A \wedge\ \omega_X + \frac{s}{2} d\mathrm{vol} \tag{267}$$

The next theorem asserts that this differential equation indeed has a (unique) solution.

**Theorem 20.** *Given a compact Riemannian manifold* $X$*, the differential equation*

$$\Delta\lambda(x) + F(x, \lambda(x)) = g(x) \tag{268}$$

*where* $g : X \to \mathbb{R}$ *and* $F : X \times \mathbb{R} \to \mathbb{R}$ *are smooth functions, has a unique solution* $\lambda \in C^\infty(X)$ *if* $F$ *satisfies the following conditions:*

1. $F(x, \cdot) : \mathbb{R} \to \mathbb{R}$ *is strictly increasing for all* $x \in M$ *besides on a null set, in which we assume* $F$ *only to be increasing.*

2. *We have*

$$\lim_{\lambda \to \infty} F(x, \lambda) = \infty \ \textit{and} \ \lim_{\lambda \to -\infty} F(x, \lambda) = -\infty$$

   *for all* $x \in M$ *besides on a null set.*

*Proof.* See Appendix C. □

Thus, we are done if we prove that $\lambda \mapsto \mu_I \circ u_\alpha^\lambda$ satisfies these two conditions. Recall the sets $U_k(\lambda) := U_k$ defined by the section $u_\alpha^\lambda$. If we pick a point $x \in U_k(\lambda) \backslash N$, which means that $x$ satisfies

$$e^{\sum_{l=0}^{k} -W_l'} |\phi(x)|^{2k+3} < e^\lambda |\alpha(x)|^2 < e^{\sum_{l=0}^{k-2} -W_l'} |\phi(x)|^{2k-1}, \tag{269}$$

we have the local description

$$\mu_I \circ u_\alpha^\lambda(x) = \frac{1}{2}\Big(e^{2W_{reg,k}} \cdot e^\lambda |\alpha_k|^2 - e^{-2W_{reg,k}} \cdot e^{-\lambda} |\frac{\phi}{\alpha_k}|^2\Big)(x) \tag{270}$$

For a small variation of $\lambda + \delta\lambda$ of $\lambda$, the point $x$ will still be in $U_k(\lambda + \delta\lambda)$ and the local description is still valid. The function $W_{\mathrm{reg},k}$ is the integral over a positive function up to $\mu_I \circ u_\alpha^\lambda$. From this we see, that $\mu_I \circ u_\alpha^\lambda$ cannot be constant if one varies $\lambda$, otherwise $W_{\mathrm{reg},k}$ would be constant as a function of $\lambda$, but the terms $e^\lambda|\alpha_k|^2$ and $-e^{-\lambda}|\frac{\phi}{\alpha_k}|^2$ would be increasing if $\lambda$ increases, so $\mu_I \circ u_\alpha^\lambda$ would be increasing, giving a contradiction. So either it is strictly increasing or decreasing. For $\lambda$ big enough, we have $e^{-W_0'}|\phi(x)|^3 < e^\lambda \alpha(x)$, meaning $\mu_I \circ u_\alpha^\lambda > a_0$, while for $\lambda$ small enough $\mu_I \circ u_\alpha^\lambda < a_{n-1}$, which means the function must be strictly increasing. With the same argument, using the local description for $\mu_I$, we also see that the second condition is fulfilled on $X \backslash N$.

□

Now given a pair $(\overline{\partial}, \alpha)$ of a holomorphic structure on $L$ and a section $\alpha$ of $L$ which is holomorphic to $\overline{\partial}$, such that $\frac{\phi^n}{\alpha}$ is holomorphic and $\frac{\phi}{\alpha}$ has an empty indeterminacy set, it determines a solution $(A, u_\alpha^\lambda)$ to the perturbed generalized Seiberg-Witten equation up to Gauge equivalence. Two such pairs determine the same solution up to Gauge equivalence if and only if the holomorphic structures are isomorphic and there is a holomorphic isomorphism of $L$ mapping one holomorphic section into a scalar multiple of the other[15]. In conclusion:

**Theorem 21.** *There is a bijection between the Moduli space* $\mathcal{M}_L^\phi = \{$ *Solutions to the perturbed gen. SW with perturbations* $(\phi, is \cdot \omega_X)\}/\mathcal{G}$ *for the target manifold a MEH space with* $n$ *singularities and the set*

$$\Big\{ \textit{effective Divisors } D \textit{ on } X, \textit{s.t. } c_1(D) = c_1(L),$$
$$n \cdot (\phi) - D \geq 0, (k \cdot (\phi) - D)_0 \cap (k \cdot (\phi) - D)_\infty = \emptyset \textit{ for all } k = 1, ..., n \Big\}.$$

## 4.3 Solving the perturbed Equations Part 2

We give an alternative description of the solution space, arising via the description of the MEH-spaces as hyperKähler reductions. The following considerations are due to V. Pidstrygach.

Recall the description of MEH- spaces as a hyperKähler reduction

$$M_\epsilon = \left(\mu^{\mathbb{T}^{n-1}}\right)^{-1}(\epsilon) \big/ \mathbb{T}^{n-1}. \tag{271}$$

[15]This is a similar argument to the one used to solve the unperturbed equations, see [28].

We denote

$$Y_\epsilon := \left(\mu^{\mathbb{T}^{n-1}}\right)^{-1}(\epsilon) \xrightarrow{\pi_1} M_\epsilon, \tag{272}$$

which is a $\mathbb{T}^{n-1}$ prinipal bundle over $M_\epsilon$, carrying a natural connection[16] induced by the surjective submersion. We will now relate solutions to the generalized Seiberg-Witten equations with $MEH$ spaces as a target space to solutions to the gSW equations with target space $\mathbb{H}^n$, where the generalized spinor takes values in $Y_\epsilon \subseteq \mathbb{H}^n$.

Let $Q$ be a $U^{\mathbb{C}}(2)$-structure over $X$ and let $(u, a)$ be a solution to the GSW equations with target space $M_\epsilon$. We have the following diagram:

$$\begin{array}{ccc} u^*Y_\epsilon & \xrightarrow{\widehat{u}} & Y_\epsilon \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ Q & \xrightarrow{u} & M_\epsilon \end{array}$$

The bundle $u^*Y_\epsilon$ is a $U^{\mathbb{T}^n}(2)$-prinical bundle over $X$, where

$$U^{\mathbb{T}^n}(2) := (\mathbb{T}^n \times U(2))_\pm \ . \tag{273}$$

Denoting the connection on $Y_\epsilon$ induced by the Riemannian metric by $A^\perp$, and the connection on $Q$ obtained by lifting $a$ together with the Levi-Civita connection by $A$, we obtain a connection on $u^*Y_\epsilon$ by

$$\widehat{A} := \pi_2^*(A) \oplus \widehat{u}^*(A^\perp) \ . \tag{274}$$

We write $(a_1, ..., a_n)$ for the $\mathbb{T}^n$-part of $\widehat{A}$, and $\widehat{u} = (u_1, ..., u_n)$.

**Theorem 22** ([23])**.** *Solutions to the perturbed GSW equations with perturbation $\phi$ and target space $M_\epsilon$ up to $S^1$-gauge equivalence are in 1-1 correspondence with solutions $(\widehat{u}, \widehat{A})$ of the system*

$$\not{D}_{\widehat{A}}\widehat{u} = 0 \tag{275}$$

$$\mu^{\mathbb{T}^{n-1}}(\widehat{u}) = \epsilon \tag{276}$$

$$\mu^{S^1}(\widehat{u}) = pr_{S^1} \circ F^+_{\widehat{A}} + \phi \tag{277}$$

*up to $\mathbb{T}^n$- gauge equivalence.*

[16] To be more precise, this connection is given by taking the orthogonal complement of the vertical subspace.

The above system of equations reads in components:

$$\not{D}_{A_s} u_s = 0 \ , \ \ s = 1, ..., n \tag{278}$$

$$\sigma(u_{s+1}) - \sigma(u_s) = \epsilon_s \cdot \omega \ , \ \ s = 1, ..., n-1 \tag{279}$$

$$\sum_{s=1}^{n} F^+_{a_s} = \sum_{s=1}^{n} \sigma(u_s) + \phi \tag{280}$$

where $A_s$ is the lift of $a_s$ together with the Levi-Civita connection, and $\sigma$ is the moment map of the standard $S^1$-action on $\mathbb{H}$. Writing $u_s = (\alpha_s, \beta_s)$, one can solve these equations more explicitly. In fact, it turns out that $(\alpha_s, \beta_s)$ are holomorphic for all $s$, therefore the moduli space can be identified with certain divisors:

**Theorem 23** ([23])**.** *Solutions to the system above up to $\mathbb{T}^n$-gauge equivalence are in 1-1 correspondence to the following space:*

$$\Big\{ \textit{effective Divisors } (\alpha_k)_0 = \sum_{i=k}^{n} A_i, \ (\beta_k)_0 = \sum_{i=0}^{k-1} A_i, \ k = 1, ..., n,$$
$$\sum_{i=0}^{n} A_i = (\phi), (\alpha_{k+1})_0 \cap (\beta_k)_0 = \emptyset \textit{ for all } k = 1, ..., n, \ \sum_{k=1}^{n} c_1(\alpha_k) = c_1(L) \Big\}$$

Unsurprisingly, we can relate those two descriptions of our moduli space:

**Theorem 24.** *Fix $\phi \in H^0(X, K_X)$. The systems*

$$\Big\{ \textit{effective divisors } D, \ c_1(D) = c_1(L), \ D \leq n \cdot (\phi),$$
$$(k \cdot (\phi) - D)_0 \cap (k \cdot (\phi) - D)_\infty = \emptyset \textit{ for all } k = 1, ..., n \Big\}$$

*and*

$$\Big\{ \textit{effective Divisors } (\alpha_k)_0 = \sum_{i=k}^{n} A_i, \ (\beta_k)_0 = \sum_{i=0}^{k-1} A_i, \ k = 1, ..., n,$$
$$\sum_{i=0}^{n} A_i = (\phi), (\alpha_{k+1})_0 \cap (\beta_k)_0 = \emptyset \textit{ for all } k = 1, ..., n, \ \sum_{k=1}^{n} c_1(\alpha_k) = c_1(L) \Big\}$$

*are equivalent.*

*Proof.* Given $D$, it defines a holomorphic section $\zeta$ of $L$ such that $\frac{\phi^n}{\zeta}$ is holomorphic and $(\frac{\phi^k}{\zeta})_0 \cap (\frac{\zeta}{\phi^k})_0 = \emptyset$. Set $(\alpha_k)_0 = (\frac{\zeta}{\phi^{(k-1)}})_0 - (\frac{\zeta}{\phi^k})_0$ and $(\beta_k)_0 = (\frac{\phi^k}{\zeta})_0 - (\frac{\phi^{(k-1)}}{\zeta})_0$. The condition $(\alpha_{k+1})_0 \cap (\beta_k)_0 = \emptyset$ is clearly satisfied, also $\sum_{i=0}^n A_i = (\alpha_k)_0 + (\beta_k)_0 = (\phi)$.

A small calculation shows that the divisors $A_k$ are effective and give decompositions of $\alpha_k$ and $\beta_k$ as desired.
On the other hand, given a collection of divisors $\alpha_k$ and $\beta_k$ satisfying the conditions above, define $(\zeta)_0 = \sum_{k=1}^n \alpha_k$. This defines a holomorphic section of $L$, since $c_1(\zeta) = \sum_{k=1}^n c_1(\alpha_k) = c_1(L)$. Furthermore $\sum_{k=1}^n (\beta_k)_0 - (\alpha_k)_0 = \sum_{i=0}^n A_i = n \cdot (\phi)$, so $\sum_{k=1}^n \beta_k = \frac{\phi^n}{\zeta}$ is effective as well. We are left to show that

$$(k \cdot (\phi) - D)_0 \cap (k \cdot (\phi) - D)_\infty = \emptyset.$$

We compute:

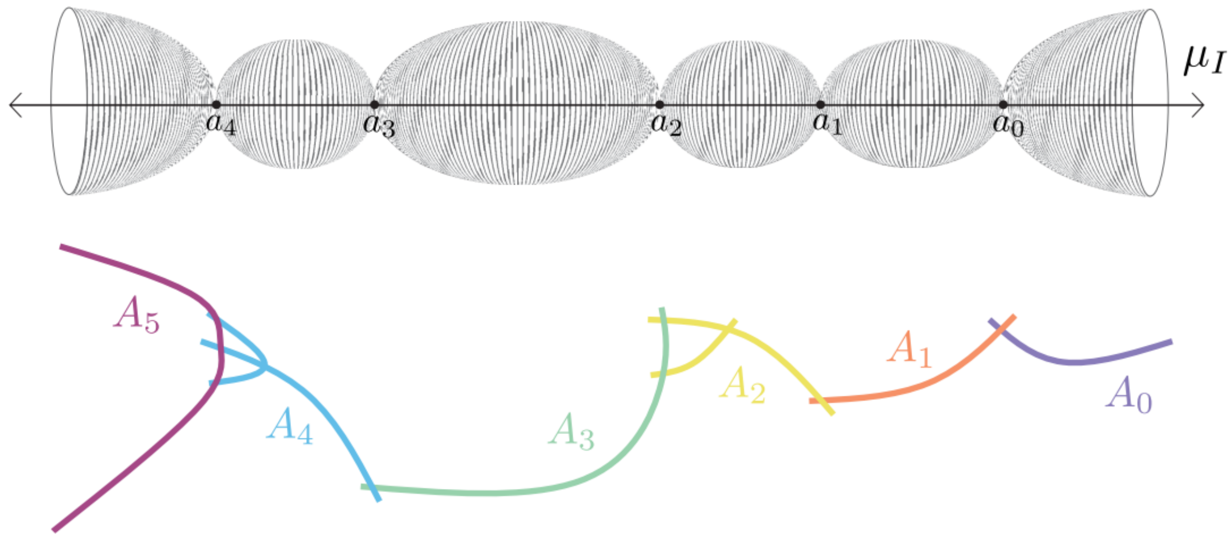
$$k \cdot (\phi) - \zeta = \sum_{j=0}^n k \cdot A_j - \sum_{k=1}^n \alpha_k = \sum_{j=0}^n k \cdot A_j - \sum_{j=1}^n j \cdot A_j = \sum_{j \le k-1} (k-j) \cdot A_j - \sum_{j \ge k+1} (j-k) A_j$$

But since $(\alpha_{k+1})_0 \cap (\beta_k)_0 = \left(\sum_{j\ge k+1} A_j\right) \cap \left(\sum_{j \le k-1} A_j\right) = \emptyset$ the $A_i$'s are all effective divisors, also their positive weighted sums do not intersect, so indeed,

$$(k \cdot (\phi) - D)_0 \cap (k \cdot (\phi) - D)_\infty = \emptyset.$$

□

This gives a nice geometric description of our solutions: A solution is equivalent to decomposing $(\phi)$ into $n+1$ divisors $A_i, i = 0, ..., n$ such that each $A_i$ only intersects $A_{i-1}$ and $A_{i+1}$. The map $u$ is then constructed in such a way that these divisors are mapped into the $\mu_{\mathbb{C}}^{-1}(0)$ variety, where $A_0$ and $A_n$ are mapped into the outer copies of $\mathbb{C}$ respectively, and each $A_i$ is mapped into the $i$-th sphere.




## 4.4 Virtual Dimension

We now have an explicit description of the points in the moduli space, but we are also interested in its topological structure. For further investigations, recall how the moduli space was constructed[17]:

For $u \in \mathrm{Map}(Q, M)^{U^{\mathbb{C}}(2)}$, we label the following Banach manifolds[18]:

$$\mathcal{X} := \mathrm{Map}(Q, M)^{U^{\mathbb{C}}(2)} \times \mathcal{A}_0, \tag{281}$$

$$\mathcal{Y} := \mathrm{Map}(Q, u^*TM^-)^{U^{\mathbb{C}}(2)} \times \Omega^2_+(X) \tag{282}$$

We then defined the perturbed Seiberg-Witten map

$$sw_{(\cdot)} : \mathcal{X} \times \mathcal{H}^2_+(X, \mathbb{R}) \to \mathcal{Y}. \tag{283}$$

The gauge group $\mathcal{G}$ acts on its zero set, and thus we can define the parametrized moduli space

$$\mathcal{PM} := sw^{-1}_{(\cdot)}(\{0\})/\mathcal{G}. \tag{284}$$

This contains all possible solutions for all possible perturbations modulo gauge

[17]The following discussion is very similar to the linear case, for more details see [22].

[18]Again, technically the second space depends on some element in the first space, but this is not important for the considerations here.

equivalence. It has an obvious projection onto the last component

$$\mathfrak{p} : \mathcal{PM} \to \mathcal{H}^2_+(X, \mathbb{R}), [(u, a, s \cdot \omega, \phi)] \mapsto (s \cdot \omega, \phi) \tag{285}$$

and the preimage of a fixed perturbation is precisely the moduli space we have investigated so far:

$$\mathfrak{p}^{-1}(\{(s \cdot \omega, \phi)\}) = \mathcal{M}^{s,\phi} \; . \tag{286}$$

By Lemma (24), the differential $Tsw_{(\cdot)}$ of the perturbed Seiberg-Witten map is surjective for all solutions, so we can use the infinite dimensional version of the regular value theorem to give $sw^{-1}_{(\cdot)}(\{0\})$ the structure of an infinite dimensional manifold. Away from $\mathfrak{p}^{-1}(\{0\})$, the action of the Gauge group is free[19], so we expect $\mathcal{PM}$ to be a smooth manifold there. Its dimension is equal to the dimension of its tangent space, which is given by

$$T\mathcal{PM} \simeq \ker Tsw_{(\cdot)}/\mathrm{im} T\mathcal{G} \tag{287}$$

By Sard's theorem, for a generic perturbation $(s \cdot \omega, \phi)$, the preimage $\mathfrak{p}^{-1}(\{(s \cdot \omega, \phi)\})$ will either be empty or a smooth manifold of dimension $d_{vir} = \dim \mathcal{PM} - \dim b_2^+$, called the *virtual dimension of the moduli space.* We compute dim $\mathcal{PM}$:

Denote by $L_0$ the line bundle such that $\pi_! u^* TM \simeq L_0 \oplus L_0 \otimes K_X^{-1}$, where $u : Q \to M$ is an equivariant map, $Q \to X$ a $U^{\mathbb{C}}(2)$-structure given by a line bundle $L$. As noted in corollary (8), the linearizations of the generalized Seiberg-Witten equations and the linearization of the linearized equations agree up to order zero terms. Hence we obtain the following result:

**Lemma 40** ([22], p. 229)**.** *The virtual dimension of the Moduli space of the linear Seiberg-Witten equations with* $U^{\mathbb{C}}(2)$*-structure* $L_0 \bullet Q_{can}$ *is given by*

$$v_{dim} = \frac{1}{4}\Big((2L_0 - K_X)^2 - 2\chi(X) + 3\tau(X)\Big) \tag{288}$$

$$= \frac{1}{4}\Big((4L_0^2 + K_X^2 - 4L_0 \cdot K_X) - K_X^2\Big) = L_0 \cdot (L_0 - K_X) \tag{289}$$

*Here,* $\chi(X)$ *and* $\tau(X)$ *are the Euler characterstic and the signature of* $X$ *respectively.*

*Proof.* We would like to compute the dimension of

$$T\mathcal{PM} \simeq \ker Tsw_{(\cdot)}/\mathrm{im} T\mathcal{G} \; . \tag{290}$$

[19]A fixed point of the gauge group action must satisfy $u \equiv 0$.

Notice that $T_1\mathcal{G} \simeq \Omega^0(X, \mathbb{R})$ and the differential of the gauge group action is given by an operator $K$, which is the sum of the deRham differential

$$d : \Omega^0(X, \mathbb{R}) \to \Omega^1(X, \mathbb{R}) \tag{291}$$

and a term of order zero in the first factor[20].

Assuming the spinor $u$ is non-constant, the kernel of $K$ is injective. Using the adjoint, we can characterize $\operatorname{im} T\mathcal{G}$ as the kernel of $K^*$, which is given by $d^*$ plus a term, which is the adjoint of an order zero operator, which again is of order zero. Furthermore, $K^*$ is surjective. We therefore only need to compute the dimension of the kernel of the map

$$Tsw_{(\cdot)} + K^* = \not{D}^{lin} + d^+ + d^* + \text{terms of order zero.} \tag{292}$$

This is a surjective Fredholm operator, therefore the dimension of its kernel is equal to its index. The index is invariant under perturbation of terms or order zero, and therefore

$$\dim T\mathcal{PM} = \operatorname{index}(\not{D}^{lin} + d^+ + d^*) + b_2^+ \tag{293}$$

The last term comes from the fact that we have artifically enlarged the domain of the map from $\mathcal{X}$ to $\mathcal{X} \times \mathcal{H}^2_+(X, \mathbb{R})$. The index can be computed using the Atiyah-Singer index theorem, which, as noticed, gives the same dimension as in the linear Seiberg-Witten case, which is computed in ([22], 3.2) and gives the number above. □

Notice that the virtual dimension can be computed as $L_0 \cdot (L_0 - K_X) = c_2(\pi_! u^* TM) = e(\pi_! u^* TM)$, the Euler class of the vector bundle[21]. It turns out that we have an explicit section of this bundle:

$$\mathcal{K}(x) := [p, K_i^M(u(p))] \in \Gamma(X, \pi_! u^* TM) \simeq \Gamma(X, Q \times_H u^* TM)\ , p \in Q, \pi(p) = x \tag{294}$$

where $K_i^M$ is the fundamental vector field of the hyperKähler $S^1$ action on $M$. Now, let $u$ be defined by a collection of divisors $(A_0, A_1, ..., A_n)$, such that they are the components of $\phi \in H^0(X, K_X)$, or, equivalently, $u$ is defined by a section $\alpha \in H^0(X, L)$ such that $\alpha$ divides $\phi^n$.

We distinguish between two cases:

[20] The first factor is given by the fundamental vector field, described by the vector field $\mathcal{K}(x)$ below.

[21] To be precise, we identify top differential forms with numbers by integrating over M.

*Case 1:* All the divisors meet transversally and $\mathcal{K}$ meets the zero section transversally.
In this case we can compute the Euler class by counting the zeroes of $\mathcal{K}$ with suitable orientations. We observe that the zeroes of $\mathcal{K}$ are prescisely those points, where $u(p) = a_i$ for some singularity $a_i$, since these are the fixed points of the $S^1$ action. On the other hand, $u(p) = a_i \Leftrightarrow \pi(p) \in A_i \cap A_{i+1}$.
*Case 2:* Not all divisors meet transversally (i.e. some of them have a common component), or $\mathcal{K}$ does not intersect the zero section transversally.
We notice that the isomorphism class of the vector bundle $\pi_! u^* TM$ depends only on the homotopy type of $u$. We therefore perturb $u$ to a smooth section $u'$ such that both conditions are guaranteed. The divisors $A_i$ are perturbed into zero sets of smooth sections, which we will denote by $A'_i$. To achieve this, we look at the local picture. The map $u$ can locally be given as $u = (a, \overline{b})$ for maps $a, b : U \to \mathbb{C}$ on some neighborhood $U \subseteq X$, such that $a^{-1}(\{0\}) = A_{i|U}$ and $b^{-1}(\{0\}) = A_{i+1|U}$. Consequently, the intersection $\left(A_i \cap A_{i+1}\right)_{|U}$ corresponds to $u = 0$, or equivalently $a = 0$ and $b = 0$[22]. If the intersection is not transversal, we perturb $a$ and $b$ into smooth maps $a', b'$, such that the intersection of their zeroes is transversal. Changing $u$ locally in this fashion yields a smooth section $u'$ with all corresponding intersections being transversal.

As above, $\mathcal{K}(u')(x)$ vanishes if and only if $x \in A'_i \cap A'_{i+1}$. We use the fact here that the intersection number is really a topological invariant, and can be computed using the intersection of generic smooth sections which zero sets meet transversally, in particular we have

$$A_i \cdot A_{i+1} = A'_i \cdot A'_{i+1} \quad \text{for all } i = 0, ..., n-1. \tag{295}$$

We determine the corresponding orientations at these intersection points:

**Lemma 41.** *The orientation of some point $x \in A'_i \cap A'_{i+1}$ is always $(-1)$. Therefore the Euler class and hence the virtual dimension is given by*

$$v_{\dim} = -\sum_{i=0}^{n-1} A_i \cdot A_{i+1} \tag{296}$$

*Proof.* Let $x \in A'_i \cap A'_{i+1}$. Denote $L_i := \mathcal{O}(A_i)$ and $L_{i+1} := \mathcal{O}(A_{i+1})$. By the transversality assumption $A'_i$ and $A'_{i+1}$ are two dimensional submanifolds of $X$ around

[22] Geometrically in the local picture, the divisors $A_i$ and $A_{i+1}$ are mapped to the two coordinate axes in $\mathbb{C}^2$, and their intersection is precisely what is mapped into zero.

the point $x$. They inherit orientations from $A_i$ and $A_{i+1}$, by parallel transport via the perturbation (although $A_i$ and $A_{i+1}$ might have singularities, these are isolated, so the orientations can be lifted at these points, as soon as they are perturbed to smooth points). Take a small neighborhood $V \simeq \mathbb{C}^2$ around the singularity $a_i \in M$, on which we have holomorphic coordinates and $u'^{-1}_{|V}\{a_i\} = x$, and let $U$ be the connected component containing $x$ of the preimage $u'^{-1}(V)$. Then, on $U$, the map $u'$ is given by

$$(a, \overline{b}) \in \Gamma(U, (L_i \oplus L_{i+1}^{-1})_{|U}), \text{ such that} \tag{297}$$

$$a^{-1}(\{0\}) = A_i' \text{ and } b^{-1}(\{0\}) = A_{i+1}'. \tag{298}$$

Now, $\mathcal{K}_{|U} = (ia, -i\overline{b})$, and the point $x$ is precisely where $a$ and $b$ vanish simultaneously, therefore we have we can compute the intersection number at $x$ via the eulerclass of $L_i \oplus L_{i+1}^{-1}$

$$e\Big((L_i \oplus L_{i+1}^{-1})_{|U}\Big) = c_2\Big((L_i \oplus L_{i+1}^{-1})_{|U}\Big) = L_{i|U} \cdot \Big(-L_{i+1|U}\Big) = -A_{i|U} \cdot A_{i+1|U} \tag{299}$$

□

The previous lemma actually gives us a lot of information about the moduli space:

Assume for a moment that all the intersection numbers $A_i \cdot A_{i+1}$ are non-negative. Then the virtual dimension of the Moduli space is non-positive, and it is only equal to zero if all the intersections satisfy $A_i \cdot A_{i+1} = 0$. When the virtual dimension has negative dimension, we know the moduli space must be empty for a generic perturbation $\phi$, therefore we could conclude that the moduli space is always zero dimensional (which also concides with the actual dimension) or empty, and this only happens when all the $A_i's$ are non-intersecting. But what if the intersection number $A_i \cdot A_{i+1}$ is negative for some $i$, so $A_i = A_{i+1}$?[23] It turns out that this can never happen:

**Proposition 4.** Every intersection number $A_i \cdot A_{i+1}$ is non-negative.

*Proof.* Indeed, assume $A_k$ and $A_{k+1}$ have negative-self intersection. Then they contain a maximal common component $C$ with negative self-intersection. Since $A_k$

[23]This is a feature of complex algebraic geometry: Transverse intersections are always positively oriented, so negative intersections can only occur if one intersects the curve "with itself". One example of this is the exceptional divisor of a blow up, which has negative self-intersection.

and $A_{k+1}$ are both components of $(\phi)$, we write

$$A_k = A'_k + C, \; A_{k+1} = A'_{k+1} + C \text{ and} \tag{300}$$

$$(\phi) = \sum_{j \neq k,k+1} A_j + A'_k + A'_{k+1} + 2C. \tag{301}$$

We know $A_j$ does not intersect $A_{k+1}$ for $j \leq k-1$, in particular it does not intersect $C$. Also $A_j$ does not intersect $A_k$ for $j \geq k+2$, so it does not intersect $C$. Furthermore, by the maximality of C,

$$A'_k \cdot A'_{k+1} \geq 0.$$

By the Adjunction formula

$$2p_a(C) - 2 = C \cdot (K_X + C) = C \cdot \Big((\phi) + C\Big) = C \cdot \Big(\sum_j A_j + C\Big) = \tag{302}$$

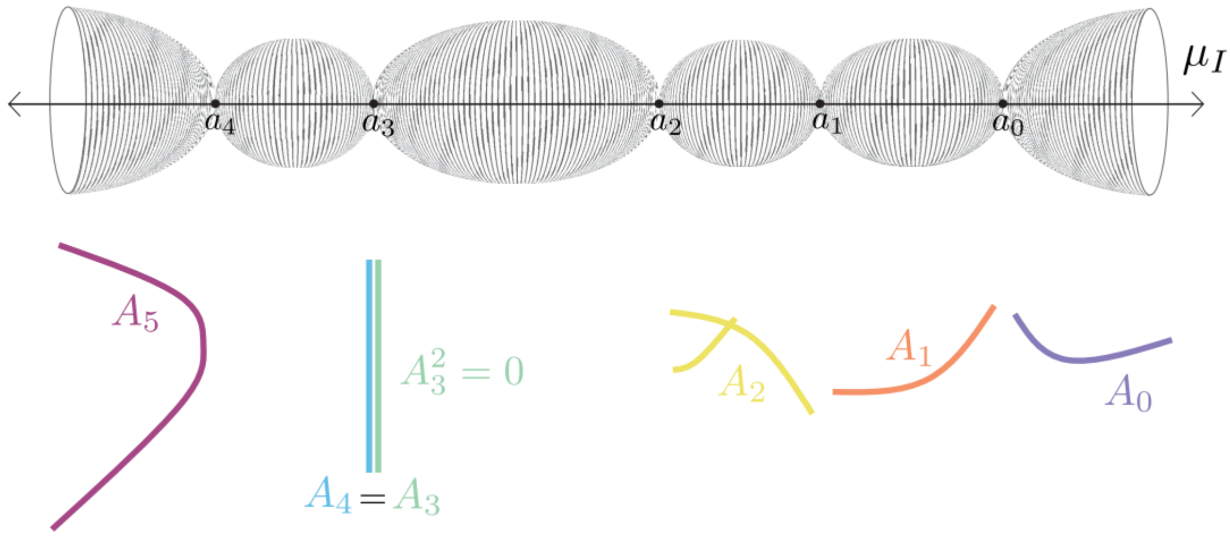
$$C \cdot \Big(A_k + A_{k+1} + C\Big) = 2C^2 + C \cdot A'_k + C \cdot A'_{k+1} + C^2 \leq \tag{303}$$

$$A'_k \cdot A'_{k+1} + C^2 + C \cdot A'_k + C \cdot A'_{k+1} + 2C^2 = A_k \cdot A_{k+1} + 2C^2 \leq -3 \tag{304}$$

implying that $2p_a(C) \leq -1$, which is a contradiction, as the arithmetic genus is a non-negative number. Therefore such a curve cannot exist. □

What might happen though is that $A_i = A_{i+1} = C$ with $C^2 = 0$. This case also deserves special attention, as by varying the perturbation $\phi$, this solution might "bifurcate" into two different solutions, giving rise to compactness problems. This will be considered in the next chapter. We summarize:

**Corollary 11.** *If the moduli space for a generic perturbation $\phi$ is non-empty, then the virtual dimension is zero. Furthermore all the divisors $A_i$ are (algebraically) disjoint.*




## 4.5 Orientation

*"One does not speak about orientations in public."*

- an anonymous mathematican

Since we model our moduli space as a manifold, and we would like to extract invariants by integrating over our moduli space (in our case, since the Moduli space is always zero dimensional, it comes down to counting the number of points), we need to assign them orientations ( in our case, a $\pm 1$ for each point). This is done in the following, more general framework: Assume we have a (maybe infinite dimensional) Frechet manifold $\mathcal{X}$, and a possibly infinite dimensional Lie group $\mathcal{G}$ acting on $\mathcal{X}$. Now assume that we have a map $\mathsf{F} : \mathcal{X} \to \mathcal{Y}$ to another possibly infinite dimensional Hilbert space, such that $0$ is a regular value of $\mathsf{F}$, $\mathsf{F}$ is $\mathcal{G}$-invariant, and the $\mathcal{G}$-action is proper and free on $\mathsf{F}^{-1}(0)$, so the quotient $\mathsf{F}^{-1}(0)/\mathcal{G}$ is a manifold. Linearizing this equation at some $x \in \mathcal{X}$ with $\mathsf{F}(x) = 0$ gives the following complex, where $\mathtt{Lie}(\mathcal{G})$ denotes the Lie algebra of $\mathcal{G}$:

$$0 \to \mathtt{Lie}(\mathcal{G}) \xrightarrow{\mathsf{K}_x} \mathsf{T}_x\mathcal{X} \xrightarrow{d\mathsf{F}_x} \mathcal{Y} \to 0 \tag{305}$$

or if we dualise the complex by using the adjoint map:

$$\mathsf{T}_x\mathcal{X} \xrightarrow{d\mathsf{F}_x \oplus (\mathsf{K}_x)^*} \mathcal{Y} \oplus \mathtt{Lie}(\mathcal{G})^* \to 0 \tag{306}$$

The tangent space of the quotient manifold $\mathsf{F}^{-1}(0)/\mathcal{G}$ at the point $[x]$ is modeled as the kernel of the map $dF_x \oplus (K_x)^*$ [24] In our explicit setting, given a solution to the perturbed Seiberg-Witten equations $(u, a, is \cdot \omega, \phi)$, this is given by

$$\Omega^0(X, i\mathbb{R}) \xrightarrow{K} \Gamma(Q, u^*TM^+)^{U^{\mathbb{C}}(2)} \times \Omega^1(X, i\mathbb{R}) \xrightarrow{sw^{lin}_{(u,A,\phi,is\cdot\omega)}} \tag{307}$$

$$\Gamma(Q, u^+TM^-)^{U^{\mathbb{C}}(2)} \times \Omega^2(X, i\mathbb{R})_+ \tag{308}$$

or if we wrap the complex:

$$\Gamma(Q, u^*TM^+)^{U^{\mathbb{C}}(2)} \times \Omega^1(X, i\mathbb{R}) \xrightarrow{sw^{lin}_{(u,A,\phi,is\cdot\omega)} \oplus K^*} \tag{309}$$

$$\Gamma(Q, u^+TM^-)^{U^{\mathbb{C}}(2)} \times \Omega^2(X, i\mathbb{R})_+ \times \Omega^0(X, i\mathbb{R}) \tag{310}$$

where

$$F_1 := sw^{lin}_{(u,A,\phi,is\cdot\omega)} \oplus K^*(\psi, \alpha) = \left(\not{D}^{lin,A}\psi + cl(K^M_\alpha), d^+\alpha - d\mu \circ \psi, d^*\alpha + \langle K^M_i, \psi\rangle\right) \tag{311}$$

One observes that by getting rid of the zero order terms, the map will be given by $F_0 = (\not{D}^{lin,A}, d^+ \oplus d^*)$. The kernel of this can be described very explicitly: It is given by $\ker \not{D}^{lin,A} \oplus \mathcal{H}^1(X, i\mathbb{R})$. Furthermore the cokernel is $\mathcal{H}^2_+(X, i\mathbb{R}) \oplus \mathcal{H}^0(X, i\mathbb{R})$. All these spaces carry canonical orientations.[25]

The idea is now to "transport" orientations from these well known spaces via a homotopy between $F_0$ and $F_1$ to orientations on the kernel of $F_1$. This can be done using the following Theorem:

**Theorem 25** ([22], Corollary 1.5.7)**.** *Let* $F_t,\ t \in [0, 1]$ *be a continiously varying family of Fredholm operators, and denote*

$$\det F_t := \det(\ker F_t) \otimes \det(\operatorname{coker} F_t) \tag{312}$$

*Then* $\det(F_0) \simeq \det(F_1)$*, and any orientation on* $\det(F_0)$ *induces canonically an orientation on* $\det(F_1)$.

Usually, finding this orientation transport means going into the nitty-gritty of the case at hand. Luckily, most of the work has been done for us already. But first, lets simplify a little bit: We write $\mu = \mu_I + \mu_{\mathbb{C}}$ as before, and observe that

[24] The regular value theorem is applied here.

[25] The kernel of $\not{D}^A_{lin}$ carries a complex structure, which induces a natural orientation. $\mathcal{H}^0(X)$ is the space of constant functions, so we just choose the constant 1-function to be positively oriented, and $H^2_+(X)$ splits as the real span of the symplectic form and a complex vector space.

$d\mu_I \circ \psi = \omega_I(K_i^M, \psi)$, $d\mu_{\mathbb{C}} \circ \psi = \omega_{\mathbb{C}}(K_i^M, \psi)$. Further we identify $i \cdot \mathbb{R} \cdot \omega_X \simeq i \cdot \underline{\mathbb{R}}$ and view the latter as a subbundle of the complexification of $\underline{\mathbb{R}}$. Also, we identify $\Omega^1(X, \mathbb{R}) \simeq \Omega^{0,1}(X)$ and $\pi_! u^* TM^* \simeq L_0 \oplus L_0 \otimes \Lambda^{0,2}(X)$ and denote the section $K_i^M = (K_1, K_2) \in \Gamma\big(X, L_0 \oplus L_0 \otimes \Lambda^{0,2}(X)\big)$, similarly $\psi = (\psi_1, \psi_2)$. Last but not least, let $B$ be the connection on $L_0$ satisfying $\not{D}^A_{lin} = \not{D}^B_{L_0} = \overline{\partial}_B \oplus \overline{\partial}^*_B$. Putting all this together we get:

$$F_1(\psi, \alpha) = \Big(\not{D}^A_{lin}\psi + cl(K^M_\alpha), d^+\alpha - d\mu \circ \psi, d^*\alpha + \langle K_i^M, \psi \rangle\Big) \tag{313}$$

$$= \begin{pmatrix} \overline{\partial}_B\psi_1 + \overline{\partial}^*_B\psi_2 + \sqrt{2}(\alpha \wedge K_1 - \overline{\alpha} \angle K_2) \\ \overline{\partial}\alpha - \omega_{\mathbb{C}}(K_i^M, \psi) \\ \overline{\partial}^*\alpha + h(K_i^M, \psi) \end{pmatrix} \tag{314}$$

We simplified the last term using the hermitian form $h(\cdot,\cdot) = \langle \cdot,\cdot \rangle + i\omega_I(\cdot,\cdot)$ on $TM$. In this form, the linearization of the generalized equation take the same form as the linearization of the linearized equations[26].

Thus, we can save us some work and use the same technique as in ([22], 3.3), where the linear case is proven:

**Lemma 42.** *Given a $U^{\mathbb{C}}(2)$-structure on $X$ defined by a line bundle $L_0$ and a solution $(u, a, is \cdot \omega, \phi)$ to the linear Seiberg-Witten equations defined by some $\alpha \in |L_0|$, then the orientation of the corresponding point in the moduli space is given by $(-1)^{\left(\dim H^0(X, L_0) - 1\right)}$[27].*

The idea of the proof of the lemma above is to notice that

$$F_0(\psi, \alpha) = \begin{pmatrix} \overline{\partial}_B\psi_1 + \overline{\partial}^*_B\psi_2 \\ \overline{\partial}\alpha \\ \overline{\partial}^*\alpha \end{pmatrix} \tag{315}$$

and then split the path to $F_1$ into one where one only has complex linear operators, which preserve the orientation, and one involving only complex antilinear operators which "flip" the orientation depending on the dimensions of the spaces involved, and compute these dimensions explicitly. For more details, consult [22].

**Corollary 12.** *Given a $U^{\mathbb{C}}(2)$-structure on $X$ defined by a line bundle $L$ and a solution $(u, a, \phi, is \cdot \omega)$ to the generalized Seiberg-Witten equations, let $L_0$ be the line*

[26]The only difference being that we use the hermitian form $h$ and the fundamental vector field on $M$, not the standard one on $\mathbb{C}^2$.

[27]In the reference, the proof is only given for elliptic surfaces, but the proof can easily be generalized for arbitrary projective surfaces.

*bundle such that* $\pi_! u^* TM^+ \simeq L_0 \otimes W^+$*. Then the orientation of the corresponding point in the moduli space is given by* $(-1)^{\left(\dim H^0(X,L_0)-1\right)}$.

## 4.6 Compactness

In the following section, we take a closer look at the topology of the moduli space with respect to variations of the Kähler structure.

Fix a smooth manifold $X$ and a smooth path of Kähler structures $(g_t, I_t), t \in [0,1]$ on X. Let $L$ be a line bundle over $X$, such for each complex structure $I_t$ it admits a holomorphic structure $\overline{\partial}_t$ , or equivalently, a path of connections $a_t$ such that $F^{2,0}_{a_t} = 0$, where the $(2,0)$-part is with respect to $I_t$. Furthermore we fix a path of perturbations $\phi_t \in \Gamma(X, K_X)$, $t \in [0,1]$, such that $\phi_t$ is holomorphic with respect to the complex structure $I_t$. Recall that for a fixed $t$, we identified the solutions to the generalized Seiberg-Witten equations with $\overline{\partial}_t$-holomorphic sections of $L$ (denoted by $|L|_t$) dividing $\phi_t^n$ and satisfying certain restrictions on indeterminacy sets. Denote by $|L|_{[0,1]} := \cup_{t\in[0,1]}|L|_t$ the collection of all the $I_t$-holomorphic sections with varying complex structures and endow it with the topology on $\Gamma(X, L)$ given by the supremum-norm.[28] As this is precisely the moduli space of the linear Seiberg-Witten equations with perturbation $\phi = 0$, we can use the following crucial fact from classical Seiberg-Witten theory:

**Theorem 26** ([22])**.** *The space* $|L|_{[0,1]}$ *is compact.*

Now, fix the metric on $M$ induced by the Riemannian metric. Define a topology on the space of solutions via the metric induced by the supremum norm, i.e. the metric is given by

$$d((u,a),(u',a')) = \sup_{x\in X}|u(x) - u'(x)| + |(a - a')(x)| \tag{316}$$

This topology does not depend on the specific metric on $X$ and $M$, respectively. Furthermore, it defines a topology on the moduli space via

$$d([u,a],[u',a']) = \inf_{\gamma_1,\gamma_2\in\mathcal{G}} \left\{ \sup_{x\in X} |\gamma_1(x)\cdot u(x) - \gamma_2(x)\cdot u'(x)| + |(\gamma_1^* a - \gamma_2^* a')(x)| \right\} \tag{317}$$

**Proposition 5.** The moduli space equipped with this topology is homeomorphic to the subset $|L|_{[0,1]}$ described above, equipped with the usual subset topology. In order to prove this, we need the following two lemmas:

[28] For this, we need to fix a metric on $L$, but the resulting topology does not depend on the choice of the metric.

**Lemma 43.** *Let $\alpha_l$ be a sequence of $\overline{\partial}_t$-holomorphic sections satisfying the conditions on the indeterminacy set described in Lemma (38), converging to $\alpha$, also satisfying the conditions. Then the maps $u_l$ into $M$ defined by $\alpha_l$ as described in Lemma (38) converge uniformly to $u$, the map defined by $\alpha$. On the other hand, if $u_l \to u$ is a sequence of $\overline{\partial}_t$-holomorphic maps, then the corresponding sections $\alpha_l$ converge uniformly to $\alpha$, the corresponding section to $u$.*

*Proof.* For each $x \in X$, we know that $x \in \overline{U'_k}$ for some $k = 1, ..., n$, where $U'_k$ is defined as in Lemma (38) for the section $\alpha$. Since $\alpha_l$ converges uniformly to $\alpha$, for $l$ big enough, there is a neighborhood around $x$ which is inside the set $U'_{k,l} := {U'_k}^{(u_{\alpha_l})}$, defined by $\alpha_l$ as. It follows directly from the local description of $u_l$ that $u_l \to u$ uniformly on this local neighborhoood and by compactness $u_l \to u$ on the whole of $X$. The other direction is similar. □

**Lemma 44.** *Assume $u_l \to u$ uniformly, where $u_l$ is holomorphic with respect to $\overline{\partial}_{t_l}$, while $u$ is holomorphic with respect to $\overline{\partial}_{t_0}$ and they satisfy the complex moment map condition $\mu_{\mathbb{C}} \circ u_l = \phi_{t_l}$ and $\mu_{\mathbb{C}} \circ u = \phi_{t_0}$ respectively.*

*Then the unique real gauge transformation $\lambda_l$ taking $(u_l, a_{t_l})$ to a solution to the perturbed GSW converges to $\lambda$ smoothly, the unique real gauge transformation taking $(u, a_t)$ to a solution.*

*Proof.* See Lemma (50) in the Appendix. □

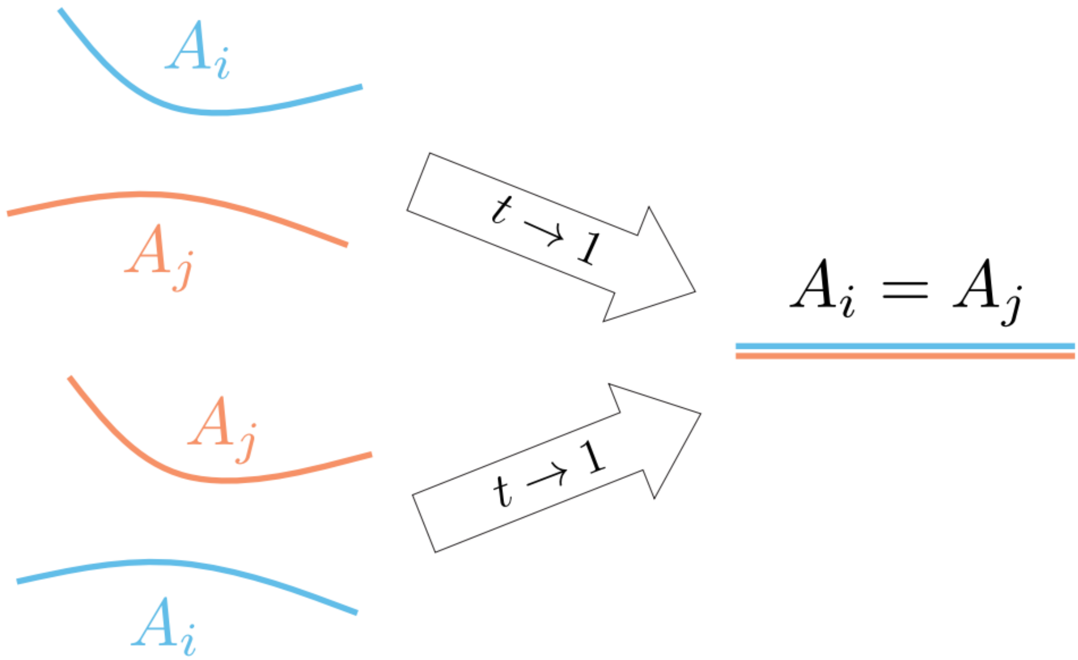
We can now prove the proposition:

*Proof.* Assume a sequence of elements $\alpha_l \in |L|_{[0,1]}$ converges to $\alpha$. Then $\alpha_l$ defines a map $u_l$ into $M$ as described in Lemma (38) converging smoohtly to $u$, and there exists a unique real gauge $\lambda_l$ such that $(\lambda_l.u_l, \lambda_l^* a_{t_l})$ defines a solution to the perturbed GSW. By the previous lemma, $\lambda_l \to \lambda$ smoothly, where $\lambda$ is the unique real gauge transforming $(u, a_{t_0})$ into a solution.
For the other direction, given a sequence of solutions $(u_l, a_l) \to (u, a)$, we know that $\alpha_l \to \alpha$ by the penultimate lemma. □

From this we see that the moduli space is not necessarily compact; The condition on the indeterminacy set is an open one. Indeed, it might happen that by varying $\phi_t$ the zero set of $\frac{\phi_t^k}{u_t}$ tends toward to the zero set of $\frac{u_t}{\phi_t^k}$ and if they intersect in the limit, it seizes to be a solution.

Let us be more precise: Assume that a collection of divisors $(A_0, ..., A_n)$ define a solution for the perturbation $\phi_0$, and assume they are pairwise disjoint. Further, let $L'$ be a line bundle over $X$ and assume that $\mathcal{O}(A_i) = \mathcal{O}(A_j) = L'$ for some $i \neq j$. Let $\gamma : [0, 1] \to |L|$ be path with $\gamma(0) = A_i, \gamma(1) = A_j$ such that $\gamma(t)$

does not intersect $A_i$ and $A_j$ for all $t \in (0,1)$. Now, the collection of divisors $(A_0, ..., A_{i-1}, \gamma(t), A_{i+1}, ..., A_n)$ defines a solution for a perturbation $\phi_t$, where $\phi_t$ is defined by replacing the $A_i$ component by $\gamma(t)$. But by switching $\gamma(t)$ and $A_j$, i.e. taking the collection $(A_0, ..., A_{i-1}, A_j, A_{i+1}, ..., A_{j-1}, \gamma(t), A_{j+1}, ..., A_n)$ we also obtain a solution for the perturbation $\phi_t$. In the limit $t \to 1$, these two different solutions either do not define a solution anymore (when $i$ and $j$ are not adjacent) or come together and define a solution given by $(A_0, ..., A_i, A_i, ..., A_n)$ when $i$ and $j$ are adjacent. Both cases give rise to compactness problems.




But this can be dealt by using only perturbations $\phi$ which are "generic enough":

**Corollary 13.** *The set*

$$\left\{(\phi) \in |K_X|,\ (\phi) = \sum_{i=0}^{n} A_i : A_i = A_j \text{ for some } i \neq j \text{ with } \dim H^0(X, \mathcal{O}(A_i)) \geq 2\right\} \tag{318}$$

*has real codimension at least 2 in* $|K_X|$.

*Proof.* This follows directly from the fact that the diagonal in $|\mathcal{O}(A_i)| \times |\mathcal{O}(A_i)|$ is at least a complex codimension 1 submanifold, so at least real codimension 2. □

Now, we only allow perturbations $\phi$ in the complement of this subset, and since it is of codimension 2, any generic path between two such perturbations will also miss the subset.

If $(\phi)$ has a fixed multiple component, i.e. there is some $A_i$ which appears twice and has $\dim H^0(X, \mathcal{O}(A_i)) = 1$, we allow all corresponding solutions where $A_i$ appears twice, as this component cannot bifurcate into different divisors by definition.

We can apply the whole discussion to the case where we vary the Kähler structure, with one small caveat:

Can we guarantee that for a path of perturbations $\phi_t = \sum_{i=0}^{n} (A_i)_t$ the numbers $\dim H^0(X, \mathcal{O}((A_i)_t))$ stay constant? Because if $\dim H^0(X, \mathcal{O}((A_i)_t)) \geq 2$ for all $t < t_0$ and $\dim H^0(X, \mathcal{O}((A_i)_{t_0})) \leq 1$, then the case described above is unavoidable!

As we will see in the next chapter, by looking at every possible case, this can not happen by examining each case individually.

We are now ready to define the invariants:

**Definition 37.** Let $\mathcal{M}'_n$ the moduli space of the perturbed generalized Seiberg-Witten equations with target space a MEH space with $n$ singularities. We define the **generalized Seiberg-Witten Invariants** of the Kähler manifold $X$ by

$$sw_n(X, Q) = \sum_{p \in \mathcal{M}'_n} \underbrace{\pm 1}_{\text{orientation of p}}$$

if $K_X$ admits a non-zero holomorphic section $\phi$, and set it to zero otherwise.

# 5 Invariants

In this chapter, we will explicitly compute the invariants arising from all the machinery we have developed so far. Again, we fix a simply connected, four dimensional, projective Kähler manifold $X$, where projective means it arises as a complex submanifold of some projective space, and the Kähler structure is the one induced by the standard Kähler structure on projective space. Such manifolds are also interesting from the viewpoint of algebraic geometry, since these are precisely the simply connected, two dimensional, complex, projective, smooth varieties. For the general facts stated in this chapter, we refer to [1] and [6]. It turns out that we have a rough algebraic classification of those, using the so called *Kodaira dimension*:

**Definition 38.** Denote by $P_n = \dim H^0(K_X^{\otimes n})$ the *n-th plurigena* of $X$. If $P_n = 0$ for all $n \in \mathbb{N}$ we set the *Kodaira dimension* $\kappa(X)$ to be $-\infty$, otherwise it is defined by

$$\kappa(X) = \text{ the smallest } \kappa \in \mathbb{N} \text{ such that } \frac{P_n}{n^\kappa} \text{ is bounded.} \tag{319}$$

In fact, surfaces can only have Kodaira dimensions equal to $-\infty, 0, 1, 2$. The restriction to simply-connected ones leaves us with a rather concrete picture, as we will see in a moment.

We call a complex surfaces *minimal* if it contains no smooth complex submanifolds biholomorphic to $\mathbb{CP}^1$ with self-intersection $-1$. These are precisely the surfaces which cannot be obtained as a blow-up of another surface.

**Theorem 27.** *([22], 3.1) Every complex surface $S$ with $\kappa(S) \geq 0$ has a unique minimal model $M$, i.e. there exists a minimal surface $M$ with $\kappa(M) = \kappa(S)$, such that $S$ is obtained by finitely many blow-ups of $M$.*

Therefore it makes sense to restrict ourselves to minimal surfaces first, and then see how blow-ups change the game.

## 5.1 Rational surfaces

Simply-connected surfaces with $\kappa(X) = -\infty$ are called rational surfaces.

These are surfaces which are birationally equivalent[29] to $\mathbb{CP}^2$, for example the surfaces $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. Since we demand that $h^0(K_X) > 1$ for our deformation of the generalized Seiberg-Witten equations, we cannot define invariants in this case, as these have $h^0(K_X^{\otimes n}) = 0$ for all $n \in \mathbb{N}$.

[29]Two surfaces are birationally equivalent if they are algebraically isomorphic up to a lower-dimensional subset.

## 5.2 K3-surfaces

A minimal simply-connected surface of Kodaira dimension zero is called a K3-surface. These satisfy $\mathcal{O}_X \simeq \underline{\mathbb{C}}$ which make the invariants very simple. Since the only holomorphic sections of the trivial bundle are constant, we choose the perturbation $\phi \equiv c, \; c \neq 0$. The only possibilty for a section of some line bundle $L$ to divide $\phi$ is if $L$ is trivial itself, since $\phi$ does not have a zero set. therefore the invariants for a K3-surface are

$$sw(X, L) = \begin{cases} 1 \text{ if } L = K_X = \underline{\mathbb{C}} \\ 0 \text{ else} \end{cases}$$

## 5.3 Elliptic Surfaces

We follow [6]: An elliptic surface is a complex surface $X$ together with a proper morphism $\pi : X \to C$, where $C$ is a curve and the generic fibre is a smooth curve of genus one, aka an elliptic curve. The simply-connected condition gives us a direct classification:

**Theorem 28.** *A minimal elliptic surface $X$ is simply connected if and only if $C = \mathbb{P}^1$ and there are at most two singular fibres which are multiple fibres, and if there are two multiple fibres, their multiplicities are coprime.*

A multiple fibre can be characterized as follows: Let $X_t := \pi^{-1}(t)$ be a singular fibre, and write $X_t = \sum n_i E_i$ as a divisor. We call the fibre $X_t$ a *multiple fibre* if the greatest common divisor of the coefficients $n_i$ is $m \geq 2$, and then $m \cdot X_t = f$, where $f$ is a generic fibre, and $m$ is called the *mutliplicity* of the fibre $X_t$. For the reader who does not like to involve themselves in the language of algebraic geometry, the following picture suffices:

Consider the map $z \mapsto z^m$ in the complex plane; a generic fibre is just a point with multiplicity $1$, but the fibre at $0$ has multiplicity $m$.

Furhermore, the canonical bundle can be described explicitly:

**Lemma 45.** *For an elliptic surface over $\mathbb{P}^1$ with at most two multiple fibres $F_1$ and $F_2$ with multiplicities $m_1, m_2$ respectively, and denoting by $f$ the generic fibre, the canonical bundle is given by*

$$K_X = \mathcal{O}_X((p_g - 1)f + (m_1 - 1)F_1 + (m_2 - 1)F_2) \tag{320}$$

*where* $p_g = h^{2,0}(X) = \frac{b_2^+(X)-1}{2}$.

A general fact about such simply-connected elliptic surfaces, which we will need later:

**Theorem 29** ([22], 3.3.11). *There exists a primitive class* $F \in H_2(X;\mathbb{Z})$ *such that*

$$f = m_1 m_2 F, \; F_1 = m_2 F, \; F_2 = m_1 F. \tag{321}$$

As discussed in the previous chapter, we choose

$$(\phi) = \sum_{j=1}^{p_g - 1} \pi^{-1}(b_j) + (m_1 - 1)F_1 + (m_2 - 1)F_2 \tag{322}$$

where $b_j \in \mathbb{P}^1$ are distinct points such that $\pi^{-1}(b_j)$ are regular fibres, and $F_1$ and $F_2$ are the two multiple fibres respectively.[30] Any other choice for $(\phi)$ is equivalent to choosing different $(p_g - 1)$ regular fibres, which does not change the invariants. It is clear that all possible solutions which might appear for any line bundle are decompositions of $\phi$ into $(n+1)$, possibly empty, algebraically disjoint pieces. To get all these possible configurations, notice that we can map the regular fibres $\pi^{-1}(b_j)$ arbitrarily into any of the generalized spheres, because they are set-wise disjoint to all the other divisors. For the multiple fibres, as discussed in the previous chapter, the only possibility is that $A_{l_1} = b_1' F_1$ and $A_{l_1+1} = b_1 F_1$ for some $l_1$, where $b_1' + b_1 = (m_1 - 1)$, similarly $A_{l_2} = b_2' F_2$ and $A_{l_2+1} = b_2 F_2$ for some $l_2$ and with $b_2' + b_2 = (m_2 - 1)$. Thus, in total there are

$$\underbrace{(n+1)^{(p_g-1)}}_{(a)} \cdot \underbrace{n m_1}_{(b)} \cdot \underbrace{n m_2}_{(c)}$$

solutions, where

(a) is the number of possibilities to distribute the $(p_g - 1)$ regular fibres given by $\pi^{-1}(b_1), \dots, \pi^{-1}(b_{p_g-1})$ into $n+1$ generalized spheres.

(b) is the the number of possibilities to distribute the $m_1 - 1$ singular fibres $F_1$ into two neighboring generalized spheres [31].

(c) is the the number of possibilities to distribute the $m_2 - 1$ singular fibres $F_2$ into two neighboring generalized spheres.

Now, to compute the invariant for a fixed line bundle $L$ turns out to be quite the combinatorial exercise: Knowing the explicit form of $\phi$, we know that $L$ must be of

[30] $\dim H^0(X, \mathcal{O}_X(F_1)) = \dim H^0(X, \mathcal{O}_X(F_2)) = 1$, so there is no freedom of choice.

[31] There are $n$ of them, each pair intersecting in one of the $n$ singularties.

the form

$$L = \mathcal{O}_X(c \cdot F),$$

where $c \in \mathbb{N}$ and $F$ is the primitive class. Using the defining properties of the primtive class, we see that

$$K_X = \mathcal{O}_X(((p_g-1)m_1m_2+2m_1m_2-m_1-m_2)F) = \mathcal{O}_X(((p_g+1)m_1m_2-m_1-m_2)F),$$

therefore we know that

$$0 \leq c \leq n \cdot (p_g+1)m_1m_2 - m_1 - m_2. \tag{323}$$

Suppose we have some solution for some $U^{\mathbb{C}}(2)$-structure defined by a line bundle $L$. We then denote by $a_k$ the number of regular fibres which are mapped into the $k-th$ generalized sphere. Since there are $p_g - 1$ total regular fibres, we have a partition $(a_0, a_1, ..., a_n)$ of $p_g - 1$, i.e. $\sum_{k=0}^{n} a_k = p_g - 1$. Furthermore, assume that $b_1'$ of the $(m_1 - 1)$ singular fibres $F_1$ are mapped into the $l_1 - th$ generalized sphere, and the remaining $b_1 = (m_1 - 1) - b_1'$ are mapped into the $(l_1 + 1) - th$ sphere, similarly denote the same numbers for $F_2$ by $l_2, b_2'$ and $b_2$ respectively. The corresponding line bundle $L$ will then be given by

$$L = \mathcal{O}_X(c \cdot F) \text{ where} \tag{324}$$

$$c = \sum_{k=0}^{n} k \cdot a_k \cdot m_1m_2 + \Big(l_1(m_1-1)+b_1\Big)m_2 + \Big(l_2(m_2-1)+b_2\Big)m_1 \ , \tag{325}$$

where we have simplified $l_j \cdot b_j' + (l_j + 1)b_j = l_j \cdot (m_j - 1) + b_j$ for $j = 1, 2$. Putting all this together, we have

**Corollary 14.** *Given* $c \in \mathbb{N}$ *with*

$$0 \leq c \leq n \cdot (p_g+1)m_1m_2 - m_1 - m_2,$$

*the number of solutions for the line bundle* $L = \mathcal{O}_X(c \cdot F)$*, where* $F$ *is the primitive* *class, is given by the number of solutions to the following system:*

$$a_k, b_j, l_j \in \mathbb{N}, \qquad k = 0, ..., n \;\; j = 1, 2 \tag{326}$$
$$0 \le a_k \le p_g - 1 \qquad 0 \le b_1 \le m_1 - 1 \tag{327}$$
$$0 \le b_2 \le m_2 - 1 \qquad 0 \le l_1, l_2 \le n \tag{328}$$
$$\sum_{k=0}^{n} a_k = p_g - 1 \tag{329}$$
$$c = \sum_{k=0}^{n} k \cdot a_k \cdot m_1 m_2 + \Big(l_1(m_1 - 1) + b_1\Big)m_2 + \Big(l_2(m_2 - 1) + b_2\Big)m_1 \tag{330}$$

As this involves counting numbers of partitions satisfying certain further relations, we stop here and leave more explicit computations to the future.

## 5.4 Surfaces of general type

We begin with a couple of useful properties:

**Theorem 30** (([1], 2.2))**.** *If* $X$ *is a minimal surface of general type, then* $K_X^2 > 0$.

**Proposition 6** (([1], 6.1))**.** If $X$ is a minimal surface of general type and $(\phi) \in |K_X|$, then $(\phi)$ is 1-connected, i.e. it cannot be written as as sum of effective divisors $(\phi) = D_1 + D_2$ such that $D_1 \cdot D_2 = 0$.

This means that the only possible decompositions of $(\phi)$ into non-intersecting divisors is $A_l = (\phi)$ for some $l$, and $A_j = 0$ for $j \ne l$. Therefore, there are precisely $n + 1$ solutions for a minimal surface of general type:

$$sw(X, L) = \begin{cases} (-1)^{h^0(X,L)-1} \text{ if } L = K_X^{\otimes l}\,,\; l = 0, ..., n \\ 0 \text{ else} \end{cases}$$

## 5.5 Blow-Ups

Taking into account blow-ups, we have to distinguish between two cases:

1. $|K_X|$ is not base point free, i.e. there is some $x_0 \in X$ such that $\phi(x_0) = 0$ for all $\phi \in |K_X|$ and we blow up at this particular $x_0$. For example, this happens if $X$ is elliptic and $x_0$ lies in one of the singular fibres.

2. We blow up at any point $y_0$ not lying in the base locus of $|K_X|$. Then for a generic $\phi \in |K_X|$, $\phi(y_0) \ne 0$.

In both cases, we have the following:

**Proposition 7** ([6])**.** The canonical bundle of the blow-up $\pi : \widehat{X} \to X$ is given by

$$K_{\widehat{X}} = \pi^* K_X + \mathcal{O}(E) \tag{331}$$

where $E$ denotes the exceptional divisor, satisfying $E^2 = -1$. Furthermore $\pi^* D \cdot E = 0$, where $D$ is any divisor on $X$.

We also have to take care of the sign in the invariants, which is determined by the underlying surface:

**Proposition 8** ([6])**.** Let $\pi : \widehat{X} \to X$ be a blow-up at a single point, then one has for $n \in \mathbb{N}$:

$$h^0(X, K_X^{\otimes n}) = h^0(\widehat{X}, K_{\widehat{X}}^{\otimes n}) \tag{332}$$

Furthermore $\mathcal{O}(mE)$ only has one section for all $m \in \mathbb{N}$.

Now in the second case mentioned above, let $(\phi)$ be any divisor representing $K_X$. Then $K_{\widehat{X}}$ can be represented by $(\pi^*\phi) + E$, which is a decomposition into disconnected components, therefore all solutions are of the following form[32]

$$sw(\widehat{X}, L) = \begin{cases} sw(X, L') \text{ if } L = \pi^* L' \otimes \mathcal{O}(mE) \ , \ m = 0, ..., n \\ 0 \text{ else} \end{cases}$$

In the first case we have a restriction: Although their algebraic intersection is zero, the set theoretic intersection of $(\pi^*\phi)$ and $E$ is always non-empty. Let $C$ be the component of $(\pi^*\phi)$ intersecting $E$. We only obtain solutions if $C$ and $E$ are mapped into neighboring spheres, or in other words, if there is a solution which contributes to the count $sw(X, L')$ and $C$ is mapped into the $l$'th sphere, then this solution contributes once to each of the counts for $sw(\widehat{X}, \pi^* L' \otimes \mathcal{O}(mE))$ with $m = l - 1, l, l$. If $C$ is mapped into the rightmost copy of $\mathbb{C}$, then it contributes to $sw(\widehat{X}, \pi^* L')$ and $sw(\widehat{X}, \pi^* L' \otimes \mathcal{O}(E))$ once, and if its mapped to the leftmost copy of $\mathbb{C}$, it contributes to $sw(\widehat{X}, \pi^* L' \otimes \mathcal{O}((n+1)E))$ and $sw(\widehat{X}, \pi^* L' \otimes \mathcal{O}(nE))$. For general type surfaces, this count is quite straightforward, while for elliptic surfaces this involves some combinatorics again.

[32] We take a decomposition of $(\phi)$, and then we have the freedom to map the exceptional divisor to any of the $(n+1)$ spheres.

# Conlusion and Outlook

We have finally computed the invariants, which essentially depend on the Kodaira dimension, the multiplicity of the fibres in the elliptic case, and the number and type of $(-1)$-curves in the surface, or equivalently, the information on how to obtain this surface from a minimal model by iterative blow-ups.

Of course the story does not end here; one of the crucial assumptions was that our line bundles $L$ carry a holomorphic structure. A first step would be to generalize the results above to non-holomorphic line bundles, entering the realm of non-integrable complex structures. Another step in this direction is to consider not only Kähler manifolds, but symplectic ones carrying a non-integrable complex structure. It was shown by Taubes (see [29]) that in the linear case, one obtains substantially the same result for symplectic manifolds when considering a limiting case of the perturbations we are using. Therefore, it is not a long shot to speculate a little bit:

**Conjecture 1.** Let $X$ be a symplectic manifold with a compatible almost complex structure $J$, and let $K_X$ be the corresponding canonical bundle. If $C$ is any pseudo-holomorpic curve[33] representing $K_X$, then the number of decompositions of $C$ into algebraically disjoint components is an invariant of $X$.

The results by Taubes are obtained by proving certain estimates on the curvatures and derivatives of spinors involved, which is hard to do in the non-linear case. This is related to the compactness problem; in fact we have proven compactness of the moduli space by explicitly computing it and going through every single possible case. It would be nice to have a more general abstract argument as in the linear case( where compactness is proven by showing that the norm of the spinor is bounded by the scalar curvature, and then applying a bootstrapping argument), shedding more light on what is happening in the symplectic case as well.

Yet another avenue which can be explored is the one of Furuta-Bauer invariants, see [7] and [2]:

Instead of only focusing on the solutions space, which is given by $sw^{-1}(0)$ and trying to extract information from it, one can investigate the map $sw$ itself. Again, the non-linearity makes this Ansatz much more delicate compared to the linear case.

[33]A pseudo-holomorphic curve $C \subset X$ is an embedded complex surface $f : C \to X$ such that the embedding map $f$ commutes with the complex structures.

Another interesting consideration arises if one views the MEH-space as a hyper-Kähler quotient. As we have seen, the solutions we obtain correspond to particular solutions of the multi-spinor Seiberg-Witten equations. As the overall solution space and its compactness problems in the multi-spinor case are not well explored yet, it would be interesting to investigate this correspondence further.

# Appendix A: Weitzenböck Formula

We follow [4]:

Let us compute the Dirac operator locally in the Kähler case. Let $e_1, e_2, e_3, e_4$ be an local orthonormal frame of TX, $\{e^i\}_{i=1,2,3,4}$ the corresponding dual frame such that $dz_1 = e^1 + ie^2$ and $dz_2 = e^3 + ie^4$. Let $\{\tilde{e}_i\}_{i=1,2,3,4}$ be the unique lifts to $TQ$ with respect to a connection $A \in \mathcal{A}$. Given a generalized spinor $u$, we have $\nabla^A u = \sum_{i=1}^4 e^i \otimes Tu(\tilde{e}_i)$, which we identify with

$$\nabla^A u = \sum_{i=1}^{4} e^i \otimes \left(\pi^{1,0}(Tu(\tilde{e}_i)) \otimes 1 - \pi^{1,0}(I_2 Tu(\tilde{e}_i))\right) \otimes d\bar{z}_1 \wedge d\bar{z}_2 \,. \tag{333}$$

Then applying Clifford multiplication yields

$$\not{D}^A u = \sum_{i=1}^{4} \pi^{1,0}(Tu(\tilde{e}_i)) \otimes \sqrt{2}\pi^{0,1}(e^i) \cdot 1 + \pi^{1,0}(I_2 Tu(\tilde{e}_i)) \otimes \sqrt{2}\pi^{0,1}(e^i) \angle d\bar{z}_1 \wedge d\bar{z}_2 \,. \tag{334}$$

Using

$$\pi^{0,1}(e^1) = \frac{1}{2} d\bar{z}_1, \pi^{0,1}(e^2) = \frac{1}{2} i d\bar{z}_1, \pi^{0,1}(e^3) = \frac{1}{2} d\bar{z}_2, \pi^{0,1}(e^4) = \frac{1}{2} i d\bar{z}_2 \tag{335}$$

we compute:

$$\begin{aligned}
&\not{D}^A u = \\
&\pi^{1,0}(Tu(\tilde{e_1})) \otimes \frac{1}{\sqrt{2}} d\bar{z}_1 + \pi^{1,0}(I_2 Tu(\tilde{e_1})) \otimes \frac{1}{\sqrt{2}} d\bar{z}_2 + \\
&\pi^{1,0}(Tu(\tilde{e_2})) \otimes \frac{i}{\sqrt{2}} d\bar{z}_1 - \pi^{1,0}(I_2 Tu(\tilde{e_2})) \otimes \frac{i}{\sqrt{2}} d\bar{z}_2 + \\
&\pi^{1,0}(Tu(\tilde{e_3})) \otimes \frac{1}{\sqrt{2}} d\bar{z}_2 - \pi^{1,0}(I_2 Tu(\tilde{e_3})) \otimes \frac{1}{\sqrt{2}} d\bar{z}_1 + \\
&\pi^{1,0}(Tu(\tilde{e_4})) \otimes \frac{i}{\sqrt{2}} d\bar{z}_2 + \pi^{1,0}(I_2 Tu(\tilde{e_4})) \otimes \frac{i}{\sqrt{2}} d\bar{z}_1 \,.
\end{aligned}$$

To obtain a norm estimate for the Dirac operator, we pull the hermitian product $h = g_M + i\omega_{I_1}$ on $TM$ back to $T^{1,0}_{I_1}M$. Using the local description of the Dirac operator and abbreviating $x_i := Tu(\tilde{e}_i)$, while making use of the relation $\pi^{1,0}(v) \otimes i \cdot z = I_1 \pi^{1,0}(v) \otimes z = \pi^{1,0}(I_1 v) \otimes z$, we obtain:

$$
\begin{aligned}
&\|\not{D}^A u\|^2_{h\otimes g_X} = \frac{1}{2}(\|x_1 + I_1x_2 - I_2x_3 + I_3x_4\|^2_h + \|x_3 + I_1x_4 + I_2x_1 - I_3x_2\|^2_h)\\
=&\frac{1}{2}\|x_1 + I_1x_2 - I_2x_3 + I_3x_4\|^2_h + \frac{1}{2}\| - I_2(x_3 + I_1x_4 + I_2x_1 - I_3x_2)\|^2_h\\
=&\frac{1}{2}\|x_1 + I_1x_2 - I_2x_3 + I_3x_4\|^2_h + \frac{1}{2}\|x_1 + I_1x_2 - I_2x_3 + I_3x_4\|^2_h\\
=&g_M(x_1 + I_1x_2 - I_2x_3 + I_3x_4, x_1 + I_1x_2 - I_2x_3 + I_3x_4)\\
=&(\sum_{i=1}^4 g_M(x_i, x_i) + 2\omega_I(x_1, x_2) + 2\omega_I(x_3, x_4) - 2\omega_J(x_1, x_3) + 2\omega_J(x_2, x_4)\\
&+ 2\omega_K(x_1, x_4) + 2\omega_K(x_2, x_3))\\
=&\|\sum_{i=1}^4 Tu(\tilde{e}_i) \otimes e^i\|^2_{g_M\otimes g_X} + 2(\langle\omega_I(\nabla^A u, \nabla^A u), \omega_X\rangle_{\Lambda^\bullet(X)}+\\
&2(\langle\omega_J(\nabla^A u, \nabla^A u), -\eta_2\rangle_{\Lambda^\bullet(X)} + 2(\langle\omega_K(\nabla^A u, \nabla^A u), \eta_3\rangle_{\Lambda^\bullet(X)}\\
=&\|\nabla^A u\|^2 + 2\omega_I(\nabla^A u, \nabla^A u) \wedge *(\omega_X)-\\
&2\,\mathrm{Re}(\langle\omega_J(\nabla^A u, \nabla^A u) + i\omega_K(\nabla^A u, \nabla^A u), \eta_2 + i\eta_3\rangle_{\Lambda^\bullet(X,\mathbb{C})})\\
=&\|\nabla^A u\|^2 + \omega_I(\nabla^A u, \nabla^A u) \wedge \omega_X-\\
&2\,\mathrm{Re}(\langle\omega_J(\nabla^A u, \nabla^A u) + i\omega_K(\nabla^A u, \nabla^A u), dz_1 \wedge dz_2\rangle_{\Lambda^\bullet(X,\mathbb{C})})\\
=&\|\nabla^A u\|^2 + \omega_I(\nabla^A u, \nabla^A u) \wedge \omega_X-\\
&2\,\mathrm{Re}((\omega_J(\nabla^A u, \nabla^A u) + i\omega_K(\nabla^A u, \nabla^A u)) \wedge d\bar{z}_1 \wedge d\bar{z}_2)
\end{aligned}
$$

Notice that we made use of $\mathrm{vol}(X) = \frac{1}{2}\omega_X \wedge \omega_X$ in the last line. To further simplify the above expression, we take a closer look at the terms $\omega_J(\nabla^A u, \nabla^A u)$ and $\omega_K(\nabla^A u, \nabla^A u)$. Since we might not have a permuting action of $Sp(1)$, the total hyperKähler-form might not be exact, but a permuting $S^1$ action will at least yield exactness of $\omega_J$ and $\omega_K$.

**Lemma 46.** *For $\zeta, \eta \in \mathfrak{sp}(1)$, such that $\zeta$ induces the fundamental vector field of the permuting $S^1$ action, we have*

$$\mathcal{L}_\zeta \omega_\eta = \omega_{[\zeta,\eta]}\,. \tag{336}$$

*Proof.*

$$\begin{aligned}\mathcal{L}_\zeta\omega_\eta &= \frac{d}{dt}_{|t=0}(\exp(-t\zeta))^* g(\cdot, I_\eta \cdot)\\ &= \frac{d}{dt}_{|t=0}\, g(d\exp(-t\zeta)\cdot, I_\eta d\exp(-t\zeta)\cdot)\\ &= \frac{d}{dt}_{|t=0}\, g(\cdot, I_{\exp(t\zeta)\cdot\eta\cdot\exp(-t\zeta)}\cdot) = g(\cdot I_{ad(\zeta)\eta}\cdot) = \omega_{[\zeta,\eta]}\end{aligned}$$

□

In particular we have $\frac{1}{2}\mathcal{L}_i\omega_J = \omega_K$ and $\frac{1}{2}\mathcal{L}_{(-i)}\omega_K = \omega_J$. Thus we compute with the Cartan formula:

$$\omega_K = \frac{1}{2}\mathcal{L}_i\omega_J = \frac{1}{2}(d\iota(K_i^M)\omega_J + \iota(K_i^M)d\omega_J) = \frac{1}{2}d\iota(K_i^M)\omega_J\,. \tag{337}$$

Similarly, $\omega_J = \frac{1}{2}d\iota(K_{-i}^M)\omega_K$, so for $\gamma_K = \frac{1}{2}\iota(K_i^M)\omega_J$ and $\gamma_J = \frac{1}{2}\iota(K_{-i}^m)\omega_K$ we have $d\gamma_K = \omega_K, d\gamma_J = \omega_J$. We can now compute $\omega_J(\nabla^A_{e_i}u, \nabla^A_{e_j}u)$.
Therefore, choose normal coordinates at some point $p \in U \subseteq X$ and denote the corresponding frame by $\{e_i\}_{i=1,2,3,4}$, and a corresponding section $\tilde{h} : U \to P_{U(2)}$. Choosing a section $g : U \to P_{\hat{G}}$, we can lift the section $\tilde{h} \times g$ to a section $h : U \to Q$. Further we denote the invariant extension of $Th(e_i)$ on $Q$ with $\hat{e_i}$, i.e. $\hat{e_i}(h(p).g) = TR_g(Th(e_i)(h(p))$. We will use the following lemma.

**Lemma 47.** *For $\alpha \in \Omega^1(Q)$, $Y$ and $Z$ vector fields on Q, we have*

$$\begin{aligned}&d\alpha(pr_{horA}(Y), pr_{horA}(Z)) = d(\alpha \circ pr_{horA})(Y, Z) + \frac{1}{2}(-K^Q_{A(Y)}(\alpha(pr_{horA}(Z)))\\ &+ K^Q_{A(z)}(\alpha(pr_{horA}(Y))) + \alpha(pr_{horA}([Y, Z]) - [pr_{horA}X, pr_{horA}Y]))\end{aligned}$$

*Proof.* On the one hand,

$$\begin{aligned}d\alpha(pr_{horA}(Y), pr_{horA}(Z)) &= \frac{1}{2}(pr_{horA}(Y)(\alpha(pr_{horA}(Z)))\\ &- pr_{horA}(Z)(\alpha(pr_{horA}(Y))) - \alpha([pr_{horA}(Y), pr_{horA}(Z)]))\end{aligned}$$

On the other hand,

$$d(\alpha \circ pr_{horA})(Y, Z) = \frac{1}{2}(Y(\alpha(pr_{horA}(Z))) - Z(\alpha(pr_{horA}(Y))) - \alpha \circ pr_{horA}([Y, Z]))$$

Further using $Y - pr_{horA}(Y) = K^Q_{A(Y)}$ yields the above formula. □

We can now compute

$$h^*\omega_J(\nabla^A_{e_i}u,\nabla^A_{e_j}u)(p) = d\gamma_J(Tu\circ pr_{horA}\mathrm{Th}(e_i), Tu\circ pr_{horA}\mathrm{Th}(e_j))(u(h(p))) =$$
$$d(u^*\gamma_J)(pr_{horA}\mathrm{Th}(e_i), pr_{horA}\mathrm{Th}(e_j))(h(p)) = d(u^*\gamma_J\circ pr_{horA})(\mathrm{Th}(e_i),\mathrm{Th}(e_j))(h(p))$$
$$+\frac{1}{2}(-K^Q_{A(\mathrm{Th}(e_i))}(u^*\gamma_J(pr_{horA}(\mathrm{Th}(e_j))) + K^Q_{A(\mathrm{Th}(e_j))}(u^*\gamma_J(pr_{horA}(\mathrm{Th}(e_i)))(h(p))+$$
$$u^*\gamma_J(pr_{horA}[\mathrm{Th}(e_i),\mathrm{Th}(e_j)] - [pr_{horA}\mathrm{Th}(e_i), pr_{horA}\mathrm{Th}(e_j)]])(h(p))$$

In the last term, $[\mathrm{Th}(e_i),\mathrm{Th}(e_j)] = \mathrm{Th}([e_i,e_j]) = 0$ and since $pr_{horA}(\mathrm{Th}(e_i)) = \tilde{e_i}$, where $\tilde{e_i}$ is the unique horizontal lift of $e_i$ with respect to A, and $pr_{horA}([\tilde{e_i},\tilde{e_j}]) = \widetilde{[e_i,e_j]} = 0$ we get that $([pr_{horA}\mathrm{Th}(e_i), pr_{horA}\mathrm{Th}(e_j)])$ is vertical, thus its equal to $-K^M_{F^A(\mathrm{Th}(e_i),\mathrm{Th}(e_j))}$.

Studying the next term, we see:

$$\begin{aligned}
&K^Q_{A(\mathrm{Th}(e_i))}(u^*\gamma_J(pr_{horA}(\mathrm{Th}(e_j)))(h(p)) = K^Q_{A(\mathrm{Th}(e_i))}(u^*\gamma_J(\tilde{e_j}))(h(p))\\
&= \frac{d}{dt}_{|t=0} u^*\gamma_J(\tilde{e_j})(h(p).\exp(tA(\mathrm{Th}(e_i))))\\
&= \frac{d}{dt}_{|t=0} u^*\gamma_J(R_{\exp(tA(\mathrm{Th}(e_i)))}\tilde{e_j})(h(p)) = \frac{d}{dt}_{|t=0} R^*_{\exp(tA(\mathrm{Th}(e_i)))}u^*\gamma_J(\tilde{e_j})(h(p))\\
&= \frac{d}{dt}_{|t=0} u^*L^*_{\exp(-tA(\mathrm{Th}(e_i)))}\gamma_J(\tilde{e_j})(h(p)) = u^*\mathcal{L}_{A(\mathrm{Th}(e_i))}\gamma_J(\tilde{e_j})(h(p))\\
&= u^*\mathcal{L}_{i\cdot A_0(\mathrm{Th}(e_i))}\gamma_J(\tilde{e_j})(h(p))\,.
\end{aligned}$$

Where $i\cdot A_0$ is the (permuting) $i\mathbb{R}$ part of the connection A and the last line follows, since $\gamma_J$ is invariant under the actions of $Sp(1)_-$ and $G$. Using Lemma (46) we get $\mathcal{L}_{i\cdot A_0(\mathrm{Th}(e_i))}\gamma_J = 2A_0(\mathrm{Th}(e_i))\cdot\gamma_K$, but $A$ is induced by the Levi-Civita connection and the $e_i$ are normal coordinates in p, thus the connection term vanishes.

To conclude, we have simplified:

$$\begin{aligned}
&h^*\omega_J(\nabla^A_{e_i}u,\nabla^A_{e_j}u)(p)\wedge d\bar{z}_1\wedge d\bar{z}_2 =\\
&d(h^*u^*\gamma_J\circ pr_{horA})(e_i,e_j)\wedge d\bar{z}_1\wedge d\bar{z}_2(p) + \frac{1}{2}h^*u^*\gamma_J(K^M_{F^A(e_i,e_j)})\wedge d\bar{z}_1\wedge d\bar{z}_2(p)\,.
\end{aligned}$$

We can simplify $\omega_K(\nabla^A_{e_i}u,\nabla^A_{e_j}u)(p)$ in the same manner. Notice that the forms are actually independent of the choice of normal coordinates, so they are defined globally. Further investigating the non-exact term, we notice that $(\gamma_J+i\gamma_K)(Tu(K^M_{F^A}))(u(p))\wedge d\bar{z}_1\wedge d\bar{z}_2 = (\gamma_J+i\gamma_K)(-K^Q_{F^A})(u(p))\wedge d\bar{z}_1\wedge d\bar{z}_2 = \langle\mu_{\mathbb{C}}\circ u, -F^A\rangle_{\mathfrak{g}\otimes\mathbb{C}}\wedge d\bar{z}_1\wedge d\bar{z}_2(p) = -\langle\mu_{\mathbb{C}}\circ u, (F_a)^{2,0}\rangle_{\mathfrak{g}\otimes\mathbb{C}}\cdot vol_X(p)$, where $\langle\cdot,\cdot\rangle_{\mathfrak{g}\otimes\mathbb{C}}$ induces a scalar product by taking its real part and $a$ is the (hyperKähler) $i\cdot\mathbb{R}$ part of the connection $A$. Integration yields the Weitzenböck formula, using Stokes theorem:

**Theorem 31** (Weitzenböck formula)**.**

$$\int_M \|\not{D}^A u\|^2 * 1 = \int_M |\nabla^A u|^2 * 1 + \omega_I(\nabla^A u, \nabla^A u) \wedge \omega_X + \langle \mu_{\mathbb{C}} \circ u, (F_a)^{2,0} \rangle * 1$$

# Appendix B: Coordinates in Gibbons-Hawking Space

For the coordinates $\psi_k$ around the singularity $a_k$ and the coordinates $\psi_{k+1}$ around $a_{k+1}$, we compute the change of coordinates:

**Lemma 48.**

$$\psi_{k+1}^{-1} \circ \psi_k : (e^{-u_{reg,k}} \cdot z_1, e^{u_{reg,k}} \cdot z_2) \mapsto (e^{-u_{reg,k}-u'_k} \cdot z_1^2 \cdot z_2, e^{u_{reg,k}+u'_k} \cdot \frac{1}{z_1}).$$

*Proof.* Given a trivialisation $(x_1, x_2, x_3, e^{it})$ of the $S^1$ principal bundle around $a_k$ inducing coordinates

$$(z_1, z_2) = \left( (|x - a_k \cdot e_1| - (x_1 - a_k))^{1/2} e^{i\varphi/2} e^{it}, (|x - a_k \cdot e_1| + x_1 - a_k)^{1/2} e^{i\varphi/2} e^{-it} \right)$$

where we used $z = x_2 + ix_3 = |z| \cdot e^{i\varphi}$ and a trivialisation $(x_1, x_2, x_3, e^{it'})$ around $a_{k+1}$ inducing coordinates

$$(z'_1, z'_2) = \left( (|x - a_{k+1} \cdot e_1| - (x_1 - a_{k+1}))^{1/2} e^{i\varphi/2} e^{it'}, (|x - a_{k+1} \cdot e_1| + x_1 - a_{k+1})^{1/2} e^{i\varphi/2} e^{-it'} \right)$$

Using the standard trivialisations, we find that $e^{it'} = e^{i(t+\varphi)}$ as this is the change of coordinates of the Hopf bundle from north to south pole.

We make the following computation:

$$\begin{aligned}
(z'_1, z'_2) &= \left( (|x - a_{k+1} \cdot e_1| - (x_1 - a_{k+1}))^{1/2} e^{i\varphi/2} e^{it'}, (|x - a_{k+1} \cdot e_1| + x_1 - a_{k+1})^{1/2} e^{i\varphi/2} e^{-it'} \right) \\
&= \left( (|x - a_{k+1} \cdot e_1| - (x_1 - a_{k+1}))^{1/2} e^{\frac{3}{2} i\varphi} e^{it}, (|x - a_{k+1} \cdot e_1| + x_1 - a_{k+1})^{1/2} e^{-i\frac{1}{2}\varphi} e^{-it} \right) \\
&= \Bigg( \frac{z \cdot z_1}{\left( (|x - a_{k+1} \cdot e_1| + (x_1 - a_{k+1}) \cdot (|x - a_k \cdot e_1| - (x_1 - a_k) \right)^{1/2}}, \\
&\quad \frac{\left( (|x - a_{k+1} \cdot e_1| + (x_1 - a_{k+1}) \cdot (|x - a_k \cdot e_1| - (x_1 - a_k) \right)^{1/2} \cdot z_2}{z} \Bigg)
\end{aligned}$$

So using $b = \Big( (|x - a_{k+1} \cdot e_1| + (x_1 - a_{k+1}) \cdot (|x - a_k \cdot e_1| - (x_1 - a_k) \Big)^{1/2}$, we compute:

$$\begin{aligned} b &= \Big( (|x - a_{k+1} \cdot e_1| + (x_1 - a_{k+1}) \cdot (|x - a_k \cdot e_1| - (x_1 - a_k) \Big)^{1/2} \\ &= \exp\Big( \frac{1}{2} \ln(|x - a_{k+1} \cdot e_1| + (x_1 - a_{k+1})) + \frac{1}{2} \ln(|x - a_k \cdot e_1| - (x_1 - a_k)) \Big) \\ &= \exp\Big( \int_{a_k}^{x_1} \frac{1}{2\|(q, x_2, x_3) - a_{k+1} \cdot e_1\|} dq - \int_{a_{k+1}}^{x_1} \frac{1}{2\|a_k \cdot e_1 - (q, x_2, x_3)\|} dq \Big) \\ &= \exp\Big( \int_{a_k}^{x_1} V_{reg,k} dq - \int_{a_{k+1}}^{x_1} V_{reg,k+1} + \int_{a_{k+1}}^{a_k} V'_{reg} dq \Big) = \exp\Big( - u_{reg,k+1} + u_{reg,k} + u'_k \Big) \end{aligned}$$

where we use $V'_{reg} = V - \frac{1}{4\pi\|x - a_{k+1} \cdot e_1\|} - \frac{1}{4\pi\|x - a_k \cdot e_1\|}$ and $u'_k := 2\pi \int_{a_{k+1}}^{a_k} V'_{reg}$ so in conclusion:

$$(z'_1, z'_2) = \Big( e^{u_{k+1} - u_k - u'_k} \cdot z_1 \cdot z, e^{-(u_{k+1} - u_k - u'_k)} \cdot \frac{z_2}{z} \Big) = \Big( e^{u_{k+1} - u_k - u'_k} \cdot z_1^2 \cdot z_2, e^{-(u_{k+1} - u_k - u'_k)} \cdot \frac{1}{z_1} \Big)$$

Using the holomorphic charts $\psi_k^{-1} : M_k \to \mathbb{C}^2, (z_1, z_2) \mapsto (e^{-u_{reg,k}} z_1, e^{u_{reg,k}} z_2)$ we compute the change of charts:

$$\psi_{k+1}^{-1} \circ \psi_k = (e^{-u_{reg,k}} z_1, e^{u_{reg,k}} z_2) \mapsto (z_1, z_2) \mapsto$$
$$\Big( e^{u_{reg,k+1} - u_{reg,k} - u'_k} \cdot z_1^2 \cdot z_2 \,,\; e^{-(u_{reg,k+1} - u_{reg,k} - u'_k)} \cdot \frac{1}{z_1} \Big) \mapsto \Big( e^{-u_{reg,k} - u'_k} \cdot z_1^2 \cdot z_2 \,,\; e^{u_{reg,k} + u'_k} \cdot \frac{1}{z_1} \Big).$$

□

# Appendix C: Non-Linear Elliptic Equation

We are considering a nonlinear- partial differential equation on a compact Riemannian manifold of the form

$$\Delta\lambda(x) + F(x, \lambda(x)) = g(x) \tag{338}$$

where $g : M \to \mathbb{R}$ and $F : M \times \mathbb{R} \to \mathbb{R}$ are smooth functions. We make the following assumptions on $F$:

1. $F(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is strictly increasing for all $x \in M$ besides on a null set, in which we assume $F$ only to be increasing.

2. We have

   $$\lim_{\lambda\to\infty} F(x, \lambda) = \infty \text{ and } \lim_{\lambda\to-\infty} F(x, \lambda) = -\infty$$

   for all $x \in M$ besides on a null set.

We denote the *mean* of a function $g$ by $\overline{g} = \int_M \frac{g}{vol(M)} dvol$. We notice that we can simplify the differential equation above:
Let $\nu$ be a solution to $\Delta\nu = \overline{g} - g$. Assume $\lambda$ solves (338). Then we have $\Delta(\lambda + \nu) = \overline{g} - F(x, \lambda) = \overline{g} - G(x, \lambda + \nu)$, where we define $G(x, u) = F(x, u - \nu)$. The function $G$ also satisfies the two conditions above, so it is sufficient to find solutions for the equation

$$\Delta\lambda + G(x, \lambda) = c \tag{339}$$

where $c \in \mathbb{R}$ is now a constant. We use the following tools from [22]:

**Theorem 32** (Method of lower and upper solutions)**.** *Suppose there are functions smooth functions $u$ and $U$ on $M$ such that*

- $u(x) \leq U(x)$ *for all* $x \in M$.
- $\Delta u \leq c - G(x, u(x))$ *for all* $x \in M$.
- $\Delta U \geq c - G(x, U(x))$ *for all* $x \in M$.

*Then there exists a solution a solution $\lambda$ to (339) with $u \leq \lambda \leq U$.*

**Theorem 33** (Comparison principle)**.** *Suppose $G$ satisfies the first property above, i.e. is strictly increasing in the second argument for almost all $x \in M$. Then*

$$\Delta u + G(x, u) \geq \Delta\nu + G(x, \nu) \implies u \geq \nu.$$

We can now show:

**Lemma 49.** *Let* $G$ *be as above, then the problem (*$339$*) has a unique solution* $\lambda$*.*

*Proof.* The function $h : \lambda \mapsto \overline{G(x,\lambda)} = \int_M \frac{G(x,\lambda)}{\mathrm{vol}(M)} d\mathrm{vol}$ is continuous and satifies $\lim_{\lambda\to\infty} h(\lambda) = \infty$ and $\lim_{\lambda\to-\infty} h(\lambda) = -\infty$ (by the second property of $G$ above). By the intermediate value theorem, there is a $\lambda_0 \in \mathbb{R}$ such that $c = h(\lambda_0) = \overline{G(x,\lambda_0)}$. Let $\nu$ be a solution of $\Delta\nu = G(x,\lambda_0) - c$. Further, choose $\lambda^+ \in \mathbb{R}$ such that $\nu(x) + \lambda^+ > \lambda_0$ and $\lambda^- < \lambda^+$ such that $\nu(x) + \lambda^- < \lambda_0$ for all $x \in M$. Then $u_\pm = \nu + \lambda^\pm$ satifies

$$\Delta u_\pm + c - G(x, u_\pm) = G(x,\lambda_0) - G(x, \nu + \lambda^\pm)$$

which is, since $G$ is increasing everywhere, bigger (respectively smaller) or equal to zero, which means that $u_+$ is an upper solution, $u_-$ a lower solution, which satisfy $u_- \leq u_+$. Therefore, there exists a solution to (339).
Its uniqueness follows directly from the comparison principle. □

We also would like to have some sort of continuity of the solution $\lambda$ depending on the function $F$:

**Lemma 50.** *Assume* $F_n \to F$ *uniformly, where each* $F_n$ *satisfies the conditions above. Then the unique solution* $\lambda_n$ *to the differential equation defined with* $F_n$ *converges to* $\lambda$*, the unique solution of the differential equation defined by* $F$*.*

*Proof.* This follows from the implicit function theorem: Define

$$T(\lambda, F) = \Delta(\lambda) + F(x,\lambda) - g$$

Then $\frac{\partial T}{\partial \lambda}(\lambda', 0) = \Delta(\lambda') + \frac{\partial F}{\partial \lambda} \cdot \lambda'$. By assumption, we have $\frac{\partial F}{\partial \lambda} \geq 0$. Asumme there is $\varphi$ such that $\varphi \perp \mathrm{Im}\,\frac{\partial T}{\partial \lambda}$, but this would imply

$$0 = \int_X \varphi \cdot \Big(\Delta(\varphi) + \frac{\partial F}{\partial \lambda}\varphi\Big) d\mathrm{vol} = \int_X |\nabla\varphi|^2 + \frac{\partial F}{\partial \lambda} \cdot |\varphi|^2 d\mathrm{vol}$$

which implies $\varphi \equiv 0$, since $\frac{\partial F}{\partial \lambda}$ is non-vanishing up to a null-set. Therefore the map is onto, and by the implicit function theorem for $(\lambda, F)$ a solution to $T(\lambda, F) = 0$, there is a neighborhood such that $\lambda = \lambda(F)$ is continious. In particular if $F_n \to F$, then $\lambda_n \to \lambda$. □

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