The distribution of rational points on some projective varieties

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Notation

In general, statements involving ϵ are assumed to hold for any sufficiently small positive values of ϵ. We use δ to denote a sufficiently small positive value but the exact value of δ may vary each time it arises. Following standard convention in analytic number theory we let $e(\alpha) = \exp(2\pi i \alpha)$. Vectors $(x_1, x_2, \ldots, x_s)$ are denoted as $\mathbf{x}$, where the dimension may vary from occasion to occasion and statements like $\mathbf{x} \leq X$ have to be read as $x_i \leq X$ for all $i = 1, 2, \ldots s$. We may write $|\mathbf{x}|_\infty$ to denote the maximum norm, i.e. $\max_{i=1}^s |x_i|$. Vinogradov's notation $\ll$ is used. For instance if $f = O(g)$, we may write $f \ll g$. The notation $\lfloor x \rfloor$ is used for the integer part of x.

1 Introduction

Given a polynomial $f(\mathbf{x})$ of degree k in s variables and integer coefficients it is a classical problem in number theory to determine whether or not the equation

$$f(\mathbf{x}) = 0$$

has integer solutions and - if so - how 'many'? To be accessible to analytic methods it is common to restrict ourself to the consideration of cases where s is larger than k. In this generic situation infinitely many solutions are likely to exist and one considers their density in boxes of size $X \geq 1$, that is by restricting the sizes of the variables: $|x_i| \leq X$. By letting X tend to infinity we get a quantitative answer to questions regarding the distribution of integer solutions.

A probability based crude heuristic predicts that the number of solutions in a box of size X should be of order of magnitude X^{s-k}. Let n be a natural number and write

$$f(\mathbf{x}) = x_1^k + x_2^k + \ldots + x_s^k \tag{1}$$

and consider the numbers of representation $r(n)$ of n as sum of s k-th powers, i.e. the number of solutions of $f(\mathbf{x}) = n$ with $x_i \leq X := n^{1/k}$. Applying the above heuristic to f we predict

$$r(n) = c(n) n^{s/k-1/k} (1 + o(1)) \tag{2}$$

solutions for n tending to infinity and for some $c(n)$, such that $c(n)$ satisfies $c < c(n) < C$ for constants c, C and all n. So if we are able to establish this asymptotic for some s large in terms of k together with the positivity of $c(n)$ we may deduce that every large enough n is representable in such a way. One may ask what is the minimal number $G(k)$ of variables s such that every sufficiently large n is representable as sum of s kth powers. An easy argument considering volumes shows that we have $G(k) > k$. A related problem is the corresponding number $g(k)$ of variables needed such that every natural number n is representable. This is known as Waring's problem and questions surrounding it are subject to active research stemming from a wide range of different branches of mathematics. Lagrange's four-square theorem from 1770 for instance may be reformulated as $g(2) = G(2) = 4$. $G(4)$

is known to equal 16 (Davenport [14]) but other exact values are unknown. Note that $G(k) \leq g(k)$ and that there are easy lower bounds on $g(k)$ (See Chapter 1 of [30]) and that there are upper bounds for $G(k)$ in terms of k and the conjectured asymptotic holds for $s \geq 2k^2 + 2k - 3$ variables which was established Wooley [33] - for a more recent improvement see Bourgain [6].

A commonly used analytic machinery to establish such asymptotic results on the zero-set of integer equations is the Hardy-Littlewood circle method. The main idea is to write the number of integer solutions to an equation as a complex integral over exponential sums which then may be approximated near rational numbers a/q. Let $g(\alpha) = \sum_{x \leq X} e\left(\alpha x^k\right)$ such that by orthogonality, (1) and the definition of $r(n)$ we have

$$r(n) = \int_0^1 g(\alpha)^s e\left(-n\alpha\right)\, d\alpha. \tag{3}$$

Due to amplification effects the problem gets easier if one increases the number of variables so that the interesting case is the one with s relatively small. Thus for s large against the degree k we can apply the circle method to the integral in (3). Trivially $g(0) = \lfloor X \rfloor$, and for $\alpha = a/q$ we may divide the summation over x into residue classes b mod q. That is

$$g\left(\frac{a}{q}\right) = \sum_{b=1}^{q} \sum_{\substack{x \leq X \\ x \equiv b \bmod q}} e\left(\frac{ax^k}{q}\right) = \frac{X}{q}\sum_{b=1}^{q} e\left(\frac{ab^k}{q}\right) + O(q).$$

Therefore if the complete exponential sum does not vanish, $g(\alpha)$ is expected to be large close to rational numbers with small denominator. Now the 'circle' $\mathbb{R}/\mathbb{Z}$ is divided into the α close to a/q with q smaller than Q, the so called major arcs $\mathfrak{M}$ and their counterpart - the minor arcs $\mathfrak{m}$. The idea is to control the contribution of the minor arcs to (3) by bounding the size of $g(\alpha)$ on $\mathfrak{m}$ combined with a mean value estimate for an appropriate m-th moment for $g(\alpha)$. For instance $\int_0^1 |g(\alpha)|^2\, d\alpha = \lfloor X \rfloor$ so we expect some cancellation. The treatment of the major arcs generalizes the idea to evaluate $g(\alpha)$ at a/q by writing $\alpha = a/q + \beta$, and then obtaining an asymptotic evaluation on $\mathfrak{M}$ which will produce the main term in (3) provided the number of variables is large enough.

Concerning Waring's problem for cubes in $s = 8$ variables the most recent result is due to Vaughan [31] where the asymptotic in (2) takes the shape (in this form with an improved log exponent due to Boklan [5])

$$r(n) = c(n)n^{5/3} + O\left(n^{5/3}\left(\log n\right)^{-3+\epsilon}\right)$$

which rests on his celebrated 8-th moment estimate on the minor arcs. One expects an asymptotic formula to hold for $s \geq 4$, but this seems far out of reach with methods currently available.

By considering equations of the type (1) we are also entering the realm of algebraic geometry which is well known to be linked with the study of rational solutions of polynomial equations. Given a number field K and a projective variety X a series of conjectures is linked with the set of K-rational points $X(K)$. If X is a Fano variety endowed with some anticanonical height function $H\colon X(K) \to \mathbb{R}$ then Manin's conjecture (cf. [17]) is concerned with linking the number of rational points of bounded height on some nice open subset $U \subset X$

$$N_{H,U}(B) = \#\{x \in U(K)) : H(x) \leq B\}$$

with the variety's inner geometry. The conjecture states that

$$N_{H,U} \sim CB(\log B)^{r-1},$$

where C is some constant and r is the rank of the Picard group of $X(K)$. An interpretation of the constant C is given by Peyre [24]. A classical result due to Birch [2] can be seen in this context, the rank of the Picard group being one in that case so the logarithmic factor is not visible.

A fruitful testing ground around the conjecture lies in bihomogeneous varieties, where Manin's conjecture has been established for complete intersections of large dimension by Schindler [27] using the circle method. Although similar to Birch's work the number of variables is rather large. Consider the family of varieties $X_k^s \subset \mathbb{P}(K)^{s-1} \times \mathbb{P}(K)^{s-1}$ given by

$$x_1 y_1^k + x_2 y_2^k + \ldots + x_s y_s^k = 0. \tag{4}$$

From this point on we may set our focus on $K = \mathbb{Q}$. Suppose we have $\mathbf{x} \in \mathbb{P}(\mathbb{Q})^{s-1}$ represented by a primitive vector $(x_1, \ldots, x_s) \in \mathbb{Z}^s$, then we may write

$$H(\mathbf{x}) = \max\{|x_i| : i = 1, \ldots, s\}$$

for the exponential height function and define a height function on X_k^s by writing

$$H(\mathbf{x}, \mathbf{y}) = H(\mathbf{x})^{s-1} H(\mathbf{y})^{s-k}$$

for a representative $(\mathbf{x}, \mathbf{y}) \in X_k^s$. The accumulating subvariety U_k^s is given by

$$x_1 x_2 \cdots x_s y_1 y_2 \cdots y_s \neq 0.$$

If $k = 1$ and $s \geq 3$ the a result which later inspired Manin's work was first proved by Bump ([12], Chapter 7) using meromorphic continuation of Eisenstein series. And subsequently it was established for $s \geq 4$ by Robbiani [26] using the circle method, which was improved upon by Spencer [28], who reduced the number of variables needed to $s \geq 3$ and work of Blomer and Brüdern [4] who achieved a second main term. For $k = 2$ and $s = 3$ there are sharp upper and lower bounds of the right order of magnitude by Le Boudec [23], who showed

$$B \log B \ll N_{U_3^2, H}(B) \ll B \log B.$$

For the case $k = 2$ and $s = 4$ there was recent progress of Browning and Heath-Brown [11], who proved Manin's conjecture for the quadric bundle

$$x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2 + x_4 y_4^2 = 0.$$

2 Main result

The next case to be considered is $k = 3$ and since an asymptotic formula (2) for Warings problem for cubes (cf. Vaughan [31]) is only attainable for $s \geq 8$ this is the most interesting and challenging case. With the introduction of some coefficients in (4) we consider a slight generalization of X_3^8. Let $\mathbf{c} \in \mathbb{Z}^8$ be a nonzero vector and consider the smooth bi-homogeneous variety X in $\mathbb{P}(\mathbb{Q})^7 \times \mathbb{P}(\mathbb{Q})^7$ given by

$$c_1 x_1 y_1^3 + c_2 x_2 y_2^3 + \ldots + c_8 x_8 y_8^3 = 0. \tag{5}$$

Let U be the subset given by $x_1 x_2 \cdots x_8 y_1 \cdots y_8 \neq 0$. We have the following result.

Theorem 1. *Let $c_1, c_2 \ldots, c_8$ be non-zero integers. Then there are positive numbers δ and C such that*

$$N_U(B) = CB \log B + O\left(B \log B (\log\log B)^{-\delta}\right).$$

That is Manin's conjecture holds for the variety X with respect to the removed subset U. It is worth noting that the constant C arising in the theorem is a product of local densities.

Before we go into details of the proof it is convenient to give a general outline of the underlying strategy and the main difficulties that need to be tackled. Following the popularity of analytic methods (namely the circle method) we follow the approach taken by Blomer and Brüdern [3] who considered the multihomogeneous variety given by

$$\sum_{j=0}^{n} a_j (x_{1,j} x_{2,j} \ldots x_{k,j})^d = 0$$

and proved a strong form of the conjecture with asymptotic expansion, i.e.

$$N_{U,H}(B) = CBQ(\log B) + O(B^{1-\delta})$$

for a suitable subset U and a polynomial Q of degree $k-1$.

The key reduction step in this paper enables us to reduce the counting problem by decoupling the height conditions. That is instead of having to deal with a condition of the type $|\mathbf{x}|_\infty |\mathbf{y}|_\infty \leq B$ we may discuss the independent conditions $|\mathbf{x}|_\infty \leq X$ and $|\mathbf{y}|_\infty \leq Y$. Then a suitable variant of [3] theorem 2.1 will produce our theorem once we can establish the corresponding asymptotic for equation (5) with $\mathbf{x}$ and $\mathbf{y}$ in independent boxes and similarly for $\mathbf{x}$ or $\mathbf{y}$ fixed and just $\mathbf{y}$ or $\mathbf{x}$ respectively in boxes. Thus our first objective is to establish an asymptotic formula for these cases. The situation with $\mathbf{x}$ fixed is essentially Waring's problem for cubes. We heavily rely on Vaughan's work [31] and use his minor arc estimate. It is worth mentioning that here is the first occasion where we find $s = 8$ to be an obstacle. According to the current state of knowledge of Waring's problem one cannot deal with 7 variables unconditionally.
The case where $\mathbf{y}$ is small, represents a traditional lattice point problem and is

dealt with accordingly. This leaves the independent box count. Here we distinguish two cases: Firstly the case when Y is small against some small power of X thereby ensuring that we can sum up the asymptotic for $|\mathbf{y}| \leq Y$. The second remaining case is where most hard work needs to be done. Although the proof is oriented along the lines of Vaughan [31] and [32] their key ingredients need to be reproduced in a two-dimensional setting. Vaughan's treatment of the cubic case in Waring's problem relies on an 8-th moment estimate for the minor arcs where he establishes logarithmic saving. Since the argument is built upon a sieving technique we need a 4-th moment estimate which is governed by diagonal solutions. Hence we need to show that for a suitable subset $\mathscr{E} \subset [1, Y]$ the number of solutions of

$$x_1 y_1^3 + x_2 y_2^3 = x_3 y_3^3 + x_4 y_4^3$$

with $x_i \leq X$ and $y_i \in \mathscr{E}$ is up to some small power of logarithms bounded by $|\mathscr{E}|^2 X^3$. For comparison in the case without the $\mathbf{x}$ variables i.e. the number of $y_i \in \mathscr{E}$ such that

$$y_1^3 + y_2^3 = y_3^3 + y_4^3 \tag{6}$$

is for $|\mathscr{E}|$ moderately large, bounded by $O(|\mathscr{E}|^2)$ since the number of solutions to (6) not lying on rational lines is $O\left(Y^{2-\delta}\right)$ for some $\delta > 0$. The particular shape of the set $\mathscr{E}$ will be the subset of numbers in the interval $[1, Y]$ that do not have prime divisors in a certain prescribed interval. The size of this set will save a logarithm over the trivial bound Y. We will obtain the required 4-th moment estimate by viewing the number of solutions counted by (1) as a weighted divisor sum. This is based on ideas of Wolke and Erdös while using a result of Pollack [25].

The teatment of the minor arcs is closely mimicking the proofs of Vaughan in [31] and [32], where we subdivide the cubic exponential sum on the minor arcs into certain classes and show that largest potential contribution actually comes from the sum over $\mathscr{E}$, that is

$$g_E(\alpha) = \sum_{x \leq X} \sum_{y \in \mathscr{E}} e\left(\alpha x y^3\right). \tag{7}$$

A reduction step due to Boklan [5] will then show that the minor arc contribution is bounded by

$$\int_0^1 |g_E(\alpha)|^8 \, d\alpha,$$

which is treated as in Vaughan [32].

Another crucial step in the analysis performed in Vaughan's argument is the availability of a major arc approximation for the exponential sum $g(\alpha) = \sum_{y \leq Y} e\left(\alpha y^3\right)$ with a good error term which is of use even on the minor arcs. This is done by Poisson summation together with square-root cancellation on average in shifts of the corresponding complete exponential sum $\sum_{y \bmod q} e\left(a y^3 / q\right)$. This then is coupled with the use of Hooley's delta function Δ to prove an analogue of Weyl's inequality for g that produces just a log factor instead of the Y^ϵ present in the application of the classical variant of the inequality. Of course these features have to be reproduced in our case.

The last step is to adapt the proof of the hyperbola type argument in [3] in order to be able to deal with our situation. Due to the adaptations necessary and the fact that we are in the situation where log savings have to suffice we are only able to secure a log log saving in the final theorem.

2.1 Lattices

We start this section by recording some results concerning lattices from Chapter 4.2 of Browning [10] on the geometry of numbers. For a general account on lattices we refer to Cassels [13]. Adapting the notation in [10], we say a lattice $\Lambda \subseteq \mathbb{Z}^n \subset \mathbb{R}^n$ is *primitive* if it has a basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$ that can be extended to a basis of $\mathbb{Z}^n$. For our purpose the notion of a *dual lattice* is of importance. Given a vector $\mathbf{x} \in \mathbb{R}^n$ we write $||\mathbf{x}||$ for the usual euclidean norm $||\mathbf{x}|| = \sqrt{x_1^2 + \ldots + x_n^2}$ and given another vector $\mathbf{y} \in \mathbb{R}^n$ we write $\mathbf{x}.\mathbf{y}$ for the standard scalar product.

Let $\Lambda \subseteq \mathbb{Z}^n$ be a primitive lattice of dimension r, then the *dual lattice* Λ^* is defined to be the lattice

$$\Lambda^* = \{\mathbf{x} : \mathbf{x}.\mathbf{y} = 0 \quad \forall \mathbf{y} \in \Lambda\}.$$

The lattice Λ^* is primitive and of dimension $n - r$. A particularly interesting case is the dual lattice corresponding to the 1-dimensional lattice spanned (over $\mathbb{Z}$) by a fixed primitive vector $\mathbf{a} \in \mathbb{Z}^n$.

Lemma 1. *Let $\mathbf{a}$ be a primitive vector, then the set*

$$\Lambda_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{Z}^n : \mathbf{x}.\mathbf{a} = 0\}$$

is a lattice of dimension $n - 1$ and determinant $||\mathbf{a}||$.

Proof. This is Lemma 4.4 from [10]. □

We may also cite Lemma 4.5 of Browning [10], which gives a bound on the number of lattice points inside a box of size R.

Lemma 2. *Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice of dimension r. Then we have*

$$\#\{\mathbf{x} \in \Lambda : |\mathbf{x}|_\infty \leq R\} \ll R^{r-1} + \frac{R^r}{\det \Lambda},$$

for any $R \geq 1$.

This however has the disadvantage of being just an upper bound and does not provide an asymptotic. Let $\mathbf{a} \in \mathbb{Z}^8$ be primitive and consider the 7-dimensional lattice $\Lambda_{\mathbf{a}}$ which is contained in the subspace

$$V = \{\mathbf{x} \in \mathbb{R}^8 : \mathbf{x}.\mathbf{a} = 0\} \subset \mathbb{R}^8$$

. Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_7$ be a positively oriented orthonormal basis for V and denote by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_7$ the standard basis on $\mathbb{R}^7$. Consider the isomorphism $\phi \colon V \to \mathbb{R}^7$

with $\phi(\mathbf{b}_i) = \mathbf{e}_i$ for $i = 1, \ldots, 7$. Then $\phi(\Lambda_{\mathbf{a}})$ is a lattice of full rank in $\mathbb{R}^7$ and determinant $||\mathbf{a}||$.

There are numerous results on the number of lattice points in a given domain D. One expects approximately $\mathrm{vol}(D)/\det(\Lambda)$ points in $\Lambda \cap D$. In our case we need good control over the error terms in order to perform a summation over $y_1, \ldots, y_8$. This is provided by a result of Thunder [29]. Given a subspace $W \subset \mathbb{R}^n$, let $D(W)$ be the orthogonal projection of D onto W and let $V_m(D) = \max(\mathrm{vol}_m(D(W))$ where the maximum is taken over all m-dimensional subspaces W.

Lemma 3 ([29] Theorem 4)**.** *Let $D \subset \mathbb{R}^n$ be a compact domain such that any line intersects D in at most s intervals. Let Λ be an n-dimensional lattice in $\mathbb{R}^n$. Then*

$$\left| \# (D \cap \Lambda) - \frac{\mathrm{vol}(D)}{\det(\Lambda)} \right| \ll_{s,n} \sum_{m=0}^{n-1} \frac{V_m(D)\lambda_1\lambda_2 \cdots \lambda_m}{\det(\Lambda)},$$

where λ_i are the successive minima of Λ.

Let D be the intersection of the box $\{\mathbf{x} \in \mathbb{R}^8 : |\mathbf{x}|_\infty \leq R\}$ with the hyperplane V. Then D has 7-dimensional volume (see [16])

$$\mathrm{vol}(D) = 2^7 R^7 \frac{||\mathbf{a}||}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{8} \frac{\sin(a_i t)}{a_i t}\, dt. \tag{8}$$

Let $N(\Lambda, R)$ be the number of integer points in $D \cap \phi(\Lambda)$, then by combining lemma 3 and (8) we deduce

$$N(\Lambda_{\mathbf{a}}, R) = \frac{2^7 R^7}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{8} \frac{\sin(a_i t)}{a_i t}\, dt + O\left(R^6\right).$$

Fix $\mathbf{y}, \mathbf{c} \in \mathbb{Z}^n \setminus \{0\}$ and write $d(\mathbf{y}) = (c_1 y_1^3, c_2 y_2^3, \ldots, c_8 y_8^3)$ for their greatest common divisor. Evidently the equation

$$\frac{c_1 y_1^3}{d(\mathbf{y})} x_1 + \frac{c_2 y_2^3}{d(\mathbf{y})} x_2 + \ldots + \frac{c_8 y_8^3}{d(\mathbf{y})} x_8 = 0$$

now by lemma 1 defines a 7-dimensional primitive lattice with determinant

$$\frac{1}{d(\mathbf{y})} \left((c_1 y_1^3)^2 + (c_2 y_2^3)^2 + \ldots + (c_8 y_8^3)^2 \right)^{1/2}.$$

Let $M(\mathbf{y}, X)$ denote the number of solutions to (5) with $|\mathbf{x}|_\infty \leq X$. Thus we have shown the following

Lemma 4. *Let $\mathbf{y}, \mathbf{c} \in \mathbb{Z}^n \setminus \{0\}$ be fixed, then*

$$M(\mathbf{y}, X) = c(\mathbf{y}, \mathbf{c}) X^7 + O\left(X^6\right),$$

where

$$c(\mathbf{y},\mathbf{c}) = \frac{2^7}{\pi d(\mathbf{y})^6}\int_{-\infty}^{\infty}\prod_{i=1}^{8} p(c_i y_i^3 t)\, dt$$

with

$$p(t) = \frac{\sin(t)}{t}.$$

Fix some small $\delta > 0$ and write $N(X,Y)$ for the number of solutions to (5) with $|\mathbf{x}|_\infty \leq X$ and $|\mathbf{y}|_\infty \leq Y$. Since

$$N(X,Y) = \sum_{|\mathbf{y}|_\infty \leq Y} M(\mathbf{y},X),$$

we have for $Y \leq X^{1/3-\delta/3}$,

$$N(X,Y) = \frac{2^7}{\pi}X^7 \sum_{|\mathbf{y}|_\infty \leq Y} d(\mathbf{y})^{-6}\int_{-\infty}^{\infty}\prod_{i=1}^{8} p(c_i y_i^3 t)\, dt + O\left(X^7 Y^{5-\delta}\right). \tag{9}$$

We may order the summation in the main term in (9) according to the value of $k = d(\mathbf{y})$. Thus

$$\sum_{|\mathbf{y}|_\infty \leq Y} d(\mathbf{y})^{-6}\int_{-\infty}^{\infty}\prod_{i=1}^{8} p(c_i y_i^3 t)\, dt = \sum_{k \leq |\mathbf{c}|_\infty Y^3} k^{-6} \sum_{\substack{|\mathbf{y}|_\infty \leq Y \\ k | c_i y_i^3 \\ (c_1 y_1^3/k,\ldots,c_8 y_8^3/k)=1}} \int_{-\infty}^{\infty}\prod_{i=1}^{8} p(c_i y_i^3 t)\, dt$$

$$= \sum_{k \leq |\mathbf{c}|_\infty Y^3} k^{-6} \sum_{d \leq |\mathbf{c}|_\infty Y^3/k} \mu(d) \sum_{\substack{|\mathbf{y}|_\infty \leq Y \\ kd | c_i y_i^3}} \int_{-\infty}^{\infty}\prod_{i=1}^{8} p(c_i y_i^3 t)\, dt.$$

Consider the sum

$$\sum_{\substack{y \leq Y \\ k | c_i y^3}} p(c_i y^3 t).$$

Applying partial summation then gives rise to

$$\sum_{\substack{y \leq Y \\ k | c_i y^3}} p(cy^3 t) = p(c_i Y^3 t) \sum_{\substack{y \leq Y \\ k | c_i y^3}} 1 - \int_1^Y \frac{\partial}{\partial \xi} p(c_i \xi^3 t) \Big(\sum_{\substack{y \leq \xi \\ k | c_i y^3}} 1 \Big)\, d\xi. \tag{10}$$

Let $\rho_i(k)$ denote the number of solutions of the congruence $c_i y^3 = 0 \bmod k$ with $y \bmod k$, then

$$\sum_{\substack{y \leq \xi \\ k | c_i y^3}} 1 = \frac{\rho_i(k)\xi}{k} + O\left(\rho_i(k)\right). \tag{11}$$

Since

$$\frac{\partial}{\partial \xi} p(c_i \xi^3 t) = \frac{3\cos(c_i \xi^3 t)}{\xi} - \frac{3\sin(c_i \xi^3 t)}{\xi^4 t},$$

applying (11) on the right-hand-side of (10) and integration by parts gives

$$\sum_{\substack{y\le Y\\ k|c_iy^3}} p(cy^3t) = \frac{\rho_i(k)}{k}\int_1^Y p(c_iy^3t)\,dy + O\left(\rho_i(k)\log Y\right). \tag{12}$$

For $k\ge 2$ we have by Ball [1] Lemma 3

$$\int_{-\infty}^{\infty} |p(t)|^k\,dt \le \frac{\pi\sqrt{2}}{\sqrt{k}}$$

and as an easy consequence the bound

$$\int_{-\infty}^{\infty} |p(c_iy^3t)|^k\,dt \ll \frac{1}{c_iy^3}.$$

This can be combined with Hölder's inequality to estimate for instance

$$\int_{-\infty}^{\infty}\prod_{i=2}^{8} p(c_iy_i^3t)\,dt \ll (y_1y_2\cdots y_8)^{-3/7}$$

or similar terms.

By combining this with (12) repeatedly, we obtain from (9)

$$N(X,Y) = \frac{2^7}{\pi}X^7\sum_{kd\le|\mathbf{c}|_\infty Y^3}\frac{\mu(d)\mathfrak{R}(kd)}{k^{14}d^8}\int_{-\infty}^{\infty}\int_{[1,Y]^8}\prod_{i=1}^{8}p(c_iy_i^3t)\,d\mathbf{y}dt + O\left(X^7Y^{5-\delta}\right),$$

where we have written $\mathfrak{R}(k) = \rho_1(k)\rho_2(k)\cdots\rho_8(k)$. By extending the range of integration and the substitution $y_i\mapsto Yy_i$ we deduce

$$\int_{-\infty}^{\infty}\int_{[1,Y]^8}\prod_{i=1}^{8}p(c_iy_i^3t)\,d\mathbf{y}dt = Y^8\int_{-\infty}^{\infty}\int_{[0,1]^8}\prod_{i=1}^{8}p(c_iy_i^3Y^3t)\,d\mathbf{y}dt.$$

Finally the substitution $Y^3t\mapsto t$ shows the main term of $N(X,Y)$ to be equal to

$$\frac{2^7}{\pi}X^7Y^5\sum_{kd\le|\mathbf{c}|_\infty Y^3}\frac{\mathfrak{R}(kd)}{k^{14}d^8}\int_{-\infty}^{\infty}\int_{[0,1]^8}\prod_{i=1}^{8}p(c_iy_i^3t)\,d\mathbf{y}dt.$$

It remains to extend the summation over k and d to deduce:

Lemma 5. *For $Y\le X^{1/3-\delta}$, we have for some constant C,*

$$N(X,Y) = CX^7Y^5 + O\left(X^7Y^{5-\delta}\right). \tag{13}$$

2.2 Counting cubes

The goal of this section is to establish an asymptotic expansion for the number of solutions $N_{\mathbf{c}}(\mathbf{x}, Y)$ to (5) for fixed $\mathbf{x}$ and $|\mathbf{y}| \le Y$. Note that the number of variables such that an asymptotic formula is available cannot be reduced with current technology. For technical reasons it is convenient to also define $N_{\mathbf{c}}^{+}(\mathbf{x}, Y)$ as the number of solution to (5) with $1 \le y_i \le Y$.

Proposition 1. *Let $\eta > 0$ be sufficiently small, then we have uniformly in $|\mathbf{x}|_\infty \ll Y^\eta$,*

$$N_{\mathbf{c}}^{+}(\mathbf{x}, Y) = A^{+}(\mathbf{x}, \mathbf{c}) Y^5 + O\left(Y^5 (\log Y)^{-3+\epsilon}\right).$$

with some non-negative constant $A^{+}(\mathbf{x}, \mathbf{c})$. Furthermore we have

$$N_{\mathbf{c}}(\mathbf{x}, Y) = A(\mathbf{x}, \mathbf{c}) Y^5 + O\left(Y^5 (\log Y)^{-3+\epsilon}\right) \tag{14}$$

with $A(\mathbf{x}, \mathbf{c})$ non-negative.

This is achieved by an application of the Hardy-Littlewood Circle method. Let

$$g(\alpha) = \sum_{y \le Y} e\left(\alpha y^3\right),$$

such that by orthogonality

$$N_{\mathbf{c}}^{+}(\mathbf{x}, Y) = \int_0^1 g(c_1 x_1 \alpha) g(c_2 x_2 \alpha) \cdots g(c_8 x_8 \alpha)\, d\alpha. \tag{15}$$

Consider the minor arcs from Vaughan [31]

$$\mathfrak{t} = \left\{\alpha \in [0,1] : |q\alpha - a| \le Y^{-9/4} \quad \text{with } (a,q) = 1, \text{ implies} \quad q > Y^{3/4}\right\}.$$

By an adaption of Boklan [5][Proof of Corollary I] we have

$$\int_{\mathfrak{t}} |g(\alpha)|^8\, d\alpha \ll Y^5 (\log Y)^{\epsilon - 3}.$$

It is convenient to write $Q_1 = |\mathbf{c}|_\infty |\mathbf{x}|_\infty Y^{3/4}$. Define the major arcs $\mathfrak{N}$ as the union of the intervals $\{\alpha \in [0,1] : |q\alpha - a| \le Q_1 Y^{-3}\}$ with $1 \le a \le q \le Q_1, (q,a) = 1$ and let $\mathfrak{n} = [0,1] \setminus \mathfrak{N}$. To deal with the minor arcs note that (cf. Chapter 8 of [9]) $\{\alpha' \in [0,1] : \alpha'/|c_i x_i| \in \mathfrak{n}\} \subset \mathfrak{t}$ such that by periodicity of $g(\alpha)$,

$$\int_{\mathfrak{n}} g(c_1 x_1 \alpha) g(c_2 x_2 \alpha) \cdots g(c_8 x_8 \alpha)\, d\alpha \ll \sum_{i=1}^{8} \int_{\mathfrak{n}} |g(c_i x_i \alpha)|^8\, d\alpha$$
$$= \sum_{i=1}^{8} \frac{1}{|c_i x_i|} \int_{\{\alpha' : \alpha'/|c_i x_i| \in \mathfrak{n}\}} |g(\alpha')|^8\, d\alpha' \ll \int_{\mathfrak{t}} |g(\alpha)|^8\, d\alpha \ll Y^5 (\log Y)^{-3+\epsilon}. \tag{16}$$

The treatment of the major arcs follows a well known routine. Introduce

$$S_1(q,a) = \sum_{n \bmod q} e\left(\frac{an^3}{q}\right)$$
$$v_1(\beta) = \int_0^Y e\left(\beta y^3\right)\, d\beta.$$

For $\alpha \in \mathfrak{N}$ write $\alpha = \frac{a}{q} + \beta$ for some co prime $1 \le a \le q$ and recall the assumptions made about $\mathbf{x}$ such that by [30][Theorem 7.2]

$$g(c_i x_i \alpha) = q^{-1} S_1(q, a x_i c_i) v_1(c_i x_i \beta) + O\left(q\left(1 + |c_i x_i \beta| Y^3\right)\right). \tag{17}$$

Using (17) and standard bounds we have the following approximation on the major arcs

$$\int_{\mathfrak{N}} \prod_{i=1}^{8} g(c_i x_i \alpha)\, d\alpha = \sum_{q \le Q_1} T_1(q) \int_{\frac{-Q_1}{qY^3}}^{\frac{Q_1}{qY^3}} \prod_{i=1}^{8} v_1\left(c_i x_i \beta\right)\, d\beta + O\left(Y^{5-\delta}\right),$$

where

$$T_1(q) = \sum_{\substack{a=1 \\ (q,a)=1}}^{q} q^{-8} S_1(q, c_1 x_1 a) S_1(q, c_2 x_2 a) \cdots S_1(q, c_8 x_8 a).$$

Introduce

$$v_0(\beta) = \int_0^1 e\left(\beta y^3\right)\, dy$$

and recall the bound

$$v_0(\beta) \ll \min\left(1, |\beta|\right)^{-\frac{1}{3}}.$$

Thus we have

$$\int_{\frac{-Q_1}{q}}^{\frac{Q_1}{q}} v_0\left(c_1 x_1 \beta\right) \cdots v_0\left(c_8 x_8 \beta\right)\, d\beta = \mathfrak{I}'(\mathbf{x}, \mathbf{c}) + O\left(|\mathbf{x}|_\infty^3 \left(\frac{Q_1}{q}\right)^{-\frac{5}{3}}\right), \tag{18}$$

where we have introduced the singular integral

$$\mathfrak{I}'(\mathbf{x}, \mathbf{c}) = \int_{-\infty}^{\infty} v_0\left(c_1 x_1 \beta\right) v_0\left(c_2 x_2 \beta\right) \cdots v_0\left(c_8 x_8 \beta\right)\, d\beta. \tag{19}$$

Note that by substitution

$$\int_{\frac{-Q_1}{qY^3}}^{\frac{Q_1}{qY^3}} v_1\left(c_1 x_1 \beta\right) v_1\left(c_2 x_2 \beta\right) \cdots v_1\left(c_8 x_8 \beta\right)\, d\beta$$
$$= Y^5 \int_{\frac{-Q_1}{q}}^{\frac{Q_1}{q}} v_0\left(c_1 x_1 \beta\right) v_0\left(c_2 x_2 \beta\right) \cdots v_0\left(c_8 x_8 \beta\right)\, d\beta.$$

Turning our attention to the singular series, define

$$\mathfrak{S}'(\mathbf{x},\mathbf{c}) = \sum_{q=1}^{\infty} T_1(q). \tag{20}$$

As for $a/q = a'/q'$ we have $S_1(q,a) = (q/q')\, S_1(q',a')$ and since by Theorem 4.2 of [30] the bound

$$S_1(q,a) \ll q^{2/3}$$

holds for $(q,a) = 1$, we may deduce

$$T_1(q) \ll q^{1-8/3}(q,c_1x_1)^{1/3}(q,c_2x_2)^{1/3}\cdots(q,c_8x_8)^{1/3} \ll q^{-5/3}|\mathbf{c}|_\infty^3|\mathbf{x}|_\infty^3.$$

Hence we may complete the sum over T_1, since

$$\sum_{q\le Q_1} T_1(q) = \mathfrak{S}'(\mathbf{x},\mathbf{c}) + O\left(|\mathbf{x}|_\infty^3|\mathbf{c}|_\infty^3 Q_1^{-2/3}\right). \tag{21}$$

The above calculation also shows that

$$\mathfrak{S}'(\mathbf{x},\mathbf{c}) \ll |\mathbf{c}|_\infty^3|\mathbf{x}|_\infty^3.$$

Following Lemma 2.11 in Vaughan [30] one shows that $T_1(q)$ is multiplicative.

Lemma 6. *If q and r are co prime integers, we have*

$$T_1(qr) = T_1(q)T_1(r).$$

Proof. This kind of argument is widely used when dealing with exponential sums. Note that we may write a residue class $\bmod qr$ uniquely as $n = tr + uq$ with $t \bmod q$ and $u \bmod q$. Suppose we have $(q,a) = (b,r) = (q,r) = 1$ then by the definition of $S_1(q,a)$ we have

$$\begin{aligned}
S_1\left(qr, c_ix_i(ar+bq)\right) &= \sum_{n \bmod qr} e\left(\frac{c_ix_i(ar+bq)n^3}{qr}\right)\\
&= \sum_{t \bmod q}\sum_{u \bmod r} e\left(\frac{c_ix_i(ar+bq)\,(tr+uq)^3}{qr}\right)\\
&= \sum_{t \bmod q}\sum_{u \bmod r} e\left(\frac{c_ix_iac_ix_it^3}{q} + \frac{bc_ix_iu^3}{r}\right)\\
&= S_1(q,ac_ix_i)S_1(r,bc_ix_i).
\end{aligned}$$

With this relation in hand we can readily establish the multiplicativity of $T_1(q)$. Let $(q,r) = 1$ then

$$T_1(qr) = \sum_{\substack{a=1\\(qr,a)=1}}^{qr} (qr)^{-8} S_1(qr,c_1x_1a)\cdots S_1(qr,c_8x_8a).$$

By an application of the Chinese remainder theorem we may write the above sum as

$$T_1(qr) = \sum_{\substack{a=1 \\ (q,a)=1}}^{q} \sum_{\substack{b=1 \\ (r,b)=1}}^{r} (qr)^{-8} S_1(qr, c_1x_1(ar+bq)) \cdots S_1(qr, c_8x_8(ar+bq))$$

which by the calculation for S_1 factors. Thus $T_1(qr) = T_1(q)T_1(r)$. □

Since T_1 is multiplicative we may write (20) as an Euler product and interpret the factors arising at each prime p as local densities. That is

$$\mathfrak{S}'(\mathbf{x}, \mathbf{c}) = \prod_p E_p'(\mathbf{x}, \mathbf{c})$$

where

$$E_p'(\mathbf{x}, \mathbf{c}) = \sum_{l=0}^{\infty} T_1(p^l) = \lim_{L\to\infty} p^{-7L} \Phi_{\mathbf{x},\mathbf{c}}'(p^L)$$

and $\Phi_{\mathbf{x},\mathbf{c}}'(q)$ denotes the number of solutions to

$$c_1x_1y_1^3 + c_2x_2y_2^3 + \ldots + c_8x_8y_8^3 = 0 \tag{22}$$

modulo q. To justify this expression we show:

Lemma 7. *For a natural number q we have*

$$\sum_{d|q} T_1(q) = q^{-7}\Phi_{\mathbf{x},\mathbf{c}}'(q).$$

Note that for $q = p^\ell$, the left-hand side is

$$\sum_{h=0}^{\ell} T_1(p^h)$$

and thus, by definition,

$$E_p'(\mathbf{x}, \mathbf{c}) = \lim_{L\to\infty} p^{-7L}\Phi_{\mathbf{x},\mathbf{c}}'(p^L).$$

Proof. By orthogonality we may write

$$\Phi_{\mathbf{x},\mathbf{c}}'(q) = \frac{1}{q}\sum_{r=1}^{q}\sum_{n_1=1}^{q} \cdots \sum_{n_8=1}^{q} e\left(\frac{r\,(c_1x_1n_1^3 + c_2x_2n_2^3 \ldots + c_8x_8n_8^3)}{q}\right).$$

Splitting the sum over r in terms corresponding to $d = q/(r,q)$ we deduce

$$\Phi_{\mathbf{x},\mathbf{c}}'(q) = \frac{1}{q}\sum_{d|q}\sum_{\substack{a=1 \\ (a,d)=1}}^{d}\left(\frac{q}{d}\right)^8 \sum_{n_1=1}^{d} \cdots \sum_{n_8=1}^{d} e\left(\frac{r\,(c_1x_1n_1^3 + c_2x_2n_2^3 \ldots + c_8x_8n_8^3)}{q}\right).$$

Comparing this to the definition of $T_1(q)$ we get the relations claimed. □

Lemma 8. *Assume (22) admits a non-trivial p-adic solution, then $E_p'(\mathbf{x}, \mathbf{c})$ is non-negative.*

Proof. Let $\mathbf{r}$ be a solution with not all r_i divisible by p. We may assume that $p \nmid r_1$. A classic result provides the existence of a natural number $\gamma = \gamma(\mathbf{x})$ such that, if the congruence $cy^3 = b \bmod p^\gamma$ has a solution with $p \nmid y$, then the congruences $cy^3 = b \bmod p^L$ are also soluble for $L \geq \gamma$ with $p \nmid y$. Since we assume the existence of a solution we have

$$c_1x_1r_1^3 + c_2x_2r_2^3 + \ldots + c_8x_8r_8^3 = 0 \mod p^\gamma.$$

Now choose $y_2, \ldots, y_8$ subject to $y_i = r_i \bmod p^\gamma$ and $0 < y_i \leq p^L$. This is possible in $p^{7(L-\gamma)}$ ways. Pick y_1 such that

$$c_1x_1y_1^3 = -c_2x_2y_2^3 - \ldots - c_8x_8y_8^3 \mod p^L$$

which is possible by assumption since

$$-c_2x_2y_2^3 - \ldots - c_8x_8y_8^3 = c_1x_1r_1^3 \mod p^\gamma.$$

This shows that $\Phi'_{\mathbf{x},\mathbf{c}}(p^L) \geq C_p p^{7(L-\gamma)}$ for a positive C_p. □

Note that convergence of the singular series can be easily shown by working along the lines of Davenport [15]. Thus we have established

Lemma 9. *The singular series (20) is real and non-negative. If (22) admits non-trivial p-adic solutions for all primes p the singular series is positive.*

As convergence is easily shown by standard bounds we now may turn our attention to the singular integral and develop its positivity. Following the argument in Davenport [15] chapter 8 one now establishes

$$v_0(\beta) = \int_0^1 e\left(\beta y^3\right) dy = \frac{1}{3}\int_0^1 t^{-2/3} e\left(\beta t\right) dt.$$

This is done by using the above inside (19) to deduce the identity

$$\mathfrak{J}'(\mathbf{x}, \mathbf{c}) = 3^{-8} \int_{-\infty}^{\infty} \left(\int_{[0,1]^8} \left(t_1t_2 \cdots t_8\right)^{-2/3} e\left(\beta(c_1x_1t_1 + c_2x_2t_2 + \ldots + c_8x_8t_8)\right) d\mathbf{t} \right) d\beta.$$

With the substitution

$$c_1x_1t = c_1x_1t_1 + c_2x_2t_2 \ldots + c_8x_8t_8$$

this is readily transformed into

$$\mathfrak{J}'(\mathbf{x}, \mathbf{c}) = 3^{-8} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(t) e\left(c_1x_1\beta t\right) dt d\beta, \tag{23}$$

where

$$B(t) = \int_{\mathfrak{B}(t)} \left(\frac{c_1x_1t - c_2x_2t_2 - \ldots - c_8x_8t_8}{c_1x_1} \right)^{-2/3} \left(t_2 \cdots t_8\right)^{-2/3} d\mathbf{t} \tag{24}$$

and the region of integration is given by

$$\mathfrak{B}(t) = \left\{ (t_2, \ldots, t_8) \in [0,1]^7 : 0 \leq \frac{c_1 x_1 t - c_2 x_2 t_2 - \ldots - c_8 x_8 t_8}{c_1 x_1} \leq 1 \right\}.$$

By Fourier inversion we deduce from (23), $\mathfrak{J}'(\mathbf{x}, \mathbf{c}) = 3^{-8} |c_1 x_1|^{-1} B(0)$ and since the integrand in (24) is non-negative. Hence we deduce

Lemma 10. *The singular integral* (19) *is real and non-negative.*

Note that from (24), if not all coefficients $c_1 x_1, \ldots, c_8 x_8$ have the same sign, $\mathfrak{B}(0)$ will contain a box of positive 7-dimensional volume and therefore we may indeed deduce that $\mathfrak{J}'(\mathbf{x}, \mathbf{c})$ is positive.

Collecting (15), (16), (17), (18) and (21) we have uniformly in $|\mathbf{x}|_\infty \leq Y^\eta$

$$N_{\mathbf{c}}^+(Y) = \mathfrak{S}'(\mathbf{x}, \mathbf{c}) \mathfrak{J}'(\mathbf{x}, \mathbf{c}) Y^5 + O\left(Y^5 (\log Y)^{-2}\right). \tag{25}$$

Together with Lemma 9 and Lemma 10, (25) implies the first part of Proposition 1 by putting

$$c^+(\mathbf{c}, \mathbf{x}) = \mathfrak{S}'(\mathbf{x}, \mathbf{c}) \mathfrak{J}'(\mathbf{x}, \mathbf{c}).$$

To deduce the second half of the Proposition we note that there is a correspondence of non-negative solutions to integer solutions since -1 is a third power. Therefore

$$N_{\mathbf{c}}(Y) = \sum_{\substack{\epsilon_i \in \{\pm 1\} \\ 1 \leq i \leq 8}} N_{\epsilon \mathbf{c}}^+(Y)$$

and since $S_1(q, -a) = S_1(q, a)$ we have $\mathfrak{S}'(\mathbf{x}, \mathbf{c}) = \mathfrak{S}'(\mathbf{x}, \epsilon \mathbf{c})$ we have

$$c(\mathbf{x}, \mathbf{c}) = \mathfrak{S}'(\mathbf{x}, \mathbf{c}) \sum_{\substack{\epsilon_i \in \{\pm 1\} \\ 1 \leq i \leq 8}} \mathfrak{J}'(\mathbf{x}, \epsilon \mathbf{c}).$$

By (19) we may write

$$\sum_{\substack{\epsilon_i \in \{\pm 1\} \\ 1 \leq i \leq 8}} \mathfrak{J}'(\mathbf{x}, \epsilon \mathbf{c}) = \mathfrak{J}_1(\mathbf{x}, \mathbf{c})$$

where

$$\mathfrak{J}_1(\mathbf{x}, \mathbf{c}) = \int_{-\infty}^{\infty} \int_{[-1,1]^8} e\left(c_1 x_1 \beta y_1^3 + \ldots + c_8 x_8 \beta y_8^3\right) d\mathbf{y} d\beta.$$

Hence

$$c(\mathbf{x}, \mathbf{c}) = \mathfrak{S}'(\mathbf{x}, \mathbf{c}) \mathfrak{J}_1(\mathbf{x}, \mathbf{c})$$

finishing the proof of the proposition.

3 Circle method

Recall that $N_{\mathbf{c}}(X,Y)$ denotes the number of solutions to (5) with $1 \leq |x_i| \leq X$ and $1 \leq |y_i| \leq Y$. Let $N_{\mathbf{c}}^{+}(X,Y)$ denote the number of solutions with all x_i and y_i positive. The goal of this section is to establish an asymptotic formula for $N_{\mathbf{c}}^{+}(X,Y)$ using the Hardy-Littlewood circle method. This time we work in a 'two-dimensional' setting with more or less independent box sizes. We will only require that $Y^3 \geq X^{1-\delta}$ and $X \geq (\log Y)^{12}$. The corresponding asymptotic formula for $N_{\mathbf{c}}(X,Y)$ will then be derived from the corresponding one with positive solutions.

Theorem 2. *Let $Y \geq X^{\frac{1}{3}-\delta/3}$, $X \geq (\log Y)^{12}$ and assume $\mathbf{c} \in \mathbb{Z}^8 \setminus \{0\}$ then there are real numbers $\mathcal{J}(\mathbf{c})$ and $\mathcal{J}^{+}(\mathbf{c})$ with*

$$N_{\mathbf{c}}(X,Y) = \mathcal{J}(\mathbf{c}) X^7 Y^5 + O\left(X^7 Y^5 (\log Y)^{-2+\epsilon}\right)$$

and

$$N_{\mathbf{c}}^{+}(X,Y) = \mathcal{J}^{+}(\mathbf{c}) X^7 Y^5 + O\left(X^7 Y^5 (\log Y)^{-2+\epsilon}\right), \tag{26}$$

where the constant $\mathcal{J}(\mathbf{c})$ is positive. The constant $\mathcal{J}^{+}(\mathbf{c})$ is positive if the coefficients c_i are not all of the same sign.

Fix a small positive η and let $\mathfrak{M}(q,a)$ denote the set of $\alpha \in [0,1]$ such that we have $\left|\alpha - \frac{a}{q}\right| \leq \frac{Y^{3/4+\eta}}{Y^3 X q}$ and define $\mathfrak{M}$ to be the union of all $\mathfrak{M}(q,a)$ for $(a,q) = 1$ and $q \leq Y^{3/4+\eta}$. As usual denote by $\mathfrak{m} = \mathfrak{m}(Y)$ the complementary set in the unit interval.

Write

$$f(\alpha) := \sum_{x \leq X} \sum_{y \leq Y} e\left(\alpha x y^3\right)$$

and define for $1 \leq x \leq X$

$$f_x(\alpha) = \sum_{y \leq Y} e\left(\alpha x y^3\right).$$

The notation is chosen to highlight the one-dimensional nature of the argument to follow.

3.1 A Weyl inequality

The course of action now is a careful adaption of the innovative reduction technique in [31] leading to a suitable moment estimate on the minor arcs $\mathfrak{m}$. The first step is to establish a version of [30] Theorem 4.1.

Lemma 11. *Let $(a,q) = 1$, then*

$$S_x(q,a,b) := \sum_{n \bmod q} e\left(\frac{axn^3 + bn}{q}\right) \ll (b,q) q^{\frac{1}{2}+\epsilon}. \tag{27}$$

Proof. It is sufficient to consider the case of q a prime power. Note that if $(x, q) = 1$ also $(ax, q) = 1$ and the claim follows by [30, Lemma 4.1] Now let $q = p$ be prime. Assume $(x, p) = p$, then $S_x(p, a, b)$ is zero if $(p, b) = 1$ or p, if $(p, b) = p$. In either case (27) holds. Let $q = p^\ell$ and $x^\theta \| b$ with $\theta \geq 0$. If $\theta = 0$ write $x' = x/p$ and $n = yp^{\ell-1} + z$ with y mod p and z mod $p^{\ell-1}$. Thus

$$\begin{aligned} S_x(p^\ell, a, b) &= \sum_{y \bmod p} \sum_{z \bmod p^{\ell-1}} e\left(\frac{ax'p\left(yp^{\ell-1}+z\right)^3 + b\left(yp^{\ell-1}+z\right)}{p^\ell}\right) \\ &= \sum_{y \bmod p} e\left(\frac{by}{p}\right) \sum_{z \bmod p^{\ell-1}} e\left(\frac{ax'z^3+bz}{p^{\ell-1}}\right) = 0. \end{aligned}$$

Assume $\theta \geq 1$ and let $p^\tau \| b$ with $\tau \geq 1$ and write $n = yp^{\ell-\tau} + z$ with y mod p^τ and z mod $p^{\ell-\tau}$. If $\theta \geq \tau$ then for $x' = x/p^\tau$ and $b' = b/p^\theta$ we have

$$\begin{aligned} S_x(p^\ell, a, b) &= \sum_{y \bmod p^\tau} \sum_{z \bmod p^{\ell-\tau}} e\left(\frac{ax'p^\tau\left(yp^{\ell-\tau}+z\right)^3 + b\left(yp^{\ell-\tau}p^\theta b + zp^\theta b\right)}{p^\ell}\right) \\ &= \sum_{y \bmod p^\tau} \sum_{z \bmod p^{\ell-\tau}} e\left(\frac{az^3x' + p^{\theta-\tau}b'z}{p^{\ell-\theta}}\right) \\ &\ll p^\tau\left(p^{\theta-\tau}b', p^{\ell-\tau}\right)p^{(\ell-\tau)/2+\epsilon} \leq p^{\ell/2+\epsilon}p^\theta. \end{aligned}$$

If $\tau \geq \theta$ a similar calculation as in the first case shows $S_x(p^\ell, a, b) = 0$.

□

Write

$$v_x(\beta) := \int_0^Y e\left(\beta x y^3\right)\, dy$$

and set

$$S_x(q, a) = S_x(q, a, 0).$$

It is useful to record here the bound (c.f. [30], Chapter 4)

$$S_x(q, a) \ll q^{2/3}(q, x)^{1/3}.$$

Lemma 12. *Suppose* $(a, q) = 1$ *and write* $\alpha = \frac{a}{b} + \beta$*, then*

$$f_x(\alpha, Y) - q^{-1}S_x(q, a)v_x(\beta) \ll q^{\frac{1}{2}+\epsilon}\left(1 + xY^3|\beta|\right)^{\frac{1}{2}}. \tag{28}$$

If further $|\beta| \leq \left(6qY^2X\right)^{-1}$*, then*

$$f_x(\alpha) - q^{-1}S_x(q, a)v_x(\beta) \ll q^{\frac{1}{2}+\epsilon}.$$

Proof. This is essentially the same as in [30][Theorem 4.1]. □

Lemma 13. *Assume* $Y \geq X^{\frac{1}{3}-\delta}$ *then uniformly for* $\alpha \in \mathfrak{m}$*, we have*

$$f(\alpha) \ll XY^{\frac{3}{4}}\left(\log Y\right)^{1/4+\epsilon}. \tag{29}$$

Proof. Let $\alpha \in \mathfrak{m}$ and for $\delta > 0$ sufficiently small pick co prime integers $(a,q) = 1$ with $q \leq Y^{2-\delta}X$ and $\left|\alpha - \frac{a}{q}\right| \leq q^{-1}Y^{\delta-2}X^{-1}$. Then we have

$$f_x(\alpha) \ll q^{-\frac{1}{3}}(x,q)^{\frac{1}{3}}Y\left(1 + xY^3\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{3}} + q^{\frac{1}{2}+\epsilon}\left(1 + xY^3\left|\alpha - \frac{a}{q}\right|\right)^{\frac{1}{2}}.$$

If $q \leq Y^{\frac{3}{2}-\delta}$ this gives

$$f_x(\alpha) \ll q^{-\frac{1}{3}}(x,q)^{\frac{1}{3}}Y\left(1 + xY^3\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{3}} + Y^{\frac{3}{4}}$$

As $f(\alpha) = \sum_{x \leq X} f_x(\alpha)$ the contribution of the second term is negligible. For $q > Y^{\frac{3}{4}+\eta}$, we have

$$\begin{aligned}\sum_{x \leq X} f_x(\alpha) &\ll q^{-\frac{1}{3}}Y\sum_{x\leq X}(x,q)^{\frac{1}{3}} \ll Y^{\frac{3}{4}-\frac{\eta}{3}+\epsilon}\sum_{d|q}d^{\frac{1}{3}}\sum_{\substack{x\leq X\\ d|x}}1\\ &\ll Y^{\frac{3}{4}}X.\end{aligned}$$

If $\left|\alpha - \frac{a}{q}\right| > q^{-1}Y^{-\frac{9}{4}+\eta}X^{-1}$, the contribution is also $O\left(Y^{\frac{3}{4}}X\right)$. Thus we may assume $q \geq Y^{\frac{3}{2}-\delta}$, that is $\left|\alpha - \frac{a}{q}\right| \leq Y^{2\delta-\frac{7}{2}}X^{-1}$. Since now

$$f_x(\alpha) - f_x\left(\frac{a}{q}\right) \ll xY^4\left|\alpha - \frac{a}{q}\right| \ll Y^{\frac{1}{2}+2\delta}$$

we have in this case

$$f(\alpha) = f\left(\frac{a}{q}\right) + O\left(XY^{\frac{3}{4}}\right).$$

Following the proof of Weyl's inequality we are lead to considering

$$\begin{aligned}\left|f\left(\frac{a}{q}\right)\right|^4 &\ll X^3\sum_{x\leq X}\left|f_x\left(\frac{a}{q}\right)\right|^4\\ &\ll X^4Y^3 + YX^3\sum_{x\leq X}\sum_{h_1,h_2\ll Y}\min\left(Y, \left\|\frac{axh_1h_2}{q}\right\|^{-1}\right).\end{aligned}$$

The relevant sum is

$$\sum_{|b|\leq\frac{1}{2}q}\min\left(Y, \frac{q}{|b|}\right)\sum_{\substack{h_1,h_2\ll Y\\ x\leq X\\ axh_1h_2 = b \bmod q}}1.$$

Hooley's delta function

$$\Delta_r(n) = \max_{u_1,\dots,u_{r-1}}\sum_{\substack{d_1\cdots d_{r-1}|n\\ u_i<d_i\leq eu_i}}1$$

was introduced by Hooley in [21], where he provided a mean value estimate for $\Delta(n) = \Delta_2(n)$,

$$\sum_{n\leq x} \Delta(n) \ll x(\log x)^{4/\pi-1}.$$

Subsequent impovement by Hall and Tenenbam [19][Theorem 70] for $\Delta_3(n)$, that is

$$\sum_{n\leq x} \Delta_3(n) \ll x(\log x)^{\epsilon},$$

may be combined with Hooley [21][Theorem 3]. As by our assumption on $Y \geq X^{1/3-\delta}$, $q \ll (Y^2X)^{1-\delta'}$ and we conclude that

$$\begin{aligned}
&\sum_{|b|\leq \frac{1}{2}q} \min\left(Y, \frac{q}{|b|}\right) \sum_{\substack{h_1,h_2\ll Y\\ x\leq X\\ axh_1h_2=b \bmod q}} 1 \\
&\ll \sum_{|b|\leq \frac{1}{2}q} \min\left(Y, \frac{q}{|b|}\right) d((q,b))q^{-1}XY^2(\log Y)^{\epsilon} \\
&\ll XY^2(\log Y)^{\epsilon}\left(Yd(q)q^{-1} + \sum_{r|q}\frac{d(r)}{r}\sum_{m\leq q/r} m^{-1}\right) \\
&\ll XY^2(\log Y)^{\epsilon}(1 + (\log\log q)^2\log q).
\end{aligned}$$

Thus we deduce the bound

$$O\left(Y^3X^4(\log Y)^{1+\epsilon}\right)$$

for the sum in question which finishes the proof.

□

3.2 A fourth moment estimate

A successful application of the Hardy-Littlewood circle method crucially depends on the availability of good bounds for some even integer moment. Following the scheme of things, we are therefore interested in providing a rather sharp (in the sense that we do not give up too many logarithms, let alone powers) bound for the fourth moment of $f(\alpha)$. A reasonable start for our venture is the second moment for $f(\alpha)$, that is the number of solutions to

$$x_1y_1^3 = x_2y_2^3$$

with $x_i \leq X$ and $y_i \leq Y$. A first crude approach would be to pick x_1 and y_1 such that the right side is now determined up to a divisor function. This would give a bound of $O(XY^{1+\epsilon})$ which is already too bad for our purpose. However this can be easily removed by a more careful treatment.

Lemma 14. *We have*

$$\int_0^1 |f(\alpha)|^2 \, d\alpha \ll XY. \tag{30}$$

Proof. Write $g = (x_1, x_2)$, $h = (y_1, y_2)$ and introduce $v_i = x_i/g$ and $w_i = y_i/h$. The integral (30) now corresponds to the sum

$$\sum_{g \leq X} \sum_{h \leq Y} \sum_{\substack{w_i \leq \min((X/g)^{1/3}, Y/h) \\ (w_1, w_2) = 1}} 1$$

which is

$$\begin{aligned} &\leq \sum_{g \leq X} \sum_{h \leq Y (g/X)^{1/3}} \frac{X^{2/3}}{g^{2/3}} + \sum_{h \leq Y} \sum_{g \leq X (h/Y)^3} \frac{Y^2}{h^2} \\ &\leq \sum_{g \leq X} Y \frac{X^{1/3}}{g^{1/3}} + \sum_{h \leq Y} X \frac{h}{Y} \ll XY. \end{aligned}$$

□

We may define the set $\mathscr{E}$ that appeared in the introduction as follows: Let

$$\mathscr{E} = \left\{ y \leq Y : p \mid y \Rightarrow p \notin [(\log Y)^{80}, Y^{1/7}] \right\}$$

and recall the definition of g_E in (7).

Lemma 15. *Let $\epsilon > 0$, then we have the following estimates:*

$$\int_0^1 |f(\alpha)|^4 \, d\alpha \ll X^3 Y^2 (\log XY)^2 \tag{31}$$

and

$$\int_0^1 |g_E(\alpha)|^4 \, d\alpha \ll X^3 Y^2 (\log XY)^{\epsilon}. \tag{32}$$

Before we start with the proof of this important lemma, we record an easy consequence for the 8-th moment.

Lemma 16. *Let $\epsilon > 0$, then we have*

$$\int_0^1 |f(\alpha)|^8 \, d\alpha \ll X^7 Y^5 (\log XY)^{3+\epsilon}. \tag{33}$$

Proof. Obviously we have the decomposition

$$\int_0^1 |f(\alpha)|^8 \, d\alpha = \int_{\mathfrak{M}} |f(\alpha)|^8 \, d\alpha + \int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha.$$

Hence either

$$\int_0^1 |f(\alpha)|^8 \, d\alpha \ll \int_{\mathfrak{M}} |f(\alpha)|^8 \, d\alpha$$

or

$$\int_0^1 |f(\alpha)|^8 \, d\alpha \ll \int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha.$$

The first case, the major arc contribution dominates and is calculated in the corresponding section later on and is negligible. Thus we may assume that the second case holds. Thus by using (29) and extending the range of integration,

$$\int_0^1 |f(\alpha)|^8 \, d\alpha \ll \int_{\mathfrak{m}} |f(\alpha)|^8 \ll X^4 Y^3 (\log Y)^{1+\epsilon} \int_0^1 |f(\alpha)|^4 \, d\alpha.$$

The lemma follows by invoking (31). □

Proof of Lemma 15. We begin our treatment of $|f(\alpha)|^4$ by applying the Cauchy-Schwarz inequality to the x summation in one square:

$$\int_0^1 |f(\alpha)|^4 \, d\alpha \leq X \int_0^1 \sum_{x \leq X} \left| \sum_{y \leq Y} e\left(xy^3\alpha\right) \right|^2 |f(\alpha)|^2 \, d\alpha.$$

The Integral on the right hand side corresponds to the number of solutions of

$$x(z_1^3 - z_2^3) = x_1 y_1^3 - x_2 y_2^3 \tag{34}$$

subject to the conditions $x, x_i \leq X$ and $y_i, z_i \leq Y$. In view of (30) the solutions with $z_1 = z_2$ contribute $O(X^2Y^2)$ to (34) which implies a total contribution to (31) and (32) of $O(X^3Y^2)$ which is acceptable. It remains to treat the solutions with $z_1 \neq z_2$ and we note that up to now there was no effect of the y_i, z_i belonging to the full Interval $[1, Y]$ or to the subset $\mathscr{E}$. Thus we may replace $f(\alpha)$ by $g_E(\alpha)$. Let $\psi(m, X, Y)$ denote the number of solutions of

$$x_1 y_1^3 - x_2 y_2^3 = m \tag{35}$$

with $x_i \leq X$ and $y_i \leq Y$. Furthermore let $\psi_{\mathscr{E}}(m, X, Y)$ denote the solutions to (35) with the additional constraint $y_i \in \mathscr{E}$. Hence we have

$$\int_0^1 |f(\alpha)|^4 \, d\alpha \ll X \sum_{m \geq 1} \psi(m, X, Y) d_3(m) + O(X^3Y^2)$$

and

$$\int_0^1 |g_E(\alpha)|^4 \, d\alpha \ll X \sum_{m \geq 1} \psi_{\mathscr{E}}(m, X, Y) d_3(m) + O(X^3Y^2).$$

In order to estimate the sums over m we follow the approach taken in the proof of lemma 5 in [8]. Let $\psi^*(m, X, Y)$ and $\psi^*_{\mathscr{E}}(m, X, Y)$ denote the number of solutions counted by $\psi(m)$ and $\psi_{\mathscr{E}}(m, X, Y)$ respectively where the additional condition $(y_1, y_2) = 1$ holds.

Let $d \leq (XY)^{1/3}$ and introduce

$$A_d = \sum_{\substack{m \geq 1 \\ m = 0 \bmod d}} \psi^*(m, X, Y) \qquad \text{and} \qquad A_d(\mathscr{E}) = \sum_{\substack{m \geq 1 \\ m = 0 \bmod d}} \psi^*_{\mathscr{E}}(m, X, Y).$$

This then is bounded by the number of solutions to

$$x_1y_1^3 - x_2y_2^3 = 0 \bmod d$$

subject to $x_i \leq X, y_i \leq Y$ (or $y_i \in \mathscr{E}$ for $A_d(\mathscr{E})$) and $(y_1, y_2) = 1$.

It is convenient to consider three cases corresponding to the size of d relative to X and Y. If $d \leq \min(X, Y)$ we may order the solutions according to their residue classes, that is by writing $\lambda(a_1, a_2, b_1, b_2)$ ($\lambda_{\mathscr{E}}(a_1, a_2, b_1b_2)$ respectively) for the number of $x_i \leq X$ and $y_i \leq Y$ ($y_i \in \mathscr{E}$ respectively) with $x_i = a_i \bmod d$ and $y_i = b_i \bmod d$. This implies

$$A_d = \sum_{a_1,a_2,b_1,b_2} \lambda(a_1, a_2, b_1, b_2) \qquad \text{and} \qquad A_d(\mathscr{E}) = \sum_{a_1,a_2,b_1,b_2} \lambda_{\mathscr{E}}(a_1, a_2, b_1, b_2) \tag{36}$$

where the summations are taken over the tuples (a_1, a_2, b_1, b_2) satisfying

$$a_1b_1^3 - a_2b_2^3 = 0 \bmod d \tag{37}$$

with $0 \leq a_i, b_i \leq d$ and $(b_1, b_2, d) = 1$. Denote the number of these tuples with $r(d)$.

Lemma 17. *The number of solutions $r(q)$ satisfies the bound*

$$r(q) \leq q^3. \tag{38}$$

Proof. By the Chinese Remainder theorem the function $r(q)$ is multiplicative and it is therefore sufficient to consider the case $q = p^\ell$, with p prime. Since we imposed the condition $(b_1, b_2, p^\ell) = 1$, not both b_1 and b_2 can be divisible by p. If $p \nmid b_1$, we may pick b_1, b_2 and a_2. There are $\varphi(p^\ell)p^{2\ell}$ choices. This will fix a_1. Similarly, for $p \mid b_1$ and $p \nmid b_2$ there are at most $\varphi(p^\ell)p^{2\ell-1}$ solutions. Thus we have $r(p^\ell) \leq p^{3\ell}$. □

We have

$$\lambda(a_1, a_2, b_1, b_2) = \sum_{\substack{x_i \leq X \\ x_i = a_i \bmod d}} \sum_{\substack{y_i \leq Y \\ y_i = b_i \bmod d}} 1$$

and

$$\lambda_{\mathscr{E}}(a_1, a_2, b_1, b_2) = \sum_{\substack{x_i \leq X \\ x_i = a_i \bmod d}} \sum_{\substack{y_i \in \mathscr{E} \\ y_i = b_i \bmod d}} 1.$$

Recalling (36) and the definition of $r(d)$ together with (38) we may infer that for $d \leq \min(X, Y)$,

$$A_d \ll X^2Y^2d^{-1}. \tag{39}$$

For the corresponding sum involving $\mathscr{E}$ we refer to Lemma 2 in [32], to deduce

$$A_d(\mathscr{E}) \ll X^2Y^2(\log Y)^{-2+\epsilon}d^{-1}. \tag{40}$$

Now consider the case $d > Y$. We may order the x_1 and x_2 according to their residue class mod d to deduce the bound

$$A_d \ll \frac{X^2}{d^2} r_1(d),$$

where $r_1(d)$ is the number of solutions to (37) with $a_i \leq d$, $b_i \leq Y$ and $(b_1, b_2) = 1$. We may choose b_1, b_2 and a_2 which are $O(Y^2 d)$ choices and will fix a_2 such that this implies $r_1(d) \ll Y^2 d$. Note that if b_i belong to $\mathscr{E}$ we save the factor $(\log Y)^{2-\epsilon}$. Thus we have (39) and (40) available for $d > Y$.

It remains to consider the situation in which $d > X$ and therefore only y_1, y_2 should be ordered by residue classes mod d. This time we obtain a bound of the shape

$$A_d \ll \frac{Y^2}{d^2} r_2(d),$$

where $r_2(d)$ is the number of solutions to (37) with $a_1, a_2 \leq X$, $b_1, b_2 \leq d$ and $(b_1, b_2) = 1$. Since $a_i \leq X < d$ we have $a_i \neq 0 \bmod d$ and thus picking $a_1, a_2 \leq X$ and $y_2 \leq d$ gives the bound $r_2(d) \ll X^2 d$. By recalling the above cited Lemma 2 in [32] this implies (39) and (40) also for $d > X$.

With these estimates available for $d \leq (XY)^{1/3}$, we may use [25][Theorem 1.2] in junction with (38) to obtain the bounds

$$\Psi(X, Y) := \sum_{m \geq 1} \psi^*(m, X, Y) d_3(m) \ll X^2 Y^2 (\log XY)^2 \tag{41}$$

and

$$\Psi(X, \mathscr{E}) = \sum_{m \geq 1} \psi^*(m, X, \mathscr{E}) d_3(m) \ll X^2 Y^2 (\log XY)^{\epsilon}.$$

The bounds for the sums without coprimality restriction follow now from elementary manipulations. By writing $w = (y_1, y_2)$ and arrange the solutions counted by $\psi(m, X, Y)$ according to the value of w, we infer that

$$\psi(m, X, Y) = \sum_{\substack{w \leq Y \\ w^3 | m}} \psi^* \left(\frac{m}{w^3}, X, \frac{Y}{w} \right),$$

and therefore

$$\begin{aligned} \sum_{m \geq 1} \psi(m, X, Y) d_3(m) &= \sum_{w \leq Y} \sum_{\substack{m \geq 1 \\ w^3 | m}} \psi^* \left(\frac{m}{w^3}, X, \frac{Y}{w} \right) d_3(m) \\ &\leq \sum_{w \leq Y} d_3(w^3) \Psi \left(X, \frac{Y}{w} \right). \end{aligned}$$

With (41), this gives

$$\sum_{m \geq 1} \psi(m, X, Y) d_3(m) \ll \sum_{w \leq Y} d_3(w^3) \frac{X^2 Y^2}{w^2} (\log XY)^2 \ll X^2 Y^2 (\log XY)^2,$$

which shows (31).

To deduce an analogous result for $\psi(m, X, \mathscr{E})$ an inspection of the argument leading up to the estimation of $A_d(\mathscr{E})$ in (41) and $\Psi(X, \mathscr{E})$ reveals that one may replace $\mathscr{E}$ by the set $\frac{\mathscr{E}}{w}$ which consists of integers y such that $wy \in \mathscr{E}$. Indeed this gives

$$\begin{aligned}\sum_{m\geq 1}\psi(m,X,\mathscr{E})d_3(m) &= \sum_{w\leq Y}\sum_{\substack{m\geq 1\\ w^3|m}}\psi^*\left(\frac{m}{w^3},X,\frac{\mathscr{E}}{w}\right)d_3(m)\\ &\leq \sum_{w\leq Y} d_3(w^3)\Psi\left(X,\frac{\mathscr{E}}{w}\right)\\ &\ll \sum_{w\leq Y} d_3(w^3)\frac{X^2Y^2}{w^2}(\log XY)^\epsilon \ll X^2Y^2(\log XY)^\epsilon.\end{aligned}$$

This then implies (32).

□

3.3 Differencing

This section is concerned with establishing a bound for a certain 6-th moment comparable to the 'differencing lemma' [31][Lemma 5]. It serves as preparation to a reduction step when discussing the minor arcs.

Let

$$F(\beta) = \sum_{y\leq Y} e\left(\beta y^2\right).$$

Lemma 18. *Assume α satisfies $\left|\alpha - \frac{a}{q}\right| \leq q^{-2}$ for some coprime a and q. Then we have for $H \leq Y$*

$$\sum_{x\leq N}\sum_{h\leq H}|F(3\alpha hx)|^2 \ll Y^\epsilon\left(HY^2Nq^{-1} + HNY + q\right). \tag{42}$$

Proof. By the standard proof of Weyl's inequality we have

$$\begin{aligned}|F(3\alpha hx)|^2 &= \sum_{y_1,y_2\leq Y} e\left(3\alpha hx(y_1^2-y_2^2)\right)\\ &\ll Y + \sum_{0<h_1\leq Y}\sum_{y\ll Y} e\left(3\alpha hxh_1(2y+h_1^2)\right)\\ &\ll Y + \sum_{0<h_1\leq Y}\min\left(Y,\left\|\frac{6\alpha hh_1x}{q}\right\|^{-1}\right).\end{aligned}$$

Summing that over x and h gives the bound

$$\sum_{x\leq N}\sum_{h\leq H}|F(3\alpha hx)|^2 \ll NHY + Y^\epsilon\sum_{u\ll NHP}\min\left(P,\left\|\frac{\alpha u}{q}\right\|^{-1}\right)$$

which is bounded using [30][Lemma 2.2]. □

Lemma 19. *Let $H \leq Y$, $\delta > 0$ small and define $\mathfrak{k}$ to be the set of α in the unit interval such that $(a, q) = 1$ and $\left|\alpha - \frac{a}{q}\right| \leq q^{-1}H^{-1}X^{-1}Y^{\delta-1}$ implies $q \geq Y^{1+\delta}$. Then for $\alpha \in \mathfrak{k}$ we have*

$$\sum_{x \leq N} \sum_{h \leq H} |F(3\alpha hx)|^2 \ll NHY(\log Y)^{1+\epsilon}.$$

Proof. Following the proof of [31][Lemma 4] with Lemma 18 in hand this easily follows. □

Lemma 20. *Suppose that $M \leq Y^{\frac{1}{7}}, N \leq X$, $Q = \frac{Y}{M}$ and $0 \leq k \leq 3$, and let S denote the number of solutions of*

$$x(z_1^3 - z_2^3) = p^k(x_1y_1^3 - x_2y_2^3 + x_3y_3^3 - x_4y_4^3) \tag{43}$$

subject to

$$x \leq N, \quad x_i \leq X, \quad z_i \leq Y, \quad y_i \leq Q, \quad M < p \leq 2M \quad \textit{prime}, \quad (xz_i, p) = 1.$$

Then

$$S \ll NX^3Y^{\frac{7}{2}}(\log XY)^3M^{-k-\frac{3}{2}}.$$

Proof. The proof of this lemma is a slight adaptation of the ideas of Vaughan in [31]. By (31) the solutions with $z_1 = z_2$ in (43) contribute

$$\ll NX^3Y^{1+\epsilon}MQ^2,$$

which is sufficient. Thus, by symmetry it is enough to count solutions to (43) with $z_1 > z_2$. Fix p temporarily and note that any solution to (43) satisfies $xz_1^3 = xz_2^3 \bmod p^k$. As x is coprime to p, this also implies that $z_1^3 = z_2^3 \bmod p^k$. To give an upper bound it is therefore enough (cf. Vaughan [31][Lemma 5]) to give a bound on the number S_1 of solutions to (43) with $z_1 > z_2$ and $z_1 = z_2 \bmod p^k$. On writing $h = (z_1 - z_2)/p^k$, (43) becomes

$$xh(3(2z_2 + hp^k)^2 + h^2p^{2k}) = 4\left(x_1y_1^3 - x_2y_2^3 + x_3y_3^3 - x_4y_4^3\right).$$

Then, summing over p, it suffices to bound the number S_2 of solutions to

$$xh(3z^2 + h^2p^{2k}) = 4\left(x_1y_1^3 - x_2y_2^3 + x_3y_3^3 - x_4y_4^3\right)$$

subject to

$$z \leq 2Y, \quad x \leq N, \quad x_i \leq X, \quad y_i \leq Y \quad h \leq \frac{Y}{M^k}, \quad M < p \leq 2M.$$

Write $H = \frac{Y}{M^k}$ and define

$$f(\alpha, Q) = \sum_{x \leq X} \sum_{y \leq Q} e\left(\alpha xy^3\right)$$

and

$$G(\beta) = \sum_{M<p\le 2M} e\left(\beta p^{2k}\right).$$

Then, by orthogonality,

$$S_2 = \int_0^1 \sum_{x\le N}\sum_{h\le H} F(3\alpha xh)G(\alpha xh^3)\left|f(\alpha,Q)\right|^4\, d\alpha.$$

Let $\delta = 10^{-4}$,

$$\mathfrak{K}(q,a) := \{\alpha : \left|\alpha - \frac{a}{q}\right| \le \left(qY^{1-\delta}HX\right)^{-1}\},$$

and define $\mathfrak{K}$ to be the union of such $\mathfrak{K}(q,a)$ with $q \le Y^{1-\delta}$ and $(a,q) = 1$. Then for $\alpha \in \mathfrak{K}(q,a)$, we have by (42),

$$\sum_{x\le N}\sum_{h\le H} \left|F(3\alpha hx)\right|^2 \ll HNY^{2+\epsilon}q^{-1}$$

and consequently

$$\sum_{x\le N}\sum_{h\le H} F(3\alpha hx)G(\alpha xh^3) \ll MHNY^{1+\epsilon}q^{-\frac{1}{2}}. \tag{44}$$

Let $\alpha \in \mathfrak{K}(q,a)$, then for fixed $x \le X$,

$$f_x(\alpha) \ll q^{-\frac{1}{3}}(x,q)^{\frac{1}{3}}Q\left(1 + xQ^3\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{3}} + Y^{\frac{1}{2}+\frac{\delta}{2}+\epsilon}\left(\frac{x}{X}\right)^{\frac{1}{2}}.$$

Summing over $x \le X$ gives for $\alpha = \frac{a}{q} + \beta \in \mathfrak{K}(q,a)$,

$$f(\alpha,Q) \ll Qq^{-1/3}\sum_{d|q} d^{1/3} \sum_{\substack{x\le X\\ d|x}} \frac{1}{\left(1 + xQ^3|\beta|\right)^{1/3}} + Y^{\frac{1}{2}+\frac{\delta}{2}+\epsilon}X$$

and thus

$$\left|f(\alpha,Q)\right|^4 \ll Q^4q^{-4/3+\epsilon}X^3\sum_{d|q} d^{4/3-3} \sum_{x\le X/d} \frac{1}{\left(1 + xdQ^3|\beta|\right)^{4/3}} + X^4Y^{2+2\delta+\epsilon}. \tag{45}$$

By (44) and (45) the contribution of $\mathfrak{K}$ is

$$\begin{aligned} I_{\mathfrak{K}} &= \int_{\mathfrak{K}} \sum_{x\le N}\sum_{h\le H} F(3\alpha xh)G(\alpha xh^3)\left|f(\alpha,Q)\right|^4\, d\alpha \\ &\ll \sum_{q\le Y^{1+\delta}} MHNY^{1+\delta}q^{1/2+\epsilon}\left(q^{-4/3+\epsilon}X^3(\log X)Q + X^4Y^{2+2\delta+\epsilon}(qXHY^{1-\delta})^{-1}\right) \\ &\ll \sum_{q\le Y^{1+\delta}} X^3NY^{3+\delta}M^{-k+1}q^{-5/3+\epsilon} + q^{-1/2+\epsilon}MX^3NY^{2+4\delta+\epsilon} \\ &\ll X^3NY^{3+1/6+2\delta}M^{-k+1} + MX^3NY^{2+1/2+5\delta}. \end{aligned}$$

Hence for $M \le Y^{1/7}$, we have

$$I_{\mathfrak{K}} \ll X^3NY^3M^k \left(Y^{1/2}M^{-9/2}+1\right). \tag{46}$$

For the minor arcs $\mathfrak{k} := [Y^{1-\delta}HX, 1+Y^{1-\delta}HX]\setminus\mathfrak{K}$, we have after an application of the Cauchy-Schwarz inequality,

$$I_{\mathfrak{k}} = \int_{\mathfrak{k}} \sum_{h\le H}\sum_{x\le N} F(3\alpha hx)G(\alpha h^3x)\left|f(\alpha,Q)\right|^4 \, d\alpha \ll (I_1)^{\frac{1}{2}}(I_2)^{\frac{1}{2}}, \tag{47}$$

where

$$I_1 = \int_{\mathfrak{k}} \sum_{h\le H}\sum_{x\le N} \left|F(3\alpha hx)\right|^2\left|f(\alpha,Q)\right|^4 \, d\alpha$$

and

$$I_2 = \int_{\mathfrak{k}} \sum_{h\le H}\sum_{x\le N} \left|G(\alpha hx)\right|^2\left|f(\alpha,Q)\right|^4 \, d\alpha.$$

Note that for $\alpha \in \mathfrak{k}$, with Lemma 19, we have

$$\sum_{h\le H}\sum_{x\le N} \left|F(3\alpha hx)\right|^2 \ll HNY\left(\log Y\right)^{1+\epsilon}.$$

Using this in conjunction with Lemma 31 we deduce

$$I_1 \ll NX^3Q^2HY\left(\log XY\right)^{3+\epsilon}. \tag{48}$$

Introduce

$$E(\beta) = \sum_{x\le N}\sum_{h\le H} e\left(\beta x h^3\right),$$

then by opening the square we have the bound

$$\sum_{h\le H}\sum_{x\le N}\left|G(\alpha x h^3)\right|^2 \ll HNM + \sum_{M<p_1<p_2\le 2M}\left|E(\alpha(p_2^6-p_1^6)\right|.$$

Thus we have

$$I_2 \ll HNX^3M\left(\log XY\right)^2Q^2 + \sum_{M<m_1<m_2\le 2M} K(m_1^6-m_2^6), \tag{49}$$

where we have introduced

$$K(d) = \int_0^1 \left|E(\alpha d)\right|\left|f(\alpha,Q)\right|^4 \, d\alpha$$

for an integer d. Define $\mathfrak{N}(q,a) = \{\alpha : \left|\alpha - \frac{a}{q}\right| \le (6qH^2X)^{-1}\}$ and denote by $\mathfrak{N}$ the union of such for $1 \le a \le qd$ and $q \le H$ with $(a,q)=1$. Let $\mathfrak{V} = [(6qH^2X)^{-1}, d+(6qH^2X)^{-1}]$ and set $\mathfrak{n} = \mathfrak{V}\setminus\mathfrak{N}$. Note that minor arcs $\mathfrak{n}$ are contained in $\mathfrak{m}(H)$ (and therefore a variant of lemma 13 can be used) such that we may deduce for $\beta \in \mathfrak{n}$ the bound

$$E(\beta) \ll H^{\frac{3}{4}} N (\log H)^{\frac{1}{4}+\epsilon}.$$

Hence we may rewrite the integral using again (31) to get

$$\begin{aligned}\int_0^1 |E(d\alpha)|\,|f(\alpha,Q)|^4\, d\alpha &= d^{-1}\int_0^d |E(\beta)| \left| f\left(\frac{4\beta}{d},Q\right)\right|^4 d\alpha \\ &= d^{-1}\int_{\mathfrak{N}} |E(\beta)| \left| f\left(\frac{4\beta}{d},Q\right)\right|^4 d\alpha + O\left(H^{\frac{3}{4}} N X^3 (\log XY)^{2+\frac{1}{4}+\epsilon} Q^2\right)\end{aligned} \tag{50}$$

Let

$$J = d^{-1}\int_{\mathfrak{N}} |E(\beta)| \left| f\left(\frac{4\beta}{d},Q\right)\right|^4 d\alpha.$$

We apply the Hölder inequality on J to deduce the bound

$$J \le L(d)^{1/4} M_4(d)^{1/2} M_8(d)^{1/4}, \tag{51}$$

where we have written

$$L(d) = d^{-1}\int_{\mathfrak{N}} |E(\beta)|^4\, d\beta$$

and

$$M_k(d) = d^{-1}\int_0^d \left| f\left(\frac{4\beta}{d},Q\right)\right|^k d\beta.$$

By a standard estimate on the major arcs $\mathfrak{N}$ we deduce that for $\epsilon > 0$, $L(d)$ satisfies

$$L(d) \ll N^3 H^{1+\epsilon}.$$

Moreover by a change of variables , (16) and (15) we have the following bounds for $M_k(d)$:

$$M_4(d) \ll X^3 Q^{2+\epsilon} \qquad \text{and} \quad M_8(d) \ll X^7 Q^{5+\epsilon}. \tag{52}$$

Equipped with (50),(51) and (52) we readily deduce

$$K(d) \ll N^{3/4} X^{13/4} Q^{9/4} H^{1/4} Y^{\epsilon} + H^{3/4} N X^3 (\log XY)^{2+1/4+\epsilon} Q^2.$$

Recalling (49), we may bound, writing $L = \log XY$,

$$\begin{aligned}&I_2 \ll HNX^3ML^2Q^2 + M^2(N^{3/4}X^{13/4}Q^{9/4}H^{1/4}Y^{\epsilon} + H^{3/4}NX^3L^{2+\frac{1}{4}+\epsilon}Q^2) \\ &= NX^3Y^3M^{-k-1}L^2 + N^{3/4}X^{13/4}Y^{2+1/2+\epsilon}M^{-1/4-k/4} + NX^3Y^{2+3/4}M^{-3k/4}L^{2+1/4+\epsilon}.\end{aligned}$$

Therefore, by (48) and (47), we have for the contribution of $\mathfrak{k}$ to S_2,

$$\begin{aligned}|I_{\mathfrak{k}}|^2 \ll\, & N^2X^6Y^7M^{-2k-3}L^{5+\epsilon} + N^{7/4}X^{25/4}Y^{6+1/2+\epsilon}M^{-9/4-5k/4}L^{3+\epsilon} \\ &+ N^2X^6Y^{6+3/4}M^{-2-7k/4}L^{5+1/4+\epsilon}.\end{aligned}$$

Thus, $I_{\mathfrak{k}} = O(NX^3Y^{7/2}M^{-k-3/2}L^3)$. But since

$$S_2 = I_{\mathfrak{K}} + I_{\mathfrak{k}},$$

this together with (46) finishes the proof.

□

3.4 The minor arcs

The goal of this section is to establish the following central lemma:

Lemma 21. *Provided $Y \geq X^{\frac{1}{3}-\delta/3}$, we have*

$$\int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha \ll X^7 Y^5 (\log Y)^{-2+\epsilon}.$$

The treatment of the minor arcs relies on the following lemma which reduces the task of bounding the 8-th moment of $f(\alpha)$ on the minor arcs to the one of $g_E(\alpha)$.

Lemma 22. *Provided $Y \geq X^{1/3-\delta/3}$ we have,*

$$\int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha \ll \int_0^1 |g_E(\alpha)|^8 \, d\alpha + X^7 Y^5 (\log Y)^{-3}.$$

For the sake of brevity write $L := \log XY$ and denote by $m(y)$ the smallest prime factor p of y with $p > L^{80}$ if such factor exists and set $m(y)$ to be $+\infty$ otherwise. Define the sets of ordered pairs

$$\begin{aligned}
\mathfrak{C}_d &= \{(y,z) : y, z \leq Y, (y,z) = d\} \\
\mathfrak{C} &= \bigcup_{d > L^{80}} \mathfrak{C}_d \\
\mathfrak{D} &= \{(y,z) : (y,z) < L^{80}, m(z) \leq Y^{1/7}\} \\
\mathfrak{E} &= \{(y,z) : (y,z) < L^{80}, m(y) \leq Y^{1/7}, m(z) > Y^{1/7}\} \\
\mathfrak{G} &= \{(y,z) : (y,z) < L^{80}, m(y), m(z) > Y^{1/7}\}.
\end{aligned}$$

It might be worth mentioning that the value 80 appearing in the exponent of the logarithm might be replaced by some large integer B.

Let $\mathfrak{B}$ be a set of the above and write

$$I(\mathfrak{B}) = \int_{\mathfrak{m}} \sum_{\substack{x_i \leq X \\ y_i \leq Y \\ (y_1, y_2) \in \mathfrak{B}}} e\left(\alpha\left(x_1 y_1^3 - x_2 y_2^3\right)\right) |f(\alpha)|^6 \, d\alpha,$$

so that we have the decomposition

$$I = \int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha = I(\mathfrak{C}) + I(\mathfrak{D}) + I(\mathfrak{E}) + I(\mathfrak{G}).$$

We start with discussing the contribution of $I(\mathfrak{C}_d)$. By Hölder's inequality and (33) we have

$$\begin{aligned}
I(\mathfrak{C}_d) &\leq \left(\int_0^1 \Bigg| \sum_{\substack{x_i \leq X \\ y_i \leq Y \\ (y_1, y_2) \in \mathfrak{C}_d}} e\left(\alpha\left(x_1 y_1^3 - x_2 y_2^3\right)\right) \Bigg|^4 d\alpha \right)^{1/4} \left(\int_0^1 |f(\alpha)|^8 \, d\alpha \right)^{3/4} \\
&\ll X^7 Y^5 L^{1+\epsilon} d^{-5/4}.
\end{aligned}$$

By summing over $d > L^{80}$ we deduce that for $A \le 18$

$$I(\mathfrak{C}) = \sum_{d > L^{80}} I(\mathfrak{C}_d) \ll X^7 Y^5 L^{-A}.$$

The treatment of either $I(\mathfrak{D})$ or $I(\mathfrak{E})$ is similar and relies on the fact that we may extract a prime divisor p between L^{80} and $Y^{1/7}$. Thus let $\mathfrak{B}$ denote either $\mathfrak{D}$ or $\mathfrak{E}$ and apply lemma 13 once to deduce

$$I(\mathfrak{B}) \ll XYL^{1/4+\epsilon} I_1, \tag{53}$$

where we have written

$$I_1 = \int_0^1 \Big| \sum_{\substack{x_i \le X \\ y_i \le Y \\ (y_1, y_2) \in \mathfrak{B}}} e\left(\alpha\left(x_1 y_1^3 - x_2 y_2^3\right)\right) \Big| \, |f(\alpha)|^5 \, d\alpha.$$

An application of the Cauchy-Schwarz inequality gives

$$I_1 \ll \left(\int_0^1 \Big| \sum_{\substack{x_i \le X \\ y_i \le Y \\ (y_1, y_2) \in \mathfrak{B}}} e\left(\alpha\left(x_1 y_1^3 - x_2 y_2^3\right)\right) \Big|^2 |f(\alpha)|^2 \, d\alpha \right)^{1/2} \left(\int_0^1 |f(\alpha)|^8 \, d\alpha \right)^{1/2} \tag{54}$$

and the first integral is bounded by the number of solutions V to the system

$$xy_1^3 - p_1^3 x_1 z_1^3 + x_2 w^3 = x' y_2^3 - p_2^3 x_3 z_2^3 + x_4 (w')^3$$

subject to $x, x', x_i \le X$, $z_i \le Y/p_i$, $w, w' \le Y$ and $L^{80} < p_i \le Y^{1/7}$. Note that by the definition of $\mathfrak{B}$ we also have $(y_i, p_i) = 1$. Thus by (54) and (53) we have

$$I(\mathfrak{B}) \ll X^5 Y^{13/4} L^{3/4+\epsilon} V^{1/2}. \tag{55}$$

We may divide the solutions counted by V into dyadic intervals. Define

$$\mathscr{M} = \left\{ M : M = 2^k L^{80}, k = 0, 1, \dots ; M \le Y^{1/7} \right\}$$

and let

$$g_p(\alpha) = \sum_{x \le X} \sum_{\substack{y \le Y \\ (y,p)=1}} e\left(\alpha x y^3\right),$$

then

$$V \le \int_0^1 \Big| \sum_{M \in \mathscr{M}} \sum_{M < p \le 2M} g_p(\alpha) f\left(-\alpha p^3, Y/M\right) \Big|^2 |f(\alpha)|^2 \, d\alpha.$$

By Cauchy-Schwarz and Hölder's inequality we get

$$\begin{aligned} V &\le L \sum_{M \in \mathscr{M}} \int_0^1 \sum_{M < p \le 2M} |g_p(\alpha)|^2 \left| f\left(-\alpha p^3, Y/M\right) \right|^2 |f(\alpha)|^2 \, d\alpha. \\ &\ll L \sum_{M \in \mathscr{M}} M V_1(M)^{1/2} V_2(M)^{1/4} V_3(M)^{1/4}, \end{aligned}$$

where we have introduced

$$V_1(M) = \int_0^1 \sum_{M<p\le 2M} |g_p(\alpha)|^2 \left| f\left(-\alpha p^3, Y/M\right)\right|^4 d\alpha$$
$$V_2(M) = \int_0^1 \sum_{M<p\le 2M} |g_p(\alpha)|^4 \, d\alpha$$
$$V_3(M) = \int_0^1 M\,|f(\alpha)|^8 \, d\alpha.$$

In order to apply lemma 20 to $V_1(M)$ we need to first apply Cauchy-Schwarz to get

$$V_1(M) \le X \int_0^1 \sum_{M<p\le 2M} \sum_{x\le X} \Bigg| \sum_{\substack{y\le Y \\ (y,p)=1}} e\left(\alpha x y^3\right)\Bigg|^2 \left| f\left(-\alpha p^3, Y/M\right)\right|^4 d\alpha,$$

which corresponds to the number of solutions to the equation

$$x(z_1^3 - z_2^3) = p^3(x_1 y_1^3 - x_2 y_2^3 + x_3 y_3^3 - x_4 y_4^3)$$

subject to the constraints $x, x_i \le X, z_i \le Y, y_i \le Y/M$ and $(p, z_i) = 1, M < p \le 2M$. Since p might still divide x in this setup, we divide the $x \le X$ according to the highest power $0 \le k \le 3$ that divides x. Upon writing $N = X/M^{3-k}$ these classes are now subject of lemma 20, which now produces the bound

$$V_1(M) \ll X^5 Y^{7/2} L^3 M^{-9/2}. \tag{56}$$

Recalling lemma 15 and 16, equations (33) and (31) are easily applied to bound $V_2(M)$ and $V_3(M)$ respectively. Thus

$$V_2(M) \ll M X^3 Y^2 L^2 \tag{57}$$

and

$$V_3(M) \ll M X^7 Y^5 L^{3+\epsilon}. \tag{58}$$

Collecting (56), (57) and (58) now gives rise to a bound for V, that is

$$V \ll X^5 Y^{7/2} L^6 \sum_{M\in\mathscr{M}} M^{-3/4} \ll X^5 Y^{7/2} L^{-54}.$$

Inserting this in (55) finally shows

$$I(\mathfrak{B}) \ll X^7 Y^5 L^{-A}.$$

The final step in the proof of lemma 22 is a slight variation of an adaptation due to Boklan [5] of Vaughan's approach in [31].

Recall the decomposition

$$I = \int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha = I(\mathfrak{C}) + I(\mathfrak{D}) + I(\mathfrak{E}) + I(\mathfrak{G}),$$

such that the above arguments imply that

$$I = I(\mathfrak{G}) + O\left(X^7Y^5L^{-A}\right).$$

Furthermore we notice that by the definition of the set E, we have

$$I(\mathfrak{G}) = \int_{\mathfrak{m}} |g_E(\alpha)|^2 |f(\alpha)|^6 \, d\alpha - \sum_{d>L^{80}} \int_{\mathfrak{m}} \sum_{\substack{x_i \le X \\ y_i \le Y \\ (y_1,y_2)\in\mathfrak{G}}} e\left(\alpha\left(x_1y_1^3 - x_2y_2^3\right)\right) |f(\alpha)|^6 \, d\alpha.$$

The sum over $d > L^{80}$ can be dealt with the same way as in the treatment of $I(\mathfrak{C})$, which leaves

$$I \ll \int_{\mathfrak{m}} |g_E(\alpha)|^2 |f(\alpha)|^6 \, d\alpha + O\left(X^7Y^5L^{-A}\right).$$

An application of Hölder's inequality now shows that

$$I \ll \left(\int_{\mathfrak{m}} |g_E(\alpha)|^8 \, d\alpha\right)^{1/4} \left(\int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha\right)^{3/4} + O\left(X^7Y^5L^{-A}\right),$$

and thus

$$I \ll \int_{\mathfrak{m}} |g_E(\alpha)|^8 \, d\alpha + O\left(X^7Y^5L^{-A}\right).$$

We may now extend the range of integration to deduce lemma 22.

3.5 An eighth moment bound

In view of lemma 22 the treatment of the minor arcs is reduced to finding a suitable estimate for the 8-th moment of $g_E(\alpha)$. This follows mainly the approach taken in [32] with slight adaptations.

Let

$$\Lambda(\mathbf{x},\mathbf{y}) = x_1y_1^3 - x_2y_2^3 + x_3y_3^3 - x_4y_4^3.$$

For further reference we record two divisor sum estimated related to Λ.

Lemma 23. *There is a positive number λ such that*

$$\sum_{\substack{\mathbf{x}\le X \\ \mathbf{y}\le Y \\ \Lambda(\mathbf{x},\mathbf{y})\neq 0}} d\left(|\Lambda(\mathbf{x},\mathbf{y})|\right)^8 \ll X^4Y^4 (\log Y)^\lambda \tag{59}$$

and a positive number δ such that

$$\sum_{\substack{\mathbf{x}\le X \\ \mathbf{y}\le Y \\ \Lambda(x,\mathbf{y})\neq 0}} d\left(|\Lambda(\mathbf{x},\mathbf{y})|\right)^8 e^{\delta\Omega(|\Lambda(x,\mathbf{y})|)} \ll X^4Y^4 (\log Y)^\lambda. \tag{60}$$

This is by [22][Theorem 3].

Lemma 24. *We have*

$$\int_0^1 |g_E(\alpha)|^8 \, d\alpha \ll X^7 Y^5 (\log Y)^{\epsilon - 2}.$$

By applying Hölder's inequality together with an application of the Weyl technique we get

$$|g_E(\alpha)|^2 \le X \sum_{x \le X} \left| \sum_{y \in \mathscr{E}} e\left(\alpha x y^3\right) \right|^2 = X \sum_{x \le X} \sum_{y_1, y_2 \in \mathscr{E}} e\left(\alpha x (y_1^3 - y_2^3)\right)$$
$$= X^2 |\mathscr{E}| + \sum_{x \le X} \sum_{y \in \mathscr{E}} \sum_{\substack{h \le Y \\ y + h \in \mathscr{E}}} e\left(\alpha x h (3y^2 + 3hy + h^2)\right).$$

Therefore

$$|g_E(\alpha)|^4 \ll X^4 Y |\mathscr{E}|^2 + X^3 Y \sum_{x \le X} \sum_{h_1 \ll Y} \sum_{h_2 \ll Y} \sum_{y \in \mathscr{E}} e\left(\alpha x h_1 h_2 \Theta_{h_1, h_2}(y)\right),$$

where the summation is subject to the conditions $y \pm h_i \in \mathscr{E}$ and $\Theta_{h_1,h_2}(y)$ is a linear polynomial. We now multiply with $|g_E(\alpha)|^4$ and integrate, recalling (32), to get

$$\int_0^1 |g_E(\alpha)|^8 \, d\alpha \ll X^7 |\mathscr{E}|^2 L^\epsilon Y^3 + X^3 Y \sideset{}{'}\sum_{\substack{\mathbf{x} \le X \\ \mathbf{y} \\ \Lambda(\mathbf{x}, \mathbf{y}) \ne 0}} r\left(|\Lambda(\mathbf{x}, \mathbf{y})|\right), \tag{61}$$

with $r(n)$ being the number of solutions to the equation

$$x h_1 h_2 \ell = n \tag{62}$$

subject to $x \le X$, $h_i \le Y$ and $\ell \ll Y$. To bound the sum in (61) we pick an integer Q with $1 \le Q \le Y$ and consider the contribution of $\mathbf{x}$ and $\mathbf{y}$ with $|\Lambda(\mathbf{x}, \mathbf{y})| \le Q Y^2 X$. By Cauchy's inequality we have

$$\Psi_1 := \sum_{\substack{\mathbf{x} \le X \\ \mathbf{y} \le Y \\ \Lambda(x, \mathbf{y}) \ne 0 \\ |\Lambda| \le Q Y^2 X}} r\left(|\Lambda(x, \mathbf{y})|\right) \le \Lambda_1^{1/2} \Lambda_2^{1/2}, \tag{63}$$

with

$$\Lambda_1 = \sideset{}{'}\sum_{\substack{\mathbf{x} \le X \\ \mathbf{y} \le Y \\ |\Lambda| \le Q Y^2 X}} 1$$

and

$$\Lambda_2 = \sideset{}{'}\sum_{\substack{\mathbf{x} \le X \\ \mathbf{y} \le Y \\ |\Lambda| \le Q Y^2 X}} d_4\left(|\Lambda(\mathbf{x}, \mathbf{y})|\right)^2.$$

Recalling (59) and (62) we may bound the right-hand-side of (63) by

$$O\left(\Lambda_1^{1/2} X^2 Y^2 (\log Y)^{\lambda/2}\right) \tag{64}$$

and we are left to deal with Λ_1.

Lemma 25. *We have for* $1 \leq Q \leq Y$

$$\Lambda_1 \ll X^4 Y^4 \left(\frac{Q}{Y}\right)^{\frac{1}{3}} (\log X). \tag{65}$$

Proof. We start by dividing the x_1 range into dyadic interval and fix $N \leq x_1 < 2N$, $x_2, x_3, x_4 \leq X$ and $y_2, y_3, y_4 \leq Y$. Assume that there is a $y_1 > Y\left(\frac{Q}{Y}\right)^{\frac{1}{3}}$ with $|\Lambda| \leq QY^2X$. If there is another y_1' with the same property then we have

$$N|y_1 - y_1'|Y^2 \left(\frac{Q}{Y}\right)^{\frac{2}{3}} \leq x_1|y_1^3 - (y_1')^3| \leq 4QY^2X$$

implying that

$$|y_1 - y_1'| \ll \frac{X}{N} Y \left(\frac{Q}{Y}\right)^{\frac{1}{3}}.$$

Thus there are $\frac{X}{N}Y\left(\frac{Q}{Y}\right)^{\frac{1}{3}}$ choices for y_1. Therefore, after accounting for the NX^3Y^3 variables fixed in the beginning and summing over the dyadic ranges,

$$\Lambda_1 \ll X^4 Y^4 \left(\frac{Q}{Y}\right)^{\frac{1}{3}} (\log X).$$

□

Collecting (63), (64) and (65) we conclude that

$$\Psi_1 \ll X^4 Y^4 (\log Y)^{\frac{\lambda}{2}+1} \left(\frac{Q}{Y}\right)^{\frac{1}{6}}.$$

Upon choosing $Q = Y(\log Y)^{-A}$ where A is a sufficiently large number we deduce

$$\Psi_1 \ll X^4 Y^4 (\log Y)^{-3}. \tag{66}$$

It remains to show that the contribution of

$$\Psi_2 := \sum_{\substack{\mathbf{x} \leq X \\ \mathbf{y} \leq Y \\ \Lambda(\mathbf{x},\mathbf{y}) \neq 0 \\ |\Lambda| > QY^2X}} r(|\Lambda(\mathbf{x},\mathbf{y})|)$$

is not too big either. Notice that if $n > 4XY^3$ the definition of $r(n)$ in (62) implies that $r(n) = 0$. So for $QY^2X < n < 4XY^3$ we disect the ranges of the variables in $r(n)$ into intervals of length $\log(Y)$ and deduce

$$r(n) \ll (\log\log Y)^4 \Delta_4(n),$$

where $\Delta_4(n)$ is Hooley's delta function introduced in chapter 3. Therefore

$$\Psi_2 \ll (\log\log Y)^4 \sum_n \Delta_4(n) F(n) \tag{67}$$

where $F(n)$ is the number of solutions of

$$\Lambda(\mathbf{x},\mathbf{y}) = n$$

with $y \in \mathscr{E}$. Let μ be a small positive number and define

$$Z = \exp\left(\frac{\delta\mu \log Y}{(\lambda+10)\log\log Y}\right)$$

and

$$n^* = \prod_{\substack{p^t \| n \\ p \leq Z}} p^t.$$

When $n^* > Y^\mu$ we have $\Omega(n)\log Z \geq \log n^* > \mu \log Y$ and hence

$$\delta\Omega(n) > (\lambda+10)\log\log Y.$$

By (60) we may bound

$$\begin{aligned}\sum_{\substack{n \\ n^* > Y^\mu}} \Delta_4(n)F(n) &\leq \sum_n d(n)^4 F(n) e^{\delta\Omega(n)} (\log Y)^{-\lambda-10} \\ &\ll X^4 Y^4 (\log Y)^{-10},\end{aligned}$$

thereby reducing a bound for (67) to bounding

$$\Psi_3 := \sum_{\substack{n \\ n^* \leq Y^\mu}} \Delta_4(n)F(n).$$

Let

$$\mathcal{N} := \{n \in \mathbb{N} : p \mid n \Rightarrow p \leq Z\}, \qquad \mathcal{M} := \{n \in \mathbb{N} : p \mid n \Rightarrow p > Z\}.$$

By factorizing and applying the basic inequality $\Delta_4(nm) \leq d_4(n)\Delta_4(m)$ we get

$$\begin{aligned}\Psi_3 &= \sum_{\substack{m \in \mathcal{M} \\ m \leq Y^\mu}} \sum_n \Delta_4(mn)F(mn) \\ &\ll \sum_{\substack{m \in \mathcal{M} \\ m \leq Y^\mu}} \Delta_4(m) \sum_n d_4(n)F(mn). \qquad (68)\end{aligned}$$

As in [9][Lemma 7], following the lines of [31][p. 16 f], we may observe that for $n \ll XY^3$ there exists a number L depending only on μ such that for ever n there is a divisor $n_1 \leq Y^\mu$ of n, such that $d_4(n) \ll 2^{L\Omega(n_1)}$. We may apply this in the inner sum in (68) to get

$$\sum_{\substack{n \\ n^* \leq Y^\mu}} \Delta_4(n)F(n) \ll \sum_{\substack{m \in \mathcal{M} \\ m \leq Y^\mu}} \Delta_4(m) \sum_{\substack{n_1 \in \mathcal{N} \\ n_1 \leq Y^\mu}} 2^{L\Omega(n_1)} \sum_{\substack{n \\ mn_1 | n}} F(n). \qquad (69)$$

Further progress depends on information about

$$\sum_{\substack{n\\ mn_1|n}} F(n).$$

For convenience write $d = mn_1$.

Assume $d \leq \min(X, Y^{2\mu})$. Then by dividing the variables in $F(n)$ into residue classes mod d we have

$$\sum_{\substack{n\\ d|n}} F(n) \ll Y^4 X^4 (\log Y)^{\epsilon-4} \frac{S(d)}{d^8}$$

where $S(d)$ is the number of solutions of $\Lambda(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod d$. Evidently we have

$$\begin{aligned} S(d) &= d^{-1} \sum_{a=1}^{d} \left| \sum_{x=1}^{d} \sum_{n=1}^{d} e\left(\frac{axn^3}{d}\right) \right|^4 \\ &= d^{-1} \sum_{q|d} \sum_{\substack{a=1\\ (a,d)=q}}^{d} \left| \sum_{x=1}^{d} \sum_{n=1}^{d} e\left(\frac{axn^3}{d}\right) \right|^4 \\ &= d^{-1} \sum_{q|d} \left(\frac{d}{q}\right)^8 \sum_{\substack{a=1\\ (a,q)=1}}^{q} |S(q,a)|^4 \end{aligned}$$

where we have used

$$S(q,a) = \sum_{x=1}^{q} \sum_{n=1}^{q} e\left(\frac{axn^3}{q}\right).$$

Thus

$$S(d) = d^7 \sum_{q|d} A_0(q),$$

where

$$A_0(q) = q^{-8} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |S(q,a)|^4.$$

This implies

$$\sum_{\substack{n\\ d|n}} F(n) \ll Y^4 X^4 (\log Y)^{\epsilon-4} d^{-1} \sum_{q|d} A_0(q). \tag{70}$$

Suppose now that $X < d$. Then we have to argue slightly differently. Consider $S_1(d)$, defined to be the number of solutions of $\Lambda(\mathbf{x}, \mathbf{y}) \equiv 0 \pmod d$ with y_i mod d and $x_i \leq X < d$. We still might divide the y variables into residue classes mod d, such that we have

$$\sum_{\substack{n\\ d|n}} F(n) \ll Y^4 (\log Y)^{\epsilon-4} \frac{S_1(d)}{d^4}.$$

By orthogonality

$$S_1(d) = d^{-1} \sum_{b=1}^{d} \left| \sum_{x \le X} \sum_{n=1}^{d} e\left(\frac{axn^3}{d}\right) \right|^4$$
$$= d^{-1} \sum_{q|d} q^4 \sum_{\substack{a=1 \\ (a,d/q)=1}}^{d/q} \left| \sum_{x \le X} \sum_{n=1}^{d/q} e\left(\frac{axn^3}{d/q}\right) \right|^4$$

Since $(a, d/q) = 1$, we have

$$\sum_{x \le X} \sum_{n=1}^{d/q} e\left(\frac{axn^3}{d/q}\right) \ll \sum_{x \le X} \left(\frac{d}{q}\right)^{2/3} \left(x, \frac{d}{q}\right)^{1/3}$$
$$\ll \left(\frac{d}{q}\right)^{2/3} \sum_{\ell | d/q} \ell^{1/3} \sum_{\substack{x \le X \\ \ell | x}} 1$$
$$\ll X \left(\frac{d}{q}\right)^{2/3} d\left(\frac{d}{q}\right).$$

This gives

$$S_1(d) \ll X^4 d^{5/3} \sum_{q|d} q^{4/3} \varphi(d/q) d(d/q)^4$$
$$= X^4 d^3 \sum_{q|d} q^{-4/3} \varphi(q) d(q)^4$$

and hence for $A_1(q) = q^{-4/3}\varphi(q)d(q)^4$,

$$\sum_{\substack{n \\ d|n}} F(n) \ll Y^4 X^4 (\log Y)^{\epsilon - 4} d^{-1} \sum_{q|d} A_1(q).$$

Note that $A_0(q) \le q^{-4/3}\varphi(q)$ such that $A_1(q), A_0(q) \ll q^{-4/3+\epsilon}\varphi(q) = A(q)$, say. Combining the above discussion with equation (70) the left-hand side of (69) becomes

$$\ll X^4 Y^4 (\log Y)^{\epsilon - 4} \sum_{\substack{m \in \mathcal{M} \\ m \le Y^\mu}} \frac{\Delta_4(m)}{m} \sum_{\substack{n_1 \in \mathcal{N} \\ n_1 \le Y^\mu}} \frac{2^{L\Omega(n_1)}}{n_1} \sum_{q | mn_1} .A(q)$$

The well known inequality $\Delta_4(r\nu) \le d_4(r)\Delta_4(\nu)$ implies

$$\ll XY^4 (\log Y)^{\epsilon - 4} \sum_{\substack{r \in \mathcal{M} \\ r \le Y^\mu}} \sum_{\substack{s \in \mathcal{N} \\ s \le Y^\mu}} \frac{A(rs)}{rs} d_4(r) 2^{L\Omega(s)} \sum_{\nu \le Y^\mu} \frac{\Delta_4(\nu)}{\nu} \sum_{\substack{y \in \mathcal{N} \\ y \le Y^\mu}} \frac{2^{L\Omega(y)}}{y}.$$

By Hall and Tenenbaum [19][Theorem 70]

$$\frac{1}{x} \sum_{n \le x} \Delta_4(n) \ll (\log x)^\epsilon,$$

such that an application of partial summation shows the sum $\sum_{\nu\leq Y}\frac{\Delta_4(\nu)}{\nu}$ to be $\ll (\log Y)^{1+\epsilon}$. Since $y\in\mathcal{N}$, the sum over y is at most

$$\prod_{Z<p\leq Y^\mu}\left(1+\sum_{t=1}^{\infty}2^{Lt}p^{-t}\right)\ll (\log\log Y)^{2^L}.$$

Hence

$$\sum_{\substack{n\\ n^*\leq Y^\mu}}\Delta_4(n)F(n)\ll X^4Y^4(\log Y)^{\epsilon-3}\sum_{\substack{r\in\mathcal{M}\\ r\leq Y^\mu}}\sum_{\substack{s\in\mathcal{N}\\ s\leq Y^\mu}}\frac{A(rs)}{rs}d_4(r)2^{L\Omega(s)}.$$

Note that $A(q)$ is multiplicative. Futhermore the series

$$\sum_{q=1}^{\infty}\frac{A(q)}{q}d_4(q),\quad \sum_{q\in\mathcal{N}}^{\infty}\frac{A(q)}{q}2^{L\Omega(q)} \tag{71}$$

converge and the second series is bounded by a constant independet of Y.

Therefore, by (71)

$$\sum_{\substack{n\\ n^*\leq Y^\mu}}\Delta_4(n)F(n)\ll X^4Y^4(\log Y)^{\epsilon-3}.$$

And finally by (61), (66) and (67), lemma 24 follows. We note that this together with lemma 22 proves the main result for the minor arcs, that is lemma 21.

3.6 The major arcs

Since lemma 21 deals with the minor arc we are now set to prove Theorem 2. Assume $(c_1,c_2,\ldots,c_8)=1$ and and recall that $X\geq L^{12}$. Let

$$\mathcal{F}(\alpha)=\prod_{i=1}^{8}f(c_i\alpha),$$

such that by orthogonality

$$N_{\mathbf{c}}^{+}(X,Y)=\int_0^1\mathcal{F}(\alpha)\,d\alpha. \tag{72}$$

Following the standard approach in using the Hardy-Littlewood circle method we define

$$S(q,a)=\sum_{m \bmod q}\sum_{n \bmod q}e\left(\frac{amn^3}{q}\right)$$
$$v(\beta)=\int_0^X\int_0^Y e\left(\beta xy^3\right)\,dydx$$

to obtain a suitable approximation for $f(\alpha)$ on the major arcs. Recall the definitions from chapter 2:

$$S_1(q,a) = \sum_{n \bmod q} e\left(\frac{an^3}{q}\right)$$

and

$$v_1(\beta) = \int_0^Y e\left(\beta y^3\right)\, dy$$

and introduce

$$f^*(\alpha) = \sum_{x\le X} \frac{S_1(q,ax)}{q} v_1(\beta x).$$

Let $Q \ge 1$ and consider for co prime integers a, q with $q \le Q$ the sets

$$\mathfrak{M}(q,a) = \left\{\alpha \in [0,1] : |q\alpha - a| \le \frac{Q}{XY^3}\right\}$$

and define

$$\mathfrak{M}(Q) = \bigcup_{q\le Q} \bigcup_{\substack{a=1\\(q,a)=1}}^{q} \mathfrak{M}(q,a).$$

Note that $\mathfrak{M} = \mathfrak{M}(Y^{3/4+\eta})$. We wish to repace $f(\alpha)$ in (72) by $f^*(\alpha)$. For $\alpha \in \mathfrak{M}(Q)$ let $\alpha = \frac{a}{q} + \beta$ with co-prime a and q such that by (28),

$$\begin{aligned} f\left(\frac{a}{q}+\beta\right) &= \sum_{x\le X}\left(\frac{S_1(q,ax)}{q} v_1(\beta x) + O\left(q^{1/2+\epsilon}(1+Y^3x|\beta|)^{1/2}\right)\right) \\ &= f^*(\alpha) + O\left(XQ^{1/2+\epsilon}\right). \end{aligned} \tag{73}$$

We may now replace two $f(c_i\alpha)$ by the corresponding $f^*(c_i\alpha)$ since by (73) and the trivial bound $f^*(\alpha) \ll XY$ the integral over $\mathfrak{M}$ in (72) is equal to

$$\int_{\mathfrak{M}} f(c_3\alpha)\cdots f(c_8\alpha)\left(f^*(c_1\alpha)+O(XY^{3/8+\eta/2})\right)\left(f^*(c_2\alpha)+O(XY^{3/8+\eta/2})\right)\, d\alpha$$

$$\begin{aligned} = \int_{\mathfrak{M}} & f(c_3\alpha)\cdots f(c_8\alpha) f^*(c_1\alpha) f^*(c_2\alpha)\, d\alpha \\ & + O\left(X^2Y^{11/8+\eta/2}\int_0^1 |f(\alpha)|^6\, d\alpha\right). \end{aligned} \tag{74}$$

By

$$\int_0^1 |f(\alpha)|^6\, d\alpha \ll X^5Y^{7/2+\epsilon} \tag{75}$$

the error term in (74) is $O(X^7Y^{5-\delta})$ and thus we have

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha)\, d\alpha = \int_{\mathfrak{M}} f^*(c_1\alpha)f^*(c_2\alpha)f(c_3\alpha)\cdots f(c_8\alpha)\, d\alpha + O(X^7Y^{5-\delta}).$$

The familiar bound $q^{-1}S_1(q,ax) \ll q^{-1/3}(q,x)^{1/3}$ together with [30] Lemma 2.8 provides us with the upper bound

$$f^*(\alpha) \ll \sum_{x\le X} \frac{(q,x)^{1/3}}{q^{1/3}} Y\left(1+Y^3x|\beta|\right)^{-1/3} \tag{76}$$

for $f^*(\alpha)$ on $\mathfrak{M}(q,a)$.

Lemma 26. *We have*

$$\int_{\mathfrak{M}(Q)} |f^*(\alpha)|^6 \, d\alpha \ll Y^3X^5 \log Q. \tag{77}$$

Proof. To ease notation we introduce

$$s(x,\beta) = \left(1+Y^3x|\beta|\right)^{-1/3}.$$

From the definition of $\mathfrak{M}(Q)$ and (76) we have to bound

$$\int_{\mathfrak{M}(Q)} |f^*(\alpha)|^6 \, d\alpha = \sum_{q\le Q} \sum_{\substack{a=1\\(q,a)=1}}^{q} \int_{\mathfrak{M}(q,a)} |f^*(\alpha)|^6 \, d\alpha$$

$$\ll Y^6 \sum_{x_i\le X} \sum_{q\le Q} \sum_{\substack{a=1\\(q,a)=1}}^{q} \frac{(q,x_1)^{1/3}\cdots(q,x_6)^{1/3}}{q^2} \int_{\mathfrak{M}(q,a)} s(x_1,\beta)\cdots s(x_6,\beta)\, d\beta.$$

$$\ll Y^6 \sum_{x_i\le X} \sum_{q\le Q} q^{-1}(q,x_1)^{1/3}\cdots(q,x_6)^{1/3} \int_{|\beta|\le \frac{Q}{qXY^3}} s(x_1,\beta)\cdots s(x_6,\beta)\, d\beta.$$

By an application of Hölder's inequality and completion of the β integration shows

$$\int_{|\beta|\le \frac{Q}{qXY^3}} s(x_1,\beta)\cdots s(x_6,\beta)\, d\beta$$

$$\ll \left(\int_{-\infty}^{\infty} s(x_1,\beta)^6\, d\beta\right)^{1/6} \cdots \left(\int_{-\infty}^{\infty} s(x_6,\beta)^6\, d\beta\right)^{1/6}.$$

By the definition of $s(x_i,\beta)$ the corresponding integral is $O(Y^{-1/2}x_i^{-1/6})$ such that we are left with

$$\int_{\mathfrak{M}(Q)} |f^*(\alpha)|^6 \, d\alpha \ll Y^3 \sum_{q\le Q} q^{-1} \left(\sum_{x\le X} \frac{(q,x)^{1/3}}{x^{1/6}}\right)^6.$$

A short calculation confirms that

$$\sum_{x\le X} \frac{(q,x)^{1/3}}{x^{1/6}} \ll X^{5/6}\sigma_{-2/3}(q),$$

where $\sigma_s(q) = \sum_{d|n} d^s$ and finishes the proof, as $\sum_{q\le Q} \sigma_{-2/3}^6(q)q^{-1} \ll \log Q$. □

Since the bound supplied by (77) is superior to the one supplied by (75), we may repeat the argument leading up to (74) in conjunction with Hölder's inequality to replace all $f(\alpha)$ by $f^*(\alpha)$. Thus

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha)\, d\alpha = \int_{\mathfrak{M}} \mathcal{F}^*(\alpha)\, d\alpha + E^*_{\mathfrak{M}} \tag{78}$$

with $\mathcal{F}^*(\alpha) = f^*(c_1\alpha)\cdots f^*(c_8\alpha)$ and acceptable error $E^*_{\mathfrak{M}} \ll X^7Y^{5-\delta}$.

The next step consists in a pruning of the major arcs to achieve a suitable approximation to $f^*(\alpha)$ on $\mathfrak{Y} = \mathfrak{M}(Q^*)$, with $Q^* = L^6$. A very similar argument that lead to (77) shows that the contribution of $\mathfrak{M}\setminus\mathfrak{Y}$ is negligible. Indeed

$$\begin{aligned}\int_{\mathfrak{M}\setminus\mathfrak{Y}} |f^*(\alpha)|^8\, d\alpha &\ll Y^5 \sum_{Q^*<q\le Q} q^{-5/3}\left(\sum_{x\le X}\frac{(q,x)^{1/3}}{x^{1/8}}\right)^8\\ &\ll Y^5X^7(Q^*)^{-2/3}\log Q.\end{aligned}$$

Thus we infer that

$$\int_{\mathfrak{M}} \mathcal{F}^*(\alpha)\, d\alpha = \int_{\mathfrak{Y}} \mathcal{F}^*(\alpha)\, d\alpha + O(X^7Y^5L^{-3}). \tag{79}$$

On $\mathfrak{Y}$ we now have q small against X, such that dividing x into residue classes mod q is now a reasonable approach.

Lemma 27. *For $\alpha = \frac{a}{q}+\beta \in \mathfrak{Y}$ we have*

$$f^*(\alpha) = q^{-2}S(q,a)v(\beta) + O\left(YQ^*\right). \tag{80}$$

Proof. For $q \le X$ consider the sum

$$\begin{aligned}\sum_{\substack{x\le X\\ x=b \bmod q}} e\left(\gamma x\right) &= \sum_{1\le z\le X/q} e\left(\gamma b\right) e\left(\gamma q z\right) + O(1)\\ &= e\left(\gamma b\right)\left(\int_0^{X/q} e\left(\gamma q z\right)\, dz + O(1+X|\gamma|)\right).\end{aligned}$$

Put $\gamma = \beta y^3$ such that $e\left(\gamma b\right) = 1 + O(Q^*/X)$. Therefore the above line is after an obvious substitution

$$= (1+O(Q^*/X))\left(\frac{1}{q}\int_0^X e\left(\gamma x\right)\, dx + O(1+X|\gamma|)\right).$$

Since $X|\beta|y^3 \le Q^*/q$, which is of the same order of magnitude as $Q^*(qX)^{-1}\int_0^X e\left(\gamma x\right)\, dx$, this implies that

$$\sum_{\substack{x\le X\\ x=b \bmod q}} e\left(\beta x y^3\right) = \frac{1}{q}\int_0^X e\left(\beta x y^3\right)\, dx + O\left(1+\frac{Q^*}{q}\right).$$

Integration from 0 to Y gives

$$\sum_{\substack{x\le X\\ x=b \bmod q}} v_1(\beta x) = \frac{1}{q}\int_0^X e\left(\beta x y^3\right)\,dx + O\left(Y + \frac{YQ^*}{q}\right).$$

But now

$$\begin{aligned} f^*(\alpha) &= \sum_{x\le X} q^{-1} S_1(q, ax) v_1(\beta x) = \sum_{b=1}^{q} q^{-1} S_1(q, ab) \sum_{\substack{x\le X\\ x=b \bmod q}} v_1(\beta x) \\ &= q^{-2} S(q,a) v(\beta) + O(Q^* Y). \end{aligned}$$

□

Hence by (78) and (79) the contribution of $\mathfrak{M}$ to (72) may be written as

$$\int_{\mathfrak{Y}} \mathcal{F}(\alpha)\,d\alpha = \sum_{q\le Q^*} T(q) \int_{\frac{-Q^*}{qXY^3}}^{\frac{Q^*}{qXY^3}} \prod_{i=1}^{8} v(c_i\beta)\,d\beta + E_{\mathfrak{M}} \tag{81}$$

where

$$T(q) = \sum_{\substack{a=1\\ (a,q)=1}}^{q} q^{-16} \prod_{i=1}^{8} S(q, c_i a)$$

and

$$E_{\mathfrak{M}} = \sum_{q\le Q^*} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \int_{\frac{-Q^*}{qXY^3}}^{\frac{Q^*}{qXY^3}} \left(\mathcal{F}^*\left(\frac{a}{q}+\beta\right) - q^{-2}\prod_{i=1}^{8} S(q, c_i a) v(c_i\beta)\right) d\beta.$$

Lemma 28. *Let $(a,q)=1$, then we have*

$$q^{-2} S(q,a) \ll q^{-\frac{1}{3}} \tag{82}$$

and

$$v(\beta) \ll XY\left(1 + |\beta| XY^3\right)^{-\frac{1}{3}}. \tag{83}$$

Proof. The first part follows from a straight forward evaluation of $S(q,a)$:

$$\begin{aligned} S(q,a) &= \sum_{m \bmod q} \sum_{n \bmod q} e\left(\frac{amn^3}{q}\right) \\ &= q|\{n \bmod q : an^3 \equiv 0 \bmod q\}| \\ &\ll q^{5/3}. \end{aligned}$$

as claimed. The second part follows from [30] Lemma 2.8 via integration.

□

By (82), (83) and (80) the error term $E_{\mathfrak{M}}$ in (81) is bounded by

$$\ll (Q^*)^8 Y^8 \frac{(Q^*)^2}{XY^3} + (Q^*Y) \sum_{q\le Q^*} q^{-7/3} X^7 Y^7 \int_0^\infty \frac{1}{(1+|\beta|XY^3)^{7/3}}\, d\beta$$
$$\ll Y^5 X^7 X^{-8} (Q^*)^{10} + X^7 Y^5 Q^*.$$

As $X \ge (Q^*)^2$ this is $O(X^7Y^5L^{-4})$, which is fine. By (81) and lemma 21 we have

$$\int_0^1 \mathcal{F}(\alpha)\, d\alpha = \sum_{q\le Q^*} T(q) \int_{\frac{-Q^*}{qXY^3}}^{\frac{Q^*}{qXY^3}} \prod_{i=1}^{8} v(c_i\beta)\, d\beta + O\left(X^7Y^5L^{-3+\epsilon}\right). \tag{84}$$

As usual the next objective is the completion of the integral and the summation over q in (81). Write

$$I(\beta) = \int_0^1 \int_0^1 e\left(\beta x y^3\right)\, dxdy$$

and define the singular integral

$$\mathfrak{I}^+(\mathbf{c}) = \int_{-\infty}^{\infty} I(c_1\beta) I(c_2\beta) \cdots I(c_8\beta)\, d\beta \tag{85}$$

and the singular series

$$\mathfrak{S}^+(\mathbf{c}) = \sum_{q=1}^{\infty} T(q). \tag{86}$$

For $I(\beta)$ we may use the bound $\int_0^1 e\left(\gamma y^3\right)\, dy \ll |\gamma|^{-1/3}$ and integrate to deduce

$$I(\beta) \ll |\beta|^{-1/3}. \tag{87}$$

From the bound (87) we infer, that the integral in (85) does indeed converge and

$$\int_{\frac{-Q}{q}}^{\frac{Q}{q}} I(c_1\beta)\cdots I(c_8\beta)\, d\beta = \mathfrak{I}^+(\mathbf{c}) + O\left(((Q^*)/q)^{-5/3}\right). \tag{88}$$

Note that by (82) we have

$$\sum_{q\le Q} T(q) = \mathfrak{S}^+(\mathbf{c}) + O\left((Q^*)^{-2/3}\right) \tag{89}$$

which implies the convergence of $\mathfrak{S}^+(\mathbf{c})$. Since by an obvious substitution

$$v(c_i\beta) = XY I(c_i XY^3 \beta),$$

we may write

$$\int_{\frac{-Q^*}{qXY^3}}^{\frac{Q^*}{qXY^3}} v(c_1\beta)\cdots v(c_8\beta)\, d\beta = (XY)^8 \int_{\frac{-Q^*}{qXY^3}}^{\frac{Q^*}{qXY^3}} I(c_1XY^3\beta)\cdots I(c_8XY^3\beta)\, d\beta.$$

Yet another substitution confirms that

$$(XY)^8 \int_{\frac{-Q^*}{qXY^3}}^{\frac{Q^*}{qXY^3}} I'(c_1XY^3\beta)\cdots I'(c_8XY^3\beta)\,d\beta = X^7Y^5 \int_{\frac{-Q^*}{q}}^{\frac{Q^*}{q}} I(c_1\beta)\cdots I(c_8\beta)\,d\beta.$$

Thus the main term in (84) becomes

$$X^7Y^5 \sum_{q\le Q} T(q) \int_{-\infty}^{\infty} I(c_1\beta)\cdots I(c_8\beta)\,d\beta \tag{90}$$

and by (88) the arising error is bounded by

$$X^7Y^5(Q^*)^{-5/3} \sum_{q\le Q^*} qq^{-8/3}q^{5/3} \ll X^7Y^5(Q^*)^{-2/3}.$$

In view of (89) we may also complete to summation in (90) over q to accommodate the singular series. Therefore

$$X^7Y^5 \sum_{q\le Q} T(q)\mathfrak{J}^+(\mathbf{c}) = X^7Y^5\mathfrak{S}^+(\mathbf{c})\mathfrak{J}^+(\mathbf{c}) + O\left(X^7Y^5L^{-4}\right). \tag{91}$$

Let us turn our attention to the analysis of the singular series first. Similar to Lemma 9 we have:

Lemma 29. *The singular series (86) is real and non-negative.*

Proof. The convergence of the singular series was already discussed. To show the positivity of (86) we follow the approach taken in the proof of Lemma 9. A routine argument shows that $T(q)$ is multiplicative and the singular series $\mathfrak{S}^+(\mathbf{c})$ can hence be written as Euler product. That is

$$\mathfrak{S}^+(\mathbf{c}) = \prod_p E_p(\mathbf{c}),$$

where the local densities $E_p(\mathbf{c})$ are given as

$$E_p(\mathbf{c}) = \sum_{\ell=0}^{\infty} T(p^\ell) = \lim_{L\to\infty} p^{-14L}\Phi_{\mathbf{c}}(p^L)$$

and $\Phi_{\mathbf{c}}(q)$ denotes the number of incongruent solution of

$$c_1x_1y_1^3 + \ldots + c_8x_8y_8^3 = 0 \mod q.$$

This shows the non-negativity of the singular series. To establish the positivity of the singular series we may use the linear part of the above equation. Here we note that it suffices to consider the reduced equation

$$c_1x_1y_1^3 + c_2x_2y_2^3 = 0 \mod p.$$

where we may suppose that $p \nmid c_1, c_2$. Picking values for $y_1, y_2 \neq 0 \pmod p$ we may solve for x_1 and x_2. This now following the argument following lemma 8 produces

$$E_p(\mathbf{c}) > 0.$$

□

Lemma 30. *The singular integral (85) is real and non-negative. If the c_i are not all of the same sign, then the singular integral is positive.*

Proof. Substituting $xy^3 = \lambda$ for x readily gives

$$I(\beta) = \int_0^1 \int_0^{y^3} y^{-3} e(\beta\lambda)\, d\lambda dy$$

and writing

$$\phi(\lambda) = \int_{\{y\in[0,1]:y^3\geq\lambda\}} y^{-3}\, dy$$

produces the identity

$$I(\beta) = \int_0^1 \phi(\lambda) e(\beta\lambda)\, d\lambda.$$

Now by (85), we have

$$\mathfrak{J}^+(\mathbf{c}) = \int_{-\infty}^{+\infty} \int_{[0,1]^8} \phi(\lambda_1)\phi(\lambda_2)\cdots\phi(\lambda_8) e(\beta \mathbf{c}\cdot\boldsymbol{\lambda})\, d\boldsymbol{\lambda} d\beta.$$

Then a similar argument as in the proof of Lemma 4.3 in [3] gives the desired conclusion. □

Writing $\mathfrak{J}^+(\mathbf{c}) = \mathfrak{S}^+(\mathbf{c})\mathfrak{I}^+(\mathbf{c})$ and by invoking equations (72), (91), (90) and the preceding lemma, this proves the second part of Theorem 2. To deduce the first part a similar maneuver as in the previous application of the circle method is used. We have the correspondence (cf. Brüdern and Blomer [3] Chapter 4.4)

$$N_{\mathbf{c}}(X,Y) = 2 \sum_{\substack{\epsilon_i\in\{\pm1\}\\ 1\leq i\leq 8}} N^+_{\boldsymbol{\epsilon}\mathbf{c}}(X,Y).$$

Again the singular series remains unchanged by the transition from $\mathbf{c}$ to $\boldsymbol{\epsilon}\mathbf{c}$ such that with $\mathfrak{S}(\mathbf{c}) = \mathfrak{S}^+(\mathbf{c})$,

$$\mathfrak{J}(\mathbf{c}) = 2\,\mathfrak{S}(\mathbf{c}) \sum_{\substack{\epsilon_i\in\{\pm1\}\\ 1\leq i\leq 8}} \mathfrak{I}^+(\boldsymbol{\epsilon}\mathbf{c}).$$

Note that the sum of the singular integrals will be positive. This finishes the proof of Theorem 2.

4 Closing the gap

4.1 Small X

As Theorem 2 requires $X \geq L^{12}$ we have to argue in the remaining range in a different fashion. As now X is really small against Y we may use the asymptotic for fixed $\mathbf{x}$ uniformly. Indeed a variant of the treatment in Chapter 2 confirms that one might sum the minor arc error $O(Y^5L^{-3+\epsilon})$ for $x_i \leq X \leq L^{12}$ introducing an error of size $(\log L)^8$, which is sufficient. Chapter 3 and 4 in [7] show that on might sum the leading constants depending on $\mathbf{x}$ in (14)

$$\sum_{\mathbf{x}\leq X} A(\mathbf{x},\mathbf{c}) \ll X^7.$$

That is for $X \leq L^{12}$, we have the upper bound

$$N_{\mathbf{c}}(X,Y) \ll X^7Y^5$$

at our disposal.

4.2 Weighted hyperbolic counting

This paragraph is concerned with the adaptation of the weighted hyperbola count first introduced by Blomer and Brüdern in [3]. Let $h : \mathbb{N}^2 \to [0.\infty)$ be an arithmetical function. Consider the associated summatory function

$$H(L,M) = \sum_{\ell\leq L}\sum_{m\leq M} h(\ell,m).$$

Fix real C and positive parameters $\delta > 2, \beta_1$ and β_2 such that h satisfies

$$H(L,M) = CL^{\beta_1}M^{\beta_2} + O\left(L^{\beta_1}M^{\beta_2}\left(\log\min\left(L,M\right)\right)^{-\delta}\right) \tag{92}$$

for $L > (\log M)^A$, where A is some positive number. For $L \leq (\log M)^A$ assume that we have instead

$$H(L,M) \ll L^{\beta_1}M^{\beta_2}. \tag{93}$$

Assume further that there are functions $c_1, c_2 : \mathbb{N} \to [0,\infty)$ such that

$$\sum_{\ell\leq L} h(\ell,m) = c_1(m)L^{\beta_1} + O\left(m^D L^{\beta_1-\delta}\right) \tag{94}$$

uniformly in $m \leq L^\nu$ for some $T, D, \nu > 0$. Further assume that similarly uniformly in $\ell \leq (\log M)^T$ we have

$$\sum_{m\leq M} h(\ell,m) = c_2(\ell)M^{\beta_2} + O\left(M^{\beta_2}\left(\log M\right)^{-\delta}\right). \tag{95}$$

Lemma 31. *Let h satisfy the above conditions, then we have*

$$\sum_{\ell\leq L} c_2(\ell) \ll L^{\beta_1}$$

and

$$\sum_{m\leq M} c_1(m) = CM^{\beta_2}\left(1 + O\left(M^{-\delta}\right)\right).$$

Proof. By (94), for $L \geq 1$ and $M \leq L^{\nu}$, we have

$$H(L,M) = \sum_{m \leq M} \left(\sum_{\ell \leq L} h(\ell, m) \right)$$
$$= L^{\beta_1} \left(\sum_{m \leq M} c_1(m) \right) + O\left(M^{D+1} L^{\beta_1} (\log L)^{-\delta} \right).$$

But by (92) we may write

$$H(L,M) = C L^{\beta_1} M^{\beta_2} + O\left(L^{\beta_1} M^{\beta_2} \left(\log \min(L,M) \right)^{-\delta} \right)$$

thus by taking $L = M^K$ for K large we deduce the claimed asymptotic. For the first part of the lemma argue the same but use (93).

□

Lemma 31 suffices to show that the contribution of terms in the 'spikes' are negligible.

Lemma 32. *For $\mu > 0$ sufficiently small define the sum*

$$T_1 = \sum_{\ell \leq (\log P)^A} \sum_{P^{\frac{1}{2}} < m^{\beta_2} \leq P \ell^{-\beta_1}} h(\ell, m),$$

then we have

$$T_1 \ll P \log \log P.$$

Proof. Following the proof of Lemma 9.3 in [27] we have

$$T_1 = \sum_{\ell \leq (\log P)^A} \left(P \frac{c_2(\ell)}{\ell^{\beta_1}} + O\left(P \ell^{-\beta_1} \left(\log P + \log \ell \right)^{-\delta} \right) \right) - H\left((\log P)^A, P^{1/(2\beta_2)} \right).$$

Thus

$$T_1 = P \sum_{\ell \leq (\log P)^A} \frac{c_2(\ell)}{\ell^{\beta_1}} - H\left((\log P)^A, P^{1/(2\beta_2)} \right) + O\left(P (\log P)^{-\vartheta} \right).$$

The sum over ℓ is evaluated by summation by parts using Lemma 31. This yields

$$\sum_{\ell \leq (\log P)^A} \frac{c_2(\ell)}{\ell^{\beta_1}} \ll \log \log P.$$

□

We record that the same holds for the sum with ℓ and m interchanged, where one uses the much stronger conclusion from lemma 31.

Lemma 33. *Assume h satisfies condition (92) and define the sum*

$$T_2 = \sum_{(\log P)^A \le \ell \le P^{\frac{1}{2\beta_1}}} \sum_{P^{\frac{1}{2}} < m^{\beta_2} \le P\ell^{-\beta_1}} h(\ell, m),$$

then one has

$$T_2 = C(1/2) P \log P + O\left(P \log P \left(\log\log P\right)^{-\delta/2}\right).$$

Proof. Choose some large integer J and define $\theta > 0$ by

$$(1+\theta)^J = P^{1/(2\beta_1)} (\log P)^{-A}.$$

Consider $(\log P)^A \le L < L' = (1+\theta)L \le P^{1/(2\beta_1)}$ and define the slice

$$V(L) = \sum_{L < \ell \le L'} \sum_{p^{1/2} < m^{\beta_2} \le P\ell^{-\beta_1}} h(\ell, m),$$

and the corresponding sums

$$V_{-}(L) = \sum_{L < \ell \le L'} \sum_{p^{1/2} < m^{\beta_2} \le P(L')^{-\beta_1}} h(\ell, m)$$

and

$$V_{+}(L) = \sum_{L < \ell \le L'} \sum_{p^{1/2} < m^{\beta_2} \le PL^{-\beta_1}} h(\ell, m).$$

Note that since $h \ge 0$ we have $V^{-}(L) \le V(L) \le V^{+}(L)$. Following [27] we evaluate $V_{+}(L)$ as

$$V_{+}(L) = H(L', P^{1/(2\beta_2)} L^{-\beta_1/\beta_2}) - H(L', P^{1/(2\beta_2)}) - H(L, P^{1/\beta_2} L^{-\beta_1/\beta_2}) + H(L, P^{1/(2\beta_2)}).$$

This is equal to

$$C\beta_1 \theta P + C\beta_1 \theta L^{\beta_1} P^{1/2} + O(\theta^2 P) + O(P(\log\log P)^{-\delta}).$$

The same asymptotic holds for $V_{-}(L)$ hence also for $V(L)$. We may apply this to the disection $L_j = (\log P)^A (1+\theta)^j$ for $0 \le j < J$ such that

$$\begin{aligned} T_2 &= \sum_{0 \le j < J} V(L_j) \\ &= C\beta_1 (J\theta) P + C\beta_1 \theta P^{1/2} \sum_{0 \le j < J} L_J^{\beta_1} + O(J\theta^2 P) + O(JP(\log\log P)^{-\delta}). \end{aligned}$$

A short calculation confirms that

$$\sum_{0 \le j < J} L_J^{\beta_1} \ll P^{1/2} + P^{1/2}\theta.$$

Now choose J as largest integer smaller than $\log P(\log\log P)^{\delta/2}$. By the definition of θ we have

$$\theta = J^{-1}\frac{1}{2\beta_1}\log P - \frac{A}{J}\log\log P + O(J^{-2}\log P)$$

which implies

$$J\theta = \frac{1}{2\beta_1}\log P + O(\log\log P)$$

and

$$\theta \ll (\log\log P)^{-\delta/2}.$$

□

Let

$$T_h(P) = \sum_{\ell^{\beta_1} m^{\beta_2}\le P} h(\ell,m).$$

Note that

$$T_h(P) = \sum_{\substack{\ell^{\beta_1} m^{\beta_2}\le P\\ m^{\beta_2}>P^{1/2}}} h(\ell,m) + \sum_{\substack{\ell^{\beta_1} m^{\beta_2}\le P\\ \ell^{\beta_1}>P^{1/2}}} h(\ell,m) + H\left(P^{1/(2\beta_1)}, P^{1/(2\beta_2)}\right)$$

and

$$\sum_{\substack{\ell^{\beta_1} m^{\beta_2}\le P\\ m^{\beta_2}>P^{1/2}}} h(\ell,m) = T_1 + T_2.$$

Hence by lemma 32 and 33 for η sufficiently small

$$\sum_{\substack{\ell^{\beta_1} m^{\beta_2}\le P\\ m^{\beta_2}>P^{1/2}}} h(\ell,m) = (1/2)CP\log P + O\left(P\log P\,(\log\log P)^{-\eta}\right).$$

Furthermore by symmetry, the same asymptotic holds for the sum with $\ell^{\beta_1} > P^{1/2}$, this leads to:

$$T_h(P) = CP\log P + O\left(P\log P\,(\log\log P)^{-\eta}\right) \tag{96}$$

4.3 Proof of Theorem 1

To deduce the theorem we apply (96) to the function $h(\ell,m)$ defined by the number of integer vectors $\mathbf{x},\mathbf{y}\in\mathbb{Z}^8$, such that $|\mathbf{x}| = \ell$ and $|\mathbf{y}| = m$, that satisfy (5). We now have to verify that conditions (94) and (95) as well as (92) are met for $\beta_1 = 7$ and $\beta_2 = 5$. By definition, we have by lemma 4 uniformly in $m \le L^\nu$, for some $\nu > 0$

$$\sum_{\ell\le L} h(\ell,m) = \sum_{|\mathbf{y}|=m} M_{\mathbf{c}}(\mathbf{y},L) = c_1(m)L^7 + O\left(m^7L^{7-\delta}\right),$$

where we have defined

$$c_1(m) = \sum_{|\mathbf{y}|=m} c(\mathbf{y},\mathbf{c}),$$

which shows condition (94) with $D = 7$. A similar argument shows that (95) holds, by invoking Proposition 1, that is (14), by setting

$$c_2(\ell) = \sum_{|\mathbf{x}|=\ell} A(\mathbf{x}, \mathbf{c}).$$

Finally condition (92) is covered by combining (13) with (26). Therefore we may apply (96). This is essentially $N_{H,U}(P)$, except we have to take care of the coprimality condition. This is done as in [10] Chapter 1. Let $N_{U,H}(B, e_1, e_2)$ be the number of $(\mathbf{x}, \mathbf{y}) \in U$ with $|\mathbf{x}|^7 |\mathbf{y}|^5 \leq B$ and $e_1 \mid \mathbf{x}, e_2 \mid \mathbf{y}$. Hence

$$N_{U,H}(B, e_1, e_2) = T_h \left(\frac{B}{e_1^7 e_2^5} \right)$$

and thus

$$\begin{aligned} N_{U,H}(B) =& \frac{1}{4} \sum_{e_1^7 e_2^5 \leq B} \mu(e_1)\mu(e_2) N_{U,H}(B, e_1, e_2) \\ &= \frac{1}{4} \sum_{e_1^7 e_2^5 \leq B} \mu(e_1)\mu(e_2) T_h \left(\frac{B}{e_1^7 e_2^5} \right). \end{aligned}$$

We now can apply (96). And one easily checks that the summation over e_1, e_2 can be extended to infinity establishing Theorem 1.

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Declaration of Authorship

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