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Generalizations of Quandles to Multi-Linkoids

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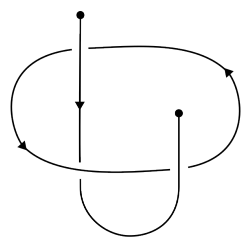
Abstract

In this thesis, we define the fundamental quandle of knotoids and linkoids and prove that it is invariant under the under forbidden-move and hence encodes only the information of the underclosure of the knotoid.
We then introduce n-pointed quandles, which generalize quandles by specifying n elements as ordered basepoints. This leads to the notion of fundamental pointed quandles of linkoids, which enhances the fundamental quandle. Using 2-pointed quandle allows us to distinguish 1-linkoids with equivalent under-closures and leads to a couple of 1-linkoid invariants. In particular we study implications on the 2-cocyle invariant. We then define n-pointed biquandles in a similar way to use biquandle colorings to distinguish 1-linkoids.
We also generalize the notion of homogeneity of quandles to n-homogeneity of quandles. We classify all ∞-homogeneous, finite quandles.

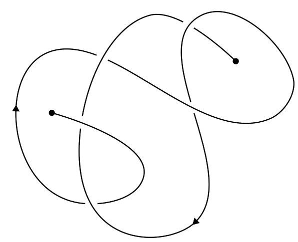
Contents

1 Introduction

Knotoids were introduced by Turaev in 2010 [Tur10]. Knotoids can be understood as knot diagrams with distinct open endpoints which cannot be moved over or under other strands in the diagram. Since then, a lot of research on knotoids was done, as for example in [GK17], [Mol22] or [BBHL19]. Similarly, Turaev introduced 1-linkoids (under the name multi-knotoids) as knotoids with additional closed components.
This was generalized in [GG22] to linkoids (under the name of multi-linkoids), which are a union of multiple open and closed components.



(a) A 1-linkoid with two components



(b) a knotoid

Because knots, and hence linkoids as well, are very hard to distinguish in general, multiple approaches for invariants were developed to discriminate between different knots. One important approach is to color each arc in a diagram. This leads to the algebraic structure called *quandles*, which is designed to have exactly the properties we need to color a knot diagram with so that it only depends on the knot and not the diagram. Quandles where introduced independently by Joyce in [Joy82] and Matveev in [Mat84] (under the name distributive groupoids). Both proved that quandles can distinguish all knots up to mirror image with reversed orientation. On the other hand, this means that quandles are as complex as knots themselves.

Since then, a lot on research of quandles was done. Some research on the algebraic structure can be found for example in [EMRL10], [Nel02] and [BLRY10]. Research regarding methods to distinguish different knots and links using quandles can be found for example in [CJK$^+$01], [CN18] and [CESY14].

There is, however, only very little research explicitly on studying knotoids and linkoids using quandles. Especially the notion of the fundamental quandle of a linkoid as a direct generalization of the fundamental quandle of a link is not studied yet. There is some research on coloring 1-tangles with quandles, for example in [CSV16] and [CDS16, Chapter 3]. These can be understood as knot-type knotoids, which means knotoids with both endpoints in the same region.
There are also approaches to transfer shadow quandle colorings to knotoids, for example in [Caz22]. Biquandle colorings of knotoids, a generalization of quandle

colorings were studied in [GN18].

In this thesis we define the fundamental quandle of a linkoid and study its basic properties. We show that the fundamental quandle is invariant under moving an endpoint under another arc. This is called an under forbidden move. It is named forbidden, because it can change the linkoid to a non-equivalent one. To detect this move, we then introduce a new generalization of quandles, called n-pointed quandles. These are quandles with n ordered basepoints. With these basepoints, we can fix the color of the endpoints in a linkoid. This leads to the new concept of a fundamental pointed quandle of a linkoid. In section 3.4, we provide classes of knotoids that cannot be distinguished using regular quandles but can be distinguished using pointed quandles.

A quandle X is homogeneous if the group of quandle automorphisms $Aut(X)$ of the quandle act transitively on it. With our new definition of n-pointed quandles, we introduce the notion of quandles being n-homogeneous, which means that all pointed quandles, with the same underlying quandle, are isomorphic whenever „possible“. In Theorem 4.25, we classify all finite quandles that are k-homomgeneous, where $k = |X|$ is the cardinality of the quandle.

A cohomology theory for quandles was introduced in [CJK$^+$01]. This leads to the 2-cocycle invariant for links. This uses a 2-cocycle to give weights to the crossings in a link diagram. Then one sums the weighted colorings to compute the 2-cocycle invariant. We define the 2-cocycle invariant for linkoids and show that is does on the explicit cocycle, instead of its cohomology class as it is the case for links. We then use the structure of pointed quandles to introduce an enhancement the 2-cocyclce invariant for linkoids.

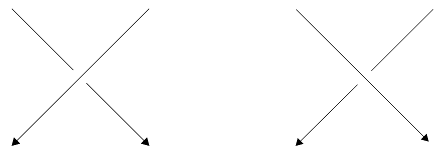
Finally, in Section 6 we review biquandles and introduce pointed biquandles, similar to pointed quandles. This way, we can distinguish linkoids that cannot be distinguished using biquandle colorings or pointed quandle colorings.

2 Fundamental notions of knots and linkoids

In this section, we study knotoids and linkoids. We will begin with the definition of a linkoid and see knotoids as a special case of linkoids.

Definition 2.1. An (oriented) *linkoid diagram* in S^2 is a generic immersion of a finite number of unit intervals $[0, 1]$ and unit circles S^1 into S^2 with finitely many transverse double points. Each such double point is endowed with over/under-crossing data.

We call a component open, if it is the image of $[0, 1]$ and closed if it is the image of S^1. Every component has an orientation, where $[0, 1]$ is oriented from 0 to 1. There are two types of crossings with respect to the orientation, called positive and negative crossing as specified in Figure 2 below.



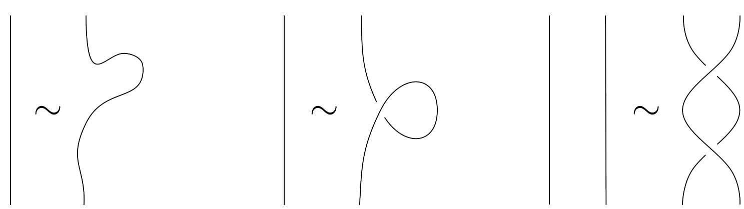
(a) Positive crossing (b) negative crossing

Figure 2

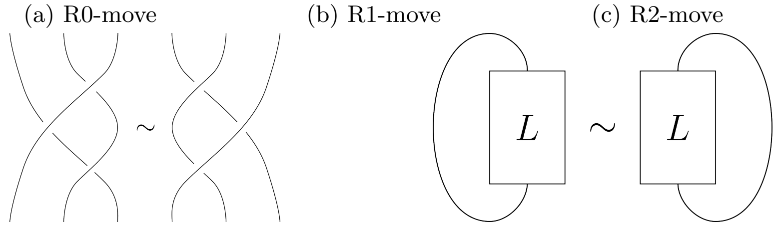
In an open component, we call the image of 0 the *leg* (or tail), and the image of 1 the *head* of the component.

Remark 2.2. Linkoids can in general be defined as generic immersions into any orientable surface. We will only consider spherical linkoids in this thesis. More general surfaces lead to the notion of virtual knot(oid) and link(oid) diagrams. These were first introduced in [Kau98] and further studied for instance in [GK17], [KR01] or [FJSK04].

Definition 2.3. Two linkoid diagrams are *equivalent*, if one can be moved into the other by a finite sequence of local oriented Reidemeister moves R0 - R3 depicted in Figure 3. These moves happen away from the endpoints and cannot move endpoints over or under a strand. The equivalence classes of these diagrams are called *linkoids.*



(a) R0-move (b) R1-move (c) R2-move



(d) R3-move (e) S-move

Figure 3: Reidemeister moves (a) - (d) and the spherical move (e)

Remark 2.4. In Figure 3 above unoriented Reidemeister moves are depicted. That means the orientation of the strands is not taken into account. If we choose an orientation for each strand in the diagram, there are four oriented versions of R1, four of R2, and eight types of R3. As shown in [Pol10, Thm 1.1], all eight R1 and R2 moves together with the R3 move with only positive crossings generate all oriented Reidemeister moves. This set of generating moves is not minimal. A minimal generating set of oriented Reidemeister moves has five elements by [Pol10, Thm 1.2]

Because we consider diagrams in S^2, the equivalence class doesn't change under the so-called spherical move (S-move) as shown in Figure 3e. It is simply a planar isotopy (R0-move) where we pull the strand around the S^2. Note that if a linkoid is a link, we can pull the strand over every crossing using a sequence of $R2$ and $R3$ moves, instead of pulling it around the "back" of S^2. So there is a one to one correspondence for equivalence classes of link diagrams in $\mathbb{R}^2$ and in S^2. However, this is in general not true once the diagram has at least one open component, because we cannot pull an arc over the endpoints in this case.

Because pulling an endpoint over or under an adjacent strand can potentially change the linkoid, these moves as depicted in Figure 4 are called the *forbidden moves* Ω_+ and Ω_-.

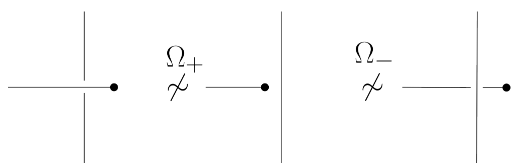


Figure 4: The over and under forbidden under

In the context of this thesis we find it useful to introduce the notation of an n-linkoid.

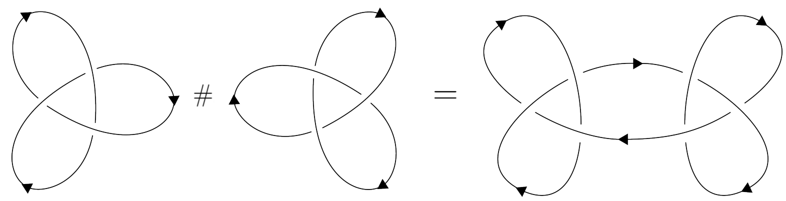
Definition 2.5. A (non-empty) linkoid with n open components (and any number of closed components) is also called an n-linkoid. In particular a

- 0-linkoid is called a *link*.
- 0-linkoid with only one component is called a *knot*.
- 1-linkoid with no closed component is called a *knotoid*.
- linkoid with no closed components is called a *full linkoid*.

Remark 2.6. In Turaev's paper [Tur10], a 1-linkoid is called a multi-knotoid. A full linkoid is called a linkoid in [GG22].

A linkoid that is equivalent to a linkoid with zero crossings is called *trivial*. The trivial knot is called the *unknot*.

There is an operation to combine two knots K and K' into one single knot $K\#K'$. This is called the *connected sum*. We do this by cutting open both knots and connecting the open ends such that in matches the orientation.



A knot that can not be written as a sum of two non-trivial knots is called *prime*.

The above definition of a knot (or link) is exactly definition of an equivalent class of knot diagrams as we know it from usual knot theory. That is, we can also view (tame) knots as simple closed curves in three-dimensional space. Here two knots are equivalent, if they are ambient isotopic. That is, two knots K_0 and K_1 are equivalent if there exists a continuous map $H\colon \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $H(K_0, 0) = K_0$, and $H(K_0, 1) = K_1$ and $H(x,t)$ is injective for every $t \in [0,1]$. We can think of this as moving the knot around in space without tearing the knot or the surrounding space apart. Reidemeister [Rei27] and Alexander and Briggs [AB26] showed independently that these both definitions give the same classes of knots. We mostly work with the diagrammatic point of view, because it is a bit more complicated to describe the corresponding three-dimensional representation for linkoids.

Remark 2.7. In [Tur10] it is shown that a knotoid is in one-to-one correspondence to so-called *simple theta-graphs*. These are embeddings of a graph with two vertices and three edges between them into three-space. So the graph looks a bit like the symbol Θ. The term *simple* means that if we remove the middle edge, the graph is homotopic to S^1.

We can think of this correspondence as considering the diagram close to the horizontal plane in 3D and the endpoints of a knotoid diagram as vertices. We then add edges connecting both endpoints. One “over” the diagram and one “under” the diagram. See Figure 5 for a diagram of the theta graph corresponding to the knotoid in Figure 1b.

For linkoids in S^2 there is a similar representation in three-space that was introduced and studied in [GG22]. There, every component has such a closing edge over and under the diagram. However, there are two extra vertices, one over and one under the diagram, where all the closing edges meet on the corresponding side of the diagram. We will not use these concepts here, but they might be helpful to keep in mind for a better intuition.

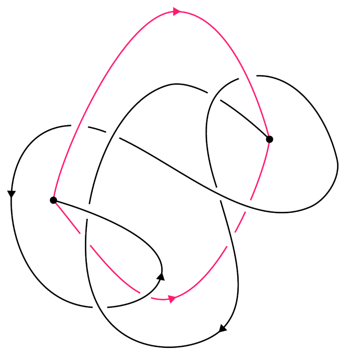


Figure 5: The theta graph of the knotoid shown in Figure 1b

A linkoid diagram divides S^2 into *regions*, the areas between the arcs.

A knotoid K with both endpoints in the same region is called a *knot-type knotoid*. We call a 1-linkoid L with both endpoints in the same region a *link-type 1-linkoid*.

There are some transformations that look similar to the forbidden moves Ω_+ or Ω_-, but can be obtained by a sequence of Reidemeister or spherical moves. For example on a given *link-type 1-linkoid diagram*, we can pull both endpoints under (or over) an adjacent strand simultaneously as in Figure 6. This move can be obtained by pulling the strand to the outside region of the diagram using R2 and R3 moves, then using the S-move to pull the strand around the S^2 and finally using again R2 and R3 moves to pull the strand back to the region with the endpoints. We call this move *under (or over) fake forbidden move* and denote it by F_- (or F_+). Note that such a move does not exist for n-linkoids with $n \geq 2$, because then there are more endpoints, which we cannot pull strands over (or under).

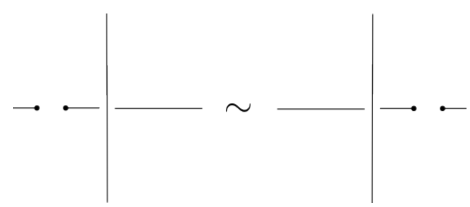


Figure 6: The under fake forbidden move F_-

Given a 1-linkoid L, we can obtain a link as follows. Choose an in S^2 from the head to the leg of L that intersects L only transversely at a finite set of points. We call this arc a *shortcut* of the 1-linkoid. If this arc goes under every

strand of L, the resulting link $L_- := L \cup a$ is called the *under closure* of L. If it goes over every strand of L it is called the *over closure* of M and denoted by L_+.
We say the 1-linkoid L *represents* the link L_-. If L is a link-type linkoid then $L_- = L_+$ and we say L is of type L_-.

Lemma 2.8. *Two 1-linkoids L and L' represent the same link, if they can be transformed into each other by a finite sequence of Reidemeister moves and under forbidden moves Ω_-.*

Proof. We will transform the link L_- to the link L'_- using the moves of given sequence of moves. Every R-move on L can be performed on L_- as well, since the shortcut of L crosses under every arc. This might need some extra R2 or R3 moves to pass over the shortcut. For the forbidden moves we move the „endpoint position“ of the link along the arc. This does not change the diagram. This way we end up with a diagram of L'_- as L combined with its shortcut. □

Proposition 2.9. *Any two knot-type knotoids of the same type are equivalent.*

Proof. Let K and K' be two knot-type knotoids of the same type. So their closures are related by a finite sequence of R-moves. We will perform these moves on K. Whenever we need to move an arc under or over the endpoints we perform a fake forbidden move F_- or F_+. The resulting diagram looks equal to K' but can have the endpoints in a different region. We can again use F_- or F_+ moves to pull both endpoints along the diagram. This transforms the knotoid K to K'. □

On the other hand, given a knot, we can obtain a knotoid removing an arc which is disjoint from any crossing. If we choose different arcs in the diagram, we might get different knotoid diagrams. With the observation in Proposition 2.9 we see that all of these diagrams are equivalent.

This proves the following.

Corollary 2.10. *There is a one-to-one correspondence between knots and knot-type knotoids.*

Because we can transform any knotoid into a knot-type knotoid using under forbidden moves, Corollary 2.10 together with Lemma 2.8 proves the next corollary.

Corollary 2.11. *Two knotoids represent the same knot if and only if they can be transformed into each other by a finite sequence of Reidemeister moves and under forbidden moves Ω_-.*

This is not true for links and 1-linkoids because we can choose any component to open the link, therefore the resulting 1-linkoids can be non-equivalent.

3 A review on quandles

3.1 Basic quandle definitions

In this section we learn about quandles, see some instructive examples and define properties of quandles we need later.

Definition 3.1. A *quandle* is a set $(X, \triangleright)$ equipped with an operation $\triangleright : X \times X \to X$ such that

(1) $x \triangleright x = x$ for all $x \in X$. (idempotent)

(2) For any $y \in X$, the map $\beta_y : X \to X$ defined as $\beta_y(x) = x \triangleright y$ is bijective. (right invertible)

(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all $x, y, z \in X$ (right self-distributive).

We often only write X for the quandle $(X, \triangleright)$, if the structure is clear. We write $x \triangleright^{-1} y := \beta_y^{-1}(x)$ and see immediately

$$(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$$

for all $x, y \in X$. We think of $x \triangleright y$ as y "acting" on x. For $x \triangleright y$ we read it as "x quandle y" or "we quandle x with y". A quandle is in general neither commutative nor associative. This means it is important to write parentheses.

Remark 3.2. Different authors use different conventions for the quandle operation. Most texts related to knot theory use $\triangleright$ as the operation symbol. Texts with a more algebraic focus often use $*$ as operation symbol.

A particular kind of quandles is a so-called kei. It is mainly used for unoriented links.

Definition 3.3. A quandle X with $\beta_y^{-1} = \beta_y$ for all $y \in X$ is called a *kei*.

Next, let us provide a few explicit examples.

Example 3.4.

- Let X be any set and $x \triangleright y = x$ for all $x \in X$. This defines a quandle structure. It is called the *trivial quandle*. If $|X| = n$ is finite, it is denoted by T_n. In particular a trivial quandle can have any number of elements, unlike for example the trivial group, which has only one element.

- For any group G we can define a quandle structure on G using conjugation: $g \triangleright h := h^n g h^{-n}$. Here $g \triangleright^{-1} h = h^{-n} g h^n$. This is called the *$n$-conjugation quandle* and denoted by $Conj_n(G)$ or $Conj(G)$ if $n = 1$. Note that if G is abelian, this is again a trivial quandle.

- Now let $n \in \mathbb{N}$ and $X = \mathbb{Z}/n\mathbb{Z}$ or $X = \mathbb{Z}$. We can define a quandle structure on X by $x \triangleright y := 2y - x (\text{mod } n)$. To see that this is indeed a quandle we check the first quandle axiom. We see $x \triangleright x = 2x = x = x$ for all x. Next we want to find the inverse map to β_y. Note that

$$\beta_y \circ \beta_y(x) = \beta_y(2y - x) = 2y - (2y - x) = x$$

for all $x, y \in X$. So $\beta_y^{-1} = \beta_y$. To check the third quandle axiom we compute

$$(x \triangleright y) \triangleright z = 2z - (2y - x) = 2(2z - y) - (2z - x) = (x \triangleright z) \triangleright (y \triangleright z).$$

This quandle is called the *dihedral quandle* and denoted as R_n. In particular, it is a kei.

For (small) finite quandles it is convenient to write them down as operation tables. For example the dihedral quandle R_4 is given by

$\triangleright$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

Definition 3.5. A *quandle homomorphism* between two quandles $(X, \triangleright_X)$ and $(Y, \triangleright_Y)$ is a map $f\colon X \to Y$ such that

$$f(x_1 \triangleright_X x_2) = f(x_1) \triangleright_Y f(x_2).$$

We denote the category of quandles by **Qnd**. A bijective quandle homomorphism is called an isomorphism.

For example if $X = \mathbb{Z}$ and $x \triangleright y = 2y - x$ and $a \in \mathbb{Z}$. Then the maps $f\colon x \mapsto ax$ and $g\colon x \mapsto x + a$ are both quandle homomorphisms. We see that f is only bijective for $a = \pm 1$ while g is bijective, giving us examples of automorphisms. We denote the group of automorphisms of a quandle X by $Aut(X)$.

Another example for an automorphism is the maps β_y for $y \in X$. The fact that these maps are indeed quandle homomorphisms follows from the third quandle axiom:

$$\beta_y(x_1 \triangleright x_2) = (x_1 \triangleright x_2) \triangleright y = (x_1 \triangleright y) \triangleright (x_2 \triangleright y) = \beta_y(x_1) \triangleright \beta_y(x_2).$$

The subgroup of $Aut(X)$ generated by these maps is called the group of *inner automorphisms* and denoted by $Inn(X)$.

Definition 3.6. A quandle X is *faithful*, if the map $x \mapsto \beta_x$ is injective.

For example R_4 is not faithful, because in the operation table above, the zeroth and second column are identical, hence $\beta_0 = \beta_2$.

Definition 3.7. The *algebraic components* of a quandle Q are the orbits under action of the inner automorphism group $Inn(Q)$.

Definition 3.8. A quandle X is *connected*, if it has only one component. So for each $x, y \in X$ there are $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$such that

$$(\ldots((x \triangleright^{\varepsilon_1} x_1) \triangleright^{\varepsilon_2} x_2) \ldots) \triangleright^{\varepsilon_n} x_n = y.$$

Definition 3.9. A quandle X is *homogeneous*, if $Aut(X)$ acts transitively on X. So for each $x, y \in X$ there is an $f \in Aut(X)$ with $f(x) = y$.

Any connected quandle is homogeneous, but the converse is not true. For example any trivial quandle T is homogeneous, because every permutation is an automorphism. But it is not connected for $|T| > 1$.

Recall the conjugate quandle $Conj(G)$ in Example 3.4 for a given group G. This is indeed a functor $Conj\colon \mathbf{Grp} \to \mathbf{Qnd}$. To see that any group homomorphism $f\colon G \to H$ respects the quandle structures, we compute for $x, y \in Conj(G)$

$$f(x \triangleright y) = f(y^{-1}xy) = f(y)^{-1}f(x)f(y) = f(x) \triangleright f(y) \in Conj(H).$$

This functor has a left adjoint functor, which was also introduced in [Joy82, Chapter 6].

Definition 3.10. Let X be a quandle. Then the *associated group* $As(X)$ is defined as

$$As(X) = \langle e_x, x \in X | e_y^{-1} e_x e_y = e_{x \triangleright y}, x, y \in X \rangle.$$

This group is sometimes called "conjugate group" ([Win11]) or "enveloping group" ([Car10]). For a quandle homomorphism $f\colon X \to Y$ let $As(f)\colon As(X) \to As(Y)$ be the group homomorphism defined by $As(f)(e_x) = e_{f(x)}$ on generators.

For example if T is a trivial quandle, then $As(T) = \mathbb{Z}^{|T|}$. In particular these groups do not need to be finite even for finite quandles.

3.2 Combinatorial definition of the fundamental quandles of a link

Fundamental quandles are so far mostly used to understand knots and links. Even though Turaev suggested defining fundamental quandles for knotoids in [Tur10], we are not aware of any resource defining it. There are papers related to this idea, for example Cazet examines shadow quandle colorings of knotoids in [Caz22] and Gügümcü and Nelson look at biquandle colorings of knotoids in [GN18]. These are both generalizations or enhancements of quandle colorings. We discuss some ideas and results of fundamental quandles of knots and links in this section. In the next section we further look at linkoids and how quandle colorings interact with endpoints.

Let D be an oriented link diagram and X a finite quandle. An *arc* of the diagram is a continuous strand in the diagram from one under crossing to the next under crossing. An arc can cross over several other arcs. We can *color* D with X by assigning an element of X to every arc in D, which we call color. The coloring has to be done in a way such that at every crossing in D the colors satisfy the *crossing relation* shown in Figure 7 below. Note that both relations are equivalent if we only consider the orientation of the over-crossing arc, since $(x \triangleright^{-1} y) \triangleright y = x$.

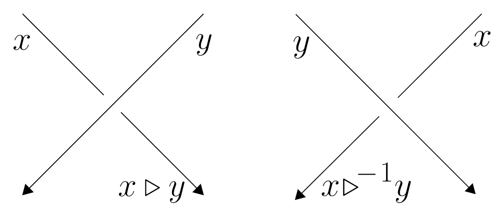


Figure 7: Quandle relation given on a crossing

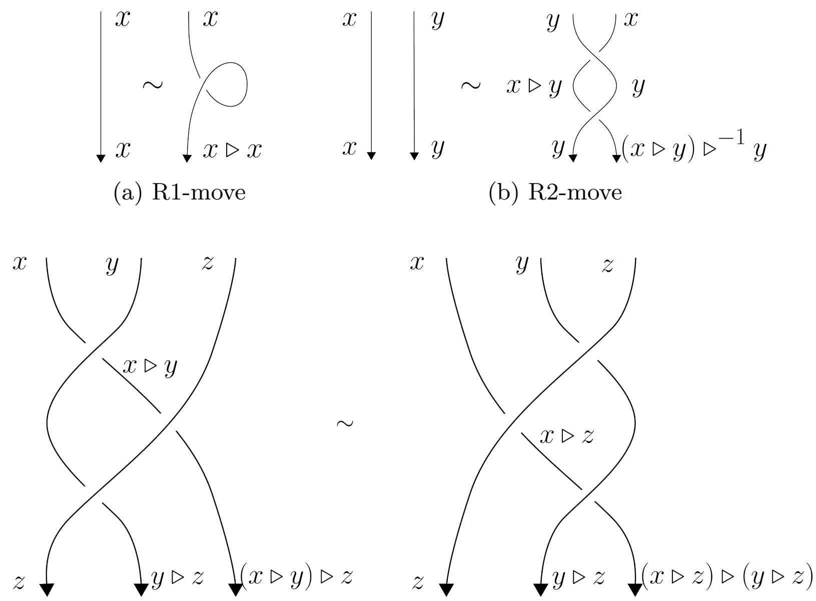
We can always color a linkoid with only one color because the first quandle axiom ensures that all the relations are satisfied. A coloring like this is called *trivial.*

Now we examine what happens with these relations when we change the diagram by Reidemeister moves. Assume we have a given coloring of a diagram. We see in Figure 8a how the diagram changes under an R1-move. Because $x \triangleright x = x$ by the first quandle axiom, the outgoing arcs on the left and right have the same label. So adding a kink gives a unique coloring that is consistent with the rest of the coloring.

If we change the diagram by an R2-move as in Figure 8b, the outgoing arcs on the left and right have the same label assigned to them as the corresponding arc after the R2-move, because $(x \triangleright y) \triangleright^{-1} y = x$ by the second quandle axiom. Because the coloring on the right side is completely determined by x and y, there is again a one-to-one correspondence between colorings before and after the R2-move.

Similarly, if we change the diagram by the R3-move as in Figure 8c, the outgoing arcs have the same label before and after the move. This is due to the third quandle axiom $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

This shows that if we have two equivalent diagrams D and D', every coloring of D gives a unique coloring of D', because we can turn them into each other by a finite sequence of Reidemeister moves. So the above described coloring only depends on the link and not on the diagram we chose.



(a) R1-move

(b) R2-move

(c) R3-move

Figure 8: Quandle relations on Reidemeister moves

With this in mind we now define the fundamental quandle of a linkoid, sometimes also called the *knot/link quandle*

Definition 3.11 ([Joy82, Chp. 15], [Mat84])**.** Let L be an oriented link diagram and $A(L)$ the set of arcs in the diagram. Then the *fundamental quandle* of L is defined as

$$Q(L) \coloneqq Q\langle x \in A(L) \quad | \quad r_\tau \text{ for all crossings } \tau\rangle,$$

where the quandle consists of words in $A(L)$ modulo the quandle axiom relations and the relations r_τ given by each crossing τ as in Figure 7.

Theorem 3.12. *The fundamental quandle is an invariant for links in the sense that it is invariant under Reidemeister moves.*

Our observation above illustrates the proof of this Theorem. For a complete proof with all variants of oriented Reidemeister moves see for example [Joy82, Thm 15.1].

Example 3.13. Let L be the unlink with n components. Then $Q(L) = Q\langle a_1, \dots, a_n\rangle$ is the free quandle with n generators. In particular if L is the unknot then $Q(L) = T_1$, the trivial quandle with one element.

Example 3.14. Let X be the trefoil knot 3_1 as in Figure 9. Its fundamental quandle has three generators (since there are three arcs) and three relations, one

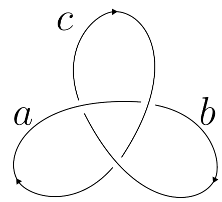


Figure 9: The trefoil knot 3_1

for each crossing. Namely, $a = b \triangleright c$, $b = c \triangleright a$ and $c = a \triangleright b$. So the fundamental quandle is

$$Q(3_1) = Q\langle a, b, c | b = c \triangleright a, a = b \triangleright c, c = a \triangleright b \rangle$$

The presentation of this quandle is not unique. For example, we can replace c by $a \triangleright b$ in every relation and get the isomorphic presentation

$$Q\langle a, b | a = b \triangleright (a \triangleright b), b = a \triangleright (b \triangleright a) \rangle \cong Q\langle a, b | a = (b \triangleright a) \triangleright b, b = (a \triangleright b) \triangleright a \rangle$$

where the second isomorphism uses the first and third quandle axiom, so $b \triangleright (a \triangleright b) = (b \triangleright b) \triangleright (a \triangleright b) = (a \triangleright b) \triangleright b$. This quandle is infinite, because for example $((a \triangleright b) \triangleright b \ldots) \triangleright b \in Q(3_1)$ cannot be simplified further. This turns out to be true for nearly all links.

Theorem 3.15 ([CHMS19, Prop. 2])**.** *The fundamental quandle $Q(L)$ of an oriented link L is finite if and only if L is either the unknot or the Hopf link.*

Lemma 3.16 ([FR92])**.** *Let L be a link. There is a one-to-one correspondence between components of L and algebraic components of $Q(L)$.*

We will see a proof of this in Lemma 3.22 since links are 0-linkoids, so a special case of linkoids.

With the notion of the fundamental quandle we can understand the coloring of a link L with a quandle X as a quandle homomorphism $\varphi\colon Q(L) \to X$ which is defined by mapping the generators of $Q(L)$ to the color the corresponding arc is colored with. With this understanding, we can now give the following definition.

Definition 3.17. The space of all such homomorphisms $\mathbf{Qnd}(Q(L), X)$ is called the *coloring space*. The number of colorings for a given finite quandle X is called the *quandle counting invariant* with respect to X, which we denote as

$$\Phi_X^{\mathbb{Z}}(L) \coloneqq |\mathbf{Qnd}(Q(L), X)|.$$

Because we can always color the link trivially, we see immediately $\Phi_X^{\mathbb{Z}}(L) \geq |X|$.

Example 3.18. We compute the quandle counting invariant $\Phi^{\mathbb{Z}}_{R_3}(3_1)$ for the trefoil with the dihedral quandle $R_3 = \{0, 1, 2\}$ with $x \triangleright y = 2y - x \pmod 3$. Therefore, we write

$$Q(3_1) = Q\langle a, b | a = (b \triangleright a) \triangleright b, b = (a \triangleright b) \triangleright a\rangle$$

as before. If we examine the relations in R_3 we get $(b \triangleright a) \triangleright b = 2b - (b \triangleright a) = 2b - (2a - b) = 3b - 2a \equiv a (\mathrm{mod}\ 3)$. So these relations are always satisfied. Hence, we can choose any color for a and b. In particular $\Phi^{\mathbb{Z}}_{R_3}(3_1) = 3 \cdot 3 = 9$.

Now, to compute $\Phi^{\mathbb{Z}}_{R_4}(3_1)$, we again look at

$$(b \triangleright a) \triangleright b = 2b - (2a - b) = 3b - 2a \equiv 2a - b (\mathrm{mod}\ 4) = b \triangleright a.$$

The relation in $Q(3_1)$becomes $a = (b \triangleright a) \triangleright b = b \triangleright a$. In the same way, we compute $b = a \triangleright b$. Combining these two gives $a = b \triangleright a = (a \triangleright b) \triangleright a = b$. This means, we can color the trefoil only trivially with R_4 and hence, $\Phi^{\mathbb{Z}}_{R_3}(3_1) = 4$.

3.3 Geometric definition of the fundamental quandle of a link

So far we have considered fundamental quandles as a structure defined on the diagram of a linkoid. For links, there is also a geometric definition that gives an equivalent definition for the fundamental quandle. It was introduced together with the notion of quandles in [Joy82] and [Mat84]. We follow the approach as in [EN15, Chapter 4]. For a more detailed and in depths description see for instance [Win11, Section 3.3]. For the proof that these quandle definitions are indeed isomorphic see [Joy82] or [Mat84].

We define the fundamental quandle as homotopy classes of paths in S^3. Let K be a thickened knot in S^3. Then the boundary of $S^3 \setminus K$ looks like a (knotted) torus. Now choose a basepoint $*$ in the interior of $S^3 \setminus K$. We consider paths $x\colon [0, 1] \to S^3 \setminus K$ from the basepoint $*$ to the boundary of $S^3 \setminus K$, that is to the knot K, as depicted in Figure 10.

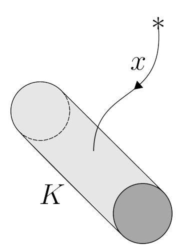


Figure 10: Path x from the basepoint to K

In this context, a homotopy H_t between two paths x and y has to be such that $H_t(0) = *$ for all t and $H_t(1) \in \partial(S^3 \setminus K)$ for all t, staying on the boundary. Note that the endpoint of the path can be moved around the boundary during the homotopy.

Now for any point $p \in \partial(S^3 \setminus K)$ we define the *meridian* m_p at p as the circle around K which links with the (non-thickened) knot once. With this we can define a quandle operation for paths x and y by $x \triangleright y := y m_y^{-1} y^{-1} x$. So we first walk the path y, then around the knot at the endpoint of y, back to the basepoint and finally walk x. This gives the path to $x \triangleright y$ in Figure 11 via the homotopy that drags the endpoint along the undergoing arc.

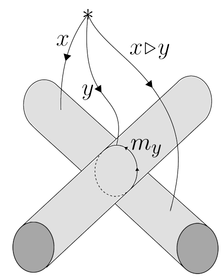


Figure 11: The path $x \triangleright y = y m_y^{-1} y^{-1} x$

With this construction Joyce and Matveev independently showed the following theorem.

Theorem 3.19 ([Joy82] and [Mat84]). *Let K and K' be two tame knots. Then $Q(K) \cong Q(K')$ if and only if K and K' are weakly equivalent.*

For a given oriented link L we define the reversed link $r(L)$ as L with reversed orientation of every component and the *mirror link* $m(L)$ where we change every over crossing to an under crossing and vice versa. We write $rm(L)$ for the reversed mirror of L.
We call two links L and L' *weakly equivalent* if either $L \sim L'$ or $L \sim rm(L')$.

The fact that $Q(K) \cong Q(rm(K))$ is easily seen from the diagrammatic definition. If we change the orientation and the type of a given crossing (and flip the labels), then we get the exact same relation for each crossing and hence the same presentation for the quandles.

The theorem above means that the fundamental quandle is, in a way, the algebraic version of the knot. It should be possible to derive all knot invariants from the fundamental quandle.

For example can we get the knot group $\pi_1(S^3 \setminus K)$ as the associated group $As(Q(K))$ of the fundamental quandle of a knot (or link). To see this, note that the definition of the associated group and the relations in the fundamental quandle directly give the Wirtinger presentation of the knot group (see [Joy82]).

We know that the knot group can distinguish prime knots, but it can fail to distinguish composite knots. For example the square knot and granny knot in Figure 12 have isomorphic knot groups, but non-isomorphic fundamental quandles. This is the case because loops around the connecting arcs (here the two horizontal arcs in the middle) are homotopic by pulling them over one component of the knot. However, these arcs generally have different labels in the fundamental quandle.

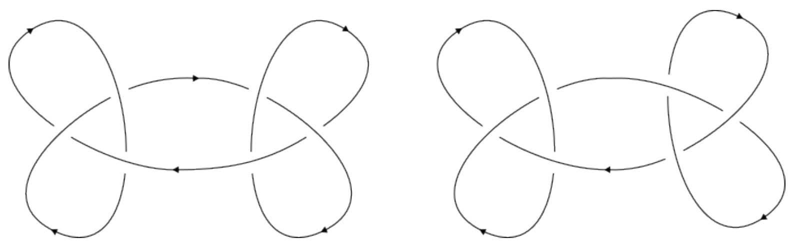


Figure 12: The square knot (left) and granny knot (right)

3.4 The fundamental quandle of a linkoid

In this section we generalize fundamental quandles to linkoids and examine how the fundamental quandle behaves with endpoints and on knotoids.

Definition 3.20. Let L be an oriented linkoid diagram and $A(L)$ the set of arcs in the diagram. Then the *fundamental quandle* of L is defined as

$$Q(L) \coloneqq Q\langle x \in A(L) \quad | \quad r_\tau \text{ for all crossings } \tau \rangle,$$

where the quandle consists of words in $A(L)$ modulo the quandle axiom relations and the relations given by each crossing as in Figure 7.

Theorem 3.21. *The fundamental quandle is invariant for linkoids.*

Proof. Two equivalent diagrams can be transformed into each other with a finite sequence of Reidemeister moves. Because changing a diagram by a Reidemeister move gives an isomorphic fundamental quandle, the fundamental quandle does only depend on the linkoid and not on the diagram. ☐

Lemma 3.22. *Let L be a linkoid. There is a one-to-one correspondence between components of L and algebraic components of* $Q(L)$*.*

Proof. Let a, b be labels of arcs in L. Assume a and b lie in the same component of L, then we can "walk" through the diagram from a to b. At each crossing where we go under an arc, we need to quandle with the label of the over-crossing arc c_i, where i counts the times we pass a crossing this way. This corresponds to the inner isomorphism β_{c_i}. After completing the walk from a to b we get a morphism $f = \beta_{c_n} \circ \cdots \circ \beta_{c_1} \in Inn(Q(L))$ with $f(a) = b$.

Now let a and b be two labels of arcs in L and assume there is a series of generators of $Q(L)$ such that $((((a \triangleright c_1) \triangleright c_2) \triangleright \dots) \triangleright c_n) = b$. For every crossing relation $x = y \triangleright z$ the arcs with label x and y lie in the same component of the linkoid and the quandle axioms do not change the element input of the inner homomorphisms, therefore a and b lie in the same component of L.

This proves that there is a one-to-one correspondence between components of L and algebraic components of $Q(L)$. □

The main difference between links and linkoids with open components are the endpoints. We will now see what happens at or with the endpoints in the fundamental quandle.

Lemma 3.23. *The fundamental quandle of a linkoid is invariant under the forbidden move Ω_-.*

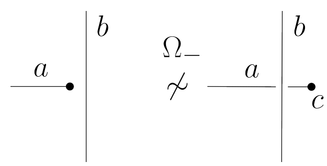


Figure 13: The under forbidden move with labeled arcs

Proof. Let L and L' be two linkoids that differ only by one forbidden under move Ω_-. So the presentation of the fundamental quandle of one linkoid, let's say $Q(L)$, has one extra generator c and one extra relation $a \triangleright^{\varepsilon} b = c$ as depicted in figure 13. Here $\varepsilon = \pm 1$ is the sign of the crossing that got added by the Ω_--move. Note that c appears in no other relation in the presentation of $Q(L)$. This means the map $\varphi\colon Q(L) \to Q(L')$ defined by $\varphi(x) = x$ for all generators $x \neq c \in Q(L)$ and $\varphi(c) = a \triangleright^{\varepsilon} b$ (and extended to the quandle) satisfies all relations in $Q(L')$ (because the relations in both quandles are the same). Hence, φ is a quandle homomorphism. Now φ has the inverse map given by $\varphi^{-1}(x) = x \in Q(L)$, so it is an isomorphism. This shows $Q(L) \cong Q(L')$. □

Together with Corollary 2.11 the following Corollary follows.

Corollary 3.24. *If two knotoids K and K' represent the same knot, then $Q(K) \cong Q(K')$.*

This means that when we consider a coloring as a map in $\mathbf{Qnd}(Q(K), X)$ for some finite quandle X, the coloring does not depend on the specific knotoid we choose to color, but only on the knot it represents. We will enhance the fundamental quandle for linkoids in Section 4 in a way that it can track the color of end-arcs.

On the other hand, Corollary 3.24 means we can study knotoids to understand fundamental quandles of knot-type knotoids (and vice versa), which only differ by one relation from knot quandles. But a knotoid can potentially have fewer crossings and arcs, hence the quandle has fewer generators and relations. Specifically for computational purposes, this can be very helpful.

We will now examine how the quandle of a knotoid corresponds to the quandle of the knot it represents.

First, let K be a knot-type knotoid representing the knot K_-. Let's denote the labels of the arcs connected to the leg and head of K by l and $h \in Q(K)$. The presentation of the fundamental quandle of K and K_- only differs by the extra relation $l = h$ in $Q(K_-)$ (because both end-arcs are connected). So we can write $Q(K_-) \cong Q(K) \, / \, (l = h)$ where the quotient is defined by adding the new relation to the quandle.
In particular if $l = h \in Q(K_-)$ then $Q(K) \cong Q(K_-)$. However, this is generally not the case.

Lemma 3.25. *Let K_- be a knot that is not equivalent to its reversed mirror $rm(K_-)$ and let K be a knot-type knotoid representing K_-. Denote the labels corresponding to the end arcs in $Q(K)$ by l and h. Then $l \neq h \in Q(K)$.*

Remark 3.26. When we talk about the mirror knotoid or mirror linkoid of a knotoid or linkoid, we have to be careful. There are two variants of mirroring a knotoid. We can either change the sign of every crossing (imagine holding the mirror behind the linkoid), or by reflecting it on an axis in the plane, outside the diagram. For links both of these are equivalent by turning the link around 180° in three-dimensional space. However, we are not allowed to do this for linkoids, due to the endpoints. Here, $m(L)$ refers to the second concept of a mirror image. By $rm(L)$ we denote this mirror linkoid with reversed orientation.

Proof of Lemma 3.25. By Theorem 3.19 we know $Q(K) \cong Q(rm(K))$. The knot K_- can be written as $K \cup \alpha$ for some arc α. Now we take the connected sum of K and K or K and $rm(K)$ along this arc α connecting the endpoints. The composite knots $K_-\#K_-$ and $K_-\#rm(K_-)$ can be seen as two copies of K (or of K and $rm(K)$) connected on the endpoints, one time head to leg and one time head to head (and reversed orientation). See Figure 14 below. So by assumption $rm(K_-\#K_-) \not\sim K_-\#K_- \not\sim K_-\#rm(K_-)$.

We will examine the fundamental quandles of the composite knots $K_-\#K_-$

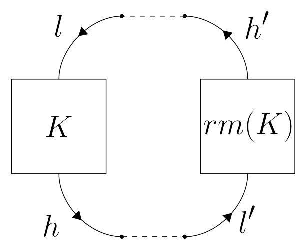


Figure 14: The knot $K_-\#rm(K_-)$ as sum of K and $rm(K)$

and $K_-\#rm(K_-)$. Therefore, let

$$Q(K) = Q\langle l, h, x_i | r_j \rangle$$

be the presentation of $Q(K)$ given by the diagram of K, where l and h are the leg and head of K. The generators and relations of $K_-\#K_-$ are given by

$$Q(K_-\#K_-) = Q\langle l, h, x_i, l', h', x_i, x_i' | r_j, r_j', l = l', h = h' \rangle$$

where l, h, x_i and r_i are coming from the left and l', h', x_i' and r_j' right side of the composite knot. Similarly,

$$Q(K_-\#rm(K_-)) = Q\langle l, h, x_i, l', h', x_i, x_i' | r_j, r_j', l = h', h = l' \rangle.$$

Now assume $l = h \in Q(K)$. Then $Q(K\#K) \cong Q(K\#rm(K))$ which contradicts the assumption that K and $rm(K)$ are not equivalent by using Theorem 3.19. □

In particular $Q(K) \ncong Q(K_-)$ in the situation of the lemma above. Together with Corollary 3.24 we see

Corollary 3.27. *Let K_- be a knot that is not equivalent to its reversed mirror $rm(K_-)$ and let K be a knotoid that represents K_-. Then $Q(K) \ncong Q(K_-)$.*

Let K be a knotoid that represents the trivial knot. Then we see immediately that $Q(K) \cong T_1 \cong Q(K_-)$. So some condition on K or is necessary for the above corollary. Maybe a more precise condition can be found in future work.

Remark 3.28. Recall that *knot* quandles can be defined as homotopy classes of paths from a chosen basepoint to the knot (or to a tubular neighborhood of the knot) as described in Section 3.3. In this setting the expression $x \triangleright y$ means to first walk a loop around the arc corresponding to y and then walk (the path to) x.

If we now consider the knot $K_-\#K_- \in S^3$, we see that the loop around the arcs l and h are homotopic by simply dragging the loop around one of the component knots. This, in fact, does not depend on K_- at all. For any knot-type knotoid, the loops around the end-arcs (at least if we consider the knotoid

as part of a composite knot, so we have a three-dimensional representation of it) are homotopic. So we could assume $\beta_a = \beta_b$.
Now this argument does not work, because our three-dimensional representation of a knotoid as summarized in Remark 2.7 does not give us a geometric quandle definition that coincides with the combinatorial one.

Nevertheless, the intuition in the remark above proves to lead to the correct statement.

Lemma 3.29. *Let L be a link-type 1-linkoid and denote the labels corresponding to the end-arcs in $Q(L)$ by l and h. Then $\beta_l = \beta_h$. In particular $Q(L)$ is not faithful.*

The proof can be found in [Nos11, Lemma 5.6] and uses shadow quandle colorings of 1-tangles. Shadow quandle coloring means we do not only color the arcs of the diagram but also the regions around the diagram.

Corollary 3.30. *Let L be a link-type 1-linkoid, X a faithful quandle and $f \in$ $\mathbf{Qnd}(Q(L), X)$ any coloring. Then $f(l) = f(h)$ where l and $h \in Q(L)$ denote the end-arc labels.*

Remark 3.31. Let L be a link-type 1-linkoid. If for a given quandle X, every coloring assigns the same color to both end-arcs, the pair (L, X) is called *end monochromatic*. By the corollary above this is always the case if X is faithful. But there are also end monochromatic pairs with non-faithful quandles. This is studied for example in [CSV16] and [CDS16] to better understand quandle colorings of composite knots.

4 Pointed quandles

In this section we introduce pointed quandles. These are quandles with marked points which we call basepoints. This way we are able to remember the labels of the endpoints. This lets us detect the under forbidden move on linkoids.

4.1 Introduction of n-pointed quandles

Definition 4.1. An *n-pointed quandle* $(X, x_1, \dots, x_n)$ is an ordered tuple consisting of a quandle X together with n (ordered) elements $x_1, \dots, x_n$ of X. We call $x_1, \dots, x_n$ the *basepoints* of the pointed quandle.

Note that a 0-pointed quandle is again a quandle.

Definition 4.2. A homomorphism between two n-pointed quandles

$$\varphi \colon (X, x_1, \dots, x_n) to (Y, y_1, \dots, y_n)$$

is a quandle homomorphism $\varphi \colon X \to Y$ such that $\varphi(x_i) = (y_i)$ for $i = 1, \dots, n$. We denote the category of n-pointed quandles by $\mathbf{PQnd_n}$.

There is a forgetful functor $U_n\colon \mathbf{PQnd_n} \to \mathbf{Qnd}$ mapping $(X, x_1, \dots, x_n)$ to X, forgetting the basepoints. The set of all n-pointed quandles with underlying quandle X is denoted by $U_n^{-1}(X)$.

For readability, we denote pointed quandles by calligraphic letters, for example $\mathcal{X} = (X, x_1, \dots, x_n)$.

Remark 4.3. For a given pointed quandle $\mathcal{X} = (X, x_1, \dots, x_n)$ every (unpointed) quandle homomorphism $(\varphi\colon X \to Y) \in \mathbf{Qnd}(X, Y)$ gives a pointed quandle homomorphism $\mathcal{X} \to (Y, \varphi(x_1), \dots, \varphi(x_n))$. We write

$$\varphi(\mathcal{X}) := (Y, \varphi(x_1), \dots, \varphi(x_n)).$$

The map $\varphi\colon X \to Y$ is a quandle isomorphism if and only if $\varphi\colon \mathcal{X} \to \varphi(\mathcal{X})$ is a pointed quandle isomorphism.

Definition 4.4. An *ordered n-linkoid* is an n-linkoid with a given ordering of the open components.

Note that the ordering is invariant under Reidemeister moves.

Definition 4.5. Given an ordered n-linkoid L, we define the *fundamental pointed quandle of* L as the $2n$-pointed quandle $P(L) := (Q(L), l_1, h_1, \dots, l_n, h_n)$, where $Q(L)$ is the fundamental quandle of L as defined in 3.11 and l_i and h_i are the generator corresponding to the arc attached to the leg and head of the i-th component of L.

Theorem 4.6. *The fundamental pointed quandle is invariant under Reidemeister moves and the spherical move.*

Proof. This follows directly from the fact that Reidemeister moves happen away from the endpoints. So the quandle isomorphism between the fundamental quandle before and after a Reidemeister move maps the endpoint labels to endpoint labels, hence it is a pointed quandle isomorphism. □

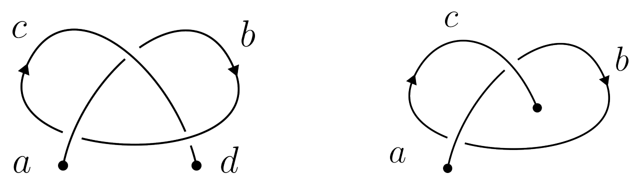
If L is a link (that is a 0-linkoid) then the fundamental pointed quandle is a 0-pointed quandle which is simply a quandle. So $P(L) = Q(L)$. In this sense the fundamental pointed quandle is a generalization of the fundamental quandle.

We will now see an example of fundamental pointed quandle of two knotoids that represent the same knot.

Example 4.7. Consider the knotoids K_1 and K_2 in Figure 15. Because both knotoids represent the trefoil knot their fundamental quandles are isomorphic by Proposition 3.24. We explicitly compute them as

$$\begin{aligned} Q(K_1) &\cong Q\langle a, b, c, d | b = a \triangleright c, c = b \triangleright a, d = c \triangleright b\rangle \\ &\cong Q\langle a, b, c | b = a \triangleright c, c = b \triangleright a\rangle \cong Q(K_2). \end{aligned}$$

Their fundamental n-pointed quandles are then $P(K_1) = (Q(K_1), a, c \triangleright b)$ and $P(K_2) = (Q(K_1), a, c)$. To see that these are not isomorphic as pointed quandles let $\mathcal{X} = ((R_3), 0, 0)$ be a pointed dihedral quandle and consider a pointed



(a) K_1 (b) K_2

Figure 15: Two knotoids representing the same knot

quandle homomorphism $f\colon P(K_1) \to \mathcal{X}$. Because f maps basepoints to basepoints we know $f(a) = 0 = f(c \triangleright b)$. Assume $f(b) = 1$, then $f(c) = f(b \triangleright a) = f(b) \triangleright f(a) = 1 \triangleright 2$. We need to check the other relation in $Q(K_1)$: $f(b) = 1 = 0 \triangleright 2 = f(a \triangleright 2)$. So this is a non-constant 2-pointed quandle homomorphism.

Now consider a pointed quandle homomorphism $g\colon P(K_2) \to \mathcal{X}$. So $g(a) = 0 = f(c)$. Then $g(b) = g(a \triangleright c) = 0 \triangleright 0 = 0$. Hence, the constant homomorphism $g(x) = 0$ is the only such morphism. It follows that $P(K_1) \ncong P(K_2)$.

This shows that the fundamental pointed quandle is not invariant under the under forbidden move and that it can potentially distinguish knotoids or 1-linkoids that represent the same knot or link.

Remark 4.8. Note that if a knotoid K represents the unknot, then $Q(K) = T_1$ has only one element. This means its fundamental pointed quandle can only have this element as basepoints. Hence, the fundamental pointed quandle has the same information as the fundamental quandle in this case (which is only very little).

Using the fundamental pointed quandle we will now see a sufficient condition for two 1-linkoids to represent the same link.

Lemma 4.9. *Let L, L' be two link-type 1-linkoids such that $P(L) \cong P(L')$ and let L_- and L'_- be the closure of L and L', respectively. Then $Q(L_-) \cong Q(L'_-)$.*

Proof. Let $\varphi\colon P(L) = (Q(L), l, h) \to P(L') = (Q(L'), l', h')$ be the pointed quandle isomorphism. In particular, φ is a quandle isomorphism. Because $\varphi(h) = h'$ and $\varphi(l) = l'$, also the map on the quotients

$$\tilde{\varphi}\colon Q(L) \big/ (h = l) \to Q(L') \big/ (h' = l')$$

is a quandle isomorphism. Now $Q(L_-) \cong Q(L) \,/\, (h = t)$, which completes the proof. $\square$

Now let L be any 1-linkoid (not necessarily link-type) representing the link L_- with fundamental pointed quandle $(Q(L), l, h)$. Consider the link-type 1-linkoid $L_\sim$ which is derived from L by moving the head of (the open component of) L to the region with the leg using forbidden under moves and denote its end-arcs by $\tilde{h}$ and $l \in Q(L) \cong Q(L_\sim)$. Now write $\tilde{h} = f(h)$ where $f \in Inn(Q(L))$ is the map we derive from the series of consecutive under forbidden moves. Then

$$Q(L_-) \cong Q(L_\sim) \big/ (\tilde{h} = l) \cong Q(L) \big/ (f(h) = l) = Q(L) \big/ r_L.$$

The relation $f(h) = l$ in this setting is called the *closing relation* of L and denoted by r_L. Note that for link-type 1-linkoids the closing relation is simply $l = h$.

Example 4.10. Consider again the knotoid K_2 in Figure 15b. Its closing relation is $r_{K_2} \colon a = c \triangleright b = \beta_b(c)$ in $Q(K_2) = Q\langle a, b, c | b = a \triangleright c, c = b \triangleright a\rangle$, because we need to move the head under the arc labeled b to move it into the region with the leg.

With the closing relation we can now write the fundamental pointed quandle of $L_\sim$ as $(Q(L), l, f(h))$. Together with Lemma 4.9 this shows the following corollary.

Corollary 4.11. *Let L and L' be two 1-linkoids with closing relations $r_L \colon f(h) = l$ and $r_{L'} \colon g(h') = l'$ as above. If there exists a pointed quandle isomorphism $\varphi \colon (Q(L), l, f(h)) \to (Q(L'), l', g(h'))$, then $Q(L_-) \cong Q(L'_-)$.*

In the case that the 1-linkoid is a knotoid, we can use theorem 3.19 to prove the following corollary.

Corollary 4.12. *Let K and K' be two knotoids with closing relations $r_K \colon f(h) = l$ and $r_{K'} \colon g(h') = l'$. If there exists a pointed quandle isomorphism*

$$\varphi \colon (Q(K), l, f(h)) \to (Q(K'), l', g(h')),$$

then K_- and K'_- represent weakly equivalent knots.

4.2 Isomorphism classes of pointed quandles

For a quandle X, $n \in \mathbb{N}$ and the forgetful functor U_n, we define

$$P_n(X) := U_n^{-1}(X) \big/ Aut(X),$$

the classes of pointed n-quandles with underlying quandle X under isomorphy. Here $\mathcal{X} \sim \mathcal{Y} \in P_n(X)$ if they are isomorphic n-pointed quandles. By Remark 4.3 this is exactly the case if there exists $\varphi \in Aut(X)$ with $\mathcal{Y} = \varphi(\mathcal{X})$. We denote the number of such classes by $d_n(X) := |P_n(X)|$.

Because $U_n^{-1}(X) = \{(X, x_1, \ldots, x_n) | x_i \in X\} \cong X^n$ as sets we only write $(x_1, \ldots, x_n)$ instead of $(X, x_1, \ldots, x_n)$, if X is clear from the context.

Let us compute the isomorphism classes of quandles of some examples:

Example 4.13.

1. Let $X = R_3$ be the dihedral quandle $\{0, 1, 2\}$ with $x \triangleright y \equiv 2y - x \pmod 3$. Note that the automorphism group $Aut(R_3)$ is a subgroup of S_3, the symmetric group. This means we can describe inner automorphisms as permutations. In particular $(01) = \beta_2$, $(12) = \beta_0$ and $(02) = \beta_1$. So $S_3 = Inn(R_3) \subseteq Aut(R_3)$.
   In $P_1(R_3)$ we see
   $$(R_3, 0) = (0) \overset{(01)}{\sim} (1) \overset{(12)}{\sim} (2),$$
   so $P_1(R_3) = \{[(0)]\}$ and $d_1(R_3) = 1$.

   For $P_2(R_3)$, we see
   $$(0,0) \overset{(01)}{\sim} (1,1) \overset{(12)}{\sim} (2,2)$$
   and
   $$(0,1) \overset{(12)}{\sim} (0,2) \overset{(01)}{\sim} (1,2) \overset{(02)}{\sim} (1,0) \overset{(12)}{\sim} (2,0) \overset{(01)}{\sim} (2,1)$$
   so $P_2(R_3) = \{[(0,0)], [(0,1)]\}$ has two equivalent classes.

2. Let $X = T_3$ be the trivial quandle. We see that $Inn(T_3) = \{id\}$, since $\beta_x = id$ for all $x \in T_3$. For any $f \in S_3$, we notice that $f(a \triangleright b) = f(a) = f(a) \triangleright f(b)$ so $f \in Aut(T_3)$. This shows that $Aut(T_3) = S_3$ and hence $P_2(T_3) = \{[(0,0)], [(0,1)]\}$.

3. There are only three non-isomorphic quandles with three elements. So now let V_3 be the remaining quandle with three elements, that is $\beta_0 = (12)$ and $\beta_1 = \beta_2 = id$. Let $f \in Aut(V_3)$. If $f(0) = 0$ then either $f = id$ or $f = (12)$. In both cases $f \in Inn(V_3)$. Now assume $f(0) = 1$. Then $f(2) = f(1) \triangleright f(0) = f(1) \triangleright 1 = f(1)$ which is a contradiction to f being a bijection. Similarly, if we assume $f(0) = 2$. This shows $Aut(V_3) = Inn(V_3) = \{id, (12)\}$.
   So $(1) \sim (2)$ and $P_1(V_3) = \{[(0), (1)]\}$ with $d_1(V_3) = 2$.

   For $P_2(V_3)$, we observe
   $$\begin{aligned}(1,1) &\sim (2,2)\\ (0,1) &\sim (0,2)\\ (1,0) &\sim (2,0)\\ (1,2) &\sim (2,1),\end{aligned}$$

   but the other combinations are not equivalent. This gives
   $$P_2(V_3) = \{[(0,0)], [(1,0)], [(0,1)], [(1,1)], [(1,2)]\}.$$
   In particular $d_2(V_3) = |P_2(V_3)| = 5$.

Now, we study which values $d_n(X)$ can have. For a finite quandle X with $k = |X|$, we immediately see $1 \leq d_n(X) \leq k^n$ because $\{(X, x_1, \dots, x_n) | x_i \in X\} \cong X^n$. Our goal is to find a better lower bound for $d_n(X)$ for an arbitrary finite quandle X with k elements.

Because $Aut(X)$ is a subgroup of the symmetric group S_k we will count the number of orbits of X^n under the diagonal action of S_k. That is $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ if there is a permutation $\pi \in S_k$ such that $x_i = \pi(y_i)$ for all $i = 1, \dots, n$. This is a lower bound for $d_n(X)$. We denote this number by $d_{0,n,k}$ and compute it in the following part.

For example $d_{0,1,k} = 1$ for all k, since for every $x, y \in X$ the transposition $(xy) \in S_k$ maps x to y, so all elements lie in the same orbit. If $k = 1$ then $d_{0,n,1} = 1$ because there is only one element in X^n.

If $k \geq 2$ then $d_{0,2,k} = 2$ because $(x, x) \sim (y, y)$ for all $x, y \in X$ as in the case $n = 1$. To see this, we note $(x_1, x_2) \sim (y_1, y_2)$ for all $x_1 \neq x_2$ and $y_1 \neq y_2$ using the permutation $(x_2y_2)(x_1y_1)$ if $x_2 \neq y_1$ and the permutation $(x_1y_1x_2)$ if $x_2 = y_1$. But $(x, x) \not\sim (y_1, y_2)$ with $y_1 \neq y_2$ because permutations are bijections.

This is the case for any n:

Lemma 4.14. *Two n-tuples, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in X^n$, lie in the same orbit under the action of S_k if and only if $x_i = x_j \Leftrightarrow y_i = y_j$ for all $i, j = 1, \dots, n$, i.e. they have equal entries in the same positions.*

Proof. Assume $x \sim y$ so there exists $\pi \in S_k$ with $\pi(x_i) = \pi(y_i)$ for all i. Because π is a bijection, we see $x_i = x_j \iff y_i = \pi(x_i) = \pi(x_j) = y_j$.
On the other hand if $x_i = x_j \iff y_i = y_j$, we can define a permutation on the set $\{x_1, \dots x_n, y_1, \dots, y_n\}$ of unique elements in the tuples with $\pi(x_i) = y_i$. So $\pi(x) = y$ and hence $x \sim y$. $\square$

Note that the statement in the lemma does not depend on k. By the lemma above we need to count the possibilities of how many entries are equal in a tuple and in which position these entries are. We will count these recursively over the number of entries in the tuple.

We extend our notation to $d_{m,n,k}$ for $m, n \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ and $m \leq k$ which denotes the number of equivalence classes of tuples $(x_1, \dots, x_{m+n}) \in X^{m+n}$ where $x_1, \dots x_m$ are fixed *unique* elements. Of course this is not possible if we would allow $m > k$.

For instance, $d_{1,1,k} = 2$ for $k \geq 2$ because given any fixed element x_1, we can either have $x_1 = x_2$ or $x_1 \neq x_2$. These are all equivalence classes. We immediately observe

$$d_{m,0,k} = 1,$$

since all entries are already fixed.

The reason we only allow unique elements in the fixed entries is that there are the same number of classes completing the tuple $(x_1, \dots, x_m, -, \dots, -) \in X^{m+n}$ as there are completing the tuple $(x_1, x_1, \dots, x_m, -, \dots, -) \in X^{m+1+n}$. So while counting the orbits, we can remove a fixed element if it is already fixed in another entry.

Now assume $k > m$. For a given a tuple $(x_1, \dots, x_m, -, \dots, -)$, we count the number of non-equivalent possibilities for the $(n+1)$-th entry.
We can either choose one of the distinct elements $x_1, \dots, x_m$. There are m such choices. For each choice there are now $d_{m,n-1,k}$ possibilities to complete the tuple, having no new fixed element in the tuple.
Or we choose a new element. Then there are $d_{m+1,n-1,k}$ many ways to complete the tuple. This leads us to

$$d_{m,n,k} = m \cdot d_{m,n-1,k} + d_{m+1,n-1,k}.$$

If $k = m$, then $\{x_1, \dots x_m\} = X$. This means we can only choose elements that are already in the tuple. This proves the following theorem.

Theorem 4.15. *Let X be a set with $|X| = k \in \mathbb{N} \cup \{\infty\}$. Let $d_{m,0,k} = 1$ and*

$$d_{m,n,k} = \begin{cases} d_{m,n,k} = m \cdot d_{m,n-1,k} + d_{m+1,n-1,k} & \text{if } m < k \\ d_{m,n,k} = m \cdot d_{m,n-1,k} & \text{if } m = k. \end{cases}$$

Then $|(X^n \,/\, S_k)| = d_{0,n,k}$.

Example 4.16. With this result we can explicitly compute $d_{m,n,k}$ for $n+m \leq k$:

- $n = 1$: We see that

  $$d_{m,1,k} = m d_{m,0,k} + d_{m+1,0,k} = m + 1.$$

  This shows $d_{0,1,k} = 1$, as we have seen before.

- $n = 2$:

  $$d_{m,2,k} = m d_{m,1,k} + d_{m+1,1,k} = m(m+1) + m + 2 = m^2 + 2m + 2$$

  So as expected $d_{0,2,k} = 2$.

- $n = 3$:

  $$d_{m,3,k} = m d_{m,2,k} + d_{m+1,2,k} = \ldots = m^3 + 3m^2 + 5m + 5$$

  and $d_{0,3,k} = 5$.

- Further $d_{0,4,k} = 15$, $d_{0,5,k} = 52$ and $d_{0,6,k} = 203$.

On the other hand for $n = 3$ and $k = 2$ we compute

$$d_{0,3,2} = d_{1,2,2} = 1 \cdot d_{1,1,2} + d_{2,1,2} = (d_{1,0,2} + d_{2,0,2}) + 2 \cdot d_{2,0,2} = 4.$$

Thinking about triplets in $\{a, b\}^3$, there are exactly the four equivalent classes $[(a, a, a)]$, $[(a, a, b)]$, $[(a, b, a)]$ and $[(b, a, a)]$. In $\{a, b, c\}^3$ there is one more class, namely $[(a, b, c)]$.

4.3 n-homogeneous quandles

Recall that in Section 4.2 we defined

$$d_n(X) := |P_n(X)| = \left| \left(\{(X, x_1, \ldots, x_n) | x_i \in X\} \big/ Aut(X) \right) \right|.$$

We give quandles with minimal $d_n(X)$ a special name and study them in more detail.

Definition 4.17. Let X be a quandle and $n \in \mathbb{N}$. We say X is *n-homogeneous* if $d_n(X) = d_{n,0,|X|}$. We say X is *uniform* or *∞-homogeneous* if X is n-homogeneous for all $n \in \mathbb{N}$.

Proposition 4.18. *A quandle is 1-homogeneous if and only if it is homogeneous (as defined in Definition 3.9).*

Proof. Observe that $d_1(X) = 1$ means $(X, x_1) \sim (X, x_2)$ for all $x_1, x_2 \in X$. Hence, there is a quandle automorphism $f \in Aut(X)$ with $x_2 = f(x_1)$. This is exactly the definition of being homogeneous. □

Remark 4.19. In [TAM13] the concept of a *two-pointed homogeneous quandle* is introduced. A quandle X is two-pointed homogeneous if the action of $Inn(X)$ on $U_2^{-1}(X)$ has two orbits. There we act on $U_2^{-1}(X)$ only with automorphisms that are inner automorphisms. This implies X being connected. Every two-pointed homogeneous quandle is of course 2-homogeneous as in our definition above. The opposite is not true.

Lemma 4.20. *Let X be a quandle, $n \in \mathbb{N}$. The following are equivalent.*

(1) X is n-homogeneous.

(2) X is $(n-1)$-homogeneous and if $x_i \neq x_j$ and $y_i \neq y_j$ for all $i \neq j$ then $(X, x_1, \ldots, x_n) \cong (X, y_1, \ldots, y_n)$.

(3) Any two n-pointed quandles $\mathcal{X} = (X, x_1, \ldots, x_n)$ and $\mathcal{Y} = (X, y_1, \ldots, y_n)$ with underlying quandle X are isomorphic if and only if $x_i = x_j \Leftrightarrow y_i = y_j$ for all $i, j = 1, \ldots, n$.

Proof. We first proof (1) ⇔ (3) and use this to show (2) ⇔ (3).

(1) ⇒ (3) Let X be n-homogeneous. If $\mathcal{X} \cong \mathcal{Y}$, then the right-hand side of (3) follows immediately.
Let's assume $x_i = x_j \Leftrightarrow y_i = y_j$ for all $i, j = 1, \ldots, n$ holds. By Lemma 4.14 $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ lie in the same orbit under the action of S_k. Because $Aut(X)$ is a subgroup of S_k, the orbit $Aut(X) \cdot x \subseteq S_k \cdot x$ for all $x \in X$. Now X is n-homogeneous, meaning there are the same number of orbits under the action of $Aut(X)$ and S_k. Since orbits are either equal or disjoint the orbits must be all equal. So $Aut(X) \cdot x = S_k \cdot x$. Hence, indeed $\mathcal{X} \sim_{Aut(X)} \mathcal{Y} \in P_n(X)$ which shows $\mathcal{X} \cong \mathcal{Y}$.

(3) ⇒ (1) Assume (3). So $\mathcal{X} \cong \mathcal{Y}$ if and only if $(x_1, \ldots, x_n) \sim_{S_k} (y_1, \ldots, y_n)$. This means $S_k \cdot x = Aut(X) \cdot x$ for all $x \in X^n$. Hence, X is n-homogeneous.

(2) ⇒ (3) Let $\mathcal{X} = (X, x_1, \ldots, x_n)$ and $\mathcal{Y} = (X, y_1, \ldots, y_n)$ with $x_i = x_j \Leftrightarrow y_i = y_j$ for all $i, j = 1, \ldots, n$. If all x_i are different (and hence all y_i), then $\mathcal{X} \cong \mathcal{Y}$ by the second part of (2). If there are $i \neq j$ with $x_i = x_j$ (and therefore $y_i = y_j$), then $(X, x_1, \ldots, \hat{x_j}, \ldots, x_n) \cong (X, y_1, \ldots, \hat{y_j}, \ldots, x_n)$ are isomorphic $(n-1)$-pointed quandles because X is $(n-1)$-homogeneous and (1) ⇔ (3). The same quandle isomorphism gives $\mathcal{X} \cong \mathcal{Y}$.

(3) ⇒ (2) Assume (3), so $(X, x_1, \ldots x_{n-1}, x_{n-1}) \cong (X, y_1, \ldots, y_{n-1}, y_{n-1})$ if and only if $x_i = x_j \Leftrightarrow y_i = y_j$ for all $i, j = 1, \ldots, n-1$. This shows that X is indeed $(n-1)$-homogeneous. The second part of (2) also follows immediately.

□

In particular 4.20(2) implies the following corollary.

Corollary 4.21. *If a quandle X is n-homogeneous, then it is m-homogeneous for all $m \leq n$.*

Lemma 4.20 above says that a quandle X is n-homogeneous if and only if all n-pointed quandles $\mathcal{X}$ and $\mathcal{Y}$ with $U_n(\mathcal{X}) = X = U_n(\mathcal{Y})$ are isomorphic whenever possible. Here possible means $x_i = x_j \Leftrightarrow y_i = y_j$ for all $i, j = 1, \ldots, n$.

Proposition 4.22. *Let X be a quandle and $k = |X| \in \mathbb{N} \cup \{\infty\}$.*

(1) X is uniform if and only if X is k-homogeneous.

(2) X is uniform if and only if X is $(k-1)$-homogeneous.

(3) If $Aut(X) \cong S_k$, then X is uniform.

(4) If $k < \infty$ is finite and X uniform, then $Aut(X) \cong S_k$.

Proof. (1) A uniform quandle is obviously k-homogeneous. Let X be k-homogeneous. If $k = \infty$, we are done. So let $k \in \mathbb{N}$ be finite. By Lemma 4.20 (2), it is n-homogeneous for all $n \leq k$. On the other hand, if X is n-homogeneous with $n \geq k$, there is no pointed $(n+1)$ quandle $(X, x_1, \ldots x_{n+1})$ with $y_i \neq y_j$ for all $i \neq j \in \{1, \ldots, n+1\}$. So again Lemma 4.20 (2) shows that X is $(n+1)$-homogeneous. Because X is k-homogeneous, it is by induction uniform.

(2) Let X be a $(k-1)$-homogeneous quandle. We want to show that X is k-homogeneous and use (1). Again by Lemma 4.20 it is enough to show $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_k)$ for $x_i \neq x_j$ and $y_i \neq y_j$ for all $i \neq j$. We know that there exists $f \in Aut(X)$ with $f(x_i) = y_i$ for $i = 1, \ldots, k-1$ since X is $(k-1)$-homogeneous. Because $k = |X|$ and f is a bijection we find $f(x_k) = y_k$, so indeed $(x_1, \ldots, x_k) \overset{f}{\sim} (y_1, \ldots, y_k)$.

(3) Follows immediately from the definition together with 4.15.

(4) Let $X = \{x_1, \ldots, x_k\}$ be uniform and assume $Aut(X) \subsetneq S_k$. Then there exists $f \in S_k \setminus Aut(X)$. This means $(x_1, \ldots, x_k) \underset{Aut(X)}{\nsim} (f(x_1), \ldots, f(x_k))$ by assumption. This contradicts X being k-homogeneous because of 4.20(2).

□

Example 4.23.

- All trivial quandles are uniform, since $Aut(T_k) = S_k$ for all $k \in \mathbb{N} \cup \{\infty\}$. Note that trivial quandles are not two-point homogeneous as described in Remark 4.19.
- R_3 is uniform as we have seen in Example 4.13.
- V_3 in Example 4.13 is not 2-homogeneous.
- Let us consider the regular tetrahedron quandle $X = \{0, 1, 2, 3\}$ with

$$\beta_0 = (123), \qquad \beta_1 = (032), \qquad \beta_2 = (013), \qquad \beta_3 = (021).$$

  Note that X is connected, so in particular 1-homogeneous. Now $(x, y) \overset{\beta_x^i}{\sim} (x, z)$ for some i for all $y \neq x \neq z$. This way we can reach every tuple which does not have equal entries. Here we see for example

$$(0, 1) \overset{\beta_0}{\sim} (0, 2) \overset{\beta_2}{\sim} (1, 2) \overset{\beta_1}{\sim} (1, 0).$$

  So X is 2-homogeneous. Assume there exists $f \in Aut(X)$ such that $(0, 1, 2) \overset{f}{\sim} (1, 0, 2)$, so $1 = f(0) = f(2 \triangleright 1) = f(2) \triangleright f(1) = 2 \triangleright 0 = 3$ which is a contradiction. Hence, $(0, 1, 2) \nsim (1, 0, 2)$. This means X is not 3-homogeneous and in particular not uniform.

The last example can be generalized to a wider set of examples. A finite quandle X with $k = |X|$ is called of *cyclic type* if for every $x \in X$, the map β_x acts on $X \setminus \{x\}$ as a cyclic permutation of order $(k-1)$ as defined in [TAM13, Def. 3.5]. We immediately see that for example the regular tetrahedron quandle is of cyclic type.

Proposition 4.24 ([TAM13, Prop. 3.6])**.** *Every finite quandle of cyclic type is 2-homogeneous.*

Now we classify all uniform quandles.

Theorem 4.25. *Let X be a uniform and finite quandle. Then either $X \cong R_3$ or X is trivial.*

Proof. Let X be a nontrivial uniform quandle. This means there is $a_0, a_1, a_2 \in X$ with $a_0 \triangleright a_1 = a_2$ and $a_0 \neq a_2$. Then also $a_0, a_2 \neq a_1$. Since X is uniform, $Aut(X) \cong S_{|X|}$ by Proposition 4.22, in particular are the transpositions $(a_i a_j)$

in $Aut(X)$. So $a_1 = (a_1a_2)(a_2) = (a_1a_2)(a_0 \triangleright a_1) = a_0 \triangleright a_2$, similarly $a_0 = a_2 \triangleright a_1$ and therefore

$$a_i \triangleright a_j = a_k = (a_ia_j)(a_k) = (a_ia_j)(a_i \triangleright a_j) = a_j \triangleright a_i$$

for all $\{i, j, k\} = \{0, 1, 2\}$. This computes the following section of the operation table of X.

$\triangleright$	a_0	a_1	a_2	$\dots$
a_0	a_0	a_2	a_1	
a_1	a_2	a_1	a_0	?
a_2	a_1	a_0	a_2	
$\vdots$		?		

If X has only three elements, this means it is isomorphic to R_3.
Suppose X has another element $x \neq a_0, a_1, a_2$ and assume $a_i \triangleright x \neq a_i$ for some i. Then choose $a_j \neq a_i, a_i \triangleright x$ (which exists since we have three elements to choose from). Now we see

$$a_i \triangleright x \underset{a_i \triangleright x \neq a_i, a_j}{=} (a_ia_j)(a_i \triangleright x) = a_j \triangleright x,$$

which is a contradiction to β_x being bijective. So $a_i \triangleright x = a_i$ for all i.
But now we see, using the transposition $(a_0x) \in Aut(X)$, that $(a_0x)(a_1 \triangleright x) = (a_0x)(a_1) = a_1$ but $(a_0x)(a_1) \triangleright (a_0x)(x) = a_1 \triangleright a_0 = a_2$. But this is a contradiction to (a_0x) being a homomorphism. This shows that X either has only three elements or is trivial. $\square$

For $x \in X$, denote $Aut(X)_x := \{f \in Aut(X) | f(x) = x\}$ the stabilizer subgroup with respect to x. We follow the proof of [TAM13, Proposition 3.3] to prove the next Proposition.

Proposition 4.26. *Let X be a quandle with $|X| \geq 3$. Then the following are equivalent:*

(1) X is 2-homogeneous.

(2) For every $x \in X$, the action of $Aut(X)_x$ on $X \setminus \{x\}$ is transitive.

(3) X is homogeneous and there exists $x \in X$ such that the action of $Aut(X)_x$ on $X \setminus \{x\}$ is transitive.

Proof.

(1) $\Rightarrow$ (2) Let X be 2-homogeneous. For arbitrary $x \in X$, we know that $(x, y_1) \overset{f}{\sim} (x, y_2)$ for any $y_1, y_2 \neq x$ for some $f \in Aut(X)$. So $f \in Aut(X)_x$ and $f(y_1) = y_2$. This shows that the action of $Aut(X)_x$ is transitive on $X \setminus \{x\}$.

(2) $\Rightarrow$ (3) We only need to show that X is homogeneous. For any $y_1, y_2 \in X$ we can choose $x \in X \setminus \{y_1, y_2\}$ because $|X| \geq 3$. By assumption, there exists $f \in Aut(X)_x \subseteq Aut(X)$ with $f(y_1) = y_2$.

(3) $\Rightarrow$ (1) Assume (3). So X is 1-homogenous. By Lemma 4.20, it is enough to show that $(x_1, x_2) \sim (y_1, y_2)$ for all $x_1 \neq x_2$ and $y_1 \neq y_2$. By assumption, there exists $x \in X$ such that $Aut(X)_x$ acts transitive on $X \setminus \{x\}$. Because X is (1-)homogeneous, there exists $f, g \in Aut(X)$ with $f(x_1) = x$ and $g(y_1) = x$. Because $x_1 \neq x_2$ and $y_1 \neq y_2$ and f and g are bijective, we see $f(x_2), g(y_2) \neq x$. Hence, there exists $h \in Aut(X)_x$ with $h(f(x_2)) = g(y_2)$. Now we can see

$$g^{-1} \circ h \circ f(x_1, x_2) = g^{-1} \circ h(x, f(x_2)) = g^{-1}(x, g(y_2)) = (y_1, y_2)$$

which shows $(x_1, x_2) \sim (y_1, y_2)$.

□

4.4 Pointed quandle counting invariant and quandle counting matrix

We now turn our attention back towards linkoids and introduce n-pointed quandle colorings of linkoids.

Definition 4.27. Let $\mathcal{X}$ be a $2n$-pointed quandle and L an n-linkoid. We define the *pointed quandle counting invariant* as

$$\Phi^{\mathbb{Z}}_{\mathcal{X}}(L) := |\mathbf{PQnd_{2n}}(P(L), \mathcal{X})|.$$

Theorem 4.28. *The pointed quandle counting invariant is invariant under Reidemeister moves.*

Proof. This follows immediately from the fact that $P(L)$ is a linkoid invariant. □

Note that the (unpointed) quandle counting invariant with respect to a quandle X is always at least 1 because we can color the diagram trivially. For the pointed quandle counting invariant this is in general not true. If not all basepoints of a pointed quandle $\mathcal{X}$ are equal, we cannot color the diagram trivially. This means $\Phi^{\mathbb{Z}}_{\mathcal{X}}(L)$ can be zero.

We can combine all pointed quandles in $U_2^{-1}(X)$, the set of 2-pointed quandles with underlying quandle X, into one matrix.

Definition 4.29. Let L be a 1-linkoid and $X = \{1, \dots, k\}$ a finite quandle. We define the *quandle counting matrix* $\Phi^{M_k}_X(L)$ of L with respect to X by the $k \times k$ matrix

$$\left(\Phi^{M_k}_X(L)\right)_{i,j} := \Phi^{\mathbb{Z}}_{(X,i,j)}(L),$$

with the pointed quandle counting invariant for each possible combination of basepoints as entries.

Theorem 4.30. *Let X be a finite quandle, L a 1-linkoid. The quandle counting matrix $\Phi^{M_k}_{(X,i,j)}(L)$ is an invariant of L.*

Proof. This follows immediately from the fact that the pointed quandle counting invariant is invariant under Reidemeister moves. □

We first collect some basic properties and observations and then see some examples. Therefore, we use the fact that for two isomorphic $2n$-pointed quandles $\mathcal{X}$ and $\mathcal{Y}$ the pointed quandle counting invariant is equal, that is $\Phi^{\mathbb{Z}}_{\mathcal{X}}(L) = \Phi^{\mathbb{Z}}_{\mathcal{Y}}(L)$ for any n-linkoid.

Proposition 4.31. *Let L be a 1-linkoid and X a finite quandle, $i, j \in X$. Then*

(1) $(\Phi^{M_k}_X(L))_{i,j} \geq 0$.

(2) $(\Phi^{M_k}_X(L))_{i,i} \geq 1$.

(3) $\Phi^{M_k}_X(L) = I_k$ *the identity matrix if and only if L is only trivially colorable by X.*

(4) $\sum_{i,j=1}^{k}(\Phi^{M_k}_X(L))_{i,j} = \Phi^{\mathbb{Z}}_X(L) = |\mathbf{Qnd}(Q(L), X)|$ *the (unpointed) quandle coloring counting invariant.*

(5) If X is homogeneous, then $(\Phi^{M_k}_X(L))_{i,i} = (\Phi^{M_k}_X(L))_{j,j}$.

(6) If X is 2-homogeneous, then $(\Phi^{M_k}_X(L))_{i_1,j_1} = (\Phi^{M_k}_X(L))_{i_2,j_2}$ *for all $i_1 \neq j_1$ and $i_2 \neq j_2 \in X$.*

Proof. (1) is obvious, (2) and (3) follow from the fact that every knotoid is trivially colorable. (4) is the fact that the quandle coloring counting invariant counts all possible colorings, independent of the endpoint colors.
To see (5) and (6), note that if $(i_1, j_1) \sim (i_2, j_2) \in X^2 \,/\, Aut(X) \cong P_2(X)$, then $\Phi^{\mathbb{Z}}_{(X,i_1,j_1)}(L) = \Phi^{\mathbb{Z}}_{(X,i_2,j_2)}(L)$ since the pointed quandles are isomorphic. So if X is homogeneous then $(i, i) \sim (j, j)$ for all $i, j \in X$ and if X is 2-homogeneous $(i_1, j_1) \sim (i_2, j_2)$ for all $i_1 \neq j_1$ and $i_2 \neq j_2 \in X$.

□

Note that any 2-homogeneous quandle is also homogeneous, so all values on the diagonal are also equal, but not necessarily equal to the values not on the diagonal.

Lemma 4.32. *Let L be a 1-linkoid and X a finite quandle. If $i, j \in X$ are in the same algebraic component of X (that is in the same orbit under the action of $Inn(X)$ on X), then* $(\Phi^{M_k}_X(L))_{i,i} = (\Phi^{M_k}_X(L))_{j,j}$.

Proof. If i, j lie in the same component of X then there exists an inner automorphism $f \in Inn(X) \subseteq Aut(X)$ with $f(i) = j$. So $(i, i) \overset{f}{\sim} (j, j)$. □

Proposition 4.33. *Let L be a link-type 1-linkoid and X a finite quandle, $i, j \in X$. Then:*

(1) *The trace* $tr\left(\Phi_X^{M_k}(L)\right) = \Phi_X^{\mathbb{Z}}(L_-)$*, the (unpointed) quandle counting invariant of the under-closure of* L*.*

(2) *If* X *is faithful, then* $\Phi_X^{M_k}(L))_{i,j} = 0$ *for all* $i \neq j$

Proof. To see (1), we note that

$$\mathbf{Qnd}(Q(K_-), X) \cong \mathbf{Qnd}\left({}^{(Q(K))}\Big/_{(h = l)}, X\right) \cong \bigcup_{i=1}^{k} \mathbf{PQnd_2}(P(K), (X, i, i))$$

as sets. The first bijection follows from the fact that $Q(K_-) \cong Q(K) \, / \, (h = l)$ where h and l are the arcs attached to the head and leg of X. To see the second bijection we note that a quandle homomorphism $\varphi\colon (Q(K)) \, / \, (h = l) \to X$ is the same as a quandle homomorphism $Q(K) \to X$ with $\varphi(h) = \varphi(l)$ and therefore an element in $\mathbf{PQnd_2}(P(K), (X, \varphi(l), \varphi(h)))$. And of course every pointed homomorphism in $\psi \in \mathbf{PQnd_2}(P(K), (X, i, i)$ satisfies $\psi(h) = i = \psi(l)$ and so is an element of $\mathbf{Qnd}\left((Q(K)) \, / \, (h = l), X\right)$.
(2) follows immediately from Corollary 3.30 which states that every coloring by a faithful quandle assigns the same color to both endpoints. Hence, $\Phi_X^{M_k}(K)_{i,j} = 0$ for all $i \neq j$. ☐

Combining Proposition 4.31 (4) and 4.33 proves the following corollary.

Corollary 4.34. *Let* L *be a link-type 1-linkoid and* X *a finite, faithful quandle. Then* $\Phi_X^{\mathbb{Z}}(L) = \Phi_X^{\mathbb{Z}}(L_-)$*.*

We now compute some quandle counting matrices.

Example 4.35. Let K_1 be the knotoid from Figure 15. So

$$\begin{aligned} Q(K_1) &\cong Q\langle a, b, c, d | b = a \triangleright c, c = b \triangleright a, d = c \triangleright b \rangle \\ &\cong Q\langle a, b, c | b = a \triangleright c, c = b \triangleright a \rangle \end{aligned}$$

and $P(K_1) = (Q(K_1), a, c \triangleright b)$.

Consider the pointed quandle $X = (R_3, 0, 0)$. A homomorphism $P_2(K_1) \to X$ maps $a, d \mapsto 0$. For any given value of $f(b)$ we compute $f(c) = f(b \triangleright a) = f(b) \triangleright 0 = 2 \cdot 0 - f(b) = -f(b) \pmod 3$. We need to check the other relations in $Q(K_1)$, to see which values give indeed a quandle homomorphism. That is $f(a \triangleright c) = 0 \triangleright f(c) = 2 \cdot 0 - f(c) = f(c) \pmod 3$ and $f(c \triangleright b) = f(c) \triangleright f(b) = 2f(b) - f(c) = 3f(b) = 0 \pmod 3$. So every value for $f(b)$ determines exactly on coloring by $\mathcal{X}$. This shows $\Phi_X^{\mathbb{Z}}(K_1) = 3$.

Now we can compute the quandle coloring matrix of K_1. Since R_3 is 2-homogeneous, all diagonal entries are equal by Proposition 4.31. By Proposition

4.33 all other entries are zero, since K_1 is of knot type. In total, this gives

$$\Phi_{R_3}^{M_3}(K_1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Computing the trace $tr(\Phi_{R_3}^{M_3}(K_1)) = 9 = \Phi_{R_3}^{\mathbb{Z}}(K_-)$, we again get the quandle counting invariant that we already computed in Example 3.18.

Let now K_2 be the other knotoid in Figure 15. Because $Q(K_1) \cong Q(K_2)$ we write $P(K_2) = (Q(K_1), a, c)$. A homomorphism $f\colon P_2(K_2) \to (R_3, i, j)$ maps $a \mapsto i$ and $c \mapsto j$. So $f(b) = f(a \triangleright c) = i \triangleright j$ hence f is already completely determined. Because $f(b \triangleright a) = (i \triangleright j) \triangleright i = 2i - (i \triangleright j) = 2i - 2j + i \equiv j = f(c) \pmod 3$ every such map is a quandle homomorphism. Then $\Phi_{(R_3,i,j)}^{\mathbb{Z}}(K_2) = 1$ for all i, j and

$$\Phi_{R_3}^{M_3}(K_2) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

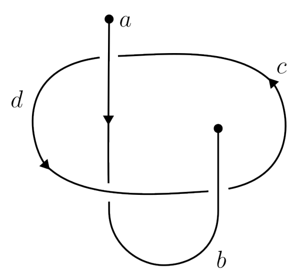


Figure 16

As a last example consider the 1-linkoid L in Figure 16 and the quandle V_3 with three elements and $\beta_0 = (12)$ and $\beta_1 = \beta_2 = id$. We have seen in Example 4.13 that $d_2(V_3) = 5$. This means there are five isomorphism classes of pointed quandles with underlying quandle V_3. So we have to compute five pointed coloring counting invariants for the coloring matrix.
The fundamental quandle of L is given by

$$Q(L) = Q\langle a, b, c, d | c = d \triangleright b, a = b \triangleright c, d = c \triangleright a \rangle.$$

We now compute the pointed quandle counting invariant for all five isomorphism classes of pointed quandles with underlying quandle V_3.

- A pointed quandle homomorphism $f\colon P(L) \to (V_3, 0, 0)$ maps $a, b \mapsto 0$. If $f(c) = 0$ then $f(d) = f(0 \triangleright 0) = 0$ gives only the trivial coloring. If $f(c) = 1$ then $f(d) = f(c \triangleright a) = 1 \triangleright 0 = 2$. This satisfies $f(d \triangleright b) = 2 \triangleright 0 = 1 = f(c)$ and $f(b \triangleright c) = 0 \triangleright 1 = 0 = f(a)$. So $\Phi_{(V_3,0,0)}^{\mathbb{Z}}(L) = 3$.

- Now consider a map $f\colon P(L) \to (V_3, 0, 1)$. Then $0 = f(a) = f(b \triangleright c) = 1 \triangleright f(b)$ which cannot happen.
- Similarly, for a map $f\colon P(L) \to (V_3, 1, 0)$ we have $0 = f(b) = f(a \triangleright^{-1} c) = 1 \triangleright^{-1} f(b)$ which is also not possible.
- A homomorphism $f\colon P(L) \to (V_3, 1, 1)$ map $f(c) = f(d \triangleright b) = f(d) \triangleright b = f(d)$. Now $f(a) = f(b) \triangleright f(c)$ gives $1 = 1 \triangleright f(c)$ so $f(c) = 1, 2$. Both values satisfy the third relation $f(d) = f(c \triangleright a) = fa(c)$. This shows $\Phi^{\mathbb{Z}}_{(V_3,0,0)}(L) = 2$.
- Lastly let $f\colon P(L) \to (V_3, 1, 2)$. Because $1 = f(a) = f(b) \triangleright f(c) = 2 \triangleright f(c)$ we obtain $f(c) = 2$. Then $f(d) = 2 \triangleright 1 = 2$. This satisfies $f(d \triangleright b) = 2 \triangleright 1 = 2 = f(c)$. So $\Phi^{\mathbb{Z}}_{(V_3,1,2)}(L) = 1$.

This yields the quandle counting matrix

$$\Phi^{M_3}_{V_3}(L) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

5 Cohomology and the quandle 2-cocycle invariant

In [CJK+01] a (co-)homology theory for quandles was introduced. We review the definition and then focus on 2-cocycles to review the definition of the *2-cocycle invariant* associated to a given 2-cocyle. Then we introduce an enhancement of the 2-cocycle invariant which we name *pointed quandle 2-cocycle invariant*. This uses pointed quandles we studied in the previous section.

There are also applications for 3-cocycles on links that can be generalized to linkoids, for example in [Caz22] for knotoids. We do not study 3-cocycles and their applications to linkoids here.

5.1 Review of quandle (co-)homology

We begin with the definition of a chain complex following [CJK+01, Chapter 3]. Let X be a quandle. Define $C^R_n(X)$ as the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of X. We write $C^R_n(X)$ additively for now. Let $C^D_n(X) \subseteq C^R_n(X)$ be the subgroup generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in 1 \ldots n-1$ if $n \geq 2$ and $C^D_1(X) = 0$. Finally, let $C_n(X) \coloneqq C^R_n(X) \,/\, C^D_n(X)$.

An element in $C_n(X)$ is a linear combination $\sum_\alpha n_\alpha(x_1, \ldots, x_n)$ for some $n_\alpha \in \mathbb{Z}$. Note that even though $(x_1, \ldots, x_n)$ looks like a vector, it is not. We cannot simplify terms like $(x_1, \ldots, x_n) + (y_1, \ldots, y_n)$ for $x_i, y_i \in X$ unless both terms are equal. For example let $X = R_3$ then $(1,0) + 2(1,0) = 3(1,0)$, but $(1,0) + (0,1) \neq (1,1) = (0,0)$ in $C_2(R_3)$.

Now define the boundary map $\partial_n \colon C_n^R \to C_{n-1}^R$ on generators by

$$\partial_n(x_1,\ldots,x_n) = \sum_{i=2}^{n}(-1)^i\big[(x_1,\ldots,\widehat{x_i},\ldots,x_n) \\ - (x_1 \triangleright x_i, x_2 \triangleright x_i, \ldots, x_{i-1} \triangleright x_i, x_{i+1}, \ldots, x_n)\big]$$

for $n > 2$ and $\partial_n = 0$ for $n \leq 1$, where $\widehat{x_i}$ means x_i is removed from the tuple.

Theorem 5.1. *It holds that $\partial_{n-1} \circ \partial_n = 0$, and $\partial_n(C_n^D(X)) \subseteq C_{n-1}^D(X)$. So the above construction is a chain complex.*

We will see the proof of this theorem after defining homology groups and seeing some examples.

As usual in homology theory, we can allow coefficients in any fixed abelian group A. We do this by defining

$$C_n(X;A) \coloneqq C_n(X) \otimes A \quad \text{and} \quad \partial = \partial \otimes id$$

We can think of elements in $C_n(X;A)$ as linear combinations $\sum_\alpha n_\alpha(x_1,\ldots,x_n)$ but now with coefficients $n_\alpha \in A$.

Definition 5.2. We define cycle and boundary groups as $Z_n(X;A) \coloneqq ker(\partial_n)$ and $B_n(X;A) \coloneqq im(\partial_{n+1})$ and homology groups

$$H_n(X;A) \coloneqq {Z_n(X;A)}\big/{B_n(X;A)}.$$

For cohomology, we define the cochain groups as

$$C^n(X;A) \coloneqq Hom(C_n(X),A),$$

the dual groups. The coboundary map $\delta^n \colon C^n(X;A) \to C^{n+1}(X;A)$ is the precomposition with the boundary map ∂_{n+1} from above. Because $\partial_{n-1}\partial_n = 0$ also $\delta^n\delta^{n-1} = 0$. Following the usual concepts of homology, we define:

Definition 5.3. The *cocycle groups* are given by $Z^n(X;A) \coloneqq ker(\delta^n)$ and the *coboundary groups* by $B^n(X;A) \coloneqq im(\delta^{i-1})$. We define cohomology groups as

$$H^n(X;A) \coloneqq {Z^n(X;A)}\big/{B^n(X;A)}.$$

For now, we are mostly interested in cocycles.

Example 5.4. A *quandle 1-cocycle* $f \in Z^1(X;A) = ker(\delta^1) \subseteq C^1$ is a linear function $A[X] \to A$ or equivalently, a function $\varphi\colon X \to A$ that extends linearly to $A[X]$. For a 1-cocycle $\varphi \in ker(\delta^1)$, we see $\delta^1 \circ \varphi = 0 \in C^2(X; A)$. This gives for $x, y \in X$ the following condition.

$$0 = (\delta^1 \circ \varphi)(x, y) = (\varphi \circ \partial_2)(x, y) = \varphi((x) - (x \triangleright y)) = \varphi(x) - \varphi(x \triangleright y).$$

A *1-cocycle* is a function $\varphi \colon X \to A$ that satisfies

$$\varphi(x) - \varphi(x \triangleright y) = 0$$

for all $x, y \in X$.

Example 5.5. Let $\varphi \in Z^2(X; A)$ be a 2-cocycle. So we have a function $\varphi \colon X \times X \to A$ that we extend linearly as before. Here there is a bit of abuse of notation, because in $Z^2(X; A)$ we write $\varphi(n_{(x_1,x_2)}(x_1, x_2))$ but as a map $X \times X \to A$ we write $n_{(x_1,x_2)}\varphi(x_1, x_2)$ for $x_1, x_2 \in X$, and all $n_\alpha \in A$.
Because $(x, x) = 0 \in C_2(X)$ for all x, we immediately see $\varphi(x, x) = \varphi(0) = 0$ for all $x \in X$ since φ is linear. The formula for ∂_3 from above is

$$\partial_3(x, y, z) = [(x, z) - (x \triangleright y, z)] - [(x, y) - (x \triangleright z, y \triangleright z)].$$

Inserting it to $\varphi \in ker(\delta^2) \iff \varphi \circ \partial_3 = 0$ gives

$$\begin{aligned} 0 = \varphi \circ \partial_3(x, y, z) &= \varphi([(x, z) - (x \triangleright y, z)] - [(x, y) - (x \triangleright z, y \triangleright z)]) \\ &= [\varphi(x, z) - \varphi(x \triangleright y, z)] - [\varphi(x, y) - \varphi(x \triangleright z, y \triangleright z)] \end{aligned}$$

for all $x, y, z \in X$. Rearranging that, we can summarize that a *2-cocycle* of the quandle X with coefficients in A is a function $\varphi \colon X \times X \to A$ such that for all $x, y, z \in X$ the conditions

$$\varphi(x, x) = 0$$
$$\varphi(x, z) + \varphi(x \triangleright z, y \triangleright z) = \varphi(x, y) + \varphi(x \triangleright y, z)$$

are satisfied.

Example 5.6. We now want to understand 2-coboundaries $\varphi \in B^2(X, A)$. Given a function $f \in C^1$ we compute

$$(\delta_1 \circ f)(x, y) = (f \circ \partial_2)(x, y) = f(x - x \triangleright y) = f(x) - f(x \triangleright y)$$

Hence, 2-coboundaries are exactly the 2-cocyles given by $\varphi(x, y) = f(x) - f(x \triangleright y)$ for any function $f \colon X \to A$.

Proof of Theorem 5.1. For the first statement $\partial_{n-1} \circ \partial_n = 0$, we want to compute

$$\begin{aligned} \partial_{n-1} \circ \partial_n(x_1, \ldots, x_n) = &\sum_{i=2}^{n} (-1)^i \partial_{n-1}(x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_n) \\ &- \sum_{i=2}^{n} (-1)^i \partial_{n-1}(x_1 \triangleright x_i, x_2 \triangleright x_i, \ldots, x_{i-1} \triangleright x_i, x_{i+1}, \ldots, x_n). \end{aligned}$$

We compute both sums separately.

$$\begin{aligned}
&\sum_{i=2}^{n}(-1)^i\partial_{n-1}(x_1,\dots,\widehat{x_i},\dots,x_n)\\
&=\sum_{i=2}^{n}\Bigg(\sum_{j=2}^{i}(-1)^i(-1)^j\big[(x_1,\dots,\widehat{x_j},\dots,\widehat{x_i},\dots,x_n)\\
&\qquad -(x_1\triangleright x_j,\dots,x_{i-1}\triangleright x_j,x_{i+1},\dots,\widehat{x_i},\dots,x_n)\big]\\
&+\sum_{j=i+1}^{n}(-1)^i(-1)^{j-1}\big[(x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_n)\\
&\qquad -(x_1\triangleright x_j,\dots,\widehat{x_i\triangleright x_j},\dots,x_{j-1}\triangleright x_j,x_{j+1},\dots,x_n)\big]\Bigg).
\end{aligned}$$

Here we used an index shift on j in the second sum. Similarly,

$$\begin{aligned}
&\sum_{i=2}^{n}(-1)^i\partial_{n-1}(x_1\triangleright x_i,\dots,x_{i-1}\triangleright x_i,x_{i+1},\dots,x_n)\\
&=\sum_{i=2}^{n}\Bigg(\sum_{j=2}^{i}(-1)^i(-1)^j\big[(x_1\triangleright x_i,\dots,\widehat{x_j\triangleright x_i},\dots,x_{i-1}\triangleright x_i,x_{i+1},\dots,x_n)\\
&\qquad -\big((x_1\triangleright x_i)\triangleright(x_j\triangleright x_i),\dots,(x_{j-1}\triangleright x_i)\triangleright(x_j\triangleright x_i),\\
&\qquad\qquad x_{j+1}\triangleright x_i,\dots,x_{i-1}\triangleright x_i,x_{i+1},\dots,x_n\big)\big]\\
&+\sum_{j=i+1}^{n}(-1)^i(-1)^{j-1}\big[(x_1\triangleright x_i,\dots,x_{i-1}\triangleright x_i,x_{i+1},\dots,\widehat{x_j},x_n)\\
&\qquad -\big((x_1\triangleright x_i)\triangleright x_j,\dots,(x_{i-1}\triangleright x_i)\triangleright x_j,\\
&\qquad\qquad x_{i+1}\triangleright x_j,\dots,x_{j-1}\triangleright x_j,x_{j+1},\dots x_n\big)\big]\Bigg).
\end{aligned}$$

We now see that the terms with matching colors cancel since after switching i and j in one of them, it is the negative of the other. In the blue term we also used the third quandle axiom $(x_k\triangleright x_j)\triangleright x_i=(x_k\triangleright x_i)\triangleright(x_j\triangleright x_i)$.

For the second statement in the theorem let $(x_1,\dots,x_n)$ be such that $x_j=x_{j+1}=a$ for some j. Then for each $i\neq j,j+1$ the corresponding summand is again in $C^D_{n-1}(X)$. If $j=1$, then the first term $(i=2)$is

$$(-1)^2((a,x_3,\dots,x_n)-(a\triangleright a,x_3,\dots,x_n))=0.$$

For $j>2$ the summands for $i=j$ and $i=j+1$ are equal but with sign $(-1)^j$ and $(-1)^{j+1}$. Hence, both cancel. The total sum is therefore a linear combination of elements in $C^D_{n-1}(X)$. $\square$

5.2 Quandle 2-cocycle invariant

From now on we write A multiplicatively to simplify the notation. Let X be a *finite* quandle, L a linkoid and $f\colon Q(L) \to X$ a coloring. We give weights to each crossing to enhance the quandle counting invariant as defined in Definition 3.17. Therefore, let $\varphi \in Z^2(X; A)$ be a 2-cocycle. We give every crossing a (Boltzmann) weight that depends on the chosen coloring f as shown in Figure 17. Note that the inputs for φ are the labels of the arcs on the left of the diagram, unlike in the diagrams we used to define quandle relations on a given crossing. Note also the position of x and y in the negative crossing.

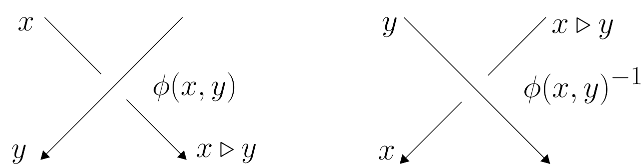


Figure 17: Weights for positive and negative crossings

With these we can define the product over all crossings.

Definition 5.7. The *quandle 2-cocycle invariant* associated with the 2-cocycle φ of a linkoid diagram L is defined as the formal sum

$$\Phi_\varphi(L) \coloneqq \sum_{f \in \mathbf{Qnd}(Q(L),X)} \prod_\tau \varphi(x,y)^{\varepsilon_\tau}$$

where the product is taken over all crossings τ. Here $\varepsilon_\tau = \pm 1$ denotes the sign of the crossing and the sum is taken over all colorings of L with X. So x, y are the colors of the arcs in the crossing τ.

The quandle 2-cocycle invariant was introduced in [CJK$^+$01] under the name state-sum invariant and partitioning function.

Because $\varphi(x,x) = 0$ by definition of a 2-cocycle, adding or removing a R1 move does not change the quandle 2-cocycle invariant. Similarly, an R2 move multiplies the quandle 2-cocycle invariant by $\varphi(x,y)\varphi(x,y)^{-1} = 1$, hence it also is not changing the quandle 2-cocycle invariant.

If we change the diagram by an R3-move, we replace the factor

$$\varphi(x,y)\varphi(y,z)\varphi(x \triangleright y, z) \text{ with } \varphi(x,z)\varphi(y,z)\varphi(x \triangleright z, y \triangleright z)$$

as we can see in Figure 18 below. If we cancel $\varphi(y,z)$ on both sides, this is exactly one of the conditions for a 2-cocycle we found in Example 5.5 written now in multiplicative notation.

This sketches the proof of the following lemma.

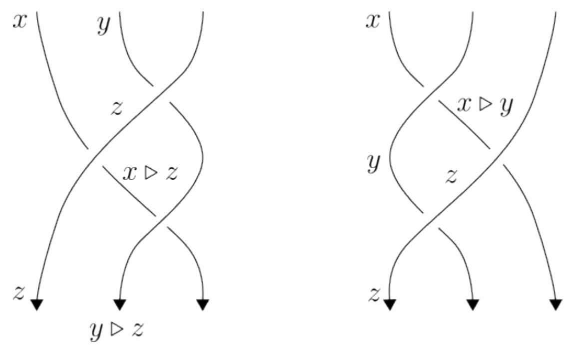


Figure 18: Inputs of φ for an R3-move

Lemma 5.8. *Let φ be a 2-cocycle. For a given coloring of a diagram, the product $\prod_\tau \varphi(x,y)^{\varepsilon_\tau}$, using the notation from above, is invariant under Reidemeister moves.*

A rigid proof can be found, for example, in [CJK$^+$01, Thm 4.4]. A direct consequence of this is the following theorem.

Theorem 5.9. *The quandle 2-cocycle invariant associated with a given 2-cycle is invariant under Reidemeister moves and hence, is an invariant for linkoids.*

Lemma 5.10. *Let L be a linkoid and denote its fundamental pointed quandle by $P(L) = (Q(L), l_1, h_1, \dots l_n, h_n)$. Let X a finite quandle and $\varphi(x,y) = g(x)g(x \triangleright y)^{-1} \in B^2(X;A)$ be a coboundary. Then*

$$\Phi_\varphi(L) = \sum_{f \in \mathbf{Qnd}(Q(L),X)} \prod_{i=1}^{n} g(l_i)g(h_i)^{-1}.$$

To prove this lemma, we follow the ideas of the proof of [CJK$^+$01, Proposition 4.5] which is here Corollary 5.12 below.

Proof. For a given coloring f we can interpret the weight $\varphi(x,y)$ of a positive crossing as weights $g(x)$ and $g(x \triangleright y)^{-1}$ on the corresponding strands of the under-arc (which have labels x and $x \triangleright y$). Similarly, the weights for a negative crossing are $g(x)^{-1}$ and $g(x \triangleright y)$. In both cases, the incoming strand has a ”positive” weight and the outgoing arc has an inverted weight of its label.
If we now take the product over all crossings, an arc x that is not attached to an endpoint contributes $g(x)g(x)^{-1} = 1$ to the product. Every l_i contributes $g(l_i)$ and every h_i contributes $g(h_i)^{-1}$ because these only appear in exactly one crossing. ☐

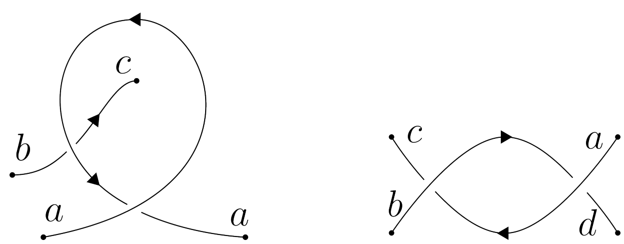
The lemma above means, for a given coloring, any closed component of a linkoid contributes 1 to the sum. Summing over all possible colorings proves:

Corollary 5.11. *Let L be a link and φ a coboundary. Then $\Phi_\varphi(L) = \Phi_X^{\mathbb{Z}}(L) \in \mathbb{Z}$ is the coloring counting invariant.*

Here we write $1 = e \in A$ as the neutral element.

Corollary 5.12. *Let L be a link and $\varphi, \varphi' \in Z^2(X; A)$ 2-cocycles that are cohomologuos, that is $\varphi = \varphi'\delta\psi$ for a 1-cochain ψ. Then $\Phi_\varphi(L) = \Phi_{\varphi'}(L)$.*

On the other hand, Lemma 5.10 means that the quandle 2-cocycle invariant for linkoids really depends on the associated 2-cocyle and not only on the equivalence class in the cohomology group.



(a) L_1 (b) L_2

Figure 19: Two 2-linkoid with two crossings

Example 5.13. Let L_1 and L_2 be the 2-linkoids in Figure 19. We immediately see that we can transform both into trivial 2-linkoids by under forbidden moves, so both fundamental quandles are free quandles with two generators, that is

$$Q(L_1) \cong Q(L_2) \cong Q\langle a, b\rangle.$$

We can see this isomorphism explicitly by noting that $c = b \triangleright a$ in $Q(L_1)$ and $c = a \triangleright b, d = b \triangleright a$ in $Q(L_2)$.

Now we want to color both linkoids with the quandle $R_4 = \{0, 1, 2, 3\}$ with $x \triangleright y \equiv 2y - x \pmod 4$. With our observation above, we can choose two colors freely. This then determines the coloring. So the quandle counting invariants are

$$\Phi_{R_4}^{\mathbb{Z}}(L_1) = \Phi_{R_4}^{\mathbb{Z}}(L_2) = 4^2 = 16.$$

To distinguish these linkoids, we will use the 2-cocycle invariant. Assume $A = \mathbb{Z} = \langle t\rangle$ (in multiplicative notation). Consider the 2-cocylce

$$\varphi(x, y) = \begin{cases} t & \text{if } (x, y) = (0, 1) \text{ or } (0, 3) \\ 1 & \text{otherwise.} \end{cases}$$

In L_1, we see that in one crossing both inputs are equal. So this does not contribute to the 2-cocycle. The other crossing is a positive crossing. This means we can write

$$\Phi_\varphi(L_1) = \sum_{f \in \mathbf{Qnd}(Q(L_1), X)} \prod_\tau \varphi(x, y) = \sum_{f \in \mathbf{Qnd}(Q(L_1), X)} \varphi(f(b), f(a)).$$

Because every combination of colors for a and b gives exactly one coloring of the diagram, there are two summands with value t while the other 14 have value 1. In total, we see

$$\Phi_\varphi(L_1) = 2t + 14.$$

Now in L_2 both crossings are positive. We compute

$$\Phi_\varphi(L_2) = \sum_{f \in \mathbf{Qnd}(Q(L_2), X)} \prod_\tau \varphi(x, y) = \sum_{f \in \mathbf{Qnd}(Q(L_2), X)} \varphi(f(a), f(b)) \cdot \varphi(f(b), f(a)),$$

where $\varphi(f(a), f(b))$ comes from the left crossing and $\varphi(f(b), f(a))$ from the right crossing in the diagram. Here, four summands are equal to t, namely if we color (a, b) by $(0, 1), (1, 0), (0, 3)$ or $(3, 0)$. The total sum is

$$\Phi_\varphi(L_1) = 4t + 12.$$

In particular this shows that L_1 and L_2 are non-equivalent 2-linkoids.

As we did for the quandle counting invariant we can define the 2-cocycle invariant for pointed quandles by simply summing only over the pointed colorings.

Definition 5.14. Let $\mathcal{X}$ be a $2n$-pointed quandle. The *pointed quandle 2-cocycle invariant*, (associated with the 2-cocycle φ), of an n-linkoid diagram L is defined as

$$\Phi_{\varphi, \mathcal{X}}(L) \coloneqq \sum_{f \in \mathbf{PQnd_{2n}}(P(L), \mathcal{X})} \prod_\tau \varphi(x, y)^{\varepsilon_\tau}$$

with notation as in Definition 5.7.

Theorem 5.15. *The pointed quandle 2-cocylce invariant associated with a given 2-cocycle is an invariant for linkoids.*

Proof. This follows immediately from Lemma 5.8. □

The advantage for pointed quandle 2-cocylce invariants, similar to pointed quandle counting invariants etc. is that we do not need to compute the sum over all colorings, but only over those where we might suspect the linkoids differ. Especially for large linkoids, this might simplify the computations significantly.

6 Biquandles for links and linkoids

6.1 Review on biquandles

Biquandles are a generalization of quandles. They were introduced in [FJSK04] to study virtual knots. We will see that biquandles can detect the under forbidden move on linkoids.

Definition 6.1. (as in [NOR15]) A *biquandle* is a set X with two binary operations $\underline{\triangleright}$, $\overline{\triangleright}\colon X \times X \to X$ satisfying for all $x, y, z \in X$

(1) $x\underline{\triangleright}x = x\overline{\triangleright}x$.

(2) The maps $\alpha_y, \beta_y \colon X \to X$ and $S\colon X\times X \to X\times X$ defined by $\alpha_y(x) = x\overline{\triangleright}y$, $\beta_y(x) = x\underline{\triangleright}y$ and $S(x, y) = (y\overline{\triangleright}x, x\underline{\triangleright}y) = (\alpha_a(b), \beta_b(a))$ are invertible.

(3) The following *exchange laws* are satisfied

$$\begin{aligned}
(x\underline{\triangleright}y)\underline{\triangleright}(z\underline{\triangleright}y) &= (x\underline{\triangleright}z)\underline{\triangleright}(y\overline{\triangleright}z)\\
(x\underline{\triangleright}y)\overline{\triangleright}(z\underline{\triangleright}y) &= (x\overline{\triangleright}z)\underline{\triangleright}(y\overline{\triangleright}z)\\
(x\overline{\triangleright}y)\overline{\triangleright}(z\overline{\triangleright}y) &= (x\overline{\triangleright}z)\overline{\triangleright}(y\underline{\triangleright}z)
\end{aligned}$$

We write $x\overline{\triangleright}^{-1}y$ for $\alpha_y^{-1}(x)$ and similarly $x\underline{\triangleright}^{-1}y$ for $\beta_y^{-1}(x)$.

Example 6.2.

- Let X be a set with a bijection $\sigma\colon X \to X$. Then we define $x\underline{\triangleright}y = x\overline{\triangleright}y = \sigma(x)$. This is called the *constant action biquandle*. Note that we cannot define a quandle in this way, because the first axiom is different. If $\sigma = id$, then this is the trivial quandle.

- Any quandle X is a biquandle with $x\underline{\triangleright}y \coloneqq x \triangleright y$ and $x\overline{\triangleright}y \coloneqq x$ for all $x, y \in X$. Inserting these operations into the three biquandle axioms gives exactly the quandle axioms.

- Let R be a commutative ring with identity and X be an $R[s^{\pm 1}, t^{\pm 1}]$-module. We define the *Alexander biquandle* as

$$x\underline{\triangleright}y = tx + (s - t)y \text{ and } x\overline{\triangleright}y = sx$$

  For $s = 1$, this is a quandle, called *Alexander quandle*.

Lemma 6.3. *[IIK+17, Lemma 2.8] Let X be a biquandle.*

(1) For $x, y \in X$, if $x\underline{\triangleright}y = y\overline{\triangleright}x$, then $x = y$.

(2) For any $x \in X$ there exists a unique $y \in X$ such that $y\underline{\triangleright}y = y\overline{\triangleright}y = x$.

Proof. To proof (1) let $x\underline{\triangleright}y = y\overline{\triangleright}x$. Then

$$\begin{aligned} x\overline{\triangleright}x &\overset{(1)}{=} x\underline{\triangleright}x \overset{(2)}{=} (x\underline{\triangleright}x)\underline{\triangleright}(y\overline{\triangleright}x)\underline{\triangleright}^{-1}(y\overline{\triangleright}x) \overset{(3)}{=} (x\underline{\triangleright}y)\underline{\triangleright}(x\underline{\triangleright}y)\underline{\triangleright}^{-1}(y\overline{\triangleright}x) \\ &\overset{(1)}{=} (y\overline{\triangleright}x)\underline{\triangleright}(y\overline{\triangleright}x)\underline{\triangleright}^{-1}(y\overline{\triangleright}x) = (y\overline{\triangleright}x). \end{aligned}$$

Here, the numbers indicate which of the biquandle axioms we use. Because α_x is a bijection, this shows $x = y$.
To proof (2) consider $(y_1, y_2) \coloneqq S^{-1}(x, x)$. Note that this tuple is unique. So

$$(x, x) = S(y_1, y_2) = (y_2\overline{\triangleright}y_1, y_1\underline{\triangleright}y_2).$$

Using the first part of this lemma gives $y_1 = y_2$. □

Definition 6.4. A *biquandle homomorphism* is a map $f \colon X \to Y$ between two biquandles X and Y such that for all $x, y \in X$ we have

$$f(x\underline{\triangleright}y) = f(x)\underline{\triangleright}f(y) \text{ and } f(x\overline{\triangleright}y) = f(x)\overline{\triangleright}f(y).$$

This forms the locally small category of biquandles which we denote by **Bqnd**.

We can color linkoid diagrams with biquandles. Instead of coloring each arc, we divide the diagrams into semiarcs. Remember, an arc is the portion of a diagram between two under-crossings. That is, one continuous line in the diagram. For *semiarcs* we divide the diagram on every crossing, instead of only on under-crossings. To color a diagram with a biquandle X, we assign to each semiarc an element of X. We do that in such a way that on each crossing, the relations in Figure 20 are satisfied. So if an arc with color x goes *above* y, then the semiarc on the other side of the diagram has color $x\overline{\triangleright}y = \alpha_y(x)$ and if it goes *below* y, then it has color $x\underline{\triangleright}y = \beta_y(x)$.

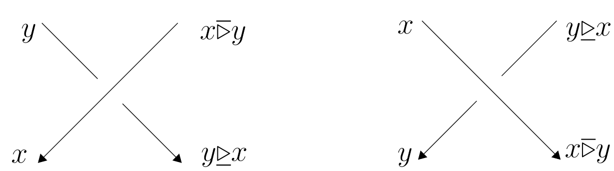


Figure 20: Biquandle relations for positive and negative crossings

Remark 6.5. Note that the inputs for α and β in Figure 20 are on the left of both crossings, unlike for quandles. When introduced in [FJSK04], biquandles were defined so the inputs are at the top. Even though the definitions are equivalent, it turned out that the notation of biquandles and the corresponding calculations are easier and clearer the way we defined it here. Especially the third biquandle axiom, which is derived from a R3 move, is a lot simpler. (See [EN15, Chapter 3] for a more detailed discussion of this).

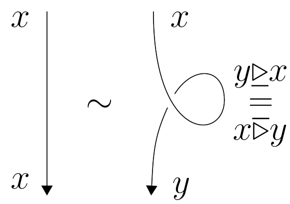


Figure 21: A biquandle coloring under R1 move

We again observe what happens under Reidemeister moves.

Consider the situation in Figure 21. Lemma 6.3 shows that $x = y$ on the right site. On the other hand, if we add a kink, then $x = y$, so $x \underline{\triangleright} x = x \overline{\triangleright} x$ by the first biquandle axiom. This shows that there is a one to one correspondence between colorings before and after adding or removing a kink via a R1 move. The other types of oriented R1 moves are similar.

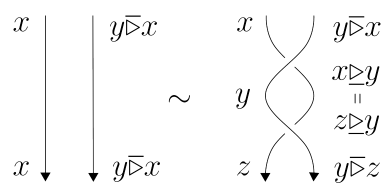


Figure 22: A biquandle coloring under an R2 move

Now, we examine the situation for R2 moves. There are two types of R2-moves which use different biquandle properties. First, consider the situation in Figure 22. Note that on the left side, we can write the label of the right strand as $y \overline{\triangleright} x = \alpha_x(y)$ since α_x is a bijection.
If we add an R2-move, then $z = x$ on the right side, so we immediately see that this gives a unique and valid coloring.
Now assume that we have the situation on the right-hand side. Then, because $\beta_y(x) = \beta_y(z)$, we find that $(x = z)$.

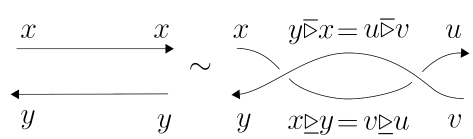


Figure 23: A biquandle coloring under another R2 move

Consider an R2-move where the strands have opposite orientations, as in Figure 23. Adding an R2-move again gives a unique coloring. If we start with the right-hand side, we use the map S from the second biquandle axiom. So

$$S(x,y) = (y\overline{\triangleright}x, x\underline{\triangleright}y) = (u\overline{\triangleright}v, v\underline{\triangleright}u) = S(u,v).$$

Because S is a bijection, this shows $(x,y) = (u,v)$.

Lastly, when we change the diagram with the R3-move in Figure 24 we see that the labels on the top and bottom are equal on both sides, using the third biquandle axiom. Similarly, the equation on the right side in the middle hold, using the last relation in the third biquandle axiom.

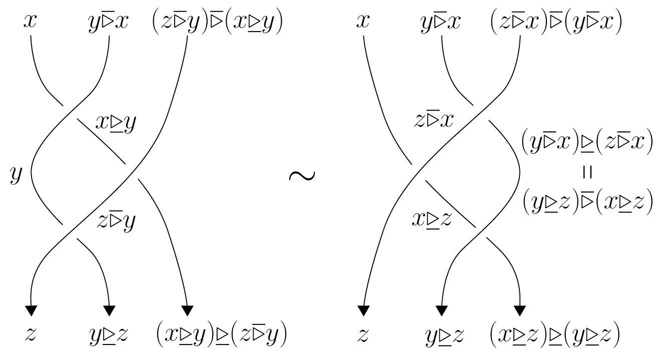


Figure 24: A biquandle coloring under an R3 move

This means if L and L' are equivalent linkoid diagrams, then every coloring of L with a given biquandle X gives a unique coloring of L'.

Similar to quandles we can define a fundamental biquandle.

Definition 6.6. Let L be an oriented linkoid diagram with a set of semiarcs $S(L)$. Then the *fundamental biquandle* of L is defined as

$$B(L) \coloneqq B\langle x \in S(L) \quad | \quad r_\tau \text{ for all crossings } \tau\rangle,$$

the biquandle consisting of words in $S(L)$ modulo the biquandle axiom relations and the relations r_τ given by each crossing as shown in Figure 20.

Theorem 6.7. *The fundamental biquandle $B(L)$ is an invariant of the linkoid L.*

Proof. In the discussion above, we have seen that changing a diagram by Reidemeister moves give isomorphic fundamental quandles. The missing oriented Reidemeister moves work in exactly the same way as the ones we discussed above. $\square$

Remark 6.8. In [Hor20] a topological construction for biquandles of knots was studied. It is similar to the one for quandles described in Section 3.3. The resulting biquandle is in general not isomorphic to the fundamental biquandle, but to a quotient of the fundamental biquandle of the knot.

We can now understand a coloring of L with a finite biquandle X as a biquandle homomorphism $\varphi\colon B(L) \to X$.

Note that unlike in the quandle coloring case, there is not always a trivial biquandle coloring of a knotoid because in general $x \trianglerighteq x = x \overline{\triangleright} x \neq x$.

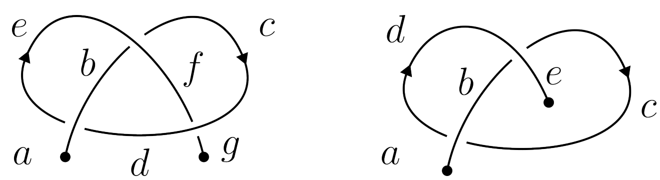
Definition 6.9. Let X a finite biquandle. The *biquandle counting invariant* for a linkoid L is defined as $\Phi_X^{\mathbb{Z}}(L) \coloneqq |\mathbf{Bqnd}(B(L), X)|$, the number of colorings.

Theorem 6.10. *The biquandle counting invariant is an invariant of linkoids.*

Proof. This follows directly from Theorem 6.7, since $B(L)$ is the only part that depends on L. ☐

We compute some biquandle counting invariants in the following examples.

Example 6.11. Let K_1 be the knotoid in Figure 25a.



(a) K_1 (b) K_2

Figure 25: Two knotoids with labeled semiarcs

These are the same diagrams we studied in Example 4.7. Now we assigned a label to every semiarc, instead of every arc. The biquandle relations given by the crossings are

$$\begin{aligned} e &= d \trianglerighteq b \quad &&\text{and} \quad && a = b \overline{\triangleright} d, \\ c &= b \trianglerighteq f \quad &&\text{and} \quad && e = f \overline{\triangleright} b, \\ g &= f \trianglerighteq d \quad &&\text{and} \quad && c = d \overline{\triangleright} f. \end{aligned}$$

Here, the two relations in the same line belong to the same crossing in the diagram. Because a, c and e only appear on the left side of the relations, these are merely alternative names for the element on the right. So the fundamental biquandle is

$$\begin{aligned} B(K_1) &= B\langle a, b, c, d, e, f, y \quad | \quad r_\tau \text{ for all crossings } \tau \rangle \\ &\cong B\langle b, d, f \quad | \quad d \trianglerighteq b = f \overline{\triangleright} b,\, b \trianglerighteq f = d \overline{\triangleright} f \rangle \\ &\cong B\langle b, f \quad | \quad (f \overline{\triangleright} b) \trianglerighteq^{-1} b = (b \trianglerighteq f) \overline{\triangleright}^{-1} f \rangle. \end{aligned}$$

Let $X = \{1, 2, 3\}$ be the biquandle with $\beta_1 = (12)$, $\beta_2 = (23)$, $\beta_3 = (13)$ and $\alpha_* = (123)$ for all $* \in X$. In other words, X has the following operation tables.

$\trianglerighteq$	1	2	3
1	2	1	3
2	1	3	2
3	3	2	1

$\overline{\triangleright}$	1	2	3
1	2	2	2
2	3	3	3
3	1	1	1

Note that $\alpha_*^{-1} = (132)$. For $x, y, z \in X$ we can check that the relations in $B(L)$ always hold in X:

$$(y \overline{\triangleright} x) \trianglerighteq^{-1} x = \alpha_*^{-1} \circ \beta_x(y) = \left. \begin{cases} y & \text{if x = y} \\ z \neq x, y & \text{if x} \neq \text{y} \end{cases} \right\} = \beta_x \circ \alpha_*(y) = (y \trianglerighteq x) \overline{\triangleright}^{-1} x$$

where z is the third element in X. This means, any choice of two elements $x, y \in X$ gives a biquandle homomorphism $\varphi \colon B(K_1) \to X$ by defining $\varphi(b) = x$ and $\varphi(f) = y$. In particular, $\varphi_X^{\mathbb{Z}}(K_1) = 9$.

Example 6.12. Consider the knotoid K_2 in Figure 25b. The corresponding crossing relations with respect to the given labeling are

$$d = c \trianglerighteq b \quad \text{and} \quad a = b \overline{\triangleright} c,$$
$$c = b \trianglerighteq e \quad \text{and} \quad d = e \overline{\triangleright} b.$$

So its fundamental biquandle is given by

$$\begin{aligned} B(K_2) &= B\langle a, b, c, d, e \quad | \quad r_\tau \text{ for all crossings } \tau \rangle \\ &\cong B\langle b, c, e \quad | \quad c \trianglerighteq b = e \overline{\triangleright} b, c = b \trianglerighteq e \rangle \\ &\cong B\langle b, e \quad | \quad (b \trianglerighteq e) \trianglerighteq b = e \overline{\triangleright} b \rangle. \end{aligned}$$

Let X be the biquandle above. We will compute $\Phi_X^{\mathbb{Z}}(K_2)$. Therefore, we need to check for which $x, y \in X$ the equation $(x \trianglerighteq y) \trianglerighteq x = x \overline{\triangleright} y$ holds.
If $x = y$, then $(x \trianglerighteq x) \trianglerighteq x = x$ because $\beta_x^2 = id$, but $x \overline{\triangleright} x \neq x$. Checking the remaining six combinations for x, y we obtain the following table.

x	y	$(x \trianglerighteq y) \trianglerighteq x = y \overline{\triangleright} x$?
1	2	$1 \neq 3$
1	3	$2 \neq 1$
2	1	$1 \neq 2$
2	3	$3 \neq 1$
3	1	$1 \neq 2$
3	2	$2 \neq 3$

This means, there are no such homomorphisms and the biquandle counting invariant $\Phi_X^{\mathbb{Z}}(K_2) = 0$.

The last two examples show in particular that the fundamental biquandle can distinguish some 1-linkoids that represent the same link. So, unlike quandles, the fundamental biquandle might change under the under forbidden move.

This is especially helpful when studying linkoids that can be transformed into trivial linkoids using only under forbidden-moves. By Lemma 3.23 the fundamental quandle of these linkoids are free quandles with one generator for each component. So if we consider the pointed fundamental quandles, then all basepoints are equal to the corresponding generator of the component, hence it too cannot carry any information about the linkoid (besides the number of components).

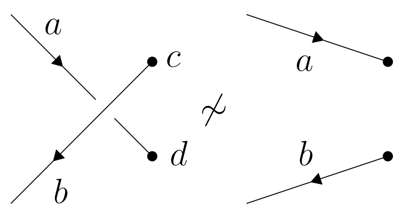


Figure 26: Twisting move with labeled semiarcs

In the next part we consider the *twisting* move on two endpoints as in Figure 26 above. Of course a twisting move can only happen in an n-linkoid with $n \geq 1$. If it takes place in a 1-linkoid, this is a fake forbidden move as in Figure 6 together with an R1-move. If it takes place in an n-linkoid with $n \geq 2$, then this can yield a non-equivalent linkoid. We will see some examples for this later in Example 6.20.

Lemma 6.13. *The fundamental biquandle is invariant under the twisting move.*

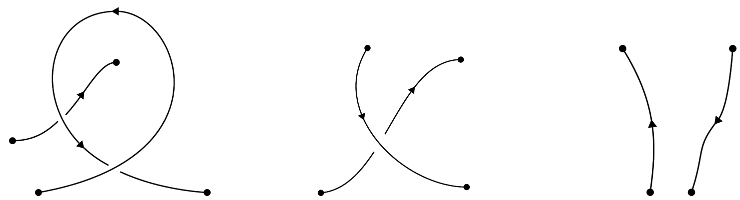
Proof. Let L and L' be two linkoid diagrams such that L' can be derived from L by removing one twist using the twisting move. We can, without loss of generality, assume the orientation as in Figure 26 and will use its labeling. If we look at the fundamental biquandles, we see that $B(L)$ has two extra generators, namely c and d as well as two additional relations, namely $c = b \overline{\triangleright} a$ and $d = a \underline{\triangleright} c$. So c and d are labels for elements that are already in the biquandle $B(L')$. In other words, the biquandle homomorphism $\varphi\colon B(L) \to B(L')$ defined as $\varphi(x) = x$ for all generators $x \neq c, d \in B(L)$, $\varphi(c) = b \overline{\triangleright} a$ and $\varphi(d) = a \underline{\triangleright} b$ is an isomorphism. $\square$

If we look at the 2-linkoids in Figure 27 above, we immediately know by Lemma 6.13 that the three linkoids all isomorphic fundamental biquandles, namely

$$B(L_1) \cong B(L_2) \cong B(L_3) \cong B\langle a, b \rangle,$$

the free biquandle with two generators.

Remark 6.14. Because quandles are a special case of biquandles, Lemma 6.13 shows that the fundamental quandle of a linkoid is also invariant under the twisting move. But the twisting move can be obtained by an under forbidden



(a) L_1 (b) L_2 (c) L_3

Figure 27: Three 2-linkoids with isomorphic fundamental biquandle

move. This means, the fact that the fundamental quandle is invariant under the twisting move is already implied by Lemma 3.23.

6.2 Pointed biquandles

Similar to the definition of pointed quandles in 4.1 we now introduce pointed biquandles. These are able to detect twisting moves.

Definition 6.15. An *n-pointed biquandle* $\mathcal{X} = (X, x_1, \ldots, x_n)$ is an ordered tuple consisting of a biquandle X together with n elements $x_1, \ldots, x_n \in X$. A homomorphism between two n-pointed biquandles is a biquandle homomorphism $\varphi\colon (X, x_1, \ldots, x_n) \to (Y, y_1, \ldots, y_n)$ such that $\varphi(x_i) = y_i$ for all $i = 1, \ldots, n$. This forms the category of n-pointed biquandles $\mathbf{PBqnd_n}$.

We use calligraphic letters for pointed biquandles.

Definition 6.16. Let L be an ordered n-linkoid. The *fundamental pointed biquandle* of L is the $2n$-pointed biquandle

$$PB(L) \coloneqq (B(L), l_1, h_1, \ldots, l_n, h_n),$$

where l_i and h_i are the labels of the semiarcs attached to the leg and head of the i-th component of L.

Theorem 6.17. *The fundamental pointed biquandle is an invariant of an ordered linkoid.*

Proof. This follows directly from the fact that Reidemeister moves happen away from the endpoints. So the biquandle isomorphism between the fundamental biquandle before and after a Reidemeister move maps the endpoint labels to endpoint labels and hence is a pointed biquandle isomorphism. □

Following our strategy for (pointed) quandles, we count the number of colorings for a given pointed biquandle.

Definition 6.18. Let $\mathcal{X}$ be a $2n$-pointed biquandle and L an n-linkoid. Define the *pointed biquandle counting invariant* as

$$\Phi^{\mathbb{Z}}_{\mathcal{X}}(L) \coloneqq |\mathbf{PBqnd_{2n}}(PB(L), \mathcal{X})|$$

Theorem 6.19. *For a given $2n$-pointed biquandle, the pointed biquandle counting invariant is an invariant of n-linkoids.*

Proof. This is a direct consequence of Theorem 6.17. $\square$

Example 6.20. Let L_1, L_2 and L_3 be again the linkoids in Figure 27 and order the components as the under-passing component first in L_1 and L_2, and the left component first in L_3. By only using the labels of the end-semiarcs, as in Figure 28, we see that $h_1 = l_1 \trianglerighteq (h_2 \trianglerighteq l_2)$ and $l_2 \overline{\triangleright} h_2 = (h_2 \trianglerighteq l_2) \overline{\triangleright} l_1$ in $B(L_1)$.

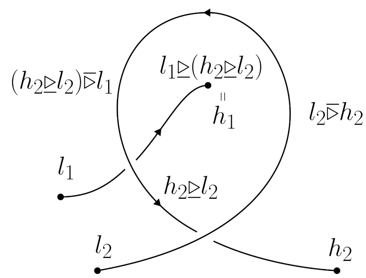


Figure 28: A 2-linkoid L_1 with biquandle labels

For a pointed biquandle homomorphism $\varphi \colon PB(L_1) \to \mathcal{X}$ to exist, we need the same relations to hold for $\mathcal{X} = (X, x_1, x_2, x_3, x_4)$ with respect to its corresponding endpoints.

Let X be the biquandle from Example 6.11. Since α_* does not depend on $* \in X$, the equations are equivalent to $x_3 = x_4 \trianglerighteq x_3$ and $x_1 = x_2 \trianglerighteq x_4$. This holds for example for $\mathcal{X}_1 = (X, 2, 1, 1, 2)$, since $1 = 2 \trianglerighteq 1$ and $2 = 1 \trianglerighteq 2$. Now the colors of the end-arcs determine the coloring completely, so this shows $\Phi^{\mathbb{Z}}_{\mathcal{X}_1}(L_1) = 1$. If we consider $\mathcal{X}_2 \coloneqq (X, 1, 2, 1, 2)$, then we see that $x_1 = 1 \neq 3 = 2 \trianglerighteq 2 = x_2 \trianglerighteq x_4$. So $\Phi^{\mathbb{Z}}_{\mathcal{X}_2}(L_1) = 0$.

Now consider L_2, the second linkoid in Figure 27. Here we have the equations $h_1 = l_1 \trianglerighteq l_2$ and $h_2 = l_2 \overline{\triangleright} l_1$. For $\mathcal{X}_1$ we see that $x_2 = 1 \neq 3 = 2 \trianglerighteq 2 = x_1 \trianglerighteq x_4$, so $\Phi^{\mathbb{Z}}_{\mathcal{X}_1}(L_2) = 0$. But for $\mathcal{X}_2$ we have $2 = 1 \trianglerighteq 1$ and $2 = 1 \overline{\triangleright} 1$, so $\Phi^{\mathbb{Z}}_{\mathcal{X}_2}(L_1) = 1$.

Lastly, in $PB(L_3)$ we have $l_1 = h_1$ and $l_2 = h_2$, so $\Phi^{\mathbb{Z}}_{\mathcal{X}_1}(L_3) = \Phi^{\mathbb{Z}}_{\mathcal{X}_2}(L_3) = 0$. But for $\mathcal{X}_3 = (X, 1, 1, 1, 1)$ we see that $\Phi^{\mathbb{Z}}_{\mathcal{X}_3}(L_3) = 1$. Whereas $\Phi^{\mathbb{Z}}_{\mathcal{X}_3}(L_1) = \Phi^{\mathbb{Z}}_{\mathcal{X}_3}(L_2) = 0$, since $1 \trianglerighteq 1 = 2 \neq 1$.

Similar to the concept for pointed quandles, we can collect all pointed biquandle coloring invariants of a linkoid into one matrix.

Definition 6.21. Let L be a 1-linkoid and $X = \{1, \dots, k\}$ a finite biquandle. We define the *biquandle counting matrix* $\Phi^{M_k}_X(L)$ of L with respect to X as the

$k \times k$ matrix given by

$$\left(\Phi_X^{M_k}(L)\right)_{i,j} \coloneqq \Phi^{\mathbb{Z}}_{(X,i,j)}(L).$$

So the entries are the pointed biquandle counting invariants with respect to the pointed biquandle with the given basepoints. In particular all entries are integers.

Theorem 6.22. *The biquandle counting matrix is an invariant of 1-linkoids.*

Proof. This follows directly from the fact that the pointed biquandle coloring invariant, that is, every entry in the biquandle counting matrix, is an invariant of linkoids (6.19). $\square$

This definition coincides with the definition of the biquandle counting matrix in [GN18, Definition 4]. The definition there is only for knotoids and does not use fundamental quandles or pointed quandles, but explicitly defines the entries to be the number of colorings (as in, giving labels to each semiarc) with fixed colors on the end-semiarcs.

Example 6.23. Consider the following two knotoids 5_{17} and 5_{28} in Figure 29 from the knotoid table in [Bar21].

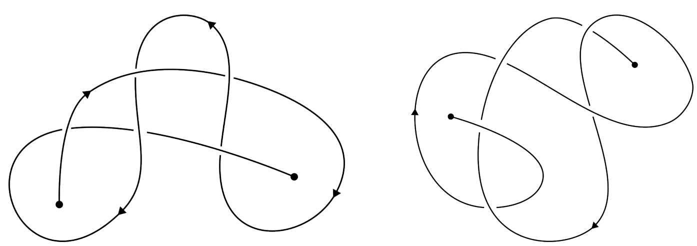


Figure 29: The knotoids 5_{17} and 5_{28}

Let X again be the quandle from Example 6.11. Then

$$\Phi_X^{M_3}(5_17) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \Phi_X^{M_3}(5_28) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{1}$$

as computed in [GN18, Example 6].

7 Discussion

We end the thesis with some open questions that emerged in the thesis and possible future directions for the research of linkoids.

(1) Are there 1-linkoids that represent different links but have isomorphic fundamental quandles?

(2) In our examples of non-equivalent linkoids with isomorphic fundamental pointed quandle we only considered linkoids that could be transformed to trivial linkoids using under forbidden moves. Are there other pairs of linkoids with non-equivalent fundamental quandles? And if so, what can we say about such a pair?

(3) Computing quandle coloring matrices of 1-linkoids is a hard task when done as direct computations by hand. There is a lot of potential for developing tools to compute examples and as well as developing a tool to compute a list of all n-homogeneouos quandles for some n.

(4) In Section 4.2 we computed a lower bound for the number of isomorphism classes of n-pointed quandles for a given underlying quandle X. We only gave the trivial upper bound $|X|^n$. What upper bounds can we find for this?

(5) In theorem 4.15 we gave a recursive formula for $|X^n \,/\, S_k|$. Is there a direct formula for this?

(6) We have introduced the forgetful functor $U_n \colon \mathbf{PQnd_n} \to \mathbf{Qnd}$ in Section 4.1. Is there an adjoint functor of U_n?.

References

[AB26] Alexander, J. W. ; Briggs, G. B.: On Types of Knotted Curves. In: *Annals of Mathematics* 28 (1926), Nr. 1/4, 562–586. `http://www.jstor.org/stable/1968399`. – ISSN 0003486X

[Bar21] Bartholomew, Andrew: *Andrew Bartholomew's Mathematics Page*. `https://www.layer8.co.uk/maths/knotoids/index.htm`. Version: 02 2021. – Accessed: 2023-11-07

[BBHL19] Barbensi, Agnese ; Buck, Dorothy ; Harrington, Heather A. ; Lackenby, Marc: *Double branched covers of knotoids*. 2019

[BLRY10] Bunch, Eric ; Lofgren, P. ; Rapp, A. ; Yetter, David: ON quotients of quandles. In: *Journal of Knot Theory and its Ramifications* 9 (2010), 09. `http://dx.doi.org/10.1142/S021821651000839X`. – DOI 10.1142/S021821651000839X

[Car10] Carter, J. S.: *A Survey of Quandle Ideas*. 2010

[Caz22] Cazet, Nicholas: *The shadow quandle cocycle invariant of knotoids*. 2022

[CDS16] Clark, W. E. ; Dunning, Larry A. ; Saito, Masahico: *Computation of quandle 2-cocycle knot invariants without explicit 2-cocycles*. http://dx.doi.org/10.48550/ARXIV.1607.04348. Version: 2016

[CESY14] Clark, W. E. ; Elhamdadi, Mohamed ; Saito, Masahico ; Yeatman, Timothy: Quandle colorings of knots and applications. In: *Journal of Knot Theory and Its Ramifications* 23 (2014), may, Nr. 06, 1450035. http://dx.doi.org/10.1142/s0218216514500357. – DOI 10.1142/s0218216514500357

[CHMS19] Crans, Alissa S. ; Hoste, Jim ; Mellor, Blake ; Shanahan, Patrick D.: *Finite n-quandles of torus and two-bridge links*. 2019

[CJK+01] Carter, J. S. ; Jelsovsky, Daniel ; Kamada, Seiichi ; Langford, Laurel ; Saito, Masahico: *Quandle Cohomology and State-sum Invariants of Knotted Curves and Surfaces*. 2001

[CN18] Cho, Karina ; Nelson, Sam: *Quandle Coloring Quivers*. http://dx.doi.org/10.48550/ARXIV.1807.10465. Version: 2018

[CSV16] Clark, W. E. ; Saito, Masahico ; Vendramin, Leandro: Quandle coloring and cocycle invariants of composite knots and abelian extensions. In: *Journal of Knot Theory and Its Ramifications* 25 (2016), apr, Nr. 05, 1650024. http://dx.doi.org/10.1142/s0218216516500243. – DOI 10.1142/s0218216516500243

[EMRL10] Elhamdadi, Mohamed ; Macquarrie, Jennifer ; Restrepo López, Ricardo: Automorphism groups of Quandles. In: *Journal of Algebra and Its Applications* 11 (2010), 12. http://dx.doi.org/10.1142/S0219498812500089. – DOI 10.1142/S0219498812500089

[EN15] Elhamdadi, M. ; Nelson, S.: *Quandles: An Introduction to the Algebra of Knots*. American Mathematical Society, 2015 (Student Mathematical Library). – ISBN 9781470422134

[FJSK04] Fenn, Roger ; Jordan-Santana, Mercedes ; Kauffman, Louis: Biquandles and virtual links. In: *Topology and its Applications* 145 (2004), Nr. 1, 157-175. http://dx.doi.org/https://doi.org/10.1016/j.topol.2004.06.008. – DOI https://doi.org/10.1016/j.topol.2004.06.008. – ISSN 0166–8641

[FR92] Fenn, Roger ; Rourke, Colin: Racks and Links in Codimension Two. In: *Journal of Knot Theory and Its Ramifications* 01 (1992), Nr. 04, 343-406. http://dx.doi.org/10.1142/S0218216592000203. – DOI 10.1142/S0218216592000203

[GG22] Gabrovšek, Boštjan ; Gügümcü, Neslihan: *Invariants of multi-linkoids*. http://dx.doi.org/10.48550/ARXIV.2204.11234. Version: 2022

[GK17] Gügümcü, Neslihan ; Kauffman, Louis H.: New invariants of knotoids. In: *Eur. J. Comb.* 65 (2017), S. 186–229. http://dx.doi.org/10.1016/j.ejc.2017.06.004. – DOI 10.1016/j.ejc.2017.06.004. – ISSN 0195–6698

[GN18] Gügümcü, Neslihan ; Nelson, Sam: *Biquandle Coloring Invariants of Knotoids*. http://dx.doi.org/10.48550/ARXIV.1803.11308. Version: 2018

[Hor20] Horvat, Eva: Constructing biquandles. In: *Fundamenta Mathematicae* 251 (2020), Nr. 2, 203–218. http://dx.doi.org/10.4064/fm715-12-2019. – DOI 10.4064/fm715–12–2019

[IIK+17] Ishii, Atsushi ; Iwakiri, Masahide ; Kamada, Seiichi ; Kim, Jieon ; Matsuzaki, Shosaku ; Oshiro, Kanako: *A multiple conjugation biquandle and handlebody-links*. 2017

[Joy82] Joyce, David: A classifying invariant of knots, the knot quandle. In: *Journal of Pure and Applied Algebra* 23 (1982), S. 37–65

[Kau98] Kauffman, Louis H.: Virtual Knot Theory. In: *Eur. J. Comb.* 20 (1998), 663-691. https://api.semanticscholar.org/CorpusID:5993431

[KR01] Kauffman, Louis H. ; Radford, David E.: *Bi-oriented Quantum Algebras, and a Generalized Alexander Polynomial for Virtual Links*. 2001

[Mat84] Matveev, Sergei: DISTRIBUTIVE GROUPOIDS IN KNOT THEORY. In: *Mathematics of The Ussr-sbornik* 47 (1984), 73-83. https://api.semanticscholar.org/CorpusID:56069452

[Mol22] Moltmaker, Wout: Framed Knotoids and Their Quantum Invariants. In: *Communications in Mathematical Physics* 393 (2022), Juni, Nr. 2, 1035–1061. http://dx.doi.org/10.1007/s00220-022-04382-1. – DOI 10.1007/s00220–022–04382–1. – ISSN 1432–0916

[Nel02] Nelson, Sam: Classification of Finite Alexander Quandles. (2002). http://dx.doi.org/10.48550/ARXIV.MATH/0202281. – DOI 10.48550/ARXIV.MATH/0202281

[NOR15] Nelson, Sam ; Orrison, Michael E. ; Rivera, Veronica: *Quantum Enhancements and Biquandle Brackets*. http://dx.doi.org/10.48550/ARXIV.1508.06573. Version: 2015

[Nos11] Nosaka, Takefumi: On homotopy groups of quandle spaces and the quandle homotopy invariant of links. In: *Topology and its Applications* 158 (2011), Nr. 8, 996-1011. http://dx.doi.org/https://doi.org/10.1016/j.topol.2011.02.006. – DOI https://doi.org/10.1016/j.topol.2011.02.006. – ISSN 0166–8641

[Pol10] Polyak, Michael: *Minimal generating sets of Reidemeister moves.* 2010

[Rei27] Reidemeister, Kurt van: Elementare Begründung der Knotentheorie. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 5 (1927), 24-32. https://api.semanticscholar.org/CorpusID:120149796

[TAM13] TAMARU, Hiroshi: Two-point homogeneous quandles with prime cardinality. In: *Journal of the Mathematical Society of Japan* 65 (2013), oct, Nr. 4. http://dx.doi.org/10.2969/jmsj/06541117. – DOI 10.2969/jmsj/06541117

[Tur10] Turaev, Vladimir: Knotoids. In: *Osaka Journal of Mathematics* 49 (2010), 02

[Win11] Winker, Steven: Quandles knot invariants and the N-fold branched cover. (2011), 05