Nocommutative structures in quantum field theory

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Lada Peksová

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Thesis committee:

Prof. Dr. Dorothea Bahns
Universität Göttingen, Mathematisches Institut

Prof. Dr. Chenchang Zhu
Universität Göttingen, Mathematisches Institut

Doc. Ing. Branislav Jurčo, CSc., DSc.
Charles University, Mathematical Institute

Members of the Examination Board:

Reviewer: Prof. Dr. Dorothea Bahns
Universität Göttingen, Mathematisches Institut

Second Reviewer: Prof. Dr. Chenchang Zhu
Universität Göttingen, Mathematisches Institut

Additional Reviewer: Dr. Roman Golovko, Ph.D.
Charles University, Mathematical Institute

Further members of the Examination Board:

Doc. Ing. Branislav Jurčo, CSc., DSc.
Charles University, Mathematical Institute

Doc. RNDr. Roman Lávička, Ph.D.
Charles University, Mathematical Institute

Prof. Dr. Viktor Pidstrygach
Universität Göttingen, Mathematisches Institut

Prof. RNDr. Vladimír Souček, DrSc.
Charles University, Mathematical Institute

Date of the oral examination: October 22, 2020

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Lada Peksová

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Dedicated to my father.

Contents

Introduction

This thesis discusses several different applications of the theory of operads in quantum field theories.

The theory of operads was developed to understand algebraic or topological structures. Operads, as objects, model operations (in certain categories) with several inputs and one output. The operations, as it is usual in a mathematical context, could be composed and their variables could be permuted. One possible point of view is to look at them as the directed rooted trees.

As such, they can be generalized in the context of graphs as "flowcharts" in two possible ways. The undirected graphs with several inputs lead to the notion of modular operads, whereas the connected directed graphs with several inputs and several outputs lead to the notion of properads. Both of these structures have some use in mathematical physics.

Modular operads became very useful in various string theories. To see the basic idea of why one should use such a complicated mathematical tool, let us first look at the cases of closed and open strings.

Closed strings, i.e., strings without any loose ends, could be naively depicted as disjoint circles embedded into Euclidean complex plane. The interaction of several closed strings is then interpreted as a Riemann surfaces with punctures in the interior corresponding to the set of strings. The space of all interactions correspond to the moduli space of Riemann surfaces with marked points that could be (up to non-trivial sign factor) freely permuted among themselves.

Open strings, i.e., each string has two loose ends, could be depicted as 1-dimensional objects, or "intervals". Their interaction will take place again on some Riemann surface, but the string itself would be this time just a small part of one boundary component. We can talk about the marked points on the boundary or, similarly as Zwiebach [45], about half-disks.[1] Such punctures cannot be freely permuted, their positions fix their order on the boundary. But (up to sign factor) we are allowed to permute the punctures on one boundary in a "cycle". And we can also freely permute the whole boundaries among themselves.

Zwiebach in his work [44] and [45] showed, that this moduli space with gluing two punctures of two distinct surfaces or gluing two punctures of a single surface[2] correspond to the *twisted modular operad*. The benefit of modular operads, when compared to cyclic operads, is given by the additional self-composition corresponding to the gluing two punctures of a single surface.

The morphism of twisted modular operads, provided by conformal field theory, then maps this "huge" twisted modular operad of moduli space to twisted endomorphism operad. This algebra over the modular operad[3] can be encoded into the following data: an anti-bracket $\{\cdot,\cdot\}$ derived from the composition of

[1] For every puncture, there is an analytic map from $\{|u| \leq 1, \mathrm{Im}(u) \geq 0\}$ to the neighborhood of the puncture such that the center of the unit disk is mapped to the puncture.

[2] For "closed" punctures this means identification of local coordinates z and w such that $zw = 1$ and for "open" punctures the identification $zw = -1$. We always glue closed puncture with closed puncture and open puncture with open puncture.

[3] One can say its "representation" on some differential graded vector space

two surfaces, the square-zero differential operator Δ (in the next called Laplace operator) derived from the self-composition, and the elements S such that the *quantum master equation*

$$dS + \Delta S + \frac{1}{2}\{S, S\} = 0$$

is satisfied. Naively, we associate to our geometric picture an algebraic one.

The resulting structure is the *Batalin-Vilkovisky algebra* with the "Maurer-Cartan elements".[4] This structure will be in the next called *quantum homotopy algebras*. It could be also equivalently encoded as a differential operator $x_S = d + \Delta + \{S, \cdot\}$ such that $x_S^2 = 0$.

If one is interested in open-closed strings, it is easy to combine the former mentioned cases into moduli spaces of Riemann surfaces with punctures in the interior and on the boundaries. By introducing a 2-colored modular operad we simply get the appropriate mathematical tool to handle the non-trivial symmetry of this case.

Although the original work of Zwiebach was done for moduli spaces of Riemann surfaces, in the following we simplify our situation a bit. One can precompose the above-mentioned morphism of twisted modular operads by another morphism of twisted modular operads. The source of this map will be defined by the Feynman transform of a modular operad $\mathcal{P}$. We consider two special operads: *quantum closed operad* and *quantum open operad* corresponding to the case of closed strings and open strings, respectively. These modular operads are given by only homeomorphism classes of Riemann surfaces. Therefore we are interested only in the "combinatorial" part of the problem.

To one's surprise, we can define morphism directly from Feynman transform of $\mathcal{P}$ to twisted endomorphism operad (i.e., construct a particular solution to the Batalin-Vilkovisky master equation) such that this newly defined morphism is equivalent to the composition of morphisms going through the "huge" twisted modular operad of moduli space of Riemann surfaces.

Thanks to this simplification we are now able to provide a partial answer to the following problem. The morphism from Feynman transform of $\mathcal{P}$ to twisted endomorphism operad on vector space V (equivalent to solutions of the quantum master equation) describes the decomposition of Riemann surfaces into "interacting vertices" and propagators connecting these vertices.[5] The symmetry of the vertices is given by operad $\mathcal{P}$[6] and the "free ends" of the graphs are decorated by elements of graded dual of V. One may be interested if it is possible to transfer this solution constructed on the vectors space V to its cohomology $H(V)$. In other words, are we able to construct a minimal model of this quantum homotopy algebra?

[4]Usually, by Maurer-Cartan elements are meant the solutions of Maure-Cartan equation. Since the quantum master equation is the "incarnation" of the Maurer-Carten equation for modular operads, we inaccurately call its solution also as Maurer-Cartan elements. We discuss the various *master equation* in section 2.3.

[5]The resemblance of the terms Feynman transform and Feynman graphs is not accidental.

[6]For example, for closed strings we use quantum closed operad and thus all vertices have trivial symmetry.

The homological perturbation lemma, as one instance of homotopy transfer, turns out to be a very useful tool. However, to be able to explicitly construct the solutions we need to be able to define an exponential of an element. Since our motivational examples still have nice geometrical interpretation we define the missing product as the connected sum of the surfaces. Interestingly, this decision surprisingly leads to *Beilinson-Drinfeld algebra*.

The combinatorial structure of homeomorphism classes of Riemann surfaces with punctures within the interior or on the boundary can be also encoded by properads. This framework is related to, for example, (equivariant) string topology, symplectic field theory and Lagrangian Floer theory of higher genus. In this case, the punctures are unambiguously divided into "incoming" and "outgoing", and the composition always connects two surfaces – from one surfaces it takes only outgoing punctures and identifies them with incoming punctures from the second surface. Notice, that this time we can compose several punctures at once and the underlying graphs are directed.

Introduced by Vallette in [41], properads are a restriction to the connected part of an even more general structure called PROPs. Despite this, the properad's setting is sufficient to encode important examples like Lie bialgebras or Hopf algebras. And, unlike PROPs, it is possible to define the Koszul duality for properads.

The well-known example of properads is a (closed) Frobenius properad. It can be also represented as Riemann surfaces with punctures in the interior with trivial symmetry and "commutative" gluing of the surfaces. The algebra over its cobar complex is the minimal model thanks to Koszulness in sense of [41] and leads to the notion of IBL_∞-algebra. A newly defined example of Open Frobenius properad gives a similar structure that we call IBA_∞-algebra.

As we will see in chapter 6, the IBL_∞-algebras can be encoded as a nilpotent differential operator $d + L$.

Thus we can interpret the quantum master equation for modular operads as a special IBL_∞-algebra. But there is more we can say about the relation of modular operads and properads. Interestingly, it is related to Kontsevich's reformulation of the deformation quantization problem.

The deformation quantization problem can be formalized as follows: for a given Poisson algebra $\mathcal{A} = (C^\infty(M), \cdot, \{,\})$ on a manifold M and a formal parameter μ, construct bidifferential μ-linear map $* : \mathcal{A}[|\mu|] \otimes \mathcal{A}[|\mu|] \to \mathcal{A}[|\mu|]$

$$f * g = \sum_{k=0}^{\infty} B_k(f, g)\mu^k$$

such that $f \cdot g = B_0(f, g)$, and the Possion bracket $\{f, g\} = B_1(f, g) - B_1(g, f)$.

The deformations are controlled by the subcomplex of Hochschild cochains of the polydifferential operators $D_{poly}(M)$. And the obstructions are given by the Hochschild cohomology given as the polyvector fields $T_{poly}(M)$. In [27] Kontsevich reformulated this problem in a homotopy algebraic set-up and constructed an L_∞-morphism between these two DGLAs

$$T_{poly}(M) \xrightarrow{L_\infty} D_{poly}(M).$$

Let us consider the 2-colored modular operad and split the corresponding Maurer-Cartan element S into a closed part S_c and open-closed part $S_{oc} = S - S_c$. Correspondingly we can split the operator Δ and the bracket $\{,\}$ and, as we mentioned, we obtain an IBL_∞-algebra determined by $S_c, \Delta_c, \{,\}_c$.

The open-closed part then determines the IBL_∞-morphism to the cyclic Hochschild complex, cyclic A_∞. We can visualize the components of this morphism as the corresponding Riemann surfaces. The source is given by the punctures in the interior and the target by the punctures on the boundaries.

This interesting construction by Münster and Sachs is briefly summarized in [38] with some comments on background independence. Or can be found with all details in [39]. The "classical version" of this point of view formulated for 2-colored operads and its intriguing connection with Kontsevich's work can be found in [24] and [25] by Kajiura and Stasheff. A similar nice interpretation of IBA_∞ algebras and morphisms, unfortunately, wasn't found yet.

The text of the thesis is based on two articles - *Properads and Homotopy Algebras Related to Surfaces* and *Quantum Homotopy Algebras and Homological Perturbation Lemma.* Part of the first one was already published as *Properads and homological differential operators related to surfaces* and part of the second one as *Modular operads with connected sum and Barannikov's theory*, both in Archivum Mathematicum. However, some technical details are added and the computations are shown with all key steps (in contrast with the very abbreviated form appearing many times in the articles).

Because some parts of the texts were taken from these articles, I decided to write also the rest of the thesis in the "we" form. This may seem strange but I was concerned that I would miss correcting the subject everywhere and it would cause a distraction to the reader. Also, I hope this form would more involve the reader.

The structure of the thesis is following:

In the first chapter, we introduce operads, modular operads, and properads. We start with operads to demonstrate the basic principles. We present the three main examples of operads, Ass, Com, and Lie. The first two of them will later reappear also in the context of modular operads and properads. We also present operads in the language of monads. In the follow-up section, we introduce modular operads. We enrich modular operads by a new graded-commutative associative product, that we call *connected sum*, and present examples of modular operads with connected sum – the Quantum Closed modular operad $\mathcal{QC}$ (as the analog of Com) and Quantum Open modular operad $\mathcal{QO}$ (as the analog of Ass). We compared connected sum with similar structures that already appeared in the literature and present the formulation of connected sum also as an algebra over monad. We close this chapter with properads. We present the well-known example of Frobenius properad (the commutative case) and introduce the new example of Open Frobenius properad (the associative case).

In the second chapter, we first look at the Cobar complex and Feynman transform from two perspectives. The first perspective gives us general outlines of the procedure, the necessary conditions, and the form of the result. The second part discusses the explicit construction of the relevant graph complex and of the coboundary operator. The reason to talk about the Cobar complex (or Feynman transform) is that it provides a quasi-free resolution of the original operad. In some cases (e.g., quadratic Koszul dual cooperad) it is also the minimal one in the sense of model categories. In the third part of this chapter, we recall the results of Barannikov for modular operads and mimic them for properads. Broadly speaking the algebra over the Cobar complex (or Feynman transform) corresponds to the solutions of some *(quantum) master equation.*

The thesis at this point definitely splits into the part devoted to the modular operads (chapters 3, 4, and 5) and to the part following the properads (chapter 6).

We start the third chapter with a very short adventure to the realm of physics to motivate the following. Afterward, we introduce the Batalin-Vilkovisky algebras and prove that the combination of the modular operad with the odd modular operad, both equipped with the connected sum, offers us the structure of Batalin-Vilkovisky algebra. Thanks to Barannikov's theory we know we can encode the algebras over Feynman transform as solutions of the quantum master equation. And, as we show at the end of the section 3.2.1, such solutions can be expressed also as some $(d+\Delta)$-closed elements. We define the quantum homotopy algebras as algebras over Feynman transform. We introduce space $\mathrm{Fun}(\mathcal{P}, V)$ containing the solutions of quantum master equation. This space can be seen as the Batalin-Vilkovisky algebra[7] of functions with "generalized" symmetry (given by the arbitrary modular operad $\mathcal{P}$). We check that the restriction to the commutative operad gives us the "standard" symmetric tensor algebra. And we close this chapter with technical tools necessary for the following chapter – namely, we recall the special deformation retracts and Hodge decomposition.

Chapter four talks about the Homological perturbation lemma. With its help we are able to transfer the structure of quantum homotopy algebras from the space $\mathrm{Fun}(\mathcal{P}, V)$ to $\mathrm{Fun}(\mathcal{P}, H(V))$. One can consider two possible perturbations. The perturbation by BV-Laplacian Δ gives us an effective action W on the cohomology satisfying the appropriate master equation. And we show that the projection P_2 given by the second perturbation has similar properties as a path integral from the physical motivation. At the end of this chapter, we show how looks the effective action for space $\mathrm{Fun}(\mathcal{QC}, V)$ when one does not have any product.

In the follow-up chapter 5, we define the homotopy between two solutions of quantum master equation and introduce the quantum homotopy algebra morphism. This gives us three equivalent definitions of the homotopy. We also present an example of $\mathrm{logBer}(\Phi)$.

In chapter 6 we first recall IBL operad and its version up to "higher homotopy", the IBL_∞ operad. The theorem identifying the algebras over the cobar complex of Frobenius properad with the IBL_∞ algebras is then rephrased in our convention. This is followed by the theorem for the "relatives" of the IBL_∞ – we introduce the IBA_∞-algebras for the associative version up to homotopy and the IB-homotopy algebra for the open-closed case. As of last, we sketch the idea of an application of the Homological perturbation lemma for IB-homotopy

[7] Or more precisely Beilinson-Drinfeld algebra.

algebras.

List of Abbreviations

For us, the field $\mathbb{K}$ is always of characteristic 0. To avoid problems with duals, we assume that all our vector spaces are $\mathbb{Z}$-graded and, unless stated otherwise, degree-wise finite-dimensional. If we consider a dual vector space, we always consider only the graded dual, denoted by V^*. We use a cohomological convention.

1. *dg* means differential graded
2. $|\cdot|$ denotes a degree of an element of a graded vector space (e.g. for differential d in the cohomological convention, $|d| = 1$)
3. $\sqcup$ is a disjoint union
4. For set C and element $c \in C$ we abbreviate $C - c := C \setminus \{c\}$
5. $[n]$ is the set $\{1, 2, \ldots n\}$
6. $\mathtt{card}(A)$ is the cardinality of the set A (e.g. $\mathtt{card}([n]) = n$)
7. Σ_n denotes the symmetric group of $[n]$
8. $\kappa(\sigma)$ is the Koszul sign (or parity) of a permutation $\sigma \in \Sigma_n$
9. For $n \in \mathbb{N}_0$ and a set $\{a_1, a_2, \ldots\}$ of natural numbers,
$$n + \{a_1, a_2, \ldots\} \equiv \{n + a_1, n + a_2, \ldots\}$$
10. $\bar{\cdot}$ denotes the skeletal version. For example ${}_a\bar{\circ}_b$ denotes the skeletal version of operadic composition
11. $\xrightarrow{\cong}$ denotes isomorphism (or bijection between two sets)
12. $\xrightarrow{\sim}$ denotes quasi-isomorphism, i.e. morphism that induces isomorphism on (co)homology
13. $\uparrow$ is a suspension (i.e. $(\uparrow V)_i = V_{i-1}$)
14. Vect is the category of graded vector spaces with homogenous linear maps of arbitrary degrees
15. $\prod^{\star}_{i \in [n]} x_i = x_1 \star \ldots \star x_n$ for $\star$-product defined in 125

Let us call Σ-module a collection of (right) Σ_n-modules for $n \geq 0$. Similarly Σ-bimodule is a collection of (Σ_m, Σ_n)-modules for $m, n \geq 0$ which are left Σ_m and right Σ_n and the left action commutes with the right.

We sometimes write for element a_i of homogeneous basis of vector space just $(-1)^i$ instead of $(-1)^{|a_i|}$. Similarly to shorten formulas, we write for the dual basis ϕ^i, $\phi(a_j) = \delta^i_j$, just $(-1)^i$. Obviously, this doesn't cause a problem since $|a_i| = -|\phi^i|$ and $(-1)^{|a_i|} = (-1)^{-|a_i|+2|a_i|}$.

Sometimes the notation will unfortunately collide across the different sections.[8]

[8]Since there is only finite number of letters and possibly infinite number of concepts.

1. Operads, Modular Operads, and Properads

Operads are objects that model operations with several inputs and one output. As such, they can be generalized in the context of graphs in two possible ways. The undirected graphs with several inputs lead to the notion of cyclic or modular operads, whereas the connected directed graphs with several inputs and several outputs lead to the notion of properads.

The main purpose of this chapter is to introduce modular operads and properads. But before we deep dive into more complicated definitions, let us start lightly with the classical operads. Readers already familiar with operads will find nothing new in the Section 1.1 and may skip it completely. This section only serves as a simplified version of more complicated definitions introduced later in sections 1.2 and 1.3.

Also let us point out that from the beginning we introduce only operads in the category of (differential graded) vector spaces, i.e., (dg) Vect. It is possible to define operads in any symmetric monoidal category. For instance, one can define operads in the category of sets, simplicial sets, topological sets, etc. For the reader interested in those we recommend section 5.3.9 in Loday and Vallette [30] as a first point where to look.

A basic references of this chapter are Loday and Vallette [30], Markl [33], and Markl with Shnider and Stasheff [35].

1.1 Operads

To start with, we need to know what an operad is. We have several different options on how to define it.

The most "compact" definition uses monads. It can be very easily generalized to any type of operads and it also simultaneously gives a prescription of all possible compositions that don't favor any specific arity. But for practical computations and direct verification, this is rather inconvenient.

On the other hand, the more explicit the definition is, the more complicated the involved axioms become when we want to generalize the definition to modular operads or properads.

We decide to choose the golden mean and, similarly as in [11], most of the time use the *biased* definition with collections $\mathcal{P}(C)$ indexed by finite sets C. This definition is slightly more general than the *classical* one but it is still biased toward the unary and binary operations.[1]

To fully understand the nuances of these three definitions, we start with the classical Definition 1.1.1, where everything could be shown explicitly. Subsequently, we compare it with a biased Definition 9 and henceforth use the biased. And since both operad and the connected sum can be defined as algebras over

[1] A binary operation is, for example, the composition $\circ_i : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$ in Definition 1. An example of unary operation can be seen later in the Definition 19 of modular operad as composition $\circ_{ab} : \mathcal{P} \to \mathcal{P}$.

monads, we recall monads (also known as triples) in the Section 1.1.2. The *combinatorial* definition can be seen as a particular example of algebra over some monad and allow us to better understand the phrase “generalization in the context of graphs”.

1.1.1 Classical definition

Although the most common definition of operads is probably the classical definition by P. May, we introduce the *partial* definition by M. Markl. The advantage of the partial definition is in the fact that we automatically obtain the “linearised” version and are immediately prepared to talk about the differential graded operads.

Since in this definition we need to describe only how to compose two operations to describe the whole operad, we explicitly show what we mean by associativity and equivariance. Then we show how this definition is translated into *biased* version which will be used later for modular operads and properads.

Definition 1. *(classical definition)* An **operad** in the category of $\mathbb{K}$-modules is a collection of right $\mathbb{K}[\Sigma_n]$-modules (Σ-module) $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ together with $\mathbb{K}$-linear maps called operadic composition

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1)$$

(where $1 \leq i \leq m$ and $0 \leq n$) such that the following two axioms are satisfied:

1. *Equivariance:* For each $1 \leq i \leq m$, $0 \leq n$, $\pi \in \Sigma_m$ and $\sigma \in \Sigma_n$ let $\pi \circ_i \sigma$ be the permutation where pairs

$$(i, \pi \circ_i \sigma(i)), (i+1, \pi \circ_i \sigma(i+1)), \dots (i+n, \pi \circ_i \sigma(i+n))$$

corresponds to σ inserted on i-th place of π^2. Then for $p \in \mathcal{P}(m), q \in \mathcal{P}(n)$ we require

$$(p\pi) \circ_i (q\sigma) = \left(p \circ_{\pi(i)} q\right)(\pi \circ_i \sigma),$$

where the action of $\pi \in \Sigma_m$ on an element $p \in \mathcal{P}(m)$ is denoted as $p\pi$.

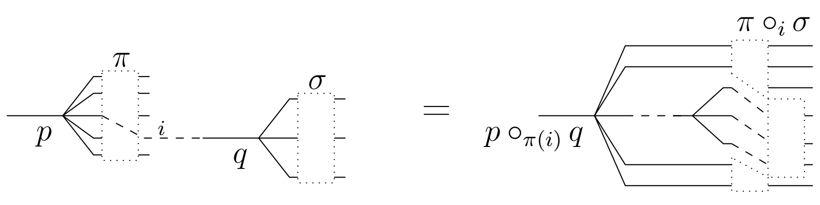


Figure 1.1: Axiom of equivariance pictorially.

2. *Associativity:* For each $1 \leq j \leq m$, $0 \leq n, 0 \leq k$ and $p \in \mathcal{P}(m)$, $q \in \mathcal{P}(n)$, $r \in \mathcal{P}(k)$

$$(p \circ_i q) \circ_j r = \begin{cases} (p \circ_j r) \circ_{i+k-1} q & \text{if } 1 \leq j < i \\ p \circ_i (q \circ_{j-i+1} r) & \text{if } i \leq j < n+i \\ (p \circ_{j-n+1} r) \circ_i q & \text{if } i+n \leq j \leq m+n-1. \end{cases}$$

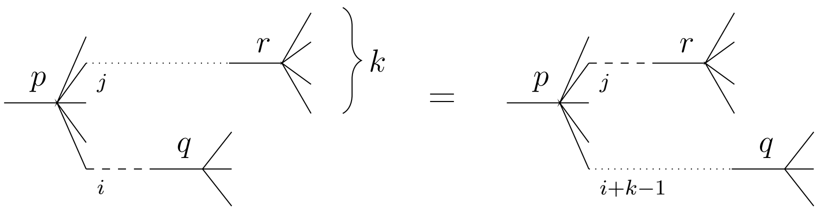


Figure 1.2: Axiom of associativity pictorially for the case $1 \leq j < i$.

For **unital operads** there is one more axiom[3]

3. *Unitality:* There exists $u \in P(1)$ such that $p \circ_i u = p$ for $p \in P(m)$ and $1 \leq i \leq m$, $u \circ_1 q = q$ for $q \in P(n)$.

One may recognize operads as abstractions of collections of composable functions. To get familiar, let us show a few examples. And let us start with the toy-model, endomorphism operad.

Example 2. For any vector space V, the **endomorphism operad** is defined as $End_V(n) = \mathrm{Hom}_{\mathbb{K}}\left(V^{\otimes n}, V\right)$. The right action[4] is defined as

$$(f\sigma)\left(v_1, v_2, \ldots v_n\right) = f\left(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \ldots v_{\sigma^{-1}(n)}\right),$$

where $v_i \in V$ and $\sigma \in \Sigma_n$. The composition is defined as $f \circ_i g = f(1_V^{i-1} \otimes g \otimes 1_V^{n-i})$

Now, we could continue with a list of interesting and important operads. Let us limit to three basic examples, called by B. Vallette as "the three graces of operads". Two of them will be important for us since we will meet their modified versions later in the language of modular operads and properads. The notation of the examples is adopted from [33].

Example 3. A **commutative operad** is a collection $Com = \{Com(n)\}_{n\geq 1}$ such that $Com(n) = \mathbb{K}$ with trivial Σ_n-action for every n.

Notice, that for a commutative operad it does not matter which index i have been used in the operadic composition $\circ_i$.

Example 4. An **associative operad** is $Ass = \{Ass(n)\}_{n\geq 1} = \{\mathbb{K}\left[\Sigma_n\right]\}_{n\geq 1}$. If α denotes a generator of regular representation $\mathbb{K}[\Sigma_2]$ and $e = \alpha^2$ is the identity permutation. Then all elements of $\mathbb{K}[\Sigma_3]$ are in the linear span of: $e \circ_1 e = e \circ_2 e = (123)$, $e \circ_1 \alpha = (213)$, $e \circ_2 \alpha = (132)$, $\alpha \circ_1 e = (312)$, $\alpha \circ_2 e = (231)$, $\alpha \circ_1 \alpha = (321)$ with the relation

$$\alpha \circ_1 \alpha = \alpha \circ_2 \alpha \tag{1.1}$$

as one would expect.

[2] For example if we take permutation $\pi = (4, 1, 3, 2) \in \Sigma_4$ and $\sigma = (2, 1, 3) \in \Sigma_3$ and insert σ as second argument of π we get $\pi \circ_2 \sigma = (2, 5, 4, 6, 3, 1) \in \Sigma_6$.

[3] In [35] the operads without unit are called *pseudo-operads*. In the following, we call as operads also those without unit.

[4] Notice that the right action on operad is induced by left action on $V^{\otimes n}$.

Notice that this operad can be nicely described also as a free operad generated by regular representation of Σ_2 (only arity 2 is nontrivial) factorized by ideal[5] generated by a linear span of the relation (1.1).

Example 5. A **Lie operad**. Let us use the handy approach mentioned in the previous example. A *Lie* operad is generated by a Σ module

$$E_{Lie} = \begin{cases} \Bbbk\beta & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

where β is signum representation of Σ_2. The ideal is generated by the Jacobi identity

$$\beta \circ_1 \beta + (\beta \circ_1 \beta)(123) + (\beta \circ_1 \beta)(132) = 0.$$

The first straightforward modification of definition of operad is to consider a notion of **differential graded operad** $(\mathcal{P}, d_{\mathcal{P}})$ in the category of differential graded $\mathbb{K}$-modules (i.e. dg vector spaces), dgVect.

Recall, the usual action of $\sigma \in \Sigma_n$ is

$$\sigma(v_1 \otimes \ldots \otimes v_n) = \kappa(\sigma) \cdot v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)},$$

where $\kappa(\sigma)$ stands for the *Koszul sign.*

Definition 6. Let $(\mathcal{P}, d_{\mathcal{P}})$, $(\mathcal{Q}, d_{\mathcal{Q}})$ be two dg operads. The **homomorphism of dg operads** $\alpha : \mathcal{P} \to \mathcal{Q}$ is a collection of degree 0 maps $\alpha_n : \mathcal{P}(n) \to \mathcal{Q}(n)$ such that these maps are equivariant, commute with operadic composition[6] and

$$d_{\mathcal{Q}} \circ \alpha = \alpha \circ d_{\mathcal{P}}.$$

An **algebra over** $\mathcal{P}$ (or $\mathcal{P}$-algebra) on V is a homomorphism of operads

$$\alpha : \mathcal{P} \to End_V.$$

It is now easy to verify that the algebras over Com are ordinary commutative algebras, the algebras over Ass are associative algebras and the algebras over Lie are Lie algebras. One can consider algebras over an operad also as a "representations" of the operad.

As a next generalization, we consider instead of sets with fixed ordering the "categorified sets". Let us first recall the invariants and coinvariants.

Definition 7. Let W be a (dg) vector space with a linear action of a finite group G. Denote by W^G the submodule of **invariants**

$$W^G = \{w \in W \,|\, \forall g \in G : g \cdot w = w\},$$

and W_G the quotient of **coinvariants**

$$W_G = W/\langle w - g \cdot w \,|\, w \in W, g \in G\rangle.$$

[5]An ideal I in operad $\mathcal{P}$ is a collection $I = \{I(n) \,|\, I(n) \subset \mathcal{P}(n)\}_{n\geq 0}$ of Σ_n-invariant subspaces such that for all $f, g \in \mathcal{P}$, $f \circ_i g$ is in I if $f \in I$ or $g \in I$.

[6]For unital operads also preserve the unit.

There are mutually inverse isomorphisms of vector spaces $\gamma_G : W^G \to W_G$ and $\gamma^G : W_G \to W^G$.

$$\gamma_G(w) = \frac{1}{|G|}[w]$$

for $w \in W^G \subset W$ where $[w] \in W_G$ denotes the equivalence class of w, and

$$\gamma^G([w]) = \sum_{g \in G} g \cdot w.$$

Remark 8. Formally, the Σ-module $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ can be viewed as functor from the groupoid of symmetric groups Γ_Σ to the category dgVect. However, category Γ_Σ is skeleton of the category Bij of finite sets and their bijections. Let us extend the functor to category Bij.[7] For C a finite set, $\texttt{card}(C) = n$, consider the coinvariant space

$$\mathcal{P}(C) = \left(\bigoplus_{f:[n] \xrightarrow{\cong} C} \mathcal{P}(n) \right)_{\Sigma_n}$$

where the right action for $p \in \mathcal{P}(n)$ is given as $\sigma(f, p) = (f\sigma, p\sigma)$.

Definition 9. *(biased definition)* An **operad** $\mathcal{P}$ consists of a collection

$$\{\mathcal{P}(C) \,|\, C \in \mathsf{Bij}\}$$

of dg vector spaces and two collections of degree 0 morphisms of dg vector spaces

$$\{\mathcal{P}(\rho) : \mathcal{P}(C) \to \mathcal{P}(C') \mid \rho : C \to C' \text{ a morphism in } \mathsf{Bij}\},$$
$$\{\circ_a : \mathcal{P}(C \sqcup a) \otimes \mathcal{P}(D) \to \mathcal{P}(C \sqcup D) \mid C, D \in \mathsf{Bij}\}.$$

These data are required to satisfy the following axioms:

1. $\mathcal{P}(1_C) = 1_{\mathcal{P}(C)}, \quad \mathcal{P}(\rho\sigma) = \mathcal{P}(\sigma)\ \mathcal{P}(\rho)$,
2. $\mathcal{P}(\rho|_C \sqcup \sigma)\ \circ_{\rho^{-1}(a)} = \circ_a\ \mathcal{P}(\rho) \otimes \mathcal{P}(\sigma)$,
3. $\circ_a(\circ_b \otimes 1) = \circ_b(1 \otimes \circ_a)$
   (respectively $\circ_a(\circ_b \otimes 1) = \circ_b(\circ_a \otimes 1)(1 \otimes \tau)$ where τ is a monoidal symmetry from category of vector spaces).

Whenever the expressions make sense.

Remark 10. Equivalently to the axiom 1., we can say there is a functor $\mathcal{P}$ from $\mathsf{Bij} to \mathsf{dgVect}$. If we consider only this axiom, the resulting structure would be called a dg Σ-module. Obviously, by forgetting the composition map, an operad gives rise to its underlying Σ-module.

All these notions are equivalent to their counterparts in Definition 1. For example, axiom 1. stands for the right Σ-actions, 2. expresses the equivariance and 3. expresses the associativity of the structure maps.

In the following, we will sometimes need a special type of permutations – shuffles and unshuffles. This is maybe a good moment to recall their definitions.

[7]In [30] this extension is called *linear species*.

Definition 11. A **shuffle** σ of type (p, q) is an element of Σ_{p+q} such that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p), \qquad \text{and} \qquad \sigma(p+1) < \ldots < \sigma(p+q).$$

Similarly an **unshuffle** ρ of type (p, q) is an element of Σ_{p+q} such that we have $\rho(i_j) = j$ for some $i_1 < i_2 < \ldots < i_p$, $i_{l+1} < \ldots < i_k$.

Hence, ρ is an unshuffle if ρ^{-1} is a shuffle.

1.1.2 Monoidal definition and combinatorial definition

Monads offer remarkably economical way of formalizing the notion of various "algebraic theories". There are monads corresponding to the theory of rings, theory of topological groups, etc. For more details and examples see Leinster [29]. The convention used here comes mainly from [35].

Monad on a category C can be defined as monoid in the monoidal category of endofunctors on C, formally:

Definition 12. Let C be a category and $(End(\mathsf{C}), \circ, 1_{\mathsf{C}})$ be a strict symmetric monoidal category of endofunctors on C.

A **monad** in C is a triple (M, μ, η) of a functor $M : \mathsf{C} \to \mathsf{C}$ together with two natural transformations $\mu : M \circ M \Rightarrow M$ and $\eta : 1_{\mathsf{C}} \Rightarrow M$ respecting associativity and unitality properties, i.e., $\forall x \in \mathsf{C} : \mu_x \circ \mu_{M(x)} = \mu_x \circ (M\mu_x)$, respective $\mu_x \circ M(\eta_x) = \mu_x \circ \eta_{M(x)} = 1_{M(x)}$.

Definition 13. An **algebra** (x, φ) **over monad** (M, μ, η) is an object $x \in \mathsf{C}$ together with a map $\varphi : M(x) \to x$ such that $\varphi \circ \eta_x = 1_x$ and $\varphi \circ \mu_x = \varphi \circ (M\varphi)$

As Hilger and Poncin [22] advise, there are now two possible ways how to apply this abstract definition. We can choose as a category C a category Vect, define operad $\mathcal{P}$ to be a monad on this category and the $\mathcal{P}$-algebra to be the algebra over this monad. See the following Remark 14. This is what is called the *monoidal definition* in Loday and Vallette [30].

Or we can choose as C a category of Σ-modules $\Sigma\text{-}\mathsf{Mod}_{\mathbb{K}}$ and define operad as algebra $(\mathcal{P}, \phi)$ over the tree monad (T, μ_T, η_T). See Remark 15. This gives us, as Loday and Vallette call it, a *combinatorial definition.*

Remark 14. Let us consider Σ-module $\{\mathcal{P}(n)\}_{n \geq 0}$ and a generic vector space V. Let us define

$$\mathcal{P}(V) = \bigoplus_n \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

Obviously, $\mathcal{P}$ is an endofunctor on the category of vector spaces Vect, known as *Schur functor.* The operad is then defined as Schur functor $\mathcal{P}$ with the composition map[8] $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and the unit map $\eta : I \to \mathcal{P}$ (where I is identity functor) which make $\mathcal{P}$ into monoid. For technical details (how to compose two Schur functors etc.) see section 5.1 in [30].

[8] It is important to note that here one must use May's definition of operad. May's definition of the composition $\gamma(i_1, \ldots i_n) : \mathcal{P}(n) \otimes \mathcal{P}(i_1) \otimes \ldots \mathcal{P}(i_n) \to \mathcal{P}(i_1 + \ldots i_n)$ may be recovered from the partial composition as

$$\gamma(i_1, \ldots i_n) = (- \circ_1 (\cdots (- \circ_{n-1} (- \circ_n -))\cdot))$$

Remark 15. Choose C to be a category $\Sigma\text{-}\mathsf{Mod}_{\mathbb{K}}$ of $\mathbb{K}[\Sigma]$-modules. The objects are collections $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ and the morphisms are maps $\alpha : \mathcal{P} \to \mathcal{Q}$ such that $\alpha_n : \mathcal{P}(n) \to \mathcal{Q}(n)$ are Σ_n-equivariant. For every $\mathcal{P}(n)$ we can consider its categorified version (see Remark 8). Let us denote the $\mathbb{K}[\Sigma]$-module for a general finite set A, $\mathtt{card}(A) = n$, as $\mathcal{P}((A))$.

Next, we need to introduce the category of graphs Tree_n. The object of this category are rooted non-planar trees with n leaves. In other words, the half-edges adjacent to every vertex could be split by their orientation. Exactly one of them is “outgoing” and the rest is “incoming”. Let us call the adjacent half-edges of vertex v as **legs** and denote by $iLeg(v)$ its *incoming legs*.

The outgoing half-edge of the whole graph is called root and the set of incoming half-edges of the graph are called leaves. The non-planarity means there is no specific embedding of the tree into the plane given (i.e. there is no specific ordering of incoming half-edges). The morphisms in this category preserve the labeling of the leaves. For a graph Γ in this category, we denote by $Vert(\Gamma)$ a set of its vertices.

Finally, we can define the endofunctor $T : \Sigma\text{-}\mathsf{Mod}_{\mathbb{K}} \to \Sigma\text{-}\mathsf{Mod}_{\mathbb{K}}$ as

$$(T\mathcal{P})(n) = \operatorname*{colim}_{\Gamma \in \mathrm{Iso}\mathsf{Tree}_n} \bigotimes_{v \in Vert(\Gamma)} \mathcal{P}((iLeg(v))),$$

where $\mathrm{Iso}\mathsf{Tree}_n$ is a subcategory of Tree_n where all morphisms are isomorphisms, i.e., core of Tree_n.

This endofunctor carries the structure of monad. The proof could be found in [30] as Lemma 5.6.2.[9]

Remark 16. Few additional remarks. We called the structure maps in Definition 1 “operadic composition” to distinguish it from the classical composition of functors. In the next, we sometimes omit the word ‘operadic’.

Notice, that for any Σ-module $\mathcal{P}$, $T(\mathcal{P})$ is a free operad. Any operad could be defined as a quotient of free operad by operadic ideal. In fact, the free operad functor $F : \Sigma\text{-}\mathsf{Mod}_{\mathbb{K}} \to \mathsf{Operad}$ is a left adjoint to the forgetful functor $\mathsf{Operad} \to \Sigma\text{-}\mathsf{Mod}_{\mathbb{K}}$. In general, any pair of adjoint functors give rise to a monad[10] and the composition of these two functors allows one to define the underlying monad of the combinatorial definition.

Taking operads in the same way as associative algebras, but in the different monoidal category, allows one to translate many of the constructions for associative algebras to operads. This applies to, for example, Koszul duality. More on this in Chapter 2.

1.2 Modular operads

As we have seen in the previous section, operads can be defined as algebras over monad of rooted trees. To define modular operads, trees are replaced by graphs. There is no chosen orientation on the edges of these graphs and the loops

[9] Roughly speaking - natural transformation μ is given by replacing vertices of a given tree by another trees with matching number of incoming half-edges. The natural transformation η is given by maping Σ_n-modules to tree with only one vertex and appropriate number of leaves.

[10] But not every monad is given by a pair of adjoint functors.

are allowed. But one needs to introduce labeling of vertices satisfying certain stability condition.

We begin this section with the biased definition and, for a moment, the specific properties of the "underlying" graphs can be only anticipated. The explicit definition will be given in 48.

Since modular operads were introduced by Getzler and Kapranov [17], we use many of their arguments. Nevertheless, the convention and notation are mostly taken from Doubek, Jurčo and Münster [11].

Definition 17. Denote Cor the **category of stable corollas**: the objects are pairs (C, G) with C a finite set and G a non-negative integer such that the **stability condition** is satisfied:

$$2(G-1)+\mathtt{card}(C)>0.$$

A morphism $(C, G) \to (D, G')$ is defined only if $G = G'$ and it is just a bijection $C \xrightarrow{\cong} D$.

Remark 18. The condition of *stability*, introduced by Getzler and Kapranov in [17], has its name from the theory of moduli spaces of curves. Recall, the moduli space of Riemann surfaces of genus g is **stable** if it does not admit an infinitesimal automorphisms. As we can see, the excluded combinations $(\mathtt{card}(C), G)$ are $(0,0), (1,0), (2,0), (0,1)$ corresponding to the spheres with less than three marked points and the torus without any marked point.

At first, this condition may seem a bit artificial for our purposes. One possible motivation could be found in physics. As we will see later, the stability condition could be understood as looking only on the interaction part of kinetic energy and ignoring the free part of the actional functional.

There is another motivation. The arguments in Remark 92 show that we need to "reduce" the category of operads to be able to define the cobar complex. From this point of view, the definition of modular operads 19 is already reduced and we are ready to construct "cobar complex for modular operads" (i.e., Feynman transform). This should explain the absence of unit to the reader.

Definition 19. A **modular operad** $\mathcal{P}$ consists of a collection

$$\{\mathcal{P}(C,G) \mid (C,G) \in \mathsf{Cor}\}$$

of dg vector spaces and three collections of degree 0 morphisms of dg vector spaces

$$\begin{gathered}\{\mathcal{P}(\rho):\mathcal{P}(C,G)\to\mathcal{P}(D,G)\mid \rho:(C,G)\to(D,G)\text{ a morphism in }\mathsf{Cor}\},\\ \{\,{}_a\circ_b:\mathcal{P}(C_1\sqcup\{a\},G_1)\otimes\mathcal{P}(C_2\sqcup\{b\},G_2)\to\mathcal{P}(C_1\sqcup C_2,G_1+G_2)\mid\\ (C_1,G_1),(C_2,G_2)\in\mathsf{Cor}\},\\ \{\,\circ_{ab}:\mathcal{P}(C\sqcup\{a,b\},G)\to\mathcal{P}(C,G+1)\mid (C,G)\in\mathsf{Cor}\}.\end{gathered}$$

These data are required to satisfy axioms

1. ${}_a\circ_b(x\otimes y)=(-1)^{|x||y|}\,{}_b\circ_a(y\otimes x)$
   for any $x\in\mathcal{P}(C_1\sqcup\{a\},G_1), y\in\mathcal{P}(C_2\sqcup\{b\},G_2)$,

2. $\mathcal{P}(1_C) = 1_{\mathcal{P}(C)}, \quad \mathcal{P}(\rho\sigma) = \mathcal{P}(\rho)\, \mathcal{P}(\sigma)$
   for any morphisms ρ, σ in Cor,
3. $(\mathcal{P}(\rho|_{C_1} \sqcup \sigma|_{C_2}))\ {}_a\circ_b = {}_{\rho(a)}\circ_{\sigma(b)}\ (\mathcal{P}(\rho) \otimes \mathcal{P}(\sigma))$
4. $\mathcal{P}(\rho|_C)\ \circ_{ab} = \circ_{\rho(a)\rho(b)}\, \mathcal{P}(\rho)$
5. $\circ_{ab}\ \circ_{cd} = \circ_{cd}\ \circ_{ab}$
6. $\circ_{ab}\ {}_c\circ_d = \circ_{cd}\ {}_a\circ_b$
7. ${}_a\circ_b\ (\circ_{cd} \otimes 1) = \circ_{cd}\ {}_a\circ_b$
8. ${}_a\circ_b\ (1 \otimes {}_c\circ_d) = {}_c\circ_d\ ({}_a\circ_b \otimes 1)$

whenever the expressions make sense.

Remark 20. As before, Axiom 2. stands for Σ-action, 3., 4. express the equivariance and 5., 6., 7. and 8. the associativity.

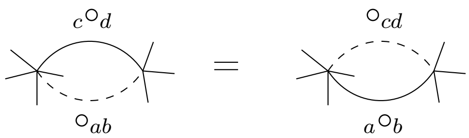


Figure 1.3: Axiom 6. pictorially.

We show some examples of modular operads in the next Section 1.2.1 when we introduce a *connected sum*. Now, let us make few remarks.

Remark 21. We assume for simplicity that all spaces $\mathcal{P}(C, G)$ are finite dimensional in each degree.

In the Chapter 2 we want to define the Feynman transform of modular operads. As was explained in [17] it is necessary to introduce a certain *twist*. For this reason we, similarly as in [11], consider also a special case of *twisted* modular operads, an **odd modular operad**. For the definition and more details about the twisting see Section 2.2.2. The operadic compositions of twisted modular operad, denoted by ${}_a\bullet_b$ and $\bullet_{ab}$, have degree 1 and the axioms 5.-8. are changed accordingly

5. $\bullet_{ab}\ \bullet_{cd} = -\bullet_{cd}\ \bullet_{ab}$
6. $\bullet_{ab}\ {}_c\bullet_d = -\bullet_{cd}\ {}_a\bullet_b$
7. ${}_a\bullet_b\ (\bullet_{cd} \otimes 1) = -\bullet_{cd}\ {}_a\bullet_b$
8. ${}_a\bullet_b\ (1 \otimes {}_c\bullet_d) = -{}_c\bullet_d\ ({}_a\bullet_b \otimes 1)$

We will also sometimes need a **skeletal version** of (odd) modular operad, $\overline{\mathcal{P}}$. The definition can be obtained be restriction of the underlying category Cor to corollas of the form $([n], G)$ (as we mentioned in Remark 8). We will also write just $\mathcal{P}(n, G)$ (instead of $\mathcal{P}([n], G)$). The explicit formulas of operadic compositions and corresponding axioms are inconveniently complicated, so we restrain from their explicit formulation (for more details see Section D in [11]).

1.2.1 Connected sum and examples of modular operads

We "enhance" the modular operads by a connected sum. This gives us a graded commutative associative product. The elegant geometrical interpretation in terms of homeomorphism classes of bordered Riemann surfaces from Zwiebach [45] will be still preserved as we will see in examples 25 and 28.

Definition 22. A **modular operad with connected sum** is a modular operad $\mathcal{P}$ equipped with a collection of degree 0 chain maps called connected sum defined on two components as

$$\#_2 : \mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \to \mathcal{P}(C \sqcup C', G + G' + 1) \tag{1.2}$$

and on one component as

$$\#_1 : \mathcal{P}(C, G) \to \mathcal{P}(C, G + 2) \tag{1.3}$$

such that

(CS1) $(\sigma \sqcup \sigma')\#_2 = \#_2(\sigma \otimes \sigma')$, $\sigma\#_1 = \#_1\sigma$ for all bijections $\sigma : C \to D$, $\sigma' : C' \to D'$,

(CS2) $\#_2\tau = \#_2$, where τ is monoidal symmetry (from category of vector spaces),

(CS3) $\#_2(1 \otimes \#_2) = \#_2(\#_2 \otimes 1)$, $\qquad \#_2(\#_1 \otimes 1) = \#_1\#_2$

(CS4) As maps $\mathcal{P}(C, G) \to \mathcal{P}(C - \{i, j\}, G + 3)$

$$\circ_{ij} \#_1 = \#_1 \circ_{ij}$$

(CS5a) As maps $\mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \to \mathcal{P}(C \sqcup C' - \{i, j\}, G + G' + 2)$,

$$\circ_{ij} \#_2 = \begin{cases} \#_2(\circ_{ij} \otimes 1) & \dots\ i, j \in C \\ \#_2(1 \otimes \circ_{ij}) & \dots\ i, j \in C' \\ \#_1\, {}_i\circ_j & \dots\ i \in C, j \in C' \\ \#_1\, {}_j\circ_i & \dots\ j \in C, i \in C' \end{cases}$$

(CS5b) As maps $\mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \to \mathcal{P}(C \sqcup C' - \{i, j\}, G + G' + 2)$,

$${}_i\circ_j(\#_1 \otimes 1) = \#_1\, {}_i\circ_j \qquad \dots\ i \in C, j \in C'$$

(CS6) As maps $\mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \otimes \mathcal{P}(C'', G'') \to \mathcal{P}(C \sqcup C' \sqcup C'' - \{i, j\}, G + G' + G'' + 1)$,

$${}_i\circ_j(1 \otimes \#_2) = \begin{cases} \#_2({}_i\circ_j \otimes 1) & \dots j \in C' \\ \#_2(1 \otimes {}_i\circ_j)(\tau \otimes 1) & \dots j \in C'' \end{cases}$$

where the map $(\tau \otimes 1) : \mathcal{P}(C, G) \otimes \mathcal{P}(C', G') \otimes \mathcal{P}(C'', G'') \to \mathcal{P}(C', G') \otimes \mathcal{P}(C, G) \otimes \mathcal{P}(C'', G'')$ switches the first two tensor factors, and $i \in C$.

Remark 23. The *connected sum for twisted modular operad* is defined precisely as in the normal, i.e., untwisted case. # is again a degree 0 operation. To make the distinction between twisted and untwisted case more explicit, we write the axioms (CS5a) and (CS6) evaluated on elements for our case, i.e., for odd modular operad.

If $p \in \mathcal{P}(C,G), p' \in \mathcal{P}(C',G')$ and $p'' \in \mathcal{P}(C'',G'')$, then in the untwisted case (CS5a)

$$\circ_{ij}(p \#_2 p') = \begin{cases} (\circ_{ij} p) \#_2 p' & \dots\ i,j \in C \\ p \#_2 (\circ_{ij} p') & \dots\ i,j \in C' \\ \#_1 (p\, {}_i\circ_j\, p') & \dots\ i \in C, j \in C' \\ \#_1 (p\, {}_j\circ_i\, p') & \dots\ j \in C, i \in C', \end{cases}$$

and in the odd case

$$\bullet_{ij}(p \#_2 p') = \begin{cases} (\bullet_{ij} p) \#_2 p' & \dots\ i,j \in C \\ p \#_2 (\bullet_{ij} p')(-1)^{|p|} & \dots\ i,j \in C' \\ \#_1 (p\, {}_i\bullet_j\, p') & \dots\ i \in C, j \in C' \\ \#_1 (p\, {}_j\bullet_i\, p') & \dots\ j \in C, i \in C'. \end{cases}$$

Axiom (CS6) in untwisted case

$$p\, {}_a\circ_b (p' \#_2 p'') = \begin{cases} (p\, {}_a\circ_b\, p') \#_2 p'' & \dots\ b \in C' \\ p' \#_2 (p\, {}_a\circ_b\, p'') & \dots\ b \in C'', \end{cases}$$

and in the odd case

$$p \bullet_{ab} (p' \#_2 p'') = \begin{cases} (p\, {}_a\bullet_b\, p') \#_2 p'' & \dots\ b \in C' \\ (-1)^{|p||p'|+|p'|} p' \#_2 (p\, {}_a\bullet_b\, p'') & \dots\ b \in C''. \end{cases}$$

Definition 24. The **skeletal version**

$$\overline{\#_2} : \overline{\mathcal{P}}(n_1, G_1) \otimes \overline{\mathcal{P}}(n_2, G_2) \to \overline{\mathcal{P}}(n_1 + n_2, G_1 + G_2 + 1)$$

of connected sum $\#_2$ is defined as

$$\overline{\#_2} \equiv (\theta_1 \sqcup \theta_2{}')^{-1} \#_2 (\theta_1 \otimes \theta_2),$$

where $\theta_1 : [n_1] \to C_1$ and $\theta_2 : [n_2] \to C_2$ are arbitrary bijections and $\theta_2{}'$ is a composition of order preserving map $n_1 + [n_2] \to [n_2]$ followed by θ_2.

One easily verifies that the definition of $\overline{\#_2}$ is independent of the choice of θ_1 and θ_2. The skeletal version of $\#_1$ is defined trivially as

$$\#_1(\overline{\mathcal{P}}(n,G)) = \overline{\mathcal{P}}(n, G+1).$$

The following two examples are taken from [11]. For a fuller treatment we refer the reader to ibid. Let us here just recall some of the basic properties, the geometrical interpretation, and their newly introduced connected sum.

Example 25. The Quantum Closed operad $\mathcal{QC}$. The components are given as one dimensional spaces $\mathcal{QC}(C,G) := \mathrm{Span}_{\Bbbk}\{C^G\}$, where C^G is a symbol of degree 0 and G satisfy that $\frac{G}{2} - \frac{\mathtt{card}(C)}{4} + \frac{1}{2}$ is integer. The connected sum is defined simply as

$$\begin{aligned} C_1^{G_1} \#_2 \, C_2^{G_2} &= (C_1 \sqcup C_2)^{G_1+G_2+1}, \\ \#_1 \left(C^G\right) &= C^{G+2}. \end{aligned}$$

In its geometrical interpretation, each component is a homeomorphism class of a connected compact orientable surface of genus g and set C of punctures in the interior, s.t. $G = 2g + \frac{\mathtt{card}(C)}{2} - 1$ is half-integer.[11] Obviously, we can permute the punctures freely among themselves. This operad would be also sometimes incorrectly called commutative since $\mathcal{QC}$ is the modular envelope of the cyclic operad Com.

The operadic composition ${}_i\circ_j$ corresponds to "sewing" the i-th puncture of one surface with the j-th puncture of the second surface. In the same manner, operadic self-composition $\circ_{ij}$ corresponds to sewing the i-th puncture with the j-th puncture of the same surface.

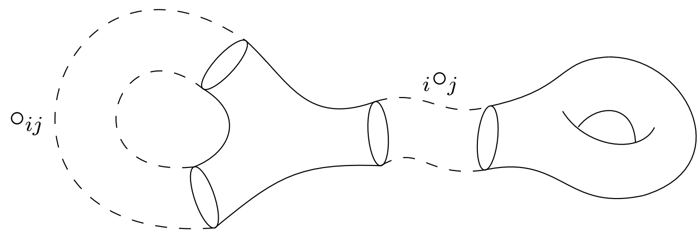


Figure 1.4: Operadic composition on Quantum closed operad.

The connected sum $\#_2$ corresponds to gluing a new "handle" between two surfaces. If we consider a surface with genus g_1 and punctures C_1 (in the component $\mathcal{QC}(C_1, G_1)$) and a surface with genus g_2, and punctures C_2 (in $\mathcal{QC}(C_2, G_2)$), then the resulting surface has

$$G = 2(g_1 + g_2) + \frac{\mathtt{card}(C_1) + \mathtt{card}(C_2)}{2} - 1.$$

In other words, the new surface is in the component $\mathcal{QC}(C_1 \sqcup C_2, G_1 + G_2 + 1)$.

And similarly, the connected sum $\#_1$ corresponds in this geometrical interpretation to gluing a new handle on one surface. The geometrical genus of the surface increase by one, i.e., the G of the new surface is given as $2(g+1) + \frac{\mathtt{card}(C)}{2} - 1$.

Remark 26. There is another possible definition of the Quantum Closed operad. In that case the components are simply given as $\mathcal{QC}(C,G) := \mathrm{Span}_{\Bbbk}\{C^G\}$ without

[11] In our definition, the components of an operad were indexed by integer. In general, we need just something isomorphic with integers.

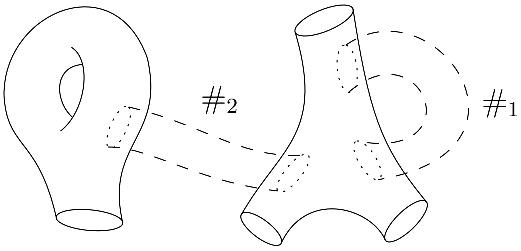


Figure 1.5: Connected sum on Quantum closed operad.

any restriction on G. The connected sum could be again defined, but we don't have its nice geometrical interpretation.

Nevertheless, as algebras both cases looks the same. The reason will be obvious from the definition of endomorphism modular operad (in Section 1.2.3).

Before we give the next example, we need to introduce a notion of a cycle.

Definition 27. The **cycle** in a set C is an equivalence class $(\!(x_1, \ldots, x_n)\!)$ of an n-tuple $(x_1, \ldots, x_n)$ of several distinct elements of C under the equivalence

$$(x_1, \ldots, x_n) \sim \sigma(x_1, \ldots, x_n),$$

where $\sigma \in \Sigma_n$ is the cyclic permutation $\sigma(i) = i + 1$ for $1 \leq i \leq n - 1$, and $\sigma(n) = 1$. In other words,

$$(\!(x_1, \ldots, x_n)\!) = \cdots = (\!(x_{n-i+1}, \ldots, x_n, x_1, \ldots, x_{n-i})\!) = \cdots = (\!(x_2, \ldots, x_n, x_1)\!) .$$

We call n the length of the cycle. We also admit the empty cycle $(\!()\!)$, which is a cycle in any set.

For a bijection $\rho : C \xrightarrow{\cong} D$ and a cycle $(\!(x_1, \ldots, x_n)\!)$ in C, define a cycle in D as

$$\rho\,(\!(x_1, \ldots, x_n)\!) := (\!(\rho(x_1), \ldots, \rho(x_n))\!) .$$

Example 28. The Quantum Open operad $\mathcal{QO}$. The components are given as

$$\mathcal{QO}(O, G) := \mathrm{Span}_{\Bbbk}\{\{\mathbf{o}_1, \ldots \mathbf{o}_b\}^g \mid b, g \in \mathbb{N}_0, \mathbf{o}_i \text{ cycle in } O, \bigsqcup_{i=1}^{b} \mathbf{o}_i = O,$$
$$G = 2g + b - 1\}.$$

The connected sum is defined in this case as

$$\begin{aligned} \{\mathbf{o}_1, \ldots \mathbf{o}_{b_1}\}^{g_1} \#_2 \{\mathbf{o}'_1, \ldots \mathbf{o}'_{b_2}\}^{g_2} &= \{\mathbf{o}_1, \ldots \mathbf{o}_{b_1}, \mathbf{o}'_1 \ldots \mathbf{o}'_{b_2}\}^{g_1+g_2}, \\ \#_1 (\{\mathbf{o}_1, \ldots \mathbf{o}_b\}^g) &= \{\mathbf{o}_1, \ldots \mathbf{o}_b\}^{g+1}. \end{aligned}$$

In geometrical interpretation, each element of $\mathcal{QO}$ is a homeomorphism class of a connected compact orientable surface with genus g, b boundaries and punctures O distributed on the boundaries according to the cycle-structure. The operadic composition ${}_i\circ_j$ and self-composition $\circ_{ij}$ are again defined as sewing punctures. Of course, only sewings resulting in orientable surfaces are allowed.

The connected sum again corresponds two gluing a new handle.

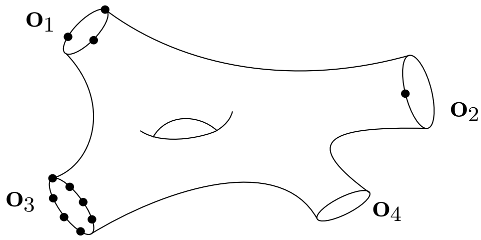


Figure 1.6: Element of $\mathcal{QO}(O, G)$ with $b = 4$ and $g = 1$.

Notice that the operadic structure of $\mathcal{QO}$ is not commutative but is (strictly) associative.

Remark 29. With the example of $\mathcal{QO}$ in the mind, we can give a nice justification for the shifts of the G-grading in the definition of the connected sum (1.2) and the reason for introducing $\#_1$ in (1.3).

The operadic self-composition $\circ_{ab}$ of quantum open part $\mathcal{QO}$ could be acting on elements on two different boundaries and so raise the geometrical genus of the surface by one. But it could also act on two elements on the same boundary, in which case the geometrical genus doesn't change but the number of boundaries increase by one. To keep our geometrical interpretation of $\mathcal{QO}$, we can simply introduce the grading by $G = \alpha g + \frac{\alpha}{2} b + \left(\frac{\alpha}{4} - \frac{1}{2}\right) \mathtt{card}(O) + 1 - \alpha$, where $\alpha \in \mathbb{N}$. Similarly, $\mathcal{QC}$ gives us $G = \alpha g + \frac{\alpha - 1}{2} \mathtt{card}(C) + 1 - \alpha$. We choose $\alpha = 2$ to have the same normalization as Zwiebach in [45] for the open part.

But for the Quantum closed modular operad, the output of the connected sum $\#_2$ on $\mathcal{P}(C, G) \otimes \mathcal{P}(C', G')$ with $G = 2g_1 + b_1 - 1$, $G' = 2g_2 + b_2 - 1$ is a Riemann surface in $\mathcal{P}(C \sqcup C', G'')$ with $G'' = 2(g_1 + g_2) + (b_1 + b_2) - 1 = G + G' + 1$.

From the axioms of connected sum, the map $\circ_{ab} \#_2$ should be equivalent to $\#_1 \, {}_a\circ_b$ for $a \in C$, $b \in C'$. The first map will obviously raise the grading in G by two from the definition of the modular operad and previous argument. Hence the map $\#_1$ also must raise the index by two.

After these two examples, one may get a misleading impression, that this geometrical interpretation of connected sum always work. In general, the connected sum of two components should be seen more like a tensor product with special behaviour with respect to the G-grading.

In Section 1.2.4 we show the appropriate combinatorial object on which the connected sum is based.

Example 30. These two examples can be easily combined into the third example a 2-coloured modular operad $\mathcal{QOC}$ introduced in [11].

It is necessary to first replace the category Cor by Cor_2 where the objects are triples (O, C, G) with O, C finite sets and G non-negative half-integer such that

$$2(G - 1) + \mathtt{card}(C) + \mathtt{card}(O) > 0.$$

Morphisms $(O, C, G) \to (O', C', G')$ are given by a pair of bijections $O \to O'$, $C \to C'$ such that $G = G'$.

All other definitions now adapt in obvious sense. For example components of the $\mathcal{QOC}$ operad are given by

$$\mathcal{QOC}(O,C,G) := \mathrm{Span}_{\Bbbk}\{\{\mathbf{o}_1,\dots\mathbf{o}_b\}^g_C \mid b,g\in\mathbb{N}_0, \mathbf{o}_i \text{ cycle in } O, \bigsqcup_{i=1}^{b}\mathbf{o}_i = O,$$
$$G = 2g+b+\frac{\mathtt{card}(C)}{2}-1\}.$$

For details about the composition see section 6.2 in [11].

Let us make a few remarks to compare our approach with others in the literature.

Remark 31. Note that when restricting to the $\mathcal{QC}$ we are in the case of [12] by Doubek, Jurčo and Pulmann.

In the commutative case, both G and g are preserved by ${}_i\circ_j$ and the structure map $\circ_{ij}$ raises them both by one. Therefore in [12], they could choose the grading by g. In their case, the grading is "recorded" by exponents of formal parameter $\hbar$, and the stability condition, imposed in our case by the definition of the modular operad, is forced by the lower bound of *weight grading*: $w = 2g+n>0$.

If we restrict ourselves to the trivial case of $\mathcal{QC}$, then connected sum $\#_2$ in (1.2) could be interpreted as the usual symmetric tensor product.[12] The connected sum $\#_1$ in (1.3) raise the genus g by one in the "geometrical" case and we can interpret it as multiplying by $\hbar$.

But for general operad, we required the consistent shifts in grading so we need to introduce the grading by G.

Remark 32. The structure defined in [40] by Schwarz may resemble the modular operads with connected sum we just defined.

The structure MO defined there is "almost equivalent" to the notion of modular operads. The operation $\sigma^{(m)}: P_m \to P_{m-2}$ corresponds to $\circ_{ij}$, the maps $\nu_{m,n}: P_m\times P_n \to P_{m+n}$ remind us the connected sum, and the composition $\sigma^{(m+n)}\circ(\rho,\tau)\circ\nu_{m,n}: P_m\times P_n \to P_{m+n-2}$ with appropriate permutations $\rho\in\Sigma_m, \tau\in\Sigma_n$ is equivalent to our ${}_i\circ_j$. The parallels could be found also in similar geometrical motivation (moduli spaces of Riemann surfaces of all genera). One subtle difference is, that instead of grading by G, [40] uses the grading by Euler characteristic $\chi = 2g+\mathtt{card}(C)-2$.

Nevertheless, the substantial difference is the absence of the map $\#_1$.

Remark 33. A parallel can be found also in [26] by Kaufmann, Ward and Zuniga. The structure defined there corresponds to the disjoint union of surfaces since it does not affect the grading by G and the map $\#_1$ is missing.

For the reader interested in the explicit comparison we recommend to see Remark 49 with the monoidal definition of connected sum since the *horizontal composition* of [26] is also given as a monad.

Notice, that in the open case the grading G has the following relation to Euler characteristic

$$G = 2g+b-1 = (2g+b-2)+1 = 1-\chi.$$

[12] Mentioned later in Remark 123.

From properties of Euler characteristic we know

$$\chi(M \sqcup N) = \chi(M) + \chi(N), \qquad \chi(M\#N) = \chi(M) + \chi(N) - \chi(S^n)$$

for M, N n-manifolds (the symbol # denotes standard connected sum), In our case $n = 2$, i.e., $\chi(S^n) = 1 - 0 + 1 = 2$. Therefore

$$G(M \sqcup N) = G(M) + G(N) - 1, \tag{1.4}$$

$$G(M\#N) = G(M) + G(N) + 1. \tag{1.5}$$

The equation (1.4) corresponds to approach of [26]. The second equation corresponds to our connected sum on modular operads. This subtle detail will affect the following and we will see its consequences in Section 3.2.1.

1.2.2 Unordered tensor product

Before we pay attention to the most important example of twisted modular operads – the twisted endomorphism modular operad, let us make a few technical observations, which will be useful for both twisted endomorphism modular operad, and later also for endomorphism properad.

In the next, we want to define a tensor product of a collection $\{V_c\}_C$ of graded vector spaces indexed by some set C. Since C is not ordered by default[13], we want also a tensor product that would not depend on any chosen order.

Definition 34. For any set $C, \mathtt{card}(C) = n$ and the vector space V we define the **unordered tensor product** as

$$\bigotimes_C V = \left(\bigoplus_{\psi : C \to [n]} V^{\otimes n} \right)_{\Sigma_n}$$

with the identifications are given as

$$(\psi, v_1 \otimes \ldots \otimes v_n)^\sigma = (\sigma\psi, \kappa(\sigma) v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)})$$

where $\sigma \in \Sigma_n$ and $\kappa(\sigma)$ is a Koszul sign of the permutation σ.

We denote by $i_\psi : V^{\otimes n} \hookrightarrow \bigoplus_{\psi : C \to [n]} V^{\otimes n}$ the canonical inclusion into the ψ-th summand.

Let us recall few useful lemmas about the unordered tensor product from Markl [32].

Lemma 35. Let $f : C \to D$ be an isomorphism of finite sets, $\psi : C \to [n]$, $\{V_c\}_{c \in C}$ and $\{W_d\}_{d \in D}$ collections of graded vector spaces, $V_c = W_d = V$ for all $c \in C$, $d \in D$. Then the assignment

$$\bigotimes_{c \in C} V_c \ni \left[v_{\psi^{-1}(1)} \otimes \cdots \otimes v_{\psi^{-1}(n)} \right] \longmapsto \left[w_{f\psi^{-1}(1)} \otimes \cdots \otimes w_{f\psi^{-1}(n)} \right] \in \bigotimes_{d \in D} V_d$$

with $w_{f\psi^{-1}(i)} = v_{\psi^{-1}(i)} \in V_{f\psi^{-1}(i)}$, $1 \le i \le n$, defines a natural map

$$F : \bigotimes_{c \in C} V_c \to \bigotimes_{d \in D} V_d$$

of unordered products

[13] As we have seen in Remark 8.

Proof. A direct verification. □

Lemma 36. For disjoint finite sets C_1, C_2, one has a canonical isomorphism

$$\bigotimes_{c\in C_1} V_c \otimes \bigotimes_{c\in C_2} V_c \cong \bigotimes_{c\in C_1\sqcup C_2} V_c.$$

Proof. Each $\psi_1 : C_1 \xrightarrow{\cong} [n]$ and $\psi_2 : C_2 \xrightarrow{\cong} [m]$ determine an isomorphism

$$\psi_1 \sqcup \psi_2 : C_1 \sqcup C_2 \xrightarrow{\cong} [n+m]$$

by the formula

$$(\psi_1 \sqcup \psi_2)^{-1}(i) := \begin{cases} \psi_1^{-1}(i), & \text{if } 1 \le i \le n \\ \psi_2^{-1}(i-n), & \text{if } n < i \le n+m. \end{cases}$$

The isomorphism of the lemma is then given by the assignment

$$\begin{aligned}&[v_{\psi_1^{-1}(1)} \otimes \cdots \otimes v_{\psi_1^{-1}(n)}] \otimes [v_{\psi_2^{-1}(1)} \otimes \cdots \otimes v_{\psi_2^{-1}(m)}] \\ &\qquad\qquad \mapsto [v_{(\psi_1\sqcup\psi_2)^{-1}(1)} \otimes \cdots \otimes v_{(\psi_1\sqcup\psi_2)^{-1}(n+m)}].\end{aligned}$$

□

Example 37. Let $C = \{c_1, \ldots, c_n\}$. By iterating Lemma 36 one obtains a canonical isomorphism

$$\bigotimes_{c\in C} V_c \cong V_{c_1} \otimes \cdots \otimes V_{c_n}$$

which, crucially, depends on the order of elements of C. In particular, for $C = [n]$, $V_c = V$, $c \in C$, we have an isomorphism $i_\psi : V^{\otimes n} \to \bigotimes_{[n]} V$ for every permutation ψ. In particular, we have the isomorphism $i_n := i_{1_{[n]}}$ corresponding to the natural ordering on the set $[n]$.

Example 38. An **endomorphism operad**. Let V be finite-dimensional dg vector space with a non-degenerate symmetric pairing of degree 0. The collection

$$\mathcal{EN}_V = \left\{ \bigotimes_C V \mid (C, G) \in \mathsf{Cor} \right\}$$

with operadic structure given by contracting indices give the simplest example of a modular operad. Notice that the grading by G is here purely formal.

1.2.3 Twisted endomorphism modular operad

Without endomorphism operad one is not able to talk about algebra over the modular operad. However, our motivation is to introduce algebras over the Feynman transform. As we show later in Section 2.2.2, Feynman transform produces twisted modular operads and thus the "ordinary" endomorphism modular operad from Example 38 is not sufficient. We need to introduce the *twisted endomorphism modular operad*. It turns out, that for our purposes is enough to introduce the endomorphism odd modular operad.

Before we give its definition, let us recall some properties of symplectic vector space and introduce some notation to shorten the formulas.

Definition 39. Let (V, d) be a dg vector space which is degre-wise finite. An **odd symplectic form** $\omega : V \otimes V \to \mathbb{K}$ of degree -1 is a nondegenerate graded-antisymmetric bilinear map[14]. If $d(\omega) = 0$, in other words

$$\omega(d \otimes 1_V + 1_V \otimes d) = 0$$

we call (V, d, ω) a **dg symplectic vector space**.

Remark 40. The condition $d(\omega) = \omega(d \otimes 1_V + 1_V \otimes d) = 0$ ensures that cohomology of d inherits a symplectic structure. We use this property later in Hodge decomposition (see Remark 149).

If $\{a_l\}$ is a homogeneous basis of V, define

$$b_k = \sum_l (-1)^{|a_l|} \omega^{kl} a_l = \sum_l (-1)^{|a_l|} (\omega(a_k, a_l))^{-1} a_l. \tag{1.6}$$

The fact that ω is degree -1 gives $|b_k| = 1 - |a_k|$. The basis $\{\phi^k\}$ of graded dual vector space V^*, dual to $\{a_k\}$, is defined by $\phi^k(a_l) = \delta_l^k$.

Remark 41. In finite dimensional vector spaces the non-degeneracy of ω gives an isomorphism $X : V \to V^*$, $a \mapsto \omega(a, \cdot)$. From this isomorphism, it is possible to define $\omega^* : V^* \otimes V^* \to \mathbb{K}$, $\omega^*(\alpha, \beta) = \omega(X^{-1}(\alpha), X^{-1}(\beta))$ such that matrix of ω^* is the inverse matrix of ω, i.e., $\omega_{ij} \cdot \omega^{jk} = \delta_i^k$.

In the infinite-dimensional case, this became a bit more complicated since ω^* is, in general, an element of $(V \otimes V)^{**}$ – a space that is much "bigger" than $V \otimes V$. But thanks to our assumptions, we can guess the inverse of ω.

First, let us fix the basis of $V = \bigoplus_i V_i$. Since each V_i is finite-dimensional, we can order the basis of V as $\{\{a_i\}_0, \{a_i\}_1, \{a_i\}_{-1}, \ldots \{a_i\}_k, \{a_i\}_{-k}, \ldots\}$ where $\{a_i\}_k$ is a basis of V_k and each of these basis can be picked in such a way that ω has a form

$$\begin{pmatrix} 0 & A_1 & 0 & 0 & \cdots \\ -A_1^T & 0 & 0 & 0 & \\ 0 & 0 & 0 & A_2 & \\ 0 & 0 & -A_2^T & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$$

where A_k is the regular matrix corresponding to the (non-degenerate) pairings of elements from V_k with elements from V_{-k+1}.

Therefore ω^{ij} as the components of the matrix inverse of $\omega_{ij} = \omega(a_i, a_j)$ are well-defined. So, as it is usual in the mathematical physics, we can consider instead of ω^* an element $s \in V \otimes V$ such that $\omega(s) = 1$.

Remark 42. Let us introduce a convenient notation. Recall the canonical inclusion $i_\psi : V^{\otimes n} \hookrightarrow \bigoplus_{\psi: C \to [n]} V^{\otimes n}$ into the ψ-th summand.

Let $I_\psi : V^{\otimes n} \to \bigotimes_C V$ denote the inclusion i_ψ followed by the natural projection. For $F \in \bigotimes_C V^* \subseteq (\bigotimes_C V)^*$ we denote the "ψ"-th component as

$$(F)_\psi = F \circ I_\psi : V^{\otimes n} \to \mathbb{K}^{\otimes n} \cong \mathbb{K}.$$

Hence $(F)_\psi = (F)_{\sigma\psi} \circ \sigma$ for any $\sigma \in \Sigma_n$.

[14]Note, that this means $\omega(u, v) \neq 0$ implies $|u| + |v| = 1$ and $\omega(v, u) = (-1)^{|v| \cdot |u| + 1} \omega(u, v)$.

Definition 43. The **endomorphism odd modular operad** $\mathcal{E}_V$ is a collection of dg vector spaces

$$\mathcal{E}_V(C, G) = \bigotimes_C V^*$$

with Σ-module structure $\mathcal{E}_V(\rho) : \mathcal{E}_V(C, G) \to \mathcal{E}_V(D, G)$ defined for any bijection $\rho : C \to D$ and an element $f \in \mathcal{E}_V(C, G)$ as $(\mathcal{E}_V(\rho)(f))_\psi = (f)_{\psi\rho}$ where we have the bijection $\psi : D \to [\mathtt{card}(D)]$.

Let us define the operadic composition: For any sets C_1, C_2, $\mathtt{card}(C_1) = n_1$, $\mathtt{card}(C_2) = n_2$, let $f \in \mathcal{E}_V(C_1 \sqcup \{i\}, G_1) \cong \bigotimes_{C_1 \sqcup \{i\}} V^*$, $g \in \mathcal{E}_V(C_2 \sqcup \{j\}, G_2)$ and $\psi : C_1 \sqcup C_2 \to [n_1 + n_2]$. Then

$$(f\ {}_i\bullet_j\ g)_\psi = \sum_k (-1)^{|f|+|g|}((f)_{\psi_1} \cdot (g)_{\psi_2})\ \Psi^{-1}(1^{\otimes n_1+n_2} \otimes a_k \otimes b_k)$$

(the symbol $\cdot$ denotes the concatenation product) where we first consider an extension of ψ as $\tilde{\psi} : C_1 \sqcup C_2 \sqcup \{i, j\} \to [n_1 + n_2 + 2]$, $\tilde{\psi}(c) = \psi(c)$ for any $c \in C_1 \sqcup C_2$ and $\tilde{\psi}(i) = n_1 + n_2 + 1, \tilde{\psi}(j) = n_1 + n_2 + 2$.

Then we define shuffle[15] $\Psi \in Sh(n_1 + 1, n_2 + 1)$ as

$$\Psi|_{[n_1]} = \tilde{\psi}\Big|_{C_1},\ \Psi(n_1+1) = \tilde{\psi}(i) \text{ and } \Psi|_{n_1+1+[n_2]} = \tilde{\psi}\Big|_{C_2},\ \Psi(n_1+n_2+2) = \tilde{\psi}(j).$$

Finally, define ψ_1, ψ_2 as compositions

$$\psi_1 : C_1 \xrightarrow{\tilde{\psi}|_{C_1}} \psi(C_1) \xrightarrow{\text{o.p.}} [n_1 + 1] \quad \text{and} \quad \psi_2 : C_2 \xrightarrow{\tilde{\psi}|_{C_2}} \psi(C_2) \xrightarrow{\text{o.p.}} [n_2 + 1]$$

where "o.p." means order preserving. Similarly, operadic self-composition for $f \in \bigotimes_{C \sqcup \{i,j\}} V^*$, $\mathtt{card}(C) = n$ and $\psi : C \to [n]$ is defined as

$$(\bullet_{ij}\ f)_\psi = \sum_k (-1)^{|f|}(f)_{\tilde{\psi}}(1^{\otimes n} \otimes a_k \otimes b_k)$$

where $\tilde{\psi}$ is an extension defined as $\tilde{\psi}(c) = \psi(c)$ for all $c \in C$, $\tilde{\psi}(i) = n + 1$, $\tilde{\psi}(j) = n + 2$.

This operad is equipped with differential given for $f \in \bigotimes_C V^*, \mathtt{card}(C) = n$ by

$$(df)_\psi = \sum_{i=0}^{n} (-1)^{|f|}(f)_\psi(1^{\otimes i} \otimes d \otimes 1^{n-i-1}).$$

Remark 44. There is a simple trick

$$\omega(x, y) = (-1)^{|x|} B(x, y)$$

how to get from antisymmetric bilinear form ω a symmetric form B.

Let $f \in \bigotimes_{[n+2]} V^*$, $\psi(i) = i$ for $i = 1, \ldots n$. Then from (1.6) we trivially have

$$\sum_k f(\cdots \otimes a_k \otimes b_k) = \sum_k f(\cdots \otimes a_k \otimes a_l)(-1)^{|a_k|}\omega^{kl} =$$
$$= \sum_l f(\cdots \otimes a_k \otimes a_l)(-1)^{|a_k|}\omega^{lk}(-1)^{|a_k|\cdot|a_l|+1} = \sum_l f(\cdots \otimes b_l \otimes a_l)(-1)^{|a_k|\cdot|a_l|}.$$

So although we introduced antisymmetric form, we are in fact "twisting" with the symmetric form. This remark will be useful later in Section 2.3.2.

[15] See Definition 11.

Definition 45. Let $\mathcal{P}$ be a twisted modular operad. An **algebra over twisted modular operad** $\mathcal{P}$ on a dg symplectic vector space V is a twisted modular operad morphism $\alpha : \mathcal{P} \to \mathcal{E}_V$, i.e., it is a collection

$$\{\alpha(C,G) : \mathcal{P}(C,G) \to \mathcal{E}_V(C,G) | (C,G) \in \mathsf{DCor}\}$$

of dg vector space morphisms such that (in the sequel, we drop the notation (C,G) at $\alpha(C,G)$ for brevity)

1. $\alpha \circ \mathcal{P}(\rho) = \mathcal{E}_V(\rho) \circ \alpha \quad$ for any bijection ρ,
2. $\alpha \circ ({}_a\circ_b)_{\mathcal{P}} = ({}_a\circ_b)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha)$,
3. $\alpha \circ (\circ_{ab})_{\mathcal{P}} = (\circ_{ab})_{\mathcal{E}_V} \circ \alpha$.

Remark 46. As was pointed in [32] by Markl, there exist two monoidal structures on Vect-enriched categories. For homogenous maps $f : V_1 \to V_2$, $g : W_1 \to W_2$ and homogenous elements $v \in V_1$, $w \in W_1$ one defines

$$(f \otimes g)(v \otimes w) = (-1)^X f(v) \otimes g(w)$$

where X equals to $|g| \cdot |v|$ or $|f| \cdot |w|$ (follow from the Koszul sign rule if we are applying the morphisms "from the left" or "from the right"). The first option is considered as the *standard monoidal structure*.

We use here a little modified definition of endomorphism operad then it is usual. The natural question is, whether we still have the standard monoidal structure.

As we mentioned in Remark 41, we can consider instead of ω an element $s \in V \otimes V$, $|s| = 1$. In [32] the operadic composition in twisted modular operad is converted to "expanding indexes" using s. The interpretation, which leads to standard monoidal structure uses the surprising inclusion $V^* \otimes W^* \hookrightarrow (V \otimes W)^*$.

Let us apply this in our modified version of endomorphism operad. In other words, ${}_a\bullet_b$-composition of $f \in \bigotimes_{S_1 \sqcup \{a\}} V^*$, $g \in \bigotimes_{S_2 \sqcup \{b\}} V^*$ is in our case, thanks to lemmas 35 and 36, given by composition

$$\bigotimes_{S_1 \sqcup \{a\}} V^* \otimes \bigotimes_{S_2 \sqcup \{b\}} V^* \xrightarrow{\cong} \bigotimes_{S_1} V^* \otimes \bigotimes_{\{a,b\}} V^* \otimes \bigotimes_{S_2} V^* \hookrightarrow$$
$$\hookrightarrow \bigotimes_{S_1} V^* \otimes \left(\bigotimes_{\{a,b\}} V\right)^* \otimes \bigotimes_{S_2} V^* \xrightarrow{1 \otimes s^* \otimes 1} \bigotimes_{S_1} V^* \otimes \bigotimes_{S_2} V^*.$$

Therefore $f\, {}_a\bullet_b\, g = (1 \otimes s^* \otimes 1)(f \otimes g)$ and similarly $\bullet_{ab} f = (1 \otimes s^*) f$.

The evaluation of, for example, ${}_a\bullet_b(1 \otimes {}_c\bullet_d) + {}_c\bullet_d({}_a\bullet_b \otimes 1)$ on arbitrary elements $x \otimes y \otimes z$ is then

$$(-1)^{|x|} x\, {}_a\bullet_b (y\, {}_c\bullet_d\, z) + (x\, {}_a\bullet_b\, y)\, {}_c\bullet_d\, z.$$

The motivation for our convention is that we later want to encode our "Quantum homotopy algebras", similarly as Barannikov in [3], as solutions S of the *quantum master equation*. In [3] these solutions encode the algebra over the Feynman transform such that the components (n, G) of the endomorphism operad are given by the finite tensor product. In the same time, we want to interpret these solutions S as the action. Therefore the solutions should be defined with elements of the dual space V^*.

Hence instead of $(\bigotimes_C V)^* = \mathrm{Hom}_{\mathbb{K}}(V^{\otimes C}, \mathbb{K})$ we choose to define the components as $\mathcal{E}_V(C,G) = \bigotimes_C V^*$. It is fortunate that this choice also leads to the standard monoidal structure of the composition of morphisms as we already discussed above.

In classical definition one puts $\mathcal{E}_V(C,G) = \bigotimes_C V$ but this leads to "non-standard" monoidal structure of composition of morphisms (see [32]). In mathematical physics, on the other hand, endomorphism operad is given by $\mathcal{E}_V(C,G) = (\bigotimes_C V)^*$. But even if V is a degree-wise finite-dimensional space, the components of $\mathcal{E}_V(C,G)$ would be in general infinite-dimensional.

Finally, the connected sum for endomorphism operad.

Theorem 47. Let $f \in \mathcal{E}_V(C_1,G_1) \cong \bigotimes_{C_1} V^*$, $g \in \mathcal{E}_V(C_2,G_2) \cong \bigotimes_{C_2} V^*$, where $n_1 = \texttt{card}(C_1)$, $n_2 = \texttt{card}(C_2)$ and $\psi : C_1 \sqcup C_2 \to [n_1+n_2]$. Then

$$(f \#_2 g)_\psi = ((f)_{\psi_1} \cdot (g)_{\psi_2})\Psi^{-1} \tag{1.7}$$

where $\Psi \in Sh(n_1,n_2)$ is defined as

$$\Psi|_{[n_1]} = \psi|_{C_1} \qquad \text{and} \qquad \Psi|_{n_1+[n_2]} = \psi|_{C_2}$$

and ψ_1, ψ_2 as compositions

$$\psi_1 : C_1 \xrightarrow{\psi|_{C_1}} \psi(C_1) \xrightarrow{\text{o.p.}} [n_1] \qquad \text{and} \qquad \psi_2 : C_2 \xrightarrow{\psi|_{C_2}} \psi(C_2) \xrightarrow{\text{o.p.}} [n_2]$$

where "o.p." means order preserving. Then $\mathcal{E}_V$ with the above defined operation $\#_2$ is odd modular operad with connected sum.

Proof. The proof itself is technical and rather tedious so we restrain from it and just mention two observations. First, it is not necessary to define $(\#_1 f)_\psi$ for $f \in \bigotimes_C$ since $\#_1$ in doesn't change the set C, only rise the G by two.

Second, it may seems that connected sum defined in (1.7) is not commutative as in Definition 22 in (CS2). But notice that here we are using shuffle $\Psi \in Sh(n_1,n_2)$. For $(-1)^{|f|\cdot|g|}(g \#_2 f)_\psi$ we would use shuffle $\Psi' \in Sh(n_2,n_1)$ defined as

$$\Psi'|_{[n_2]} = \psi|_{C_2} \qquad \text{and} \qquad \Psi'|_{n_2+[n_1]} = \psi|_{C_1}$$

The sign $(-1)^{|f|\cdot|g|}$ from monoidal symmetry will be canceled out by the Koszul signs of the shuffles and the choosen monoidal structure discussed in Remark 46. □

We introduce skeletal version of twisted endomorphism modular operad and its connected sum later in Section 3.3.1.

1.2.4 Connected sum as algebra over monad

Modular operads can be also defined as algebras over some monads. While the "combinatorial" structure of operads is captured by oriented trees, for modular operads it is given by a *stable graphs*. The following paragraphs are written in the same spirit as the Remark 15 and again, we use the notation and convention introduced in [35].

Remark 48. Let $\Sigma\text{-}\mathsf{MMod}_{\mathbb{K}}$ denote a category of modular stable Σ-modules. The objects are collections $\mathcal{P} = \{\mathcal{P}(n,G)\}$, where $\mathcal{P}(n,G)$ are $\mathbb{K}$-modules with a right Σ_n-action and satisfy $\mathcal{P}(n,G) = 0$ if $2G + n - 2 \leq 0$. The morphisms are maps $\alpha : \mathcal{P} \to \mathcal{Q}$ such that $\alpha(n,G) : \mathcal{P}(n,G) \to \mathcal{Q}(n,G)$ are Σ_n-equivariant.

Starting from a Σ_n-module $\mathcal{P}(n,G)$ we can easily define a right $Aut(S)$-module with an action of automorphisms of a general finite set S, $\mathtt{card}(S) = n$, i.e., with the use of categorified sets. We will denote such modules as $\mathcal{P}((S,G))$.

Next, let us introduce the category $\mathsf{MGr}(n,G)$ of stable (connected) labelled graphs. We label every vertex v by a non-negative integer $G(v)$. Graph Γ is then *stable* if for every vertex v holds

$$2(G(v) - 1) + \mathtt{card}(Leg(v)) > 0$$

where $Leg(v)$ is a set of half-edges adjacent to vertex v. A **genus of the graph** Γ is defined as

$$G(\Gamma) \equiv b_1(\Gamma) + \sum_{v \in Vert(\Gamma)} G(v),$$

where $b_1(\Gamma) \equiv \dim_{\mathbb{Q}} H_1(\Gamma, \mathbb{Q})$ is the first betti number and $Vert(\Gamma)$ is a set of vertices.[16] Objects in the category $\mathsf{MGr}(n,G)$ are stable graphs of genus G with n legs. A morphism $f : \Gamma_0 \to \Gamma_1$ of stable graphs is a morphism of the underlying graphs such that genus of vertex v of Γ_1 is equal to the genus of $f^{-1}(v)$.

Finally, we can define the endofunctor $T : \Sigma\text{-}\mathsf{MMod}_{\mathbb{K}} \to \Sigma\text{-}\mathsf{MMod}_{\mathbb{K}}$ as

$$(T\mathcal{P})(n,G) = \underset{\Gamma \in \mathrm{Iso}\mathsf{MGr}(n,G)}{\mathrm{colim}} \bigotimes_{v \in Vert(\Gamma)} P((Leg(v), G(v))),$$

where $\mathrm{Iso}\mathsf{MGr}(n,G)$ is a subcategory of $\mathsf{MGr}(n,G)$ where all of its morphisms are isomorphisms, i.e., core of $\mathsf{MGr}(n,G)$, and we used the previously mentioned categorification of sets. This functor carries also a monad structure which we will shortly denote as (T, μ_T, η_T). The construction and the proof could be found in [17] or with even more technical details in section 5.3 of [35]. The modular operads are algebras $(\mathcal{P}, \phi)$ over this monad.

Having this example in our mind we see, we can define the connected sum also as the algebra over some particular monad. Let us define the category $\mathsf{ColCor}(n,G)$, the category of collections of stable corollas.

Definition 49. A **collection of corollas** (one-vertex stable graphs) Λ is an unordered finite set

$$\{\Gamma_1, \ldots \Gamma_k \mid \Gamma_i \in \mathsf{MGr}(n_i, G_i), \mathtt{card}(Vert(\Gamma_i)) = 1\}.$$

A morphism $f : \Lambda_0 \to \Lambda_1$ of collectionsis a surjection $f_V : Vert(\Lambda_0) \to Vert(\Lambda_1)$ and a bijection $f_L : Leg(\Lambda_0) \to Leg(\Lambda_1)$ such that if a half-edge t is adjacent to a vertex $w \in Vert(\Lambda_0)$, then $f_L(t)$ is adjacent to a vertex $f_V(w) \in Vert(\Lambda_1)$.

Let $\mathsf{ColCor}(n,G)$ denote the **category of collections of stable corollas**. The objects are

$$\Lambda = \{\Gamma_1, \ldots, \Gamma_k \mid \Gamma_i \in \mathsf{MGr}(n_i, G_i);\ \mathtt{card}(Vert(\Gamma_i)) = 1;\ n_1 + \ldots + n_k = n;$$
$$G_1 + \ldots + G_k = G - k + 1 - 2s, s \in \mathbb{N}_0\}.$$

[16] We define genus of the graph by this formula also for the disconnected graphs, i.e., for graphs with $b_o(\Gamma) \geq 1$.

A morphism $f : \Lambda_0 \to \Lambda_1$ of labelled collections is a morphism of underlying collections such that for all $\Gamma_i \in \Lambda_1$:

$$G(\Gamma_i) = G(f^{-1}(\Gamma_i)) + b_0(f^{-1}(\Gamma_i)) - 1 + 2s, s \in \mathbb{N}_0.$$

Remark 50. Since all the corollas in the collections of $\mathsf{ColCor}(n, G)$ are stable, we have upper bound on the number of elements in the collections:

$$k < \frac{2G + n}{2}.$$

It follows, there is a finite number of isomorphism classes of objects in this category, which we will denote Iso$\mathsf{ColCor}(n, G)$.

Therefore, we can define an endofunctor $S : \Sigma\text{-}\mathsf{MMod}_{\mathbb{K}} \to \Sigma\text{-}\mathsf{MMod}_{\mathbb{K}}$ as

$$(S\mathcal{P})(n, G) = \operatorname*{colim}_{\Lambda \in \mathrm{Iso}\mathsf{ColCor}(n,G)} \bigotimes_{\Gamma_i \in \Lambda} \mathcal{P}((n_i, G_i))$$

This functor carries the monad structure (S, μ_S, η_S). In order to prove this, one uses in analogy with the construction in section 3 of [17] the nerve of the category $\mathsf{ColCor}(n, G)$. The natural transformations μ_S, η_S are again induced by the face functor ∂_1 : Nerve$_1(\mathsf{ColCor}(n, G)) \to$ Nerve$_0(\mathsf{ColCor}(n, G))$ and by inclusion of the terminal object (corolla of genus G with n legs) into $(S\mathcal{P})(n, G)$. One can consider an algebra over this monad $(\mathcal{P}, \psi)$.

Before discussing the compatibility of the modular operad and the connected sum, let us show that this definition is equivalent to the connected sum introduced in Definition 22.

Definition 51. Following [17] and their notion of contraction of graph, we introduce **collapse of collection**.

Let $\Lambda \in \mathsf{ColCor}(n, G)$ and let $J_1, J_2, \dots J_k \subset Vert(\Lambda)$ be a pairwise disjoint subsets of vertices of Λ. Then there is unique collection $\Lambda/\{J_1, \dots J_k\}$ with the following properties

(i) $Vert(\Lambda/\{J_1, \dots J_k\})$ is obtained from $Vert(\Lambda)$ by replacing the set J_i by a new vertex v_i;

(ii) $\Lambda \to \Lambda/\{J_1, \dots J_k\}$ is a morphism of labelled collections.

Any morphism of labelled collections is isomorphic to a morphism of this form.

It was shown in [17] that for the modular operad $\mathcal{P}$, seen as stable Σ-modules with maps[17] $\mu_\Gamma : \mathcal{P}(\Gamma) \to T\mathcal{P}(n, G) \xrightarrow{\phi} \mathcal{P}(n, G)$ for $\Gamma \in \mathsf{MGr}(n, G)$, one can define natural maps $\mu_{\Gamma \to \Gamma \setminus I} : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma \setminus I)$ where $I \subset Edge(\Gamma)$.

Let us denote $\mu_\Lambda : \mathcal{P}(\Lambda) \to S\mathcal{P}(n, G) \xrightarrow{\psi} \mathcal{P}(n, G)$ for $\Lambda \in \mathsf{ColCor}(n, G)$. In analogy with [17], one can define maps $\mu_{\Lambda \to \Lambda \setminus J} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda \setminus J)$. The special case when $k = 1$, $\texttt{card}(J_1) = 2$, and $G(v_1) = G(J_1) + 2 - 1 + 0$ corresponds to the map $\#_2$ from Definition 22. And similarly, the special case when $k = 1$, $\texttt{card}(J_1) = 1$, and $G(v_1) = G(J_1) + 1 - 1 + 2$ corresponds to the map $\#_1$. It follows:

[17] Given by composition of the universal map into the colimit and the structure map ϕ.

Theorem 52. An algebra $(\mathcal{P}, \psi)$ over monad (S, μ_S, η_S) gives rise to a functor $\mathcal{P}$ from ColCor to the category of dg vector spaces with connected sum satisfying axioms (CS1), (CS2), and (CS3) from the Definition 22.

Proof. Thanks to Definition 51 we are in the same situation as in Theorem 3.7 of [17] and the same arguments applies here. Since some of them will be used in the proof of the following theorem, we refrain from repeating them twice. Let us only make two comments. First, for one-component collection with $s = 0$ the natural morphism ψ corresponds to the identity morphism in the category $\mathsf{ColCor}(n, G)$.

Second, the case of collapsing two disjoint one-element subsets[18] J_1, J_2 corresponds in the notation of Definition 22 to $\#_1 \otimes \#_1$. Independence of the ordering of the subsets J_1, J_2 correspond to the condition

$$\#_1 \otimes \#_1 \equiv (\#_1 \otimes 1)(1 \otimes \#_1) = (1 \otimes \#_1)(\#_1 \otimes 1).$$

This property is hidden in Definition 22 in the fact that we work over 2-category of $Vect$-enriched categories. And similarly collapsing two disjoint two-element subsets corresponds to condition $\#_2 \otimes \#_2 = (\#_2 \otimes 1)(1 \otimes 1 \otimes \#_2)$. □

The natural question that arises is whether there is the compatibility required by the rest of the axioms in Definition 22 between the modular operad structure and the connected sum. In other words, since we have two possible options of how to compose functors T and S one may ask if there are natural transformations $\alpha(\mathcal{P}) : ST(\mathcal{P}) \to TS(\mathcal{P})$ and $\beta(\mathcal{P}) : TS(\mathcal{P}) \to ST(\mathcal{P})$ between these two compositions corresponding to the axioms (CS4)-(CS6).

Intuitively, we can understand applying functor T (defining a modular operad) on a Σ-module $\mathcal{P}$ as creating all possible connected graphs with vertices decorated by elements from the appropriate component of $\mathcal{P}$.[19] Applying functor S could similarly be seen as making all possible collections of stable corollas. A class of graphs underlying to $ST(\mathcal{P})$ then contains, for example, a collection of stable graphs K.[20] A Σ-module $ST(\mathcal{P})(K)$ is of the form

$$\bigotimes_{\substack{k\text{-th graph} \\ \text{of collection}}} \bigotimes_{\substack{i\text{-th vertex} \\ \text{of graph}}} \mathcal{P}(n_i^k, G_i^k). \tag{1.8}$$

Similarly, a class of graphs underlying to $TS(\mathcal{P})$ contains a graph L with vertices replaced by collections of corollas[21] with corresponding Σ-module

$$\bigotimes_{\substack{j\text{-th vertex} \\ \text{of graph}}} \bigotimes_{\substack{l\text{-th graph} \\ \text{of collection}}} \mathcal{P}(n_j^l, G_j^l). \tag{1.9}$$

Let us consider the natural transformation $\alpha(\mathcal{P}) : ST(\mathcal{P}) \to TS(\mathcal{P})$ that reorders the tensor products of (1.8) to obtain tensor products of the form (1.9) and

[18] In proof of [17], this would be analogical to the case of two loop-edges without a common vertex.

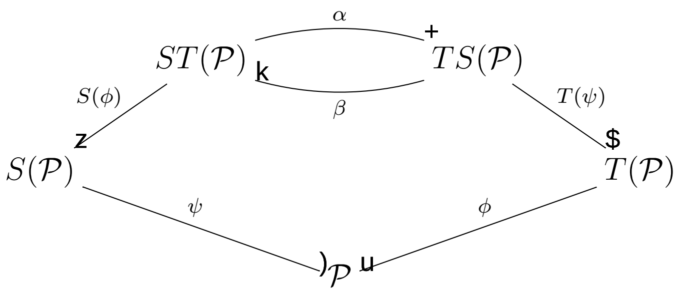
[19] For vertex of genus G with n legs we use some element from component $\mathcal{P}(n, G)$.

[20] I.e. unordered finite set of stable graphs $\Gamma_i \in \mathsf{MGr}(n_i, G_i)$ such that $G_1 + \dots G_k = G - k + 1 - 2s$, $s \in \mathbb{N}_0$ but without any condition on number of vertices of Γ_i. Examples of such graphs can be seen in the left upper corners of figures 1.7, 1.8, 1.9, 1.10.

[21] Examples of such graphs can be seen in the right upper corners of 1.7, 1.8, 1.9, and 1.10.

change (reinterpret) the underlying collection K of stable graphs to the graph L with vertices replaced by collections by pulling the edges of the graphs out of the collection and considering vertices of the graphs as independent corollas.[22] Similarly the natural transformation $\beta(\mathcal{P}) : TS(\mathcal{P}) \to ST(\mathcal{P})$ reorders tensor products and change the underlying graph L of collection to the collection K of graphs by joining all disjoint collections with $s_1, s_2, \ldots, s_k$ together into one collection with $s_1 + s_2 + \ldots + s_k$ and putting all edges inside this new collection. More formally:

Theorem 53. Let $(\mathcal{P}, \phi)$ be an algebra over the monad (T, μ_T, η_T), i.e., a modular operad. Assume that $(\mathcal{P}, \psi)$ carries also the structure of an algebra over the monad (S, μ_S, η_S) defined above, i.e., stable Σ-module equipped with a connected sum. Let α and β be natural transformations described above. Then the commutativity of two pentagons in the following diagram is equivalent to the compatibility of the connected sum and operadic compositions given by axioms in Definition 22.



Proof. The axioms (CS4)-(CS6) give us the commutativity of two pentagons:

Let us consider an element in $ST\mathcal{P}(n, G)$ given by some collection of stable graphs K. The map $\mathcal{P}(K) \to \mathcal{P}(n, G)$ is equivalent to composition

$$\mathcal{P}(K) \xrightarrow{\mathcal{P}(\pi_I)} \mathcal{P}(\Lambda) \xrightarrow{\mathcal{P}(\pi_J)} \mathcal{P}(n, G)$$

where I are subsets of edges (π_I corresponds to contractions of subgraphs) and J is a subset of connected components (π_j corresponds to collapse of collection). By arguments of Theorem 3.7 of [17], both $\mathcal{P}(\pi_I)$ and $\mathcal{P}(\pi_J)$ are independent of the ordering of the elements of I and J. Let us write this, very inaccurately, as

$$\mathcal{P}(K_1) \xrightarrow{\circ} \ldots \xrightarrow{\circ} \mathcal{P}(K_{m_1}) \xrightarrow{\circ} \mathcal{P}(\Lambda_1) \xrightarrow{\#} \mathcal{P}(\Lambda_2) \xrightarrow{\#} \ldots \xrightarrow{\#} \mathcal{P}(\Lambda_{m_2}) \xrightarrow{\#} \mathcal{P}(n, G)$$

where K_i are collections of graphs, $\Lambda_i \in \mathsf{ColCor}(n, G)$, arrows $\xrightarrow{\circ}$ represent the contractions of edges by maps ${}_i\circ_j$ or $\circ_{ij}$, and arrows $\xrightarrow{\#}$ represent the maps $\#_2$ or $\#_1$. Let us look closely on possible cases of composition $\mathcal{P}(K_{m_1}) \xrightarrow{\circ} \mathcal{P}(\Lambda_1) \xrightarrow{\#} \mathcal{P}(\Lambda_2)$. If the map $\#$ is not acting on the vertex resulting from $\circ$, we can obviously interchange their order. If the map $\#$ acts on the vertex resulting from $\circ$, we are in one of the following possible situations

$$\#_1 \circ_{ij} \qquad \#_1\, {}_i\circ_j \qquad \#_2(\circ_{ij} \otimes 1) \qquad \#_2({}_i\circ_j \otimes 1).$$

[22]This is possible since every graph has only finitely vertices and every collection has only finitely many components.

By axioms (CS4), (CS5a), (CS5b), (CS6) we can interchange the order of the maps $\#$, $\circ$ and consider the map $\mathcal{P}(K) \to \mathcal{P}(n, G)$ as a map

$$\mathcal{P}(K) \to STST\mathcal{P}(n, G) \to \mathcal{P}(n, G).$$

A class of graphs underlying to $STST\mathcal{P}$ is composed of collections of stable graphs where vertices are replaced by collections of stable graphs. Obviously, K_i are trivially contained in this class.

$$\mathcal{P}(K_1) \xrightarrow{\circ} \ldots \xrightarrow{\circ} \mathcal{P}(K_{m_1}) \xrightarrow{\#} \mathcal{P}(K_1') \xrightarrow{\circ} \mathcal{P}(\Lambda_2) \xrightarrow{\#} \mathcal{P}(\Lambda_3) \xrightarrow{\#} \ldots \xrightarrow{\#} \mathcal{P}(n, G)$$

Now we can repeat the same argument for $\mathcal{P}(K_1') \xrightarrow{\circ} \mathcal{P}(\Lambda_2) \xrightarrow{\#} \mathcal{P}(\Lambda_3)$ until we get to

$$\mathcal{P}(K_1) \xrightarrow{\circ} \ldots \xrightarrow{\circ} \mathcal{P}(K_{m_1-1}) \xrightarrow{\circ} \mathcal{P}(K_{m_1}) \xrightarrow{\#} \mathcal{P}(K_1'') \xrightarrow{\#} \ldots \xrightarrow{\#} \mathcal{P}(K_{m_2}'') \xrightarrow{\circ} \mathcal{P}(n, G)$$

seen as a map $\mathcal{P}(K) \to TST\mathcal{P}(n, G) \to \mathcal{P}(n, G)$. We can describe this procedure as "choosing an edge and pulling it out of all collections".

We repeat the same arguments for $\mathcal{P}(K_{m_1-1}) \xrightarrow{\circ} \mathcal{P}(K_{m_1}) \xrightarrow{\#} \mathcal{P}(K_1'')$ and so on. At the end we get

$$\mathcal{P}(K_1) \xrightarrow{\#} \mathcal{P}(K_2''') \xrightarrow{\#} \ldots \xrightarrow{\#} \mathcal{P}(K_{m_2}''') \xrightarrow{\circ} \mathcal{P}(\Gamma_1) \xrightarrow{\circ} \ldots \xrightarrow{\#} \mathcal{P}(\Gamma_{m_1}) \xrightarrow{\circ} \mathcal{P}(n, G).$$

This map can be seen as $\mathcal{P}(K) \to TS\mathcal{P}(n, G) \to \mathcal{P}(n, G)$. One may say "all edges were pulled out of all collections". As we have shown, using axioms (CS4)-(CS6) we can reinterpret the underlying graph and get equivariant map $\alpha(\mathcal{P}) : ST(\mathcal{P}) \to TS(\mathcal{P})$. For $\beta(\mathcal{P}) : TS(\mathcal{P}) \to ST(\mathcal{P})$ consider completely analogical method of putting all edges inside one collection (and joining all collections together).

Obviously, in both cases we need to consider at most $TSTST\mathcal{P}$. We could, of course, choose a different procedure, but this way we are not getting to absurdly long chaining of functors S and T.[23]

The commutativity of two pentagons gives the axioms (CS4)-(CS6): To illustrate the arguments of this part, we demonstrate them separately on the simplest examples. The general case will then be a combination of these four cases. We visualise the collection by a circle marked with an index $s \in \mathbb{N}_0$.

- Axiom (CS4): For an element of $ST(\mathcal{P})(n, G+3)$ think of a collection which consist of one-vertex graph of $\mathsf{MGr}(n, G)$ with one edge and vertex decorated by $a \in \mathcal{P}(n+2, G)$. The natural transformation $S(\phi)$ gives us collection of one one-vertex graph without any edges, decorated by element $\circ_{ij} a \in \mathcal{P}(n, G+1)$. Consequently ψ gives us one-vertex graph decorated by $\#_1 \circ_{ij} a \in \mathcal{P}(n, G+1+2)$.

  The natural transformation $\alpha(\mathcal{P})$ gives us element of $TS(\mathcal{P})(n, G+3)$ which is a collection of one one-vertex graph decorated by $a \in \mathcal{P}(n+2, G)$ with one edge attached to this collection. Application of the natural transformation $T(\psi)$ gives us one-vertex graph with one edge, decorated by element $\#_1 a \in \mathcal{P}(n+2, G+2)$. Finally, applying ϕ gives us one-vertex graph decorated by element $\circ_{ij} \#_1 a \in \mathcal{P}(n, G+2+1)$ which is equivalent to $\#_1 \circ_{ij} a$ under the axiom (CS4).

[23] We already got to Monty Python-like situation: *"They've taken everything we had, and not just from us, from our fathers, and from our fathers' fathers. And from our fathers' fathers' fathers. And from our fathers' fathers' fathers' fathers..."*

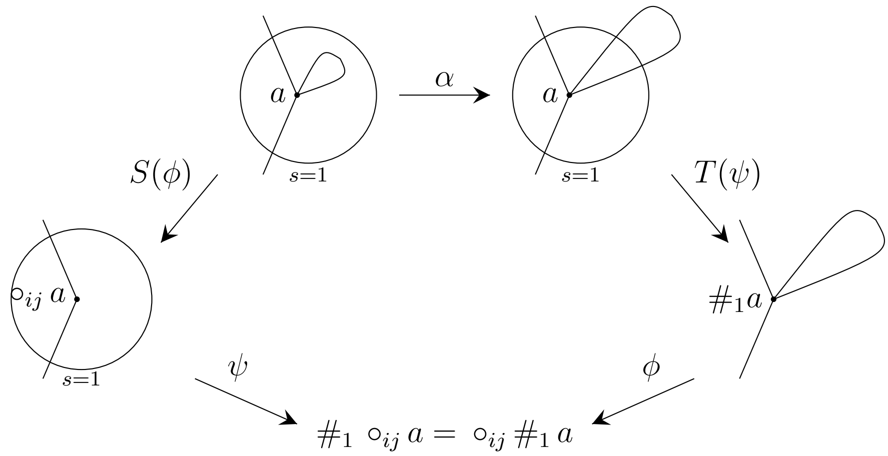


Figure 1.7: Commutativity of the diagram with $\alpha(\mathcal{P}) : ST(\mathcal{P})(n, G+3) \to TS(\mathcal{P})(n, G+3)$ for graph with one vertex and one edge is equivalent to axiom (CS4).

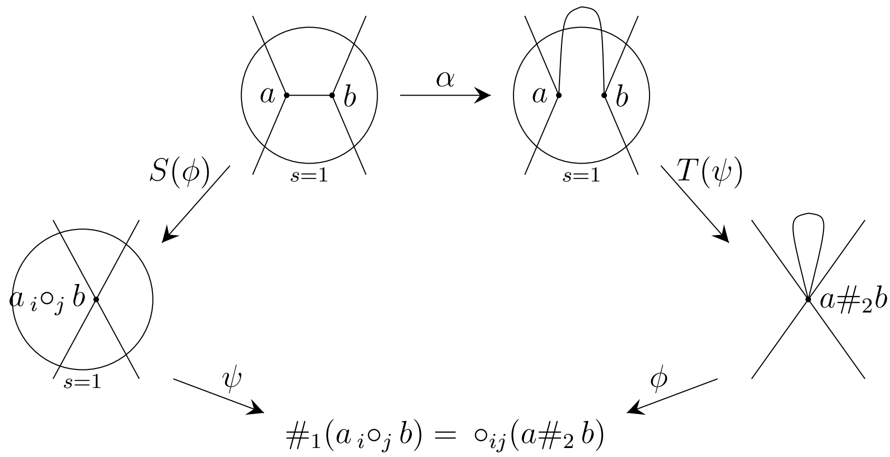


Figure 1.8: Commutativity of the diagram with $\alpha(\mathcal{P}) : ST(\mathcal{P})(n, G+2) \to TS(\mathcal{P})(n, G+2)$ for graph with two vertices connected by one edge is equivalent to axiom (CS5a).

- Axiom (CS5a): For an element of $ST(\mathcal{P})(n_1+n_2, G_1+G_2+2)$ think of a graph of $\mathsf{MGr}(n, G)$ with two vertices connected by one edge and decorated by $a \in \mathcal{P}(n_1+1, G_1), b \in \mathcal{P}(n_2+1, G_2)$. Natural transformation $S(\phi)$ gives us one-vertex graph decorated by $a\,{}_i\circ_j\, b \in \mathcal{P}(n_1+n_2, G_1+G_2)$ and ψ gives us one-vertex graph decorated by $\#_1(a\,{}_i\circ_j\, b) \in \mathcal{P}(n_1+n_2, G_1+G_2+2)$.

  $\alpha(\mathcal{P})$ gives us an element of $TS(\mathcal{P})(n_1+n_2, G_1+G_2+2)$ which is a collection of two one-vertex graphs and there is one edge attached to this collection. Natural transformation $T(\psi)$ gives us one-vertex graph with one edge. The vertex is decorated by $a\#_2\, b \in \mathcal{P}(n_1+n_2+2, G_1+G_2+1)$. Finally, applying ϕ gives us one-vertex graph decorated by element $\circ_{ij}(a\#_2\, b) \in \mathcal{P}(n_1+n_2, G_1+G_2+2)$ which is equivalent to $\#_1(a\,{}_i\circ_j\, b)$ under the axiom (CS5a).[24]

[24] The rest of the cases of (CS5a) are done similarly. For example, $(\circ_{ij}\, a)\#_2\, b = \circ_{ij}(a\#_2\, b)$ is given by the commutativity of the diagram for a collection of two one-vertex graphs such that one of them has one edge.

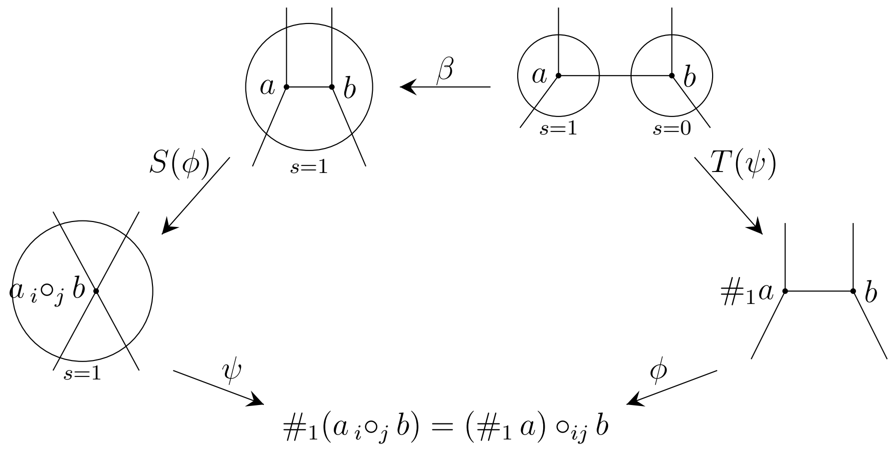


Figure 1.9: Commutativity of the diagram with $\beta(\mathcal{P}) : TS(\mathcal{P})(n, G+2) \to ST(\mathcal{P})(n, G+2)$ for graph with two vertices connected by one edge is equivalent to axiom (CS5b).

- Axiom (CS5b): For an element of $TS(\mathcal{P})(n_1+n_2, G_1+G_2+2)$ think of two collections connected by one edge. In both collection is just one one-vertex graph with vertex decorated by $a \in \mathcal{P}(n_1+1, G_1)$, and $b \in \mathcal{P}(n_2+1, G_2)$, respectively. The natural transformation $T(\psi)$ gives us one graph with two vertices connected by one edge and decorated by $\#_1\, a \in \mathcal{P}(n_1+1, G_1+2)$ and b. The natural transformation ϕ gives us one-vertex graph decorated by $(\#_1\, a)\,{}_i{\circ}_j\, b$.

  Natural transformation $\beta(\mathcal{P})$ gives us element of $ST(\mathcal{P})(n_1+n_2, G_1+G_2+2)$ that is a collection of a graph with two vertices connected by one egde. The vertices are decorated again by a, b. $S(\phi)$ gives us a collection of one-vertex graph decorated by $a\,{}_i{\circ}_j\, b$. Finally, applying ψ gives us one-vertex graph decorated by $\#_1(a\,{}_i{\circ}_j\, b) \in \mathcal{P}(n_1+n_2, G_1+G_2+2)$ which is equivalent to $(\#_1\, a)\,{}_i{\circ}_j\, b$ under the axiom (CS5b).

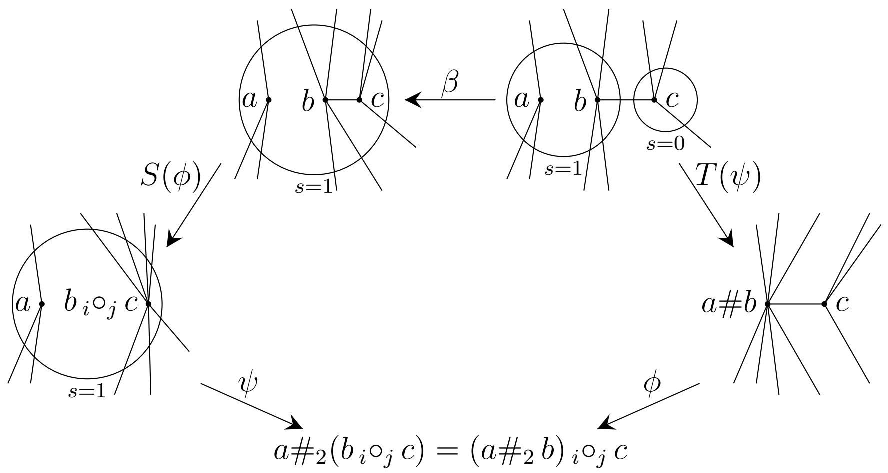


Figure 1.10: Commutativity of the diagram with $\beta(\mathcal{P}) : TS(\mathcal{P})(n, G+1) \to ST(\mathcal{P})(n, G+1)$ for graph with three vertices, two of them connected by one edge, is equivalent to axiom (CS6).

- Axiom (CS6): For an element of $TS(\mathcal{P})(n_1+n_2+n_3, G_1+G_2+G_3+1)$ think of two collections connected by one edge. In the first collection are two one-vertex graphs with vertices decorated by $a \in \mathcal{P}(n_1, G_1)$ and $b \in \mathcal{P}(n_2+1, G_2)$. In the second collection is just one one-vertex graph with decoration $c \in \mathcal{P}(n_3+1, G_3)$. The natural transformation $T(\psi)$ gives us one graph with two vertices connected by one edge and decorated by $(a \#_2 b) \in \mathcal{P}(n_1+n_2+1, G_1+G_2+1)$ and $c \in \mathcal{P}(n_3+1, G_3)$. Upon this, ϕ gives us one-vertex graph decorated by $(a\#_2 b)\, {}_i\circ_j\, c \in \mathcal{P}(n_1+n_2+n_3, G_1+G_2+G_3+1)$.

  Natural transformation $\beta(\mathcal{P})$ gives us element of $ST(\mathcal{P})(n_1+n_2+n_3, G_1+G_2+G_3+1)$ which is a collection of two graphs, one of them one-vertex decorated by $a \in \mathcal{P}(n_1, G_1)$ and the other two vertices connected by one edge and decorated by $b \in \mathcal{P}(n_2+1, G_2), c \in \mathcal{P}(n_3+1, G_3)$. $S(\phi)$ gives us a collection of two one-vertex graphs decorated by a and $b\, {}_i\circ_j\, c \in \mathcal{P}(n_2+n_3, G_2+G_3)$. Finally, applying ψ gives us one-vertex graph decorated by $a\#_2(b\, {}_i\circ_j\, c) \in \mathcal{P}(n_1+n_2+n_3, G_1+G_2+G_3+1)$ which is equivalent to $(a\#_2 b)\, {}_i\circ_j\, c$ under the axiom (CS6).

□

1.3 Properads

The framework of operads (or modular operads) is too narrow to treat structures like (Lie) bialgebras, or Hopf algebras. To model the operations with several inputs and several outputs one needs to introduce a more general object, PROP.

Algebras over PROPs then also accommodate the "coproduct-like" operations and the operads can be seen as just a special kind of PROPs. We say a bit more about PROPs in the last section, 1.3.4. But for our purposes it is sufficient to restrict ourselves to the "connected part" of PROP introduced by Vallette in [41] under the name *properad*.

In the case of properads, we formulate only the biased definition. It is possible to state also the definition with monad (called in Section 1.1.2 as the combinatorial definition), but the "strategy" of the definition would be basically the same as in the case of operads – we have to define the appropriate category of graphs with morphisms preserving the labeling of the external half-edges, and then fix the value of Σ-bimodule on each graph. An obvious missing piece, one needs for the combinatorial definition, is to specify the type of graphs. We do this in the Section 2.2.1 when discussing the cobar complex of properads. We don't specify the rest of the details since we will not use the combinatorial definition in the next.

We want to use the results from [41]. But our notation and convention sometimes slightly differ. For example, we use the biased definition in the convention, which is closest to the one in [21] from Hackney, Robertson, and Yau.

Definition 54. Denote by $\mathsf{DCor} := \mathsf{Cor} \times \mathsf{Cor}$ the category of directed corollas: the objects are pairs (C, D) with C and D finite sets which are called the outputs and inputs.

A morphism $(\rho, \sigma) : (C, D) \to (C', D')$ is a pair of bijections $\rho : C \xrightarrow{\cong} C'$, $\sigma : D \xrightarrow{\cong} D'$.

Definition 55. A properad $\mathcal{P}$ consists of a collection

$$\{\mathcal{P}(C,D) \mid (C,D) \in \mathsf{DCor}\}$$

of dg vector spaces and two collections of degree 0 morphisms of dg vector spaces

$$\{\mathcal{P}(\rho,\sigma) : \mathcal{P}(C,D) \to \mathcal{P}(C',D') \mid (\rho,\sigma) : (C,D) \to (C',D')\}$$
$$\left\{ {}_B\overset{\eta}{\circ}_A : \mathcal{P}(C_1, D_1 \sqcup B) \otimes \mathcal{P}(C_2 \sqcup A, D_2) \to \mathcal{P}(C_1 \sqcup C_2, D_1 \sqcup D_2) \mid \eta : B \xrightarrow{\cong} A \right\}$$

where A, B are arbitrary isomorphic finite nonempty sets. These data are required to satisfy the following axioms:

1. $\mathcal{P}((1_C, 1_D)) = 1_{\mathcal{P}(C,D)}, \quad \mathcal{P}((\rho\rho', \sigma'\sigma)) = \mathcal{P}((\rho,\sigma))\ \mathcal{P}((\rho',\sigma'))$

2. $(\mathcal{P}((\rho_1 \sqcup \rho_2|_{C_2}, \sigma_1|_{D_1} \sqcup \sigma_2))\ {}_B\overset{\eta}{\circ}_A = {}_{\sigma_1(B)}\overset{\rho_2\eta\sigma_1^{-1}}{\circ}_{\rho_2(A)}\ (\mathcal{P}((\rho_1,\sigma_1)) \otimes \mathcal{P}((\rho_2,\sigma_2))$

3. ${}_{B_2 \sqcup B_3}\overset{\epsilon}{\circ}_{A_2 \sqcup A_3}\ ({}_{B_1}\overset{\tilde{\eta}}{\circ}_{A_1} \otimes 1) = {}_{B_1 \sqcup B_3}\overset{\eta}{\circ}_{A_1 \sqcup A_3}\ (1 \otimes\ {}_{B_2}\overset{\tilde{\epsilon}}{\circ}_{A_2})$
   where $\tilde{\eta}, \tilde{\epsilon}$ are restrictions of η, ϵ to pairs of nonempty sets A_1, B_1 and A_2, B_2, respectively.

   For A_1, B_1 empty sets,

   ${}_{B_3}\overset{\eta}{\circ}_{A_3}\ (1 \otimes\ {}_{B_2}\overset{\tilde{\epsilon}}{\circ}_{A_2}) = {}_{B_2}\overset{\tilde{\epsilon}}{\circ}_{A_2}\ (1 \otimes\ {}_{B_3}\overset{\eta}{\circ}_{A_3})(\tau \otimes 1).$

   For A_2, B_2 empty sets,

   ${}_{B_3}\overset{\epsilon}{\circ}_{A_3}\ ({}_{B_1}\overset{\tilde{\eta}}{\circ}_{A_1} \otimes 1) = {}_{B_1}\overset{\tilde{\eta}}{\circ}_{A_1}\ ({}_{B_3}\overset{\epsilon}{\circ}_{A_3} \otimes 1)(1 \otimes \tau).$

where τ is the monoidal symmetry. Whenever the expressions make sense.

We will denote the category of properads with the obvious morphisms by $\mathsf{Pr}_{\mathsf{DCor}}$.

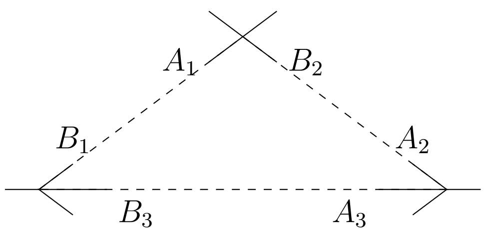


Figure 1.11: Axiom 3. pictorially.

Remark 56. If we consider only Axiom 1., the resulting structure is called a Σ-bimodule. Obviously, by forgetting the composition map, a properad gives rise to its underlying Σ-module.

All these notions are equivalent to their usual counterparts in [41]. For example, Axiom 1. stands for the left and right Σ-actions on C, D respectively, 2. expresses the equivariance and 3. expresses the associativity of the operadic composition.

Remark 57. Without any additional filtration, the components $\mathcal{P}(C, D)$ for a fixed $\texttt{card}(C)$, $\texttt{card}(D)$ would be huge. For a simple illustrative example of the possible problem, one can think of arbitrary bialgebra with product and coproduct. The appropriate properad contains infinitely many different elements in $\mathcal{P}(C, D)$ for $\texttt{card}(C) = 1 = \texttt{card}(D)$ given by series of compositions of "product-coproduct-product-…" elements.

Therefore similarly as in the case of modular operads we consider only properads such that the dg vector spaces $\mathcal{P}(C, D)$ have an additional $\mathbb{N}_0$ grading which will be denoted by G.

The differential and both left and right Σ-actions are assumed to preserve the degree G-components $\mathcal{P}(C, D, G)$. For operations ${}_B\overset{\eta}{\circ}_A$, we assume that they map the components with respective degrees G_1 and G_2 into the component of the degree $G(G_1, G_2, A, B, \eta)$ which is determined, in general, by the degrees G_1, G_2, by the sets A, B and their identification η. This choice might seem surprising at this moment, so let us show a few examples to illustrate how the G of the new component depend on the composition.

Remark 58. Similarly as before in Definition 17 we will assume, unless explicitly mentioned otherwise, the **stability condition**

$$2(G-1) + \texttt{card}(C) + \texttt{card}(D) > 0$$

In particular, this means that for $G = 0$, $\texttt{card}(C) + \texttt{card}(D) \geq 3$ and for $G = 1$, $\texttt{card}(C) + \texttt{card}(D) \geq 1$. For $G > 1$, there is no restriction on the number of inputs and outputs.

Here we should mention that we use slightly different conventions then Vallette in [41], where it is assumed that the sets C and D are always non-empty, i.e., there is always at least one input and one output. Also, in [41], one input and one output are allowed for $G = 0$. We will comment on this further when describing the cobar complex and algebras over it.

1.3.1 Examples of properads

Our two main examples can be again interpreted in terms of 2-dimensional Riemann surfaces, with boundaries and punctures.

Example 59. The (closed) Frobenius properad $\mathcal{F}$. For each $(C, D) \in \mathsf{DCor}$ and G s.t. the condition of stability is satisfied, put $\mathcal{F}(C, D, G) = \mathbb{K}$, i.e., the linear span on one generator $p_{C,D,G}$ in degree zero. The differential is trivial, as well as the Σ-bimodule structure. The operations ${}_B\overset{\eta}{\circ}_A$ do not depend on sets A, B and η,

$${}_B\overset{\eta}{\circ}_A \colon p_{C_1, D_1 \sqcup B, G_1} \otimes p_{C_2 \sqcup A, D_2, G_2} \mapsto p_{C_1 \sqcup C_2, D_1 \sqcup D_2, G}$$

where $G = G_1 + G_2 + \texttt{card}(A) - 1$.

Geometrically, this properad consists of homeomorphism classes of 2-dimensional compact oriented surfaces with two kinds of labeled boundary components, the inputs and outputs. Here, $G = g$, is the geometric genus of the surface. Bijections act by relabeling the inputs and outputs independently. The operation ${}_B\overset{\eta}{\circ}_A$ for a non-trivial pair of sets (A, B) consists of gluing surfaces along the inputs in B and outputs in A identified according to η.

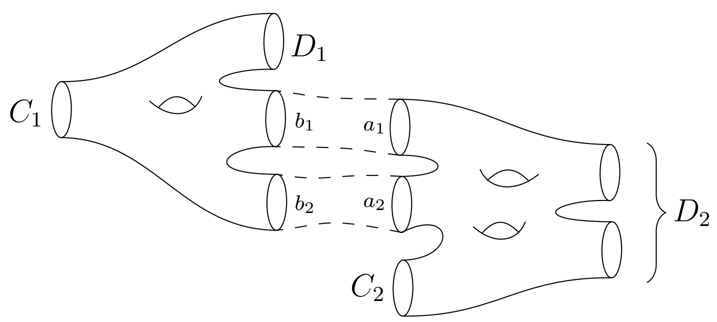


Figure 1.12: $\,_B\overset{\eta}{\circ}_A$, where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $\eta(b_1) = a_1, \eta(b_2) = a_2$.

Remark 60. This example motivates us to introduce the **Euler characteristic**[25]

$$\chi = 2G - 2 + |C| + |D|$$

The stability condition then simply says $\chi > 0$.

Consider two elements $p_{C_1, D_1 \sqcup B, G_1}$ and $p_{C_2 \sqcup A, D_2, G_2}$ of Frobenius properad $\mathcal{F}$. The operation $\,_B\overset{\eta}{\circ}_A$ gives us a surface with Euler characteristic

$$\chi = 2(g_1 + g_2 + |A| - 1) - 2 + \mathtt{card}(C_1) + \mathtt{card}(C_2) + \mathtt{card}(D_1) + \mathtt{card}(D_2) = \chi_1 + \chi_2$$

Hence, the Euler characteristic χ is additive for the Frobenius properad $\mathcal{F}$.

Obviously we can switch to grading by χ and use the notation $\mathcal{P}(C, D, \chi)$ for $\mathcal{P}(C, D, G)$ with $2G = \chi - |C| - |D| + 2 \geq 0$. In the next, we prefer the grading by χ despite the fact, that it is not additive for arbitrary properad as we will see in the next example.

Example 61. The open Frobenius properad $\mathcal{OF}$.

$$\mathcal{OF}(C, D, \chi) := \mathrm{Span}_{\mathbb{K}} \left\{ \{\mathbf{c}_1, \mathbf{c}_2, \dots \mathbf{c}_p, \mathbf{d}_1, \mathbf{d}_2 \dots, \mathbf{d}_q\}^g \mid p, q \in \mathbb{N},\ g \in \mathbb{N}_0 \right\},$$

where $\mathbf{c}_i, \mathbf{d}_j$ are cycles[26] in C and D, respectively, $\bigsqcup_{i=1}^p \mathbf{c}_i = C, \bigsqcup_{j=1}^q \mathbf{d}_j = D$, and the components are stable. Also, $\{\mathbf{c}_1, \mathbf{c}_2, \dots \mathbf{c}_p, \mathbf{d}_1, \mathbf{d}_2 \dots, \mathbf{d}_q\}^g$ is a symbol of degree 0, formally being a pair consisting of $g \in \mathbb{N}_0$ and a set of cycles in (C, D) with the above properties.[27]

For a pair of bijections $(\rho, \sigma) : (C, D) \overset{\cong}{\rightrightarrows} (C', D')$, let

$$\begin{aligned} \mathcal{OF}(\rho, \sigma))(\{\mathbf{c}_1, \mathbf{c}_2, \dots \mathbf{c}_p, \mathbf{d}_1, \mathbf{d}_2 \dots, \mathbf{d}_q\}^g) &:= \\ = \{\rho(\mathbf{c}_1), \dots \rho(\mathbf{c}_p), \sigma^{-1}(\mathbf{d}_1), \dots \sigma^{-1}(\mathbf{d}_q)\}^g. \end{aligned}$$

Although this example is first time defined by us, the formal definition of the operations $\,_B\overset{\eta}{\circ}_A$ is very clumsy, so we refrain from it. Instead, we illustrate it on the geometric interpretation of this properad.

The component $\mathcal{OF}(C, D, \chi)$ of the properad is spanned by homeomorphism classes of 2-dimensional compact oriented stable surfaces with genus g, p output boundaries and q input boundaries. The input boundaries can be permuted freely among themselves, as well as the output boundaries. We put $\chi = 2(2g + b - 1) + \mathtt{card}(C) + \mathtt{card}(D) - 2$, i.e, $G = 2g + b - 1$, with $b = p + q$.

[25]We define the Euler characteristic with the opposite sign than it is usual to shorter later formulas of ubiquitous signs.

[26]As in Definition 27.

[27]Trivially: $\mathbf{c}_i \cap \mathbf{d}_j = \emptyset$, $\mathbf{c}_i \cap \mathbf{c}_j = \emptyset$ and $\mathbf{d}_i \cap \mathbf{d}_j = \emptyset$ for all i, j.

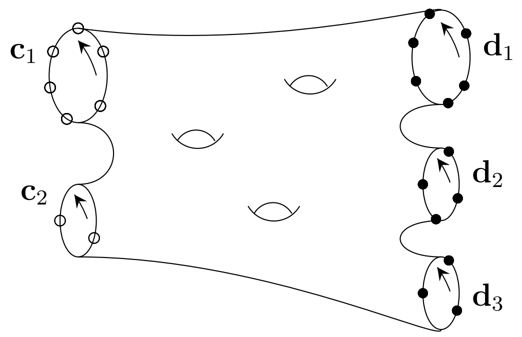


Figure 1.13: Element of $\mathcal{OF}(C, D, \chi)$ with $p = 2$, $q = 3$ and $g = 3$ (arrows indicate the orientations of the boundaries).

The result of ${}_B\overset{\eta}{\circ}_A$ is obtained in two steps. The first step is the (orientation preserving) gluing of two surfaces along the inputs in B and outputs in A identified according to η. Such a gluing creates a new surface which might contain mixed cycles, i.e., cycles containing both inputs and outputs. Such mixed cycles are subsequently split, within the resulting surface (and in an orientation preserving way), into pairs of cycles containing either inputs or outputs only.

Such general description wouldn't be enough to fully understood the "definition" of ${}_B\overset{\eta}{\circ}_A$. Let us, therefore, show it for two cases – first for a case when $\texttt{card}(A) = 1$ and then for the case $\texttt{card}(A) > 1$. To elucidate the second case, we show it also on an explicit example.

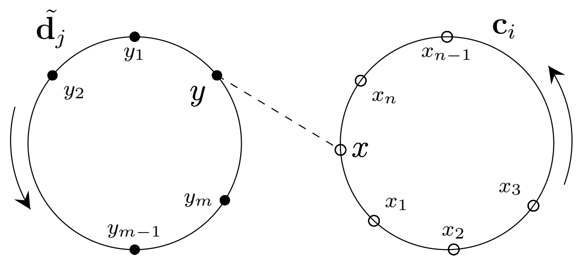


Figure 1.14: Connecting a puncture x from boundary b_i with a puncture y from boundary $\tilde{b_j}$. The output punctures are depicted as black circles and the input punctures as white circles.

We start with the simplest example of gluing two surfaces along one output and one input. We want to glue together an output puncture x of the cycle $\mathbf{c}_i = (\!(x, x_1, x_2, \dots x_n)\!)$ of the boundary b_i together with an input puncture y of the cycle $\tilde{\mathbf{d}}_j = (\!(y_1, y_2, \dots y_m, y)\!)$ of the boundary $\tilde{b_j}$.[28] Obviously, we can consider only boundaries on which the punctures (that we are gluing together) are positioned and ignore the rest.

According to the above description, there are two steps. In the first step, a new mixed cycle $(\!(y_1, y_2, \dots y_m, x_1, x_2, \dots x_n)\!)$ is created. This new cycle is obtained by

[28]Note, that using the cyclic symmetry of the cycles, we can always move the punctures x and y to these positions within the respective cycles.

identifying of x with y, removing the resulting point, and joining the remaining parts of the original cycles, so that the resulting orientation is still compatible with the induced orientation of the boundaries.

However, we want to get again boundaries with inputs or outputs only. This leads to the second step, where we split the new cycle into two cycles of just outputs $\langle\!\langle x_1, x_2, \ldots x_n \rangle\!\rangle$ and inputs $\langle\!\langle y_1, y_2, \ldots y_m \rangle\!\rangle$.[29]

Let us now turn our attention to the more general case, when on each of the two surfaces there are several punctures on several boundaries that have to be glued together. Let us describe in words a simple algorithm how to glue the punctures in order to obtain the mixed cycles (composed of both inputs and outputs), i.e., the "new cycles" from step 1 above.

We can choose an arbitrary puncture on one of these boundaries which has to be glued.[30] Following the orientation of its cycle, we write down the punctures of this cycle until we meet another puncture which has to be glued to another puncture of a boundary on the other surface. We do not write down this puncture nor its "glued partner", but instead we move to this partner along the gluing and continue in recording the punctures according to the orientation of the partner's cycle. We continue this procedure until we get back to the point where we started. The recorded sequence gives a new mixed cycle. To find all these mixed cycles, we choose another puncture which wasn't written yet and start the procedure again.[31]

This gives us cycles with mixed outputs and inputs, but all of them could be splitted again into cycles of inputs and of outputs only by omitting the punctures of the other type. We should be cautious with the following: if in course of this procedure an empty cycle arises, we have to split it too into an "output" and an "input" cycle.

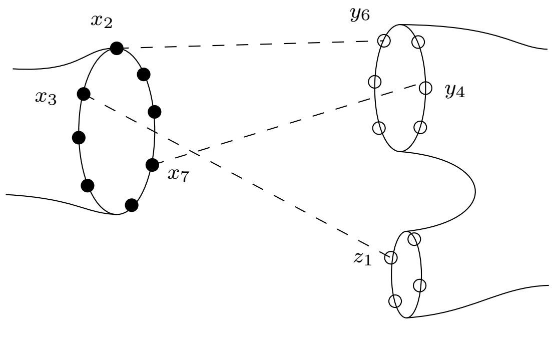


Figure 1.15: Connecting punctures x_2 with y_6, x_3 with z_1 and x_7 with y_4.

An illustrative example could be gluing together punctures x_2 with y_6, x_3 with z_1 and x_7 with y_4 of cycle $\langle\!\langle x_1, x_2, \ldots x_8 \rangle\!\rangle$ of output punctures of one surface and of cycles $\langle\!\langle y_1, y_2, \ldots y_6 \rangle\!\rangle$, $\langle\!\langle z_1, z_2, \ldots z_4 \rangle\!\rangle$ of input punctures of an another one.

[29]This step may look bit trivial in this case but it gives a nontrivial result in the general case.

[30]It can be either an input or output puncture.

[31]Now already within the newly created surface.

Let us choose one arbitrary puncture, for example y_1. Following the orientation we write in a sequence y_1, y_2, y_3. The following puncture y_4 is glued so we do not write it, nor its glued partner x_7 but we continue from the position of x_7 according to orientation, i.e., with x_8, x_1. Then again, x_2 is glued with y_6 so we move to position of y_6 without recording this glued couple and continue according orientation. By this we get again into the position of y_1 where we started. One of the mixed cycles is therefore $(\!(y_1, y_2, y_3, x_8, x_1)\!)$.

To obtain another mixed cycle we choose, for example, x_4 and by following the orientation we get a beginning of the sequence x_4, x_5, x_6 which eventually gives us a mixed cycle $(\!(x_4, x_5, x_6, y_5, z_2, z_3, z_4)\!)$.

These two mixed cycles are later splitted into cycles $(\!(x_8, x_1)\!)$, $(\!(x_4, x_5, x_6)\!)$ of input punctures and into cycles $(\!(y_1, y_2)\!)$, $(\!(y_5, z_2, z_3, z_4)\!)$ of output punctures.

Remark 62. One can check that the above algorithm is independent of the choices we made.

But now, the Euler characteristic, in contrary to the closed Frobenius properad, is not additive anymore. Concerning the genus of the resulting surface, it is given by a sum of genera of the original surfaces and the number of distinct pairs of boundaries which were "glued together". For instance, in the last illustrative example, there are only two distinct pairs of boundaries which were glued together although we glued together three pairs of punctures.

Despite this, we stick with Euler characteristics since in the case of Frobenius properad the results can be used without crucial modification.

Finally, we can combine the above two properads in a rather simple way to obtain a 2-colored properad $\mathcal{OCF}$, which we call open-closed Frobenius properad.

Definition 63. Let DCor_2 be the category of 2-colored directed corollas. The objects are pairs $((O_1, O_2,)(C_1, C_2), G)$, where (O_1, O_2) and (C_1, C_2) are pairs of finite sets and G is a non-negative half-integer, i.e., of the form $G = \frac{N}{2}$ for a non-negative integer N. Elements of O are called open, elements of C are called closed.

A morphism $((O_1, O_2), (C_1, C_2), G) \to ((O'_1, O'_2), (C'_1, C'_2), G')$ is defined only for $G = G'$ and it is a quadruple of bijections $O_1 \xrightarrow{\cong} O'_1$, $O_2 \xrightarrow{\cong} O'_2$, $C_1 \xrightarrow{\cong} C'_1$ and $C_2 \xrightarrow{\cong} C'_2$.

To define a 2-colored properad, we replace in Definition 55 the category DCor by DCor_2, the characteristic χ is now

$$\chi = 2G + \mathtt{card}(O_1) + \mathtt{card}(O_2) + \mathtt{card}(C_1) + \mathtt{card}(C_2) - 2$$

and also we consider only operations of form

$$\begin{aligned}&{}_{(B_o,B_c)}\circ^{(\eta_o,\eta_c)}_{(A_o,A_c)}:\\ &((O_1, O_2 \sqcup B_o), (C_1, C_2 \sqcup B_c), \chi_1) \otimes ((O'_1 \sqcup A_o, O'_2), (C'_1 \sqcup A_c, C'_2), \chi_2)\\ &\qquad\qquad \to ((O_1 \sqcup O'_1, O_2 \sqcup O'_2), (C_1 \sqcup C'_1, C_2 \sqcup C'_2), \chi),\end{aligned}$$

for bijections $\eta_o : B_o \xrightarrow{\cong} A_o$ and $\eta_c : B_c \xrightarrow{\cong} A_c$.[32] The modification of axioms is obvious, we leave it to the reader to fill in the details.

[32] Subscripts o and c again correspond to open and closed, respectively.

Example 64. The open-closed Frobenius properad $\mathcal{OCF}$. For the 2-colored properad $\mathcal{OCF}$, the degree 0 vector space $\mathcal{OCF}((O_1, O_2), (C_1, C_2), G)$ is generated by homeomorphism classes of 2-dimensional compact oriented stable surfaces with genus g, $\mathtt{card}(O_1)$ open outputs and $\mathtt{card}(O_2)$ open inputs distributed over b_1 and b_2 open boundaries respectively and $\mathtt{card}(C_1)$ closed outputs and $\mathtt{card}(C_2)$ closed inputs in the interior, $G = 2g + b + (\mathtt{card}(C_1) + \mathtt{card}(C_2))/2 - 1$ with $b = b_1 + b_2$. The Σ-action preserves the colors and the operations are defined by gluing open/closed inputs into open/closed outputs.

Remark 65. A small sidenote. So far, we discussed only linear properads, i.e., properads in the category of (differential graded) vector spaces Vect. It follows from the definitions that all our examples discussed so far are linearizations of properads in sets. For example, the (closed) Frobenius properad $\mathcal{F}$ is a linearization of the terminal Set-properad. This can be compared to the modular operad $\mathtt{Mod}(Com)$, the modular envelope of the cyclic operad Com. This modular operad is a linearization of $\mathtt{Mod}(*_C)$[33], the terminal modular operad in Set [31]. In [31], Markl also formulates the following Terminality principle:

For a large class of geometric objects there exists a version of modular operads such that the set of isomorphism classes of these objects is the terminal modular Set-*operad of a given type.*

It could be interesting to formulate a similar principle also in the world of properads.

Remark 66. In Section 5 of [9] Costello introduced a non-Σ properad similar to our $\mathcal{OCF}$ properad. Applying functor *Sym*, similar to functor introduced by Markl also in [31], would give us its symmetric version. It could be interesting to compare these two properads.

1.3.2 Skeletal version of properads

The biased definition is easier to formulate. But for explicit computation is many times clumsy. It will prove useful to consider the skeletal version of properads.

Definition 67. $\mathbf{\Sigma}$ is the skeleton of category DCor consisting of corollas of the form $([m], [n])$, $m, n \in \mathbb{N}_0$. Σ-bimodule is a functor from $\mathbf{\Sigma}$ to dg vector spaces.

Before giving the following definitions, let us introduce the following convenient notation "how to shift a subset by a fixed number".

For $n \in \mathbb{N}_0$ and a set $\{a_1, a_2, \ldots\}$ of natural numbers, define

$$n + \{a_1, a_2, \ldots\} := \{n + a_1, n + a_2, \ldots\}.$$

Given $N \subset [n_1 + \mathtt{card}(N)]$, and $M \subset [m_2 + \mathtt{card}(M)]$ define bijections

$$\rho_{n_2} : [n_1 + \mathtt{card}(N)] - N \to n_2 + [n_1],$$

$$\rho_{m_1} : [m_2 + \mathtt{card}(M)] - M \to m_1 + [m_2]$$

by requiring them to be increasing. Numbers n_2, m_1 correspond in the following definition to $\mathtt{card}(D_2)$, $\mathtt{card}(C_1)$, respectively.

[33] The modular envelope of the terminal cyclic operad $*_C$ in Set

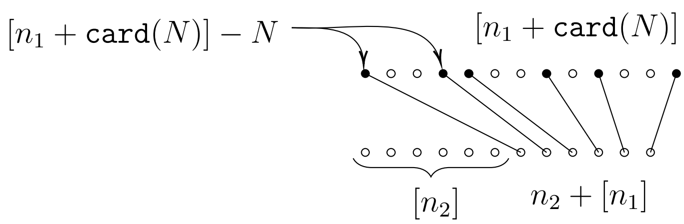


Figure 1.16: Bijection ρ_{n_2}.

Definition 68. Given a properad $\mathcal{P}$ with structure morphisms ${}_B\overset{\eta}{\circ}_A$, define $\bar{\mathcal{P}}$ to consist of a collection

$$\left\{\bar{\mathcal{P}}(m,n) \mid ([m],[n]) \in \mathsf{DCor}\right\}$$

of dg $\Sigma_m \times \Sigma_n$-bimodules and a collection

$$\{ {}_N\overset{\xi}{\circ}_M : \bar{\mathcal{P}}(m_1, n_1 + \mathtt{card}(N)) \otimes \bar{\mathcal{P}}(m_2 + \mathtt{card}(M), n_2) \to \bar{\mathcal{P}}(m_1 + m_2, n_1 + n_2) \mid \\ \xi : N \overset{\simeq}{\rightarrow} M\}$$

of a degree 0 morphisms of dg vector spaces determined by formulas

$$\bar{\mathcal{P}}(m,n) := \mathcal{P}([m],[n])$$

$${}_N\overset{\xi}{\circ}_M := \mathcal{P}(\kappa_1^{-1} \sqcup \rho_{m_1}\kappa_2^{-1}|_{C_2}, \rho_{n_2}\lambda_1^{-1}|_{D_1} \sqcup \lambda_2^{-1})\ {}_B\overset{\eta}{\circ}_A\ (\mathcal{P}(\kappa_1,\lambda_1) \otimes \mathcal{P}(\kappa_2,\lambda_2)),$$

where $\kappa_1 : [m_1] \to C_1$, $\lambda_1 : [n_1 + \mathtt{card}(B)] \to D_1 \sqcup B$, $\kappa_2 : [m_2 + \mathtt{card}(A)] \to C_2 \sqcup A$ and $\lambda_2 : [n_2] \to D_2$ are arbitrary bijections such that $C_1 \cap C_2 = D_1 \cap D_2 = \emptyset$ and $\xi = \kappa_2^{-1}\eta\lambda_1$. Also, $M = \kappa_2^{-1}A$ and $N = \lambda_1^{-1}B$.

Remark 69. Obviously, the definition of ${}_N\overset{\xi}{\circ}_M$ doesn't depend on bijections $\kappa_1, \lambda_1, \kappa_2, \lambda_2$. Hence, sometimes, it might be useful, to make some simplifying choices of these. If, e.g., $C_1 \cup C_2 = [m], D_1 \cup D_2 = [n]$, κ_1, λ_2 as well as $\lambda_1|_{[n_1+|B|]-B}$ and $\kappa_2|_{[m_2+|A|]-A}$ are increasing, then $(\kappa_1^{-1} \sqcup \rho_M\kappa_2^{-1}|_{C_2})$ and $(\rho_N\lambda_1^{-1}|_{D_1} \sqcup \lambda_2^{-1})$ are (m_1, m_2) and (n_2, n_1)-unshuffles, respectively.

The operations ${}_N\overset{\xi}{\circ}_M$ satisfy properties analogous to the axioms of Definition 55. Hence, we can introduce a new category $\mathsf{Pr}_{\boldsymbol{\Sigma}}$ of Σ-bimodules with operations ${}_N\overset{\xi}{\circ}_M$. Obviously, categories $\mathsf{Pr}_{\mathsf{DCor}}$ and $\mathsf{Pr}_{\boldsymbol{\Sigma}}$ are equivalent. Although the axioms for operations ${}_N\overset{\xi}{\circ}_M$ in $\mathsf{Pr}_{\boldsymbol{\Sigma}}$ is a way too complicated for practical purposes. Nevertheless, as we will see, the description of endomorphism properads $\mathcal{E}_V$ in the category $\mathsf{Pr}_{\boldsymbol{\Sigma}}$ is nice and simple.

1.3.3 Endomorphism properad

In the following, we again use the observations about the unordered tensor product we made in section 1.2.2.

Definition 70. For $(C, D) \in \mathsf{DCor}$, $\chi > 0$ define

$$\mathcal{E}_V(C, D, \chi) := \mathrm{Hom}_{\mathbb{K}}(\bigotimes_D V, \bigotimes_C V).$$

Let $\bar{f} \in \mathrm{Hom}_{\mathbb{K}}(V_{d_1} \otimes \cdots \otimes V_{d_n}, V_{c_1} \otimes \cdots \otimes V_{c_m})$ correspond to an element $f \in \mathrm{Hom}_{\mathbb{K}}(\bigotimes_D V, \bigotimes_C V)$, under the above isomorphism in Example 37. Then the differential on $\mathcal{E}_V$ is given, by abuse of notation, as

$$d(\bar{f}) = \sum_{i=0}^{m-1} (1^{\otimes i} \otimes d \otimes 1^{\otimes m-i-1})\bar{f} - (-1)^{|\bar{f}|} \sum_{i=0}^{n-1} \bar{f}(1^{\otimes i} \otimes d \otimes 1^{\otimes n-i-1}) \tag{1.10}$$

Given a morphism $(\rho, \sigma) : (C, D) \to (C', D')$ in DCor, define

$$\begin{aligned} \mathcal{E}_V(\rho, \sigma) : \mathcal{E}_V(C, D, \chi) &\to \mathcal{E}_V(C', D', \chi) \\ f &\mapsto \overline{\rho}\, f\, \overline{\sigma}, \end{aligned}$$

for $f \in \mathrm{Hom}_{\mathbb{K}}(\bigotimes_D V, \bigotimes_C V) \in \mathcal{E}_V(C, D, \chi)$ and $\overline{\rho}$, $\overline{\sigma}$ as in Lemma 35.

For $f \in \mathcal{E}_V\Big(C_2 \sqcup A, D_2, \chi_2\Big)$ and $g \in \mathcal{E}_V\Big(C_1, D_1 \sqcup B, \chi_1\Big)$ let

$$g \ {}_B\overset{\eta}{\circ}_A\ f \in \mathcal{E}_V\Big(C_1 \sqcup C_2, D_1 \sqcup D_2, \chi\Big)$$

be the composition

$$\begin{aligned} \bigotimes_{d \in D_1 \sqcup\ D_2} V_d &\xrightarrow{\simeq} \bigotimes_{d \in D_1} V_d \otimes \bigotimes_{d' \in D_2} V_{d'} \xrightarrow{1 \otimes f} \bigotimes_{d \in D_1} V_d \otimes \bigotimes_{c \in C_2 \sqcup A} V_c \\ &\xrightarrow{\simeq} \bigotimes_{d \in D_1} V_d \otimes \bigotimes_{a \in A} V_a \otimes \bigotimes_{c \in C_2} V_c \xrightarrow{1 \otimes \eta^{-1} \otimes 1} \bigotimes_{d \in D_1} V_d \otimes \bigotimes_{b \in B} V_b \otimes \bigotimes_{c \in C_2} V_c \\ &\xrightarrow{\simeq} \bigotimes_{d \in D_1 \sqcup B} V_d \otimes \bigotimes_{c \in C_2} V_c \xrightarrow{g \otimes 1} \bigotimes_{c \in C_1} V_c \otimes \bigotimes_{c' \in C_2} V_{c'} \xrightarrow{\simeq} \bigotimes_{c \in C_1 \sqcup\ C_2} V_c \end{aligned}$$

in which the isomorphisms are easily identified with those of Lemma 36.

It follows that the collection

$$\mathcal{E}_V = \{\mathcal{E}_V(C, D, \chi) | (C, D) \in \mathsf{DCor}, \chi > 0\}$$

with the above operations is a properad.

It is now straightforward to describe the skeletal version $\bar{\mathcal{E}}_V$ of the endomorphism properad $\mathcal{E}_V$

$$\bar{\mathcal{E}}_V(m, n, \chi) = \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V^{\otimes m}) \cong V^{\otimes m} \otimes (V^*)^{\otimes n},$$

where the last isomorphism is explicitly for

$$v_1 \otimes \ldots \otimes v_m \otimes \alpha_1 \otimes \ldots \otimes \alpha_n \in V^{\otimes m} \otimes (V^*)^{\otimes n}$$

given as

$$v_1 \otimes \ldots \otimes v_m \otimes \alpha_1 \otimes \ldots \otimes \alpha_n : w_n \otimes \ldots \otimes w_1 \mapsto \alpha_1(w_1) \ldots \alpha_n(w_n) v_1 \otimes \ldots \otimes v_m$$

The Σ-bimodule structure for $(\rho, \sigma) \in \Sigma_m \times \Sigma_n$,

$$(\rho, \sigma) : v_1 \otimes \ldots \otimes v_m \otimes \alpha_1 \otimes \ldots \otimes \alpha_n \mapsto \pm\, v_{\rho^{-1}(1)} \otimes \ldots \otimes v_{\rho^{-1}(m)} \otimes \alpha_{\sigma(1)} \otimes \ldots \otimes \alpha_{\sigma(n)},$$

where $\pm$ is the product of the respective Koszul signs corresponding to permutations ρ and σ.

The differential d is given by the natural extension of d on V, as a degree one derivation, to $V^{\otimes m} \otimes (V^*)^{\otimes n}$.[34]

Finally, the properadic compositions ${}_N\overset{\xi}{\circ}_M$ are described as follows. Let N be a set $N = n_1 + [\mathtt{card}(N)] \subset [n_1 + \mathtt{card}(N)]$, $M = [\mathtt{card}(N)] \subset [m_2 + \mathtt{card}(N)]$ and ξ a bijection $\xi(n_1 + \mathtt{card}(N) - i + 1) = i$ for $i = 1, \dots \mathtt{card}(N)$ then ${}_N\overset{\xi}{\circ}_M$ is defined by the following assignment:

$$
\begin{aligned}
{}_N\overset{\xi}{\circ}_M : (v_1 \otimes \dots v_{m_1} \otimes \alpha_1 \otimes \dots \alpha_{n_1+\mathtt{card}(N)}) \otimes (w_1 \otimes \dots w_{m_2+\mathtt{card}(N)} \otimes \beta_1 \otimes \dots \beta_{n_2}) \\
\mapsto \pm \prod_{i=1}^{\mathtt{card}(N)} \alpha_{n_1+\mathtt{card}(N)-i+1}(w_i) \cdot v_1 \otimes \dots \otimes v_{m_1} \otimes w_{\mathtt{card}(N)+1} \otimes \dots \otimes w_{m_2+\mathtt{card}(N)} \otimes \\
\otimes \alpha_1 \otimes \dots \otimes \alpha_{n_1} \otimes \beta_1 \otimes \dots \otimes \beta_{n_2}
\end{aligned}
$$

where $\pm$ is the Koszul sign, coming from commuting consecutively the vectors $w_{\mathtt{card}(N)+1} \otimes \dots \otimes w_{m_2+\mathtt{card}(N)}$ trough the one-forms $\alpha_{n_1} \otimes \dots \otimes \alpha_1$. The general case is then easily determined by the equivariance of the operations ${}_N\overset{\xi}{\circ}_M$.

Remark 71. The above introduced skeletal version of the endomorphism properad is equivalent to the one which uses unordered tensor products $\otimes_{[n]} V$ instead of ordinary ones $V^{\otimes n}$. This is possible due to Example 37 according to which we have the canonical isomorphism $\otimes_{[n]} V \cong V^{\otimes n}$ corresponding to the natural ordering on $[n]$.

Finally, we briefly discuss the 2-colored version of the endomorphism properad.

Definition 72. Let $\mathcal{E}_{V_o \oplus V_c}$ be an abbreviation for the direct sum of dg vector spaces (V_o, d_o) and (V_c, d_c). Let

$$
\mathcal{E}_{V_o \oplus V_c}((O_1, O_2), (C_1, C_2), \chi) := \mathrm{Hom}_{\mathbb{K}}(\bigotimes_{O_2} V_o \otimes \bigotimes_{C_2} V_c, \bigotimes_{O_1} V_o \otimes \bigotimes_{C_1} V_c).
$$

The Σ-action and the operations are defined analogously to the 1-colored case.

Definition 73. Let $\mathcal{P}$ be a properad. An **algebra over properad** $\mathcal{P}$ on a dg vector space V is a properad morphism

$$
\alpha : \mathcal{P} \to \mathcal{E}_V,
$$

i.e., it is a collection of dg vector space morphisms

$$
\{\alpha(C, D, \chi) : \mathcal{P}(C, D, \chi) \to \mathcal{E}_V(C, D, \chi) \mid (C, D) \in \mathsf{DCor}, \chi > 0\}
$$

such that (in the sequel, we drop the notation (C, D, χ) at $\alpha(C, D, \chi)$, for brevity)

1. $\alpha \circ \mathcal{P}(\rho, \sigma) = \mathcal{E}_V(\rho, \sigma) \circ \alpha$

   for any morphism (ρ, σ) in DCor

[34] Recall, $(d\alpha)(v) = (-1)^{|\alpha|}\alpha(dv)$.

2. $\alpha \circ ({}_B\overset{\eta}{\circ}_A)_{\mathcal{P}} = ({}_B\overset{\eta}{\circ}_A)_{\mathcal{E}_V} \circ (\alpha \otimes \alpha)$

   Algebra over a 2-colored properad is again defined by replacing DCor by DCor_2.

In practice, however, one is rather interested in skeletal version of α's, i.e., $\Sigma_m \times \Sigma_n$-equivariant maps

$$\alpha(m, n, \chi) : \bar{\mathcal{P}}(m, n, \chi) \to \bar{\mathcal{E}}_V(m, n, \chi)$$

intertwining between the respective ${}_N\overset{\xi}{\circ}_M$ operations.

Remark 74. Note that the above formula 2. is compatible with any composition law for the degree G, or equivalently for the Euler characteristic χ. This is because, for fixed values of m and n, the vector spaces $\mathcal{E}_V(m, n, \chi)$ are independent of the actual value of χ. So we always can choose the composition law for χ in the endomorphism properad $\mathcal{E}_V$ so that it respects the one for $\mathcal{P}$.

1.3.4 PROPs

Similarly as in the case of modular operads, one may wish to introduce some product for properads $\mathcal{P}$ to make sense of elements e^p for $p \in \mathcal{P}$. It turns out such a structure already exists inside the notion of PROPs.

Roughly speaking we can say, that one adds a tensor product called as *horizontal product* to our properads and allows also the non-connected components.[35] The following definition is taken from [33].

This product is, however, different from the connected sum introduced for modular operads by the absence of map $\#_1$. The "nice" geometrical picture is a little bit distorted in the interpretation of Σ-actions on punctures of a product of two surfaces. It could be interesting to introduce the connected sum also for properads.

Definition 75. A ($\Bbbk$-linear) **PROP** is a symmetric strict monoidal category $\mathcal{P} = (\mathcal{P}, \odot, S, 1)$ enriched over $Mod_{\Bbbk}$ such that

- the objects are indexed by (or identified with) the set N of natural numbers
- the product $\odot$ satisfies $m \odot n = m + n$

Remark 76. The foregoing composition $\circ$ is called **vertical** and the monoidal product $\odot$ induces a **horizontal composition**

$$\otimes : \mathcal{P}(m_1, n_1) \otimes \ldots \otimes \mathcal{P}(m_k, n_k) \to \mathcal{P}(m_1 + \ldots + m_k, n_1 + \ldots + n_k)$$

Remark 77. As we will see in the next chapter, the Koszul duality theory is essential in some constructions, for us namely for the construction of minimal resolutions given by the cobar complex construction. Although there have been some works trying to generalize Koszul duality theory (for $\frac{1}{2}$-PROPs and dioperads), the consistent work was done only for properads in [41].

Therefore we set aside some of the ideas.

Remark 78. One possible modification of the problem is to restrict the vertical composition to just one edge and add the horizontal composition. This approach was done in [26] by Kaufmann, Ward, and Zuniga.

[35] Our choice of the orientation of pictures is done so they better fit on the paper.

2. Cobar complex, Feynman transform and master equations

As we have seen in Section 1.1.2 (monoidal definition) the operads can be defined as monad in certain linear category, category Vect. The advantage of this approach is that one can hope to extend the notion of *bar* and *cobar* construction from algebras to operads.

After introducing the cobar complex for operads we show its simple generalization to properads and analogical construction of Feynman transform for modular operads. We can formulate here the "contours" of this general principle and then clarify aspects of our approach.

A lot of details will be missed – among others, we skip introducing the twisting morphisms and Koszul morphisms, many examples of twisting coboundaries will be missed, etc. We cherry-pick here only the necessary terms and notions. For full treatment, we refer the reader to sections 6.4 – 6.7 in Loday and Vallette [30] as the main source of this chapter. A nice summary could be also found in [37] by Nasuda. The "pedestrian way" is taken from Markl, Shnider, Stasheff [35], and for modular operads we draw from Barannikov [3]. For us, the inspiring material is also Kaufmann, Ward, and Zuniga [26] because of their nice unifying point of view.

Afterward, we comment on the particular aspects for the cobar complex for properads and the analog of cobar complex for the modular operads, the Feynman transform.

2.1 Very abstract point of view

Remark 79. In the following, we will be mentioning the term **cooperad**. Roughly speaking, cooperad $\mathcal{C}$ is a dual notion to the operad with the decomposition map $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and the counit map $\epsilon : \mathcal{C} \to I$. Although we should define it more properly, we will show in the following that in the end, it is not necessary to consider it in our construction of cobar complex.

Remark 80. Let us remind the combinatorial definition from Remark 15. There we constructed the endofunctor $T : \Sigma\text{-}\mathsf{Mod}_{\mathbb{K}} \to \Sigma\text{-}\mathsf{Mod}_{\mathbb{K}}$. $T(\mathcal{P})$ for any Σ-module $\mathcal{P}$ is a free operad.

Let us define the **weight grading of the free operad** $T(\mathcal{P})^{(w)}$ as the number w of generating operations needed in the construction of a given element of the free operad. In the language of trees, this corresponds to the number of vertices in the tree.

Definition 81. An **augmented** dg operad is a dg operad $\mathcal{P}$ equipped with a morphism $\epsilon : \mathcal{P} \to I$ of dg operads. In this case $\mathcal{P} = \overline{\mathcal{P}} \oplus I$.

Similarly dg cooperad $\mathcal{C}$ is **coaugmented** if there is a morphism of dg cooperads $\eta : I \to \mathcal{C}$ and $\mathcal{C} = \overline{\mathcal{C}} \oplus I$.

Similarly as in the setting of algebras and coalgebras, there is a pair of adjoint functors, bar B and cobar Ω

$$\{\text{aug. dg operads}\} \underset{\Omega}{\overset{B}{\rightleftarrows}} \{\text{coaug. dg cooperads}\} \tag{2.1}$$

between augmented dg operads and coaugmented dg cooperads. Since we are used to working with operads, we point out to the counit of the adjunction (2.1). It is an operad morphism $\Omega B(\mathcal{P}) \to \mathcal{P}$.

From *fundamental theorem of operadic twisting morphism* (theorem 6.6.2 in [30]) follows that the counit is in fact a quasi-isomorphism $\Omega B(\mathcal{P}) \xrightarrow{\sim} \mathcal{P}$. Let us now recall from [30] just the necessary facts about the cobar complex.

Remark 82. The cobar complex $\Omega\mathcal{C}$ is an augmented dg operad defined as the free operad $T(\uparrow \overline{\mathcal{C}})$ over the suspension of $\overline{\mathcal{C}}$. Therefore the elements of this free operad can be represented by trees with vertices "decorated" by elements of $\uparrow \overline{\mathcal{C}}$.

The differential of this operad is given as $d = d_1 + d_2$ where d_1 is the differential induced from the differential $d_{\mathcal{C}}$ of the dg cooperad $\mathcal{C}$ and d_2 is induced by Δ_1 on $\uparrow \overline{C}$.[1] The differential d is completely characterized by the image of the generators. Notice that as a graded object (i.e. forgetting the differential) this operad is free but as a differential graded object it is not. Such operads are called **quasi-free**.

Hence the counit of the adjunction provides a resolution of dg operads. It is quasi-free resolution (cofibrant replacement) but it is not *minimal* in general. This object is in fact "huge". Similarly as $\Omega(\mathcal{C})$ is free operad, $B(\mathcal{P})$ is a free cooperad.[2] The elements in $\Omega B(\mathcal{P})$ therefore can be perceived as trees composed of trees.

We would like to find an object, which would be minimal in the sense that all other quasi-free objects quasi-isomorphic to $\mathcal{P}$ will factorize through it. In other words, we are looking for a *minimal model* in the model category of dg operads.

Remark 83. By definition, a **minimal operad** is a quasi-free operad $(T(E), d)$ whose differential is decomposable, i.e., $d : E \to T(E)^{(\geq 2)}$, and the generating graded Σ-module E admits a decomposition into $E = \bigoplus_{k\geq 1} E^{(k)}$ satisfying

$$d\left(E^{(k+1)}\right) \subset T\left(\bigoplus_{i=1}^{k} E^{(i)}\right)$$

A **minimal model** for the dg operad $\mathcal{P}$ is the minimal operad $(T(E), d)$ with quasi-isomorphism of dg operads $(T(E), d) \overset{\sim}{\ } \mathcal{P}$

It turns out that for a special class of operads, *quadratic Koszul operads*, we are able to construct the minimal model explicitly.

Definition 84. An operad $\mathcal{P}$ is **quadratic** if it has a presentation $\mathcal{P} = T(E)/(R)$, where the ideal $R \subseteq T(E)^{(2)}$. In other words, $\mathcal{P}$ is universal among the quotients of $T(E)$ such that the composite

$$R \hookrightarrow T(E) \quad \mathcal{P}$$

[1] Thanks to the suspension, the differential is truly of degree 1.

[2] See definition 5.8.7 in [30] for details.

is zero. Let us denote the data as $\mathcal{P}(E, R)$.

Similarly **quadratic cooperad** $\mathcal{C}(E, R)$ is a sub-cooperad of the cofree cooperad $T^c(E)$ which is universal such that the composite

$$\mathcal{C} \hookrightarrow T^c(E) \quad\quad T^c(E)^{(2)}/R$$

is zero.

Remark 85. We already have seen examples of quadratic operads. All *Com*, *Ass*, and *Lie* are quadratic.

The presentation of operad *Com* is given by $E_{Com} = \mathbb{K} \cdot \mu$ (trivial representation of Σ_2) with $R_{Com} = Span_{\mathbb{K}}\{\mu \circ_1 \mu - \mu \circ_2 \mu\}$. The presentation of operad *Ass* is given in Example 3 by Σ-module $E_{Ass} = \mathbb{K} \cdot \Sigma_2$ (regular presentation of Σ_2 with generator α) and the ideal generated by relations $R_{Ass} = Span_{\mathbb{K}}\{\alpha \circ_1 \alpha - \alpha \circ_2 \alpha\}$. The presentation of *Lie* is written explicitly in Example 4.

Definition 86. By definition, **Koszul dual cooperad** of the quadratic operad $\mathcal{P}(E, R)$ is the quadratic cooperad $\mathcal{P}^{\text{i}} = \mathcal{C}(\uparrow E, \uparrow^2 R)$.

Finally:

Definition 87. A quadratic operad $\mathcal{P}$ is **quadratic Koszul operad** if there is a quasi-isomorphism $\Omega(\mathcal{P}^{\text{i}}) \xrightarrow{\sim} \mathcal{P}$ of dg operads.

Let us make few remarks to this very abstract point of view.

Remark 88. Since algebras over cofibrant operads are homotopy invariant, the algebras over the minimal model of the operad $\mathcal{P}$ are called the *strongly homotopy* $\mathcal{P}$*-algebras*, shortly $\mathcal{P}_\infty$-algebras. For example $\Omega(Ass^{\text{i}}) \to \mathcal{E}_V$ corresponds to A_∞-algebra and $\Omega(Com^{\text{i}}) \to \mathcal{E}_V$ corresponds to L_∞-algebra.

Remark 89. By proposition 10.1.3 in [30], the $\mathcal{P}_\infty$-algebras, i.e., operad morphisms $\text{Hom}(\Omega(\mathcal{P}^{\text{i}}), \mathcal{E}_V)$, are in bijection with Σ-module morphisms $\alpha : \mathcal{P}^{\text{i}} \to \mathcal{E}_V$, $|\alpha| = 1$ such that $\alpha(1) = d_V$ (where 1 is in the image of coaugmentation map η) and $\alpha \star \alpha = 0$ with the convolution product

$$\alpha_1 \star \alpha_2 = \circ_1(\alpha_1 \otimes \alpha_2)\Delta_1 \tag{2.2}$$

where Δ_1 is the decomposition map in $\mathcal{P}^{\text{i}}$ and $\circ_1$ is the composition map in $\mathcal{E}_V$.[3] The space $\text{Hom}_\Sigma(\mathcal{P}^{\text{i}}, \mathcal{E}_V)$ can be equipped with differential

$$d(\alpha) = d_V \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_{\mathcal{P}^{\text{i}}}$$

and we can introduce the bracket $\{f \star g\} = f \star g - (-1)^{|f| \cdot |g|} g \star f$.

Then $\mathcal{P}_\infty$-algebra can be encoded as looking for elements $\alpha \in \text{Hom}_\Sigma(\mathcal{P}^{\text{i}}, \mathcal{E}_V)$, $|\alpha| = 1$ such that they solve the **Maurer-Cartan equation**

$$d(\alpha) + \alpha \star \alpha = d(\alpha) + \frac{1}{2}\{\alpha \star \alpha\} = 0. \tag{2.3}$$

[3]Maps α are in fact the *twisting morphisms*.

Remark 90. Looking at ∞-algebras as dg operad morphism from $\Omega(\mathcal{C})$ to $\mathcal{P}$ serves well when we want to define the ∞-algebra. The definition via degree 1 morphism $\alpha : \mathcal{P}^{\text{i}} \to \mathcal{E}_V$ becomes handy for a deformation theory. For ∞-morphism of ∞-algebras best suits the definition via codifferential, see "Rosetta stone" in Vallette's lecture notes [42].

Remark 91. Now, we already mentioned three equivalent ways how to define the ∞-algebras. There is one more, which is missing, and it uses the other side of the adjunction – we can define the ∞-algebras as dg cooperad morphisms from $\mathcal{C}$ to $B(\mathcal{E}_V)$. This definition is the most suitable for extracting the **Homotopy Transfer Theorem** (for the details see again [30, 42]): Let $\mathcal{P}$ is a Koszul operad, chain complex (H, d_H) is a *homotopy retract*[4] of (A, d_A)

$$k \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H),$$

$$id_A - i \circ p = d_A \circ k + k \circ d_A, \qquad i \text{ is quasi} - \text{isomorphism.}$$

Then any $\mathcal{P}_\infty$-algebra structure on A can be transferred into $\mathcal{P}_\infty$-algebra structure on H such that it extends to an ∞-quasi-isomorphism.

A first example, that may come to one's mind, is the case when A is dg associative algebra (trivial case of A_∞-algebra) and $H = H(A, d_A)$ is a (co)homology of A. In that case, $H(A)$ is also A_∞-algebra with a trivial differential, $d_H = 0$.

In the following we don't want to talk about the minimality in the sense of model categories. But inspired by the previous example, we "content" ourselves with a minimality in the sense of the **decomposition theorem**, see Kajiura [23]: We call $\mathcal{P}_\infty$-algebra **minimal** if the differential is trivial, i.e., $m_1 = 0$, and **contractible** if all higher operations are trivial, i.e., $m_k = 0$ for $k \geq 2$. Any A_∞-algebra is ∞-isomorphic to the direct sum of a minimal A_∞-algebra and a linear contractible A_∞-algebra.

The minimal model theorem follows from the decomposition theorem, although the form of the minimal model is not explicit.

Later, in a similar fashion, we will call algebras over the cofibrant replacement of modular operads as "quantum homotopy algebras" and consider the structure transferred to cohomology as their minimal model.

The *homological perturbation lemma*, introduced in Chapter 4, is then a set of techniques convenient to transfer the structures from the decomposable object to its minimal part up to homotopy.

2.2 Pedestrian way to cobar complex

In the previous section, we defined the cobar complex as the functor Ω from cooperads to operads. In this sections, we show that under some assumptions we can define cobar complex as a functor C from operads to operads which moreover gives us a minimal model.

[4]Defined in 146.

The glimpses of the idea were already seen in Remark 82. We can think about the cobar complex as *graph complex* and the cooperadic decomposition as a coboundary map "expanding" a vertex into an edge.[5] This approach was developed in [18] by Ginzburg, and Kapranov where the notion of graph complex was generalized to the case of an arbitrary operad $\mathcal{P}$ and called the cobar complex $C(\mathcal{P})$.

We follow here [18], sometimes with slight changes similar to [35]. Some details are also influenced by [3] and [26] since we later want to compare the cobar complex construction with the analogous one for the modular operads. In Remark 108 we show how both constructions can be made in the same spirit.

The underlying idea is this: if one considers a graded dual of $\mathcal{P}$, then the dual maps $(\circ_i)^*$ define a collection of dg maps

$$(\circ_i)^* : \mathcal{P}(n)^* \longrightarrow \sum_{k+l=n+1} \mathcal{P}(k)^* \otimes \mathcal{P}(l)^*$$

that have the same properties as we need from the decomposition map in cooperads (i.e. Σ-equivariance and coassociativity).

It reminds us of the coboundary map, but to define the graph complex properly, this map should be of degree 1. So similarly as in Remark 82 and Definition 86 we need to "shift the degree" of the components. Instead of suspending the vector spaces $\mathcal{P}(n)^*$ we use the *determinant cocycle*, see Definition 94.

The operad $C(\mathcal{P})$ itself is then a collection of trees with vertices v_i decorated by elements $p \otimes \uparrow v_i$ where $p \in P(n)^*$ (for vertex v_i with n incoming half-edges) and $\uparrow v_i$ is a formal element of degree 1. The operadic composition is defined as grafting the trees (according to the orientation of half-edges).

Remark 92. Obviously, this construction can be made only under some assumption: We consider only operads $\mathcal{P}$ such that all components $\mathcal{P}(n)$ are finite dimensional vector spaces. Obviously then there is no problem with considering the linear dual $\mathcal{P}(n)^*$.

Similarly as in [18] we limit ourselves to the operads $\mathcal{P}$ such that $\mathcal{P}(0) = \mathcal{P}(1) = 0$. It is necessary to ensure that the free operad is still composed only of the connected trees (triviality of $\mathcal{P}(0)$) and that there is no ambiguity caused by identification of trees with a different number of vertices and therefore different degrees (triviality of $\mathcal{P}(1)$).[6]

Remark 93. Later we want to adapt this approach also to properads and modular operads. Notice that in the case of modular operads we already prepared by assuming finiteness of components $\mathcal{P}(C, G)$ in 21 and stability condition in 17.

In the case of properads we also assume the stability condition, see 58. To avoid problems with duals, we assume that the dg vector space $\mathcal{P}(C, D, \chi)$ is finite dimensional for any triple (C, D, χ) whenever $C\mathcal{P}$ appears.

Definition 94. The **determinant**. For I a finite set, define

$$Det(I) = \bigwedge_{i \in I} (\uparrow \mathbb{K}^i)$$

[5] Or several edges in the case of properads.

[6] It would be sufficient to consider absence of the unit. But in our examples the component $\mathcal{P}(1)$ contains only unit.

And with notation from [35]:

Definition 95. The **cobar complex** of operad $\mathcal{P}$ is a dg Σ-module $C(\mathcal{P})$ with differential $d_{C(\mathcal{P})}$

$$\mathcal{P}(n)^* \xrightarrow{d_{C(\mathcal{P})}} \operatorname*{colim}_{Vert(\Gamma)=1} P(\Gamma)^* \otimes Det(Vert(\Gamma)) \xrightarrow{d_{C(\mathcal{P})}} \dots$$

$$\dots \xrightarrow{d_{C(\mathcal{P})}} \operatorname*{colim}_{Vert(\Gamma)=n-1} P(\Gamma)^* \otimes Det(Vert(\Gamma))$$

where $\Gamma \in \mathrm{Iso}\mathsf{Tree}_n$ is implicit in all colimits and the differential is given as sum

$$d_{C(\mathcal{P})} = d_{\mathcal{P}^*} \otimes 1 + \sum_{\substack{\Gamma_1,\Gamma_2 \\ \Gamma_1 \circ_i \Gamma_2}} (\circ_i)^* \otimes (\uparrow v \wedge -)$$

($\Gamma_1 \circ_i \Gamma_2$ is short for grafting tree Γ_2 into i-th leaf of tree Γ_1).

Remark 96. Notice that the cobar complex is a double complex with grading from $\mathcal{P}$ and the "tree degree" given by the number of vertices.

One usually consider the cobar complex of operad with the trivial internal differential, i.e., $d_{\mathcal{P}^*} = 0$.

2.2.1 Cobar complex of properads

Since operads are just special cases of properads the generalization of the cobar complex will be pretty straightforward in this case.

The cobar complex of a properad $\mathcal{P}$ is a properad denoted by $C\mathcal{P}$. It is the free properad generated by the suspended dual of $\mathcal{P}$, with the differential induced by the duals of structure maps. As we have seen in previous sections, the cobar complex of a properad $\mathcal{P}$ is in fact a double complex with the differentials being the two terms in the formula (2.4). Each component $C\mathcal{P}(C, D, \chi)$ is given by a colimit of $(\bigwedge_{i=1}^{n} \uparrow V_i) \otimes \mathcal{P}(C, D, \chi)$ over all isomorphism classes of directed connected graphs Γ with n vertices with $\mathtt{card}(D)$ inputs and $\mathtt{card}(C)$ outputs.

Roughly speaking, $C\mathcal{P}$ is spanned by directed graphs with no directed circuits and its vertices are decorated with elements of $\mathcal{P}^*$.

To ensure the following will be unambiguous and to pay off a debt of the missing combinatorial definition of properads, let us first specify the "underlying" category of graphs.[7]

Definition 97. A **graph** Γ consists of vertices and half-edges. Exactly one end of every half-edge is attached to a vertex. The other end is either unattached (such a half-edge is called a leg) or attached to the end of another half-edge (in that case, these two half-edges form an edge). Every end is attached to at most one vertex/end. The half-edge structure for vertex V_1 of the graph Γ is indicated on the following picture on the left.

Definition 98. In a **directed graph**, every half-edge has assigned an orientation such that two half-edges composing one edge have the same orientation. The half-edges attached to each vertex are partitioned into incoming and outgoing half-edges.

A **directed circuit** in such graph is a set of edges such that we can go along them following their orientation and get back to the point where we started.

[7]In, e.g., [26] these graphs are called *connected directed graphs without wheels.*

We require that to every vertex V_i a nonnegative integer G_i is assigned. We define

$$G := \dim_{\mathbb{Q}} H_1(\Gamma, \mathbb{Q}) + \sum_i G_i$$

to be the genus of the graph. The stable graphs then fulfill the condition

$$\chi_i := 2(G_i - 1) + \mathtt{card}(C_i) + \mathtt{card}(D_i) > 0,$$

for every vertex V_i, where $\mathtt{card}(C_i)$ and $\mathtt{card}(D_i)$ denotes the number of outgoing resp. incoming half-edges attached to V_i.

Consider a finite directed graph Γ with no directed circuits and with integers G_i assigned to each vertex as is indicated on the picture on the right.

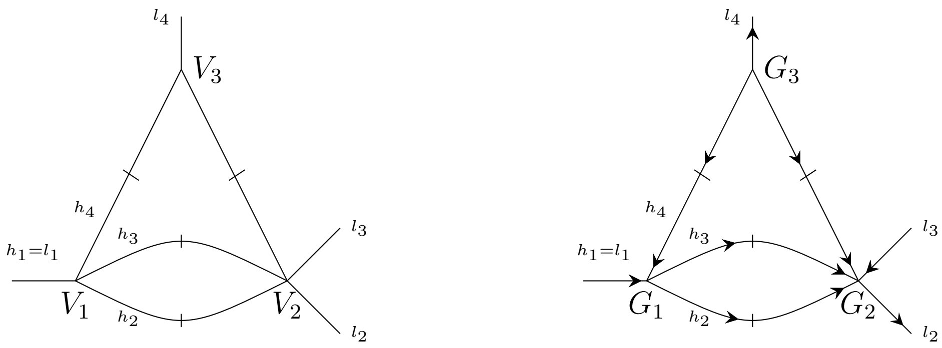


Figure 2.1: Half-edge structure of a graph and a directed graph with assigned G_i

Finally, we require that the incoming legs of Γ are in bijection with the set D and outgoing legs with C.[8] The graph Γ is "decorated" by an element

$$(\uparrow V_1 \wedge \cdots \wedge \uparrow V_n) \otimes (P_1 \otimes \cdots \otimes P_n),$$

where $V_1, \ldots V_n$ are all vertices of Γ, $\uparrow V_i$'s are formal elements of degree $+1$, $\wedge$ stands for the graded symmetric tensor product and $P_i \in \mathcal{P}(C_i, D_i, \chi_i)^*$, for every vertex V_i.

Then the isomorphism class of Γ together with $(\uparrow V_1 \wedge \cdots \wedge \uparrow V_n) \otimes (P_1 \otimes \cdots \otimes P_n)$ is an actual element of $C\mathcal{P}(C, D, \chi)$.

The operation $({}_B\overset{\eta}{\circ}_A)_{C\mathcal{P}}$ is defined by grafting of graphs, attaching together $\mathtt{card}(A)$ pairs of incoming and outgoing legs with a suitable orientation so that no directed circuits are formed.

The differential $d_{C\mathcal{P}}$ on $C\mathcal{P}$ is the sum of the differential $d_{\mathcal{P}^*}$ and of the differential given by the dual of $({}_B\overset{\eta}{\circ}_A)$ which adds one vertex V, $\mathtt{card}(A)$ edges attached to it and modifies the decoration of Γ. For an explicit formula, it is enough to consider a graph Γ with one vertex. On such a graph we have

$$d_{C\mathcal{P}} = d_{\mathcal{P}^*} \otimes 1 + \tag{2.4}$$
$$+ \sum_{\substack{C_1 \sqcup C_2 = C \\ D_1 \sqcup D_2 = D \\ \chi = \chi(\chi_1, \chi_2, A, B, \eta) \\ \chi_1, \chi_2 > 0}} \frac{1}{\mathtt{card}(A)!} \left({}^{(C_1, D_1 \sqcup B, \chi_1)}{}_B\overset{\eta}{\circ}_A{}^{(C_2 \sqcup A, D_2, \chi_2)} \right)^*_{\mathcal{P}} \otimes (\uparrow V \wedge \cdot)$$

[8]In [17], it is shown that the number of isomorphism classes of (ordinary) stable graphs with legs labeled by the set $[n]$ and with the fixed genus G is finite. The additional conditions on graphs, i.e., being directed with no directed circuits, will obviously not change this.

where

$$\left({}^{(C_1,D_1\sqcup B,\chi_1)}\ {}_{B}{\overset{\eta}{\circ}}_{A}\ {}^{(C_2\sqcup A,D_2,\chi_2)}\right)^*_{\mathcal{P}} : \mathcal{P}(C,D,\chi)^* \\ \to \mathcal{P}(C_1, D_1 \sqcup B, \chi_1)^* \otimes P(C_2 \sqcup A, D_2, \chi_2)^* \tag{2.5}$$

for stable vertices $(C_1, D_1 \sqcup B, \chi_1)$ and $(C_2 \sqcup A, D_2, \chi_2)$. For a general stable graph, the differential extends by the Leibniz rule.

Remark 99. Here we should clarify the used notation. The sum is over pairs of sets C_1, C_2 and D_1, D_2 as indicated and also over characteristics χ_1, χ_2 and the bijection[9] η such that $\chi_1, \chi_2 > 0$ and the result of ${}^{(C_1,D_1\sqcup B,\chi_1)}\ {}_{B}{\overset{\eta}{\circ}}_{A}\ {}^{(C_2\sqcup A,D_2,\chi_2)}$ gives a component of the given characteristic χ. Such sum is obviously finite.

For example, in the case of closed Frobenius properad where the Euler characteristic is additive, the sum is just over G_1, G_2, η such that $1 \le \mathtt{card}(A) \le G+1$, $G_1 + G_2 + \mathtt{card}(A) - 1 = G$ for a given G.

We will use this shortened notation also in the following.

Remark 100. In the above formula, we should make a choice of the "new vertex" V out of the two vertices created by the splitting of the original one. Since we consider only connected directed graphs with no directed circuits, the new $\mathtt{card}(A)$ edges in the resulting graph will necessarily start in one vertex and end in the other one.

We can choose any of them as the new one but once the choice is made, we have stick to it consistently when extending the differential using the Leibniz rule. The decoration by graded symmetric product of degree-one elements then ensures that the $d_{C\mathcal{P}}$ is really a differential.

Remark 101. The notion of cobar complex of a 2-colored properad is defined using a suitable definition of 2-colored directed graphs. We leave it to the reader to fill in the details.

2.2.2 Feynman transform

At first sight, the definition of graph complex for Feynman transform F differs from the definition of cobar complex C for properads. The reason is given by different structure maps. In the case of properads, the maps $(\ {}_A\circ_B)^*$ add several edges but exactly one vertex. Therefore the degree of the elements in the free operad $C(\mathcal{P})$ must be tied to the number of vertices. Whereas in the case of modular operads, both $(\ {}_i\circ_j)^*$ and $(\ \circ_{ij})^*$ add exactly one edge but in the latter case no vertex. Therefore we must use a "different cocycle".

We start by clarifying what is the cocycle and what we mean by a twist. We follow here Barannikov [3], or Getzler and Kapranov [17]. Then the Feynman transform is introduced with the possible unifying point of view of [26].

Remark 102. The graded vector space V is **invertible**, if there exist another graded vector space V^{-1} such that $V \otimes V^{-1} \cong \mathbb{K}$.

Obviously V is invertible if and only if it is of the form $\uparrow^n \mathbb{K}$ for some $n \in \mathbb{Z}$. Then $V^{-1} = \downarrow^n \mathbb{K}$.

[9] Notice, that by giving η we also identify the sets A, B and their size.

Definition 103. A **cocycle** is a functor $D : \mathsf{MGr}(n,g) \to \mathsf{gVect}$ which assigns to a stable graph Γ a graded one-dimensional vector space $D(\Gamma)$ that is invertible, and to any morphism of stable graphs $f : \Gamma_0 \to \Gamma_1$ the linear isomorphism

$$D(f) : D(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} D(f^{-1}(v)) \longrightarrow D(\Gamma_0)$$

satisfying the natural associativity condition, see 4.1.3 in [17].

Moreover, we assume D maps the graphs with only one vertex and no edges, i.e., corollas $\Gamma = *_{n,g}$, to $\mathbb{K}$.[10]

Example 104. An important example of cocycle is

$$K(\Gamma) = Det(Edge(\Gamma))$$

where $Edge(\Gamma)$ is the unordered set of egdes of graph Γ. In [17] named as the *dualizing cocycle*, in [26] as the *twist*.

Let s be an Σ-module such that each object $s(n,G)$ is invertible. Then s defines a natural structure of cocycle

$$D_s(\Gamma) = s(n,G) \otimes \bigotimes_{v \in Vert(\Gamma)} s^{-1}((leg(v), G(v))$$

called *coboundary* of s. Notice that tensoring with s defines a functor on Σ-modules.

Definition 105. Let modular operad $\mathcal{P}$ be an algebra over the monad (T, μ, η). The **twisted modular D-operad** is an alegbra $\mathcal{P}_D$ over the twisted monad (T_D, μ_D, η_D)

$$\begin{aligned} (T_D\mathcal{P})(n,g) &= \operatorname*{colim}_{\Gamma \in \mathrm{Iso}\mathsf{MGr}(n,g)} \mathcal{P}^*(\Gamma) \otimes D(\Gamma) = \\ &= \operatorname*{colim}_{\Gamma \in \mathrm{Iso}\mathsf{MGr}(n,g)} \bigotimes_{v \in Vert(\Gamma)} \mathcal{P}^*((leg(v), g(v))) \otimes D(\Gamma) \end{aligned}$$

The unit η_D of this monad is defined by $D(*_{n,g}) \cong \mathbb{K}$ and the identification of $\mathcal{P}^*(n,g)$ with $\mathcal{P}^*(*_{n,g})$ (graphs with no edges). The natural transformation $\mu_D : T_D \circ T_D \to T_D$ is given by the identity

$$(T_D^2\mathcal{P})(n,g) = \operatorname*{colim}_{[\Gamma_0 \xrightarrow{f} \Gamma_1] \in \mathrm{Iso}\mathsf{MGr}(n,g)} \mathcal{P}^*(\Gamma_0) \otimes D(\Gamma_1) \otimes \bigotimes_{v \in Vert(\Gamma_1)} D(f^{-1}(v))$$

and the associativity of composition given by the definition of cocycle.

Remark 106. The **odd modular operad** is a modular operad twisted by $K(\Gamma)$ from Example 104.

Now we can explain the need for "mysterious" signs in Definition 21 of odd modular operad. In the odd version of modular operads, each edge gets weight 1 and so permutations of the edges give rise to signs.

If s defines the coboundary, then tensoring the underlying Σ-module with s defines equivalence of the category of algebras over monad T_D and over monad $s \circ T_D \circ s^{-1} \cong T_{D \otimes D_s}$. For more details about coboundaries see section 4.4 in [17].

[10]The terminal object of $\mathsf{MGr}(n,g)$ is mapped to the unit object of gVect.

Definition 107. The **Feynman transform** of modular operad $\mathcal{P}$ is the free twisted modular K-operad $F(\mathcal{P})$. As Σ-module (forgetting the differential) $F(\mathcal{P})$ is a free modular K-operad generated by stable Σ-modules $\{\mathcal{P}(n,G)^*\}$.

The differential $d_{F(\mathcal{P})}$ is the sum of the differential $d_{\mathcal{P}^*}$ and of the differential given by the dual of the structure maps. For an explicit formula, it is enough to consider a stable graph Γ with one vertex. On such a graph we have

$$\begin{aligned} d_{F(\mathcal{P})} &= d_{\mathcal{P}^*}+ \\ &+\left(\sum_{G_3+1=G} ({}^{(C\sqcup\{i,j\},G_3)} \circ_{ij})^*_{\mathcal{P}} + \frac{1}{2} \sum_{\substack{C_1\sqcup C_2=C\\ G_1+G_2=G}} ({}^{(C_1\sqcup\{i\},G_1)}{}_i\circ_j {}^{(C_2\sqcup\{j\},G_2)})^*_{\mathcal{P}} \right) \otimes (\uparrow e \wedge \cdot) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} ({}^{(C_1\sqcup\{i\},G_1)}{}_i\circ_j {}^{(C_2\sqcup\{j\},G_2)})^*_{\mathcal{P}} &: \mathcal{P}(C,G)^* \to \mathcal{P}(C_1 \sqcup \{i\}, G_1)^* \otimes \mathcal{P}(C_2 \sqcup \{j\}, G_2)^* \\ ({}^{(C\sqcup\{i,j\},G_3)} \circ_{ij})^*_{\mathcal{P}} &: \mathcal{P}(C,G)^* \to \mathcal{P}(C \sqcup \{i,j\}, G-1)^* \end{aligned} \tag{2.7}$$

$\uparrow e \in Det(\{e\})$ and the factor $\frac{1}{2}$ appears since the edges are not oriented.

Let us make few remarks about possible generalizations.

Remark 108. In [3] the Feynman transform is presented in more general form for any twisted modular D-operad. In that case, the Feynman transform produces twisted modular KD^{-1}-operad.

If we consider the modified definition of properads mentioned in Remark 78, then all "operad-like structures" have composition maps contracting each time exactly one edge. Let us fix monad T encoding the operad-like structure (i.e. operad, modular operad, "Kaufmann-Ward-Zuniga-version" of properad from [26], PROP, ...).

Then, in the fashion of [26], we see that Feynman transform is more general, unifying notion. The Feynman transform is a functor

$$F : T\text{-algebras} \longrightarrow T_K\text{-algebras}$$

such that as Σ-module $F(\mathcal{P})$ is defined by the free algebra over twisted monad T_K with Σ-module given by the graded linear dual of $\mathcal{P}$. The differential is again given by the sum of the internal differential and from the duals of composition maps.

This agrees with cobar complex defined above "modulo coboundaries" from Example 104.

Remark 109. One detail we need to fill in is the question of Koszulness of modular operads. Although it is not obvious what quadratic means in the context of modular operads – see for example discussion in [46], in [43] Ward has showed that modular operads are Koszul.

2.3 Algebras over the transforms, Barannikov's theory

The mantra of this section can be summarized in the following:

> There is a natural bijective correspondence between the algebra over "the transform" and the solutions of the "equation".

By transform we mean here the cobar complex or the Feynman transform. As we have seen in the previous sections, they have many properties in common. Therefore it would be convenient to formulate the general strategy in some universal language so we wouldn't have to repeat it twice.

By "equation" we mean its various incarnations that appear under different names

$\{S,S\}=0$	$dS+\frac{1}{2}\{S,S\}=0$	$dS+\Delta(S)+\frac{1}{2}\{S,S\}=0$
Master eq.	*Maurer-Cartan eq.*	*Quantum Master eq.*

(where one always has to specify where S lives and what are the definitions of Δ and $\{-,-\}$).

For (ordinary) operads this mantra is a classical result. In, for example, Kajiura and Stasheff [24] is shown how multilinear maps encoding Maurer-Cartan equation from (2.3) for A_∞ can be nicely packed into a coderivation $m \in \mathrm{Coder}T^cV$ on tensor coalgebra such that $[m,m]=0$ (section 2.2) or similarly L_∞ as $l \in S^cV$, a coderivation differential on graded symmetric (sub)coalgebra (section 2.5).

The case of the modular operads was done by Barannikov in [3]. Since we follow his construction closely also in the case of properads, we decided to call this technical procedure as the "Barannikov's theory". We first phrase all the statements as the algebras over Feynman transform $F(\mathcal{P})$ and later we adapt the notation for the case of properads but the underlying idea will be the same.

2.3.1 Algebra over Feynman transform

The following arguments can be found in [3] in section 4.

The algebra on V over Feynman transform is by definition a morphism of twisted modular operads

$$\alpha : F(\mathcal{P}) \to \mathcal{E}_V$$

Since as a graded object (forgetting the differential) $F(\mathcal{P})$ is free as operad generated by stable Σ-module $\{\mathcal{P}(n,G)^*\}$, the $F(\mathcal{P})$-algebra structure on V is determined by the Σ_n-equivariant degree 0 maps $\alpha_{n,G} : F(\mathcal{P})(n,G) \to \mathcal{E}_V(n,G)$. Trivially we can extend this into collection of maps $\alpha_{C,G}$ for arbitrary finite set C, $\mathtt{card}(C)=n$.

From construction of $F(\mathcal{P})$, the elements of $F(\mathcal{P})(C,G)$ correspond to one-vertex graphs of genus G with set C of legs. An arbitrary element of the free operad $F(\mathcal{P})$ is then a result of composition maps (grafting of stable graphs) acting on elements of the form $\underset{\Gamma\in\mathsf{IsoMGr}(n,g)}{\mathrm{colim}} \mathcal{P}^*(\Gamma)\otimes D(\Gamma)$.

For α to be truly a morphism of twisted modular operads, it is necessary

$$d_{\mathcal{E}_V} \circ \alpha = \alpha \circ d_{F(\mathcal{P})} \tag{2.8}$$

Since both differentials are compatible with composition maps, it is sufficient to check condition (2.8) on the generators (elements of $F(\mathcal{P})(C,G)$). The condition (2.8) is thanks to the explicit formula in (2.6) equivalent to

$$d_{\mathcal{E}_V(n,G)} \circ \alpha_{n,G} = \alpha_{n,G} \circ d_{\mathcal{P}^*} + \sum_{G_3+1=G} (\circ_{ij})_{\mathcal{E}_V} \circ \alpha(C \sqcup \{i,j\}, G_3) \circ (\circ_{ij})^*_{\mathcal{P}} \tag{2.9}$$

$$+ \sum_{\substack{C_1 \sqcup C_2 = C \\ G_1+G_2=G}} ({}_i\circ_j)_{\mathcal{E}_V} \circ (\alpha(C_1 \sqcup \{i\}, G_1) \otimes \alpha(C_2 \sqcup \{j\}, G_2)) \circ ({}_i\circ_j)^*_{\mathcal{P}} +$$

with $({}_i\circ_j)^*_{\mathcal{P}}$, $(\circ_{ij})^*_{\mathcal{P}}$ given by (2.7).

Remark 110. We can recognize in the maps $({}_i\circ_j)_{\mathcal{E}_V}$ and $({}_i\circ_j)^*_{\mathcal{P}}$ of (2.9) analogs of the operadic composition $\circ_1$ and cooperadic decomposition Δ_1 from Remark 89. Therefore the sum can be understood as the convolution product (2.2).

2.3.2 Barannikov's theory for modular operads

In [3] the results are stated in full generality. But our aim is only to work with a special case of twisted modular operads – the modular operads twisted by the "determinant-of-edges" coefficient system (as in Doubek, Jurčo, Münster [11]).

We already adjusted our Definition 107 to this case. In Section 1.2.3 we introduced the endomorphism odd modular operad $\mathcal{E}_V$ with odd symplectic form ω. Also, let us recall our earlier Remark 44 where we observed we are twisting by the symmetric inner product. Thus we can apply results of section 3.2 of [3].

There is an isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{\Sigma_n}(\mathcal{P}(n,G)^*, \mathcal{E}_V(n,G)) & \overset{\cong}{\to} & (\mathcal{P}(n,G) \otimes \mathcal{E}_V(n,G))^{\Sigma_n} \\ \alpha_{n,G} & \mapsto & \sum_i p_i \otimes \alpha_{n,G}(p_i^*) \end{array}$$

Let us denote the elements of $\prod_{n,G} (\mathcal{P}(n,G), \mathcal{E}_V(n,G))^{\Sigma_n}$ corresponding to the collection of $\alpha_{n,G}$ under this isomorphism as S_n^G.

The equation (4.8) of [3] then says

$$(d_{\mathcal{P}} \otimes 1 - 1 \otimes d_{\mathcal{E}_V}) S_n^G + ((\circ_{ij})_{\mathcal{P}} \otimes (\bullet_{ij})_{\mathcal{E}_V})\, S_{n+2}^{G-1} +$$
$$+ \frac{1}{2} \sum_{\substack{G_1+G_2=G \\ C_1 \sqcup C_2 = [n]}} (({}_i\circ_j)_{\mathcal{P}} \otimes ({}_i\bullet_j)_{\mathcal{E}_V})\, S^{G_1}_{C_1 \sqcup \{i\}} \otimes S^{G_2}_{C_2 \sqcup \{j\}} = 0$$

The section 6 ibid shows how to equivalently state this equation in terms of degree 1 differential Δ and degree 1 bracket $\{\cdot,\cdot\}$, combining all the pieces of knowledge in the following theorem:

Theorem 1, [3]. *The modular $F(\mathcal{P})$-algebra structure on chain complex V with antisymmetric inner product ω of degree -1 where $\mathcal{P}$ is arbitrary modular operad is in one-to-one correspondence with solutions of the quantum master equation*

$$dS + \Delta S + \frac{1}{2}\{S,S\} = 0$$

in the space $\prod_{n,G} (\mathcal{P}(n,G) \otimes V^*)^{\Sigma_n}$

We will see more on this in Section 3.2.

2.3.3 Algebra over the cobar complex of properad

The following theorem is essentially the only thing we need from the theory of the cobar transform for properads. The term algebra over the cobar complex of properad appeared already in Vallette [41] under the name *$\mathcal{P}$-gebra up to homotopy* at the end of 8-th section. Thanks to the explicit construction of cobar complex $C(\mathcal{P})$ we can give also explicit formulation of algebra over cobar complex.

In order to describe an algebra over the cobar complex, it is enough to consider graphs with one vertex.

Theorem 111. An algebra over the cobar complex $C\mathcal{P}$ of a properad $\mathcal{P}$ on a dg vector space V is uniquely determined by a collection of degree 1 linear maps

$$\{\alpha(C,D,\chi) : \mathcal{P}(C,D,\chi)^* \to \mathcal{E}_V(C,D,\chi) \mid (C,D) \in \mathsf{DCor}, \chi > 0\},$$

(no compatibility with differential on $\mathcal{P}(C,D,\chi)^*$!) such that

$$\mathcal{E}_V(\rho,\sigma)\circ\alpha(C,D,\chi) = \alpha(C',D',\chi)\circ\mathcal{P}(\rho^{-1},\sigma^{-1})^*$$

for any pair of bijections $(\rho,\sigma) : (C,D) \xrightarrow{\sim} (C',D')$ and

$$\begin{aligned} d\circ\alpha(C,D,\chi) &= \alpha(C,D,\chi)\circ d_{\mathcal{P}^*} + \\ &+ \sum_{\substack{C_1\sqcup C_2=C\\ D_1\sqcup D_2=D\\ \chi=\chi(\chi_1,\chi_2,A,B,\eta)\\ \chi_1,\chi_2>0}} \frac{1}{|A|!}(\,{}_B\overset{\eta}{\circ}{}_A)_{\mathcal{E}_V}\circ(\alpha(C_1,D_1\sqcup B,\chi_1)\otimes\alpha(C_2\sqcup A,D_2,\chi_2))\circ(\,{}_B\overset{\eta}{\circ}{}_A)^*_{\mathcal{P}} \end{aligned} \tag{2.10}$$

where $(\,{}_B\overset{\eta}{\circ}{}_A)^*_{\mathcal{P}}$ is a shorthand notation for $({}^{(C_1,D_1\sqcup B,\chi_1)}\,{}_B\overset{\eta}{\circ}{}_A\,{}^{(C_2\sqcup A,D_2,\chi_2)})^*_{P}$ from (2.5)

$$(\,{}_B\overset{\eta}{\circ}{}_A)^*_{\mathcal{P}} : \mathcal{P}(C,D,\chi)^* \to \mathcal{P}(C_1,D_1\sqcup B,\chi_1)^*\otimes\mathcal{P}(C_2,D_2\sqcup A,\chi_2)^*.$$

Proof. The arguments are the same as in the Section 2.3.1. □

It will also be useful to have the skeletal version of the above theorem.

Lemma 112. Algebra over the cobar complex $C\mathcal{P}$ of a properad $\mathcal{P}$ on a dg vector space V is uniquely determined by a collection

$$\left\{\bar\alpha(m,n,\chi) : \bar{\mathcal{P}}(m,n,\chi)^* \to \bar{\mathcal{E}}_V(m,n,\chi) \mid ([n],[m]) \in \mathsf{DCor}\right\}$$

of degree 1 linear maps (no compatibility with differential on $\bar{\mathcal{P}}(m,n,\chi)^*$!) such that[11]

$$\bar{\mathcal{E}}_V(\rho,\sigma)\bar\alpha = \bar\alpha\bar{\mathcal{P}}(\rho^{-1},\sigma^{-1})^*$$

for any pair $(\rho,\sigma) \in \Sigma_m\times\Sigma_n$ and

$$\begin{aligned} d\bar\alpha = \bar\alpha d_{\bar{\mathcal{P}}^*} + \sum_{\substack{C_1\sqcup C_2=[m]\\ D_1\sqcup D_2=[n]\\ \chi=\chi(\chi_1,\chi_2,A,B,\eta)\\ \chi_1,\chi_2>0}} &\mathcal{E}_V(\kappa_1\sqcup\kappa_2\rho_A^{-1},\lambda_1\rho_B^{-1}\sqcup\lambda_2)(\,{}_{\lambda_1^{-1}(B)}\overset{\kappa_2^{-1}\eta\lambda_1}{\circ}{}_{\kappa_2^{-1}(A)}\,)_{\mathcal{E}_V}(\bar\alpha\otimes\bar\alpha) \\ &(\mathcal{P}(\kappa_1,\lambda_1)^*\otimes\mathcal{P}(\kappa_2,\lambda_2)^*)\,(\,{}_B\overset{\eta}{\circ}{}_A)^*_{\mathcal{P}} \end{aligned} \tag{2.11}$$

[11] In the sequel, we simplify the notation a bit further: the (m,n,χ) at $\bar\alpha(m,n,\chi)$ is usually omitted and so is the symbol $\circ$ for composition of maps.

where

$$\begin{aligned}
\kappa_1 &: [\mathtt{card}(C_1)] \xrightarrow{\sim} C_1 \\
\kappa_2 &: [\mathtt{card}(C_2) + \mathtt{card}(A)] \xrightarrow{\sim} C_2 \sqcup A \\
\lambda_1 &: [\mathtt{card}(D_1) + \mathtt{card}(B)] \xrightarrow{\sim} D_1 \sqcup B \\
\lambda_2 &: [\mathtt{card}(D_2)] \xrightarrow{\sim} D_2 \\
\rho_A &: [\mathtt{card}(C_2) + \mathtt{card}(A)] - A \xrightarrow{\sim} \mathtt{card}(C_1) + [\mathtt{card}(C_2)] \\
\rho_B &: [\mathtt{card}(D_1) + \mathtt{card}(B)] - B \xrightarrow{\sim} \mathtt{card}(D_2) + [\mathtt{card}(D_1)]
\end{aligned}$$

are arbitrary bijections.

Remark 113. The above discussion straightforwardly carries over to the 2-colored case, the reader can easily fill in the details.

2.3.4 Barannikov's theory for properads

As we have seen in Section 2.3.2 the algebra over the Feynman transform of modular operad $\mathcal{P}$ is equivalently described as a solution of a certain master equation in an algebra succinctly defined in terms of $\mathcal{P}$, cf. also Theorem 20 in [11].

In [34], by Markl, Merkulov, and Shadrin, was given similar result for wheeled PROP in theorem 3.4.3. Here, we formulate the corresponding theorem for properads in our formalism and then adapt it to our applications.

Assume $C_1, D_1, C_2, D_2, \kappa_1, \lambda_1, \kappa_2, \lambda_2$ are given as in Lemma 112.

Definition 114. For a properad $\mathcal{P}$, define

$$\mathfrak{P}(m,n,\chi) := {}^{\Sigma_m}\left(\mathcal{P}([m],[n],\chi) \otimes \mathcal{E}_V([m],[n],\chi)\right)^{\Sigma_n}$$

$$\mathfrak{P} := \prod_{\substack{n\geq 0, m\geq 0\\ \chi>0}} \mathfrak{P}(m,n,\chi)$$

with $\mathfrak{P}(m,n,\chi)$ being the space of invariants under the diagonal $\Sigma_m \times \Sigma_n$ action on the tensor product.

Let $\mathfrak{P}$ be equipped with a differential, given for $f \in \mathfrak{P}(m,n,\chi)$, by

$$d(f) := \left(d_{\mathcal{P}([m],[n],\chi)} \otimes 1_{\mathcal{E}_V([m],[n],\chi)} - 1_{\mathcal{P}([m],[n],\chi)} \otimes d_{\mathcal{E}_V([m],[n],\chi)}\right)(f), \qquad (2.12)$$

The composition $\circ$ is described as follows: Assume $g \in \mathfrak{P}(m_1, n_1 + \mathtt{card}(B), \chi_1)$, $h \in \mathfrak{P}(m_2 + \mathtt{card}(A), n_2, \chi_2)$ and $\mathtt{card}(A) = \mathtt{card}(B)$, then the component $(m = m_1 + m_2, n = n_1 + n_2, \chi = \chi(\chi_1, \chi_2, A, B, \eta))$ of the composition $g \circ h$ is given by

$$\sum \left(({}_B\overset{\eta}{\circ}_A)_{\mathcal{P}} \otimes ({}_B\overset{\eta}{\circ}_A)_{\mathcal{E}_V} \right) \sigma_{23}(\mathcal{P}(\kappa_1,\lambda_1) \otimes \mathcal{E}_V(\kappa_1,\lambda_1) \otimes \mathcal{P}(\kappa_2,\lambda_2) \otimes \mathcal{E}_V(\kappa_2,\lambda_2))(g \otimes h).$$

The differential and the composition are extended by infinite linearity to the whole $\mathfrak{P}$. Here the sum is over $C_1 \sqcup C_2 = [m], D_1 \sqcup D_2 = [n]$, $\mathtt{card}(C_1) = m_1$, $\mathtt{card}(C_2) = m_2$, $\mathtt{card}(D_1) = n_1$, $\mathtt{card}(D_2) = n_2$ and σ_{23} is the flip excahning the two middle factors. Recall that $\kappa_1, \kappa_2, \lambda_1, \lambda_2$ depend on C_1, C_2, D_1, D_2.

Remark 115. Since the above definition of the composition $\circ$ doesn't depend on the choice of maps $\kappa_1, \kappa_2, \lambda_1, \lambda_2$ it might be sometimes useful to make a convenient choice of these.

Without loss of generality we can assume $A \subset [m_2 + \mathtt{card}(A)]$ and $B \subset [n_1 + \mathtt{card}(B)]$ and hence relabel them as M and N respectively, just to follow our conventions from Remark 69. Let κ_1, λ_2 be increasing as well as λ_1 when restricted to $[n_1 + \mathtt{card}(N)] - N$ and κ_2 when restricted to $[m_2 + \mathtt{card}(M)] - M$. Then the $(m = m_1 + m_2, n = n_1 + n_2, \chi(\chi_1, \chi_2, M, N, \xi))$ component of the above composition $g \circ h$ can be rewritten as

$$\sum (\mathcal{P}(\rho,\sigma) \otimes \mathcal{E}_V(\rho,\sigma)) \left(({}_N\overset{\xi}{\circ}_M)_{\mathcal{P}} \otimes ({}_N\overset{\xi}{\circ}_M)_{\mathcal{E}_V} \right) \sigma_{23}(g \otimes h), \tag{2.13}$$

with the sum running over all (m_1, m_2)-shuffles ρ and (n_2, n_1)-shuffles σ.

Theorem 116. Algebra over the cobar complex $C\mathcal{P}$ on a dg vector space V is equivalently given by a degree 1 element $L \in \mathfrak{P}$ satisfying the master equation

$$d(L) + L \circ L = 0. \tag{2.14}$$

Sketch of proof. Consider the isomorphism

$$\begin{aligned} \mathrm{Hom}_{\Sigma_C \times \Sigma_D}(\mathcal{P}(C,D,\chi)^*, \mathcal{E}_V(C,D,\chi)) \overset{\cong}{\to} {}^{\Sigma_C}(\mathcal{P}(C,D,\chi) \otimes \mathcal{E}_V(C,D,\chi))^{\Sigma_D} \\ \alpha \mapsto \sum_i p_i \otimes \alpha(p_i^*) \end{aligned} \tag{2.15}$$

where $\{p_i\}$ is a $\mathbb{K}$-basis of $\mathcal{P}(C,D,\chi)$ and $\{p_i^*\}$ is its dual basis. Under this isomorphism, (2.11) becomes the (χ, m, n)-component of the master equation of this theorem.

□

By (2.15), any $L \in \mathfrak{P}$ can be written in the form

$$L = \sum_{n,m,\chi} L_{m,n,\chi} = \sum_{n,m,\chi} \sum_i p_i \otimes \alpha(p_i^*)$$

for some collection α of $\Sigma_m \times \Sigma_n$-equivariant maps of degree 1

$$\alpha([m],[n],\chi) : \mathcal{P}([m],[n],\chi)^* \to \mathcal{E}_V([m],[n],\chi).$$

Let p_i be a basis of $\mathcal{P}([m],[n],\chi)$ and p_i^* the dual one. Put $f_{p_i} := \bar{\alpha}(p_i^*) : V^{\otimes n} \to V^{\otimes m}$. Also, pick a homogeneous basis $\{a_i\}$ of V and denote $f_{p_i I}^J$ the respective coordinates of f_{p_i}, where $I := (i_1, \dots, i_n)$ and $J := (j_1, \dots, j_m)$ are multi-indices in $[\dim V]^{\times n}$ and in $[\dim V]^{\times m}$, respectively.

Hence, we have an isomorphism Y:

$$\begin{aligned} Y : {}^{\Sigma_m}(\mathcal{P}([m],[n]\chi) \otimes \mathcal{E}_V([m],[n],\chi))^{\Sigma_n} \cong \mathcal{P}([m],[n],\chi)_{\Sigma_m} \otimes_{\Sigma_n} (V^{\otimes m} \otimes (V^*)^{\otimes n}) \\ \sum_i p_i \otimes \alpha(p_i^*) \mapsto \frac{1}{m! \cdot n!} \sum_{i,I,J} f_{p_i}^J (p_i \,{}_{\Sigma_m}\!\otimes_{\Sigma_n} (a_J \otimes \phi^I)) \end{aligned} \tag{2.16}$$

and the right hand side is the space of coinvariants with respect to the diagonal $\Sigma_n \times \Sigma_m$ action on the tensor product. Here, $\{\phi^i\}$ is the basis dual to $\{a_i\}$. The coefficient $\frac{1}{n!m!}$ is purely conventional. In particular, we have

$$L = \sum_{n,m,\chi} \frac{1}{m!n!} \sum_{i,I,J} f^J_{p_i I}(p_i\, {}_{\Sigma_m}\otimes_{\Sigma_n} (a_J \otimes \phi^I)) \tag{2.17}$$

The obvious inverse Y^{-1} is

$$Y^{-1} : p\, {}_{\Sigma_m}\otimes_{\Sigma_n} (a_J \otimes \phi^I) \mapsto \sum_{(\rho,\sigma)\in\Sigma_m\otimes\Sigma_n} \mathcal{P}(\rho,\sigma)(p) \otimes \mathcal{E}_V(\rho,\sigma)(a_J \otimes \phi^I).$$

Denote

$$\tilde{\mathfrak{P}}(m,n,\chi) := \left(\mathcal{P}([m],[n],\chi)\, {}_{\Sigma_m}\otimes_{\Sigma_n} (V^{\otimes m} \otimes (V^*)^{\otimes n}\right) \tag{2.18}$$

$$\tilde{\mathfrak{P}} := \prod_{m,n,\chi} \tilde{\mathfrak{P}}(m,n,\chi). \tag{2.19}$$

Then $\mathfrak{P} \cong \tilde{\mathfrak{P}}$ and we can transfer the operations d and $\circ$ from $\mathfrak{P}$ to $\tilde{\mathfrak{P}}$. We start with the differential $\tilde{d}$ on $\tilde{\mathfrak{P}}$, which is obvious

$$\tilde{d}\left(p\, {}_{\Sigma_m}\otimes_{\Sigma_n} (a_J \otimes \phi^I)\right) = d_{\mathcal{P}}(p)\, {}_{\Sigma_m}\otimes_{\Sigma_n} (a_J \otimes \phi^I) - (-1)^{|p|} p\, {}_{\Sigma_m}\otimes_{\Sigma_n} d_{\mathcal{E}_V}(a_J \otimes \phi^I) \tag{2.20}$$

Concerning the composition $\tilde{\circ}$, this is a bit more complicated, but also straightforward.

$$\begin{array}{ccc} \mathfrak{P} \otimes \mathfrak{P} & \xrightarrow[\cong]{Y \otimes Y} & \tilde{\mathfrak{P}} \otimes \tilde{\mathfrak{P}} \\ \circ \downarrow & & \downarrow \tilde{\circ} \\ \mathfrak{P} & \xrightarrow[\cong]{Y} & \tilde{\mathfrak{P}} \end{array}$$

Chasing the above commutative diagram, we obtain:

$$\left(p_1\, {}_{\Sigma_{m_1}}\otimes_{\Sigma_{n_1}} (a_{J_1} \otimes \phi^{I_1})\right) \tilde{\circ} \left(p_2\, {}_{\Sigma_{m_2}}\otimes_{\Sigma_{n_2}} (a_{J_2} \otimes \phi^{I_2})\right) = \tag{2.21}$$

$$= \sum_{M,N,\xi} \left(({}_N\overset{\xi}{\tilde{\circ}}_M)_{\mathcal{P}}(p_1 \otimes p_2) \right) {}_{\Sigma_{m_1+m_2-|M|}}\otimes_{\Sigma_{n_1+n_2-|M|}} \left(({}_N\overset{\xi}{\tilde{\circ}}_M)_{\mathcal{E}_V}(a_{J_1} \otimes \phi^{I_1}) \otimes (a_{J_2} \otimes \phi^{I_2}) \right),$$

where the sum runs over all pairs of nonempty subsets $M \subset [m_2]$, $N \subset [n_1]$ with $\mathtt{card}(M) = \mathtt{card}(N) \leq \min\{m_2, n_1\}$ and all isomorphisms ξ between N and M.

Remark 117. Let us denote by $(x \,\tilde{\circ}\, y) \,\tilde{\circ}\, z$ the sum of all possible compositions of three elements in $\tilde{\mathfrak{P}}$ spanned by graphs on picture 2.2. Let us denote by $x \,\tilde{\circ}\, (y, z)$ the summands spanned by the first graph on picture 2.2. In this notation $x \,\tilde{\circ}\, (y, z) = x \,\tilde{\circ}\, (z, y)$.

Similarly, $x \,\tilde{\circ}\, (y \,\tilde{\circ}\, z)$ is spanned by graphs on picture 2.3 and let us denote by $(x, y) \,\tilde{\circ}\, z$ the summands spanned by the first graph. Obviously

$$(x \,\tilde{\circ}\, y) \,\tilde{\circ}\, z - x \,\tilde{\circ}\, (y \,\tilde{\circ}\, z) = x \,\tilde{\circ}\, (y, z) - (x, y) \,\tilde{\circ}\, z \tag{2.22}$$

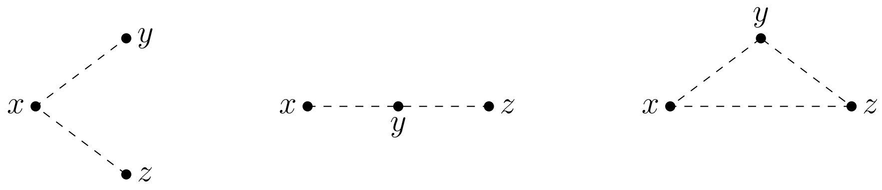


Figure 2.2: Compositions $(x \,\tilde{\circ}\, y) \,\tilde{\circ}\, z$ in $\tilde{\mathfrak{P}}$

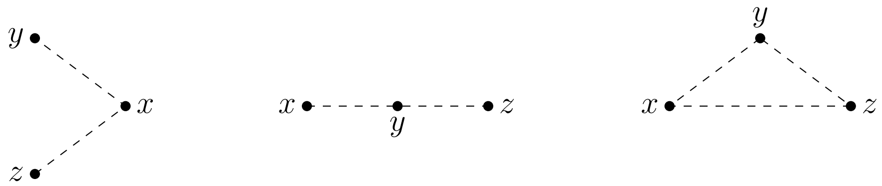


Figure 2.3: Compositions $x \,\tilde{\circ}\, (y \,\tilde{\circ}\, z)$ in $\tilde{\mathfrak{P}}$

Lemma 118. $(\tilde{\mathfrak{P}}, \tilde{\circ})$ forms a Lie-admissible algebra.

Proof. Let us recall what means Lie-admissible: A graded vector space A with a binary product $\circ$ is **Lie-admissible algebra** if one has the *associator* $As(\cdot,\cdot,\cdot)$ such that

$$\sum_{\sigma \in \Sigma_3} \mathrm{sgn}(\sigma) As(\cdot,\cdot,\cdot)^\sigma = 0$$

where, for instance, $As(x,y,z)^\sigma$ for $\sigma = (23)$ is $(-1)^{|y|\cdot|z|}((x \circ z) \circ y) - (x \circ (z \circ y))$.

Similarly as in Proposition 4 in [41] let us consider two subgroups $H = \{\mathrm{id}, (23)\}$, $K = \{\mathrm{id}, (12)\}$ of Σ_3. Trivially from (2.22) we get

$$\begin{aligned} \sum_{\sigma \in \Sigma_3} \mathrm{sgn}(\sigma) As(\cdot,\cdot,\cdot)^\sigma &= \sum_{\sigma \in \Sigma_3} \mathrm{sgn}(\sigma) \left(((\cdot \,\tilde{\circ}\, \cdot) \,\tilde{\circ}\, \cdot)^\sigma - (\cdot \,\tilde{\circ}\, (\cdot \,\tilde{\circ}\, \cdot))^\sigma \right) = \\ = \sum_{\substack{\tau \in \Sigma_3 \\ \tau H = \Sigma_3}} \mathrm{sgn}(\tau) & \underbrace{\Big((\cdot \,\tilde{\circ}\, (\cdot,\cdot))^\tau - (\cdot \,\tilde{\circ}\, (\cdot,\cdot))^{\tau(23)} \Big)}_{0} \\ & - \sum_{\substack{\rho \in \Sigma_3 \\ \rho K = \Sigma_3}} \mathrm{sgn}(\rho) \underbrace{\Big(((\cdot,\cdot) \,\tilde{\circ}\, \cdot))^\rho - ((\cdot,\cdot) \,\tilde{\circ}\, \cdot)^{\rho(12)} \Big)}_{0} = 0 \end{aligned}$$

□

Let us assume that for each object $([m],[n]) \in \mathsf{DCor}$ there is a basis $\{p_i\}$ of $\mathfrak{P}([m],[n],\chi)$ which is preserved by the $\Sigma_m \times \Sigma_n$-action and the operations ${}_B\overset{\eta}{\circ}_A$. This is obviously satisfied, e.g., for the closed Frobenius properad considered in 59. With these choices, the coordinates $f^J_{p_i I}$ have the following simple invariance property

$$f^J_{p_i I} = \pm f^{\rho(J)}_{\mathcal{P}(\rho,\sigma)(p_i)\sigma^{-1}(I)},$$

where $\pm$ is product of respective Koszul signs corresponding to $\rho(J)$ and $\sigma(I)$.

We can decompose $\{p_i\}$ into $\Sigma_m \times \Sigma_n$-orbits indexed by r and choose a representative p_r for each r. Denote $O(p_r) := \Sigma_m \times \Sigma_m / \mathrm{Stab}(p_r)$ and also fix a section $\Sigma_m \times \Sigma_n/\mathrm{Stab}(p_r) \hookrightarrow \Sigma_m \times \Sigma_n$ of the natural projection, thus viewing $O(p_r)$ as a subset of $\Sigma_m \times \Sigma_n$. Hence the orbit of p_r in $\mathcal{P}([m],[n],\chi)$ is $\{\mathcal{P}(\rho,\sigma)p_r \mid (\rho,\sigma) \in O(p_r)\}$ and it has $|O(p_r)| = \frac{n!m!}{|\mathrm{Stab}(p_r)|}$ elements. Hence, we can get an expression for elements of $\tilde{\mathfrak{P}}$ involving p_r's only:

$$\frac{1}{m!n!}\sum_{i,I,J} f^J_{p_i I}(p_i \,{}_{\Sigma_m}\otimes_{\Sigma_n} (a_J \otimes \phi^I)) = \tag{2.23}$$
$$= \sum_r \frac{1}{|\mathrm{Stab}(p_r)|}\sum_{I,J} f^J_{p_r I}(p_r \,{}_{\Sigma_m}\otimes_{\Sigma_n} (a_J \otimes \phi^I))$$

Thus the generating operator $L \in \tilde{\mathfrak{P}}$ can be expressed as

$$L = \sum_{m,n,\chi}\sum_{r,I,J} \frac{1}{|\mathrm{Stab}(p_r)|} f^J_{p_r I}\ (p_r \,{}_{\Sigma_m}\otimes_{\Sigma_n} (a_J \otimes \phi^I)). \tag{2.24}$$

2.3.5 Master equation of properads as homological differential operators

It can be useful to have the following interpretation of the operation $\tilde{\circ}$. Here we shall assume the our corollas have always at least one input and one output, i.e., we assume $\mathcal{P}(C,D,\chi)$ to be nontrivial only if both C and D are non-empty and $m+n>2$, for $G=0$. In this case, we introduce, similarly to [11], positional derivations

$$\frac{\partial^{(k)}}{\partial a_j}(a_{i_1}\otimes\ldots\otimes a_{i_m}) = (-1)^{|a_j|(|a_{i_1}|+\ldots|a_{i_{k-1}}|)}\,\delta^{i_k}_j\,(a_{i_1}\otimes\ldots\otimes\widehat{a_{i_k}}\otimes\ldots\otimes a_{i_m}) \tag{2.25}$$

and for sets $J=\{j_1,\ldots j_{|N|}\}$ and $K=\{k_1,\ldots k_{|N|}\}$

$$\frac{\partial^{(K)}}{\partial a_J} = \frac{\partial^{(k_1)}}{\partial a_{j_1}}\cdots\frac{\partial^{(k_{|N|})}}{\partial a_{j_{|N|}}}.$$

Although the formula defining the positional derivative might seem obscure at the first sight, its usefulness will be obvious from the forthcoming formula (2.26).

The meaning of the positional derivative $\frac{\partial^{(k)}}{\partial a_j}$ is simple. Applied to a tensor product like $a_{i_1}\otimes\ldots\otimes a_{i_m}$ it is zero unless there is a tensor factor a_j at the k-th position, in which case it cancels this factor and produces the relevant Koszul sign. We have introduced it because, in contrary to the left derivative familiar from the supersymmetry literature, here we do not have a rule how to commute the tensor factor a_j to the left. The "inputs" from $(V^*)^{\otimes n_1}$ in equation (2.21) can then be interpreted as the partial derivations acting on the "outputs" from $V^{\otimes m_2}$, and hence we can interpret elements of $\tilde{\mathfrak{P}} = \prod_{m,n,\chi}\tilde{\mathfrak{P}}(m,n,\chi)$ as differential operators acting on $\tilde{\mathfrak{P}}_+ := \prod_k \tilde{\mathfrak{P}}(k,0,\chi)$ as

$$p_1\,{}_{\Sigma_{m_1}}\otimes_{\Sigma_{n_1}}(a_{J_1}\otimes\phi^{I_1}) : p_2\,{}_{\Sigma_{m_2}}\otimes a_{J_2} \mapsto \tag{2.26}$$
$$\pm\sum_{M,N,\xi}\frac{\partial^{\xi(N)}}{\partial a_N}(a_M)\,({}_N\overset{\xi}{\circ}_M)_{\mathcal{P}}(p_1\otimes p_2)\,{}_{\Sigma_{m_1+m_2-|M|}}\otimes_{\Sigma_{n_1-|M|}} a_{J_1}a_{J_2-M},$$

where the sign $\pm$ is given as in (2.25). Hence, in the master equation $\tilde{d}\tilde{L}+\tilde{L}\tilde{\circ}\tilde{L}=0$ where $\tilde{L}=Y(L)$ with Y being the isomorphism (2.16), the operation $\tilde{\circ}$ becomes the composition of differential operators. For this, recall that $\tilde{L}$ is of degree 1 so we can write $\tilde{L}\,\tilde{\circ}\,\tilde{L}=\frac{1}{2}[\tilde{L}\overset{\circ}{,}\tilde{L}]$ as the graded commutator.

Let us remind the combinatorial definition from Remark 15. There we constructed the endofunctor $T:\Sigma\text{-}\mathsf{Mod}_{\mathbb{K}}\to\Sigma\text{-}\mathsf{Mod}_{\mathbb{K}}$. $T(\mathcal{P})$ for any Σ-module $\mathcal{P}$ is a free operad.

3. Batalin-Vilkovisky algebras

A Batalin-Vilkovisky algebra (BV-algebra for short) is a graded commutative algebra equipped with a second-order odd differential operator that squares to zero, and odd Poisson bracket. In this chapter, we show that modular operads with the connected sum (we defined in 22) form the BV-algebras.

Since BV-algebras appear in various contexts as algebraic topology or differential geometry, but most importantly, in mathematical physics, let us start with a small "motivation" from physics.

3.1 Motivation from physics

This short preview is by no means complete and serves only to motivate the way we choose some of the conditions.[1] Most of it was taken from [1] by Albert, Bleile, and Fröhlich.

Similarly as in [16] by Fiorenza let us start with a problem familiar to all mathematicians: to evaluate the line integral of an analytic function we employ the *residues theorem*. The strategy can be rephrased as the following steps:

1. We want to compute $\int_M \Phi$ where M is an n-dimensional manifold and Φ is a top form on M.

2. We embed M into $2n$-dimensional manifold N and extend Φ to a closed n-form Ω.

3. Since Ω is closed, we can choose to integrate Ω over another cycle M_0 that is in the same homology class as M.

4. The cycle M_0 is chosen in such a way that Ω has a power series expansion in a neighborhood of M_0.

The physicist would call the condition $d\Omega = 0$ as *gauge invariance*, the embedding of M as *gauge fixing*, and the change of M to M_0 as *change of gauge.* The basic idea behind BV-formalism would be the same.

Let us think of the "toy model" of finite-dimensional configuration space that has a structure of finite-dimensional manifold M. The dynamics are described by a set of equations encoded into action S_0. The solutions of the equations of motion determine a subspace of the configuration space, $E \subset M$. If the system has gauge symmetries, i.e., there exist, one-parameter groups of transformations of solutions, these solutions are mapped to new solutions. In other words, these transformations correspond to vector fields $P \subset \Gamma(TM)$.

The observables are elements $f \in C^\infty(E)$. If the system has some symmetries, we should not be able to distinguish between solutions on the same orbit, i.e.,

[1] E.g., our different definition of endomorphism modular operad.

$X(f) = 0$ for $X \in P$. Quantization of the theory then means calculating the formal path-integral

$$< f >= \frac{\int_M f \; e^{iS_0/\hbar}\Omega_0}{\int_M e^{iS_0/\hbar}\Omega_0}$$

where Ω_0 is a formal measure that respects the symmetry, i.e., $\mathrm{div}_{\Omega_0} X = 0$ for all $X \in P$. Due to gauge symmetry both of the integrals may diverge and one needs to replace it by a *gauge-fixed version*.

If the infinitesimal local gauge symmetries do not form a Lie algebra, one needs a help of BV method.[2] Let us extend M to a graded manifold $\mathcal{M}$ by adding auxiliary even and odd fields, the so-called *ghosts*, as dictated by the symmetry P. Let us denote the coordinates on $\mathcal{M}$ as z^i.

Next we enlarge the space even more by introducing an *antifield* $z_i^\dagger$ for every field (including ghosts). These fields are of the opposite statistics.[3] In other words, we extended $\mathcal{M}$ to its odd cotangent bundle $\mathcal{E} = \Pi T^*\mathcal{M} \xrightarrow{\pi_0} M$. This bundle is naturally equipped with the odd symplectic structure $\omega = dz^i \wedge dz_i^\dagger$.

The gauge-fixing is then given by the choice of suitable Lagrangian submanifold $\mathcal{L}$ (in general $(k, m-k)$-dimensional). In order to define the expectation value $< f >$ we need to find a suitable semidensity $[s]$ on $\mathcal{E}$, where semidensities are, roughly speaking, cohomology classes of $(\omega \wedge \cdot)$. For $f \in C^\infty(M)$ then

$$< f >= Z^{-1} \int_{\mathcal{L}} \pi_0^* f \cdot [s] \qquad \text{where } Z = \int_{\mathcal{L}} [s].$$

The semidensity is chosen to be of a form

$$[s] = [\exp\left(\frac{i}{\hbar} S\right)\Omega]$$

where Ω is a pull-back of measure on $\mathcal{E}$ and S is an extension of S_0:

$$S = S_0 + \hbar \cdot \text{“ghost terms''} + \text{“higher terms of } \hbar\text{''}$$

and its “suitability” is equivalent to the condition

$$\frac{1}{2}\{S, S\} - i\hbar\Delta S = 0.$$

According to theorem 2.9 of [1], the value $< f >$ is gauge-invariant (i.e. invariant under Hamiltonian variations of $\mathcal{L}$) if $(\{S, \cdot\} - i\hbar\Delta)(\pi_0^* f) = 0$.

Let us make a few remarks linking this physical motivation with what follows.

Remark 119. We denote the space of fields and antifields as V and assume it decomposes into $V = V' \oplus V''$ as well as $\omega = \omega' \otimes \omega''$. Then we integrate out the fields in V'' by chosing Lagrangian subspace $L'' \subset V''$.

Our assumption that in each degree $\dim(V_i) < \infty$ encodes the fact, that we consider only space-time composed of “few” points (0-dimensional spaces).

In the following, we drop the factor i to simplify the formulas. The role of $\hbar$ will be played by { as we will see in Lemma 131.

[2] Although it is possible to employ BV also in the case when they do form a Lie algebra.

[3] In our language, of the opposite degree.

The effective action is defined as

$$e^{W/\hbar} = \int_{L''} e^{S/\hbar}$$

and satisfies the master equation in the BV-algebra of $\mathrm{Fun}(\mathcal{P}, V')$.

In the graded geometry, linear transformations are given by supermatrices. Therefore one needs the generalization of the determinant – a superdeterminant also known as *Berezinian*. The semidensities then transform with a square root of Berezinian. We don't want to go much into details so we only show a special case in Remark 183.

3.2 Definition of BV-algebras and generalized BV-algebras

Definition 120. A **BV algebra** is a graded commutative associative algebra on graded vector space $\mathcal{F}$ with a bracket $\{,\} : \mathcal{F}^{\otimes 2} \to \mathcal{F}$ of degree 1 that satisfies

$$\begin{aligned} \{X, Y\} &= -(-1)^{(|X|+1)(|Y|+1)}\{Y, X\}\,, \\ \{X, \{Y, Z\}\} &= \{\{X, Y\}, Z\} + (-1)^{(|X|+1)(|Y|+1)}\{Y, \{X, Z\}\}\,, \\ \{X, YZ\} &= \{X, Y\}Z + (-1)^{(|X|+1)|Y|}Y\{X, Z\}, \end{aligned} \tag{3.1}$$

and a square zero operator called BV Laplacian $\Delta : \mathcal{F} \to \mathcal{F}$ of degree 1 such that

$$\Delta(XY) = (\Delta X)Y + (-1)^{|X|}X\Delta Y + (-1)^{|X|}\{X, Y\}\,. \tag{3.2}$$

For algebras with unit 1, we will require $\Delta(1) = 0$.

Remark 121. The conditions $\Delta^2 = 0$ and (3.2) give us the compatibility between Δ and $\{,\}$

$$\Delta\{X, Y\} = \{\Delta X, Y\} + (-1)^{|X|+1}\{X, \Delta Y\}.$$

Remark 122. It is possible to consider a case of graded vector space with bracket and Laplacian defined as in Definition 120 but without associative algebra structure. In this way, we obtain what will be referred to as *generalized BV algebra*. Besides preserving graded Jacobi identity and $\Delta^2 = 0$ it also preserves the compatibility shown in the previous Remark 121.

But for the case with associative algebra structure, the condition (3.2) tells us we need to define the bracket in some specific way. This could be seen later in Section 3.2.1.

Remark 123. As will be easily see in a moment, if we consider symmetric tensor algebra $Sym(V^*)$ with symmetric tensor product defined as

$$\phi^{i_1} \odot \ldots \odot \phi^{i_n} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma(\phi^{i_1} \otimes \ldots \otimes \phi^{i_n})$$

of dg symplectic vector space V, both bracket, and BV Laplacian can be defined directly from the symplectic form. This is the underlying idea of, e.g., Doubek, Jurčo, Pulmann in [12].

Therefore for the quantum closed operad $\mathcal{QC}$it is not necessary to introduce the connected sum to define the BV algebra. One may ask if the results are therefore the same as ours when introducing the connected sum. We look closely at this in section 3.3.2.

Our goal now is to use the properties of twisted endomorphism modular operad, to define bracket and BV Laplacian also for a vector space with an arbitrary symmetry (given by modular operad $\mathcal{P}$).

Let us consider the skeletal version and modular operads $\mathcal{P}$, $\mathcal{Q}$, such that $\mathcal{P}$ is a dg modular operad and $\mathcal{Q}$ an odd dg modular operad, both of finite type.[4] Let us define

$$\begin{aligned} \mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G) &= \left(\mathcal{P}(n,G)\otimes\mathcal{Q}(n,G)\right)^{\Sigma_n}, \\ \mathrm{Con}(\mathcal{P},\mathcal{Q}) &= \prod_{n\geq 0}\prod_{G\geq 0}\mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G). \end{aligned}$$

There are degree 1 operations (defined component-wise)

$$\begin{aligned} d: &\quad \mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G)\to\mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G), \\ \Delta: &\quad \mathrm{Con}(\mathcal{P},\mathcal{Q})(n+2,G)\to\mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G+1), \\ \{-,-\}: &\quad \mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1+1,G_1)\otimes\mathrm{Con}(\mathcal{P},\mathcal{Q})(n_2+1,G_2) \\ &\qquad\qquad\to\mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1+n_2,G_1+G_2), \end{aligned}$$

defined by

$$d = d_{\mathcal{P}}\otimes 1 - 1\otimes d_{\mathcal{Q}}, \tag{3.3}$$

$$\Delta = (\,\circ_{ij}\otimes\bullet_{ij})(\theta\otimes\theta), \tag{3.4}$$

for arbitrary bijection $\theta:[n+2]\xrightarrow{\sim}[n]\sqcup\{i,j\}$, and

$$\{X,Y\} = (-1)^{|X|}\cdot 2\sum_{C_1,C_2}(\,{}_i\circ_j\otimes{}_i\bullet_j)(\theta_1\otimes\theta_2\otimes\theta_1\otimes\theta_2)(1\otimes\tau\otimes 1)(X\otimes Y) \tag{3.5}$$

where τ is the monoidal symmetry and we sum over all disjoint decompositions $[n_1+n_2]=C_1\sqcup C_2$, such that $\mathtt{card}(C_1)=n_1$, $\mathtt{card}(C_2)=n_2$, the bijections[5] $\theta_1:[n_1+1]\xrightarrow{\sim}C_1\sqcup\{i\}$, $\theta_2:[n_2+1]\xrightarrow{\sim}C_2\sqcup\{j\}$ are chosen arbitrarily.

These operations extend in the usual way to $\mathrm{Con}(\mathcal{P},\mathcal{Q})\to\mathrm{Con}(\mathcal{P},\mathcal{Q})$ or $\mathrm{Con}(\mathcal{P},\mathcal{Q})\otimes\mathrm{Con}(\mathcal{P},\mathcal{Q})\to\mathrm{Con}(\mathcal{P},\mathcal{Q})$.

Remark 124. The compatibility properties of d,Δ and $\{,\}$ were proven in [11] in theorem 20[6]:

$$\begin{aligned} d^2 &= 0, \\ d\{,\}+\{,\}(d\otimes 1+1\otimes d) &= 0, \\ \Delta^2 &= 0, \\ \Delta\{,\}+\{,\}(\Delta\otimes 1+1\otimes\Delta) &= 0, \\ \Delta d+d\Delta &= 0, \end{aligned}$$

and the Jacobi identity

$$\{f,\{g,h\}\} = \{\{f,g\},h\}+(-1)^{(|f|+1)(|g|+1)}\{g,\{f,h\}\}.$$

We obtain what is referred to as *generalized Batalin-Vilkovisky algebra* in [11].

[4] I.e. $\mathcal{P}(n,G)$, $\mathcal{Q}(n,G)$ are finite dimensional vector spaces for all n.

[5] No summation over those.

[6] In a bit different sign convention.

3.2.1 BV-algebras for modular operads with connected sum

Let us provide the missing piece – the graded commutative associative product.

Definition 125. Let $\mathcal{P}$ and $\mathcal{Q}$ be a dg modular operad defined as above, both with connected sum. A **product**

$$\star : \mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1,G_1) \otimes \mathrm{Con}(\mathcal{P},\mathcal{Q})(n_2,G_2) \to \mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1+n_2,G_1+G_2+1)$$

is defined as

$$\star = \sum_{C_1,C_2} (\#_2 \otimes \#_2)(\theta_1 \otimes \theta_2 \otimes \theta_1 \otimes \theta_2)(1 \otimes \tau \otimes 1) \tag{3.6}$$

where, as before, τ is the monoidal symmetry, the sum runs over all disjoint decompositions $C_1 \sqcup C_2 = [n_1+n_2]$, such that $\mathtt{card}(C_1) = n_1$, $\mathtt{card}(C_2) = n_2$, the bijections $\theta_1 : [n_1] \xrightarrow{\sim} C_1$, $\theta_2 : [n_2] \xrightarrow{\sim} C_2$ are chosen arbitrarily.

An **operator** $\sharp : \mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G) \to \mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G+2)$ is defined as

$$\sharp = \#_1 \otimes \#_1. \tag{3.7}$$

Lemma 126. The definition of the product $\star$ in (3.6) doesn't depend on the choice of θ_1, θ_2.

Proof. Every $\theta_1 : [n_1] \xrightarrow{\sim} C_1$ corresponds to composition $\theta_1 = \psi|_{C_1} \circ \tilde{\theta}_1$ where $\psi \in UnSh(n_1,n_2)$ is an unshuffle and $\tilde{\theta}_1 \in \Sigma_{n_1}$. Since $\mathrm{Con}(\mathcal{P},\mathcal{Q})(n_1,G_1)$ is a space of invariants under the action of Σ_{n_1}, the premutation $\tilde{\theta}_1$ won't play any role. Similarly for θ_2. Therefore we sum only over decompositions $C_1 \sqcup C_2$. □

Theorem 127. If $\mathcal{P}$ is dg modular operad with the connected sum, and $\mathcal{Q}$ an odd dg modular operad with the connected sum, then $\mathrm{Con}(\mathcal{P},\mathcal{Q})$ with operations d, Δ, $\{-,-\}$ and $\star$ from Definition 125 is a BV algebra as in Definition 120, i.e.,

(1) $\star$ is a commutative associative product, i.e., on elements:

$$X \star Y = (-1)^{|X|\cdot|Y|} Y \star X, \qquad \text{and} \qquad (X \star Y) \star Z = X \star (Y \star Z). \tag{3.8}$$

(2) $\Delta\star = \star(\Delta \otimes 1) + \star(1 \otimes \Delta) + \sharp\{-,-\}$, i.e., on elements:

$$\Delta(X \star Y) = (\Delta X) \star Y + (-1)^{|X|} X \star (\Delta Y) + (-1)^{|X|} \sharp\{X,Y\}. \tag{3.9}$$

(3) $\{-,-\}(1 \otimes \star) = \star(\{-,-,\} \otimes 1) + \star(1 \otimes \{-,-,\})(\tau \otimes 1)$, i.e., on elements:

$$\{X, Y \star Z\} = \{X,Y\} \star Z + (-1)^{|X|\cdot|Y|+|Y|} Y \star \{X,Z\}. \tag{3.10}$$

Proof. Let us consider $X = \sum_i x^i_{\mathcal{P}} \otimes x^i_{\mathcal{Q}} \in \mathrm{Con}(\mathcal{P},\mathcal{Q})$, where $x^i_{\mathcal{P}} \in \mathcal{P}(n_x,G_x)$ and $x^i_{\mathcal{Q}} \in \mathcal{Q}(n_x,G_x)$. In sake of brevity, we will omit the summation over i (including the index) from the notation. Hence we will write $X = x_{\mathcal{P}} \otimes x_{\mathcal{Q}}$. Similarly, we have $Y = \sum_i y^i_{\mathcal{P}} \otimes y^i_{\mathcal{Q}} = y_{\mathcal{P}} \otimes y_{\mathcal{Q}}$ and $Z = \sum_i z^i_{\mathcal{P}} \otimes z^i_{\mathcal{Q}} = z_{\mathcal{P}} \otimes z_{\mathcal{Q}}$ where $y^i_{\mathcal{P}} \in \mathcal{P}(n_y,G_y)$ etc. The calculations are straightforward.

(1) follows from (CS1), (CS2) and (CS3). For commutativity:

$$
\begin{aligned}
X \star Y &= \sum_{C_1,C_2} (-1)^{|x_\mathcal{Q}|\cdot|y_\mathcal{P}|}(\theta_1 x_\mathcal{P} \#_2\, \theta_2 y_\mathcal{P}) \otimes (\theta_1 x_\mathcal{Q} \#_2\, \theta_2 y_\mathcal{Q}) \\
Y \star X &= \sum_{C_1,C_2} (-1)^{|x_\mathcal{P}|\cdot|y_\mathcal{Q}|}(\theta_1 y_\mathcal{P} \#_2\, \theta_2 x_\mathcal{P}) \otimes (\theta_1 y_\mathcal{Q} \#_2\, \theta_2 x_\mathcal{Q}) = \\
&\overset{(CS2)}{=} \sum_{C_1,C_2} (-1)^{|x_\mathcal{P}||y_\mathcal{Q}|+|x_\mathcal{P}||y_\mathcal{P}|+|x_\mathcal{Q}||y_\mathcal{Q}|}(\theta_2 x_\mathcal{P} \#_2\, \theta_1 y_\mathcal{P}) \otimes (\theta_2 x_\mathcal{Q} \#_2\, \theta_1 y_\mathcal{Q}) = \\
&= \quad (-1)^{|X|\cdot|Y|} X \star Y.
\end{aligned}
$$

And associativity:

$$
\begin{aligned}
(X \star Y) \star Z \quad &= \quad \sum_{C_3,C_4} (-1)^{|x_\mathcal{Q}|\cdot|y_\mathcal{P}|} \left((\theta_3 x_\mathcal{P} \#_2\, \theta_4 y_\mathcal{P}) \otimes (\theta_3 x_\mathcal{Q} \#_2\, \theta_4 y_\mathcal{Q})\right) \star Z = \\
&= \quad \sum_{\substack{C_1,C_2,\\ C_3,C_4}} (-1)^{(|(x_\mathcal{Q}|+|y_\mathcal{Q}|)\cdot|z_\mathcal{P}|}(-1)^{|x_\mathcal{Q}|\cdot|y_\mathcal{P}|} \\
&\qquad \cdot(\theta_1(\theta_3 x_\mathcal{P} \#_2\, \theta_4 y_\mathcal{P}) \#_2\, \theta_2 z_\mathcal{P}) \otimes (\theta_1(\theta_3 x_\mathcal{Q} \#_2\, \theta_4 y_\mathcal{Q}) \#_2\, \theta_2 z_\mathcal{Q})
\end{aligned}
$$

where $C_1 \sqcup C_2 = [n_x + n_y + n_z]$ and $C_3 \sqcup C_4 = [n_x + n_y]$, $\theta_1 : [n_x + n_y] \xrightarrow{\sim} C_1$, $\theta_2 : [n_z] \xrightarrow{\sim} C_2, \theta_3 : [n_x] \xrightarrow{\sim} C_3, \theta_4 : [n_y] \xrightarrow{\sim} C_4$ are chosen arbitrarily. From (CS1) (with convention as in Definition 24) we get

$$
\theta_1(\theta_3 x_\mathcal{P} \#_2\, \theta_4 y_\mathcal{P}) = \theta_1(\theta_3 \sqcup \theta_4)(x_\mathcal{P} \#_2\, y_\mathcal{P})
$$

where $(\theta_3 \sqcup \theta_4) : [n_x] \sqcup (n_x + [n_y]) \xrightarrow{\sim} C_3 \sqcup C_4 = [n_x + n_y]$ and similarly for $\mathcal{Q}$-part.Therefore we can rewrite the sums over decompositions $C_1 \sqcup C_2$ and $C_3 \sqcup C_4$ and actions of θ's as

$$
\sum_{E_1 \sqcup E_2 \sqcup E_3} (-1)^A (\psi_1 \sqcup \psi_2 \sqcup \psi_3)((x_\mathcal{P} \#_2\, y_\mathcal{P}) \#_2\, z_\mathcal{P}) \otimes \\
\otimes\ (\psi_1 \sqcup \psi_2 \sqcup \psi_3)((x_\mathcal{Q} \#_2\, y_\mathcal{Q}) \#_2\, z_\mathcal{Q})
$$

where $A = (|(x_\mathcal{Q}| + |y_\mathcal{Q}|) \cdot |z_\mathcal{P}| + |x_\mathcal{Q}| \cdot |y_\mathcal{P}|$ and $\psi_1 : [n_x] \xrightarrow{\sim} E_1$, $\psi_2 : [n_y] \xrightarrow{\sim} E_2$, $\psi_3 : [n_z] \xrightarrow{\sim} E_3$ and we are using notation as in Definition 24. Similarly one gets

$$
\begin{aligned}
X \star (Y \star Z) \quad &= \quad \sum_{D_3,D_4} (-1)^{|y_\mathcal{Q}|\cdot|z_\mathcal{P}|} \left(X \star (\phi_3 y_\mathcal{P} \#_2\, \phi_4 z_\mathcal{P}) \otimes (\phi_3 y_\mathcal{Q} \#_2\, \phi_4 z_\mathcal{Q})\right) = \\
&= \quad \sum_{\substack{D_1,D_2,\\ D_3,D_4}} (-1)^{|x_\mathcal{Q}|\cdot(|y_\mathcal{P}|+|z_\mathcal{P}|)}(-1)^{|y_\mathcal{Q}|\cdot|z_\mathcal{P}|} \\
&\qquad \cdot(\phi_1 x_\mathcal{P} \#_2\, \phi_2(\phi_3 y_\mathcal{P} \#_2\, \phi_4 z_\mathcal{P}) \otimes (\phi_1 x_\mathcal{Q} \#_2\, \phi_2(\phi_3 y_\mathcal{Q} \#_2\, \phi_4 z_\mathcal{Q}))
\end{aligned}
$$

where $\phi_1 : [n_x] \xrightarrow{\sim} D_1$, $\phi_2 : [n_y + n_z] \xrightarrow{\sim} D_2$, $\phi_3 : [n_y] \xrightarrow{\sim} D_3$, $\phi_4 : [n_z] \xrightarrow{\sim} D_4$. And after analogical adjustments we get

$$
\sum_{E_1 \sqcup E_2 \sqcup E_3} (-1)^A (\psi_1 \sqcup \psi_2 \sqcup \psi_3)(x_\mathcal{P} \#_2\, (y_\mathcal{P} \#_2\, z_\mathcal{P})) \otimes (\psi_1 \sqcup \psi_2 \sqcup \psi_3)(x_\mathcal{Q} \#_2\, (y_\mathcal{Q} \#_2\, z_\mathcal{Q})).
$$

And by (CS3) we finally get $(X \star Y) \star Z = X \star (Y \star Z)$.

(2) follows from (CS5a). The left side of the required equality is:

$$\Delta(X\star Y)=\sum_{C_1,C_2}\circ_{ij}\,\phi(\theta_1 x_{\mathcal{P}}\#_2\,\theta_2 y_{\mathcal{P}})\otimes\ \bullet_{ij}\,\phi(\theta_1 x_{\mathcal{Q}}\#_2\,\theta_2 y_{\mathcal{Q}})(-1)^B$$

where $B=|x_{\mathcal{Q}}||y_{\mathcal{P}}|+|x_{\mathcal{P}}|+|y_{\mathcal{P}}|$, $C_1\sqcup C_2=[n_x+n_y]$ and we can choose $\phi=1_{[n_x+n_y]}$ (i.e. $i=n_x+n_y-1, j=n_x+n_y$). Now we split the sum by distinguishing four cases according to position of i,j in the decomposition $C_1\sqcup C_2$ (as in axiom (CS5a)):

$$\begin{aligned}\Delta(X\star Y)\ &=\ \sum_{\substack{C_1,C_2\\ i,j\in C_1}}(-1)^B\,(\circ_{ij}\,\theta_1 x_{\mathcal{P}})\#_2\,\theta_2 y_{\mathcal{P}}\otimes(\bullet_{ij}\,\theta_1 x_{\mathcal{Q}})\#_2\,\theta_2 y_{\mathcal{Q}}\ +\\ &+\ \sum_{\substack{C_1,C_2\\ i,j\in C_2}}(-1)^{B+|x_{\mathcal{Q}}|}\,\theta_1 x_{\mathcal{P}}\#_2\,(\circ_{ij}\,\theta_2 y_{\mathcal{P}})\otimes\theta_1 x_{\mathcal{Q}}\#_2\,(\bullet_{ij}\,\theta_2 y_{\mathcal{Q}})\ +\\ &+\ \sum_{\substack{C_1,C_2\\ i\in C_1,j\in C_2}}(-1)^B\,\#_1\,(\theta_1 x_{\mathcal{P}}\,{}_i\circ_j\,\theta_2 y_{\mathcal{P}})\otimes\#_1\,(\theta_1 x_{\mathcal{Q}}\,{}_i\bullet_j\,\theta_2 y_{\mathcal{Q}})\ +\\ &+\ \sum_{\substack{C_1,C_2\\ i\in C_2,j\in C_1}}(-1)^B\,\#_1\,(\theta_1 x_{\mathcal{P}}\,{}_j\circ_i\,\theta_2 y_{\mathcal{P}})\otimes\#_1\,(\theta_1 x_{\mathcal{Q}}\,{}_j\bullet_i\,\theta_2 y_{\mathcal{Q}})\,.\end{aligned}$$

It is easy to verify that the third and fourth lines give the same result. We compare the previous calculation with

$$(\Delta X)\star Y=\sum_{C_1,C_2}(\theta_1\circ_{ij}\phi x_{\mathcal{P}})\#_2\,\theta_2 y_{\mathcal{P}}\otimes(\theta_1\bullet_{ij}\phi x_{\mathcal{Q}})\#_2\,\theta_2 y_{\mathcal{Q}}\ (-1)^{|x_{\mathcal{P}}|+(1+|x_{\mathcal{Q}}|)|y_{\mathcal{P}}|}$$

where $C_1\sqcup C_2=[n_x+n_y-2]$ and we can choose $\phi=1_{[n_x]}$ and $i=n_x-1, j=n_x$.

$$(-1)^{|X|}X\star(\Delta Y)=\sum_{C_1,C_2}(-1)^{B+|x_{\mathcal{Q}}|}\,\theta_1 x_{\mathcal{P}}\#_2\,\theta_2(\circ_{ij}\,\phi y_{\mathcal{P}})\otimes\theta_1 x_{\mathcal{Q}}\#_2\,\theta_2(\bullet_{ij}\,\phi y_{\mathcal{Q}})$$

where we can choose $\phi=1_{[n_y]}$ and $i=n_y-1, j=n_y$, and

$$(-1)^{|X|}\sharp\{X,Y\}=2\sum_{C_1,C_2}(-1)^B\,\#_1\,(\theta_1 x_{\mathcal{P}}\,{}_i\circ_j\,\theta_2 y_{\mathcal{P}})\otimes\#_1\,(\theta_1 x_{\mathcal{Q}}\,{}_i\bullet_j\,\theta_2 y_{\mathcal{Q}})\,.$$

It is now easy to see that required equality holds.

(3) follows from (CS1) and (CS6). First observe that:

$$\{X,Y\star Z\}=$$
$$=2\sum_{\substack{C_1,C_2\\ D_1,D_2}}(-1)^E\ (\phi_1 x_{\mathcal{P}}\,{}_i\circ_j\,\phi_2(\theta_1 y_{\mathcal{P}}\#_2\,\theta_2 z_{\mathcal{P}}))\otimes(\phi_1 x_{\mathcal{Q}}\,{}_i\bullet_j\,\phi_2(\theta_1 y_{\mathcal{Q}}\#_2\,\theta_2 z_{\mathcal{Q}}))$$

where $E=|y_{\mathcal{Q}}|\cdot|z_{\mathcal{P}}|+|x_{\mathcal{Q}}|\cdot(|y_{\mathcal{P}}|+|z_{\mathcal{P}}|)+|x_{\mathcal{P}}|+|y_{\mathcal{P}}|+|z_{\mathcal{P}}|+|X|$, the sum is over all decompositions $C_1\sqcup C_2=[n_x+n_y]$, $D_1\sqcup D_2=[n_x+n_y+n_z-2]$, and $\theta_1:[n_y]\xrightarrow{\sim}C_1$, $\theta_2:[n_z]\xrightarrow{\sim}C_2$, $\phi_1:[n_x]\xrightarrow{\sim}D_1\sqcup\{i\}$, $\phi_2:[n_y+n_z]\xrightarrow{\sim}D_2\sqcup\{j\}$ are arbitrary bijections. We split the sum into two according to the position of j (inaccurately, if $j\in\phi_2(C_1)$ or $j\in\phi_2(C_2)$) and compare to the following calculations:

$$\{X,Y\}\star Z=$$
$$=2\sum(-1)^F\ (\theta_1(\phi_1 x_{\mathcal{P}}\,{}_i\circ_j\,\phi_2 y_{\mathcal{P}})\#_2\,\theta_2 z_{\mathcal{P}})\otimes(\theta_1(\phi_1 x_{\mathcal{Q}}\,{}_i\bullet_j\,\phi_2 y_{\mathcal{Q}})\#_2\,\theta_2 z_{\mathcal{Q}})$$

where $F = |x_\mathcal{Q}| \cdot |y_\mathcal{P}| + |x_\mathcal{P}| + |y_\mathcal{P}| + |z_\mathcal{P}| \cdot (|x_\mathcal{Q}| + |y_\mathcal{Q}| + 1) + |X|$ and we sum over all decompositions $C_1 \sqcup C_2 = [n_x + n_y + n_z - 2]$, $D_1 \sqcup D_2 = [n_x + n_y]$ and $\phi_1 : [n_x] \xrightarrow{\sim} D_1$, $\phi_2 : [n_y] \xrightarrow{\sim} D_2$, $\theta_1 : [n_x + n_y - 2] \xrightarrow{\sim} C_1$, $\theta_2 : [n_z] \xrightarrow{\sim} C_2$ are arbitrary bijections, and i, j are arbitrary integers (so that expression makes sense).

$$(-1)^{|X|\cdot|Y|+|Y|} Y \star \{X, Z\} =$$
$$= 2 \sum (-1)^G \ (\theta_1 y_\mathcal{P} \#_2\, \theta_2(\phi_1 x_\mathcal{P}\, {}_i{\circ}_j\, \phi_2 z_\mathcal{P})) \otimes (\theta_1 y_\mathcal{Q} \#_2\, \theta_2(\phi_1 x_\mathcal{Q}\, {}_i{\bullet}_j\, \phi_2 z_\mathcal{Q}))$$

where $G = |X| \cdot |Y| + |Y| + |x_\mathcal{Q}| \cdot |z_\mathcal{P}| + |x_\mathcal{P}| + |z_\mathcal{P}| + |y_\mathcal{Q}| \cdot (|x_\mathcal{P}| + |z_\mathcal{P}|) + |X|$ and we sum over all decompositions $C_1 \sqcup C_2 = [n_x + n_y + n_z - 2]$, $D_1 \sqcup D_2 = [n_x + n_z]$ and $\phi_1 : D_1 \xrightarrow{\sim} [n_x]$, $\phi_2 : D_2 \xrightarrow{\sim} [n_z]$, $\theta_1 : C_1 \xrightarrow{\sim} [n_y]$, $\theta_2 : C_2 \xrightarrow{\sim} [n_x + n_z - 2]$ are arbitrary bijections, and i, j arbitrary integers (so that expression makes sense).

Using (CS1) and (CS6)[7], it is easy to see that

$$\{X, Y \star Z\} = \{X, Y\} \star Z + (-1)^{|X|\cdot|Y|+|Y|} Y \star \{X, Z\}.$$

□

Let us make a few remarks to the convention at this place.

Remark 128. As first, note that we constructed a non-unital BV algebra. However in the following we define the **exponential** of $A \in \mathrm{Con}(\mathcal{P}, \mathcal{Q})$ as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{A \star \ldots \star A}_{n-\text{times}} = \underbrace{1 \otimes 1}_{\text{in } \mathbb{K}\otimes\mathbb{K}} + \underbrace{A + \sum_{n=2}^{\infty} \frac{1}{n!} A \star \ldots \star A}_{\text{in } \mathrm{Con}(\mathcal{P},\mathcal{Q})}$$

where $\mathbb{K} \otimes \mathbb{K}$ is a tensor product of two 1-dimensional vector spaces. The space $\mathbb{K} \otimes \mathbb{K}$ is not a subspace of $\mathrm{Con}(\mathcal{P}, \mathcal{Q})$ but we can define $1 \otimes 1$ to be an element of $\mathrm{Con}(0, -1)$ and to play the role of **unit**, i.e., for any element $A \in \mathrm{Con}(\mathcal{P}, \mathcal{Q})(n, G)$, the element $A \star (1 \otimes 1) \in \mathrm{Con}(\mathcal{P}, \mathcal{Q})(n, G)$. We can extend the maps d and Δ naturally to $\mathrm{Con}(\mathcal{P}, \mathcal{Q}) \oplus (\mathbb{K} \otimes \mathbb{K})$ as $d(1 \otimes 1) = 0$ and $\Delta(1 \otimes 1) = 0$.

Element $1 \otimes 1$ has moreover a nice geometrical interpretation for both closed and open modular operad – it corresponds to the surface without any punctures or boundaries and geometrical genus $g = 0$, the sphere.

Remark 129. The bracket defined in (3.5) could seem bit strange at first but now wee see the factor $(-1)^{|X|} \cdot 2$ ensures the required compatibility in (3.9). The BV-algebra we constructed is a bit different from the one defined in [12]. The correct name of the this structure is **Beilinson-Drinfeld algebra**. More details on this can be found in book of Costello and Gwilliam [10]. The role of the operator $\sharp$ in our calculations will be explained in a moment.

Remark 130. In the next, we will sometimes omit the symbol $\star$ from the calculations as it is usual with the normal product. We will use it explicitly only in the situations when we want to stress that we defined the product with a special property of raising G by 1.

Also let us denote

$$\underbrace{A \star \ldots \star A}_{n-\text{times}} = \prod_n^{\star} A$$

to distinguish it from the direct sum.

[7]We have to be careful about the signs.

Lemma 131. For $\#_1$ injective on both $\mathcal{P}$ and $\mathcal{Q}$, let us introduce the space

$$\mathrm{Con}_{\{}(\mathcal{P},\mathcal{Q}) = \prod_{n\geq 0, G\geq 0} (\mathbb{K}((\{)) \otimes \mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G)) \Big/ \sim$$

where $((\{))$ are formal Laurent series and the equivalence $\sim$ is given by for any element $\mathrm{Con}_{\{}(\mathcal{P},\mathcal{Q})$ as

$$\sharp f \sim \{ f. \tag{3.11}$$

If S is a degree 0 element of $\mathrm{Con}_{\{}(\mathcal{P},\mathcal{Q})$, then

$$(d+\Delta)\, \exp\left(\frac{S}{\{}\right) = \frac{1}{\{}\left(dS + \Delta S + \frac{1}{2}\{S,S\}\right)\, \exp\left(\frac{S}{\{}\right).$$

Proof. The arguments are the same as in [12]. It is a simple consequence of equations (3.9) and (3.10) that

$$\Delta S^n = nS^{n-1}\Delta S + \frac{n(n-1)}{2}\sharp\{S,S\}S^{n-2}$$

Thus for a power series $f(S) = \sum_{n\geq 0} f_n S^n$, we have after identification

$$\Delta(f(S)) = \sum_{n\geq 0} f_n \left(nS^{n-1}\Delta S + \frac{n(n-1)}{2}\{\,\{S,S\}S^{n-2}\right) =$$
$$= f'(S)\Delta S + \frac{1}{2} f''(S)\{\,\{S,S\}.$$

□

Remark 132. In Section 2.3.2 we recall the results of [3] that every dg operad morphism from Feynman transform of $\mathcal{P}$ to $\mathcal{Q}$, i.e., $F(\mathcal{P}) \to \mathcal{Q}$, is equivalently given by a degree 0 solution $S \in \mathrm{Con}(\mathcal{P},\mathcal{Q})$ of the *quantum master equation*

$$dS + \Delta S + \frac{1}{2}\{S,S\} = 0. \tag{3.12}$$

If we look closely on the case $\mathcal{Q} = \mathcal{E}_V$, the equation (4.8) in [3] is[8]

$$(d_{\mathcal{P}} + d_V)S_n^G + \Delta S_{n+2}^{G-1} + \frac{1}{2}\sum_{\substack{G_1+G_2=G \\ I_1\sqcup I_2=[n]}} \{S_{I_1\sqcup\{i\}}^{G_1}, S_{I_2\sqcup\{j\}}^{G_2}\} = 0$$

where $S_n^G \in \mathrm{Con}(\mathcal{P},\mathcal{Q})(n,G)$. Thanks to the modified definition of BV-algebra in (3.9) where we used the operator $\sharp$ we get the same terms with respect to the G-grading. Therefore having algebra over Feynman transform of $\mathcal{P}$ is equivalent with the condition that $\exp\left(\frac{S}{\{}\right)$ is $(d+\Delta)$-closed in the space $\mathrm{Con}_{\{}(\mathcal{P},\mathcal{E}_V)$.

Remark 133. We will call algebras over Feynman transform of modular operads (with connected sum) with the element S satisfying (3.12) as **quantum homotopy algebras**.

Since d is a proper differential and the quantum master equation doesn't contain any "0-ary" operation, we will call these algebras *flat*.[9]

[8] Up to a sign convention.

[9] The term flatness is motivated from geometry, where the master equation of the form

$$d\Theta + \frac{1}{2}[\Theta,\Theta] = 0$$

expresses the flatness of a principal connection Θ.

3.3 Space of formal functions $\mathrm{Fun}(\mathcal{P}, V)$

In the following, we will specialize to the case $\mathcal{Q} = \mathcal{E}_V$ and we identify

$$\mathrm{Con}(\mathcal{P}, \mathcal{E}_V)(n, G) = (\mathcal{P}(n, G) \otimes (V^*)^{\otimes n})^{\Sigma_n}.$$

Let us fix this convention in the following definition.

Definition 134. The **space of formal functions** on V is a space of invariants under the diagonal action

$$\mathrm{Fun}(\mathcal{P}, V) = \prod_{n \geq 1} \prod_{G \geq 0} (\mathcal{P}(n, G) \otimes (V^*)^{\otimes n})^{\Sigma_n} \tag{3.13}$$

where $\mathcal{P}(n, G)$ is a component of an operad $\mathcal{P}$.

The number n is usually called the polynomial degree. Similarly as in Lemma 131 we also define the space $\mathrm{Fun}(\mathcal{P}, V)_{\{}$.

Remark 135. Notice the space $\mathrm{Fun}(\mathcal{P}, V)$ has three different gradings corresponding to cohomological grading, polynomial grading by n, and "genus" grading by G.

We can equivalently define the formal functions as coinvariants since there are mutually inverse isomorphisms between the space of invariants and coinvariants (see Definition 7). Coinvariants better capture the idea of commuting variables. However, operad theory produces invariants so we stick to them.

Let us introduce positional derivations and positional multiplications.

Definition 136. The **positional derivation** of $\phi^{I_1 \dots I_n} \in (V^*)^{\otimes n}$ is

$$\frac{\partial^{(i)}}{\partial \phi^{I_i}} \phi^{I_1 \dots I_n} = \kappa \phi^{I_1 \dots \hat{I}_i \dots I_n}$$

where κ is a Koszul sign of permutation transforming $\phi^{I_1 \dots I_i \dots I_n}$ to $\phi^{I_i I_1 \dots \dots I_n}$.

The **positional multiplication** of ϕ^k and $\phi^{I_1 \dots I_n} \in (V^*)^{\otimes n}$ is

$$\phi^k_{(i)} \phi^{I_1 \dots I_n} = \kappa \phi^{I_1 \dots I_{i-1} k I_i \dots I_n}$$

where κ is a Koszul sign of permutation transforming $\phi^k \cdot \phi^{I_1 \dots I_n}$ to $\phi^{I_1 \dots I_{i-1} k I_i \dots I_n}$.

3.3.1 Skeletal version of twisted endomorphism operad

Before we start comparing our results with the results in [12], let us introduce the skeletal version of the twisted modular operad. We slightly change the convention from [11] to ensure the same sign convention and coefficients as in [12].

Definition 137. The **skeletal version of odd endomorphism operad** is dg vector space

$$\overline{\mathcal{E}}_V(n, G) = (V^*)^{\otimes n}$$

with operadic composition, self-composition, Σ-action by $\rho \in \Sigma_n$, and the differential defined after an identification $f \in \overline{\mathcal{E}}_V(n,G) = \bigotimes_{[n]} V^*$ with $(f)_{1_{[n]}}$ as

$$
\begin{aligned}
{}_i\overline{\bullet}_j(f \otimes g) &:= \frac{-1}{2}\sum_k (-1)^{|f||b_k|} f(\cdots \otimes \underbrace{a_k}_{i\text{-th}} \otimes \cdots)\cdot g(\cdots \otimes \underbrace{b_k}_{j\text{-th}} \otimes \cdots)\\
\overline{\bullet}_{ij}(f) &:= \frac{-1}{2}\sum_k f(\cdots \otimes \underbrace{a_k}_{i\text{-th}} \otimes \cdots \otimes \underbrace{b_k}_{j\text{-th}} \otimes \cdots)\\
\overline{\mathcal{E}}_V(\rho)(f) &= f \circ \rho^{-1}\\
d(f) \equiv (-1)^{|f|} f \circ d_{V^{\otimes n}} &= \sum_{i=1}^{n} (-1)^{|f|} f(\ldots \otimes d \otimes \ldots)
\end{aligned}
$$

where again $b_k = \sum_l (-1)^{|a_l|}\omega^{kl} a_l$.

Now we give a simple formula for skeletal version of connected sum $\overline{\#_2}$.

Lemma 138. Let $f \in \overline{\mathcal{E}}_V(n_1,G_1) \cong \bigotimes_{[n_1]} V^*, g \in \overline{\mathcal{E}}_V(n_2,G_2) \cong \bigotimes_{[n_2]} V^*$. Then

$$f \overline{\#_2}\, g = (f \#_2\, g)_{1_{[n_1+n_2]}} = (f)_{1_{[n_1]}} \cdot (g)_{1_{[n_2]}}.$$

The connected "self"-sum $\#_1$ is just the identity.

Proof. In Definition 24 we defined for $f \in \overline{\mathcal{P}}(n_1,G_1), g \in \overline{\mathcal{P}}(n_2,G_2)$:

$$f \overline{\#_2}\, g = \Big((\theta_1 \sqcup \theta_2{}')^{-1} \#_2 (\theta_1 \otimes \theta_2)\Big)(f \otimes g) = ((f \circ \theta_1^{-1}) \#_2 (g \circ \theta_2^{-1}))(\theta_1 \sqcup \theta_2{}')$$

Let us choose $C_1 = [n_1]$, $C_2 = n_1 + [n_2], \theta_1 = 1_{[n_1]}, \theta_2 = 1_{[n_2]}$. Then $\theta_1 \sqcup \theta_2{}' = 1_{[n_1+n_2]}$ and

$$f \overline{\#_2}\, g = (f \#_2\, g) \circ 1_{[n_1+n_2]} = (f \#_2\, g)_{1_{[n_1+n_2]}}.$$

Using (1.7) with $\psi_1 = \theta_1, \psi_2 = \theta_2$ and $\overline{\psi} = 1_{[n_1+n_2]}$ leads to desired formula. □

Remark 139. To be more precise, we should distinguish $F \in \mathcal{E}_V([n]) = \bigotimes_{[n]} V^*$ and $f \in (V^*)^{\otimes n}$ which represents F, i.e., $(F)_{1_{[n]}} = f$. We denote both by the same symbol for simplicity. In this condensed notation, the above lemma reads simply $f \overline{\#_2}\, g = f \cdot g$.

3.3.2 BV-algebra for Quantum closed modular operad

In Lemma 138 we will show that the connected sum for commutative operad corresponds to the symmetric tensor product and, after small change in the convention in endomorphism operad, we show the equivalence of BV-Laplacian and BV-bracket in remarks 142 and 144.

Remark 140. In (3.4) we defined BV Laplace as $\Delta = (\circ_{ij} \otimes \bullet_{ij})(\theta \otimes \theta)$. The numerous discussions in [11] in section 3.8 about the choice of arbitrary bijections θ and indices show that we can choose some fixed indices. Lemma 19 ibid says we can even choose the indices in bracket differently for each shuffle π in the formula (3.17) for BV-bracket. We make the choice $i = 1, j = 2$ to better handle the signs which can possibly arise.

Let us recall we denote by $\{a_l\}$ a homogeneous basis of V and $\{\phi^k\}$ a dual basis. Remember also, we write for better readability $(-1)^m$ instead of $(-1)^{|a_m|}$ for elements of basis of V (and similarly for elements of the dual basis).

The symbol κ_σ in the following computation is a Koszul sing of permutation σ (taking monomial $\phi^{i_1} \otimes \ldots \otimes \phi^{i_n}$ to $\phi^{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes \phi^{i_{\sigma^{-1}(n)}}$) and I denotes a multi-index. We denote by $\mathtt{card}(I) = n$ the length of the multi-index.

Example 141. Let us consider an element of $(\mathcal{P}(n, G) \otimes (V^*)^{\otimes n})^{\Sigma_n}$

$$X = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma p \otimes \sigma(\phi^I) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma p \otimes \sigma(\phi^{i_1} \otimes \ldots \otimes \phi^{i_n}).$$

Then BV-Laplacian Δ evaluated on this element is

$$\begin{aligned} \Delta(X) &= \Delta\left(\frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma p \otimes \sigma(\phi^I)\right) = \\ &= \frac{(-1)^{|p|}}{n!} \sum_{\sigma \in \Sigma_n} (\,\bar{\circ}_{12}\, \sigma p) \otimes \kappa_\sigma\, \bar{\bullet}_{12}(\phi^{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes \phi^{i_{\sigma^{-1}(n)}}) = \frac{(-1)^{|p|}}{n!} \sum_{\sigma \in \Sigma_n} (\,\bar{\circ}_{12}\, \sigma p) \otimes \\ &\otimes \left(\kappa_\sigma \frac{-1}{2} \sum_{k,l} \phi^{i_{\sigma^{-1}(1)}}(a_k) \phi^{i_{\sigma^{-1}(2)}}((-1)^l \omega^{kl} a_l) \phi^{i_{\sigma^{-1}(3)}} \otimes \ldots \phi^{i_{\sigma^{-1}(n)}}\right). \end{aligned}$$

If we use the positional derivation we can make the following interpretation[10]

$$\begin{aligned} &\phi^{i_{\sigma^{-1}(1)}}(a_k) \otimes \phi^{i_{\sigma^{-1}(2)}}(a_l) \otimes \phi^{i_{\sigma^{-1}(3)}} \otimes \ldots \otimes \phi^{i_{\sigma^{-1}(n)}} = \\ &\qquad = (-1)^{k \cdot l} \frac{\partial^{(1)}}{\partial \phi^k} \frac{\partial^{(2)}}{\partial \phi^l} (\phi^{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes \phi^{i_{\sigma^{-1}(n)}}) \end{aligned}$$

ω^{kl} is non-zero only for $|a_k| + |a_l| = 1$, therefore $(-1)^{k \cdot l} = 1$ and we get

$$\Delta(X) = \left(\bar{\circ}_{12} \otimes \sum_{k,l} \frac{(-1)^k}{2} \omega^{kl} \frac{\partial^{(1)}}{\partial \phi^k} \frac{\partial^{(2)}}{\partial \phi^l}\right)(X)\,. \tag{3.14}$$

Remark 142. Let us compare this with the results of [12]. Their results are in the language of coinvariants so let us first recall the pair of isomorphism between invariants and coinvariants

$$\begin{array}{rcl} (\mathcal{P}(n, G) \otimes (V^*)^{\otimes n})^{\Sigma_n} & \cong & \mathcal{P}(n, G) \otimes_{\Sigma_n} (V^*)^{\otimes n} \\ \sum_i p_i \otimes \phi^{I_i} & \longmapsto & \frac{1}{n!} \sum_{i,J} p_i \otimes_{\Sigma_n} \phi^{I_i}(a_J) \phi^J \\ \sum_{\sigma \in \Sigma_n} \sigma p \otimes \sigma(\phi^I) & \longleftarrow & p \otimes_{\Sigma_n} \phi^I \end{array}$$

where $a_J = a_{j_1} \otimes \ldots \otimes a_{j_n} \in V^{\otimes n}$ and $\{a_i\}$ is a basis dual to $\{\phi^i\}$.

[10] To be precise, the self-composition $\bar{\bullet}_{12}$ not only "erase" $\phi^{i_{\sigma^{-1}(1)}}, \phi^{i_{\sigma^{-1}(2)}}$ but also map the element ϕ^I to the component with higher G. The positional derivations lack this kind of information but for general case we keep the track of this information in the self-composition on the $\mathcal{P}$-part.

The results of [12] are considered only for commutative operad, so let us also restrict to this case. And the results in [12] are formulated in terms of left derivation, so let us remind it. The *left derivation* is defined as

$$\frac{\partial_L}{\partial \phi^k}\phi^I = \sum_{i=1}^{\mathtt{card}(I)} \frac{\partial^{(i)}}{\partial \phi^k}\phi^I \tag{3.15}$$

Now we need two key observation.

The positional derivation on invariants corresponds (thanks to mutual isomorphisms) to the left derivation on coinvariants.

$$\begin{array}{ccc} \sum_{\rho\in\Sigma_n}\rho\phi^I & \longleftarrow\!\!\shortmid & \phi^I \\ \Big| & & \vdots \\ \frac{\partial^{(1)}}{\partial\phi^k}\sum_{\rho\in\Sigma_n}\rho\phi^I & \longmapsto & \frac{\partial_L}{\partial\phi^k}\phi^I \end{array}$$

The positional derivation on invariants corresponds to

$$\sum_{\rho\in\Sigma_n}\frac{\partial^{(1)}}{\partial\phi^k}\rho\phi^I = \sum_{\rho\in\Sigma_n}\tilde{\rho}\frac{\partial^{\rho^{-1}(1)}}{\partial\phi^k}\phi^I = \sum_{\sigma\in\Sigma_{n-1}}\sigma\left(\sum_{i=1}^{n}\frac{\partial^{(i)}}{\partial\phi^k}\phi^I\right) = \sum_{\sigma\in\Sigma_{n-1}}\sigma\frac{\partial_L}{\partial\phi^k}\phi^I \tag{3.16}$$

where $\tilde{\rho}$ is ρ restricted to $[n]-\{\rho^{-1}(1)\}$. Now this element is mapped by lower horizontal arrow in the diagram back to coinvariants as

$$\frac{1}{(n-1)!}\sum_{\substack{J\\ \mathtt{card}(J)=n-1}}\left(\sum_{\sigma\in\Sigma_{n-1}}\sigma\frac{\partial_L}{\partial\phi^k}\phi^I\right)(a_J)\phi^J = \frac{1}{(n-1)!}\sum_{\sigma\in\Sigma_{n-1}}\sum_J\frac{\partial_L}{\partial\phi^k}\phi^I\,(a_J)\,\phi^J.$$

Notice that all the sumands do not depend on the permutation σ and there are exactly $(n-1)!$ permutations in Σ_{n-1}.

Repeating this argument with appropriate coefficients will give us

$$\frac{1}{(n-2)!}\sum_{\substack{J\\ \mathtt{card}(J)=n-2}}\left(\sum_{\rho\in\Sigma_n}\sum_{k,l}\frac{(-1)^k}{2}\omega^{kl}\frac{\partial^{(1)}}{\partial\phi^k}\frac{\partial^{(2)}}{\partial\phi^l}(\rho\phi^I)\right)(a_J)\phi^J =$$

$$= \sum_{k,l}\frac{(-1)^k}{2}\omega^{kl}\frac{\partial_L}{\partial\phi^k}\frac{\partial_L}{\partial\phi^l}\phi^I.$$

Therefore after changing the sign convention and coefficients in Definition 137 we get the same expression of Δ as in the setting of coinvariants in [12] in the case of commutative operad.

Example 143. Similarly we want to express the BV-bracket. First note that the sum over decompositions $C_1 \sqcup C_2 = [n+m]$ in the definition of bracket in (3.5) corresponds to the sum over all shuffles $\pi \in Sh(n-1, m-1)$. Evaluated on elements we get

$$\{X,Y\} = \{\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\sigma p\otimes\sigma(\phi^I),\ \frac{1}{m!}\sum_{\rho\in\Sigma_m}\rho q\otimes\rho(\phi^J)\} = \tag{3.17}$$

$$= \frac{(-1)^{|p|+I}\cdot(-1)^{|q|\cdot I+|p|+|q|}\cdot 2}{n!\cdot m!}\sum_{\substack{\sigma\in\Sigma_n\\ \rho\in\Sigma_m\\ \pi\in Sh(n-1,m-1)}}\pi(\sigma p\,{}_1\bar{\circ}_1\,\rho q)\otimes\pi(\sigma\phi^I\,{}_1\bar{\bullet}_1\,\rho\phi^J) =$$

$$=(-1)^{|X|}\sum_{\pi\in Sh(n-1,m-1)}\left(\pi({}_1\bar{\circ}_1)\otimes\sum_{k,l}(-1)^k\omega^{kl}\pi\left(\frac{\partial^{(1)}}{\partial\phi^k}\otimes\frac{\partial^{(1)}}{\partial\phi^l}\right)\right)(1\otimes\tau\otimes 1)(X\otimes Y).$$

Remark 144. Similarly as in Remark 142 let us restrict to commutative operad.

$$\{\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\sigma(\phi^I),\frac{1}{m!}\sum_{\rho\in\Sigma_m}\rho(\phi^J)\} =$$

$$= \frac{(-1)^I}{n!m!}\sum_{\substack{\sigma\in\Sigma_n\\ \rho\in\Sigma_m\\ \pi\in Sh(n-1,m-1)}}\sum_{k,l}(-1)^{k+I\cdot l}\omega^{kl}\pi\left(\frac{\partial^{(1)}}{\partial\phi^k}\sigma\phi^I\otimes\frac{\partial^{(1)}}{\partial\phi^l}\rho\phi^J\right) =$$

$$= \frac{(-1)^I}{n!m!}\sum_{\pi\in\Sigma_{n+m-2}}\sum_{k,l}(-1)^{k+I\cdot l}\omega^{kl}\pi\left(\frac{\partial_L\phi^I}{\partial\phi^k}\cdot\frac{\partial_L\phi^J}{\partial\phi^l}\right)$$

where we used (3.16) and the fact that all possible permutations σ,ρ on two components $\frac{\partial_L\phi^I}{\partial\phi^k},\frac{\partial_L\phi^J}{\partial\phi^l}$ followed by all possible shuffles π between these two components are exactly all possible permutations.

To compare with results of [12] let us recall the *right derivation*

$$\frac{\partial_R}{\partial\phi^k}\phi^I = (-1)^{k\cdot(I+1)}\frac{\partial_L}{\partial\phi^k}\phi^I.$$

Since ω is odd symplectic form $(-1)^I\cdot(-1)^{k+I\cdot l}\cdot(-1)^{k\cdot(I+1)} = 1$. By similar arguments as before we get

$$\frac{1}{(n+m-2)!}\sum_{\substack{K,\mathtt{card}(K)=n+m-2\\ \pi\in\Sigma_{n+m-2}}}\sum_{k,l}\omega^{kl}\pi\left(\frac{\partial_R\phi^I}{\partial\phi^k}\cdot\frac{\partial_L\phi^J}{\partial\phi^l}\right)(a_K)\phi^K =$$

$$= \sum_{k,l}\omega^{kl}\frac{\partial_R\phi^I}{\partial\phi^k}\cdot\frac{\partial_L\phi^J}{\partial\phi^l}.$$

This agree with the BV-bracket in [12].

Example 145. Continuing on previous examples, let us finally describe the product $\star$. For $X = \frac{1}{n!}\sum_{\sigma\in\Sigma_n}\sigma p\otimes\sigma(\phi^I)$, $\phi^I = \phi^{i_1\dots i_n}$, and $Y = \frac{1}{m!}\sum_{\rho\in\Sigma_m}\rho q\otimes\rho(\phi^J)$, $\phi^J = \phi^{j_1\dots j_m}$

$$X\star Y = \frac{(-1)^{|q|\cdot I}}{n!\cdot m!}\sum_{\pi\in Sh(n,m)}\sum_{\sigma,\rho}\pi(\sigma p\ \#_2\ \rho q)\otimes\pi(\sigma\phi^I\ \#_2\ \rho\phi^J) =$$

$$\overset{(CS1)}{=}\frac{(-1)^{|q|\cdot I}}{n!\cdot m!}\sum_{\gamma\in\Sigma_{n+m}}\gamma(p\ \#_2\ q)\otimes\gamma(\phi^{i_1\dots i_n j_1\dots j_m}).$$

3.3.3 Special deformation retracts

Definition 146. A **special deformation retract (SDR)** is a pair (V, d) and (W, e) of dg vector spaces, a pair p and i of their morphisms and a homotopy $k : V \to V$ between ip and 1_V

$$k \circlearrowright (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (W, e)$$

$$\begin{aligned}
&d^2 = 0, \quad e^2 = 0, \quad |d| = |e| = 1, && \ldots \textit{differentials}\\
&pd = ep, \quad ie = di, \quad |p| = |i| = 0, && \ldots \textit{chain maps}\\
&ip - \mathrm{id}_V = kd + dk, \quad |k| = -1, && \ldots \textit{homotopy map}\\
&pi - \mathrm{id}_W = 0, && \ldots \textit{deformation retract}\\
&pk = 0, \quad ki = 0, \quad k^2 = 0 && \ldots \textit{special deformation retract.}
\end{aligned}$$

Remark 147. It is possible to consider only the first three conditions, i.e., chain maps i, p between chain complexes $(V, d), (W, e)$ with homotopy k. In that case, one gets the so-called *standard situation*. When considering also the fourth condition one gets the *deformation retract*. But in the next, we will always consider the SDR.

Starting with SDR on a chain complex, there is a process inducing the SDR on its tensor powers. The original construction was made in [14] by Eilenberg and Mac Lane. Here we rephrase it in our conventions for the tensor product of two chain complexes. The general tensor power can be then defined by the iteration of this process.

Lemma 148. (Tensor trick) Given two SDR

$$k_1 \circlearrowright (V_1, d_1) \underset{i_1}{\overset{p_1}{\rightleftarrows}} (W_1, e_1) \qquad\qquad k_2 \circlearrowright (V_2, d_2) \underset{i_2}{\overset{p_2}{\rightleftarrows}} (W_2, e_2) \tag{3.18}$$

then there is a SDR on their tensor product, i.e.,

$$\tilde{k} \circlearrowright (V_1 \otimes V_2, \tilde{d}) \underset{\tilde{i}}{\overset{\tilde{p}}{\rightleftarrows}} (W_1 \otimes W_2, \tilde{e})$$

where $\tilde{i} = i_1 \otimes i_2, \tilde{p} = p_1 \otimes p_2, \tilde{d} = d_1 \otimes \mathrm{id} + \mathrm{id} \otimes d_2, \tilde{e} = e_1 \otimes \mathrm{id} + \mathrm{id} \otimes e_2$, and $\tilde{k} = \mathrm{id} \otimes k_2 + k_1 \otimes i_2 p_2$.

Proof. From the observation $|k_i| = -1$ and $|d_i| = 1$ we see that, for example, the member given as $(\mathrm{id} \otimes d_2) \circ (k_1 \otimes i_2 p_2) = -k_1 \otimes d_2 i_2 p_2$ has the opposite sign as the one given by $(k_1 \otimes i_2 p_2) \circ (1 \otimes d_2)$. The rest of the proof is straightforward computation. □

Notice that $\tilde{k}$ can be defined both as $\tilde{k} = \mathrm{id} \otimes k_2 + k_1 \otimes i_2 p_2$ or as $\tilde{k} = k_1 \otimes \mathrm{id} + i_1 p_1 \otimes k_2$.

Remark 149. For any dg vector space (V, d) it is possible to construct SDR since there is always a decomposition

$$V \cong H(V) \oplus \mathrm{Im}(d) \oplus W$$

Such decomposition is known as *harmonious Hodge decomposition.* The homotopy map k is defined as follows

$$k|_{H(V)\oplus W} = 0 \qquad\qquad k|_{\mathrm{Im}(d)} = (d|_W)^{-1}$$

As Chuang and Lazarev showed in [6], in case we have the symplectic form on V, it is possible to choose the decomposition compatible with this form. This is ensured by the previously mentioned condition $d(\omega) = 0$.

Also, since we have a field of characteristic not dividing the order of Σ_n for any n, by Maschke's theorem we can choose decomposition that is compatible with Σ_n-action.

One may asks if this decomposition could be also made on every component $\mathcal{P}(n, G)$ of operad $\mathcal{P}$ so it will be compatible with operad structure maps ${}_a\circ_b$ and $\circ_{ab}$. Obviously modular operads with trivial differential are examples satisfying the required condition. Unfortunately, we are not aware of any non-trivial examples.

3.3.4 Cohomology of formal functions

In the following, we want to consider the perturbation of the space of generalized formal functions. A natural question that arises is if the cohomology of generalized formal functions is equal to generalized formal functions on cohomology, i.e., $H(\mathrm{Fun}(\mathcal{P}, V)) = \mathrm{Fun}(H(\mathcal{P}), H(V))$. Thanks to Remark 149 we can rephrase this in the terms of SDR.

In general, the differential on the space $\mathrm{Fun}(\mathcal{P}, V)$ is given as a sum of the differential on $\mathcal{P}$ and dual of differential on V. Since in most of the examples is the first one trivial, i.e., $d_{\mathcal{P}} = 0$, we first solve this case in the following lemma and devote the subsequent Remark 151 to the case with nontrivial $d_{\mathcal{P}}$.

Lemma 150. Let $\mathcal{P}$ be operad with trivial differential. If

$$k \circlearrowright (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (H(V), 0)$$

is a SDR, then

$$K \circlearrowright (\mathrm{Fun}(\mathcal{P}, V), D) \underset{I}{\overset{P}{\rightleftarrows}} (\mathrm{Fun}(\mathcal{P}, H(V)), 0) \tag{3.19}$$

is SDR, where

$$D = \sum_{n\geq 1}\sum_{i=1}^{n} \mathrm{id}_{\mathcal{P}} \otimes (\mathrm{id}^{\otimes i-1} \otimes d^* \otimes \mathrm{id}^{\otimes n-i}),$$

$$I = \sum_{n\geq 1} \mathrm{id}_{\mathcal{P}} \otimes (p^*)^{\otimes n}, \qquad\qquad P = \sum_{n\geq 1} \mathrm{id}_{\mathcal{P}} \otimes (i^*)^{\otimes n},$$

$$K = \sum_{n\geq 1}\sum_{\sigma\in\Sigma_n}\sum_{i=1}^{n} \frac{\sigma}{n!} \mathrm{id}_{\mathcal{P}} \otimes (\mathrm{id}^{\otimes i-1} \otimes k^* \otimes (p^* i^*)^{\otimes n-i}). \tag{3.20}$$

Proof. The only nontrivial identity from Definition 146 to verify is $KD + DK = IP - 1_{\mathrm{Fun}(\mathcal{P},V)}$.

To simplify the computation let us consider just the invariants of the form $(\pi_j \otimes \phi^{I_j})$ where $\pi_j \in \mathcal{P}(n,G)$ with G fixed and ϕ^{I_j} are monomials in $(V^*)^{\otimes n}$, where we use the standard abbreviation $\phi^{I_j} = \phi^{i_1} \otimes \cdots \otimes \phi^{i_n}$ for any multiindex $I_j = (i_1, \ldots i_n)$. The operator K restricted to this space could be written as

$$K|_{(\mathcal{P}(n,G)\otimes(V^*)^{\otimes n})^{\Sigma_n}} = \underbrace{\sum_{\sigma\in\Sigma_n} \frac{\sigma}{n!}}_{S_{n,G}} \mathrm{id}_{\mathcal{P}} \otimes \underbrace{\sum_{i=1}^{n} (\mathrm{id}^{\otimes i-1} \otimes k^* \otimes (p^* i^*)^{\otimes n-i})}_{K_{n,G}}$$

Similarly denote the restrictions of D, I, and P on this space as $\mathrm{id}_{\mathcal{P}} \otimes D_{n,G}$, $\mathrm{id}_{\mathcal{P}} \otimes I_{n,G}$, and $\mathrm{id}_{\mathcal{P}} \otimes P_{n,G}$, respectively. It turns out, to prove the desired identity we need to show

$$S_{n,G}(\mathrm{id}_{\mathcal{P}} \otimes K_{n,G} D_{n,G}) + (\mathrm{id}_{\mathcal{P}} \otimes D_{n,G}) S_{n,G}(\mathrm{id}_{\mathcal{P}} \otimes K_{n,G}) = \mathrm{id}_{\mathcal{P}} \otimes I_{n,G} P_{n,G} - \mathrm{id}$$

We want to know if we can interchange the permutation $\sigma \in \Sigma_n$ and the operator $D_{n,G}$.[11] The following observation is a key to prove this identity, unfortunately, it is rather technical and bit messy in the notation of indices.

Consider a monomial $\phi^I = \phi^{i_1} \otimes \ldots \phi^{i_n}$. For every $k = 1, \ldots n$ we can write it as $\phi^I = \phi^{I_1}\phi^{i_k}\phi^{I_2}$ where I_1, I_2 are the corresponding multiindices such that $I = (I_1, i_k, I_2)$ is a multiindex. Applying permutation σ on monomial ϕ^I we can rewrite as

$$\sigma(\phi^I) = (-1)^{i_k(J_1+J_2)} \cdot \sigma_1(\phi^{I_1-J_1}\phi^{J_2})\phi^{i_k}\sigma_2(\phi^{J_1}\phi^{I_2-J_2})$$

Then

$$\begin{aligned} &\sigma\left(D_{n,G}(\phi^I)\right) = \\ &= \sigma\left(D_{n_1,G}(\phi^{I_1})\phi^{i_k}\phi^{I_2} + (-1)^{I_1}\phi^{I_1} d^*(\phi^{i_k})\phi^{I_2} + (-1)^{I_1+i_k}\phi^{I_1}\phi^{i_k} D_{n_2,G}(\phi^{I_2})\right) \end{aligned}$$

where for short $D_{1,G} = d^*$ and $n_1 = \mathtt{card}(I_1), n_2 = \mathtt{card}(I_2)$. So the only element with $d^*(\phi^{i_k})$ is

$$(-1)^{I_1} \cdot (-1)^{(J_1+J_2)\cdot(i_k+1)} \sigma_1(\phi^{I_1-J_1}\phi^{J_2}) \cdot d^*(\phi^{i_k}) \cdot \sigma_2(\phi^{J_1}\phi^{I_2-J_2})$$

On the other hand

$$\begin{aligned} D_{n,G}(\sigma(\phi^I)) &= (-1)^{i_k(J_1+J_2)} \cdot \left(D_{n_3,G}(\sigma_1(\phi^{I_1-J_1}\phi^{J_2})) \otimes \phi^{i_k}\sigma_2(\phi^{J_1}\phi^{I_2-J_2})\right) + \\ &+ (-1)^{i_k(J_1+J_2)+(I_1-J_1+J_2)} \cdot \left(\sigma_1(\phi^{I_1-J_1}\phi^{J_2}) \otimes d^*(\phi^{i_k}) \otimes \sigma_2(\phi^{J_1}\phi^{I_2-J_2})\right) + \\ &+ (-1)^{i_k(J_1+J_2)+(I_1-J_1+J_2+i_k)} \cdot \left(\sigma_1(\phi^{I_1-J_1}\phi^{J_2})\phi^{i_k} \otimes D_{n_4,G}(\sigma_2(\phi^{J_1}\phi^{I_2-J_2}))\right) \end{aligned}$$

where $n_3 = \mathtt{card}(I_1) - \mathtt{card}(J_1) + \mathtt{card}(J_2)$ and $n_4 = n - n_3 - 1$. It is evident both of the expressions contain the same term with $d^*(\phi^{i_k})$. Repeating this argument for all indices $k = 1, \ldots n$ we get the required identity

$$(\mathrm{id}_{\mathcal{P}} \otimes D_{n,G}) S_{n,G} = S_{n,G}(\mathrm{id}_{\mathcal{P}} \otimes D_{n,G})$$

Finally, we need to show $S_{n,G}(\mathrm{id}_{\mathcal{P}} \otimes K_{n,G} D_{n,G} + \mathrm{id}_{\mathcal{P}} \otimes D_{n,G} K_{n,G}) = \mathrm{id}_{\mathcal{P}} \otimes I_{n,G} P_{n,G} - \mathrm{id}$ but this follows from the tensor trick introduced in Lemma 148 and the fact that right hand side is invariant under the action of $S_{n,G}$. $\square$

[11] Obviously, up to the sign this is true. But we want equality with the specific sign.

Remark 151. To our question about nontrivial differential on $\mathcal{P}$. Following Remark 149, we can construct SDR for any component (n, G). From two SDR's

$$k_P \circlearrowright (\mathcal{P}(n,G), d_P) \underset{i_P}{\overset{p_P}{\rightleftarrows}} (H(\mathcal{P}(n,G)), 0)$$

and

$$k_V \circlearrowright ((V^*)^{\otimes n}, d_V) \underset{i_V}{\overset{p_V}{\rightleftarrows}} (H(V^*)^{\otimes n}), 0)$$

we obtain by Lemma 148 the following SDR

$$\begin{array}{c} {}_{1\otimes k_V + k_P \otimes i_V p_V} \circlearrowright (\mathcal{P}(n,G) \otimes (V^*)^{\otimes n}, d_P \otimes \mathrm{id} + \mathrm{id} \otimes d_V) \\ {}_{i_P\otimes i_V}\uparrow \downarrow{}_{p_P\otimes p_V} \\ (H(\mathcal{P}(n,G)) \otimes H(V^*)^{\otimes n}, 0) \end{array}$$

Our aim is to find cohomology of formal functions $\mathrm{Fun}(\mathcal{P}, V)$. To do this we first need to restrict ourselves to the subset of invariants.

From the second part of Remark 149 we can choose decomposition of every component $\mathcal{P}(n,G)$ (that is finite dimensional from the assumption) in such a way that p_P, i_P and k_P are equivariant.

When we restrict ourselves to the subset of invariants $\pi_i \otimes \phi^{I_i} \in (\mathcal{P}(n,G) \otimes (V^*)^{\otimes n})^{\Sigma_n}$ we get for arbitrary $\sigma \in \Sigma_n$

$$(d_P \otimes \mathrm{id} + 1 \otimes d_V)(\pi_i \otimes \phi^{I_i}) = (d_P \otimes \mathrm{id} + 1 \otimes d_V)(\sigma\pi_i \otimes \sigma\phi^{I_i}) =$$
$$= d_P \sigma\pi_i \otimes \sigma\phi^{I_i} + (-1)^{|\pi_i|}\sigma\pi_i \otimes d_V \sigma\phi^{I_i} = \sigma d_P \pi_i \otimes \sigma\phi^{I_i} + (-1)^{|\pi_i|}\sigma\pi_i \otimes \sigma d_V \phi^{I_i} =$$
$$= (\sigma \otimes \sigma)(d_P \otimes \mathrm{id} + 1 \otimes d_V)(\pi_i \otimes \phi^{I_i})$$

thanks to the equivariance of differential d_P and from the technical part of the previous lemma. Hence $d_P \otimes \mathrm{id} + 1 \otimes d_V$ goes from invariants to invariants. Similarly, thanks to equivariance of p_P, p_V, i_P, i_V clearly $p_P \otimes p_V$ and $i_P \otimes i_V$ map invariants to invariants.

To make the homotopy map going also to invariants, we need to modify it a bit as in previous lemma

$$\sum_{\rho \in \Sigma_n} \frac{\rho}{n!}(1 \otimes k_V) + k_P \otimes i_V p_V = K + k_p \otimes i_V p_V$$

where K is defined in 3.20. One can easily check that subspaces of invariants together with these maps give SDR. In other words

$$(H(\mathcal{P}(n,G)) \otimes H(V^*)^{\otimes n})^{\Sigma_n} = H((\mathcal{P}(n,G) \otimes (V^*)^{\otimes n})^{\Sigma_n})$$

Extending this linearly we get required equivalence

$$H(\mathrm{Fun}(\mathcal{P}, V)) = \mathrm{Fun}(H(\mathcal{P}), H(V))$$

Remark 152. Let us set for the unit that we "artificially" added in Remark 128

$$D(1 \otimes 1) = 0 \qquad K(1 \otimes 1) = 0$$
$$I(1 \otimes 1) = 1 \otimes 1 \qquad P(1 \otimes 1) = 1 \otimes 1$$

Obviously, the condition $IP - 1 = KD + DK$ is satisfied also for the element $1 \otimes 1$.

3.3.5 Hodge decomposition of $\mathrm{Fun}(\mathcal{P}, V)$

Consider the symplectit Hodge decomposition of Remark 149 and its graded dual. Let us show the explicit form of the maps D, I, P and K.

Lemma 153. There is a SDR

$$K \circlearrowright (\mathrm{Fun}(\mathcal{P}, V), D) \underset{I}{\overset{P}{\rightleftarrows}} (\mathrm{Fun}(\mathcal{P}, H(V)), 0) \tag{3.21}$$

and a basis $\{\alpha^k\}$ of $H(V)^*$, a basis $\{\beta^k\}$ of $(\mathrm{Im}d)^*$ and a basis $\{\gamma^k\}$ of W^* such that in these bases $d^*(\beta^l) = -\sum_k D^l_k \gamma^k$ and the maps are of the form

$$I = \sum_{n\geq 1} \mathrm{id}_{\mathcal{P}} \otimes (p^*)^{\otimes n} \qquad\qquad P = \sum_{n\geq 1} \mathrm{id}_{\mathcal{P}} \otimes (i^*)^{\otimes n}$$

and for $X \in (\mathcal{P}(n, G) \otimes (V^*)^{\otimes n})^{\Sigma_n} \subset \mathrm{Fun}(\mathcal{P}, V)$

$$D(X) = -\left(\mathrm{id}_{\mathcal{P}} \otimes \sum_{i=1}^{n} \sum_{k,l} D^l_k \gamma^k_{(i)} \frac{\partial^{(i)}}{\partial \beta^l}\right)(X)$$

where we use the positional derivation and positional multiplication introduced in Definition 136. And if $X = \frac{1}{n!}\sum_{\rho\in\Sigma_n} \rho p \otimes \rho(\phi^I)$

$$K(X) = \frac{1}{\eta_{\beta,\gamma}(X)}\left(1_{\mathcal{P}} \otimes \sum_{i=1}^{n}\sum_{k,l} (D^{-1})^k_l \; \beta^l_{(i)} \frac{\partial^{(i)}}{\partial \gamma^k}\right)(X) \tag{3.22}$$

where the symbol $\eta_{\beta,\gamma}(X)$ denotes the number of occurrences of β^k and γ^k in ϕ^I. When $\eta_{\beta,\gamma}(X) = 0$ then $K(X) = 0$.

Proof. Again, we only need to check $KD + DK = IP - 1_{\mathrm{Fun}(\mathcal{P},V)}$. The right hand side of this equation could be expressed on the element $X = \frac{1}{n!}\sum_{\rho\in\Sigma_n} \rho p \otimes \rho(\phi^I)$ as

$$(IP - 1_{\mathrm{Fun}(\mathcal{P},V)})X = (1_{\mathcal{P}} \otimes ((p^*i^*)^n - 1^n))\, X$$

Obviously, if $\phi^I = \phi^{i_1} \otimes \cdots \otimes \phi^{i_n}$ contains only α^k then $(p^*i^*)^n\phi^I = \phi^I$.

On the other hand, if for some i, $\phi^i = \beta^k$ or $\phi^i = \gamma^k$ then $(p^*i^*)^n\phi^I = 0$. Therefore the right hand side corresponds to

$$(IP - 1_{\mathrm{Fun}(\mathcal{P},V)})X = \frac{1}{n!}\sum_{\rho\in\Sigma_n} \rho p \otimes \frac{-1}{\eta_{\beta,\gamma}(X)} \sum_{i=1}^{n}\sum_{k} \left(\beta^k_{(i)} \frac{\partial^{(i)}}{\partial \beta^k} + \gamma^k_{(i)} \frac{\partial^{(i)}}{\partial \gamma^k}\right) \rho\phi^I$$

Using a technical observation we can rewrite the left hand side as

$$(KD + DK)X =$$

$$= \sum_{\substack{\sigma\in\Sigma_n \\ \rho\in\Sigma_n}} \frac{\sigma}{n!\cdot n!} \sum_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} \rho p \;\otimes \Big((1^{j-1} \otimes k^* \otimes (p^*i^*)^{n-j})(1^{i-1} \otimes d^* \otimes 1^{n-i})\Big)\, \rho(\phi^I) +$$

$$+ \sum_{\substack{\sigma\in\Sigma_n \\ \rho\in\Sigma_n}} \frac{\sigma}{n!\cdot n!} \sum_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} \rho p \;\otimes \Big((1^{i-1} \otimes d^* \otimes 1^{n-i})(1^{j-1} \otimes k^* \otimes (p^*i^*)^{n-j})\Big)\, \rho(\phi^I)$$

By closer look, we see we can separate this into three types of terms.

If $i < j$ then we always get two terms of the form

$$\sum_{\substack{\sigma\in\Sigma_n\\ \rho\in\Sigma_n}} \frac{\sigma}{n!\cdot n!} \sum_{1\le i<j\le n} \rho p \otimes \left(1^{i-1} \otimes d^* \otimes 1 \otimes \ldots \otimes 1 \otimes k^* \otimes (p^* i^*)^{n-j}\right) \rho(\phi^I)$$

but with the signs $+1$ and $(-1)^{|d^*|\cdot|k^*|}$. So they canceled out.

If $i > j$ then we always get two terms of the form

$$\sum_{\substack{\sigma\in\Sigma_n\\ \rho\in\Sigma_n}} \frac{\sigma}{n!\cdot n!} \sum_{1\le j<i\le n} \rho p \otimes \left(1^{j-1} \otimes k^* \otimes (p^* i^*) \otimes \ldots \otimes d^*(p^* i^*) \otimes (p^* i^*)^{n-i}\right)$$

and since d^* on $\mathrm{Im}(p^*)$ is always 0, we again get zero. The last type are the terms of the form

$$\sum_{\substack{\sigma\in\Sigma_n\\ \rho\in\Sigma_n}} \frac{\sigma}{n!\cdot n!} \sum_{1\le j\le n} \rho p \otimes \left(1^{j-1} \otimes (k^* d^* + d^* k^*) \otimes (p^* i^*)^{n-j}\right) \rho(\phi^I) =$$
$$= \sum_{\substack{\sigma\in\Sigma_n\\ \rho\in\Sigma_n}} \frac{\sigma}{n!\cdot n!} \sum_{1\le j\le n} \rho p \otimes \left(1^{j-1} \otimes (p^* i^* - 1) \otimes (p^* i^*)^{n-j}\right) \rho(\phi^I)$$

The only non-zero $\rho(\phi^I)$ for this map are of the form

$$\sum_{j=1}^{n-1} \sum_{k_j,\ldots k_n} \phi^{i_{\rho^{-1}(1)}} \otimes \ldots \phi^{i_{\rho^{-1}(j)}} \otimes \beta^{k_j} \otimes \alpha^{k_{j+1}} \otimes \ldots \alpha^{k_n}$$

and

$$\sum_{j=1}^{n-1} \sum_{k_j,\ldots k_n} \phi^{i_{\rho^{-1}(1)}} \otimes \ldots \phi^{i_{\rho^{-1}(j)}} \otimes \gamma^{k_j} \otimes \alpha^{k_{j+1}} \otimes \ldots \alpha^{k_n}$$

Therefore if ϕ^I contains only α^k also $KD + DK$ gives zero. From $d^*(\beta^l) = -\sum_k D^l_k \gamma^k$ we trivially get

$$D|_{(\mathcal{P}(n,G)\otimes(V^*)^{\otimes n})^{\Sigma_n}} = \sum_{i=1}^{n} \mathrm{id}_{\mathcal{P}} \otimes (\mathrm{id}^{\otimes i-1} \otimes d^* \otimes \mathrm{id}^{\otimes n-i}) = -\mathrm{id}_{\mathcal{P}} \otimes \sum_{i=1}^{n} \sum_{k,l} D^l_k\ \gamma^k_{(i)} \frac{\partial^{(i)}}{\partial \beta^l}$$

From this we finally see the form of K is as in equation (3.22). □

Remark 154. The decomposition of vector space $V = H(V) \oplus (\mathrm{Im}(d) \oplus W)$ as in Remark 149 and the explicit form of Δ in (3.14) allow us to do the same decomposition $\Delta = \Delta_\alpha + \Delta_{\beta\gamma}$ of BV Laplacian on $\mathrm{Fun}(\mathcal{P}, V)$ as in [12] in Lemma 5.

We split the symplectic form in the basis $\{\{a_i\}, \{b_i\}, \{c_i\}\}$ of $H(V)\oplus\mathrm{Im}(d)\oplus W$ to $\omega' = (a_i, a_j)$ and $\omega'' = (b_i, c_j)$. Then in the dual basis $\{\{\alpha^i\}, \{\beta^i\}, \{\gamma^i\}\}$ of $H(V)^* \oplus (\mathrm{Im}(d))^* \oplus W^*$

$$\Delta_\alpha = \frac{1}{2}\left(\overline{\circ}_{12} \otimes \sum_{k,l} (-1)^k (\omega')^{kl} \frac{\partial^{(1)}}{\partial \alpha^k} \frac{\partial^{(2)}}{\partial \alpha^l} \right)$$
$$\Delta_{\beta\gamma} = \frac{1}{2}\left(\overline{\circ}_{12} \otimes \sum_{k,l} (-1)^k (\omega'')^{kl} \left(\frac{\partial^{(1)}}{\partial \beta^k} \frac{\partial^{(2)}}{\partial \gamma^l} + \frac{\partial^{(1)}}{\partial \gamma^k} \frac{\partial^{(2)}}{\partial \beta^l}\right) \right)$$

And similarly from (3.17) we get $\{\cdot,\cdot\} = \{\cdot,\cdot\}_\alpha + \{\cdot,\cdot\}_{\beta\gamma}$ where

$$\{X,Y\}_\alpha = (-1)^{|X|} \sum_{Sh} \left(\pi({}_1\bar{\circ}_1) \otimes \sum_{k,l} (-1)^k (\omega')^{kl} \pi \left(\frac{\partial^{(1)}}{\partial \alpha^k} \otimes \frac{\partial^{(1)}}{\partial \alpha^l} \right) \right) \tau\,(X \otimes Y)$$

$$\{X,Y\}_{\beta\gamma} =$$
$$= \frac{(-1)^{|X|}}{2} \sum_{Sh} \left(\pi({}_1\bar{\circ}_1) \otimes \sum_{k,l} (-1)^k (\omega'')^{kl} \pi \left(\frac{\partial^{(1)}}{\partial \beta^k} \otimes \frac{\partial^{(1)}}{\partial \gamma^l} + \frac{\partial^{(1)}}{\partial \gamma^k} \otimes \frac{\partial^{(1)}}{\partial \beta^l} \right) \right) \tau\,(X \otimes Y)$$

where Sh is a shorthand notation for all relevant shuffles and τ denotes the monomial symmetry $(p \otimes \phi^I \otimes q \otimes \phi^J \to (-1)^{|q|\cdot I} p \otimes q \otimes \phi^I \otimes \phi^J)$.

4. Homological perturbation lemma

Definition 155. Let (V, d) be a dg vector space. A **perturbation** $\delta : V \to V$ of the differential d is a linear map of degree 1 such that

$$(d + \delta)^2 = 0.$$

Equivalently,

$$\delta^2 + \delta d + d\delta = 0.$$

Theorem 156 (Perturbation lemma)**.** Consider

$$k \circlearrowright (V, d) \underset{i}{\overset{p}{\rightleftarrows}} (W, e). \tag{4.1}$$

Let δ be a perturbation of d which is small in the sense that

$$(1 - \delta k)^{-1} \equiv \sum_{i=0}^{\infty} (\delta k)^i$$

is a well defined linear map $V \to V$. Denote $A \equiv (1 - \delta k)^{-1}\delta$ and

$$\begin{aligned}
d' &\equiv d + \delta, \\
e' &\equiv e + pAi = e + p(1 - \delta k)^{-1}\delta i, \\
i' &\equiv i + kAi = i + k(1 - \delta k)^{-1}\delta i, \\
p' &\equiv p + pAk = p(1 - \delta k)^{-1}, \\
k' &\equiv k + kAk = k(1 - \delta k)^{-1},
\end{aligned} \tag{4.2}$$

$$k' \circlearrowright (V, d') \underset{i'}{\overset{p'}{\rightleftarrows}} (W, e')\,.$$

Then if (4.1) is an SDR, then (4.2) is an SDR.

We now apply this theorem to our situation. Consider the SDR of Theorem 150

$$K \circlearrowright (\mathrm{Fun}_{\{}(\mathcal{P}, V), D) \underset{I}{\overset{P}{\rightleftarrows}} (\mathrm{Fun}_{\{}(\mathcal{P}, H(V)), 0).$$

In this case, there are two possible perturbations we can consider:

- A perturbation by $\delta_1 = \Delta$.
- A perturbation by $\delta_2 = \boldsymbol{\{} \Delta + \{S, -\}$.

The first one can be considered only in the case when we have defined graded commutative associative algebra on $\mathrm{Fun}_{\{}(\mathcal{P}, V)$, as we did in Definition 125. The second one can be considered also in the cases, when we don't have any such product. We discuss this "subcase" in Section 4.3.

Remark 157. One may ask if the perturbation by $\delta_1 + \delta_2$ gives the same result as consecutive perturbation of δ_1 followed by perturbation by δ_2. The answer is positive and the detailed computation can be found in section 2.5 of [15].

4.1 Perturbation by Δ of $(\mathrm{Fun}_{\{}(\mathcal{P}, V), \star)$

Remark 158. Thanks to the explicit formula for the map K in (3.22) and to the decomposition of Δ in Remark 154 we can make two helpful observations:

$$K\Delta I = 0 \qquad \text{and} \qquad P(\Delta_\alpha K)^i = 0 \tag{4.3}$$

and rearrange the perturbed maps as

$$\begin{aligned}
D_1 &= D + \Delta, \\
E_1 &= \Delta_\alpha, \\
I_1 &= I, \\
P_1 &= P\sum_{i\geq 0}(\Delta_{\beta\gamma}K)^i = P + P(\Delta_{\beta\gamma}K) + P(\Delta_{\beta\gamma}K)^2 + \dots, \\
K_1 &= K\sum_{i\geq 0}(\Delta_{\beta\gamma}K)^i = K + K(\Delta_{\beta\gamma}K) + K(\Delta_{\beta\gamma}K)^2 + \dots.
\end{aligned}$$

Since $K^2 = 0$, $\Delta^2 = 0$ and $PK = 0$, we can also consider the forms

$$\begin{aligned}
P_1 &= P\sum_{i\geq 0}\left([\Delta_{\beta\gamma}, K]\right)^i, \\
K_1 &= K\sum_{i\geq 0}\left([\Delta_{\beta\gamma}, K]\right)^i.
\end{aligned}$$

Remark 159. Notice, that K always "makes" one β out of one γ and $\Delta_{\beta\gamma}$ acts on a pair of β and γ. Since P projects all β to zero, to have no nontrivial result we have to start with monomials generated only by α's and even number of γ's.

Moreover, thanks to $\dfrac{1}{\eta_{\beta,\gamma}(X)}$ in the map K we get for each K a numerical factor

$$\frac{1}{2n\cdot(2n-2)\dots 2} = \frac{1}{2^n\cdot n!}.$$

In the next we will write $K(X) = \dfrac{1}{\eta_{\beta,\gamma}(X)}\cdot K_0(X)$ where

$$K_0(X) = \left(1_{\mathcal{P}} \otimes \sum_{i=1}^{n}\sum_{k,l}(D^{-1})^k_l\ \beta^l_{(i)}\frac{\partial^{(i)}}{\partial\gamma^k}\right)(X).$$

And to shorten the formulas: $\partial_K = [\Delta_{\beta\gamma}, K_0]$.

Definition 160. The **effective action** W is defined by

$$\exp\left(\frac{W}{\{}\right) = P_1\left(\exp\left(\frac{S}{\{}\right)\right) \tag{4.4}$$

where $S \in \mathrm{Fun}_{\{}(\mathcal{P}, V)$ is the solution of quantum master equation from Remark 132.

Remark 161. We define the **logarithm** for $A \in \mathrm{Fun}(\mathcal{P}, V)$ (or $\mathrm{Fun}_{\{}(\mathcal{P}, V)$) as

$$\log(1\otimes 1 + A) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\underbrace{A\star\dots\star A}_{n-\text{times}}. \tag{4.5}$$

Obviously, $\log(\exp(A))$ is well defined and the condition $\log(\exp(A)) = A$ is still satisfied.

Theorem 162. (Transfer theorem) The effective action W is a well-defined element of $\mathrm{Fun}(\mathcal{P}, H(V))$.

Moreover, W satisfies the master equation on $\mathrm{Fun}(\mathcal{P}, H(V))$

$$\Delta_\alpha W + \frac{1}{2}\{W, W\}_\alpha = 0.$$

Proof. The technical observations from Remark 159 help us to to rearrange the right hand side of (4.4) as

$$P_1\left(\exp\left(\frac{S}{\{}\right)\right) = P\sum_{i\geq 0}\frac{1}{i!}\left(\frac{\partial_K}{2}\right)^i \exp\left(\frac{S}{\{}\right) = P\exp\left(\frac{\partial_K}{2}\right)\exp\left(\frac{S}{\{}\right).$$

Using this form of P_1 we can use a standard arguments of Quantum Field theory. In the following, we will be therefore inaccurately talking about elements of $\mathrm{Fun}(\mathcal{P}, V)$ as of stable graphs with legs labelled by elements of V^*. The arguments are similar as in the beginning of section 3 in Costello [8].

Since $S = \sum_{\substack{n,G\\ stable}} S_n^G$ we can write

$$\exp\left(\frac{S}{\{}\right) = \exp\left(\sum_{n=1}^{\infty}\sum_{\substack{G\geq 0,\\ 2G+n>2}}\frac{S_n^G}{\{}\right) = 1\otimes 1 + \sum_{\overrightarrow{u}_n^G}\prod^{\star}_{\substack{n,G\\ \text{stable}}}\left(\frac{S_n^G}{\{}\right)^{u_n^G}\cdot\frac{1}{u_n^G!}$$

where $\prod^\star$ was defined in 130 and u_n^G of $\overrightarrow{u}_n^G$ tells us how many S_n^G appear in one component of $\exp(S/\{)$.[1]

Looking at one term of this sum with $2m$ γ's, we need to act with the term $P\dfrac{(\partial_K)^m}{2^m\cdot m!}$ of P_1. In $P\dfrac{(\partial_K)^m}{2^m\cdot m!}\prod^\star_{\substack{n,G\\ \text{stable}}}\left(\dfrac{S_n^G}{\{}\right)^{l_n^G}\dfrac{1}{u_n^G!}$ the operator ∂_K^m gives $(2m)!$ different terms corresponding to all possible permutations of γ's (since it acts as second-order differential operator).

Since elements $S_n^G \in \mathrm{Fun}(\mathcal{P}, V)(n, G)$ can be described by a stable one-vertex graphs with labelled legs, the terms in $P(\partial_K)^m\prod^\star_{\substack{n,G\\ \text{stable}}}\left(\dfrac{S_n^G}{\{}\right)^{l_n^G}\dfrac{1}{u_n^G!}$ are stable graphs with labelled legs, where the edge composed of two legs means ∂_K acts on the corresponding indices of S_n^G. Of these, $2^m m!$ are equal because 2^m choices of "orientation" of edges and $m!$ "labelings" of edges give the same result.[2]

Let us denote by $\mu_\Gamma^{\mathcal{P}} : \mathcal{P}(\Gamma) \to \mathcal{P}(n, G)$ the composition in $\mathcal{P}$. In particular if Γ' has two vertices – the first vertex with set of legs $I_1 \sqcup \{i\}$ and genus G_1, the second vertex with set of legs $I_2 \sqcup \{j\}$ and genus G_2 and the edge is composed of legs i, j. Then

$$\mu_{\Gamma'}^{\mathcal{P}} : \mathcal{P}((I_1 \sqcup \{i\})) \otimes \mathcal{P}((I_2 \sqcup \{j\})) \to \mathcal{P}(\mathtt{card}(I_1) + \mathtt{card}(I_2), G_1 + G_2)$$

corresponds to our usual ${}_i\circ_j$-composition. So we can associate to a stable graph Γ a map

$$\mu_\Gamma^{\mathcal{P}} : \bigotimes_{v\in Vert(\Gamma)} \mathcal{P}((leg(v), G(v))) \to \mathcal{P}(n, G).$$

[1] If we think of l_n^G as an entry on the position n, G of some infinite matrix, then we can form a vector by (for example) zig-zag walking through this matrix.

[2] This follows from the symmetry of ∂_K and the axioms of the modular operad.

Moreover, when connecting two vertices by ∂_K which were already "connected" by $\star$, we get the operator $\sharp$ acting on the result.[3]

Thus we get

$$P\frac{(\partial_K)^m}{2^m \cdot m!} \prod_{\substack{n,G\\ \text{stable}}}^{\star} \left(\frac{S_n^G}{\{}\right)^{u_n^G} \cdot \frac{1}{u_n^G!} =$$

$$= P\left(\sum_{\substack{\Gamma \text{ stable}\\ \text{graphs with}\\ \text{labelled legs}}} \frac{N_\Gamma}{\prod u_n^G!} \prod_{\substack{\text{connected}\\ \text{comp. } \Gamma_c\\ \text{of } \Gamma}}^{\star} \frac{\sharp^{vert(\Gamma_c)-1}}{\{^{\,vert(\Gamma_c)}} (\mu_{\Gamma_c}^{\mathcal{P}} \otimes \mu_{\Gamma_c}^{\mathcal{E}_v}) \left(\bigotimes_{v \in Vert(\Gamma)} S_n^G \right) \right)$$

where N_Γ is the number of stable graphs with labelled legs and vertices giving the same graph without labelled vertices, i.e.,

$$\frac{N_\Gamma}{\prod u_n^G!} = \frac{|\mathrm{Orbit}_{G(\Gamma)}(\Gamma)|}{|G(\Gamma)|} = \frac{1}{|\mathrm{Aut}(\Gamma)|}$$

with $G(\Gamma) = \Sigma_{u_{n_1}^{G_1}} \times \Sigma_{u_{n_2}^{G_2}} \times \ldots \times \Sigma_{u_{n_v}^{G_v}}$ acting by permuting the vertices of the same "type". It follows that $W = \{\ \log P_1 e^{S/\{}$ can be written as a sum over connected graphs without any powers of $\{$.

For the second part, note that

$$0 = P_1(D+\Delta)\left(\exp\left(\frac{S}{\{}\right)\right) = P_1 D_1 \left(\exp\left(\frac{S}{\{}\right)\right) = E_1 P_1 \left(\exp\left(\frac{S}{\{}\right)\right) =$$
$$= \Delta_\alpha P_1 \left(\exp\left(\frac{S}{\{}\right)\right) = \Delta_\alpha \left(\exp\left(\frac{W}{\{}\right)\right)$$

since $\exp\left(\frac{S}{\{}\right)$ is $(D+\Delta)$-closed and after perturbation we still have SDR. Similar arguments to those in Lemma 131 give us

$$\Delta_\alpha(\exp\left(\frac{W}{\{}\right)) = (\Delta_\alpha W + \frac{1}{2}\{W, W\}_\alpha) \exp\left(\frac{W}{\{}\right).$$

□

4.2 Perturbation by $\Delta + \{S, -\}$ of $(\mathrm{Fun}(\mathcal{P}, V), \star)$

The case without the product will be considered in the next section, 4.3.

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[3]We are very vaguely talking about the last term on the right hand side of:

$$\partial_K(A \star B) = \partial_K(A) \star B + A \star \partial_K(B) \pm K_0 A \star \Delta_{\beta\gamma} B \pm \Delta_{\beta\gamma} A \star K_0 B \pm \sharp\left(\{K_0 A, B\} + \{A, K_0 B\}\right)$$

The perturbed maps are

$$\begin{aligned}
D_2 &= D + \delta_2 = D + \Delta + \{S, -\}, \\
E_2 &= P \sum_{i \geq 0} (\delta_2 K)^i \delta_2 I, \\
I_2 &= I + K \sum_{i \geq 0} (\delta_2 K)^i \delta_2 I = \sum_{i \geq 0} (K \delta_2)^i I, \\
P_2 &= P \sum_{i \geq 0} (\delta_2 K)^i, \\
K_2 &= K \sum_{i \geq 0} (\delta_2 K)^i.
\end{aligned} \tag{4.6}$$

Remark 163. From Section 3.3.5 we know the explicit form of the map K

$$K(X) = \frac{1}{\eta_{\beta,\gamma}(X)} \left(1_{\mathcal{P}} \otimes \sum_{i=1}^{n} \sum_{k,l} (D^{-1})_l^k \; \beta_{(i)}^l \frac{\partial^{(i)}}{\partial \gamma^k} \right) (X).$$

Let us consider elements $f = \sum_j p_j \otimes \phi^{I_j} \in \mathrm{Fun}(\mathcal{P}, V)$ where ϕ^{I_j} are monomials in variables α, β and γ. Obviously the map K will add one variable β. Therefore $K_2(f)$ will have at least one variable β.

Theorem 164. Let us define a map $Z : \mathrm{Fun}(\mathcal{P}, V) \to \mathrm{Fun}(\mathcal{P}, H(V))$ called a **path integral** as

$$Z(f) = \exp\left(\frac{-W}{\{}\right) \cdot P_1 \left(\exp\left(\frac{S}{\{}\right) \cdot f \right).$$

This map is equal to P_2, i.e.,

$$\exp\left(\frac{-W}{\{}\right) \cdot P \sum_i (\Delta_{\beta\gamma} K)^i \left(\exp\left(\frac{S}{\{}\right) \cdot f \right) = P \sum_i \left((\Delta + \{S, -\}) K \right)^i f.$$

Proof. The same reasoning as in [12] applies here – to prove this theorem we make three observations.

As first, let us consider the arguments from Remark 159. Both $P_1(f)$ and $P_1(\exp(S/\{) \cdot f)$ are zero if $f = p_j \otimes \phi^{I_j}$ where ϕ^{I_j} are monomials containing at least on β. Trivially also $Z(f) = 0$ if ϕ^{I_j} are monomials containing at least on β.

For the second observation, if one considers $f = \sum_j p_j \otimes \phi^{I_j} \in \mathrm{Fun}(\mathcal{P}, H(V))$ then $I(f)$ has still no variables β or γ. So $\Delta_{\beta\gamma} K$ does not act on it. Thus

$$P(\Delta_{\beta\gamma} K)^i \left(\exp\left(\frac{S}{\{}\right) \cdot I(f) \right) = P(\Delta_{\beta\gamma} K)^i \left(\exp\left(\frac{S}{\{}\right) \right) \cdot PI(f),$$

and

$$\begin{aligned}
Z(I(f)) =& \exp\left(\frac{-W}{\{}\right) \cdot \sum_{i \geq 0} P(\Delta_{\beta\gamma} K)^i \left(\exp\left(\frac{S}{\{}\right) \right) \cdot PI(f) = \\
=& \exp\left(\frac{-W}{\{}\right) \cdot \exp\left(\frac{W}{\{}\right) \cdot f = f.
\end{aligned}$$

As third, let us look closely on $ZD_2(f)$.

$$\begin{aligned}
ZD_2(f) = & \exp\left(\frac{-W}{\{}\right) \cdot P_1 \left(\exp\left(\frac{S}{\{}\right) \cdot D_2 f \right) = \\
& = \exp\left(\frac{-W}{\{}\right) \cdot P_1 \left(\exp\left(\frac{S}{\{}\right) \cdot (Df + \Delta f + \{S, f\}) \right).
\end{aligned} \tag{4.7}$$

The right hand side evokes an idea. From (3.10) we know that

$$\{\frac{S^k}{k!}, f\} = \frac{1}{k}\left(\frac{S^{k-1}}{(k-1)!}\cdot\{S,f\} + \{\frac{S^{k-1}}{(k-1)!}, f\}\cdot S\right) = \frac{S^{k-1}}{(k-1)!}\cdot\{S,f\}.$$

Therefore

$$\{\exp\left(\frac{S}{\{}\right), f\} = \exp\left(\frac{S}{\{}\right)\cdot\{\frac{S}{\{}, f\}. \tag{4.8}$$

Combining

$$\begin{aligned}(D+\Delta)&\left(\exp\left(\frac{S}{\{}\right)\cdot f\right) = \\ &= (D+\Delta)\left(\exp\left(\frac{S}{\{}\right)\right)\cdot f + \exp\left(\frac{S}{\{}\right)\cdot\left(Df + \Delta f + \{\{\frac{S}{\{}, f\}\right)\end{aligned}$$

with the quantum master equation (3.12) we get

$$(D+\Delta)\left(\exp\left(\frac{S}{\{}\right)\cdot f\right) = \exp\left(\frac{S}{\{}\right)\cdot(Df + \Delta f + \{S,f\}).$$

This is exactly on the right hand side of (4.7). Therefore

$$\begin{aligned}ZD_2(f) =&\exp\left(\frac{-W}{\{}\right)\cdot P_1\left(D_1\left(\exp\left(\frac{S}{\{}\right)\cdot f\right)\right) = \\ =&\exp\left(\frac{-W}{\{}\right)\cdot E_1\left(P_1\left(\exp\left(\frac{S}{\{}\right)\cdot f\right)\right)\end{aligned}$$

where

$$P_1\left(\exp\left(\frac{S}{\{}\right)\cdot f\right) = \exp\left(\frac{W}{\{}\right)\cdot\exp\left(\frac{-W}{\{}\right)\cdot P_1\left(\exp\left(\frac{S}{\{}\right)\cdot f\right) = \exp\left(\frac{W}{\{}\right)\cdot Z(f).$$

Together

$$ZD_2(f) = \exp\left(\frac{-W}{\{}\right)\cdot\Delta_\alpha\left(\exp\left(\frac{W}{\{}\right)\cdot Z(f)\right) \tag{4.9}$$

Finally, let us apply the map Z on the equality $I_2P_2 - 1 = K_2D_2 + D_2K_2$ and evaluate it on $f \in \mathrm{Fun}(\mathcal{P}, V)$:

$$ZI_2P_2(f) - Z(f) = ZK_2D_2(f) + ZD_2K_2(f).$$

From the arguments in Remark 163 we know that $K_2D_2(f)$ contains at least one β and the observation from the first step of this proof give us $ZK_2D_2(f) = 0$. By similar argument about adding β and the second observation we get $ZI_2P_2(f) = ZIP_2(f) = P_2(f)$. And once again, since K_2 adds at least one β we have thanks to first observation

$$ZD_2K_2(f) = \exp\left(\frac{-W}{\{}\right)\cdot\Delta_\alpha\left(\exp\left(\frac{W}{\{}\right)\cdot ZK_2(f)\right) = 0$$

This finishes the proof. □

Theorem 165.

$$E_2 = \Delta_\alpha + \{W, -\}_\alpha.$$

Proof. We already did most of the work. Let us look closely on the right hand side of (4.9):

$$\Delta_\alpha\left(\exp\left(\frac{W}{\{}\right)\cdot Z(f)\right)=$$
$$=\Delta_\alpha\left(\exp\left(\frac{W}{\{}\right)\right)\cdot Z(f)+\exp\left(\frac{W}{\{}\right)\cdot\Delta_\alpha(Z(f))+\{\{\exp\left(\frac{W}{\{}\right),Z(f)\}.$$

First term on the right hand side is zero from master equation on $\mathrm{Fun}(\mathcal{P},H(V))$ and on the third term we can use (4.8) and get

$$\Delta_\alpha\left(\exp\left(\frac{W}{\{}\right)\cdot Z(f)\right)=\exp\left(\frac{W}{\{}\right)\cdot\left(\Delta_\alpha Z(f)+\{W,Z(f)\}_\alpha\right).$$

Therefore

$$ZD_2(f)=\exp\left(\frac{-W}{\{}\right)\cdot\exp\left(\frac{W}{\{}\right)\cdot\left(\Delta_\alpha Z(f)+\{W,Z(f)\}_\alpha\right).$$

By previous theorem $ZD_2=P_2D_2$, from homological perturbation lemma $P_2D_2=E_2P_2$, and from surjectivity of P_2 the proof is complete. □

4.3 Perturbation by $\Delta+\{S,-\}$ of $\mathrm{Fun}(\mathcal{P},V)$ without connected sum

The perturbed maps are the same as in (4.6). But without a definition of the product, we cannot introduce the exponential of any element. Surprisingly in the commutative case, it is possible to introduce another formula for W. Its generalization to the non-commutative case, unfortunately, isn't successful.

Remark 166. Let us start with some very trivial observation to simplify the next. In this section we should write, for example, $S\in\mathrm{Fun}(\mathcal{P},V)$ as $S=\sum_{n,G}S(n,G)$, $S(n,G)=\sum_i s_i\otimes\phi^{I_i}$ where ϕ^{I_i} are some monomials. But without the connected sum the individual components of $S(n,G)$ can't affect the rest. Therefore we can move the sum over n,G,i, in front and work with the summands individually. We thus take the liberty to simplify the formulas and write just S.

Remark 167. The perturbation Lemma 156 yields

$$E_2=P(1-\delta_2K)^{-1}\delta_2I=\sum_{m=0}^{\infty}P(\delta_2K)^m(\Delta_\alpha+\Delta_{\beta\gamma}+\{S,-\}_\alpha+\{S,-\}_{\beta\gamma})I.$$

From Remark 154 we see

$$\Delta_{\beta\gamma}I=0=\{S,-\}_{\beta\gamma}I.$$

Together with $K\Delta I=0$ we get

$$E_2=P\Delta_\alpha I+\sum_{m=0}^{\infty}P(\delta_2K)^m\{S,-\}_\alpha I.$$

Let us consider elements $\sum_j\phi^{I_j}\in\mathrm{Fun}(\mathcal{QC},H(V))$ where ϕ^{I_j} are monomials in variables α. Since the map P projects all β to 0, a term $P(\delta_2K)^m\{S,I(\sum_j\phi^{I_j})\}_\alpha$

can be nonzero only if every β added by K is removed by some δ_2. Obviously only the part $\Delta_{\beta\gamma} + \{S, -\}_{\beta\gamma}$ remove β. Let us denote it as

$$\delta_{\beta\gamma} = \Delta_{\beta\gamma} + \{S, -\}_{\beta\gamma}.$$

Hence

$$E_2 = P\Delta_\alpha I + \sum_{m=0}^{\infty} P(\delta_{\beta\gamma}K)^m \{S, I(-)\}_\alpha. \tag{4.10}$$

To further simplify the formulas, let us denote in the computations

$$\sum_{m=0}^{\infty} P(\delta_{\beta\gamma}K)^m \frac{\partial^{(1)} S}{\partial \alpha^k} = W^k$$

where W^k are elements of degree $1 + |\alpha^l| = -|\alpha^k|$.

Lemma 168.

$$\left(\frac{\partial^{(a)}}{\partial \alpha^p} \sum_m P(\delta_{\beta\gamma}K)^m \frac{\partial^{(1)} S}{\partial \alpha^q} \otimes -\right) = (-1)^{p\cdot q} \left(\frac{\partial^{(a)}}{\partial \alpha^q} \sum_m P(\delta_{\beta\gamma}K)^m \frac{\partial^{(1)} S}{\partial \alpha^p} \otimes -\right)$$

where $(-1)^{p\cdot q}$ denotes $(-1)^{|\alpha^p|\cdot|\alpha^q|}$.

Proof. Let $X \in \mathrm{Fun}(\mathcal{QC}, V)$. From $(E_2)^2 = 0$ we get

$$\begin{aligned} 0 =& 0 + \frac{1}{2} P \sum_{\pi \in Sh} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k} (\omega')^{ij} (\omega')^{kl} \frac{\partial^{(1)}}{\partial \alpha^i} \frac{\partial^{(2)}}{\partial \alpha^j} \pi \left(W^k \otimes \frac{\partial^{(c)}}{\partial \alpha^l} X\right) + \\ &+ \frac{1}{2} P \sum_{\pi \in Sh} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k} (\omega')^{ij} (\omega')^{kl} \pi \left(W^k \otimes \frac{\partial^{(1)}}{\partial \alpha^l} \left(\frac{\partial^{(1)}}{\partial \alpha^i} \frac{\partial^{(2)}}{\partial \alpha^j} X\right)\right) + \\ &+ P \sum_{\pi, \tilde{\pi} \in Sh} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k} (\omega')^{ij} (\omega')^{kl} \pi \left(W^k \otimes \frac{\partial^{(1)}}{\partial \alpha^l} \tilde{\pi} \left(W^i \otimes \frac{\partial^{(d)}}{\partial \alpha^j} X\right)\right) \end{aligned} \tag{4.11}$$

where we used observations from Remark 140 to choose some arbitrary indices c, d. Let us look closely on the second term on the right hand side.[4] There will be four kinds of shuffles π:

- $\pi^{-1}(1)$, $\pi^{-1}(2)$ both originally from W^k ... denote π_1
- $\pi^{-1}(1)$, $\pi^{-1}(2)$ both originally from $\dfrac{\partial^{(c)}}{\partial \alpha^l} X$ ... denote π_2
- $\pi^{-1}(1)$ from W^k, $\pi^{-1}(2)$ from $\dfrac{\partial^{(c)}}{\partial \alpha^l} XI$ ... denote π_3
- $\pi^{-1}(1)$ from $\dfrac{\partial^{(c)}}{\partial \alpha^l} X$, $\pi^{-1}(2)$ from W^k ... denote π_4

[4] The only non-trivial on the first line of (4.11).

For the kind π_2 of shuffles let $a = \pi_2^{-1}(1)$, $b = \pi_2^{-1}(2)$

$$\frac{\partial^{(a)}}{\partial\alpha^i}\frac{\partial^{(b)}}{\partial\alpha^j}\left(W^k \otimes \frac{\partial^{(c)}}{\partial\alpha^l}X\right) = (-1)^{(i+j)\cdot(k+l)}W^k \otimes \frac{\partial^{(c)}}{\partial\alpha^l}\frac{\partial^{(a)}}{\partial\alpha^i}\frac{\partial^{(b)}}{\partial\alpha^j}X$$

where $|\alpha^i| + |\alpha^j| = -1 = |\alpha^k| + |\alpha^l|$. We see that the second kind of shuffles cancels out with the second term of (4.11).

By technical observation we can see that π_3 and π_4 give us in fact the same terms.[5]

Now let us focus on the last term of (4.11). There are two kinds of shuffles $\tilde{\pi}$.

- $\tilde{\pi}^{-1}(1)$ is from W^i ... denote $\tilde{\pi}_1$
- $\tilde{\pi}^{-1}(1)$ is from $\dfrac{\partial^{(1)}}{\partial\alpha^j}\sigma\phi^I$ ... denote $\tilde{\pi}_2$

For the second one denote $e = \tilde{\pi}_2^{-1}(1)$

$$\sum_{\pi,\tilde{\pi}_2}\sum_{\substack{i,j\\k,l}}(-1)^{i+k+l\cdot i}(\omega')^{ij}(\omega')^{kl}\pi(1\otimes\tilde{\pi}_2)\left(W^k \otimes W^i \otimes \frac{\partial^{(e)}}{\partial\alpha^l}\frac{\partial^{(d)}}{\partial\alpha^j}X\right). \tag{4.12}$$

Notice, that $\pi(1\otimes\tilde{\pi}_2)$ is a shuffle of type $(\mathtt{card}(W^k), \mathtt{card}(W^i), \mathtt{card}(X)-2)$.[6] Since we sum over permutations $\pi, \tilde{\pi}_2$, we get all possible shuffles of this type. Notice that (4.12) is equivalent to the following sum over all permutations $\ddot{\pi} \in Sh(\mathtt{card}(W^i), \mathtt{card}(W^k), \mathtt{card}(X)-2)$

$$\sum_{\ddot{\pi}}\sum_{\substack{i,j\\k,l}}(-1)^{i+k+l\cdot i+i\cdot k+j\cdot l}(\omega')^{ij}(\omega')^{kl}\ddot{\pi}\left(W^i \otimes W^k \otimes \frac{\partial^{(d)}}{\partial\alpha^j}\frac{\partial^{(e)}}{\partial\alpha^l}X\right)$$

Now let us switch the labels of the indices and positions[7] as $j \leftrightarrow l$, $i \leftrightarrow k$, and $d \leftrightarrow e$:

$$\sum_{\ddot{\pi}}\sum_{\substack{i,j\\k,l}}(-1)^{i+k+j\cdot k+k\cdot i+l\cdot j}(\omega')^{ij}(\omega')^{kl}\ddot{\pi}\left(W^k \otimes W^i \otimes \frac{\partial^{(e)}}{\partial\alpha^l}\frac{\partial^{(d)}}{\partial\alpha^j}X\right).$$

This is exactly the term in (4.12) but with opposite sign. Therefore the term in (4.12) must be zero.

Finally, let us look on what we know about the (4.11).

$$0 = \text{terms of } \pi_1 + 2\cdot\text{terms of } \pi_3 + \text{terms of } \tilde{\pi}_1$$

Notice that both "terms of π_1" and "terms of $\tilde{\pi}_1$" are first-order differential operators acting on X and the "terms of π_3" are second order. Since this identity holds

[5] After the switch of the labels i, j of indices we use the fact that $\omega^{ji} = (-1)^{ij+1}\omega^{ij}$.

[6] A shuffle σ of type $(p_1, \ldots, p_n)$ is an element of $\Sigma_{p_1+\ldots+p_n}$ such that $\sigma(1) < \ldots < \sigma(p_1)$, $\sigma(p_1+1) < \ldots < \sigma(p_2)$, ..., and $\sigma(p_{n-1}+1) < \ldots < \sigma(p_n)$.

[7] We also use the fact, that $X \in \mathrm{Fun}(\mathcal{QC}, V)$ is an invariant. Therefore the choice of position e, d was arbitrary.

for all X, we get two separate identities (for first and for second-order differential operators). Obviously the second-order differential operator has to be zero.

$$0 = \sum_{\pi_2} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k} (\omega')^{ij} (\omega')^{kl} \frac{\partial^{(1)}}{\partial \alpha^i} \frac{\partial^{(2)}}{\partial \alpha^j} \pi \left(W^k \otimes \frac{\partial^{(c)}}{\partial \alpha^l} X \right) = \tag{4.13}$$

$$= \sum_{\pi_2} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k+k\cdot j} (\omega')^{ij} (\omega')^{kl} \pi \left(\frac{\partial^{(a)}}{\partial \alpha^i} W^k \otimes \frac{\partial^{(b)}}{\partial \alpha^j} \frac{\partial^{(c)}}{\partial \alpha^l} X \right)$$

where $a = \pi^{-1}(1), b = \pi^{-1}(2)$. Obviously from (4.13)

$$\sum_{\pi_2} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k+k\cdot j} (\omega')^{ij} (\omega')^{kl} \pi \left(\frac{\partial^{(a)}}{\partial \alpha^i} W^k \otimes \frac{\partial^{(b)}}{\partial \alpha^j} \frac{\partial^{(c)}}{\partial \alpha^l} X \right) =$$

$$= \sum_{\pi_2} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k+k\cdot j+l\cdot j} (\omega')^{ij} (\omega')^{kl} \pi \left(\frac{\partial^{(a)}}{\partial \alpha^i} W^k \otimes \frac{\partial^{(c)}}{\partial \alpha^l} \frac{\partial^{(b)}}{\partial \alpha^j} X \right) =$$

$$= \sum_{\pi_2} \sum_{\substack{i,j \\ k,l}} (-1)^{i+k+l} (\omega')^{il} (\omega')^{kj} \pi \left(\frac{\partial^{(a)}}{\partial \alpha^i} W^k \otimes \frac{\partial^{(b)}}{\partial \alpha^j} \frac{\partial^{(c)}}{\partial \alpha^l} X \right)$$

where we first switch the order of the positional derivations and then we switched the labels of indices $l \leftrightarrow j$ and positions $c \leftrightarrow b$. Since (4.13) is zero, we can take it twice and contract both sums with $(\omega')_{jp}, (\omega')_{lq}$

$$\sum_{\pi \in Sh} \sum_{\substack{i,j \\ k,l}} \left((-1)^{i+k+k\cdot j} \delta^i_p \delta^k_q \ + (-1)^{i+k+k\cdot l+l\cdot j} \delta^i_q \delta^k_p \ \right) \pi \left(\frac{\partial^{(a)}}{\partial \alpha^i} W^k \otimes \frac{\partial^{(b)}}{\partial \alpha^j} \frac{\partial^{(c)}}{\partial \alpha^l} X \right) = 0$$

where δ is Kronecker delta. This gives the required identity. □

Remark 169. The same way as we introduced $\eta_{\beta\gamma}(X)$ in (3.22) for elements of the form $X = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma p \otimes \sigma(\phi^I)$ we can introduce $\eta_\alpha(X)$ as the number of occurences of α's in ϕ^I. We can express it for $\phi^I \in (V^*)^{\otimes n}$ also as

$$\eta_\alpha(X) \cdot X = \left(1 \otimes \sum_{t=1}^{n} \sum_h \alpha^h_{(t)} \frac{\partial^{(t)}}{\partial \alpha^h} \right) (X).$$

Obviously it still holds that $\eta_\alpha(X) = 0$ if ϕ^I doesn't contain any α. If $\eta_\alpha(X)$ is 0, we define $\frac{1}{\eta_\alpha(X)}$ also to be 0. Let us denote $\eta_\alpha(X) \cdot X$, respective $\frac{1}{\eta_\alpha(X)} \cdot X$ shortly by $\eta_\alpha X$, respective $\frac{1}{\eta_\alpha} X$.

Definition 170. We define **effective action** $\tilde{W} \in \mathrm{Fun}(\mathcal{P}, H(V))$ (without connected sum) as

$$\tilde{W} = \sum_{m=0}^{\infty} \frac{1}{\eta_\alpha} (P(\delta_{\beta\gamma} K)^m \eta_\alpha S) = \sum_{m=0}^{\infty} \frac{1}{\eta_\alpha} (P((\Delta_{\beta\gamma} + \{S, -\}_{\beta\gamma}) K)^m \eta_\alpha S). \tag{4.14}$$

Remark 171. Obviously $\{I(\tilde{W}), F\} = \{I(\tilde{W}), F\}_\alpha$ for any $F \in \mathrm{Fun}(\mathcal{P}, V)$. Since $|\delta_{\beta\gamma}| = 1$, $|K| = -1$ we easily get $|\tilde{W}| = 0$.

To avoid confusion let us remind that $\frac{1}{\eta_\alpha}$ takes as an argument the whole term $(P((\Delta_{\beta\gamma} + \{S, -\}_{\beta\gamma})K)^n \eta_\alpha S)$. Notice that we can't simply cancel $\frac{1}{\eta_\alpha}$ with η_α because $\{S, -\}_{\beta\gamma}$ possibly changed the number of occurrences of α.

Theorem 172. $P\{I(\tilde{W}), -\}$ has the same form as $\sum_{m=0}^{\infty} P(\delta_{\beta\gamma}K)^m \{S, I(-)\}_\alpha$.

Proof. In the commutative case, we can omit the part corresponding to $\mathcal{P}$. Also let us denote shortly $I(Q) = Q$.

$$\{\tilde{W}, Q\} = \{\tilde{W}, Q\}_\alpha = \sum_{\pi \in Sh} \sum_{k,l} (-1)^k (\omega')^{kl} \pi \left(\frac{\partial^{(a)}}{\partial \alpha^k} \otimes \frac{\partial^{(1)}}{\partial \alpha^l} \right) \left(\tilde{W} \otimes Q \right)$$

Obviously the positional derivation $\frac{\partial^{(a)}}{\partial \alpha^k}$ lower the number of occurrences of α in W by 1:

$$\sum_{\pi \in Sh} \frac{1}{\eta_\alpha + 1} \pi \left(\sum_{k,l} (-1)^k \omega^{kl} \frac{\partial^{(a)}}{\partial \alpha^k} \left(\sum_m P(\delta_{\beta\gamma}K)^m \sum_h \alpha^h_{(t)} \frac{\partial^{(t)}}{\partial \alpha^h} S \right) \otimes \frac{\partial^{(1)} Q}{\partial \alpha^l} \right).$$

If K or $\delta_{\beta\gamma}$ act on position where is $\alpha^h_{(t)}$, we trivially get zero. So the only way these maps can possibly influence $\alpha^h_{(t)}$ is a change of its position (by the shuffles from $\{S, -\}_{\beta\gamma}$). Let us denote the position of $\alpha^h_{(t)}$ after $\sum_m P(\delta_{\beta\gamma}K)^m$ as u.[8]

Now, $\frac{\partial^{(a)}}{\partial \alpha^k}$ will either act on $\alpha^h_{(u)}$ and we get condition $h = k$. Or the positional derivation acts on the rest. But since a is arbitrary position, we can always choose $a = u + 1$.

$$\begin{aligned}\{\tilde{W}, Q\}_\alpha = &\sum_{\pi \in Sh} \sum_{k,l} \frac{(-1)^k \omega^{kl}}{\eta_\alpha + 1} \delta^h_k \pi \left(\sum_m P(\delta_{\beta\gamma}K)^m \frac{\partial^{(t)} S}{\partial \alpha^k} \otimes \frac{\partial^{(1)} Q}{\partial \alpha^l} \right) + \\ &+ \sum_{\pi \in Sh} \sum_{t,h} \sum_{k,l} \frac{(-1)^{k + h \cdot k} \omega^{kl}}{\eta_\alpha + 1} \pi \left(\alpha^h_{(u)} \frac{\partial^{(u)}}{\partial \alpha^k} \sum_m P(\delta_{\beta\gamma}K)^m \frac{\partial^{(t)} S}{\partial \alpha^h} \otimes \frac{\partial^{(1)} Q}{\partial \alpha^l} \right).\end{aligned}$$

Now we can use the result of Lemma 168 and get

$$\{\tilde{W}, Q\}_\alpha = \sum_{\pi \in Sh} \sum_h \sum_{k,l} \frac{(-1)^k \omega^{kl}}{\eta_\alpha + 1} \pi \left((\eta_\alpha + 1) \sum_m P(\delta_{\beta\gamma}K)^m \frac{\partial^{(t)} S}{\partial \alpha^h} \otimes \frac{\partial^{(1)} Q}{\partial \alpha^l} \right).$$

∎

[8] The position u still depends on t.

5. Homotopies

Most of the arguments in this section are the same as in the [12] by Doubek, Jurčo, and Pulmann.

Definition 173. By $\Omega([0,1])$ we mean the algebra of smooth differential forms on the unit interval $[0,1]$. Elements of this algebra can be written as $f(t)+g(t)\mathrm{d}t$, the differential d_{dR} sends such element to $\dfrac{\partial}{\partial t} f(t)\mathrm{d}t$.

The tensor product $\mathrm{Fun}(\mathcal{P},V)\otimes\Omega([0,1])$ is defined as

$$\mathrm{Fun}\otimes\Omega([0,1]) = \prod_{n\geq 1}\prod_{G\geq 0}\left((\mathcal{P}(n,G)\otimes(V^*)^{\otimes n})^{\Sigma_n}\right)\otimes\Omega([0,1]).$$

Definition 174. We say that $(A(t)+B(t)\mathrm{d}t)/\{ \in \mathrm{Fun}(\mathcal{P},V)\otimes\Omega([0,1])$ is a **homotopy** between $A(0)$ and $A(1)$ if A(t) is of degree 0, $B(t)$ is of degree -1 and

$$(D+\Delta+\mathrm{d}_{dR})\left(\exp\left(\frac{A(t)+B(t)\mathrm{d}t}{\{}\right)\right)=0. \tag{5.1}$$

This is equivalent to saying that $A(t)$ solves the quantum master equation for every t and that

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t}+D(B(t))+\{A(t),B(t)\}+\Delta B(t)=0. \tag{5.2}$$

Theorem 175. Let us take two action $S_0,S_1\in\mathrm{Fun}(\mathcal{P},V)$. Then the following are equivalent:

1. There exists a homotopy in the sense of Definition 174 connecting S_0 and S_1.
2. There exists $F\in\mathrm{Fun}(\mathcal{P},V)$ such that

$$\exp\left(\frac{S_0}{\{}\right)-\exp\left(\frac{S_1}{\{}\right)=(D+\Delta)F.$$

Proof. The implication 1. ⇒ 2. is simple. Equation (5.1) says that

$$\frac{\partial}{\partial t}\left(\exp\left(\frac{A(t)}{\{}\right)\right)dt=-(D+\Delta)\left(\exp\left(\frac{A(t)+B(t)\mathrm{d}t}{\{}\right)\right).$$

Since $A(t)$ solves quantum master equation we get

$$\frac{\partial}{\partial t}\left(\exp\left(\frac{A(t)}{\{}\right)\right)=-(D+\Delta)\ \left(\exp\left(\frac{A(t)}{\{}\right)\cdot\frac{B(t)}{\{}\right)$$

i.e. the required F is

$$F=\exp\left(\frac{A(t)}{\{}\right)\cdot\frac{B(t)}{\{}.$$

The implication 2. ⇒ 1. From the two given actions we can define $\exp\left(A(t)/\{\right)$ as their convex combination

$$\exp\left(\frac{A(t)}{\{}\right)=(1-t)\cdot\exp\left(\frac{S_0}{\{}\right)+t\cdot\exp\left(\frac{S_1}{\{}\right). \tag{5.3}$$

Elements $A(t)$ are well defined since the right hand side of (5.3) starts with 1. Obviously $A(t)$ solves the quantum master equation. Define $B(t)$ as

$$B(t) = \{ \cdot \exp\left(\frac{-A(t)}{\{}\right) \cdot F.$$

Obviously $B(t)$ is still an element of $\mathrm{Fun}(\mathcal{P}, v)$. Since both S_0, S_1 are of degree 0 and using the fact that

$$\begin{aligned} 0 = \quad & (D+\Delta)(1) = (D+\Delta)\left(\exp\left(\frac{-A(t)}{\{}\right) \cdot \exp\left(\frac{A(t)}{\{}\right)\right) = \\ = \quad & (D+\Delta)\left(\exp\left(\frac{-A(t)}{\{}\right)\right) \cdot \exp\left(\frac{A(t)}{\{}\right) - \frac{1}{\{}\{A(t), A(t)\}. \end{aligned}$$

There is no problem to directly verify that equation (5.2) holds:

$$\begin{aligned} & \frac{\mathrm{d}A(t)}{\mathrm{d}t} + (D+\Delta)(B(t)) + \{A(t), B(t)\} = \\ = \frac{\{ \left(\mathrm{e}^{S_0/\{} - \mathrm{e}^{S_1/\{}\right)}{(1-t)\cdot \mathrm{e}^{S_0/\{} + t \cdot \mathrm{e}^{S_1/\{}} & + \{ (D+\Delta)\left(e^{-A(t)/\{} F\right) + \{ \{A(t), e^{-A(t)/\{} F\} = \\ & = \{ \left((D+\Delta)e^{-A(t)/\{}\right) \cdot F - \{A(t), A(t)\} \cdot e^{-A(t)/\{} \cdot F. \end{aligned}$$

□

Remark 176. From this theorem, one can easily see that

$$\begin{aligned} \exp\left(\frac{W_0}{\{}\right) - \exp\left(\frac{W_1}{\{}\right) = P_1\left(\exp\left(\frac{S_0}{\{}\right) - \exp\left(\frac{S_1}{\{}\right)\right) = \\ = P_1(D+\Delta)F = E_1 P_1 F = \Delta_\alpha P_1 F, \end{aligned}$$

i.e., homotopic solutions S_0, S_1 of quantum master equation on $\mathrm{Fun}(\mathcal{P}, V)$ give homotopic effective actions W_0, W_1.

Remark 177. Now we are able to construct a homtopy between $\mathrm{e}^{W/\{}$ and $\mathrm{e}^{S/\{}$.

From SDR in the first perturbation we have

$$I_1 P_1\left(\exp\left(\frac{S}{\{}\right)\right) - \exp\left(\frac{S}{\{}\right) = K_1(D+\Delta)\exp\left(\frac{S}{\{}\right) + (D+\Delta)K_1 \exp\left(\frac{S}{\{}\right)$$

using quantum master equation and the definition of effective action (4.4)

$$\exp\left(\frac{I(W)}{\{}\right) - \exp\left(\frac{S}{\{}\right) = (D+\Delta)K_1 \exp\left(\frac{S}{\{}\right).$$

The second condition of Theorem 175 tells us

$$F = K_1 \exp\left(\frac{S}{\{}\right).$$

And so we have a homotopy connecting $I(W)$ and S.

5.1 Morphisms

Remark 178. We want to define a morphism between two quantum homotopy algebras. As we mentioned in Remark 133 our quantum homotopy algebras do not contain any curvature element m_0 (i.e., deviation of $m_1 = d$ to be a differential). However, the triviality of the curvature elements does not imply triviality of the "curved quantum homotopy algebra morphisms".

In the following definition, we refer to triviality of the 0-ary component of the morphism as *fixing the origin*.

Definition 179. Given two symplectic vector spaces $(U, \omega_U), (V, \omega_V)$ and solutions of master equations $S_U \in \mathrm{Fun}(\mathcal{P}, U), S_V \in \mathrm{Fun}(\mathcal{P}, V)$, we say that a map $\Phi : \mathrm{Fun}(\mathcal{P}, U) \to \mathrm{Fun}(\mathcal{P}, V)$ fixing the origin is a **quantum homotopy algebra morphism** if it is a Poisson map, i.e.,

$$\{\Phi(f), \Phi(g)\}_{\mathrm{Fun}(\mathcal{P},V)} = \Phi \circ \{f, g\}_{\mathrm{Fun}(\mathcal{P},U)}$$

for any $f, g \in \mathrm{Fun}(\mathcal{P}, U)$, and if

$$\Phi \circ (\Delta_{\mathrm{Fun}(\mathcal{P},U)} f + \{S_U, f\}_{\mathrm{Fun}(\mathcal{P},U)}) = \Delta_{\mathrm{Fun}(\mathcal{P},V)}(\Phi f) + \{S_V, \Phi f\}_{\mathrm{Fun}(\mathcal{P},V)} \tag{5.4}$$

for any $f \in \mathrm{Fun}(\mathcal{P}, U)$.

Remark 180. Let us consider homotopy between S_0 and S_1. Similarly as Costello in [8] let

$$X(t) = \{-B(t), \text{-}\}$$

be a one-parameter family of Hamiltonian vector fields associated to the functionals $B(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(f) = -X(t)\Phi_t(f) = \{B(t), \Phi_t(f)\} \tag{5.5}$$

where the flow Φ_t of this Hamiltonian vector field is an automorphism of the space $\mathrm{Fun}(\mathcal{P}, V)$ with $\Phi_t \in C^\infty(\mathbb{R}, \mathrm{End}(\mathrm{Fun}(\mathcal{P}, V)))$.

Then we get a third condition for Theorem 175.

Lemma 181. Let $M \in \mathrm{Fun}(\mathcal{P}, V)$ be such that $\exp(\{M, \cdot\}) = \Phi_1$ is a Poisson map. Φ_1 is a quantum homotopy algebra isomorphism between $(\mathrm{Fun}(\mathcal{P}, V), S_0)$ and $(\mathrm{Fun}(\mathcal{P}, V), S_1)$ if and only if there exists a homotopy between S_0 and S_1.

Proof. Let Φ_1 be a quantum homotopy algebra isomorphism (in the connected component of identity). Let X be the generating vector field, $\Phi_t = \exp(tX)$, and the corresponding Hamiltonian B, $X = \{B, \text{-}\}$, is defined up to a constant.

Differentiating

$$c(t) = \Phi_t \Delta \Phi_{-t}(g) + \{\Phi_t(S_0), g\} - \Delta(g)$$

with respect to t gives

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t\right)\Delta\Phi_{-t}g + \Phi_t\Delta\left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{-t}\right)g + \{\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(S_0), g\} =$$
$$=\{\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(S_0) - \Phi_t\Delta\Phi_{-t}B, g\}.$$

Thus $c(t) = \{\tilde{A}(t), \text{-}\}$ where $\tilde{A}(t)$ is unique up to some $K(t) \in \mathrm{Fun}(\mathcal{P}, V)(0, G)$. Now consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t\left(Df + \Delta f + \{S_0, f\}\right) = \{B, \Phi_t\left(Df + \Delta f + \{S_0, f\}\right)\}. \tag{5.6}$$

From properties of Φ_t, the right hand side is equal to

$$\{B, D\Phi_t(f) + \Delta\Phi_t(f) + \{\tilde{A}(t), \Phi_t(f)\}\}$$

and the left hand side to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(D\Phi_t f + \Delta\Phi_t f + \{\tilde{A}(t), \Phi_t f\}\right) =& \\
= \left(D + \Delta + \{\tilde{A}(t), \text{-}\}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t f\right) + \{\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}(t), \Phi_t f\} =& \\
= \left(D + \Delta + \{\tilde{A}(t), \text{-}\}\right)(\{B, \Phi_t f\}) + \{\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}(t), \Phi_t f\} =& \\
= \{(D+\Delta)B, \Phi_t f\} + \{\tilde{A}(t), \{B, \Phi_t\}\} + \{B, (D+\Delta)\Phi_t f\} + \{\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}(t), \Phi_t f\} =& \\
= \{(D+\Delta)B + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}(t), \Phi_t f\} + \{\{\tilde{A}(t), B\}, \Phi_t f\} + \{B, \{A(t), \Phi_t f\} + (D+\Delta)\Phi_t f\}.&
\end{aligned}$$

Comparing the left hand side with ride hand side we get

$$0 = \{(D+\Delta)B + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}(t) + \{\tilde{A}(t), \text{-}\}, \Phi_t f\}. \tag{5.7}$$

This implies

$$\tilde{K}(t) = (D+\Delta)B(t) + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}(t) + \{\tilde{A}(t), B(t)\}$$

where $\tilde{K}(t) \in \mathrm{Fun}(\mathcal{P}, V)(0, G)$. Let us choose $K(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = \tilde{K}(t).$$

This gives $A(t)$ up to constant, fixed by $A(0) = S_0$. Since $A(0)$ is a a solution of quantum master equation, from definition of Φ_t also all $A(t)$ are solutions. And we proved $A(t) + B(t)\mathrm{d}t$ is homotopy.

For the opposite implication let $A(t) + B(t)\mathrm{d}t$ be a homotopy and let Φ_t be a flow of $\{B(t), \text{-}\}$. As above, we define $c(t) = \Phi_t\Delta\Phi_{-t}(g) + \{\Phi_t(S_0), g\} - \Delta(g)$ and we differentiate $c(t)$ with respect to t. Hence there is $\tilde{A}(t)$ such that $c(t) = \{\tilde{A}(t), \text{-}\}$ and $\tilde{A}(0) = S_0$. By the same arguments as above we get (5.7) and the freedom in $\tilde{A}(t)$ can be use to

$$0 = (D+\Delta)B(t) + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}(t) + \{\tilde{A}(t), B(t)\}.$$

Since $A(t)$ and $\tilde{A}(t)$ solve the same differential equation with the same initial condition, we get $A(t) = \tilde{A}(t)$. □

Remark 182. To integrate flow between $(\mathrm{Fun}(\mathcal{P}, V), S_0)$ and $(\mathrm{Fun}(\mathcal{P}, V), S_1)$ we can use *Magnus expansion* (see section 3.4.1 in [4]) which gives

$$\Phi_t = \exp\left(\{M(t), \text{-}\}\right)$$

where $M(t) = \sum_{i=1}^{\infty} M_i$ is degree -1 element of $\mathrm{Fun}(\mathcal{P}, V) \otimes \Omega([0,1])$. The first term of this sum is given as

$$\begin{aligned} M_1(t) = -\int_0^t B(t)\mathrm{d}t = -\int_0^t \{ \cdot \exp\left(\frac{-A(t)}{\{}\right) \cdot F \ \mathrm{d}t = \\ = -\{ \int_0^t \frac{1}{e^{S_0/\{} + t(e^{S_1/\{} - e^{S_0/\{})}\mathrm{d}t \cdot F = \\ = \frac{\{}{e^{S_0/\{} - e^{S_1/\{}} \cdot \log\left(1 + t(e^{(S_1 - S_0)/\{} - 1)\right) \cdot F. \end{aligned}$$

Remark 183. Equation (5.4) for flow Φ_t applied on element $f = \Phi_t(g)$ gives us

$$\Phi_t \Delta \Phi_{-t} = \Delta + \{A(t) - \Phi_t(S_0), \text{-}\}. \tag{5.8}$$

We want to interpret $A(t) - \Phi_t(S_0)$ as special case of $\mathrm{logBer}(\Phi_t)$. Thanks to Remark 182

$$\Phi_t \Delta \Phi_{-t} = \exp\left(\{M(t), \text{-}\}\right) \Delta \exp\left(-\{M(t), \text{-}\}\right)$$

and with help of Baker–Campbell–Hausdorff formula we get

$$\begin{aligned} \Phi_t \Delta \Phi_{-t} = \Delta - \{\Delta M(t), \text{-}\} + \\ + \frac{1}{2}\{\{\Delta M(t), M(t)\}, \text{-}\} - \frac{1}{3!}\{\{\{\Delta M(t), M(t)\}, M(t)\}, \text{-}\} + \dots \end{aligned}$$

where we use

$$\Delta\{M(t), Y\} = \{\Delta M(t), Y\} + (-1)^{|M(t)|+1}\{M(t), \Delta Y\}$$

(where $|M(t)| = -1$) and

$$\{F, \{M(t), Y\}\} = \{\{F, M(t)\}, Y\} + (-1)^{(|F|+1)(|M(t)|+1)}\{M(t), \{F, Y\}\}.$$

Together it gives us

$$\mathrm{logBer}(\Phi_t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\{\{\cdots\{\Delta \underbrace{M(t), M(t)\}, \cdots M(t)}_{n-\text{times}}\}, \text{-}\}.$$

6. IBL_∞-algebras and their cousins

Many bialgebras satisfy a condition called *involutivity.* A properad IBL of involutive Lie bialgebras will capture the involutive relation for elements $p \in IBL(1,2,0)$, $q \in IBL(2,1,0)$ as

$$0 = p\,{}_{1,2}\circ_{1,2}\, q \in IBL(1,1,1)$$

Besides this condition, the IBL properad will be also defined by the relations of graded commutativity and cocomutativity, Jacobi and coJacobi identity, and the 5-term identity for elements of $IBL(2,2,0)$. For more details see example 2.2 by Campos, Merkulov, and Willwacher in [5].

For a very basic example of algebra over such properad, one may think of cyclic words on vector space V equipped with a skew-symmetric pairing that admits a canonical involutive Lie bialgebra structure. See, for example, section 4 in [19] by Gonzalez.

In [5] was proven that the Frobenius properad is Koszul. Its minimal resolutions is particularly nice and can be done explicitly. This gives us a minimal model (as disscused in Section 2.1). It turns out that the algebras over the cobar complex of Frobenius properad are then the well-known involutive Lie bialgebras up to homotopy, shortly IBL_∞-algebras. The relation of involutivity holds up to homotopy as we will see in the explicit description of corresponding properad, the IBL_∞ properad.

6.1 IBL_∞-algebras

Let us first recall the notion of IBL_∞-properad.[1]

Definition 184. A properad IBL_∞ is a properad generated by degree 1 elements $p \in IBL_\infty(1,2,0)$, $q \in IBL_\infty(2,1,0)$ such that

$$\begin{aligned} (1,\sigma)\ p &= -p, & (\sigma,1)\ q &= -q, \\ p\,{}_1\circ_1\, p &= -p\,{}_2\circ_1\, p, & q\,{}_1\circ_1\, q &= -q\,{}_1\circ_2\, q, \\ & q\,{}_1\circ_1\, p = -p\,{}_2\circ_1\, q, \end{aligned} \tag{6.1}$$

where we use for the $\Sigma_2 \times \Sigma_1$ and $\Sigma_1 \times \Sigma_2$ actions the permutations $1 \in \Sigma_1$ and $\sigma = (12) \in \Sigma_2$.

Remark 185. The first two relations in (6.1) can be seen as a modification of commutativity and cocommutatitvity, the next two relations as a modification of associativity and coassociativity. The last one is shown pictorially on Figure 6.1.

[1] Note that our conventions are slightly different. Usually, for IBL-algebras one assumes that $n \geq 1$, $m \geq 1$, $G \geq 0$. We will comment on this later.

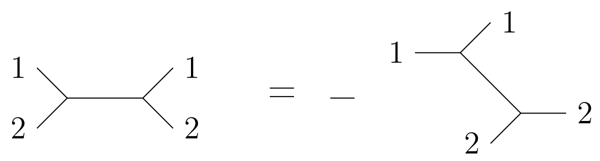

Figure 6.1: Relation $q\,{}_1{\circ}_1\,p = -p\,{}_2{\circ}_1\,q$ of IBL_∞ operad

Thanks to these relations, any element of $IBL_\infty(k,l,n)$ is of the form

$$\underbrace{(q\,{}_1{\circ}_1\,(q\,{}_1{\circ}_1\ldots(q\,{}_1{\circ}_1\,q)))}_{k-1}\;{}_1{\circ}_1\;(p\,{}_{1,2}{\circ}_{1,2}\,q)^n\;{}_1{\circ}_1\;\underbrace{(((p\,{}_1{\circ}_1\,p)\,{}_1{\circ}_1\ldots p)\,{}_1{\circ}_1\,p)}_{l-1}$$

where n is the G-degree of the element.

The following statement appeared in [13] by Drummond-Cole, Terilla, and Tradler, cf. also [7] by Cieliebak, Fukaya, and Latschev.

Theorem 186. The algebras over the cobar complex $\mathcal{CF}$ of the (closed) Frobenius properad are IBL_∞-algebras.

Proof. Let us rephrase the arguments in our notation. To prove the theorem, recall the definition of the Frobenius properad $\mathcal{F}$ from Example 59. Each stable $\bar{\mathcal{F}}(m,n,\chi)$ is a trivial $\Sigma_m\times\Sigma_n$-bimodule spanned on one generator $p_{m,n,\chi}$. Hence,

$$\bar{\mathcal{F}}(m,n,\chi)\,{}_{\Sigma_m}\otimes_{\Sigma_n}(V^{\otimes m}\otimes V^{*\otimes n})\cong S^m(V)\otimes S^n(V^*)$$

is the tensor product of the respective symmetric powers. It follows that formula (2.24) for the generating operator $L\in\tilde{P}$ is simplified to the form[2]

$$L=\sum_{m,n,\chi}\sum_{I,J}\frac{1}{m!n!}f_I^{\chi,J}(a_J\otimes\phi^I)$$

where $f_I^{\chi,J}=(\bar{\alpha}(p^*_{m,n,\chi}))^J_I$.

Further, the algebra over cobar complex $\mathcal{CF}$ is given by (2.11). The differential $d_{\mathcal{P}^*}$ is for Frobenius properad trivial and the differential on $\mathcal{E}_V$ is given by (1.10). What is left is to exhibit the second term of the right hand side of (2.11).

As in formula (2.11), assume $A\subset[m_2+\mathtt{card}(A)]$, $B\subset[n_1+\mathtt{card}(B)]$, relabel them as M and N respectively, and assume $N=\{n_1+1,\ldots,n_1+\mathtt{card}(N)\}$, $M=\{1,\ldots,\mathtt{card}(N)\}$, $\xi(n_1+k)=k$. The second term of the right hand side in (2.11) evaluated on the generator $p_{m,n,\chi}$ gives

$$\sum_{\substack{m_1+m_2=m\\ n_1+n_2=n}}\;\sum_{\mathtt{card}(N)=1}^{\frac{1}{2}(\chi-m-n)+2}\;\sum_{\chi_1}\sum_{\rho,\sigma}\rho\;({}_N\overset{\xi}{\circ}_M)_{\mathcal{E}_V}\,(\alpha_{m_1,n_1,\chi_1}\otimes\alpha_{m_2,n_2,\chi_2})\,\sigma^{-1},$$

where $\max\{m_1+n_1+\mathtt{card}(N)-2,1\}\le\chi_1\le\min\{\chi-m_2-n_2-\mathtt{card}(N)+2,\chi-1\}$ by stability condition, $\alpha_{m_1,n_1,\chi_1}:=\bar{\alpha}(p^*_{m_1,n_1,\chi_1})$, $\alpha_{m_2,n_2,\chi_2}:=\bar{\alpha}(p^*_{m_2,n_2,\chi_2})$ and the

[2]Note, the invariance property $f_I^{\chi,J}=\pm f^{\chi,\rho(J)}_{\sigma^{-1}(I)}$, $\pm$ being the product of Koszul signs corresponding to permutations ρ and σ.

last sum runs over shuffles ρ, σ of type (m_1, m_2) and (n_2, n_1), respectively. If we denote the differential d of dg vector space as $\alpha_{1,1,0}$ then together we get

$$0 = \sum_{\substack{m_1+m_2=m \\ n_1+n_2=n}} \sum_{\mathtt{card}(N)=1}^{\frac{1}{2}(\chi-m-n)+2} \sum_{\chi_1} \sum_{\rho,\sigma} \rho \left({}_N\overset{\xi}{\circ}_M \right)_{\mathcal{E}_V} \left(\alpha_{m_1,n_1,\chi_1} \otimes \alpha_{m_2,n_2,\chi_2}\right) \sigma^{-1},$$

where the sum over χ_1 is given as $m_1 + n_1 + \mathtt{card}(N) - 2 \leq \chi_1 \leq \chi - m_2 - n_2 - \mathtt{card}(N) + 2$.[3]

This is one of the equivalent descriptions of an IBL_∞-algebra. In Baranikov's formalism this equation corresponds to the master equation, in Theorem 116, for L given above. □

Remark 187. In the above theorem we allow all stable values of (m, n, χ). In this case the corresponding IBL_∞-algebras are referred to as "generalized" ones, cf. [7]. If we assume only non-zero values of m and n and $m+n > 2$, for $G = 0$, there is another interpretation [7], [13] of an IBL_∞-algebra in terms of a "homological differential operator", cf. Section 2.3.5. Obviously, for the Frobenius properad, the respective discussion simplifies a lot. The assignment $\phi^i \mapsto \partial_{a_i}$,

$$\partial_{a_i} a_j - (-1)^{|a_i||a_j|} a_j \partial_{a_i} = \delta_i^j$$

turns the generating element $L \in \tilde{P}$ into a differential operator on $S(V)$,[4]

$$L = \sum_{m,n,\chi} \sum_{I.J} \frac{1}{m!n!} f_I^{\chi,J} a_J \frac{\partial}{\partial a^I}.$$

Finally note, that the differential d on $S(V) \otimes S(V^*)$ can be thought of as an element in $V \otimes V^*$ and hence as a first order differential operator on $S(V)$ with coefficients linear in a_i's. Obviously, the derivatives ∂_{a_i} have the meaning of the left derivatives $\partial^L_{a_i}$, well known from the supersymmetry literature.

All in all, on $S(V)$, we have a degree one differential operator $d + L$, squaring to 0,

$$(d + L) \circ (d + L) = (d + L)^2 = 0.$$

The last remark: For a formal definition of an IBL_∞-algebra, one can simply consider any degree one differential operator on $S(V)$ squaring to zero. This would accommodate IBL_∞-algebras within the framework of BV formalism [39].

6.2 IBA_∞-algebras and open-closed IB-homotopy algebras

Here we consider the cases of the open and open-closed Frobenius properads we introduced in Section 1.3. In view of the proof of the above Theorem 186, the following two theorems are straightforward. Their proofs are rather technical, but can be easily reconstructed by following the proofs of the corresponding theorems

[3] χ_2 is in this case then uniquely given from additivity of Euler characteristic.

[4] P_+ as introduced before is only a subspace of $S(V)$, but there is no problem in extending L to the whole symmetric algebra.

for modular operads [11].

Let us consider set $[m]$ of outputs and set $[n]$ of inputs distributed over $b = p+q$ boundaries of a genus g 2-dimensional oriented surface.

More formally, we have a set of cycles $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p, \mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_q\}$, of respective lengths $(k_1, k_2, \ldots, k_p, l_1, l_2, \ldots l_q)$. In $[m]$ we have cycles $\mathbf{c}_1 = (\!(i_1 \cdots i_{k_1})\!)$, $\mathbf{c}_2 = (\!(i_{k_1+1} \cdots i_{k_1+k_2})\!)$, $\ldots$, $\mathbf{c}_p = \left(\!\left(i_{k_1+\ldots k_{p-1}+1} \cdots i_m\right)\!\right)$. And similarly in $[n]$ we have cycles $\mathbf{d}_1 = (\!(j_1 \cdots j_{l_1})\!)$, $\mathbf{d}_2 = (\!(j_{l_1+1} \cdots j_{l_1+l_2})\!)$, $\ldots$ $\mathbf{d}_q = \left(\!\left(j_{l_1+\ldots l_{q-1}+1} \cdots j_n\right)\!\right)$.

Now, let each of the indices $j_1, \ldots j_n$ and $i_1, \ldots i_m$ take values in the set $[\mathrm{dim}V]$ and group them into respective multi-indices

$$I := i_1 \cdots i_{k_1} | i_{k_1+1} \cdots i_{k_1+k_2} | \cdots | i_{k_1+\ldots k_{p-1}+1} \cdots + i_m.$$

$$J := j_1 \cdots j_{l_1} | j_{l_1+1} \cdots j_{l_1+l_2} | \cdots | j_{l_1+\ldots l_{q-1}+1} \cdots + j_n$$

We will use the following, hopefully self-explanatory, notation for these indices: $I = I_1 | I_2 | \cdots | I_p$. And similarly for $J = J_1 | J_2 | \cdots | J_q$.

Concerning the coinvariants (2.16), consider elements in the tensor algebra $T(V) \otimes T(V^*)$ of the form

$$a_{I_1|I_2|\cdots|I_p} \otimes \phi^{J_1|J_2|\cdots|J_q}$$

where we identify, up to the corresponding Kozsul sign, tensors which differ by cyclic permutations of outputs/inputs within the boundaries, i.e., within the individual multi-indices I_i and J_j and also under permutations of output/input boundaries, i.e., independent permutations of multi-indices (I_i) and (J_i). We will denote the subspace of $T(V) \otimes T(V^*)$ spanned by these elements as $T^{\mathrm{cyc}}(V) \otimes T^{\mathrm{cyc}}(V^*)$.

Further, consider coefficients $f^{(g,p,q)I_1|I_2|\cdots|I_p}_{J_1|J_2|\cdots|J_q}$ possessing the corresponding invariance, up the Koszul sing, under cyclic permutations of outputs/inputs within the boundaries and also under independent permutations of output/input boundaries.

Put,

$$L = \sum_{p,q,g} \sum_{\substack{I_1|I_2|\cdots|I_p \\ J_1|J_2|\cdots|J_q}} \frac{1}{p!q! \prod'_s l_s k_s} f^{(g,p,q)I_1|I_2|\cdots|I_p}_{J_1|J_2|\cdots|J_q} a_{I_1|I_2|\cdots|I_p} \otimes \phi^{J_1|J_2|\cdots|J_q}, \tag{6.2}$$

where $\prod'$ is the product of nonzero l_s's and k_s's and where I_s runs over all elements of $[\mathrm{dim}V]^{\times k_s}$ and similarly J_s runs over all elements of $[\mathrm{dim}V]^{\times l_s}$. Also, we included the differential into L as an element corresponding to the cylinder with one input and one output.

Theorem 188. Algebra over the cobar complex $\mathcal{COF}$ is described by a degree one element L (6.2) of $T^{\mathrm{cyc}}(V) \otimes T^{\mathrm{cyc}}(V^*)$ such that $L \circ L = 0$.

Remark 189. A remark completely analogous to the above Remark 187 can be made. In particular, we can think of ϕ_i as being represented by a "left" derivative $\partial^L_{a_i}$. This is possible because in any monomial of the form $a_{I_1|I_2|\cdots|I_p}$ one can always get any of the variables a_{i_k} to the left by a permutation of boundaries and a cyclic permutation within the respective boundary. Hence, if we consider for *any* collection of multi-indicies $I_1|I_2|\cdots|I_p$ the tensor product $V^{\otimes I_1}\otimes\ldots\otimes V^{\otimes I_p}$ modulo the respective symmetry relations, on the direct product over all such multi-indices, we have again a homological differential operator L.

Finally, let us concern the cobar complex $\mathcal{COCF}$ of the two-colored properad $\mathcal{OCF}$.

To describe coinvarinats, consider elements of $T^{\text{cyc}}(V_{\underline{o}})\otimes S(V_{\underline{c}})\otimes T^{\text{cyc}}(V_{\underline{o}}^*)\otimes S(V_{\underline{c}}^*)$ of the form $a_{I_1|I_2|\cdots|I_p;I}\otimes\phi^{J_1|J_2|\cdots|J_q;J}$ where $a_{I_1|I_2|\cdots|I_p;I}:=a_{I_1|I_2|\cdots|I_p}\otimes a_I$ and $\phi^{J_1|J_2|\cdots|J_q;J}:=\phi^{J_1|J_2|\cdots|J_q}\otimes\phi^J$. Correspondingly, consider the coefficients $f^{(g,p,q)I_1|I_2|\cdots|I_p;I}_{J_1|J_2|\cdots|J_q;J}$ with the obvious symmetry properties. Put,

$$L=\sum_{m,n,p,q,g}\sum_{\substack{I_1|I_2|\cdots|I_p;I\\J_1|J_2|\cdots|J_q;J}}\frac{1}{m!n!p!q!\prod_s' l_sk_s}f^{(g,p,q)I_1|I_2|\cdots|I_p;I}_{J_1|J_2|\cdots|J_q;J}a_{I_1|I_2|\cdots|I_p;I}\otimes\phi^{J_1|J_2|\cdots|J_q;J}, \tag{6.3}$$

where, as before, $\prod'$ is the product of nonzero k_s's and l_s's and where I_s runs over all elements of $[\dim V_{\underline{o}}]^{\times k_s}$ and J_s runs over all elements of $[\dim V_{\underline{o}}]^{\times l_s}$. The closed multi-index I runs over all elements of $[\dim V_{\underline{c}}]^{\times m}$, similarly J runs over all elements of $[\dim V_{\underline{c}}]^{\times n}$. Also, we included the open and closed differentials into L as elements corresponding to the cylinder with one input and one output and to sphere with one input and one output, respectively.

Theorem 190. Algebra over the cobar complex $\mathcal{COCF}$ is described by degree one element L (6.3) of $T^{\text{cyc}}(V_{\underline{o}})\otimes S(V_{\underline{c}})\otimes T^{\text{cyc}}(V_{\underline{o}}^*)\otimes S(V_{\underline{c}}^*)$, such that $L\circ L=0$.

Finally, remarks 187 and 189 apply correspondingly.

6.3 HPL for IB-homotopy algebras

In (2.18) we introduced

$$\mathfrak{P}_V(m,n,\chi):={}^{\Sigma_m}\left(\mathcal{P}([m],[n],\chi)\otimes\mathcal{E}_V([m],[n],\chi)\right)^{\Sigma_n}$$
$$\mathfrak{P}_V:=\prod_{\substack{n\geq 0,m\geq 0\\ \chi>0}}\mathfrak{P}(m,n,\chi)$$

We can recognize a space similar to $\mathrm{Fun}(\mathcal{P},V)$ introduced for modular operads. Since this space contains the element L such that

$$d(L)+L\circ L=0$$

we can construct a perturbed differential

$$(d+L\circ)^2X=d^2X+d(L\circ X)+L\circ dX+L\circ(L\circ X)=$$
$$=dL\circ X+(-1)^{|L|}L\circ dX+L\circ dX+(L\circ L)\circ X=(dL+L\circ L)X=0$$

for homological perturbation lemma and construct SDR

$$K\circlearrowright(\mathfrak{P}_V,d)\underset{I}{\overset{P}{\rightleftarrows}}(\mathfrak{P}_{H(V))},0) \tag{6.4}$$

However, the construction of "effective action" as in the case of modular is not yet possible since the exponential of the element is not well defined.

Bibliography

[1] C. Albert, B. Bleile, J. Fröhlich. *Batalin–Vilkovisky Integrals in Finite Dimensions*. Journal of Mathematical Physics, vol. 51 (2010). arXiv: 0812.0464

[2] S. Barannikov. *Solving the Noncommutative Batalin-Vilkovisky Equation*. Letters in Mathematical Physics, Vol. 103.6 (2013). arXiv: 1004.2253

[3] S. Barannikov. *Modular operads and Batalin-Vilkovisky geometry*. International Mathematics Research Notices, Oxford University Press, Vol. 2007 (2006). arXiv: 1710.08442

[4] S. Blanes, F. Casas, J.A. Oteo, J. Ros. *The Magnus expansion and some of its applications*. Physics Reports, Vol. 470, no. 5–6 (2009). arXiv: 0810.5488

[5] R. Campos, S. Merkulov, T. Willwacher *The Frobenius properad is Koszul*. Duke Mathematical Journal, vol. 165 (2016). arXiv:1402.4048

[6] J. Chuang and A. Lazarev. *Abstract Hodge Decomposition and Minimal Models for Cyclic Algebras*. Letters in Mathematical Physics, 89.1 (2009). arXiv: 0810.2393.

[7] K. Cieliebak, K. Fukaya, J. Latschev. *Homological algebra related to surfaces with boundaries*. (2015). arXiv:1508.02741v3

[8] K. Costello. *Renormalization and Effective Field Theory*. Mathematical surveys and monographs, American Mathematical Society, (2011). ISBN: 978-0821852880

[9] K. Costello. *Topological conformal field theories and gauge theories*. Geomtery & Topology, Vol. 11, (2006). arXiv: math/0605647v5

[10] K. Costello *Factorization Algebras in Quantum Field Theory: Volume 1*. Cambridge University Press (2016). ISBN: 978-1-107-16310-2

[11] M. Doubek, B. Jurčo, K. Münster. *Modular operads and the quantum open-closed homotopy algebra*. Journal of High Energy Physics, Vol. 2015, No. 12 (2015). arXiv: 1308.3223v2

[12] M. Doubek, B. Jurčo, J. Pulmann. *Quantum L_∞ Algebras and the Homological Perturbation Lemma*. Communications in Mathematical Physics, Vol 367 (2017). arXiv: 1712.02696

[13] G. C. Drummond-Cole, J. Terilla, T. Tradler. *Algebras over Cobar(coFrob)*. J. Homotopy Relat. Struct., Vol. 5 Issue 1 (2010) arXiv: 0807.1241

[14] S. Eilenberg, S. Mac Lane. *On the Groups $H(\Pi, n)$*. Annals of Mathematics Second Series, Vol. 58, No. 1 (1953).

[15] H. Erbin, C. Maccaferri, M. Schnabl, J. Vošmera. *Classical algebraic structures in string theory effective actions.* arXiv:2006.16270

[16] D. Fiorenza *An introduction to the Batalin-Vilkovisky formalism.* arXiv: math/0402057v2

[17] E. Getzler, M. M. Kapranov. *Modular operads.* Compositio Mathematica, Vol. 110 (1998). arXiv: dg-ga/9408003

[18] V. Ginzburg, M.M. Kapranov. *Koszul duality for operads.* Duke Math. J 76 (1994). arXiv: 0709.1228v1

[19] Ana González. *The Lie bialgebra structure of the vector space of cyclic words.* Boletin de la Sociedad Matematica Mexicana,vol. 2 (2016). arXiv:1111.4709

[20] F. Guillén Santos, V. Navarro, P. Pascual and A. Roig. *Moduli spaces and formal operads*, Duke Mathematical Journal, Vol. 129, No. 2 (2005).arXiv: math/0402098

[21] P. Hackney, M. Robertson, D. Yau. *Infinity Properads and Infinity Wheeled Properads.* Lecture Notes in Mathematics vol. 2147, Springer International Publishing, (2015). ISBN: 978-3-319-20547-2

[22] P. Hilger, N. Poncin. *Lectures on Algebraic Operads.* Course notes, University of Luxembourg (2011). https://orbilu.uni.lu/bitstream/10993/14381/1/LecturesAlgebraicOperads.pdf

[23] H. Kajiura. *Noncommutative homotopy algebras associated with open strings.* Reviews in Mathematical Physics, Vol. 19 (2007) arXiv:math/0306332

[24] H. Kajiura, J. Stasheff. *Homotopy algebras inspired by classical open-closed string field theory.* Commun. Math. Phys. 263 (2006). arXiv:math/0410291

[25] H. Kajiura, J. Stasheff. *Open-closed homotopy algebra in mathematical physics.* Journal of Mathematical Physics, vol. 47 (2006). arXiv:hep-th/0510118

[26] R. M. Kaufmann, B. C. Ward, J. J. Zuniga. *The odd origin of Gerstenhaber brackets, Batalin-Vilkovisky operators, and master equations.* J. of Mathematical Physics, Vol. 56 Issue 10 (2015). arXiv: 1208.5543

[27] M. Kontsevich. *Deformation quantization of Poisson manifolds, I* Letters in Mathematical Physics, vol. 66 (2003). math.QA/9709040.

[28] M. Kontsevich, Y. Manin *Gromov-Witten classes, quantum cohomology, and enumerative geometry.* Commun.Math. Phys. 164 (1994). arXiv: hep-th/9402147

[29] T. Leinster. *Homotopy Algebras for Operads.* arXiv: math/0002180

[30] J.-L. Loday, B. Vallette. *Algebraic operads.* Grundlehren Math Wiss, vol. 346. Springer, Heidelberg (2012). ISBN: 978-3-642-30362-3

[31] M. Markl. *Modular envelopes, OSFT and nonsymmetric (non-Σ) modular operads.* J. of Noncommutative Geom., vol. 10 (2016). arXiv: 1410.3414

[32] M. Markl. *Odd structures are odd.* Advances in Applied Clifford Algebras, Vol. 27 (2017). arXiv: 1603.03184

[33] M. Markl. *Operads and PROPs.* Handbook of Algebra, vol. 5, (2006). arXiv: math/0601129

[34] M. Markl, S. Merkulov, S. Shadrin. *Wheeled PROPs, graph complexes and the master equation.* Journal of Pure and Applied Algebra, vol. 213, (2009). arXiv: math/0610683

[35] M. Markl, S. Shnider, J. Stasheff. *Operads in algebra, topology and physics.* Vol. 96 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI (2002). ISBN: 978-0-8218-4362-8

[36] M. Markl, A. A. Voronov. *The MV formalism for IBL_∞- and BV_∞-algebras.* (2015). arXiv: 1511.01591.

[37] N. Masuda. *Bar-Cobar and Koszul duality for algebraic Operads.* (2019).

[38] K. Münster, I. Sachs. *On Homotopy Algebras and Quantum String Field Theory.* Miskolc Mathematical Notes, vol. 14 (2013). arXiv: 1303.3444

[39] K. Münster, I. Sachs. *Quantum Open-Closed Homotopy Algebra and String Field Theory.* Comm. Math. Phys., vol. 321 Issue 3 (2013) arXiv: 1109.4101

[40] A. Schwarz. *Grassmannian and String Theory.* Communications in Mathematical Physics, Vol. 199 (1998). arXiv: hep-th/9610122

[41] B. Vallette. *A Koszul duality for props.* Trans. Amer. Math. Soc. 359, no. 10 (2007). arXiv:math/0411542

[42] B. Vallette. *Algebra + Homotopy = Operad.* Math. Sci. Res. Inst. Publ., vol. 62 (2012). arXiv:1202.3245

[43] B. C. Ward. *Massey products for graph homology.* (2019). arXiv:1903.12055

[44] B. Zwiebach. *Closed string field theory: Quantum action and the B-V master equation.* Nucl. Phys. B, vol. 390, 33–152 (1993). arXiv:hep-th/9206084v1

[45] B. Zwiebach. *Oriented open-closed string theory revisited.* Annals Phys.267, 193–248(1998). arXiv:hep-th/9705241v1

[46] `https://mathoverflow.net/questions/105835/koszul-duality-for-modular-operads/178911#178911`

List of Figures

List of publications

i. L. Peksová, *Properads and homological differential operators related to surfaces*, Archivum Mathematicum : Masaryk University, 2018. p. 299 – 312.

ii. M. Doubek, B. Jurčo, L. Peksová, *Properads and Homotopy Algebras Related to Surfaces*, arXiv:1708.01195.

iii. L. Peksová, *Modular operads with connected sum and Barannikov's theory*, Archivum Mathematicum : Masaryk University, 2020. p. 287 – 300.

iv. M. Doubek, B. Jurčo, L. Peksová, J. Pulmann *Quantum Homotopy Algebras and Homological Perturbation Lemma*, in preparation.