A real effective version of the Freĭman-Scourfield Theorem

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Contents

1 Introduction

1.1 Problems and results

There is a long-standing interest in solving equations in natural numbers as they occur naturally in a range of different contexts. The expressions considered are called *Diophantine equations*. One of the simpler examples entails the question whether for a given exponent $k \in \mathbb{N}$ there always exists a finite number of summands $s \in \mathbb{N}$ such that the equation

$$n = x_1^k + \cdots + x_s^k \tag{1}$$

is solvable in natural numbers $x_i \in \mathbb{N}$ $(1 \leq i \leq s)$ for all natural numbers $n \in \mathbb{N}$. This is *Waring's problem*. Often, reducing the number s is hindered by obstructions involving only comparably small integers. Therefore, one variant of this problem is to consider (1) only for *sufficiently large* numbers, meaning for all numbers $n \in \mathbb{N}$ larger than some fixed threshold $N \in \mathbb{N}$. A detailed survey of Waring's problem, including developments over time and numerous results, can be found in Vaughan and Wooley [23].
How many solutions are there to be expected for these problems under the assumption that the attained values are roughly equidistributed? For estimating this heuristically, multiply the sizes of all available ranges for the variables and then divide the expression by the measure of the attained values. From (1), we see that the size of the variables is naturally bounded by $n^{1/k}$. Plugging this into the equation, we note that the values attained are *asymptotically* n many, meaning constant factors are negligible. Therefore, we would expect an order of magnitude of

$$n^{-1+s/k}$$

for the number of solutions. To guarantee that there exist solutions, it is hence natural to expect that more than k many summands are needed for Waring's problem with k-th powers. There exist countless variants of this problem, modifying the equation or the permitted set of solutions.

1.1.1 Mixed exponents

One example of this is to consider an equation of mixed type, involving different exponents $k_i \in \mathbb{N}$. To facilitate notation, we order them by size as $k_1 \leq k_2 \leq \ldots$ and collect them in a sequence $(k_i)_{i\in\mathbb{N}}$. The question of whether an equation with these exponents is solvable with a finite number of summands for sufficiently large n depends on the exact shape of the sequence of exponents. How large are the gaps between them? Are the attained values dense enough to cover all large natural numbers? Freĭman proposed in [12] the following characterisation for a sequence, where $k_1 \geq 2$. For any 'start index' $j \in \mathbb{N}$ we find a 'stop index' $s(j) =: s \in \mathbb{N}$ such that

$$n = x_j^{k_j} + x_{j+1}^{k_{j+1}} + \cdots + x_s^{k_s}, \qquad x_i \in \mathbb{N} \ \ (j \leq i \leq s) \tag{2}$$

is solvable for sufficiently large $n \in \mathbb{N}$ if and only if the infinite series $\sum k_i^{-1}$ diverges. As we want (2) to hold for any j and respective s, the condition has to depend on the shape of the whole sequence of exponents. The mentioned connection between the density of attained values and the exponents is also encoded in this condition, as we see by the following argument. If we think about the heuristic for the number

of solutions discussed for Waring's problem, we deduce that here, we would, as a power of n, expect a magnitude of

$$-1 + \sum_{j \le i \le s} k_i^{-1}.$$

If the series of all reciprocals diverges, this number is thus reasonably large for each partial sum starting at some $j \in \mathbb{N}$, by choosing s adequately. Testing the condition, for example with $(k_i)_{i\in\mathbb{N}} = \mathbb{N}$, we note that it would be satisfied, as opposed to inserting only square numbers for k_i, as here the gaps between the exponents would be too large. Freĭman's statement was formally proved by Scourfield [19], therefore we reference it as the *Freĭman-Scourfield Theorem.*

The novelty of our work is to consider problems of this type for real exponents. Firstly, we state a result cognate to the classic Freĭman-Scourfield Theorem. The adjustment to real exponents involves flooring in the Diophantine equation. We adapt the usual notation for the floor function $\lfloor x \rfloor := \max\{n \in \mathbb{N} : n \le x\}$ for $x \in \mathbb{R}^+$, which is complemented by $\{x\} := x - \lfloor x \rfloor$, denoting the fractional part.

Theorem 1.1 *Let $k_i \in \mathbb{R}$ with $0 < k_1 \le k_2 \le \ldots$. Then, for any $j \in \mathbb{N}$ there exists $s \in \mathbb{N}$ such that all sufficiently large integers n are representable as*

$$n = \lfloor x_j^{k_j} \rfloor + \cdots + \lfloor x_s^{k_s} \rfloor, \qquad x_i \in \mathbb{N} \ (j \le i \le s)$$

if and only if $\sum_{i\in\mathbb{N}} k_i^{-1}$ is divergent.

Now, the condition on the series over the reciprocals of the exponents gives no indication of the number s of summands one might need, for some specific j. This was only recently made explicit by Brüdern and Wooley [8]. An arbitrary index j is fixed and for ease of notation set to $j = 1$. The *effective* version of the Freĭman-Scourfield Theorem for integer exponents reads as follows.

Theorem 1.2 (Brüdern-Wooley) *Let $k_i \in \mathbb{N}$ $(i > 1)$ satisfy $2 \le k_1 \le k_2 \le \ldots$. Suppose that s is a natural number for which*

$$\sum_{3 \le i \le s} \frac{1}{k_i} > 2 \log k_1 + \frac{1}{k_2} + 3.20032. \tag{3}$$

Then all sufficiently large natural numbers n are represented in the form

$$n = x_1^{k_1} + \cdots + x_s^{k_s}, \qquad x_i \in \mathbb{N} \ (1 \le i \le s).$$

Likewise, we establish a version of this effective result for real exponents.

Theorem 1.3 *Given a sequence of real exponents $0 < k_1 \le k_2 \le \ldots$, fix $s \in \mathbb{N}$ and let ℓ be the smallest index $1 \le \ell \le s$ such that $k_\ell \notin \mathbb{Z}$. If*

$$\sum_{\substack{1 \le i \le s \\ i \ne \ell}} \frac{1}{k_i} > 8 \log k_\ell + 9.72 \tag{4}$$

holds, then for any sufficiently large natural number $n \in \mathbb{N}$, the equation

$$n = \lfloor x_1^{k_1} \rfloor + \cdots + \lfloor x_s^{k_s} \rfloor, \qquad x_i \in \mathbb{N} \ (1 \le i \le s) \tag{5}$$

is always solvable.

As is evident from the theorem, the first non-integer exponent k_ℓ plays a crucial role. If it appears only late in the sequence, more specifically, if (3) holds for $s = \ell - 1$, we refer to the work of Brüdern and Wooley [8], as then all k_i $(1 \leq i < \ell)$ are integers. Furthermore, we note that with more effort the lower bound in (4) can certainly be improved. The purpose of this work is to demonstrate how to approach an effective version of Freĭman-Scourfield's theorem for real exponents. We will highlight opportunities for improvement where applicable.
Furthermore, the paper of Brüdern and Wooley [8] contains results on a more specific setting for the exponents. For k, r non-negative integers with $k \geq 2$ the equation with exponents in arithmetic progression takes the shape

$$n = x_1^k + x_2^{k+r} + \cdots + x_s^{k+r(s-1)}, \qquad x_i \in \mathbb{N} \ (1 \leq i \leq s). \tag{6}$$

In this context, the number $R(k, r)$ is defined as the least number for s such that (6) is solvable for sufficiently large integers $n \in \mathbb{N}$. The second result of the paper is to bound uniformly in k and r the amount by

$$R(k, r) \leq r^{-1} 25^r (r + 1)^{r+1} (k + 1)^{r+1}.$$

If $r \geq k$ holds, the estimate $R(k, r) \leq (6k + 6)^{2r}$ is possible. In this generality, there only existed a comparable result by Scourfield [19] beforehand. We will now present some selected results for specific values of k and r.

1.1.2 Special sequences of exponents

Firstly, we turn our attention to the case $R(2, 1)$ with the visually pleasing equation

$$n = x_1^2 + x_2^3 + x_3^4 + \cdots + x_s^{s+1}, \qquad x_i \in \mathbb{N} \ (1 \leq i \leq s). \tag{7}$$

This case was originally examined by Roth [18] and it was shown that $R(2, 1) \leq 50$ holds. There were many improvements on this, see Brüdern and Wooley [8] for an exact listing. We want to stress an improvement by Brüdern [5] to $R(2, 1) \leq 17$, as it introduces the technique of *pruning* in Lemma 2, which will be further explained in subsection 1.2. The current record for (7) is held by Liu and Zhao [16] with $R(2, 1) \leq 13$.
The second problem we want to focus on is even ascending powers in the form

$$n = x_1^2 + x_2^4 + x_3^6 + \cdots + x_s^{2s}, \qquad x_i \in \mathbb{N} \ (1 \leq i \leq s). \tag{8}$$

Setting $s = 4$, a quantitative density result was proved by Brüdern [7]. It is stated that almost all $n \in \mathbb{N}$ not divisible by eight are represented as a sum of a square, a fourth, a sixth and an eighth power. Here, we note that in some cases, there are p-adic obstructions to solving an equation for almost all natural numbers. In contrast to density results, findings about the solvability of (8) for sufficiently large $n \in \mathbb{N}$ require considerably more summands. In [15] Kuan, Lesesvre and Xiao state that $R(2, 2) \leq 133$ holds.
There are countless other results on different combinations of mixed powers, some overview can be found in Vaughan and Wooley [23, Table 5]. In order to recall it later, we give a result of Hooley [14] for squares, cubes and exponents a, b with $(1/a) + (1/b) > 1/3$. For the equation

$$n = x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^a + x_6^b \tag{9}$$

he shows an asymptotic formula for the number of solutions.

As our work implements some techniques from Diophantine inequalities, we continue by giving some context on this.

1.1.3 Diophantine inequalities

An analogue of Waring's problem concerning Diophantine inequalities is studied by Davenport and Heilbronn [10] and is also explained in chapter 20 of Davenport [9]. As we use a similar approach, we want to outline it here and in subsection 1.2 in some detail. Related to Waring's problems for squares, they examine the quadratic form

$$Q(\mathbf{x}) \coloneqq \lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2, \tag{10}$$

with $\lambda_i \in \mathbb{R}$, not all of the same sign, all non-zero and at least one of the ratios λ_i/λ_j $(1 \leq i, j \leq 5)$ irrational. Then, by the theorem of Davenport and Heilbronn, for any $\varepsilon > 0$ there exist $x_1, \ldots, x_5$, not all zero, such that

$$|Q(\mathbf{x})| < \varepsilon. \tag{11}$$

Similar to equation (10) mimicking Waring's problem for squares, there is interest in studying problems known as Diophantine equations in their inequality version. In the foreword of Davenport [9] by D. E. Freeman, a survey is given. As an example for work on mixed exponents, we revisit equation (9). This was first studied by Baker and Harman [2] and improved in Brüdern [6]. Let $\lambda_1, \ldots, \lambda_6$ be non-zero, real and such that at least one of λ_i/λ_j $(1 \leq i, j \leq 6)$ is irrational. If $(1/a)+(1/b) > 1/3$, then there exists a real $\Delta > 0$ such that for any real number μ the inequality

$$|\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^3 + \lambda_4 x_4^3 + \lambda_5 x_5^a + \lambda_6 x_6^b - \mu| < \max(|x_i|)^{-\Delta}$$

has infinitely many integer solutions. We want to mention another result from the same paper about the familiar equation (7). We assume $\lambda_2, \ldots, \lambda_{17}$ to be all non-zero, real and at least one of λ_i/λ_j irrational. Then the values taken by

$$\sum_{1 \leq i \leq 16} \lambda_{i+1} x_i^{i+1} \tag{12}$$

at integer points for the variables x_i are dense on the real line. In general, it seems reasonable to expect somewhat similar results for Diophantine inequalities compared to their counterparts for equations.
In the intersection of Diophantine inequalities and dealing with non-integer exponents lies the work of Poulias [17]. There, an expression reminiscent of Waring's problem but with a real non-integer exponent $\theta > 2$ and coefficients is analysed. For fixed non-zero real numbers λ_i $(1 \leq i \leq s)$, not all of the same sign, define

$$F(\mathbf{x}) \coloneqq \lambda_1 x_1^\theta + \cdots + \lambda_s x_s^\theta.$$

Then, for any fixed real $\varepsilon > 0$ an asymptotic for the integer solutions to $|F(\mathbf{x})| < \varepsilon$ follows if $s \geq (\lfloor 2\theta \rfloor + 1)(\lfloor 2\theta \rfloor + 2) + 1$ holds.
This section addressed, among others, non-integer exponents in Diophantine inequalities. If we, on the other hand, wish to insert such exponents in Diophantine equations, we need to modify the equation in order to represent integer values still. For our work, we chose the option of flooring each summand. This was also already studied for Waring's problem, which we will cover in the next section.

1.1.4 Waring with non-integer exponents

For real non-integer $k > 2$ the problem reads as

$$n = \lfloor x_1^k \rfloor + \cdots + \lfloor x_s^k \rfloor, \qquad x_i \in \mathbb{N} \ (1 \leq i \leq s). \tag{13}$$

As in Waring's problem, one quest is for the smallest number of summands s such that all sufficiently large natural numbers $n \in \mathbb{N}$ are represented by (13). Denote this number by $G(k)$. A first result was presented by Segal [20], showing that $G(k) < k^2 2^k$. This was followed up by Deshouillers [11] who improved the bound to $4k(\log k + 0.5 \log\log k + 3.5)$. His work was taken up by Arkhipov and Zhitkov in [1], who proved under the assumptions $k > 12$ and $s > 22k^2(\log k + 4)$ an asymptotic formula for the number of solutions to (13). Combining the methods of Poulias [17] with those of Bourgain, Demeter and Guth [4] could make a reduction to $s \gg k^2$ possible.

1.2 Methods

After presenting some contextualising results for our research, we give an overview of the employed techniques.

1.2.1 Hardy-Littlewood circle method

A standard technique for approaching Diophantine equations and related problems is the *Hardy-Littlewood circle method*. An extensive description of the circle method and various techniques can be found in Davenport [9] or Vaughan [22]. In this introduction, we only refer to some aspects that are relevant for discussing the different ideas we want to shed light on and that ultimately lead to our result. Central is the use of exponential sums that often take a form similar to

$$f(\alpha) \coloneqq \sum_{x \in I} e^{2\pi i \alpha x^k},$$

with some suitable set I and $\alpha \in \mathbb{R}$ a real variable. They are sometimes also called *Weyl sums*. For Waring's problem, the factor x^k in the exponent represents an occurring summand and can be adapted depending on the considered equation. To encode the whole equation (1), we multiply s many of these sums and note that the exponential function recovers the additive structure of the Diophantine equation. Further, we use a Fourier transform detecting n to count the number $R(n)$ of solutions to the equation (1) with variables restricted to I by the expression

$$R(n) = \int_0^1 f(\alpha)^s e^{-2\pi i \alpha n} \mathrm{d}\alpha. \tag{14}$$

This integral is then further evaluated to show that it is large and positive for n sufficiently large, which would allow the conclusion that there indeed exist solutions to (1). Here, it would be reasonable to expect the order of magnitude $n^{-1+s/k}$, as explained earlier.
To show that the integral in (14) is of this size, we divide its range into different parts, depending on how large $f(\alpha)$ is for certain intervals of α. Inspired by the shape

of the complex integral being a circle, we call these segments *arcs* and the whole process *circle method*. The parts that are larger in absolute value are denoted *major arcs* and carry the main contribution. To this end, we register how far α is from some specific rational number a/q and write $\alpha = \beta + a/q$. Then, we replace the sum $f(\alpha)$ by the product $q^{-1}S_{a,q}I(\beta)$ with some acceptable error. For (1) to be solvable in integers, the existence of solutions in real numbers and modulo each prime power is a necessary condition. This is encoded in these factors. The factor $I(\beta)$ covers the real case by integrating over the whole real range of the variable. Analogously, the factors $q^{-1}S_{a,q}$ represent the p-adic condition. Then, it is demonstrated that the substitution leads to the main term and represents the real and p-adic condition there.
In contrast, the complement to the major arcs is usually called *minor arcs* and the integral over this region is bounded from above to show that the main term dominates it. Part of the trick here is to find pointwise estimates for $f(\alpha)$ on a certain subset, often referred to as *Weyl-type*, and for multiple of these sums in mean.

In general, a great advantage of the method is its flexibility, as it can be adapted to various equations through changes in the integral. In the following, we want to shed some light on different ways to achieve the estimates for the exponential sums.

1.2.2 Techniques of estimation for mixed exponents

The circle method is also applied in the original proof of Freĭman-Scourfield's theorem in Scourfield [19], with different exponential sums. There, for retrieving an estimate of the exponential sums in mean, the technique of *diminishing ranges* is adopted from Hardy and Littlewood [13]. The idea is to shorten the ranges of some variables in a way that grants more control and knowledge over the attained values. Simultaneously, the ranges preserve enough freedom to solve equation (2). In Scourfield [19], the technique of diminishing ranges is applied to some of the exponential sums in mean. The saving, compared to the trivial estimate, is of the shape

$$n^{-1+\rho},$$

where ρ is a parameter in $(0, 1)$ that depends on s and the exponents k_i for indices $j \leq i \leq s$. The divergence of the series summing over all reciprocals ensures that this parameter converges to zero with growing s. That saving then needs to be balanced with some pointwise estimates applied to the remaining sums to achieve the bound on the minor arcs. This technique will also find application in subsection 4.1.

Another restriction on the variables is used for the modern version of Freĭman-Scourfield's theorem by Brüdern and Wooley [8]. They employ *smooth numbers*, which are numbers that are only divisible by primes smaller than some defined threshold. This again allows for more control and information on the values attained and how to factor them. The exponential sums are then defined over this set and a new Weyl-type estimate is established. As before, this is combined with a mean value estimate for the smooth exponential sums.
For the result on ascending powers in [16], Liu and Zhao manage to combine the technique of diminishing ranges with a mean value estimate for smooth Weyl sums, among other methods.

We still owe an explanation for the mentioned method of pruning. It was introduced by Brüdern in [5] and combines estimates available for exponential sums on parts of the major arcs to achieve an upper bound there. Hence, the sizes of the major arcs are gradually pruned until they are manageable. This is especially helpful for mixed powers with different ranges for the variables.
Pruning was combined with a mix of smooth exponential sums and 'normal' ones by Kuan, Lesesvre and Xiao [15], studying even ascending powers.

1.2.3 The Davenport-Heilbronn method

As we borrow some techniques from a version of the circle method designed for Diophantine inequalities, which is called *Davenport-Heilbronn method*, we go into some detail here. In the language of Fourier analysis, the classical circle method uses the transform on $\mathbb{R}/\mathbb{Z}$, whereas for inequalities we employ this over the whole real line.
We demonstrate the technique with the example of the inequality version of Waring's problem for squares illustrated in (10) and (11). By rescaling, it suffices to show this for $\varepsilon = 1$. It is crucial for this problem to have an indicator function that is positive for $|Q(\mathbf{x})|< 1$ and zero on the rest of the real axis. If we consider the Fourier transform

$$\int_{-\infty}^{\infty} \mathrm{e}(\eta\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 \mathrm{d}\alpha = \max(0, 1 - |\eta|),$$

we see that the function on the right is one of the possible indicator functions for this problem. Its Fourier transform is called the *kernel* from now on. In subsection 3.2, we use a similar variant, but once convoluted with itself. If we plug in $\eta = Q(\mathbf{x})$ in the equation above and sum over all ranges for the variables $x_1, \dots, x_5$, we again recover an integral counting solutions to (11) involving exponential sums. Once more, now with the real axis, the integral is separated into different arcs. On the tails of the integral, the decaying behaviour of the kernel dominates. We note that for Diophantine inequalities, there exists no p-adic condition, only a real one. This is also evident here, as the major arcs are simply an interval around zero, and the exponential sums $f(\lambda_i\alpha)$ are only replaced with an integral term. We follow the same tactic, as can be observed in Lemma 5.2.
By developing a version of the pruning lemma adapted to the Davenport-Heilbronn method, the result in Brüdern [6] is achieved. This modified pruning lemma, presented in Lemmata 2 and 3, gives rise to many applications, for example the result involving the inequality version of ascending powers (12).

1.2.4 Weyl-type estimates with real exponents

Located in the intersection of Diophantine inequalities and the world of non-integer exponents lies the work of Poulias [17]. There, exponential sums of length P with non-integer degree k occur, for which he uses a Weyl-type estimate with a saving of $P^{-4^{-k}}$. This would be too weak for our application, as this saving strongly shapes the effective bound. Therefore, we show our version of a Weyl-type estimate for non-integer exponents in subsection 4.2, which is based on work by Baker and Kolesnik [3, Proposition 1].
We further want to mention work of Deshouillers [11] that contains estimates for

exponential sums with non-integer exponents and also a floor function in the argument. He shows for both versions a pointwise bound for a non-integer exponent $k > 12$. These results could be further developed to improve the Weyl-type estimate that is used in our work.

1.3 Survey of this work

Our result combines an equation of Freĭman-Scourfield-type with real exponents and flooring. We will make extensive use of the Davenport-Heilbronn method, so the theory around Diophantine inequalities is also applied. If one compares the bounds in (4) and (3), it is evident that even for $\ell = 1$ the bound in the integer case is way less restrictive. The main reason for this are the current shortcomings of a Weyl-type estimate for non-integer exponents. As discussed in Brüdern and Wooley [8], the true limit for the effective condition is expected at 4, or with further restrictions at 2 or even 1. Relating our results to those of Waring-type (13) by setting $k = k_1 = k_2 = \cdots = k_s$ and assuming all to be non-integer, we retrieve the condition $s > k(8 \log k + 9.72) + 1$, which is unsurprisingly weaker than the current state of the art, since it holds in more generality.

It remains to give an overview of the following chapters. The result in Theorem 1.3 is proved by an application of the Hardy-Littlewood circle method, more specifically by a variant of the Davenport-Heilbronn method. We begin with some basic notation in section 2, before we present the transformation into a Diophantine inequality using a kernel function in section 3. This is where the Davenport-Heilbronn method comes into play. After establishing an integral counting the solutions to the inequality, we separate it into different arcs in subsection 3.4. In this context, we also elaborate on the building blocks of the method, moving on to analysing them in section 4. This involves a mean value estimate utilising diminishing ranges in subsection 4.1 and proving a novel Weyl-type estimate for exponential sums with a non-integer exponent in subsection 4.2. These results are implemented on the arcs in section 5, leading to the calculation of the effective bound in section 6.

2 Fundamental notation

We simplify the notation considerably by reordering the exponents such that we assign the first non-integer exponent k_ℓ the index $i = 0$, so we set $k_0 \coloneqq k_\ell$. The order of the remaining exponents stays the same, and their index is shifted by one to the left for $i > \ell$. In other words, we map

$$(k_1, k_2, \dots, k_\ell, \dots, k_s) \mapsto (k_0, k_1, k_2, \dots, k_{s-1}).$$

We continue by introducing the objects at the heart of the argument. Writing $v_i \coloneqq k_i^{-1}$, we note that the variables x_i are roughly of size $P_i \coloneqq n^{v_i}$. Here $n \in \mathbb{N}$ refers to the integer we want to represent by (5). From now on, it shall be regarded as fixed but large.
For technical reasons, we divide the variables x_i into different blocks. The first exponent with $i = 0$ is singled out. Furthermore, we chose an integer $r \in \mathbb{N}$ with $1 < r < s$ that we specify in section 6. The first block contains the indices $1 \leq i < r$, whereas the second block collects the remaining indices with $r \leq i < s$. The parameter s is also treated as fixed from now on.
We now set the ranges of the variables in the exact way that later enables the use of diminishing ranges for the two blocks of variables. For these, we construct the ranges of definition in a way such that the variables with the smallest indices, i.e. $i = 1$ and $i = r$, have the respectively longest ranges compared to the rest. With increasing index, the ranges become shorter and shorter. In contrast, the first variable x_0 has a 'full length' range. In mathematical terms, for the domains with 'full length', this transfers to setting the shortening factor

$$\pi_i \coloneqq 1$$

for indices $i = 0, 1, r$. For the shortened intervals, we define this factor as

$$\pi_i \coloneqq \prod_{1 \leq j < i} (1 - v_j) \quad (1 < i < r),$$
$$\pi_i \coloneqq \prod_{r \leq j < i} (1 - v_j) \quad (r < i < s)$$

and subsequently set the ranges

$$I_i \coloneqq [\tfrac{1}{2} P_i^{\pi_i}, P_i^{\pi_i}] \quad (0 \leq i < s).$$

From now on, $\alpha \in \mathbb{R}$ always denotes a real integration variable. As is customary, for any argument $z \in \mathbb{R}$ we further set $\mathrm{e}(z) \coloneqq \mathrm{e}^{2\pi i z}$. With this, we proceed to the introduction of the exponential sums

$$f_i(\alpha) \coloneqq \sum_{x_i \in I_i} \mathrm{e}(\alpha x_i^{k_i}),$$

with real exponents which occur in this shape only for $i = 0$ for technical reasons. In contrast, we also use exponential sums that contain the floor function from equation (5), regardless of whether the respective exponent k_i is non-integer or not. Writing

$$g_i(\alpha) \coloneqq \sum_{x_i \in I_i} \mathrm{e}(\alpha \lfloor x_i^{k_i} \rfloor),$$

we find the building components of the blocks, so we introduce

$$T_1(\alpha) \coloneqq \prod_{1\le i<r} g_i(\alpha) \quad \text{and} \quad T_2(\alpha) \coloneqq \prod_{r\le i<s} g_i(\alpha)$$

as an abbreviated notation. Furthermore, we shall use the standard Landau and Vinogradov symbols, where the underlying parameter is n and the implicit constants may depend on r, s and k_i $(0 \le i < s)$.

3 The overall approach

In Theorem 1.3 we require only that the exponents k_i $(0 \le i < s)$ are positive. As for the range $0 < k_i \le 1$ only one summand suffices to cover all natural numbers, we will from now on always assume $1 < k_i$ for all $0 \le i < s$.

3.1 Transformation into a Diophantine inequality

For the first summand, we want to avoid the floor function in the argument in the Weyl sums, as can already be seen by the definition of the sum f_0. To achieve this, instead of testing the solvability of (5), we show the existence of integers that satisfy

$$n \le x_0^{k_0} + \lfloor x_1^{k_1} \rfloor + \cdots + \lfloor x_{s-1}^{k_{s-1}} \rfloor < n+1. \tag{15}$$

The error made by flooring here is $0 \le x_0^{k_0} - \lfloor x_0^{k_0} \rfloor < 1$ and therefore solutions for (15) also adhere to (5) and vice versa. For brevity, we shall write $n_0 \coloneqq n + 1/2$, the vector of all variables as

$$\mathbf{x} \coloneqq (x_0, \dots, x_{s-1}) \in \mathbb{N}^s$$

and define

$$Q(\mathbf{x}) \coloneqq x_0^{k_0} + \lfloor x_1^{k_1} \rfloor + \cdots + \lfloor x_{s-1}^{k_{s-1}} \rfloor - n_0. \tag{16}$$

With this, we transform (15) into the condition $Q(\mathbf{x}) \in (-1/2, 1/2)$, which allows the treatment presented in the next chapter.

3.2 Encoding the inequality

The implementation of this inequality is carried out via a weighted indicator function and its Fourier transform. Firstly, we define the kernel

$$K(\alpha) \coloneqq \frac{1}{4\pi^3} \left(\frac{\sin(\pi\alpha/4)}{\alpha} \right)^4. \tag{17}$$

Secondly, we use the version of the Fourier transform reading as

$$\hat{K}(x) \coloneqq \int_{-\infty}^{\infty} K(\alpha) \mathrm{e}(-\alpha x) \mathrm{d}\alpha.$$

In our case, we calculate by convolution that

$$\hat{K}(x) = \int_{-\infty}^{\infty} \max(1 - |4t|, 0) \max(1 - |4(x-t)|, 0) \, \mathrm{d}t \tag{18}$$

holds, which is positive for $x \in (-1/2, 1/2)$ and equal to zero on the rest of the real axis. Another important property of the function $\hat{K}$ is the symmetry with respect to zero, so we have $\hat{K}(-x) = \hat{K}(x)$. This also holds for the kernel function K. Moreover, concerning the size and behaviour of K, we note that, on one hand by Jordan's inequality, we have for $\alpha \in [-2, 2]$ that

$$K(\alpha) \asymp 1$$

holds. On the other hand, the behaviour of K for α outside of this interval is dominated by α^{-4}. We summarise this to the estimate

$$K(\alpha) \ll (1 + |\alpha|)^{-4} \tag{19}$$

that holds for all $\alpha \in \mathbb{R}$. If convenient, we further use the notation $\mathrm{d}_K\alpha \coloneqq K(\alpha)\mathrm{d}\alpha$.

3.3 The Fourier transform

The kernel shall be used to convert the inequality into a tangible integral by the usual transformation from the Davenport-Heilbronn method. To this end, we introduce a function that counts in the box

$$\mathbf{I} \coloneqq \bigtimes_{0 \le i < s} I_i$$

the number of solutions

$$R(n) \coloneqq \#\{\mathbf{x} \in \mathbf{I} : \sum_{0 \le i < s} \lfloor x_i^{k_i} \rfloor = n\}$$

to the original equation (5). By the argument in subsection 3.1, we see that also

$$R(n) = \#\{\mathbf{x} \in \mathbf{I} : Q(\mathbf{x}) \in (-1/2, 1/2)\}$$

holds. We find, considering the weights of the indicator function, that

$$R(n) \gg \sum_{\mathbf{x} \in \mathbf{I}} \hat{K}(Q(\mathbf{x}))$$

holds. Further, by symmetry one has

$$\hat{K}(Q(\mathbf{x})) = \hat{K}(-Q(\mathbf{x})),$$

with which we avoid dragging along the sign in the Fourier transform. Hence, we calculate

$$\begin{aligned}
\sum_{\mathbf{x} \in \mathbf{I}} \hat{K}(-Q(\mathbf{x})) &= \sum_{\mathbf{x} \in \mathbf{I}} \int_{-\infty}^{\infty} \mathrm{e}(\alpha Q(\mathbf{x})) K(\alpha) \mathrm{d}\alpha \\
&= \int_{-\infty}^{\infty} \left(\sum_{\mathbf{x} \in \mathbf{I}} \mathrm{e}(\alpha x_0^{k_0}) \mathrm{e}(\alpha \lfloor x_1^{k_1} \rfloor) \cdot \ldots \cdot \mathrm{e}(\alpha \lfloor x_{s-1}^{k_{s-1}} \rfloor) \right) \mathrm{e}(-\alpha n_0) \mathrm{d}_K\alpha \\
&= \int_{-\infty}^{\infty} f_0(\alpha) T_1(\alpha) T_2(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K\alpha. \qquad (20)
\end{aligned}$$

Introducing some more notation, we write

$$T_{1,2}(\alpha) \coloneqq T_1(\alpha)T_2(\alpha)$$

to group the first and second block. If we wish to be very brief, the notation

$$A(\alpha) \coloneqq f_0(\alpha)T_{1,2}(\alpha)$$

is used. For estimating the blocks trivially, we define

$$\tau \coloneqq \sum_{1\le i<r} v_i \prod_{1\le j<i}(1-v_j) + \sum_{r\le i<s} v_i \prod_{r\le j<i}(1-v_j),$$

originating from the definitions of the shortened intervals. By convention, we set the empty product equal to one. Note that hence

$$T_{1,2}(0) \asymp n^{\tau}$$

holds. In this context, we moreover define $\theta \coloneqq v_0 + \tau$. To finish the chapter, we deduce from the recent equations

$$R(n) \gg \sum_{\mathbf{x}\in\mathbf{I}} \hat{K}(Q(\mathbf{x})) = \int_{-\infty}^{\infty} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha. \tag{21}$$

As our objective is to show that $R(n)$ is positive for $n \in \mathbb{N}$ large enough, we set out to prove

$$\int_{-\infty}^{\infty} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha \gg n^{\theta-1} \tag{22}$$

throughout the rest of the paper. This bound stems from the usual heuristic from the circle method for the expected number of solutions as mentioned in the introduction. Note here, that we consider (5) with $\mathbf{x} \in \mathbf{I}$. Collecting the possible values for all variables x_i $(0 \le i < s)$ gives a magnitude of n^{θ}. Simultaneously, the values attained by

$$\lfloor x_0^{k_0} \rfloor + \cdots + \lfloor x_{s-1}^{k_{s-1}} \rfloor$$

are of size n. If we assume that these values are roughly equidistributed, we thus get the prediction $n^{\theta-1}$.

3.4 Definition of the arcs

For further analysis, we split the range of the integral in (21) into different arcs, some of which provide the main term, resulting in the estimate (22). To this end, we introduce the real parameters

$$D \coloneqq 0.01 \quad \text{and} \quad \delta \coloneqq Dv_0^4/2.$$

In section 6 we demonstrate why we defined δ this way. Now, we establish the shape of the arcs by considering the behaviour of the sums in the integral. Crucial is the first factor $f_0(\alpha)$, which peaks when α approaches zero. Since it is defined with a non-integer exponent, this constitutes the only peak and we shall later show a

Weyl-type inequality that holds for a range of values far enough away from zero. Consequently, we define the major arcs, where the integral is large, as

$$\mathfrak{M} \coloneqq \{\alpha \in \mathbb{R} : |\alpha| \leq \tfrac{v_0}{2} n^{-1+v_0}\}.$$

The contribution of the other arcs is comparably small and absorbed in the error term. We separate them into two cases. The trivial arcs consist of the tails of the integral, defined as

$$\mathfrak{t} \coloneqq \{\alpha \in \mathbb{R} : |\alpha| > n^{\delta}\},$$

where the decaying behaviour of the kernel dominates. The rest of the integral is contained in the minor arcs

$$\mathfrak{m} \coloneqq \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{t}),$$

whose contribution is also negligible. Before we present the detailed treatment of the arcs, we set out to prove some necessary auxiliary results.

4 Tools

This chapter contains estimates for each building block of the integral. We start with the technique of diminishing ranges to retrieve a non-trivial estimate for the blocks $T_{1,2}$. Then, we prove a novel Weyl-type estimate for the sum f_i containing non-integer exponents. Balancing these will enable us to show the desired asymptotic estimate for the whole integral.

4.1 Diminishing ranges

This chapter follows the outline of the technique of diminishing ranges in Scourfield [19, Lemma 13]. Here, the aim is to achieve a non-trivial estimate for $|T_{1,2}|$ in mean. Therefore, we now analyse the first and second block, which contain variables with shortened ranges designed for applying the technique of diminishing ranges. In this context, we redefine with more flexibility than before for arbitrary indices $\ell, m \in \mathbb{N}$ the products from the shortening factors of section 2 as

$$\Pi_{\ell,m} \coloneqq \prod_{\ell \leq i < m} (1 - v_i).$$

Furthermore, we encounter in this chapter the parameter

$$\rho \coloneqq (\Pi_{1,r} + \Pi_{r,s})/2, \tag{23}$$

which is key to calculating the effective lower bound later on. The main result of this chapter reads as follows.

Theorem 4.1 *For the blocks the estimate*

$$\int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \mathrm{d}_K \alpha \ll n^{\tau-1+\rho}$$

is true.

As we split up this expression by Schwarz's inequality, as seen later in the proof of Theorem 4.1, we now give two lemmata concerning the sums T_i $(i = 1, 2)$ separately.

Lemma 4.2 *For the first and second block, respectively*

$$\int_{-\infty}^{\infty} |T_i(\alpha)|^2 \mathrm{d}_K\alpha \ll T_i(0)$$

holds for $i = 1, 2$.

Proof. We demonstrate the estimate for T_1, the proof for T_2 works analogously. For this proof, we therefore always consider $1 \leq i < r$. Setting $\mathbf{y}$ analogously to $\mathbf{x}$, define the term

$$S_i(\mathbf{x}, \mathbf{y}) \coloneqq \sum_{i \leq j < r} \lfloor x_j^{k_j} \rfloor - \sum_{i \leq j < r} \lfloor y_j^{k_j} \rfloor,$$

starting at some index i. The integral

$$\int_{-\infty}^{\infty} |T_1(\alpha)|^2 K(\alpha) \mathrm{d}\alpha$$

is asymptotically equal to the number of $x_i, y_i \in I_i$ $(1 \leq i < r)$ for which

$$S_1(\mathbf{x}, \mathbf{y}) \in (-1/2, 1/2)$$

holds. Since all summands are integers, this is equivalent to solving

$$S_1(\mathbf{x}, \mathbf{y}) = 0.$$

The aim is now to show gradually that for the solutions x_i and y_i for any index $1 \leq i < r-1$ the bound $|x_i - y_i| \ll 1$ holds. Then we use this to reduce the problem further to counting solutions to

$$S_{i+1}(\mathbf{x}, \mathbf{y}) = 0, \tag{24}$$

until only $\lfloor x_{r-1}^{k_{r-1}} \rfloor = \lfloor y_{r-1}^{k_{r-1}} \rfloor$ remains. So we fix some index i with $1 \leq i < r-1$ and count solutions $x_i, y_i \in I_i$ to the equation

$$S_i(\mathbf{x}, \mathbf{y}) = 0, \tag{25}$$

which is equivalent to

$$S_{i+1}(\mathbf{x}, \mathbf{y}) = \lfloor y_i^{k_i} \rfloor - \lfloor x_i^{k_i} \rfloor. \tag{26}$$

From now on, we assume that $x_i \neq y_i$ holds, as otherwise $|x_i - y_i| \ll 1$ is already satisfied. Furthermore, we assume without loss of generality $x_i < y_i$, as the argument is symmetrical. Firstly, we observe that in (26), the size of the left-hand side $S_{i+1}(\mathbf{x}, \mathbf{y})$ is dominated by the values of x_{i+1}, y_{i+1} due to the shortening ranges of definition. Therefore, we estimate

$$|S_{i+1}(\mathbf{x}, \mathbf{y})| \ll n^{k_{i+1} v_{i+1} \Pi_{1,i+1}}.$$

From this, deduce that

$$\lfloor y_i^{k_i} \rfloor - \lfloor x_i^{k_i} \rfloor \ll n^{\Pi_{1,i+1}} \tag{27}$$

is true. Secondly, we note that $\lfloor y_i^{k_i} \rfloor - \lfloor x_i^{k_i} \rfloor \asymp y_i^{k_i} - x_i^{k_i}$ holds. To evaluate this further, we apply the mean value theorem to the function $z \mapsto z^{k_i}$. For any combination of $x_i, y_i \in I_i$ this constitutes a continuos and differentiable function on the interval $[x_i, y_i] \subset [\frac{1}{2}P_i^{\pi_i}, P_i^{\pi_i}]$. Hence, there exists a point $z_i \in (x_i, y_i)$ such that

$$k_i z_i^{k_i-1} = \frac{y_i^{k_i} - x_i^{k_i}}{y_i - x_i} \tag{28}$$

is satisfied. We can estimate the left-hand side further as

$$k_i z_i^{k_i-1} \asymp n^{(k_i-1)v_i\Pi_{1,i}} = n^{\Pi_{1,i+1}}.$$

Combining this with (27), and rearranging (28), we conclude

$$|y_i - x_i| \ll \frac{n^{\Pi_{1,i+1}}}{n^{\Pi_{1,i+1}}} = 1,$$

hence the distance between solutions is bounded for some constant $C > 0$ as $|y_i - x_i| < C$. To summarise, we have

$$\#\{x_i, y_i \in I_i : (25) \text{ holds}\} \leq C\#I_i \ll \#I_i. \tag{29}$$

Now we reduce the further counting problem to (24). For this, we fix some values for x_i and y_i and set $h \coloneqq \lfloor y_i^{k_i} \rfloor - \lfloor x_i^{k_i} \rfloor$. This recovers the equation

$$S_{i+1}(\mathbf{x}, \mathbf{y}) = h. \tag{30}$$

The count of solutions x_{i+1}, y_{i+1} to this is transformed by the argument from the beginning into an integral and is further bounded as

$$\int_{-\infty}^{\infty} \Big| \prod_{i<j<r} g_j(\alpha) \Big|^2 \mathrm{e}(-\alpha h) \mathrm{d}_K\alpha \ll \int_{-\infty}^{\infty} \Big| \prod_{i<j<r} g_j(\alpha) \Big|^2 \mathrm{d}_K\alpha \tag{31}$$

by triangle inequality. The integral on the right-hand side in turn corresponds to the number of solutions to (24). The estimate in (31) shows that in order to count solutions $x_{i+1}, y_{i+1} \in I_{i+1}$ to (30), it suffices to count those to (24).
Therefore, we have shown that indeed for all $1 \leq i < r-1$ the estimate (29) holds and leads to reducing the counting problem to (24). In the remaining case $\lfloor x_{r-1}^{k_{r-1}} \rfloor = \lfloor y_{r-1}^{k_{r-1}} \rfloor$ we also easily deduce the bound $\#I_{r-1}$ for the number of solutions x_{r-1}, y_{r-1}. Consequently, we bound

$$\int_{-\infty}^{\infty} |T_1(\alpha)|^2 K(\alpha) \mathrm{d}\alpha \ll \# \bigtimes_{1 \leq i < r} I_i \asymp T_1(0).$$

□

The complementary step to this Lemma is to rearrange the counts $T_1(0)$ and $T_2(0)$.

Lemma 4.3 *We have for the first block*

$$T_1(0) \asymp n^{1-\Pi_{1,r}}$$

and similarly for the second

$$T_2(0) \asymp n^{1-\Pi_{r,s}}.$$

Proof. Once more, we demonstrate this for T_1. Looking at the definition, we see

$$T_1(0) \asymp \prod_{1\le i<r} n^{v_i\Pi_{1,i}} = n^{\sum\limits_{1\le i<r} v_i\Pi_{1,i}}.$$

To show that these powers of n do indeed coincide, we carry out a proof by induction with respect to some integer $m \in \mathbb{N}$. The aim is to demonstrate

$$\sum_{1\le i\le m} v_i\Pi_{1,i} = 1 - \Pi_{1,m+1} \tag{32}$$

for $m \ge 1$. The identity (32) is obviously true for indices $m = 1$ and $m = 2$. We assume it for a fixed $m \in \mathbb{N}$ and step from m to $m+1$ by calculating, for the second equality using (32), that

$$\begin{aligned}\sum_{1\le i\le m+1} v_i\Pi_{1,i} &= \sum_{1\le i\le m} v_i\Pi_{1,i} + v_{m+1}\Pi_{1,m+1}\\ &= 1 - \Pi_{1,m+1} + v_{m+1}\Pi_{1,m+1}\\ &= 1 - \Pi_{1,m+2}\end{aligned}$$

by definition holds. Inserting $m = r - 1$ into (32) gives the desired statement. The proof for T_2 works completely analogously. It is also possible to show this by multiplying everything out and grouping it differently, which could be helpful for a deeper understanding but is more tedious. □

We now derive the theorem from the two results.

Proof of Theorem 4.1. For applying the lemmata, we wish to separate the blocks, which is carried out by Schwarz's inequality and results in

$$\int_{-\infty}^{\infty} |T_{1,2}(\alpha)|\mathrm{d}_K\alpha \ll \left(\int_{-\infty}^{\infty} |T_1(\alpha)|^2\mathrm{d}_K\alpha\right)^{1/2}\left(\int_{-\infty}^{\infty} |T_2(\alpha)|^2\mathrm{d}_K\alpha\right)^{1/2}.$$

For these factors we use Lemma 4.2 and Lemma 4.3 to get

$$\int_{-\infty}^{\infty} |T_1(\alpha)|^2\mathrm{d}_K\alpha \ll T_1(0) \asymp T_1(0)^2 n^{-1+\Pi_{1,r}}$$

and respectively for the second block the result

$$\int_{-\infty}^{\infty} |T_2(\alpha)|^2\mathrm{d}_K\alpha \ll T_2(0)^2 n^{-1+\Pi_{r,s}}.$$

Merging this, we arrive at

$$\int_{-\infty}^{\infty} |T_{1,2}(\alpha)|\mathrm{d}_K\alpha \ll T_{1,2}(0) n^{-1+\rho}$$

and, with recalling $T_{1,2}(0) \asymp n^{\tau}$, we have proved the theorem.

□

Remark 1 *We recover Theorem 4.1 for the square root of the kernel as*

$$\int_{-\infty}^{\infty} |T_{1,2}(\alpha)|^2 \sqrt{K(\alpha)} \mathrm{d}\alpha \ll n^{\tau-1+\rho}.$$

Proof. We have

$$\int_{-\infty}^{\infty} |T_1(\alpha)|^2 \sqrt{K(\alpha)} \mathrm{d}\alpha \ll \int_{-\infty}^{\infty} |T_1(\alpha)|^2 K(\alpha) \mathrm{d}\alpha \tag{33}$$

by the following argument. We see by the definition of the kernel $K(\alpha)$ in (17) that it is a fittingly scaled version of $((\sin \alpha)/\alpha)^4$ such that its Fourier transform $\hat{K}(x)$ is a weighted indicator function of $(-1/2, 1/2)$. By considering the square root $\sqrt{K}$ we reverse the convolution described in (18) and thus have the transform

$$\widehat{\sqrt{K}}(x) = \max(0, 1 - |4x|)$$

that detects values in $(-1/4, 1/4)$. Relating this to the proof of Theorem 4.1, we note that the condition on $S_1(\mathbf{x}, \mathbf{y})$ that the integral on the left-hand side encodes, is more restrictive and has hence at most equally many solutions. The weights only differ by constants. This proves (33), and by the arguments of Theorem 4.1 and Lemma 4.3, the whole claim follows.

□

Note that the bounds of this remark and of Theorem 4.1 also hold on any subset of the real line as all factors in the integral are positive.

4.2 Weyl-type lemma

As a counterpart to the mean value estimate for $T_{1,2}$, we present a pointwise bound for the exponential sum f_i, where the corresponding exponent k_i is non-integer. For this section, we shall omit the index i, hence writing k instead of k_i and so forth. Moreover, we use some local notation in the proof of this lemma that will partially be redefined for a different use in later chapters.

Lemma 4.4 *Assume for non-integer $k \notin \mathbb{Z}$ with $k > 1$ and real α we have*

$$\tfrac{v}{2} n^{-1+v} < |\alpha| \leq n^{Dv^4/2},$$

with $D = 0.01$, then we estimate

$$|f(\alpha)| \ll n^v (|\alpha| n)^{-Dv^3}.$$

The proof is split into the case of large values for $k > 11.5$ which is based on Baker and Kolesnik [3, Proposition 1] but uses a more modern version of the Vinogradov mean value estimate. Additionally, the calculations differ in the latter part of the proof. For the remaining $1 < k \leq 11.5$, we apply a result by Segal [21, Lemma 2] and include it at the end. Combining both, the resulting bound as displayed above is comparably weak, due to its generality. An improvement of this Weyl-type estimate, for example by following Deshouillers [11], would directly lead to a better effective bound overall.

Proof. Firstly, assume $k > 11.5$. For n sufficiently large one has

$$n^{-1+v/2} \leq \tfrac{v}{2} n^{-1+v}$$

and we use the left-hand side as a lower bound for the rest of the proof. When we show the result for this bound, it naturally also follows for the bound on the right hand side. For convenience write $|\alpha| P^k = P^{\lambda k}$ with

$$v/2 \leq \lambda \leq 1 + Dv^4/2.$$

From now on, we treat the parameter α as fixed and hence suppress it in the notation. We first want to show that for any integer $B \geq 1$, we have

$$|f| \ll B^{-2} \sum_{P/2 \leq x \leq P} |W(x)| + B^2, \tag{34}$$

where we set

$$W(x) \coloneqq \sum_{1 \leq z \leq B} \sum_{1 \leq y \leq B} \mathrm{e}(\alpha(x+yz)^k). \tag{35}$$

If $B^2 > P/2$ holds, then we estimate

$$|f| \ll P \ll B^2,$$

so we get the estimation against the second term. Otherwise, for any $B \leq \sqrt{P/2}$ and any integer $1 \leq y, z \leq B$ we split

$$\begin{aligned}
\sum_{x=P/2}^{P} \mathrm{e}(\alpha x^k) &= \left(\sum_{x=P/2}^{(P/2)+yz-1} + \sum_{x=\frac{1}{2}P+yz}^{P} + \sum_{x=P+1}^{P+yz} - \sum_{x=P+1}^{P+yz} \right) \mathrm{e}(\alpha x^k) \\
&= \sum_{x=(P/2)+yz}^{P+yz} \mathrm{e}(\alpha x^k) + \mathcal{O}(B^2) \\
&= \sum_{x=P/2}^{P} \mathrm{e}(\alpha(x+yz)^k) + \mathcal{O}(B^2).
\end{aligned}$$

This we average over $1 \leq y, z \leq B$ by applying

$$B^{-2} \sum_{1 \leq y \leq B} \sum_{1 \leq z \leq B}$$

to both sides. Hence, we have shown that (34) holds. Next, for some choice of x, while respecting $P/2 \leq x \leq P$, we have

$$|f| \ll PB^{-2} |W(x)| + B^2. \tag{36}$$

As we have fixed the variable x to this choice, we will drop it from the notation going forward. Let

$$p \coloneqq \lfloor \sigma k \rfloor + 1 \quad \text{and} \quad q \coloneqq \lfloor p\omega \rfloor + 1 \tag{37}$$

be set parameters. The positive numbers $\sigma \geq 2 > \omega > 0$ are later assigned numerical values. Writing

$$A_j \coloneqq \frac{(\alpha x^k)^{(j)}}{j!}$$

for the j-th derivative with respect to x, Taylor's formula gives

$$\alpha(x+yz)^k - \sum_{0\le j\le p} A_j(yz)^j \ll A_{p+1}B^{2p+2} \tag{38}$$

in (35). Note that we know for every $j \in \mathbb{N}$

$$|A_j| \asymp |\alpha|P^{k-j} \asymp P^{\lambda k-j}, \tag{39}$$

where the implied constants may depend on j, which is unproblematic since we use this asymptotic only for finitely many j. Furthermore, here the condition that k is non-integer is crucial. If we had $k = j$ for some $j \in \mathbb{N}$, starting with A_{j+1}, the derivates were all zero and hence in no dependency of P. Combining (37), (38) and (39) and observing that $\mathrm{e}(x) - 1 \ll |x|$, we have

$$W \ll |U| + B^2 A_{p+1} B^{2p+2} \ll |U| + B^2 P^{\lambda k}(PB^{-2})^{-(p+1)}, \tag{40}$$

where

$$U \coloneqq \sum_{1\le y\le B} \sum_{1\le z\le B} \mathrm{e}(A_1 yz + \cdots + A_p(yz)^p). \tag{41}$$

We choose B to have the value

$$B \coloneqq \lfloor P^{(\sigma-\lambda)/(2\sigma)} \rfloor, \tag{42}$$

then using (37) we gather

$$P^{\lambda k}(PB^{-2})^{-(p+1)} \ll P^{\lambda k - \lambda(\sigma k+1)/\sigma} \ll P^{-\lambda/\sigma}. \tag{43}$$

Combining (36), (40), (42) and (43), we arrive at

$$|f| \ll PB^{-2}|U| + P^{1-\lambda/\sigma}. \tag{44}$$

Now we estimate the sum U. Applying Hölder's inequality to the sum over y in (41), we have

$$\begin{aligned} |U|^{2pq} &\ll B^{2pq-1} \sum_{1\le y\le B} \Big| \sum_{1\le z\le B} \mathrm{e}(A_1 yz + \cdots + A_p(yz)^p) \Big|^{2pq} \\ &= B^{2pq-1} \sum_{1\le \boldsymbol{z}\le B} \sum_{1\le y\le B} \mathrm{e}(A_1 yZ_1 + \cdots + A_p y^p Z_p). \end{aligned} \tag{45}$$

Here we write $\boldsymbol{z} \coloneqq (z_1, \dots, z_{2pq})$, additionally

$$Z_j \coloneqq Z_j(\boldsymbol{z}) \coloneqq z_1^j + \dots z_{pq}^j - (z_{pq+1}^j + \cdots + z_{2pq}^j), \tag{46}$$

and $\sum_{1\le \boldsymbol{z}\le B}$ as a shorthand notation for the summation over $1 \le z_1, \dots, z_{2pq} \le B$. For any lattice point $\boldsymbol{\eta} \in \mathbb{Z}^p$ we define the indicator function

$$\mathbb{1}_{\boldsymbol{\eta}}((Z_1(\boldsymbol{z}), \dots, Z_p(\boldsymbol{z}))) \coloneqq \begin{cases} 1 & (Z_1(\boldsymbol{z}), \dots, Z_p(\boldsymbol{z})) = \boldsymbol{\eta} \\ 0 & \text{otherwise} \end{cases}$$

and with this, the sum

$$E(\boldsymbol{\eta}) \coloneqq \sum_{1\le \boldsymbol{z}\le B} \mathbb{1}_{\boldsymbol{\eta}}((Z_1(\boldsymbol{z}),\dots,Z_p(\boldsymbol{z})))$$

is counting how often $\boldsymbol{\eta}$ is a solution to the equations defined in (46). Now we again use Hölder's inequality, this time to the sum over $\boldsymbol{z}$ in (45) and get

$$\begin{aligned}|U|^{4p^2q^2} &\le (B^{2pq-1})^{2pq}(B^{2pq})^{2pq-1}\sum_{1\le \boldsymbol{z}\le B}\Big|\sum_{1\le y\le B}\mathrm{e}(A_1Z_1y+\cdots+A_pZ_py^p)\Big|^{2pq}\\ &= B^{p^2q^2-4pq}\sum_{\boldsymbol{\eta}}E(\boldsymbol{\eta})\Big|\sum_{1\le y\le B}\mathrm{e}(A_1\eta_1y+\cdots+A_p\eta_py^p)\Big|^{2pq}. \qquad (47)\end{aligned}$$

Noting the possible range for $\boldsymbol{\eta}$, we only need to sum over the box

$$\Lambda \coloneqq \{\boldsymbol{\eta}\in\mathbb{Z}^p : -pB^j < \eta_j < pB^j,\quad (j=1,\dots,p)\}.$$

Using a device of Hua, one has

$$E(\boldsymbol{\eta}) = \int_{\mathcal{U}_p}\Big|\sum_{1\le z\le B}\mathrm{e}(t_1z+\cdots+t_pz^p)\Big|^{2pq}\mathrm{e}(-(t_1\eta_1+\cdots+t_p\eta_p))\mathrm{d}\boldsymbol{t},$$

where $\mathcal{U}_p$ denotes the p-dimensional unit cube. So, Parseval's relation yields

$$\sum_{\boldsymbol{\eta}\in\Lambda}E(\boldsymbol{\eta})^2 = \int_{\mathcal{U}_p}\left|\sum_{1\le z\le B}\mathrm{e}(t_1z+\cdots+t_pz^p)\right|^{4pq}\mathrm{d}\boldsymbol{t} \eqqcolon J_{2pq,p}(B).$$

With Bourgain, Demeter and Guth [4, Theorem 1.1], we get the bound

$$J_{2pq,p}(B) \ll B^{4pq-p(p+1)/2} \qquad (48)$$

if $4pq > p(p+1)$. From the definition of q, we thus retrieve the condition

$$\omega > (1+1/p)/4 \qquad (49)$$

for the future choice of ω. Using Cauchy's inequality in (47), we obtain

$$\begin{aligned}|U|^{8p^2q^2} &\le B^{16p^2q^2-8pq}\sum_{\boldsymbol{\eta}\in\Lambda}E(\boldsymbol{\eta})^2\sum_{\boldsymbol{\eta}\in\Lambda}\Big|\sum_{1\le y\le B}\mathrm{e}(A_1\eta_1y+\cdots+A_p\eta_py^p)\Big|^{4pq}\\ &\ll B^{16p^2q^2-4pq-p(p+1)/2}\sum_{\boldsymbol{\eta}\in\Lambda}\Big|\sum_{1\le y\le B}\mathrm{e}(A_1\eta_1y+\cdots+A_p\eta_py^p)\Big|^{4pq}. \qquad (50)\end{aligned}$$

To evaluate the inner sum, we want to apply Lemma 5.3. in Vaughan [22]. Therefore, we divide up the integers in Λ, defining some auxiliary parameters. We start by defining m to be the largest integer with

$$A_m \gg 1.$$

Since $A_m \asymp P^{\lambda k-m}$, we want to choose m in a way such that $0 \le \lambda k - m \le 1$ is satisfied, hence we set

$$m \coloneqq \lfloor \lambda k \rfloor.$$

Moreover, define u to be the largest integer with

$$2pB^u A_u \gg 1.$$

From

$$B^u A_u \asymp P^{u(\sigma-\lambda)/(2\sigma)+\lambda k-u},$$

we gather similarly the choice

$$u \coloneqq \left\lfloor \frac{2\lambda\sigma k}{\lambda+\sigma} \right\rfloor. \tag{51}$$

With these parameters, we now structure the values in Λ component-wise further. For an index $1 \le j \le p$, we divide up the integers in the interval $(-pB^j, pB^j)$ into M_j many sets $S(j,i)$ as follows.

(i) If $m = 0$, consider only (ii) and (iii). If $m > 0$, then for $1 \le j \le m$, every set $S(j,i)$ consists of a single element of $(-pB^j, pB^j)$.

(ii) For $m < j \le u$, the set $S(j,i)$ is the set of integers in an interval of length smaller than $(2A_j)^{-1}$. This may be accomplished with

$$M_j \ll B^j A_j + 1 \ll B^j P^{\lambda k - j}, \tag{52}$$

where the last estimate follows from (51).

(iii) For $u < j \le p$, the set $S(j,i)$ is the intersection of $(-pB^j, pB^j)$ with a residue class modulo $\lfloor B^{-j} A_j^{-1} \rfloor + 1$. In this case, we know

$$M_j \ll B^{-j} A_j^{-1} + 1 \ll B^{-j} P^{j-\lambda k} \tag{53}$$

from the definition of u.

Let $1 \le i(j) \le M_j$ for $j = 1, \dots, p$ be any choice of p indices and define the set

$$\gamma \coloneqq \{(A_1\eta_1, \dots, A_p\eta_p) \in \mathbb{R}^p : \eta_j \in S(j, i(j)) \quad (j = 1, \dots, p)\}.$$

Now we assert that γ satisfies the hypothesis of Vaughan [22, Lemma 5.3] with values of Δ_j satisfying

$$\Delta_j \gg B^{-j} \quad (j = 1, \dots, p). \tag{54}$$

To see this, let $(A_1\zeta_1, \dots, A_p\zeta_p)$ and $(A_1\eta_1, \dots, A_p\eta_p)$ be distinct points of γ. Then $\zeta_j \ne \eta_j$ with $j > m$. On the one hand, if $j \le u$, then, from (ii), we have

$$A_j(2A_j)^{-1} > |A_j\zeta_j - A_j\eta_j| \ge A_j \gg B^{-j}.$$

The last inequality holds because of (51). It follows

$$\|A_j\zeta_j - A_j\eta_j\| \gg B^{-j}. \tag{55}$$

On the other hand, if $j > u$, then from (iii), we deduce

$$1/2 > 2pB^j A_j > |A_j\zeta_j - A_j\eta_j| \ge A_j(\lfloor B^{-j} A_j^{-1} \rfloor + 1) \gg B^{-j}.$$

The first inequality follows from the definition of u. Again, we have (55), leading to (54). Now we apply Vaughan [22, Lemma 5.3] for

$$\mathrm{S}(\boldsymbol{\beta}) \coloneqq \left\{ \sum_{1 \le y \le B} \mathrm{e}(\beta_1 y + \beta_2 y^2 + \cdots + \beta_p y^p) \right\}^{2pq}$$

and $\mathrm{N}_j \coloneqq 2pqB^j$ for $j = 1, \ldots, p$. This gives

$$\begin{aligned} \sum_{\eta_1 \in S(1,i(1))} \cdots \sum_{\eta_p \in S(p,i(p))} \left| \sum_{1 \le y \le B} \mathrm{e}(A_1\eta_1 y + \cdots + A_p\eta_p y^p) \right|^{4pq} &\ll B^{p(p+1)/2} \int_{\mathcal{U}_p} |\mathrm{S}(\beta)|^2 \mathrm{d}\beta \\ &\ll B^{4pq} \end{aligned} \tag{56}$$

by (54) and an application of (48). Summing over all sets of indices $i(1) \le M_1, \ldots, i(p) \le M_p$, we find from (50), (56) that

$$\begin{aligned} |B^{-2}U|^{8p^2q^2} &\ll B^{-p(p+1)/2} M_1 \cdot \ldots \cdot M_p \\ &\ll B^{-p(p+1)/2} \prod_{1 \le j \le m} B^j \prod_{m < j \le u} B^j P^{\lambda k - j} \prod_{u < j \le p} B^{-j} P^{j - \lambda k} \end{aligned} \tag{57}$$

holds, in view of (52) and (53). The last expression in (57) is bounded by

$$\ll B^{-p(p+1)+u(u+1)} P^{\lambda k(2u-p-m)+(p(p+1)/2)+(m(m+1)/2)-u(u+1)} \ll P^{-\mu}, \tag{58}$$

here setting

$$\mu \coloneqq \frac{\sigma - \lambda}{2\sigma} \left(p(p+1) - u(u+1)\right) - \lambda k(2u - p - m) - \frac{p(p+1)}{2} - \frac{m(m+1)}{2} + u(u+1).$$

We note the condition $\sigma > \lambda$ for the parameter σ, which is guaranteed since we have $\lambda < 2$. We rewrite μ as

$$\mu = -\frac{\lambda}{2\sigma} p(p+1) + u(u+1)\left(\frac{1}{2} + \frac{\lambda}{2\sigma}\right) - \lambda c(2u - p - m) - \frac{m(m+1)}{2}.$$

We move on by deriving a lower bound for μ by splitting terms into their fractional and integer part. This entails

$$\begin{aligned} \xi_1 &\coloneqq \{\sigma k\}, \\ \xi_2 &\coloneqq \left\{ \frac{2\lambda\sigma k}{\lambda + \sigma} \right\} \quad \text{and} \\ \xi_3 &\coloneqq \{\lambda k\}. \end{aligned}$$

With this, we rephrase the corresponding variables as

$$\begin{aligned} p &= \lfloor \sigma k \rfloor + 1 = \sigma k + 1 - \xi_1, \\ u &= \left\lfloor \frac{2\lambda\sigma k}{\lambda + \sigma} \right\rfloor = \frac{2\lambda\sigma k}{\lambda + \sigma} - \xi_2 \quad \text{and} \\ m &= \lfloor \lambda k \rfloor = \lambda k - \xi_3. \end{aligned}$$

Plugging this into μ and rearranging, we get

$$\begin{aligned}\mu = &-\frac{\lambda}{2\sigma}(\sigma^2k^2+3\sigma k+2-\xi_1(2\sigma k+3)+\xi_1^2)\\ &+\left(\frac{1}{2}+\frac{\lambda}{2\sigma}\right)\left(\frac{(2\lambda\sigma k)^2}{(\lambda+\sigma)^2}+\frac{2\lambda\sigma k}{\lambda+\sigma}-\xi_2\left(\frac{4\lambda\sigma k}{\lambda+\sigma}+1\right)+\xi_2^2\right)\\ &-\lambda k\left(\frac{4\lambda\sigma k}{\lambda+\sigma}-\sigma k-\lambda k-1+\xi_1-2\xi_2+\xi_3\right)\\ &-\frac{1}{2}\left(\lambda^2k^2+\lambda k-\xi_3(2\lambda k+1)+\xi_3^2\right).\end{aligned}$$

Firstly, we only collect the terms containing fractional parts and summarise them to the term

$$\frac{\lambda}{2\sigma}\Big(3\xi_1-\xi_1^2-\xi_2+\xi_2^2\Big)+\frac{1}{2}\Big(-\xi_2+\xi_2^2+\xi_3-\xi_3^2\Big). \tag{59}$$

Taking into account the values ξ_i $(i=1,2,3)$ attain and considering the respective minima and maxima, it follows that (59) is bounded from below by

$$-\frac{\lambda+\sigma}{8\sigma}. \tag{60}$$

Secondly, condensing the remaining terms without fractional part to

$$\frac{\lambda(\lambda-\sigma)^2k^2}{2(\lambda+\sigma)}-\frac{\lambda}{\sigma} \tag{61}$$

we conclude

$$\mu\geq\frac{\lambda(\lambda-\sigma)^2k^2}{2(\lambda+\sigma)}-\frac{\lambda}{\sigma}-\frac{\lambda+\sigma}{8\sigma}=\frac{\lambda(\lambda-\sigma)^2k^2}{2(\lambda+\sigma)}-\left(\frac{9\lambda}{8\sigma}+\frac{1}{8}\right). \tag{62}$$

As an interlude, we assert that the term containing μ dominates the estimate for f. From (44), (57) and (58) we got that

$$|f|\ll P^{1-\mu/(8p^2q^2)}+P^{1-\lambda/\sigma} \tag{63}$$

holds. Hence, it remains to show

$$\frac{\mu}{8p^2q^2}\leq\frac{\lambda}{\sigma}. \tag{64}$$

For this, we again consider the decomposition of μ into the terms containing fractional parts and by a similar argument as displayed in (59)-(60), we bound these terms by

$$\frac{9\lambda}{8\sigma}+\frac{1}{8}.$$

Combining this with (61), we have

$$\mu\leq\frac{\lambda(\lambda-\sigma)^2k^2}{2(\lambda+\sigma)}-\frac{\lambda}{\sigma}+\frac{9\lambda}{8\sigma}+\frac{1}{8}\leq\frac{\lambda(\lambda-\sigma)^2k^2+1}{4},$$

in the last estimate using that $\sigma \geq 2$ holds. By definition, we have $p \geq \sigma k$ and by assumption (49) we know $q > (p+1)/4$, consequently

$$8p^2q^2 > \sigma^4 k^4/2$$

follows. From this, we conclude (64), hence the term containing μ is the relevant one for the proof of the lemma. Therefore, we further estimate $\mu/(8p^2q^2)$ from below, with use of (62). By definition, note that

$$p \leq k\left(\sigma + \frac{1}{k}\right) \quad \text{and} \quad q \leq k\left(\omega\sigma + \frac{\omega+1}{k}\right)$$

holds. We now also employ the information $k > 11.5$, whenever useful. For example, for $\sigma \coloneqq 2$ we set $\omega \coloneqq 0.261$ to satisfy (49). With this, for the constant

$$d \coloneqq 8\left(2 + \frac{1}{11.5}\right)^2 \left(0.261 \cdot 2 + \frac{0.261+1}{11.5}\right)^2,$$

we estimate the denominator as $8p^2q^2 \leq dk^4$. Thus, to arrive at the bound displayed in Lemma 4.4, we need to show

$$\frac{\frac{\lambda(\lambda-2)^2k^2}{2(\lambda+2)} - \left(\frac{9\lambda}{16} + \frac{1}{8}\right)}{dk^4} \geq \frac{D\lambda}{k^2},$$

which is equivalent to

$$\frac{(\lambda-2)^2}{2(\lambda+2)d} - \frac{9}{16dk^2} - \frac{1}{8\lambda dk^2} - D \geq 0, \tag{65}$$

taking $k > 11.5$ and $\lambda \in [1/(2k), 1 + D/(2k^4)]$ into account to divide by λ/k^2. Making further use of these constraints, and deducing $k\lambda \geq 1/2$, (65) is implied if we showed

$$\frac{(\lambda-2)^2}{2(\lambda+2)d} - \frac{9}{16 \cdot 11.5^2 d} - \frac{1}{4 \cdot 11.5 d} - D \geq 0. \tag{66}$$

This term is monotonically decreasing as a function of one variable for increasing λ. Note that the largest possible value for λ in its respective ranges for all $k > 11.5$ is

$$\lambda = 1 + \frac{D}{2 \cdot 11.5^4},$$

for which (66) is indeed satisfied. Therefore, we deduce $\mu/8p^2q^2 \geq D\lambda v^2$ and combining this with (63) and (64), we have

$$|f| \ll P^{1-D\lambda v^2}. \tag{67}$$

At the beginning of the proof, we rewrote $|\alpha|P^k = P^{\lambda k}$. To translate this back, we rearrange it to the form

$$P = |\alpha|^{1/(\lambda k)} P^{1/\lambda},$$

which then gives the desired estimate

$$|f(\alpha)| \ll P(|\alpha|n)^{-Dv^3}.$$

To finish, we add the proof for the case $1 < k \leq 11.5$. This is an application of Segal [21, Lemma 2], which we now state for the case of the $\lceil k \rceil + 1$-th derivative and adapted to our notation.

Lemma 4.5 *Let the real function αx^k have in the interval $P/2 \leq x \leq P$ the derivatives $(\alpha x^k)', (\alpha x^k)'', \ldots, (\alpha x^k)^{(\lceil k \rceil+1)}$. If $(\alpha x^k)^{(\lceil k \rceil+1)}$ be positive in the named interval $P/2 \leq x \leq P$ and*

$$(\alpha x^k)^{(\lceil k \rceil+1)} \asymp V^{-1},$$

where V is > 0 and does not depend on x, then we have

$$|f(\alpha)| \ll P\Big(V^{\frac{-1}{2^{\lceil k \rceil+1}-2}} + P^{\frac{-1}{2^{\lceil k \rceil}}} + V^{\frac{1}{2^{\lceil k \rceil}}} P^{\frac{-(\lceil k \rceil+1)}{2^{\lceil k \rceil}}}\Big).$$

We treat the negative range for α here, the positive range follows by considering $-\alpha x^k$ in Lemma 4.5 and processing the resulting complex conjugation in $f(\alpha)$. We check that the conditions of this lemma are satisfied in our case. The derivatives of αx^k do indeed exist for the range

$$\alpha \in \left(-n^{Dv^4/2}, -\tfrac{v}{2} n^{-1+v}\right]. \tag{68}$$

The derivative in question is

$$(\alpha x^k)^{(\lceil k \rceil+1)} = \alpha k(k-1)\ldots(k-\lceil k \rceil) x^{k-\lceil k \rceil-1},$$

which is positive in the proposed range for x and $\alpha < 0$ and the asymptotic

$$(\alpha x^k)^{(\lceil k \rceil+1)} \asymp V^{-1},$$

for $V \coloneqq |\alpha|^{-1} n^{-1+(\lceil k \rceil+1)/k}$ holds. Thus, we get by the lemma the estimate

$$|f(\alpha)| \ll P\left((|\alpha| n^{1-\frac{\lceil k \rceil+1}{k}})^{\frac{1}{2^{\lceil k \rceil+1}-2}} + n^{\frac{-1}{k 2^{\lceil k \rceil}}} + (|\alpha| n)^{\frac{-1}{2^{\lceil k \rceil}}}\right), \tag{69}$$

plugging in our definitions. To prove the claim of Lemma 4.4, this is estimated against the upper bound

$$P(|\alpha| n)^{-Dv^3}.$$

For the first summand, being the most restrictive, we have to show

$$|\alpha|^{\frac{1}{2^{\lceil k \rceil+1}-2}+\frac{D}{k^3}} \ll n^{(-1+\frac{\lceil k \rceil+1}{k})\frac{1}{2^{\lceil k \rceil+1}-2}-\frac{D}{k^3}}.$$

Using $|\alpha| < n^{Dv^4/2}$ and estimating further, we arrive at the condition

$$\frac{D}{2k^4}\left(\frac{1}{2^k}+\frac{D}{k^3}\right) - \frac{1}{k 2^{k+2}} + \frac{D}{k^3} \leq 0 \tag{70}$$

that would guarantee this. Note that the left-hand side is monotonically increasing as a function of one variable in k for $1 < k \leq 11.5$. Inserting the values $D = 0.01$ and $k = 11.5$, the inequality (70) is satisfied and hence also holds for all small values of $1 < k \leq 11.5$. The other two summands in (69) are treated in a similar manner, leading to less restrictive conditions, which are thus also satisfied. Hence, also for the remaining values $1 < k \leq 11.5$, the bound

$$|f(\alpha)| \ll P(|\alpha| n)^{-Dv^3}$$

holds, which concludes the proof. □

This lemma finds application on the minor arcs. In addition, the shape of the bound displayed in this lemma directly impacts the form of the effective bound, as shown in section 6.

5 Treatment of the arcs

In this chapter, the previous results are put into effect for the different arcs. We begin with the trivial arcs, then proceed to the minor arcs, and finally address the major arcs. This results in conditions on the parameter ρ, which are then transformed into an effective bound in section 6.

5.1 The trivial arcs

The trivial arcs, defined as

$$\mathfrak{t} \coloneqq \{\alpha \in \mathbb{R} : |\alpha| > n^{\delta}\},$$

cover the tails of the integral, where the form of the kernels keeps the absolute value of the integral small. We deduce from Remark 1 that

$$\int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \sqrt{K(\alpha)} \mathrm{d}\alpha \ll n^{\tau-1+\rho}$$

is true.
Hence, we estimate the first sum f_0 trivially and split the kernel to receive

$$\begin{aligned}\left| \int_{\mathfrak{t}} A(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha \right| &\ll n^{v_0} \int_{n^{\delta}}^{\infty} |T_{1,2}(\alpha)| \alpha^{-2} \sqrt{K(\alpha)} \mathrm{d}\alpha \\ &\ll n^{v_0-2\delta} \int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \sqrt{K(\alpha)} \mathrm{d}\alpha \\ &\ll n^{\theta-1+\rho-2\delta} = \mathcal{o}(n^{\theta-1})\end{aligned}$$

if we assume

Condition 1 $\rho < 2\delta$.

We see in section 6 that this is ensured by the condition in Theorem 1.3.

5.2 The minor arcs

Parallel to the last chapter, we prove that the contribution of the integral on the minor arcs is also negligible, which means

$$\left| \int_{\mathfrak{m}} A(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha \right| = \mathcal{o}(n^{\theta-1}).$$

We begin with estimating the sum f_0 by Lemma 4.4, which holds exactly on the minor arcs. Here, we achieve a saving compared to the trivial estimate of

$$(|\alpha| n)^{-Dv_0^3},$$

in which we insert the smaller endpoint of the integral $|\alpha| = \frac{v_0}{2} n^{-1+v_0}$. It follows

$$\begin{aligned}\left| \int_{\mathfrak{m}} f_0(\alpha) T_{1,2}(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha \right| &\ll n^{v_0 - D v_0^4} \int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \mathrm{d}_K \alpha \\ &\ll n^{\theta - 1 + \rho - D v_0^4} \\ &= \mathcal{O}(n^{\theta-1})\end{aligned}$$

if

Condition 2 $\rho < D v_0^4$

is presupposed.

5.3 The $\mathfrak{M}$ajor arcs

The purpose of this section is to prove that the major arcs constitute the main term for the circle method approach to the problem. To this end, we also assume the two conditions from the minor and trivial arcs and resort to them.

Theorem 5.1 *With the notation from before, we have*

$$\int_{\mathfrak{M}} A(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha \gg n^{\theta-1}.$$

5.3.1 The overall idea

A common technique on the major arcs is to replace exponential sums with integrals and to show that the error is negligible. Here, we realise this for the first factor f_0. Hence we set

$$\varphi_0(t) \coloneqq \int_{I_0} \mathrm{e}(t x_0^{k_0}) \mathrm{d}x_0$$

and note that this is trivially estimated by $|\varphi_0| \ll P_0$. Now, we are ready to state the two main steps for proving Theorem 5.1. We establish that replacing the first sum with an integral produces a negligible error.

Lemma 5.2 *With the notation from before, we have*

$$\int_{\mathfrak{M}} f_0(\alpha) T_{1,2}(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha = \int_{\mathfrak{M}} \varphi_0(\alpha) T_{1,2}(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha + \mathcal{O}(n^{\theta-1}).$$

Furthermore, this modified term constitutes the main term.

Lemma 5.3 *We find the lower bound*

$$\int_{\mathfrak{M}} \varphi_0(\alpha) T_{1,2}(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha \gg n^{\theta-1}.$$

The following subsections contain the proofs for these steps, together with auxiliary results.

5.3.2 Replacing sums by integrals

The first goal is to prove Lemma 5.2, for which we replace the first sum f_0 with the corresponding integral φ_0 with an acceptable error. Before we prove this, we include one result vital to substituting sums by integrals.

Lemma 5.4 *Whenever* $|\alpha| \leq \frac{v_0}{2} n^{-1+v_0}$ *holds, we have*

$$f_0(\alpha) = \varphi_0(\alpha) + \mathcal{O}(1).$$

Proof. This follows by applying van der Corput's lemma, as described for example in Davenport [9, Lemma 9.1] with, in regard to their notation, $\mathrm{f}(x_0) \coloneqq \alpha x_0^{k_0}$ and sum parameters $\mathrm{A} \coloneqq P_0/2$ and $\mathrm{B} \coloneqq P_0$.
We first apply this lemma only to the range $0 \leq \alpha \leq \frac{v_0}{2} n^{-1+v_0}$. Then we have

$$\mathrm{f}'(x_0) = \alpha k_0 x_0^{k_0-1},$$

which is estimated as

$$0 \leq \mathrm{f}'(x_0) \leq \tfrac{v_0}{2} n^{-1+v_0} k_0 n^{v_0(k_0-1)} = 1/2$$

and thus meets this prerequisite of the lemma. Further, we calculate

$$\mathrm{f}''(x_0) = \alpha k_0(k_0-1)x^{k_0-2} \geq 0$$

and hence conclude

$$f_0(\alpha) = \varphi_0(\alpha) + \mathcal{O}(1)$$

for the non-negative range of α. The proof for the negative range works in an analogous manner by setting $\mathrm{f}(x_0) \coloneqq -\alpha x_0^{k_0}$. □

Now, we have acquired everything we need to prove Lemma 5.2.

Proof of Lemma 5.2. Using Theorem 4.1, we conclude

$$\begin{aligned}\int_{\mathfrak{M}} |f_0(\alpha) - \varphi_0(\alpha)||T_{1,2}(\alpha)| \mathrm{d}_K\alpha &\ll \int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \mathrm{d}_K\alpha \\ &\ll n^{\theta-1+\rho-v_0}.\end{aligned}$$

We note that Condition 2 guarantees $\rho < v_0$. Then, one has

$$\int_{\mathfrak{M}} |f_0(\alpha) - \varphi_0(\alpha)||T_{1,2}(\alpha)| \mathrm{d}_K\alpha = o(n^{\theta-1}).$$

□

For the proof of Lemma 5.3, another ingredient is required, namely a bound for the factor φ_0.

Lemma 5.5 *For all values of* α *we have*

$$|\varphi_0(\alpha)| \ll n^{v_0}(1+n|\alpha|)^{-1}.$$

Proof. Firstly, substituting $u \coloneqq x_0^{k_0}$ we get

$$|\varphi_0(\alpha)| = \left| \int_{n/2^{k_0}}^{n} v_0 \mathrm{e}((\alpha)u) u^{v_0-1} \mathrm{d}u \right|,$$

by partial integration, we obtain secondly

$$\begin{aligned} |\varphi_0(\alpha)| &= \left| \frac{v_0}{2\pi i \alpha} \Big(\Big[\mathrm{e}(\alpha u) u^{v_0-1} \Big]_{n/2^{k_0}}^{n} - (v_0 - 1) \int_{n/2^{k_0}}^{n} \mathrm{e}(\alpha u) u^{v_0-2} \mathrm{d}u \Big) \right| \\ &\ll |\alpha|^{-1} n^{v_0-1}. \end{aligned}$$

This constitutes an improvement on the trivial estimate as long as $|\alpha| > n^{-1}$. Therefore, the desired result holds. □

5.3.3 Gaining the main term

We subdivide this argument into two steps. First, we consider the integral over the whole real line.

Lemma 5.6 *For the full integral over $\mathbb{R}$ we have*

$$\int_{-\infty}^{\infty} \varphi_0(\alpha) T_{1,2}(\alpha) \mathrm{e}(-\alpha n_0) \mathrm{d}_K \alpha \gg n^{\theta-1}.$$

Finally, we check that the contribution outside of $\mathfrak{M}$ is smaller than the desired main term.

Lemma 5.7 *The contribution outside of the major arcs is bounded by*

$$\int_{\mathbb{R}\setminus\mathfrak{M}} |\varphi_0(\alpha) T_{1,2}(\alpha)| \mathrm{d}_K \alpha = \mathcal{O}(n^{\theta-1}).$$

We begin by proving that overall, we achieve the main term with the integral containing φ_0.

Proof of Lemma 5.6. For the sake of the proof, we require some detailed definitions for the blocks of variables in use. In particular, we want to collect all variables of shortened ranges and all of the first and second block as well. Imitating the previously introduced definitions of $\mathbf{x}$ and $\mathbf{I}$, we now set

$$\mathbf{x}_1 \coloneqq (x_1, \dots, x_{s-1}) \quad \text{and} \quad \check{\mathbf{x}}_2 \coloneqq (x_2, \dots, x_{r-1}, x_{r+1}, x_{s-1}).$$

In line with this, we denote the ranges of definition

$$\mathbf{I}_1 \coloneqq \bigtimes_{1 \le i < s} I_i \quad \text{and} \quad \check{\mathbf{I}}_2 \coloneqq \bigtimes_{\substack{2 \le i < s \\ i \ne r}} I_i.$$

Furthermore, the sum over the shortened variables is written as

$$\sigma(\check{\mathbf{x}}_2) \coloneqq \sum_{\substack{2\leq i<s \\ i\neq r}} \lfloor x_i^{k_i} \rfloor$$

and we recall from (16) the sum with all variables

$$Q(\mathbf{x}) = x_0^{k_0} + \lfloor x_1^{k_1} \rfloor + \lfloor x_r^{k_r} \rfloor + \sigma(\check{\mathbf{x}}_2) - n_0.$$

In order to handle the expression in Lemma 5.6, we want to rediscover an identity given by a Fourier transform. For the expression consisting of sums, we showed in (21) that

$$\int_{-\infty}^{\infty} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha = \sum_{\mathbf{x}\in\mathbf{I}} \hat{K}(Q(\mathbf{x}))$$

is true. To get a similar expression for the terms in Lemma 5.6 we collect the summands into $Q(\mathbf{x})$ and deduce the 'sibling identity' to equation (21) as

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi_0(\alpha)T_{1,2}(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha &= \int_{I_0} \sum_{\mathbf{x}_1\in\mathbf{I}_1} \int_{-\infty}^{\infty} \mathrm{e}(\alpha Q(\mathbf{x}))\mathrm{d}_K\alpha\mathrm{d}x_0 \\ &= \sum_{\mathbf{x}_1\in\mathbf{I}_1} \int_{I_0} \hat{K}(Q(\mathbf{x}))\mathrm{d}x_0.\end{aligned}$$

With this, we proceed to the proof and substitute in the expression on the right the variable $y_0 \coloneqq x_0^{k_0}$. If we set

$$\tilde{Q}((y_0, \mathbf{x}_1)) \coloneqq y_0 + \lfloor x_1^{k_1} \rfloor + \lfloor x_r^{k_r} \rfloor + \sigma(\check{\mathbf{x}}_2) - n_0,$$

then the result of the substitution is

$$v_0 \sum_{\mathbf{x}_1\in\mathbf{I}_1} \int_{n/2^{k_0}}^{n} \hat{K}(\tilde{Q}((y_0, \mathbf{x}_1)))y_0^{v_0-1}\mathrm{d}y_0. \tag{71}$$

The goal now is to extract a region for the variables $(y_0, \mathbf{x}_1)$ in which the kernel $\hat{K}(\tilde{Q}((y_0, \mathbf{x}_1)))$ is positive, say larger than some positive real constant c. Due to the structure of the kernel, we choose a constant $a \in \mathbb{R}$ with $0 < a < 1/2$, such that

$$\tilde{Q}((y_0, \mathbf{x}_1)) \in (-a, a)$$

holds if and only if

$$\hat{K}(\tilde{Q}((y_0, \mathbf{x}_1))) > c.$$

To achieve this, we restrict the ranges of definition further, gaining more control over the attained values. Reducing the size of the domains and hoping for a lower bound is valid, as all factors occurring in equation (71) are non-negative. As a first step we restrict I_1 and I_r such that

$$\lfloor x_i^{k_i} \rfloor \in [n/2^{k_i}, n/3] \quad (i = 1, r)$$

holds. This is realised if the values of x_1 and x_r are in the intervals

$$x_i \in [P_i/2, P_i/3^{v_i}] \quad (i = 1, r).$$

In contrast, y_0 keeps the range of $[n/2^{k_0}, n]$. For the remaining variables, note that by the construction of $\check{\mathbf{I}}_2$, we know that for each block, the first factor attains the largest value. If we then compare these bounds n^{1-v_1} and n^{1-v_r}, we get by $k_1 \leq k_r$ that

$$0 \leq \sigma(\check{\mathbf{x}}_2) \ll n^{1-v_r}$$

holds. Hence,

$$n_0 - \sigma(\check{\mathbf{x}}_2) = n + \tfrac{1}{2} + \mathcal{O}(n^{1-v_r}),$$

which leads to

$$n_0 - \sigma(\check{\mathbf{x}}_2) \geq \tfrac{11}{12}n + \tfrac{1}{2}$$

for n large enough. Thus we have

$$n_0 - \sigma(\check{\mathbf{x}}_2) \in \left[\tfrac{11}{12}n + \tfrac{1}{2}, n + \tfrac{1}{2}\right].$$

For the inner integral with respect to y_0, the values of $\mathbf{x}_1$ are fixed at some point in their respective ranges. For any value that they sum up to, we choose

$$y_0 = -\left(\lfloor x_1^{k_1} \rfloor + \lfloor x_r^{k_r} \rfloor + \sigma(\check{\mathbf{x}}_2) - n_0\right)$$

to balance them such that $\tilde{Q}((y_0, \mathbf{x}_1)) \in (-a, a)$ holds. The values that y_0 attains for this balance then lie in the interval

$$\left[\tfrac{1}{4}n + \tfrac{1}{2}, (1 - \tfrac{1}{2^{k_1}} - \tfrac{1}{2^{k_r}})n + \tfrac{1}{2}\right] \subset \left[\tfrac{1}{2^{k_0}}n, n\right],$$

so this is always possible. Consequently, $\tilde{Q}((y_0, \mathbf{x}_1)) \in (-a, a)$ always holds for these modified intervals for $(y_0, \mathbf{x}_1)$, so we retrieve equivalently

$$\hat{K}(\tilde{Q}((y_0, \mathbf{x}_1))) > c.$$

Furthermore, the range for y_0, where this inequality is satisfied, is bounded by $2a$, hence of size $\mathcal{O}(1)$, due to $\mathbf{x}_1$ being fixed from the perspective of y_0. All other variables can range completely through their domains as chosen here, since we balance them by y_0. Their modified intervals are still of the same asymptotic measure, hence remain $\gg n^{\tau}$. Further, note that by definition

$$y_0^{v_0-1} \gg n^{v_0-1}$$

is true. By collecting this, we reach our goal to see

$$\begin{aligned} v_0 \sum_{\mathbf{x}_1 \in \mathbf{I}_1} \int_{n/2^{k_0}}^{n} \hat{K}(\tilde{Q}((y_0, \mathbf{x}_1))) y_0^{v_0-1} \mathrm{d}y_0 \\ \gg n^{\tau} n^{v_0-1} = n^{\theta-1}. \end{aligned}$$

□

As the final step, we set out to prove that the values taken outside of $\mathfrak{M}$ are too small to cancel the achieved main term.

Proof of Lemma 5.7. We consider cases analogous to the minor and trivial arcs. So we divide the complement of $\mathfrak{M}$ up into the familiar sets

$$\begin{aligned}\mathfrak{m} &= \{\alpha \in \mathbb{R} : \tfrac{v_0}{2} n^{-1+v_0} < |\alpha| \leq n^{\delta}\} \\ \mathfrak{t} &\coloneqq \{\alpha \in \mathbb{R} : |\alpha| > n^{\delta}\}.\end{aligned}$$

For $\mathfrak{m}$ we use Lemma 5.5, which gives for $|\alpha| > \frac{v_0}{2} n^{-1+v_0}$ that

$$|\varphi_0(\alpha)| \ll n^{v_0-1} |\alpha|^{-1} \ll 1$$

holds. We thus estimate with Theorem 4.1 the terms as

$$\int_{\mathfrak{m}} |\varphi_0(\alpha) T_{1,2}(\alpha)| \mathrm{d}_K \alpha \ll \int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \mathrm{d}_K \alpha \ll n^{\theta-1+\rho-v_0}.$$

Finally, we conclude with Condition 2 that

$$\int_{\mathfrak{m}} |\varphi_0(\alpha) T_{1,2}(\alpha)| \mathrm{d}_K \alpha = \mathcal{O}(n^{\theta-1}),$$

as proposed. Lastly, for $\mathfrak{t}$ we use the kernel K, for which we note similarly to the trivial arcs that

$$\int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \sqrt{K(\alpha)} \mathrm{d}\alpha \ll n^{\tau-1+\rho}$$

is the case. With this, estimating φ_0 trivially, we calculate

$$\begin{aligned}\int_{\mathfrak{t}} |\varphi_0(\alpha) T_{1,2}(\alpha)| \mathrm{d}_K \alpha &\ll n^{v_0} \int_{n^{\delta}}^{\infty} |T_{1,2}(\alpha) \alpha^{-2} \sqrt{K(\alpha)}| \mathrm{d}\alpha \\ &\ll n^{v_0-2\delta} \int_{-\infty}^{\infty} |T_{1,2}(\alpha)| \sqrt{K(\alpha)} \mathrm{d}\alpha \\ &\ll n^{\theta-1-2\delta+\rho} = \mathcal{o}(n^{\theta-1}),\end{aligned}$$

due to Condition 1. Hence, we conclude the proof of Lemma 5.7. □

As we also presented proofs to all steps, which together yield Theorem 5.1, we have completed the aim of this chapter.

6 Becoming effective

From the calculations on the arcs, we obtained the two conditions

$$\rho < 2\delta \quad \text{and} \quad \rho < Dv_0^4. \tag{72}$$

To balance these we have set $\delta = Dv_0^4/2$. Define the value of the balanced conditions as

$$\chi \coloneqq Dv_0^4,$$

then we want to deduce from the condition in (4) in the main theorem that $\rho < \chi$ holds. Now we specify the index r with $1 < r < s$ to be the smallest integer such that

$$\sum_{1\le i<r} v_i > 4\log(k_0) + \log\left(D^{-1}\right)$$

is satisfied. With (4), by observing $v_{r-1} \le 1/2$ and using that by construction

$$\sum_{1\le i<r-1} v_i \le 4\log(k_0) + \log\left(D^{-1}\right),$$

we then also deduce for the second block of exponents

$$\begin{aligned}\sum_{r\le i<s} v_i &> 8\log(k_0) + 9.72 - v_{r-1} - 4\log(k_0) - \log\left(D^{-1}\right)\\ &> 4\log(k_0) + \log\left(D^{-1}\right).\end{aligned}$$

So, we now have bounded the sums $\sum_{1\le i<r} v_i$ and $\sum_{r\le i<s} v_i$ from below by

$$4\log(k_0) + \log\left(D^{-1}\right) = \log(\chi^{-1}).$$

By rearranging and estimating against the Taylor expansion, we obtain for the first block

$$\log(\chi) > -\sum_{1\le i<r} v_i > -\sum_{1\le i<r}\sum_{1\le \ell\le\infty} \frac{v_i^\ell}{\ell} = \sum_{1\le i<r} \log(1-v_i) = \log\Big(\prod_{1\le i<r}(1-v_i)\Big).$$

This transfers to $\chi > \Pi_{1,r}$ and by the same argument to $\chi > \Pi_{r,s}$ for the second block. Hence, we have shown that from the condition in (4) and by definition of ρ in (23) indeed

$$\chi > (\Pi_{1,r} + \Pi_{r,s})/2 = \rho$$

follows.

7 Conclusion

Summarising the previous work, we conclude Theorem 1.3. From (21) we have for the count of the solutions of (5) that

$$R(n) \gg \int_{-\infty}^{\infty} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha$$

holds. We split this into

$$\int_{-\infty}^{\infty} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha = \left(\int_{\mathfrak{t}} + \int_{\mathfrak{m}} + \int_{\mathfrak{M}}\right) A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha$$

and deduce with section 5 that the minor and trivial arcs are bounded as

$$\left|\int_{\mathfrak{t}} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha + \int_{\mathfrak{m}} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha\right| = \mathcal{O}(n^{\theta-1})$$

and the major arc contribution

$$\int_{\mathfrak{M}} A(\alpha)\mathrm{e}(-\alpha n_0)\mathrm{d}_K\alpha \gg n^{\theta-1}$$

holds under assumption (4). Therefore, in summary we have that

$$\lfloor x_0^{k_0}\rfloor + \cdots + \lfloor x_{s-1}^{k_{s-1}}\rfloor = n$$

is solvable in integers for sufficiently large $n \in \mathbb{N}$.

From Theorem 1.3 also the weaker Theorem 1.1 follows. For the missing implication of the equivalence stated there, namely that the series over the reciprocals has to diverge, we refer to the proof in Scourfield [19], as it works completely analogously.

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