Index Theory and Positive Scalar Curvature

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Abstract

The aim of this dissertation is to use relative higher index theory to study questions of existence and classification of positive scalar curvature metrics on manifolds with boundary. First we prove a theorem relating the higher index of a manifold with boundary endowed with a Riemannian metric which is collared at the boundary and has positive scalar curvature there, to the relative higher index as defined by Chang, Weinberger and Yu. Next, we define relative higher rho-invariants associated to positive scalar curvature metrics on manifolds with boundary, which are collared at boundary. In order to do this, we define variants of Roe and localisation algebras for spaces with cylindrical ends and use this to obtain an analogue of the Higson-Roe analytic surgery sequence for manifolds with boundary. This is followed by a comparison of our definition of the relative index with that of Chang, Weinberger and Yu. The higher rho-invariants can be used to classify positive scalar curvature metrics up to concordance and bordism. In order to show the effectiveness of the machinery developed here, we use it to give a simple proof of the aforementioned statement regarding the relationship of indices defined in the presence of positive scalar curvature at the boundary and the relative higher index. We also devote a few sections to address technical issues regarding maximal Roe and structure algebras and a maximal version of Paschke duality, whose solutions was lacking in the literature.

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To Hadi

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Chapter 1

Introduction and Synopsis

The aim of this dissertation is to study and develop techniques which help investigate the questions of existence and classification of positive scalar curvature metrics on smooth manifolds. Concretely, given a smooth manifold M, does it admit a metric with positive scalar curvature and what can be said about the space of such metrics?

In the following we will mainly focus on manifolds with boundary. The motivation for the above questions and why we are not interested, for example, in metrics with negative scalar curvature is the following theorem of Kazdan and Warner

Theorem 1.0.1. *Let M be a closed manifold with* $\dim M \geq 3$. *Let f be a smooth function on M with $f(x_0) < 0$ for some $x_0 \in M$. Then there exists a Riemannian metric g on M, with* $\mathrm{scal}(g) = f$.

There are three approaches one can use to determine whether a given manifold "does not" admit a positive scalar curvature (psc) metric:

- The index theory approach
- The minimal hypersurface approach
- The Seiberg-Witten approach

As the title of the dissertation suggests, we are here interested in the index theory approach. For a survey of all the above approaches see [32]. The index theory approach relies heavily on the spin Dirac operator or its Clifford linear version. Therefore, we will restrict our attention to spin manifolds. It is not an exaggeration to claim that the index theory approach to positive scalar curvature is based on the Schrödinger-Lichnerowicz formula

$$\not{D}_g^2 = \nabla^*\nabla + \frac{\mathrm{scal}(g)}{4}$$

and its refinements, where $\not{D}_g$ denotes the spin Dirac operator, $\nabla^*\nabla$ denotes the connection Laplacian on the spinor bundle and $\mathrm{scal}(g)$ denotes the scalar curvature function of g. Denote by $\not{S}$ the spinor bundle on M. If M is even-dimensional, the spinor bundle comes with a $\mathbb{Z}_2$-grading $\not{S} = \not{S}^+ \oplus \not{S}^-$. The Dirac operator can then be seen as an unbounded self-adjoint operator on $L^2(\not{S}) = L^2(\not{S}^+) \oplus L^2(\not{S}^-)$. Here L^2 denotes the square-integrable sections of a given bundle. The Dirac operator is odd with respect to the grading on $L^2(\not{S})$. Denote by $\not{D}^\pm$ the restriction of $\not{D}$ to $L^2(\not{S}^\pm)$. On a compact manifold the kernel and cokernel of $\not{D}$ are finite dimensional. Denote by $\operatorname{ind}\not{D}$ the Fredholm index of $\not{D}^+$. By abuse of language, we call $\operatorname{ind}\not{D}$ the index of $\not{D}$. If g has positive scalar curvature, the Schrödinger-Lichnerowicz formula implies that $\not{D}^2$ is a strictly positive operator and thus has a trivial kernel. Noting that

$$\operatorname{ind}\not{D} = \dim\ker\not{D}^+ - \dim\operatorname{coker}\not{D}^+ = \dim\ker\not{D}^+ - \dim\ker\not{D}^-$$

we get that $\operatorname{ind}\not{D}$ vanishes. Combining this with the observation that the index of the Dirac operator does not depend on the metric, we get that the nonvanishing of the index is an obstruction to the existence of a positive scalar curvature metric. In order to use this to produce examples of manifolds which do not admit a psc metric one has to be able to compute the index. However, this computation is a special case of the Atiyah-Singer index theorem and the index of the spin Dirac operator is computed to be the $\widehat{A}$-genus of the manifold. There are many spin manifolds with nonvanishing $\widehat{A}$-genus and one obtains in this way examples of manifolds which do not admit any psc metric. However, the observation that the index is given by the $\widehat{A}$-genus also shows the limitations of the index as an obstruction to the existence of psc metrics, as the $\widehat{A}$-genus vanishes for all manifolds whose dimension is not divisible by 4. The index also fails to provide any information on manifolds with a trivialisable tangent bundle such as tori. Therefore it is natural to look for possible refinements of the classical notion of index. A successful refinement has come about by bringing in the fundamental group of the manifold. We first discuss this refinement in the case the fundamental group is finite. Consider the Dirac operator $\widetilde{\not{D}}_{\widetilde{g}}$ on the universal cover $\widetilde{M}$ of a compact spin manifold M where $\widetilde{g}$ is a metric on $\widetilde{M}$ which is invariant under the action of $\pi_1(M)$ by deck transformations. In this case $\ker\widetilde{\not{D}}^+$ and $\ker\widetilde{\not{D}}^-$ will be finite dimensional $\pi_1(M)$-representations or equivalently finitely generated projective $\mathbb{C}\pi_1(M)$-modules and $[\ker\widetilde{\not{D}}^+] - [\ker\widetilde{\not{D}}^-]$ defines a class in $K_0(\mathbb{C}\pi_1(M))$. The point here is that even though $\ker\widetilde{\not{D}}^+$ and $\ker\widetilde{\not{D}}^-$ may have the same dimension and be isomorphic as vector spaces and thus represent the same class in $K_0(\mathbb{C})$, they need not be isomorphic as $\pi_1(M)$-representations and their difference might thus be a nontrivial element of $K_0(\mathbb{C}\pi_1(M))$. More generally ($\pi_1(M)$ not necessarily finite), the spin Dirac operator gives rise to a so-called fundamental class in the K-homology groups $K_*(M)$ of the manifold and the higher index of the Dirac operator is defined to be the image of the fundamental class under the equivariant assembly map

$$\mu^{\pi_1(M)} : K_*(M) \to K_*(C_r^*(\pi_1(M))),$$

where $C_r^*(\pi_1(M))$ denotes the reduced group C^*-algebra of $\pi_1(M)$. Here one can replace the reduced group C^*-algebra by other completions of the group ring to obtain variants of the index map. We will discuss this later in more detail as it turns out to be useful for our purposes. Before giving a quick description of the index map, we will quickly discuss the relationship with the numerical index. On $C_r^*(\pi_1(M))$ one can define a trace by extending the functional

$$\mathbb{C}\pi_1(M) \to \mathbb{C}$$
$$\sum_{\gamma\in\pi_1(M)} a_\gamma \cdot \gamma \mapsto a_e$$

by continuity. This induces a map $K_0(C_r^*(\pi_1(M))) \to \mathbb{C}$, which maps the higher index to the numerical index of the Dirac operator. This is a consequence of the Atiyah L^2-index theorem (see e.g. [33]).

Now we quickly describe the definition of the equivariant index map. There are many equivalent approaches to the definition of the index map. We will use the coarse geometric approach (see Roe for comparison of the latter approach with the original definition of Kasparov). One of the main applications of coarse geometry in index theory was the possibility of defining indices of Dirac operators on noncompact manifolds. On noncompact manifolds the Dirac operator is not in general Fredholm and it is thus not always possible to define the numerical index. Another way to see this is that the (bounded transform of) the Dirac operator is not invertible modulo compact operators. Two ways to deal with this problem are to either set some conditions on the scalar curvature at infinity to force the Dirac operator to be Fredholm or to consider a suitable enlargement of the algebra of compact operators modulo which the (bounded transform of the) Dirac operator is always invertible. In the latter case, a standard construction in K-theory then provides an "index" in the K-theory of the aforementioned algebra. One fruitful choice is to consider the Roe algebra. Let X be a not necessarily compact, even-dimensional spin manifold, endowed with a free and proper action of a discrete group Γ by spin structure preserving isometries. We will later define the Roe algebra for general locally compact metric spaces. One then also gets a unitary representation of Γ on $L^2(\not{S}^+)$.

Definition 1.0.2. The equivariant Roe algebra of X is defined as the closure of the $*$-algebra of finite propagation and locally compact operators on $L^2(\not{S}^+)$, which are further fixed by the Γ-action. It will be denoted by $C^*(X)^\Gamma$.

Roughly speaking, an operator is called a finite propagation operator if it does not move the support of sections too much. An operator is called locally compact, if after cutting it down to compact regions one obtains compact operators. If the Γ-action on X is cocompact one has the following

Proposition 1.0.3. *Suppose the action of Γ on X is cocompact. Then $C^*(X)^\Gamma$ is Morita equivalent to $C^*_r(\Gamma)$. In particular $K_*(C^*(X)^\Gamma) \cong K_*(C^*_r(\Gamma))$.*

Setting X to be the universal cover of a compact spin manifold, we thus obtain the right hand side of the index map using the language of coarse geometry. Now we discuss how to find a model for K-homology using Roe algebras.

Definition 1.0.4. The equivariant localisation algebra of X is defined to be the completion with respect to the supremum norm of the $*$-algebra of uniformly continuous functions $f : [1, \infty) \to C^*(X)^\Gamma$ for which the propagation of $f(t)$ vanishes as t tends to infinity.

We will later give a more precise definition of the notion of propagation of an operator. The important point here is that the K-theory of the localisation algebra provides a model for K-homology.

Proposition 1.0.5. *There is an isomorphism $K_*(X)^\Gamma \cong K_*(C^*_L(X)^\Gamma)$, where $K_*(X)^\Gamma$ denotes the equivariant K-homology group of X.*

Now for spaces of our interest the equivariant K-homology of the space is isomorphic to the nonequivariant K-homology of the quotient by the group action. In particular, we have an isomorphism $K_*(M) \cong K_*(C^*_L(\widetilde{M}))^{\pi_1(M)}$.

Definition 1.0.6. The equivariant index map $\mu^{\pi_1(M)}$ is defined as the composition

$$K_*(M) \cong K_*(C^*_L(\widetilde{M})^{\pi_1(M)}) \xrightarrow{(\mathrm{ev}_1)_*} K_*(C^*(\widetilde{M})^{\pi_1(M)}) \cong K_*(C^*_r(\pi_1(M))).$$

Recall that the higher index of the Dirac operator was defined as the image of the fundamental class of the Dirac operator under the index map.

The above definitions of the index map and the higher index allow us to give a natural and useful proof of the fact that the nonvanishing of the higher index is an obstruction to the existence of a positive scalar curvature metric on M. The short exact sequence

$$0 \to C^*_{L,0}(\widetilde{M})^{\pi_1(M)} \to C^*_L(\widetilde{M})^{\pi_1(M)} \xrightarrow{\mathrm{ev}_1} C^*(\widetilde{M})^{\pi_1(M)} \to 0,$$

where $C^*_{L,0}(\widetilde{M})^{\pi_1(M)}$ denotes the kernel of $\mathrm{ev}_1 : C^*_L(\widetilde{M})^{\pi_1(M)} \to C^*(\widetilde{M})^{\pi_1(M)}$ gives rise to a long exact sequence of K-theory groups

$$\ldots \to K_*(C^*_{L,0}(\widetilde{M})^{\pi_1(M)}) \to K_*(M) \xrightarrow{\mu^{\pi_1(M)}} K_*(C^*_r(\pi_1(M))) \to \ldots.$$

The positivity of the scalar curvature implies the existence of a gap around 0 in the spectrum of the Dirac operator on $\widetilde{M}$, which can be used to define a canonical lift $\rho^{\pi_1(M)}(g)$ of the fundamental class in $K_*(C^*_{L,0}(\widetilde{M})^{\pi_1(M)})$. The existence of such a lift and the exactness of the latter sequence imply the vanishing of the index. The usefulness of this proof lies in the fact that the "reason" $\rho^{\pi_1(M)}(g)$ for the vanishing of the index can be used to classify positive scalar curvature metrics (up to concordance, bordism, etc.).

In order to use the higher index to detect whether a given closed manifold does not admit a metric of positive scalar curvature, one needs to be able to compute it or at least to figure out whether the higher index vanishes. In [10], the authors introduced the notion of enlargeability and used it to answer the question whether Tori admit metrics of positive scalar curvature in the negative. Hanke and Schick showed in [13], that enlargeability implies the nonvanishing of the (maximal) higher index; i.e. the (maximal) higher index detects enlargeability. The maximal higher index is given as the image of the fundamental class under the maximal equivariant index map

$$\mu^{\pi_1(M)}_{\max} : K_*(M) \to K_*(C^*_{\max}(\pi_1(M))),$$

where $C^*_{\max}\pi_1(M)$ denotes the maximal group C^*-algebra. Using the language of coarse geometry, the maximal equivariant index map can be defined analogously to the usual index map by replacing the Roe algebra by the maximal Roe algebra; i.e. the completion of the $*$-algebra of locally compact, equivariant and finite propagation operators in the universal C^*-norm. The maximal higher index is a finer invariant than the "reduced" higher index. One can obtain the reduced higher index as the image of the maximal higher index under the map

$$K_*(C^*_{\max}(\pi_1(M))) \to K_*(C^*_r(\pi_1(M)))$$

which is induced by the canonical projection $C^*_{\max}(\pi_1(M)) \to C^*_r(\pi_1(M))$. Because of the better functoriality properties of the maximal group C^*-algebra, it is sometimes advantageous to use the maximal higher index. In the first part of Chapter 2, we will discuss this and also the subtleties appearing in the noncompact setting when one deals with the maximal index.

Now we turn to manifolds with boundary and pose the following questions: does a given manifold with boundary admit a psc metric which is collared at the boundary and what can we say about the space of such metrics? Again, we will only discuss the index theoretic approach to these questions. Since the Dirac operator on a spin manifold with nonempty boundary is not essentially self-adjoint one usually starts with attaching an infinite half-cylinder at the boundary and extending the metric on the manifold thus obtained by using the product metric on the half-cylinder and the standard metric on $\mathbb{R}_+$. The question of existence and classification of psc metrics on a manifold with boundary which are collared at the boundary then becomes equivalent to the question of existence and classification of psc metrics on the manifold obtained by attaching a half-cylinder at the boundary, which have product structure on the cylindrical end. The new issue one has to deal with is however that due to the noncompactness of the manifold with cylindrical end, the Dirac operator is not Fredholm without further assumptions and the numerical index is not always defined. As pointed out above, one can always define in this case an index in the K-theory of the Roe algebra of the manifold with cylindrical end. However, if the original manifolds with boundary is assumed to be compact, then the K-theory of the Roe algebra of the manifold with cylindrical end vanishes and the so called "coarse index" does not give any information. If the metric is assumed to have positive scalar curvature on the boundary, the metric on the manifold with cylindrical end will then have positive scalar curvature outside a compact set. In [10], Gromov and Lawson showed that in this case the Dirac operator is Fredholm. The numerical index however will depend on the metric at the boundary. If the metric has psc everywhere, the numerical index vanishes. Using the same condition on the metric at the boundary one can use, for example, the coarse geometric machinery to define an "absolute" index in $K_*(C^*_r(\pi_1(M)))$. This higher index will again depend on the metric at the boundary and will vanish if the metric has psc everywhere. All of this leaves open the question whether one can define a higher index for the Dirac operator on a compact manifold with boundary without any assumptions on the metric at the boundary.

Let M be a compact spin manifold with boundary N. In [2] Chang, Weinberger and Yu define a relative index map

$$\mu^{\pi_1(M),\pi_1(N)} : K_*(M,N) \to K_*(C^*(\pi_1(M),\pi_1(N))),$$

where the left hand side is the relative K-homology and $C^*(\pi_1(M), \pi_1(N))$ is a C^*-algebra measuring the difference between $C^*(\pi_1(M))$ and $C^*(\pi_1(N))$ and is called the relative group C^*-algebra. Here one cannot always use the reduced group C^*-algebra and thus one has to work with other completions of the involved group rings. For now we will not specify the chosen completion. The relative index map fits into a commutative diagram of long exact sequences

$$\begin{array}{ccccc} \to K_*(N) & \longrightarrow & K_*(M) & \longrightarrow & K_*(M,N) \to \\ \downarrow{\scriptstyle \mu^{\pi_1(N)}} & & \downarrow{\scriptstyle \mu^{\pi_1(M)}} & & \downarrow{\scriptstyle \mu^{\pi_1(M),\pi_1(N)}} \\ \to K_*(C^*(\pi_1(N))) & \longrightarrow & K_*(C^*(\pi_1(M))) & \xrightarrow{j} & K_*(C^*(\pi_1(M),\pi_1(N))) \to . \end{array}$$

The relative higher index is then defined as the image of the relative fundamental class under the relative index map. With an eye to the above discussion for closed manifolds, the first order of business is to establish the vanishing of the relative higher index in the presence of a psc metric which is collared at the boundary. Now a metric g which has psc at the boundary can be extended to a metric on the manifold obtained by attaching a half-cylinder which has positive scalar curvature outside a compact set and as mentioned above one can define an index in $K_*(C^*(\pi_1(M)))$ which we will here denote by $\mathrm{ind}_g^{\pi_1(M)}(M)$ where the subscript g is there to remind us that the index depends on the metric at the boundary. The main result of Chapter 2 is that $\mathrm{ind}_g^{\pi_1(M)}(M)$ is mapped to $\mu^{\pi_1(M),\pi_1(N)}([D_{M,N}])$ under the map $j : K_*(C^*(\pi_1(M))) \to K_*(C^*(\pi_1(M), \pi_1(N)))$. Here $[D_{M,N}]$ denotes the relative fundamental class of the Dirac operator on M. This at once implies that the relative index vanishes if g has positive scalar curvature everywhere. In [4], Deeley and Goffeng obtain a similar result using the language of geometric K-homology. Even though the latter result proves the vanishing theorem and relates previously defined indices to the more recently defined relative higher index it still leaves open the question of classification of psc metrics on manifolds with boundary. Recall from above that one approach of tackling these questions in the closed case is to define secondary invariants (e.g. the higher rho-invariant). Now, the relative index map fits in a long exact sequence

$$\ldots \to K_*(SC_{\psi_{L,0}}) \to K_*(M,N) \to K_*(C^*(\pi_1(M), \pi_1(N))) \ldots,$$

where the C^*-algebra $SC_{\psi_{L,0}}$ will be defined in the following chapters. Analogous to the closed case we would like to use the positivity of the scalar curvature to lift the relative fundamental class to $K_*(SC_{\psi_{L,0}})$ in a way that the lift has the invariance properties which make it useful for the classification of psc metrics up to concordance, bordism .... Furthermore such a lift would give a very natural proof of the vanishing theorem. One of the objectives of Chapter 3 is the definition of the "relative higher rho-invariant". We develop machinery which we think is the right one to use for the coarse geometric approach to index theory on manifolds with boundary and which allows one to adapt the proofs of well-known theorems for closed manifolds to prove their counterparts for manifolds with boundary. More precisely, we define variants of Roe algebras for spaces with cylindrical ends and discuss the existence and classification of psc metrics on such manifolds. We then discuss how the results can be used in the study of psc metrics on manifolds with boundary and relate our approach to the one of Chang, Weinberger and Yu. Using our machinery, we can easily define higher rho-invariants for psc metrics on manifolds with cylindrical ends. We produce the desired lift of the relative fundamental class in $K_*(SC_{\psi_{L,0}})$ by pushing the higher rho-invariant for the manifold with cylindrical end to $K_*(SC_{\psi_{L,0}})$ using a canonical homomorphism of K-theory groups. In order to demonstrate the efficiency of the machinery developed in Chapter 3 we also give a simple proof of the main theorem in Chapter 2.

We further note that Chapter 2 is made public in preprint form on the arXiv (arXiv:1811.08142v1) as joint work with Thomas Schick and has been submitted for publication. Chapter 3 is thematically connected to Chapter 2 and I plan to submit it for publication soon.

Chapter 2

On an index Theorem of Chang, Weinberger and Yu[1]

2.1 Introduction

In [2] Chang, Weinberger and Yu define a relative index of the Dirac operator on a compact spin manifold M with boundary N as an element of $K_*(C^*(\pi_1(M), \pi_1(N)))$, where this relative K-theory group measures the difference between the two fundamental groups. The main geometric theorem of [2] then says that the existence of a positive scalar curvature metric on M which is collared at the boundary implies the vanishing of this index. The argument for this vanishing theorem is rather complicated and indeed contains a gap. We address this gap in this paper. After the first version of the present article was made public, [11] was posted, which also attempts to fix this gap.

More explicitly, the K-theory groups of the absolute and relative group C^*-algebras of the manifold and its boundary fit in a long exact sequence

$$\to K_*(C^*(\pi_1(N))) \to K_*(C^*(\pi_1(M))) \xrightarrow{j} K_*(C^*(\pi_1(M), \pi_1(N))) \to \dots . \tag{2.1}$$

The relative index $\mu([M, N])$ is defined as the image of a relative fundamental class $[M, N] \in K_{\dim M}(M, N)$ under a relative index map $\mu\colon K_*(M, N) \to K_*(C^*(\pi_1(M), \pi_1(N)))$. Here, $K_*(M, N)$ is the relative K-homology and $[M, N]$ is constructed with the help of the Dirac operator on M. Indeed, in this paper we mainly deal with a small variant of the construction of [2] by choosing a slightly different C^*-completion. We discuss this in more detail below, throughout the introduction, we work with this modification.

[1]This paper is joint work with Thomas Schick. It can be found on arXiv (see [34]). Furthermore, it has been submitted to a journal and is under review.

Our main goal is to better understand the vanishing theorem of Chang, Weinberger and Yu, and to prove a strengthening of it, at the same time giving a new and more conceptual proof.

For our approach, recall that one has a perfectly well defined K-theoretic index of the Dirac operator on a Riemannian manifold with boundary provided the boundary operator is invertible, for example if the metric is collared and of positive scalar curvature near the boundary (see e.g. [24]). This index takes values in $K_*(C^*(\pi_1(M)))$ and explicitly depends on the boundary operator (i.e. on the positive scalar curvature metric g of the boundary). In the latter case we denote it by $\mathrm{Ind}^{\pi_1(M)}(g) \in K_*(C^*(\pi_1(M)))$. Our main result states that a slight variant of the relative index of Chang-Weinberger-Yu is the image of the absolute index defined with invertible boundary operator under the natural homomorphism j of (2.1) (whenever this absolute index is defined):

Theorem 2.1.1.

$$j(\mathrm{Ind}^{\pi_1(M)}(g)) = \mu([M, N]).$$

The absolute index $\mathrm{Ind}^{\pi_1(M)}(g)$ vanishes whenever we have positive scalar curvature on all of M, implying immediately the corresponding vanishing result for the relative index of Chang, Weinberger, and Yu.

Relative index theory has recently been the subject of considerable activity. In [4], Deeley and Goffeng define a relative index map using geometric K-homology instead of coarse geometry and prove index and vanishing results similar to the main result of our paper. However, this relies and uses the full package of higher Atiyah-Patodi-Singer index theory (like [21]), which we consider technically very demanding and somewhat alien to the spirit of large scale index theory. Indeed, in [4] it is not even proved in general that the constructions coincide with the ones of [2]. Yet another approach to relative index theory and the results of [2] is given by Kubota in [19]. There, the new concepts of relative Mishchenko bundles and Mishchenko-Fomenko index theory are introduced, and heavy use is made of the machinery of KK-theory. In [19], a careful identification of the different approaches is carried out.

The main point of our paper is its very direct and rather easy approach to the index theorems as described above. We work entirely in the realm of large scale index theory, and just rely on the basic properties of the Dirac operator (locality, finite propagation of the wave operator, ellipticity). We avoid APS boundary conditions and we avoid deep KK-techniques. Such a direct approach is relevant also because it is more likely to allow for the construction of secondary invariants, to be used for classification rather than obstruction purposes.

In [2], fundamental use is made of the *maximal* Roe and localisation algebras to obtain the required functoriality needed e.g. in the sequence (2.1). The identification of its K-theory with K-homology of the space is needed for the *maximal* localisation algebra and reference is given to [27] for the proof. However, that reference only deals with the *reduced* setting. Working out the details to extend the known results to the maximal setting turned out to be rather non-trivial. The first part of the present paper is devoted to the careful development of foundational issues of maximal Roe and localisation algebras. For us, this complete and careful discussion of the properties of maximal completions in the context of coarse index theory is the second main contribution of this paper. Our results on this are used e.g. in [4].

The maximal Roe algebra is defined in a rather ad hoc and ungeometric way: one comes up with the (somewhat arbitrary) algebraic Roe algebra, a $*$-subalgebra of bounded operators on a Hilbert space which is *not* closed, and then passes to the maximal C^*-closure. This is hard to control and to compute (there are very few cases of actual computation), and geometric arguments are very delicate. It required the whole additional unpublished preprint [11], which appeared after the first version of this paper was posted, to prove the claim of [2] that the Schrödinger-Lichnerowicz vanishing theorem applies also to in the maximal Roe algebra. This claim was unjustified in [2], as the authors of [11] also observe.

Our approach is going in a different direction. We propose to use instead of the ad hoc maximal completion a much more geometric completion C^*_q, which we introduce in Section 2.3. Problems with the standard (reduced) Roe algebra arise in the equivariant setting of the group Γ acting on the space X due to lack of functoriality. Our completion takes all normal quotients Γ/N acting on X/N into account. This restores full functoriality, but is completely geometric. The Schrödinger-Lichnerowicz formula and other geometric arguments apply effortless.

The precise formulation of Theorem 2.1.1 and of (2.1) requires to specify which completion is used. In our approach, this becomes $C^*_q(\pi_1(M), \pi_1(N))$, involving the completions of the group algebras in the direct sum of the regular representation of all its quotients. Formally, the relative index in this K-theory group is weaker than the relative index obtained by using the maximal completion. However, not a single case is known where extra information on obstructions and classification has been obtained from the difference of the K-theory of the maximal and the reduced group C^*-algebras, and the Novikov conjecture suggests that this should not be possible. In any event, it seems extremely hard to exploit such a difference for geometric means. So we believe that our approach and our completion is a very good choice: full functoriality, no extra effort for geometric arguments, in practice no loss of information.

Remark 2.1.2. Our approach works for arbitrary, also non-cocompact situations. In the cocompact case, there is another way for geometric constructions: one works with the compact space, and with the infinite dimensional Mishchenko bundle. Here, one has the choice to use arbitrary group algebra completions, including the maximal one, which is used in [4] and [19].

Remark 2.1.3. We present details of the construction and manipulation of the relative index and the vanishing theorem only in the case that the dimension of the manifold is even. We chose to do this because this is the most classical setup, and the constructions are particularly explicit and direct. This also means that we remain close to the original treatment of [2].

We discuss in Remark 2.5.3 how one can reduce the general case to the even dimensional situation. We also discuss there how one could use the techniques of Zeidler [41] combined with our setup to uniformly treat all dimensions and even the case of real C^*-algebras.

In parts of the present paper we give missing arguments for some of the results of the master thesis of Seyedhosseini [35].

2.1.1 Structure of the paper

In Section 2.2 we present our foundational results on *maximal* Roe algebras. In Section 2.3, we introduce our geometric functorial completed Roe algebra and establish its main properties. Section 2.4 recalls the construction of the relative index, following [2]. We try to motivate the construction, give additional details and fix small glitches in [2]. Section 2.5 gives the proof of Theorem 2.1.1.

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2.2 The Maximal Roe Algebra

In the following, we will only consider separable and proper metric spaces with bounded geometry. We recall that a locally compact metric X space has bounded geometry if one can find a discrete subset Y of X such that:

- There exists $c > 0$ such that every $x \in X$ has distance less than c to some $y \in Y$.
- For all $r > 0$ there is N_r such that $\forall x \in X$ we have $|Y \cap B_r(x)| \leq N_r$.

A covering of a compact Riemannian manifold with the lifted metric obviously has bounded geometry.

2.2.1 Roe Algebras

Let X be a separable and proper metric space endowed with a free and proper action of a discrete group Γ by isometries. In this section, we will recall the definition of the Roe algebra associated to X. Let $\rho\colon C_0(X) \to L(H)$ be an ample representation of $C_0(X)$ on some separable Hilbert space H. A representation of $C_0(X)$ is called ample if no non-zero element of $C_0(X)$ acts as a compact operator on H. The representation ρ is called covariant for a unitary representation $\pi\colon \Gamma \to U(H)$ of Γ if $\rho(f_\gamma) = \mathrm{Ad}_{\pi(\gamma)}\, \rho(f)\ \forall \gamma \in \Gamma$. Here f_γ denotes the function $x \mapsto f(\gamma^{-1}x)$.

From now on we will assume that ρ is an ample and covariant representation of $C_0(X)$ as above. By an abuse of notation we will denote $\rho(f)$ simply by f. We will later use representations of $C_0(X)$ which are an infinite direct sum of copies of an ample representation. Such representations are called very ample.

Definition 2.2.1. An operator $T \in L(H)$ is called a finite propagation operator if there exists an $r > 0$ such that $fTg = 0$ for all those $f, g \in C_0(X)$ with the property $d(\mathrm{supp}(f), \mathrm{supp}(g)) \geq r$. The smallest such r is called the propagation of T and is denoted by $\mathrm{prop}\, T$. An operator $T \in L(H)$ is called locally compact if Tf and fT are compact for all $f \in C_0(X)$.

Definition 2.2.2. Denote by $\mathbb{R}_\rho(X)^\Gamma$ the $*$-algebra of finite propagation, locally compact operators in $L(H)$ which are furthermore invariant under the action of the group Γ. We will call $\mathbb{R}_\rho(X)^\Gamma$ the algebraic Roe algebra of X. The maximal Roe algebra associated to the space X is the maximal C^*-completion of $\mathbb{R}_\rho(X)^\Gamma$, i.e. the completion of $\mathbb{R}_\rho(X)^\Gamma$ with respect to the supremum of all C^*-norms. This supremum is finite for spaces of bounded geometry by Proposition 2.2.3. It will be denoted by $C^*_{\rho,\max}(X)^\Gamma$. The reduced Roe algebra is the completion of the latter $*$-algebra using the norm in $L(H)$. We denote this algebra by $C^*_{\rho,\mathrm{red}}(X)^\Gamma$.

Proposition 2.2.3. *Suppose X has bounded geometry. For every $R > 0$ there is a constant C_R such that for every $T \in \mathbb{R}_\rho(X)^\Gamma$ with propagation less than R and every $*$-representation $\pi\colon \mathbb{R}(X)^\Gamma \to L(H')$ we have*

$$||\pi(T)||_{L(H')} \leq C_R ||T||_{C^*_{\rho,\mathrm{red}}(X)^\Gamma}.$$

In particular, $||T||_{C^*_{\rho,\max}(X)^\Gamma} \leq C_R ||T||_{C^*_{\rho,\mathrm{red}}(X)^\Gamma}$ and the bounded geometry assumption on X implies that the maximal Roe algebra is well-defined.

Proof. This follows from [9, Lemma 3.4] and [7, Theorem 2.7]. □

Note that Proposition 2.2.3 implies that restricted to the subset of operators of propagation bounded by R, the reduced and the maximal norms are equivalent.

Proposition 2.2.4. *The K-theory groups of the reduced and maximal Roe algebra are independent of the chosen ample and covariant representation up to a canonical isomorphism.*

Proof. In the reduced case, this is the content of [16, Corollary 6.3.13]. For the maximal case we just note that conjugation by the isometries of the kind handled in [16, Section 6.3] gives rise to $*$-homomorphisms of the *algebraic* Roe algebra and thus extend to morphisms of the maximal Roe algebras. Up to stabilisation, any two such morphisms can be obtained from each other by conjugation by a unitary making the induced map in K-theory canonical. □

Remark 2.2.5. As a consequence of Proposition 2.2.4 we will drop ρ in our notation for the Roe algebras. Later we will introduce a new completion of $\mathbb{R}(X)^\Gamma$, which sits between the reduced and maximal completions and denote it by $C_q^*(X)^\Gamma$. Moreover, if Γ is the trivial group, we will denote the Roe algebra by $C_d^*(X)$, where d stands for the chosen completion.

Proposition 2.2.6. *The K-theory of the maximal Roe algebra is functorial for coarse maps between locally compact metric spaces.*

Proof. The proof is similar to that of Proposition 2.2.4 and makes use of it. In the reduced case, this is proved by constructing an appropriate isometry between the representation spaces. Conjugation with the latter isometry gives rise to a $*$-homomorphisms of the *algebraic* Roe algebra and thus extends to a morphism of the reduced and maximal Roe algebra. The latter then gives rise to homomorphisms of the K-theory groups of the Roe algebra. As in the proof of Proposition 2.2.4, the induced map in K-theory is canonical which also implies functoriality. See [16, Section 6.3] for a more detailed discussion. □

In the case where Γ acts cocompactly on X, we have the following theorem.

Theorem 2.2.7. *Suppose that Γ acts cocompactly on X. Then $K_*(C^*_{\max}(X)^\Gamma) \cong K_*(C^*_{\max}(\Gamma))$.*

Proof. See [9, Section 3.12 & 3.14] for the isomorphism $C^*_{\max}(|\Gamma|)^\Gamma \cong C^*_{\max}(\Gamma)\otimes K(H)$, where $C^*_{\max}(|\Gamma|)^\Gamma$ is the equivariant Roe algebra of Γ seen as a metric space using some word metric. The action of Γ on itself is given by left multiplication. Since the action of Γ on X is cocompact, the Γ-space X is coarsely equivalent to Γ. This implies that $K_*(C^*_{\max}(X)^\Gamma) \cong K_*(C^*_{\max}(\Gamma)^\Gamma)$. The claim then follows from the stability of K-theory. □

For a Γ-invariant closed subset Y of X, we would like to define its Roe algebra relative to X as a closure of a space of operators in $C^*_{\max}(X)^\Gamma$, which are suitably supported near Y. The next two definitions make this precise.

Definition 2.2.8. For an operator $T \in L(H)$ we define the support $\operatorname{supp} T$ of T as the complement of the union of all open sets $U_1 \times U_2 \subset X \times X$ with the property that $fTg = 0$ for all f and g with $\operatorname{supp} f \subset U_1$ and $\operatorname{supp} g \subset U_2$. T is said to be *supported near* $Y \subset X$ if there exists $r > 0$ such that $\operatorname{supp} T \subset B_r(Y) \times B_r(Y)$. Here and afterwards $B_r(Y)$ denotes the open r-neighbourhood of Y.

Definition 2.2.9. For a Γ-invariant closed subset Y of X as above, denote by $\mathbb{R}(Y \subset X)^\Gamma$ the $*$-algebra of operators in $\mathbb{R}(X)^\Gamma$ which are supported near Y. The relative Roe algebra of Y in X is defined as the closure of $\mathbb{R}(Y \subset X)^\Gamma$ in $C^*_{\max}(X)^\Gamma$ and is an ideal inside the latter C^*-algebra. It is denoted by $C^*_{\max}(Y \subset X)^\Gamma$.

Since Y is a locally compact metric space with an action of Γ, it has its own (absolute) equivariant Roe algebra $C^*_{\max}(Y)^\Gamma$. Theorem 2.2.11 identifies the K-theory of the relative and absolute equivariant Roe algebras in the case, where the action of Γ on the subset is cocompact. However, for its proof we need further conditions on the group action.

Definition 2.2.10. Let Γ act freely and properly by isometries on X. Γ is said to *act conveniently* if there exists a fundamental domain F for the action of Γ satisfying:

- For each $R > 0$, there exist $\gamma_1, \dots \gamma_{N_R} \in \Gamma$ such that $B_R(F) \subset \bigcup_{i=1}^{N_R} \gamma_i \cdot F$
- For each $\gamma \in \Gamma$ and $R > 0$ there exists $S(R,\gamma) > 0$ such that $\gamma^{-1} B_R(x) \cap F \subset B_{S(R,\gamma)}(x)$.

Theorem 2.2.11. *Let Y and X be as above and suppose that Γ acts conveniently on X and cocompactly on Y. The inclusion $Y \to X$ induces an isomorphism $K_*(C^*_{\max}(Y)^\Gamma) \cong K_*(C^*_{\max}(Y \subset X)^\Gamma)$.*

Remark 2.2.12. A representation $\rho\colon C_0(X) \to L(H_X)$ gives rise to a spectral measure which can be used to extend ρ to the C^*-algebra $B_\infty(X)$ of bounded Borel functions on X (see [22, Theorem 2.5.5]). Given $Z \subset X$, we get a representation $C_0(Z) \to L(\chi_Z H_X)$. This is what is meant in the following Lemma 2.2.13 by "compressing the representation space of $C_0(X)$ in order to obtain a representation of $C_0(Z)$". Given Z as above we can choose ρ such that it and its compression to Z are both ample; for example, by choosing the ample representation of X to be given by multiplication of functions with square summable sequences on some countable dense subset of X whose intersection with Y is a dense subset of Y. We will need Lemma 2.2.13 for the proof of Theorem 2.2.11. Indeed, the novel difficulty in Theorem 2.2.11 is to relate the $*$-representations used in the definition of $C^*_{\max}(Y)^\Gamma$ with the $*$-representations used to define $C^*_{\max}(X)^\Gamma$ —of which $C^*_{\max}(Y \subset X)^\Gamma$ by definition is an ideal. Note that, at the moment, we only manage to do this if Y is cocompact and the Γ-action is convenient. It is an interesting challenge to generalise Theorem 2.2.11 to arbitrary pairs (X, Y) and arbitrary free and proper actions.

Lemma 2.2.13. *Let Γ act conveniently on X and $Z \subset X$ be Γ-invariant and suppose that the action of Γ on Z is cocompact. Construct $\mathbb{R}(Z)^\Gamma$ by compressing the representation space of $C_0(X)$, so that $\mathbb{R}(Z)^\Gamma$ is naturally a $*$-subalgebra of $\mathbb{R}(X)^\Gamma$. Then an arbitrary non-degenerate $*$-representation of $\mathbb{R}(Z)^\Gamma$ on a Hilbert space can be extended to a non-degenerate $*$-representation of $\mathbb{R}(X)^\Gamma$. In particular, the inclusion $\mathbb{R}(Z)^\Gamma \to \mathbb{R}(X)^\Gamma$ extends to an injection $C^*_{\max}(Z)^\Gamma \to C^*_{\max}(X)^\Gamma$.*

Proof. Choose an ample representation $\rho\colon C_0(X) \to L(H_X)$. By compressing the Hilbert space H_X and restricting the representation, we obtain a very ample representation of $C_0(Z)$, i.e. $\rho|_{C_0(Z)}\colon C_0(Z) \to L(H_Z)$, where H_Z denotes the space $\chi_Z H_X$. Choose $D_Z \subset D_X$ fundamental domains of Z and X for the action of Γ. Similarly to the proof of [16, Lemma 12.5.3] one has $\mathbb{R}(Z)^\Gamma \cong \mathbb{C}[\Gamma] \odot K(\tilde{H}_Z)$, where $\tilde{H}_Z = \chi_{D_Z} H_Z$. The latter isomorphism is obtained using the isomorphisms $H_Z \cong \bigoplus_{\gamma\in\Gamma} \tilde{H}_Z \cong l^2(\Gamma) \otimes \tilde{H}_Z$. Denote by $\tilde{H}_X$ the Hilbert space $\chi_{D_X} H_X$. The isomorphism constructed in the proof can be extended to an injective map $\mathbb{C}[\Gamma] \odot L(\tilde{H}_X) \to L(H_X)$. The convenience of the action implies that its image contains the algebra $\mathbb{F}(X)^\Gamma$ of finite propagation Γ-invariant operators on X. This injection makes the diagram

$$\begin{array}{ccc} \mathbb{C}[\Gamma] \odot K(\tilde{H}_Z) & \xrightarrow{\cong} & \mathbb{R}(Z)^\Gamma \\ \downarrow & & \downarrow \\ \mathbb{C}[\Gamma] \odot L(\tilde{H}_X) & \longrightarrow & L(H_X) \end{array}$$

commutative. We show that an arbitrary non-degenerate $*$-representation of $\mathbb{C}[\Gamma] \odot K(\tilde{H}_Z)$ on a Hilbert space H_0 can be extended to a non-degenerate $*$-representation of $\mathbb{C}[\Gamma] \odot L(\tilde{H}_X)$. This implies the lemma since $\mathbb{R}(X)^\Gamma \subset \mathbb{F}(X)^\Gamma$. Suppose that $\pi \colon \mathbb{C}[\Gamma] \odot K(\tilde{H}_Z) \to L(H_0)$ is a non-degenerate $*$-representation of $\mathbb{C}[\Gamma] \odot K(\tilde{H}_Z)$ on a Hilbert space H_0. The representation π extends to a representation of $C^*_{\max}(\Gamma) \otimes K(\tilde{H}_Z)$ which we denote by π. Note that since the C^*-algebra of compact operators is nuclear, the C^*-algebra tensor product above is unique. $C^*_{\max}(\Gamma) \otimes K(\tilde{H}_Z)$ is a C^*-subalgebra of $C^*_{\max}(\Gamma) \otimes K(\tilde{H}_X)$ and π can thus be extended to a non-degenerate representation of $C^*_{\max}(\Gamma) \otimes K(\tilde{H}_X)$ on a possibly bigger Hilbert space H, which we denote by $\tilde{\pi}$. From [22, Theorem 6.3.5], it follows that there exist unique non-degenerate representations $\tilde{\pi}_1$ and $\tilde{\pi}_2$ of $C^*_{\max}(\Gamma)$ and $K(\tilde{H}_X)$ on H respectively, such that $\tilde{\pi}(a \otimes b) = \tilde{\pi}_1(a)\tilde{\pi}_2(b) = \tilde{\pi}_2(b)\tilde{\pi}_1(a)$ for all $(a, b) \in C^*_{\max}(\Gamma) \times K(\tilde{H}_X)$. The representation $\tilde{\pi}_2$ can be extended to a representation $\hat{\pi}_2$ of $L(\tilde{H}_X)$ on H by [6, Lemma 2.10.3] and from the same lemma it follows that $\tilde{\pi}_2(K(\tilde{H}_X))$ is strongly dense in $\hat{\pi}_2(L(\tilde{H}_X))$. From the double commutant theorem, it follows that the commutant of a C^*-subalgebra of $L(H)$ is strongly closed. This in turn implies that $\tilde{\pi}_1(a)\hat{\pi}_2(b) = \hat{\pi}_2(b)\tilde{\pi}_1(a)$ for $(a, b) \in C^*_{max}(\Gamma) \times L(\tilde{H}_X)$. Now restrict $\tilde{\pi}_1$ to $\mathbb{C}[\Gamma]$. From [22, Remark 6.3.2], it follows that there is a unique $*$-representation $\hat{\pi} \colon \mathbb{C}[\Gamma] \odot L(\tilde{H}_X) \to L(H)$ with the property $\hat{\pi}(a \otimes b) = \tilde{\pi}_1(a)\hat{\pi}(b)$. It is clear that $\hat{\pi}$ is an extension of π. □

Proof of Theorem 2.2.11. The proof is analogous to that of [17, Lemma 5.1]. As in Lemma 2.2.13, construct the algebras $C^*(\overline{B_n(Y)})^\Gamma$ by compressing the representation space of $C_0(X)$. The inclusions $\mathbb{R}(\overline{B_r(Y)})^\Gamma \to \mathbb{R}(\overline{B_R(Y)})^\Gamma$ for $r \leq R$ induce maps $C^*_{\max}(\overline{B_r(Y)})^\Gamma \to C^*_{\max}(\overline{B_R(Y)})^\Gamma$. We will show that $\varinjlim C^*_{\max}(\overline{B_r(Y)})^\Gamma = C^*_{\max}(Y \subset X)^\Gamma$. Let A be a C^*-algebra and let $\phi_r \colon C^*_{\max}(\overline{B_r(Y)})^\Gamma \to A$ be C^*-algebra morphisms such that all the diagrams of the form

$$\begin{array}{ccc} C^*_{\max}(\overline{B_r(Y)})^\Gamma & \longrightarrow & C^*_{\max}(\overline{B_R(Y)})^\Gamma \\ \downarrow & \swarrow & \\ A & & \end{array}$$

with $r < R$ commute. The above compatibility condition implies the existence of a unique morphism of $*$-algebras $\phi : \mathbb{R}(Y \subset X)^\Gamma \to A$, such that all the diagrams

$$\begin{array}{ccc} \mathbb{R}(\overline{B_r(Y)})^\Gamma & \longrightarrow & \mathbb{R}(Y \subset X)^\Gamma \\ \downarrow & \swarrow & \\ A & & \end{array}$$

are commutative. Lemma 2.2.13 then implies that the map ϕ is continuous if $\mathbb{R}(Y \subset X)^\Gamma$ is endowed with the norm of $C^*(X)^\Gamma$. To see this note that Lemma 2.2.13 implies that for $a \in \mathbb{R}(\overline{B_r(Y)})$, $||a||_{C^*_{\max}(\overline{B_r(Y)})^\Gamma} = ||a||_{C^*_{\max}(X)^\Gamma}$. Hence, $||\phi(a)|| = ||\phi_r(a)|| \leq ||a||_{C^*_{\max}(\overline{B_r(Y)})^\Gamma} = ||a||_{C^*_{\max}(X)^\Gamma}$. Thus, ϕ can be extended uniquely to a morphism $C^*(Y \subset X)^\Gamma \to A$ of C^*-algebras. The universal property of the direct limit of C^*-algebras, implies that $\varinjlim C^*_{\max}(\overline{B_r(Y)})^\Gamma = C^*_{\max}(Y \subset X)^\Gamma$. The claim of the theorem then follows from the continuity of K-theory and the coarse equivalence of $\overline{B_r(Y)}$ and $\overline{B_R(Y)}$ for arbitrary $r, R \in \mathbb{N}$ (recall that the K-theory groups of the Roe algebras of coarsely equivalent spaces are isomorphic). □

2.2.2 The Structure Algebra and Paschke Duality

Let X be as in the previous section. A representation $\rho\colon C_0(X) \to L(H)$ of $C_0(X)$ is called very ample if it is an infinite sum of copies of an ample representation. Construct $\mathbb{R}(X)^\Gamma$ and $C^*(X)^\Gamma$ using some very ample representation. In this section we will define a C^*-algebra associated to X which contains $C^*_{\max}(X)^\Gamma$ as an ideal and such that the K-theory of the quotient provides a model for K-homology of X.

Definition 2.2.14. We recall that an operator $T \in L(H)$ is called pseudolocal if it commutes with the image of ρ up to compact operators; i.e., $[f, T] \in K(H)$ for all $f \in C_0(X)$.

Definition 2.2.15. Denote by $\mathbb{S}_\rho(X)^\Gamma$ the $*$-algebra of finite propagation, pseudolocal operators in $L(H)$ which are furthermore invariant under the action of the group Γ. The maximal structure algebra associated to the space X is the maximal C^*-completion of $\mathbb{S}_\rho(X)^\Gamma$. It will be denoted by $D^*_{\rho,\max}(X)^\Gamma$. The reduced structure algebra is the completion of the latter $*$-algebra using the norm in $L(H)$. We denote this algebra by $D^*_{\rho,\text{red}}(X)^\Gamma$.

Remark 2.2.16. From now on, we will drop ρ from our notation. Later we will introduce a new completion of $\mathbb{S}(X)^\Gamma$, which sits between the reduced and maximal completions and denote it by $D^*_q(X)^\Gamma$. If the action of Γ is trivial, we denote the structure algebra by $D^*_d(X)$, where d stands for the chosen completion.

In comparison to the well known $D^*_{\text{red}}(X)^\Gamma$, the definition and properties of the maximal structure algebra $D^*_{\max}(X)^\Gamma$ are trickier than one might think in the first place. First of all, one has to establish its existence; i.e. an upper bound on the C^*-norms. Secondly, we want that $C^*_{\max}(X)^\Gamma$ is an ideal in $D^*_{\max}(X)^\Gamma$ and for this one has to control the a priori different C^*-representations which are used in the definitions. Only then does it make sense to form $D^*_{\max}(X)/C^*_{\max}(X)$. Paschke duality states that its K-theory is canonically isomorphic to the locally finite K-homology of X. All of this will be done in the remainder of this section. We now introduce the so-called dual algebras, which are larger counterparts of the Roe and structure algebra.

Definition 2.2.17. Denote by $\mathfrak{C}^*(X)^\Gamma$ the C^*-algebra of Γ-invariant locally compact operators in $L(H)$. Denote by $\mathfrak{D}^*(X)^\Gamma$ the C^*-algebra of Γ-invariant pseudolocal operators in $L(H)$.

It is clear that $\mathfrak{C}^*(X)^\Gamma$ is an ideal of $\mathfrak{D}^*(X)^\Gamma$. We have the following

Theorem 2.2.18. *There is an isomorphism $K_{*+1}(\frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}) \cong K^{lf}_*(X)$, where the right-hand side is the locally finite K-homology of X, given as the Kasparov group $KK_*(C_0(X), \mathbb{C})$.*

Proof. This is proven in [36, Proposition 3.4.11]. □

Lemma 2.2.19. *The map $\frac{\mathbb{S}(X)}{\mathbb{R}(X)} \to \frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}$ induced by the inclusion $\mathbb{S}(X) \to \mathfrak{D}^*(X)$ is an isomorphism. In particular, $\frac{\mathbb{S}(X)}{\mathbb{R}(X)}$ is a C^*-algebra. The corresponding statement holds for the Γ-equivariant versions.*

Proof. In [16, Lemma 12.3.2], the isomorphism $\frac{D^*_{\text{red}}(X)}{C^*_{\text{red}}(X)} \cong \frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}$ is proven. The truncation argument used in the proof shows that $\mathfrak{D}^*(X) = \mathbb{S}(X) + \mathfrak{C}^*(X)$, which implies the surjectivity of the map $\frac{\mathbb{S}(X)}{\mathbb{R}(X)} \to \frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}$. Injectivity is clear. An analogous argument using a suitable invariant open covering and partition of unity gives the isomorphism $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \cong \frac{\mathfrak{D}^*(X)^\Gamma}{\mathfrak{C}^*(X)^\Gamma}$. □

Proposition 2.2.20. *For $a \in \mathbb{S}(X)^\Gamma$ there exists $C_a > 0$ such that, for an arbitrary non-degenerate representation π of $\mathbb{S}(X)^\Gamma$ we have $||\pi(a)|| \leq C_a$.*

We need a few lemmas before proving Proposition 2.2.20. This proposition shows that the maximal structure algebra is well-defined. Since the structure algebra depends on both the coarse and topological structure of the space, the coarse geometric property of having bounded geometry alone does not guarantee the existence of the maximal structure algebra. This is where the properness of the metric is needed. More precisely, this is used in Lemma 2.2.19, which is itself used in the proof of Proposition 2.2.20.

Lemma 2.2.21. *There exists a C^*-algebra $A \subset \mathbb{R}(X)^\Gamma$ which contains an approximate identity for $C^*_{\max}(X)^\Gamma$.*

Proof. Let D be a fundamental domain for the action of Γ on X. Choose a discrete subset Y_D of D as provided by the bounded geometry condition. Denote the set obtained by transporting Y_D by the action of Γ by Y. Y is then clearly Γ-invariant. By [7, Proposition 2.7], extended straightforwardly to the equivariant case, it suffices to show that there exists a C^*-algebra $B \subset \mathbb{R}(Y)^\Gamma$ which contains an approximate identity for $C^*_{\max}(Y)^\Gamma$. Here, as the representation space we choose $l^2(Y) \otimes l^2(\mathbb{N})$, where the action of $C_0(Y)$ is given by multiplication. By [7, Proposition 2.19], $l^\infty(Y; C_0(\mathbb{N}))^\Gamma \subset \mathbb{R}(Y)^\Gamma$ is a C^*-algebra which contains an approximate unit of $\mathbb{R}(Y)$ endowed with the reduced norm and, by Proposition 2.2.3, of $\mathbb{R}(Y)$ endowed with the maximal norm. The claim then follows from density of $\mathbb{R}(Y)^\Gamma$ in $C^*(Y)^\Gamma$. □

Lemma 2.2.22. *Let ρ be an arbitrary non-degenerate $*$-representation of $\mathbb{R}(X)^\Gamma$ on some Hilbert space H. It extends in a unique way to a $*$-representation of $\mathbb{S}(X)^\Gamma$ on H.*

More generally, let $\mathbb{M}(X)^\Gamma$ be the algebra of bounded multipliers of $\mathbb{R}(X)^\Gamma$, i.e. all bounded operators on the defining Hilbert space which preserve $\mathbb{R}(X)^\Gamma$ by left and right multiplication. Note that $\mathbb{M}(X)^\Gamma$ contains $\mathbb{S}(X)^\Gamma$. The representation ρ extends in a unique way to a $*$-representation of $\mathbb{M}(X)^\Gamma$.

Proof. Let $\pi\colon \mathbb{R}(X)^\Gamma \to L(H)$ be a non-degenerate $*$-representation of $\mathbb{R}(X)^\Gamma$. It extends to a non-degenerate representation of $C^*_{\max}(X)^\Gamma$. Pick a C^*-subalgebra A of $C^*_{\max}(X)^\Gamma$ which contains an approximate identity for $C^*_{\max}(X)^\Gamma$ and sits inside $\mathbb{R}(X)^\Gamma$. The restriction of π to A is thus also non-degenerate. It follows from the Cohen-Hewitt factorisation theorem ([14, Theorem 2.5]) that, for all $w \in H$, there exist $T \in A$ and $v \in H$ with $\pi(T)v = w$. Furthermore, $\pi(S)v = 0$ for all $S \in \mathbb{R}(X)^\Gamma$ implies that v is in the orthogonal complement of $\pi(\mathbb{R}(X))H$; hence, $v = 0$ by the nondegeneracy of π. It follows from [8, Proposition IV.3.18] that $\hat{\pi}(T)(\pi(S)v) := \pi(TS)v$ for $T \in \mathbb{S}(X)^\Gamma$ gives a well-defined algebraic representation $\hat{\pi}\colon \mathbb{M}(X)^\Gamma \to \mathbb{L}(H)$. Here $\mathbb{L}(H)$ denotes the vector space of linear maps on H. It is clear that $\hat{\pi}$ is an extension of π. We show that $\hat{\pi}$ is actually a $*$-representation of $\mathbb{M}(X)^\Gamma$. The equalities

$$\langle \hat{\pi}(T)(\pi(S)v), \pi(S')v' \rangle = \langle \pi(TS)v, \pi(S')v' \rangle = \langle \pi((S^*T^*)^*)v, \pi(S')v' \rangle$$

$$= \langle v, \pi(S^*T^*S')v' \rangle = \langle \pi(S)v, \pi(T^*S')v' \rangle = \langle \pi(S)v, \hat{\pi}(T^*)(\pi(S')v') \rangle$$

imply that the operator $\hat{\pi}(T)$ is formally self-adjoint if T is self-adjoint. Furthermore, since $\hat{\pi}(T)$ is defined everywhere on H, it follows from the Hellinger-Toeplitz theorem that it is bounded. Since every element of a $*$-algebra is a linear combination of self-adjoint elements, this implies that the image of $\hat{\pi}$ is actually contained in $L(H)$. The previous computation then shows that $\hat{\pi}$ respects the involution; thus, it is a $*$-representation. Uniqueness of the extension follows from the fact that every extension $\hat{\pi}$ of π has to satisfy $\hat{\pi}(T)(\pi(S)v) = \pi(TS)v$ for $T \in \mathbb{M}(X)^\Gamma$ and $S \in \mathbb{R}(X)^\Gamma$, but this determines $\hat{\pi}$ since all elements of H are of the form $\pi(S)v$ for some $S \in \mathbb{R}(X)^\Gamma$ and $v \in H$. □

Lemma 2.2.23. *An arbitrary non-degenerate $*$-representation π of $\mathbb{S}(X)^\Gamma$ can be decomposed as $\pi = \pi_1 \oplus \pi_2$, where both π_1 and its restriction to $\mathbb{R}(X)^\Gamma$ are non-degenerate representations on some Hilbert space H_1 and π_2 is a non-degenerate representation of $\mathbb{S}(X)^\Gamma$ vanishing on $\mathbb{R}(X)^\Gamma$.*

Proof. This follows from Lemma 2.2.22 and the discussion prior to [1, Theorem 1.3.4]. □

Proof of Proposition 2.2.20. We denote by S the set of cyclic representations of $\mathbb{S}(X)^\Gamma$ on some Hilbert space with the property that their restriction to $\mathbb{R}(X)^\Gamma$ is a non-degenerate representation of $\mathbb{R}(X)^\Gamma$ on the same space. For $\pi \in S$, denote by $\pi_\mathbb{R}$ its restriction to $\mathbb{R}(X)^\Gamma$. The bounded geometry condition on X (see Proposition 2.2.3) implies that $\bigoplus_{\pi\in S} \pi_\mathbb{R}$ is a well-defined non-degenerate representation of $\mathbb{R}(X)^\Gamma$. Lemma 2.2.22 implies that $\Pi = \bigoplus_{\pi\in S} \pi$ is a well-defined Hilbert space representation of $\mathbb{S}(X)^\Gamma$. For $a \in \mathbb{S}(X)^\Gamma$ set $C_1^a = ||\Pi(a)||$. It is shown in Lemma 2.2.19 that $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$ is a C^*-algebra. Set $C_2^a = ||[a]||_{\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}}$ and $C_a = \max\{C_1^a, C_2^a\}$. Now let π be an arbitrary non-degenerate representation of $\mathbb{S}(X)^\Gamma$ with a decomposition $\pi_1 \oplus \pi_2$ as provided by Lemma 2.2.23. Obviously $||\pi(a)|| \leq \max\{||\pi_1(a)||, ||\pi_2(a)||\}$. The claim now follows from the facts that π_1 is a subrepresentation of Π and π_2 factors through $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$. □

Proposition 2.2.24. *As with the Roe algebra, the K-theory groups of the structure algebra are independent of the choice of the very ample representation. Furthermore, the assignment $X \mapsto K_*(D^*_{\max}(X)^\Gamma)$ is functorial for uniform (i.e. coarse and continuous) maps.*

Proof. See the discussion in [16, Chapter 12.4] □

Lemma 2.2.22 immediately implies the following

Proposition 2.2.25. *$C^*_{\max}(X)^\Gamma$ is an ideal of $D^*_{\max}(X)^\Gamma$.*

Proposition 2.2.26. *The inclusion $\mathbb{S}(X)^\Gamma \to D^*_{\max}(X)^\Gamma$ gives rise to an isomorphism $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \cong \frac{D^*_{\max}(X)^\Gamma}{C^*_{\max}(X)^\Gamma}$.*

Proof. Since $D^*_{\max}(X)^\Gamma$ is the maximal C^*-completion of $\mathbb{S}(X)^\Gamma$, the projection $\mathbb{S}(X)^\Gamma \to \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$ gives rise to a morphism of C^*-algebras $D^*_{\max}(X)^\Gamma \to \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$. Continuity of this map and the fact that its kernel contains $\mathbb{R}(X)^\Gamma$ implies that it induces a morphism $\frac{D^*_{\max}(X)^\Gamma}{C^*_{\max}(X)^\Gamma} \to \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$. The composition $\frac{D^*_{\max}(X)^\Gamma}{C^*_{\max}(X)^\Gamma} \to \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \to \frac{D^*(X)^\Gamma}{C^*(X)^\Gamma}$ is the identity on the set of classes of $\frac{D^*_{\max}(X)^\Gamma}{C^*_{\max}(X)^\Gamma}$ which have a representative from $\mathbb{S}(X)^\Gamma$. Since the latter set is dense, it follows that the composition is injective. On the other hand, by construction the composition $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \to \frac{D^*_{\max}(X)^\Gamma}{C^*_{\max}(X)^\Gamma} \to \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$ is the identity and the claim follows. □

Corollary 2.2.27. There is an isomorphism $K_{*+1}(\frac{D^*_{\max}(X)}{C^*_{\max}(X)}) \cong K^{lf}_*(X)$.

2.2.3 Yu's Localisation Algebras and K-homology

Definition 2.2.28 ([27, Section 2])**.** Let A be a normed $*$-algebra. By $\mathfrak{T}A$ denote the normed $*$-algebra of functions $f\colon [1,\infty) \to A$ which are bounded and uniformly continuous.

Clearly, if A is a C^*-algebra, so is $\mathfrak{T}A$. Important examples for us will be the algebras $\mathfrak{T}D^*_{\max}(X)$ and $\mathfrak{T}C^*_{\max}(X)$ defined using some very ample representation of $C_0(X)$. Now we are in the position to define the localisation algebra associated to a locally compact metric space X.

Definition 2.2.29 ([27, Section 2])**.** The C^*-algebra generated by functions $f \in \mathfrak{T}C^*_{\max}(X)^\Gamma$ with the properties

- $\operatorname{prop} f(t) < \infty$ for all $t \in [1,\infty)$
- $\operatorname{prop} f(t) \to 0$ as $t \to \infty$

is called the localisation algebra of X and is denoted by $C^*_{L,\max}(X)^\Gamma$.

Remark 2.2.30. In analogy to the fact that $C^*_{\max}(X)^\Gamma$ is contained as an ideal in the C^*-algebra $D^*_{\max}(X)^\Gamma$, one can define a C^*-algebra denoted by $D^*_{L,\max}(X)^\Gamma$, which contains $C^*_{L,\max}(X)^\Gamma$ as an ideal. This is the C^*-algebra generated by the elements in $\mathfrak{T}D^*_{\max}(X)^\Gamma$ with the two properties of Definition 2.2.29.

Yu's theorem states that the K-theory groups of the localisation algebra are isomorphic to the locally finite K-homology groups.

Theorem 2.2.31 ([27, Theorem. 3.4]). *Let X be a locally compact metric space and suppose $C^*_{L,\max}(X)$ is defined using a very ample representation. Then the local index map* $\mathrm{ind}_L\colon K^{lf}_*(X) \to K_*(C^*_{L,\max}(X))$ *of* [27, Definition 2.4] *is an isomorphism. Furthermore, the diagram*

$$\begin{array}{ccc} K^{lf}_*(X) & \xrightarrow{\mathrm{ind}_L} & K_*(C^*_{L,\max}(X)) \\ & \searrow^{\mu} & \downarrow (\mathrm{ev}_1)_* \\ & & K_*(C^*_{\max}(X)) \end{array}$$

is commutative. Here μ denotes the index map $K^{lf}_*(X) \cong K_{*+1}(\frac{D^*_{\max}(X)}{C^*_{\max}(X)}) \to K_*(C^*_{\max}(X))$.

Proof. First note that the local index map as defined in [27] can be defined analogously in the maximal case. In [27, Theorem. 3.4] the theorem is proven for the reduced localisation algebra and uses the isomorphism $K_{*+1}(\frac{D^*_{\mathrm{red}}(X)}{C^*_{\mathrm{red}}(X)}) \cong K^{lf}_*(X)$. However, Corollary 2.2.27 states that the isomorphism still holds if we replace the reduced Roe and structure algebra with the maximal ones. Thus, the argument of [27] can be used literally. □

Having the above theorem in mind, we will, from now on, use the notation $K^L_*(X)$ for the group $K_*(C^*_{L,\max}(X))$. Given a closed subset Y of X, we are now going to define the relative K-homology groups using localisation algebras and discuss the existence of a long exact sequence for pairs. Chang, Weinberger and Yu define the relative groups by using a concrete very ample representation, which we will now describe.

Let $Y \subset X$ be as above. Choose a countable dense set Γ_X of X such that $\Gamma_Y := \Gamma_X \cap Y$ is dense in Y. Define $C^*_{L,\max}(X)$ and $C^*_{L,\max}(Y)$ using the very ample representations $H_X = l^2(\Gamma_X) \otimes l^2(\mathbb{N})$ and $H_Y = l^2(\Gamma_Y) \otimes l^2(\mathbb{N})$ respectively. The constant family of isometries $V_t := \iota$, where $\iota\colon H_Y \to H_X$ is the inclusion covers the inclusion $Y \to X$ in the sense of [27, Def. 3.1]. Hence, applying $\mathrm{Ad}(V_t)$ pointwise we obtain a C^*-algebra morphism $C^*_{L,\max}(Y) \to C^*_{L,\max}(X)$, which we will denote by $\iota(X,Y)$. Note that on elements with finite propagation this map for each t is just the extension by zero of an operator on H_Y to an operator on H_X. We get a map $\iota(X,Y)_*\colon K^L_*(Y) \to K^L_*(X)$.

Now denote by $K^L_*(X,Y)$ the group $K_{*-1}(C_{\iota(X,Y)})$, where S denotes the suspension and $C_{\iota(X,Y)}$ the mapping cone of $\iota(X,Y)$. The short exact sequence

$$0 \to SC^*_{L,\max}(X) \to C_{\iota(X,Y)} \to C^*_{L,\max}(Y) \to 0$$

gives rise to a long exact sequence

$$\ldots \to K_*(C^*_{L,\max}(Y)) \to K_{*-1}(SC^*_{L,\max}(X)) \to K_{*-1}(C_{\iota(X,Y)}) \to \cdots$$

of K-theory groups. Using the canonical isomorphism $K_{*-1}(S(\cdot)) = K_*(\cdot)$ this sequence becomes the desired long exact sequence of a pair

$$\ldots \to K^L_*(Y) \to K^L_*(X) \to K^L_*(X,Y) \to \ldots,$$

constructed solely using localisation algebras.

Relative Localisation Algebra

Let X and Y be as above. We would like to extend $C^*_{L,\max}(Y)^\Gamma \subset C^*_{L,\max}(X)^\Gamma$ to an ideal with the same K-theory.

Definition 2.2.32. Denote by $C^*_{L,\max}(Y \subset X)^\Gamma$ the ideal in $C^*_{L,\max}(X)^\Gamma$ generated by functions $f \in \mathfrak{T}C^*(X)^\Gamma$ such that for all $t \in [1,\infty)$, $f(t)$ is supported in an $S(t)$-neighbourhood of Y, where $S\colon [1,\infty) \to \mathbb{R}$ is some function with $S(t) \to 0$ as $t \to \infty$.

Lemma 2.2.33 ([40, Lemma 1.4.18])**.** *Let Y and X be as above. The inclusion $Y \to X$ induces isomorphisms $K_*(C^*_{L,\max}(Y)^\Gamma) \cong K_*(C^*_{L,\max}(Y \subset X)^\Gamma)$.*

Proof. In [40, Lemma 1.4.18], this is proven in the reduced case. However in light of the discussion in Section 2.2.1, the modification of the arguments for use in the maximal setting is straightforward. □

2.2.4 Relative Group C^*-algebra

Let X be a proper path-connected metric space and Y a path-connected subset of X. The inclusion $Y \to X$ induces a map $\pi_1(Y) \to \pi_1(X)$, where we choose a point $y_0 \in Y$ to construct the fundamental groups and the latter map. This map in turn induces a morphism $\varphi\colon C^*_{\max}(\pi_1(Y)) \to C^*_{\max}(\pi_1(X))$. The *relative group C^*-algebra* is defined as

$$C^*_{\max}(\pi_1(X), \pi_1(Y)) := SC_\varphi.$$

The short exact sequence

$$0 \to SC^*_{\max}(\pi_1(X)) \to C_\varphi \to C^*_{\max}(\pi_1(Y)) \to 0$$

and the Bott periodicity isomorphism gives a long exact sequence

$$\to K_*(C^*_{\max}(\pi_1(Y))) \to K_*(C^*_{\max}(\pi_1(X))) \to K_*(C^*_{\max}(\pi_1(X), \pi_1(Y))) \to .$$

Remark 2.2.34. Note that the above C^*-algebras are independent of the chosen point y_0 up to an isomorphism which is well defined up to conjugation by a unitary and therefore is canonical on K-theory.

Remark 2.2.35. Recall that unless $\varphi\colon \pi_1(Y) \to \pi_1(X)$ is injective it does not necessarily induce a map of the reduced group C^*-algebras. Thus, the relative group C^*-algebra does not always have a reduced counterpart.

2.2.5 The Relative Index Map

The index of Chang, Weinberger and Yu is the image of a fundamental class in $K_*^L(X, Y)$ under a mapping $\mu\colon K_*^L(X, Y) \to K_*(C^*_{\max}(\pi_1(X), \pi_1(Y)))$, which they call the relative Baum-Connes map. In this subsection we present the definition of this map along the lines of [2, Section 2]. There the authors relate the K-theory groups of the localisation algebras and their equivariant counterparts and exploit Theorem 2.2.7 to relate the latter K-theory groups with those of the group C^*-algebras of the fundamental groups.

Let X be a locally compact, path-connected, separable metric space and Y be a closed path-connected subset of X. We suppose that the universal coverings $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{Y} \to Y$ of these spaces exist (e.g. suppose X and Y are CW-complexes) and are endowed with an invariant metric and that the metrics on X and Y are the pushdowns of these metrics, i.e. the projections are local isometries. In the case of smooth manifolds we can start with Riemannian metrics on X and Y and take their pullbacks to be the invariant Riemannian metrics on $\widetilde{X}$ and $\widetilde{Y}$. Pick countable dense subsets Γ_X and Γ_Y of X and Y such that $\Gamma_Y \subset \Gamma_X$ as before. Denote by $\Gamma_{\tilde{X}}$ and $\Gamma_{\tilde{Y}}$ the preimages of Γ_X and Γ_Y, respectively. Construct the (equivariant) Roe algebras and the (equivariant) localisation algebras using the representations $l^2(\Gamma_\cdot) \otimes l^2(\mathbb{N})$. We recall that the equivariant algebras are constructed using the action of fundamental groups by deck transformations.

Proposition 2.2.36 ([2, Proposition 2.8])**.** *Let X and $\tilde{X}$ be as above. Suppose furthermore that X is compact. Then there exists an $\epsilon > 0$ depending on X such that for finite propagation locally compact operators T with* $\operatorname{prop}(T) < \epsilon$*, the kernel $\tilde{k}$ defined in the following defines an element of $C^*_{\max}(\tilde{X})^{\pi_1(X)}$, which we will denote by $L(T)$.*

Observe for the definition of $L(T)$ that a finite propagation locally compact operator T on $l^2(\Gamma_X) \otimes H$ with $\operatorname{prop}(T) = r$ *is given by a matrix $\Gamma_X \times \Gamma_X \xrightarrow{k} K(H)$ such that $k(x, x')$ is 0 for all $(x, x') \in \Gamma_X \times \Gamma_X$ with $d_X(x, x') \geq r$. Define the lifted operator on $l^2(\Gamma_{\tilde{X}}) \otimes H$ using the matrix $(\tilde{x}, \tilde{x}') \overset{\tilde{k}}{\mapsto} k(p(\tilde{x}), p(\tilde{x}'))$ if $d_{\tilde{X}}(\tilde{x}, \tilde{x}') < r$ and 0 otherwise.*

Vice versa, every equivariant kernel $\tilde{T} \in C^*_{\max}(\tilde{X})^{\pi_1(X)}$ of propagation $< \epsilon$ is such a lift, and this in a unique way, defining the push-down $\pi(\tilde{T}) \in C^*_{\max}(X)$ *as the inverse of the lift.*

For the appropriate choice of ϵ, the covering $\tilde{X} \to X$ should be trivial when restricted to balls say of radius 2ϵ.

Remark 2.2.37. Later we will need a slight generalisation of Proposition 2.2.36 for manifolds obtained by attaching an infinite cylinder to a compact manifold with boundary. It is evident that the ϵ obtained for the manifold with boundary also works for the manifold with the infinite cylinder attached, and then the construction indeed goes through without any modification.

Definition 2.2.38. Let $T\colon s \mapsto T_s$ be an element of $\mathbb{R}_L(X)$, i.e. T_s is locally compact and has finite propagation which tends to 0 as $s \to \infty$. Therefore $\operatorname{prop}(T_s) < \epsilon$ for all $s \geq s_T$ with some $s_T \in [1, \infty)$. Define the lift

$$L(T)\colon s \mapsto \begin{cases} L(T_{s_T}); & s \leq s_T \\ L(T_s); & s \geq s_T \end{cases}$$

to obtain an element in $C^*_{L,\max}(\tilde{X})^{\pi_1(X)}$.

Similarly, for $\tilde{T}\colon s \mapsto \tilde{T}_s$ an element of $\mathbb{R}_L(\tilde{X})^{\pi_1(X)}$ such that $\tilde{T}_s$ is locally compact, equivariant and has finite propagation which tends to 0 as $t \to \infty$ (in particular $\operatorname{prop}(\tilde{T}_s) < \epsilon$ for all $s \geq s_{\tilde{T}}$ for some $s_{\tilde{T}} \in [1, \infty)$) define its *push-down*

$$\pi(\tilde{T})\colon s \mapsto \begin{cases} \pi(\tilde{T}_{s_{\tilde{T}}}); & s \leq s_{\tilde{T}} \\ \pi(\tilde{T}_s); & s \geq s_{\tilde{T}}. \end{cases}$$

Proposition 2.2.39. *Set $C^*_0(\widetilde{X})^{\pi_1(X)} := C_0([1,\infty), C^*_{\max}(\tilde{X})^{\pi_1(X)})$, the ideal of $C^*_{L,\max}(\widetilde{X})^{\Gamma}$ consisting of functions whose norm tends to 0 as $s \to \infty$. The assignments of Definition 2.2.38 give rise to continuous $*$-homomorphisms*

$$L\colon \mathbb{R}_L(X) \to C^*_{L,\max}(\tilde{X})^{\pi_1(X)}/C^*_0(\tilde{X})^{\pi_1(X)}$$
$$\pi\colon \mathbb{R}_L(\tilde{X})^{\pi_1(X)} \to C^*_{L,\max}(X)/C^*_0(X),$$

where we use that the algebra of functions vanishing at ∞ is an ideal of the localisation algebra. Being continuous, they extend to the C^*-completions, and they evidently map the ideal $C_0([1,\infty), C^*_{\max}(X))$ or $C_0([1,\infty), C^*_{\max}(\tilde{X})^{\pi_1(X)})$ to 0, so that we get C^*-algebra homomorphisms

$$L\colon C^*_{L,\max}(X)/C^*_0(X) \to C^*_{L,\max}(\tilde{X})^{\pi_1(X)}/C^*_0(\tilde{X})^{\pi_1(X)}$$
$$\pi\colon C^*_{L,\max}(\tilde{X})^{\pi_1(X)}/C^*_0(\tilde{X})^{\pi_1(X)} \to C^*_{L,\max}(X)/C^*_0(X).$$

By construction these two homomorphisms are inverse to each other.

Being cones, $C_0([1,\infty), C^*_{\max}(\tilde{X})^{\pi_1(X)})$ *and* $C_0(1,\infty), C^*_{\max}(X))$ *have vanishing K-theory and by the 6-term exact sequence the projections induce isomorphisms in K-theory*

$$K_*(C^*_{L,\max}(\tilde{X})^{\pi_1(X)}) \to K_*(C^*_{L,\max}(\tilde{X})^{\pi_1(X)}/C^*_0(\tilde{X})^{\pi_1(X)}),$$
$$K_*(C^*_{L,\max}(X)) \to K_*(C^*_{L,\max}(X)/C^*_0(X)).$$

We therefore get a well defined induced isomorphism in K-theory

$$L_*\colon K^L_*(X) = K_*(C^*_{L,\max}(X)) \to K_*(C^*_{L,\max}(\tilde{X})^{\pi_1(X)})$$

with inverse π_*.

The proof of Proposition 2.2.39 is not trivial, as we have to come to grips with the potentially different representations which enter the definition of the maximal C^*-norms for $C^*_{\max}(X)$ and $C^*_{\max}(\tilde{X})^{\pi_1(X)}$. To do this, we use the following lemma.

Lemma 2.2.40. *Let* ϵ *be as in Proposition 2.2.36. There exists* $K \in \mathbb{N}$*, such that for all* $T \in \mathbb{R}(X)$ *and* $\tilde{T} \in \mathbb{R}(\tilde{X})^{\pi_1(X)}$ *with propagation less than* ϵ *we have* $||L(T)||_{C^*_{\max}(\tilde{X})^{\pi_1(X)}} \leq K||T||_{C^*_{\max}(X)}$ *and* $||\pi(\tilde{T})||_{C^*_{\max}}(X) \leq K||\tilde{T}||_{C^*_{\max}(\tilde{X})^{\pi_1(X)}}$.

Proof. By assumption, X has bounded geometry. Consequently, we can and do choose for some fixed $c > 0$ a c-dense uniformly discrete subset D of Γ_X and denote by $C^*_{\max}(D)$ and $C^*_{\max}(\tilde{D})^{\pi_1(X)}$ the Roe algebras of D constructed using $l^2(D) \otimes H$ and $l^2(\tilde{D}) \otimes H$ as before. The proof of [9, Lemma 3.4] guarantees the existence of a $K \in \mathbb{N}$ such that for all $T \in C^*_{\max}(D)$ with $\mathrm{prop}(T) < \epsilon$ there exist operators $T_{i\in\{1,\dots,K\}} \in C^*_{\max}(D)$ such that $||T_i|| \leq ||T||$, $T_i^*T_i \in l^\infty(D; K(H))$, i.e. $T_i^*T_i$ are operators of propagation 0, and such that $\sum T_i = T$. Moreover, the lift $\tilde{T}_i$ satisfies that $\tilde{T}_i^*\tilde{T}_i = \widetilde{T_i^*T_i} \in l^\infty(\tilde{D}; K(H))^{\pi_1(X)} \overset{L}{\cong} l^\infty(D; K(H))$. Hence the norm of $\tilde{T}_i^*\tilde{T}_i$ is exactly $||T_i||^2$.

We thus have $||L(T)|| \leq K||T||$. With a completely analogous argument we get $||\pi(\tilde{T})|| \leq K||\tilde{T}||$.

Note that there are isomorphisms

$$C^*_{\max}(X) \to C^*_{\max}(D),$$
$$C^*_{\max}(\tilde{X})^{\pi_1(X)} \to C^*_{\max}(\tilde{D})^{\pi_1(X)}$$

which can be constructed explicitly (compare [9, Section 4.4]). These isomorphisms can be chosen so as to make the diagrams

$$\begin{array}{ccc} \mathbb{R}(\tilde{X})^{\pi_1(X)}_\epsilon & \longrightarrow & \mathbb{R}(\tilde{D})^{\pi_1(X)}_\epsilon \\ {\scriptstyle L}\uparrow & & {\scriptstyle L}\uparrow \\ \mathbb{R}(X)_\epsilon & \longrightarrow & \mathbb{R}(D)_\epsilon \end{array} \qquad \begin{array}{ccc} \mathbb{R}(\tilde{X})^{\pi_1(X)}_\epsilon & \longrightarrow & \mathbb{R}(\tilde{D})^{\pi_1(X)}_\epsilon \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R}(X)_\epsilon & \longrightarrow & \mathbb{R}(D)_\epsilon \end{array}$$

commute. Here the subscript ϵ means that we are only considering operators with propagation less than ϵ.

The latter commutative diagrams complete the proof. □

Proof of Proposition 2.2.39. Recall that for $(\tilde{T}\colon s \to \tilde{T}_s) \in C^*_{L,\max}(\tilde{X})^{\pi_1(X)}$ we use the supremum norm: $||\tilde{T}|| = \sup_{s\in[1,\infty)} ||\tilde{T}_s||$. It follows that the norm of the image of $\tilde{T}$ in $C^*_{L,\max}(\tilde{X})^{\pi_1(X)}/C_0([1,\infty); C^*(\tilde{X})^{\pi_1(X)})$ under the projection map is $||[\tilde{T}]|| = \limsup_{s\in[1,\infty)} ||\tilde{T}_s||$ (specifically, multiplication of $\tilde{T}$ with a cutoff function $\rho\colon [1,\infty) \to [0,1]$ which vanishes on $[1,R]$ and is identically 1 on $[R+1,\infty)$ produces representative of $[\tilde{T}]$ whose norm in $C^*_L(\tilde{X})^{\pi_1(X)}$ approaches $\limsup_{s\in[1,\infty)} ||\tilde{T}_s||$ as $R \to \infty$).

The assertion then follows immediately from Lemma 2.2.40. □

Until the end of Section 2.5 we are going to suppose that X is compact and that Y is a closed subset of X. Recall that φ denotes the map $\pi_1(Y) \to \pi_1(X)$ induced by the inclusion. Following the notation introduced in [2, Section 2], we denote by Y' the set $p^{-1}(Y)$ and by $p''\colon Y'' \to Y$ the covering of Y associated to the subgroup $\ker\varphi$; hence, $Y' = \pi_1(X) \times_{\pi_1(Y)/\ker\varphi} Y''$. Now construct the equivariant Roe and localisation algebras for Y' and Y'' using the sets $p^{-1}(\Gamma_Y)$ and $(p'')^{-1}(\Gamma_Y)$ similarly as before.

Theorem 2.2.41 ([2, Lemma 2.12])**.** *There is a map*

$$\psi''\colon C^*_{\max}(\tilde{Y})^{\pi_1(Y)} \to C^*_{\max}(Y'')^{\pi_1(Y)/\ker\varphi}$$

with the property that there exists $\epsilon > 0$ such that given an operator $T \in C^*_{\max}(\tilde{Y})^{\pi_1(Y)}$ with $\mathrm{prop}(T) < \epsilon$ and kernel k on $(p')^{-1}(\Gamma_Y)$ the pushdown of k gives a unique well-defined kernel k_Y on Γ_Y and $\psi''(T)$ is given by the kernel $(x,y) \mapsto k_Y(p''(x), p''(y))$ for $x, y \in Y''$ with $d_{Y''}(x,y) < \epsilon$.

Remark 2.2.42. It can be observed from the proof of Theorem 2.2.41, that the result can be generalised to obtain a map $C^*_{\max}(Z)^{\Gamma} \to C^*_{\max}(Z/N)^{\Gamma/N}$, where Z is a bounded geometry space satisfying the properties mentioned in the beginning of the paper, Γ is a discrete group acting freely and properly on Z via isometries, $N \subset \Gamma$ is a normal subgroup and there exists an ϵ such that the coverings $Z \to Z/N'$ are trivial when restricted to ϵ-balls for any normal subgroup $N' \subset \Gamma$.

Remark 2.2.43. For the proof of Theorem 2.2.41, Chang, Weinberger and Yu use that the push-down of operators with small propagation as defined in Definition 2.2.38 can be extended to an honest $*$-homomorphism. Doing it partially gives a morphism of $*$-algebras $\psi''\colon \mathbb{R}(\tilde{Y})^{\pi_1(Y)} \to \mathbb{R}(Y'')^{\pi_1(Y)/\ker\varphi}$

and then maximality of the norms provides the extension to the desired C^*-homomorphism $C^*_{\max}(\tilde{Y})^{\pi_1(Y)} \to C^*_{\max}(Y'')^{\pi_1(Y)/\ker\varphi}$. Note that, in general, this is not possible if we use the reduced equivariant Roe algebras.

Using $Y' = Y'' \times_{\pi_1(Y)/\ker\varphi} \pi_1(X)$, we get a C^*-algebra morphism

$$\psi' \colon C^*_{\max}(Y'')^{\pi_1(Y)/\ker\varphi} \to C^*_{\max}(Y')^{\pi_1(X)} \subset C^*_{\max}(\tilde{X})^{\pi_1(X)}$$

where the first map repeats the operators on the different copies of Y'' inside Y'. Composing ψ' and ψ'' we obtain the map $\psi \colon C^*_{\max}(\tilde{Y})^{\pi_1(Y)} \to C^*_{\max}(\tilde{X})^{\pi_1(X)}$. Application of the maps pointwise defines the corresponding maps for localisation algebras, which we denote with the same symbols with subscript L.

Theorem 2.2.44. *The constructions just described fit into the following commutative diagram of C^*-algebras, where the composition in the third row is the map ψ_L, in the forth row is ψ, and in the last row is φ. The projection maps in the second row of vertical maps are K-theory isomorphism. The last vertical maps induce the canonical isomorphism in K-theory of Theorem 2.2.7. The Roe and localisation algebras are constructed using the maximal completion.*

$$\begin{array}{ccccccc}
C^*_{L,\max}(Y) & = & C^*_{L,\max}(Y) & = & C^*_{L,\max}(Y) & \xrightarrow{\iota} & C^*_{L,\max}(X) \\
\downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} \\
\dfrac{C^*_{L,\max}(\tilde{Y})^{\pi_1(Y)}}{C^*_0(\tilde{Y})^{\pi_1(Y)}} & \xrightarrow{\psi''_L} & \dfrac{C^*_{L,\max}(Y'')^{\pi_1(Y)/\ker\varphi}}{C^*_0(Y'')^{\pi_1(Y)/\ker\varphi}} & \xrightarrow{\psi'_L} & \dfrac{C^*_{L,\max}(Y')^{\pi_1(x)}}{C^*_0(Y')^{\pi_1(x)}} & \xrightarrow{\subset} & \dfrac{C^*_{L,\max}(\tilde{X})^{\pi_1(X)}}{C^*_0(\tilde{X})^{\pi_1(X)}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
C^*_{L,\max}(\tilde{Y})^{\pi_1(Y)} & \xrightarrow{\psi''_L} & C^*_{L,\max}(Y'')^{\pi_1(Y)/\ker\varphi} & \xrightarrow{\psi'_L} & C^*_{L,\max}(Y')^{\pi_1(X)} & \xrightarrow{\subset} & C^*_{L,\max}(\tilde{X})^{\pi_1(X)} \\
\downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} \\
C^*_{\max}(\tilde{Y})^{\pi_1(Y)} & \xrightarrow{\psi''} & C^*_{\max}(Y'')^{\pi_1(Y)/\ker\varphi} & \xrightarrow{\psi'} & C^*_{\max}(Y')^{\pi_1(X)} & \xrightarrow{\subset} & C^*_{\max}(\tilde{X})^{\pi_1(X)} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
C^*_{\max}(\pi_1(Y)) & \xrightarrow{pr_*} & C^*_{\max}(\pi_1(Y)/\ker\varphi) & \longrightarrow & C^*_{\max}(\pi_1(X)) & = & C^*_{\max}(\pi_1(X)).
\end{array}$$

Proof. If in the first row $C^*_L(Y)$ is replaced by $\mathbb{R}_L(Y)$, then the definition of L, the behaviour of the push-down map ψ'' and the (trivial) lifting map ψ' on operators of small propagation and the definition of ι and $\subset$ imply the commutativity of the first two rows of the diagram. The continuity of the involved maps then implies the commutativity of the first two rows. In order to show the commutativity of the last two rows we recall the isomorphisms

$K_*(C^*_{\max}(\pi_1(\cdot))) \to K_*(C^*_{\max}(\tilde{\cdot})^{\pi_1(\cdot)})$. For this we need the isomorphisms $C^*_{\max}(\pi_1(\cdot)) \otimes K(H) \xrightarrow{\cong} C^*_{\max}(\tilde{\cdot})^{\pi_1(\cdot)}$. Here we modify the proof of [16, Lemma 12.5.3] slightly to suit our choice of the representation space. Choose a countable dense subset D of the fundamental domain of $\tilde{Y}$ such that D and gD are disjoint for $g \neq e$ in $\pi_1(Y)$. With $\Gamma_{\tilde{Y}} = \bigsqcup_{g \in \pi_1(Y)} gD$, we get an isomorphism $l^2(\Gamma_{\tilde{Y}}) \otimes l^2(\mathbb{N}) \cong l^2(\pi_1(Y)) \otimes (\bigoplus_{n \in \mathbb{N}} l^2(D))$. Using this isomorphism we then obtain a $*$-isomorphism between $\mathbb{C}[\pi_1(Y)] \otimes K(\bigoplus_{n \in \mathbb{N}} l^2(D))$ and the algebra of invariant, finite propagation and locally compact operators. This induces the desired isomorphism $C^*_{\max}(\pi_1(Y)) \otimes K(\bigoplus_{n \in \mathbb{N}} l^2(D)) \xrightarrow{\cong} C^*_{\max}(\tilde{Y})^{\pi_1(Y)}$. Furthermore we note (see [31, Proposition 6.4.1 and Proposition 8.2.8]) that the standard isomorphisms $K_p(A) \to K_p(A \otimes K(H))$ for a C^*-algebra A and a separable infinite dimensional Hilbert space H is induced by the morphism $a \mapsto a \otimes p$, with p a rank one projection. Now consider the rank one projection $p_{x_0} \otimes p_1$ on $\bigoplus_{n \in \mathbb{N}} l^2(D) \cong l^2(D) \otimes l^2(\mathbb{N})$ for some $x_0 \in D$ and p_1 the operator on $l^2(\mathbb{N})$ projecting to the first component. The composition gives the desired map $C^*_{\max}(\pi_1(Y)) \to C^*_{\max}(\tilde{Y})^{\pi_1(Y)}$ which induces the K-theory isomorphism of Theorem 2.2.7. We can perform the same procedure for $Y'' = \tilde{Y}/(\ker \varphi)$. Considering the above D (or rather its image under $\tilde{Y} \to Y''$) as a subset of Y'' and using $\Gamma_{Y''} = \bigsqcup_{g \in \frac{\pi_1(Y)}{\ker \varphi}} gD$, we get the corresponding isomorphism $l^2(\Gamma_{Y''}) \otimes l^2(\mathbb{N}) \cong l^2(\frac{\pi_1(Y)}{\ker \varphi}) \otimes (\bigoplus_{n \in \mathbb{N}} l^2(D))$. Choosing the same p as above our procedure defines the desired $C^*_{\max}(\frac{\pi_1(Y)}{\ker \varphi}) \to C^*_{\max}(Y'')^{\frac{\pi_1(Y)}{\ker \varphi}}$ which is a K-theory isomorphism and which makes the lower left corner of the diagram of Theorem 2.2.44 commutative. Similarly we construct the corresponding map for Y', which is the associated bundle to Y'' with fibre $\pi_1(X)$ (we can consider the above D as a subset of Y'). The construction gives rise to the morphism $C^*_{\max}(\pi_1(X)) \to C^*_{\max}(Y')^{\pi_1(X)}$ which is a K-theory isomorphism and which makes the lower middle square of the diagram of Theorem 2.2.44 commutative. Finally, considering D as a subset of Y' and extending it to a dense subset of a fundamental domain of $\tilde{X}$, we obtain, similarly as above, a corresponding map for $\tilde{X}$, the morphism $C^*_{\max}(\pi_1(X)) \to C^*_{\max}(\tilde{X})^{\pi_1(X)}$ which is a K-theory isomorphism such that also the lower right corner of the diagram of Theorem 2.2.44 commutes. This finishes the proof of the said Theorem. □

Definition 2.2.45. The commutative diagram of Theorem 2.2.44 defines a zig-zag of maps between the mapping cones of the compositions of the maps from left to right. Using in addition that the two wrong way vertical maps induce isomorphisms in K-theory, we obtain the map

$$\mu\colon K_*(SC_{\iota(X,Y)}) \to K_*(SC_\varphi) \overset{\text{Def}}{=} K_*(C^*_{\max}(\pi_1(X), \pi_1(Y))),$$

which we call the *relative index map.* In [2] it is called the *maximal relative Baum-Connes map.*

2.3 A Geometric and Functorial Completion of the Equivariant Roe Algebra

2.3.1 Maximal Roe Algebra and Functions of the Dirac Operator

Before describing our geometric completion of the algebraic Roe algebra, we discuss issues arising in coarse index theory when one uses maximal completions of the relevant C^*-algebras, which lead to gaps in [2]. A crucial role in coarse index theory is played by functions of the Dirac operator (via functional calculus). If we work with the usual (reduced) Roe algebras, the latter are defined as algebras of bounded operators on L^2-spinors, and the Dirac operator is an unbounded operator on the same Hilbert space. Ellipticity and finite propagation of the wave operator then are used to show that certain functions of the Dirac operator satisfy the defining conditions for the reduced Roe algebra and of the reduced structure algebra.

However, if one uses the maximal versions this is highly non-trivial:

1. The functions $f(D)$ which do have finite propagation are by the very definition elements of the algebraic Roe algebra (if f vanishes at infinity) or of the algebraic structure algebra (if f is a normalising function). The wave operators e^{itD} are bounded multipliers of the maximal Roe algebra and by Lemma 2.2.22 act as bounded operators on the defining representation of the maximal Roe algebra.

2. However, it is not obvious at all that the one parameter group $t \mapsto e^{itD}$ is strongly continuous on the defining representation of the maximal Roe algebra, i.e. is obtained from an (unbounded) self-adjoint operator D on that Hilbert space. This one needs to have a reasonable definition of $f(D)$ in the maximal Roe and structure algebra for f without a compactly supported Fourier transform.

3. Even if one manages to construct the self-adjoint unbounded operator D on the maximal representation, it remains to show that this

maximal Dirac operator is invertible if the underlying manifold has uniformly positive scalar curvature: one has to make sense also of a (geometric) Schrödinger-Lichnerowicz formula for this non-geometric representation?

Chang, Weinberger, and Yu's article [2] taks all these necessary constructions and properties for granted, without any justification. We propose a way around by passing to a slightly different and much more convenient completion. Later, Guo, Xie, and Yu posted the preprint [11] where they also identify these gaps in [2] and propose positive answers the above questions.

2.3.2 The Quotient Completion

Our suggestion to overcome the problems addressed in Section 2.3.1 is to work with another functorial completion of the equivariant Roe algebra which is more geometric. We are studying the case that a group Γ acts freely and properly discontinuously by isometries on a proper metric space X.

For every normal subgroup $N \subset \Gamma$ we then can form the metric space X/N on which the quotient group $Q := \Gamma/N$ acts as before. Indeed, typically we obtain X as a Γ-covering of a space X/Γ and the X/N are then other normal coverings of X/Γ.

In the usual way, the purely algebraically defined algebras $\mathbb{R}(X)^\Gamma$ and $\mathbb{S}(X)^\Gamma$ act via their images in $\mathbb{R}(X/N)^{\Gamma/N}$ and $\mathbb{S}(X/N)^{\Gamma/N}$ on all these quotients (see Theorem 2.2.41), and we complete with respect to all these norms at once. Denote the corresponding completions by $C^*_q(X)^\Gamma$ and $D^*_q(X)^\Gamma$. It is clear that the former is an ideal in the latter. It is also clear that this has the usual functoriality properties for Γ-equivariant maps for fixed Γ, but now in addition is functorial (this is built in) for the quotient maps $X \to X/N$, giving $C^*_q(X)^\Gamma \to C^*_q(X/N)^{\Gamma/N}$ and $D^*_q(X)^\Gamma \to D^*_q(X/N)^{\Gamma/N}$.

Finally, for inclusion of groups $\iota\colon \Gamma \to G$ induces an *induction map* $C^*_q(X)^\Gamma \to C^*_q(X \times_\Gamma G)^G$, because for every quotient G/N we get the associated induction

$$\mathbb{R}(X/\Gamma \cap N)^{\Gamma/(\Gamma\cap N)} \to \mathbb{R}(X/(\Gamma \cap N) \times_{\Gamma/\Gamma\cap N} G/N)^{G/N} = \mathbb{R}(X \times_\Gamma G/N)^{G/N}.$$

The corresponding construction works for D^*_q and for the localisation algebras.

Putting this together, we get the expected functoriality of C^*_q and D^*_q and the localisation algebras for maps equivariant for any homomorphism $\alpha\colon \Gamma_1 \to \Gamma_2$.

Lemma 2.3.1. *Suppose Γ acts cocompactly on X. Then $C^*_q(X)^\Gamma$ is isomorphic to $C^*_q(\Gamma) \otimes K(H)$. Here, $C^*_q(\Gamma)$ is the C^*-completion of $\mathbb{C}[\Gamma]$ in the* *representation* $\bigoplus_{N \lhd \Gamma} l^2(\Gamma/N)$, *where the sum is over all normal subgroups* N *of* Γ.

Proof. The proof is precisely along the lines of the one of Theorem 2.2.7. □

Proposition 2.3.2. *Let* X/Γ *be a complete Riemannian spin manifold with* Γ*-covering* X. *The Dirac operator on the different normal coverings* X/N *for the normal subgroups* N *of* Γ *gives rise to a self-adjoint unbounded operator in the defining representation of* $C_q^*(X)^\Gamma$. *If* $f \in C_0(\mathbb{R})$ *we get* $f(D) \in C_q^*(X)^\Gamma$, *if* $\Psi\colon \mathbb{R} \to [-1,1]$ *is a normalising function, we get* $\Psi(D) \in D_q^*(X)^\Gamma$.

This construction is functorial for the quotient maps $X \to X/N$ *for normal subgroups* $N \lhd \Gamma$.

The Schrödinger-Lichnerowicz argument applies: if X/Γ *has uniformly positive scalar curvature then the spectrum of the operator* D *in the defining representation of* $C_q^*(X)^\Gamma$ *does not contain* 0.

Let $A \subset X$ *be a* Γ*-invariant measurable subset. Then* χ_A, *the operator of multiplication with the characteristic function of* A *is an element of* $D_q^*(X)^\Gamma$, *in particular a multiplier of* $C_q^*(X)^\Gamma$. *Under the quotient map* $X \to X/N$ *for a normal subgroup* $N \lhd \Gamma$ *it is mapped to* $\chi_{A/N}$. *Similarly, a function of the Dirac operator on* X *is mapped to the same function of the Dirac operator on* X/N.

Proof. The statements about the Dirac operator are just an application of the usual arguments to all the quotients X/N simultaneously, using Lemma 3.3.3.

The statement about χ_A is a direct consequence of the definitions. □

Remark 2.3.3. We note that all the statements in Section 2.2 have a counterpart when we use the quotient completion instead of the maximal completion of the equivariant algebras and their proofs are completely analogous to (and often easier than) the proofs for the maximal completions. In particular, we have a relative index map in this case. Furthermore we would like to emphasise that Theorem 2.2.11 holds for the quotient completion. Given a map $\phi\colon \Gamma \to \pi$, we get by functoriality a morphism $\phi : C_q^*(\Gamma) \to C_q^*(\pi)$, and $C_q^*(\pi, \Gamma)$ will denote SC_ϕ as before.

2.4 Higher Indices of Dirac Operators on Manifolds with Boundary

2.4.1 Construction of the Relative Index

Throughout this section, we consider only even dimensional spin manifolds. We define the relative index of the Dirac operator of a manifold M with boundary N in the following groups:

- in $C^*_{\max}(\pi_1(M), \pi_1(N))$,
- in $C^*_q(\pi_1(M), \pi_1(N))$ and
- in $C^*_{\mathrm{red}}(\pi_1(M), \pi_1(N))$ if $\pi_1(N) \to \pi_1(M)$ is injective.

In what follows the subscript d stands for one of the mentioned completions. Before defining the relative index of the Dirac operator on a manifold with boundary, we recall the explicit image of the fundamental class under the local index map. Given a complete Riemannian spin manifold X with a free and proper action of Γ by isometries, denote by D_X the Dirac operator on X. Let Ψ_t be a sup-norm continuous family of normalising functions, i.e. each Ψ_t is an odd, smooth function $\Psi_t \colon \mathbb{R} \to [-1, 1]$ such that $\Psi_t(s) \xrightarrow{s\to\infty} 1$. Suppose furthermore that for $t \geq 1$ the distributional Fourier transform of Ψ_t is supported in a $\frac{1}{t}$-neighbourhood of 0. Choose an isometry α between $L^2(\not{S}^+)$ and $L^2(\not{S}^-)$ induced from a measurable bundle isometry, set $\Psi_t(D_X)^+ := \Psi_t(D_X)|_{L^2(\not{S}^+)}$ and $F_X(t) := \alpha^* \circ \Psi_t(D_X)^+$. Set $e_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{22} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Note that the presence of α^* implies that $F_X(t)$ is an operator on $L^2(\not{S}^+)$.

Definition 2.4.1. In the above situation, the (locally finite) fundamental class $[D_X] \in K_0(C^*_{L,d}(X)) = K_0^L(X)$ is given explicitly by $[P_X] - [e_{11}]$, with

$$P_X := \begin{pmatrix} FF^* + (1 - FF^*)FF^* & F(1 - F^*F) + (1 - FF^*)F(1 - F^*F) \\ (1 - F^*F)F^* & (1 - F^*F)^2 \end{pmatrix}.$$

In this formula F denotes $F_X(\cdot)$ and P_X is an idempotent in $M_2(C^*_{L,d}(X)^+)$. Here, A^+ denotes the unitalisation of A.

Remark 2.4.2. Note that since Ψ_t is assumed to have compactly supported Fourier transform, $\Psi_t(D_X)$ has finite propagation which means that P_X is a matrix over the unitalisation of $\mathbb{R}^*_L(X) \subset C^*_{L,\max}(X)$.

Now let M be a compact spin manifold with boundary N. Denote by N_∞ the cylinder $N \times [0, \infty)$ and by M_∞ the manifold $M \cup_N N_\infty$. Given a Riemannian metric on M which is collared at the boundary, we will equip N_∞ with the product metric. Taking the image of $[D_{M_\infty}]$ in $K_*^L(M_\infty, N_\infty)$ and then under the excision isomorphism defines the relative fundamental class $[M, N] \in K_*^L(M, N)$. For the index calculations which we have to carry out we need an explicit representative of this class, and this in the model of relative K-homology as the K-theory of the mapping cone algebra $C_{\iota(M,N)}$. Therefore, we recall the construction of [2], referring for further details to [2] —see also [16, Proposition 4.8.2] and [16, Proposition 4.8.3].

As the relative K-homology groups are constructed as mapping cones which come with a built-in shift of degree, we have to use Bott periodicity to shift the fundamental class to the suspension algebra (with degree shift). To implement this, denote by v the Bott generator of $K_1(C_0(\mathbb{R}))$. Following [2] define the invertible element

$$\tau_D := v \otimes P_{M_\infty} + I \otimes (I - P_{M_\infty})$$

in a matrix algebra over $C(S^1) \otimes C^*_{L,d}(M_\infty)^+$ with inverse given by $\tau_D^{-1} = v^{-1} \otimes P_{M_\infty} + I \otimes (I - P_{M_\infty})$ (see [16, Proposition 4.8.3] for more details). Next, we map to the relative K-homology of the pair (M, N), which requires applying the inverse of the excision isomorphism $K_*(M, N) \to K_*(M_\infty, N_\infty)$. This is implemented for our K-theory cycles by multiplication with a cut-off. For technical reasons, we observe that instead of $N \subset M$ we can use the homeomorphic $N_R := N \times \{R\} \subset M_R := M \cup N \times [0, R]$ for each $R \geq 0$. We use localisation algebras, and then we can use the K-theory isomorphism $C^*_{L,d}(M_R) \to C^*_{L,d}(M \subset M_\infty)$ and work with $C^*_{L,d}(M \subset M_\infty)$ which is independent of R. Similarly, we use the K-theory isomorphism $C^*_{L,d}(N_R) \to C^*_{L,d}(N \subset N_\infty)$ and replace $C^*_{L,d}(N_R)$ by the R-independent $C^*_{L,d}(N \subset N_\infty)$. This causes slight differences to the construction of [2].

For the cut-off, set $\chi_R := \chi_{M_R}$, the characteristic function of M_R. Consider

$$\begin{aligned}\tau_{D,R} :=& v \otimes (\chi_R P_{M_\infty} \chi_R + (1 - \chi_R)\, e_{11} (1 - \chi_R)) \\ &+ I \otimes (I - (\chi_R P_{M_\infty} \chi_R + (1 - \chi_R) e_{11} (1 - \chi_R)))\end{aligned}$$

and define $\tau_{D,R}^{-1}$ in the same way with v replaced by v^{-1}. Note that these two operators are in general *not* inverse to each other. Define, for $s \in [0, 1]$,

$$w_{D,R}(s) := \begin{pmatrix} I & (1-s)\tau_{D,R} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -(1-s)(\tau_D)^{-1}_M & I \end{pmatrix} \begin{pmatrix} I & (1-s)\tau_{D,R} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Finally set

$$q_{D,R}(s) := w_{D,R}(s)\; e_{11}\, w_{D,R}(s)^{-1}. \tag{2.2}$$

Applying the same procedure not to τ_D but to $v \otimes e_{11} + I \otimes e_{22}$, we obtain a curve $q_p(s)$. Note that by construction of $\tau_{D,R}$, all operators, in particular $q_{D,R}(s)$, are diagonal for the decomposition $L^2(M_\infty) = L^2(M_R) \oplus L^2(N \times [R, \infty))$ and are of standard form on $L^2(N \times [R, \infty))$. This summand does not appear in [2] but has to be there to construct the appropriate operators in $C^*_{L,d}(M \subset M_\infty)$.

Lemma 2.4.3. *Assume that the operator $F_{M_\infty}(t)$ has propagation $\leq L$ for some $L \in [0,\infty)$. Then $q_{D,R}(s)(t)$ (recall that we always have an additional $t \in [1,\infty)$-dependency) has propagation $\leq 30L$. It is diagonal with respect to the decomposition $L^2(M_\infty) = L^2(M_R) \oplus L^2(N \times [R,\infty))$ and coincides with $q_p(s)$ on $L^2(N \times [R,\infty))$. It is obtained via finitely many algebraic operations (addition, composition) from $\Psi_t(D_{M_\infty})$, the measurable bundle isometry α, the Bott element v and χ_R.*

If $R > 30L$ then $q_{M,R}(0)(t)$ differs from $q_p(0)(t)$ by an operator Q supported on $N \times [0,R]$. More precisely, for suitable operators A, B,

$$Q = \chi_R A \circ I \otimes [\chi_R, P_\infty] \circ B \chi_R$$

where the commutator $[\chi_R, P_\infty]$ is supported on $N \times [R-5L, R+5L]$ and Q has propagation $\leq 30L$.

Like $q_{D,R}(s)(t)$, the operator $Q(t)$ is obtained via finitely many algebraic operations from $\Psi_t(D_{M_\infty})$, α, v, v^{-1}, and χ_R.

Due to the local nature of all constructions and because of the support property of the commutator $[\Psi_R, P_\infty]$ (using Lemma 3.3.3 for $\Psi_t(D)$), the operator Q on $L^2(N \times [0,R])$ is equal *to the operator constructed correspondingly, where D_{M_∞} is replaced by $D_{N\times\mathbb{R}}$ and χ_R by $\chi_{N\times(-\infty,R]}$.*

Proof. The explicit formulas show that $q_{D,M}(s)(t)$ is an algebraic combination of $\Psi_t(D_{M_\infty})$, α, etc. as claimed, where all building blocks either have propagation 0 or are $\Psi_t(D_{M_\infty})$, and we compose at most 30 of the latter. The claim about the propagation follows.

As it can be seen from the formula in the proof of [2, Claim 2.19], $q_{D,R}(0)$ would be equal to $q_p(0)$ if $\tau_{D,R}$ was invertible with inverse $\tau_{D,R}^{-1}$, which would happen if $\chi_R P_{M_\infty} \chi_R$ was an idempotent. To compare with this situation one has to commute P_{M_∞} and χ_R which produces the shape of Q as claimed. The rest then follows as for $q_{D,R}(s)$. □

Denote by ι'_R the inclusion of $C^*_{L,d}(N \subset N_\infty)$ in $C^*_{L,d}(M \subset M_\infty)$, the image consisting of those operators which act only on $L^2(N_\infty)$.

The relative fundamental class $[M,N] \in K_0(C_{S\iota'_R}) \cong K_0(SC_{\iota'}) \cong K_0^L(M,N)$ is defined as

$$[M,N] := [(q_{D,R}(0), q_{D,R}(\cdot))] - [(q_p(0), q_p(\cdot))]\,. \tag{2.3}$$

It is implicit in [2] that the K-theory class is independent of R and the family of normalising functions Ψ_t.

Definition 2.4.4 (The Relative Index)**.** The relative index of the Dirac operator is defined as

$$\mu([M,N]) \in K_0(C^*_d(\pi_1(M), \pi_1(N))).$$

The explicit K-theory cycle defining $[M,N]$ and the description of the map μ of Definition 2.2.45 gives us an explicit cycle for the relative index:

We have to lift the operators $q_{D,M}(s)$ involved in the construction of $[M,N]$ to equivariant operators on the $\pi_1(M)$-cover $\widetilde{M_\infty}$ and those involved in $q_{D,M}(0)$ to equivariant operators on the $\pi_1(N)$-cover $\tilde{N}_\infty$. This is possible here and the operators are given as the corresponding functions of the Dirac operator on the coverings. For this, we use that by Lemma 2.4.3 the operators $q_{D,M}(t)$ is obtained as an expression in functions of the Dirac operator which lift to the corresponding functions of the Dirac operator by Lemma 3.3.3.

Similarly, by Lemma 2.4.3 and if $R > 30L$, where the propagation of $\Psi_t(D)$ is bounded by L for all $t \in [1,\infty)$, the operator $q_{D,R}(0)$ is obtained as an algebraic combination of functions of $D_{N\times\mathbb{R}}$ and the cut-off function $\chi_{N\times(-\infty,R]}$ which lift by Lemma 3.3.3 to $\pi_1(N)$-equivariant operators on $\tilde{N} \times [0,\infty)$ defined by the same expressions. Thus if we denote by $\tilde{q}_{D,R}$ the element constructed as above using the Dirac operator of $\widetilde{M_\infty}$ and $\chi_{\widetilde{M_R}}$ and by $\tilde{q}^N_{D,R}$ the element constructed using the Dirac operator on $\widetilde{N} \times \mathbb{R}$ and $\chi_{N\times(-\infty,R]}$ then we have the following

Lemma 2.4.5. *The expression* $[(\tilde{q}^N_{D,R}(0), \tilde{q}_{D,R}(\cdot))] - [(q_p(0), q_p(\cdot))]$ *defines an element of* $K_0(SC_{C^*_{L,d}(\tilde{N}\subset\tilde{N}_\infty)^{\pi_1(N)} \to C^*_{L,d}(\widetilde{M}\subset\widetilde{M}_\infty)^{\pi_1(M)}})$ *which identifies under the canonical isomorphism of the latter group with* $K_0(M,N)$ *with* $[M,N]$.

Hence under these conditions on R and the propagation of $\Psi_t(D)$, the relative index is the obtained by evaluation at $t=1$, or by homotopy invariance at any $t \geq 1$:

$$\begin{aligned} &\mu([M,N]) = [(\tilde{q}^N_{D,R}(0)(t), \tilde{q}_{D,R}(\cdot)(t))] - [(q_p(0), q_p(\cdot))] \in \\ &K_0(SC_{C^*(\tilde{N}\subset\tilde{N}_\infty)^{\pi_1(N)} \to C^*_d(\widetilde{M}\subset\widetilde{M}_\infty)^{\pi_1(M)}}) \cong K_0(C^*_d(\pi_1(M), \pi_1(N))). \end{aligned} \tag{2.4}$$

As $q_p(\cdot)$ is independent of t, we omit specifying the evaluation at t here.

2.4.2 The Localised Fundamental Class and Coarse Index

Suppose X is a smooth even dimensional spin manifold with free and proper action by Γ. Let Z be a closed Γ-invariant subset of X. Suppose that there exists a complete Γ-invariant Riemannian metric on X which has uniformly positive scalar curvature outside Z. In [28] and in more detail in [30], Roe defines a localised coarse index of the Dirac operator in $K_*(C^*_{\mathrm{red}}(Z \subset X)^\Gamma)$. In the course of the proof of [12, Theorem 3.11], the construction of the latter localised index is generalised to the case of a Dirac operator twisted with a

Hilbert C^*-module bundle. In [40, Chapter 2], Zeidler defines this index using localisation algebras. There, he also shows that under certain assumptions on a manifold X with boundary Y, the localised coarse index can be used to define an obstruction to the extension of a uniformly positive scalar curvature metric on the boundary to a uniformly positive scalar curvature metric on the whole manifold. In this section we follow the approach in [40] to define the localised fundamental class and coarse index.

Definition 2.4.6. Denote by $C^*_{L,0,d}(X)^\Gamma$ the kernel of the evaluation homomorphism $\mathrm{ev}_1\colon C^*_{L,d}(X)^\Gamma \to C^*_d(X)^\Gamma$. Denote by $C^*_{L,Z,d}(X)^\Gamma$ the preimage of $C^*_d(Z \subset X)^\Gamma$ under ev_1. The symbol d here stands for the chosen completion (red, max, or q).

Suppose that g is a Γ-invariant metric on X with uniformly positive scalar curvature outside of a Γ-invariant set Z. In [40, Definition 2.2.6], in this situation the so-called partial ρ-invariant $\rho^\Gamma_{Z,\mathrm{red}}(g) \in C^*_{L,Z,\mathrm{red}}(X)^\Gamma$ is constructed, which is a lift of $[D_X]$ under the morphism $K_*(C^*_{L,Z,\mathrm{red}}(X)^\Gamma) \to K_*(C^*_{L,\mathrm{red}}(X)^\Gamma)$ induced by the inclusion.

Recall the explicit representative for $[D_X] \in K_0(C^*_{L,d}(X)^\Gamma)$ of Section 2.4. We next recall the construction of [40, Definition 2.2.6] and show that it also works for C^*_q.

Lemma 2.4.7. *If $f_2 \in C_b(\mathbb{R})$ has Fourier transform with support in $[-r, r]$ then $f_2(D)$ is r-local and depends only on the r-*local geometry *in the following sense: if $A \subset X$ is a Γ-invariant measurable subset then $\chi_A f_2(D)(1 - \chi_{\overline{B_r(A)}}) = 0$ and $\chi_A f_2(D)$ depends only on the Riemannian metric on $\overline{B_r(A)}$.*

Proof. This is the usual unit propagation statement in the form that $f_2(D)$ is the integral of $\hat{f}_2(t)e^{itD}$ where e^{itD} not only has propagation $|t|$ but also is well known to depend only on the r-local geometry. The latter fact is a consequence of [16, Corollary 10.3.4]. □

Lemma 2.4.8 ([30, Lemma 2.3], [12, Proposition 3.15])**.** *Suppose as above that the scalar curvature of g outside Z is bounded from below by $4\epsilon^2$. If $f \in C_0(\mathbb{R})$ has support in $(-\epsilon, \epsilon)$, then $f(D)$ lies in $C^*_d(Z \subset X)^\Gamma$.*

Proof. By [12, Proposition 3.15] the statement holds for all quotients X/N and their reduced Roe algebra, which implies by definition of the quotient completion that it holds for $C^*_q(X)^\Gamma$. □

Because of the geometric nature of the completion of the Roe algebra we use, Lemmas 3.3.3 and 2.4.8 allow to define the localised coarse index using the completion C^*_q as follows.

Definition 2.4.9. Choose a sup-norm continuous family of normalising functions Ψ_t for $t \geq 1$ such that $\Psi_1^2 - 1$ has support in $(-\epsilon, \epsilon)$, the Fourier transform of Ψ_t has compact support for each $t > 1$ and the Fourier transform of Ψ_t has support in $[-\frac{1}{t}, \frac{1}{t}]$ for $t \geq 2$. Note that the support condition on Ψ_1 implies that its Fourier transform is *not* compactly supported. For the existence note that we have to approximate the Fourier transform of Ψ_1 by compactly supported functions (with a singularity at 0) such that the error is small in L^1-norm. This is possible, as can be seen from the discussion in the proof of [12, Lemma 3.6].

Define $F_X(t)$ and P_X as in Section 2.4. Observe, however, that by Lemma 3.3.3 $F_X(1)F_X(1)^* - 1 \in C_q^*(Z \subset X)^\Gamma$. It follows that now the cycle $[P_X] - [e_{11}]$ defines a class

$$\rho_Z^\Gamma(g) \in K_0(C^*_{L,Z,d}(X)^\Gamma)$$

which is of course a lift of $[D_X]$.

Corollary 2.4.10. The construction shows that if we have uniform positive scalar curvature not only on $X \setminus Z$ but on all of X there is a further lift of $\rho_Z^\Gamma(g)$ to $\rho^\Gamma(g) \in K_0(C^*_{L,0,d}(X)^\Gamma)$, the usual rho-invariant.

Definition 2.4.11. Let $Z \subset X$ and g be as above. Suppose furthermore that the action of Γ on Z is cocompact so that Lemma 2.2.11 holds for Z. The equivariant localised coarse index $\mathrm{Ind}_Z^\Gamma(g)$ of g with respect to Z is defined as the image of $\rho_Z^\Gamma(g)$ under the composition

$$K_0(C^*_{L,Z,d}(X)^\Gamma) \to K_0(C^*_d(Z \subset X)^\Gamma) \cong K_0(C^*_d(Z)^\Gamma),$$

where the first map is induced by evaluation at 1.

The long exact sequence in K-theory associated to the short exact sequence

$$0 \to C^*_{L,0}(X)^\Gamma \to C^*_{L,Z}(X)^\Gamma \to C^*(Z \subset X)^\Gamma \to 0,$$

along with Corollary 2.4.10 imply that if g has uniformly positive scalar curvature on all of X, then $\mathrm{Ind}_Z^\Gamma(g)$ vanishes.

2.4.3 Application to the Case of a Compact Manifold with Boundary

Suppose M is compact even-dimensional spin manifold with boundary N. In this case we cannot directly define an index for the Dirac operator on M with value in $K_*(C_q^*(\pi_1(M)))$. However given a metric g with positive scalar curvature and product structure near the boundary, we can use the above localised coarse index to define an index in $K_0(C_q^*(\widetilde{M})^{\pi_1(M)}) \cong K_0(C_q^*(\pi_1(M)))$.

Note that this index *does* in general depend on the chosen metric of positive scalar curvature near the boundary. Let us review the construction of the latter index.

As in Section 2.4, denote by N_∞ the cylinder $N \times [0,\infty)$ and by M_∞ the manifold $M \cup_N N_\infty$. Denote by $[D_{M_\infty}]$ the fundamental class of the Dirac operator in $K_*(C^*_{L,q}(M))$ associated to some metric g on M_∞ (not necessarily collared on the cylindrical end) and by $[\widetilde{D_{M_\infty}}]$ the fundamental class of the Dirac operator in $K_*(C^*_{L,q}(\widetilde{M})^{\pi_1(M)})$ on $\widetilde{M_\infty}$ associated to the pullback of g, which we denote by $\tilde{g}$. As observed in Remark 2.2.37, Proposition 2.2.36 extends to M_∞ and the pointwise lifting procedure of operators with small propagation gives rise to an isomorphism $K^L_*(M_\infty) \cong K_*(C^*_L(\widetilde{M_\infty})^{\pi_1(M)})$ under which $[D_{M_\infty}]$ is mapped to $[\widetilde{D_{M_\infty}}]$. If g has positive scalar curvature on N, then its pullback has uniformly positive scalar curvature on $N'_\infty \subset \widetilde{M_\infty}$, i.e. outside the cocompact subset $\widetilde{M}$ of $\widetilde{M_\infty}$. This allows us to the define the localised coarse index $\mathrm{Ind}^{\pi_1(M)}(g) := \mathrm{Ind}^{\pi_1(M)}_{\widetilde{M}}(\tilde{g}) \in K_0(C^*(\widetilde{M})^{\pi_1(M)}) \cong K_0(C^*(\pi_1(M)))$. The latter index is an obstruction to $\tilde{g}$, and thus g, having positive scalar curvature.

2.5 Statement and Proof of the Main Theorem

Finally we are in the position to state the main theorem of this paper. Throughout this section we will assume all the manifolds and their boundary to be path-connected.

Theorem 2.5.1. *Let M be a compact spin manifold with boundary N. We have the commutative diagram*

$$\begin{array}{ccccc} \to K^L_*(N) & \longrightarrow & K^L_*(M) & \longrightarrow & K^L_*(M,N) \to \\ \big\downarrow{\scriptstyle \mu_N} & & \big\downarrow{\scriptstyle \mu_M} & & \big\downarrow{\scriptstyle \mu} \\ \to K_*(C^*_q(\pi_1(N))) & \longrightarrow & K_*(C^*_q(\pi_1(M))) & \xrightarrow{\ j\ } & K_*(C^*_q(\pi_1(M),\pi_1(N))) \to \end{array}$$

where the vertical maps are the index maps and relative index maps.

Assume that M has a metric g which is collared at the boundary and has positive scalar curvature there. Then

$$j(\mathrm{Ind}^{\pi_1(M)}(g)) = \mu([M,N])$$

under the canonical map $j\colon K_*(C^*_q(\pi_1(M))) \to K_*(C^*_q(\pi_1(M),\pi_1(N)))$.

The above theorem has as a corollary the following vanishing theorem of Chang, Weinberger and Yu for the relative index constructed in the mapping cone of the quotient completion of the group ring:

Theorem 2.5.2. *Let M be a compact spin manifold with boundary N. Suppose that M admits a metric of uniformly positive scalar curvature which is collared at the boundary. Then $\mu([M,N]) = 0$.*

Proof of the Theorem 2.5.1. Proposition 2.2.44 implies the commutativity of the diagram. To see this, note that the discussion there relies only on the functoriality properties of the maximal completions which are also satisfied by the quotient completions. It remains to show that given a metric with positive scalar curvature at the boundary, $\mathrm{Ind}^{\pi_1(M)}(g)$ is mapped to $\mu([M,N])$ under the canonical map. Let us analyse the situation with the strategy of proof and the difficulties involved. For the notation used we refer to Sections 2.4 and 3.3.2 on the relative index and the localised coarse index.

Both index classes are defined using explicit expressions involving functions of the Dirac operator. For $\mathrm{Ind}^{\pi_1(M)}(g)$, we only use the manifold $\widetilde{M}$ and $\pi_1(M)$-equivariant constructions, which, however, are necessarily non-local to make use of the invertibility of the Dirac operator on the boundary. For $\mu([M,N])$, on the other hand, one has to use a $\pi_1(M)$-equivariant operator on $\widetilde{M}$ and a further lift to a $\pi_1(N)$-equivariant operator on $\tilde{N}$, which is only possible if all the functions of the Dirac operator involved are sufficiently local. To show that the two classes are mapped to each other, we need to reconcile these two points.

First, observe that in the construction of the relative fundamental class and relative index we use the explicit implementation of the Bott periodicity map. We apply this now to our representative of the local index: with our choice of Ψ_1, $P_{\widetilde{M}_\infty}(1)$ is an idempotent in $C_q^*(\tilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$ representing $\mathrm{Ind}^{\pi_1(M)}(g) \in K_0(C_q^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}) \cong K_0(C^*(\pi_1(M)))$. Next,

$$\tau := v \otimes P_{\widetilde{M}_\infty}(1) + I \otimes (I - P_{\widetilde{M}_\infty}(1))$$

is the invertible element in $C_0(\mathbb{R}) \otimes C^*(\widetilde{M} \subset \tilde{M}_\infty)^{\pi_1(M)}$ representing the K_1-class corresponding to the localised index under the suspension isomorphism. Finally, if we define $q(s)$ as in Equation (2.2) with $\tau_{D,R}$ replaced by τ then

$$a := [q(0)(1), q(\cdot)(1)] - [q_p(0), q_p(\cdot)] \in K_0(SC_{\{0\}\to C^*(\tilde{M}\subset\widetilde{M}_\infty)^{\pi_1(M)}})$$

defines the class corresponding to $\mathrm{Ind}^{\pi_1(M)}(g)$ under the Bott periodicity isomorphism, where we use that the cone of the inclusion of $\{0\}$ into A is the suspension of A. Of course, here $q(0)(1) = q_p(0)$.

We now have to show that, under the canonical map to the suspension of the cone of $C^*(\tilde{N} \subset \tilde{N}_\infty)^{\pi_1(N)} \to C^*(\widetilde{M} \subset \tilde{M}_\infty)^{\pi_1(M)}$ induced by the inclusion $\{0\} \to C^*(\tilde{N}, \tilde{N}_\infty)^{\pi_1(N)}$, the class a is mapped to the relative index $\mu[M,N]$. Recall from (2.4) that the latter is represented by any cycle of the form

$$[\tilde{q}^N_{D,R_t}(0)(t), \tilde{q}_{D,R_t}(\cdot)(t)] - [(q_p(0), q_p(\cdot))]$$

for $t > 1$, such that the support of $\widehat{\Psi}_t$ is contained in $[-L_t, L_t]$ for $L_t \in \mathbb{R}$ and therefore $\Psi_t(D)$ has propagation $\leq L_t$, where we must choose $R_t > 30L_t$. The construction of $\tilde{q}_{D,R_t}(\cdot)(t)$ involves the same steps as the one of $q(\cdot)$, but we use $\Psi_t(D)$ instead of $\Psi_1(D)$ and moreover apply cut-off with χ_{R_t}. Note that now $\tilde{q}^N_{D,R_t}(0)(t) - q_p(0) \neq 0$, but rather $\tilde{q}^N_{D,R_t}(0)(t) - q_p(0) \in C^*(\tilde{N} \subset \tilde{N}_\infty)^{\pi_1(N)}$, so that this is not a class in the suspension of $SC^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$ but in the mapping cone.

We claim now that for each $\epsilon > 0$ there is (t_ϵ, R_ϵ) such that

$$||\tilde{q}^N_{D,R_\epsilon}(0)(t_\epsilon) - q_p(0)|| + ||\tilde{q}_{D,R_\epsilon}(\cdot)(t_\epsilon) - q(\cdot)(1)|| \leq \epsilon. \tag{2.5}$$

This implies by standard properties of the K-theory of Banach algebras the desired result (as $q(0)(1) = q_p(0)$),

$$\mu([M,N]) = c(\mathrm{Ind}^{\pi_1(M)}(g)).$$

To prove (2.5) we make use of Lemma 2.4.3 which explicitly describes the operators involved. This implies

$$||\tilde{q}_{D,R}(\cdot)(t) - \tilde{q}_{D,R}(\cdot)(1)|| \xrightarrow{t \to 1} 0 \tag{2.6}$$

uniformly in R, as the two expressions are obtained via algebraic operations involving $\Psi_t(D)$, and by the sup-norm continuity of Ψ_t, $\Psi_t(D)$ converges to $\Psi_1(D)$ in norm (and this again uniformly, independent of the complete Riemannian manifold for which D is considered).

Next by the uniformly positive scalar curvature on N_∞ we have $P_{\widetilde{M}_\infty}(1) - e_{11} \in C^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$. This implies (convergence in norm)

$$\chi_R(P_{\widetilde{M}_\infty}(1) - e_{11})\chi_R \xrightarrow{R \to \infty} P_{\widetilde{M}_\infty}(1) - e_{11}$$

or equivalently

$$\chi_R P_{\widetilde{M}_\infty}(1)\chi_R + (1-\chi_R)\, e_{11}\, (1-\chi_R) \xrightarrow{R \to \infty} P_{\widetilde{M}_\infty}(1). \tag{2.7}$$

Because of Lemma 2.4.3, (2.7) implies that

$$||\tilde{q}_{D,R}(\cdot)(1) - q(\cdot)(1)|| \xrightarrow{R \to \infty} 0 \tag{2.8}$$

as these operators are obtained as a fixed algebraic expression of either

$$\chi_R P_{\widetilde{M}_\infty}(1)\chi_R + (1-\chi_R)\, e_{11}\,(1-\chi_R) \quad \text{or } P_{\widetilde{M}_\infty}(1).$$

Next, (2.6) together with (2.8) imply the assertion of (2.5) for the second summand. Here, we can and have to choose R_ϵ depending on t_ϵ such that $R_\epsilon > R_{t_\epsilon}$ (depending on the propagation of $\Psi_{t_\epsilon}(D)$).

Then, the lift $\tilde{q}^N_{D,R_\epsilon}(0)(t_\epsilon)$ to $C^*(\tilde{N} \subset \tilde{N}_\infty)^{\pi_1(N)}$ actually exists, is defined in terms of the Dirac operator on $\tilde{N} \times \mathbb{R}$, and we have to show that by choosing t_ϵ sufficiently close to 1 it is close to $q_p(0)$.

This, as we already showed, it is a special case of (2.6) and (2.8), now applied to the Dirac operator on $\tilde{N} \times \mathbb{R}$. Note that because of the invertibility of the Dirac operator on $N \times \mathbb{R}$ and our appropriate choice of the normalising function Ψ_1, we have on the nose

$$\tilde{q}^N(0)(1) = q_p(0),$$

where q^N is defined like q but using the Dirac operator on $\tilde{N} \times \mathbb{R}$. This finishes the proof of (2.5) and therefore of our main Theorem 2.5.1. □

Remark 2.5.3. We decided to present the details of the index constructions and proofs only for even dimensional manifolds.

The case of odd dimensional manifolds can easily be reduced to this case via a "suspension construction", as also done in [2]. More precisely, if we have an odd dimensional compact manifold M, we pass to the even dimensional manifold $M \times S^1$. Correspondingly, the covering space $\tilde{M}$ with action by $\pi_1(M)$ is replaced by $\tilde{M} \times \mathbb{R}$ with action of $\pi_1(M) \times \mathbb{Z}$.

It is now a standard result that we have Künneth isomorphisms for the K-theory groups relevant to us, in particular for a group homomorphism $\Lambda \to \Gamma$

$$K_0(C^*_d(\Gamma \times \mathbb{Z}, \Lambda \times \mathbb{Z})) \xrightarrow{\cong} K_0(C^*_d(\Gamma, \Lambda)) \oplus K_1(C^*_d(\Gamma, \Lambda)). \tag{2.9}$$

The ad hoc definition of the relative index $\mu(M, N) \in K_1(C^*_d(\pi_1(M), \pi_1(N))$, generalizing Definition 2.4.4 to odd dimensional M, is now just the image of $\mu([M \times S^1, N \times S^1])$ under the Künneth map (2.9) (and indeed, the K_0-component is zero).

Because positive scalar curvature of M implies positive scalar curvature of $M \times S^1$, Theorem 2.5.2 for odd dimensiona M follows from its version for the even dimensional $M \times S^1$.

In the same way, using Künneth and suspension isomorphisms for the whole diagram of Theorem 2.5.1 (using along the way e.g. [41, Section 5]), the statement and proof of Theorem 2.5.1 for odd dimensional M follows from the corresponding one for the even dimensional $M \times S^1$.

More systematically, Zeidler [41] develops a setup of Cl_n-linear Roe algebras and localisation algebras and Cl_n-equivariant Dirac operators on n-dimensional spin manifolds. Our constructions and arguments should carry through in this setup, given a uniform treatment for all dimensions, and working with real group C^*-algebras. As this requires a bit more notation and additional concepts, and as we were striving for a down to earth exposition, we decided to stick to the classical setup and leave it to the interested reader to work out the details of such an approach.

Chapter 3

A Variant of Roe Algebras for Spaces with Cylindrical Ends with Applications in Relative Higher Index Theory

3.1 Introduction

The question whether a given manifold admits a metric of positive scalar curvature has spurred much activity in recent years. One of the main approaches to partially answer this question is index theory. On a closed spin manifold M the Schrödinger-Lichnerowicz formula implies that the nonvanishing of the Fredholm index of the Dirac operator is an obstruction to the existence of positive scalar curvature metric. However, this does not tell the whole story, since there exist spin manifolds with vanishing Fredholm index of the Dirac operator, which however do not admit metrics with positive scalar curvature. One way to obtain more refined invariants from the Dirac operator is to not only consider the dimensions of its kernel and cokernel, but also to consider the action of the fundamental group on them. This gives rise to a higher index for the Dirac operator which is an element of the K-theory of the group C^*-algebra of the fundamental group. In general, one can associate a class in the K-homology of the manifold to the spin Dirac operator and the higher index is obtained as the image of this class under the index map

$$\mu^{\pi_1(M)} : K_*(M) \to K_*(C^*(\pi_1(M))).$$

The nonvanishing of the higher index gives an obstruction to the existence of positive scalar curvature metrics. In order to prove this one can use the fact that the index map fits in the Higson-Roe exact sequence

$$\ldots \to S_*^{\pi_1(M)}(M) \to K_*(M) \to K_*(C^*(\pi_1(M))) \to \ldots$$

and that the positivity of the scalar curvature allows the definition of a lift of the fundamental class in $S_*^{\pi_1(M)}(M)$. Given two positive scalar curvature metrics on M, one can also define an index difference in $K_{*+1}(C^*(\pi_1(M)))$. These secondary invariants can then also be used for classification of positive scalar curvature metrics up to concordance and bordism. More concretely, in [37] and [38] the authors use these invariants to prove concrete results on the size of the space of positive scalar metrics on closed manifolds.

In [2] Chang, Weinberger and Yu recently considered the question on compact spin manifolds with boundary. Let M be a compact spin manifold with boundary N. They constructed a relative index map

$$\mu^{\pi_1(M),\pi_1(N)} : K_*(M,N) \to K_*(C^*(\pi_1(M),\pi_1(N))),$$

where $K_*(M,N)$ and $C^*(\pi_1(M),\pi_1(N))$ denote the relative K-homology group and the so called relative group C^*-algebra. One can define a relative class for the Dirac operator on M in the relative K-homology group. The relative index is then the image of the latter relative class under the relative index map. Given a positive scalar curvature metric on M which is collared at the boundary, it was shown in [2] that the relative index vanishes. A general Riemannian metric which is collared at the boundary and has positive scalar curvature there, also defines an index in $K_*(C^*(\pi_1(M)))$, which vanishes if the metric has positive scalar curvature everywhere. It was shown in [4] and [34] that the latter index maps to the relative index under a certain group homomorphism. Apart from relating previously defined indices to the relative index, this fact also gives a conceptual proof that the relative index is an obstruction to the existence of positive scalar curvature metrics which are collared at the boundary.

The relative index map fits in an exact sequence

$$\ldots \to S_*^{\pi_1(M),\pi_1(N)}(M,N) \to K_*(M,N) \to K_*(C^*(\pi_1(M),\pi_1(N))) \to \ldots,$$

where $S_*^{\pi_1(M),\pi_1(N)}(M,N)$ is the relative analytic structure group and has different realisations. The main aim of the following paper is to answer the following natural question: given a positive scalar curvature metric, which is collared at the boundary, can one define a secondary invariant in $S_*^{\pi_1(M),\pi_1(N)}(M,N)$ which lifts the relative fundamental class and is useful for classification purposes? Using the machinery we develop in this paper, we will be able to answer the latter question in the positive. Furthermore, the same machinery allows us to define a higher index difference associated to positive scalar curvature metrics on manifolds with boundary. The definition of such secondary invariants paves the way for generalisations of the known results, such as those of [37] and [38], on the size of the space of positive scalar curvature metrics to manifolds with boundary.

Closely related to the question of existence and classification of positive scalar curvature metrics on manifolds with boundary which are collared at the boundary, is the question of existence and classification of positive scalar curvature metrics on manifolds with cylindrical ends, which are collared on the cylindrical end. The usual coarse geometric approach to index theory cannot be applied in this case, since the Roe algebras of spaces with cylindrical ends tend to have vanishing K-theory. We deal with this problem by introducing a variant of Roe algebras for such spaces with more interesting K-theory. The operators in the new Roe algebras are required to be asymptotically invariant in the cylindrical direction. Such operators can then be evaluated at infinity in a sense to be described later. Let X be a space with cylindrical end and denote by Y_∞ its cylindrical end. Let Λ and Γ be discrete groups and $\varphi : \Lambda \to \Gamma$ a group homomorphism. φ then induces a map $B\Lambda \to B\Gamma$ of the classifying spaces of the groups which we can assume to be injective. Given a map $(X, Y_\infty) \to (B\Gamma, B\Lambda)$ of pairs we construct a long exact sequence

$$\cdots \to K_*(C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to \cdots .$$

In the above sequence $\widetilde{X}$ denotes the Γ-cover of X associated to the map $X \to B\Gamma$ and $C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$ consists, roughly, of operators which are asymptotically invariant and whose evaluation at infinity results in operators admitting Λ-invariant lifts. For a spin manifold X we associate a fundamental class to the Dirac operator in $K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})$. The index of the Dirac operator on the manifolds with cylindrical end is then defined as the image of the latter class under the map $K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})$. Given a positive scalar curvature metric on X which is collared on Y_∞, we define a lift of the fundamental class in $K_*(C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})$, which proves that the nonvanishing of the new index is an obstruction to the existence of positive scalar metrics on X and paves the way for classification of such metrics. By removing Y_∞ we obtain a manifold with boundary, which we denote by $\overline{X}$. We prove that there is a commutative diagram of exact sequences

$$\begin{array}{ccccc}
K_*(C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \\
\downarrow & & \downarrow & & \downarrow \\
S^{\Gamma,\Lambda}_*(\overline{X}, \partial\overline{X}) & \longrightarrow & K_*(\overline{X}, \partial\overline{X}) & \longrightarrow & K_*(C^*(\Gamma,\Lambda)),
\end{array}$$

where the lower sequence is the relative Higson-Roe sequence mentioned above. Furthermore, we show that the fundamental class of $\widetilde{X}$ maps to the relative fundamental class under the middle vertical map. This shows that the relative index can be obtained from the new index defined in $K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})$ and allows us to define secondary invariants in $S_*^{\Gamma,\Lambda}(\overline{X},\partial\overline{X})$.

As another application of the machinery developed here we give a short proof the main statement of [34].

The paper is organised as follows. The second section is a very short reminder of the picture of K-theory for graded C^*-algebras due to Trout. In the third section we recall basic notions from coarse geometry and the coarse geometric approach to index theory on manifolds with and without boundary. In the fourth section we introduce variants of Roe algebras for spaces with cylindrical ends and cylinders and define the evaluation at infinity map, which plays an important role in the rest of the paper. In the final sections, we define indices for Dirac operators on manifolds with cylindrical ends and discuss applications to the existence and classification problem for metrics with positive scalar curvature on such manifolds. This is followed by a discussion of the relationship with the relative index for manifolds with boundary and a short proof of a statement on the relationship between the relative index and indices defined in the presence of a positive scalar curvature metric on the boundary.

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3.2 K-theory for Graded C^*-algebras

In this paper we will use the approach of Trout to K-theory of graded C^*-algebras. This description of K-theory was used by Zeidler in [41], where he proves product formulas for secondary invariants associated to positive scalar curvature metrics. We quickly recall the basics, and refer the reader to [41, Section 2] for more details.

Let H be a Real $\mathbb{Z}_2$-graded Hilbert space and denote by $\mathbb{K}$ the Real C^*-algebra of compact operators on H. The $\mathbb{Z}_2$-grading on H induces a $\mathbb{Z}_2$ grading on $\mathbb{K}$ by declaring the even and odd parts to be the set of operators preserving and exchanging the parity of vectors respectively. The Clifford algebra $\mathrm{Cl}_{n,m}$ will be the C^*-algebra generated by $\{e_1,\dots,e_n,\epsilon_1,\dots,\epsilon_m\}$ subject to the relations $e_ie_j+e_je_i=-2\delta_{ij}, \epsilon_i\epsilon_j+\epsilon_j\epsilon_i=2\delta ij, e_i\epsilon_j+\epsilon_je_i=0, e_i^*=-e_i$ and $\epsilon_i^*=\epsilon_i$. The Real structure and the $\mathbb{Z}_2$-grading of $\mathrm{Cl}_{n,m}$ are defined by declaring these generators to be real and odd. Denote by $\mathcal{S}$ the C^*-

algebra $C_0(\mathbb{R})$ endowed with a Real structure given by complex conjugation and a $\mathbb{Z}_2$-grading defined by declaring the odd and even parts to be the set of odd and even functions. Given Real, $\mathbb{Z}_2$-graded C^*-algebras A and B denote by $\mathrm{Hom}(A, B)$ the space of C^*-algebra homomorphism between A and B respecting the Real structures and the $\mathbb{Z}_2$-gradings, by $[A, B]$ the set $\pi_0(\mathrm{Hom}(A, B))$ and by $A\widehat{\otimes}B$ their maximal graded tensor product. The n-th K-theory group of the Real graded C^*-algebra A is defined to be

$$\widehat{K}_n(A) \coloneqq \pi_n(\mathrm{Hom}(\mathcal{S}, A\widehat{\otimes}\mathbb{K}))$$

and turns out to be isomorphic to $[\mathcal{S}, \Sigma^n A\widehat{\otimes}\mathbb{K}]$, where $\Sigma^n A$ denotes the n-th suspension of A. Any Real graded homomorphism of C^*-algebras $\varphi : \mathcal{S} \to A$ gives rise to a class $[\varphi] \coloneqq [\varphi\widehat{\otimes}e_{11}] \in \widehat{K}_0(A)$ with e_{11} some rank one projection.

Denote by $\mathcal{S}(-\epsilon, \epsilon)$ the Real graded C^*-subalgebra of $\mathcal{S}$ consisting of functions vanishing outside $(-\epsilon, \epsilon)$. For our discussion of secondary invariants we will make use of the fact that the inclusion $\mathcal{S}(-\epsilon, \epsilon) \to \mathcal{S}$ is a homotopy equivalence.

3.3 Roe Algebras and the Relative Index Map

Throughout this section X, Y and Z will denote locally compact metric spaces with bounded geometry.

3.3.1 Roe Algebras

Let Γ be a discrete group acting freely and properly on Z by isometries. Pulling back functions along the action gives rise to an action $\alpha : \Gamma \to \mathrm{Aut}(C_0(Z))$. Let (ρ, U) be an ample covariant representation of the C^*-dynamical system$(C_0(Z), \Gamma, \alpha)$ on a Hilbert space H. Here ample means that no non-zero element of $C_0(Z)$ acts as a compact operator. The space H will be referred to as a Z-module. We will also make use of Cl_n-linear Z-modules which are defined analogously by replacing the Hilbert space H with a Real, graded Hilbert Cl_n-module $\mathfrak{H}$ and by requiring the representation ρ to be by adjointable operators. In the following we will denote $\rho(f)$ simply by f.

Definition 3.3.1. An operator $T \in L(H)$ is called *locally compact* if for all $f \in C_0(Z)$ both Tf and fT are compact. T is called a *finite propagation operator* if there exists $R > 0$ with the property that $fTg = 0$ for all $f, g \in C_0(Z)$ with $\mathrm{dist}(\mathrm{supp}\, f, \mathrm{supp}\, g) > R$. The smallest such R is called the *propagation* of T and is denoted by $\mathrm{prop}\, T$. T is called Γ-equivariant if $T =$

$U_\gamma^* T U_\gamma$ for all $\gamma \in \Gamma$. Similarly, one defines the notions of local compactness and finite propagation for adjointable operators on $\mathfrak{H}$.

Definition 3.3.2. The *equivariant algebraic Roe algebra* is the $* - algebra$ of locally compact, finite propagation, Γ-equivariant operators on H and is denoted by $\mathbb{R}(Z)^\Gamma_\rho$. The *equivariant Roe algebra* is a C^*-completion of the algebraic Roe algebra and is denoted by $C^*_{(d)}(Z)^\Gamma_\rho$. Here (d) is a placeholder for the chosen completion. Similarly, one defines the Cl_n*-linear equivariant (algebraic) Roe algebra* by using finite propagation, locally compact and equivariant operators on $\mathfrak{H}$. These algebras will be denoted by $\mathbb{R}(Z; \mathrm{Cl}_n)^\Gamma_\rho$ and $C^*_{(d)}(Z; \mathrm{Cl}_n)^\Gamma_\rho$.

Remark 3.3.3. It follows from Proposition 3.3.9 below that the K-theory groups of the Roe algebra are independent of the chosen ample representation. We will therefore drop ρ form the notation.

Remark 3.3.4. Examples of possible completions are

- the reduced completion $C^*_{\mathrm{red}}(Z)^\Gamma$; i.e. the closure of $\mathbb{R}(Z)^\Gamma$ in $L(H)$,
- the maximal completion $C^*_{\max}(Z)^\Gamma$ obtained by taking the completion using the universal C^*-norm and
- the quotient completion $C^*_q(Z)^\Gamma$ introduced in [34].

In the following we will denote the Roe algebras obtained by the quotient completion simply by $C^*(Z)^\Gamma$ and $C^*(Z; \mathrm{Cl}_n)$. Most of what will follow will be valid for all of the above completions, however we will state all of our results only for the quotient completion.

Later in the paper we will introduce variants of Roe algebras which are suitable for spaces with cylindrical ends and show that the K-theory groups of these algebras define functors on a certain category of spaces. Our proofs of the functoriality of the K-theory of the new Roe algebras and their independence from the chosen ample modules makes use of the analogues of these results for the classical Roe algebras. Hence, we quickly recall the latter results in the following. Analogues of the results mentioned below hold for the Cl_n-linear versions of the algebras introduced and we will later make use of them.

Definition 3.3.5 (See [28, Chapter 2]). Let X and Y be locally compact separable proper metric spaces endowed with a free and proper action of a discrete group Γ by isometries. A map $f : X \to Y$ is called coarse if the inverse image of each bounded set of Y under f is bounded and for each $R > 0$ there exists $S > 0$ such that $d_X(x, x') < R$ implies $d_Y(f(x), f(x')) < S$.

Definition 3.3.6. Let X and Y be as in Definition 3.3.5. Let H and H' denote an X and Y-module respectively. The *support* of an operator $T : H \to H'$ is the complement of the union of all sets $V \times U \subset Y \times X$ with the property that $fTg = 0$ for all $f \in C_0(V)$ and $g \in C_0(U)$. It will be denoted by $\mathrm{Support}(T)$.

Definition 3.3.7. Let X and Y be as in Definition 3.3.5. Let $f : X \to Y$ be a coarse map. Let H and H' denote an X and Y-module respectively. An isometry $V : H \to H'$ is said to *cover* f if there exists an $R > 0$ such that $d_Y(f(x), y) < R$ for all $(y, x) \in \mathrm{Support}(T)$.

Lemma 3.3.8 ([16, Lemma 6.3.11])**.** *Let f, X, Y, H and H' be as in Definition 3.3.7. If an isometry V covers f, then $T \mapsto VTV^*$ defines a map from $\mathbb{R}(X)^\Gamma$ to $\mathbb{R}(Y)^\Gamma$, which extends to a map $C^*(X)^\Gamma \to C^*(Y)^\Gamma$.*

Proposition 3.3.9 ([16, Proposition 6.3.12])**.** *Let f, X, Y, H and H' be as in Definition 3.3.7. There exists an isometry which covers f and thus induces a map $K_*(C^*(X)^\Gamma) \to K_*(C^*(Y)^\Gamma)$. The latter map is independent of the choice of the isometry covering f. In particular, the $K_*(C^*(X)^\Gamma)$ is independent of the choice of the X-module up to a canonical isomorphism.*

For the rest of the section we consider a space Z with a chosen Z-module H. In the case the action of Γ on Z is cocompact we have the following

Proposition 3.3.10. *If the action of Γ on Z is cocompact, then $K_*(C^*(Z)^\Gamma) \cong K_*(C^*_q(\Gamma))$, where $C^*_q(\Gamma)$ is the quotient completion of the group ring of Γ as introduced in* [34]*.*

Proof. In the proof of [16, Lemma 12.5.3] an isomorphism $\mathbb{R}(X)^\Gamma \cong \mathbb{C}[\Gamma] \odot K(H')$ is given. Here $K(H')$ denotes the algebra of compact operators on a suitable Hilbert space H'. This isomorphism becomes an isometry if the left hand side is endowed with the norm of $C^*(X)^\Gamma$ and the right hand side is endowed with the norm of $C^*_q(\Gamma) \otimes K(H')$ and thus extends to an isomorphism of the latter two algebras. The claim then follows from the stability of K-theory. □

Given a Γ-invariant subset $S \subset Z$ it will be useful to look at the $*$-algebra of operators in $\mathbb{R}(Z)^\Gamma$ which are supported near S in the sense of the following

Definition 3.3.11. Given a subset $S \subset Z$, T is said to be supported near S if there exists an $R > 0$ with the property that $\mathrm{supp}\, T \subset U_R(S) \times U_R(S)$. Here $U_R(S)$ denotes the open R-neighbourhood of S.

Definition 3.3.12. Let S be a Γ-invariant subset of Z. The equivariant *algebraic Roe algebra of S relative to Z* is the subalgebra of $\mathbb{R}(Z)^\Gamma$ consisting of operators supported near S and will be denoted by $\mathbb{R}(S \subset Z)^\Gamma$. The equivariant *Roe algebra of S relative to Z* is the closure of $\mathbb{R}(S \subset Z)^\Gamma$ in $C^*(Z)^\Gamma$ and is denoted by $C^*(S \subset Z)^\Gamma$.

Since S is itself a Γ-space, it has its own Roe algebra. This is related to the Roe algebra of S relative to Z by the following

Proposition 3.3.13 ([17, Section 5, Lemma 1])**.** $K_*(C^*(S)^\Gamma) \cong K_*(C^*(S \subset Z)^\Gamma)$.

We will also need the notion of support of a vector in H.

Definition 3.3.14. Let $v \in H$. The support of v is the complement of the union of all open subsets U with the property that $fv = 0$ for all $f \in C_0(U)$.

3.3.2 Yu's Localisation Algebras

Given a C^*-algebra A denote by $\mathfrak{T}A$ the C^*-algebra of all uniformly continuous functions $f : [1, \infty) \to A$ endowed with the supremum norm.

Definition 3.3.15. The *equivariant localisation algebra* of Z is defined to be the C^*-subalgebra of $\mathfrak{T}C^*(Z)^\Gamma$ generated by elements f satisfying

- $\operatorname{prop} f(t) < \infty$ for all $t \in [1, \infty)$
- $\operatorname{prop} f(t) \xrightarrow{t\to\infty} 0$.

It will be denoted by $C_L^*(Z)^\Gamma$.

The K-theory of the localisation algebra provides a model for the equivariant locally finite K-homology. Yu constructed an isomorphism $\mathrm{Ind}_L : K_*^\Gamma(Z) \to K_*(C_L^*(Z)^\Gamma)$, where $K_*^\Gamma(Z)$ denotes the equivariant KK-group $KK_*^\Gamma(C_0(Z), \mathbb{C})$.

Definition 3.3.16. A Γ-cover Z of a locally compact metric space M is called *nice* if there exists an $\epsilon > 0$ such that the restriction of Z to every ϵ-ball in M is trivial.

Proposition 3.3.17. *Let $Z \to M$ be a nice Γ-cover. Then there is an isomorphism $K_*(C_L^*(Z)^\Gamma \cong K_*(C_L^*(M))$ induced by lifting operators on M with small propagation to equivariant operators on Z. In particular* Ind_L *gives rise to an isomorphism $K_*(M) \cong K_*(C_L^*(Z)^\Gamma)$.*

Remark 3.3.18. In the following we will assume all covers to be nice.

Given a Γ-invariant subset S of Z it will be useful to define the localisation algebra of S relative to Z.

Definition 3.3.19. The *equivariant localisation algebra of S relative to Z* is defined as the C^*-subalgebra of $C_L^*(Z)^\Gamma$ generated by elements f with the property that there exists a continuous function $B : [1,\infty) \to \mathbb{R}$ vanishing at infinity such that $\operatorname{prop} f(t) < B(t)$. It will be denoted by $C_L^*(S \subset Z)^\Gamma$.

Proposition 3.3.20 ([41, Lemma 3.7])**.** $K_*(C_L^*(S)^\Gamma) \cong K_*(C_L^*(S \subset Z)^\Gamma)$

Definition 3.3.21. The *equivariant structure algebra* of Z is the C^*-subalgebra of $C_L^*(Z)^\Gamma$ consisting of $C^*(Z)^\Gamma$-valued functions f on $[1,\infty)$ with $f(1) = 0$. It is denoted by $C_{L,0}^*(Z)^\Gamma$.

Given a Γ-cover $Z \to M$ induced by a map $M \to B\Gamma$, with M compact, the index map $\mu^\Gamma : K_*(M) \to K_*(C^*(\Gamma))$ can be defined by

$$K_*(M) \cong K_*(C_L^*(Z)^\Gamma) \xrightarrow{(\mathrm{ev}_1)_*} K_*(C^*(Z)^\Gamma) \cong K_*(C_q^*(\Gamma))$$

. Clearly, it fits into a long exact sequence

$$\ldots \to S_*^\Gamma(M) \to K_*(M) \to K_*(C^*(\Gamma)) \to \ldots,$$

where $S_*^\Gamma(M)$ denotes $K_*(C_{L,0}^*(Z)^\Gamma)$ and is called the *analytic structure group*. This long exact sequence is called the *Higson-Roe analytic surgery sequence*.

Fundamental Class of Dirac Operators

Now suppose that Z is an n-dimensional spin manifold. We assume that Γ acts by spin structure preserving isometries. Denote by $\not{\mathfrak{S}} = P_{\mathrm{Spin}}(Z) \times_{\mathrm{Spin}} \mathrm{Cl}_n$ the Cl_n-spinor bundle on Z. Recall that the Cl_n-linear Dirac operator on Z (acting on sections of $\not{\mathfrak{S}}$) gives rise to a class in $K_*(Z)^\Gamma$. Under the isomorphism of 3.3.17, this class corresponds to the class $[\not{D}_Z] \in \widehat{K}_0(C_L^*(Z; \mathrm{Cl}_n)^\Gamma) \cong K_n(C_L^*(Z)^\Gamma)$ defined by $\varphi_{\not{D}} : \mathcal{S} \to C_L^*(Z; \mathrm{Cl}_n)^\Gamma$ sending $f \in \mathcal{S}$ to $(t \mapsto f(\frac{1}{t}\not{D})) \in C_L^*(Z; \mathrm{Cl}_n)^\Gamma$.

3.3.3 The Relative Index Map

Let Λ and Γ be discrete groups and $\varphi : \Lambda \to \Gamma$ a group homomorphism. The homomorphism φ gives rise to a continuous map $B\varphi : B\Lambda \to B\Gamma$. It also induces a map $\varphi : C_{\max}^*(\Lambda) \to C_{\max}^*(\Gamma)$. We can and will assume that

$B\varphi$ is injective. Given a compact space X, a subset $Y \subset X$ and a map $f : (X, Y) \to (B\Gamma, B\Lambda)$ Chang, Weinberger and Yu ([2]) define a relative index map $\mu^{\Gamma,\Lambda} : K_*(X, Y) \to K_*(C^*_{\max}(\Gamma, \Lambda))$. Here $C^*_{\max}(\Gamma, \Lambda) \coloneqq SC_\varphi$ denotes the suspension of the mapping cone of φ and is called the (maximal) relative group C^*-algebra. If X is not compact, then their construction gives rise to a relative index map with target the K-theory group of a relative Roe algebra. Here, we quickly recall the construction of the relative index map. Denote by $\widetilde{X}$ and $\widetilde{Y}$ the Γ and Λ coverings of X and Y associated to f and $f\restriction Y$ respectively. Denote by Y' the restriction of $\widetilde{X}$ to Y. Using particular $\widetilde{X}, Y'$ and $\widetilde{Y}$-modules Chang, Weinberger and Yu construct a morphism of C^*-algebras

$$\psi : C^*_{\max}(\widetilde{Y})^\Lambda \to C^*_{\max}(Y')^{\frac{\Lambda}{\ker\phi}} \hookrightarrow C^*_{\max}(\widetilde{X})^\Gamma.$$

We will later discuss the morphism ψ in more detail. Applying ψ pointwise we obtain a morphism

$$\psi_L : C^*_{L,\max}(\widetilde{Y})^\Lambda \to C^*_{L,\max}(Y')^{\frac{\Lambda}{\ker\phi}} \hookrightarrow C^*_{L,\max}(\widetilde{X})^\Gamma.$$

Analogous to the absolute case, there is a map $\mathrm{Ind}_L^{\mathrm{rel}} : K_*(X, Y) \to K_*(SC_{\psi_L})$.

Proposition 3.3.22. $\mathrm{Ind}_L^{\mathrm{rel}}$ *is an isomorphism. If, furthermore, X is compact, then $K_*(SC_\psi) \cong K_*(C^*_{\max}(\Gamma, \Lambda))$.*

Evaluation at 1 gives rise to morphisms $\mathrm{ev}_1 : C^*_{L,\max}(\widetilde{Y})^\Lambda \to C^*_{\max}(\widetilde{Y})^\Lambda$ and $\mathrm{ev}_1 : C^*_{L,\max}(\widetilde{X})^\Gamma \to C^*_{\max}(\widetilde{X})^\Gamma$. The diagram

$$\begin{array}{ccc} C^*_{L,\max}(\widetilde{Y})^\Lambda & \xrightarrow{\mathrm{ev}_1} & C^*_{\max}(\widetilde{Y})^\Lambda \\ \downarrow{\psi_L} & & \downarrow{\psi} \\ C^*_{L,\max}(\widetilde{X})^\Gamma & \xrightarrow{\mathrm{ev}_1} & C^*_{\max}(\widetilde{X})^\Gamma \end{array}$$

is commutative. Hence, the evaluation at 1 maps give rise to a morphism $SC_{\psi_L} \to SC_\psi$, which we also denote by ev_1.

Definition 3.3.23. The *relative index map* $\mu^{\Gamma,\Lambda}$ is defined to be the composition

$$K_*(X, Y) \xrightarrow{\mathrm{Ind}_L^{\mathrm{rel}}} K_*(SC_{\psi_L}) \xrightarrow{(\mathrm{ev}_1)_*} K_*(SC_\psi).$$

Remark 3.3.24. If X is compact, the isomorphism $K_*(SC_\psi) \cong K_*(C^*_{\max}(\Gamma, \Lambda))$ allows us to consider $\mu^{\Gamma,\Lambda}$ as a map with values in the K-theory of the relative group C^*-algebra.

Remark 3.3.25. Instead of the maximal completion of the group rings and the Roe algebras, one can consider the quotient completion introduced in [34] and obtain a similar relative index map. If the group homomorphism $\varphi : \Lambda \to \Gamma$ is injective, then one can also use the reduced completion of the group rings and Roe algebras.

Analogous to the absolute case the relative index map fits into a long exact sequence. The map ψ_L gives rise, by restriction, to a map $\psi_{L,0} : C^*_{L,0}(\widetilde{Y})^\Lambda \to C^*_{L,0}(\widetilde{X})^\Gamma$. We have a short exact sequence of C^*-algebras

$$0 \to SC_{\psi_{L,0}} \to SC_{\psi_L} \xrightarrow{\mathrm{ev}_1} SC_\psi \to 0,$$

which gives rise to a long exact sequence of K-theory groups

$$\cdots \to K_*(SC_{\psi_{L,0}}) \to K_*(X,Y) \xrightarrow{\mu^{\Gamma,\Lambda}} K_*(SC_\psi) \to \cdots .$$

Remark 3.3.26. Similarly, one defines maps $C^*_{(L)}(\widetilde{Y};\mathrm{Cl}_n)^\Lambda \to C^*_{(L)}(\widetilde{X};\mathrm{Cl}_n)^\Gamma$, which we will also denote by $\psi_{(L)}$.

The Relative Index of Dirac Operators on Manifolds with Boundary

Given a compact spin manifold M with boundary N with a metric on M which is collared at the boundary, consider the manifold M_∞ obtained by attaching $N_\infty := N \times [0,\infty)$ to M along N. Extend the metric on M to a metric on M_∞ using the product metric on the half-cylinder (the metric on $\mathbb{R}_+$ is the usual one). Denote by $[\not{D}_{M_\infty}]$ the fundamental class of the Dirac operator on M_∞ in $K_*(M_\infty)$. Given a map $f : (M,N) \to (B\Gamma, B\Lambda)$, the construction of the previous section gives rise to a relative index map $\mu^{\Gamma,\Lambda} : K_*(M,N) \to K_*(C^*(\Gamma,\Lambda))$.

Definition 3.3.27. The relative index of the Dirac operator on M is defined to be the image of $[\not{D}_{M_\infty}]$ under the composition

$$K_*(M_\infty) \to K_*(M_\infty, N_\infty) \xrightarrow{\cong} K_*(M,N) \xrightarrow{\mu^{\Gamma,\Lambda}} K_*(C^*(\Gamma,\Lambda)).$$

where the isomorphism $K_*(M_\infty, N_\infty) \xrightarrow{\cong} K_*(M,N)$ is given by excision.

The nonvanishing of the relative index obstructs the existence of positive scalar curvature metrics on M.

Proposition 3.3.28 ([2, Proposition 2.18],[34, Theorem 5.1],[4, Theorem 4.12])**.** *If there exists a positive scalar curvature metric on M which is collared at the boundary, then the relative index of the Dirac operator on M vanishes.*

3.4 Coarse Spaces with Cylindrical Ends

Let X be a locally compact metric space with a free and proper action of a discrete group Γ by isometries. For a Γ-invariant subset Y of X we can endow $Y \times \mathbb{R}$ with a Γ-action by setting $\gamma(y,t) = (\gamma y, t)$.

Definition 3.4.1. Let X and Y be as above. The space X is said to have a cylindrical end with base Y if there exists a Γ-equivariant isometry $\iota : Y \times [0,\infty) \to X$ satisfying

- $\iota((y,0)) = y$
- $\lim_{R\to\infty} \operatorname{dist}(\iota(Y \times [R,\infty)), X - Y_\infty) = \infty$

Here Y_∞ denotes $\iota(Y \times [0,\infty))$ and $Y \times [0,\infty)$ is endowed with the product metric.

Definition 3.4.2. Let (X, Y, ι) and (X', Y', ι') be spaces with cylindrical ends. a map $f : X \to X'$ is called a coarse map of spaces with cylindrical ends if it is a coarse map and satisfies

- $f(X \setminus Y_\infty) \subset X' \setminus Y'_\infty$ and
- $f(\iota(y,t)) = \iota'(g(y),t)$ with $g := f\restriction Y$.

3.4.1 Roe Algebras for Spaces with Cylindrical Ends

Using the isometry ι one can define an action of $\mathbb{R}_+$ on $C_0(Y_\infty)$ by setting $L_s(f)(\iota((y,t))) = f(\iota(y,t-s))$ for $s \in \mathbb{R}_+$. We would like to define a variant of Roe algebras for spaces with cylindrical ends. In order to do this we use modules which are equipped with an action of $\mathbb{R}_+$ by partial isometries, which is compatible with the action of $\mathbb{R}_+$ on $C_0(Y_\infty)$. Before making this precise we introduce some notation. Let H_Y be a Y-module. The Hilbert space $L^2(\mathbb{R}_+; H_Y)$ can be endowed with the structure of Y_∞-module in a natural way. On $L^2(\mathbb{R}_+; H_Y)$ one can define a family of partial isometries P_s^{st} by $P_s^{\mathrm{st}}(f)(t) = f(t-s)$ for $t \geq s$ and $P_s^{\mathrm{st}}(f)(t) = 0$ otherwise.

Definition 3.4.3. Let (X, Y, ι) be a space with cylindrical end. A Hilbert space is called an X-module tailored to the end if there is a tuple $(\rho, U, \{P_s\})$ satisfying the following properties:

- (ρ, U) is a covariant ample representation of $C_0(X)$ on H.
- P_s is a strongly continuous family of partial isometries on H satisfying

- $P_{-s} = P_s^*$
- $P_s^* P_s = \widetilde{\rho}(\chi_{\iota(Y\times[0,\infty))})$ for all $s > 0$
- $P_s P_s^* = \widetilde{\rho}(\chi_{\iota(Y\times[s,\infty))})$ for all $s > 0$
- $\rho(f)P_s = P_s\rho(L_s(f))$ for all $f \in C_0(Y_\infty)$.

- For some Y-module H_Y, there is a Γ-equivariant unitary: $W : \chi_{Y_\infty} H \to L^2(\mathbb{R}_+; H_Y)$ which covers the identity and satisfies $WP_s = P_s^{\mathrm{st}}W$.

Here, the tuple $(\rho, U, \{P_s\})$ is part of the structure of the X-module and $\widetilde{\rho}$ is the extension of the representation ρ to the bounded Borel functions.

Similarly, one can define Cl_n-linear modules tailored to the end. The following definitions generalise in an obvious manner to the Cl_n-linear context. In the rest of the section (X, Y, ι) will be a space with cylindrical end (endowed with a Γ-action) and H will denote an X-module tailored to the end. We will construct a variant of Roe algebras for spaces with cylindrical ends. Since H is in particular an X-module, it can be used to construct the usual equivariant algebraic Roe algebra $\mathbb{R}(X)^\Gamma$.

Definition 3.4.4. An operator $T \in L(H)$ is called *asymptotically* $\mathbb{R}_+$-*invariant* if

$$\lim_{R\to\infty} \sup_{s>0} ||(P_{-s}TP_s - T)\chi_{\iota(Y\times[R,\infty))}|| = 0.$$

Lemma 3.4.5. *The set of operators in* $\mathbb{R}(X)^\Gamma$*, which are asymptotically* $\mathbb{R}_+$*-invariant is a* $*$*-subalgebra.*

Proof. Let $S, T \in \mathbb{R}(X)^\Gamma$ be asymptotically $\mathbb{R}_+$-invariant. Set $R_0 := \operatorname{prop} T$. In the following χ_R will denote $\chi_{\iota(Y\times[R,\infty))}$. Since $P_sP_{-s} = \chi_s$ (for all $s > 0$) and elements in the image of $TP_s\chi_R$ are supported in $\iota(Y \times [R - R_0 + s, \infty))$ we have for $R > R_0$

$$(P_{-s}STP_s)\chi_R = (P_{-s}SP_sP_{-s}TP_s)\chi_R.$$

Furthermore, since elements in the image of $(P_{-s}TP_s)\chi_R$ are supported in $\iota(Y \times [R - R_0, \infty))$ we have

$$(P_{-s}SP_sP_{-s}TP_s)\chi_R = (P_{-s}SP_s\chi_{R-R_0}P_{-s}TP_s)\chi_R.$$

From the asymptotic $\mathbb{R}_+$-invariance, it follows that $P_{-s}SP_s\chi_{R-R_0} = S\chi_{R-R_0} + E_{R-R_0,s}(S)$ and $P_{-s}TP_s\chi_R = T\chi_R + E_{R,s}(T)$ with

$$\lim_{R\to\infty} \sup_{s>0} ||E_{R-R_0,s}(S)|| = 0 = \lim_{R\to\infty} \sup_{s>0} ||E_{R,s}(T)||. \qquad (*)$$

Therefore $(P_{-s}STP_s - ST)\chi_R$ is equal to

$$S\chi_{R-R_0}T\chi_R + S\chi_{R-R_0}E_{R,s}(T) + E_{R-R_0,s}(S)T\chi_R + E_{R-R_0,s}(S)E_{R,s}(T) - ST\chi_R$$

$$= S\chi_{R-R_0}E_{R,s}(T) + E_{R-R_0,s}(S)T\chi_R + E_{R-R_0,s}(S)E_{R,s}(T).$$

The latter equality and $(*)$ imply that ST is asymptotically $\mathbb{R}_+$-invariant. We now show that T^* is also asymptotically $\mathbb{R}_+$-invariant. We have

$$(P_{-s}T^*P_s - T^*)\chi_R = (\chi_R(P_{-s}TP_s - T))^*.$$

Furthermore, since the propagation of T is R_0 the right hand side is equal to $(\chi_R(P_{-s}TP_s - T)\chi_{R-R_0})^* = (\chi_R E_{R-R_0,s}(T))^*$. This shows that T^* is asymptotically $\mathbb{R}_+$-invariant. The fact that the set of asymptotically $\mathbb{R}_+$-invariant operators is closed under addition is clear. □

Definition 3.4.6. The equivariant algebraic Roe algebra of X tailored to the end is the $*$-subalgebra of $\mathbb{R}(X)^\Gamma$ consisting of asymptotically $\mathbb{R}_+$-invariant operators. It will be denoted by $\mathbb{R}(X)^{\Gamma,\mathbb{R}_+}$. The Roe algebra of X tailored to the end is the closure of $\mathbb{R}(X)^{\Gamma,\mathbb{R}_+}$ in $C^*_{(d)}(X)^\Gamma$ and will be denoted by $C^*_{(d)}(X)^{\Gamma,\mathbb{R}_+}$. Similarly, using a Cl_n-module tailored to the end, one defines $\mathbb{R}(X;\mathrm{Cl}_n)^{\Gamma,\mathbb{R}_+}$ and $C^*_{(d)}(X;\mathrm{Cl}_n)^{\Gamma,\mathbb{R}_+}$.

Remark 3.4.7. Note that the algebraic and C^*-algebraic Roe algebras defined above depend, a priori, on the chosen modules tailored to the end. We will see later, that the K-theory groups of the C^*-algebras defined using different modules are canonically isomorphic.

Remark 3.4.8. The equivariant Roe algebra of X tailored to the end obtained by using the quotient completion will simply be denoted by $C^*(X)^{\Gamma,\mathbb{R}_+}$. In the following we will only make use of the quotient completion; however, most of the results are also valid for the reduced and maximal completions.

Let (X', Y', ι') be another space with a cylindrical end and H' an X'-module tailored to the end given by the data $(\rho', U', \{P'_s\})$.

Definition 3.4.9. Let $f : X \to X'$ be a map of spaces with cylindrical ends. An isometry $V : H \to H'$ is said to cover f if it covers f in the sense of [16, Definition 6.3.9] and satisfies $VP_s = P'_sV$.

Lemma 3.4.10. *Let f and V be as in Definition 3.3.7. Then $T \to VTV^*$ defines a map $C^*(X)^{\Gamma,\mathbb{R}_+} \to C^*(X')^{\Gamma,\mathbb{R}_+}$.*

Proof. The fact that conjugation by V gives a map $C^*(X)^\Gamma \to C^*(X')^\Gamma$ is the content of [16, Lemma 6.3.11]. We show that if $T \in \mathbb{R}(X)^\Gamma$ is asymptotically $\mathbb{R}_+$-invariant, then so is VTV^*. In the following $\widetilde{\rho}$ and $\widetilde{\rho}'$ will denote the extension of ρ and ρ' to the bounded Borel functions on X and X' respectively. Using the fact that V intertwines the families $\{P_s\}$ and $\{P'_s\}$ we get

$$(P'_{-s}VTV^*P'_s - VTV^*)\rho'(\chi_R) = V(P_{-s}TP_s - T)V^*\rho'(\chi_R) =$$

$$V(P_{-s}TP_s - T)V^*P'_sP'_{-s} = V(P_{-s}TP_s - T)P_sP_{-s}V^* = V(P_{-s}TP_s - T)\widetilde{\rho}(\chi_R)V^*,$$

which proves the claim. □

Proposition 3.4.11. *Let $f : X \to X'$ be a map of spaces with cylindrical ends. Then there is an isometry $V : H \to H'$ which covers f. Conjugation by V induces a homomorphism $K_*(C^*(X)^{\Gamma,\mathbb{R}_+}) \to K_*(C^*(X')^{\Gamma,\mathbb{R}_+})$ which does not depend on the choice of the covering isometry V. In particular, $K_*(C^*(X)^{\Gamma,\mathbb{R}_+})$ does not depend on the choice of the X-module tailored to the end up to a canonical isomorphism.*

Proof. We prove the existence of an isometry covering f. The proof that the induced map on the K-theory groups by conjugation with V does not depend on the choice of V is the same as that of [16, Lemma 5.2.4]. We have $H \cong \chi_{X\setminus Y_\infty}H \oplus \chi_{Y_\infty}H \cong \chi_{X\setminus Y_\infty}H \oplus (H_Y \otimes L^2(\mathbb{R}_+))$. Similarly $H' \cong \chi_{X'\setminus Y'_\infty}H' \oplus (H'_{Y'} \otimes L^2(\mathbb{R}_+))$. By Proposition 3.3.9, there are isometries $V_1 : \chi_{X\setminus Y_\infty}H \to \chi_{X\setminus Y_\infty}H'$ and $V_2 : H_Y \to H'_{Y'}$ covering the restrictions of f to $X \setminus Y_\infty$ and Y respectively. We use the above decompositions of H and H' and set $V = V_1 \oplus (V_2 \otimes \mathrm{Id})$. Since the isomorphisms $\chi_{Y_\infty}H \cong H_Y \otimes L^2(\mathbb{R}_+)$ and $\chi_{Y'_\infty}H' \cong H'_{Y'} \otimes L^2(\mathbb{R}_+)$ cover the identity maps on Y_∞ and Y'_∞ respectively, V, seen as an isometry from H to H', covers f in the sense of Definition 3.3.7. Furthermore, the latter isomorphisms intertwine the families $\{P_s\}$ and $\{P'_s\}$ with the standard families of partial isometries $\{P_s^{\mathrm{st}}\}$ on $H_Y \otimes L^2(\mathbb{R}_+)$ and $H_{Y'} \otimes L^2(\mathbb{R}_+)$, which implies that V intertwines $\{P_s\}$ and $\{P'_s\}$. Thus, V covers f in the sense of Definition 3.4.9. □

One can also define localisation and structure algebras tailored to the end.

Definition 3.4.12. The *equivariant localisation algebra* of X tailored to the end is defined to be the C^*-algebra of $\mathfrak{T}C^*(X)^{\Gamma,\mathbb{R}_+}$ generated by elements f satisfying

- $\operatorname{prop} f(t) < \infty$ for all $t \in [1, \infty)$
- $\operatorname{prop} f(t) \xrightarrow{t\to\infty} 0$.

It will be denoted by $C^*_L(X)^{\Gamma,\mathbb{R}_+}$. The *equivariant structure algebra* of X is defined to be the subalgebra of $C^*_L(X)^{\Gamma,\mathbb{R}_+}$ generated by f which further satisfy $f(1) = 0$. It will be denoted by $C^*_{L,0}(X)^{\Gamma,\mathbb{R}_+}$.

Remark 3.4.13. One can also prove the existence of families of isometries covering a given map in a suitable sense and inducing maps between localisation and structure algebras tailored to the end. One can then deduce an analogue of Proposition 3.4.11 for structure and localisation algebras tailored to the end. These statements can be proved by using the approach of the proof of Proposition 3.4.11 and slight modifications of the proofs for the classical structure and localisation algebras.

3.4.2 Roe algebras for Cylinders

One of our main goals in the following is to evaluate asymptotically $\mathbb{R}_+$-invariant operators on a space (X, Y, ι) with cylindrical end and obtain $\mathbb{R}$-invariant operators on the cylinder over Y. In this section we define a Roe algebra for cylinders which will be the target of the aforementioned "evaluation at infinity map". In the following Y will denote a locally compact separable metric space endowed with a free and proper action of a discrete group Γ by isometries. Endow $Y \times \mathbb{R}$ with the product metric. Furthermore, $L'_s(f)(y,t) = f(y, t-s)$ defines an action of $\mathbb{R}$ on $C_0(Y \times \mathbb{R})$. Let H_Y be a Y-module. The space $L^2(\mathbb{R}, H_Y)$ can then be endowed with the structure of a $Y \times \mathbb{R}$-module. There is a family $\{Q^{\mathrm{st}}_s\}$ of unitaries on $L^2(\mathbb{R}, H_Y)$ given by the shift of functions in the $\mathbb{R}$-direction

Definition 3.4.14. A Hilbert space H is called a cylindrical $Y \times \mathbb{R}$-module if there is a tuple $(\rho, U, \{Q_s\})$ satisfying the following properties:

- (ρ, U) is a covariant ample representation of $C_0(Y \times \mathbb{R})$ on H.
- $\{Q_s\}$ is a strongly continuous group of unitaries commuting with the representation U of Γ on H and satisfying $\rho(f)Q_s = Q_s\rho(L'_s(f))$.
- For some Y-module H_Y, there is a unitary isomorphism $W : H \to L^2(\mathbb{R}, H_Y)$ which covers the identity map of $Y \times \mathbb{R}$ in the sense of Definition 3.3.7, intertwines the families $\{Q_s\}$ and $\{Q^{\mathrm{st}}_s\}$ and which does not shift the support of vectors in the $\mathbb{R}$-direction.

A cylindrical $Y \times \mathbb{R}$-modules is in particular a $Y \times \mathbb{R}$-module and allows us to define the usual Roe algebras $\mathbb{R}(Y \times \mathbb{R})^\Gamma$ and $C^*(Y \times \mathbb{R})^\Gamma$

Definition 3.4.15. An operator $T \in \mathbb{R}(Y \times \mathbb{R})^\Gamma$ is called $\mathbb{R}$-*invariant* if

$$Q_{-s}TQ_s - T = 0$$

for all $s \in \mathbb{R}$. The closure of the $*$-algebra of such elements in $C^*(Y \times \mathbb{R})^\Gamma$ will be denoted by $C^*(Y \times \mathbb{R})^{\Gamma \times \mathbb{R}}$. Similarly, using a cylindrical $Y \times \mathbb{R}$-Cl_n-module, one defines $C^*(Y \times \mathbb{R}; \mathrm{Cl}_n)^{\Gamma, \mathbb{R}_+}$.

Now let Y' be another space. Let $f : Y \times \mathbb{R} \to Y' \times \mathbb{R}$ be a coarse map, which is the suspension of a map $g : Y \to Y'$. Let H and H' be cylindrical $Y \times \mathbb{R}$ and $Y' \times \mathbb{R}$-modules respectively. A slight modification of the proof of Proposition 3.4.11, proves the following

Proposition 3.4.16. *Let f, H and H' be as above. There exists an isometry $V : H \to H'$ which covers f in the sense of Definition 3.3.7 and intertwines the families $\{Q_s\}$ and $\{Q'_s\}$. Conjugation by V induces a homomorphism $K_*(C^*(Y \times \mathbb{R})^{\Gamma \times \mathbb{R}}) \to K_*(C^*(Y' \times \mathbb{R})^{\Gamma \times \mathbb{R}})$. The latter homomorphism is independent of the choice of the isometry V satisfying the above properties. In particular, $K_*(C^*(Y \times \mathbb{R})^{\Gamma \times \mathbb{R}})$ does not depend on the chosen cylindrical $Y \times \mathbb{R}$-module.*

3.4.3 The Evaluation at Infinity Map

Let (X, Y, ι) be a space with cylindrical end on which Γ acts as above. Asymptotically $\mathbb{R}_+$-invariant operators can be "evaluated at infinity" in the sense of Propositions 3.4.19 and 3.4.20 to give $\mathbb{R}$-invariant operators on $Y \times \mathbb{R}$. In order to do this we first introduce the notion of (X, Y, ι) modules, which is given by a pair consisting of an X-module tailored to the end and a cylindrical $Y \times \mathbb{R}$-module which are related in a special way.

Definition 3.4.17. Let (X, Y, ι) be a space with cylindrical end. A pair (H, H') of Hilbert spaces is called a (X, Y, ι)-*module*, if there is a tuple $(\rho, \rho', U, U', \{P_s\}, \{Q_s\}, i)$ satisfying the following properties:

- $(\rho, U, \{P_s\})$ and $(\rho', U', \{Q_s\})$ endow H and H' with the structure of an X-module tailored to the end and a cylindrical $Y \times \mathbb{R}$-module respectively.

- i is a unitary $\chi_{Y_\infty} H \to \chi_{Y \times \mathbb{R}_+} H'$ intertwining the Γ-representations and the representations of $C_0(Y_\infty)$ and $C_0(Y \times \mathbb{R}_+)$ on $\chi_{Y_\infty} H$ and $\chi_{Y \times \mathbb{R}_+} H'$ respectively.

- $Q_s \circ i = i \circ P_s\restriction_{\chi_{Y_\infty} H}$ for all $s > 0$.

Remark 3.4.18. Note that $\rho'(f)Q_s = Q_s\rho'(L'_s(f))$ in particular implies that Q_s applied to vectors in H' which are supported in $Y \times [R,\infty)$, results in vectors with support in $Y \times [R+s,\infty)$. This observation will be used in the proof of Proposition 3.4.19.

In the following we will call an element $v \in H'$ compactly supported if its support in the sense of Definition 3.3.14 is a compact subset of $Y \times \mathbb{R}$. The nondegeneracy of ρ' implies that compactly supported vectors are dense in H'.

Proposition 3.4.19. *For $T \in \mathbb{R}(\widetilde{X})^{\Gamma,\mathbb{R}_+}$ and a compactly supported vector $v \in H'$ the limit $T^\infty v := \lim_{s\to\infty} Q_{-s}iTi^*Q_s v$ exists in H' and the mapping $v \mapsto T^\infty v$ extends to a continuous linear map T^∞ on H'. Furthermore, the operator T^∞ defined in this way is an element of $\mathbb{R}(Y\times\mathbb{R})^{\Gamma\times\mathbb{R}}$.*

Proof. In the following χ_R will denote $\chi_{\iota(Y\times[R,\infty))}$ and will be seen as an operator on H. χ'_R will denote $\chi_{Y\times[R,\infty)}$ and will act as on operator on H'.

- The limit exists: for $\epsilon > 0$ choose $\widetilde{R}$ such that $\sup_{s>0}||(P_{-s}TP_s - T)\chi_R|| < \epsilon$ for all $R \geq \widetilde{R}$. Let $\widetilde{s}$ be such that $Q_s(v)$ is supported on $Y \times \mathbb{R}_+$ for all $s \geq \widetilde{s}$. Set $s_0 = \widetilde{R}+\widetilde{s}$. Then we have
$$||Q_{s_0+s}^{-1}iTi^*Q_{s_0+s}v - Q_{s_0}^{-1}iTi^*Q_{s_0}v|| = ||Q_{s_0}^{-1}(Q_s^{-1}iTi^*Q_s - iTi^*)Q_{s_0}v||.$$
Note that $(Q_s^{-1}iTi^*Q_s - iTi^*)Q_{s_0}v = i(P_{-s}TP_s - T)\chi_{\widetilde{R}}i^*Q_{s_0}v$; hence
$$||Q_{s_0}^{-1}(Q_s^{-1}iTi^*Q_s - iTi*)Q_{s_0}v|| < ||(P_{-s}TP_s - T)\chi_{\widetilde{R}}||||v||,$$
where we use that Q_{s_0} is a unitary. The latter inequality shows that $\{Q_s^{-1}iTi^*Q_sv\}_{s\geq\widetilde{s}}$ is a Cauchy net and thus has a limit.

- T^∞ is a bounded operator on H': we clearly have $||T^\infty v|| \leq ||T||||v||$ for all compactly supported v which shows that $v \mapsto T^\infty v$ is a bounded operator on the dense subspace of compactly supported vectors in H' and thus extends to a bounded operator on H'.

- T^∞ is an $\mathbb{R}$ and Γ-invariant operator: for $t \in \mathbb{R}$ we have
$$Q_{-t}T^\infty Q_t v = Q_{-t}(\lim_{s\to\infty} Q_{-s}iTi^*Q_sQ_tv) = \lim_{s\to\infty} Q_{-s-t}iTi^*Q_{s+t}v =$$
$$\lim_{s\to\infty} Q_{-s}iTi^*Q_sv = T^\infty v$$
for all compactly supported v. Therefore $Q_{-t}T^\infty Q_t = T^\infty$. A similar computation and the fact that the $\mathbb{R}$-action and the Γ-action on H' commute proves the Γ-invariance.

- T^∞ is locally compact: We show that for $\psi \in C_c(Y \times \mathbb{R})$ ψT^∞ is compact. The proof of the compactness of $T^\infty \psi$ is similar and even more straightforward. There exists $M > 0$ such that the support of ψ is contained in $Y \times [-M, \infty)$. Set $R_0 := \operatorname{prop} T$. If v is compactly supported with support in $Y \times (-\infty, -M-R_0)$, then $\psi Q_{-s} i T i^* Q_s v = 0$. We thus have a commutative diagram

$$\begin{array}{ccc} H' & \xrightarrow{\psi T^\infty} & H' \\ \big\downarrow{\scriptstyle \chi'_{-M-R_0}} & & \big\uparrow \\ \chi'_{-M-R_0} H' & \xrightarrow{\psi T^\infty} & \chi'_{-M-R_0} H'. \end{array}$$

Therefore, it suffices to show that the restriction of ψT^∞ to $\chi'_{-M-R_0} H'$ is compact. First we show that $\{\chi'_{-M-R_0} Q_{-s} i T i^* Q_s \chi'_{-M-R_0}\}_{s \geq M+R_0}$ is a norm convergent net of operators on $\chi'_{-M-R_0} H'$. Set $s_1 := \widetilde{R} + M + R_0$. Then, similar to the above computation, we have

$$||\chi'_{-M-R_0} Q^{-1}_{s_1+s} i T i^* Q_{s_1+s} \chi'_{-M-R_0} - \chi'_{-M-R_0} Q^{-1}_{s_1} i T i^* Q_{s_1} \chi'_{-M-R_0}|| =$$
$$||\chi'_{-M-R_0} Q^{-1}_{s_1} (Q^{-1}_s i T i^* Q_s - i T i^*) Q_{s_1} \chi'_{-M-R_0}||.$$

Furthermore,

$$(Q^{-1}_s i T i^* Q_s - i T i^*) Q_{s_1} \chi'_{-M-R_0} = i(P_{-s} T P_s - T) \chi_{\widetilde{R}} i^* Q_{s_1} \chi'_{-M-R_0},$$

which implies

$$||\chi'_{-M-R_0} Q^{-1}_{s_1} (Q^{-1}_s i T i^* Q_s - i T i^*) Q_{s_1} \chi'_{-M-R_0}|| < \epsilon.$$

Hence, $\{\psi \chi'_{-M-R_0} Q_{-s} i T i^* Q_s \chi'_{-M-R_0}\}_{s \geq M+R_0}$ is norm convergent and converges strongly to ψT^∞ in $L(\chi'_{-M-R_0} H')$. Thus ψT^∞ restricted to $\chi'_{-M-R_0} H'$ is actually the norm limit of

$$\psi \chi'_{-M-R_0} Q_{-s} i T i^* Q_s \chi'_{-M-R_0} = \chi'_{-M-R_0} Q_{-s} L'_s(\psi) i T i^* Q_s \chi'_{-M-R_0}$$
$$= \chi'_{-M-R_0} Q_{-s} i L_s(\psi) T i^* Q_s \chi'_{-M-R_0}$$

as s tends to infinity. The compactness of $\psi T^\infty \restriction_{\chi'_{-M-R_0} H'}$ then follows from that of $L_s(\psi) T$.

□

Proposition 3.4.20. *The map* $\mathrm{ev}_\infty : \mathbb{R}(\widetilde{X})^{\Gamma, \mathbb{R}_+} \to \mathbb{R}(Y \times \mathbb{R})^{\Gamma \times \mathbb{R}}$ *given by* $T \mapsto T^\infty$ *is continuous if the domain and target space are endowed with the norms of* $C^*(\widetilde{X})^{\Gamma, \mathbb{R}_+}$ *and* $C^*(Y \times \mathbb{R})^{\Gamma \times \mathbb{R}}$ *respectively. Thus it gives rise to a morphism of* C^**-algebras* $\mathrm{ev}_\infty : C^*(\widetilde{X})^{\Gamma, \mathbb{R}_+} \to C^*(Y \times \mathbb{R})^{\Gamma \times \mathbb{R}}$.

Proof. If we endow $\mathbb{R}(\widetilde{X})^{\Gamma,\mathbb{R}_+}$ with the reduced norm, the continuity of the map $\mathrm{ev}_\infty : \mathbb{R}(\widetilde{X})^{\Gamma,\mathbb{R}_+} \to C^*_{\mathrm{red}}(Y\times\mathbb{R})^{\Gamma\times\mathbb{R}}$ follows from the proof of the previous proposition. Indeed, we already saw that this map is a contraction. The continuity of this map for the quotient completion follows from its continuity for the reduced completion, the commutativity of the the diagram

$$\begin{array}{ccc} \mathbb{R}(X)^{\Gamma,\mathbb{R}_+} & \xrightarrow{\mathrm{ev}_\infty} & \mathbb{R}(Y\times\mathbb{R})^{\Gamma\times\mathbb{R}} \\ \downarrow & & \downarrow \\ \mathbb{R}(X/N)^{\Gamma/N,\mathbb{R}_+} & \xrightarrow{\mathrm{ev}_\infty} & \mathbb{R}(Y/N\times\mathbb{R})^{\Gamma/N\times\mathbb{R}} \end{array}$$

for all normal subgroups N of Γ and the definition of the quotient completion in [34, Section 4]. It remains to show that it is a morphism of $*$-algebras. Let S and T be in $\mathbb{R}(\widetilde{X})^{\Gamma,\mathbb{R}_+}$ and let $v \in H'$ be compactly supported. We have

$$(TS)^\infty v = \lim_s Q_s^{-1} iTSi^* Q_s v = \lim_s Q_s^{-1} iTi^* Q_s Q_s^{-1} iSi^* Q_s v =$$

$$\lim_s Q_s^{-1} iTi^* Q_s (S^\infty v + E(s)) = T^\infty(S^\infty v).$$

The last equality follows from the fact that $||Q_s^{-1} iTi^* Q_s(E(s))|| \leq ||T||||E(s)||$. The rest is clear. □

Thus an (X,Y,ι)-module allows us to define an evaluation at infinity map. Next we will prove a functoriality result, which in particular shows that the induced map on K-theory is independent of the chosen (X,Y,ι)-module. Let $(\widehat{X},\widehat{Y},\widehat{\iota})$ be another space with cylindrical end (and a Γ-action). Let $(\widehat{H},\widehat{H}')$ be an $(\widehat{X},\widehat{Y},\widehat{\iota})$-module. Let $f : (X,Y,\iota) \to (\widehat{X},\widehat{Y},\widehat{\iota})$ be a map of spaces with cylindrical ends. In particular, the suspension of the restriction of f to Y defines a map $Y\times\mathbb{R} \to \widehat{Y}\times\mathbb{R}$. In this situation we have the following

Proposition 3.4.21. *There are isometries $V : H \to \widehat{H}$ and $V' : H' \to \widehat{H'}$ which satisfy the conditions of Definition 3.4.9 and Proposition 3.4.16 respectively and which make the diagram*

$$\begin{array}{ccc} C^*(X)^{\Gamma,\mathbb{R}_+} & \xrightarrow{\mathrm{ev}_\infty} & C^*(Y\times\mathbb{R})^{\Gamma\times\mathbb{R}} \\ \downarrow{\scriptstyle \mathrm{Ad}_V} & & \downarrow{\scriptstyle \mathrm{Ad}_{V'}} \\ C^*(\widehat{X})^{\Gamma,\mathbb{R}_+} & \xrightarrow{\mathrm{ev}_\infty} & C^*(\widehat{Y}\times\mathbb{R})^{\Gamma\times\mathbb{R}} \end{array}$$

commutative. In particular, the map $(ev_\infty)_* : K_*(C^*(X)^{\Gamma,\mathbb{R}_+}) \to K_*(C^*(Y\times\mathbb{R})^{\Gamma\times\mathbb{R}})$ does not depend on the choice of the (X,Y,ι)-module up to the usual canonical isomorphisms.

Proof. Let $V' : H' \to \widehat{H'}$ satisfy the conditions of Proposition 3.4.16 and such that V' and $(V')^*$ map vectors which are supported in $Y \times \mathbb{R}_+$ and $\widehat{Y} \times \mathbb{R}_+$ to vectors which are supported in $\widehat{Y} \times \mathbb{R}_+$ and $Y \times \mathbb{R}_+$ respectively. We have decompositions $H \cong \chi_{X \setminus Y_\infty} H \oplus \chi_{Y_\infty} H$ and $\widehat{H} = \chi_{\widehat{X} \setminus \widehat{Y}_\infty} \widehat{H} \oplus \chi_{\widehat{Y}_\infty} \widehat{H}$. Using these decompositions we define V to be the isometry

$$\begin{pmatrix} V_1 & 0 \\ 0 & \widehat{i}^* V' i \end{pmatrix},$$

where $V_1 : \chi_{X \setminus Y_\infty} H \oplus \chi_{Y_\infty} H \to \chi_{\widehat{X} \setminus \widehat{Y}_\infty} \widehat{H}$ is any isometry covering the restriction of f to $X \setminus Y_\infty$ and $\widehat{i}$ is the unitary from the definition of an $(\widehat{X}, \widehat{Y}, \widehat{\iota})$-module identifying $\chi_{\widehat{Y}_\infty} \widehat{H}$ and $\chi_{\widehat{Y} \times \mathbb{R}_+} \widehat{H}'$. Now we show that for $T \in \mathbb{R}(X)^{\Gamma, \mathbb{R}_+}$, $(\mathrm{Ad}_{V'} \circ \mathrm{ev}_\infty)(T) = (\mathrm{ev}_\infty \circ \mathrm{Ad}_V)(T)$. This then finishes the proof of the proposition. Let $v \in \widehat{H}'$ be compactly supported. We have

$$(\mathrm{Ad}_{V'} \circ \mathrm{ev}_\infty)(T)v = V' \lim_s Q_{-s} i T i^* Q_s V'^* v = \lim_s \widehat{Q}_{-s} V' i T i^* V'^* \widehat{Q}_s v.$$

On the other hand $(\mathrm{ev}_\infty \circ \mathrm{Ad}_V)(T)v = \lim_s \widehat{Q}_{-s} \widehat{i} V T V^* \widehat{i}^* \widehat{Q}_s v$. Set $R_0 = \operatorname{prop} T$. For s sufficiently large $\widehat{Q}_s v$ is supported in $Y \times (R_0, \infty)$. Therefore

$$\lim_s \widehat{Q}_{-s} \widehat{i} V T V^* \widehat{i}^* \widehat{Q}_s v = \lim_s \widehat{Q}_{-s} \widehat{i}\widehat{i}^* V' i T i^* V'^* \widehat{i}\widehat{i}^* \widehat{Q}_s v = \lim_s \widehat{Q}_{-s} V' i T i^* V'^* \widehat{Q}_s v.$$

Hence, $(\mathrm{Ad}_{V'} \circ \mathrm{ev}_\infty)(T) = (\mathrm{ev}_\infty \circ \mathrm{Ad}_V)(T)$. □

3.4.4 (Γ, Λ)-equivariant Roe Algebras

Let (X, Y, ι) be a space with cylindrical end. We do not assume the existence of an action of Γ on X. Let Λ, Γ and φ be as in Section 3.3.3. Suppose there exists a map of pairs $\eta : (X, Y_\infty := \iota(Y \times \mathbb{R}_+)) \to (B\Gamma, B\Lambda)$ satisfying $\eta(\iota((y,t))) = \eta(\iota((y,0)))$ for all $t \in \mathbb{R}_+$. This allows us to define Γ-coverings $\widetilde{X}, Y'_{(\infty)}$ of $X, Y_{(\infty)}$ and a Λ-covering $\widetilde{Y_{(\infty)}}$ of $Y_{(\infty)}$. We obtain in this way new spaces with cylindrical ends $(\widetilde{X}, Y', \iota')$ and $(\widetilde{Y}_\infty, \widetilde{Y}, \widetilde{\iota})$. In this section the Roe algebras will be constructed using fixed $(\widetilde{X}, Y', \iota')$ and $(\widetilde{Y_\infty}, \widetilde{Y}, \widetilde{\iota})$-modules. The construction of the previous section gives rise to evaluation at infinity maps $C^*(\widetilde{Y_\infty})^{\Lambda, \mathbb{R}_+} \to C^*(\widetilde{Y} \times \mathbb{R})^{\Lambda \times \mathbb{R}}$ and $C^*(\widetilde{X})^{\Gamma, \mathbb{R}_+} \to C^*(Y' \times \mathbb{R})^{\Gamma \times \mathbb{R}}$. Chang, Weinberger and Yu constructed a map $C^*(\widetilde{Y} \times \mathbb{R})^\Lambda \to C^*(Y' \times \mathbb{R})^\Gamma$ [1]. It is easy to see that this map respects the $\mathbb{R}$-invariance and asymptotic $\mathbb{R}_+$-invariance of operators. Thus we get, by restriction, a map $\varphi : C^*(\widetilde{Y} \times \mathbb{R})^{\Lambda \times \mathbb{R}} \to C^*(Y' \times \mathbb{R})^{\Gamma \times \mathbb{R}}$.

[1] They constructed the map between the maximal Roe algebras. In [34] the quotient completion was introduced and it was shown, that one has a similar map between the quotient completions of the equivariant algebraic Roe algebras.

Definition 3.4.22. $T \in C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+}$ is called *asymptotically Λ-invariant* if $\mathrm{ev}_\infty(T)$ is contained in the image of φ. The pullback of $C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+}$ and $C^*(\widetilde{Y}\times\mathbb{R})^{\Lambda\times\mathbb{R}}$ along ev_∞ and φ, is called the (Γ,Λ)*-equivariant Roe algebra of* X and will be denoted by $C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$.

Remark 3.4.23. By definition, we have a commutative diagram

$$\begin{array}{ccc} C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda} & \longrightarrow & C^*(\widetilde{Y}\times\mathbb{R})^{\Lambda\times\mathbb{R}} \\ \big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+} & \xrightarrow{\mathrm{ev}_\infty} & C^*(Y'\times\mathbb{R})^{\Gamma\times\mathbb{R}} \end{array}$$

and elements of $C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$ are given by pairs (S,T) with $S\in C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+}$, $T\in C^*(\widetilde{Y}\times\mathbb{R})^{\Lambda\times\mathbb{R}}$ with $ev_\infty(S)=\varphi(T)$.

Definition 3.4.24. The (Γ,Λ)*-equivariant localisation algebra*((Γ,Λ)*-equivariant structure algebra*) of X is defined to be the pullback of the following diagram

$$\begin{array}{ccc} & & C^*_{L,(0)}(\widetilde{Y}\times\mathbb{R})^{\Lambda\times\mathbb{R}} \\ & & \big\downarrow{\scriptstyle\varphi} \\ C^*_{L,(0)}(\widetilde{X})^{\Gamma,\mathbb{R}_+} & \xrightarrow{\mathrm{ev}_\infty} & C^*_{L,(0)}(Y'\times\mathbb{R})^{\Gamma\times\mathbb{R}} \end{array}$$

It will be denoted by $C^*_{L,(0)}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$.

We obtain an analogue of the Higson-Roe sequence for spaces with cylindrical ends: the short exact sequence

$$0\to C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}\to C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}\to C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}\to 0$$

gives rise to a long exact sequence

$$\cdots\to K_*(C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})\to K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})\to K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})\to\cdots.$$

3.5 Index of Dirac Operators on Manifolds with Cylindrical Ends

Let X be an n-dimensional spin manifold with a cylindrical end with base Y. By this we mean that (X,Y,ι) is a space with cylindrical end, ι is smooth and

$X \setminus \iota(Y \times (0,\infty))$ is a smooth codimension zero submanifold with boundary Y. We fix a map $\eta : (X, Y_\infty := \iota(Y \times \mathbb{R}_+)) \to (B\Gamma, B\Lambda)$ satisfying $\eta(\iota((y,t))) = \eta(\iota((y,0)))$ for all $t \in \mathbb{R}_+$ which gives rise to certain covers of X and Y, which we will denote as in the previous section. Denote by $L^2(\slashed{S}_{\widetilde{X}}), L^2(\slashed{S}_{Y' \times \mathbb{R}}), L^2(\slashed{S}_{\widetilde{Y_\infty}})$ and $L^2(\slashed{S}_{\widetilde{Y} \times \mathbb{R}})$ the square integrable sections of the Cl_n-spinor bundles on $\widetilde{X}, Y' \times \mathbb{R}, \widetilde{Y_\infty}$ and $\widetilde{Y} \times \mathbb{R}$ respectively. The pairs $(L^2(\slashed{S}_{\widetilde{X}}), L^2(\slashed{S}_{Y' \times \mathbb{R}}))$ and $(L^2(\slashed{S}_{\widetilde{Y_\infty}}), L^2(\slashed{S}_{\widetilde{Y} \times \mathbb{R}}))$ can be given the structure of an $(\widetilde{X}, Y', \iota')$ Cl_n-module and an $(\widetilde{Y_\infty}, \widetilde{Y}, \widetilde{\iota})$ Cl_n-module in the natural way respectively. In particular, the families of unitaries on $L^2(\slashed{S}_{Y' \times \mathbb{R}})$ and $L^2(\slashed{S}_{\widetilde{Y} \times \mathbb{R}})$ needed in the definition of cylindrical $Y' \times \mathbb{R}$ and $\widetilde{Y} \times \mathbb{R}$-modules will be given by the shift of sections in the $\mathbb{R}$-direction and will be denoted by $\{Q'_s\}$ and $\{\widetilde{Q}_s\}$ respectively. We will use these modules to construct the relevant C^*-algebras in the following section. As in Section 3.3.2, we obtain classes $[\slashed{D}_{\widetilde{X}}]$ and $[\slashed{D}_{\widetilde{Y} \times \mathbb{R}}]$ in $\widehat{K}_0(C^*_L(\widetilde{X}; \mathrm{Cl}_n)^\Gamma)$ and $\widehat{K}_0(C^*_L(\widetilde{Y} \times \mathbb{R}; \mathrm{Cl}_n)^\Lambda)$ respectively. Note that $\widetilde{Y} \times \mathbb{R}$ is a manifold with cylindrical end with base $\widetilde{Y}$. In the following we will define a fundamental class for the Dirac operators on X and its cylindrical end in the K-theory groups of the (Γ, Λ)-equivariant localisation algebra and discuss indices and secondary invariants obtained from it. We will need the following

Lemma 3.5.1. *The following diagrams are commutative*

$$
\begin{array}{ccc}
\mathcal{S} & \longrightarrow & C^*(\widetilde{X}; \mathrm{Cl}_n)^{\Gamma, \mathbb{R}_+} \\
& \searrow & \big\downarrow \mathrm{ev}_\infty \\
& & C^*(Y' \times \mathbb{R}; \mathrm{Cl}_n)^{\Gamma \times \mathbb{R}}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{S} & \longrightarrow & C^*(\widetilde{Y} \times \mathbb{R}; \mathrm{Cl}_n)^{\Lambda \times \mathbb{R}} \\
& \searrow & \big\downarrow \varphi \\
& & C^*(Y' \times \mathbb{R}; \mathrm{Cl}_n)^{\Gamma \times \mathbb{R}}.
\end{array}
$$

Here $\mathcal{S} \to C^*(\widetilde{X}; \mathrm{Cl}_n)^{\Gamma, \mathbb{R}_+}$, $\mathcal{S} \to C^*(Y' \times \mathbb{R}; \mathrm{Cl}_n)^{\Gamma \times \mathbb{R}}$ *and* $\mathcal{S} \to C^*(\widetilde{Y} \times \mathbb{R}; \mathrm{Cl}_n)^{\Lambda \times \mathbb{R}}$ *denote the functional calculi for* $\slashed{D}_{\widetilde{X}}$, $\slashed{D}_{Y' \times \mathbb{R}}$ *and* $\slashed{D}_{\widetilde{Y} \times \mathbb{R}}$ *respectively.*

Proof. First note that the isometry ι' allows us to identify the Cl_n-spinor bundles over $Y' \times \mathbb{R}_+$ and Y'_∞, which in turn gives rise to the unitary $i' : \chi_{Y'_\infty} L^2(\slashed{S}_{\widetilde{X}}) \to \chi_{Y \times \mathbb{R}_+} L^2(\slashed{S}_{Y' \times \mathbb{R}})$. Let $v \in L^2(\slashed{S}_{Y' \times \mathbb{R}})$ be compactly supported. For $f \in \mathcal{S}$ whose Fourier transform is supported in $(-r, r)$, it is well known that $f(\slashed{D}_{\widetilde{X}})$ and $f(\slashed{D}_{Y' \times \mathbb{R}})$ have propagation less than r and depend on the r-local geometry in the sense that $f(\slashed{D}_{\widetilde{X}})w$ and $f(\slashed{D}_{Y' \times \mathbb{R}})v$ depend only on the Riemannian metric in the r-neighbourhood of the supports of w and v respectively. For $v \in L^2(\slashed{S}_{Y' \times \mathbb{R}})$ with compact support pick s_0 such that $Q'_s v$ is supported in $Y' \times [2r, \infty)$ for all $s > s_0$. The previous observation then implies that $if(\not{D}_{\widetilde{X}})i^*Q'_sv = f(\not{D}_{Y'\times\mathbb{R}})Q'_sv$ for all $s > s_0$. Hence

$$\lim_s Q'_{-s}if(\not{D}_{\widetilde{X}})i^*Q'_sv = \lim_s Q'_{-s}f(\not{D}_{Y'\times\mathbb{R}})Q'_sv.$$

However, because the Riemannian metric on $Y'\times\mathbb{R}$ is $\mathbb{R}$-invariant, Q'_s commutes with the Dirac operator and its functions. This implies that $Q'_{-s}f(\not{D}_{Y'\times\mathbb{R}})Q'_sv = f(\not{D}_{Y'\times\mathbb{R}})v$ and shows that for f with compactly supported Fourier transform $\mathrm{ev}_\infty(f(\not{D}_{\widetilde{X}})) = f(\not{D}_{Y'\times\mathbb{R}})$. The commutativity of the left diagram then follows from the fact that the functions in $\mathcal{S}$ with compactly supported Fourier transform form a dense subset.

Now we show the commutativity of the right diagram. First we need to recall one of the main properties of the map $\varphi : C^*(\widetilde{Y}\times\mathbb{R};\mathrm{Cl}_n)^\Lambda \to C^*(Y'\times\mathbb{R};\mathrm{Cl}_n)^\Gamma$. Since all the covers are assumed to be nice one has bijections $C^*(\widetilde{Y}\times\mathbb{R};\mathrm{Cl}_n)^{\Lambda\times\mathbb{R}}_\epsilon \cong C^*(Y\times\mathbb{R};\mathrm{Cl}_n)^{\mathbb{R}}_\epsilon$ and $C^*(Y'\times\mathbb{R};\mathrm{Cl}_n)^{\Gamma\times\mathbb{R}}_\epsilon \cong C^*(Y\times\mathbb{R};\mathrm{Cl}_n)^{\mathbb{R}}_\epsilon$, where $C^*(Y\times\mathbb{R};\mathrm{Cl}_n)^{\mathbb{R}}$ is constructed using $L^2(\not{S}_{Y\times\mathbb{R}})$ as the $Y\times\mathbb{R}$-module, ϵ is a sufficiently small positive real number, and $C^*(\cdot)^{\cdot}_\epsilon$ denotes the set of elements in the corresponding Roe algebra which have propagation less than ϵ. The bijections are given by pushdowns and lifts of operators on different covers. Furthermore, φ makes the diagram

$$\begin{array}{ccc} C^*(\widetilde{Y}\times\mathbb{R};\mathrm{Cl}_n)^{\Lambda\times\mathbb{R}}_\epsilon & \xrightarrow{\varphi} & C^*(Y'\times\mathbb{R};\mathrm{Cl}_n)^{\Gamma\times\mathbb{R}}_\epsilon \\ & \searrow^{\cong} & \downarrow\cong \\ & & C^*(Y\times\mathbb{R};\mathrm{Cl}_n)^{\mathbb{R}}_\epsilon. \end{array}$$

commutative. Let $f\in\mathcal{S}$ have a Fourier transform which is supported in $(-\epsilon',\epsilon')$, with ϵ' sufficiently small. The observation that f applied to the different Dirac operators depends only on the ϵ'-local geometry and the niceness of covers imply that $f(\not{D}_{\widetilde{Y}\times\mathbb{R}}), f(\not{D}_{Y\times\mathbb{R}})$ and $f(\not{D}_{Y'\times\mathbb{R}})$ correspond to each other under the pushdown/lift maps. The commutativity of the latter diagram then implies that for f with the above property $\varphi(f(\not{D}_{\widetilde{Y}\times\mathbb{R}})) = f(\not{D}_{Y'\times\mathbb{R}})$. The commutativity of the right diagram in the claim of the lemma then follows from the fact, that the C^*-subalgebra of $\mathcal{S}$ generated by functions whose Fourier transform is supported in a fixed interval $(-C,C)$ is the whole of $\mathcal{S}$, since it separates points. $\square$

Lemma 3.5.1 allows us to make the following

Definition 3.5.2. The (Γ,Λ)-fundamental class of the X is the class $[\not{D}_{\widetilde{X},\widetilde{Y}}] \in \widehat{K}_0(C^*_L(\widetilde{X};\mathrm{Cl}_n)^{\Gamma,\mathbb{R}_+,\Lambda}) \cong K_n(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})$ defined by

$$\varphi_{\not{D}_{\widetilde{X},\widetilde{Y}}} : \mathcal{S}\to C^*_L(\widetilde{X};\mathrm{Cl}_n)^{\Gamma,\mathbb{R}_+,\Lambda}, f\mapsto (t\mapsto (f(\frac{1}{t}\not{D}_{\widetilde{X}}), f(\frac{1}{t}\not{D}_{\widetilde{Y}\times\mathbb{R}})).$$

The (Γ,Λ)-index of the Dirac operator associated to the map $\eta : (X, Y_\infty := \iota(Y \times \mathbb{R}_+)) \to (B\Gamma, B\Lambda)$ as above is defined to be the image of $[\not{D}_{\widetilde{X},\widetilde{Y}}]$ under the map $(\mathrm{ev}_1)_* : K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})$.

3.5.1 Application to Existence and Classification of Positive Scalar Curvature Metrics

Suppose that the scalar curvature of the metric g on X is bounded from below by ϵ. The same then holds for the lifts of g to various covers of X and $Y(_\infty)$. This implies that the spectra of the various Dirac operators considered here do not intersect the interval $(-\frac{\sqrt{\epsilon}}{4}, \frac{\sqrt{\epsilon}}{4})$. Let ψ be a homotopy inverse to the inclusion $\mathcal{S}(-\frac{\sqrt{\epsilon}}{4}, \frac{\sqrt{\epsilon}}{4}) \to \mathcal{S}$.

Definition 3.5.3. Let g be as above. The (Γ,Λ)-rho-invariant of g is the class in $K_0(C^*_{L,0}(\widetilde{X};\mathrm{Cl}_n)^{\Gamma,\mathbb{R}_+,\Lambda}) \cong K_n(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda})$ defined by the morphism

$$\varphi_{\not{D}_{\widetilde{X},\widetilde{Y}}} \circ \psi : \mathcal{S} \to C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$$

and will be denoted by $\rho^{\Gamma,\Lambda}(g)$.

Clearly, $\rho^{\Gamma,\Lambda}(g)$ lifts $[D_{\widetilde{X},\widetilde{Y}}]$ and by the exactness of the sequence

$$\cdots \to K_*(C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to \cdots$$

we have the following

Proposition 3.5.4. *If the metric on X has positive scalar curvature then the (Γ,Λ)-index of the Dirac operator vanishes.*

One can define a notion of concordance for positive scalar curvature metrics on manifolds with cylindrical ends. Let g and g' be such metrics on X. They are called concordant if there exist a positive scalar curvature metric G on $X\times\mathbb{R}$ and a map $j : Y \times \mathbb{R} \times \mathbb{R}_+ \to Y_\infty \times \mathbb{R}$ which makes $(X \times \mathbb{R}, Y \times \mathbb{R}, j)$ a manifold with cylindrical end and such that G restricted to $X \times (1,\infty)$ is $g + dt^2$ and restricted to $X \times (-\infty, 0)$ is $g' + dt^2$. Using the strategy of Zeidler in [41] and by replacing the usual Roe, localisation and structure algebras by their (Γ,Λ)-invariant counterparts one can without much difficulty prove a partitioned manifold index theorem for secondary invariants for manifolds with cylindrical ends and prove the concordance invariance of the (Γ,Λ)-rho-invariant. However we refrain from discussing this, since it does not entail any novelties.

More generally following the approach of [41] we define partial (Γ, Λ)-rho-invariants associated to metrics having positive scalar curvature outside of a given subset Z of X. Denote by Z' and and Z'' the preimages of Z and $(Z \cap \iota(Y \times \{1\})) \times \mathbb{R}$ under the covering maps $\widetilde{X} \to X$ and $\widetilde{Y} \times \mathbb{R} \to Y \times \mathbb{R}$ respectively. Denote by $C^*(Z' \subset \widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$ the C^*-subalgebra of $C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$ consisting of elements (T_1, T_2) with $T_1 \in C^*(Z' \subset \widetilde{X})^{\Gamma}$ and $T_2 \in C^*(Z'' \subset \widetilde{Y} \times \mathbb{R})^{\Lambda}$. Denote by $C^*_{L,Z'}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$ the preimage of $C^*(Z' \subset \widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$ under the evaluation at 1 map. The justification for the following definition is provided by [30, Lemma 2.3].

Definition 3.5.5. Given a metric g on X which is collared at the boundary whose scalar curvature is bounded below by $\epsilon > 0$ outside of a subset Z define the class $\rho_Z^{\Gamma,\Lambda}(g)$ by the morphism

$$\varphi_{\not{D}_{\widetilde{X},\widetilde{Y}}} \circ \psi : \mathcal{S} \to C^*_{L,Z'}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}.$$

Another higher index theoretic notion which has been successfully used to obtain information about the size of the space of positive scalar curvature metrics on closed manifolds is the higher index difference, which gives rise to a map from the space of positive scalar curvature metrics to the K-theory of the group C^*-algebra of the manifold. We now show that one can easily define a (Γ, Λ)-index difference of two positive scalar curvature metrics for manifolds with cylindrical ends. This becomes particularly interesting after we discuss the application of the above machinery to relative higher index theory in the next section. Let g_0 and g_1 be two metrics on X with scalar curvature bounded below by $\epsilon > 0$ which are collared on the cylindrical end. Define a metric G on $X \times \mathbb{R}$ which restricts to $g_0 \oplus dt^2$ and $g_1 \oplus dt^2$ on $X \times [0, \infty)$ and $X \times (-\infty, -1)$ respectively and which is collared on the cylindrical end in the X-direction.

Definition 3.5.6. Let g_0, g_1 and G be as above. The (Γ, Λ)-index difference of g_0 and g_1 is the image of $\rho^{\Gamma,\Lambda}_{X \times [0,1]}(G)$ under the composition

$$K_{n+1}(C^*_{L,\widetilde{X}\times[0,1]}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \xrightarrow{(\mathrm{ev}_1)_*} K_{n+1}(C^*(\widetilde{X} \times [0,1] \subset \widetilde{X} \times \mathbb{R})^{\Gamma,\mathbb{R}_+,\Lambda})$$

$$\to K_{n+1}(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}),$$

where the last map is induced by projection on $\widetilde{X}$. It will be denoted by $\mathrm{ind}^{\Gamma,\Lambda}(g_0, g_1)$.

3.5.2 Relationship to the Relative Index of Chang, Weinberger and Yu

As mentioned above, the relative index map of Chang, Weinberger and Yu for manifolds with boundary takes values in mapping cones of equivariant Roe algebras. Note that given a manifold (X, Y, ι) with cylindrical end, $\overline{X} \coloneqq X \setminus \iota(Y \times (0, \infty))$ is a manifold with boundary Y. By restriction, we obtain a map $\eta : (\overline{X}, Y) \to (B\Gamma, B\Lambda)$. Let $\psi_{(L,(0))} : C^*_{(L,(0))}(\widetilde{Y})^\Lambda \to C^*_{(L,(0))}(\widetilde{\overline{X}})^\Gamma$ denote the map introduced in Section 3.3.3. In the following, we will see that there exists a commutative diagram of exact sequences

$$\begin{array}{ccccc} K_*(C^*_{L,0}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \\ \downarrow & & \downarrow & & \downarrow \\ K_*(SC_{\psi_{L,0}}) & \longrightarrow & K_*(SC_{\psi_L}) & \longrightarrow & K_*(SC_\psi). \end{array}$$

Proposition 3.5.7. *The following is a commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\widetilde{Y} \subset \widetilde{Y}_\infty)^\Lambda & \longrightarrow & C^*(\widetilde{Y}_\infty)^{\Lambda,\mathbb{R}_+} & \xrightarrow{\mathrm{ev}_\infty} & C^*(\widetilde{Y} \times \mathbb{R})^{\Lambda\times\mathbb{R}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle (\varphi,\mathrm{ev}_\infty)} & & \downarrow{\scriptstyle (\varphi,\mathrm{id})} & & \\ 0 & \longrightarrow & C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma & \longrightarrow & C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda} & \longrightarrow & C^*(\widetilde{Y} \times \mathbb{R})^{\Lambda\times\mathbb{R}} & \longrightarrow & 0. \end{array}$$

Analogous diagrams exist when C^* is replaced by C^*_L and $C^*_{L,0}$.

Proof. We first show that the first row is exact. It follows immediately from the definition of ev_∞ that $\mathbb{R}(\widetilde{Y} \subset \widetilde{Y}_\infty)^\Lambda$ is in its kernel. By continuity we get that $C^*(\widetilde{Y} \subset \widetilde{Y}_\infty)$ is in the kernel of ev_∞. Furthermore, [12, Lemma 3.12] implies that the kernel of ev_∞ is exactly $C^*(\widetilde{Y} \subset \widetilde{Y}_\infty)^\Lambda$. It remains to show that ev_∞ is surjective. For $T \in \mathbb{R}(\widetilde{Y} \times \mathbb{R})^{\Lambda\times\mathbb{R}}$, the operator $\chi_{\widetilde{Y}\times\mathbb{R}_+} T \chi_{\widetilde{Y}\times\mathbb{R}_+}$ maps to T under ev_∞. The surjectivity then follows from the fact that the image of a homomorphism of C^*-algebras is closed. The exactness of the second row can be proven using similar arguments. However we note that the exactness in the middle uses the fact that $\lim_{R\to\infty} \mathrm{dist}(\iota(Y' \times [R, \infty)), \widetilde{X} - Y'_\infty) = \infty$ (see Definition3.4.1). As for the commutativity of the diagram we note that φ and ev_∞ commute. ☐

Proposition 3.5.8. *The inclusion*

$$C_{C^*(\widetilde{Y}\subset\widetilde{Y_\infty})^\Lambda \to C^*(\widetilde{\overline{X}}\subset\widetilde{X})^\Gamma} \to C_{C^*(\widetilde{Y_\infty})^{\Lambda,\mathbb{R}_+} \to C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}}$$

gives rise to isomorphisms of K-theory groups. Analogous statements hold when C^* is replaced by C_L^* and $C_{L,0}^*$.

Proof. Note that the mapping cone of the identity map on $C^*(\widetilde{Y}\times\mathbb{R})^{\Lambda\times\mathbb{R}}$ is contractible and thus has trivial K-theory. The statement then follows from an application of the five-lemma to the long exact sequence of K-theory groups associated to the short exact sequence of mapping cones

$$0\to C_{C^*(\widetilde{Y}\subset\widetilde{Y_\infty})^\Lambda\to C^*(\widetilde{\widetilde{X}}\subset\widetilde{X})^\Gamma}\to C_{(\varphi,\mathrm{ev}_\infty)}\to C_{\mathrm{id}}\to 0$$

. ∎

Proposition 3.5.9. *There is a commutative diagram of long exact sequences*

$$\begin{array}{ccccc}
K_*(C_{L,0}^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C_L^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \\
\downarrow & & \downarrow & & \downarrow \\
K_*(SC_{\psi_{L,0}}) & \longrightarrow & K_*(SC_{\psi_L}) & \longrightarrow & K_*(SC_\psi),
\end{array}$$

where the vertical maps are given by the compositions

$$\begin{aligned}
K_*(C_{(L,(0))}^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) &\to K_*(SC_{C_{(L,(0))}^*(\widetilde{Y_\infty})^{\Lambda,\mathbb{R}_+}\to C_{(L,(0))}^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}}) \\
&\cong K_*(SC_{C_{(L,(0))}^*(\widetilde{Y}\subset\widetilde{Y_\infty})^\Lambda\to C_{(L,(0))}^*(\widetilde{\widetilde{X}}\subset\widetilde{X})^\Gamma})\cong K_*(SC_{\psi_{(L,(0))}}).
\end{aligned}$$

Proof. The diagram in the claim of the proposition is obtained by composing the diagrams

$$\begin{array}{ccccc}
K_*(C_{L,0}^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C_L^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \\
\downarrow & & \downarrow & & \downarrow \\
K_*(SC_{(\varphi_{L,0},\mathrm{ev}_{\infty L,0})}) & \longrightarrow & K_*(SC_{(\varphi_L,\mathrm{ev}_{\infty L})}) & \longrightarrow & K_*(SC_{(\varphi,\mathrm{ev}_\infty)}),
\end{array}$$

and

$$\begin{array}{ccccc}
K_*(SC_{(\varphi_{L,0},\mathrm{ev}_{\infty L,0})}) & \longrightarrow & K_*(SC_{(\varphi_L,\mathrm{ev}_{\infty L})}) & \longrightarrow & K_*(SC_{(\varphi,\mathrm{ev}_\infty)}) \\
\cong\uparrow & & \cong\uparrow & & \cong\uparrow \\
K_*(SC_{\psi_{L,0}}) & \longrightarrow & K_*(SC_{\psi_L}) & \longrightarrow & K_*(SC_\psi),
\end{array}$$

where $(\varphi_{(L,(0))},\mathrm{ev}_{\infty(L,(0))})$ denotes the map $C_{(L,(0))}^*(\widetilde{Y_\infty})^{\Lambda,\mathbb{R}_+}\to C_{(L,(0))}^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}$ of Proposition 3.5.7. The commutativity of the first diagram is due to the naturality of the mapping cone exact sequence and the commutativity of the second diagram is clear. ∎

Denote the image of the fundamental class of the Dirac operator on $\widetilde{X}$ under the composition

$$K_*(C_L^*(\widetilde{X})^\Gamma) \to K_*(SC_{C_L^*(\widetilde{Y_\infty})^\Lambda \to C_L^*(\widetilde{X})^\Gamma}) \cong K_*(SC_{C_L^*(\widetilde{Y} \subset \widetilde{Y_\infty})^\Lambda \to C_L^*(\widetilde{\widetilde{X}} \subset \widetilde{X})^\Gamma})$$

by $[D_{\widetilde{\widetilde{X}}, \widetilde{Y}}]$.

Lemma 3.5.10. *The class $[D_{\widetilde{X}, \widetilde{Y}}]$ maps to $[D_{\widetilde{\widetilde{X}}, \widetilde{Y}}]$ under the map $K_*(C_L^*(\widetilde{X})^{\Gamma, \mathbb{R}_+, \Lambda}) \to K_*(SC_{\psi_L})$ of Proposition 3.5.9.*

Proof. We first note that the commutativity of the diagram

$$\begin{array}{ccc} K_*(C_L^*(\widetilde{Y_\infty})^{\Lambda, \mathbb{R}_+}) & \longrightarrow & K_*(C_L^*(\widetilde{X})^{\Gamma, \mathbb{R}_+, \Lambda}) \\ \downarrow & & \downarrow \\ K_*(C_L^*(\widetilde{Y_\infty})^\Lambda) & \longrightarrow & K_*(C_L^*(\widetilde{X})^\Gamma), \end{array}$$

where the second vertical map is given by the composition of the projection onto the $C_L^*(\widetilde{X})^{\Gamma, \mathbb{R}_+}$ component followed by the inclusion, implies that of

$$\begin{array}{ccc} K_*(C_L^*(\widetilde{X})^{\Gamma, \mathbb{R}_+, \Lambda}) & \longrightarrow & K_*(SC_{C_L^*(\widetilde{Y_\infty})^{\Lambda, \mathbb{R}_+} \to C_L^*(\widetilde{X})^{\Gamma, \mathbb{R}_+, \Lambda}}) \\ \downarrow & & \downarrow \\ K_*(C_L^*(\widetilde{X})^\Gamma) & \longrightarrow & K_*(SC_{C_L^*(\widetilde{Y_\infty})^\Lambda \to C_L^*(\widetilde{X})^\Gamma}). \end{array}$$

Furthermore, the diagram

$$\begin{array}{ccc} K_*(SC_{C_L^*(\widetilde{Y} \subset \widetilde{Y_\infty})^\Lambda \to C_L^*(\widetilde{\widetilde{X}} \subset \widetilde{X})^\Gamma}) & \longrightarrow & K_*(SC_{C_L^*(\widetilde{Y_\infty})^{\Lambda, \mathbb{R}_+} \to C_L^*(\widetilde{X})^{\Gamma, \mathbb{R}_+, \Lambda}}) \\ & \searrow & \downarrow \\ & & K_*(SC_{C_L^*(\widetilde{Y_\infty})^\Lambda \to C_L^*(\widetilde{X})^\Gamma}), \end{array}$$

where all the arrows are isomorphisms, is commutative. The claim then follows from the commutativity of the latter two diagrams and the fact that $[D_{\widetilde{X}, \widetilde{Y}}]$ lifts the fundamental class of $\widetilde{X}$ □

Corollary 3.5.11. The (Γ, Λ)-index of the Dirac operator associated to (X, Y, ι) maps to the relative index of the Dirac operator on $\overline{X}$ under $K_*(C^*(\widetilde{X})^{\Gamma, \mathbb{R}_+, \Lambda}) \to K_*(SC_\psi)$ defined in Proposition 3.5.9.

Combining Lemma 3.5.10 and Proposition 3.5.4 gives a new (and very natural) proof of the following

Proposition 3.5.12. *The nonvanishing of the relative index of the Dirac operator on a manifold with boundary is an obstruction to the existence of a positive scalar metric which is collared at the boundary.*

3.5.3 Localised Indices and the Relative Index

Given a metric g on X which has positive scalar curvature outside $\overline{X}$, one can define a localised coarse index in $C^*(\widetilde{\overline{X}})^\Gamma)$. In [34] it was shown that this index maps to the relative index of $\overline{X}$. We quickly recall the construction of the localised index and use the machinery developed previously to give a short proof of the latter statement.

Definition 3.5.13. Denote by $C^*_{L,\widetilde{\overline{X}}}(\widetilde{X})^\Gamma$ the preimage of $C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma$ under $\mathrm{ev}_1 : C^*_L(\widetilde{X})^\Gamma \to C^*(\widetilde{X})^\Gamma$.

Suppose that the scalar curvature of the metric restricted to the complement of $\overline{X}$ is bounded from below by $\epsilon > 0$. The following proposition is well-known. As in [41] one can define a partial ρ-invariant $\rho^\Gamma_{\overline{X}}(g) \in K_n(C^*_{L,\widetilde{\overline{X}}}(\widetilde{X})^\Gamma)$ using the morphism

$$\varphi_{D_{\widetilde{X}}} \circ \psi : \mathcal{S} \to C^*_{L,\widetilde{\overline{X}}}(\widetilde{X}; \mathrm{Cl}_n)^\Gamma.$$

Definition 3.5.14. The localised coarse index $\mathrm{ind}^\Gamma_{\widetilde{\overline{X}}}(g)$ is the image of $\rho^\Gamma_{\overline{X}}(g)$ under $(\mathrm{ev}_1)_* : K_n(C^*_{L,\widetilde{\overline{X}}}(\widetilde{X})^\Gamma) \to K_n(C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma)$.

Remark 3.5.15. Note that in the above situation we can also define $\rho^{\Gamma,\Lambda}_{\overline{X}}(g)$. Furthermore, we note that the commutativity of the diagram

$$\begin{array}{ccc} K_*(C^*_{L,\widetilde{\overline{X}}}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*_{L,\widetilde{\overline{X}}}(\widetilde{X})^\Gamma) \\ & \searrow & \big\downarrow (\mathrm{ev}_1)_* \\ & & K_*(C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma), \end{array}$$

and the fact that $\rho^{\Gamma,\Lambda}_{\overline{X}}(g)$ is a lift of $\rho^\Gamma_{\overline{X}}(g)$ under the horisontal map imply that $\mathrm{ind}^\Gamma_{\widetilde{\overline{X}}}(g)$ is the image of $\rho^{\Gamma,\Lambda}_{\overline{X}}(g)$ under the map $K_*(C^*_{L,\widetilde{\overline{X}}}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) \to K_*(C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma)$.

The following lemma is a simple observation

Lemma 3.5.16. *The following diagram is commutative*

$$\begin{array}{ccccc} K_*(C^*_{L,\widetilde{\overline{X}}}(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma) & \longrightarrow & K_*(SC_{C^*(\widetilde{Y} \subset \widetilde{Y_\infty})^\Lambda \to C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ K_*(C^*_L(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}) & \longrightarrow & K_*(SC_{C^*(\widetilde{Y_\infty})^{\Lambda,\mathbb{R}_+} \to C^*(\widetilde{X})^{\Gamma,\mathbb{R}_+,\Lambda}}). \end{array}$$

Suppose $\overline{X}$ is compact. Then $K_*(C^*(\widetilde{\overline{X}} \subset \widetilde{X})^\Gamma) \cong K_*(C^*(\Gamma))$. Using the previous remark and lemma we obtain the following corollary, which was one of the main statements of [34].

Corollary 3.5.17. Suppose $\overline{X}$ is compact. Then $\mathrm{ind}^\Gamma_{\widetilde{X}}(g)$ maps to the relative index of Chang, Weinberger and Yu under the map $K_*(C^*(\Gamma)) \to K_*(C^*(\Gamma, \Lambda))$.

Chapter 4

Overview and Outlook

In this final chapter we discuss some recent related works and some possible directions for future research.

This dissertation strongly revolves around the equivariant relative index map, which for a manifold M with boundary N takes the form

$$K_*(M, N) \to K_*(C^*(\pi_1(M), \pi_1(N))),$$

with the left hand side being the relative K-homology group and the right hand side the K-theory of the "relative group C^*-algebra". Now the usual "absolute" index map

$$K_*(M) \to K_*(C^*(\pi_1(M)))$$

and the Higson-Roe sequence

$$\ldots \to S_*^{\pi_1(M)} \to K_*(M) \to K_*(C^*(\pi_1(M))) \to \ldots$$

have many different realisations. Indeed, there are many different models for the K-homology groups and for each of these models one has a possible realisation of the index map. Using the definition of K-homology as a KK-group one has the KK-theoretic definition by Kasparov (see for example [15] for the details). Using the language of coarse geometry and Paschke duality one can, as Higson and Roe do in [16], use the K-theory of the quotient of certain C^*-algebras as a model for K-homology in which case the index map can be seen as the boundary map in K-theory of a certain short exact sequence of C^*-algebras. In this work, we used the K-theory of the so called localisation algebra as a model for K-homology, where the index map is the induced map on K-theory of a morphism of C^*-algebras. The latter approach is in spirit the same as the Higson-Roe approach. In [29], Roe showed that the Kasparov approach and the coarse geometric approach to the assembly map coincide. Yet another KK-theoretic approach uses the Mischenko-Fomenko index pairing (see [20] for a detailed description of this approach and the comparison with the definition of Kasparov.). Another useful model for K-homology is the geometric model of Baum and Douglas. In [3], the authors gave another realisation of the index map, and indeed of the whole Higson-Roe sequence in this language. Connes used the "adiabatic groupoid" to give another realisation of the index map. In [42], Zenobi showed that the Higson-Roe sequence can be identified with the long exact sequence associated to the adiabatic deformation of a certain Lie groupoid. In view of the above and the fact that each of these definition has certain merits, it is natural to ask whether one has analogous definitions for the relative index map. Recently, much work has been done to answer the latter question. In [4], Deeley and Goffeng, gave a "geometric" definition of the relative index map. In their work, they also proved a statement similar to the main Theorem of Chapter 2, relating absolute indices defined in the presence of positive scalar curvature at the boundary with the relative index. However, they could not show in full generality that their relative index map coincides with that of Chang, Weinberger and Yu. In [19], Kubota gave a definition of the relative index map as a relative Mischenko-Fomenko index pairing and showed that it coincides with both that of Chang, Weinberger and Yu and Deeley and Goffeng. Using his work, one can show that the main theorem of Chapter 2 is actually equivalent to the result of Deeley and Goffeng mentioned above. We further note that the works [5] [25] use the language of groupoids to do index theory in more general singular situations than manifolds with boundary.

Now we discuss some directions for future research. One of the main contributions of Chapter 3 is the definition of a relative higher rho-invariant for manifolds with boundary. The higher rho-invariants associated to positive scalar curvature metrics on closed manifolds have been successfully used to distinguish and make statements about the size of the moduli space of positive scalar curvature metrics (see for example [37] and [38]). A natural question is then whether the higher rho-invariant for manifolds with boundary defined in Chapter 3 can be used to prove concrete results about the moduli space of positive scalar curvature metrics on manifolds with boundary.

As a usual rule, results regarding positive scalar curvature and the Dirac operator have a counterpart regarding homotopy equivalences and the signature operator. Recently, in [18] Hou and Liu defined higher rho-inavariants associated to the above data. We believe that one can use the machinery developed in Chapter 3 to define higher rho-invariants associated to the signature operator on the union of homotopy equivalent manifolds with a given homotopy equivalence and we plan to address this in a future work.

The higher rho-invariant was also used by Higson and Roe to give a conceptual proof of a rigidity result concerning relative eta-invariants on closed manifolds. One of the results discussed there related the validity of the Baum-Cones conjecture with the vanishing of the relative eta-invariants on closed manifolds with positive scalar curvature. We plan to use the higher rho-invariant defined in this work to prove an analogous result concerning the relative Baum-Connes map and relative eta-invariants on manifolds with boundary.

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