Higher Currents for the sine-Gordon Model in perturbative Algebraic Quantum Field Theory

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Submitted by
Fabrizio Zanello
from Alessandria, Italy

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Thesis Committee

Prof. Dr. Dorothea Bahns
Mathematisches Institut, Georg-August-Universität Göttingen

Prof. Dr. Chenchang Zhu
Mathematisches Institut, Georg-August-Universität Göttingen

Prof. Dr. Ralf Meyer
Mathematisches Institut, Georg-August-Universität Göttingen

Members of the Examination Board

Reviewer: Prof. Dr. Dorothea Bahns
Mathematisches Institut, Georg-August-Universität Göttingen

Second Reviewer: Prof. Dr. Chenchang Zhu
Mathematisches Institut, Georg-August-Universität Göttingen

Further members of the Examination Board:

Prof. Dr. Wolfram Bauer
Institut für Analysis, Fakultät für Mathematik und Physik, Leibniz-Universität Hannover

Prof. Dr. Ralf Meyer
Mathematisches Institut, Georg-August-Universität Göttingen

Prof. Dr. Karl-Henning Rehren
Fakultät für Physik, Georg-August-Universität Göttingen

Prof. Dr. Max Wardetzky
Institut für Numerische und Angewandte Mathematik, Georg-August-Universität Göttingen

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– Anche le città credono d'essere opera della mente o del caso,
ma né l'una né l'altro bastano a tener su le loro mura.
D'una città non godi le sette o le settantasette meraviglie,
ma la risposta che dà alla tua domanda.

– O la domanda che ti pone obbligandoti a rispondere,
come Tebe per bocca della Sfinge.[1]

[1] Italo Calvino, *Le città invisibili*.

Abstract

The main results of this thesis are the establishment of the super-renormalizability by power counting and of the summability of an infinite number of (classically conserved) higher currents for the sine-Gordon model in the framework of perturbative Algebraic Quantum Field Theory (pAQFT). In order to achieve this, we first consider the classical theory. Combining the notion of Bäcklund transformations with Noether's Theorem, we obtain recursive formulas for the components of the higher currents and also characterize them introducing a suitable notion of degree. We then move to the pAQFT setting and, by means of some technical results, we compute explicit formulas for the unrenormalized interacting components of the currents. In the context of the Epstein and Glaser approach to renormalization, we prove a uniform bound on the scaling degree of the interacting components given by the notion of degree introduced previously, which directly implies super-renormalizability. Subsequently, we describe the concrete renormalization of the interacting components by a procedure which we call piecewise renormalization. Finally, we show that the formal power series arising as expectation values of the renormalized interacting components in a generic Gaussian state are, under suitable conditions, summable.

Contents

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Göttingen, August 2023

[2] Guy de Maupassant, *La Parure*

Introduction

Considered as a whole, the thesis presented here could be viewed as a further step in the direction of the realization of a project which, in our opinion, is quite ambitious. The project we are referring to aims in the first place at further exploring the remarkable properties of the two-dimensional massless sine-Gordon model in the context of perturbative Algebraic Quantum Field Theory, pAQFT for short, in relation to its nature as a (classical) integrable system. On a larger scale and future work, the goal is to identify the relevant structures encoding integrability (possibly also with a look at a more general level than the specific case of the sine-Gordon model) and to investigate to what extent these structures, and their quantization, can be understood in the framework of pAQFT.

The sine-Gordon model can be formulated as a classical relativistic non-linear scalar field theory (see [22]) on two-dimensional Minkowski spacetime $\mathbb{M}_2$. The Lagrangian is given by:

$$L(\varphi) = L_0(\varphi) + L_{\text{int}}(\varphi) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \cos(a\varphi), \qquad \varphi \in C^\infty(\mathbb{M}_2),$$

where $L_0 = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ is the free massless scalar Lagrangian in two dimensions, L_{int} is the interaction Lagrangian characteristic of the sine-Gordon model, and $a > 0$ is a real parameter called coupling constant. The corresponding equation of motion is the so-called sine-Gordon equation:

$$-\Box\varphi - a\sin(a\varphi) = 0.$$

During its very long history, dating back to the 1860's, the sine-Gordon model has been keeping exhibiting a great richness of properties. Since early times it was generally known to be an example of a so-called integrable system and as such it has been approached by a multitude of methods and points of view. Among the variety of features that were discovered in time, the ones we will be mostly concerned with are: the existence of an infinite number of solutions to the sine-Gordon equation, related to each other by the so-called Bäcklund transformations (see [26] for an overview), and the existence of an infinite number of conserved currents, which form an involutive Poisson algebra with respect to the Peierls bracket (see [9]).

The Bäcklund transformations are defined in an implicit way: take a solution of the sine-Gordon equation $\varphi \in C^\infty(\mathbb{M}_2)$, then we say that $\varphi' \in C^\infty(\mathbb{M}_2)$ is obtained from φ by a Bäcklund transformation of parameter $\alpha \in \mathbb{R}$ if it satisfies the following parametric system of first order PDE's

$$\frac{1}{2}(\varphi' + \varphi)_\xi = \frac{1}{\alpha} \sin\left[\frac{a}{2}(\varphi' - \varphi)\right]$$
$$\frac{1}{2}(\varphi' - \varphi)_\tau = \alpha \sin\left[\frac{a}{2}(\varphi' + \varphi)\right],$$

where (τ, ξ) denote the so-called light-cone coordinates on $\mathbb{M}_2$, and the subscripts $_\tau$ and $_\xi$ denote partial derivatives. It can be shown that φ' is then automatically a solution of the sine-Gordon equation.

Particularly relevant for our purposes is the approach, proposed in [28] and [29] for the sine-Gordon model (without coupling constant a), where Bäcklund transformations are combined with Noether's Theorem in order to obtain an infinite number of conservation laws. More concretely, the outcome of the construction presented in [29] are 1-forms, called currents, on $\mathbb{M}_2$

$$s^N(\varphi) = \big(- s_1^N(\varphi)\big) d\tau + \big(s_2^N(\varphi)\big) d\xi \qquad \forall N \in \mathbb{N},$$

that satisfy a null-divergence equation (conservation law)

$$\big(s_1^N(\varphi)\big)_\xi + \big(s_2^N(\varphi)\big)_\tau = 0,$$

whenever $\varphi \in C^\infty(\mathbb{M}_2)$ is a solution of the sine-Gordon equation ("on-shell conservation law"). We remark that the first current s^0 is in fact the stress-energy tensor of the sine-Gordon model. By convention, as a whole, the currents $(s^N)_{N\in\mathbb{N}}$ are generically referred to as the higher currents.

As a quantum physical system, that is, after a proper quantization procedure, the sine-Gordon model is known to admit a non-trivial scattering theory. More recently, it has revealed remarkable features also in the context of pAQFT. Generally speaking, the approach to quantization advocated by pAQFT does not rely on the choice of a representation space for some algebra of observables (for example, Fock space), but instead is based on the deformation quantization of such an algebra, whose representation theory may be studied at a later stage (see [10], [24]). In its perturbative spirit, pAQFT distinctly separates the linear part of a theory, which yields (formal) power series in the (formal) parameter $\hbar$ as observables (first deformation, or "quantization"), from the interaction part (i.e. a possibly nonlinear term in the equation of motion), which yields an additional formal power series expansion in a coupling constant κ (second deformation, or "perturbation theory").

Specifically, in [1], [2], the perturbative quantum scattering matrix of the two-dimensional massless sine-Gordon model in Minkowski signature was explicitly constructed in the pAQFT framework as a formal power series, and its renormalizability and summability proved, building partially on older results in Euclidean signature (e.g. [17]).

In relation to the overall view mentioned at the beginning, the further steps that we take in this work involve the extension of the renormalizability and summability results to the pAQFT-quantized higher currents coming from the classical sine-Gordon theory. In particular, we are interested in studying the interacting pAQFT-quantized higher currents. Indeed, the notion of interacting observables in pAQFT represents the quantum equivalent of the classical restriction on-shell, i.e. of classical observables evaluated only on solutions of the classical equation of motion.

Concretely, we regard the components s_1^N, s_2^N of the higher currents of the sine-Gordon model as elements of the space of fields $\mathscr{D}'(\mathbb{M}_2\,;\mathcal{F}_{\mu c})$, i.e. as distributions with values in a certain space of functionals $\mathcal{F}_{\mu c}$. In the perturbative spirit of pAQFT, the framework yields the unrenormalized interacting components $(\check{s}_{1,2}^N)_{\text{int}}$ with respect to the interaction Lagrangian κL_{int} (here κ is called "bookkeeping" coupling constant and is simply a tool to account for the order of perturbation). These are given by certain formal power series in κ and $\hbar$ with coefficients in the space of fields $\mathcal{F}_{\mu c}$ and we denote them by:

$$(\check{s}_j^N)_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n\big(L_{\text{int}}^{\otimes n}, s_j^N\big), \quad j = 1, 2\,,$$

where $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_{1,2}^N\big) \in \mathscr{D}'(\check{\mathbb{M}}_2^{n+1}\,;\mathcal{F}_{\mu c})[\![\hbar]\!]$ is the so-called unrenormalized retarded product (see formula (2.16)) and

$$\check{\mathbb{M}}_2^{n+1} = \left\{\, (x_1, \ldots, x_{n+1}) \in \mathbb{M}_2^{n+1} \,\middle|\, x_i \neq x_j \quad \forall 1 \leq i < j \leq n+1 \,\right\}$$

is the $(n+1)$-fold product of $\mathbb{M}_2$ without the "big" diagonal.

In the framework of Epstein and Glaser, which we adopt, the renormalization problem is the problem of extending the retarded product $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_{1,2}^N\big)$ to a well-defined distribution on the whole space $\mathbb{M}_2^{n+1}$ by an ingenious application of techniques from microlocal analysis. The freedom in the solution of this problem is determined by the so-called scaling degree of the retarded product and in general one has the following situation (see Chapter 2 for the details).

Definition 2.5.2 Fix fields $\kappa F \in \mathscr{D}'(\mathbb{M}_d\,;\kappa\mathcal{F}_{\mu c})$ and $G \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})$. Consider the unrenormalized interacting field $(\check{G})_{\kappa F}$, given by:

$$(\check{G})_{\kappa F} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n(F^{\otimes n}, G).$$

Let $N(F, G, \cdot)\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be the function defined in the following way:

$$\text{for} \quad n = 0, \qquad N(F, G, 0) = \max\{0, \operatorname{sd}(G) - d\},$$

and for $n \geq 1$

$$N(F, G, n) = \max\left\{0, \operatorname{sd}\left(\check{R}_n(F^{\otimes n}, G)\right) - (n+1)d - N(F, G, n-1)\right\},$$

where sd indicates the scaling degree of the corresponding distributions. We say that the unrenormalized interacting field $(\check{G})_{\kappa F}$ is:

(a) **renormalizable by power counting** if $n \mapsto N(F, G, n)$ is bounded;

(b) **super-renormalizable by power counting** if $N(F, G, n)$ is non-zero only for a finite number of $n \in \mathbb{N}$.

We remark that super-renormalizability means that the ambiguity of the renormalization process is harmless. This is particularly relevant not only from the purely theoretical point of view, but also from the point of view of the empirical implications of physical theories. Roughly speaking, super-renormalizability means that it is in principle sufficient to perform a finite number of experiments to completely determine the theory.

Coming back to the sine-Gordon model, as a first main result, we prove in Chapter 4 the following theorem.

Theorem 4.1.1 *Let us consider the unrenormalized interacting components of the higher currents of the sine-Gordon model:*

$$(\check{s}_1^N)_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big), \qquad (\check{s}_2^N)_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big),$$

where $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big), \check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big) \in \mathscr{D}'(\check{\mathbb{M}}_2^{n+1}; \mathcal{F}_{\mu c})[\![\hbar]\!]$*. Then the scaling degree of the retarded products is uniformly bounded by the degree of the components. Specifically, for every* $n \geq 1$ *it holds:*

$$\operatorname{sd}\left(\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big)\right) = \deg(s_1^N) = 2N,$$
$$\operatorname{sd}\left(\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big)\right) = \deg(s_2^N) = 2(N+1).$$

As a direct consequence, according to the previous definition, the interacting components of the higher currents of the sine-Gordon model are super-renormalizable by power counting.

We remark that, compared to other approaches, where partially analogous results were obtained considering super-symmetric formulations or Fock space representations (see for example [13]), our setting has the advantage that we can avoid the issues related to the nonexistence of the massless Wightman field in two dimensions, see also Remark 5.0.2 about this.

We also point out that our argument on the renormalizability of the interacting currents follows from well-known results on scaling-degree-preserving extensions of distributions ([7], [10], [20], [27]) and on a notion of degree that we introduce based on the concrete expressions of the classical higher currents, which allows to obtain the crucial upper bound on the scaling degree of the distributions that are to be renormalized. Contrary to other renormalization techniques, these estimates do not require the explicit computation of so-called counterterms.

Another consequence of the fact that, in the perturbative spirit of pAQFT, interacting fields are described by formal power series in κ and $\hbar$ is that no notion of convergence is a priori considered for these objects – and in fact, no physical theory with summable such objects in dimension $d = 4$ is known. However, summability of at least certain observables can be expected for integrable systems in $d = 2$ dimensions, inasmuch as it has long been established using different quantization frameworks.

Remarkably the sine-Gordon model meets these expectations and indeed, in analogy with the analysis carried out in [1] for the scattering matrix, we are able to prove that the formal power series that arise as expectation values of the renormalized interacting components $(s^N_{1,2})_{\text{int}}$ in a generic Gaussian state are in fact summable. This follows directly from the second main result of our work, proved in Chapter 5.

Theorem 5.1.1 *Let $\beta = \frac{\hbar a^2}{4\pi}$ and let $\gamma > 1$ be such that $\beta\gamma < 1$. Let $g \in \mathscr{D}(\mathbb{M}_2)$ be a cut-off function for the interaction Lagrangian L_{int} and denote $f = g^{\otimes(n+1)} \in \mathscr{D}(\mathbb{M}_2^{n+1})$. Consider the expectation values $\omega_{\varphi,H}\left(\mathcal{R}_n(L_{\text{int}}, s^N_{1,2})(f)\right)$ of the retarded products $\mathcal{R}_n(L_{\text{int}}, s^N_{1,2})$ in the state $\omega_{\varphi,H}$, with H as in formula (3.2). Then, choosing the support of g small enough, there exist two pairs of constants $\mathcal{K}^{s_1}_{\gamma,g,a\hbar,N}$, $\mathcal{C}^{s_1}_{\gamma,g}$ and $\mathcal{K}^{s_2}_{\gamma,g,a\hbar,N}$, $\mathcal{C}^{s_2}_{\gamma,g}$ such that for all $n \geq 1$ the following estimates hold:*

$$\left|\mathcal{R}_n(L_{\text{int}}, s^N_1)(f)[\varphi]\right| \leq \mathcal{K}^{s_1}_{\gamma,g,a\hbar,N} \frac{(n+1)^2 n^{2N} \left(\mathcal{C}^{s_1}_{\gamma,g}\right)^n}{\left(\left[\frac{n}{2}\right]!\right)^{1-\frac{1}{\gamma}}},$$
$$\left|\mathcal{R}_n(L_{\text{int}}, s^N_2)(f)[\varphi]\right| \leq \mathcal{K}^{s_2}_{\gamma,g,a\hbar,N} \frac{\left[\frac{n}{2}\right] n^{2N} \left(\mathcal{C}^{s_2}_{\gamma,g}\right)^n}{\left(\left[\frac{n}{2}\right]!\right)^{1-\frac{1}{\gamma}}},$$

where $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$.

To the best of our knowledge, Theorem 4.1.1 and Theorem 5.1.1 are the first complete results on renormalizability and summability of all the higher currents of the sine-Gordon model.

For a recent discussion of the renormalizability, summability, conservation and other properties of the *first* of the higher conserved currents of the sine-Gordon model, that is, its stress-energy tensor, in a framework related to pAQFT, where the counterterms are explicitly computed, we refer to [15] and [16].

Besides extending the list of remarkable properties of the sine-Gordon model in pAQFT, we believe that, as a byproduct of the analysis, the notation and technical results that we introduce along the way might represent a good foundation for the subsequent investigation of the conservation and involutivity properties of the interacting components, using tools which are independent of any a priori representation on e.g. Fock space. For more comments about these interesting questions, see the Outlook.

The material of the thesis is organized as follows.

Chapter 1 deals briefly with the classical theory of the sine-Gordon model. In view of our later purposes, we extend the construction presented in [29] to the case of the sine-Gordon model with coupling constant (so that the setting considered in [29] is recovered as a particular case for the value $a = 1$). After recalling the basic notions, we introduce the notion of Bäcklund transformations and we derive explicit recursive formulas for the solution of the extended Bäcklund transformations. We then give a brief account of the concepts entering in Noether's Theorem and use it in combination with the extended Bäcklund transformations to finally obtain the higher conserved currents. Again, we give explicit expressions for the components of the higher conserved currents. Along the way, we also introduce a notion of degree that reveals to be crucial in the discussion of the renormalization of the currents in pAQFT.

Chapter 2 opens the discussion of the quantum theory. It is a summary of the fundamentals of perturbative Algebraic Quantum Field Theory on Minkowski spacetime in arbitrary dimension.

Chapter 3 starts adapting the notions introduced in Chapter 2 to the case of the sine-Gordon theory. The final goal is to compute the explicit expressions of the unrenormalized retarded products $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_{1,2}^N\big)$. This is achieved by first proving and then applying some technical results on the time-ordered products and on the star products of fields with specific properties, inspired by those of the components of the higher currents.

In Chapter 4 we show the super-renormalizability by power counting of the unrenormalized interacting components $(\check{s}_{1,2}^N)_{\text{int}}$. This result is obtained as a corollary of Theorem 4.1.1 presented above. Subsequently, we describe in detail the concrete renormalization of the retarded product $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_{1,2}^N\big)$. We outline a procedure which we call piecewise renormalization. It consists of three steps: first, expansion the unrenormalized expressions of the retarded components to their very elementary parts, then scaling-degree-preserving extension of the elementary parts separately and finally proof that reassembling the piecewise renormalized parts all together gives a well-defined renormalized version of the initial object.

Chapter 5 is completely devoted to the proof of the summability properties of the renormalized interacting components of the higher currents. After briefly recalling the setting, in the spirit of [1], and introducing the convenient notation, the quite involved steps of the proof are worked out in full detail.

Finally, Appendices A, B and C collect useful notions and well-known results about the concepts of wavefront set of distributions and of scaling degree and about the most relevant properties of the Feynman and anti-Feynman propagators.

The main results of Chapters 1, 3 and 4 have been published in a recent preprint [30].

Chapter 1

Conserved currents in the classical theory

In this chapter we explain how the higher conserved currents for the classical sine-Gordon model can be obtained combining the so-called extended Bäcklund transformations with Noether's Theorem.

Remark 1.0.1. We point out that our exposition is far from being a complete account of all the techniques developed in the study of the sine-Gordon model as an integrable system. The literature in this respect is extremely extensive and varied. We decided to build on a specific construction because it seemed to us to be the best suited to highlight the crucial properties of the conserved currents, in view of their subsequent quantization in the framework of pAQFT. For this purpose, the most relevant result of this chapter are the recursive formulas given in Proposition 1.4.1.

Conceptually we follow the same passages as in [29]. However, we extend all the definitions and results (given there only for the standard sine-Gordon model) to the more general case of the sine-Gordon model with coupling constant (in the following referred to as the general sine-Gordon model or simply as the sine-Gordon model). In addition to this and in view of the subsequent discussion of the renormalization properties of the conserved currents in the quantum theory, we also derive explicit recursive formulas for the quantities involved.

We start introducing some notation. The sine-Gordon model is a massless relativistic non-linear scalar field theory. The role of spacetime is played by the 2-dimensional Minkowski space $\mathbb{M}_2$ (see also Section 2.1). The configuration bundle of the theory is the trivial bundle $\mathbb{M}_2 \times \mathbb{R} \longrightarrow \mathbb{M}_2$. Configurations are sections of this bundle, that is, functions $\varphi \in C^\infty(\mathbb{M}_2) := \mathscr{E}(\mathbb{M}_2)$. Adopting cartesian coordinates $(x^0 =: t, x^1 =: \vec{x})$ on $\mathbb{M}_2$, with Minkowski metric $\eta =$

$\mathrm{diag}(1,-1)$, the Lagrangian of the sine-Gordon model is written as:

$$L(\varphi)\,dt\wedge d\vec{x}=(L_0+L_{\text{int}})\,dt\wedge d\vec{x}=\Big(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+\cos(a\varphi)\Big)\,dt\wedge d\vec{x},$$

where $\eta^{\mu\nu}$ is the "inverse" of the Minkowski metric and the parameter $a>0$ is called coupling constant. The corresponding Euler-Lagrange equation, also called sine-Gordon equation, is:

$$-\Box\varphi-a\sin(a\varphi)=-\partial_t^2\varphi+\partial_{\vec{x}}^2\varphi-a\sin(a\varphi)=0.$$

As it often happens in the study of partial differential equations, the choice of independent variables with respect to which the equations are written is of crucial importance. In the sequel we will always work with the so-called light-cone coordinates (τ,ξ), which turn out to be very efficient in the description of the conservation laws of the sine-Gordon model. The relation between cartesian and light-cone coordinates is given by:

$$\begin{cases}\tau=\frac{1}{2}(\vec{x}+t)\\ \xi=\frac{1}{2}(\vec{x}-t)\end{cases}\qquad\qquad\begin{cases}\partial_t=\frac{1}{2}(\partial_\tau-\partial_\xi)\\ \partial_{\vec{x}}=\frac{1}{2}(\partial_\tau+\partial_\xi).\end{cases}$$

The sine-Gordon Lagrangian in light-cone coordinates becomes

$$L(\varphi)\,d\tau\wedge d\xi=(L_0+L_{\text{int}})\,d\tau\wedge d\xi=\Big(-\frac{1}{2}\varphi_\xi\varphi_\tau+\cos(a\varphi)\Big)\,d\tau\wedge d\xi,$$

and the sine-Gordon equation is

$$\varphi_{\xi\tau}-a\sin(a\varphi)=0, \tag{1.1}$$

where we adopted the convention that the $_\tau$ and $_\xi$ indicate partial derivation with respect to the corresponding coordinate.

Remark 1.0.2. The so-called standard sine-Gordon model, which is more often treated in the literature, for example also in [29], is the special case for $a=1$.

1.1 Bäcklund transformations

In [29], Bäcklund transformations are defined for the standard sine-Gordon model. We claim that a slight modification of the formulas presented there, namely the insertion of a dependence on the coupling constant $a>0$, yields the correct expressions of the Bäcklund transformations for the sine-Gordon model with coupling constant. More specifically, we consider the following parametric system of first order partial differential equations:

$$\frac{1}{2}(\varphi' + \varphi)_\xi = \frac{1}{\alpha}\sin\big[\frac{a}{2}(\varphi' - \varphi)\big] \tag{1.2a}$$

$$\frac{1}{2}(\varphi' - \varphi)_\tau = \alpha\sin\big[\frac{a}{2}(\varphi' + \varphi)\big], \tag{1.2b}$$

where $\varphi', \varphi \in \mathscr{E}(\mathbb{M}_2)$, φ is given, φ' is unknown and $\alpha \in \mathbb{R}$.

Confirming that our definition is sensible, we have that a necessary and sufficient condition for the Bäcklund transformations above to be integrable is that φ is a solution of the sine-Gordon equation. In other words, this means that if on $\mathbb{M}_2$ we define the 1-form θ (also called Pfaffian 1-form) as

$$\theta = \Big(\frac{1}{2}\varphi_\tau + \alpha\sin\big[\frac{a}{2}(\varphi' + \varphi)\big]\Big)d\tau + \Big(-\frac{1}{2}\varphi_\xi + \frac{1}{\alpha}\sin\big[\frac{a}{2}(\varphi' - \varphi)\big]\Big)d\xi,$$

then θ is a closed form if and only if φ is a solution of the sine-Gordon equation. This can be seen by the following elementary computations. The differential of the form θ is given by:

$$\begin{aligned} d\theta &= \Big[-\Big(\frac{1}{2}\varphi_\tau + \alpha\sin\big[\frac{a}{2}(\varphi' + \varphi)\big]\Big)_\xi \\ &\quad + \Big(-\frac{1}{2}\varphi_\xi + \frac{1}{\alpha}\sin\big[\frac{a}{2}(\varphi' - \varphi)\big]\Big)_\tau\Big]\, d\tau \wedge d\xi \\ &= \Big[-\frac{1}{2}\varphi_{\tau\xi} - \frac{a\alpha}{2}(\varphi' + \varphi)_\xi \cos\big[\frac{a}{2}(\varphi' + \varphi)\big] \\ &\quad -\frac{1}{2}\varphi_{\tau\xi} + \frac{a}{2\alpha}(\varphi' - \varphi)_\tau \cos\big[\frac{a}{2}(\varphi' - \varphi)\big]\Big]\, d\tau \wedge d\xi. \end{aligned}$$

Substituting equations (1.2a) and (1.2b), changing the sign of one of the arguments, and using the addition formula for sine, we then obtain:

$$\begin{aligned} d\theta &= \Big(-\varphi_{\tau\xi} - a\sin\big[\frac{a}{2}(\varphi' - \varphi)\big]\cos\big[\frac{a}{2}(\varphi' + \varphi)\big] \\ &\quad + a\sin\big[\frac{a}{2}(\varphi' + \varphi)\big]\cos\big[\frac{a}{2}(\varphi' - \varphi)\big]\Big)\, d\tau \wedge d\xi \\ &= \Big(-\varphi_{\tau\xi} + a\sin\big[\frac{a}{2}(\varphi - \varphi')\big]\cos\big[\frac{a}{2}(\varphi' + \varphi)\big] \\ &\quad + a\cos\big[\frac{a}{2}(\varphi - \varphi')\big]\sin\big[\frac{a}{2}(\varphi' + \varphi)\big]\Big)\, d\tau \wedge d\xi \\ &= \Big(-\varphi_{\tau\xi} + a\sin\big[\frac{a}{2}(\varphi - \varphi') + \frac{a}{2}(\varphi' + \varphi)\big]\Big)\, d\tau \wedge d\xi \\ &= \Big(-\varphi_{\tau\xi} + a\sin(a\varphi)\Big)\, d\tau \wedge d\xi. \end{aligned}$$

The condition on the Pfaffian 1-form θ of being closed or, equivalently, of equations (1.2a) and (1.2b) of being integrable ensures the existence of a unique (local) solution φ' to the Bäcklund transformations (for a proof of this statement, see for example [21, Proposition 19.28, p.510]).

As a consequence, φ' is in turn automatically a solution of the sine-Gordon equation. Indeed, if we take the derivative with respect to τ of the first equation (1.2a) and the derivative with respect to ξ of the second one (1.2b)

$$\begin{aligned} \frac{1}{2}\varphi'_{\tau\xi} + \frac{1}{2}\varphi_{\tau\xi} &= \frac{a}{2\alpha}(\varphi' - \varphi)_\tau \cos\left[\frac{a}{2}(\varphi' - \varphi)\right] \\ \frac{1}{2}\varphi'_{\tau\xi} - \frac{1}{2}\varphi_{\tau\xi} &= \frac{a\alpha}{2}(\varphi' + \varphi)_\xi \cos\left[\frac{a}{2}(\varphi' + \varphi)\right], \end{aligned}$$

substitute again equations (1.2a) and (1.2b)

$$\begin{aligned} \frac{1}{2}\varphi'_{\tau\xi} + \frac{1}{2}\varphi_{\tau\xi} &= a\sin\left[\frac{a}{2}(\varphi' + \varphi)\right] \cos\left[\frac{a}{2}(\varphi' - \varphi)\right] \\ \frac{1}{2}\varphi'_{\tau\xi} - \frac{1}{2}\varphi_{\tau\xi} &= a\sin\left[\frac{a}{2}(\varphi' - \varphi)\right] \cos\left[\frac{a}{2}(\varphi' + \varphi)\right], \end{aligned}$$

and sum up the two equations, we finally get

$$\varphi'_{\tau\xi} = a\sin\left[\frac{a}{2}(\varphi' + \varphi) + \frac{a}{2}(\varphi' - \varphi)\right] = a\sin(a\varphi').$$

Hence, as expected, Bäcklund transformations relate solutions φ', φ of the sine-Gordon equation, and we write $\varphi' = B_\alpha\varphi$.

1.2 Extended Bäcklund transformations

As noted in [29], Bäcklund transformations are not suited to be combined with Noether's Theorem because they are transformations acting only on the space of solutions of the sine-Gordon equation. On the contrary, we seek for transformations acting on generic configurations.

It turns out that the correct way to implement this necessity, without at the same time completely forgetting about the relations between solutions encoded in the Bäcklund transformations, is simply to drop one of the equations (1.2a) or (1.2b). The choice does not influence the subsequent discussion, which can be repeated in either case.

Definition 1.2.1. We say that the configuration $\varphi' \in \mathscr{E}(\mathbb{M}_2)$ is obtained from a given configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$ by an extended Bäcklund transformation $\hat{B}_\alpha$ of parameter $\alpha \in \mathbb{R}$, in notation $\varphi' = \hat{B}_\alpha\varphi$, if φ' satisfies the parametric PDE:

$$\frac{1}{2}(\varphi' + \varphi)_\xi = \frac{1}{\alpha}\sin\left[\frac{a}{2}(\varphi' - \varphi)\right] \tag{1.3}$$

The next conceptual step, following once more [29], may seem cumbersome at a first glance. It consists in assuming that φ' admits a power series expansion in the parameter α. We denote the above mentioned expansion by:

$$\varphi' = \hat{B}_\alpha \varphi = \sum_{\nu=0}^{\infty} A_\nu[a,\varphi]\alpha^\nu, \tag{1.4}$$

where the coefficients A_ν depend on both the coupling constant a and the initial configuration φ. Now we can substitute (1.4) in equation (1.3) and use the power series expansion of the sine function. Omitting the dependence of the coefficients A_ν on a and φ, we get:

$$\sum_{\nu=0}^{\infty} A_{\nu,\xi}\alpha^\nu = -\varphi_\xi + \frac{2}{\alpha}\sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{(2\mu+1)!}\left(\frac{a}{2}\right)^{2\mu+1}\left(\sum_{\nu=0}^{\infty}A_\nu\alpha^\nu - \varphi\right)^{2\mu+1}. \tag{1.5}$$

Requiring that the value for $\alpha = 0$ of the right hand side is well-defined amounts to the equation:

$$2\sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{(2\mu+1)!}\left(\frac{a}{2}\right)^{2\mu+1}(A_0-\varphi)^{2\mu+1} = 2\sin\left[\frac{a}{2}(A_0-\varphi)\right] = 0.$$

This equation implies $A_0 = \varphi + \frac{2k\pi}{a}$, $k \in \mathbb{Z}$, from which we select the particular case $A_0 = \varphi$. Implementing this condition in formula (1.5) we obtain:

$$\sum_{\nu=0}^{\infty} A_{\nu,\xi}\alpha^\nu = -\varphi_\xi + 2\sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{(2\mu+1)!}\left(\frac{a}{2}\right)^{2\mu+1}\alpha^{2\mu}\left(\sum_{\nu=0}^{\infty}A_{\nu+1}\alpha^\nu\right)^{2\mu+1}. \tag{1.6}$$

Now we start comparing the coefficients from the left hand side and the right hand side of equation (1.6) order by order:

- At order 0, we have: $A_{0,\xi} = -\varphi_\xi + 2\cdot\frac{a}{2}A_1 \quad\longrightarrow\quad A_1 = \frac{2}{a}\varphi_\xi.$
- At order 1, we get: $A_{1,\xi} = 2\cdot\frac{a}{2}A_2 \quad\longrightarrow\quad A_2 = \frac{2}{a^2}\varphi_{\xi\xi}.$

For the higher coefficients A_ν, studying more in detail the structure of equation (1.6), we can prove the following result.

Proposition 1.2.1. *For $\nu \geq 2$ the following formula holds:*

$$A_{\nu+1} = \frac{1}{a}A_{\nu,\xi} + \sum_{\beta=0}^{[\frac{\nu}{2}]-1}(-1)^\beta\left(\frac{a}{2}\right)^{2(\beta+1)}\sum_{\substack{n_0,\dots,n_{\nu-2-2\beta}\geq 0\\ n_0+\dots+n_{\nu-2-2\beta}=2\beta+3\\ 1\cdot n_1+\dots+(\nu-2-2\beta)\cdot n_{\nu-2-2\beta}=\nu-2-2\beta}}\frac{A_1^{n_0}\cdots A_{\nu-1-2\beta}^{n_{\nu-2-2\beta}}}{n_0!\cdots n_{\nu-2-2\beta}!}, \tag{1.7}$$

where $[\frac{\nu}{2}]$ denotes the integer part of $\frac{\nu}{2}$.

Proof. We rearrange the summation on the right hand side of equation (1.6) in the following way:

$$\sum_{\mu=0}^{\infty} \frac{(-1)^{\mu}}{(2\mu+1)!} \left(\frac{a}{2}\right)^{2\mu+1} \alpha^{2\mu} \Big(\sum_{\nu=0}^{\infty} A_{\nu+1}\alpha^{\nu}\Big)^{2\mu+1}$$
$$= \sum_{\mu,\rho=0}^{\infty} (-1)^{\mu} \left(\frac{a}{2}\right)^{2\mu+1} \Big(\sum_{\substack{n_0,\dots,n_\rho \geq 0 \\ n_0+\cdots+n_\rho = 2\mu+1 \\ 1\cdot n_1+\cdots+\rho\cdot n_\rho=\rho}} \frac{A_1^{n_0}\cdots A_{\rho+1}^{n_\rho}}{n_0!\cdots n_\rho!}\Big)\alpha^{\rho+2\mu}.$$

We rewrite the double summation using indices $\nu := \rho + 2\mu$ and $\beta := \mu$, this gives the expression:

$$\sum_{\nu=0}^{\infty} \alpha^{\nu} \Big(\sum_{\beta=0}^{[\frac{\nu}{2}]} (-1)^{\beta} \left(\frac{a}{2}\right)^{2\beta+1} \sum_{\substack{n_0,\dots,n_{\nu-2\beta} \geq 0 \\ n_0+\cdots+n_{\nu-2\beta} = 2\beta+1 \\ 1\cdot n_1+\cdots+(\nu-2\beta)\cdot n_{\nu-2\beta}=\nu-2\beta}} \frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}\Big),$$

where $[\frac{\nu}{2}]$ is the integer part of $\frac{\nu}{2}$. We now observe that we can decompose the coefficient of α^{ν} in two parts, one corresponding to $\beta \geq 1$ and the other for $\beta = 0$, respectively:

$$\sum_{\beta=0}^{[\frac{\nu}{2}]} (-1)^{\beta} \left(\frac{a}{2}\right)^{2\beta+1} \sum_{\substack{n_0,\dots,n_{\nu-2\beta} \geq 0 \\ n_0+\cdots+n_{\nu-2\beta} = 2\beta+1 \\ 1\cdot n_1+\cdots+(\nu-2\beta)\cdot n_{\nu-2\beta}=\nu-2\beta}} \frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}$$
$$= \sum_{\beta=1}^{[\frac{\nu}{2}]} (-1)^{\beta} \left(\frac{a}{2}\right)^{2\beta+1} \sum_{\substack{n_0,\dots,n_{\nu-2\beta} \geq 0 \\ n_0+\cdots+n_{\nu-2\beta} = 2\beta+1 \\ 1\cdot n_1+\cdots+(\nu-2\beta)\cdot n_{\nu-2\beta}=\nu-2\beta}} \frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}$$
$$+ \frac{a}{2} \sum_{\substack{n_0,\dots,n_\nu \geq 0 \\ n_0+\cdots+n_\nu = 1 \\ 1\cdot n_1+\cdots+\nu\cdot n_\nu=\nu}} \frac{A_1^{n_0}\cdots A_{\nu+1}^{n_\nu}}{n_0!\cdots n_\nu!}.$$

In particular the last term reduces to $\frac{a}{2}A_{\nu+1}$. Comparing the coefficients of the power α^{ν}, for $\nu \geq 2$, from equation (1.6), we get:

$$A_{\nu,\xi} = 2\sum_{\beta=1}^{[\frac{\nu}{2}]} (-1)^{\beta} \left(\frac{a}{2}\right)^{2\beta+1} \sum_{\substack{n_0,\dots,n_{\nu-2\beta} \geq 0 \\ n_0+\cdots+n_{\nu-2\beta} = 2\beta+1 \\ 1\cdot n_1+\cdots+(\nu-2\beta)\cdot n_{\nu-2\beta}=\nu-2\beta}} \frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}$$
$$+ aA_{\nu+1}.$$

Extracting $A_{\nu+1}$ and rescaling the summation over β, we conclude. *q.e.d.*

The computation of the first coefficients using formula (1.7) gives:

$$\begin{aligned} A_3 &= \frac{2}{a^3}\varphi_{\xi\xi\xi} + \frac{1}{3a}\varphi_\xi^3, \\ A_4 &= \frac{2}{a^4}\varphi_{4\xi} + \frac{2}{a^2}\varphi_\xi^2\varphi_{\xi\xi}. \end{aligned} \tag{1.8}$$

Remark 1.2.1. We observe that from formula (1.7) it follows that the coefficients A_ν are all polynomials in the derivatives of the configuration φ with respect only to the light-cone coordinate ξ.

Remark 1.2.2. As a consistency check, the expressions for the coefficients A_ν presented in [29] are recovered from our expressions setting $a = 1$.

Proposition 1.2.2. *The extended Bäcklund transformations enjoy the following properties*

(a) For $\alpha = 0$*,* $\hat{B}_0$ *is the identity;*

(b) Invertibility: $\hat{B}_\alpha^{-1} = \hat{B}_{-\alpha}$*;*

(c) Commutativity: $\hat{B}_{\alpha_1} \circ \hat{B}_{\alpha_2} = \hat{B}_{\alpha_2} \circ \hat{B}_{\alpha_1}$.

Remark 1.2.3. Despite these nice properties, Bäcklund transformations do not form a group, i.e. the composition $\hat{B}_{\alpha_1} \circ \hat{B}_{\alpha_2}$ is not in general again a Bäcklund transformation $\hat{B}_{\alpha_3}$, for some $\alpha_3 \in \mathbb{R}$.

We introduce now a notion that will turn out to be crucial for the subsequent discussion of the renormalization properties of the higher currents in pAQFT.

Definition 1.2.2. Consider a configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$. We assign a degree to its k-th derivative with respect to the light-cone coordinate ξ, by:

$$\deg(\varphi_{k\xi}) = k, \qquad \forall k \in \mathbb{N}.$$

We extend this definition to monomials in the derivatives of φ by additivity:

$$\deg(\varphi_{k_1\xi}\varphi_{k_2\xi}\dots\varphi_{k_N\xi}) = k_1 + k_2 + \dots + k_N.$$

We say that a polynomial in the derivatives of φ is homogeneous of degree d if all its monomial terms have degree d.

Proposition 1.2.3. *For every* $\nu \geq 0$*, the coefficient* A_ν *is homogeneous of degree equal to* ν.

Proof. The claim is trivial for $A_0 = \varphi$, $A_1 = \frac{2}{a}\varphi_\xi$ and $A_2 = \frac{2}{a^2}\varphi_{\xi\xi}$. For $\nu \geq 3$ we proceed by induction. For $\nu = 3$, using formulas (1.8), we have

$$\deg(A_3) = \deg\Big(\frac{2}{a^3}\varphi_{\xi\xi\xi} + \frac{1}{3a}\varphi_\xi^3\Big) = 3.$$

Now suppose the claim is true for $\nu \leq N$. The coefficient A_{N+1} is given by formula (1.7). The first term is $\frac{1}{a}A_{N,\xi}$ which, due to the additional derivative with respect to the coordinate ξ, has degree $N+1$. The other terms are given by products

$$A_1^{n_0} \cdots A_{N-1-2\beta}^{n_{N-2-2\beta}},$$

with the following conditions on the indices $n_0, \ldots, n_{N-2-2\beta} \in \mathbb{N}$:

$$\begin{aligned} n_0 + \cdots + n_{N-2-2\beta} &= 2\beta + 3 \\ 1 \cdot n_1 + \cdots + (N-2-2\beta) \cdot n_{N-2-2\beta} &= N - 2 - 2\beta \end{aligned}$$

From these conditions and additivity of the degree, it follows that

$$\begin{aligned} \deg\big(A_1^{n_0} \cdots A_{N-1-2\beta}^{n_{N-2-2\beta}}\big) &= 1 \cdot n_0 + \cdots + (N-1-2\beta) \cdot n_{N-2-2\beta} \\ &= n_0 + n_1 + \cdots + n_{N-2-2\beta} \\ &\quad + n_1 + \cdots + (N-2-2\beta) \cdot n_{N-2-2\beta} \\ &= 2\beta + 3 + N - 2 - 2\beta \\ &= N + 1. \end{aligned}$$

q.e.d.

1.3 Noether's Theorem

Before entering in the details of the relation between Noether's Theorem and the higher conserved currents of the sine-Gordon model, in this section we briefly recall the main features of the former in broad generality. The following review is based on [12].

A k-th order Lagrangian field theory is given by a fiber bundle $\pi\colon E \to M$, called configuration bundle, over a d-dimensional manifold M, which plays the rôle of spacetime, and a Lagrangian function L

$$\begin{array}{ccc} J^kE & \xrightarrow{\;L\;} & \Omega^d M \\ \downarrow{\scriptstyle \pi_0^k} & & \\ E & & \downarrow \\ \downarrow{\scriptstyle \pi} & & \\ M & \xrightarrow{\;\mathrm{id}\;} & M \end{array}$$

where $J^k E$ is the k-th order jet prolongation of the fiber bundle E, π_0^k is the natural projection on E, and $\Omega^d M$ is the bundle of differential d-forms on M.

Definition 1.3.1. We say that a fiber bundle automorphism (Ψ, ψ) of E

$$\begin{array}{ccc} E & \xrightarrow{\Psi} & E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\psi} & M \end{array}$$

is a symmetry of the Lagrangian L if

$$\big(j^k(\Psi, \psi)\big)^* L = L,$$

where $j^k(\Psi, \psi)$ is the prolongation to the jet of order k of the automorphism (Ψ, ψ), i.e.

$$\begin{array}{ccc} J^k E & \xrightarrow{j^k(\Psi,\psi)} & J^k E \\ {\scriptstyle \pi_0^k}\downarrow & & \downarrow{\scriptstyle \pi_0^k} \\ E & \xrightarrow{\Psi} & E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\psi} & M \end{array}$$

(Ψ, ψ) is also called a Lagrangian symmetry.

Definition 1.3.2. Let $G \subseteq \mathrm{Aut}(E)$ be a subgroup of automorphisms of the fiber bundle E. A Lagrangian of order k is G-covariant if any automorphism $(\Psi, \psi) \in G$ is a Lagrangian symmetry.

Consider now a 1-parameter subgroup (Ψ_s, ψ_s) of Lagrangian symmetries and denote by Ξ its infinitesimal generator. Then $\Xi \in \mathfrak{X}_{\text{proj}}(E)$ is a projectable vector field on E which projects onto a vector field $\xi \in \mathfrak{X}(M)$.

Theorem 1.3.1 (Noether's Theorem). *Let $(\pi\colon E \to M, L)$ be a k-th order Lagrangian field theory and Ξ be the infinitesimal generator of a 1-parameter group of Lagrangian symmetries. With this data it is possible to determine a $(d-1)$-form $\mathcal{E}(L, \Xi)$, defined on $j^{2k-1}E$, such that for every section of the configuration bundle $\rho\colon M \to E$ which is also a solution of the Euler-Lagrange equations it holds*

$$d\big((j^{2k-1}\rho)^* \mathcal{E}(L, \Xi)\big) = 0, \tag{1.9}$$

The $(d-1)$-form $\mathcal{E}(L, \Xi)$ is called a Noether current and equation (1.9) is called an on-shell conservation law.

In some cases (for example, in the sine-Gordon model, as we will see in the sequel of this section) it turns out that it is necessary to have a wider definition of symmetries. Hints about the correct generalization come from the following two observations. On the one hand, the determination of the Noether current from the infinitesimal generator of the symmetries involves the computation of the Lie derivative of the Lagrangian with respect to the infinitesimal generator. But, Lie derivatives can also be computed with respect to generalized vector fields (roughly speaking, vector fields defined on the infinity jet $J^\infty E$, for a detailed definition see [12]). On the other hand, since the final result of Noether's Theorem is to produce a $(d-1)$-form which is closed on-shell, we always have the freedom to add some exact form. These remarks lead to the following definition and theorem.

Definition 1.3.3. An infinitesimal generator of generalized symmetries for a Lagrangian field theory $(\pi\colon E \to M, L)$ of order k is a generalized projectable vector field X such that

$$\pounds_X\big((\pi_k^\infty)^* L\big) = d\beta,$$

for some $(d-1)$-form β defined on some finite order jet prolongation of E.

Theorem 1.3.2 (Generalized Noether's Theorem). *Suppose X is an infinitesimal generator of generalized symmetries for the Lagrangian L. It is possible to determine a Noether current $\mathcal{E}(L, X)$ such that an on-shell conservation law of the form (1.9) holds.*

1.4 Conserved currents via Noether's Theorem

Coming back to the sine-Gordon model, and following the analysis in [29], we observe that extended Bäcklund transformations act on the sine-Gordon Lagrangian changing it by an exact form. In other words, given an arbitrary configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$, it holds:

$$\big(L(\hat{B}_\alpha\varphi) - L(\varphi)\big)\, d\tau \wedge d\xi = dp^{[\alpha,0]} = \big(-p_{1,\xi}^{[\alpha,0]} + p_{2,\tau}^{[\alpha,0]}\big)\, d\tau \wedge d\xi, \tag{1.10}$$

where $p^{[\alpha,0]}$ is a horizontal 1-form $p^{[\alpha,0]} = p_1^{[\alpha,0]} d\tau + p_2^{[\alpha,0]} d\xi$, with components

$$p_1^{[\alpha,0]} = \frac{2\alpha}{a} \cos\Big[\frac{a}{2}\big(\hat{B}_\alpha\varphi + \varphi\big)\Big] - \frac{1}{2}\varphi_\tau(\hat{B}_\alpha\varphi - \varphi),$$
$$p_2^{[\alpha,0]} = -\frac{2}{a\alpha} \cos\Big[\frac{a}{2}\big(\hat{B}_\alpha\varphi - \varphi\big)\Big] - \frac{1}{2}\varphi_\xi(\hat{B}_\alpha\varphi - \varphi).$$

To show this, we first check that for any pair of configurations $\varphi', \varphi \in \mathscr{E}(\mathbb{M}_2)$ it holds

$$L(\varphi') - L(\varphi) = \sigma_{+,\xi}\sigma_{-,\tau} + \sigma_{-,\xi}\sigma_{+,\tau} + 2\sin(a\sigma_+)\sin(a\sigma_-), \tag{1.11}$$

where $\sigma_\pm := \frac{1}{2}(\varphi' \pm \varphi)$. Indeed:

$$\begin{aligned}
&\sigma_{+,\xi}\sigma_{-,\tau} + \sigma_{-,\xi}\sigma_{+,\tau} + 2\sin(a\sigma_+)\sin(a\sigma_-) \\
&= \frac{1}{4}\left(\varphi'_\xi\varphi'_\tau - \varphi'_\xi\varphi_\tau + \varphi_\xi\varphi'_\tau - \varphi_\xi\varphi_\tau\right) + \frac{1}{4}\left(\varphi'_\xi\varphi'_\tau + \varphi'_\xi\varphi_\tau - \varphi_\xi\varphi'_\tau - \varphi_\xi\varphi_\tau\right) \\
&\quad + 2\sin\left(\frac{a}{2}(\varphi' + \varphi)\right)\sin\left(\frac{a}{2}(\varphi' - \varphi)\right) \\
&= 2\left[\sin\frac{a\varphi'}{2}\cos\frac{a\varphi}{2} + \cos\frac{a\varphi'}{2}\sin\frac{a\varphi}{2}\right]\cdot\left[\sin\frac{a\varphi'}{2}\cos\frac{a\varphi}{2} - \cos\frac{a\varphi'}{2}\sin\frac{a\varphi}{2}\right] \\
&\quad + \frac{1}{2}\varphi'_\xi\varphi'_\tau - \frac{1}{2}\varphi_\xi\varphi_\tau \\
&= \frac{1}{2}\varphi'_\xi\varphi'_\tau - \frac{1}{2}\varphi_\xi\varphi_\tau + 2\left[\sin^2\frac{a\varphi'}{2}\cos^2\frac{a\varphi}{2} - \cos^2\frac{a\varphi'}{2}\sin^2\frac{a\varphi}{2}\right] \\
&= \frac{1}{2}\varphi'_\xi\varphi'_\tau - \frac{1}{2}\varphi_\xi\varphi_\tau + \left[\left(1 - \cos^2\frac{a\varphi'}{2}\right)\cos^2\frac{a\varphi}{2} - \left(1 - \sin^2\frac{a\varphi'}{2}\right)\sin^2\frac{a\varphi}{2}\right] \\
&\quad + \left[\sin^2\frac{a\varphi'}{2}(1 - \sin^2\frac{a\varphi}{2}) - \cos^2\frac{a\varphi'}{2}(1 - \cos^2\frac{a\varphi}{2})\right] \\
&= \frac{1}{2}\varphi'_\xi\varphi'_\tau - \frac{1}{2}\varphi_\xi\varphi_\tau + \left[\cos^2\frac{a\varphi}{2} - \sin^2\frac{a\varphi}{2}\right] + \left[\sin^2\frac{a\varphi'}{2} - \cos^2\frac{a\varphi'}{2}\right] \\
&= \left(\frac{1}{2}\varphi'_\xi\varphi'_\tau - \cos(a\varphi')\right) - \left(\frac{1}{2}\varphi_\xi\varphi_\tau - \cos(a\varphi)\right).
\end{aligned}$$

Remark 1.4.1. Our formula (1.11) slightly differs from its analogue in [29]. More precisely, we do not refer to the presence of the coupling constant a, which is a consequence of the more general situation we are working in, but to the plus sign in $\sigma_{+,\xi}\sigma_{-,\tau} + \sigma_{-,\xi}\sigma_{+,\tau}$ (which is $\sigma_{+,\xi}\sigma_{-,\tau} - \sigma_{-,\xi}\sigma_{+,\tau}$ in [29] instead). We believe that this might be a misprint in the paper.

When φ' is related to φ by a Bäcklund transformation $\varphi' = \hat{B}_\alpha\varphi$, we moreover have that the right hand sides of equations (1.10) and (1.11) actually coincide. In fact, if we expand the first one we obtain:

$$\begin{aligned}
&- p^{[\alpha,0]}_{1,\xi} + p^{[\alpha,0]}_{2,\tau} \\
&= \alpha\left(\hat{B}_\alpha\varphi + \varphi\right)_\xi \sin\left[\frac{a}{2}\left(\hat{B}_\alpha\varphi + \varphi\right)\right] + \frac{1}{2}\varphi_{\tau\xi}\left(\hat{B}_\alpha\varphi - \varphi\right) + \frac{1}{2}\varphi_\tau\left(\hat{B}_\alpha\varphi - \varphi\right)_\xi \\
&+ \frac{1}{\alpha}\left(\hat{B}_\alpha\varphi - \varphi\right)_\tau \sin\left[\frac{a}{2}\left(\hat{B}_\alpha\varphi - \varphi\right)\right] - \frac{1}{2}\varphi_{\tau\xi}\left(\hat{B}_\alpha\varphi - \varphi\right) - \frac{1}{2}\varphi_\xi\left(\hat{B}_\alpha\varphi - \varphi\right)_\tau.
\end{aligned}$$

Using equation (1.3) to rewrite the terms $\left(\hat{B}_\alpha\varphi + \varphi\right)_\xi$ and $\sin\left[\frac{1}{2}\left(\hat{B}_\alpha\varphi - \varphi\right)\right]$, we finally get

$$\begin{aligned}
-p_{1,\xi}^{[\alpha,0]} + p_{2,\tau}^{[\alpha,0]} &= 2\sin\left[\frac{a}{2}\big(\hat{B}_\alpha\varphi - \varphi\big)\right]\sin\left[\frac{a}{2}\big(\hat{B}_\alpha\varphi + \varphi\big)\right] + \frac{1}{2}\varphi_\tau(\hat{B}_\alpha\varphi)_\xi \\
&+ \frac{1}{2}\big(\hat{B}_\alpha\varphi + \varphi\big)_\xi\big(\hat{B}_\alpha\varphi - \varphi\big)_\tau - \frac{1}{2}\varphi_\xi(\hat{B}_\alpha\varphi)_\tau \\
&= 2\sin\left[\frac{a}{2}\big(\hat{B}_\alpha\varphi - \varphi\big)\right]\sin\left[\frac{a}{2}\big(\hat{B}_\alpha\varphi + \varphi\big)\right] \\
&+ \frac{1}{4}\big(\hat{B}_\alpha\varphi + \varphi\big)_\xi\big(\hat{B}_\alpha\varphi - \varphi\big)_\tau + \frac{1}{4}\big(\hat{B}_\alpha\varphi - \varphi\big)_\xi\big(\hat{B}_\alpha\varphi + \varphi\big)_\tau,
\end{aligned}$$

which is exactly formula (1.11) in the special case where $\varphi' = \hat{B}_\alpha\varphi$.

The last computations seem to suggest that extended Bäcklund transformations could be regarded as generalized Lagrangian symmetries (see Definition (1.3.3)) for the sine-Gordon Lagrangian. However the situation is more complicated. A serious objection to this picture is represented by Remark 1.2.3, namely by the fact that extended Bäcklund transformations do not form a group. The way this difficulty is overcome in [29] is by introducing the so-called infinitesimal Bäcklund transformations

$$B[\alpha,\epsilon] := \hat{B}_{\alpha+\epsilon}\circ\hat{B}_{-\alpha}, \qquad \alpha,\epsilon\in\mathbb{R}. \tag{1.12}$$

Since they are defined as compositions of extended Bäcklund transformations, it is of course still true that infinitesimal Bäcklund transformations act on the sine-Gordon Lagrangian changing it by an exact form. More precisely, we can write:

$$\begin{aligned}
L(B[\alpha,\epsilon]\varphi) - L(\varphi) &= L(\hat{B}_{\alpha+\epsilon}\hat{B}_{-\alpha}\varphi) - L(\hat{B}_{-\alpha}\varphi) + L(\hat{B}_{-\alpha}\varphi) - L(\varphi) \\
&= dp^{[\alpha+\epsilon,-\alpha]} + dp^{[-\alpha,0]} \\
&= d(p^{[\alpha+\epsilon,-\alpha]} + p^{[-\alpha,0]}),
\end{aligned}$$

where the two horizontal 1-forms $p^{[\alpha+\epsilon,-\alpha]} = (p_1^{[\alpha+\epsilon,-\alpha]}d\tau + p_2^{[\alpha+\epsilon,-\alpha]}d\xi)$ and $p^{[-\alpha,0]} = (p_1^{[-\alpha,0]}d\tau + p_2^{[-\alpha,0]}d\xi)$ have components:

$$\begin{aligned}
p_1^{[\alpha+\epsilon,-\alpha]} =& \frac{2(\alpha+\epsilon)}{a}\cos\left[\frac{a}{2}\big(\hat{B}_{\alpha+\epsilon}\hat{B}_{-\alpha}\varphi + \hat{B}_{-\alpha}\varphi\big)\right] - \\
&- \frac{1}{2}(\hat{B}_{-\alpha}\varphi)_\tau(\hat{B}_{\alpha+\epsilon}\hat{B}_{-\alpha}\varphi - \hat{B}_{-\alpha}\varphi), \\
p_2^{[\alpha+\epsilon,-\alpha]} =& -\frac{2}{a(\alpha+\epsilon)}\cos\left[\frac{a}{2}\big(\hat{B}_{\alpha+\epsilon}\hat{B}_{-\alpha}\varphi - \hat{B}_{-\alpha}\varphi\big)\right] - \\
&- \frac{1}{2}(\hat{B}_{-\alpha}\varphi)_\xi(\hat{B}_{\alpha+\epsilon}\hat{B}_{-\alpha}\varphi - \hat{B}_{-\alpha}\varphi),
\end{aligned}$$

and

$$p_1^{[-\alpha,0]} = \frac{(-2\alpha)}{a}\cos\left[\frac{a}{2}\big(\hat{B}_{-\alpha}\varphi+\varphi\big)\right] - \frac{1}{2}\varphi_\tau(\hat{B}_{-\alpha}\varphi-\varphi),$$
$$p_2^{[-\alpha,0]} = -\frac{2}{(-a\alpha)}\cos\left[\frac{a}{2}\big(\hat{B}_{-\alpha}\varphi-\varphi\big)\right] - \frac{1}{2}\varphi_\xi(\hat{B}_{-\alpha}\varphi-\varphi).$$

As a consequence of Proposition 1.2.2, we have immediately that for $\epsilon = 0$

$$B[\alpha, 0] = \text{id}.$$

In [29], the set of infinitesimal Bäcklund transformations $\{B[\alpha,\epsilon]\}_{\epsilon\in\mathbb{R}}$ for a fixed $\alpha \in \mathbb{R}$ is treated as a 1-parameter group of generalized Lagrangian symmetries and its infinitesimal generator is used to obtain the higher conserved currents via generalized Noether's Theorem.

Remark 1.4.2. Although the final results of this procedure are correct, it should be noted that the starting assumption of the whole construction is affected by a problem. In fact, due to Remark 1.2.3, neither the set $\{B[\alpha,\epsilon]\}_{\epsilon\in\mathbb{R}}$ for fixed α, nor the set of infinitesimal Bäcklund transformations in general form a group.

Nevertheless there might be an interesting way to solve this difficulty. Indeed, we observe that infinitesimal Bäcklund transformations satisfy the following partial composition rule:

$$\begin{aligned} B[\alpha+\epsilon_1,\epsilon_2]\circ B[\alpha,\epsilon_1] &= \hat{B}_{\alpha+\epsilon_1+\epsilon_2}\hat{B}_{-\alpha-\epsilon_1}\hat{B}_{\alpha+\epsilon_1}\hat{B}_{-\alpha} \\ &= \hat{B}_{\alpha+\epsilon_1+\epsilon_2}\hat{B}_{-\alpha} \\ &= B[\alpha,\epsilon_1+\epsilon_2]. \end{aligned}$$

In other words, the set of infinitesimal Bäcklund transformations $\{B[\alpha,\epsilon]\}_{\alpha,\epsilon\in\mathbb{R}}$ has the structure of a groupoid with source map $s\colon B[\alpha,\epsilon]\mapsto\alpha$ and target map $t\colon B[\alpha,\epsilon]\mapsto\alpha+\epsilon$.

The further investigation of this peculiar feature, which we defer to future research work, could provide new insights on the rôle of groupoids of symmetries in the determination of the conservation laws of integrable systems.

Despite the technical problems discussed in Remark 1.4.2, we can still limit ourselves to consider only the final result of the construction in [29]. Adapting the expressions in a natural way to the more general case of the sine-Gordon with coupling constant, we find that the one-parameter family of 1-forms $s^{(\alpha)} = -s_1^{(\alpha)}d\tau + s_2^{(\alpha)}d\xi$, with components

$$s_1^{(\alpha)} = \cos\left[\frac{a}{2}(\varphi+\hat{B}_{-\alpha}\varphi)\right] + \cos\left[\frac{a}{2}(\varphi+\hat{B}_{\alpha}\varphi)\right] \tag{1.13a}$$

$$s_2^{(\alpha)} = \frac{1}{\alpha^2}\left\{2-\cos\left[\frac{a}{2}(\varphi-\hat{B}_{-\alpha}\varphi)\right]-\cos\left[\frac{a}{2}(\varphi-\hat{B}_{\alpha}\varphi)\right]\right\}, \tag{1.13b}$$

produces a family of on-shell conservation laws. In other words, $\forall\alpha \in \mathbb{R}$ and for any solution $\varphi \in \mathscr{E}(\mathbb{M}_2)$ of the sine-Gordon equation:

$$d\big((j^\infty\varphi)^* s^{(\alpha)}\big) = \big((j^\infty\varphi)^*(\partial_\xi s_1^{(\alpha)} + \partial_\tau s_2^{(\alpha)})\big)\, d\tau \wedge d\xi = 0, \tag{1.14}$$

where $j^\infty\varphi$ just means that the prolongation of the section φ to some unspecified jet order has to be considered for the pull-back to be well-defined.

Using formula (1.4) to write $\hat{B}_{\pm\alpha}\varphi$ and the power series expansion of cosine, we can expand also $s_1^{(\alpha)}$ and $s_2^{(\alpha)}$ as power series in α. Since formulas (1.13a) and (1.13b) are symmetric in α, only even powers will appear. We denote the results of the power series expansions by:

$$s_1^{(\alpha)} = \sum_{N=0}^{\infty} s_1^N \alpha^{2N}, \qquad s_2^{(\alpha)} = \sum_{N=0}^{\infty} s_2^N \alpha^{2N}.$$

For every order in α a conserved current is obtained, which we denote by

$$s^N = -s_1^N d\tau + s_2^N d\xi.$$

For the later purposes of studying the renormalization and summability properties of the conservation laws of the sine-Gordon model, we now derive explicit formulas for the conserved currents from the construction of [29]. To the best of our knowledge, although the literature on integrable systems in general and on the sine-Gordon model in particular is extremely extensive, such explicit expression were not available.

Proposition 1.4.1. *The components s_1^N and s_2^N of the conserved currents have the following form:*

$$\begin{aligned} s_1^N = \cos(a\varphi) &\left[2\sum_{\beta=1}^{N}(-1)^\beta \left(\frac{a}{2}\right)^{2\beta} \sum_{\substack{n_1,\dots,n_{2N}\geq 0 \\ n_1+\dots+n_{2N}=2\beta \\ 1\cdot n_1+\dots+2N\cdot n_{2N}=2N}} \frac{A_1^{n_1}\cdots A_{2N}^{n_{2N}}}{n_1!\cdots n_{2N}!} \right] \\ + \sin(a\varphi) &\left[2\sum_{\beta=0}^{N-1}(-1)^{\beta+1} \left(\frac{a}{2}\right)^{2\beta+1} \sum_{\substack{n_1,\dots,n_{2N}\geq 0 \\ n_1+\dots+n_{2N}=2\beta+1 \\ 1\cdot n_1+\dots+2N\cdot n_{2N}=2N}} \frac{A_1^{n_1}\cdots A_{2N}^{n_{2N}}}{n_1!\cdots n_{2N}!} \right], \end{aligned} \tag{1.15}$$

where the coefficient of $\sin(a\varphi)$ *is defined only for* $N \geq 1$*, and*

$$s_2^N = 2\sum_{\mu=0}^{N}(-1)^\mu\Big(\frac{a}{2}\Big)^{2(\mu+1)} \sum_{\substack{n_0,\dots,n_{2(N-\mu)}\geq 0\\ n_0+\cdots+n_{2(N-\mu)}=2(\mu+1)\\ 1\cdot n_1+\cdots+2(N-\mu)\cdot n_{2(N-\mu)}=2(N-\mu)}} \frac{A_1^{n_0}\cdots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}}{n_0!\cdots n_{2(N-\mu)}!}. \tag{1.16}$$

Proof. First we introduce some notation. We define:

$$\varphi + \hat{B}_\alpha\varphi =: \sum_{\nu=0}^{\infty} A_\nu^+\alpha^\nu, \quad \text{where} \quad \begin{cases} A_0^+ = 2\varphi \\ A_\nu^+ = A_\nu \qquad \forall\nu\geq 1, \end{cases}$$

$$\varphi - \hat{B}_\alpha\varphi =: \sum_{\nu=0}^{\infty} A_\nu^-\alpha^\nu, \quad \text{where} \quad \begin{cases} A_0^- = 0 \\ A_\nu^- = -A_\nu \qquad \forall\nu\geq 1. \end{cases}$$

We use the power series expansion of cosine and substitute equations above to get the following expressions for the components of the conserved currents:

$$s_1^{(\alpha)} = \sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{(2\mu)!}\Big(\frac{a}{2}\Big)^{2\mu}\left[\left(\sum_{\nu=0}^{\infty}A_\nu^+\alpha^\nu\right)^{2\mu} + \left(\sum_{\nu=0}^{\infty}A_\nu^+(-\alpha)^\nu\right)^{2\mu}\right],$$

$$s_2^{(\alpha)} = -\frac{1}{\alpha^2}\sum_{\mu=1}^{\infty}\frac{(-1)^\mu}{(2\mu)!}\Big(\frac{a}{2}\Big)^{2\mu}\left[\left(\sum_{\nu=0}^{\infty}A_\nu^-\alpha^\nu\right)^{2\mu} + \left(\sum_{\nu=0}^{\infty}A_\nu^-(-\alpha)^\nu\right)^{2\mu}\right].$$

We remark that both formulas above are symmetric in α, so only even powers will appear. We further manipulate the two components separately.

Starting with $s_1^{(\alpha)}$, we expand $\big(\sum_{\nu=0}^{\infty}A_\nu^+\alpha^\nu\big)^{2\mu}$ and $\big(\sum_{\nu=0}^{\infty}A_\nu^+(-\alpha)^\nu\big)^{2\mu}$, collect the coefficients of the even powers $\alpha^{2\rho}$ and obtain:

$$s_1^{(\alpha)} = \sum_{\rho=0}^{\infty}\alpha^{2\rho}\left[2\sum_{\mu=0}^{\infty}(-1)^\mu\Big(\frac{a}{2}\Big)^{2\mu}\sum_{\substack{n_0,\dots,n_{2\rho}\geq 0\\ n_0+\cdots+n_{2\rho}=2\mu\\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}}\frac{(A_0^+)^{n_0}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_0!\cdots n_{2\rho}!}\right].$$

We now concentrate on the coefficient of the power $\alpha^{2\rho}$, we call it s_1^ρ:

$$s_1^\rho = 2\sum_{\mu=0}^{\infty}(-1)^\mu\Big(\frac{a}{2}\Big)^{2\mu}\sum_{\substack{n_0,\dots,n_{2\rho}\geq 0\\ n_0+\cdots+n_{2\rho}=2\mu\\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}}\frac{(A_0^+)^{n_0}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_0!\cdots n_{2\rho}!}. \tag{1.17}$$

Specifically, we want to extract the dependence of the powers of A_0^+ on μ. Introducing the index β to account for the possible values of the exponent n_0, we can rewrite formula (1.17) in the following manner:

$$2\sum_{\beta=0}^{2\rho}\sum_{\mu\geq\frac{\beta}{2}}(-1)^\mu\left(\frac{a}{2}\right)^{2\mu}\frac{(A_0^+)^{2\mu-\beta}}{(2\mu-\beta)!}\sum_{\substack{n_1,\ldots,n_{2\rho}\geq 0\\ n_1+\cdots+n_{2\rho}=\beta\\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}}\frac{(A_1^+)^{n_1}\ldots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\ldots n_{2\rho}!}.$$

Then we distinguish the cases when β is even or odd. The terms for even β can be collected in the expression:

$$2\sum_{\beta=0}^{\rho}\sum_{\mu=\beta}^{\infty}(-1)^\mu\left(\frac{a}{2}\right)^{2\mu}\frac{(A_0^+)^{2(\mu-\beta)}}{\big(2(\mu-\beta)\big)!}\sum_{\substack{n_1,\ldots,n_{2\rho}\geq 0\\ n_1+\cdots+n_{2\rho}=2\beta\\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}.$$

Renaming the index $(\mu-\beta)\to\mu$, we recognize the power series expansion of $\cos\left(\frac{1}{2}aA_0^+\right)=\cos(a\varphi)$. Hence for even β we obtain the coefficient:

$$\cos(a\varphi)\left[2\sum_{\beta=0}^{\rho}(-1)^\beta\left(\frac{a}{2}\right)^{2\beta}\sum_{\substack{n_1,\ldots,n_{2\rho}\geq 0\\ n_1+\cdots+n_{2\rho}=2\beta\\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}\right].$$

On the other hand, assuming $\rho\geq 1$, the terms for odd β are

$$2\sum_{\beta=0}^{\rho-1}\sum_{\mu=\beta+1}^{\infty}(-1)^\mu\left(\frac{a}{2}\right)^{2\mu}\frac{(A_0^+)^{2\mu-2\beta-1}}{(2\mu-2\beta-1)!}\sum_{\substack{n_1,\ldots,n_{2\rho}\geq 0\\ n_1+\cdots+n_{2\rho}=2\beta+1\\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}.$$

Rescaling the summation over μ, we recognize the power series expansion of $\sin\left(\frac{1}{2}aA_0^+\right)=\sin(a\varphi)$. Hence for odd β we obtain the coefficient:

$$\sin(a\varphi)\left[2\sum_{\beta=0}^{\rho-1}(-1)^{\beta+1}\left(\frac{a}{2}\right)^{2\beta+1}\sum_{\substack{n_1,\ldots,n_{2\rho}\geq 0\\ n_1+\cdots+n_{2\rho}=2\beta+1\\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}\right].$$

Using the fact that $A_\nu^+=A_\nu$, for $\nu\geq 1$, and changing the name of the upper index ρ to N, we finally obtain the expected result for the explicit expression of s_1^N.

Concerning $s_2^{(\alpha)}$, we use the fact that $A_0^- = 0$ to extract a power $\alpha^{2\mu}$, then we divide by α^2 and finally rewrite the sums rescaling the indices, thus obtaining:

$$s_2^{(\alpha)} = \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{(2(\mu+1))!} \left(\frac{a}{2}\right)^{2(\mu+1)} \alpha^{2\mu} \cdot \left[\left(\sum_{\nu=0}^{\infty} A_{\nu+1}^- \alpha^\nu \right)^{2(\mu+1)} + \left(\sum_{\nu=0}^{\infty} A_{\nu+1}^- (-\alpha)^\nu \right)^{2(\mu+1)} \right].$$

Expanding $\left(\sum_{\nu=0}^{\infty} A_{\nu+1}^- \alpha^\nu\right)^{2(\mu+1)}$ and $\left(\sum_{\nu=0}^{\infty} A_{\nu+1}^- (-\alpha)^\nu\right)^{2(\mu+1)}$ we see that only the even powers of α survive and they give:

$$s_2^{(\alpha)} = \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{(2(\mu+1))!} \left(\frac{a}{2}\right)^{2(\mu+1)} \alpha^{2\mu} \cdot \left[2 \sum_{\rho=0}^{\infty} \alpha^{2\rho} \sum_{\substack{n_0,\ldots,n_{2\rho} \geq 0 \\ n_0+\cdots+n_{2\rho}=2(\mu+1) \\ 1\cdot n_1+\cdots+2\rho\cdot n_{2\rho}=2\rho}} \frac{(2(\mu+1))!}{n_0!\ldots n_{2\rho}!} (A_1^-)^{n_0} \ldots (A_{2\rho+1}^-)^{n_{2\rho}} \right].$$

Collecting the powers of α, rewriting the sum using indices $N := \mu + \rho$ and μ, and recalling that $A_\nu^- = -A_\nu$ for $\nu \geq 1$, we finally obtain that the coefficient of α^{2N} can be written as:

$$s_2^N = 2 \sum_{\mu=0}^{N} (-1)^\mu \left(\frac{a}{2}\right)^{2(\mu+1)} \sum_{\substack{n_0,\ldots,n_{2(N-\mu)} \geq 0 \\ n_0+\cdots+n_{2(N-\mu)}=2(\mu+1) \\ 1\cdot n_1+\cdots+2(N-\mu)\cdot n_{2(N-\mu)}=2(N-\mu)}} \frac{A_1^{n_0} \cdots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}}{n_0! \cdots n_{2(N-\mu)}!}.$$

q.e.d.

The concrete expressions of the components of the first conserved currents, computed from formulas (1.15) and (1.16), are:

$$\begin{cases} s_1^0 = 2\cos(a\varphi), \\ s_2^0 = \varphi_\xi^2, \end{cases} \qquad \begin{cases} s_1^1 = -\varphi_\xi^2 \cos(a\varphi) - \frac{2}{a}\varphi_{\xi\xi}\sin(a\varphi), \\ s_2^1 = \frac{1}{4}\varphi_\xi^4 + \frac{2}{a^2}\varphi_\xi\varphi_{\xi\xi\xi} + \frac{1}{a^2}\varphi_{\xi\xi}^2. \end{cases}$$

Remark 1.4.3. Considering formulas (1.15) and (1.16) in view of Remark 1.2.1, it follows that the coefficients of $\cos(a\varphi)$ and of $\sin(a\varphi)$ in the expression of s_1^N, and the second components s_2^N are all polynomials in the derivatives of the configuration φ with respect to the coordinate ξ.

Remark 1.4.4. Again, as a consistency check, we have that the concrete expressions for the components of the conserved currents given in [29] are recovered from our expressions setting $a = 1$.

To conclude this section, we study the properties of the degree of the components of the higher conserved currents.

Proposition 1.4.2. *Assign by convention degree* 0 *to* $\cos(a\varphi)$ *and* $\sin(a\varphi)$*. Then we have that:*

- *The component* s_1^N *of the conserved current* s^N *is homogeneous of degree equal to* $2N$*.*
- *The component* s_2^N *of the conserved current* s^N *is homogeneous of degree equal to* $2(N+1)$*.*

Proof. The first claim follows because the coefficients of $\cos(a\varphi)$ and $\sin(a\varphi)$, in formula (1.15), are given by sums of products of the form

$$A_1^{n_1} \cdots A_{2N}^{n_{2N}},$$

with the condition $n_1 + \cdots + 2N \cdot n_{2N} = 2N$. All these products have degree

$$\deg(A_1^{n_1} \ldots A_{2N}^{n_{2N}}) = 1 \cdot n_1 + \cdots + 2N \cdot n_{2N} = 2N.$$

As for the second claim, from formula (1.16) we have that s_2^N is given by a finite sum of products of the form

$$A_1^{n_0} \cdots A_{2(N-\mu)+1}^{n_{2(N-\mu)}},$$

with the conditions

$$\begin{aligned} n_0 + \cdots + n_{2(N-\mu)} &= 2(\mu+1) \\ 1 \cdot n_1 + \ldots 2(N-\mu) \cdot n_{2(N-\mu)} &= 2(N-\mu). \end{aligned}$$

The degree of each one of these products is

$$\begin{aligned} \deg\bigl(A_1^{n_0} \ldots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}\bigr) &= n_0 + \cdots + (2(N-\mu)+1) n_{2(N-\mu)} \\ &= n_0 + n_1 + \cdots + n_{2(N-\mu)} \\ &\quad + n_1 + \cdots + 2(N-\mu) n_{2(N-\mu)} \\ &= 2(\mu+1) + 2(N-\mu) \\ &= 2(N+1). \end{aligned}$$

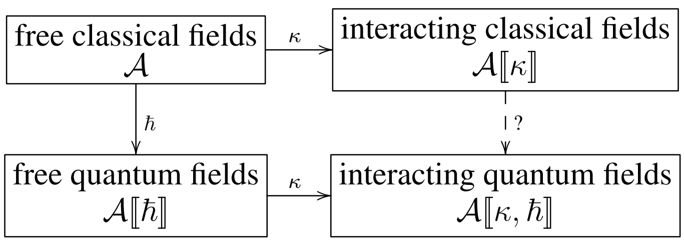
q.e.d.

Chapter 2

Basic notions of pAQFT on Minkowski spacetime

In this chapter we review the general construction of perturbative Algebraic Quantum Field Theory models on Minkowski spacetime. For more extensive treatments of the subject, we refer to [1] and [10], on which our exposition is mainly based.

The philosophy of the perturbative approach to Algebraic Quantum and Classical Field Theory can be encoded in the following diagram:



As suggested by the name, we deal with algebras of fields, which represent the observables that can be measured on our physical model. In particular, the starting point is the algebra of free classical fields $\mathcal{A}$. From this, the passage to quantum fields and the passage to interacting fields are realized as two different algebra deformations. More concretely:

- the deformation with parameter $\hbar$ represents the quantization of the algebra of classical free fields $\mathcal{A}$, for $\hbar = 0$, to the algebra $\mathcal{A}[\![\hbar]\!]$ of formal power series in $\hbar$ with coefficients in $\mathcal{A}$, for $\hbar > 0$;

- the deformation with parameter κ, called coupling constant, represents the passage from the algebra of free classical fields $\mathcal{A}$ (or free quantum fields $\mathcal{A}[\![\hbar]\!]$), for $\kappa = 0$, to the algebra of interacting classical fields $\mathcal{A}[\![\kappa]\!]$ (or interacting quantum fields $\mathcal{A}[\![\kappa, \hbar]\!]$, i.e. formal power series in κ and $\hbar$ with coefficients in $\mathcal{A}$), for $\kappa \neq 0$.

For what concerns a direct way to quantize the algebra of interacting classical fields, a general solution is still not known.

We also remark that the whole framework was developed in order to provide a mathematically rigorous formulation of (perturbative) Algebraic Quantum Field Theory on curved spacetimes, so it should not be surprising that all the structures that will be introduced in the rest of this chapter are naturally suited to be extended to any globally hyperbolic spacetime (see [7], [19], [24]).

2.1 Minkowski space and configurations

Let $\mathbb{M}_d$ be the d-dimensional Minkowski spacetime, i.e. $\mathbb{R}^d$ with flat diagonal metric $\eta = \mathrm{diag}(1, -1, \ldots, -1)$ in cartesian coordinates $x := (x^\mu)_{\mu=0,\ldots,d-1}$. With our convention on the Minkowski metric, the forward and backward light cones are defined respectively by:

$$V_+ = \left\{ x \in \mathbb{M}_d \;\middle|\; (x)^2_\eta > 0, x^0 > 0 \right\},$$
$$V_- = \left\{ x \in \mathbb{M}_d \;\middle|\; (x)^2_\eta > 0, x^0 < 0 \right\},$$

where $(x)^2_\eta = \eta(x, x)$ and $\overline{V}_+$ and $\overline{V}_-$ are the closures of these open sets. The symmetry groups of Minkwski space are $\mathcal{L}^\uparrow_+$, which is the proper orthochronous Lorentz group, and $\mathcal{P}^\uparrow_+$, which is the pertinent Poincaré group.

Configurations represent the physical fields of interest. They are in general smooth sections of some fiber bundle. But since the sine-Gordon model is a scalar field theory, we restrict our definition of the space of configurations to the following one (see also [24]).

Definition 2.1.1. Let $\mathscr{E}(\mathbb{M}_d) = C^\infty(\mathbb{M}_d, \mathbb{R})$ be the space of smooth real-valued functions on $\mathbb{M}_d$. Equivalently, configurations $\varphi \in \mathscr{E}(\mathbb{M}_d)$ can be seen as sections of the trivial bundle $\mathbb{M}_d \times \mathbb{R} \to \mathbb{M}_d$. The space $\mathscr{E}(\mathbb{M}_d)$ is endowed with the Fréchet topology generated by the family of seminorms:

$$p_{K,m}(\varphi) = \sup_{\substack{x \in K \\ |\alpha| \le m}} |\partial^\alpha \varphi(x)|,$$

where $\alpha \in \mathbb{N}^d$ is a multiindex, $m \in \mathbb{N}$ and $K \subset \mathbb{M}_d$ is a compact subset.

In other words, the above topology is the topology of uniform convergence of all the derivatives on compact subsets. We also introduce a topology on the space of test functions, which makes it a locally convex topological vecor space, but not a Fréchet space.

Definition 2.1.2. Let $\mathscr{D}(\mathbb{M}_d) = C_c^\infty(\mathbb{M}_d;\mathbb{R})$ denote the space of test functions. The fundamental system of seminorms on $\mathscr{D}(\mathbb{M}_d)$ is given by:

$$p_{\{m\},\{\epsilon\}}(\varphi) = \sup_\nu \sup_{\substack{|x|\geq\nu \\ |\alpha|\leq m_\nu}} |\partial^\alpha \varphi(x)|/\epsilon_\nu,$$

where $\alpha \in \mathbb{N}^d$ is a multiindex, $\{m_\nu\}_{\nu\in\mathbb{N}} \subseteq \mathbb{N}$ is an increasing sequence of positive numbers going to $+\infty$ and $\{\epsilon_\nu\}_{\nu\in\mathbb{N}} \subseteq \mathbb{R}$ is a decreasing sequence tending to 0.

2.2 The space of fields

One of the main advantages of the functional approach to algebraic field theory is that the fields, namely the physical observables, of both the classical and the quantum theory are defined in terms of the same space of functionals. Classical and quantum theories then differ only by the algebraic structures that are introduced on this space.

In general, fields are a certain class of smooth functionals on the space of configurations $F\colon \mathscr{E}(\mathbb{M}_d) \to \mathbb{C}$. Here smoothness is intended in the sense of Bastiani calculus on locally convex topological vector spaces (for more details see [4] or [18]). We adapt the definition to the case of our interest, where the topological vector space under consideration is $\mathscr{E}(\mathbb{M}_d)$.

Definition 2.2.1. Consider a functional $F\colon \mathscr{E}(\mathbb{M}_d) \to \mathbb{C}$. The derivative of F at $\varphi \in \mathscr{E}(\mathbb{M}_d)$ in the direction of $\psi \in \mathscr{E}(\mathbb{M}_d)$ is defined as

$$\langle F^{(1)}[\varphi]\,,\,\psi\rangle = \lim_{t\to 0} \frac{F(\varphi + t\psi) - F(\varphi)}{t}$$

whenever the limit exists. The functional F is called differentiable at φ if the quantity $\langle F^{(1)}[\varphi]\,,\,\psi\rangle$ exists for all $\psi \in \mathscr{E}(\mathbb{M}_d)$. It is continuously differentiable, in notation C^1, if the map $F^{(1)}\colon \mathscr{E}(\mathbb{M}_d) \times \mathscr{E}(\mathbb{M}_d) \to \mathbb{C}\colon (\varphi,\psi) \mapsto \langle F^{(1)}[\varphi]\,,\,\psi\rangle$ is jointly continuous. Higher derivatives are defined by

$$\langle F^{(k)}[\varphi]\,,\,\psi_1 \otimes \cdots \otimes \psi_k\rangle = \frac{\partial^k}{\partial t_1 \ldots \partial t_k} F(\varphi + t_1\psi_1 + \cdots + t_k\psi_k)\big|_{t_1=\cdots=t_k=0},$$

where $\varphi, \psi_1, \ldots, \psi_k \in \mathscr{E}(\mathbb{M}_d)$ and $\psi_1 \otimes \cdots \otimes \psi_k \in \mathscr{E}(\mathbb{M}_d^k)$ is the tensor product of smooth functions, namely $\big(\psi_1 \otimes \cdots \otimes \psi_k\big)(x_1,\ldots,x_k) = \psi_1(x_1)\cdots\psi_k(x_k)$. The functional F is said to be of class C^k if $\forall j \leq k$ the derivatives $F^{(j)}\colon \mathscr{E}(\mathbb{M}_d) \times \mathscr{E}(\mathbb{M}_d)^j \to \mathbb{C}$ are jointly continuous maps. Finally, F is Bastiani smooth if it is of class C^k for all $k \in \mathbb{N}$. We denote the space of Bastiani smooth functionals by $C^\infty(\mathscr{E}(\mathbb{M}_d),\mathbb{C})$.

Proposition 2.2.1. *Let $F\colon \mathscr{E}(\mathbb{M}_d) \to \mathbb{C}$ be a Bastiani smooth functional and $\varphi \in \mathscr{E}(\mathbb{M}_d)$. Then for any $k \geq 1$:*

(a) $F^{(k)}[\varphi]$ is a linear continuous map from $\mathscr{E}(\mathbb{M}_d)^k$ to $\mathbb{C}$.

(b) $F^{(k)}[\varphi]$ factors through a continuous map from the completed projective tensor product $\mathscr{E}(\mathbb{M}_d)^{\hat{\otimes}_\pi k} \cong \mathscr{E}(\mathbb{M}_d^k)$ to $\mathbb{C}$.

Remark 2.2.1. In view of Proposition 2.2.1, we can naturally identify derivatives of fields with field-valued distributions with compact support on copies of the spacetime $\mathbb{M}_d$. More precisely, given a smooth field $F \in C^\infty(\mathscr{E}(\mathbb{M}_d), \mathbb{C})$ and a configuration $\varphi \in \mathscr{E}(\mathbb{M}_d)$, then for every $k \geq 1$ the k-th functional derivative of F evaluated at φ defines a compactly supported distribution $F^{(k)}[\varphi] \in \mathscr{E}'(\mathbb{M}_d^k)$. On the other hand $\langle F^{(k)}[\cdot]\,,\, \psi_1 \otimes \cdots \otimes \psi_k \rangle$ is again a smooth field for any $\psi_1, \ldots, \psi_k \in \mathscr{E}(\mathbb{M}_d)$.

Notation. Regarding the derivatives of fields, we introduce also some other notation which we will come in handy in the sequel. In analogy with the usual notation for derivatives of functions depending on a finite number of variables, we write:

$$\left\langle \frac{\delta F}{\delta \phi}[\varphi], \psi \right\rangle = \left\langle F^{(1)}[\varphi], \psi \right\rangle$$

and correspondingly

$$\left\langle \frac{\delta^k F}{\delta \phi^k}[\varphi], \psi_1 \otimes \cdots \otimes \psi_k \right\rangle = \left\langle F^{(k)}[\varphi], \psi_1 \otimes \cdots \otimes \psi_k \right\rangle,$$

where as above $\varphi, \psi_1, \ldots, \psi_k \in \mathscr{E}(\mathbb{M}_d)$. Sometimes, in order to highlight the distributional character of the derivatives of fields, we will use the notation

$$\frac{\delta^k F[\varphi]}{\delta\phi(x_1)\cdots\delta\phi(x_k)} \tag{2.1}$$

and express the pairing of derivative with tensor product of configurations by the formal integral notation:

$$\begin{aligned}&\left\langle \frac{\delta^k F}{\delta \phi^k}[\varphi], \psi_1 \otimes \cdots \otimes \psi_k \right\rangle \\ &= \int_{\mathbb{M}_d^k} \frac{\delta^k F[\varphi]}{\delta\phi(x_1)\cdots\delta\phi(x_k)} \psi_1(x_1)\cdots\psi_k(x_k)\, dx_1 \cdots dx_k.\end{aligned}$$

According to the principle of locality at the core of the algebraic formulation of field theory, it is important to be able to identify observables, i.e. fields, that belong to a given region of spacetime. This can be achieved by introducing the notion of spacetime support of fields.

Definition 2.2.2. Consider the (non necessarily smooth) map $F\colon \mathscr{E}(\mathbb{M}_d) \to \mathbb{C}$. The spacetime support of F is defined by

$$\begin{aligned}\mathrm{supp}(F) = \{x \in \mathbb{M}_d \,|\, \forall\, \text{neighborhood}\, U\, \text{of}\, x,\, \exists\, \varphi, \psi \in \mathscr{E}(\mathbb{M}_d),\\ \text{with}\, \mathrm{supp}(\psi) \in U,\, \text{s.t.}\, F[\varphi+\psi] \neq F[\varphi]\}.\end{aligned}$$

It is now possible to characterize important subclasses of fields by imposing restrictions on the regularity and the support of the distributions arising as their derivatives. The most restrictive condition defines the following class.

Definition 2.2.3. A field $F \in C^\infty(\mathscr{E}(\mathbb{M}_d), \mathbb{C})$ is called regular if for all $\varphi \in \mathscr{E}(\mathbb{M}_d)$ and $k \in \mathbb{N}$, the wavefront set of $F^{(k)}[\varphi]$ is empty, i.e. $F^{(k)}[\varphi] \in \mathscr{D}(\mathbb{M}_d^k)$. The space of regular fields is denoted by $\mathcal{F}_{\text{reg}}$.

The following class of fields encompasses most (if not all) of the physical quantities that can be considered in general and in particular, for what concerns our specific case, all the quantities that will appear in the study of the sine-Gordon model.

Definition 2.2.4. A field $F \in C^\infty(\mathscr{E}(\mathbb{M}_d), \mathbb{C})$ is called local if for each $\varphi_0 \in \mathscr{E}(\mathbb{M}_d)$ there exists an open neighborhood $V \ni \varphi_0$ in $\mathscr{E}(\mathbb{M}_d)$ and $k \in \mathbb{N}$ such that for all $\varphi \in V$, it holds

$$F[\varphi] = \int_{\mathbb{M}_d} \alpha(j^k_x \varphi),$$

where $j^k_x\varphi$ is the k-th jet prolongation of φ and α is a density-valued function on the k-th order jet bundle. The space of local functionals is denoted by $\mathcal{F}_{\text{loc}}$.

Remark 2.2.2. If F is local, then $F^{(k)}[\varphi] \in \mathscr{E}'(\mathbb{M}_d^k)$ is a distribution supported on the thin diagonal

$$\Delta_k = \left\{\, (x_1, \ldots, x_k) \in \mathbb{M}_d^k \;\middle|\; x_1 = \cdots = x_k \,\right\}.$$

Moreover, the wavefront set of $F^{(k)}[\varphi]$ is conormal to $T\Delta_k$, the tangent bundle of the thin diagonal. In particular, $F^{(1)}[\varphi]$ has empty wavefront set and so it is smooth for each fixed $\varphi \in \mathscr{E}(\mathbb{M}_d)$.

Definition 2.2.5. Consider the space of local functionals $\mathcal{F}_{\text{loc}}$ endowed with the following operations:

- the commutative pointwise product

$$\begin{aligned}\mu\colon \mathcal{F}_{\text{loc}} \times \mathcal{F}_{\text{loc}} &\to C^\infty(\mathscr{E}(\mathbb{M}_d), \mathbb{C})\\ (F, G) \quad &\mapsto \quad \mu(F \otimes G) = F\,G\end{aligned} \tag{2.2}$$

  where $(F\,G)[\varphi] = F[\varphi]G[\varphi]$, for all $\varphi \in \mathscr{E}(\mathbb{M}_d)$.

- the involution operator

$$
\begin{aligned}
{}^{*}\colon \mathcal{F}_{\text{loc}} &\to \mathcal{F}_{\text{loc}} \\
F &\mapsto F^{*}
\end{aligned}
\tag{2.3}
$$

defined by complex conjugation, $F^{*}[\varphi] = \overline{F[\varphi]}$, for all $\varphi \in \mathscr{E}(\mathbb{M}_d)$.

The algebraic closure of $\mathcal{F}_{\text{loc}}$ with respect to the operations above yields a commutative $*$-algebra called algebra of multilocal fields.

In view of the operations involved in the construction of pAQFT models (see sections below), it turns out that the class of multilocal fields is still too small. Hence it is necessary to consider a bigger class of fields.

Definition 2.2.6. A functional $F \in C^{\infty}(\mathscr{E}(\mathbb{M}_d), \mathbb{C})$ is called a microcausal field if it has compact spacetime support and if moreover its derivatives satisfy the so-called microlocal condition:

$$
\mathrm{WF}(F^{(n)}[\varphi]) \subset \Xi_n, \qquad \forall n \in \mathbb{N}, \quad \forall \varphi \in \mathscr{E}(\mathbb{M}_d),
\tag{2.4}
$$

where Ξ_n is an open conic subset of the cotangent space $T^{*}\mathbb{M}_d^n$ defined as

$$
\Xi_n := T^{*}\mathbb{M}_d^n \setminus \left\{ (x_1, k_1; \ldots; x_n, k_n) \;\middle|\; (k_1, \ldots, k_n) \in (\overline{V}_+^n \cup \overline{V}_-^n)_{(x_1, \ldots, x_n)} \right\}.
$$

The space of microcausal fields is denoted by $\mathcal{F}_{\mu c}$.

Remark 2.2.3. The commutative pointwise product μ, the involution *, formulas (2.2) and (2.3), trivially extend to operations defined on the space of microcausal fields.

2.3 Star product of fields

In view of the diagram on the philosophy of the perturbative approach to algebraic field theory outlined at the beginning of this chapter, we are now ready to describe the construction of the algebra of free quantum fields. We start introducing some definitions.

Definition 2.3.1. A generalized Lagrangian on $\mathbb{M}_d$ is a map $L\colon \mathscr{D}(\mathbb{M}_d) \to \mathcal{F}_{\text{loc}}$ that satisfies the following conditions:

(i) **Additivity:** $L(f+g+h) = L(f+g) - L(g) + L(g+h)$, for any $f, g, h \in \mathscr{D}(\mathbb{M}_d)$ with $\mathrm{supp}(f) \cap \mathrm{supp}(h) = \emptyset$.

(ii) **Support:** $\mathrm{supp}\big(L(f)\big) \subseteq \mathrm{supp}(f)$, $\forall f \in \mathscr{D}(\mathbb{M}_d)$.

Remark 2.3.1. Definition 2.3.1 formalizes the idea that a generalized Lagrangian associates to a test function $f \in \mathscr{D}(\mathbb{M}_d)$ the local functional obtained by integrating f with the Lagrangian density $\mathscr{L}(x)[\varphi]$ that depends locally on the configuration $\varphi \in \mathscr{E}(\mathbb{M}_d)$. The cutoff function f is necessary to ensure the convergence of the integral.

<u>**Example.**</u> The free Lagrangian L_0 on $(\mathbb{M}_d, \eta)$ is defined as

$$L_0(f)[\varphi] := \frac{1}{2} \int_{\mathbb{M}_d} \left(\eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m^2 \varphi^2 \right) f \, dx.$$

Remark 2.3.2. The free Lagrangian just described is more than a special case, it plays a fundamental rôle in general. In fact, the Lagrangian of any pAQFT model is always given as $L = L_0 + \kappa L_{\text{int}}$, where L_{int} encodes the nature of the interactions described by the model and is treated perturbatively. The parameter $\kappa \in \mathbb{R}$ is called the "bookkeeping" coupling constant and in this case it is just a tool to account for the order of perturbation. In the sequel we will always denote generalized Lagrangians by $F(x) \in \mathscr{D}'(\mathbb{M}_d\,; \mathcal{F}_{\text{loc}})$, omitting the dependence on the test function, and simply treat them as "point-dependent" local fields.

The cutoff function of a generalized Lagrangian should be understood as an auxiliary mathematical tool, without physical meaning. Therefore the crucial structures of the theory should not depend on the choice of f. This is achieved by means of the following definition.

Definition 2.3.2. The Euler-Lagrange derivative of a generalized Lagrangian L is the smooth map $S'_L \colon \mathscr{E}(\mathbb{M}_d) \to \mathscr{D}'(\mathbb{M}_d)$ defined by:

$$\langle S'_L(\varphi), h \rangle = \left\langle L^{(1)}(f)[\varphi], h \right\rangle,$$

where $f, h \in \mathscr{D}(\mathbb{M}_d)$ and f is choosen in such a way that $f \equiv 1$ on $\operatorname{supp}(h)$.

Remark 2.3.3. Since by definition $L(f)$ is a local functional, S'_L does not depend on the choice of f.

Definition 2.3.3. The equation of motion induced by the generalized Lagrangian L is

$$S'_L(\varphi) = 0,$$

understood as a condition on $\varphi \in \mathscr{E}(\mathbb{M}_d)$.

The starting point in the construction of the algebra of free quantum fields of any pAQFT model is the differential operator associated to the equation of motion of the free Lagrangian L_0.

Definition 2.3.4. Consider the second derivative of the free Lagrangian L_0:

$$\left\langle L_0^{(2)}(f)[\varphi], h_1 \otimes h_2 \right\rangle = \int_{\mathbb{M}_d} \left(\eta^{\mu\nu}\partial_\mu h_1 \partial_\nu h_2 - m^2 h_1 h_2\right) f \, dx,$$

where $f \equiv 1$ on $\mathrm{supp}(h_1) \cup \mathrm{supp}(h_2)$. This operator extends naturally to an operator, called wave operator:

$$\begin{aligned} P \colon \mathscr{E}(\mathbb{M}_d) &\to \mathscr{D}'(\mathbb{M}_d) \\ \varphi \mapsto P\varphi &= -(\Box + m^2)\varphi = -(\partial^2_{x^0} - \partial^2_{x^1} - \cdots - \partial^2_{x^{d-1}} + m^2)\varphi, \end{aligned}$$

where φ here plays the role equivalently of h_1 or h_2 from the previous formula, which acts on test functions $h \in \mathscr{D}(\mathbb{M}_d)$ as:

$$\langle P\varphi, h\rangle = \int_{\mathbb{M}_2} -(\partial^2_{x^0}\varphi - \partial^2_{x^1}\varphi - \cdots - \partial^2_{x^{d-1}}\varphi + m^2\varphi) h \, dx.$$

The crucial assumption of pAQFT is that the wave operator P is normally hyperbolic. Hence there exist unique retarded and advanced Green's functions.

Definition 2.3.5. The retarded and advanced propagators $\Delta^R_m, \Delta^A_m \in \mathscr{D}'(\mathbb{M}_d)$ respectively, are the unique fundamental solutions of the wave operator P, i.e.

$$P\Delta^{R,A}_m = -(\Box + m^2)\Delta^{R,A}_m = \delta,$$

that moreover satisfy the following conditions on their support:

$$\mathrm{supp}(\Delta^R_m) \subseteq \overline{V}_+, \qquad \mathrm{supp}(\Delta^A_m) \subseteq \overline{V}_-.$$

The difference of the retarded and advanced propagators $\Delta_m := \Delta^R_m - \Delta^A_m$ is called causal propagator.

Remark 2.3.4. The causal propagator has the obvious properties $P\Delta_m = 0$ and $\mathrm{supp}(\Delta_m) \subseteq (\overline{V}_+ \cup \overline{V}_-)$. More importantly, it can be shown that its wavefront set is given by:

$$\begin{aligned} \mathrm{WF}(\Delta_m) = \big\{(x,k) \in T^*\mathbb{M}_d \mid (x)^2_\eta = 0,\ (k)^2_\eta = 0, \\ x = \lambda\eta^\sharp(k) \text{ for some } \lambda \in \mathbb{R},\ k^0 \neq 0\big\}, \end{aligned}$$

where $\eta^\sharp \colon T^*\mathbb{M}_2 \to T\mathbb{M}_2$ is the natural isomorphism induced by the Minkowski metric η and, due to the triviality of the bundles $T^*\mathbb{M}_2 \equiv \mathbb{M}_2 \times \mathbb{M}_2^*$, $T\mathbb{M}_2 \equiv \mathbb{M}_2 \times \mathbb{M}_2$, it descends to an isomorphism $\eta^\sharp \colon \mathbb{M}_2^* \to \mathbb{M}_2$ and $(k)^2_\eta = \eta\left(\eta^\sharp(k), \eta^\sharp(k)\right)$.

Definition 2.3.6 ([23]). There exist distributions, $W \in \mathscr{D}'(\mathbb{M}_d)$ called two-point function, and $H \in \mathscr{D}'(\mathbb{M}_d)$ called Hadamard parametrix, such that the following decomposition of the causal propagator holds:

$$\frac{i}{2}\Delta_m = W - H, \tag{2.5}$$

where the two-point function W has the following properties:

(i) The wavefront set of W is

$$\begin{aligned}\mathrm{WF}(W) = \big\{(x,k) \in T^*\mathbb{M}_d \mid (x)^2_\eta = 0,\ (k)^2_\eta = 0, \\ x = \lambda\eta^\sharp(k) \text{ for some } \lambda \in \mathbb{R},\ k^0 > 0\big\}.\end{aligned} \tag{2.6}$$

(ii) The imaginary part of W is the causal propagator, i.e. $2\mathfrak{Im}(W) = \Delta_m$.

(iii) W is a distributional solution of the wave operator, i.e. $PW = 0$.

(iv) W is positive, meaning that $\big\langle \bar{f}, W * f\big\rangle \geq 0$, where $\bar{f}$ is the complex conjugate of $f \in \mathscr{D}(\mathbb{M}_d)$ and $W * f$ denotes the convolution of a distribution with a test function.

Remark 2.3.5. The decomposition (2.5) is in general not unique, depending on the choice of the Hadamard parametrix H. The difference between two choices of Hadamard parametrices $H - H'$ is always a smooth function.

In particular, the condition on the wavefront set of the two-point function W allows us to introduce the following non-commutative product on the space of microcausal fields.

Definition 2.3.7. For a given choice of two-point function W and of Hadamard parametrix H, the star product of microcausal fields $F, G \in \mathcal{F}_{\mu c}$ is the microcausal field $F \star_H G \in \mathcal{F}_{\mu c}[\![\hbar]\!]$ defined as:

$$(F \star_H G)[\varphi] = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}[\varphi], (W^{\otimes n}) * G^{(n)}[\varphi]\right\rangle. \tag{2.7}$$

The star product $\star_H$ is extended by linearity to a map $\star_H \colon \mathcal{F}_{\mu c}[\![\hbar]\!] \times \mathcal{F}_{\mu c}[\![\hbar]\!] \to \mathcal{F}_{\mu c}[\![\hbar]\!]$.

Remark 2.3.6. In formula (2.7), the symbol $*$ denotes the convolution of distributions, while the bracket $\langle\cdot,\cdot\rangle$ indicates a special pairing between distributions with specified wavefront sets (see [5] and [6] for all the details). The wavefront set properties of microcausal fields and of the two-point function are precisely chosen in such a way that this pairing is always well-defined.

Remark 2.3.7. Formula (2.7) can equivalently be written in a very compact and useful exponential form as follows:

$$F \star_H G = \mu \circ e^{\hbar D_W}(F \otimes G),$$

where μ is the commutative pointwise product of fields (2.2) and D_W is the operator defined by

$$D_W(F \otimes G) = \left\langle F^{(1)}, W * G^{(1)} \right\rangle, \qquad \forall F, G \in \mathcal{F}_{\mu c}.$$

Remark 2.3.8. It can be proved (see for example [10]) that for two different choices of Hadamard parametrices H and H', the corresponding star products $\star_H$ and $\star_{H'}$ are intertwined by an isomorphism $\alpha_{H-H'} \colon \mathcal{F}_{\mu c}[\![\hbar]\!] \to \mathcal{F}_{\mu c}[\![\hbar]\!]$ in the sense of formal power series, namely

$$F \star_{H'} G = \alpha^{-1}_{H-H'}\big(\alpha_{H-H'}(F) \star_H \alpha_{H-H'}(G)\big),$$

where $\alpha_{H-H'} := e^{\frac{\hbar}{2}\mathcal{D}_{H-H'}}$ and, in terms of formal integral kernels, the operator $\mathcal{D}_{H-H'}$ acts on fields as

$$\begin{aligned}\mathcal{D}_{H-H'}F[\varphi] :&= \langle H - H', \frac{\delta^2}{\delta\varphi^2}\rangle\, F[\varphi] = \\ &= \int_{\mathbb{M}_d^2} \big(H(x-y) - H'(x-y)\big)\frac{\delta^2 F[\varphi]}{\delta\varphi(x)\delta\varphi(y)}\, dx\, dy.\end{aligned}$$

Remark 2.3.9. We also remark that the algebraic structure described above looks formally the same for arbitrary dimension d, but the concrete expressions of W, H and Δ_m are different in each case.

We finally have all the ingredients to spell out the following definition.

Definition 2.3.8. The space of microcausal fields $\mathcal{F}_{\mu c}[\![\hbar]\!]$, endowed with the star product $\star_H$, with the involution operation (2.3) and with the commutator $[\,,\,]_{\star_H}$ with respect to the star product, forms a Poisson $*$-algebra called the algebra of free quantum fields and denoted by $\big(\mathcal{F}_{\mu c}[\![\hbar]\!], \star_H, ^*, [\,,\,]_{\star_H}\big)$.

2.4 The interaction picture

The physical concept of evolution is encoded in the notion of interacting fields. Roughly speaking, considering interacting fields represents the quantum equivalent of the classical restriction to on-shell fields, namely observables associated to configurations that are solutions of the equation of motion (see Definition 2.3.3).

The formulation of the interaction picture in pAQFT is inspired by analogy with the interaction picture in quantum mechanics. We recall it briefly and for the details we refer to [14].

If $\kappa H_{\text{int}}(t)$ and $\psi(t)$ are respectively the interaction Hamiltonian and the wave function at time t, then the time evolution operator $U(t, t_0)$ can be introduced, relating the wave functions at different times (Schrödinger picture) by:

$$\psi(t) = U(t, t_0)\psi(t_0), \qquad \forall t, t_0 \in \mathbb{R}, \quad \forall \psi.$$

This equation is equivalent (Heisenberg picture) to the evolution equation:

$$i\hbar \frac{\partial}{\partial t} U(t, t_0) = \kappa H_{\text{int}}(t) U(t, t_0).$$

This differential equation can be solved perturbatively by the Dyson formula:

$$U(t, t_0) = \text{Id} + \sum_{n=1}^{\infty} \frac{(-i\kappa)^n}{\hbar^n} \int_{t_0 \le t_1 \le \cdots \le t_n \le t} H_{\text{int}}(t_n) \cdots H_{\text{int}}(t_1) dt_1 \dots dt_n.$$

The scattering matrix, called S-matrix, is then interpreted as the double limit:

$$S = \lim_{\substack{t \to \infty \\ s \to -\infty}} U(t, s).$$

In pAQFT the dynamics is determined by a generalized Lagrangian, always of the form $L = L_0 + \kappa L_{\text{int}}$. Heuristically then the formulation of the S-matrix can be translated in the pAQFT language by introducing suitable operators, called time-ordered products, that realize the time-ordering of the interaction Lagrangians:

$$S = 1 + \sum_{n=1}^{\infty} \frac{(i\kappa)^n}{n!\hbar^n} \int_{\mathbb{M}_d^n} T_n\big(L_{\text{int}}(x_1) \otimes \cdots \otimes L_{\text{int}}(x_n)\big) dx_1 \dots dx_n, \tag{2.8}$$

where the time-ordered product of n factors T_n is given by

$$T_n\big(L_{\text{int}}(x_1) \otimes \cdots \otimes L_{\text{int}}(x_n)\big) = L_{\text{int}}(x_{\pi(1)}) \star_H \cdots \star_H L_{\text{int}}(x_{\pi(n)}) \tag{2.9}$$

whenever $x^0_{\pi(1)} \ge \cdots \ge x^0_{\pi(n)}$, for some $\pi \in S_n$, the symmetric group of order n.

Remark 2.4.1. The time-ordered products and the scattering matrix are the mathematical tools that account for the perturbative formulation of the interaction picture in pAQFT. As pointed out in Remark 2.3.2, the interaction is always modelled by a generalized Lagrangian, which we can think of as a point-dependent local field. For the moment instead, we restrict ourselves to consider interactions which are only regular fields.

Definition 2.4.1. (Basic axioms) The n-th order time-ordered product is a map

$$T_n\colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})^{\otimes n} \to \mathscr{D}'(\mathbb{M}_d^n\,;\mathcal{F}_{\mathrm{reg}})[\![\hbar]\!]$$

which satisfies

(i) **Linearity:** T_n is linear;

(ii) **Initial condition:** $T_1\big(F(x)\big) = F(x)$ for any $F \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})$;

(iii) **Symmetry:** T_n is symmetric in all its arguments, i.e.

$$T_n\big(F_{\pi(1)}(x_{\pi(1)}) \otimes \cdots \otimes F_{\pi(n)}(x_{\pi(n)})\big) = T_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n)\big);$$

$\forall F_1, \dots, F_n \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})$ and $\forall \pi \in S_n$

(iv) **Causality:** For any $F_1, \dots, F_n \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})$, T_n satisfies the causal factorization relation

$$\begin{aligned} T_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n)\big) &= T_k\big(F_{\pi(1)}(x_{\pi(1)}) \otimes \cdots \otimes F_{\pi(k)}(x_{\pi(k)})\big) \\ &\quad \star_H T_{n-k}\big(F_{\pi(k+1)}(x_{\pi(k+1)}) \otimes \cdots \otimes F_{\pi(n)}(x_{\pi(n)})\big) \end{aligned}$$

whenever $\{\, x_{\pi(1)}, \dots, x_{\pi(k)} \,\} \cap \big(\{\, x_{\pi(k+1)}, \dots, x_{\pi(n)} \,\} + \overline{V}_-\big) = \emptyset$ for some permutation $\pi \in S_n$.

Definition 2.4.2. (Renormalization conditions) The only additional axioms for the time-ordered product that we will impose read as:

(v) **Field independence:** it ensures that field derivatives and time-ordered products can be interchanged, i.e.

$$\begin{aligned} \frac{\delta}{\delta\phi} T_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n)\big) =& T_n\left(\frac{\delta}{\delta\phi} F_1(x_1) \otimes \cdots \otimes F_n(x_n)\right) + \dots \\ & \cdots + T_n\left(F_1(x_1) \otimes \cdots \otimes \frac{\delta}{\delta\phi} F_n(x_n)\right), \end{aligned}$$

$\forall F_1, \dots, F_n \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})$.

(vi) **Translation invariance:** it is a special case of the principle of Poincaré covariance, and it simply means that the time-ordered products should be translation invariant distributions with values in microlocal fields, i.e.

$$T_n\big(F_1(x_1 - v) \otimes \cdots \otimes F_n(x_n - v)\big) = T_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n)\big),$$

$\forall v \in \mathbb{M}_d$ and for all $F_1, \dots, F_n \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})$.

We can now give a precise definition for the scattering matrix.

Definition 2.4.3. The S-matrix is a map

$$S\colon \mathscr{D}'(\mathbb{M}_d\,;\kappa\mathcal{F}_{\mathrm{reg}}) \to \mathscr{D}'(\mathbb{M}_d^\infty\,;\mathcal{F}_{\mathrm{reg}})[\![\kappa]\!]((\hbar))$$

from the space of regular generalized Lagrangians, multiplied by a coupling constant $\kappa \in \mathbb{R}$, to the space of formal power series in the coupling constant κ, Laurent series in $\hbar$, with coefficients in distributions on any finite number of copies of spacetime with values in regular fields. Concretely, the S-matrix is defined as the generating functional of the time-ordered products:

$$S(\kappa F) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{i\kappa}{\hbar}\right)^n T_n\big(F^{\otimes n}\big),$$

where the field $F \in \mathscr{D}'(\mathbb{M}_d\,;\kappa\mathcal{F}_{\mathrm{reg}})$ is interpreted as the interaction Lagrangian of the theory under consideration.

Remark 2.4.2. The causality axiom for the time-ordered product can be equivalently expressed in terms of the S-matrix by the condition:

$$S(H+G+F) = S(H+G)\star_H S(G)^{\star_H-1}\star_H S(G+F)$$

whenever $\mathrm{supp}(H)\cap(\mathrm{supp}(F)+\overline{V}_-)=\emptyset$, where the notation ${}^{\star_H-1}$ denotes the inverse with respect to the star product $\star_H$. From this relation, setting $G=0$, we obtain the equivalent of the causal factorization relation for the time-ordered product:

$$S(H+F) = S(H)\star_H S(F), \quad \text{if} \quad \mathrm{supp}(H)\cap(\mathrm{supp}(F)+\overline{V}_-)=\emptyset.$$

Using the S-matrix we can finally describe the interacting quantum fields mentioned in the diagram, at the beginning of the chapter, illustrating the general philosophy of the perturbative approach to field theory.

Definition 2.4.4. Given $\kappa F \in \mathscr{D}'(\mathbb{M}_d\,;\kappa\mathcal{F}_{\mathrm{reg}})$ and $G \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})$, the interacting field $(G)_{\kappa F} \in \mathscr{D}'(\mathbb{M}_d^\infty\,;\mathcal{F}_{\mathrm{reg}})[\![\kappa,\hbar]\!]$ with respect to the field κF is defined by the Bogoliubov formula:

$$(G)_{\kappa F} = \frac{\hbar}{i}\frac{d}{d\lambda}\bigg|_{\lambda=0} S(\kappa F)^{\star_H-1}\star_H S(\kappa F+\lambda G) = \sum_{n=0}^{\infty}\frac{\kappa^n}{n!}R_n(F^{\otimes n},G), \quad (2.10)$$

where ${}^{\star_H-1}$ indicates that we are considering the inverse with respect to the star product $\star_H$ and $R_n\colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}})^{\otimes n}\times\mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mathrm{reg}}) \to \mathscr{D}'(\mathbb{M}_d^{n+1}\,;\mathcal{F}_{\mathrm{reg}})[\![\hbar]\!]$ are separately linear maps called retarded products. It is customary in this situation to interpret κF as the interaction Lagrangian and G as the generic observable.

Proposition 2.4.1. *Substituting the definition of the S-matrix in formula (2.10), it is possible to show that the retarded product $R_n(F^{\otimes n}, G)$, $F, G \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\text{reg}})$, admits the following expression:*

$$R_n\big(F^{\otimes n}, G\big) = \Big(\frac{i}{\hbar}\Big)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} \bar{T}_{n-l}\big(F^{\otimes(n-l)}\big) \star T_{l+1}\big(F^{\otimes l} \otimes G\big), \quad (2.11)$$

where the maps $\bar{T}_k \colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\text{reg}})^{\otimes k} \to \mathscr{D}'(\mathbb{M}_d^k\,;\mathcal{F}_{\text{reg}})[\![\hbar]\!]$ are called antichronological products and are defined as the coefficients of the inverse (in the sense of formal power series) of the scattering matrix

$$S(F)^{\star_H -1} = 1 + \sum_{k=1}^{\infty} \frac{(-i)^k}{k!\hbar^k} \bar{T}_k\big(F^{\otimes k}\big).$$

Remark 2.4.3. At a first glance, the presence of the negative powers of $\hbar$ in formula (2.11) may seem in contradiction with Definition 2.4.4, where we said that the retarded product R_n takes values in formal power series in $\hbar$ with coefficients in $\mathscr{D}'(\mathbb{M}_d^{n+1}\,;\mathcal{F}_{\text{reg}})$. The situation is explained by the following fact (see [10]). The retarded product R_n admits another formula:

$$\begin{aligned} &R_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n), G(x_{n+1})\big) \\ &= \sum_{\pi \in S_n} \theta(x^0_{n+1} - x^0_{\pi(n)})\theta(x^0_{\pi(n)} - x^0_{\pi(n-1)}) \ldots \theta(x^0_{\pi(2)} - x^0_{\pi(1)}) \\ &\quad \frac{1}{i^n} \Big[F_{\pi(1)}(x_{\pi(1)}), \Big[F_{\pi(2)}(x_{\pi(2)}), \ldots, \big[F_{\pi(n)}(x_{\pi(n)}), G(x_{n+1})\big]_{\star_H} \cdots \Big]_{\star_H}\Big]_{\star_H}, \end{aligned}$$

where θ is the Heaviside step function. This formula is valid for any regular fields $F_1, \ldots, F_n, G \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\text{reg}})$, but limitedly on the subset

$$\check{\mathbb{M}}_d^{n+1} := \big\{\, (x_1, \ldots, x_{n+1}) \in \mathbb{M}_d^{n+1} \;\big|\; x_i \neq x_j \quad \forall 1 \leq i < j \leq n+1 \,\big\}\,.$$

As it can be seen from formula (2.7) for the star product, each commutator gives at least one factor $\hbar$. Hence the retarded product R_n is at least of order $\hbar^n$.

Remark 2.4.4. The antichronological products satisfy a causal factorization relation, which is analogous to the one for the time-ordered products, but with inverted order, namely for every $F_1, \ldots, F_k \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\text{reg}})$:

$$\begin{aligned} \bar{T}_k\big(F_1(x_1) \otimes \cdots \otimes F_k(x_k)\big) = \bar{T}_{k-j}&\big(F_{\pi(j+1)}(x_{\pi(j+1)}) \otimes \cdots \otimes F_{\pi(k)}(x_{\pi(k)})\big) \\ &\star_H \bar{T}_j\big(F_{\pi(1)}(x_{\pi(1)}) \otimes \cdots \otimes F_{\pi(j)}(x_{\pi(j)})\big), \end{aligned}$$

whenever $\{\, x_{\pi(1)}, \ldots, x_{\pi(j)} \,\} \cap \big(\{\, x_{\pi(j+1)}, \ldots, x_{\pi(k)} \,\} + \overline{V}_-\big) = \emptyset$ for some permutation $\pi \in S_k$. This “anticausal” factorization follows by taking the $\star_H$-inverse of the causality relation for the S-matrix, i.e.

$$S(H+F)^{\star_H -1} = S(F)^{\star_H -1} \star_H S(H)^{\star_H -1}, \quad \text{if} \quad \operatorname{supp}(H) \cap (\operatorname{supp}(F) + \overline{V}_-) = \emptyset.$$

2.5 The renormalization problem

Considering only interactions which are regular functionals is too restrictive. Usually physical interactions are modelled by generalized Lagrangians and considering that the algebra of free quantum fields is composed by microcausal fields, it would be ideal to be able to include also those in the interaction picture.

The renormalization problem is precisely the problem of extending the domain of definition of time-ordered products, scattering matrix, antichronological products and retarded products to encompass also microcausal fields.

It turns out that already the definition of the time-ordered products and of the antichronological products for regular fields imposes crucial constraints to the solution of the problem. In fact, the axioms initial condition and causality are very strong requirements and they suffice to determine $T_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n)\big)$, for any $F_1, \ldots, F_n \in \mathscr{D}'(\mathbb{M}_d\,; \mathcal{F}_{\text{reg}})$, uniquely on $\mathbb{M}_d^n \setminus \Delta_n$, where $\Delta_n = \{\, (x_1, \ldots, x_n) \in \mathbb{M}_d^n \mid x_1 = \cdots = x_n = x \,\}$ is the small diagonal. In particular, the following holds (see [10]).

Lemma 2.5.1 (Consequences of initial condition and causality axioms). *In $\mathbb{M}_d^n$, consider the subset:*

$$\check{\mathbb{M}}_d^n = \{\, (x_1, \ldots, x_n) \in \mathbb{M}_d^n \mid x_i \neq x_j \quad \forall 1 \leq i < j \leq n \,\}\,.$$

Then the following characterizations of time-ordered and antichronological products hold:

(a) On $\check{\mathbb{M}}_d^n$ the time-ordered product T_n agrees with the n-fold product $\star_{\Delta^F}$, where $\star_{\Delta^F}$ is defined by replacing the two-point function W in formula (2.7) with the Feynman propagator Δ^F:

$$T_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n)\big) = F_1(x_1) \star_{\Delta^F} \cdots \star_{\Delta^F} F_n(x_n), \tag{2.12}$$

for all $F_1, \ldots, F_n \in \mathscr{D}'(\mathbb{M}_d\,; \mathcal{F}_{\text{reg}})$.

(b) On $\check{\mathbb{M}}_d^n$, the antichronological product $\bar{T}_n$ agrees with the n-fold product $\star_{\Delta^{AF}}$, where $\star_{\Delta^{AF}}$ is defined by replacing the two-point function W in formula (2.7) with the anti-Feynman propagator Δ^{AF}:

$$\bar{T}_n\big(F_1(x_1) \otimes \cdots \otimes F_n(x_n)\big) = F_1(x_1) \star_{\Delta^{AF}} \cdots \star_{\Delta^{AF}} F_n(x_n), \tag{2.13}$$

for all $F_1, \ldots, F_n \in \mathscr{D}'(\mathbb{M}_d\,; \mathcal{F}_{\text{reg}})$.

Remark 2.5.1. The Feynman and anti-Feynman propagators $\Delta^F, \Delta^{AF} \in \mathscr{D}'(\mathbb{M}_d)$ are particular fundamental solutions of the wave operator P, with peculiar wavefront set properties (see Appendix C for more details). Their appearance in the formulation of the interaction picture in pAQFT is not a coincidence. In the foundational works [7], [23], [19], they were identified as the proper mathematical objects to encode the physical content of the theory and to provide the correct results in the calculation of the physically relevant quantities, such as correlation functions, expectation values etc...

We now take Lemma 2.5.1 as the starting point to introduce the following notation.

Notation. (cf. [10], Section 3.3) The following facts can be proved.

(i) The unrenormalized time-ordered product is a map

$$\check{T}_n\colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})^{\otimes n} \to \mathscr{D}'(\mathbb{M}_d^n \setminus \Delta_n\,;\mathcal{F}_{\mu c})[\![\hbar]\!]$$

defined, on $\check{\mathbb{M}}_d^n$, by:

$$\check{T}_n\big(F_1 \otimes \cdots \otimes F_n\big) = F_1 \star_{\Delta^F} \cdots \star_{\Delta^F} F_n, \tag{2.14}$$

for all $F_1, \ldots, F_n \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})$, which satisfies the same basic axioms and renormalization axioms as the time-ordered product for regular fields.

(ii) Analogously, the unrenormalized antichronological product is a map

$$\check{\bar{T}}_n\colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})^{\otimes n} \to \mathscr{D}'(\mathbb{M}_d^n \setminus \Delta_n\,;\mathcal{F}_{\mu c})[\![\hbar]\!]$$

defined, on $\check{\mathbb{M}}_d^n$, by:

$$\check{\bar{T}}_n\big(F_1 \otimes \cdots \otimes F_n\big) = F_1 \star_{\Delta^{AF}} \cdots \star_{\Delta^{AF}} F_n, \tag{2.15}$$

for all $F_1, \ldots, F_n \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})$.

(iii) Finally, the unrenormalized retarded product is a map

$$\check{R}_n\colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})^{\otimes n} \times \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c}) \to \mathscr{D}'(\mathbb{M}_d^{n+1} \setminus \Delta_{n+1}\,;\mathcal{F}_{\mu c})[\![\hbar]\!],$$

defined, on $\check{\mathbb{M}}_d^{n+1}$, by:

$$\check{R}_n\big(F^{\otimes n}, G\big) = \left(\frac{i}{\hbar}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} \check{\bar{T}}_{n-l}\big(F^{\otimes(n-l)}\big) \star \check{T}_{l+1}\big(F^{\otimes l} \otimes G\big), \tag{2.16}$$

$F, G \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})$, and consequently the unrenormalized interacting field $(\check{G})_{\kappa F}$ is defined as

$$(\check{G})_{\kappa F} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n(F^{\otimes n}, G) \in \mathscr{D}'(\check{\mathbb{M}}_d^{n+1}\,;\mathcal{F}_{\mu c})[\![\kappa, \hbar]\!]. \tag{2.17}$$

Remark 2.5.2. In [10] and [24] it is proved that the unrenormalized time-ordered and antichronological products can be written in a compact and useful exponential form, analogous to the one introduced in Remark 2.3.7 for the star product $\star_H$, as follows:

$$\check{T}_n\big(F_1 \otimes \cdots \otimes F_n\big) = \mu \circ e^{\hbar \sum_{1\le i<j\le n} D_F^{ij}}(F_1 \otimes \cdots \otimes F_n),$$

where μ is the commutative pointwise product of fields (2.2) and the operators $D_F^{ij}\colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})^{\otimes n} \to \mathscr{D}'(\check{\mathbb{M}}_d^n\,;\mathcal{F}_{\mu c})^{\otimes n}$ are defined by:

$$D_F^{ij}(F_1 \otimes \cdots \otimes F_n) = \Big\langle F_1 \otimes \cdots \otimes F_i^{(1)} \otimes \cdots \otimes \Big(\Delta^F * F_j^{(1)}\Big) \otimes \cdots \otimes F_n \Big\rangle .$$

In the same way we can write:

$$\check{\bar{T}}_n\big(F_1 \otimes \cdots \otimes F_n\big) = \mu \circ e^{\hbar \sum_{1\le i<j\le n} D_{AF}^{ij}}(F_1 \otimes \cdots \otimes F_n),$$

where μ is the commutative pointwise product of fields (2.2) and the operators $D_{AF}^{ij}\colon \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})^{\otimes n} \to \mathscr{D}'(\check{\mathbb{M}}_d^n\,;\mathcal{F}_{\mu c})^{\otimes n}$ are defined by:

$$D_{AF}^{ij}(F_1 \otimes \cdots \otimes F_n) = \Big\langle F_1 \otimes \cdots \otimes F_i^{(1)} \otimes \cdots \otimes \Big(\Delta^{AF} * F_j^{(1)}\Big) \otimes \cdots \otimes F_n \Big\rangle .$$

Remark 2.5.3. The reason why the definitions of time-ordered and antichronological products for microcausal fields cannot be given directly on the whole space $\mathbb{M}_d^n$ are the so-called UV-divergences of perturbative QFT. These amount to the fact that the star products (2.14) and (2.15) contain powers of Feynman and anti-Feynman propagators. In view of formula (C.2) for the wavefront set of Δ^F and of Hörmander's sufficient criterion (Theorem A.0.3), the existence of powers $(\Delta^F)^k$ can be discussed as follows:

- The wavefront set of Δ^F restricted to $\mathbb{M}_d \setminus 0$ is given by the first set of the union appearing in formula (C.2). This implies:

  $$\mathrm{WF}\big((\Delta^F|_{\mathbb{M}_d\setminus 0})^2\big) = \mathrm{WF}\big(\Delta^F|_{\mathbb{M}_d\setminus 0}\big) \oplus \mathrm{WF}\big(\Delta^F|_{\mathbb{M}_d\setminus 0}\big) = \mathrm{WF}\big(\Delta^F|_{\mathbb{M}_d\setminus 0}\big),$$

  and, by induction,

  $$\mathrm{WF}\big((\Delta^F|_{\mathbb{M}_d\setminus 0})^k\big) = \mathrm{WF}\big(\Delta^F|_{\mathbb{M}_d\setminus 0}\big), \quad \forall k \ge 2.$$

- But for $x = 0$, $\mathrm{WF}(\Delta^F) = \{0\} \times (\mathbb{R}^d \setminus 0)$, hence Hörmander's sufficient criterion is not satisfied.

As a consequence, for example, powers $\Delta^F(x_i - x_j)^k$, $k \ge 2$, exist on $\check{\mathbb{M}}_d^2$, but they cannot be defined using Hörmander's sufficient criterion for coinciding points $x_i = x_j$ (on the other hand, Hörmander's criterion is a sufficient, but not necessary criterion). This does not happen for regular fields, because by definition the field derivatives of regular fields are smooth compactly supported functions.

Definition 2.5.1. The renormalization problem in pAQFT, according to the approach proposed by Epstein and Glaser (see [11]), is the problem of extending the unrenormalized retarded product $\check{R}_n$ for microcausal fields to a well-defined $\mathcal{F}_{\mu c}[\![\hbar]\!]$-valued distributions on the whole $\mathbb{M}_d^n$.

For the practical computations of the extensions, several techniques have been introduced: differential renormalization, analytic regularization and regularization of the Feynman propagator among others. Following [7], we adopt the approach that combines the form of the wavefront sets of Δ^F and Δ^{AF} with the notion of Steinmann scaling degree (see Appendix B and [27]). The renormalization problem is solved by analyzing the scaling degree of the unrenormalized retarded product and then performing a scaling-degree-preserving extension of it according to the fundamental result of Theorem B.0.1.

Remark 2.5.4. As pointed out in Theorem B.0.1, depending on the scaling degree of the unrenormalized retarded products, the extensions may fail to be unique. Part of the indeterminacy in the extension process is usually restricted by imposing further renormalization conditions, like in Definition 2.4.2, which may encode special symmetries of the model. In general, though, the indeterminacy in the extension process cannot be completely eliminated. Nevertheless it can be described using the concept of renormalization group flow, see [10].

Definition 2.5.2. Fix fields $\kappa F \in \mathscr{D}'(\mathbb{M}_d\,;\kappa\mathcal{F}_{\mu c})$ and $G \in \mathscr{D}'(\mathbb{M}_d\,;\mathcal{F}_{\mu c})$. Consider the unrenormalized interacting field $(\check{G})_{\kappa F}$, given by:

$$(\check{G})_{\kappa F} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n(F^{\otimes n}, G).$$

Let $N(F, G, \cdot)\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ be the function defined in the following way:

$$\text{for} \quad n = 0, \qquad N(F, G, 0) = \max\{0, \operatorname{sd}(G) - d\},$$

and for $n \geq 1$

$$N(F, G, n) = \max\left\{0, \operatorname{sd}\left(\check{R}_n(F^{\otimes n}, G)\right) - (n+1)d - N(F, G, n-1)\right\},$$

where sd indicates the scaling degree of the corresponding distributions (see Appendix B). We say that the unrenormalized interacting field $(\check{G})_{\kappa F}$ is:

(a) **renormalizable by power counting** if $N(F, G, \cdot)$ is bounded;

(b) **super-renormalizable by power counting** if the number of non-vanishing values of $N(F, G, \cdot)$ is finite.

Remark 2.5.5. In most cases of physical interest, and also in the case of the sine-Gordon model, the field κF, which plays the rôle of the interaction, and the field G, which plays the rôle of generic observable, are taken to be local fields, namely $\kappa F \in \mathscr{D}'(\mathbb{M}_d\,; \kappa\mathcal{F}_{\mathrm{loc}})$ and $G \in \mathscr{D}'(\mathbb{M}_d\,; \mathcal{F}_{\mathrm{loc}})$. In this situation, it is possible to show that $N(F, G, \cdot)$ takes values in $\mathbb{N}$.

Chapter 3

The sine-Gordon model in pAQFT

In this chapter we consider in detail the formulation of the 2-dimensional massless sine-Gordon model in the framework of pAQFT. The first fundamental ingredient is the Lagrangian.

Definition 3.0.1. In pAQFT we regard the Lagrangian of the sine-Gordon model as a generalized Lagrangian $L \in \mathscr{D}'(\mathbb{M}_2\,;\mathcal{F}_{\text{loc}})$ according to Definition 2.3.1. Given a test function $f \in \mathscr{D}(\mathbb{M}_2)$ and a configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$, we have:

$$L(f)[\varphi] = \big(L_0 + L_{\text{int}}\big)(f)[\varphi] = \int_{\mathbb{M}_2} \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \cos(a\varphi)\right) f\,dx,$$

where $L_0 = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \in \mathscr{D}'(\mathbb{M}_2\,;\mathcal{F}_{\text{loc}})$ is the massless free Lagrangian and the interaction Lagrangian is $L_{\text{int}} = \cos(a\phi) \in \mathscr{D}'(\mathbb{M}_2\,;\mathcal{F}_{\text{loc}})$.

Next, we introduce one of the fundamental players in the interaction picture of the sine-Gordon model.

Definition 3.0.2. The vertex operators are fields $V_a = e^{ia\phi} \in \mathscr{D}'(\mathbb{M}_2\,;\mathcal{F}_{\text{loc}})$, with $a \in \mathbb{R}$, that act on a generic test function $f \in \mathscr{D}(\mathbb{M}_2)$ and on a generic configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$ as:

$$V_a(f)[\varphi] = \int_{\mathbb{M}_2} e^{ia\varphi}\,f\,dx.$$

Using vertex operators, we can rewrite the interaction Lagrangian of the sine-Gordon model as:

$$L_{\text{int}} = \frac{1}{2}\big(V_a + V_{-a}\big) \in \mathscr{D}'(\mathbb{M}_2\,;\mathcal{F}_{\text{loc}}).$$

Remark 3.0.1. A characteristic property of vertex operators, which turns out to be extremely useful when considering renormalization and summability issues (see

Chapters 4 and 5 below), is that field derivatives of vertex operators have the form:

$$\left\langle \frac{\delta^k}{\delta\phi^k} V_a(f)[\varphi], \psi_1 \otimes \cdots \otimes \psi_k \right\rangle = (ia)^k V_a(f\psi_1 \cdots \psi_k)[\varphi], \tag{3.1}$$

for any $f \in \mathscr{D}(\mathbb{M}_2)$ and $\phi, \psi_1, \ldots, \psi_k \in \mathscr{E}(\mathbb{M}_2)$. Thus field derivatives of vertex operators are again vertex operators, modulo constant coefficients.

We now look at propagators. According to Definition 2.3.4 the wave operator associated to the massless free Lagrangian L_0 of the sine-Gordon model takes the form:

$$P = -\Box.$$

As a consequence, all the propagators will not depend on the mass parameter. It is possible to show (cf. [1]) that the explicit expressions of the propagators in cartesian coordinates $x = (t, \vec{x}) \in \mathbb{M}_2$ are the following:

- the retarded and advanced propagators are respectively

$$\Delta^R(x) = -\frac{1}{2}\theta(t - |\vec{x}|), \quad \Delta^A(x) = -\frac{1}{2}\theta(-t - |\vec{x}|) \in \mathscr{D}'(\mathbb{M}_2);$$

- the commutator function is given by

$$\Delta(x) = \Delta^R(x) - \Delta^A(x) = \frac{1}{2}\big(- \theta(t - |\vec{x}|) + \theta(-t - |\vec{x}|)\big) \in \mathscr{D}'(\mathbb{M}_2);$$

here θ is as always the Heaviside step function. Finally, in [1] it is shown that sensible choices for the Hadamard parametrix H and the corresponding two-point function W as in formula (2.5) are:

$$H(x) = -\frac{1}{4\pi}\big(\ln|t + \vec{x}| + \ln|t - \vec{x}|\big) = -\frac{1}{4\pi}\ln\big((x)^2_\eta\big) \in \mathscr{D}'(\mathbb{M}_2) \tag{3.2}$$

and

$$W = \frac{i}{2}\Delta(x) + H(x) = -\frac{1}{4\pi}\ln\big(- (x)^2_\eta + i0t\big) \in \mathscr{D}'(\mathbb{M}_2). \tag{3.3}$$

Notation. From now on, we will always consider the Hadamard parametrix and the two-point function in the form (3.2) and (3.3). Having fixed these choices once and for all, when dealing with the star product in the sequel we will omit the subscript H and indicate it simply by $\star$.

Remark 3.0.2. A detailed proof of the formula for the two-point function W can be found in [25]. As we will see, this choice of Hadamard parametrix and consequently of the two-point function is particularly suited to study the renormalization and summability properties of the interacting higher currents of the sine-Gordon model. However, it presents some disadvantages already when considering representations of the algebra of free fields on some Hilbert space. In fact, it is shown in [2] and in particular, in [25], that, in order to obtain so-called quasifree states on the algebra of free fields, the Hadamard parametrix has to be modified.

It remains to consider the fundamental propagators of the interacting picture, the Feynman and anti-Feynman propagators. In view of the previous choices for the Hadamard parametrix and the two-point function, and using formulas (C.1) and (C.4), it can be proved that the Feynman propagator takes the form

$$\Delta^F(x) = \frac{i}{2}\big(\Delta^R(x) + \Delta^A(x)\big) + H(x) = -\frac{1}{4\pi}\ln\big((x)^2_\eta - i\varepsilon\big) \in \mathscr{D}'(\mathbb{M}_2), \quad (3.4)$$

while the anti-Feynman propagator is given by

$$\Delta^{AF}(x) = -\frac{i}{2}\big(\Delta^R(x)+\Delta^A(x)\big)+H(x) = -\frac{1}{4\pi}\ln\big((x)^2_\eta+i\varepsilon\big) \in \mathscr{D}'(\mathbb{M}_2), \quad (3.5)$$

for proofs of these formulas see [1].

Notation. From now on, we will consider the components s_1^N and s_2^N of the higher currents of the sine-Gordon model, given by formulas (1.15) and (1.16), as local fields $s_1^N, s_2^N \in \mathscr{D}'(\mathbb{M}_2\,;\mathcal{F}_{\mathrm{loc}})$. As an example, the second component $s_2^0 = \phi_\xi^2$ of the conserved current of order 0 will be intended as a local field acting on a generic test function $f \in \mathscr{D}(\mathbb{M}_2)$ and on a generic configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$ as:

$$s_2^0(f)[\varphi] = \int_{\mathbb{M}_2} \varphi_\xi^2\, f\, d\chi,$$

where $\chi = (\tau, \xi)$ indicates light-cone coordinates on $\mathbb{M}_2$.

We have now all the necessary ingredients to develop the theory. The rest of this chapter is devoted to work out explicit expressions for the unrenormalized interacting components $(\check{s}_1^N)_{\kappa L_{\mathrm{int}}}$ and $(\check{s}_2^N)_{\kappa L_{\mathrm{int}}}$, which we will simply denote by $(\check{s}_1^N)_{\mathrm{int}}$ and $(\check{s}_2^N)_{\mathrm{int}}$. In the exposition, we follow closely [30].

3.1 Some technical results

Before entering the details of the calculations of the unrenormalized interacting components of the higher currents of the sine-Gordon model, we introduce some useful technical results which hold in greater generality for time-ordered products and star products of microcausal fields with particular properties (inspired by the properties of the components of our currents).

Proposition 3.1.1. *Consider an unrenormalized time-ordered product of the form:*

$$\check{T}_{l+1}\big(A^{\otimes l} \otimes B\,C\big), \qquad A, B, C \in \mathscr{D}'(\mathbb{M}_d\,; \mathcal{F}_{\mu c})[\![\hbar]\!],$$

where C is such that $\exists\, c \in \mathbb{N}$ for which $\frac{\delta^i}{\delta\phi^i}C = 0$ whenever $i > c$, while A and B can possibly admit non-zero field derivatives of arbitrary order. Then the following equation holds:

$$\check{T}_{l+1}\big(A^{\otimes l} \otimes B\,C\big) = \sum_{\substack{j=0 \\ j_1,\ldots,j_l \geq 0 \\ j_1+\cdots+j_l=j}}^{c} \frac{\hbar^j}{j_1!\ldots j_l!} \left\langle \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}A \otimes \cdots \otimes \frac{\delta^{j_l}}{\delta\phi^{j_l}}A \otimes B\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j}C \right\rangle. \tag{3.6}$$

Proof. We start by writing the time-ordered product using the exponential form introduced in Remark 2.5.2:

$$T_{l+1}\big(A^{\otimes l} \otimes B\,C\big) = \mu \circ e^{\hbar \sum_{1\leq i<j\leq l+1} D_F^{ij}} \big(A^{\otimes l} \otimes B\,C\big).$$

We now split the exponential operators, isolating the ones that act on the product $B\,C$, as follows:

$$T_{l+1}\big(A^{\otimes l} \otimes B\,C\big) = \mu \circ e^{\hbar \sum_{1\leq i<j\leq l} D_F^{ij}} \circ e^{\hbar \sum_{i=1}^{l} D_F^{i\,l+1}} \big(A^{\otimes l} \otimes B\,C\big).$$

We concentrate on the second exponential, whose action on the fields is:

$$e^{\hbar \sum_{i=1}^{l} D_F^{i\,l+1}} \big(A^{\otimes l} \otimes B\,C\big) = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \left\langle \left(\frac{\delta}{\delta\phi_1} + \cdots + \frac{\delta}{\delta\phi_l}\right)^k \big(A^{\otimes l}\big), (\Delta^F)^{\otimes k} * \frac{\delta^k}{\delta\phi_{l+1}^k}\big(B\,C\big) \right\rangle. \tag{3.7}$$

Then we proceed studying separately the derivatives of the product $\frac{\delta^k}{\delta\phi_{l+1}^k}\big(B\,C\big)$. Applying the Leibniz rule, we have:

$$\sum_{k=0}^{\infty} \frac{\delta^k}{\delta\phi_{l+1}^k}\big(B\,C\big) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{k}{j} \frac{\delta^{k-j}}{\delta\phi_{l+1}^{k-j}}B\, \frac{\delta^j}{\delta\phi_{l+1}^j}C.$$

We can rewrite the sums using indices j and $i = k - j$. Recalling also that by hypothesis the field C admit non-zero derivatives only up to order c and using the notation (2.2) for the commutative pointwise product, we arrive at:

$$\sum_{k=0}^{\infty} \frac{\delta^k}{\delta\phi_{l+1}^k}(B\,C) = \sum_{j=0}^{c}\left(\sum_{i=0}^{\infty}\binom{i+j}{j}\frac{\delta^i}{\delta\phi_{l+1}^i}B\right)\frac{\delta^j}{\delta\phi_{l+1}^j}C$$
$$= \mu\left(\sum_{j=0}^{c}\sum_{i=0}^{\infty}\binom{i+j}{j}\frac{\delta^i}{\delta\phi_{l+1}^i}B \otimes \frac{\delta^j}{\delta\phi_{l+1}^j}C\right).$$

Omitting the operator μ from the last formula, which can be absorbed in the other operator μ appearing in the exponential formula for the time-ordered product, and substituting in equation (3.7) using the new indices i and j for the sums, we obtain:

$$e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(A^{\otimes l}\otimes B\,C\right)$$
$$= \sum_{j=0}^{c}\sum_{i=0}^{\infty}\frac{\hbar^{i+j}}{(i+j)!}\binom{i+j}{j}\Big\langle\left(\frac{\delta}{\delta\phi_1}+\cdots+\frac{\delta}{\delta\phi_l}\right)^{i+j}(A^{\otimes l}),$$
$$\left((\Delta^F)^{\otimes i} * \frac{\delta^i}{\delta\phi_{l+1}^i}B\right)\otimes\left((\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi_{l+1}^j}C\right)\Big\rangle$$
$$= \sum_{j=0}^{c}\frac{\hbar^j}{j!}\sum_{i=0}^{\infty}\frac{\hbar^i}{i!}\Big\langle\left(\frac{\delta}{\delta\phi_1}+\cdots+\frac{\delta}{\delta\phi_l}\right)^{i+j}(A^{\otimes l}),$$
$$\left((\Delta^F)^{\otimes i} * \frac{\delta^i}{\delta\phi_{l+1}^i}B\right)\otimes\left((\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi_{l+1}^j}C\right)\Big\rangle.$$

We can expand the operators $\left(\frac{\delta}{\delta\phi_1}+\cdots+\frac{\delta}{\delta\phi_l}\right)^j$ using the multinomial formula:

$$\left(\frac{\delta}{\delta\phi_1}+\cdots+\frac{\delta}{\delta\phi_l}\right)^j = \sum_{\substack{j_1,\ldots,j_l\geq 0\\ j_1+\cdots+j_l=j}}\frac{j!}{j_1!\cdots j_l!}\prod_{t=1}^{l}\frac{\delta^{j_t}}{\delta\phi_t^{j_t}},$$

where the product of field derivatives is intended as a tensor product. Substituting in the formula above, we get:

$$e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(A^{\otimes l}\otimes B\,C\right)$$
$$= \sum_{\substack{j=0\\ j_1,\ldots,j_l\geq 0\\ j_1+\cdots+j_l=j}}^{c}\frac{\hbar^j}{j_1!\cdots j_l!}\sum_{i=0}^{\infty}\frac{\hbar^i}{i!}\Big\langle\left(\frac{\delta}{\delta\phi_1}+\cdots+\frac{\delta}{\delta\phi_l}\right)^{i}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}A\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}A\right),$$
$$\left((\Delta^F)^{\otimes i} * \frac{\delta^i}{\delta\phi_{l+1}^i}B\right)\otimes\left((\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi_{l+1}^j}C\right)\Big\rangle.$$

It is now clear that the sum over the index i in the last formula corresponds to the definition of the exponential notation of Remark 2.5.2. Hence we can write:

$$e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\big(A^{\otimes l}\otimes B\,C\big) = \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{c} \frac{\hbar^j}{j_1!\cdots j_l!} \left\langle e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}A\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}A\otimes B\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi_{l+1}^j}C\right\rangle.$$

Applying the remaining operators $\mu\circ e^{\hbar\sum_{1\leq i<j\leq l} D_F^{ij}}$ we finally arrive at:

$$\mu\circ e^{\hbar\sum_{1\leq i<j\leq l} D_F^{ij}}\circ e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\big(A^{\otimes l}\otimes B\,C\big) = \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{c} \frac{\hbar^j}{j_1!\cdots j_l!}$$
$$\mu\left(\left\langle e^{\hbar\sum_{1\leq i<j\leq l} D_F^{ij}}\circ e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}A\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}A\otimes B\right),\right.\right.$$
$$\left.\left.(\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi_{l+1}^j}C\right\rangle\right)$$
$$= \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{c} \frac{\hbar^j}{j_1!\cdots j_l!} \left\langle \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}A\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}A\otimes B\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j}C\right\rangle.$$

q.e.d.

The next technical result, concerning the star product of fields, reads as follows.

Proposition 3.1.2. *Consider the product of fields* $A, B, C \in \mathscr{D}'(\mathbb{M}_d\,; \mathcal{F}_{\mu c})[\![\hbar]\!]$

$$A \star_H (B\,C),$$

where C *is such that* $\exists\, c\in\mathbb{N}$ *for which* $\frac{\delta^i}{\delta\phi^i}C = 0$ *whenever* $i > c$*, while* A *and* B *can possibly admit non-zero field derivatives of arbitrary order. Then the star product can be written in the form:*

$$A \star_H (B\,C) = \sum_{k=0}^{c} \frac{\hbar^k}{k!}\left\langle \left(\frac{\delta^k}{\delta\phi^k}A\right)\star_H B,\ (W)^{\otimes k} * \frac{\delta^k}{\delta\phi^k}C\right\rangle. \tag{3.8}$$

Proof. The claim is obtained by explicit calculation in a similar manner as in the previous proof. First we use the exponential notation introduced in Remark 2.3.7 for the star product:

$$A \star_H (B\,C) = \mu \circ e^{\hbar D_W}(A \otimes B\,C).$$

Then we expand the exponential operator and obtain:

$$e^{\hbar D_W}(A \otimes B\,C) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle \frac{\delta^n}{\delta\phi^n}A, (W)^{\otimes n} * \frac{\delta^n}{\delta\phi^n}(B\,C) \right\rangle .$$

Applying the Leibniz rule and writing the commutative pointwise product using the operator μ we get:

$$\frac{\delta^n}{\delta\phi^n}(B\,C) = \mu \left(\sum_{k=0}^{n} \binom{n}{k} \frac{\delta^{n-k}}{\delta\phi^{n-k}}B \otimes \frac{\delta^k}{\delta\phi^k}C \right) .$$

We can substitute this formula in the previous one, omitting the operator μ:

$$e^{\hbar D_W}(A \otimes B\,C)$$
$$\sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left\langle \frac{\delta^n}{\delta\phi^n}A, \left((W)^{\otimes(n-k)} * \frac{\delta^{n-k}}{\delta\phi^{n-k}}B\right) \otimes \left((W)^{\otimes k} * \frac{\delta^k}{\delta\phi^k}C\right) \right\rangle .$$

We can rewrite the double sum over indices k and $i = n - k$ and, recalling that by hypothesis the field C admits non-zero field derivatives only up to order c, we arrive at:

$$e^{\hbar D_W}(A \otimes B\,C)$$
$$= \sum_{k=0}^{c} \frac{\hbar^k}{k!} \sum_{i=0}^{\infty} \frac{\hbar^i}{i!} \left\langle \frac{\delta^{k+i}}{\delta\phi^{k+i}}A, \left((W)^{\otimes i} * \frac{\delta^i}{\delta\phi^i}B\right) \otimes \left((W)^{\otimes k} * \frac{\delta^k}{\delta\phi^k}C\right) \right\rangle .$$

In the sum over the index i we recognise the expression of the exponential operator $e^{\hbar D_W}$, hence we can write:

$$e^{\hbar D_W}(A \otimes B\,C) = \sum_{k=0}^{c} \frac{\hbar^k}{k!} \left\langle e^{\hbar D_W}\left(\frac{\delta^k}{\delta\phi^k}A \otimes B\right), (W)^{\otimes k} * \frac{\delta^k}{\delta\phi^k}C \right\rangle .$$

Finally, applying the operator μ we arrive at:

$$\mu \circ e^{\hbar D_W}(A \otimes B\,C) = \sum_{k=0}^{c} \frac{\hbar^k}{k!} \left\langle \left(\frac{\delta^k}{\delta\phi^k}A\right) \star_H B, (W)^{\otimes k} * \frac{\delta^k}{\delta\phi^k}C \right\rangle .$$

q.e.d.

3.2 Unrenormalized interacting components

We now apply the technical results just proved to compute the unrenormalized interacting components of the higher currents. According to formula (2.17) they are given by:

$$(\check{s}^N_{1,2})_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n(L^{\otimes n}_{\text{int}}, s^N_{1,2}),$$

where L_{int} is the interaction Lagrangian of the sine-Gordon model. We recall that, unlike a, which represents the truly physical coupling constant, κ is called the "bookkeeping" coupling constant and in this case it is just a tool to account for the order of perturbation.

Our goal for the rest of this section is then to work out explicit expressions for the retarded products of the interaction Lagrangian and components of the higher currents.

We start by considering the first components s^N_1. We know from formula (1.15) that the components s^N_1 are given by the sum of a homogeneous part of degree $2N$ multiplied by $\cos(a\phi)$ and another homogeneous part of degree $2N$ multiplied by $\sin(a\phi)$. We rename the two homogeneous parts q^N_1 and r^N_1 respectively, and write:

$$s^N_1 = \cos(a\phi) q^N_1 + \sin(a\phi) r^N_1. \tag{3.9}$$

Hence by linearity of the retarded product, we have:

$$\check{R}_n\big(L^{\otimes n}_{\text{int}}, s^N_1\big) = \check{R}_n\big(L^{\otimes n}_{\text{int}}, \cos(a\phi) q^N_1\big) + \check{R}_n\big(L^{\otimes n}_{\text{int}}, \sin(a\phi) r^N_1\big). \tag{3.10}$$

The two terms on the right hand side are completely analogous, so we restrict ourselves to consider only the second one. By formula (2.16), we can expand the retarded product as:

$$\begin{aligned}&\check{R}_n\big(L^{\otimes n}_{\text{int}}, \sin(a\phi) r^N_1\big)\\&= \Big(\frac{i}{\hbar}\Big)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} \check{\bar{T}}_{n-l}\big(L^{\otimes(n-l)}_{\text{int}}\big) \star \check{T}_{l+1}\big(L^{\otimes l}_{\text{int}} \otimes \sin(a\phi) r^N_1\big).\end{aligned} \tag{3.11}$$

First we consider the unrenormalized retarded product $\check{T}_{l+1}\big(L^{\otimes l}_{\text{int}} \otimes \sin(a\phi) r^N_1\big)$. We can directly apply Proposition 3.1.1, in the special case where $A = L_{\text{int}}$, $B = \sin(a\phi)$ and $C = r^N_1$. In fact, from Remark 1.4.3 and as a direct consequence of Proposition 1.4.2, we know that field derivatives of r^N_1 of order greater than $2N$

are all zero. Hence, we obtain:

$$\check{T}_{l+1}\big(L_{\text{int}}^{\otimes l}\otimes \sin(a\phi)r_1^N\big) = \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{2N} \frac{\hbar^j}{j_1!\cdots j_l!} \left\langle \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\otimes\sin(a\phi)\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j}r_1^N\right\rangle.$$

Now we substitute this equation in formula (3.11) and we get:

$$\check{R}_n\big(L_{\text{int}}^{\otimes n},\sin(a\phi)r_1^N\big) = \left(\frac{i}{\hbar}\right)^n \sum_{l=0}^n \binom{n}{l} \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{2N} \frac{(-1)^{(n-l)}\hbar^j}{j_1!\cdots j_l!}\bar{\check{T}}_{n-l}\big(L_{\text{int}}^{\otimes(n-l)}\big) \star\left\langle \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\otimes\sin(a\phi)\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j}r_1^N\right\rangle.$$

Since r_1^N and a fortiori $\frac{\delta^j}{\delta\phi^j}r_1^N$ admit non-zero field derivatives only up to order $2N$, we can apply Proposition 3.1.2 to each one of the star products appearing in the last formula. Explicitly, we get:

$$\begin{aligned}
&\bar{\check{T}}_{n-l}\big(L_{\text{int}}^{\otimes(n-l)}\big)\\
&\star\left\langle \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\otimes\sin(a\phi)\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j}r_1^N\right\rangle\\
&=\sum_{k=0}^{2N-j}\frac{\hbar^k}{k!}\Big\langle\frac{\delta^k}{\delta\phi^k}\bar{\check{T}}_{n-l}\big(L_{\text{int}}^{\otimes(n-l)}\big)\star\check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\otimes\sin(a\phi)\right),\\
&\qquad\big((W)^{\otimes k}\otimes(\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}}r_1^N\Big\rangle.
\end{aligned}$$

Then we plug this expression in equation (3.11) and obtain:

$$\begin{aligned}
\check{R}_n\big(L_{\text{int}}^{\otimes n},\sin(a\phi)r_1^N\big) &= \left(\frac{i}{\hbar}\right)^n \sum_{l=0}^n \binom{n}{l} \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{2N} \frac{(-1)^{(n-l)}\hbar^j}{j_1!\cdots j_l!}\sum_{k=0}^{2N-j}\frac{\hbar^k}{k!}\\
&\left\langle\frac{\delta^k}{\delta\phi^k}\bar{\check{T}}_{n-l}\big(L_{\text{int}}^{\otimes(n-l)}\big)\star\check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\otimes\sin(a\phi)\right),\right.\\
&\qquad\left.\big((W)^{\otimes k}\otimes(\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}}r_1^N\right\rangle.
\end{aligned} \tag{3.12}$$

Rewriting the interaction Lagrangian and the sine in terms of vertex operators, we finally arrive at:

$$\check{R}_n\big(L_{\text{int}}^{\otimes n}, \sin(a\phi)r_1^N\big) = \frac{1}{i2^{n+1}}\left(\frac{i}{\hbar}\right)^n \sum_{l=0}^{n}\binom{n}{l} \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\cdots+j_l=j}}^{2N} \frac{(-1)^{(n-l)}\hbar^j}{j_1!\cdots j_l!}\sum_{k=0}^{2N-j}\frac{\hbar^k}{k!}$$
$$\Big\langle \frac{\delta^k}{\delta\phi^k}\check{\bar{T}}_{n-l}\big((V_a+V_{-a})^{\otimes(n-l)}\big)$$
$$\star\, \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}(V_a+V_{-a})\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}(V_a+V_{-a})\otimes(V_a-V_{-a})\right),$$
$$\big((W)^{\otimes k}\otimes(\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}}r_1^N\Big\rangle. \tag{3.13}$$

A completely analogous expression occurs for the term $\check{R}_n\big(L_{\text{int}}^{\otimes n}, \cos(a\phi)q_1^N\big)$ in equation (3.10), with $\cos(a\phi)$ in place of $\sin(a\phi)$ and q_1^N in place of r_1^N.

We now pass to consider the components s_2^N. Again our goal is to give an explicit formula for the unrenormalized retarded product

$$\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big) = \left(\frac{i}{\hbar}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^{(n-l)}\check{\bar{T}}_{n-l}\big(L_{\text{int}}^{\otimes(n-l)}\big)\star\check{T}_{l+1}\big(L_{\text{int}}^{\otimes l}\otimes s_2^N\big). \tag{3.14}$$

As pointed out in Remark 1.4.3, the components s_2^N are polynomials in the derivatives of the configuration ϕ. Moreover, as a consequence of Proposition 1.4.2, we know that they admit non-zero field derivatives only up to order $2(N+1)$. We can then apply Proposition 3.1.1 in the special case where $A = L_{\text{int}}$, $B = 1$ and $C = s_2^N$. The result is:

$$\check{T}_{l+1}\big(L_{\text{int}}^{\otimes l}\otimes 1\, s_2^N\big)$$
$$\sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\cdots+j_l=j}}^{2(N+1)} \frac{\hbar^j}{j_1!\cdots j_l!}\left\langle \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\otimes 1\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j}s_2^N\right\rangle$$

In view of formula (2.14) for the unrenormalized time-ordered product, it is immediate to see that:

$$\check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\otimes 1\right) = \check{T}_l\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}L_{\text{int}}\right),$$

(for a proof of this equality using the causality axiom for the time-ordered product, see [10]). Using the axiom field independence to collect the field derivatives outside of the time-ordered product, we then get:

$$\begin{aligned}
&\check{T}_{l+1}\big(L_{\text{int}}^{\otimes l} \otimes 1\, s_2^N\big) \\
&= \sum_{\substack{j=0 \\ j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}}^{2(N+1)} \frac{\hbar^j}{j_1! \cdots j_l!} \left\langle \check{T}_l\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}} L_{\text{int}} \otimes \cdots \otimes \frac{\delta^{j_l}}{\delta\phi^{j_l}} L_{\text{int}}\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j} s_2^N \right\rangle \\
&= \sum_{j=0}^{2(N+1)} \frac{\hbar^j}{j!} \left\langle \frac{\delta^j}{\delta\phi^j} \check{T}_l\left(L_{\text{int}}^{\otimes l}\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j} s_2^N \right\rangle .
\end{aligned}$$

Substituting this formula in (3.14) we obtain:

$$\begin{aligned}
\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big) = \left(\frac{i}{\hbar}\right)^n \sum_{l=0}^{n} \binom{n}{l} \sum_{j=0}^{2(N+1)} \frac{(-1)^{(n-l)} \hbar^j}{j!} \\
\check{\bar{T}}_{n-l}\big(L_{\text{int}}^{\otimes (n-l)}\big) \star \left\langle \frac{\delta^j}{\delta\phi^j} \check{T}_l\left(L_{\text{int}}^{\otimes l}\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j} s_2^N \right\rangle .
\end{aligned}$$

Since s_2^N and a fortiori $\frac{\delta^j}{\delta\phi^j} s_2^N$ admit non-zero field derivatives only up to order $2(N+1)$, we can apply Proposition 3.1.2 to each one of the star products appearing in the last formula. Explicitly, we get:

$$\begin{aligned}
\check{\bar{T}}_{n-l}\big(L_{\text{int}}^{\otimes (n-l)}\big) \star \left\langle \frac{\delta^j}{\delta\phi^j} \check{T}_l\left(L_{\text{int}}^{\otimes l}\right), (\Delta^F)^{\otimes j} * \frac{\delta^j}{\delta\phi^j} s_2^N \right\rangle = \sum_{k=0}^{2(N+1)-j} \frac{\hbar^k}{k!} \\
\left\langle \frac{\delta^k}{\delta\phi^k} \check{\bar{T}}_{n-l}\big(L_{\text{int}}^{\otimes (n-l)}\big) \star \frac{\delta^j}{\delta\phi^j} \check{T}_l\left(L_{\text{int}}^{\otimes l}\right), \big((W)^{\otimes k} \otimes (\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}} s_2^N \right\rangle
\end{aligned}$$

Substituting in the previous formula, we finally arrive at:

$$\begin{aligned}
\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big) = \left(\frac{i}{\hbar}\right)^n \sum_{l=0}^{n} \binom{n}{l} \sum_{j=0}^{2(N+1)} \sum_{k=0}^{2(N+1)-j} \frac{(-1)^{(n-l)} \hbar^{j+k}}{j!k!} \\
\left\langle \frac{\delta^k}{\delta\phi^k} \check{\bar{T}}_{n-l}\big(L_{\text{int}}^{\otimes (n-l)}\big) \star \frac{\delta^j}{\delta\phi^j} \check{T}_l\left(L_{\text{int}}^{\otimes l}\right), \big((W)^{\otimes k} \otimes (\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}} s_2^N \right\rangle .
\end{aligned} \tag{3.15}$$

Finally, using vertex operators to write the interaction Lagrangian, we get:

$$\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big) = \Big(\frac{i}{2\hbar}\Big)^n \sum_{l=0}^{n} \binom{n}{l} \sum_{j=0}^{2(N+1)} \sum_{k=0}^{2(N+1)-j} \frac{(-1)^{(n-l)}\hbar^{j+k}}{j!k!}$$
$$\Big\langle \frac{\delta^k}{\delta\phi^k}\check{\bar{T}}_{n-l}\big((V_a + V_{-a})^{\otimes(n-l)}\big) \star \frac{\delta^j}{\delta\phi^j}\check{T}_l\left((V_a + V_{-a})^{\otimes l}\right), \tag{3.16}$$
$$\big((W)^{\otimes k} \otimes (\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}} s_2^N\Big\rangle.$$

Chapter 4

Renormalization of the interacting higher currents

In this chapter we prove that, according to Definition 2.5.2, the unrenormalized interacting components $(\check{s}_1^N)_{\text{int}}$, $(\check{s}_2^N)_{\text{int}}$ of the higher currents of the sine-Gordon model are super-renormalizable by power counting.

Before proving our main result, we introduce some useful notation. Since the concrete classical expressions of the components were obtained using light-cone coordinates, we need to work coherently in that coordinate system. We denote then by $\chi = (\tau, \xi)$ the set of light-cone coordinates on $\mathbb{M}_2$, and accordingly by $(\chi_1, \dots, \chi_n) = (\tau_1, \xi_1, \dots, \tau_n, \xi_n)$ the set of light-cone coordinates on $\mathbb{M}_2^n$.

Based on [1], we then introduce the following formulas:

(i) On the subset

$$\check{\mathbb{M}}_2^{l+1} = \left\{ (\chi_1, \dots, \chi_{l+1}) \in \mathbb{M}_2^{l+1} \,\middle|\, \chi_i \neq \chi_j, \quad \forall 1 \leq i < j \leq l+1 \right\},$$

the unrenormalized time-ordered product of vertex operators can be written in the form:

$$\begin{aligned} &\check{T}_{l+1}\big(V_{a_1}(\chi_1) \otimes \cdots \otimes V_{a_{l+1}}(\chi_{l+1})\big) \\ &= e^{i(a_1\phi(\chi_1)+\cdots+a_{l+1}\phi(\chi_{l+1}))} \check{\prod_{1\leq i<j\leq l+1}} e^{-a_i a_j \hbar \Delta^F(\chi_i - \chi_j)}, \end{aligned} \tag{4.1}$$

where the symbol $\check{\prod}_{1\leq i<j\leq l+1}$ indicates that the distributional product of exponentials of Feynman propagators is defined only on $\check{\mathbb{M}}_2^{l+1}$.

(ii) Analogously, on the subset

$$\check{\mathbb{M}}_2^{n-l} = \left\{ (\chi_{l+2}, \dots, \chi_{n+1}) \in \mathbb{M}_2^{n-l} \,\middle|\, \chi_i \neq \chi_j, \, \forall l+2 \leq i < j \leq n+1 \right\},$$

the unrenormalized antichronological product of vertex operators can be written in the form:

$$\check{\bar{T}}_{n-l}\big(V_{a_{l+2}}(\chi_{l+2}) \otimes \cdots \otimes V_{a_{n+1}}(\chi_{n+1})\big) = e^{i(a_{l+1}\phi(\chi_{l+2})+\cdots+a_{n+1}\phi(\chi_{n+1}))} \check{\prod_{l+2\leq i<j\leq n+1}} e^{-a_i a_j \hbar \Delta^{AF}(\chi_i-\chi_j)}. \tag{4.2}$$

These formulas follow immediately from the exponential notation introduced in Remark 2.5.2 and from the fact that the field derivatives of vertex operators are again vertex operators, see equation (3.1).

Finally, combining formulas (4.1) and (4.2) with the exponential notation for the star product of fields from Remark 2.3.7, we get that also the star product of the unrenormalized time-ordered product of vertex operators with the unrenormalized antichronological product of vertex operators admits a similar exponential formula. More precisely, on the subset

$$\check{\mathbb{M}}_2^{n+1} = \left\{ (\chi_1, \ldots, \chi_{n+1}) \in \mathbb{M}_2^{n+1} \;\middle|\; \chi_i \neq \chi_j, \quad \forall 1 \leq i < j \leq n+1 \right\},$$

we have that:

$$\begin{aligned} &\check{\bar{T}}_{n-l}\big(V_{a_{l+2}}(\chi_{l+2}) \otimes \cdots \otimes V_{a_{n+1}}(\chi_{n+1})\big) \star \check{T}_{l+1}\big(V_{a_1}(\chi_1) \otimes \cdots \otimes V_{a_{l+1}}(\chi_{l+1})\big) \\ &= e^{i(a_1\phi(\chi_1)+\cdots+a_{n+1}\phi(\chi_{n+1}))} \check{\prod_{1\leq i<j\leq l+1}} e^{-a_i a_j \hbar \Delta^F(\chi_i-\chi_j)} \\ &\qquad \check{\prod_{l+2\leq i<j\leq n+1}} e^{-a_i a_j \hbar \Delta^{AF}(\chi_i-\chi_j)} \check{\prod_{\substack{l+2\leq i\leq n+1\\ 1\leq j\leq l+1}}} e^{-a_i a_j \hbar W(\chi_i-\chi_j)}. \end{aligned} \tag{4.3}$$

4.1 The main theorem

We have now the elements to prove the main result of this chapter.

Theorem 4.1.1. *Let us consider the unrenormalized interacting components of the higher currents of the sine-Gordon model:*

$$(\check{s}_1^N)_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big), \qquad (\check{s}_2^N)_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big),$$

where $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big), \check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big) \in \mathscr{D}'(\check{\mathbb{M}}_2^{n+1}; \mathcal{F}_{\mu c})[\![\hbar]\!]$*. Then, the scaling degree of the retarded products is uniformly bounded by the degree of the components. Specifically, for every* $n \geq 1$ *it holds:*

$$\operatorname{sd}\left(\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big)\right) = \deg(s_1^N) = 2N,$$
$$\operatorname{sd}\left(\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big)\right) = \deg(s_2^N) = 2(N+1).$$

Proof. We show the detailed steps only for the unrenormalized interacting first component $(s_1^N)_{\text{int}}$. The same arguments apply also to the unrenormalized interacting second component $(s_2^N)_{\text{int}}$, which in fact is a special case of the former.

We start by recalling formula (3.10):

$$\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big) = \check{R}_n\big(L_{\text{int}}^{\otimes n}, \cos(a\phi)q_1^N\big) + \check{R}_n\big(L_{\text{int}}^{\otimes n}, \sin(a\phi)r_1^N\big).$$

The two terms are completely analogous, so we consider only the second one. From the previous chapter, formula (3.13), we know the explicit form of the unrenormalized retarded product $\check{R}_n\big(L_{\text{int}}^{\otimes n}, \sin(a\phi)r_1^N\big)$ on $\check{\mathbb{M}}_2^{n+1}$:

$$\begin{aligned}\check{R}_n\big(L_{\text{int}}^{\otimes n}, \sin(a\phi)r_1^N\big) = \frac{1}{i2^{n+1}}\Big(\frac{i}{\hbar}\Big)^n \sum_{l=0}^{n}\binom{n}{l} \sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{2N} \frac{(-1)^{(n-l)}\hbar^j}{j_1!\cdots j_l!} \sum_{k=0}^{2N-j}\frac{\hbar^k}{k!}\\ \Big\langle \frac{\delta^k}{\delta\phi^k}\check{\bar{T}}_{n-l}\big((V_a+V_{-a})^{\otimes(n-l)}\big)\\ \star\, \check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}(V_a+V_{-a})\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}(V_a+V_{-a})\otimes(V_a-V_{-a})\right),\\ \big((W)^{\otimes k}\otimes(\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}}r_1^N\Big\rangle.\end{aligned} \tag{4.4}$$

We extract the generic term of this sum and write it as:

$$\begin{aligned}\Big\langle \frac{\delta^k}{\delta\phi^k}\check{\bar{T}}_{n-l}\big(V_{a_{l+2}}\otimes\cdots\otimes V_{a_{n+1}}\big)\star\check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}V_{a_1}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}V_{a_l}\otimes V_{a_{l+1}}\right),\\ \big((W)^{\otimes k}\otimes(\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}}r_1^N\Big\rangle\end{aligned} \tag{4.5}$$

Our goal is to estimate the scaling degree of the distributional part of this generic term. Using formula (4.2), we can immediately compute:

$$\begin{aligned}&\frac{\delta^k}{\delta\phi^k}\check{\bar{T}}_{n-l}\big(V_{a_{l+2}}(\chi_{l+2})\otimes\cdots\otimes V_{a_{n+1}}(\chi_{n+1})\big)\\ &= \left(\frac{\delta^k}{\delta\phi^k}e^{i(a_{l+2}\phi(\chi_{l+2})+\dots+a_{n+1}\phi(\chi_{n+1}))}\right)\check{\prod_{l+2\leq i<j\leq n+1}} e^{-a_ia_j\hbar\Delta^{AF}(\chi_i-\chi_j)}.\end{aligned}$$

Using formula (4.1), we obtain:

$$\check{T}_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}V_{a_1}(\chi_1)\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}V_{a_l}(\chi_l)\otimes V_{a_{l+1}}(\chi_{l+1})\right)$$
$$=\left(\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}\right)e^{i\left(a_1\phi(\chi_1)+\cdots+a_{l+1}\phi(\chi_{l+1})\right)}\right)\check{\prod_{1\leq i<j\leq l+1}}e^{-a_ia_j\hbar\Delta^F(\chi_i-\chi_j)}.$$

We now adopt the integral kernel notation (2.1) to make more explicit the distributional character of the field derivatives appearing in the last two expressions. For the antichronological product, we obtain:

$$\frac{\delta^k}{\delta\phi(\alpha_1)\cdots\delta\phi(\alpha_k)}e^{i(a_{l+2}\phi(\chi_{l+2})+\cdots+a_{n+1}\phi(\chi_{n+1}))}$$
$$=\sum_{\substack{k_{l+2},\ldots,k_{n+1}\geq 0\\ k_{l+2}+\cdots+k_{n+1}=k}}\frac{k!(ia_{l+2})^{k_{l+2}}\cdots(ia_{n+1})^{k_{n+1}}}{k_{l+2}!\cdots k_{n+1}!}e^{i(a_{l+2}\phi(\chi_{l+2})+\cdots+a_{n+1}\phi(\chi_{n+1}))}$$
$$\delta(\alpha_1-\chi_{l+2})\cdots\delta(\alpha_{k_{l+2}}-\chi_{l+2})\cdots\delta(\alpha_{k-k_{n+1}+1}-\chi_{n+1})\cdots\delta(\alpha_k-\chi_{n+1}),$$

understood as a distribution on $\mathbb{M}_2^k$. Similarly, for the field derivatives appearing in the time-ordered product, we obtain:

$$\left(\frac{\delta^{j_1}}{\delta\phi(\beta_1)\cdots\delta\phi(\beta_{j_1})}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi(\beta_{j-j_l+1})\cdots\delta\phi(\beta_j)}\right)e^{i(a_1\phi(\chi_1)+\cdots+a_{l+1}\phi(\chi_{l+1}))}$$
$$=(ia_1)^{j_1}\cdots(ia_l)^{j_l}e^{i(a_1\phi(\chi_1)+\cdots+a_{l+1}\phi(\chi_{l+1}))}$$
$$\delta(\beta_1-\chi_1)\cdots\delta(\beta_{j_1}-\chi_1)\cdots\delta(\beta_{j-j_l+1}-\chi_l)\cdots\delta(\beta_j-\chi_l),$$

in this case understood as a distribution on $\mathbb{M}_2^j$. It remains to consider the field derivatives of r_1^N. From Remark 1.4.3 and Proposition 1.4.2, we know that r_1^N is a polynomial in the derivatives of the configuration up to order $2N$ and with homogeneous degree. Hence its field derivatives can be expressed in the form:

$$\frac{\delta^{j+k}}{\delta\phi(\sigma_1)\cdots\delta\phi(\sigma_k)\delta\phi(\rho_1)\cdots\delta\phi(\rho_j)}r_1^N(\chi_{l+1})$$
$$=\sum_{\substack{s_1,\ldots,s_k,r_1,\ldots,r_j\geq 0\\ s_1+\cdots+s_k+r_1+\cdots+r_j\leq 2N}}\frac{\partial^{j+k}r_1^N}{\partial\phi_{s_1\xi}\cdots\partial\phi_{s_k\xi}\partial\phi_{r_1\xi}\cdots\partial\phi_{r_j\xi}}(\chi_{l+1})$$
$$\partial^{s_1}_{\xi_{l+1}}\delta(\sigma_1-\chi_{l+1})\cdots\partial^{s_k}_{\xi_{l+1}}\delta(\sigma_k-\chi_{l+1})\partial^{r_1}_{\xi_{l+1}}\delta(\rho_1-\chi_{l+1})\cdots\partial^{r_j}_{\xi_{l+1}}\delta(\rho_j-\chi_{l+1}) \tag{4.6}$$

understood as a distribution on $\mathbb{M}_2^{j+k}$. We can then put together all the pieces. Omitting the field part, that is, the exponentials of configurations, we obtain that the distributional pairing of equation (4.5) reduces to:

$$\sum_{\substack{k_{l+2},\dots,k_{n+1}\geq 0\\ k_{l+2}+\cdots+k_{n+1}=k}} \sum_{\substack{s_1,\dots,s_k,r_1,\dots,r_j\geq 0\\ s_1+\cdots+s_k+r_1+\cdots+r_j\leq 2N}} \frac{k!(ia_1)^{j_1}\cdots(ia_l)^{j_l}(ia_{l+2})^{k_{l+2}}\cdots(ia_{n+1})^{k_{n+1}}}{k_{l+2}!\cdots k_{n+1}!}$$
$$\Big\langle \delta(\alpha_1-\chi_{l+2})\cdots\delta(\alpha_{k_{l+2}}-\chi_{l+2})\cdots\delta(\alpha_{k-k_{n+1}+1}-\chi_{n+1})\cdots\delta(\alpha_k-\chi_{n+1})$$
$$\delta(\beta_1-\chi_1)\cdots\delta(\beta_{j_1}-\chi_1)\cdots\delta(\beta_{j-j_l+1}-\chi_l)\cdots\delta(\beta_j-\chi_l)\,,$$
$$W(\alpha_1-\sigma_1)\cdots W(\alpha_k-\sigma_k)\Delta^F(\beta_1-\rho_1)\cdots\Delta^F(\beta_j-\rho_j)$$
$$\partial^{s_1}_{\xi_{l+1}}\delta(\sigma_1-\chi_{l+1})\cdots\partial^{s_k}_{\xi_{l+1}}\delta(\sigma_k-\chi_{l+1})\partial^{r_1}_{\xi_{l+1}}\delta(\rho_1-\chi_{l+1})\cdots\partial^{r_j}_{\xi_{l+1}}\delta(\rho_j-\chi_{l+1})\Big\rangle$$

The pairing of the two-point functions and of the Feynman propagators with the derivatives of Dirac deltas is the reason why the former inherit the derivatives. The final result is the following sum of distributions defined on $\check{\mathbb{M}}_2^{n+1}$:

$$\begin{aligned}\sum_{\substack{k_{l+2},\dots,k_{n+1}\geq 0\\ k_{l+2}+\cdots+k_{n+1}=k}} \sum_{\substack{s_1,\dots,s_k,r_1,\dots,r_j\geq 0\\ s_1+\cdots+s_k+r_1+\cdots+r_j\leq 2N}} &\frac{k!(ia_1)^{j_1}\cdots(ia_l)^{j_l}(ia_{l+2})^{k_{l+2}}\cdots(ia_{n+1})^{k_{n+1}}}{k_{l+2}!\cdots k_{n+1}!}\\
&\partial^{s_1}_{\xi_{l+1}}W(\chi_{l+2}-\chi_{l+1})\cdots\partial^{s_{k_{l+2}}}_{\xi_{l+1}}W(\chi_{l+2}-\chi_{l+1})\\
&\vdots\\
&\partial^{s_{k-k_{n+1}+1}}_{\xi_{l+1}}W(\chi_{n+1}-\chi_{l+1})\cdots\partial^{s_k}_{\xi_{l+1}}W(\chi_{n+1}-\chi_{l+1})\\
&\partial^{r_1}_{\xi_{l+1}}\Delta^F(\chi_1-\chi_{l+1})\cdots\partial^{r_{j_1}}_{\xi_{l+1}}\Delta^F(\chi_1-\chi_{l+1})\\
&\vdots\\
&\partial^{r_{j-j_l+1}}_{\xi_{l+1}}\Delta^F(\chi_l-\chi_{l+1})\cdots\partial^{r_j}_{\xi_{l+1}}\Delta^F(\chi_l-\chi_{l+1}).\end{aligned} \tag{4.7}$$

Using this last formula, omitting all the sums, the numerical coefficients and the exponentials of configurations, we finally have that the distributional part of the generic term of (4.5), and hence of $\check{R}_n\big(L_{\text{int}}^{\otimes n},\sin(a\phi)r_1^N\big)$, is given, on $\check{\mathbb{M}}_2^{n+1}$, by:

$$\begin{aligned}&\check{\prod_{1\leq i<j\leq l+1}} e^{-a_ia_j\hbar\Delta^F(\chi_i-\chi_j)}\partial^{r_1}_{\xi_{l+1}}\Delta^F(\chi_1-\chi_{l+1})\cdots\partial^{r_j}_{\xi_{l+1}}\Delta^F(\chi_l-\chi_{l+1})\\
&\check{\prod_{l+2\leq i<j\leq n+1}} e^{-a_ia_j\hbar\Delta^{AF}(\chi_i-\chi_j)} \check{\prod_{\substack{l+2\leq i\leq n+1\\ 1\leq j\leq l+1}}} e^{-a_ia_j\hbar W(\chi_i-\chi_j)}\\
&\partial^{s_1}_{\xi_{l+1}}W(\chi_{l+2}-\chi_{l+1})\cdots\partial^{s_k}_{\xi_{l+1}}W(\chi_{n+1}-\chi_{l+1}).\end{aligned} \tag{4.8}$$

This product of distributions is well-defined according to Hörmander's criterion on $\breve{\mathbb{M}}_2^{n+1}$. Hence, we can apply the properties of the scaling degree with respect to the product of distributions. In particular, the fact (see Proposition B.0.1) that the scaling degree of the product is bounded by the sum of the scaling degrees of the factors. Thus, from the condition

$$s_1 + \cdots + s_k + r_1 + \cdots + r_j \leq 2N,$$

which holds for all the generic terms of (4.5), we immediately obtain:

$$\mathrm{sd}\left(\check{R}_n\left(L_{\mathrm{int}}^{\otimes n}, \sin(a\phi) r_1^N\right)\right) \leq 2N.$$

Moreover, the maximum value $2N$ of the scaling degree is actually always attained. In fact, from formula (1.15), the classical expression of the component r_1^N always contains a term of the form A_1^{2N} which in turn, as we know from Section 1.2, is just a multiple of ϕ_ξ^{2N}. This means that at least for $j + k = 2N$ and $s_1 = \cdots = s_k = r_1 = \cdots = r_j = 1$ the scaling degree of $\check{R}_n\left(L_{\mathrm{int}}^{\otimes n}, \sin(a\phi) r_1^N\right)$ is exactly $2N$. *q.e.d.*

Corollary 4.1.1. *The unrenormalized interacting components of the higher currents of the sine-Gordon model:*

$$(s_1^N)_{\mathrm{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n\left(L_{\mathrm{int}}^{\otimes n}, s_1^N\right), \qquad (s_2^N)_{\mathrm{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \check{R}_n\left(L_{\mathrm{int}}^{\otimes n}, s_2^N\right),$$

are super-renormalizable by power counting according to Definition 2.5.2.

Proof. The interaction Lagrangian of the sine-Gordon model and the components of the higher currents are local fields, more precisely $L_{\mathrm{int}}, s_{1,2}^N \in \mathscr{D}'(\mathbb{M}_2\,; \mathcal{F}_{\mathrm{loc}})$. Hence, according to Remark 2.5.5, the function $N(L_{\mathrm{int}}, s_{1,2}^N, \cdot)$ of Definition 2.5.2 takes values in $\mathbb{N}$. The uniform bound on the scaling degree of the retarded products given by Theorem 4.1.1 directly implies:

$$\begin{aligned} N(L_{\mathrm{int}}, s_1^N, n) &= \mathrm{sd}\left(\check{R}_n(L_{\mathrm{int}}^{\otimes n}, s_1^N)\right) - 2(n+1) - N(L_{\mathrm{int}}, s_1^N, n-1) \\ &= 2N - 2(n+1) - N(L_{\mathrm{int}}, s_1^N, n-1) \\ &\leq 2N - 2(n+1), \end{aligned}$$

and

$$\begin{aligned} N(L_{\mathrm{int}}, s_2^N, n) &= \mathrm{sd}\left(\check{R}_n(L_{\mathrm{int}}^{\otimes n}, s_2^N)\right) - 2(n+1) - N(L_{\mathrm{int}}, s_2^N, n-1) \\ &= 2(N+1) - 2(n+1) - N(L_{\mathrm{int}}, s_2^N, n-1) \\ &\leq 2(N+1) - 2(n+1). \end{aligned}$$

This means that for n big enough, respectively $n \geq N-1$ and $n \geq N$, the functions $N(L_{\mathrm{int}}, s_1^N, n)$ and $N(L_{\mathrm{int}}, s_2^N, n)$ are constantly 0. *q.e.d.*

Remark 4.1.1. For a proof, in a slightly different quantization framework, of the renormalizability properties of the first of the higher interacting currents, that is, the stress-energy tensor of the sine-Gordon model, where the counterterms are explicitly computed, we refer to [16] and [15].

4.2 Piecewise renormalization

We now describe the concrete renormalization of the unrenormalized retarded products $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_{1,2}^N\big)$ for the components of the higher currents. Specifically, we carry out the detailed analysis only for the term $\check{R}_n\big(L_{\text{int}}^{\otimes n}, \sin(a\phi)r_1^N\big)$, recalling that the other term in equation (3.10) can be treated in a completely analogous way, while $\check{R}_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big)$ is a special case without the sine.

We adopt an approach which we call piecewise renormalization. It consists of three steps: expansion of the expression to be renormalized in its elementary parts, scaling-degree-preserving extension of each one of the elementary parts separately and finally showing that reassembling the extended elementary parts all together gives a well-defined result.

The rest of this section is organized as follows: in Subsection 4.2.1 we consider the time-ordered product of vertex operators and the derivatives of Feynman propagators; in Subsection 4.2.2 we consider the antichronological product of vertex operators; finally, in Subsection 4.2.3, we reassemble the extended elementary parts all together and show, by a careful study of the wavefront set of all the elements involved, that the result is in fact well-defined. In our exposition we closely follow [30].

4.2.1 Time-ordered product of vertex operators and derivatives of Feynman propagators

In all the subsequent discussion we refer to the distributional part of the generic term of $\check{R}_n\big(L_{\text{int}}^{\otimes n}, \sin(a\phi)r_1^N\big)$, described by formula (4.8). From the first line of (4.8), we consider the distributional part of the unrenormalized time-ordered product of vertex operators multiplied with derivatives of Feynman propagators as a distribution defined on $\check{\mathbb{M}}_2^{l+1}$:

$$\check{\prod_{1\le i<j\le l+1}} e^{-a_i a_j \hbar \Delta^F(\chi_i-\chi_j)} \partial_{\xi_{l+1}}^{r_1}\Delta^F(\chi_1-\chi_{l+1})\cdots\partial_{\xi_{l+1}}^{r_j}\Delta^F(\chi_l-\chi_{l+1}). \tag{4.9}$$

Using formula (4.7) we can further expand this as:

$$\check{\prod_{1\leq i<j\leq l+1}} e^{-a_i a_j \hbar \Delta^F(\chi_i-\chi_j)} \partial^{r_1}_{\xi_{l+1}} \Delta^F(\chi_1-\chi_{l+1}) \cdots \partial^{r_{j_1}}_{\xi_{l+1}} \Delta^F(\chi_1-\chi_{l+1})$$
$$\vdots$$
$$\partial^{r_{j-j_l+1}}_{\xi_{l+1}} \Delta^F(\chi_l-\chi_{l+1}) \cdots \partial^{r_j}_{\xi_{l+1}} \Delta^F(\chi_l-\chi_{l+1}).$$

Then we consider the product of exponentials of Feynman propagators and expand it as a single formal power series in $\hbar$. The coefficient of the power $\hbar^p$ is given by:

$$\sum_{\substack{\{p_{i,j}\geq 0,\, 1\leq i<j\leq l+1 \\ \text{s.t. } \sum_{i,j} p_{i,j}=p\}}} \frac{(-1)^p (a_1a_2)^{p_{1,2}} \cdots (a_l a_{l+1})^{p_{l,l+1}}}{p_{1,2}! \cdots p_{l,l+1}!}$$
$$(\Delta^F)^{p_{1,2}}(\chi_1-\chi_2) \cdots (\Delta^F)^{p_{l,l+1}}(\chi_l-\chi_{l+1}).$$

Substituting this expression in the previous formula, we obtain that the coefficient of $\hbar^p$ of the unrenormalized time-ordered product of vertex operators multiplied with derivatives of Feynman propagators is given by:

$$\begin{aligned} \sum_{\substack{\{p_{i,j}\geq 0,\, 1\leq i<j\leq l+1 \\ \text{s.t. } \sum_{i,j} p_{i,j}=p\}}} & \frac{(-1)^p (a_1a_2)^{p_{1,2}} \cdots (a_l a_{l+1})^{p_{l,l+1}}}{p_{1,2}! \cdots p_{l,l+1}!} \\ & (\Delta^F)^{p_{1,2}}(\chi_1-\chi_2) \cdots (\Delta^F)^{p_{l,l+1}}(\chi_l-\chi_{l+1}) \\ & \partial^{r_1}_{\xi_{l+1}} \Delta^F(\chi_1-\chi_{l+1}) \cdots \partial^{r_{j_1}}_{\xi_{l+1}} \Delta^F(\chi_1-\chi_{l+1}) \\ & \vdots \\ & \partial^{r_{j-j_l+1}}_{\xi_{l+1}} \Delta^F(\chi_l-\chi_{l+1}) \cdots \partial^{r_j}_{\xi_{l+1}} \Delta^F(\chi_l-\chi_{l+1}). \end{aligned} \tag{4.10}$$

We now consider each one of the factors separately, as a distribution defined on $\mathbb{M}_2 \setminus \{ 0 \}$, and denote them by:

$$\left.\begin{aligned} D_{1,2} &= (\Delta^F)^{p_{1,2}}, \\ &\vdots \\ D_{1,l+1} &= (\Delta^F)^{p_{1,l+1}} (\partial^{r_1}_{\xi_{l+1}} \Delta^F) \cdots (\partial^{r_{j_1}}_{\xi_{l+1}} \Delta^F), \\ &\vdots \\ D_{l,l+1} &= (\Delta^F)^{p_{l,l+1}} (\partial^{r_{j-j_l+1}}_{\xi_{l+1}} \Delta^F) \cdots (\partial^{r_j}_{\xi_{l+1}} \Delta^F) \end{aligned}\right\} \in \mathscr{D}'(\mathbb{M}_2 \setminus \{0\}). \tag{4.11}$$

Using the fact, see Proposition B.0.1, that each derivative of a distribution possibly increases its scaling degree by 1, the fact that the scaling degree of the product of distributions is bounded by the sum of the scaling degrees of the factors and the fact that the scaling degree of the Feynman propagator on $\mathbb{M}_2$ is 0, we immediately obtain the following values for the scaling degrees of the distributions appearing in formula (4.11) above:

$$\mathrm{sd}\,(D_{i,j}) = 0, \qquad \forall\, 1 \leq i < j \leq l, \tag{4.12}$$

while

$$\begin{aligned} \mathrm{sd}\,(D_{1,l+1}) &= r_1 + \cdots + r_{j_1} \\ &\vdots \\ \mathrm{sd}\,(D_{l,l+1}) &= r_{j-j_l+1} + \cdots + r_j. \end{aligned} \tag{4.13}$$

by the conditions on the indices $r_1, \ldots, r_j$ in formula (4.7). We can now apply Theorem B.0.1 and perform the scaling-degree-preserving extension of each one of the distributions (4.11) to the whole $\mathbb{M}_2$. We denote the results by:

$$\left.\begin{aligned} &[D_{1,2}] = [(\Delta^F)^{p_{1,2}}], \\ &\quad\vdots \\ &[D_{1,l+1}] = [(\Delta^F)^{p_{1,l+1}}(\partial^{r_1}_{\xi_{l+1}}\Delta^F)\cdots(\partial^{r_{j_1}}_{\xi_{l+1}}\Delta^F)], \\ &\quad\vdots \\ &[D_{l,l+1}] = [(\Delta^F)^{p_{l,l+1}}(\partial^{r_{j-j_l+1}}_{\xi_{l+1}}\Delta^F)\cdots(\partial^{r_j}_{\xi_{l+1}}\Delta^F)] \end{aligned}\right\} \in \mathscr{D}'(\mathbb{M}_2). \tag{4.14}$$

Remark 4.2.1. As consequences of Theorem B.0.1 and of the estimates (4.12), (4.13), we have that powers of Feynman propagators admit unique extensions, whereas for products of derivatives of Feynman propagators, when the respective sum of indices r_i's is ≥ 2, the extensions are not unique.

Remark 4.2.2. In view of formula (C.2) for the wavefront set of the Feynman propagator and as a consequence of the extension process, the wavefront set of each element of (4.14) is contained in the set:

$$\begin{aligned} \Gamma_F = &\left\{\, (x,k) \in T^*\mathbb{M}_2 \;\middle|\; (x)^2_\eta = 0,\ x \neq 0,\ x = \lambda\eta^\sharp(k),\ \lambda > 0 \,\right\} \\ &\cup \{\, (0,k) \in T^*\mathbb{M}_2 \mid k \neq 0 \,\}\,. \end{aligned} \tag{4.15}$$

We have thus completed the extension of the elementary parts of the distributional part of the unrenormalized time-ordered product of vertex operators multiplied with derivatives of Feynman propagators.

4.2.2 Antichronological product of vertex operators

We now consider the distributional part of the unrenormalized antichronological product of vertex operators, that is, the product of exponentials of anti-Feynman propagators appearing in formula (4.8), as a distribution defined on $\breve{\mathbb{M}}_2^{n-l}$:

$$\check{\prod_{l+2\leq i<j\leq n+1}} e^{-a_i a_j \hbar \Delta^{AF}(\chi_i-\chi_j)}. \tag{4.16}$$

Analogously to what we did for the distributional part of the unrenormalized time-ordered product of vertex operators, we expand the product of exponentials of anti-Feynman propagators as a single formal power series in $\hbar$. The coefficient of $\hbar^q$ is given by:

$$\sum_{\substack{\{q_{i,j}\geq 0,\, l+2\leq i<j\leq n+1 \\ \text{s.t. } \sum_{i,j} q_{i,j}=q\}}} \frac{(-1)^q (a_{l+2}a_{l+3})^{q_{l+2,l+3}} \cdots (a_n a_{n+1})^{q_{n,n+1}}}{q_{l+1,l+2}! \cdots q_{n,n+1}!} (\Delta^{AF})^{q_{l+2,l+3}}(\chi_{l+2}-\chi_{l+3}) \cdots (\Delta^{AF})^{q_{n,n+1}}(\chi_n-\chi_{n+1}). \tag{4.17}$$

We regard each one the factors in this formula as distributions defined on $\mathbb{M}_2 \backslash \{ 0 \}$ and denote them as:

$$\left.\begin{aligned} \overline{D}_{l+2,l+3} &= (\Delta^{AF})^{q_{l+2,l+3}} \\ &\vdots \\ \overline{D}_{n,n+1} &= (\Delta^{AF})^{q_{n,n+1}} \end{aligned}\right\} \in \mathscr{D}'(\mathbb{M}_2 \setminus \{ 0 \}).$$

The scaling degree of the anti-Feynman propagator, and consequently of any of its powers, is equal to 0 on $\mathbb{M}_2$. Hence by Theorem B.0.1 each on of the distributions above admits a unique scaling-degree-preserving extension to the whole $\mathbb{M}_2$. We denote them by:

$$\left.\begin{aligned} [\overline{D}_{l+2,l+3}] &= [(\Delta^{AF})^{q_{l+2,l+3}}] \\ &\vdots \\ [\overline{D}_{n,n+1}] &= [(\Delta^{AF})^{q_{n,n+1}}] \end{aligned}\right\} \in \mathscr{D}'(\mathbb{M}_2). \tag{4.18}$$

Remark 4.2.3. In view of formula (C.5) for the wavefront set of the anti-Feynman propagator and as a consequence of the extension process, we have that the wavefront set of each element of (4.18) is contained in the set:

$$\begin{aligned} \Gamma_{AF} = \Big\{ (x,k) \in T^*\mathbb{M}_2 \;\Big|\; (x)^2_\eta = 0,\ x \neq 0,\ x = \lambda \eta^\sharp(k),\ \lambda < 0 \Big\} \cup \\ \cup \{ (0,k) \in T^*\mathbb{M}_2 \mid k \neq 0 \}. \end{aligned} \tag{4.19}$$

We have thus completed the extension of the elementary parts of the distributional part of the unrenormalized antichronological product of vertex operators.

4.2.3 Discussion of well-posedness

We proceed now with the last step of the piecewise renormalization process, as explained at the beginning of Section 4.2. In particular, we first show that the piecewise renormalized versions of (4.10) and of (4.17) are well-defined distributions on the whole spaces $\mathbb{M}_2^{l+1}$ and $\mathbb{M}_2^{n-l}$, respectively. Then, by summing up the coefficients for every order in $\hbar$, we directly obtain that the piecewise renormalized versions of formulas (4.9) and (4.16) are also well defined on $\mathbb{M}_2^{l+1}$ and $\mathbb{M}_2^{n-l}$, respectively. Finally, we show the well-posedness of the product of the piecewise renormalized versions with the exponentials of two-point functions and their derivatives, hence obtaining that the piecewise renormalized version of formula (4.8) is well defined on $\mathbb{M}_2^{n+1}$. This in the end implies that the piecewise renormalized version, obtained in the way just described, of the generic term (4.5) of $\check{R}_n\left(L_{\text{int}}^{\otimes n}, \sin(a\phi) r_1^N\right)$ is a well-defined microcausal field in $\mathscr{D}'(\mathbb{M}_2^{n+1}; \mathcal{F}_{\mu c})[\![\hbar]\!]$, as desired.

Remark 4.2.4. Before proceeding with the proofs of our claims, we point out that all the arguments presented involve only the study of the wavefront sets of certain distributions. The wavefront set is a geometric object, independent of the choice of coordinate system employed to describe it. As a consequence, despite the fact that in the previous formulas the distributions under consideration were defined using light-cone coordinates, we are in fact free to consider also their expressions in cartesian coordinates.

Proposition 4.2.1. *The piecewise renormalized version of* (4.10)*, that is, of the coefficient of $\hbar^p$ of the distributional part of the unrenormalized time-ordered product of vertex operators multiplied with derivatives of Feynman propagators, given by the following expression:*

$$\begin{aligned}\sum_{\substack{\{p_{i,j}\geq 0,\, 1\leq i<j\leq l+1 \\ s.t.\ \sum_{i,j} p_{i,j}=p\}}} & \frac{(-1)^p (a_1 a_2)^{p_{1,2}} \cdots (a_l a_{l+1})^{p_{l,l+1}}}{p_{1,2}! \cdots p_{l,l+1}!} \\ & [(\Delta^F)^{p_{1,2}}](x_1 - x_2) \cdots [(\Delta^F)^{p_{1,l}}](x_1 - x_l) \cdots [(\Delta^F)^{p_{l-1,l}}](x_{l-1} - x_l) \\ & [(\Delta^F)^{p_{1,l+1}} (\partial_{\xi_{l+1}}^{r_1} \Delta^F) \cdots (\partial_{\xi_{l+1}}^{r_{j_1}} \Delta^F)](x_1 - x_{l+1}) \\ & \vdots \\ & [(\Delta^F)^{p_{l,l+1}} (\partial_{\xi_{l+1}}^{r_{j-j_l+1}} \Delta^F) \cdots (\partial_{\xi_{l+1}}^{r_j} \Delta^F)](x_l - x_{l+1}),\end{aligned} \tag{4.20}$$

is a well-defined distribution on $\mathbb{M}_2^{l+1}$.

<u>**Notation.**</u> We recall here a graph notation, introduced in [8] to describe the wavefront set of products of Feynman propagators in the context of algebraic

quantum field theory on curved spacetimes. Following the terminology of [7] and adapting it to our spacetime $\mathbb{M}_2$, we have:

- denote by $\mathcal{G}_n$ the set of oriented graphs with vertices $V = \{ 1, \dots, n \}$, and by E^G the set of edges of a given graph $G \in \mathcal{G}_n$. For any edge $e \in E^G$ between vertices $i < j$, we set source $\sigma(e) = i$ and target $\tau(e) = j$;

- an immersion of the graph $G \in \mathcal{G}_n$ into $\mathbb{M}_2$ is pair of maps $(\mathcal{X}, \mathcal{K})$ such that:
  - $\mathcal{X} \colon V \to \mathbb{M}_2$ maps vertices i of G to points $x_i \in \mathbb{M}_2$, with the condition that if the vertices $i < j$ are connected by an edge, then $(x_i - x_j)^2_\eta = \eta(x_i - x_j, x_i - x_j) = 0$;
  - $\mathcal{K} \colon E^G \to T^*\mathbb{M}_2$ with the condition that, if the vertices $i < j$ are connected by the edge $e \in E^G$, then the covector $\mathcal{K}(e) = k_e$ is:

$$\begin{cases} k_e = \lambda_{ij} \eta_\flat (x_i - x_j) & \text{for some} \quad \lambda_{ij} > 0, \qquad \text{if} \quad x_i \neq x_j, \\ k_e \in \mathbb{M}_2^* \setminus \{ 0 \}, & \text{if} \quad x_i = x_j, \end{cases}$$

  where $\eta_\flat \colon T\mathbb{M}_2 \to T^*\mathbb{M}_2$ is the natural isomorphism induced by the Minkowski metric η and due to the triviality of the bundles $T\mathbb{M}_2 \equiv \mathbb{M}_2 \times \mathbb{M}_2$, $T^*\mathbb{M}_2 \equiv \mathbb{M}_2 \times \mathbb{M}_2^*$ it descends to an isomorphism $\eta_\flat \colon \mathbb{M}_2 \to \mathbb{M}_2^*$. By convention, the covector k_e is said to be outgoing for the point x_i and incoming for the point x_j.

Proof of Proposition 4.2.1. We start introducing the following tensor product:

$$\begin{aligned} \mathrm{TOF}^p_{l+1} = & \sum_{\substack{\{p_{i,j} \geq 0,\, 1 \leq i < j \leq l+1 \\ \text{s.t. } \sum_{i,j} p_{i,j} = p\}}} \frac{(-1)^p (a_1 a_2)^{p_{1,2}} \cdots (a_l a_{l+1})^{p_{l,l+1}}}{p_{1,2}! \cdots p_{l,l+1}!} \\ & [(\Delta^F)^{p_{1,2}}](w_{1,2}) \otimes \cdots \otimes [(\Delta^F)^{p_{1,l}}](w_{1,l}) \otimes \cdots \otimes [(\Delta^F)^{p_{l-1,l}}](w_{l-1,l}) \\ & \otimes [(\Delta^F)^{p_{1,l+1}} (\partial^{r_1}_{\xi_{l+1}} \Delta^F) \cdots (\partial^{r_{j_1}}_{\xi_{l+1}} \Delta^F)](w_{1,l+1}) \otimes \cdots \\ & \cdots \otimes [(\Delta^F)^{p_{l,l+1}} (\partial^{r_{j-j_l+1}}_{\xi_{l+1}} \Delta^F) \cdots (\partial^{r_j}_{\xi_{l+1}} \Delta^F)](w_{l,l+1}), \end{aligned} \tag{4.21}$$

as a distribution in $\mathscr{D}'(\mathbb{M}_2^K)$, with $K = \binom{l+1}{2}$. We point out that the notation in the last formula is meant to indicate that we are considering the distributions $[(\Delta^F)^{p_{i,j}}]$ or $[(\Delta^F)^{p_{i,l+1}} (\partial^{r_{j_1}}_{\xi_{l+1}} \Delta^F) \cdots (\partial^{r_{j_i}}_{\xi_{l+1}} \Delta^F)]$, originally defined using light-cone coordinates, in cartesian coordinates $w_{i,j}$ on $\mathbb{M}_2^K$.

The crucial observation is that we can regard (4.20) (considered in cartesian coordinates) as the result of the pull-back of (4.21) via the map:

$$\begin{array}{rccc} s: & \mathbb{M}_2^{l+1} & \to & \mathbb{M}_2^K \\ & (x_1,\dots,x_{l+1}) & \mapsto & (w_{i,j}=x_i-x_j), \end{array} \tag{4.22}$$

for $1 \le i < j \le l+1$. The question about the well-posedness of (4.20) is then rephrased in terms of the well-posedness of the pull-back $s^*(\mathrm{TOF}^p_{l+1})$. In other words, according to Theorem A.0.1, we can consider the set

$$\Lambda_{l+1} = \mathrm{WF}\left(s^*(\mathrm{TOF}^p_{l+1})\right) = (s')^t\big(\mathrm{WF}(\mathrm{TOF}^p_{l+1})\big),$$

and we have to verify whether the condition:

$$\Lambda_{l+1} \cap \left(\mathbb{M}_2 \times \{\,0\,\}\right)^{l+1} = \emptyset \tag{4.23}$$

is satisfied. Using the graph notation introduced above, we have that Λ_{l+1} can be described as:

$$\begin{aligned} \Lambda_{l+1} = \Big\{ & (x_1,k_1,\dots,x_{l+1},k_{l+1}) \in T^*\mathbb{M}_2^{l+1} \mid \exists\, G \in \mathcal{G}_{l+1} \quad \text{and} \\ & \exists \quad \text{an immersion} \quad (\mathcal{X},\mathcal{K}) \quad \text{of} \quad G \quad \text{such that} \\ & k_i = \sum_{\substack{e\in E^G \\ \sigma(e)=i}} k_e - \sum_{\substack{f\in E^G \\ \tau(f)=i}} k_f \Big\}. \end{aligned}$$

Analyzing in detail the implications of this notation, we can show that the answer to the question whether condition (4.23) holds is indeed affirmative:

Consider first immersed graphs in $\mathbb{M}_2$ with no loops, that is, suppose that the immersion map $\mathcal{X}$ is injective. In this case the claim follows from the following argument. For every immersed vertex x_i, the corresponding covector k_i is given by a sum of covectors which are coparallel to the directions of connection of the vertex to its adjacent vertices in the immersed graph. The directions of the connections always lie on the boundary of the light-cone.

This means that, in order to have all covectors k_i equal to zero, every vertex x_i of the immersed graph has to be connected to its adjacent vertices in opposite symmetric directions. But this can never be the case. In fact, each connected component of every immersed graph has a finite number of vertices and if we consider for example, in a connected component, the vertex $\bar{x}$ with maximum time coordinate, then this vertex will be connected to its adjacent vertices only in past-directed directions. Hence the covectors over $\bar{x}$ cannot sum up to zero.

Suppose now that the immersed graph contains loops, that is, that the immersion $\mathcal{X}$ maps vertices $I = \{i_1,\dots,i_m\} \subseteq \{1,\dots,l+1\}$, $m \le l+1$, to the same point $x_I \in \mathbb{M}_2$. Let us denote by E_I^G the set of loops, that is, E_I^G is the subset of edges $e \in E^G$ such that $\sigma(e) \in I$ and $\tau(e) \in I$.

Then the conditions that the covectors $k_{i_1}, \ldots, k_{i_m}$ over the points $x_{i_1} = \cdots = x_{i_m} = x_I$ are all equal to zero can be written as:

$$\begin{cases} k_{i_1} & = \sum_{\substack{e \in E^G \setminus E_I^G \\ \sigma(e) = i_1}} k_e - \sum_{\substack{f \in E^G \setminus E_I^G \\ \tau(f) = i_1}} k_f + \sum_{\substack{e \in E_I^G \\ \sigma(e) = i_1}} k_e - \sum_{\substack{f \in E_I^G \\ \tau(f) = i_1}} k_f = 0 \\ & \vdots \\ k_{i_m} & = \sum_{\substack{e \in E^G \setminus E_I^G \\ \sigma(e) = i_m}} k_e - \sum_{\substack{f \in E^G \setminus E_I^G \\ \tau(f) = i_m}} k_f + \sum_{\substack{e \in E_I^G \\ \sigma(e) = i_m}} k_e - \sum_{\substack{f \in E_I^G \\ \tau(f) = i_m}} k_f = 0. \end{cases}$$

From these equations we see that each one of the covectors k_e associated to an edge $e \in E_I^G$ appears twice, with opposite signs. If we sum up the equations above, we are then left with the condition:

$$k_I = k_{i_1} + \cdots + k_{i_m} = \sum_{\substack{e \in E^G \setminus E_I^G \\ \sigma(e) \in I}} k_e - \sum_{\substack{f \in E^G \setminus E_I^G \\ \tau(f) \in I}} k_f = 0.$$

This corresponds to the condition that we get if we look at the immersed graph G, without considering the loops. We are then reduced to the situation discussed above and we can apply the same argument to conclude. *q.e.d.*

Completing the characterization of the piecewise renormalized coefficient of $\hbar^p$ of the distributional part of the time-ordered product of vertex operators multiplied with derivatives of Feynman propagators, formula (4.20), we have the following result.

Proposition 4.2.2. *The wavefront set Λ_{l+1} of (4.20), that is, of the piecewise renormalized version of the coefficient of $\hbar^p$ of the distributional part of the unrenormalized time-ordered product of vertex operators multiplied with derivatives of Feynman propagators, satisfies the microlocal condition (2.4), that is,:*

$$\Lambda_{l+1} \cap \left(\left(\mathbb{M}_2 \times \overline{V}_-\right)^{l+1} \cup \left(\mathbb{M}_2 \times \overline{V}_+\right)^{l+1} \right) = \emptyset.$$

Proof. For each connected component of each immersed graph, we have a vertex $\bar{x}_+$ with maximum time coordinate and another vertex $\bar{x}_-$ with minimum time coordinate. This means that $\bar{x}_+$ is connected to its adjacent vertices only by past-directed directions, and hence the covector over $\bar{x}_+$ is past-directed. Conversely the vertex $\bar{x}_-$ is connected to its adjacent vertices only by future-directed directions, and hence the covector over $\bar{x}_-$ is future-directed. This is sufficient to obtain the statement.

This situation is not affected by the presence of loops at the vertices $\bar{x}_+$ or $\bar{x}_-$. In fact, suppose that $\bar{x}_+$ is the image via the immersion map $\mathcal{X}$ of the vertices $I := \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, l+1\}$, $m \leq l+1$. Then, similarly as in the proof of Proposition 4.2.1, we have that the covectors over the immersed vertices $x_{i_1}, \ldots, x_{i_m}$ can be summed up to give:

$$k_+ = k_{i_1} + \cdots + k_{i_m} = \sum_{\substack{e \in E^G \setminus E^G_I \\ \sigma(e) \in I}} k_e - \sum_{\substack{f \in E^G \setminus E^G_I \\ \tau(f) \in I}} k_f,$$

which is precisely the expression that we get if we look at the immersed graph G, without considering the loops.

If we now assume that all covectors belong to $\overline{V}_+$, then also $k_+ \in \overline{V}_+$. But this is a contradiction, because from the first argument we know that for immersed graphs without loops the covector k_+ over $\bar{x}_+$ must belong to $\overline{V}_-$.

If we assume, on the contrary, that all covectors belong to $\overline{V}_-$ and repeat the previous reasoning for $\bar{x}_-$, we get a contradiction since we know that for immersed graphs without loops the covector over $\bar{x}_-$ must belong to $\overline{V}_+$. *q.e.d.*

Corollary 4.2.1. *The piecewise renormalized version of the distributional part of the unrenormalized time-ordered product of vertex operators multiplied with derivatives of Feynman propagators (compare with formula (4.9)), which we denote, dropping the ˇ symbol, by:*

$$\prod_{1 \leq i < j \leq l+1} e^{-a_i a_j \hbar \Delta^F(x_i - x_j)} \partial^{r_1}_{\xi_{l+1}} \Delta^F(x_1 - x_{l+1}) \cdots \partial^{r_j}_{\xi_{l+1}} \Delta^F(x_l - x_{l+1}), \tag{4.24}$$

is a well-defined distribution on $\mathbb{M}_2^{l+1}$ with wavefront set given by Λ_{l+1}.

Proof. The result follows directly from summing up the piecewise renormalized coefficients (4.20) for every order p in $\hbar$. Concerning the statement about the wavefront set, this follows immediately from point (a) of Proposition A.0.2 combined with the fact that the wavefront set of the coefficients (4.20) for every order p is always Λ_{l+1}. *q.e.d.*

Proposition 4.2.3. *The piecewise renormalized version of the coefficient of $\hbar^q$ (compare with formula (4.17)) of the distributional part of the unrenormalized antichronological product of vertex operators, given by the following expression:*

$$\sum_{\substack{\{q_{i,j} \geq 0,\, l+2 \leq i < j \leq n+1 \\ s.t.\ \sum_{i,j} q_{i,j} = q\}}} \frac{(-1)^q (a_{l+2} a_{l+3})^{q_{l+2,l+3}} \cdots (a_n a_{n+1})^{q_{n,n+1}}}{q_{l+1,l+2}! \cdots q_{n,n+1}!} [(\Delta^{AF})^{q_{l+2,l+3}}](x_{l+2} - x_{l+3}) \cdots [(\Delta^{AF})^{q_{n,n+1}}](x_n - x_{n+1}). \tag{4.25}$$

is a well-defined distribution on $\mathbb{M}_2^{n-l}$.

Proof. We repeat the same steps as in the proof of Proposition 4.2.1, with the proper modifications. We start introducing the tensor product

$$\mathrm{ACV}^q_{n-l} = \sum_{\substack{\{q_{i,j}\geq 0,\, l+2\leq i<j\leq n+1 \\ \text{s.t. } \sum_{i,j} q_{i,j}=q\}}} \frac{(-1)^q (a_{l+2}a_{l+3})^{q_{l+2,l+3}} \cdots (a_n a_{n+1})^{q_{n,n+1}}}{q_{l+2,l+3}! \cdots q_{n,n+1}!} \\ [(\Delta^{AF})^{q_{l+2,l+3}}](w_{l+2,l+3}) \otimes \cdots \otimes [(\Delta^{AF})^{q_{n,n+1}}](w_{n,n+1}), \tag{4.26}$$

as a distribution defined on $\mathbb{M}_2^{\tilde{K}}$, $\tilde{K} = \binom{n-l}{2}$. Again, we stress that we are adopting the same convention on the use of cartesian coordinates as in the proof of Proposition 4.2.1. Then we can regard (4.25) as the result of the pull-back of (4.26) via the map:

$$\begin{aligned} \tilde{s}: \qquad \mathbb{M}_2^{n-l} \quad &\rightarrow \quad \mathbb{M}_2^{\tilde{K}} \\ (x_{l+2}, \dots, x_{n+1}) &\mapsto (w_{i,j} = x_i - x_j), \end{aligned}$$

for $l+2 \leq i < j \leq n+1$. The question about the well-posedness of (4.25) is then rephrased in terms of the well-posedness of the pull-back $\tilde{s}^* \left(\mathrm{ACV}^q_{n-l}\right)$. In other words, according to Theorem A.0.1, we can consider the set

$$\tilde{\Lambda}_{n-l} = \mathrm{WF}\left(\tilde{s}^* \left(\mathrm{ACV}^q_{n-l}\right)\right) = (\tilde{s}\,')^t \big(\mathrm{WF}(\mathrm{ACV}^q_{n-l})\big)$$

and we have to verify the condition:

$$\tilde{\Lambda}_{n-l} \cap \left(\mathbb{M}_2 \times \{\,0\,\}\right)^{n-l} = \emptyset. \tag{4.27}$$

The set $\tilde{\Lambda}_{n-l}$ can be described by slightly adapting the graph notation introduced before. Considering the form of the wavefront set of the anti-Feynman propagator, formula (C.5), we simply have to change the prescription in the definition of immersion $(\tilde{\mathcal{X}}, \tilde{\mathcal{K}})$ of a graph by a sign:

$$\begin{cases} \tilde{k}_e = -\lambda_{ij}\eta_\flat(x_i - x_j) \quad \text{for some} \quad \lambda_{ij} > 0, \qquad \text{if} \quad x_i \neq x_j, \\ \tilde{k}_e \in \mathbb{M}_2 \setminus \{\,0\,\}, \qquad \text{if} \quad x_i = x_j. \end{cases}$$

We have then:

$$\begin{aligned} \tilde{\Lambda}_{n-l} = \Big\{ &(x_{l+2}, k_{l+2}, \dots, x_{n+1}, k_{n+1}) \in T^*\mathbb{M}_2^{n-l} \mid \exists\, G \in \mathcal{G}_{n-l} \quad \text{and} \\ &\exists \quad \text{an immersion} \quad (\tilde{\mathcal{X}}, \tilde{\mathcal{K}}) \quad \text{of} \quad G \quad \text{such that} \\ &k_i = \sum_{\substack{e\in E^G \\ \sigma(e)=i}} \tilde{k}_e - \sum_{\substack{f\in E^G \\ \tau(f)=i}} \tilde{k}_f \Big\}. \end{aligned}$$

This modification does not affect the validity of the arguments in the proof of Proposition 4.2.1, whose steps can be repeated to obtain the desired result. *q.e.d.*

Also the steps in the proofs of Proposition 4.2.2 and of Corollary 4.2.1 can be straightforwardly repeated to obtain the following results.

Proposition 4.2.4. *The wavefront set $\tilde{\Lambda}_{n-l}$ of (4.25), that is, of the piecewise renormalized version of the coefficient of $\hbar^q$ of the distributional part of the unrenormalized antichronological product of vertex operators, satisfies the microlocal condition (2.4), that is,:*

$$\tilde{\Lambda}_{n-l} \cap \left(\left(\mathbb{M}_2 \times \overline{V}_-\right)^{n-l} \cup \left(\mathbb{M}_2 \times \overline{V}_+\right)^{n-l} \right) = \emptyset.$$

Corollary 4.2.2. *The piecewise renormalized version of the distributional part of the unrenormalized antichronological product of vertex operators (compare with formula (4.16)), which we denote, dropping the ˇ symbol, by:*

$$\prod_{l+2 \leq i < j \leq n+1} e^{-a_i a_j \hbar \Delta^{AF}(x_i - x_j)}, \tag{4.28}$$

is a well-defined distribution on $\mathbb{M}_2^{n-l}$ with wavefront set given by $\tilde{\Lambda}_{n-l}$.

It remains now to consider the exponentials of two-point functions and their derivatives appearing in formula (4.8). Adopting again the convention of using cartesian coordinates for distributions originally defined using light-cone coordinates, we write them as:

$$\prod_{\substack{l+2 \leq i \leq n+1 \\ 1 \leq j \leq l+1}} e^{-a_i a_j \hbar W(x_i - x_j)} \left(\partial^{s_1}_{\xi_{l+1}} W\right)(x_{l+2} - x_{l+1}) \cdots \left(\partial^{s_k}_{\xi_{l+1}} W\right)(x_{n+1} - x_{l+1}). \tag{4.29}$$

We point out that, due to the form of the wavefront set of the two-point function, formula (2.6), these distributional products are always well-defined according to Hörmander's criterion. Hence no renormalization is needed. We can omit the ˇ symbol over the product and consider (4.29) directly as a distribution on $\mathbb{M}_2^{n+1}$.

In order to prove that the product of (4.29) with (4.24) and (4.28) is well-defined, we want to estimate its wavefront set. Once more, we can do this by means of the graph notation introduced above. In view of formula (2.6) for the wavefront set of W, it suffices to modify the definition of an immersion of a graph in the following way: for vertices $1 \leq j < i \leq n+1$ connected by an edge e, we set source $\sigma(e) = i$ and target $\tau(e) = j$, and define an immersion $(\hat{\mathcal{X}}, \hat{\mathcal{K}})$ by setting

$$\begin{cases} \hat{k}_e = \lambda_{ij} \eta_\flat (x_i - x_j), \ \lambda_{ij} \in \mathbb{R} \quad \text{s.t.} \quad (\hat{k}_e)^2_\eta = 0 \text{ and } \hat{k}^0_e > 0, \quad \text{if} \quad x_i \neq x_j, \\ (\hat{k}_e)^2_\eta = 0 \text{ and } \hat{k}^0_e > 0, \qquad \text{if} \quad x_i = x_j. \end{cases}$$

We then obtain the following description for the wavefront set Ω_{n+1} of (4.29):

$$\Omega_{n+1} = \Big\{(x_1, k_1, \ldots, x_{n+1}, k_{n+1}) \in T^*\mathbb{M}_2^{n+1} \mid \exists\, G \in \mathcal{G}_{n+1} \quad \text{and} \quad \exists \quad \text{an immersion} \quad (\hat{\mathcal{X}}, \hat{\mathcal{K}}) \quad \text{of} \quad G \quad \text{such that} \quad k_i = \sum_{\substack{e\in E^G \\ \sigma(e)=i}} \hat{k}_e - \sum_{\substack{f\in E^G \\ \tau(f)=i}} \hat{k}_f \Big\}.$$

Remark 4.2.5. Considering how the coordinates $(x_1, \ldots, x_{n+1})$ are distributed in formula (4.29), we see immediately that the vertices $\{ x_{l+2}, \ldots, x_{n+1} \}$ only have outgoing edges. Conversely the vertices $\{ x_1, \ldots, x_{l+1} \}$ only have incoming edges. This means that the wavefront set Ω_{n+1} of (4.29) is actually contained in the following set:

$$\begin{aligned}\Omega_{n+1} \subseteq \Big\{&(x_1, k_1, \ldots, x_{l+1}, k_{l+1}, x_{l+2}, k_{l+2}, \ldots, x_{n+1}, k_{n+1}) \in T^*\mathbb{M}_2^{n+1} \\ &\text{s.t.} \quad k_1, \ldots, k_{l+1} \in \overline{V}_- \quad \text{and} \quad k_{l+2}, \ldots, k_{n+1} \in \overline{V}_+ \Big\}.\end{aligned} \tag{4.30}$$

We have thus collected all the elements necessary to finally prove the following result.

Proposition 4.2.5. *The piecewise renormalized version of the distributional part of the unrenormalized generic term of* $\check{R}_n\big(L_{int}^{\otimes n}, \sin(a\phi) r_1^N\big)$ *(compare with formula* (4.8)*), which we denote, dropping all the* ˇ *symbols and using cartesian coordinates, by:*

$$\begin{aligned}&\prod_{1\leq i<j\leq l+1} e^{-a_i a_j \hbar \left(\Delta^F\right)(x_i - x_j)} \partial^{r_1}_{\xi_{l+1}} \Delta^F(x_1 - x_{l+1}) \cdots \partial^{r_j}_{\xi_{l+1}} \Delta^F(x_l - x_{l+1}) \\ &\prod_{l+2\leq i<j\leq n+1} e^{-a_i a_j \hbar \Delta^{AF}(x_i - x_j)} \prod_{\substack{l+2\leq i\leq n+1 \\ 1\leq j\leq l+1}} e^{-a_i a_j \hbar W(x_i - x_j)} \\ &\partial^{s_1}_{\xi_{l+1}} W(x_{l+2} - x_{l+1}) \cdots \partial^{s_k}_{\xi_{l+1}} W(x_{n+1} - x_{l+1}),\end{aligned} \tag{4.31}$$

is a well-defined distribution on $\mathbb{M}_2^{n+1}$ *and its wavefront set satisfies the microlocal condition.*

Proof. We regard formula (4.31) as the product of the distribution

$$\begin{aligned}&\left(\prod_{1\leq i<j\leq l+1} e^{-a_i a_j \hbar \left(\Delta^F\right)(x_i - x_j)} \partial^{r_1}_{\xi_{l+1}} \Delta^F(x_1 - x_{l+1}) \cdots \partial^{r_j}_{\xi_{l+1}} \Delta^F(x_l - x_{l+1})\right) \\ &\otimes \left(\prod_{l+2\leq i<j\leq n+1} e^{-a_i a_j \hbar \Delta^{AF}(x_i - x_j)}\right),\end{aligned} \tag{4.32}$$

seen as a distribution on $\mathbb{M}_2^{n+1}$, with the distribution

$$\prod_{\substack{l+2\leq i\leq n+1\\ 1\leq j\leq l+1}} e^{-a_i a_j \hbar W(x_i - x_j)} \partial^{s_1}_{\xi_{l+1}} W(x_{l+2} - x_{l+1}) \cdots \partial^{s_k}_{\xi_{l+1}} W(x_{n+1} - x_{l+1}), \quad (4.33)$$

also seen as a distribution on $\mathbb{M}_2^{n+1}$. From formula (4.30), we already know an explicit estimate on the wavefront set of (4.33). By the properties of the wavefront set with respect to the tensor product of distributions, see Theorem A.0.2, the wavefront set of (4.32) is contained in the set:

$$\begin{aligned}\Lambda_{l+1,n-l} = \Big(\Lambda_{l+1} \times \tilde{\Lambda}_{n-l}\Big) &\cup \Big(\Lambda_{l+1} \times \big(\mathbb{M}_2 \times \{0\}\big)^{n-l}\Big)\\ &\cup \Big(\big(\mathbb{M}_2 \times \{0\}\big)^{l+1} \times \tilde{\Lambda}_{n-l}\Big).\end{aligned}$$

Moreover, Proposition 4.2.2 and Proposition 4.2.4 imply:

$$\begin{aligned}\Lambda_{l+1,t-l} \cap \bigg(\Big(\big(\mathbb{M}_2 \times \overline{V}_-\big)^{l+1} \cup \big(\mathbb{M}_2 \times \overline{V}_+\big)^{l+1}\Big)\\ \times \Big(\big(\mathbb{M}_2 \times \overline{V}_-\big)^{n-l} \cup \big(\mathbb{M}_2 \times \overline{V}_+\big)^{n-l}\Big)\bigg) = \emptyset.\end{aligned} \quad (4.34)$$

According to Hörmander's sufficent criterion, the distributional product of (4.32) and (4.33) is well-defined if the set

$$\begin{aligned}\Lambda_{l+1,n-l} + \Omega_{n+1} = \big\{&(x_1, p_1 + q_1, \ldots, x_{n+1}, p_{n+1} + q_{n+1}) \in T^*\mathbb{M}_2^{n+1} \mid\\ &(x_1, p_1, \ldots, x_{n+1}, p_{n+1}) \in \Lambda_{l+1,n-l},\\ &\text{and } (x_1, q_1, \ldots, x_{n+1}, q_{n+1}) \in \Omega_{n+1}\big\}\end{aligned}$$

does not contain the null covector. Comparing formulas (4.30) and (4.34), we get immediately that this can never be the case, that is:

$$\big(\Lambda_{l+1,n-l} + \Omega_{n+1}\big) \cap \big(\mathbb{M}_2 \times \{\,0\,\}\big)^{n+1} = \emptyset.$$

Hence Hörmander's sufficient criterion is satisfied and consequently $\Lambda_{l+1,n-l} + \Omega_{n+1}$ is the wavefront set of (4.31). Finally, using again formulas (4.30) and (4.34), we obtain also the microlocal condition:

$$\big(\Lambda_{l+1,n-l} + \Omega_{n+1}\big) \cap \Big(\big(\mathbb{M}_2 \times \overline{V}_-\big)^{n+1} \cup \big(\mathbb{M}_2 \times \overline{V}_+\big)^{n+1}\Big) = \emptyset.$$

q.e.d.

Corollary 4.2.3. *The piecewise renormalized version of the unrenormalized retarded product* $\check{R}_n\big(L_{int}^{\otimes n}, \sin(a\phi) r_1^N\big)$*, which we denote, dropping the* ˇ *symbol, by:*

$$R_n\big(L_{int}^{\otimes n}, \sin(a\phi) r_1^N\big),$$

is a microcausal field $\mathscr{D}'(\mathbb{M}_2^{n+1}; \mathcal{F}_{\mu c})[\![\hbar]\!]$ *according to Definition 2.2.6.*

Proof. The claim follows straightforwardly applying Proposition 4.2.5 to all the generic terms of the form (4.8) and then summing their piecewise renormalized versions up altogether. *q.e.d.*

Remark 4.2.6. This concludes the discussion of the piecewise renormalization of the unrenormalized interacting components of the higher currents of the sine-Gordon model. In the sequel we will always consider the piecewise renormalized expressions, referring to them simply as the interacting components and denoting them, omitting the ˇ symbols, by:

$$(s_1^N)_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} R_n\big(L_{\text{int}}^{\otimes n}, s_1^N\big) \in \mathscr{D}'(\mathbb{M}_2^{n+1}; \mathcal{F}_{\mu c})[\![\kappa, \hbar]\!], \tag{4.35}$$

$$(s_2^N)_{\text{int}} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} R_n\big(L_{\text{int}}^{\otimes n}, s_2^N\big) \in \mathscr{D}'(\mathbb{M}_2^{n+1}; \mathcal{F}_{\mu c})[\![\kappa, \hbar]\!]. \tag{4.36}$$

Chapter 5

Summability of the renormalized interacting currents

In this chapter we establish the summability of the interacting components $(s_1^N)_{\text{int}}$ and $(s_2^N)_{\text{int}}$ of the higher currents of the sine-Gordon model. We proceed analogously to the discussion of the summability of the S-matrix developed in [1]. We start by recalling the setting for the latter result.

Definition 5.0.1. Fix a configuration $\varphi \in \mathscr{E}(\mathbb{M}_2)$. A Gaussian state $\omega_{\varphi,H}$, with covariance given by the Hadamard parametrix H, is a map:

$$\begin{aligned} \omega_{\varphi,H} \colon \mathcal{F}_{\mu c}[\![\kappa,\hbar]\!] &\to \mathbb{C}[\![\kappa,\hbar]\!] \\ F \quad &\mapsto \quad \omega_{\varphi,H}(F) = F[\varphi]. \end{aligned}$$

The choice $\varphi = 0$ is distinguished by the fact that $\omega_{0,H}$ is then the expectation value in the state whose two point function is given by $W = \frac{i}{2}\Delta + H$, according to formula (2.5).

Next, we slightly adapt the notation from [1] and write the renormalized S-matrix of the sine-Gordon model in the form:

$$\begin{aligned} S(\kappa L_{\text{int}}) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\kappa}{\hbar}\right)^n \left(\frac{1}{2}\right)^n T_n\big((V_a + V_{-a})^{\otimes n}\big) \\ &= \sum_{n=0}^{\infty} \kappa^n \underbrace{\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} T_n\big(V_a^{\otimes k} \otimes V_{-a}^{\otimes(n-k)}\big)}_{S_n(L_{\text{int}})} . \end{aligned} \tag{5.1}$$

With this notions at hand, we can then formulate Proposition 6 from [1] in the following way.

Proposition 5.0.1 (**Proposition 6** in [1]). *Let* $\beta = \frac{\hbar a^2}{4\pi} < 1$. *Let* $\gamma > 1$ *such that* $\beta\gamma < 1$. *Let* $g \in \mathscr{D}(\mathbb{M}_2)$ *be a function cutting off the interaction Lagrangian* L_{int}*, and denote* $f = g^{\otimes n} \in \mathscr{D}(\mathbb{M}_2^n)$*. Consider the expectation value of the* n*-th order contribution to the* S*-matrix of the sine-Gordon theory in the state* $\omega_{\varphi,H}$ *with* H *as in formula* (3.2)*. Choosing the support of* g *small enough, there exists a constant* $C = C(p, g)$ *such that for all* n*,*

$$\left|S_n(L_{\text{int}})(f)[\varphi]\right| \leq \frac{[\frac{n}{2}]C^n}{\left([\frac{n}{2}]!\right)^{1-\frac{1}{\gamma}}}, \tag{5.2}$$

where $S_n(L_{\text{int}})$ *is given by* (5.1)*.*

As a direct corollary of this estimate, under the same conditions as above, the expectation value of the S-matrix $S(\kappa L_{\text{int}})$ in the state $\omega_{\varphi,H}$, for every $\varphi \in \mathscr{E}(\mathbb{M}_2)$, is summable.

Remark 5.0.1. We remark that the "bookkeeping" coupling constant κ does not play a rôle in establishing the summability properties of the S-matrix and also it will not play a rôle in the subsequent discussion of the summability properties of the interacting components inasmuch as it is just a tool to account for the order of perturbation, without any real physical meaning.

Following this approach, we will consider the interacting components $(s_1^N)_{\text{int}}$ and $(s_2^N)_{\text{int}}$ of the higher currents of the sine-Gordon model and write them, using formulas (4.35) and (4.36), as:

$$(s_{1,2}^N)_{\text{int}} = \sum_{n=0}^{\infty} \kappa^n \underbrace{\frac{1}{n!} R_n(L_{\text{int}}^{\otimes n}, s_{1,2}^N)}_{\mathcal{R}_n(L_{\text{int}}, s_{1,2}^N)}. \tag{5.3}$$

We will show that, under the same conditions as in Proposition 5.0.1, analogous estimates as (5.2) hold for the expectation values

$$\omega_{\varphi,H}\left(\mathcal{R}_n(L_{\text{int}}, s_{1,2}^N)(f)\right) = \frac{1}{n!} R_n(L_{\text{int}}^{\otimes n}, s_{1,2}^N)(f)[\varphi].$$

Again, as a direct corollary, the expectation values $\omega_{\varphi,H}\left((s_{1,2}^N)_{\text{int}}\right)$ of the interacting components in the state $\omega_{\varphi,H}$, for every $\varphi \in \mathscr{E}(\mathbb{M}_2)$, are summable.

Remark 5.0.2. We observe that the convergence issues concerning the so-called infrared problem for massless scalar fields in 2-dimensional Minkowski spacetime are avoided in the case of the interacting components $(s_1^N)_{\text{int}}$ and $(s_2^N)_{\text{int}}$ due to the fact that they only contain derivatives of order at least one of the field ϕ (for more details, see [1]).

5.1 The main theorem

This section is entirely devoted to the proof of the following Theorem.

Theorem 5.1.1. *Let* $\beta = \frac{\hbar a^2}{4\pi}$ *and let* $\gamma > 1$ *such that* $\beta\gamma < 1$. *Let* $g \in \mathscr{D}(\mathbb{M}_2)$ *be a cut-off function for the interaction Lagrangian* L_{int} *and denote* $f = g^{\otimes(n+1)} \in \mathscr{D}(\mathbb{M}_2^{n+1})$. *Consider the expectation values* $\omega_{\varphi,H}\big(\mathcal{R}_n(L_{\text{int}}, s_{1,2}^N)(f)\big)$ *of the retarded products* $\mathcal{R}_n(L_{\text{int}}, s_{1,2}^N)$ *in the state* $\omega_{\varphi,H}$, *with* H *as in formula* (3.2). *Then, choosing the support of* g *small enough, there exist two pairs of constants* $\mathcal{K}^{s_1}_{\gamma,g,a\hbar,N}$, $\mathcal{C}^{s_1}_{\gamma,g}$ *and* $\mathcal{K}^{s_2}_{\gamma,g,a\hbar,N}$, $\mathcal{C}^{s_2}_{\gamma,g}$ *such that for all* $n \geq 1$ *the following estimates hold:*

$$\Big|\mathcal{R}_n(L_{\text{int}}, s_1^N)(f)[\varphi]\Big| \leq \mathcal{K}^{s_1}_{\gamma,g,a\hbar,N}\frac{(n+1)^2 n^{2N}\big(\mathcal{C}^{s_1}_{\gamma,g}\big)^n}{\big(\big[\frac{n}{2}\big]!\big)^{1-\frac{1}{\gamma}}},$$

$$\Big|\mathcal{R}_n(L_{\text{int}}, s_2^N)(f)[\varphi]\Big| \leq \mathcal{K}^{s_2}_{\gamma,g,a\hbar,N}\frac{\big[\frac{n}{2}\big] n^{2N}\big(\mathcal{C}^{s_2}_{\gamma,g}\big)^n}{\big(\big[\frac{n}{2}\big]!\big)^{1-\frac{1}{\gamma}}},$$

where $\big[\frac{n}{2}\big]$ *denotes the integer part of* $\frac{n}{2}$.

Proof. Once more, we can restrict ourselves to consider only the retarded product

$$\mathcal{R}_n\big(L_{\text{int}}, \sin(a\phi) r_1^N\big), \tag{5.4}$$

from formula (3.10). Indeed, the other term $\mathcal{R}_n\big(L_{\text{int}}, \cos(a\phi) q_1^N\big)$ from the same formula can be treated in a completely analogous way, while the retarded product $\mathcal{R}_n\big(L_{\text{int}}, s_2^N\big)$ can be seen as a special case where no sine or cosine appears.

We start by combining the piecewise renormalized version of formula (4.4) with formula (5.3) to write:

$$\begin{aligned}\mathcal{R}_n\big(L_{\text{int}}, \sin(a\phi) r_1^N\big) = \frac{1}{i2^{n+1}n!}\Big(\frac{i}{\hbar}\Big)^n \sum_{l=0}^{n}\binom{n}{l}\sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{2N}\frac{(-1)^{(n-l)}\hbar^j}{j_1!\cdots j_l!}\sum_{k=0}^{2N-j}\frac{\hbar^k}{k!}\\ \Big\langle \frac{\delta^k}{\delta\phi^k}\bar{T}_{n-l}\big((V_a+V_{-a})^{\otimes(n-l)}\big)\\ \star\, T_{l+1}\left(\frac{\delta^{j_1}}{\delta\phi^{j_1}}(V_a+V_{-a})\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\phi^{j_l}}(V_a+V_{-a})\otimes(V_a-V_{-a})\right),\\ \big((W)^{\otimes k}\otimes(\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}}r_1^N\Big\rangle.\end{aligned}$$

Using the properties of the field derivatives of vertex operators, equation (3.1), and the linearity of the time-ordered and antichronological products, we arrive at:

$$\mathcal{R}_n\big(L_{\text{int}}, \sin(a\phi) r_1^N\big) = \sum_{l=0}^{n} \sum_{r=0}^{n-l} \sum_{s=0}^{l} \sum_{\substack{j=0 \\ j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}}^{2N} \sum_{\substack{k=0 \\ k_1,\dots,k_{n-l}\geq 0 \\ k_1+\dots+k_{n-l}=k}}^{2N-j} \frac{1}{i2^{n+1}} \Big(\frac{i}{\hbar}\Big)^n$$

$$\frac{(-1)^{n-l} \hbar^{j+k} (ia)^{j_1+\dots+j_s} (-ia)^{j_{s+1}+\dots+j_l} (ia)^{k_1+\dots+k_r} (-ia)^{k_{r+1}+\dots+k_{n-l}}}{r!s!(n-l-r)!(l-s)!j_1!\cdots j_l! k_1!\cdots k_{n-l}!}$$

$$\Big\langle \bar{T}_{n-l}\big((V_a)^{\otimes r} \otimes (V_{-a})^{\otimes (n-l-r)}\big)$$

$$\star \Big[T_{l+1}\left((V_a)^{\otimes (s+1)} \otimes (V_{-a})^{\otimes (l-s)}\right) - T_{l+1}\left((V_a)^{\otimes s} \otimes (V_{-a})^{\otimes (l-s+1)}\right)\Big],$$

$$\big((W)^{\otimes k} \otimes (\Delta^F)^{\otimes j}\big) * \frac{\delta^{j+k}}{\delta\phi^{j+k}} r_1^N \Big\rangle.$$

Now we use the piecewise renormalized versions of formulas (4.1), (4.2) and (4.3) to write the time-ordered product of vertex operators, the antichronological product of vertex operators and their star product in exponential form. Moreover, we use (4.6) and (4.8) to expand the field derivatives of r_1^N and the whole distributional pairing. In the end the result is the following expression:

$$\mathcal{R}_n\big(L_{\text{int}}, \sin(a\phi) r_1^N\big) = \sum_{l=0}^{n} \sum_{r=0}^{n-l} \sum_{s=0}^{l} \sum_{\substack{j=0 \\ j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}}^{2N} \sum_{\substack{k=0 \\ k_1,\dots,k_{n-l}\geq 0 \\ k_1+\dots+k_{n-l}=k}}^{2N-j} \frac{1}{i2^{n+1}} \Big(\frac{i}{\hbar}\Big)^n$$

$$\frac{(-1)^{n-l} \hbar^{j+k} (ia)^{j_1+\dots+j_s} (-ia)^{j_{s+1}+\dots+j_l} (ia)^{k_1+\dots+k_r} (-ia)^{k_{r+1}+\dots+k_{n-l}}}{r!s!(n-l-r)!(l-s)!j_1!\cdots j_l! k_1!\cdots k_{n-l}!}$$

$$e^{ia(\phi(\chi_{l+2})+\dots+\phi(\chi_{l+r+1}))} e^{-ia(\phi(\chi_{l+r+2})+\dots+\phi(\chi_{n+1}))} \prod_{l+2\leq p<q\leq n+1} e^{-a_p a_q \hbar \Delta^{AF}(\chi_p - \chi_q)}$$

$$\Big(e^{ia(\phi(\chi_1)+\dots+\phi(\chi_{s+1}))} e^{-ia(\phi(\chi_{s+2})+\dots+\phi(\chi_{l+1}))} \prod_{1\leq p<q\leq l+1} e^{-a_p a_q \hbar \Delta^{F}(\chi_p - \chi_q)}$$

$$- e^{ia(\phi(\chi_1)+\dots+\phi(\chi_s))} e^{-ia(\phi(\chi_{s+1})+\dots+\phi(\chi_{l+1}))} \prod_{1\leq p<q\leq l+1} e^{-a_p a_q \hbar \Delta^{F}(\chi_p - \chi_q)}\Big) \times$$

$$\times \prod_{\substack{l+2\leq p\leq n+1\\ 1\leq q\leq l+1}} e^{-a_p a_q \hbar W(\chi_p-\chi_q)} \Bigg(\sum_{\substack{u_1,\dots,u_k,v_1,\dots,v_j\geq 0\\ u_1+\dots+v_j\leq 2N}} \frac{\partial^{j+k} r_1^N(\chi_{l+1})}{\partial\phi_{u_1\xi}\cdots\partial\phi_{u_k\xi}\partial\phi_{v_1\xi}\cdots\partial\phi_{v_j\xi}}$$
$$\partial^{v_1}_{\xi_{l+1}}\Delta^F(\chi_1-\chi_{l+1})\cdots\partial^{v_j}_{\xi_{l+1}}\Delta^F(\chi_l-\chi_{l+1})$$
$$\partial^{u_1}_{\xi_{l+1}}W(\chi_{l+2}-\chi_{l+1})\cdots\partial^{u_k}_{\xi_{l+1}}W(\chi_{n+1}-\chi_{l+1})\Bigg). \tag{5.5}$$

We pass now to estimate the expectation value of $\mathcal{R}_n\big(L_{\text{int}},\sin(a\phi)r_1^N\big)$ in the state $\omega_{\varphi,H}$ for a generic configuration $\varphi\in\mathscr{E}(\mathbb{M}_2)$. This amounts to evaluating the field-valued distribution (5.5) on the test function $f=g^{\otimes(n+1)}\in\mathscr{D}(\mathbb{M}_2^{n+1})$ and on the given configuration φ. Explicitly, we get:

$$\Big|\mathcal{R}_n\big(L_{\text{int}},\sin(a\phi)r_1^N\big)(f)[\varphi]\Big| \leq \sum_{l=0}^{n}\sum_{r=0}^{n-l}\sum_{s=0}^{l}\sum_{\substack{j=0\\ j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}^{2N}\sum_{\substack{k=0\\ k_1,\dots,k_{n-l}\geq 0\\ k_1+\dots+k_{n-l}=k}}^{2N-j}\Bigg|\frac{1}{i2^{n+1}}\left(\frac{i}{\hbar}\right)^n$$
$$\frac{(-1)^{n-l}\hbar^{j+k}(ia)^{j_1+\dots+j_s}(-ia)^{j_{s+1}+\dots+j_l}(ia)^{k_1+\dots+k_r}(-ia)^{k_{r+1}+\dots+k_{n-l}}}{r!s!(n-l-r)!(l-s)!j_1!\cdots j_l!k_1!\cdots k_{n-l}!}$$
$$\Bigg\langle e^{ia(\varphi(\chi_{l+2})+\dots+\varphi(\chi_{l+r+1}))}e^{-ia(\varphi(\chi_{l+r+2})+\dots+\varphi(\chi_{n+1}))}\prod_{l+2\leq p<q\leq n+1} e^{-a_pa_q\hbar\Delta^{AF}(\chi_p-\chi_q)}$$
$$\Bigg(e^{ia(\varphi(\chi_1)+\dots+\varphi(\chi_{s+1}))}e^{-ia(\varphi(\chi_{s+2})+\dots+\varphi(\chi_{l+1}))}\prod_{1\leq p<q\leq l+1}e^{-a_pa_q\hbar\Delta^F(\chi_p-\chi_q)}$$
$$-\,e^{ia(\varphi(\chi_1)+\dots+\varphi(\chi_s))}e^{-ia(\varphi(\chi_{s+1})+\dots+\varphi(\chi_{l+1}))}\prod_{1\leq p<q\leq l+1}e^{-a_pa_q\hbar\Delta^F(\chi_p-\chi_q)}\Bigg)$$
$$\prod_{\substack{l+2\leq p\leq n+1\\ 1\leq q\leq l+1}} e^{-a_p a_q \hbar W(\chi_p-\chi_q)} \Bigg(\sum_{\substack{u_1,\dots,u_k,v_1,\dots,v_j\geq 0\\ u_1+\dots+v_j\leq 2N}} \frac{\partial^{j+k} r_1^N(\chi_{l+1})}{\partial\varphi_{u_1\xi}\cdots\partial\varphi_{u_k\xi}\partial\varphi_{v_1\xi}\cdots\partial\varphi_{v_j\xi}}$$
$$\partial^{v_1}_{\xi_{l+1}}\Delta^F(\chi_1-\chi_{l+1})\cdots\partial^{v_j}_{\xi_{l+1}}\Delta^F(\chi_l-\chi_{l+1})$$
$$\partial^{u_1}_{\xi_{l+1}}W(\chi_{l+2}-\chi_{l+1})\cdots\partial^{u_k}_{\xi_{l+1}}W(\chi_{n+1}-\chi_{l+1})\Bigg),f\Bigg\rangle\Bigg|.$$

We can immediately see that the imaginary units i and the "$-$" signs disappear when considering the absolute value as well as all the complex exponentials of the configuration φ. Moreover we can also split the big product of distributions in formula (5.5) using the general fact that the product of distributions with specified wavefront set is hypocontinuous (see [6] and [5]). After all these steps, we obtain the following expression:

$$
\begin{aligned}
&\left|\mathcal{R}_n\left(L_{\text{int}}, \sin(a\phi) r_1^N\right)(f)[\varphi]\right| \leq \sum_{l=0}^{n} \sum_{r=0}^{n-l} \sum_{s=0}^{l} \sum_{\substack{j=0 \\ j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}}^{2N} \sum_{\substack{k=0 \\ k_1,\dots,k_{n-l}\geq 0 \\ k_1+\dots+k_{n-l}=k}}^{2N-j} \frac{1}{2^{n+1}\hbar^n} \\
&\frac{\hbar^{j+k} a^{j+k}}{r!s!(t-l-r)!(l-s)!j_1!\cdots j_l! k_1! \cdots k_{t-l}!} \\
&\left|\left\langle \prod_{l+2\leq p<q\leq n+1} e^{-a_p a_q \hbar \Delta^{AF}(\chi_p-\chi_q)} \prod_{\substack{l+2\leq p\leq n+1 \\ 1\leq q\leq l+1}} e^{-a_p a_q \hbar W(\chi_p-\chi_q)} \right.\right. \\
&\qquad \left.\left. \left(\prod_{1\leq p<q\leq l+1} e^{-a_p a_q \hbar \Delta^F(x_p-x_q)} - \prod_{1\leq p<q\leq l+1} e^{-a_p a_q \hbar \Delta^F(x_p-x_q)} \right), f \right\rangle\right| \\
&\left|\left\langle \sum_{\substack{u_1,\dots,u_k,v_1,\dots,v_j\geq 0 \\ u_1+\dots+v_j\leq 2N}} \frac{\partial^{j+k} r_1^N(\chi_{l+1})}{\partial\varphi_{u_1\xi}\cdots\partial\varphi_{u_k\xi}\partial\varphi_{v_1\xi}\cdots\partial\varphi_{v_j\xi}} \right.\right. \\
&\qquad \partial^{v_1}_{\xi_{l+1}} \Delta^F(\chi_1-\chi_{l+1}) \cdots \partial^{v_j}_{\xi_{l+1}} \Delta^F(\chi_l-\chi_{l+1}) \\
&\qquad \left.\left. \partial^{u_1}_{\xi_{l+1}} W(\chi_{l+2}-\chi_{l+1}) \cdots \partial^{u_k}_{\xi_{l+1}} W(\chi_{n+1}-\chi_{l+1}), f \right\rangle\right|.
\end{aligned} \tag{5.6}
$$

We remark that in the fourth line of equation (5.6), the two products of exponentials of Feynman propagators are not the same! The difference, as can be seen in the formula immediately before, is that for the first one $a_{s+1} = +a$, while for the second product $a_{s+1} = -a$.

We now fix all the indices l, r, s, j, j_i, k and k_i, and take into account only the first one of the terms appearing in formula (5.6), the one where $a_{s+1} = +a$ (the other one, where $a_{s+1} = -a$ can be treated in exactly the same way).

The second pairing of the distribution with the test function f depends on the indices j, j_i, k and k_i. We denote its absolute value by:

$$
\begin{aligned}
C^{FW}_{g,j,j_i,k,k_i,N} = &\left|\left\langle \sum_{\substack{u_1,\dots,u_k,v_1,\dots,v_j\geq 0 \\ u_1+\dots+v_j\leq 2N}} \frac{\partial^{j+k} r_1^N(\chi_{l+1})}{\partial\varphi_{u_1\xi}\cdots\partial\varphi_{u_k\xi}\partial\varphi_{v_1\xi}\cdots\partial\varphi_{v_j\xi}} \right.\right. \\
&\qquad \partial^{v_1}_{\xi_{l+1}} \Delta^F(\chi_1-\chi_{l+1}) \cdots \partial^{v_j}_{\xi_{l+1}} \Delta^F(\chi_l-\chi_{l+1}) \\
&\qquad \left.\left. \partial^{u_1}_{\xi_{l+1}} W(\chi_{l+2}-\chi_{l+1}) \cdots \partial^{u_k}_{\xi_{l+1}} W(\chi_{n+1}-\chi_{l+1}), f \right\rangle\right|.
\end{aligned} \tag{5.7}
$$

We now come to the crucial part of our proof: the detailed study the first factor on the right hand side of (5.6), which we recall here for convenience

$$\Bigg|\Bigg\langle \prod_{l+2\le p<q\le n+1} e^{-a_p a_q \hbar \Delta^{AF}(\chi_p-\chi_q)} \prod_{\substack{l+2\le p\le n+1\\ 1\le q\le l+1}} e^{-a_p a_q \hbar W(\chi_p-\chi_q)} \prod_{1\le p<q\le l+1} e^{-a_p a_q \hbar \Delta^{F}(x_p-x_q)}, f\Bigg\rangle\Bigg|. \tag{5.8}$$

To begin with, we express the pairing in formula (5.8) with the usual formal integral notation. Then, moving the absolute value inside the integral symbol (for the legitimacy of this step, see [1]), we get a first estimate:

$$\int_{\mathbb{M}_2^{t+1}} \Bigg| \prod_{l+2\le p<q\le n+1} e^{-a_p a_q \hbar \Delta^{AF}(\chi_p-\chi_q)} \prod_{1\le p<q\le l+1} e^{-a_p a_q \hbar \Delta^{F}(\chi_p-\chi_q)} \prod_{\substack{l+2\le p\le n+1\\ 1\le q\le l+1}} e^{-a_p a_q \hbar W(\chi_p-\chi_q)} \Bigg| \left|g^{\otimes(n+1)}\right| d^{n+1}\chi. \tag{5.9}$$

From formulas (3.3), (3.4) and (3.5) we know explicit expressions for the two-point function W, for the Feynman propagator Δ^F and for the anti-Feynman propagator Δ^{AF} on $\mathbb{M}_2$. These explicit expressions of W, Δ^F and Δ^{AF} imply that when we consider the absolute value of their exponentials, as in formula (5.9) above, the result is always the same:

$$\left|e^{-a_p a_q \hbar W(\chi_p-\chi_q)}\right| = \left|e^{-a_p a_q \hbar \Delta^{F}(\chi_p-\chi_q)}\right| = \left|e^{-a_p a_q \hbar \Delta^{AF}(\chi_p-\chi_q)}\right| = \left|(\chi_p-\chi_q)^2_\eta\right|^{\frac{a_p a_q \hbar}{4\pi}}.$$

Substituting these equations in (5.9) and adopting the notation $\tau_{pq} = \tau_p - \tau_q$, $\zeta_{pq} = \xi_p - \xi_q$, we can rewrite (5.9) as:

$$\int \prod_{l+2\le p<q\le n+1} \left|\tau_{pq}^2 - \zeta_{pq}^2\right|^{\frac{a_p a_q \hbar}{4\pi}} \prod_{1\le p<q\le l+1} \left|\tau_{pq}^2 - \zeta_{pq}^2\right|^{\frac{a_p a_q \hbar}{4\pi}} \prod_{\substack{l+2\le p\le n+1\\ 1\le q\le l+1}} \left|\tau_{pq}^2 - \zeta_{pq}^2\right|^{\frac{a_p a_q \hbar}{4\pi}} \left|g^{\otimes(n+1)}\right| d^{n+1}\chi. \tag{5.10}$$

We now show that the singular parts of these products can be recast in exactly the same form as in the proof of Proposition 6 in [1]. In order to see this, we consider the products separately. For the first one, coming from the antichronological product of vertex operators, comparing also with formula (5.5), we have:

$$\prod_{l+2\leq p<q\leq n+1} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\frac{a_p a_q \hbar}{4\pi}} = \prod_{l+2\leq p<q\leq l+r+1} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta} \prod_{\substack{l+2\leq p\leq l+r+1\\ l+r+2\leq q\leq n+1}} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{-\beta} \prod_{l+r+2\leq p<q\leq n+1} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta}, \tag{5.11}$$

where $\beta = \frac{a^2\hbar}{4\pi}$. Analogously, for the second product in formula (5.10), coming from the time-ordered product of vertex operators, we have:

$$\prod_{1\leq p<q\leq l+1} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\frac{a_p a_q \hbar}{4\pi}} = \prod_{1\leq p<q\leq s+1} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta} \prod_{\substack{1\leq p\leq s+1\\ s+2\leq q\leq l+1}} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{-\beta} \prod_{s+2\leq p<q\leq l+1} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta}. \tag{5.12}$$

Finally, for the third product in formula (5.10), coming from the star product of antichronological and time-ordered products of vertex operators, we have:

$$\prod_{\substack{l+2\leq p\leq n+1\\ 1\leq q\leq l+1}} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\frac{a_p a_q \hbar}{4\pi}} = \prod_{\substack{l+2\leq p\leq l+r+1\\ 1\leq q\leq s+1}} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta} \prod_{\substack{l+r+2\leq p\leq n+1\\ 1\leq q\leq s+1}} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{-\beta} \prod_{\substack{l+2\leq p\leq l+r+1\\ s+2\leq q\leq l+1}} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{-\beta} \prod_{\substack{l+r+2\leq p\leq n+1\\ s+2\leq q\leq l+1}} \left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta}. \tag{5.13}$$

Since what we are interested in estimating is the absolute value of a pairing of a distribution with a test function, the labels of the variables appearing in the integral notation (5.9) for this pairing do not matter. In particular, we can rename them freely. Let us denote the subsets I_+^F, I_-^F, I_+^{AF}, I_-^{AF} of the set of variables appearing in formula (5.9) by distinguishing whether they belong to antichronological products or time-ordered products and whether to vertex operators of the type V_a or vertex operators of the type V_{-a}. We have:

$$(\underbrace{x_1,\ldots,x_{s+1}}_{I_+^F},\underbrace{x_{s+2},\ldots,x_{l+1}}_{I_-^F},\underbrace{x_{l+2},\ldots,x_{l+r+1}}_{I_+^{AF}},\underbrace{x_{l+r+2},\ldots,x_{n+1}}_{I_-^{AF}}) \in \mathbb{M}_2^{n+1}.$$

We choose to rename them in the way that corresponds to reordering the subsets just introduced in the sequence: $(I_+^F, I_+^{AF}, I_-^F, I_-^{AF})$. In other words, after changing the labels of the variables, we end up with the following set of coordinates:

$$(\underbrace{x_1,\ldots,x_{s+1}}_{I_+^F},\underbrace{x_{s+2},\ldots,x_{s+r+1}}_{I_+^{AF}},\underbrace{x_{s+r+2},\ldots,x_{l+r+1}}_{I_-^F},\underbrace{x_{l+r+2},\ldots,x_{n+1}}_{I_-^{AF}})\in\mathbb{M}_2^{n+1}.$$

It is now immediate to check that, using these coordinates, the product of the right hand sides of formulas (5.11), (5.12) and (5.13) can be written all at once as:

$$\prod_{1\le p<q\le s+r+1}\left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta}\prod_{\substack{1\le p\le s+r+1\\ s+r+2\le q\le n+1}}\left|\tau_{pq}^2-\zeta_{pq}^2\right|^{-\beta}\prod_{s+r+2\le p<q\le n+1}\left|\tau_{pq}^2-\zeta_{pq}^2\right|^{\beta}. \tag{5.14}$$

This last formula is precisely the starting point of the proof of Proposition 6 in [1]. Hence, we can repeat the same arguments to obtain the following estimate for the term in formula (5.8):

$$\begin{aligned}&\Bigg|\Bigg\langle\prod_{l+2\le p<q\le n+1}e^{-a_pa_q\hbar\Delta^{AF}(\chi_p-\chi_q)}\prod_{\substack{l+2\le p\le n+1\\ 1\le q\le l+1}}e^{-a_pa_q\hbar W(\chi_p-\chi_q)}\\ &\quad\prod_{1\le p<q\le l+1}e^{-a_pa_q\hbar\Delta^F(x_p-x_q)},f\Bigg\rangle\Bigg|\\ &\le(C_{\gamma,g})^{n+1}\left(\binom{n+1-(r+s+1)}{n+1-2(r+s+1)}(r+s+1)!\right)^{\frac{1}{\gamma}}.\end{aligned} \tag{5.15}$$

We recall that here we are assuming that $r+s+1\le\left[\frac{n+1}{2}\right]$ (as pointed out in [1], this is without loss of generality because the case $r+s+1\ge\left[\frac{n+1}{2}\right]$ can be treated in the same way, just renaming the coordinates properly). We also point out that the difference between the term just discussed and the second term in formula (5.6), the one with "$-$" sign, is that while for the first one there is always at least one $a_i=+a$, in the second one there is always at least one $a_i=-a$. But according to the last observation, counting the number of $+a$'s or of $-a$'s gives the same result. Hence the two terms of formula (5.6) are completely equivalent and can be estimated by the same expression.

We now make two important observations. The first one involves the sums over indices j, j_i, k and k_i. First, the constants (5.7), which depend only on the indices j, j_i, k and k_i, can be estimated all at once by setting:

$$C_{g,N}^{FW}=\max_{\substack{j,k\\ j_1,\ldots,j_l\\ k_1,\ldots,k_{n-l}}}C_{g,j,j_i,k,k_i,N}^{FW}. \tag{5.16}$$

Having a bound on the constants (5.7), we can now estimate the remaining sums

$$\sum_{j=0}^{2N} \sum_{\substack{j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}} \sum_{k=0}^{2N-j} \sum_{\substack{k_1,\dots,k_{n-l} \geq 0 \\ k_1+\dots+k_{n-l}=k}} \frac{\hbar^{j+k} a^{j+k}}{(j_1!\cdots j_l!)(k_1!\cdots k_{n-l}!)}.$$

We can use the multinomial formula to compute the sums over indices j_i and k_i, thus obtaining:

$$\sum_{j=0}^{2N} \sum_{\substack{j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}} \sum_{k=0}^{2N-j} \sum_{\substack{k_1,\dots,k_{n-l} \geq 0 \\ k_1+\dots+k_{n-l}=k}} \frac{\hbar^{j+k} a^{j+k}}{(j_1!\cdots j_l!)(k_1!\cdots k_{n-l}!)}$$
$$= \sum_{j=0}^{2N} \sum_{k=0}^{2N-j} \frac{\hbar^{j+k} a^{j+k} l^j (n-l)^k}{j!k!}.$$

Then we can rewrite the sums over j and k using indices $m = j + k$ and j in the following manner:

$$\sum_{j=0}^{2N} \sum_{k=0}^{2N-j} \frac{\hbar^{j+k} a^{j+k} l^j (n-l)^k}{j!k!} = \sum_{m=0}^{2N} (a\hbar)^m \sum_{j=0}^{m} \frac{l^j (n-l)^{m-j}}{j!(m-j)!}$$
$$= \sum_{m=0}^{2N} (a\hbar)^m \frac{n^m}{m!} \leq \sum_{m=0}^{2N} (a\hbar)^m n^m \leq \tilde{C}_{a\hbar,N} n^{2N}.$$

Combining (5.16) with the last formulas and denoting $K^{FW}_{g,a\hbar,N} = C^{FW}_{g,N} \tilde{C}_{a\hbar,N}$, we have thus obtained that the sums over indices j, j_i, k and k_i can be estimated overall by:

$$K^{FW}_{g,a\hbar,N} n^{2N} \tag{5.17}$$

The second important observation is that formula (5.15) tells us that the estimate on the term (5.8) depends on the indices l, r and s, only via the sum $r + s$. This implies that, if we rearrange the sums over the indices l, r and s using the new index $v = r + s$ and the indices s and l, then we can directly compute the sums over s and l in formula (5.5). More concretely, what we get is:

$$\begin{aligned}
&\sum_{l=0}^{n}\sum_{r=0}^{n-l}\sum_{s=0}^{l}\frac{1}{r!s!(n-l-r)!(l-s)!}\\
&=\sum_{v=0}^{n}\sum_{s=0}^{v}\sum_{l=s}^{n-v+s}\frac{1}{s!(v-s)!(l-s)!(n-l-v+s)!}\\
&=\sum_{v=0}^{n}\sum_{s=0}^{v}\frac{1}{s!(v-s)!}\frac{2^{n-v}}{(n-v)!}\\
&=\sum_{v=0}^{n}\frac{2^{n-v}}{(n-v)!}\frac{2^{v}}{v!}=2^{n}\sum_{v=0}^{n}\frac{1}{v!(n-v)!}.
\end{aligned} \tag{5.18}$$

Finally, we can go back to formula (5.5). We substitute formulas (5.17) and (5.18) and use the appropriate form of (5.15) distinguishing the case when n is even or odd. For even n, the following estimate holds:

$$\begin{aligned}
&\left|\mathcal{R}_n\big(L_{\text{int}},\sin(a\phi)r_1^N\big)(g^{\otimes(n+1)})[\varphi]\right| \leq\\
&K^{FW}_{g,a\hbar,N}\big(C_{\gamma,g}\big)^{n+1}\frac{n^{2N}}{2\hbar^n}\left[2\sum_{v=0}^{[\frac{n}{2}]}\frac{1}{v!(n-v)!}\left(\binom{n+1-v}{n+1-2v}v!\right)^{\frac{1}{\gamma}}\right.\\
&\qquad\left.+2\sum_{v=0}^{[\frac{n}{2}]-1}\frac{1}{v!(n-v)!}\left(\binom{n+1-(v+1)}{n+1-2(v+1)}(v+1)!\right)^{\frac{1}{\gamma}}\right],
\end{aligned} \tag{5.19}$$

while for odd n, we have

$$\begin{aligned}
&\left|\mathcal{R}_n\big(L_{\text{int}},\sin(a\phi)r_1^N\big)(g^{\otimes(n+1)})[\varphi]\right| \leq\\
&K^{FW}_{g,a\hbar,N}\big(C_{\gamma,g}\big)^{n+1}\frac{n^{2N}}{2\hbar^n}\left[2\sum_{v=0}^{[\frac{n}{2}]}\frac{1}{v!(n-v)!}\left(\binom{n+1-v}{n+1-2v}v!\right)^{\frac{1}{\gamma}}\right.\\
&\qquad\left.+2\sum_{v=0}^{[\frac{n}{2}]}\frac{1}{v!(n-v)!}\left(\binom{n+1-(v+1)}{n+1-2(v+1)}(v+1)!\right)^{\frac{1}{\gamma}}\right].
\end{aligned} \tag{5.20}$$

The first type of sums can be further simplified as follows:

$$\sum_{v=0}^{[\frac{n}{2}]} \frac{1}{v!(n-v)!}\left(\binom{n+1-v}{n+1-2v} v!\right)^{\frac{1}{\gamma}} = \sum_{v=0}^{[\frac{n}{2}]} \frac{1}{v!(n-v)!}\left(\frac{(n+1-v)!v!}{(n+1-2v)!v!}\right)^{\frac{1}{\gamma}}$$
$$\leq \sum_{v=0}^{[\frac{n}{2}]} \frac{1}{(n-v)!}\big((n+1-v)!\big)^{\frac{1}{\gamma}} = \sum_{v=0}^{[\frac{n}{2}]} \frac{(n+1-v)^{\frac{1}{\gamma}}}{\big((n-v)!\big)^{1-\frac{1}{\gamma}}}$$
$$\leq \frac{\big([\frac{n}{2}]+1\big)(n+1)^{\frac{1}{\gamma}}}{\big([\frac{n}{2}]!\big)^{1-\frac{1}{\gamma}}} \leq \frac{(n+1)^2}{\big([\frac{n}{2}]!\big)^{1-\frac{1}{\gamma}}}.$$

The other two, similar, type of sums can be manipulated respectively as follows:

$$\sum_{v=0}^{[\frac{n}{2}]-1} \frac{1}{v!(n-v)!}\left(\binom{n+1-(v+1)}{n+1-2(v+1)}(v+1)!\right)^{\frac{1}{\gamma}}$$
$$= \sum_{v=0}^{[\frac{n}{2}]-1} \frac{1}{v!(n-v)!}\left(\frac{(n-v)!(v+1)!}{\big(n+1-2(v+1)\big)!(v+1)!}\right)^{\frac{1}{\gamma}}$$
$$\leq \sum_{v=0}^{[\frac{n}{2}]-1} \frac{1}{\big((n-v)!\big)^{1-\frac{1}{\gamma}}} \leq \frac{[\frac{n}{2}]}{\big([\frac{n}{2}]!\big)^{1-\frac{1}{\gamma}}} \leq \frac{(n+1)^2}{\big([\frac{n}{2}]!\big)^{1-\frac{1}{\gamma}}},$$

where by hypothesis $n \geq 2$, and analogously for the remaining one

$$\sum_{v=0}^{[\frac{n}{2}]} \frac{1}{v!(n-v)!}\left(\binom{n+1-(v+1)}{n+1-2(v+1)}(v+1)!\right)^{\frac{1}{\gamma}}$$
$$= \sum_{v=0}^{[\frac{n}{2}]} \frac{1}{v!(n-v)!}\left(\frac{(n-v)!(v+1)!}{\big(n+1-2(v+1)\big)!(v+1)!}\right)^{\frac{1}{\gamma}}$$
$$\leq \sum_{v=0}^{[\frac{n}{2}]} \frac{1}{\big((n-v)!\big)^{1-\frac{1}{\gamma}}} \leq \frac{\big([\frac{n}{2}]+1\big)}{\big([\frac{n}{2}]!\big)^{1-\frac{1}{\gamma}}} \leq \frac{(n+1)^2}{\big([\frac{n}{2}]!\big)^{1-\frac{1}{\gamma}}}.$$

In conclusion, we obtain the following estimate:

$$\left|\mathcal{R}_n\big(L_{\text{int}}, \sin(a\phi) r_1^N\big)(g^{\otimes(n+1)})[\varphi]\right| \leq \mathcal{K}^{r_1}_{\gamma,g,a\hbar,N} \frac{(n+1)^2 n^{2N} \big(\mathcal{C}^{r_1}_{\gamma,g}\big)^n}{\big([\frac{n}{2}]!\big)^{1-\frac{1}{\gamma}}},$$

where $\mathcal{K}^{r_1}_{\gamma,g,a\hbar,N} = 2K^{FW}_{g,a\hbar,N} C_{\gamma,g}$ and $\mathcal{C}^{r_1}_{\gamma,g} = \frac{C_{\gamma,g}}{\hbar}$. *q.e.d.*

With this result at hand, it is immediate to check the summability of the formal power series defining the expectation values of the interacting components of the higher currents. In other words, we directly get the following corollary.

Corollary 5.1.1. *Under the same hypothesis of Theorem 5.1.1, the expectation values of the interacting components* $(s_1^N)_{\mathrm{int}}$ *and* $(s_2^N)_{\mathrm{int}}$ *of the higher currents of the sine-Gordon model in the state* $\omega_{\varphi,H}$ *are summable.*

Remark 5.1.1. For a proof, in a slightly different quantization framework, of the summability properties of the first of the higher interacting currents, that is, the stress-energy tensor of the sine-Gordon model, we refer to [16] and [15].

Outlook

Concerning the line of research dealing with the formulation of the conservation laws of the sine-Gordon model in the framework of pAQFT, the first natural continuation is represented by the study of the conservation properties of the interacting components of the higher currents. In other words, we would like to address the question whether the ambiguities in the renormalization process of the interacting components can be fixed in such a way as to guarantee that even after quantization the corresponding conservation laws are preserved. From our point of view, some recent results, in particular, [3], where the authors considered distributions defined on $\mathbb{R}^n \setminus \{ 0 \}$ satisfying a certain set of partial differential equations and provided criteria for the existence of extensions of these distributions to the whole $\mathbb{R}^n$ that satisfy the same set of partial differential equations, seem to be particularly relevant. For the first of the higher currents, that is, the stress-energy tensor of the sine-Gordon model, conservation was shown explicitly in a slightly different renormalization framework.

A second natural research direction, related to the quantized higher currents of the sine-Gordon model, concerns their involutivity properties. As mentioned in the introduction, the classical higher currents are in involution with respect to the Peierls bracket. In the framework of pAQFT quantization, the Peierls bracket is deformed to the commutator product with respect to the star product of fields, which is typically a noncommutative product. Hence it is a sensible question whether, even after quantization, the interacting higher currents of the sine-Gordon model maintain the property of being in involution with respect to the commutator product. In general, as it happens for many physical models, one cannot expect this to be the case and this corresponds, in the physics parlance, to the appearance of so-called quantum anomalies.

Finally, a further research direction which we pursue[1] concerns the classical theory of the sine-Gordon model. Based on the observation made in Remark 1.4.2, we plan to investigate what could be the rôle of groupoids in the formulation of Bäcklund transformations and in the determination of the corresponding conservation laws.

[1]Ongoing research project with Antonio Michele Miti.

Appendix A

Wavefront set of distributions

The wavefront set is a central tool of microlocal analysis. Rougly speaking, the wavefront set of a distribution represents a refined notion of its singular support, in which also the "directions of propagation" of the singularities are taken into account. The possibility to extend a number of operations on distributions by means of a control on the wavefront sent lies at the very core of the algebraic formulation of field theories, as first noted by Radzikowski [23].

The starting point is given by the following remark.

Remark A.0.1. Consider $v \in \mathscr{E}'(\mathbb{R}^d)$, that is, a distribution on $\mathbb{R}^d$ with compact support. We can decide whether v is in $\mathscr{D}(\mathbb{R}^d)$ by examining the behaviour of its Fourier transform $\hat{v}$ at ∞. In fact, it can be shown (see [20, Theorem 7.3.1.]) that if $v \in \mathscr{D}(\mathbb{R}^d)$, then

$$|\hat{v}(\xi)| \leq C_N(1+|\xi|)^{-N}, \qquad \forall N = 1, 2, \dots, \quad \forall \xi \in \mathbb{R}^d \tag{A.1}$$

and conversely if (A.1) is satisfied, then $v \in \mathscr{D}(\mathbb{R}^d)$ by Fourier's inversion formula.

Recall the definition of singular support of a distribution.

Definition A.0.1. If $u \in \mathscr{D}'(\mathbb{R})$, then the singular support of u, denoted by the $\operatorname{sing\,supp}(u)$, is the set of points in $\mathbb{R}^d$ having no open neighborhood to which the restriction of u is a smooth function.

Similarly we can introduce the set $\Sigma(v)$ of directions $\eta \in \mathbb{R}^d \setminus \{ 0 \}$ having no conic neighborhood V such that (A.1) is valid when $\xi \in V$. It follows immediately that $\Sigma(v)$ is a closed cone in $\mathbb{R}^d \setminus \{ 0 \}$ and that $\Sigma(v) = 0$ if and only if $v \in \mathscr{D}(\mathbb{R}^d)$.

Now, on the one hand $\operatorname{sing\,supp}(v)$ describes only the location of the singularities of v and on the other hand $\Sigma(v)$ describes only the directions of non-rapid decay causing them. The two types of information can be combined by means of the following intermediate lemma.

Lemma A.0.1. *If $\phi \in \mathscr{D}(\mathbb{R}^d)$ and $v \in \mathscr{E}'(\mathbb{R}^d)$, then*

$$\Sigma(\phi v) \subseteq \Sigma(v).$$

Consider X an open subset of $\mathbb{R}^d$ and $u \in \mathscr{D}'(X)$. We set for $x \in X$

$$\Sigma(u)_x = \bigcap_{\phi} \Sigma(\phi u), \qquad \phi \in \mathscr{D}(X) \quad \text{s.t.} \quad \phi(x) \neq 0.$$

From Lemma A.0.1 it follows that

$$\Sigma(\phi u) \to \Sigma_x(u), \qquad \text{if} \quad \phi \in \mathscr{D}(X),\, \phi(x) \neq 0 \quad \text{and} \quad \operatorname{supp}(\phi) \to \{\, x \,\}\,.$$

In particular this means that $\Sigma_x(u) \neq 0$ if and only if ϕu is smooth for some $\phi \in \mathscr{D}(X)$ with $\phi(x) \neq 0$, that is, $x \notin \operatorname{sing\,supp}(u)$.

Definition A.0.2. If $u \in \mathscr{D}'(X)$, then the closed conic subset of $X \times (\mathbb{R}^d \setminus \{\, 0 \,\})$ defined by

$$\mathrm{WF}(u) = \left\{\, (x, \xi) \in X \times (\mathbb{R}^d \setminus \{\, 0 \,\}) \;\middle|\; \xi \in \Sigma_x(u) \,\right\}$$

is called the wavefront set of u.

An immediate consequence of the definition of wavefront set of a distribution u is that the projection of $\mathrm{WF}(u)$ on X is $\operatorname{sing\,supp}(u)$. Another natural result of the construction of the wavefront set is the fact that the information about the non-rapid decay directions is recovered in the following sense.

Proposition A.0.1. *If $u \in \mathscr{E}'(\mathbb{R}^d)$, then the projection of $\mathrm{WF}(u)$ to the second variable is $\Sigma(u)$.*

The following basic properties of the wavefront set follow directly from the definition.

Proposition A.0.2. *For $t, s \in \mathscr{D}'(\mathbb{R}^d)$ and $g \in \mathbb{C}^\infty(\mathbb{R}^d)$:*

(a) $\mathrm{WF}(t + s) \subseteq \mathrm{WF}(t) \cup \mathrm{WF}(s)$,

(b) $\mathrm{WF}(gt) \subseteq \big(\operatorname{supp}(g) \times (\mathbb{R}^d \setminus 0)\big) \cap \mathrm{WF}(t)$,

(c) Let $P(x) = \sum_a g_a(x)\partial^a$ be a differential operator, where $g_a \in C^\infty(\mathbb{R}^d)$ and the sum over a is finite. Then $\mathrm{WF}(Pt) \subseteq \mathrm{WF}(t)$.

As noted at the beginning, the more refined control on the singularities of distributions provided by the wavefront set allows one to extend many operations on distributions. These extensions are always done by continuity from the smooth case, so a first issue to discuss is in which sense the notion of continuity is meant for spaces of distributions with a given bound on their wavefront set.

Let X be an open subset of $\mathbb{R}^d$ and let Γ be a closed cone in $X \times (\mathbb{R}^d \setminus \{ 0 \})$. We introduce the space

$$\mathscr{D}'_\Gamma(X) = \{ u \in \mathscr{D}'(X) \mid \mathrm{WF}(u) \subseteq \Gamma \} .$$

Lemma A.0.2. *A distribution $u \in \mathscr{D}'(X)$ is in $\mathscr{D}'_\Gamma(X)$ if and only if for every $\phi \in \mathscr{D}(X)$ and every closed cone $V \subseteq \mathbb{R}^d$ with*

$$\Gamma \cap (\mathrm{supp}(\phi) \times V) = \emptyset \tag{A.2}$$

the following holds:

$$\sup_V |\xi|^N |\widehat{\phi u}(\xi)| < \infty, \qquad N = 1, 2, \ldots$$

With this result, we can now formulate a precise notion of convergence of sequences.

Definition A.0.3. We say that a sequence $u_j \in \mathscr{D}'_\Gamma(X)$ converges to $u \in \mathscr{D}'_\Gamma(X)$ if

- $u_j(\phi) \to u(\phi), \qquad \forall \phi \in \mathscr{D}(\mathbb{R}^d)$,
- $\sup_V |\xi|^N |\widehat{\phi u}(\xi) - \widehat{\phi u_j}(\xi)| \to 0, \qquad j \to \infty$,

for $N = 1, 2, \ldots$ and $\phi \in \mathscr{D}(\mathbb{R}^d)$ and V a closed cone such that (A.2) is satisfied.

We are now ready to discuss the extensions of the operations on distributions which are more relevant for our purposes.

Theorem A.0.1. *Let X and Y be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $f \colon X \to Y$ be a smooth map. Denote the set of normals of the map by*

$$N_f = \left\{ (f(x), \eta) \in Y \times \mathbb{R}^n \;\middle|\; \big(f'(x)\big)^t \eta = 0 \right\} .$$

Then the pull-back $f^* u$ can be defined uniquely for all $u \in \mathscr{D}'(Y)$ with

$$N_f \cap \mathrm{WF}(u) = \emptyset , \tag{A.3}$$

in such a way that $f^*u = u \circ f$ *when* $u \in C^\infty(Y)$*. For any closed conic subset* $\Gamma \in Y \times (\mathbb{R}^n \setminus 0)$ *with* $\Gamma \cap N_f = \emptyset$*, the pull-back* f^* *extends to a continuous (in the sequential sense described above) map* $f^* \colon \mathscr{D}'_\Gamma(Y) \to \mathscr{D}'_{f^*\Gamma}(X)$*,*

$$f^*\Gamma = \left\{ \left(x, \left(f'(x)\right)^t \eta\right) \in X \times \mathbb{R}^m \;\middle|\; (f(x), \eta) \in \Gamma \right\}.$$

In particular, for every $u \in \mathscr{D}'(Y)$ *satisfying* (A.3)

$$\mathrm{WF}(f^*u) \subseteq f^*\mathrm{WF}(u).$$

The second operation on distributions that we consider is the tensor product.

Theorem A.0.2. *If* $u \in \mathscr{D}'(X)$*,* $v \in \mathscr{D}'(Y)$*,* $X \in \mathbb{R}^m$ *and* $Y \in \mathbb{R}^n$*, then*

$$\begin{aligned}\mathrm{WF}(u \otimes v) \subseteq \big(\mathrm{WF}(u) \times \mathrm{WF}(v)\big) \cup \big((\mathrm{supp}(u) \times \{\,0\,\}) \times \mathrm{WF}(v)\big) \cup \\ \cup \big(\mathrm{WF}(u) \times (\mathrm{supp}(v) \times \{\,0\,\})\big).\end{aligned}$$

Combining these results, it is possible to define the product of distributions even in some cases where they have overlapping singularities. The basic idea comes from the observation that for smooth functions u, v on X, the product $u(x)v(x)$ can be regarded as the restriction to the diagonal of the tensor product $u(x)v(y)$ defined for $(x, y) \in X \times X$.

Theorem A.0.3 (Hörmander's sufficient criterion). *If* $u, v \in \mathscr{D}'(X)$ *then the product* uv *can be defined as the pull-back of the tensor product* $u \otimes v$ *by the diagonal map* $\delta \colon X \to X \times X$ *unless* $(x, \xi) \in \mathrm{WF}(u)$ *and* $(x, -\xi) \in \mathrm{WF}(v)$ *for some* $(x, \xi) \in T^*X$*. When the product is defined we have*

$$\mathrm{WF}(uv) \subseteq \{\, (x, \xi + \eta) \mid (x, \xi) \in \mathrm{WF}(u) \text{ or } \xi = 0,\ (x, \eta) \in \mathrm{WF}(v) \text{ or } \eta = 0 \,\}.$$

Appendix B

Scaling degree of distributions

Roughly speaking, Steinmann's scaling degree of a distribution is a measure of the strength of the singularity of the distribution at the origin. More precisely, it is given by the following definition.

Definition B.0.1. The scaling degree, with respect to the origin, of a distribution $t \in \mathscr{D}'(\mathbb{R}^d)$, or $t \in \mathscr{D}'(\mathbb{R}^d \setminus \{ 0 \})$, is given by:

$$\mathrm{sd}(t) = \inf \left\{ r \in \mathbb{R} \;\middle|\; \lim_{\rho \downarrow 0} \rho^r t(\rho x) = 0 \right\},$$

with the conventions that $\inf \emptyset = \infty$ and $\inf \mathbb{R} = -\infty$.

Example B.0.1. Consider δ, the Dirac delta supported at the origin of $\mathbb{R}^d$, and a multiindex $a \in \mathbb{N}^d$, then

$$\mathrm{sd}(\partial^a \delta) = d + |a|, \quad \text{since} \quad \partial^a \delta(\rho x) = \rho^{-d-|a|} \partial^a \delta(x).$$

Remark B.0.1. If $t \in \mathscr{D}'(\mathbb{R}^d)$ with $0 \notin \mathrm{supp}(t)$, then $\mathrm{sd}(t) = -\infty$, because for each $g \in \mathscr{D}(\mathbb{R}^d)$ there exists a $\rho_g > 0$ such that $\mathrm{supp}(t(\rho\,\cdot)) \cap \mathrm{supp}(g) = \emptyset$ for all $0 < \rho < \rho_g$. For $t \in \mathscr{D}'(\mathbb{R}^d)$, the relation $\mathrm{sd}(t) < \infty$ always holds. But for $t \in \mathscr{D}'(\mathbb{R}^d \setminus \{ 0 \})$, the value $\mathrm{sd}(t) = \infty$ is possible. A one-dimensional example is $t(x) = \theta(x) e^{1/x}$, since

$$\lim_{\rho \downarrow 0} \rho^r \int_0^\infty dx\, e^{1/x} h(x)$$

diverges $\forall r \in \mathbb{R}$ and for a suitable choice of $h \in \mathscr{D}(\mathbb{R} \setminus \{ 0 \})$ (see [10, p. 116]).

From the definition of scaling degree, it follows immediately that any extension $t \in \mathscr{D}'(\mathbb{R}^d)$ of a given $t_0 \in \mathscr{D}'(\mathbb{R}^d \setminus \{ 0 \})$ satisfies $\mathrm{sd}(t) \geq \mathrm{sd}(t_0)$.

In particular we look for extensions which do not increase the scaling degree. The following fundamental result combines techniques developed by Epstein and Glaser and Hörmander (see [11] and [20] respectively).

Theorem B.0.1. *Let $t_0 \in \mathscr{D}'(\mathbb{R}^d \setminus \{ 0 \})$, then:*

(a) If $\mathrm{sd}(t_0) < d$, there is a unique extension $t \in \mathscr{D}'(\mathbb{R}^d)$ fulfilling the condition $\mathrm{sd}(t) = \mathrm{sd}(t_0)$.

(b) If $d \leq \mathrm{sd}(t_0) < \infty$, there are several extensions $t \in \mathscr{D}'(\mathbb{R}^d)$ fulfilling the condition $\mathrm{sd}(t) = \mathrm{sd}(t_0)$. Given a particular extension $\bar{t}$, the general extension t is of the form

$$t = \bar{t} + \sum_{|a| \leq \mathrm{sd}(t_0) - d} C_a \partial^a \delta, \qquad C_a \in \mathbb{C}.$$

In case (b), the addition of a term $\sum_a C_a \partial^a \delta$ is also called a "finite renormalization" (for the proof of the Theorem see [10]).

The following properties of the scaling degree of distributions are a precise formulation of the heuristical statements that in general differentiation increases the strength of the singularities at 0, while multiplication with x^a makes the distribution less singular at 0.

Proposition B.0.1. *Consider distributions $t, t_1, t_2 \in \mathscr{D}'(\mathbb{R}^d)$, or in $\mathscr{D}'(\mathbb{R}^d \setminus \{ 0 \})$, and a generic multiindex $a \in \mathbb{N}^d$, then:*

(a) $\mathrm{sd}(\partial^a t) \leq \mathrm{sd}(t) + |a|$;

(b) $\mathrm{sd}(x^a t) \leq \mathrm{sd}(t) - |a|$;

(c) $\mathrm{sd}(t_1 \otimes t_2) = \mathrm{sd}(t_1) + \mathrm{sd}(t_2)$*, where $\otimes$ denotes the tensor product of distributions;*

(d) if $\mathrm{sd}(t_1), \mathrm{sd}(t_2) < \infty$ and their product is well-defined according to Hörmander's sufficient criterion, then $\mathrm{sd}(t_1 t_2) \leq \mathrm{sd}(t_1) + \mathrm{sd}(t_2)$.

Appendix C

Properties of Feynman and anti-Feynman propagators

Definition C.0.1. The Feynman propagator $\Delta^F \in \mathscr{D}'(\mathbb{M}_d)$ is defined in terms of the two-point function W (see formula (2.5)), by:

$$\Delta^F(x) = \theta(x^0)W(x) + \theta(-x^0)W(-x). \tag{C.1}$$

It can be shown (cf. [10]) that formula (C.1) is equivalent to:

$$\Delta^F = \frac{i}{2}(\Delta_m^R + \Delta_m^A) + H,$$

where $\frac{1}{2}(\Delta_m^R + \Delta_m^A)$ is the so-called the Dirac propagator and H is the Hadamard parametrix corresponding to W (cf. formula (2.5)).

Another equivalent characterization of the Feynman propagator uses the inverse Fourier transform for distributions and is given, in integral notation, by the following expression:

$$\Delta^F(x) = \frac{i}{(2\pi)^d} \int_{\mathbb{M}_d} \frac{e^{-ikx}}{(k)_\eta^2 - m^2 + i0}\, dk,$$

where m is the mass and $(k)_\eta^2 = \eta\left(\eta^\sharp(k), \eta^\sharp(k)\right)$, $\eta^\sharp : T^*\mathbb{M}_d \to T\mathbb{M}_d$ is the natural isomorphism induced by the Minkowski metric η which, due to the triviality of the bundles $T^*\mathbb{M}_d \equiv \mathbb{M}_d \times \mathbb{M}_d^*$, $T\mathbb{M}_d \equiv \mathbb{M}_d \times \mathbb{M}_d$, descends to an isomorphism $\eta^\sharp : \mathbb{M}_d^* \to \mathbb{M}_d$.

From all these equivalent formulations it is possible to show the following properties.

Proposition C.0.1. *The Feynman propagator Δ^F has the following properties:*

(a) it is a symmetric distribution, that is,

$$\Delta^F(x) = \Delta^F(-x);$$

(b) it is a fundamental solution of the wave operator (see Definition 2.3.4)

$$(\Box + m^2)\Delta^F(x) = -i\delta(x)$$

(c) it can be equivalently expressed using the retarded propagator Δ^R_m and the two-point function W as

$$\Delta^F(x) = i\Delta^R_m(x) + W(-x);$$

(d) the wavefront set of the Feynman propagator is given by

$$\begin{aligned}\mathrm{WF}(\Delta^F) = &\big\{(0,k) \in T^*\mathbb{M}_d \,\big|\, k \neq 0\big\} \\ &\cup \big\{\, (x,k) \in T^*\mathbb{M}_d \,\big|\, (x)^2_\eta = 0,\ x \neq 0,\ (k)^2_\eta = 0, \\ &\qquad x = \lambda\eta^\sharp(k), \textit{ for some } \lambda > 0 \,\big\};\end{aligned} \tag{C.2}$$

(e) Steinmann's scaling degree of the Feynman propagator is $(d-2)$, that is,

$$\rho^{d-2}\Delta^F(\rho x) = \Delta^F(x), \qquad \forall \rho \in \mathbb{R}_+. \tag{C.3}$$

Definition C.0.2. The anti-Feynman propagator $\Delta^{AF} \in \mathscr{D}'(\mathbb{M}_d)$ is defined as the complex conjugate of the Feynman propagator:

$$\Delta^{AF} = \overline{\Delta^F}. \tag{C.4}$$

All the properties mentioned above for the Feynman propagator hold also for the anti-Feynman propagator, after taking their complex conjugate. We only spell out explicitly the one concerning an equivalent formulation of the anti-Feynman propagator and the one concerning its wavefront set.

Proposition C.0.2. *The anti-Feynman propagator $\Delta^{AF} \in \mathscr{D}'(\mathbb{M}_d)$ admits the following equivalent expression:*

$$\Delta^{AF} = -\frac{i}{2}(\Delta^R_m + \Delta^A_m) + H,$$

and its wavefront set is given by

$$\begin{aligned}\mathrm{WF}(\Delta^{AF}) = &\big\{(0,k) \in T^*\mathbb{M}_d \,\big|\, k \neq 0\big\} \\ &\cup \big\{\, (x,k) \in T^*\mathbb{M}_d \,|\, (x)^2_\eta = 0,\ x \neq 0,\ (k)^2_\eta = 0, \\ &\qquad x = \lambda\eta^\sharp(k), \textit{ for some } \lambda < 0 \,\big\}.\end{aligned} \tag{C.5}$$

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